Lecture 8:
# Advanced Binary Trees

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Our Roadmap

- Priority Queue (binary heap)

  - Min-heap insert / delete-min

- Binary Heaps in Dynamic Arrays

  - $O(n)$ time to build min-heap

- Binary Search Tree (BST)

  - BST operators

  - Balanced BST (AVL-tree)

Priority Queue

◈ A priority queue stores a set S of n integers and supports the following operations:

  ◈ *Insert(e)*: adds a new integer to S

  ◈ *Delete-min:* removes the smallest integer in S, and returns it.

◈ Priority Queue applications:

  ◈ Artificial intelligence (A* algorithm)

  ◈ Operating systems (load balancing)

  ◈ Graph searching (Shortest path algorithms)
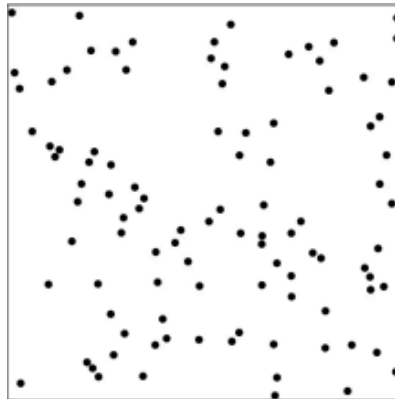
Priority Queue Example

- Suppose that the following integers are inserted into an initially empty priority queue
  - S={93, 39, 1, 26, 8, 23, 79, 54}
  - Perform Delete-min, the operation returns 1, and S ={93, 39, 26, 8, 23, 79, 54}
  - Perform Delete-min, the operation returns 8, and S ={93, 39, 26, 23, 79, 54}
  - Perform …..
- Unlike an ordinary queue (FIFO), a priority queue guarantees that the elements always leave in ascending order (or descending order with *Delete-max*), regardless of the order by which they are inserted.

Priority Queue Implementation

◈ We will implement a priority queue using a data structure called the "binary heap" to achieve the following guarantees:

  ◈ O(n) space consumption

  ◈ O(log n) insertion time

  ◈ O(log n) delete-min time

◈ The binary heap data structure is an array object that we can view as a complete binary tree.

  ◈ Level 0 to h-1 are full

  ◈ Leaf nodes in level h are "as far left as possible"

Binary Heap

- Let S be a set of n integers. A binary heap on S is a binary tree T satisfying:
  - (1) T is complete
  - (2) Every node u in T corresponds to a distinct integer in S, the integer is called the key of u (and is stored at u)
  - (3) If u is an internal node, the key of u is smaller than those of its child nodes
- Note that:
  - Condition 2 implies that T has n nodes
  - Condition 3 implies that the key of u is the smallest in the subtree of u

Binary Heap Example

◈ Two possible binary heaps on S = {93, 39, 1, 26, 8, 23, 79, 54}:
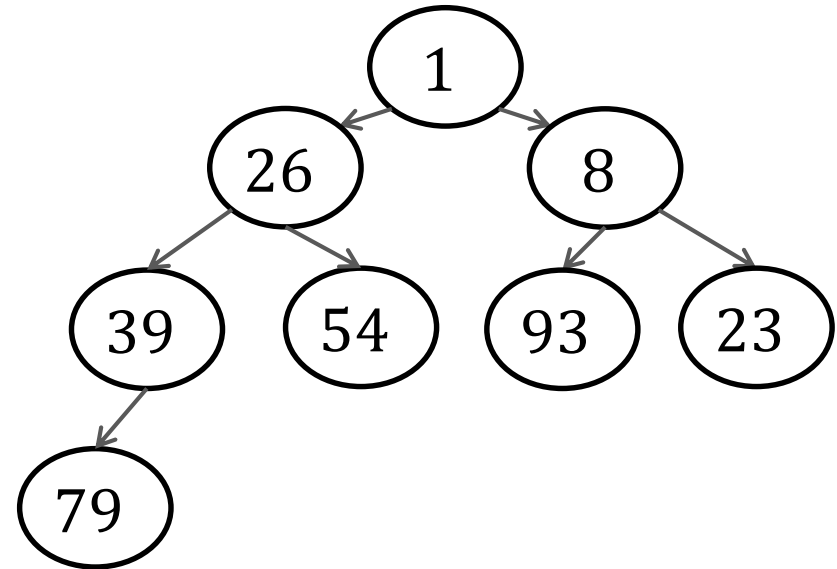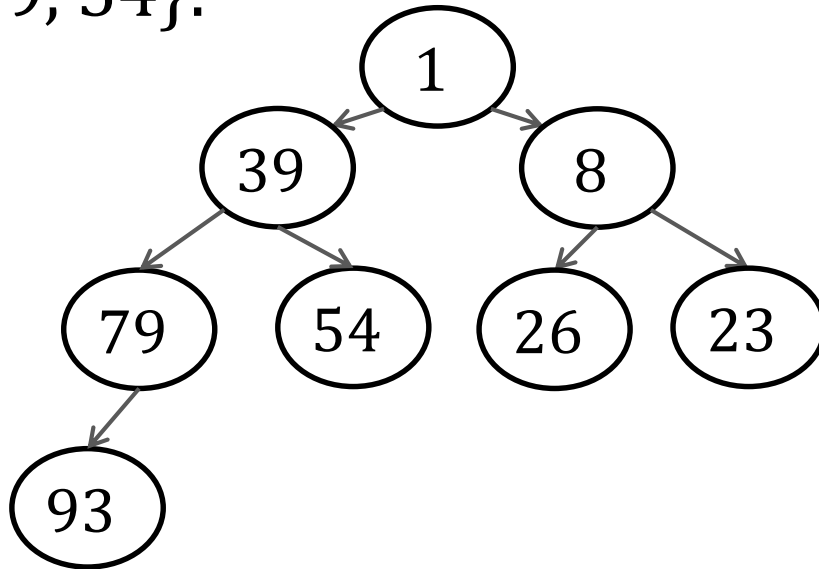
◈ The binary heaps of a set S is not unique.

◈ The smallest integer of S must be the key of the root.

Binary Heap Insertion

◈ We perform insert(e) on a binary heap T as follows:

  ◈ Step 1: Create a leaf node z with key e, while ensuring that T is a complete binary tree, it means there is only one place where z could be added.

  ◈ Step 2: Set u ← z

  ◈ Step 3: If u is the root, return.

  ◈ Step 4: If the key of u > the key of its parent p, return

  ◈ Step 5: Otherwise, swap the keys of u and p. Set u ← p, and repeat from Step 3.
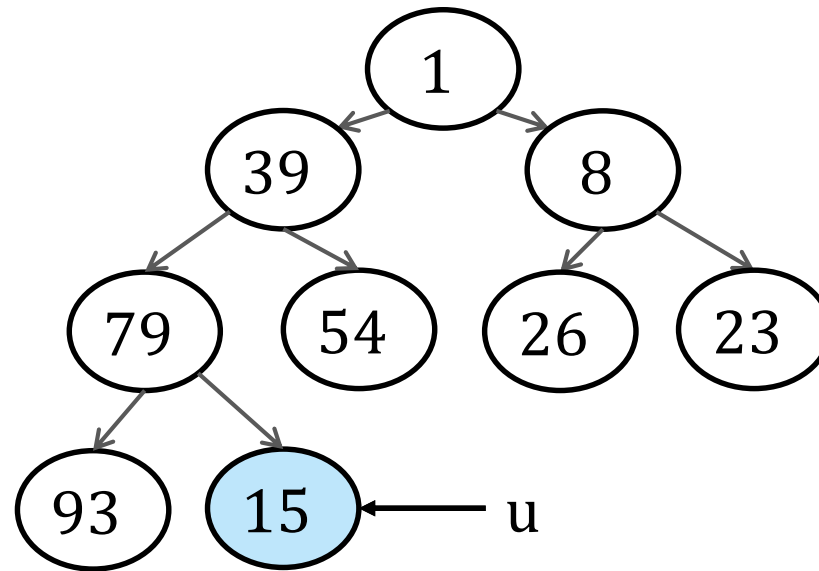
Binary Heap Insertion

◈ Suppose we want to insert 15 into the binary heap below:
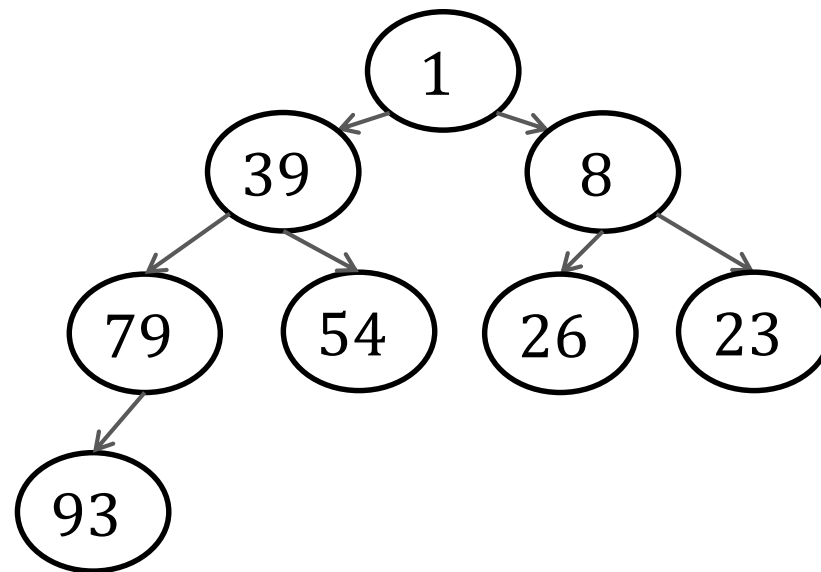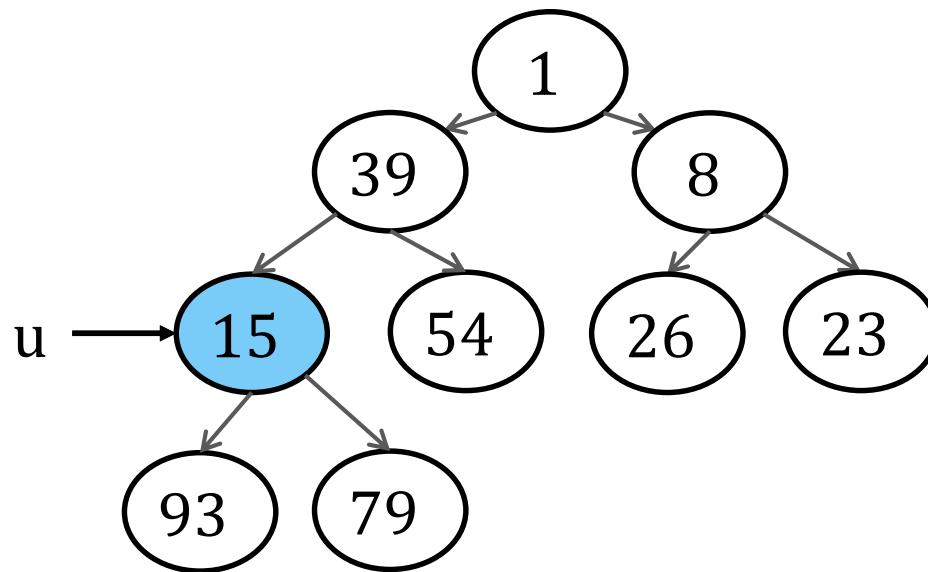
Binary Heap Insertion

◈ First add 15 as a new leaf, making sure that we still have a complete binary tree.

◈ Step 3 is not true, go to Step 4,

◈ Step 4 is not true, go to step 5.
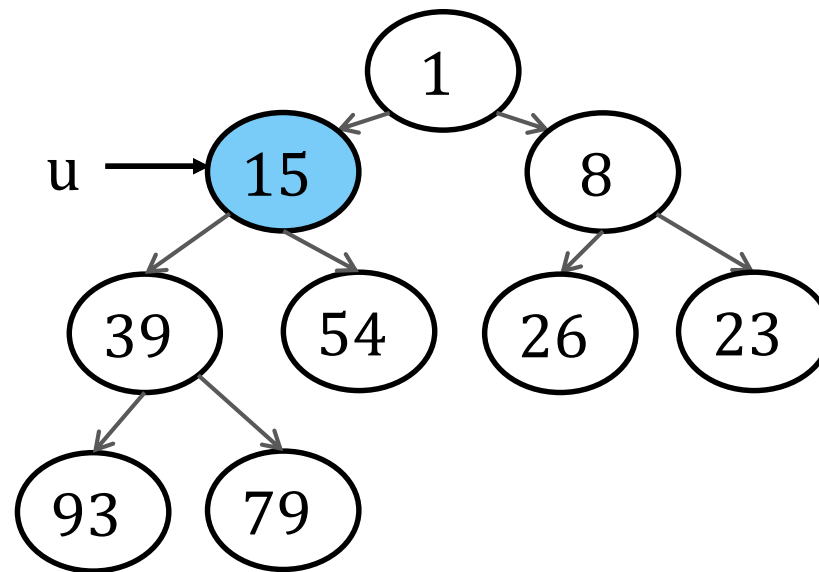
Binary Heap Insertion

◈ First add 15 as a new leaf, making sure that we still have a complete binary tree.

◈ Swap the keys of u and its parent p

◈ Set u ⭠ p, go back to Step 3)

◈ Step 3 and Step 4 are not true, go to Step 5.

Binary Heap Insertion

◈ First add 15 as a new leaf, making sure that we still have a complete binary tree.

◈ Swap the keys of u and p

◈ Set u ← p, go back to Step 3)

◈ Step 3 is not true, Step 4 is true, return. Insertion complete.

Binary Heap Delete-min

◈ We perform delete-min on a binary heap T as follows:

   ◈ Step 1: Report the key of the root

   ◈ Step 2: Identify the rightmost leaf z at the bottom level of T

   ◈ Step 3: Delete z, and store the key of z at the root

   ◈ Step 4: Set u ← the root

   ◈ Step 5: If u is leaf, return

   ◈ Step 6: If the key of u < the keys of the children of u, return

   ◈ Step 7: Otherwise, let v be the child of u with a smaller key
   Swap the keys of u and v. Set u ← v, and repeat from Step 5
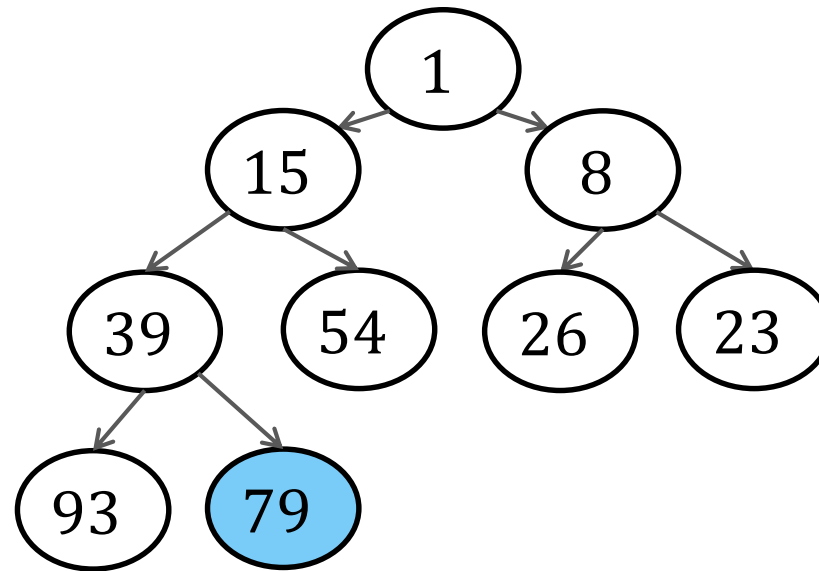
Binary Heap Delete-min

◈ Assume that we perform a delete-min from the binary heap below:
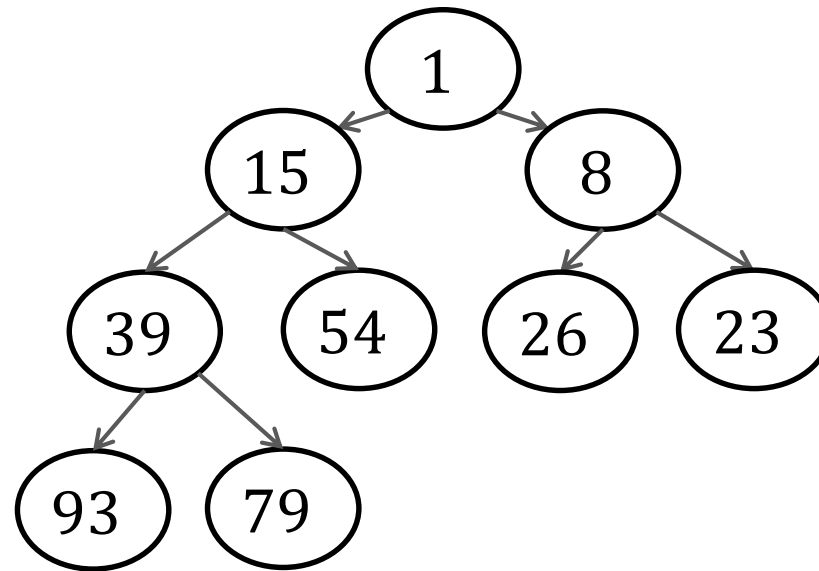
◈ Delete-min delete root node, and we should maintain the rest nodes as a complete binary tree.
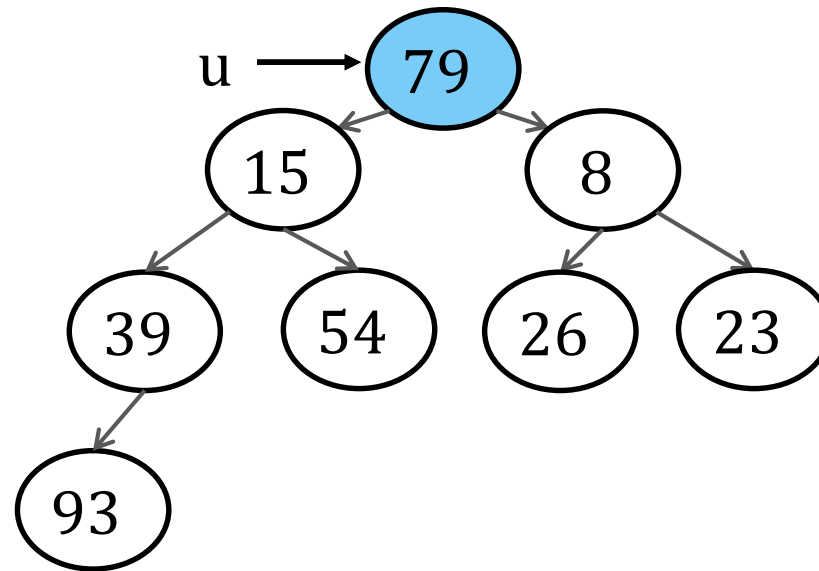
Binary Heap Delete-min

◈ Frist, find the rightmost leaf at the bottom level, it is node with key 79.

◈ Note that the tree is still a complete binary tree after removing this leaf.

Binary Heap Delete-min

◈ Remove the leaf, but place the key value 79 in the root.

u ⟶ 79
  15   8
39 54 26 23
93

◈ Step 4: set u as the root.

◈ Step 5 and 6 are not true,

◈ Go to Step 7.

Binary Heap Delete-min
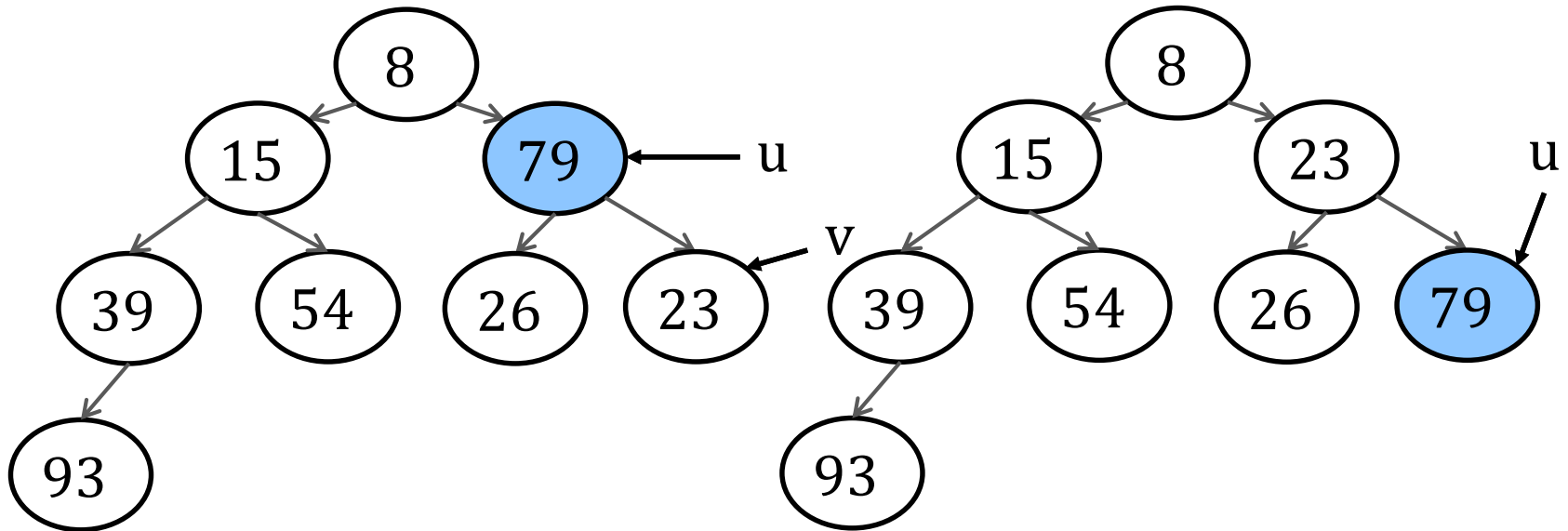
- Let v be the child of u with a smaller key.
- Swap the keys of u and v, and set u ← v
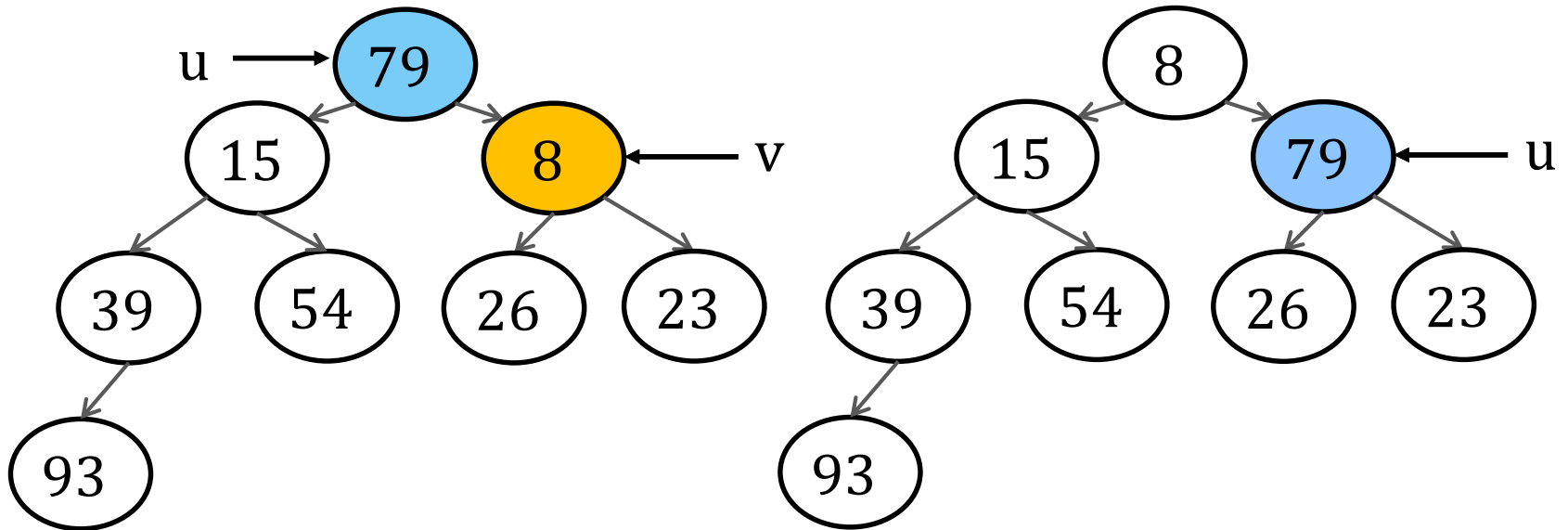
- Go to Step 5
- Step 5 and Step 6 are not true, go to Step 7

Binary Heap Delete-min

◈ Let v be the child of u with a smaller key.

◈ Swap the keys of u and v, and set u ← v

◈ Go to Step 5

◈ Step 5 is true, return. Delete-min complete.

How to find rightmost leaf?

- Before we analyzing the time complexity of insert and delete-min, let us first consider a sub-problem:

- Given a complete binary tree T with n nodes, how to identify quickly the rightmost leaf node at the bottom level of T (i.e., colored node in below tree).

  - It is Step 1 in insert algorithm, and Step 2 in delete-min algorithm
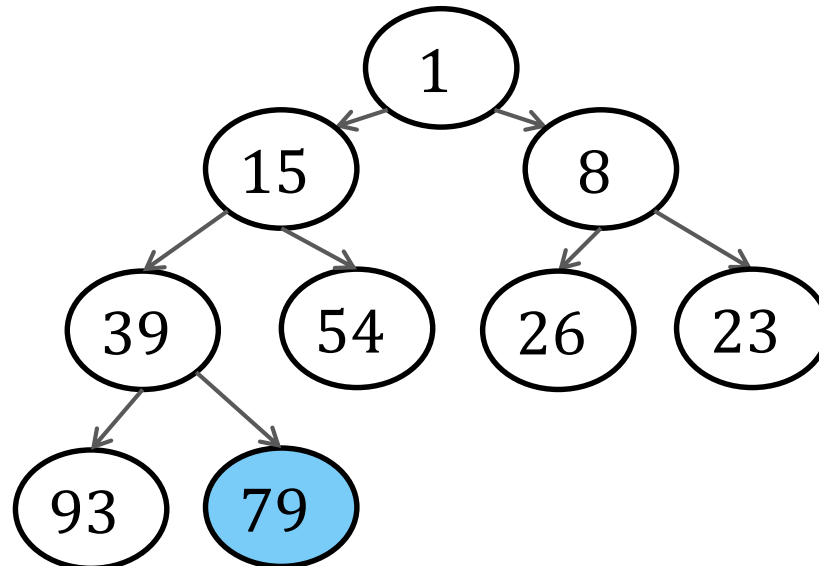
How to find rightmost leaf?

◈ We give a clever algorithm for solving the sub-problem in O(log n) time.

◈ First, write the value n in binary form. Recall the Part II question 1, in mid-term exam paper. We can do that in O(log n) time.

◈ Skip the most significant bit. We will scan the remaining bits from left to right, start from root,

  ◈ If the bit is 0, we go to the left child of the current node

  ◈ Otherwise, go to right child

Find Rightmost Leaf Example
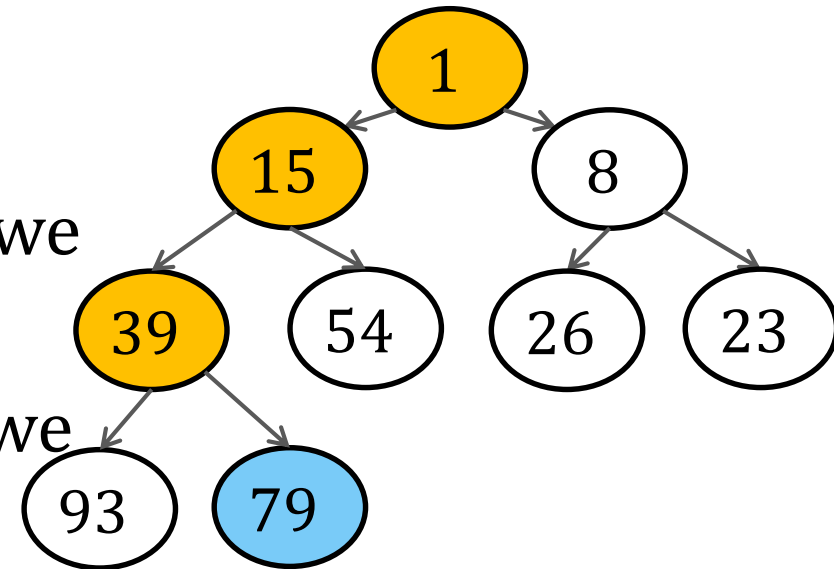
- Here n = 9, binary form: 1001
- Skip the first bit '1'
- We scan the remaining bits
- Start from root node 1.
- The 2nd leftmost bit is 0, so we visit left, and go to node 15
- The 3rd leftmost bit is 0, so we visit left, and go to node 39
- The 4th leftmost bit is 1, so we turn right, and go to node 79 (done).

Time Complexity Analysis

◈ We are now ready to prove that our insertion and delete-min algorithms finish in O(log n) time.

◈ It suffices to point out the key facts:

  ◈ Step 1 of the insertion algorithm (page 8) and Step 2 of the delete-min algorithm (page 13) can be performed in O(log n) time, using our solution to previous sub-problem

  ◈ The rest of insertion ascends a root-to-leaf path, while that of delete-min descends a root-to-leaf path. The time is O(log n) in both cases.

◈ Thus, we guarantee: (1) O(n) space consumption, (2) O(log n) insertion / delete-min operations.

Our Roadmap

◈ Priority Queue (binary heap)

  ◈ Min-heap insert / delete-min

◈ Binary Heaps in Dynamic Arrays

  ◈ O(n) time to build min-heap

◈ Binary Search Tree (BST)

  ◈ BST operators

  ◈ Balanced BST (AVL-tree)

Binary Heaps in Dynamic Arrays

◈ We have already learned that the binary heap serves as an efficient implementation of a priority queue. Our previous discussion was based on pointers (for getting a parent node connected with its children). In this lecture, we will see a "pointerless" way to implement a binary heap, which in practice achieves much lower space consumption

◈ We will also see a way to build a heap from n integers in just $O(n)$ time, improving the obvious $O(n \log n)$ bound.

Recall

- A **priority queue** stores a set S of n integers and supports the following operations:
  - *Insert(e)*: adds a new integer to S
  - *Delete-min:* removes the smallest integer in S, and returns it.

- Let S be a set of n integers. A **binary heap** on S is a binary tree T satisfying:
  - (1) T is complete
  - (2) Every node u in T corresponds to a distinct integer in S, the integer is called the key of u (and is stored at u)
  - (3) If u is an internal node, the key of u is smaller than those of its child nodes

Storing a Complete Binary Tree

◈ Storing a complete binary tree using an array

◈ Let T be any complete binary tree with n nodes, let us linearize the nodes in the following manner:

  ◈ Put nodes at a higher level before those at a lower level

  ◈ Within the same level, order the nodes from left to right

◈ Let us store the linearized sequence of nodes in an array A of length n. Example:
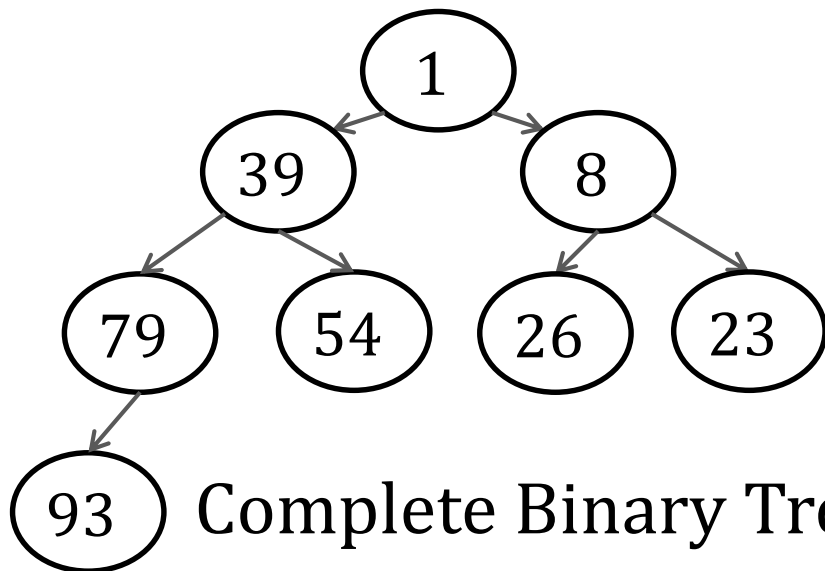
1	39	8	79	54	26	23	93

Storing in an array

Complete Binary Tree

Property 1

◈ Let us refer to the i-th element of A as A[i], for simplicity, we assume the index of A starts from 1.

◈ Lemma: Suppose that node u of T is stored at A[i]. Then, the left child of u is stored at A[2i], and the right child at A[2i+1].

◈ Observe this from the example of the previous slide

◈ Proof leaves as your homework.

◈ Hints, consider the number of nodes after u, but before its left child.

More Properties

◈ The following is an immediate corollary of the previous lemma:

◈ Corollary: Suppose that node u of T is stored at A[i]. Then, the parent of u is stored at A[⌊i/2⌋].

◈ The following is a simple yet useful fact:

◈ Lemma: the rightmost leaf node at the bottom level is stored at A[n].

◈ Now we have got everything we need to implement the insertion and delete-min algorithms on the array representation of a binary heap.

Insert 15

1	39	8	79	54	26	23	93

1	39	8	79	54	26	23	93	15

1	39	8	15	54	26	23	93	79

1	15	8	39	54	26	23	93	79

Delete-min

1	15	8	39	54	26	23	93	79

79	15	8	39	54	26	23	93

8	15	79	39	54	26	23	93

8	15	23	39	54	26	79	93

Performance Guarantees

◈ Combining our analysis on (i) binary heaps and (ii) dynamic arrays, we obtain the following guarantees on binary heap implemented with a dynamic array:

  ◈ Space consumption O(n)

  ◈ Insertion: O(log n) time amortized

  ◈ Delete-min: O(log n) time amortized

Build a binary heap in array

◈ Next we consider the problem of creating a binary heap on a set S of n integers. Obviously, we can do so in O(n log n) time by doing n insertions. However, this is an overall kill because the binary heap does not need to support any delete-min operations until all the n numbers have been inserted. This raises the question whether we can build the heap faster/

◈ The answer is positive: we will see an algorithm that does so in O(n) time.

Root-fix operator

- We are given a complete binary tree T with root r. It guaranteed that:
  - The left subtree of r is binary heap
  - The right subtree of r is a binary heap
  - However, the key of r may not be smaller than the keys of its children.

- We define the root-fix operation, it fixes the issue and makes T a binary heap.

- Root-fix can be done in O(log n) time – in the same manner as the delete-min algorithm (step 4 - 7)

Root-fix Example
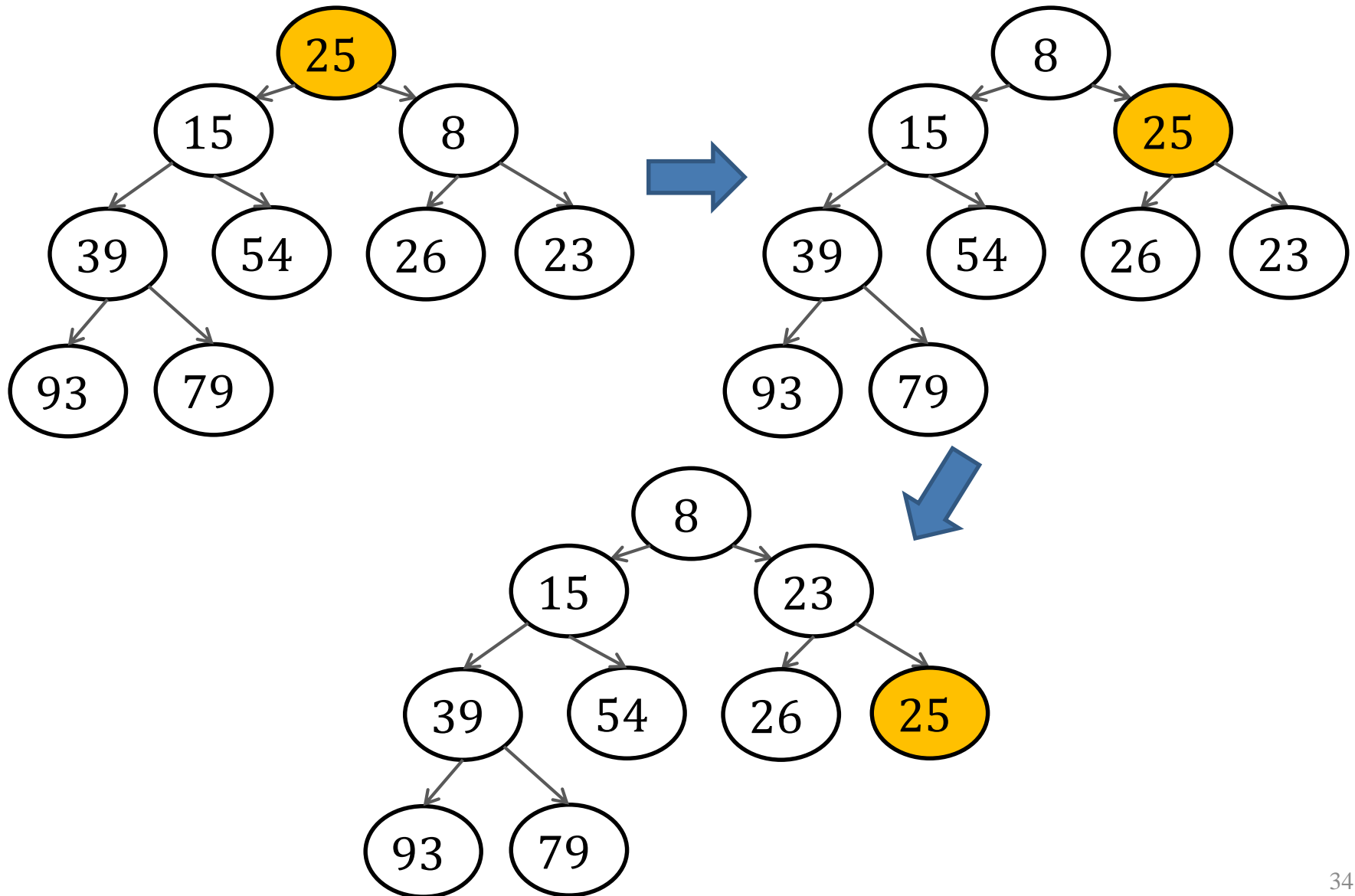
Building a Heap

◈ Create an array A that stores a set S of n integers, we can turn A into a binary heap on S using the following simple algorithm, which view A as a complete binary tree T:

◈ For each i=n downto 1:
  ◈ Perform root-fix on the subtree of T rooted at A[i]

◈ Think: why are the conditions of root-fix always satisfied?

Building a Heap example

54	26	15	93	8	1	23	**39**
54	26	15	93	8	1	**23**	39
54	26	15	93	8	**1**	23	39
54	26	15	93	**8**	1	23	39
54	26	15	**93**	8	1	23	39
54	26	**15**	39	8	1	23	93
54	**26**	1	39	8	15	23	**93**
54	8	1	39	26	**15**	**23**	93

Root-fix

54	26	15	39	8	1	23	93
54	26	1	39	8	15	23	93
56	8	1	39	26	15	23	93
1	8	15	39	26	54	23	93

Complexity Analysis

◈ Lemma: The time complexity of turn array A into a binary heap on S is O(n).

◈ Proof as follows:

  ◈ view A as a complete binary tree

  ◈ The height of T is h.

  ◈ Without loss of generality, assume that all the levels of T are full, i.e., $n=2^{h+1}-1$.

    ◆ Why no generality is lost?

  ◈ Analyze the total running time of Build heap algorithm

  ◈ Proof that $\sum_{i=0}^{h} O\left(i * 2^{h-i}\right) = O(n)$ with $n=2^{h+1}-1$.

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◈ Binary Search Tree (BST)

  ◈ BST operators

  ◈ Balanced BST (AVL-tree)

Binary Search Tree (BST)

Binary Search Tree (especially, balanced BST) is the most powerful data structure of this course. This is without a doubt one of the most important data structures in computer science.

In extreme case, BST is equivalent to a linked list, thus, we guarantee the operations performance of BST by study AVL-tree.

Dynamic Predecessor Search

◈ Let S be a set of integers. We want to store S in a data structure to support the following operations:

  ◈ A predecessor query: give an integer q, find its predecessor in S, which is the largest integer in S that does not exceed q.

  ◈ Insertion: adds a new integer to S

  ◈ Deletion: removes a integer from S

◈ Suppose that S={3,10,15,20,30,40,60,73,80}

  ◈ The predecessor of 23 is 20

  ◈ The predecessor of 15 is 15

  ◈ The predecessor of 2 does not exist

◈ Note that a predecessor query is more general than a "dictionary look-up". Why?

Binary Search Tree (BST)

- We will learn a version of the BST that guarantees:
  - O(n) space consumption
  - O(h) time per predecessor query (hence, also per dictionary lookup)
  - O(h) time per insertion
  - O(h) time per deletion
- where n = |S|, h is the height of BST, Note that all the above complexities hold in the worst case.

Binary Search Tree (BST)

- A BST on a set S of n integers in a binary tree T satisfying all the following requirements:
  - T has n nodes
  - Each node u in T stores a distinct integer in S, which is called the key of u
  - For every internal u, it holds that:
    - The key of u is larger than all the keys in the left subtree of u.
    - The key of u is smaller than all the keys in the right subtree of u.

BST Example

◇ Two possible BSTs on S = {3,10,15,20,30,40,60,73,80}
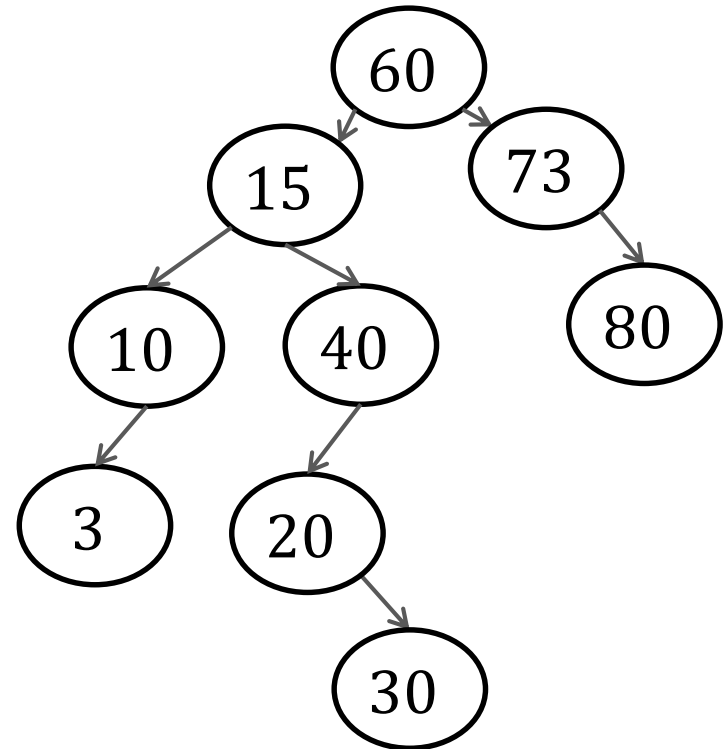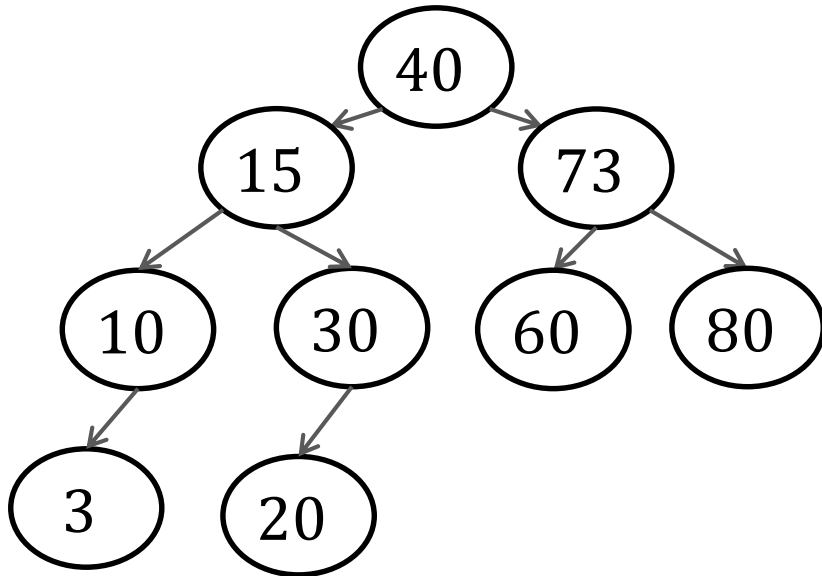
Predecessor Query

◈ Suppose that we have created a BST T on a set S of n integers. A predecessor query with search value q can be answered by descending a single root-to-leaf path:

  ◈ (1) Set p $\leftarrow -\infty$ (p will contain the final answer at the end)

  ◈ (2) Set u $\leftarrow$ the root of T

  ◈ (3) If u = nil, then return p

  ◈ (4) If key of u = q, the set p to q, and return p

  ◈ (5) If key of u > q, then set u to the left child (now u = nil if there is no left child), and repeat from Step (3)

  ◈ (6) Otherwise, set p to the key of u and u to the right child, and repeat from Step (3)

Predecessor Query Example

◈ Suppose that we want to find the predecessor of 35
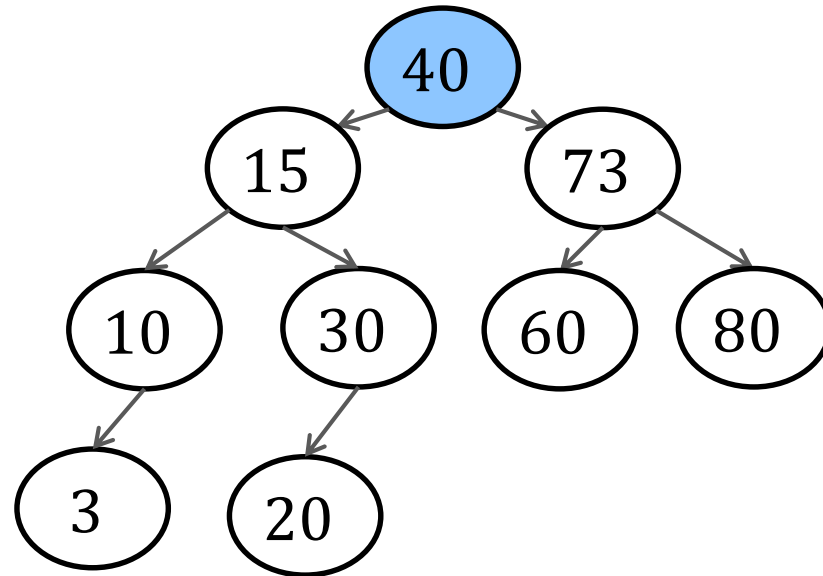
◈ Set p ⟵ −∞ , u = root 40

◈ (3) and (4) are not true, go to (5)

◈ Since 40>35, the predecessor cannot be in the right subtree of 40, so we move to the left child of 40, now u = node 15.

Predecessor Query Example

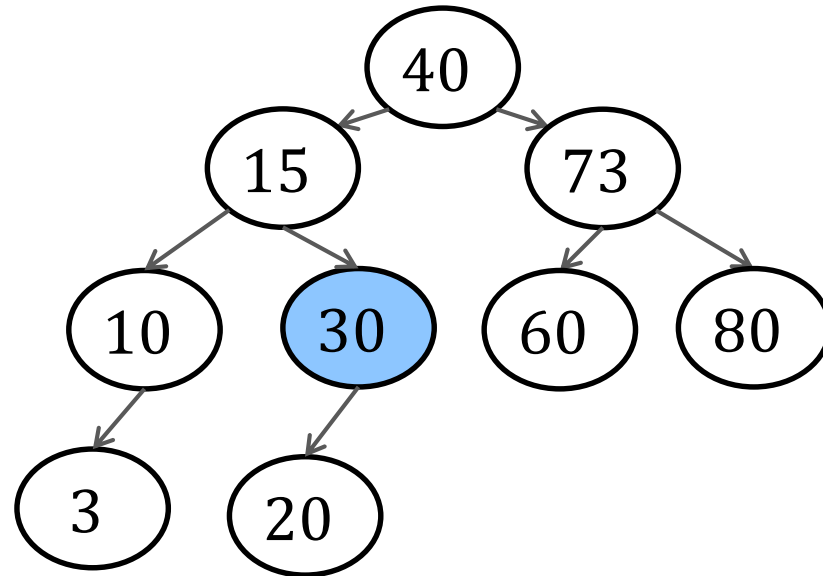◈ Suppose that we want to find the predecessor of 35
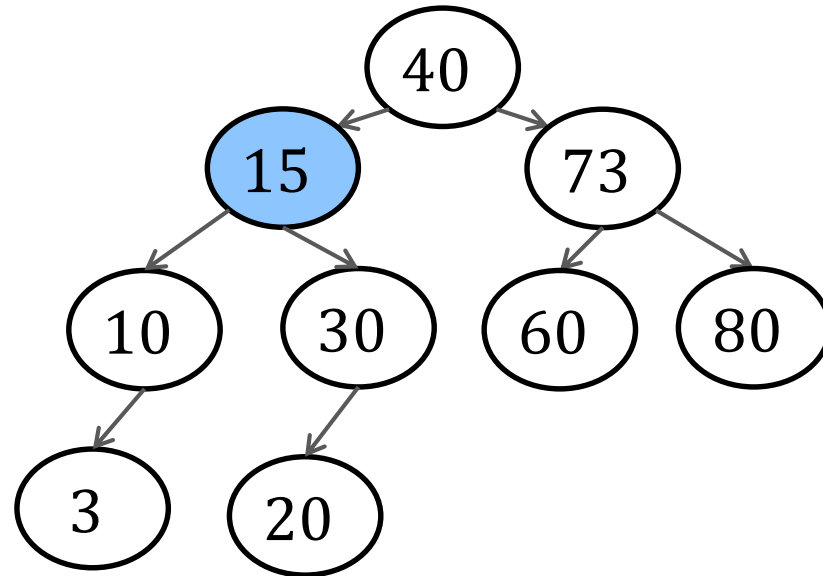
◈ (3), (4) and (5) are not true, go to (6)

◈ Since 15 < 35, p ← 15, since this is the predecessor of 35 so far.

◈ The predecessor cannot be in the left subtree of 15, so we move u to the right child, now u = node 30.

Predecessor Query Example

◈ Suppose that we want to find the predecessor of 35

◈ (3), (4) and (5) are not true, go to (6)

◈ Since 30 < 35, p ← 30, since this is the predecessor of 35 so far.

◈ The predecessor will be in the right subtree of 30, but 30 does not have a right child. So algorithm terminates here with p = 30 as the final answer.

Time complexity Analysis

◈ Obviously, we spend O(1) time at each node visited. Since the height of BST is h, therefore the total query time is O(h).

Successor Query

◈ The opposite of predecessors are successors.

◈ The successors of n integer q in S is the smallest integer in S that is no smaller than q.

◈ Suppose that S={3,10,15,20,30,40,60,73,80}

   ◈ The successor of 23 is 30

   ◈ The successor of 15 is 15

   ◈ The successor of 81 does not exist

◈ Given an integer q, a successor query returns the successor of q in S.

◈ By symmetry, we know from the earlier discussion (on predecessor queries) that a successor query can be answered using a BST in O(h) time.

BST Insertion

- Suppose that we need to insert a new integer e. First create a new leaf z storing the key e. This can be done by descending a root-to-leaf path:
  - 1. Set u ← the root of T
  - 2. If e < the key of u
    - 2.1 If u has a left child, then set u to the left child
    - 2.2 Otherwise, make z the left child of u, and done
  - 3. Otherwise:
    - 3.1 If u has a right child, then set u to the right child
    - 3.2 Otherwise, make z the right child of u, and done.
  - Repeat from Step 2.
- The total cost is proportional to the height of T, i.e., $O(h)$

BST Insertion Example

◈ Inserting 35:

◈ u is root 40, e < the key of u, u has a left child, u ← node 15

◈ u is node 15, e > the key of u u has a right child, u ← node 30

◈ u is node 30, e > the key of u, u's right child is nil, then set z as the right child of u. Done.
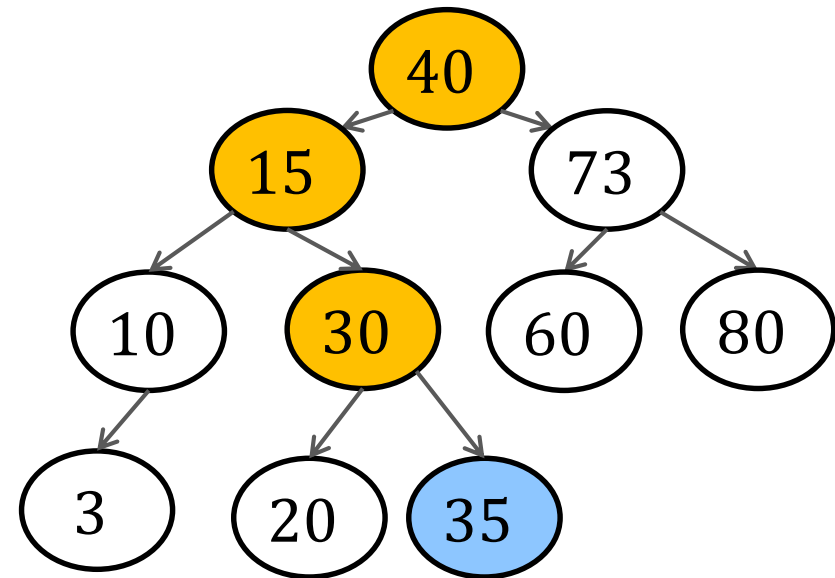
BST Deletion

◈ Suppose that we want to delete an integer e. Frist, find the node u whose key equals to e in O(h) time (through a predecessor query).

◈ Case 1: if u is a leaf node, simply remove it from T.

◈ Example: remove 60
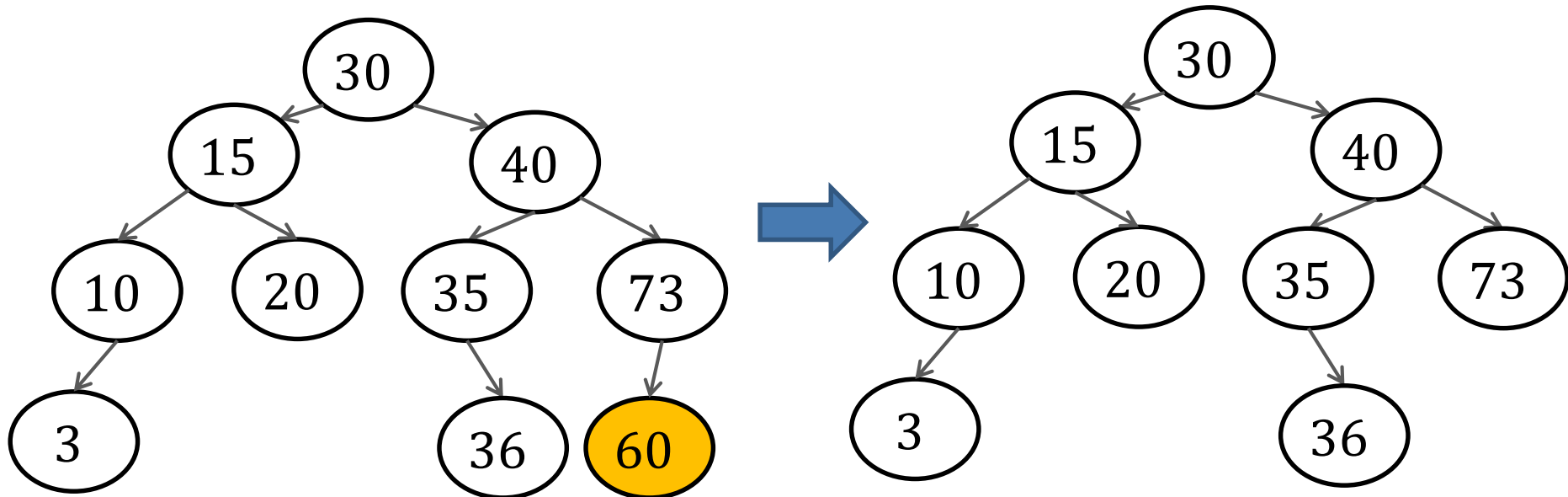
BST Deletion

- What happens if node u is not a leaf node?

- Case 2: if u has a right subtree:

  - Find the node v storing the successor s of e.

  - Set the key of u to s

  - Case 2.1: if v is a leaf node, them remove it from T

  - Case 2.2: otherwise, it must hold that v has a right child w, but not left child. Replace node v by subtree which rooted at w.

- Case 3: if u has no right subtree:

 - It must hold that u has a left child v, Replace node u by the subtree rooted at v.

Case 2.1 Example

◈ Delete 40:

◈ u has a right subtree, node v (60) is the successor of 40.

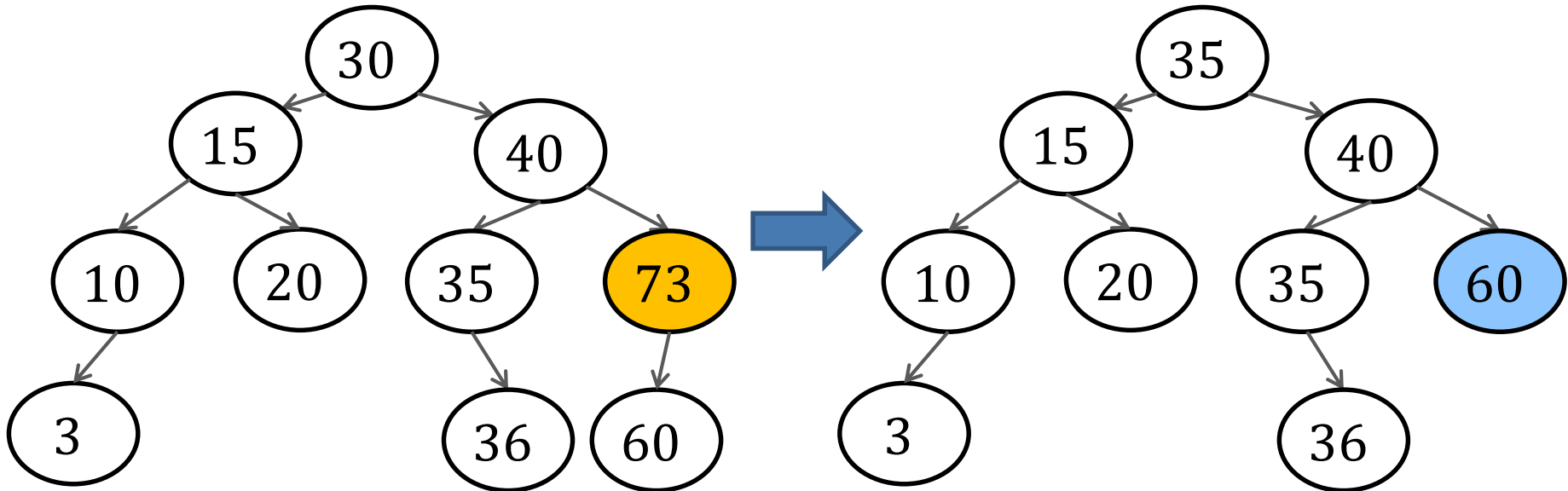◈ Set the key of u to 60

◈ v is a leaf, remove node v, done.
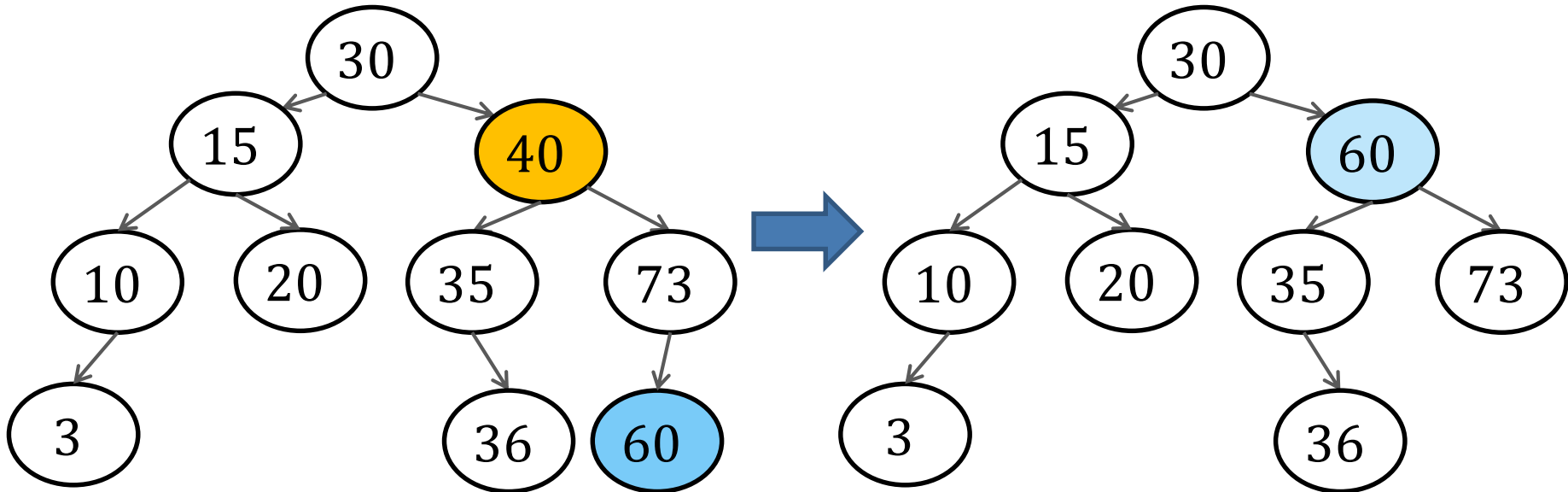
Case 2.2 Example

◈ Delete 30:

◈ u has a right subtree, node v (35) is the successor of 30.

◈ Set the key of u to 35

◈ v is not leaf node, it has right child w (36), replace node v by subtree rooted at w(36).
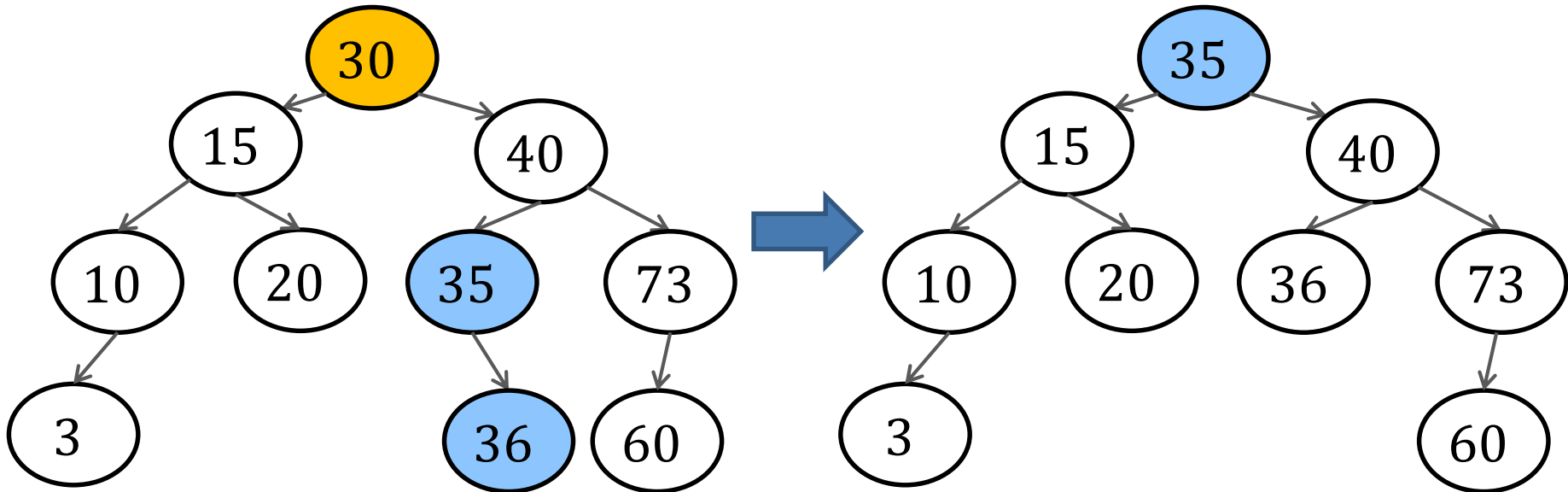
Case 3 Example

- Delete 73:

- u has no right subtree, and u must have a left child v (60), replace node u by node v(60).

- done.

BST Deletion

◈ In all above cases, we have essentially descended a root-to-leaf path (call it deletion path), and removed a leaf node.

◈ The cost so far is O(h), recall that the successor of an integer can be found in O(h) time.

◈ Given a set S of n integers, what is the maximum possible height of its BST?

⬦ h = n, why?

⬦ So what is the worst-case query cost? O(n)

⬦ However, we can guarantee h = O(log n) is the BST is balanced BST.

What is the height of tree

- Given a set S of n integers, what is the maximum possible height of its BST?
  - h = n, why?
- What is the worst-case query / insertion / deletion cost?
  - O(n) !!!
- How to achieve O(log n) time per operation?
  - Balanced Binary Search Tree

Balanced Binary Tree

◆ A binary tree T is balanced if the following holds on every internal node u of T:

  ◇ The height of the left subtree of u differs from that the right subtree of u by at most 1.

◆ If u violates the above requirement, we say that u is imbalanced.

Balanced
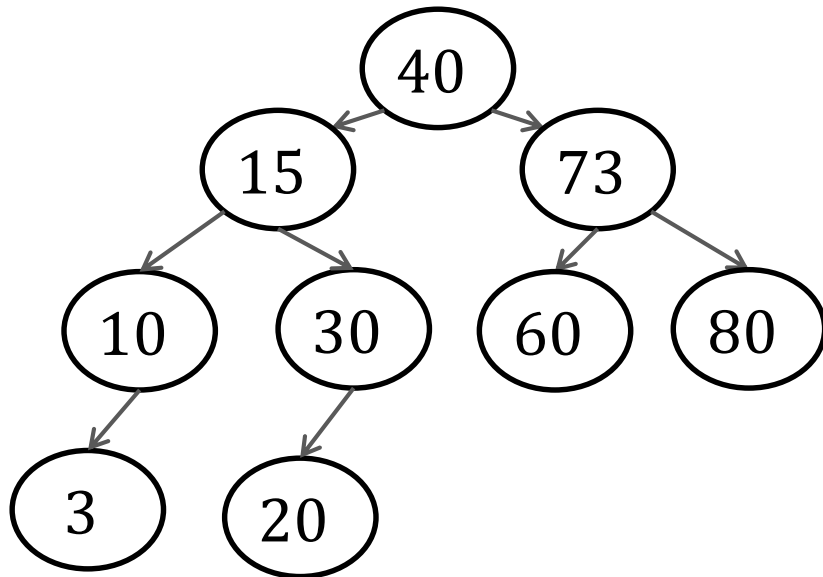
Not balanced
Node 60, 40
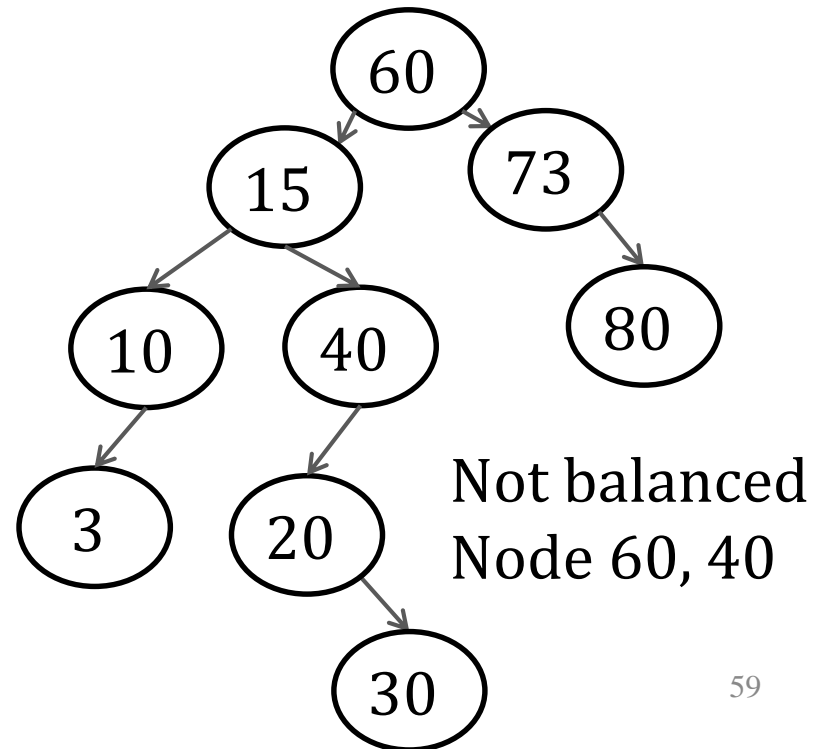
Height of a Balanced Binary Tree

- Theorem: a balanced binary tree with n nodes has height O(log n).

- Proof. (left as homework)

- Hints:
  - 1) consider minimum number of nodes in a balanced binary tree with height h
  - 2) recursive equation
  - 3) analysis two cases: case 1) h is even, case 2) h is odd.

- With the height of balanced binary tree is O(log n), we can conclude that the cost of query operation is O(log n) on a balanced binary search tree.

- How about the cost of insertion and deletion on it?

Balanced BST

◈ An AVL-tree on a set S of n integers is a balanced binary search tree T, where the following hold on every internal node u

   ◈ u stores the heights of its left and right subtrees.

◈ An AVL-tree on S = {3,10,15,20,30,40,60,73,80}

◈ For example, the number 3 and 2 near root 40 indicate that its left subtree has height 3, right subtree has height 2.

◈ By storing the subtree heights an internal node know whether it has become imbalanced
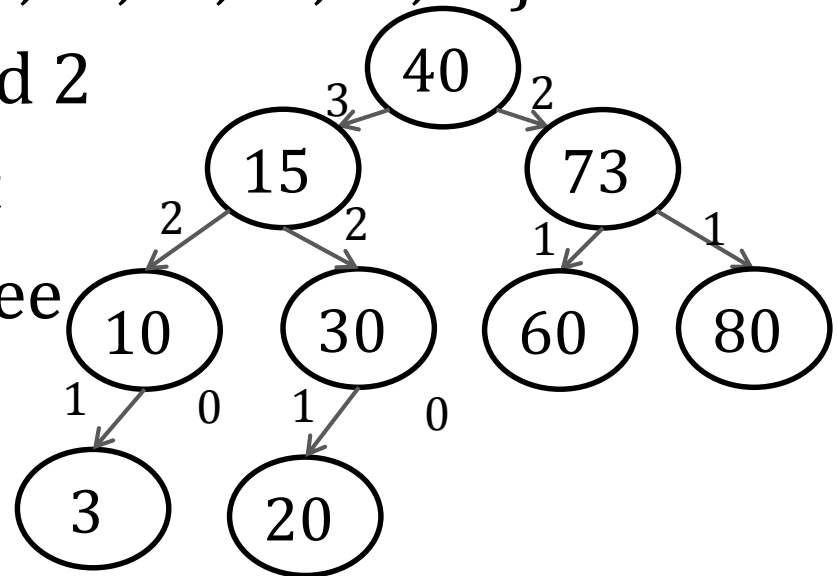
Balanced BST

- Next we will explain how to perform updates. The most important step is remedy a node u when it becomes imbalanced.

- It suffices to consider a scenario called 2-level imbalance. In this situation, two conditions apply:

    - There is a difference of 2 in the heights of the left and right subtree of u.

    - All the proper descendants of u are balanced

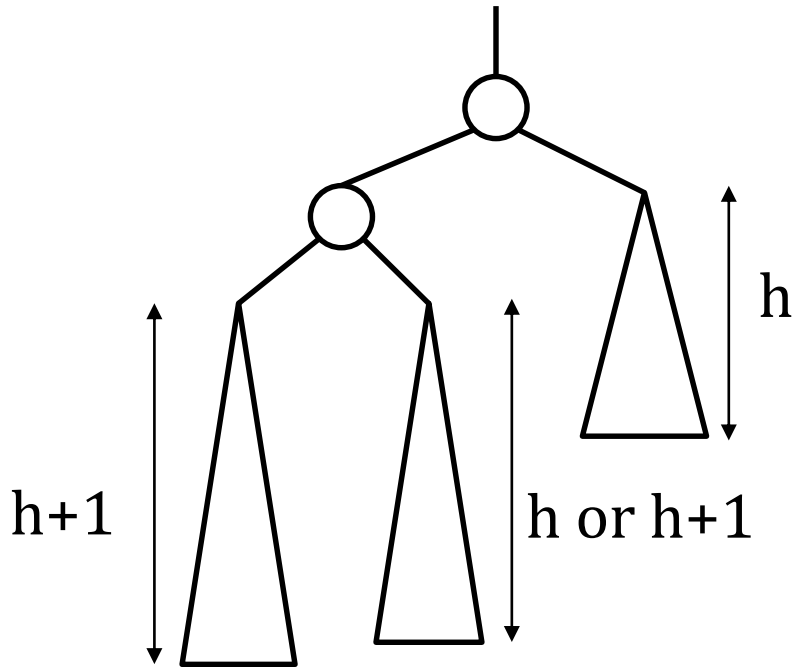- We will first explain how to rebalance u in the above situation
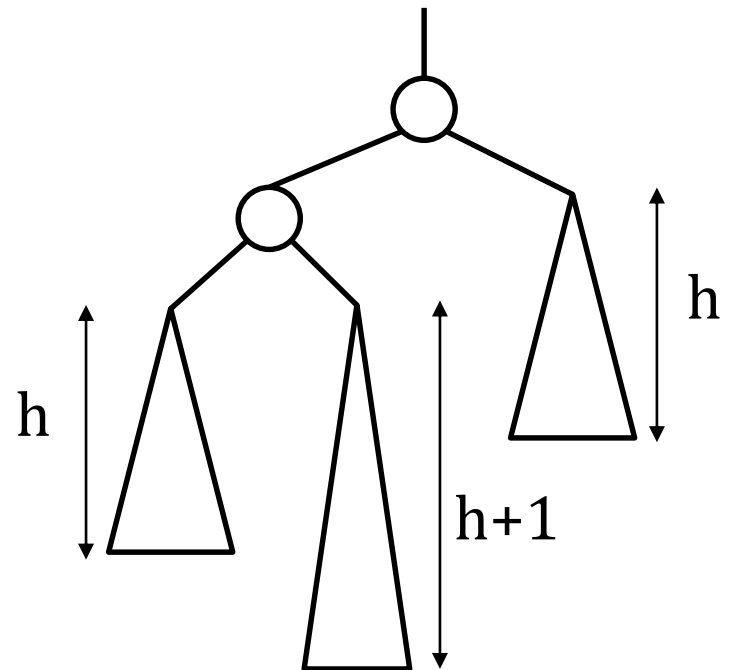
2-level imbalance
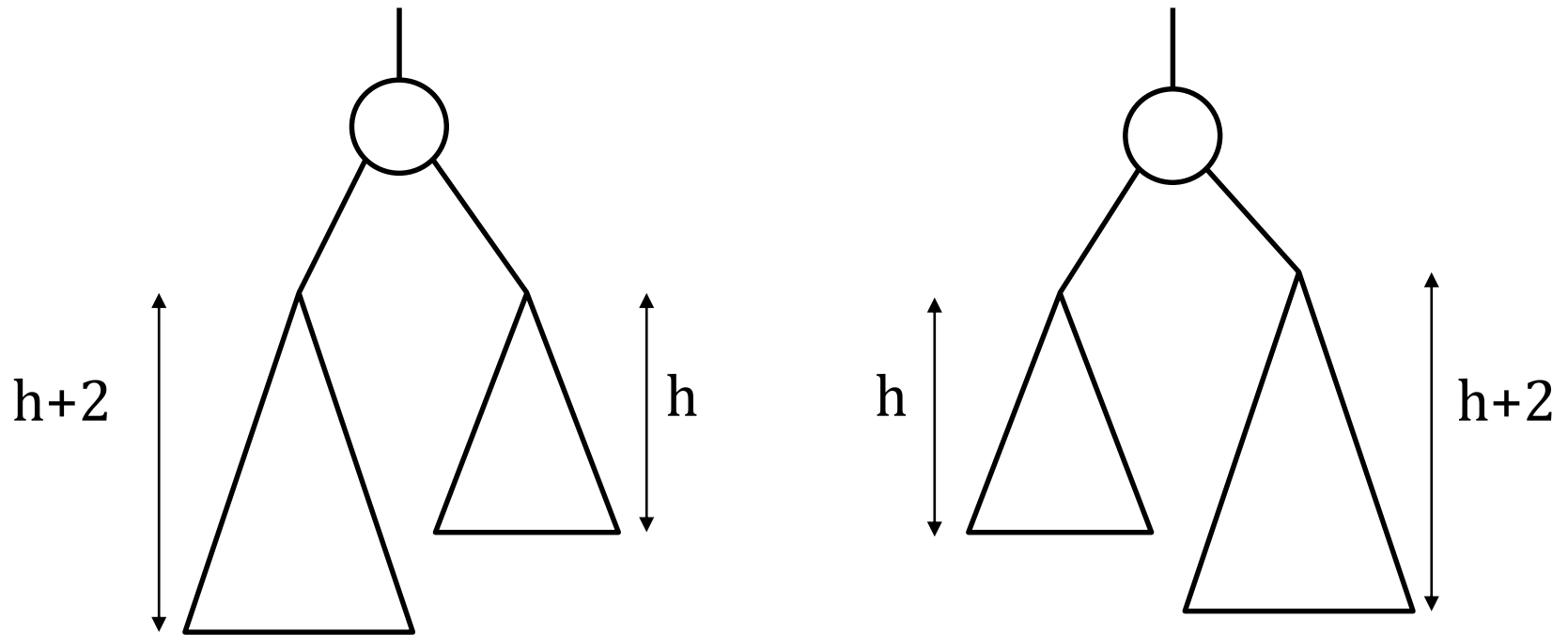
◈ There are two cases:

◈ Due to symmetry, it suffices to explain only the left case, which can be further divide to a left-left and a left-right case, as shown next.
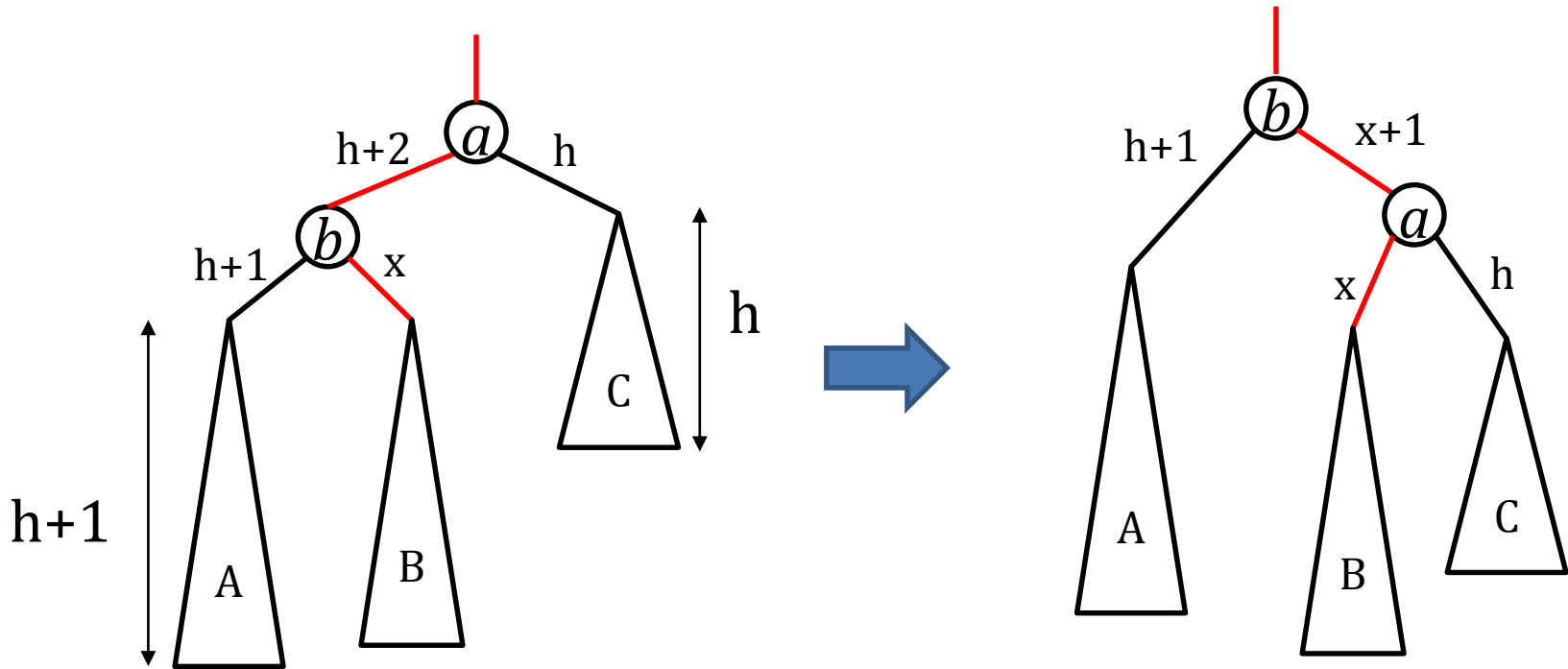
2-level imbalance

◈ There are two cases:

h+1   h or h+1   h

Left-Left case   Left-Right case

h   h+1   h

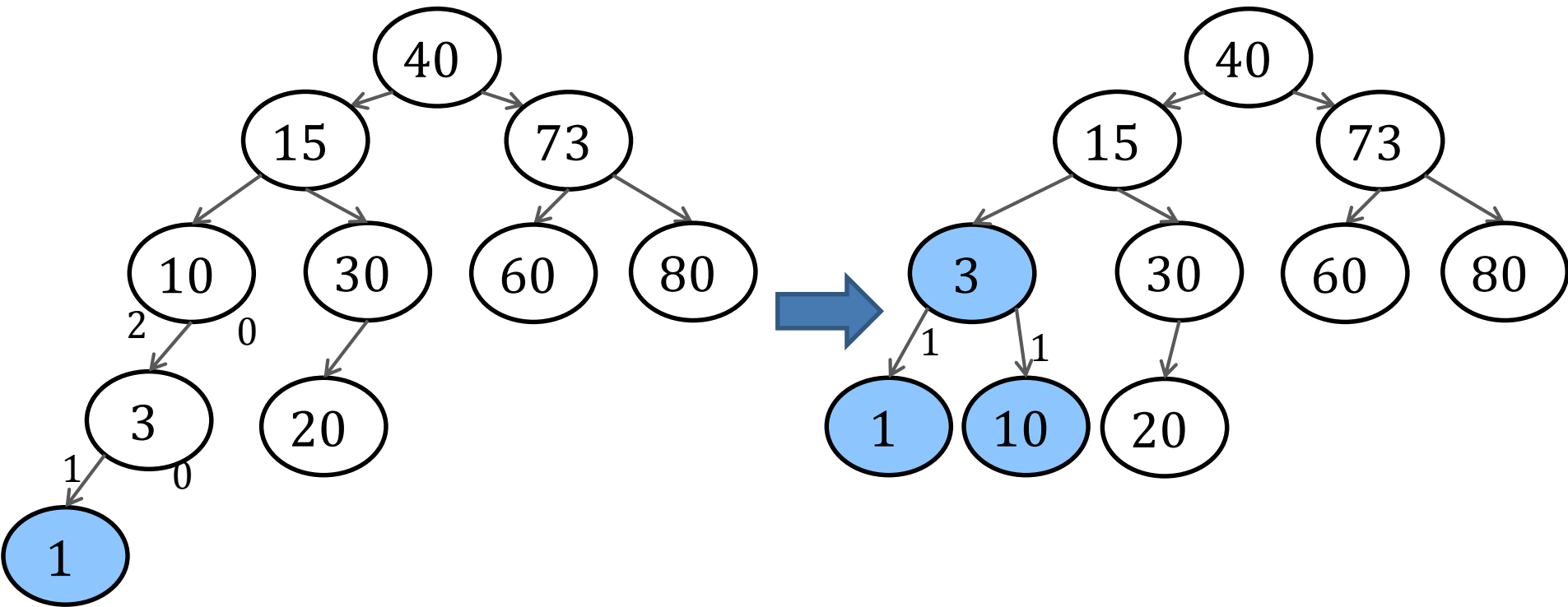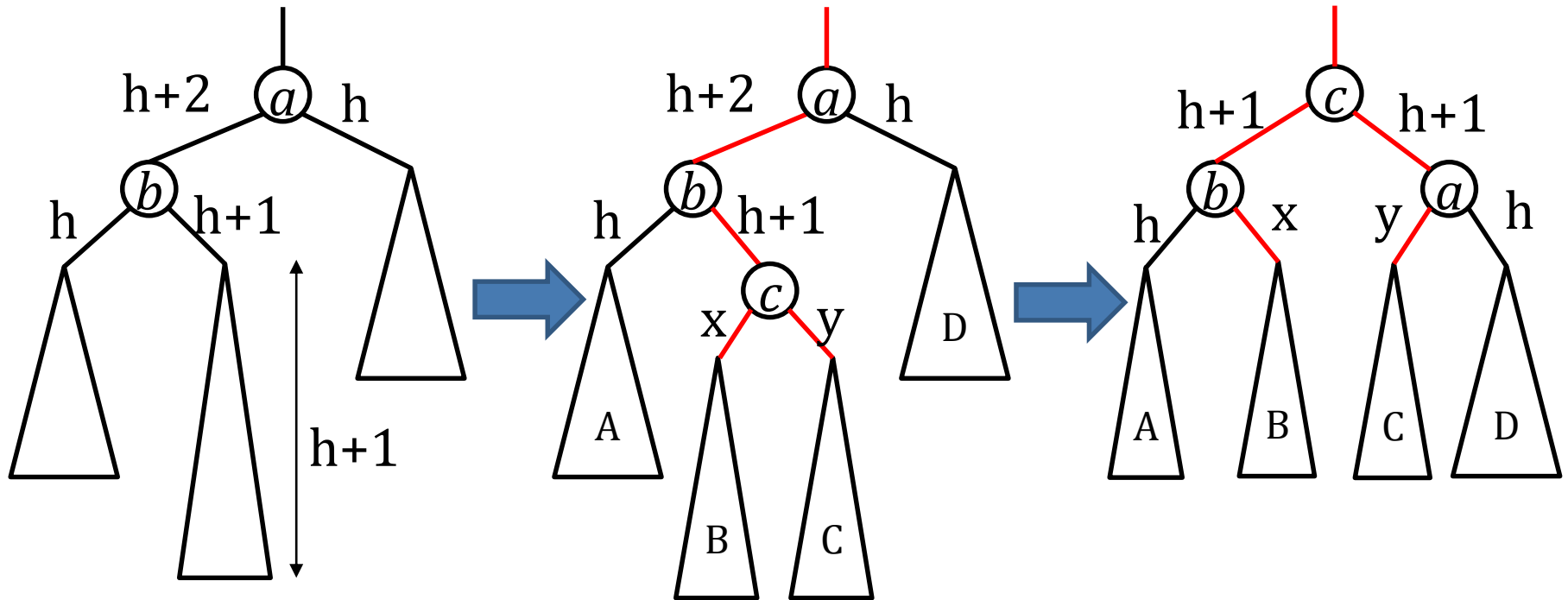# Rebalance Left-Left

◈ By a rotation:

◈ Only 3 pointers to change (the red ones). The cost is O(1).

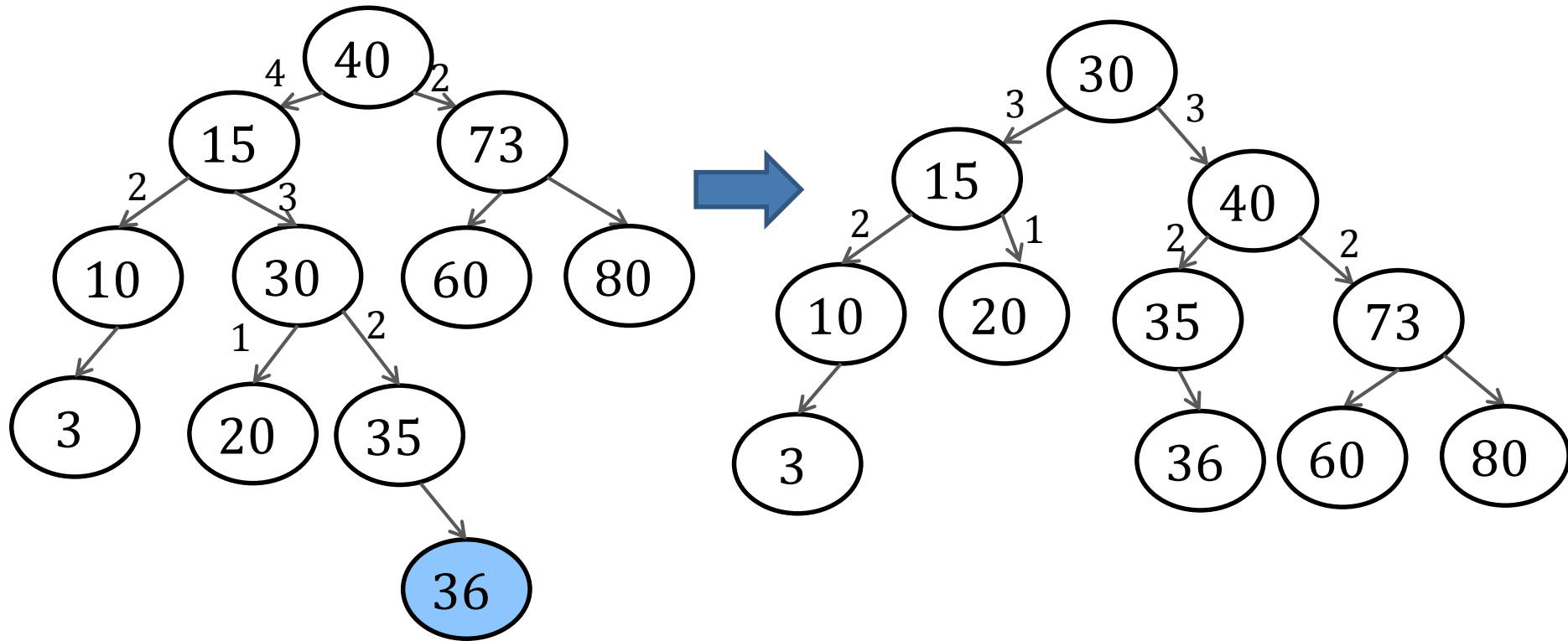◈ Recall that x = h or h+1

Rebalance Left-Left Example

Rebalance Left-Right

◈ By a double rotation:

◈ Only 5 pointers to change (see above). Hence, the cost is O(1).

◈ Note that x and y must be h or h-1. Furthermore, at least one of them must be h (why?)

Rebalance Left-Right Example

Insertion and Deletion Time

- Insertion time analysis
  - It will be left as an exercise for you to prove
    - Only 2-level imbalance can occur in an insertion
    - Once we have remedied the lowest imbalance node, all the nodes in the tree will become balanced again
  - Thus, we can conclude the insertion cost in a balanced BST is O(log n), why?
- Deletion time analysis
  - It will be left as an exercise for you to prove
    - Only 2-level imbalance can occur after a deletion
  - Thus, we can conclude the deletion cost in a balanced BST is O(log n)

Balanced BST

◈ We now conclude our discussion on the AVL-tree, which provides the following guarantees:

   ◈ O(n) space consumption

   ◈ O(log n) time per predecessor query (hence, also per dictionary lookup)

   ◈ O(log n) time per insertion

   ◈ O(log n) time per deletion

◈ All the above complexities hold in the worst case.

Thank You!