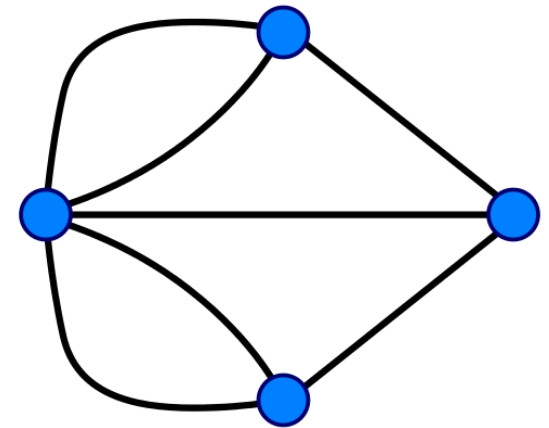
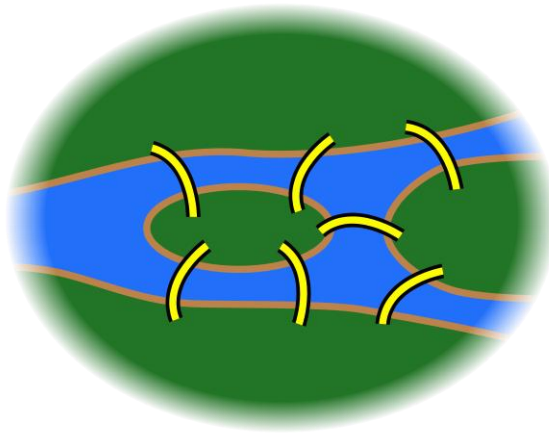
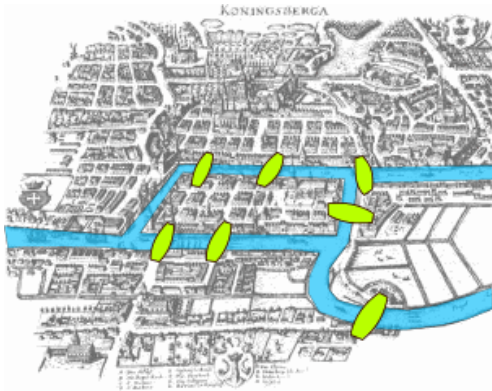


Lecture 9: Graph

Seven Bridges of Königsberg

- City A was set on both sides of the River, and included two large islands which were connected to each other by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.



- Eulerian path (In Chinese: 一笔画问题)

Our Roadmap

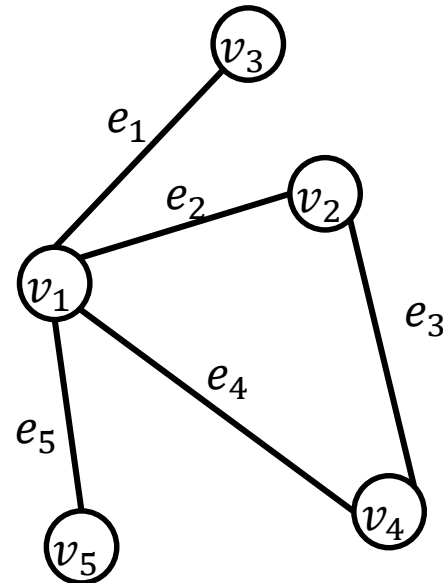
- ◆ Graph Concepts
- ◆ Graph Traversal
 - ◆ Breath First Search (SSSP)
 - ◆ Depth First Search (DAG, topological sort)
- ◆ Shortest Path Algorithm (SP)
- ◆ Minimum Spanning Tree (MST)
- ◆ Strongly Connected Component (SCC)

Undirected Graph

- ◆ An undirected graph is a pair of (V, E) where:
 - ◆ V is a set of elements, each of which called a node
 - ◆ E is a set of unordered pairs $\{u, v\}$ such that u and v are nodes
- ◆ A node may also be called a vertex. We will refer to V as the vertex set or the node set of graph, and E the edge set.

- ◆ Example:

- ◆ $V = \{v_1, v_2, v_3, v_4, v_5\}$
- ◆ $E = \{e_1, e_2, e_3, e_4, e_5\}$

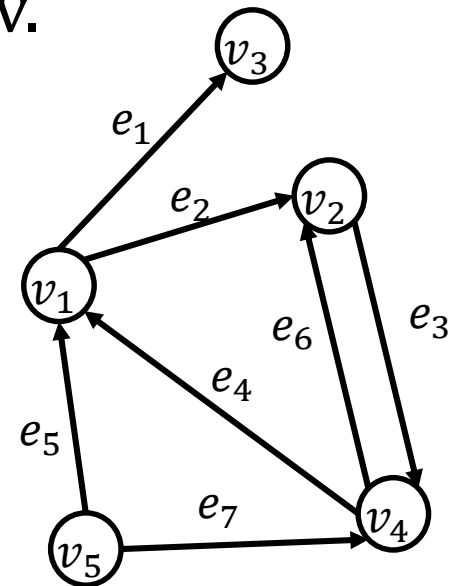


Directed Graph

- ◆ An directed graph is a pair of (V, E) where:
 - ◆ V is a set of elements, each of which called a node
 - ◆ E is a set of unordered pairs $\{u, v\}$ where u and v are nodes, we say there is a directed edge from u to v .
- ◆ A directed edge (u, v) is said to be an outgoing edge of u , and incoming edge of v . Accordingly, v is an out-neighbor of u , and u is in-neighbor of v .
- ◆ Note that every edge has a direction.

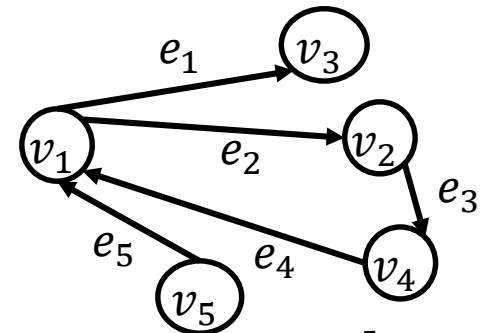
- ◆ Example:

- ◆ $V = \{v_1, v_2, v_3, v_4, v_5\}$
- ◆ $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$
- ◆ $e_3 = \{v_2, v_4\}$
- ◆ $e_6 = \{v_4, v_2\}$



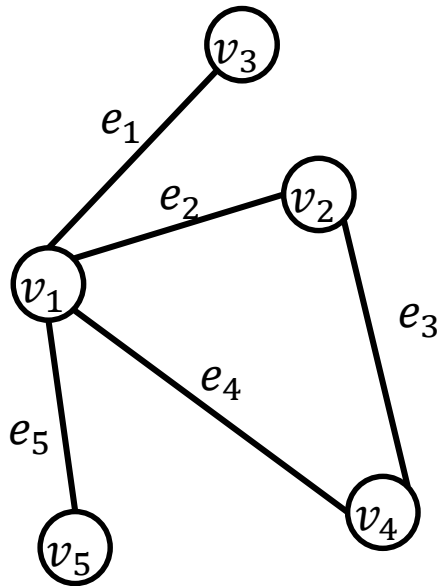
Definitions in Graph

- ◆ Let $G = (V, E)$ be a graph. A path in G is a sequence of nodes (v_1, v_2, \dots, v_k) such that
 - ◆ For every $i \in [1, k]$, there is an edge between v_i and v_{i+1} .
- ◆ A cycle in G is a path (v_1, v_2, \dots, v_k) such that $k \geq 4$ and $v_1 = v_k$.
- ◆ Example:
 - ◆ Cycle: (v_1, v_2, v_4, v_1) ; Path: (v_5, v_1, v_2, v_4)
- ◆ In an undirected graph, the degree of vertex u is the number of edges of u
- ◆ In a directed graph, the out-degree of a vertex u is the number of outgoing edges of u , and its in-degree is the number of its incoming edges

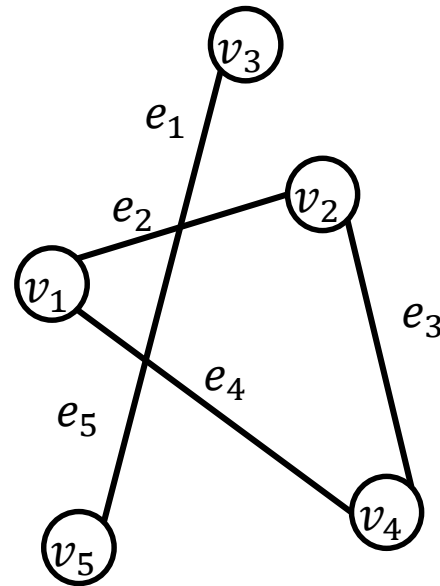


Connected Graph

- ◆ An undirected graph $G=(V,E)$ is connected if, for any two distinct vertices u and v , G has a path from u to v .



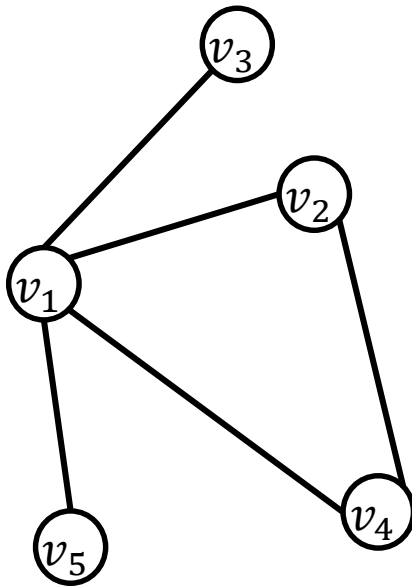
connected



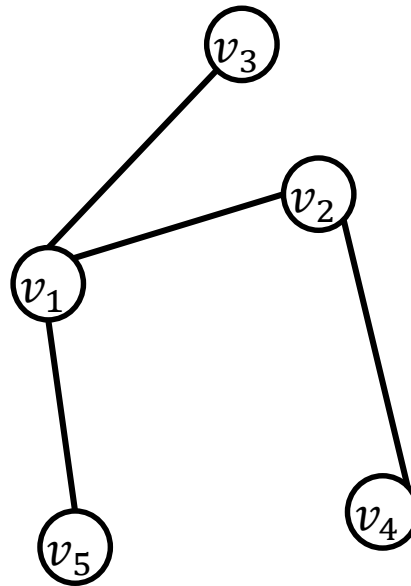
not connected

Graph vs. Tree vs. Forest

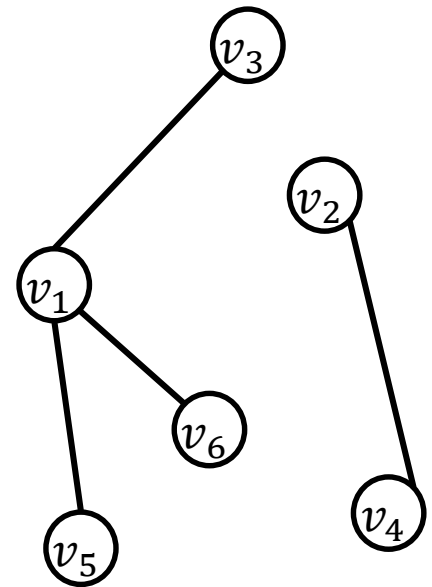
- ◆ A tree is a connected undirected graph contains no cycles.
- ◆ Forest is a set of disjoint trees.



Graph, not tree



Graph, tree



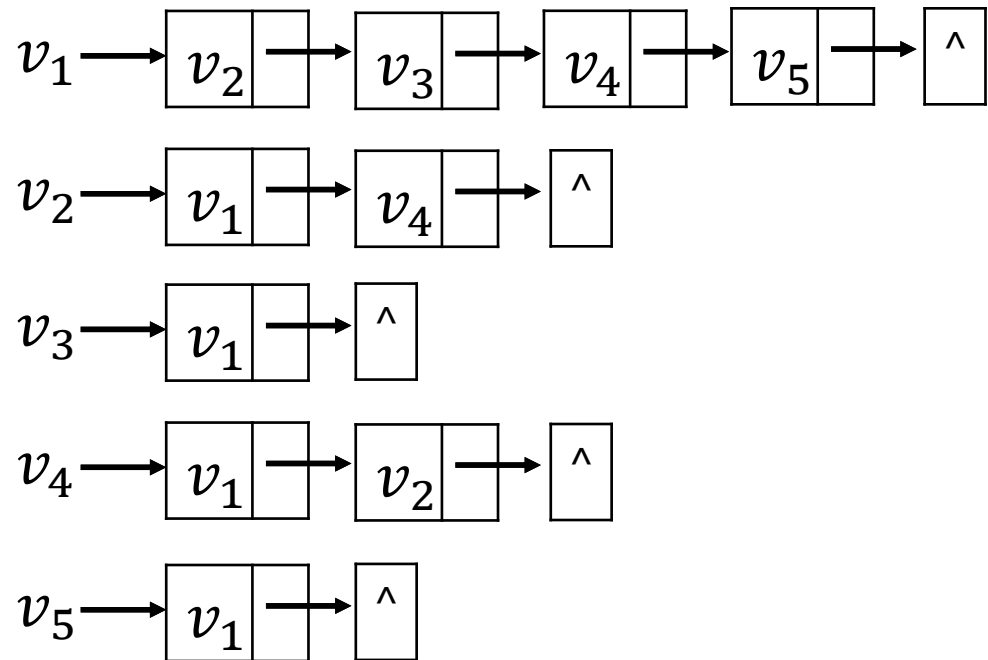
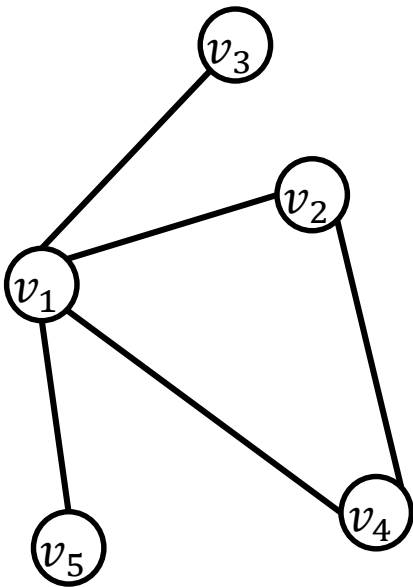
Graph, forest

Graph Representation

- ◆ We discuss two common way to store a graph:
 - ◆ Adjacency list
 - ◆ Adjacency matrix
- ◆ In both cases, we represent each vertex in V using a unique id in $1, 2, \dots, |V|$

Adjacency List: Undirected G

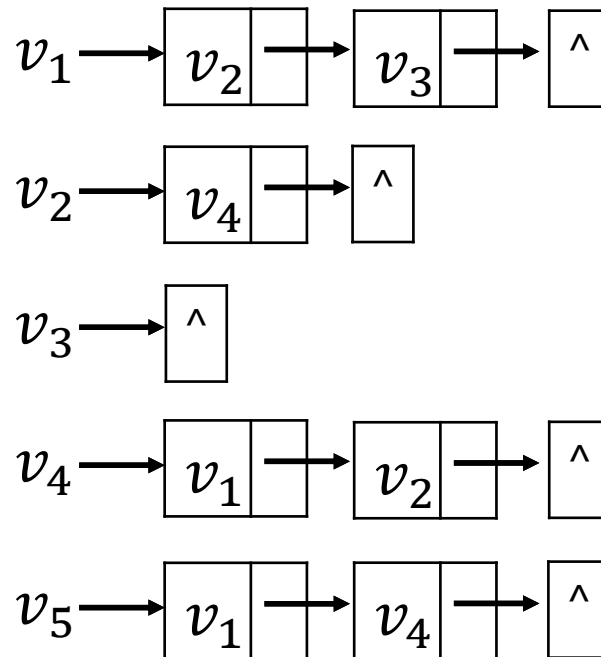
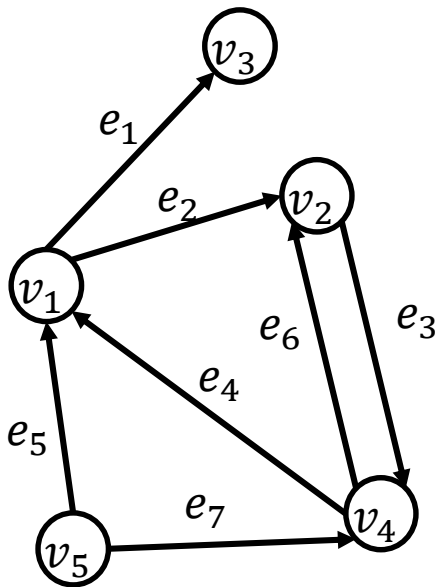
- Each vertex $u \in V$ is associated with a linked list that enumerates all the vertices that are connected to u .



- Space = $O(|V| + |E|)$

Adjacency List: Directed G

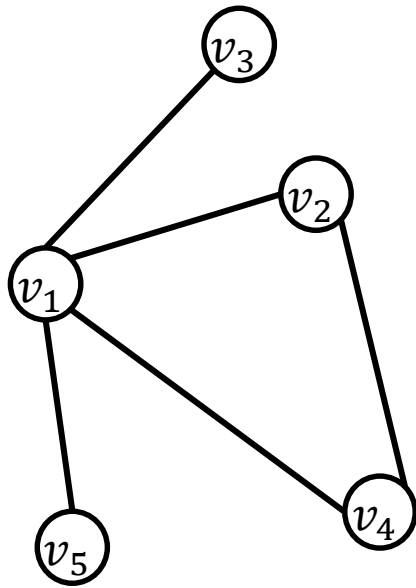
- Each vertex $u \in V$ is associated with a linked list that enumerates all the vertices $v \in V$ that there is an edge from u to v .



- Space = $O(|V| + |E|)$

Adjacency Matrix: Undirected G

- ◆ A $|V| \times |V|$ matrix A where $A[u,v] = 1$ if $(u,v) \in E$, or 0 otherwise

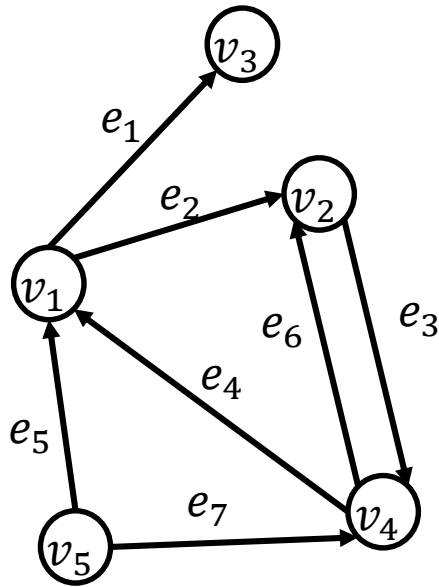


	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	1	1
v_2	1	0	0	1	0
v_3	1	0	0	0	0
v_4	1	1	0	0	0
v_5	1	0	0	0	0

- ◆ A must be symmetric
- ◆ Space = $O(|V|^2)$

Adjacency Matrix: Directed G

- Defined in the same way as in the undirected graph



	v_1	v_2	v_3	v_4	v_5
v_1	0	1	1	0	0
v_2	0	0	0	1	0
v_3	0	0	0	0	0
v_4	1	1	0	0	0
v_5	1	0	0	1	0

- A may not be symmetric.
- Space = $O(|V|^2)$

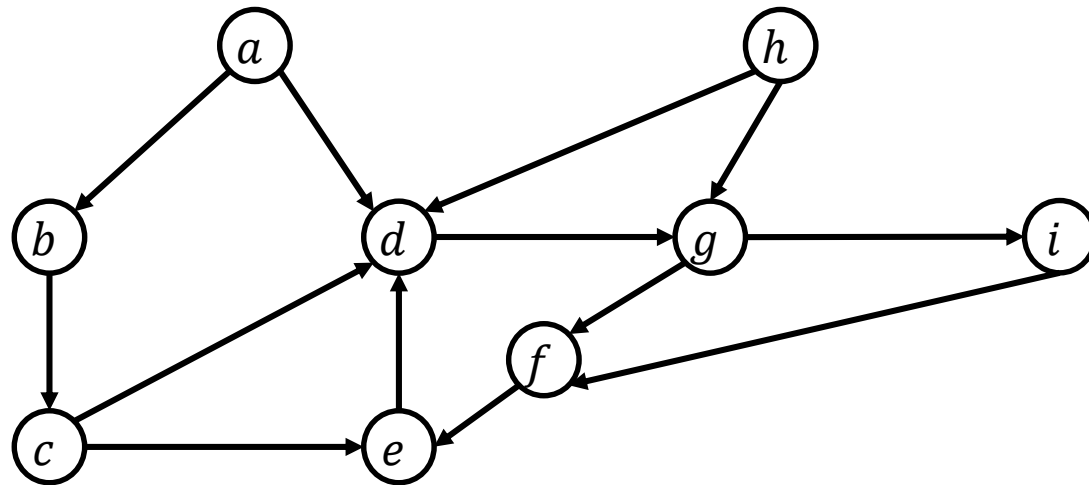
Our Roadmap

- ◆ Graph Concepts
- ◆ Graph Traversal
 - ◆ Breath First Search (SSSP)
 - ◆ Depth First Search (DAG, topological sort)
- ◆ Shortest Path Algorithms (SP)
- ◆ Minimum Spanning Tree (MST)
- ◆ Strongly Connected Component (SCC)

Shortest Path

- ◆ Let $G = (V, E)$ be a directed graph. A path in G is a sequence of nodes (v_1, v_2, \dots, v_k) such that
 - ◆ For every $i \in [1, k]$, there is an edge between v_i and v_{i+1} .
 - ◆ E.g., $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$
 - ◆ Sometimes, we also denote the path as $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$
- ◆ The path is said to be from v_1 to v_k , the length of the path is $k - 1$.
- ◆ Given two vertices $u, v \in V$, a shortest path from u to v is a path from u to v that has the minimum length among all the paths from u to v .
- ◆ If there is no path from u to v , then v is said to be unreachable from u .

Shortest Path Example



- ◆ There are several path from a to g:
 - ◆ $a \rightarrow b \rightarrow c \rightarrow d \rightarrow g$ (length 4)
 - ◆ $a \rightarrow b \rightarrow c \rightarrow e \rightarrow d \rightarrow g$ (length 5)
 - ◆ $a \rightarrow d \rightarrow g$ (length 2)
- ◆ The last one is a shortest path. In this case, the shortest path is unique.
- ◆ Note that h is unreachable from a.

Single Source Shortest Path

- ◆ Let $G=(V,E)$ be a directed graph with unit weight in each edge, and s be a vertex in V . The goal of the single source shortest path (SSSP) problem is to find, for every other vertex $t \in V \setminus \{s\}$, a shortest path from s to t , unless t is unreachable from s .
- ◆ Next, we will describe the breadth first search (BFS) algorithm to solve the problem in $O(|V|+|E|)$ time, which is clearly optimal (because any algorithm must at least see every vertex and every edge once in the worst case).

Single Source Shortest Path

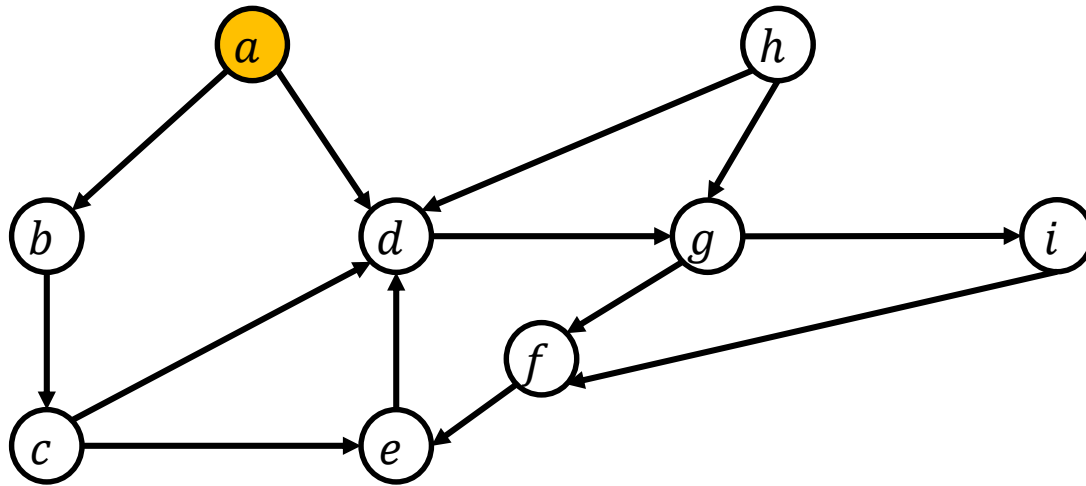
- ◆ How do you solve it?
- ◆ At first glance, this may look surprising because the total length of all the shortest paths may reach $\Omega(|V|^2)$ even when $|E|=O(|V|)$! So shouldn't the algorithm need $\Omega(|V|^2)$ time just to output all these shortest paths in the worst case?
- ◆ The answer, interestingly, is no. As will see, BFS encodes all the shortest paths in a BFS tree compactly, which uses only $O(|V|)$ space, and can be output in $O(|V|+|E|)$ time.

Breadth First Search

- ◆ At the beginning, color all vertices in graph white. And create an empty BFS tree T .
- ◆ Create a queue Q . Insert the source vertex s into Q , and color it yellow (which means “in the queue”)
- ◆ Make s the root of T .

Breadth First Search Example

- Suppose that source vertex is a .



BFS tree

a

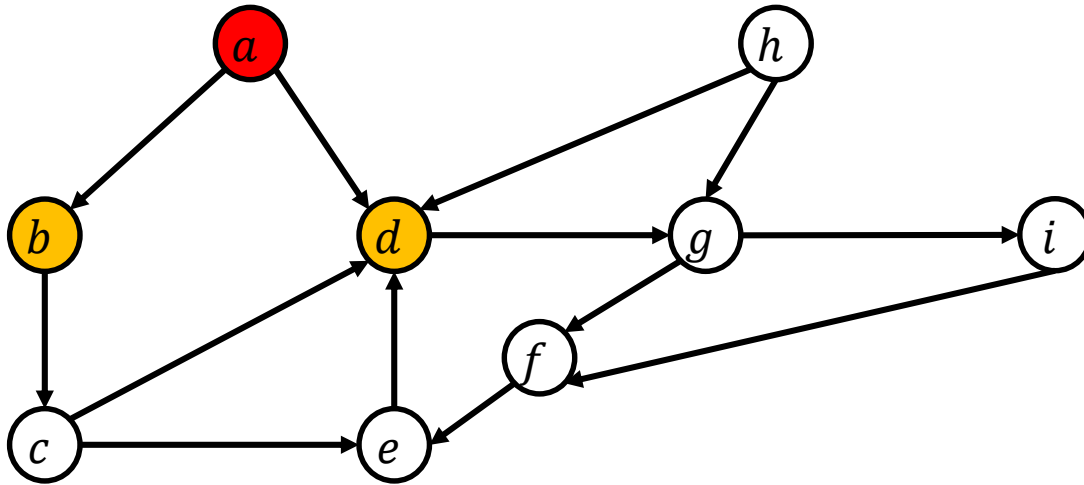
- $Q = (a)$

Breadth First Search Example

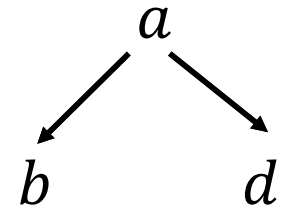
- ◈ Repeat the following until Q is empty
 - ◈ De-queue from Q the first vertex v
 - ◈ For every out-neighbor u of v that is still white
 - ◆ 2.1 Enqueue u into Q , and color u yellow
 - ◆ 2.2 Make u a child of v in the BFS tree T .
 - ◈ Color v red (meaning v is visited)

Breadth First Search Example

◆ After de-queuing a :



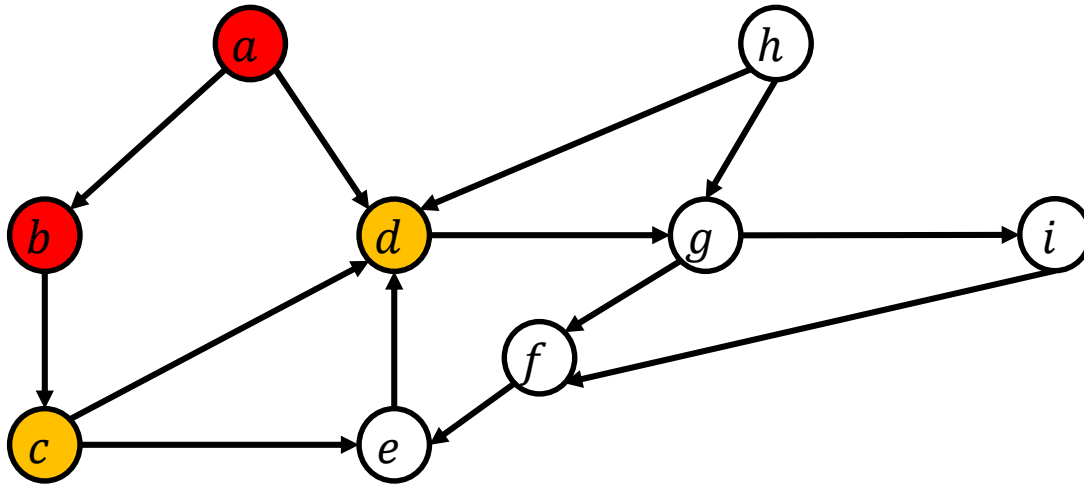
BFS tree



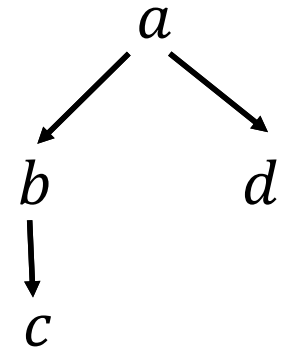
◆ $Q = (b, d)$

Breadth First Search Example

◆ After dequeuing b:



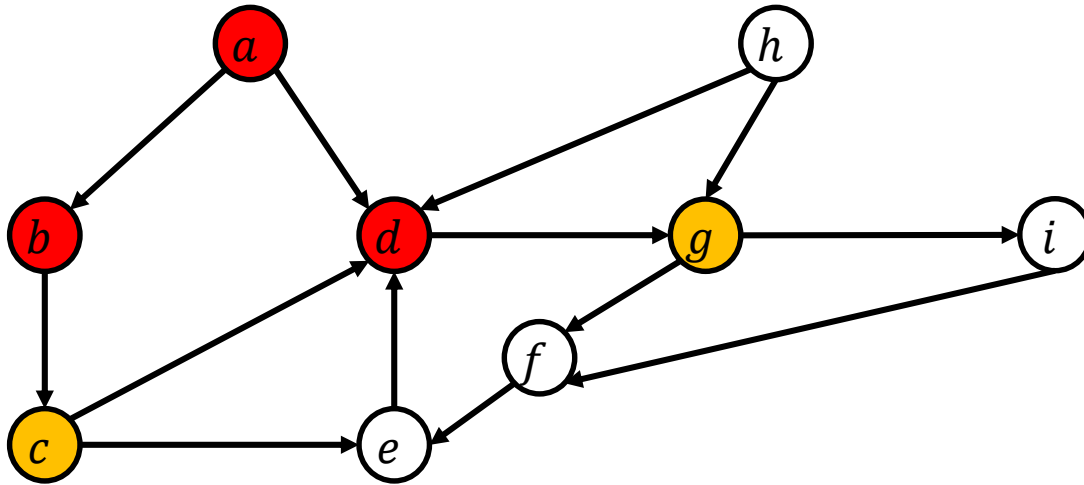
BFS tree



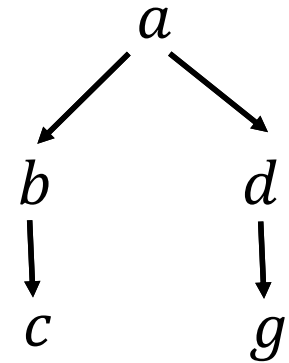
◆ $Q = (d, c)$

Breadth First Search Example

◆ After dequeuing d :



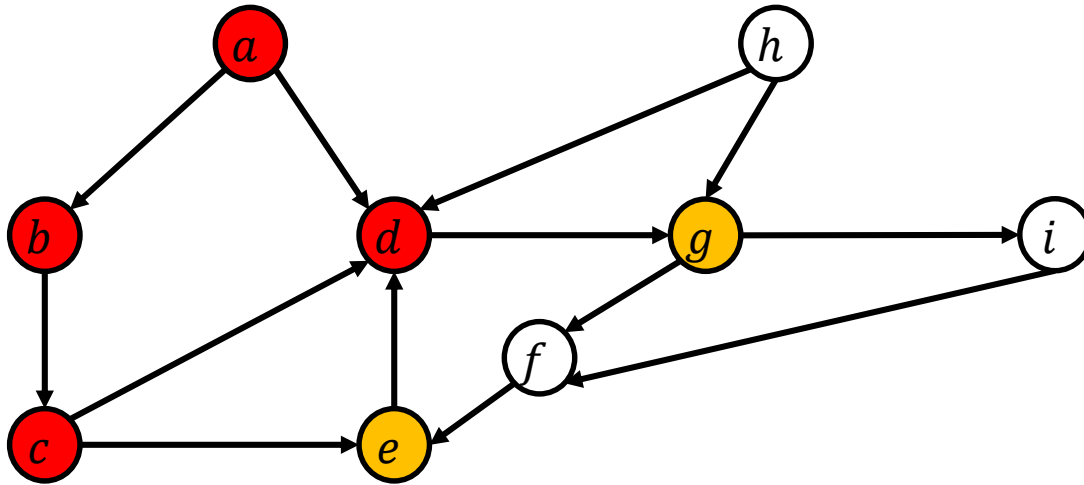
BFS tree



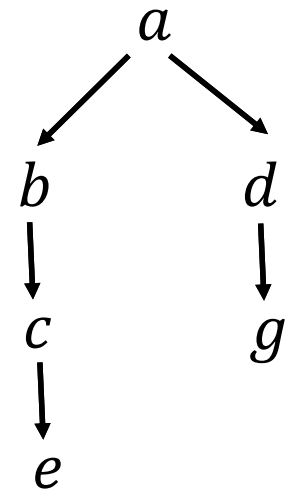
◆ $Q = (c, g)$

Breadth First Search Example

- After dequeuing c :



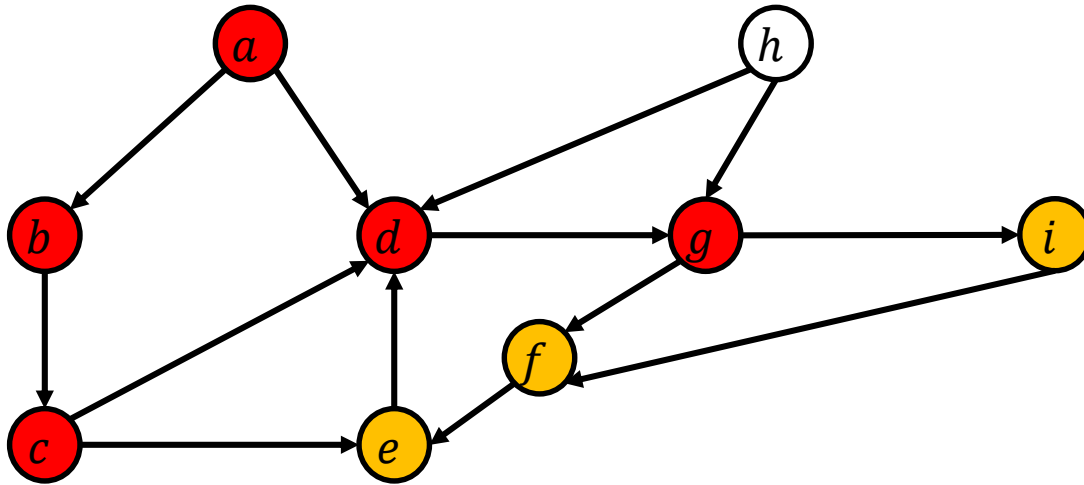
BFS tree



- $Q = (g, e)$
- d is not enqueue again as it is red now

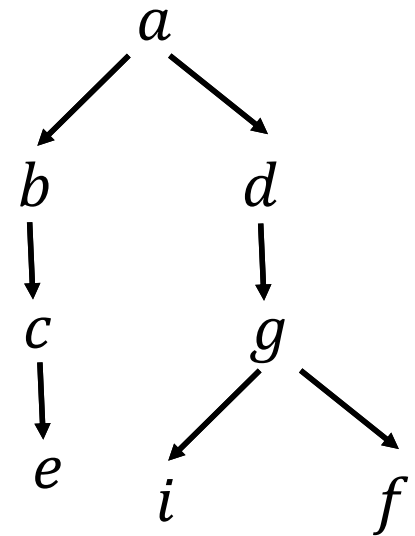
Breadth First Search Example

◆ After dequeuing g :



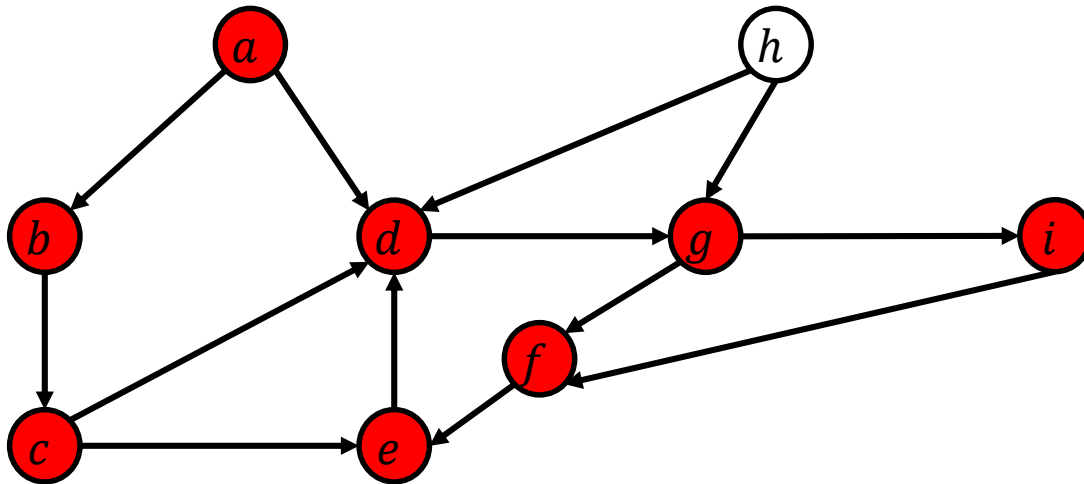
◆ $Q = (e, i, f)$

BFS tree

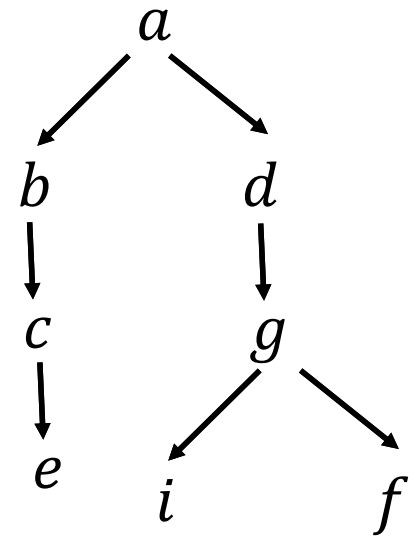


Breadth First Search Example

- After dequeuing e, i, f



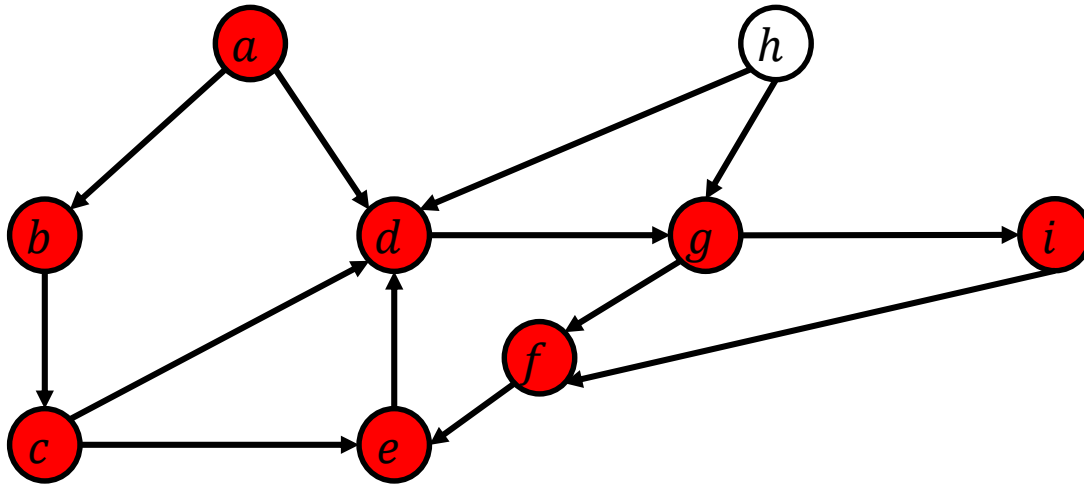
BFS tree



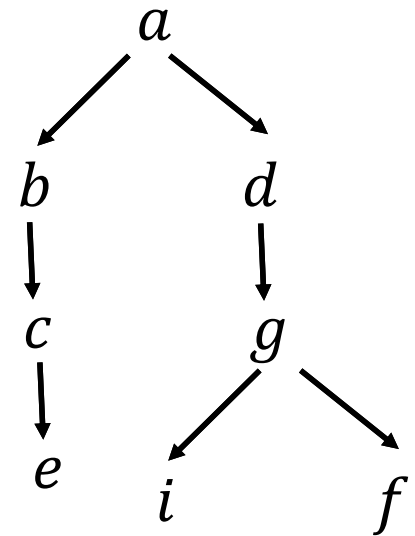
- $Q = ()$
- This is the end of BFS. Note that h remains white: we can conclude that it is not reachable from a.

SSSP solution

- Where are the shortest paths?



BFS tree



- The shortest path from a to any vertex x is simply the path from a to node x in the BFS tree!.
 - Proof?

Complexity Analysis

- ◆ When a vertex v is dequeued, we spend $O(1+d^+(v))$ time processing it, where $d^+(v)$ is the out-degree of v .
- ◆ Clearly, every vertex enters the queue at most once.
- ◆ The total running time of BFS is therefore:

$$O\left(\sum_{v \in V} (1 + d^+(v))\right) = O(|V| + |E|)$$

Our Roadmap

- ◆ Graph Concepts
- ◆ Graph Traversal
 - ◆ Breath First Search (SSSP)
 - ◆ Depth First Search (DAG, topological sort)
- ◆ Shortest Path Algorithm (SP)
- ◆ Minimum Spanning Tree (MST)
- ◆ Strongly Connected Component (SCC)

Depth First Search

- ◆ We have already learnt breadth first search (BFS). Today, we will discuss its “sister version”: the depth first search (DFS) algorithm. Our discussion will once again focus on directed graphs, because the extension to undirected graphs is straight forward.
- ◆ DFS is surprisingly powerful algorithm, and solves several classic problem elegantly. In this lecture, we will see one such problem: detecting whether the input graph contains cycles.

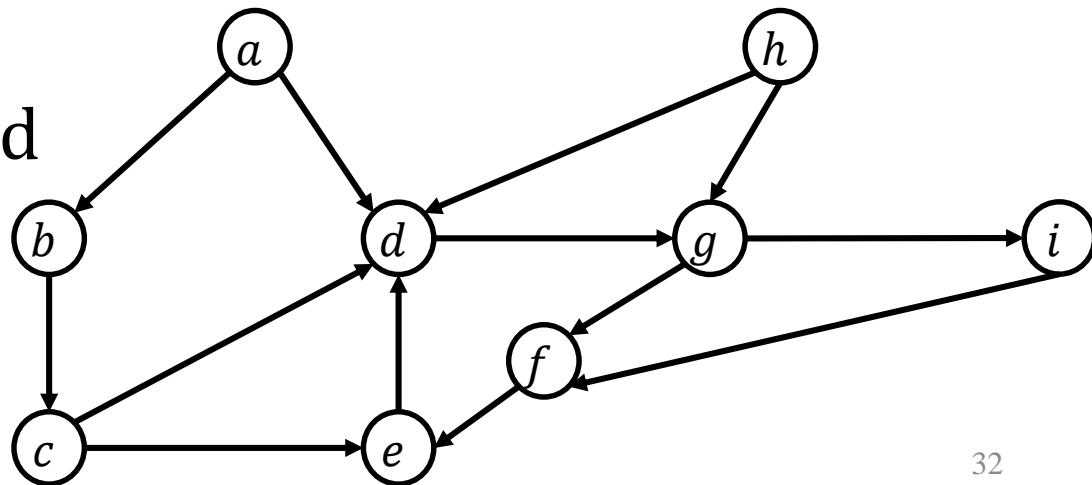
Path and Cycles

- ◆ Recall: let $G = (V, E)$ be a directed graph. A path in G is a sequence of nodes (v_1, v_2, \dots, v_k) such that
 - ◆ For every $i \in [1, k]$, there is an edge between v_i and v_{i+1} .
 - ◆ E.g., $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k)$
 - ◆ Sometimes, we also denote the path as $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$
- ◆ A cycle in G is a path (v_1, v_2, \dots, v_k) such that $k \geq 4$ and $v_1 = v_k$.

◆ Example:

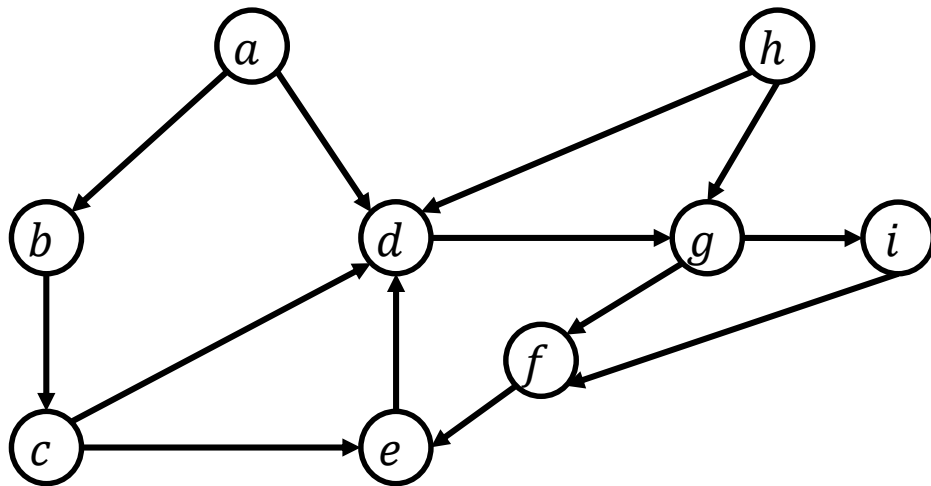
◆ $d \rightarrow g \rightarrow i \rightarrow f \rightarrow e \rightarrow d$

◆ $d \rightarrow g \rightarrow f \rightarrow e \rightarrow d$

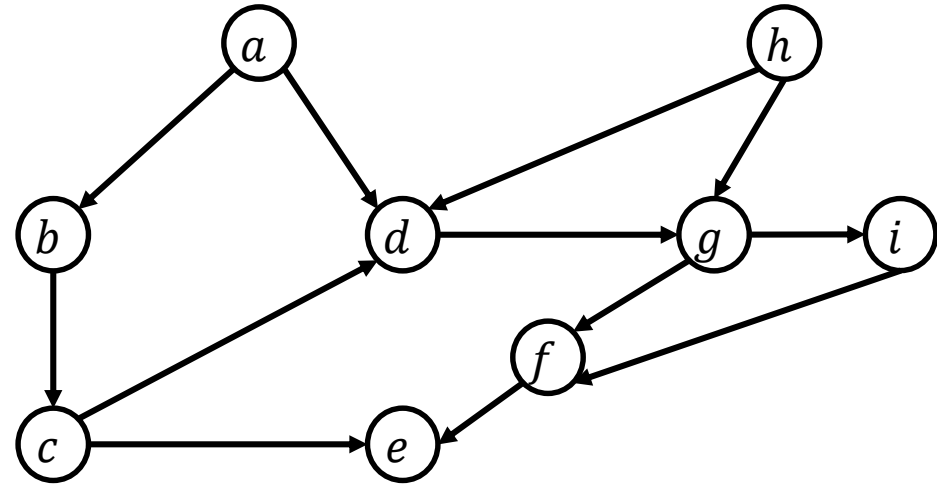


Directed Acyclic/Cyclic Graph

- ◆ If a directed graph contains no cycles, we say that it is a directed acyclic graph (DAG). Otherwise, G is Cyclic.
- ◆ DAG is extremely important concept in Computer Science, e.g., spark, tensorflow
- ◆ Example



Cyclic



DAG

The Cycle Detection Problem

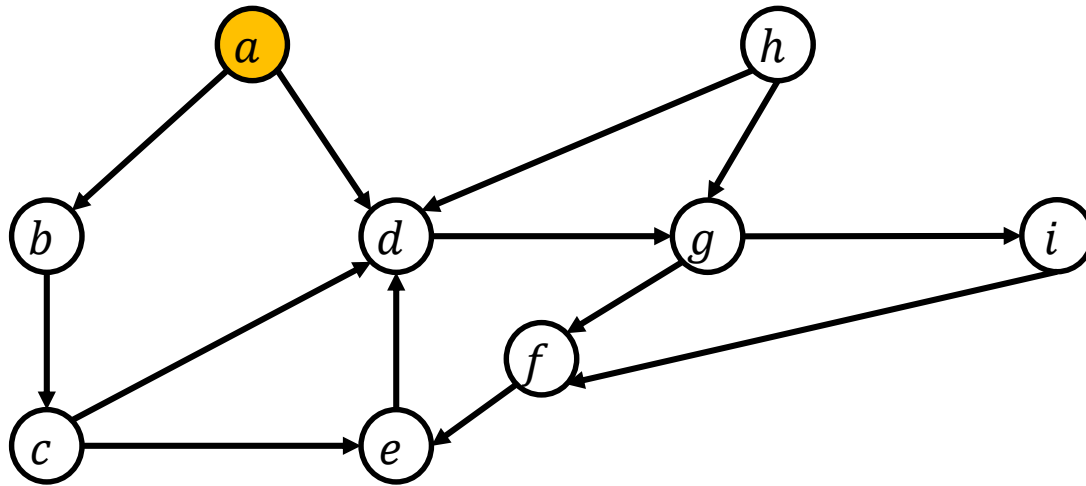
- ◆ Let $G=(V,E)$ be a directed graph. Determine whether it is a DAG.
- ◆ Next, we will describe the depth first search (DFS) algorithm to solve the problem in $O(|V|+|E|)$ time, which is optimal (because any algorithm must at least see every vertex and edge once in the worst case).
- ◆ Just like BFS, the DFS algorithm also outputs a tree, called the DFS-tree. This tree contains vital information about the input graph that allows us to decide whether the input graph is a DAG.

Depth First Search

- ◆ At the beginning, color all vertices in the graph white, and create an empty DFS tree T .
- ◆ Create a stack S . Pick an arbitrary vertex v . Push v into S , and color it yellow (which means “in the stack”)
 - ◆ What is the difference between BFS and DFS underlying data structure?
 - ◆ BFS \rightarrow Queue, DFS \rightarrow Stack
- ◆ Make v the root of T

Depth First Search Example

- Suppose we start from a .



DFS tree

a

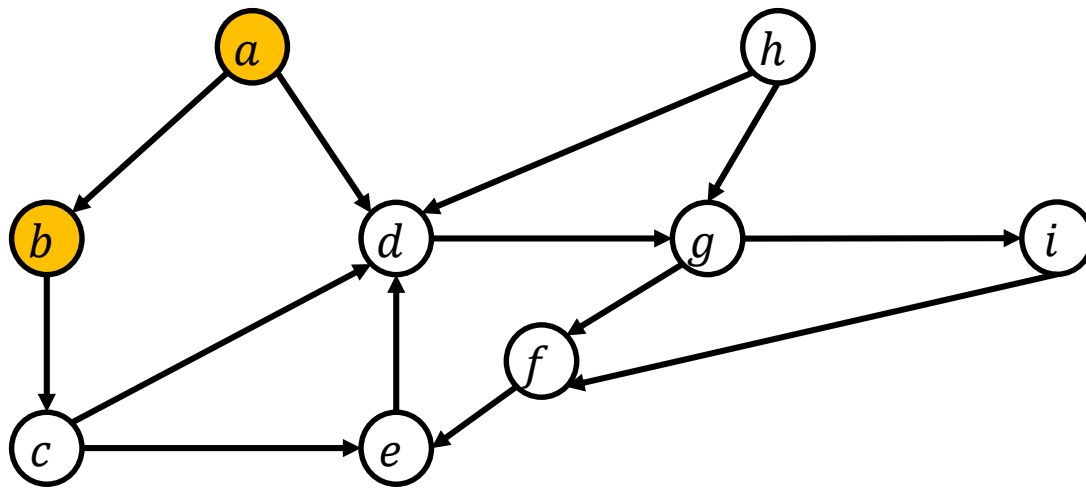
- $S = (a)$

Depth First Search Example

- ◆ Repeat the following until S is empty
 - ◆ Let v be the vertex that currently tops the stack S (do not remove v from S)
 - ◆ Does v still have a white out-neighbor
 - ◆ 2.1 If yes: let it be u .
 - ◆ Push u into S , and color u yellow
 - ◆ Make u a child of v in the DFS-tree T
 - ◆ 2.2 If no, pop v from S , and color v red (meaning v is visited)
 - ◆ If there are still white vertices, repeat the above by restarting from an arbitrary white vertex v' , creating a new DFS tree rooted at v' .

Depth First Search Example

- ◆ Top of stack: a, which has white out-neighbors b, d. Suppose we access b first. Push b into S



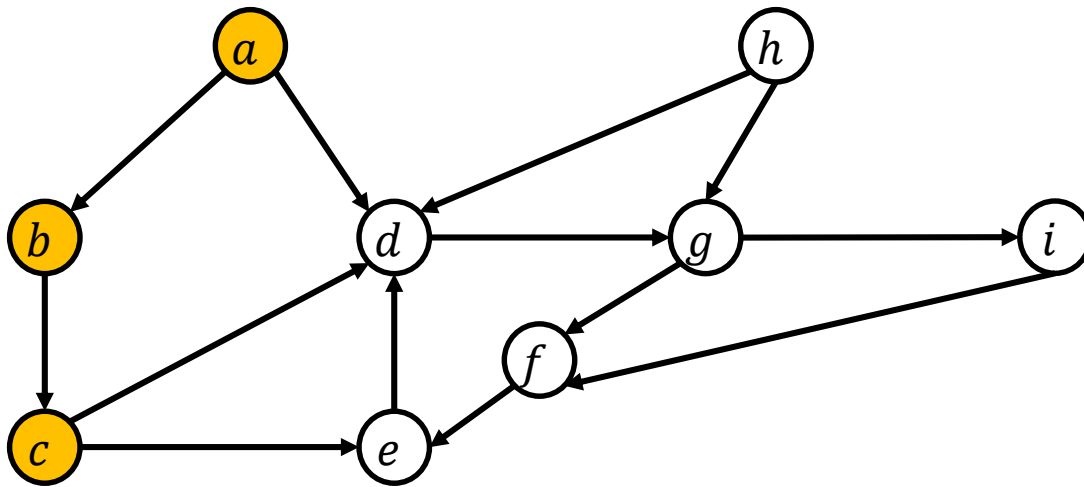
DFS tree

a
↓
 b

- ◆ $S = (a, b)$.

Depth First Search Example

- ◆ Top of stack: b, which has white out-neighbors c. Push c into S



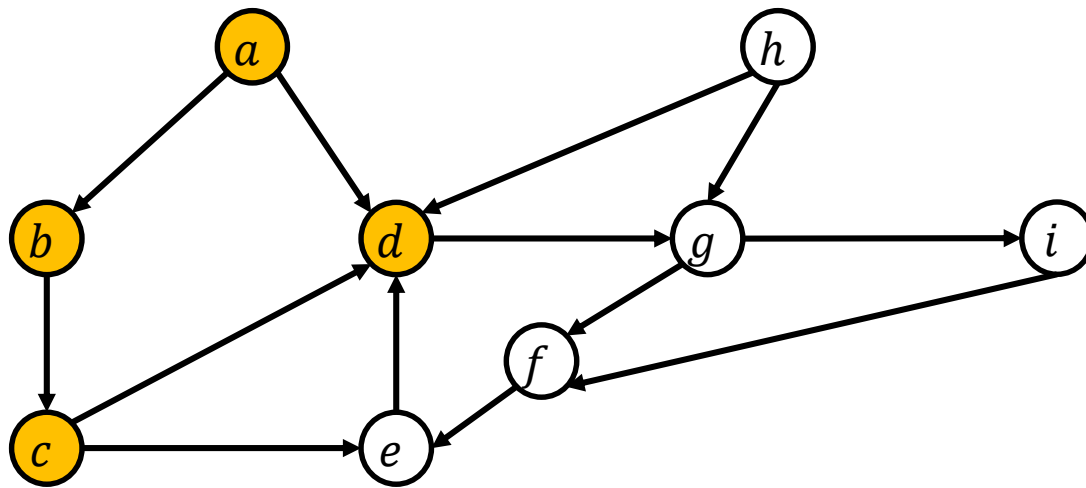
DFS tree

a
 \downarrow
 b
 \downarrow
 c

- ◆ $S = (a, b, c).$

Depth First Search Example

- ◆ Top of stack: c, which has white out-neighbors d and e. Suppose we access d first. Push d into S



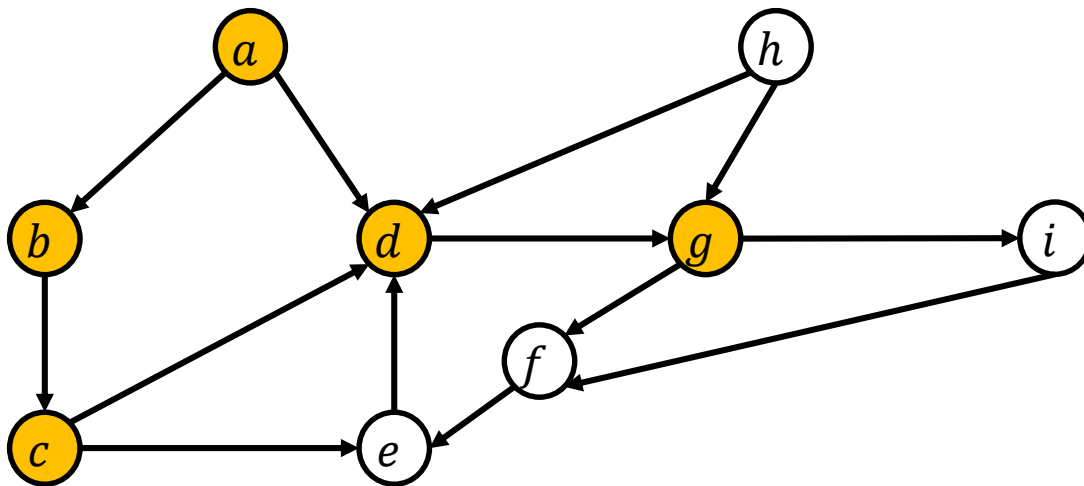
DFS tree

a
 \downarrow
 b
 \downarrow
 c
 \downarrow
 d

- ◆ $S = (a, b, c, d).$

Depth First Search Example

- ◆ Top of stack: d, which has white out-neighbors g. Push g into S



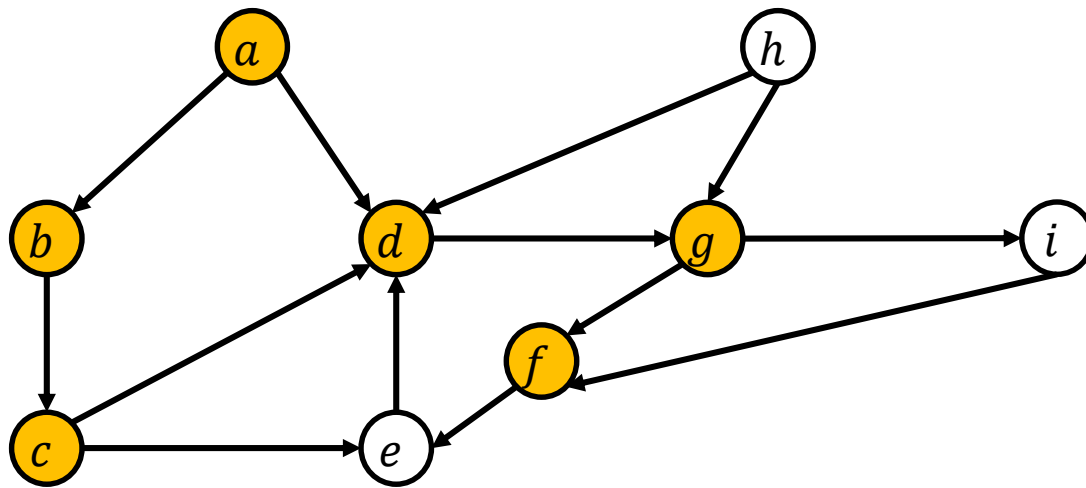
DFS tree

a
 \downarrow
 b
 \downarrow
 c
 \downarrow
 d
 \downarrow
 g

- ◆ $S = (a, b, c, d, g).$

Depth First Search Example

- Top of stack: g, which has white out-neighbors f and i. Suppose we access f first. Push f into S



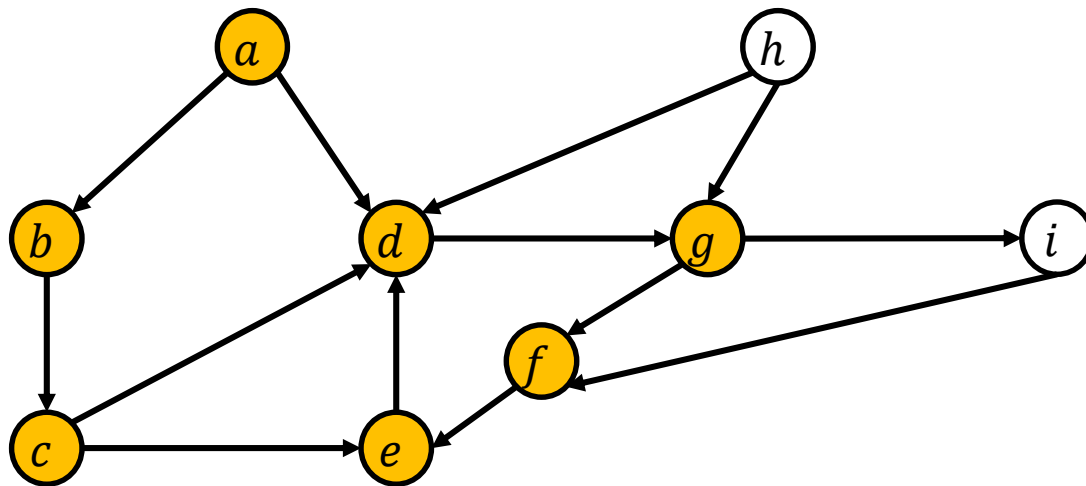
DFS tree



- $S = (a, b, c, d, g, f).$

Depth First Search Example

- Top of stack: f, which has white out-neighbors e. Push e into S



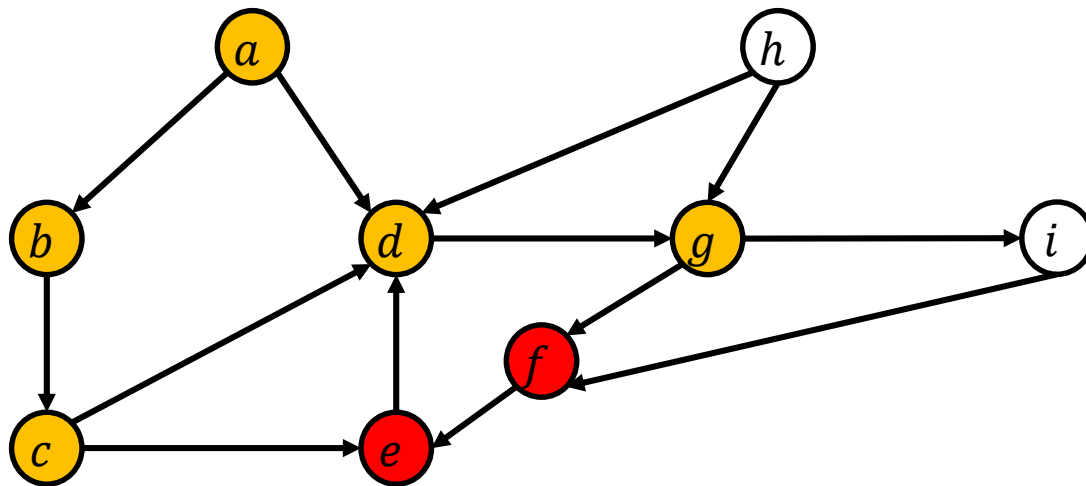
- $S = (a, b, c, d, g, f, e).$

DFS tree



Depth First Search Example

- Top of stack: e , e has no white out-neighbors. So pop it from S , and color it red. Similarly for s .



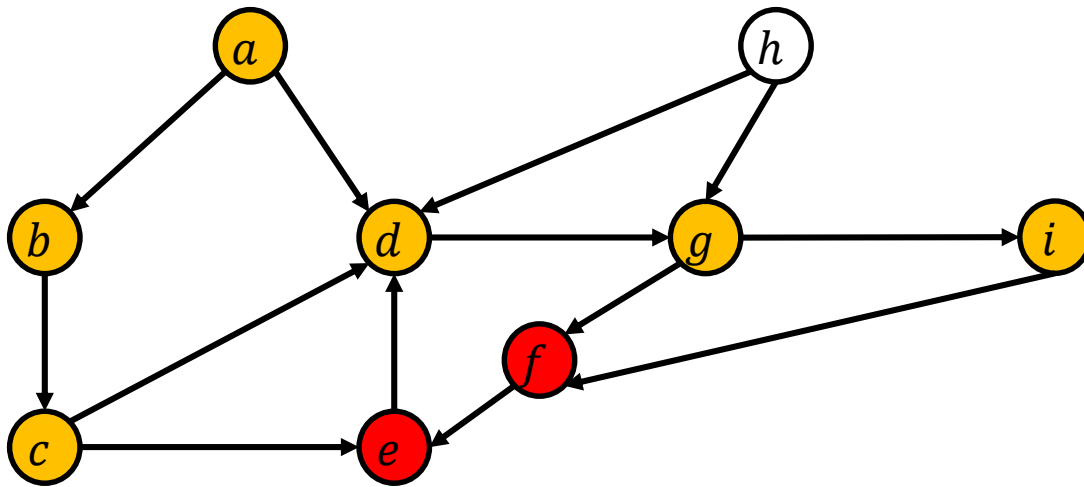
DFS tree

a
 \downarrow
 b
 \downarrow
 c
 \downarrow
 d
 \downarrow
 g
 \downarrow
 f
 \downarrow
 e

- $S = (a, b, c, d, g)$.

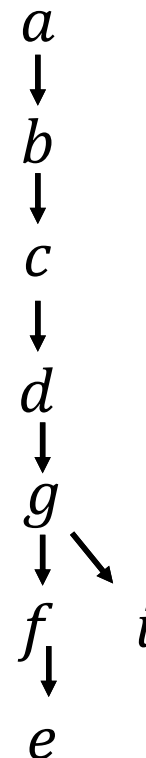
Depth First Search Example

- Top of stack: g , which still has white out-neighbors i . Push i into S



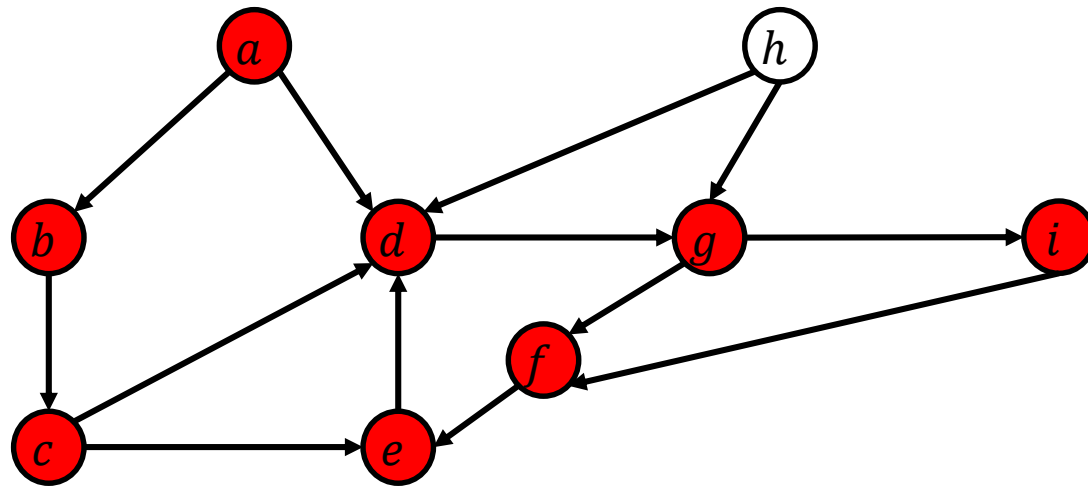
- $S = (a, b, c, d, g, i)$.

DFS tree



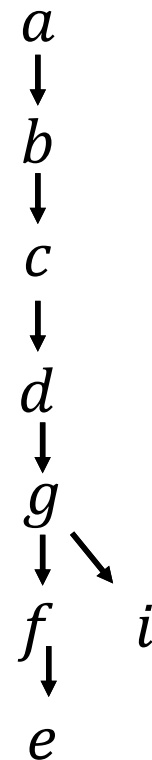
Depth First Search Example

- After popping i, g, d, c, b, a



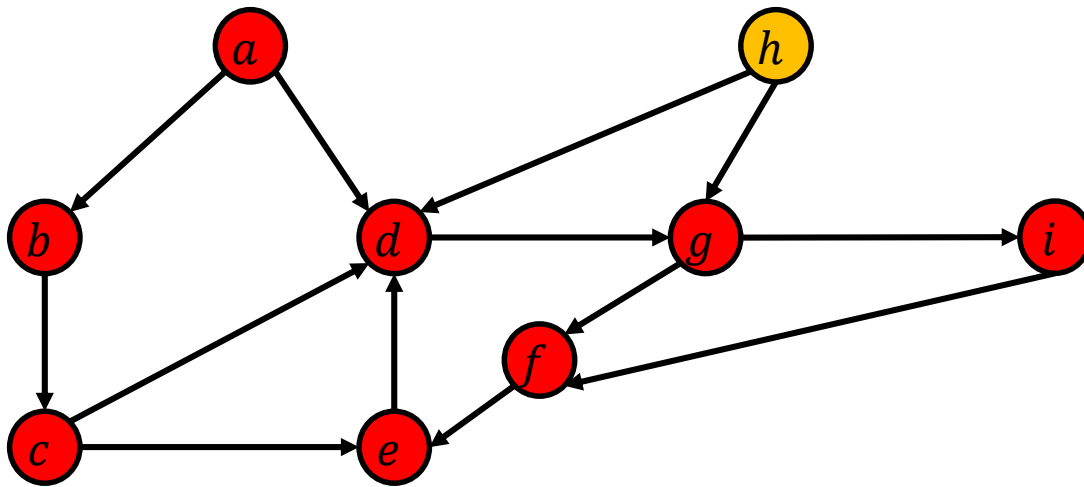
- $S = ()$.

DFS tree



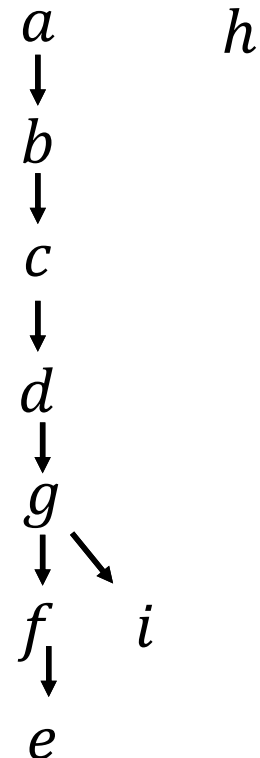
Depth First Search Example

- Now there is still a white vertex h . So we perform another DFS starting from h



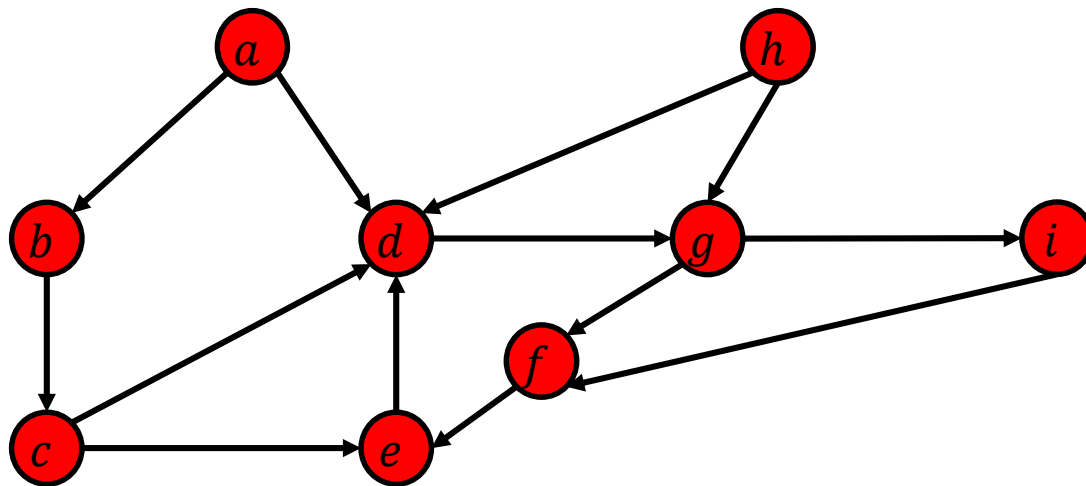
- $S = (h)$.

DFS forest



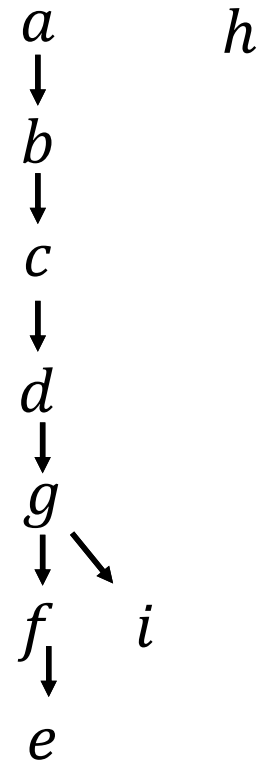
Depth First Search Example

- ◆ Pop h. The end.



- ◆ $S = ()$.
- ◆ Note that we have created a DFS-forest, Which consists of 2 DFS-trees.

DFS forest



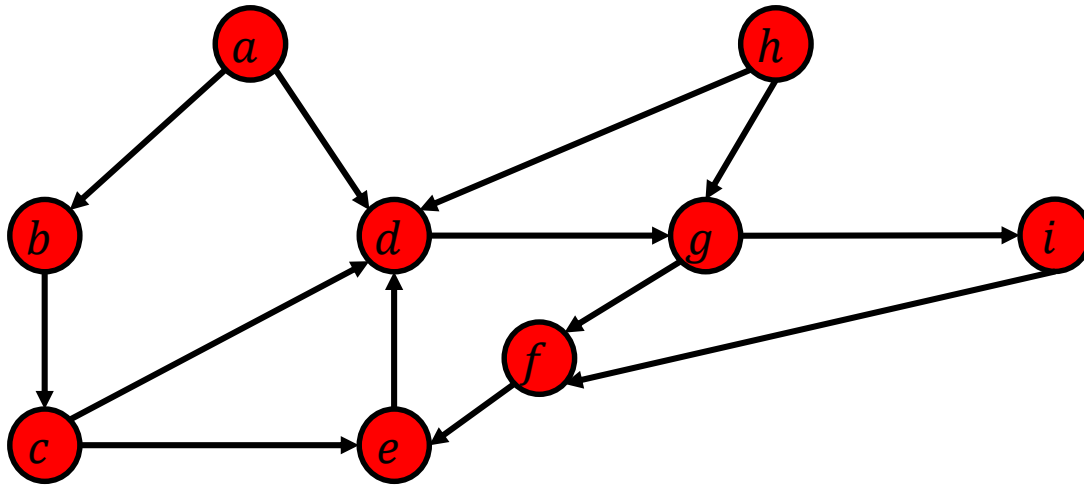
DFS Complexity Analysis

- ◆ DFS can be implemented efficiently as follows.
 - ◆ Store G in the adjacency list format
 - ◆ For every vertex v , remember the out-neighbor to explore next
 - ◆ $O(|V|+|E|)$ stack operations
 - ◆ Use an array to remember the colors of all vertices
- ◆ Hence, the total running time is $O(|V|+|E|)$.

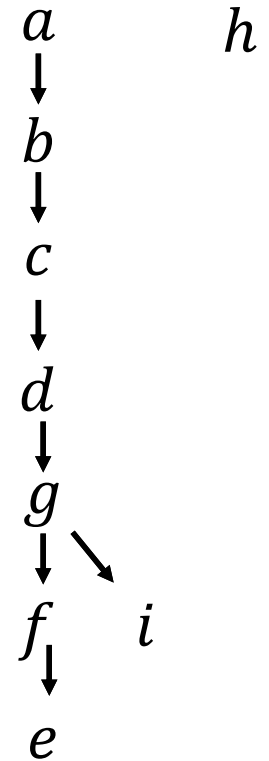
DFS Tree (Forest)

- ◆ Recall that we said earlier that the DFS-tree (well, perhaps a DFS forest) encodes information about the input graph. Next, we will make this point specific, and solve the edge detection problem.
- ◆ Edge Classification
 - ◆ Suppose we have already built a DFS-forest T .
 - ◆ Let (u,v) be an edge in G (remember that the edge is directed from u to v). It can be classified into:
 - ◆ Forward edge: u is a proper ancestor of v in a DFS-tree of T .
 - ◆ Backward edge: u is a descendant of v in a DFS-tree of T .
 - ◆ Cross edge: if neither of the above applies.

Edge Classification Example



DFS Forest

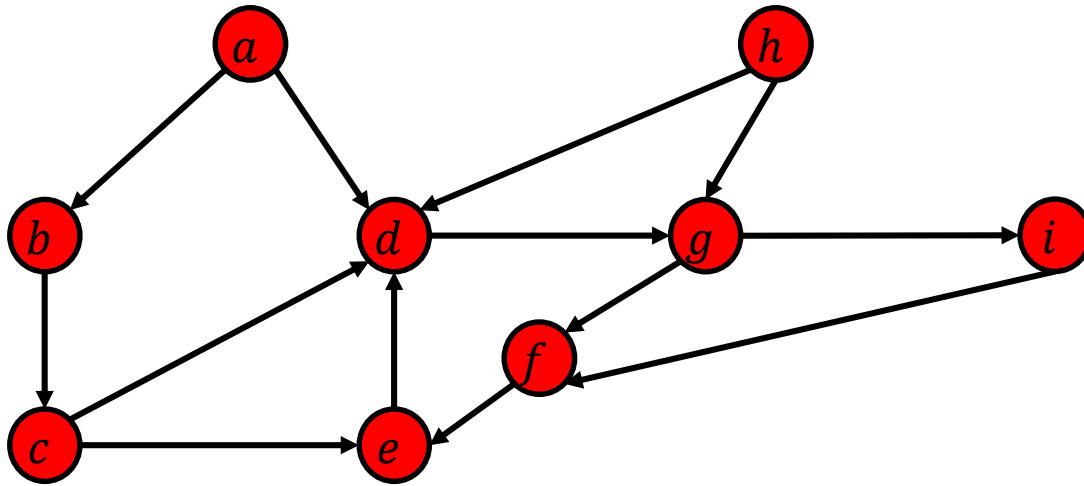


- ◆ Forward edge:
 - ◆ $(a,b), (a,d), (b,c), (c,d), (c,e), (d,g), (g,f), (g,i), (f,e)$
- ◆ Backward edge: (e,d)
- ◆ Cross edge: $(i,f), (h,d), (h,g)$

Edge Classification Example

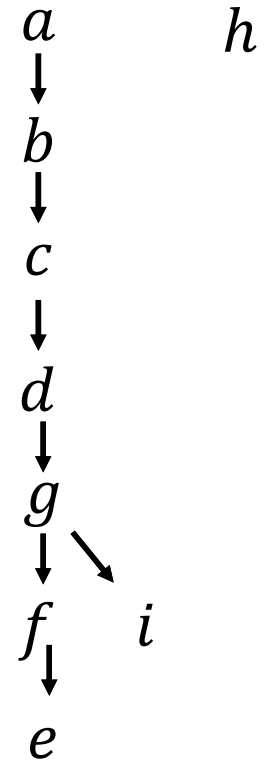
- ◆ How to determine type of each edge (u,v) by $O(1)$ cost?
 - ◆ Augmenting DFS slightly!
- ◆ Maintain a counter c , which is initially 0. Every time a push or pop is performed on the stack, we increment c by 1.
- ◆ For every vertex v , define:
 - ◆ Its discovery time $d\text{-tm}(v)$ to be the value of c right after v is pushed into the stack
 - ◆ Its finish time $f\text{-tm}(v)$ to be the value of c right after v is popped from the stack
 - ◆ Define $I(v) = [d\text{-time}(v), f\text{-tm}(v)]$
- ◆ It is straight forward to obtain $I(v)$ for all $v \in V$ by paying $O(|V|)$ extra time on top of DFS's running time.

Augment DFS algorithm



- ◆ $I(a)=[1,16], I(b)=[2,15], I(c)=[3,14]$
- ◆ $I(d)=[4,13], I(g)=[5,12], I(f)=[6,9]$
- ◆ $I(e)=[7,8], I(i)=[10,11], I(h)=[17,18]$

DFS Forest



Theorems

- ◆ **Parenthesis Theorem:** all the following are true:
 - ◆ If u is a proper ancestor of v in DFS-tree of T , then $I(u)$ contains $I(v)$.
 - ◆ If u is a proper descendant of v in DFS-tree of T , then $I(u)$ is contained in $I(v)$.
 - ◆ Otherwise, $I(u)$ and $I(v)$ are disjoint.
- ◆ **Proof:** Follows directly from the first-in-last-out property of the stack.
- ◆ **Cycle Theorem:** let T be an arbitrary DFS-forest. G contains a cycle if and only if there is a backward edge with respect to T .
- ◆ **Proof:** will left as exercise.

Cycle Detection

- ◆ Equipped with the cycle theorem, we know that we can detect whether G has a cycle easily after having obtained a DFS-forest T :
 - ◆ For every edge (u,v) , determine whether it is a backward edge in $O(1)$ time.
- ◆ If no backward edges are found, decide G to be a DAG; otherwise, G has at least a cycle.
- ◆ Only $O(|E|)$ extra time is needed
- ◆ We now conclude that the cycle detection problem can be solved in $O(|V|+|E|)$ time.

Hint of Cycle Theorem Proof

- ◆ “if” direction, (e,d) is backward edge.
- ◆ “only-if” direction:
 - ◆ White Path Theorem: let u be a vertex in G . Consider the moment when u is pushed into the stack in the DFS algorithm. Then a vertex v becomes a proper descendant of u in the DFS-forest if and only if the following is true:
 - ◆ We can go from u to v by travelling only on white vertices
- ◆ We will now prove that if G has a cycle, then there must be a backward edge in the DFS-forest.
 - ◆ Suppose the cycle is $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1$, let v_i is the first to enter the stack. Then, by white path theorem, all the other vertices in the cycle must be proper descendants of v_i in the DFS-forest. This means the edge pointing to v_i in the cycle is a backward edge.

Our Roadmap

- ◆ Graph Concepts
- ◆ Graph Traversal
 - ◆ Breath First Search (SSSP)
 - ◆ Depth First Search (DAG, topological sort)
- ◆ Shortest Path Algorithm (SP)
- ◆ Minimum Spanning Tree (MST)
- ◆ Strongly Connected Component (SCC)

Topological Sort on a DAG

- ◆ As mentioned earlier, depth first search (DFS) algorithm is surprisingly powerful. Indeed, we have already used it to detect efficiently whether a directed graph contains any cycle.
- ◆ We will use it to settle another classic problem: topological sort, in linear time.
- ◆ This algorithm is very elegant, and simple enough.

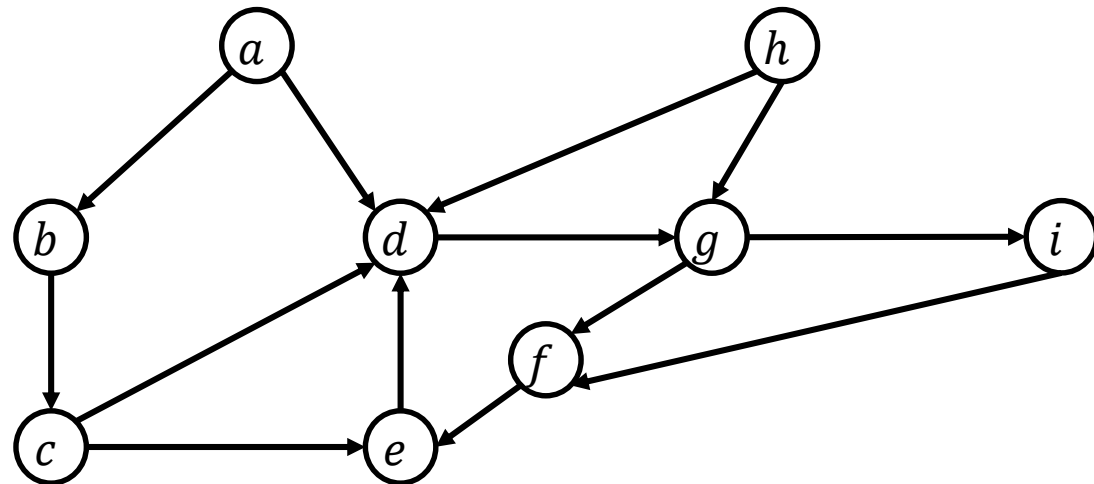
Topological Order

- Let $G=(V,E)$ be a directed acyclic graph (DAG).
- A topological order of G is an ordering of the vertices in V such that, for any edge (u,v) , it must hold that u precedes v in the ordering.
- Example: two possible topological orders:

- h, a, b, c, d, g, i, f, e

- a, h, b, c, d, g, i, f, e

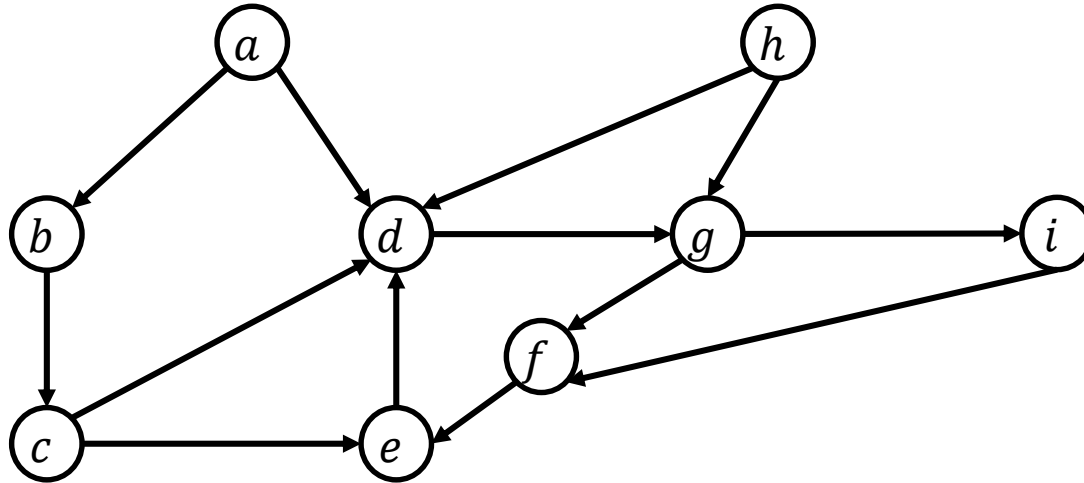
- a, h, d, b, c, g, i, f, e
is not topological order,
because of edge (c,d) .



The Topological Sort Problem

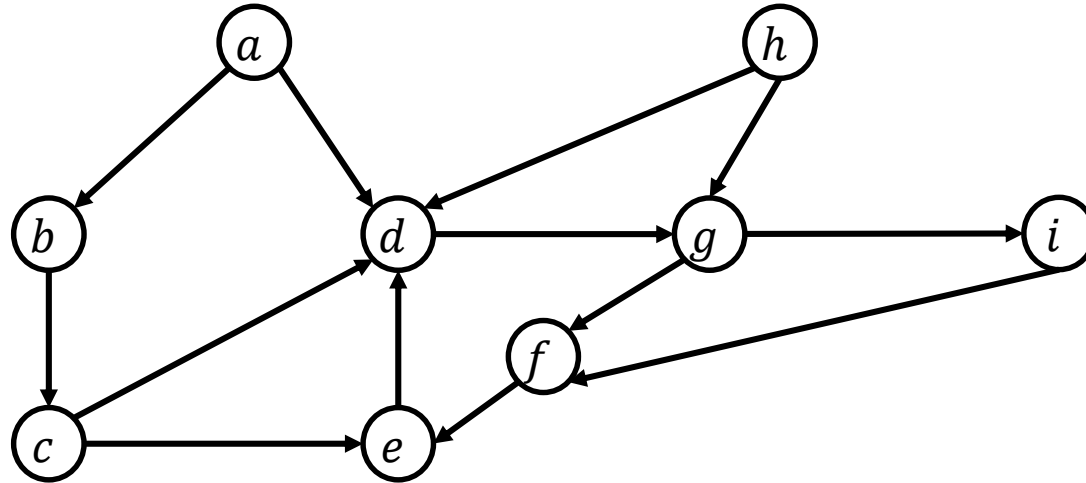
- ◆ Let $G=(V,E)$ be a directed acyclic graph (DAG). The goal of topological sort is to produce a topological order of G .
- ◆ Topological Sort Algorithm
 - ◆ Create an empty list L
 - ◆ Run DFS on G , whenever a vertex v turns red (i.e., it is popped from the stack), append it to L .
 - ◆ Output the reverse order of L
- ◆ The total running time is clearly $O(|V|+|E|)$

The Topological Sort Example



- ◆ Suppose we run DFS starting from a. The following is one possible order by which the vertices turn red:
 - ◆ e, f, i, g, d, c, b, a, h
- ◆ Therefore, we output h, a, b, c, d, g, i, f, e as a topological order.

The Topological Sort Example



- Suppose we run DFS starting from d, then restarting from h, then from a. The following is one possible order by which the vertices turn red:
 - e, f, i, g, d, h, c, b, a
- Therefore, we output a, b, c, h, d, g, i, f, e as a topological order.

Hint: Correctness Analysis

- ◆ We now prove that the algorithm is correct.
- ◆ Proof. Take any edge (u,v) . We will show that u turns red after v , which will complete the proof.
 - ◆ Consider the moment when u enters the stack, We argue that that currently v cannot be in the stack. Suppose that v was in the stack. As there must be a path chaining up all the vertices in the stack bottom up, we know that there is a path from v to u . Then, adding the edge (u,v) forms a cycle, contradicting the fact that G is a DAG.
 - ◆ v is red at this moment then obviously u will turn red after v .
 - ◆ v is white: then by the white path theorem of DFS, we know that v will become a proper descendant of u in the DFS-forest. Therefore, u will turn red after v .
- ◆ Every DAG has a topological order!

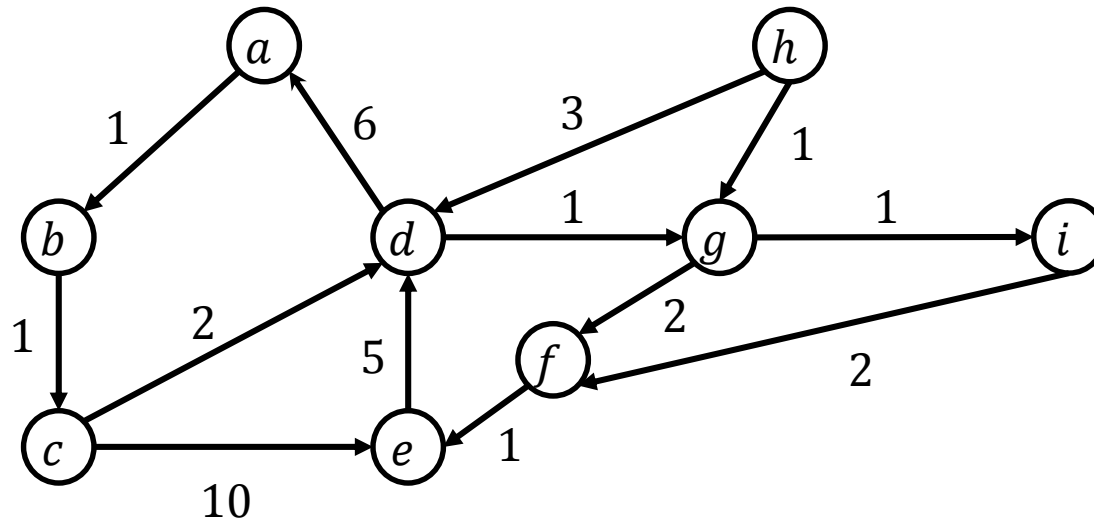
Our Roadmap

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Shortest Path

- ◆ Single source shortest path (SSSP)
 - ◆ BFS algorithm
 - ◆ All the edges have the same weight
- ◆ SSSP with arbitrary positive path (SP)
- ◆ Weight graph
 - ◆ Let $G=(V,E)$ be a directed graph. Let w be a function that maps each edge in E to a positive integer value. Specifically, for each $e \in E$, $w(e)$ is a positive integer value, which we call the weight of e .
 - ◆ A directed weighted graph is defined as the pair (G,w) .

Weighted Graph



- ◆ The integer on each edge indicates its weight. For example, $w(d,g)=1$, $w(g,f)=2$, and $w(c,e)=10$

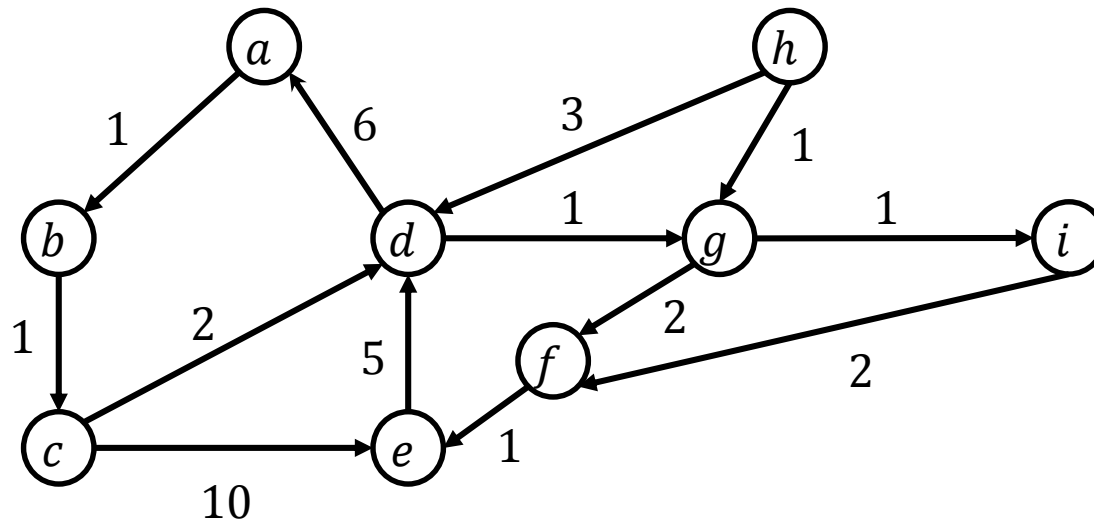
Shortest Path

- ◆ Consider a directed weighted graph defined by a directed graph $G=(V,E)$ and function w .
- ◆ Consider a path in G : $(v_1, v_2), (v_2, v_3), \dots, (v_l, v_{l+1})$, for some integer $l \geq 1$. We define the length of the path as: $\sum_{i=1}^l w(v_i, v_{i+1})$.
- ◆ Recall that we may also denote the path as: $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{l+1}$.
- ◆ Give two vertices $u, v \in V$, a shortest path from u to v is a path from u to v that has the minimum length among all the paths from u to v .
- ◆ If v is unreachable from u , then the shortest path distance from u to v is ∞ .

SSSP with Positive Weights

- ◆ Let (G, w) with $G=(V, E)$ be a directed weighted graph, where w maps every edge of E to a positive value.
- ◆ Give a vertex s in V , the goal of the SSSP problem is to find, for every other vertex $t \in V \setminus \{s\}$, a shortest path from s to t , unless t is unreachable from s .
- ◆ A subsequence property
 - ◆ Lemma: if $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{l+1}$ is a shortest path from v_1 to v_{l+1} , then for every i, j satisfying $1 \leq i \leq j \leq l + 1$, $v_i \rightarrow v_{i+1} \rightarrow \dots \rightarrow v_j$ is shortest path from v_i to v_j .
 - ◆ Proof: suppose that this is not true, then we can find a shorter path from v_i to v_j . Using that path to replace the original path from v_1 to v_{l+1} , which contradicts the fact that $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{l+1}$ is a shortest path.

Shortest Path Example



- ◆ The path $c \rightarrow e$ has length 10
- ◆ The path $c \rightarrow d \rightarrow g \rightarrow f \rightarrow e$ has length 6
- ◆ The second path is the shortest path from c to e
- ◆ We know that any subsequence of this path is also a shortest path. For example, $c \rightarrow d \rightarrow g \rightarrow f$ must be a shortest path from c to f .

Dijkstra's Algorithm

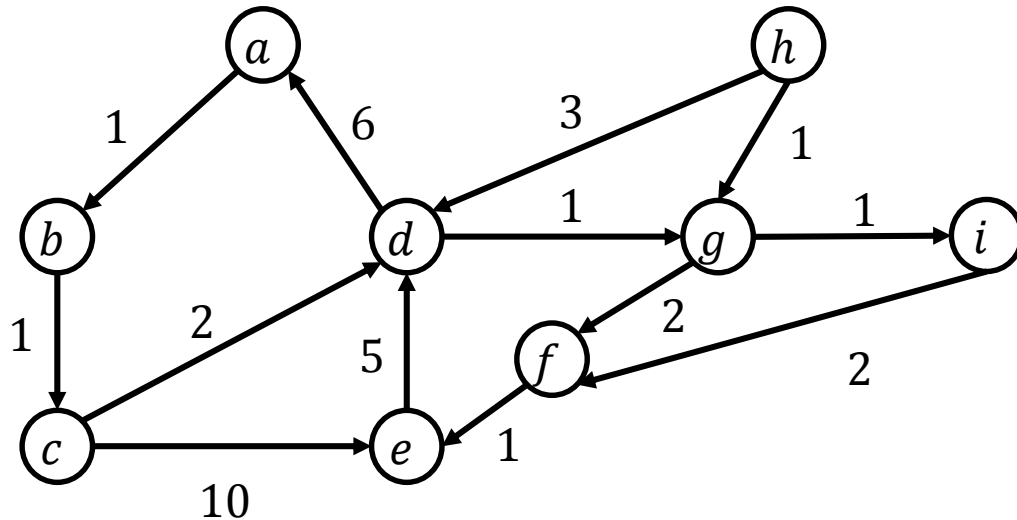
- ◆ We will first introduce the Dijkstra's algorithm for solving the SSSP with positive weights problem
- ◆ Utilizing the subsequence property, our algorithm will a shortest path tree that encodes all the shortest paths from the source vertex s .
- ◆ The edge relaxation idea
 - ◆ For every vertex $v \in V$, we will maintain a value $\text{dist}(v)$ that represents the length of the shortest path from s to v found so far.
 - ◆ At the end of the algorithm, we will ensure that every $\text{dist}(v)$ equal to the precise shortest path from s to v
 - ◆ A core operation in our algorithm is called edge relaxation. Given an edge (u,v) , we relax it as follows:
 - ◆ If $\text{dist}(v) < \text{dist}(u) + w(u,v)$, do nothing
 - ◆ Otherwise, reduce $\text{dist}(v)$ to $\text{dist}(u) + w(u,v)$

Dijkstra's Algorithm

- ◆ Set $\text{parent}(v) = \text{nil}$ for all vertices $v \in V$
- ◆ Set $\text{dist}(s) = 0$ and $\text{dist}(v) = \infty$ for all other vertices $v \in V$
- ◆ Set $S = V$
- ◆ Repeat the following until S is empty
 - ◆ Remove from S the vertex u with the smallest $\text{dist}(u)$.
/* next we relax all the outgoing edges of u^* */
 - ◆ For every outgoing edge (u,v) of u
 - ◆ If $\text{dist}(v) > \text{dist}(u) + w(u,v)$ then
 - ◆ Set $\text{dist}(v) = \text{dist}(u) + w(u,v)$, and $\text{parent}(v) = u$

Dijkstra's Algorithm Example

- Suppose that the source is c.

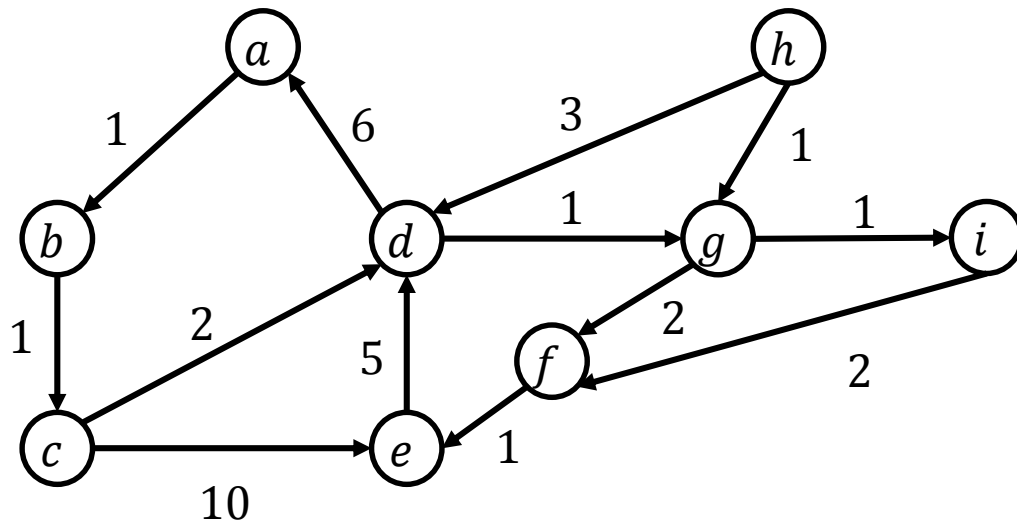


- $S = \{a, b, c, d, e, f, g, h, i\}$

Vertex v	dist(v)	parent(v)
a	∞	nil
b	∞	nil
c	0	nil
d	∞	nil
e	∞	nil
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

Dijkstra's Algorithm Example

- Relax the out-going edge of c (why is c ?)

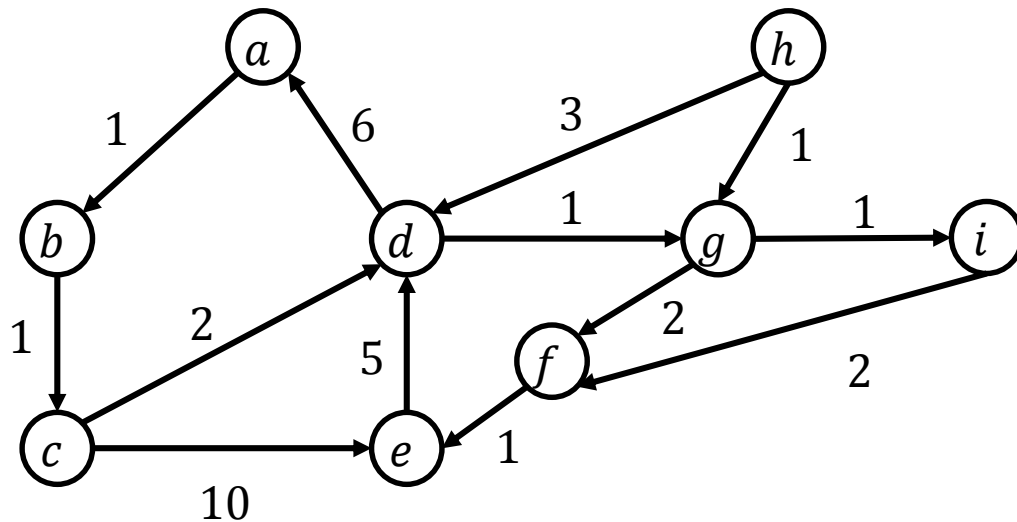


Vertex v	$\text{dist}(v)$	$\text{parent}(v)$
a	∞	nil
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	∞	nil
g	∞	nil
h	∞	nil
i	∞	nil

- $S = \{a, b, d, e, f, g, h, i\}$
- Note that c has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of d

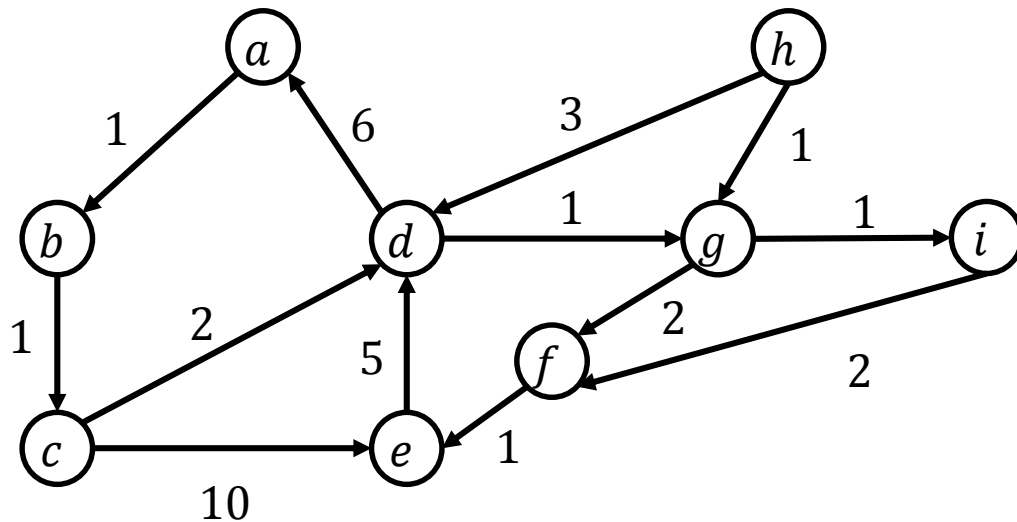


Vertex v	dist(v)	parent(v)
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	∞	nil
g	3	d
h	∞	nil
i	∞	nil

- $S = \{a, b, e, f, g, h, i\}$
- Note that d has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of g

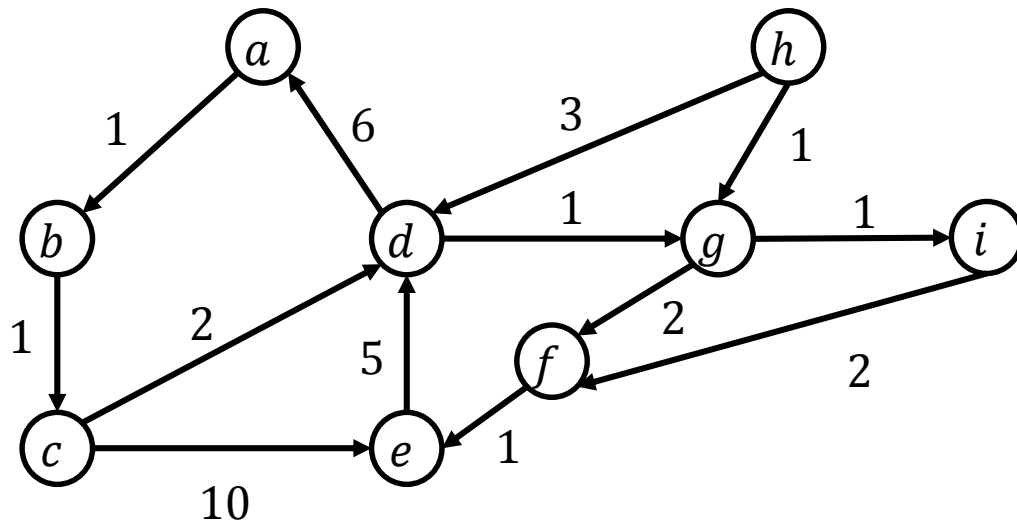


Vertex v	dist(v)	parent(v)
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	5	g
g	3	d
h	∞	nil
i	4	g

- $S = \{a, b, e, f, h, i\}$
- Note that g has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of i

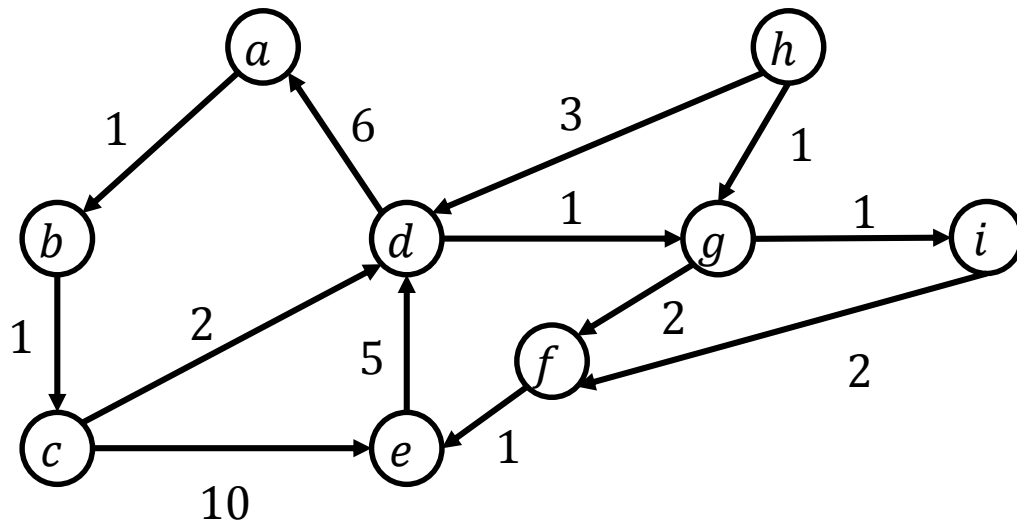


Vertex v	dist(v)	parent(v)
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	10	c
f	5	g
g	3	d
h	∞	nil
i	4	g

- $S = \{a, b, e, f, h\}$
- Note that i has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of f

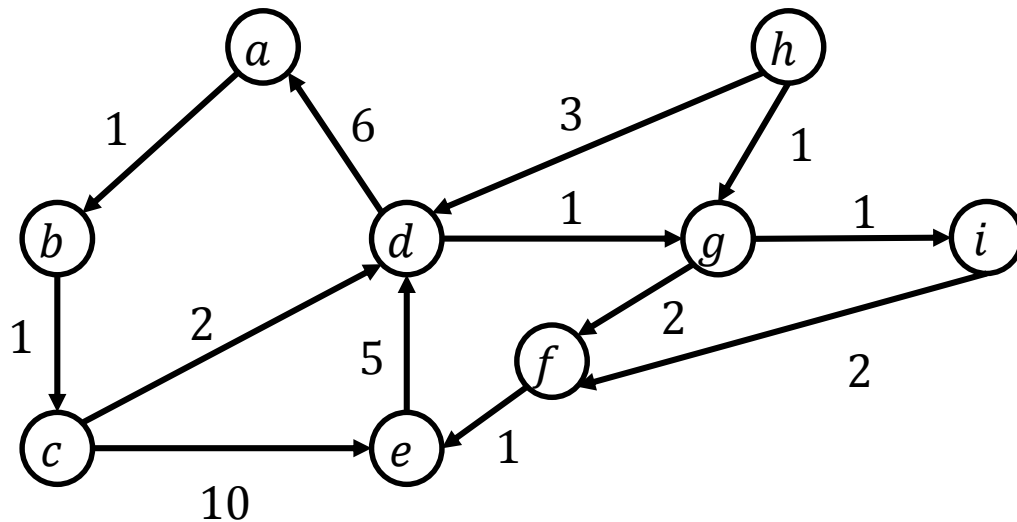


Vertex v	dist(v)	parent(v)
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

- $S = \{a, b, e, h\}$
- Note that f has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of e

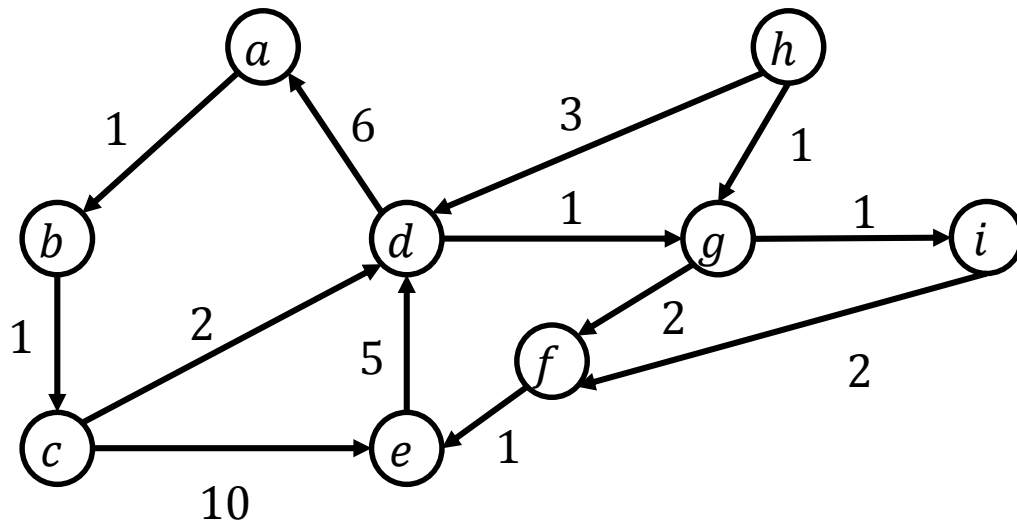


Vertex v	dist(v)	parent(v)
a	8	d
b	∞	nil
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

- $S = \{a, b, h\}$
- Note that e has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of a

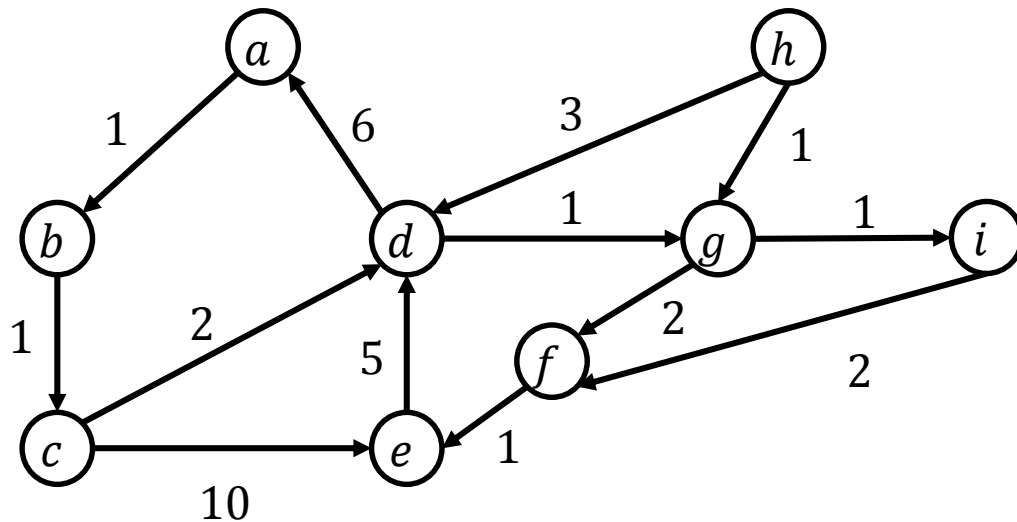


Vertex v	dist(v)	parent(v)
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

- $S = \{b, h\}$
- Note that a has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of b

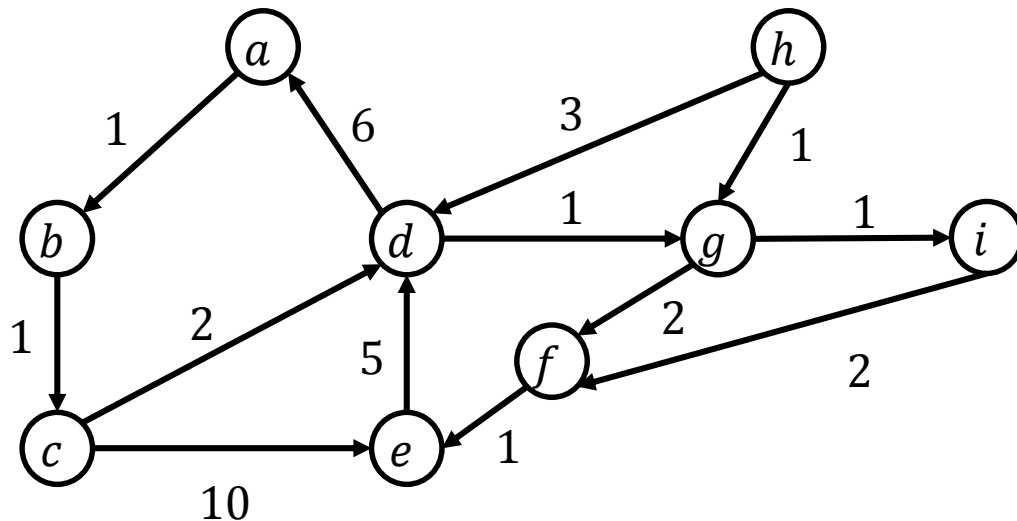


Vertex v	dist(v)	parent(v)
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

- $S = \{h\}$
- Note that b has been removed!

Dijkstra's Algorithm Example

- Relax the out-going edge of h

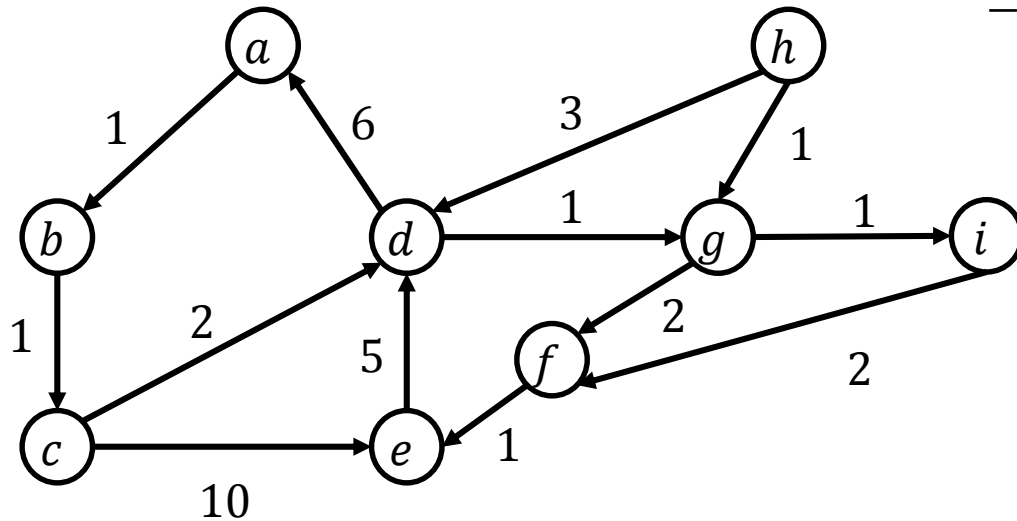


Vertex v	dist(v)	parent(v)
a	8	d
b	9	a
c	0	nil
d	2	c
e	6	f
f	5	g
g	3	d
h	∞	nil
i	4	g

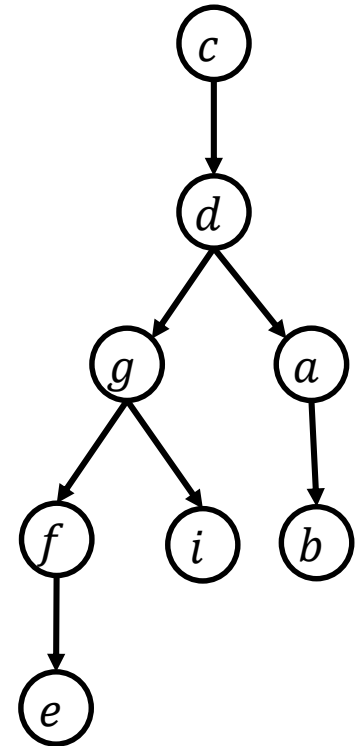
- $S = \{ \}$
- Note that h has been removed!
- All the shortest path distance are now final.

Constructing the SP Tree

- For every vertex v , if $u = \text{parent}(v)$ is not nil, then make v a child of u .



Vertex v	$\text{parent}(v)$
a	d
b	a
c	nil
d	c
e	f
f	g
g	d
h	nil
i	g



Correctness and Running Time

- ◆ It will be left as an exercise for you to prove that Dijkstra's algorithm is correct
- ◆ Just as equally instructive is an exercise for you to implement Dijkstra's algorithm in $O((|V|+|E|)\log|V|)$ time. Why?
- ◆ You have already learned all the data structure for this purpose. Now it is time to practice using them.

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Minimum Spanning Tree

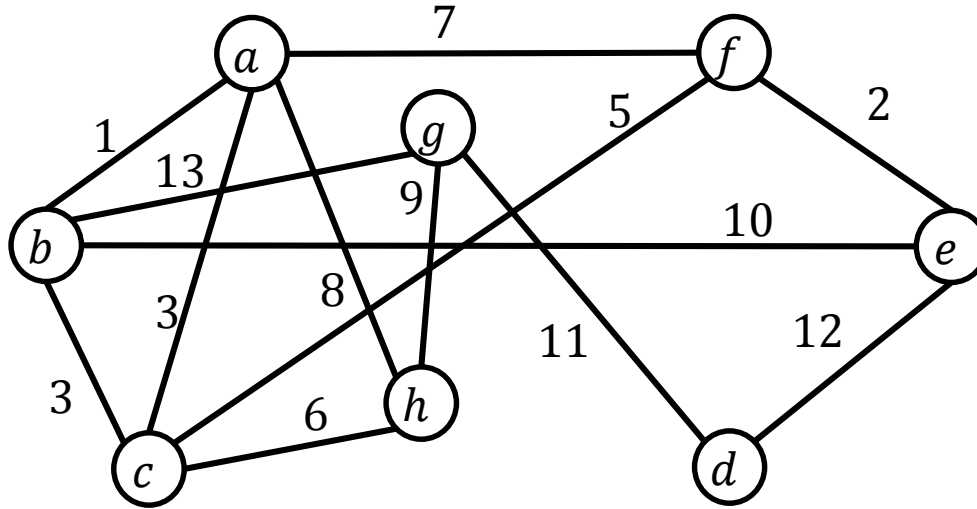
- ◆ We will study another classic problem: finding a minimum spanning tree of an undirected weighted graph.
- ◆ Interestingly, even though the problem appears rather different from SSSP (single source shortest path), it can be solved by an algorithm that is reminiscent of Dijkstra's algorithm

Undirected Weighted Graphs

- ◆ Let $G=(V, E)$ be an undirected graph. Let w be a function that maps each edge of G to a positive integer value. Specifically, for each edge e , $w(e)$ is a positive integer value, which we call the weight of e .
- ◆ An undirected weighted graph is defined as the pair (G,w)
- ◆ We will denote an edge between vertices u and v in G as $\{u,v\}$, instead of (u,v) , to emphasize that the ordering of u, v does not matter
- ◆ We consider that G is connected, namely, there is a path between any two vertices in V .

Undirected Weighted Graphs

◆ Example

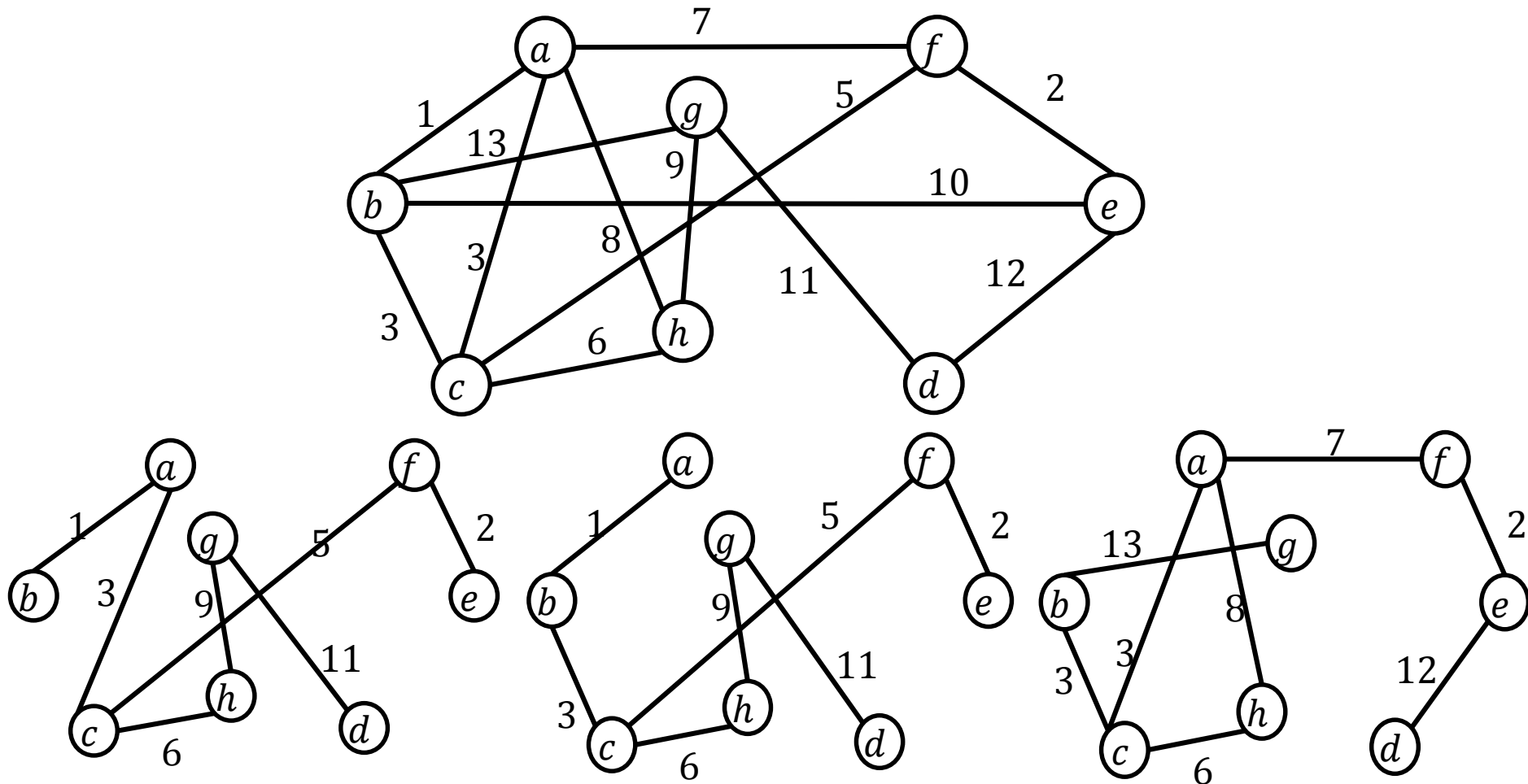


- ◆ The integer on each edge indicates its weight.
- ◆ For example, the weight of $\{g,h\}=9$,
- ◆ and that of $\{d,h\}$ is 11

Spanning Trees

- ◆ Remember that a tree is defined as a connected undirected graph with no cycles.
- ◆ Given a connected undirected weighted graph (G, w) with $G=(V, E)$, a spanning tree T is a tree satisfying the following conditions:
 - ◆ The vertex set of T is V .
 - ◆ Every edge of T is an edge of G .
- ◆ The cost of T is defined as the sum of the weights of all the edges in T (note that T must have $|V|-1$ edges)

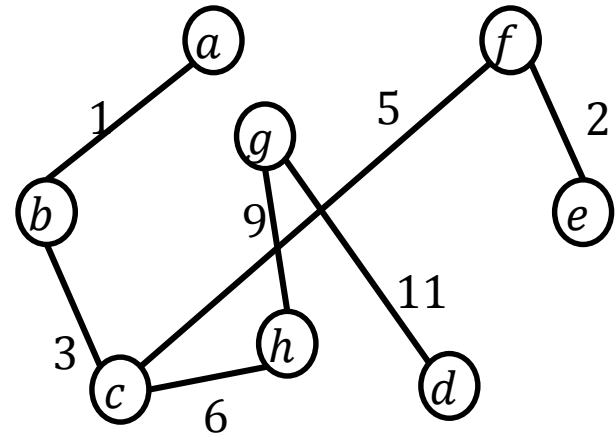
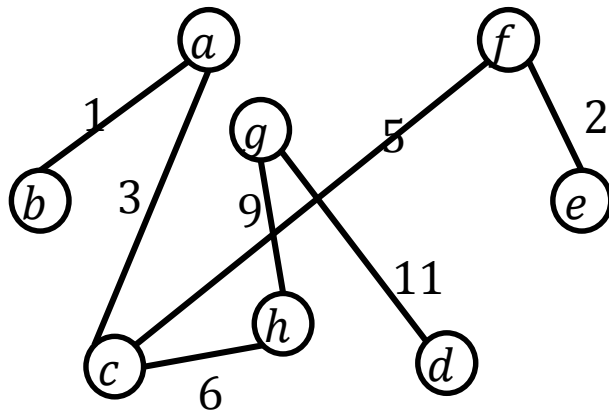
Spanning Trees Examples



- ◆ The second row shows three spanning trees. What are the costs?

Minimum Spanning Tree

- ◆ The minimum spanning tree problem
- ◆ Given a connected undirected weighted graph (G, w) with $G=(V, E)$, the goal of the minimum spanning tree (MST) problem is to find a spanning tree of the smallest cost.
- ◆ Such a tree is called an MST of (G, w)



- ◆ Both trees are MSTs. This means that MSTs may not be unique.

Prim's Algorithm

- ◆ Next, we will discuss an algorithm, called Prim's algorithm, for solving the MST problem.
- ◆ We assume that G is stored in the adjacency list format. Recall that an edge $\{u,v\}$ is represented twice: once by placing u in the adjacency list of v , and another time by placing v in the adjacency of u . The weight of $\{u,v\}$ is stored in both places.

Prim's Algorithm

- ◆ The algorithm grows a tree T_{mst} by including one vertex at a time, at any moment, it divides the vertex set V into two parts:
 - ◆ The set S of vertices that are already in T_{mst}
 - ◆ The set of other vertices: $V \setminus S$
- ◆ at the end of the algorithm, $S = V$
- ◆ If an edge connects a vertex in S and a vertex in $V \setminus S$, we call it an extension edge.
- ◆ At all times, the algorithm enforces the following lightest extension principle:
 - ◆ For every vertex $v \in V \setminus S$, it remembers which extension edge of v has the smallest weight, referred to as the lightest extension edge of v , and denoted as *best-ext*(v).

Prim's Algorithm

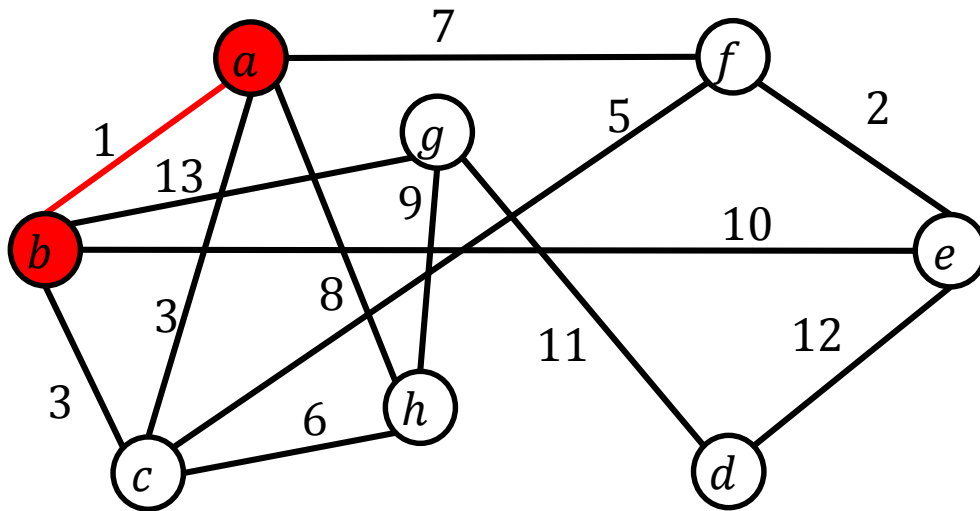
- ◆ 1. Let $\{u,v\}$ be an edge with the smallest weight among all edges
- ◆ 2. Set $S=\{u,v\}$. Initialize a tree T_{mst} with only one edge $\{u,v\}$.
- ◆ 3. Enforce the lightest extension principle:
 - ◆ For every vertex z of $V \setminus S$
 - ◆ If z is a neighbor of u , but not of v
 - ◇ $\text{best-ext}(z) = \text{edge } \{z, u\}$
 - ◆ If z is a neighbor of v , but not of u
 - ◇ $\text{best-ext}(z) = \text{edge } \{z, v\}$
 - ◆ If z is a neighbor of both u and v
 - ◇ $\text{best-ext}(z) = \text{the lighter edge between } \{z, u\} \text{ and } \{z, v\}$

Prim's Algorithm

- ◆ 4. Repeat the following until $S = V$:
 - ◆ 5. Get an extension edge of $\{u, v\}$ with the smallest weight
/* Without loss of generality, suppose $u \in S$, and */
 - ◆ 6. Add v to S , and add edge $\{u, v\}$ into T_{mst}
/* Next, we restore the lightest extension principle. */
 - ◆ For every edge $\{v, z\}$ of v :
 - ◆ If $z \notin S$ then
 - ◇ If $\text{best-ext}(z)$ is heavier than edge $\{v, z\}$ then
 - ◆ Set $\text{best-ext}(z) = \text{edge } \{v, z\}$

Prim's Algorithm Example

- Edge $\{a,b\}$ is the lightest of all. So, at the beginning $S = \{a, b\}$. The MST we are growing now has one edge $\{a,b\}$

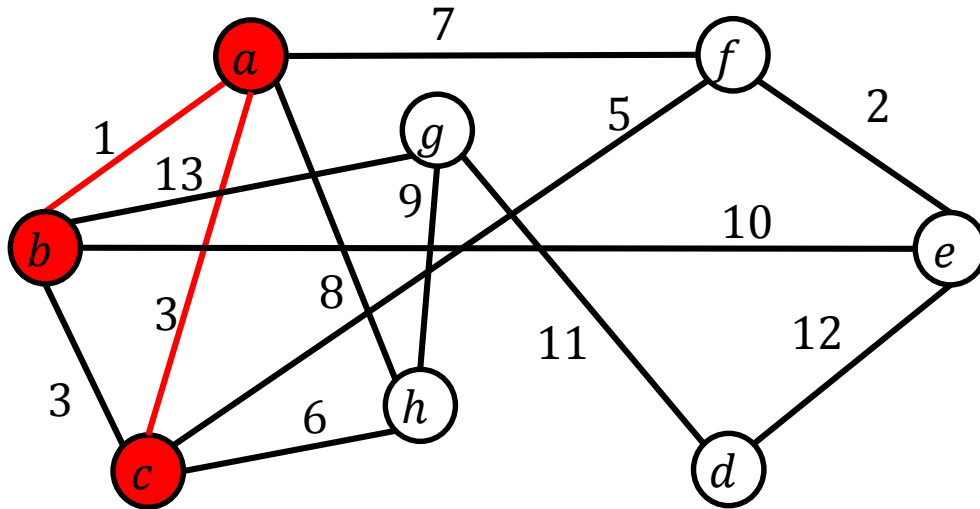


- Note: edge $\{c,a\}$ and $\{c,b\}$ have the same weight. Either of them can be $\text{best-ext}(c)$.

Vertex v	$\text{best-ext}(v)$ and weight
a	n/a
b	n/a
c	$\{c,a\}, 3$
d	nil, ∞
e	$\{e,b\}, 10$
f	$\{a,f\}, 7$
g	$\{g,b\}, 13$
h	$\{a,h\}, 8$

Prim's Algorithm Example

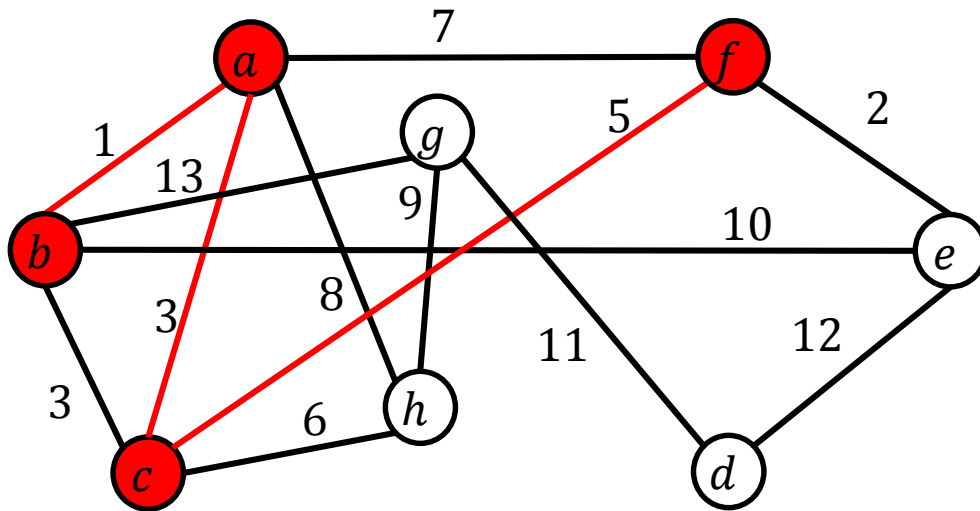
- Edge $\{c,a\}$ is the lightest extension edge. So, we add c to S , which now $S = \{a,b,c\}$, add edge $\{c,a\}$ into MST



Vertex v	best-ext(v) and weight
a	n/a
b	n/a
c	n/a
d	nil, ∞
e	$\{e,b\}, 10$
f	$\{c,f\}, 5$
g	$\{g,b\}, 13$
h	$\{c,h\}, 6$

Prim's Algorithm Example

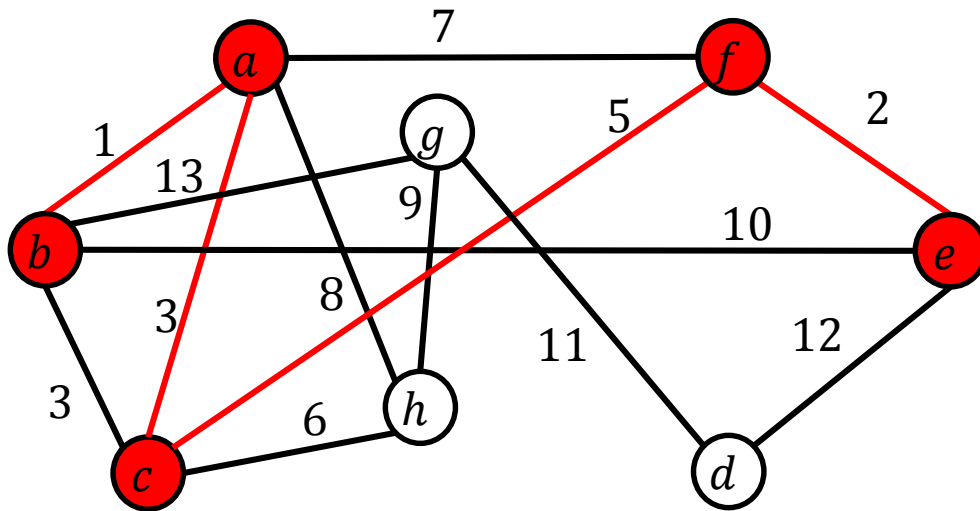
- Edge $\{c,f\}$ is the lightest extension edge. So, we add f to S , which now $S = \{a,b,c,f\}$, add edge $\{c,f\}$ into MST



Vertex v	best-ext(v) and weight
a	n/a
b	n/a
c	n/a
d	nil, ∞
e	$\{e,f\}, 2$
f	n/a
g	$\{g,b\}, 13$
h	$\{c,h\}, 6$

Prim's Algorithm Example

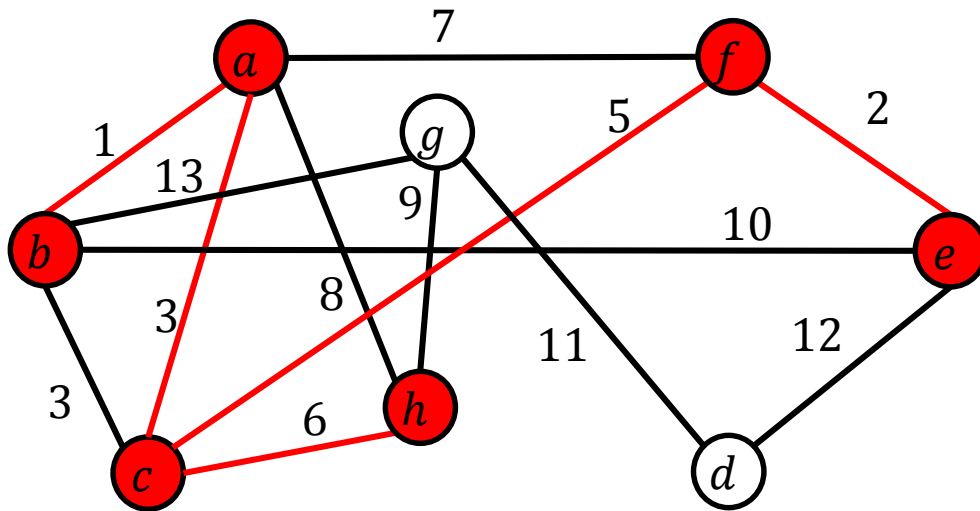
- Edge $\{e,f\}$ is the lightest extension edge. So, we add e to S , which now $S = \{a,b,c,f,e\}$, add edge $\{e,f\}$ into MST



Vertex v	best-ext(v) and weight
a	n/a
b	n/a
c	n/a
d	(e,d), 12
e	n/a
f	n/a
g	{g,b}, 13
h	{c,h}, 6

Prim's Algorithm Example

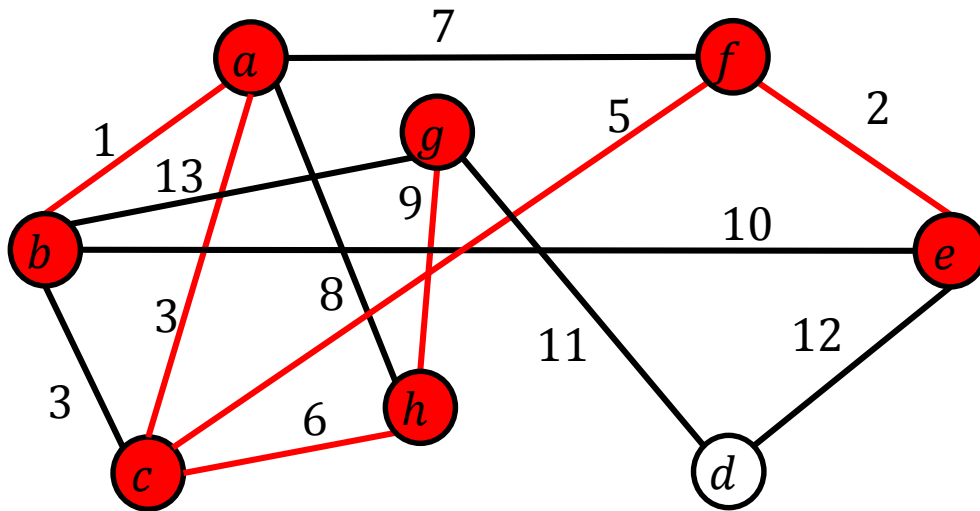
- Edge $\{c,h\}$ is the lightest extension edge. So, we add h to S , which now $S = \{a,b,c,f,e,h\}$, add edge $\{c,h\}$ into MST



Vertex v	best-ext(v) and weight
a	n/a
b	n/a
c	n/a
d	(e,d), 12
e	n/a
f	n/a
g	{g,h}, 9
h	n/a

Prim's Algorithm Example

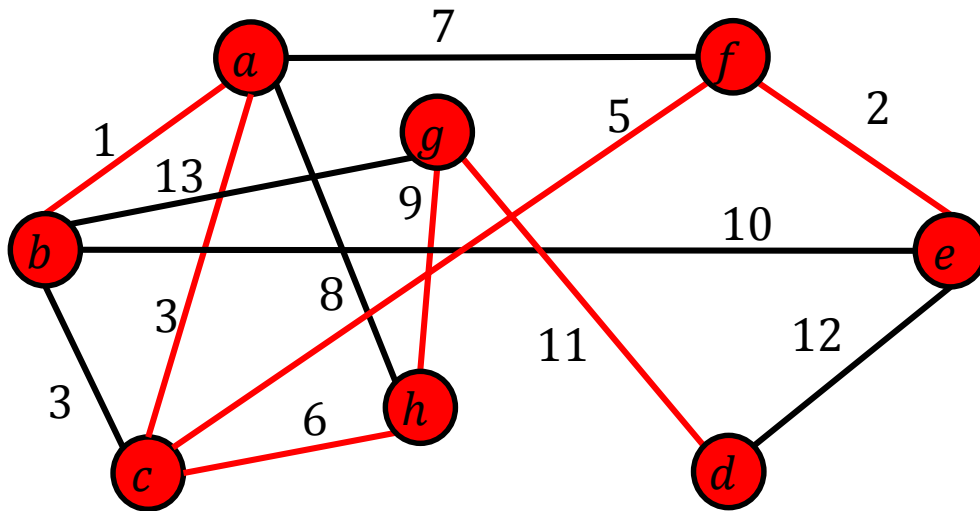
- Edge $\{g,h\}$ is the lightest extension edge. So, we add h to S , which now $S = \{a,b,c,f,e,h,g\}$, add edge $\{g,h\}$ into MST



Vertex v	best-ext(v) and weight
a	n/a
b	n/a
c	n/a
d	(g,d), 11
e	n/a
f	n/a
g	n/a
h	n/a

Prim's Algorithm Example

- Finally, edge $\{d,g\}$ is the lightest extension edge. So, we add d to S , which now $S = \{a,b,c,f,e,h,g,d\}$, add edge $\{d,g\}$ into MST



Vertex v	best-ext(v) and weight
a	n/a
b	n/a
c	n/a
d	n/a
e	n/a
f	n/a
g	n/a
h	n/a

- We have obtained our final MST.

Time Complexity Analysis

- ◆ A priority queue Q (min-heap) was employed in Prim's algorithm, what is the key of node in Q ?
- ◆ Line 1 & 2: $O(1)$
- ◆ Line 3: $O(|E|)$
- ◆ Line 4: $O(|V|)$
- ◆ Line 5: $O(|V| \log |V|)$
- ◆ Line 6: $O(|V|)$
- ◆ Line 7: $O(|E| \log |V|)$, Total: $O((|V|+|E|) \log |V|)$
- ◆ Remark: Using the Fibonacci Heap, will not cover in this course, we can improve the running time to $O(|V| \log |V| + |E|)$

Hint: Correctness Proof

- ◆ **Claim:** For any $i \in [1, |V|-1]$, there must be an MST containing all the first i edges chosen by the algorithm
- ◆ Then the algorithm's correctness follows from the above claim at $i = |V|-1$
- ◆ We prove it by induction the sequence of the edges added to the tree
- ◆ Base case: $i=1$, let $\{u,v\}$ be the edge with the smallest weight in the graph, the edge must exist in some MST
- ◆ Inductive case: the claim holds for $i \leq k-1$
- ◆ We prove it also hold for $i=k$

Our Roadmap

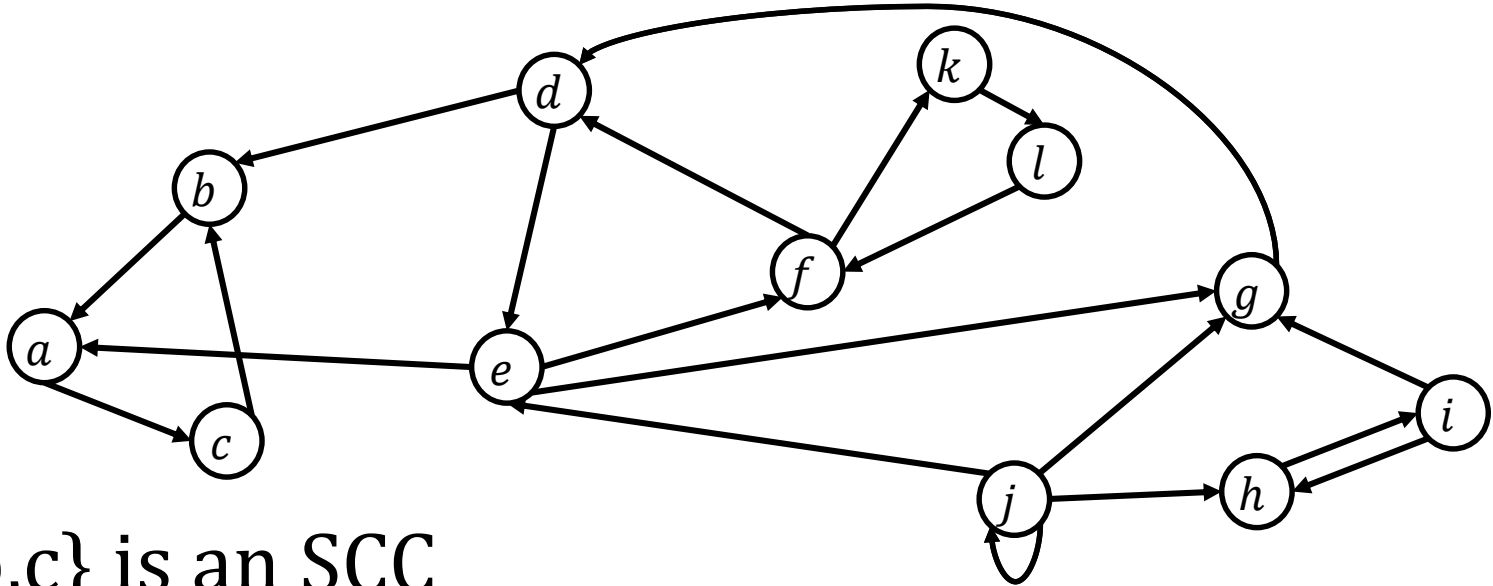
- ◆ Graph Concepts
- ◆ Graph Traversal
 - ◆ Breath First Search (SSSP)
 - ◆ Depth First Search (DAG, topological sort)
- ◆ Shortest Path Algorithm (SP)
- ◆ Minimum Spanning Tree (MST)
- ◆ Strongly Connected Component (SCC)

Strongly Connected Components

- ◆ Let $G=(V,E)$ be a directed graph.
- ◆ A strongly connected component (SCC) of G is a subset S of V such that:
 - ◆ For any two vertices $u, v \in S$, it must hold that:
 - ◆ There is a path from u to v
 - ◆ There is a path from v to u
 - ◆ S is maximal in the sense that we cannot put any more vertex into S without violating the above property
- ◆ It seems to be rather difficult at first glance, the algorithm is once again very simple, run DFS only twice.

SCC Example

- Consider the following graph:



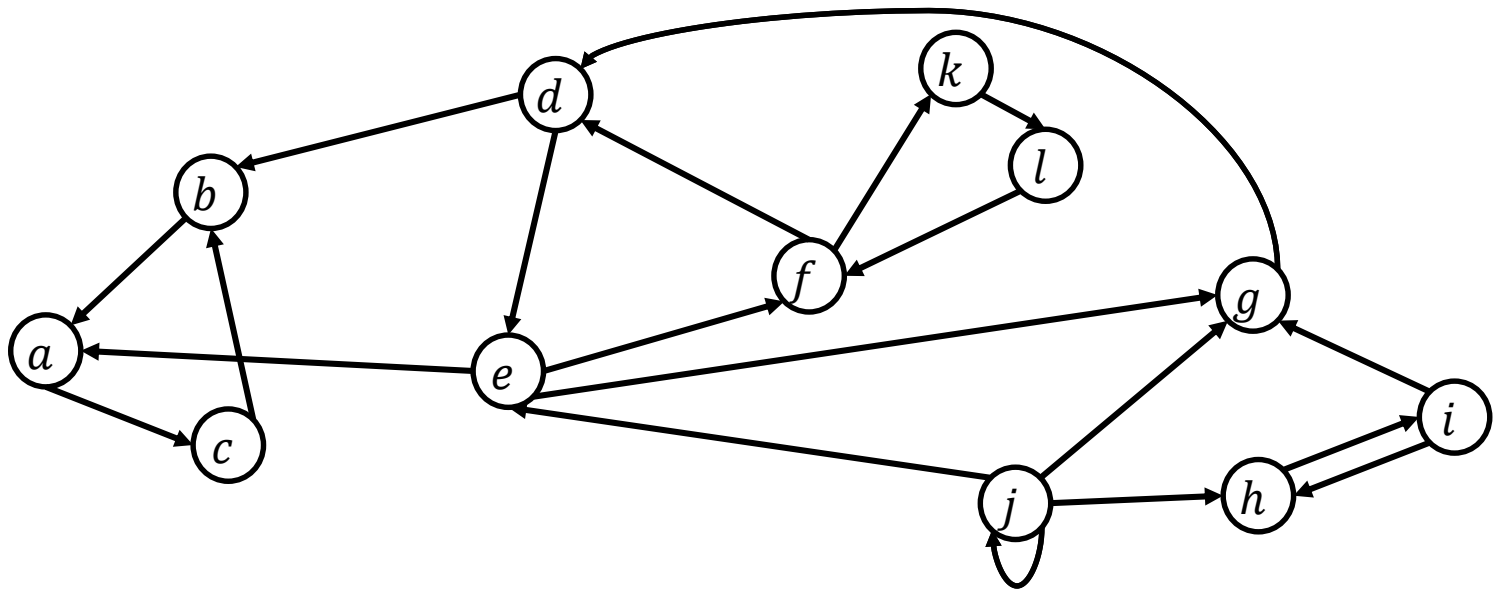
- $\{a, b, c\}$ is an SCC
- $\{a, b, c, d\}$ is not an SCC
- $\{d, e, f, k, l\}$ is not an SCC (why?)
- $\{e, d, f, k, l, g\}$ is an SCC

SCCs are Disjoint

- ◆ Theorem: Suppose that S_1 and S_2 are both SCCs of G , Then $S_1 \cap S_2 = \emptyset$
- ◆ Proof: Assume that there is a vertex v in both S_1 and S_2 . Then, for any vertex $u_1 \in S_1$ and any vertex $u_2 \in S_2$:
 - ◆ There is a path from u_1 to u_2 : we can first go from u_1 to v within S_1 , and then from v to u_2 within S_2 .
 - ◆ Likewise, there is also a path from u_2 to u_1 .Hence, neither S_1 and S_2 is maximal, contradicting the fact that they are SCCs.

Finding SCCs

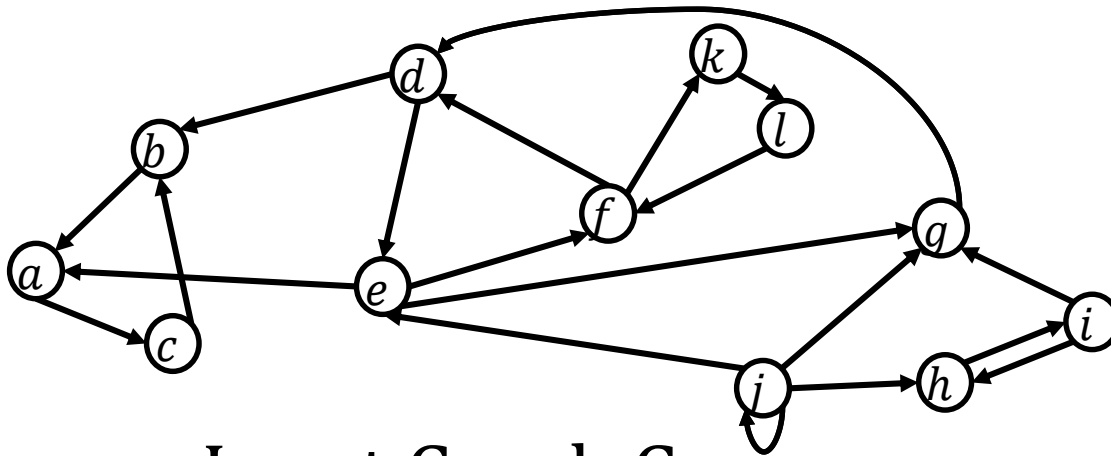
- Given a directed graph $G = (V, E)$, the goal of the finding strongly connected components problem is to divide V into disjoint subsets, each of which is an SCC.



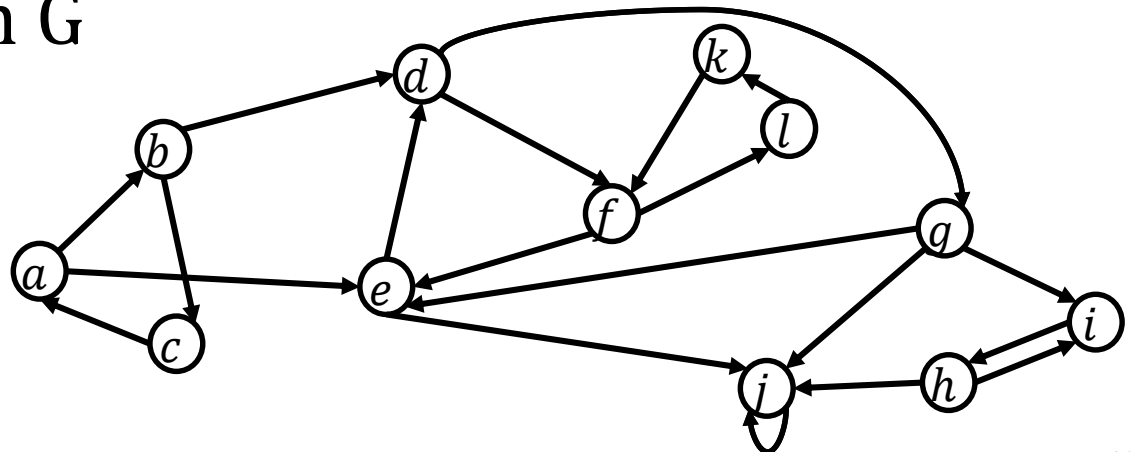
- The goal is to output the following 4 SCCs: $\{a, b, c\}$, $\{d, e, f, g, k, l\}$, $\{h, i\}$, and $\{j\}$

Finding SCCs Algorithm

- ◆ Step 1: obtain the reverse graph G^R by reversing the directions of all the edges in G .



Input Graph G



Reverse Graph G^R

Finding SCCs Algorithm

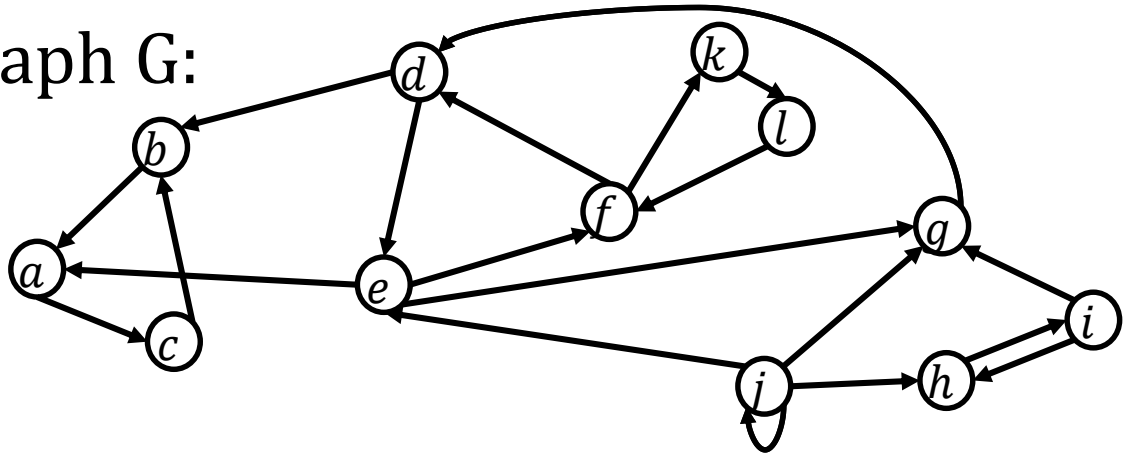
- ◆ Step 2: Perform DFS on G^R , and obtain the sequence L^R that the vertices in G^R turn red (i.e., whenever a vertex is popped out of the stack, append it to L^R)
- ◆ Obtain L as the reverse order of L^R
- ◆ We may perform DFS starting from any vertex. The following is a possible order that the vertices are discovered: f,l,k,e,j,d,g,i,h,a,b,c
- ◆ The corresponding turn-red sequence is
- ◆ $L^R = \{k,l,j,h,i,g,d,e,f,c,b,a\}$
- ◆ Hence $L = \{a,b,c,f,e,d,g,i,h,j,l,k\}$

Finding SCCs Algorithm

- ◆ Step 3: Perform DFS on the original graph G by obeying the following rules:
 - ◆ Rule 1: start the DFS at the first vertex of L
 - ◆ Rule 2: whenever a restart is needed, start from the first vertex of L that is still white.
- ◆ Output the vertices in each DFS-tree as an SCC

Finding SCCs Algorithm

- From the last step, we have $L = \{a, b, c, f, e, d, g, i, h, j, l, k\}$
- The original graph G:



- Starting DFS from a, which discovered {a,b,c}
- Restart from f, which discovered {f,k,l,d,e,g}
- Restart from i, which discovered {i,h}
- Restart from j, which discovered {j}
- The DFS returns 4 DFS-tree, whose vertex sets are as above, Each vertex set constitutes an SCC.

Running Time Analysis

- ◆ Steps 1 and 2 obviously require only $O(|V|+|E|)$ time.
- ◆ Regarding Step 3, the DFS itself takes $O(|V|+|E|)$, but how about the cost of implement Rule 2.
- ◆ Namely, whenever, DFS needs a restart, how do we find the first white vertex in L efficiently?
- ◆ It can be done in $O(|V|)$ total time.
- ◆ Hence, the overall execution time is $O(|V|+|E|)$

Hint: Correctness Proof

- ◆ Let G be the input directed graph, with SCCs S_1, S_2, \dots, S_t for some $t \geq 1$
- ◆ Let us define a SCC graph G^{SCC} as follows:
 - ◆ Each vertex in G^{SCC} is a distinct SCC in G .
 - ◆ Consider two vertices S_i and S_j , G^{SCC} has an edge from S_i to S_j if and only if:
 - ◆ $i \neq j$
 - ◆ There is a path in G from a vertex in S_i to a vertex in S_j
- ◆ G^{SCC} is a DAG, define an SCC as a sink SCC if it has no outgoing edge in G^{SCC}
- ◆ Lemma: There must be at least one sink SCC in G^{SCC}

Hint: Correctness Proof

- ◆ Let S be a sink SCC in G^{SCC} . Suppose that we perform a DFS starting from any vertex in S . Then the first DFS-tree output must include all and only the vertex in S .
- ◆ Finding SCC: The strategy
 - ◆ 1. Performing DFS from any vertex in a sink SCC S
 - ◆ 2. Delete all vertices of S from G , as well as their edges
 - ◆ 3. Accordingly, delete S from G^{SCC} , as well as its edges.
 - ◆ 4. Repeat from Step 1, until G is empty.
- ◆ Lemma: Let S_1, S_2 be SCCs such that there is a path from S_1 to S_2 in G^{SCC} . In the ordering of L , the earliest vertex in S_2 must come before the earliest vertex in S_1

Thank You!