

# General Form of Color Charge of the Quark

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## Abstract

In Maxwell theory the constant electric charge  $e$  of the electron is consistent with the continuity equation  $\partial_\mu j^\mu(x) = 0$  where  $j^\mu(x)$  is the current density of the electron where the repeated indices  $\mu = 0, 1, 2, 3$  are summed. However, in Yang-Mills theory the Yang-Mills color current density  $j^{\mu a}(x)$  of the quark satisfies the equation  $D_\mu[A]j^{\mu a}(x) = 0$  which is not a continuity equation ( $\partial_\mu j^{\mu a}(x) \neq 0$ ) which implies that the color charge of the quark is not constant where  $a = 1, 2, \dots, 8$  are the color indices. Since the charge of a point particle is obtained from the zero ( $\mu = 0$ ) component of a corresponding current density by integrating over the entire (physically) allowed volume, the color charge  $q^a(t)$  of the quark in Yang-Mills theory is time dependent. In this paper we derive the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in Yang-Mills theory in SU(3) where  $a = 1, 2, \dots, 8$ .

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## I. INTRODUCTION

The electric and magnetic phenomena in the nature originates from the electric charge. The electric charge of an electron is a fundamental quantity of the nature which is constant and has the experimentally measured value  $e = 1.6 \times 10^{-19}$  coulombs. Note that there are two types of electric charges in the nature, 1) positive (+) charge and 2) negative (-) charge. The charge of the electron is negative (-). The electric charge produces the electromagnetic force in the nature.

However, inside a hadron (such as proton and neutron) the electromagnetic force can not bind the quarks together. The force which is responsible for bound state hadron formation is called the strong force or the color force which is a fundamental force of the nature. This strong force or the color force is not produced from the electric charge of the quark but it is produced from the color charges of the quark. The color charge is a fundamental charge of the nature which exists inside hadrons.

Note that in the Maxwell theory in electrodynamics the constant electric charge  $e$  of the electron is consistent with the continuity equation

$$\partial_\mu j^\mu(x) = 0 \quad (1)$$

where  $j^\mu(x)$  is the electric current density of the electron and  $\mu = 0, 1, 2, 3$ . All the repeated indices are summed in this paper. However, in the Yang-Mills theory the Yang-Mills color current density  $j^{\mu a}(x)$  of the quark satisfies the equation [1, 2]

$$D_\mu[A]j^{\mu a}(x) = 0, \quad D_\mu^{ab}[A] = \delta^{ab}\partial_\mu + g f^{acb} A_\mu^c(x), \quad a, b, c = 1, 2, \dots, 8 \quad (2)$$

which is not a continuity equation ( $\partial^\mu j_\mu^a(x) \neq 0$ ) which implies that the color charge of the quark is not constant where  $A^{\mu a}(x)$  is the Yang-Mills potential (color potential). Since the charge of a point particle can be obtained from the zero ( $\mu = 0$ ) component of a corresponding current density by integrating over the entire (physically) allowed volume, the color charge  $q^a(t)$  of the quark in Yang-Mills theory is time dependent.

Also, it is important that the conserved color charges are not directly observable – only color representations – because of the unbroken gauge invariance of QCD. Thus, the concept of constant color charge seems unphysical. The form of the color charge of the quark can be obtained from the zero component of the color current density  $j_0^a(x)$  of the quark. For earlier works on classical Yang-Mills theory, see [3–9].

The color current density of the quark in eq. (2) is related to the quark field  $\psi_i(x)$  via the equation [1, 2]

$$j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T_{ij}^a\psi_j(x), \quad a = 1, 2, \dots, 8; \quad i, j = 1, 2, 3 \quad (3)$$

where  $T^a = \frac{\lambda^a}{2}$  are eight generators of SU(3) and  $\lambda^a$  are eight Gell-Mann matrices. Since the color current density  $j^{\mu a}(x)$  of a quark in eq. (3) has eight color components  $a = 1, 2, \dots, 8$  one finds that there are eight time dependent color charges of a quark.

We denote eight time dependent fundamental color charges of a quark by  $q^a(t)$  where  $a = 1, 2, \dots, 8$  are color indices. These eight time dependent fundamental color charges  $q^a(t)$  of a quark are independent of quark flavor, *i.e.*, a color charge  $q^a(t)$  of the  $u$  (up) quark is same as that of  $d$  (down),  $S$  (strange),  $c$  (charm),  $B$  (bottom) or  $t$  (top) quark.

It is useful to remember that the indices  $i=1,2,3=\text{RED, BLUE, GREEN}$  are not color charges of the quark but they are color indices of the quark field  $\psi_i(x)$  in eq. (3). Color charges  $q^a(t)$  of a quark are functions and they have values. This is analogous to electric charge ' $-e$ ' of the electron which is not just a ' $-$ ' sign but it has a constant value  $e = 1.6 \times 10^{-19}$  coulombs. Similarly RED, BLUE, GREEN symbols are not color charges of a quark but  $q^a(t)$  are the color charges of a quark where  $a = 1, 2, \dots, 8$ . Hence one finds that a quark does not have three color charges but a quark has eight color charges. Another argument in favour of eight color charges as opposed to three (RED, BLUE, GREEN) is that the latter depends on the representation while the former, which is the number of generators, does not. In the Maxwell theory the electric charge  $e$  produces the electromagnetic force in the nature and in the Yang-Mills theory the color charges  $q^a(t)$  produce the color force or the strong force in the nature.

In the Yang-Mills theory in SU(2) the color current density of a fermion is given by

$$j^{\mu i}(x) = g\bar{\psi}_k(x)\gamma^\mu \tau_{kn}^i\psi_n(x), \quad i = 1, 2, 3; \quad k, n = 1, 2 \quad (4)$$

where  $\tau^i = \frac{\sigma^i}{2}$  are three generators of SU(2) and  $\sigma^i$  are three Pauli matrices. Hence one finds that there are three time dependent color charges  $q_i(t)$  of a fermion in the Yang-Mills theory in SU(2) where  $i = 1, 2, 3$ . We find that the general form of three time dependent color charges of a fermion in the Yang-Mills theory in SU(2) is given by

$$q_1(t) = g \times \sin\theta(t) \times \cos\phi(t),$$

$$\begin{aligned} q_2(t) &= g \times \sin\theta(t) \times \sin\phi(t), \\ q_3(t) &= g \times \cos\theta(t) \end{aligned} \tag{5}$$

where the time dependent real phases  $\theta(t)$  and  $\phi(t)$  can not be independent of time  $t$  and the allowed ranges of  $\theta(t)$ ,  $\phi(t)$  are given by

$$\frac{\pi}{3} \leq \theta(t) \leq \frac{2\pi}{3}, \quad -\pi < \phi(t) \leq \pi. \tag{6}$$

[See eqs. (142) and (143) for the derivation of eqs. (5) and (6)].

In this paper we will derive the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in Yang-Mills theory in SU(3) where  $a = 1, 2, \dots, 8$ . We find that the general form of eight time dependent fundamental color charges of the quark in Yang-Mills theory in SU(3) is given by

$$\begin{aligned} q_1(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \cos\phi_{12}(t), \\ q_2(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \sin\phi_{12}(t), \\ q_3(t) &= g \times \cos\theta(t) \times \sin\phi(t) \\ q_4(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \cos\phi_{13}(t), \\ q_5(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \sin\phi_{13}(t), \\ q_6(t) &= g \times \sin\theta(t) \times \cos\sigma(t) \times \cos\phi_{23}(t), \\ q_7(t) &= g \times \sin\theta(t) \times \cos\sigma(t) \times \sin\phi_{23}(t), \\ q_8(t) &= g \times \cos\theta(t) \times \cos\phi(t) \end{aligned} \tag{7}$$

where the ranges of the real time dependent phases are given by

$$\begin{aligned} \sin^{-1}\left(\sqrt{\frac{2}{3}}\right) &\leq \theta(t) \leq \pi - \sin^{-1}\left(\sqrt{\frac{2}{3}}\right), & 0 \leq \sigma(t), \eta(t) \leq \frac{\pi}{2}, \\ 0 &\leq \phi(t) \leq 2\pi, & -\pi < \phi_{12}(t), \phi_{13}(t), \phi_{23}(t) \leq \pi. \end{aligned} \tag{8}$$

It can be seen that the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (7) depend on the universal coupling  $g$  which is a fundamental quantity of the nature (a physical observable) that appears in the Yang-Mills Lagrangian density [1, 2].

Since the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (7) is linearly proportional to  $g$  and the Yang-Mills potential (color potential)  $A^{\mu a}(x)$  contains

infinite powers of  $g$  (see eq. (98) or [10]) we find that the definition of the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (7) is independent of the Yang-Mills potential  $A^{\mu a}(x)$  [see section XIV for detailed discussion about this].

It can be seen that the general form of three time dependent color charges  $q_i(t)$  of a fermion in  $SU(2)$  in eq. (5) and the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in  $SU(3)$  in eq. (7) are consistent with the fact that  $SU(2)$  and  $SO(3)$  are locally isomorphic, while  $SU(3)$  and  $SO(8)$  are not [see sections IX and XV for more discussion on this].

It should be remembered that the universal coupling  $g$  (which is a physical observable, a fundamental quantity of the nature) is the only parameter (apart from the mass  $m$  of the quark) that appears in the classical Yang-Mills lagrangian density [1, 2] (see eq. (47) below). Note that in the derivation of eqs. (7) and (8) we have fixed the first Casimir (quadratic Casimir) invariant

$$q^a(t)q^a(t) = g^2 \quad (9)$$

of  $SU(3)$  to be  $g^2$  [see eqs. (61) and (157)]. Hence we find that the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in Yang-Mills theory in  $SU(3)$  in eq. (7) depend on  $g$  and seven time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  where the ranges of these seven time dependent phases are given by eq. (8). Since the first Casimir (quadratic Casimir) invariant  $q^a(t)q^a(t)$  and the second Casimir (cubic Casimir) invariant  $d_{abc}q^a(t)q^b(t)q^c(t)$  of  $SU(3)$  are two independent Casimir invariants, one expects that if the second Casimir (cubic Casimir) invariant  $d_{abc}q^a(t)q^b(t)q^c(t)$  of  $SU(3)$  corresponds to any physical observable then that physical observable should be experimentally measured and that physical observable should be different from  $g$  because the first Casimir (quadratic Casimir) invariant  $q^a(t)q^a(t)$  of  $SU(3)$  is fixed to be  $g^2$ , see eq. (9). If such a physical observable exists in the nature and is fixed to be, say  $C_3$ , [for example by experiments] where the fixed  $C_3$  is given by

$$d_{abc}q^a(t)q^b(t)q^c(t) = C_3 \quad (10)$$

then one finds that

$$\phi_{13}(t) = \phi_{12}(t) + \phi_{23}(t) + \cos^{-1}\left[\frac{1}{\sin^3\theta(t)\sin^2\sigma(t)\cos\sigma(t)\sin2\eta(t)}\right] \times \left[\frac{2C_3}{3g^3}\right]$$

$$\begin{aligned}
& - \sin^2\theta(t) \cos\theta(t) \sin\phi(t) [\sin^2\sigma(t) \sin^2\eta(t) - \cos^2\sigma(t)] \\
& - \frac{1}{\sqrt{3}} \cos\theta(t) \cos\phi(t) [3 \sin^2\theta(t) \sin^2\sigma(t) \cos^2\eta(t) + 3 \cos^2\theta(t) \sin^2\phi(t) + \frac{\cos^2\theta(t) \cos^2\phi(t)}{3} - 1]]]
\end{aligned} \tag{11}$$

[see eq. (213) for derivation of eq. (11)] in which case the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (7) depend on  $g$ ,  $C_3$  and six time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{23}(t)$  where  $\phi_{13}(t)$  is given by eq. (11). However, note that even if all the physical observables are gauge invariant but not all the gauge invariants are physical observables. Hence if there exists no physical observable in the nature which is related to the fixed value  $C_3$  as given by eq. (10) [for example if one can not find any such observable from the experiments] then the second Casimir invariant (cubic Casimir invariant)  $d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) satisfies the range  $-\frac{g^3}{\sqrt{3}} \leq d_{abc}q^a(t)q^b(t)q^c(t) \leq \frac{g^3}{\sqrt{3}}$  [see eq. (203)] in which case the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (7) depend on  $g$  and seven time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  where the ranges of these seven time dependent phases are given by eq. (8) [see section XV for detailed discussion on this].

Hence we find that the general form of eight time dependent fundamental color charges of the quark in Yang-Mills theory in SU(3) is given by eq. (7) where  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  are real time dependent phases. Note that if all of these seven phases become constants then all the eight color charges  $q^a$  become constants in which case the Yang-Mills potential  $A^{\mu a}(x)$  reduces to Maxwell-like (abelian-like) potential (see eqs. (98) and (108) or [10]). Since the abelian-like potential can not explain confinement of quarks inside (stable) proton one finds that all these seven real phases can not be constants. Hence one finds that the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark which we have derived in eq. (7) may provide an insight to the question why quarks are confined inside a (stable) proton once the exact form of these time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  are found out [see section XVI for a detailed discussion on this]. It should be remembered that the static systems in the Yang-Mills theory are, in general, not abelian-like. For example, unlike Maxwell theory where the electron at rest produces (abelian) Coulomb potential, the quark at rest in Yang-Mills theory does not produce abelian-like potential (see eq. (108) and sections VII, VIII or

[10] for details).

We will provide the derivations of eqs. (7) and (8) in this paper .

The paper is organized as follows. In section II we discuss the expression of the abelian pure gauge potential produced by the electron in Maxwell theory. In section III we discuss the charge and the charge density of a point particle using quantum mechanics. In section IV we show that the color charge of the quark in Yang-Mills theory is time dependent. The relation between the coupling constant and the color charge in Yang-Mills theory is discussed in section V. In section VI we discuss the color current density, the color charge of the quark and the general form of the Yang-Mills potential. In section VII we discuss the general form of the Yang-Mills potential (color potential) produced by the quark at rest. In section VIII we discuss the Yang-Mills color current density of the quark at rest. In section IX we describe the analogy between Maxwell theory and Yang-Mills theory to obtain the form of the fundamental charge of the fermion from the Dirac wave function. In section X we discuss the fermion color current density, the fermion wave function and the Pauli Matrices in Yang-Mills theory in SU(2). In section XI we derive the general form of three time dependent fundamental color charges  $q_i(t)$  of a fermion in Yang-Mills theory in SU(2) where  $i = 1, 2, 3$ . In section XII we discuss the Yang-Mills color current density of the quark, the quark wave function and the Gell-Mann Matrices in Yang-Mills theory in SU(3). In section XIII we derive the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in Yang-Mills theory in SU(3) where  $a = 1, 2, \dots, 8$ . In section XIV we compare our investigation with the Wong's equation. In section XV we show that the general form of eight time dependent color charges  $q^a(t)$  of the quark is consistent with the fact that there is second (cubic) casimir invariant of SU(3). The advantage of the time dependent phases in the color charge of the quark is discussed in section XVI. Section XVII contains conclusion.

## II. ABELIAN PURE GAUGE POTENTIAL IN MAXWELL THEORY

The Maxwell equation in electrodynamics is given by [11]

$$\partial_\nu F^{\nu\mu} = j^\mu, \quad \partial_\mu F_{\nu\beta} + \partial_\nu F_{\beta\mu} + \partial_\beta F_{\mu\nu} = 0 \quad (12)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (13)$$

In classical electrodynamics if the electric charge  $e$  is of a point particle whose position in the inertial frame  $K$  is  $\vec{X}(t)$  then the charge density  $\rho(t, \vec{x})$  and current density  $\vec{j}(t, \vec{x})$  of the point charge  $e$  in that frame are given by [11]

$$\begin{aligned} \rho(t, \vec{x}) &= j_0(t, \vec{x}) = e \delta^{(3)}(\vec{x} - \vec{X}(t)), \\ \vec{j}(t, \vec{x}) &= e \vec{v}(t) \delta^{(3)}(\vec{x} - \vec{X}(t)) \end{aligned} \quad (14)$$

where

$$\vec{v}(t) = \frac{d\vec{X}(t)}{dt} \quad (15)$$

is the charge's velocity in that frame  $K$ . From eq. (14) one finds

$$\begin{aligned} \frac{\partial j_0(t, \vec{x})}{\partial t} &= -e v_x(t) \delta'(x - X(t)) \delta(y - Y(t)) \delta(z - Z(t)) \\ &\quad - e v_y(t) \delta(x - X(t)) \delta'(y - Y(t)) \delta(z - Z(t)) - e v_z(t) \delta(x - X(t)) \delta(y - Y(t)) \delta'(z - Z(t)) \end{aligned} \quad (16)$$

where

$$\delta'(w) = \frac{d[\delta(w)]}{dw} \quad (17)$$

and

$$\begin{aligned} \vec{\nabla} \cdot \vec{j}(t, \vec{x}) &= e v_x(t) \delta'(x - X(t)) \delta(y - Y(t)) \delta(z - Z(t)) \\ &\quad + e v_y(t) \delta(x - X(t)) \delta'(y - Y(t)) \delta(z - Z(t)) + e v_z(t) \delta(x - X(t)) \delta(y - Y(t)) \delta'(z - Z(t)). \end{aligned} \quad (18)$$

From eqs. (16) and (18) one finds

$$\partial_\mu j^\mu(x) = \frac{\partial j^\mu(x)}{\partial x^\mu} = \frac{\partial j_0(t, \vec{x})}{\partial t} + \vec{\nabla} \cdot \vec{j}(t, \vec{x}) = 0 \quad (19)$$

which is the continuity equation in Maxwell theory. Hence one finds that the constant electric charge  $e$  of the electron satisfies the continuity eq. (19).

In the covariant formulation the current density of the electron of charge  $e$  is given by [11]

$$j^\mu(x) = \int d\tau e u^\mu(\tau) \delta^{(4)}(x - X(\tau)) \quad (20)$$

which satisfies the continuity equation

$$\partial_\mu j^\mu(x) = 0 \quad (21)$$

where

$$u^\mu(\tau) = \frac{dX^\mu(\tau)}{d\tau} \quad (22)$$

is the four-velocity of the electron and  $X^\mu(\tau)$  four-coordinate of the electron.

The solution of the inhomogeneous wave equation

$$\partial_\nu \partial^\nu A^\mu(x) = j^\mu(x) \quad (23)$$

is given by

$$A^\mu(x) = \int d^4x' D_r(x - x') j^\mu(x') \quad (24)$$

where  $D_r(x - x')$  is the retarded Greens function [11]. From eqs. (20) and (24) we find that the electromagnetic potential (Lienard-Weichert potential) produced by the constant electric charge  $e$  of the electron is given by

$$A^\mu(x) = e \frac{u^\mu(\tau_0)}{u(\tau_0) \cdot (x - X(\tau_0))} \quad (25)$$

which satisfies the Lorentz gauge condition

$$\partial_\mu A^\mu(x) = 0 \quad (26)$$

where  $\tau_0$  is obtained from the solution of the retarded equation

$$x_0 - X_0(\tau_0) = |\vec{x} - \vec{X}(\tau_0)|. \quad (27)$$

In eq. (25) the four-vector  $x^\mu$  is the time-space position at which the electromagnetic field is observed and the four-vector  $X^\mu(\tau_0)$  is the time-space position of the electron where the electromagnetic field was produced.

Hence we find that if  $A^\mu(x)$  satisfies Lorentz gauge condition  $\partial_\mu A^\mu(x) = 0$  then eq. (23) reduces to the inhomogeneous Maxwell equation

$$\partial_\nu F^{\nu\mu}(x) = j^\mu(x) \quad (28)$$

where

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \quad (29)$$

Note that an electron has non-zero mass (even if it is very small) which implies that an electron can not travel exactly at speed of light  $v = c$ . Since electromagnetic wave travels exactly at speed of light and the electron can not travel exactly at speed of light one finds that at a common time  $t = \frac{x_0}{c} = \frac{X_0(\tau)}{c}$  the observation point of the electromagnetic potential (or any gauge invariant obtained from the electromagnetic potential) is given by  $\vec{x} \neq \vec{X}(\tau)$  which implies from eq. (25) and (29) that

$$\partial_\nu F^{\nu\mu}(x) = 0 \quad (30)$$

which satisfies Maxwell equation given by eq. (28) where  $j^\mu(x)$  is given by eq. (20).

When the electron in uniform motion is at its highest speed (which is arbitrarily close to the speed of light  $v \sim c$ ) we find from eq. (25)

$$A^\mu(x) = e \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))}, \quad \beta_{\sim c}^\mu = (1, \vec{\beta}_{\sim c}), \quad \vec{\beta}_{\sim c}^2 = \frac{v^2}{c^2} \sim 1 \quad (31)$$

where  $u^\mu = c\gamma\beta^\mu$  with  $\gamma = \frac{1}{\sqrt{1-\vec{\beta}^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ . From eq. (29) we find that  $A^\mu(x)$  in eq. (31) produced by the electron in uniform motion at its highest speed (which is arbitrarily close to speed of light  $v \sim c$ ) gives [10, 12]

$$F^{\mu\nu}(x) \sim 0, \quad (32)$$

at all the time-space observation points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the electron at the time of closest approach), see section 3.1 of [10] for details. From eqs. (32) and (29) we find that at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the electron at the time of closest approach) the  $A^\mu(x)$  in eq. (31) can be written in the form [10, 12]

$$A^\mu = e \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))} \sim \partial^\mu \omega(x), \quad \text{where} \quad \omega(x) = e \ln[\beta_{\sim c} \cdot (x - X(\tau_0))]. \quad (33)$$

From eq. (33) we find that the electron in uniform motion at its highest speed (which is arbitrarily close to the speed of light  $v \sim c$ ) produces U(1) (approximate) pure gauge potential or abelian (approximate) pure gauge potential  $A^\mu \sim \partial^\mu \omega(x)$  at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the electron at the time of closest approach). We call the electromagnetic potential  $A^\mu(x)$  in eq. (33) as U(1) (approximate) pure gauge potential because an electron has non-zero mass (even if it is very small) and hence it can not travel exactly at speed of light  $\beta_c^\mu = (1, \vec{\beta}_c)$ ,  $\vec{\beta}_c^2 = \frac{v^2}{c^2} = 1$  to produce the exact U(1) pure gauge potential

$$A^\mu(x) = e \frac{\beta_c^\mu}{\beta_c \cdot (x - X(\tau_0))} = \partial^\mu \omega(x), \quad \omega(x) = e \ln[\beta_c \cdot (x - X(\tau_0))]. \quad (34)$$

From eq. (33) one finds that the  $\omega(x)$  appearing in the abelian pure gauge potential

$$A^\mu(x) = \partial^\mu \omega(x) \quad (35)$$

in Maxwell theory is linearly proportional to  $e$ , *i.e.*,

$$\omega(x) \propto e. \quad (36)$$

Similarly one finds that the  $\omega^a(x)$  appearing in the abelian-like non-abelian pure gauge potential

$$\mathcal{A}^{\mu a}(x) = \partial^\mu \omega^a(x) \quad (37)$$

in Yang-Mills theory is linearly proportional to  $g$ , *i.e.*,

$$\omega^a(x) \propto g \quad (38)$$

whereas the SU(3) pure gauge potential  $A^{\mu a}(x)$  in

$$T^a A^{\mu a}(x) = \frac{1}{ig} [\partial^\mu U(x)] U^{-1}(x), \quad U(x) = e^{ig T^a \omega^a(x)} \quad (39)$$

in SU(3) Yang-Mills theory contains infinite number of terms upto infinite powers of  $g$  and/or infinite powers of  $\omega^a(x)$ . Note that eq. (37) is only the first term in the expansion of  $A^{\mu a}(x)$  in eq. (39).

### III. CHARGE AND CHARGE DENSITY OF A POINT PARTICLE USING QUANTUM MECHANICS

In quantum mechanics the electron is described by the wave function  $\psi(x)$ . The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i\gamma^\mu\partial_\mu - m + e\gamma^\mu A_\mu(x)]\psi \quad (40)$$

from which we obtain the Dirac equation of the electron

$$[i\gamma^\mu\partial_\mu - m + e\gamma^\mu A_\mu(x)]\psi(x) = 0. \quad (41)$$

From eq. (40) we find that the current density of the electron is given by

$$j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x) \quad (42)$$

which satisfies the continuity equation

$$\partial_\mu j^\mu(x) = 0. \quad (43)$$

Note that the Dirac wave function  $\psi(x)$  in eqs. (41) and (42) is yet to be quantized in the sense of second quantization (field quantization) [2].

Note that in classical mechanics the charge density of a point charge may be described by delta function distribution, see eq. (14), which implies that the charge density at the position  $\vec{x} \neq \vec{X}(t)$  is zero where  $\vec{X}(t)$  is the spatial position of the electron and  $\vec{x}$  is the spatial position at which the current density is determined. However, in quantum mechanics the probability density  $\psi^\dagger(t, \vec{x})\psi(t, \vec{x})$  of finding a particle is not defined at a fixed position  $\vec{x} = \vec{X}(t)$  but rather it is defined in a certain (infinitesimal) volume element. Hence one finds that in quantum mechanics the charge density  $e\psi^\dagger(t, \vec{x})\psi(t, \vec{x})$  is not zero when  $\vec{x} \neq \vec{X}(t)$ .

However, when integrated over the entire (physically) allowed volume  $V = \int d^3\vec{x}$ , the charge density in quantum mechanics and the charge density in classical mechanics reproduce the same charge  $e$  of the electron, *i.e.*, for the normalized wave function

$$\int d^3\vec{x} \psi^\dagger(x)\psi(x) = 1 \quad (44)$$

we find from eq. (42)

$$\int d^3\vec{x} j_0(t, \vec{x}) = \int d^3\vec{x} e \psi^\dagger(x)\psi(x) = e. \quad (45)$$

Also from classical mechanics we find from eq. (14)

$$\int d^3\vec{x} j_0(t, \vec{x}) = \int d^3\vec{x} e \delta^{(3)}(\vec{x} - \vec{X}(t)) = e. \quad (46)$$

Eqs. (45) and (46) implies that the classical mechanics and quantum mechanics give the same value of the electric charge  $e$  of the electron even if the charge density distributions in classical mechanics and in quantum mechanics are different.

As we have seen above, since the charge of a point particle can not depend on the space coordinate  $\vec{x}$ , eqs. (42), (43), (44) and (45) suggest that quantum mechanics may provide a framework to determine the general form of the fundamental charge. For example, in electrodynamics eqs. (42), (43), (44) and (45) suggest that the fundamental electric charge  $e$  of the electron can not be time dependent but it has to be constant.

Let us apply this procedure to Yang-Mills theory to find the general form of eight time dependent fundamental color charges  $q^a(t)$  of a quark where  $a = 1, 2, \dots, 8$  are color indices.

#### IV. COLOR CHARGE OF THE QUARK IN YANG-MILLS THEORY IS TIME DEPENDENT

In Yang-Mills theory the lagrangian density of the quark in the presence of non-abelian Yang-Mills potential  $A^{\mu a}(x)$  is given by [1, 2]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a(x)F^{\mu\nu a}(x) + \bar{\psi}_i(x)[i\delta_{ij}\gamma^\mu\partial_\mu - m\delta_{ij} + gT_{ij}^a\gamma^\mu A_\mu^a(x)]\psi_j(x) \quad (47)$$

where

$$F_{\mu\nu}^a(x) = \partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + gf^{abc}A_\mu^b(x)A_\nu^c(x), \quad (48)$$

with  $a = 1, 2, \dots, 8$  and  $i, j = 1, 2, 3$  in SU(3).

From eq. (47) one finds that the Dirac equation of the quark in the presence of non-abelian Yang-Mills potential  $A^{\mu a}(x)$  is given by

$$[i\delta_{ij}\gamma^\mu\partial_\mu - m\delta_{ij} + gT_{ij}^a\gamma^\mu A_\mu^a(x)]\psi_j(x) = 0. \quad (49)$$

Similarly from eq. (47) one finds that the Yang-Mills color current density  $j^{\mu a}(x)$  generated by the color charges of the quark in Yang-Mills theory is given by [1, 2]

$$j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T_{ij}^a\psi_j(x) \quad (50)$$

which satisfies the equation

$$D_\mu[A]j^{\mu a}(x) = 0 \quad (51)$$

where

$$D_\mu^{ab}[A] = \delta^{ab}\partial_\mu + g f^{acb} A_\mu^c(x). \quad (52)$$

Since eq. (51) is not a continuity equation ( $\partial_\mu j^{\mu a}(x) \neq 0$ ) one finds from eqs. (51) and (50) that the color charge of the quark in Yang-Mills theory is not constant. As we have discussed above, since the charge of a point particle can be obtained from the zero ( $\mu = 0$ ) component of a corresponding current density after integrating over the entire (physically) allowed volume  $V = \int d^3\vec{x}$ , the color charge  $q^a(t)$  of the quark is time dependent. Since the color current density  $j^{\mu a}(x)$  of a quark in eq. (50) has eight color components  $a = 1, 2, \dots, 8$  we find that there are eight time dependent fundamental color charges of a quark.

We denote eight time dependent fundamental color charges of a quark in Yang-Mills theory in SU(3) by  $q^a(t)$  where  $a = 1, 2, \dots, 8$  are color indices. These time dependent fundamental color charges  $q^a(t)$  of the quark are independent of quark flavor, *i.e.*, a color charge  $q^a(t)$  of the  $u$  (up) quark is same as that of  $d$  (down),  $S$  (strange),  $c$  (charm),  $B$  (bottom) or  $t$  (top) quark.

## V. RELATION BETWEEN FUNDAMENTAL COUPLING CONSTANT AND FUNDAMENTAL CHARGE IN MAXWELL THEORY AND IN YANG-MILLS THEORY

Let us now ask a fundamental question in the nature. Since the strong force or the color force is produced from the color charges in the nature "what is the relation between the fundamental strong coupling constant  $\alpha_s$  and the fundamental color charges  $q^a(t)$  in the classical Yang-Mills theory" ?

One expects such a question because we know that a fundamental coupling constant is the measure of the strength of a fundamental force in the nature produced by the fundamental charges.

For example the fundamental electromagnetic force in the nature is produced from the fundamental electric charge  $\pm e$ . The strength of this fundamental electromagnetic force in

the nature is characterized by the typical value of the fundamental electromagnetic coupling constant (or fine structure constant)

$$\alpha = \frac{(\pm e)^2}{4\pi} \sim \frac{1}{137} \quad (53)$$

which is related to the fundamental electric charge  $\pm e$  in Maxwell theory.

Similarly, the fundamental strong force or the color force inside a hadron is produced from the fundamental color charges  $q^a(t)$  in the nature. Since the strength of the fundamental strong force or the color force is characterized by the value of the fundamental strong coupling constant  $\alpha_s$  one anticipates that there is a relation between the fundamental strong coupling constant  $\alpha_s$  and the fundamental color charges  $q^a(t)$  in Yang-Mills theory where  $a = 1, 2, \dots, 8$ .

It is useful to denote  $N^2 - 1$  color charges  $q^a(t)$  of a fermion in  $SU(N)$  Yang-Mills theory as components of a single color charge vector  $\vec{q}(t)$  in  $N^2 - 1$  dimensional color space (group space). For example, in  $SU(3)$  Yang-Mills theory the eight time dependent fundamental color charges  $q^a(t)$  of a quark can be denoted by a single vector  $\vec{q}(t)$  in the eight dimensional color space (group space in  $SU(3)$ ) where  $a = 1, 2, \dots, 8$ . We call  $\vec{q}(t)$  as eight-vector in color space, similar to three-vectors  $\vec{x}(t)$  and  $\vec{v}(t)$  in coordinate space. Note that although the three-vector  $\vec{x}(t)$  in coordinate space is not rotationally invariant, its magnitude  $|\vec{x}(t)|$  is rotationally invariant. Similarly, one finds that even if the eight-vector  $\vec{q}(t)$  in color space is not gauge invariant, its magnitude  $|\vec{q}(t)|$  is gauge invariant where

$$\vec{q}^2(t) = |\vec{q}(t)|^2 = \sum_{a=1}^8 q^a(t)q^a(t). \quad (54)$$

In Maxwell theory the electromagnetic coupling constant (or fine structure constant) is related to the magnitude  $e^2$  of the fundamental electric charge ' $-e$ ' of the electron via the equation

$$\alpha = \frac{(-e)^2}{4\pi}. \quad (55)$$

Similarly, since the strong coupling constant  $\alpha_s$  is gauge invariant, one finds by extending eq. (55) to Yang-Mills theory that the strong coupling constant  $\alpha_s$  is related to the magnitude  $\vec{q}^2(t)$  of the time dependent fundamental color charge ' $\vec{q}(t)$ ' of the quark via the equation

$$\alpha_s = \frac{\vec{q}^2(t)}{4\pi}. \quad (56)$$

Note that even if the fundamental color charge  $\vec{q}(t)$  of the quark is time dependent, the gauge invariant  $\vec{q}^2(t) = \sum_{a=1}^8 q^a(t)q^a(t)$  can be time independent.

In SU(2) Yang-Mills theory the color charge vector  $\vec{q}(t)$  has three components  $q_i(t)$  where  $i = 1, 2, 3$ .

Since the color charge vector  $\vec{q}(t)$  has eight components in SU(3) Yang-Mills theory we find from eq. (56)

$$\alpha_s = \frac{q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4^2(t) + q_5^2(t) + q_6^2(t) + q_7^2(t) + q_8^2(t)}{4\pi} \quad (57)$$

in SU(3) Yang-Mills theory.

Similarly, since the color charge vector  $\vec{q}(t)$  has three components in SU(2) Yang-Mills theory we find from eq. (56)

$$\alpha_s = \frac{q_1^2(t) + q_2^2(t) + q_3^2(t)}{4\pi} \quad (58)$$

in SU(2) Yang-Mills theory.

Since the universal coupling  $g$  in Yang-Mills theory and the strong coupling constant  $\alpha_s$  are related by

$$\alpha_s = \frac{g^2}{4\pi} \quad (59)$$

we find from eq. (56) that

$$g^2 = \vec{q}^2(t). \quad (60)$$

In SU(3) Yang-Mills theory we find from eq. (60)

$$g^2 = q_1^2(t) + q_2^2(t) + q_3^2(t) + q_4^2(t) + q_5^2(t) + q_6^2(t) + q_7^2(t) + q_8^2(t) \quad (61)$$

and in SU(2) Yang-Mills theory we find from eq. (60)

$$g^2 = q_1^2(t) + q_2^2(t) + q_3^2(t). \quad (62)$$

## VI. COLOR CURRENT DENSITY OF QUARK, COLOR CHARGE OF QUARK AND GENERAL FORM OF YANG-MILLS POTENTIAL

Extending eq. (14) of classical electrodynamics to time dependent color charge  $q^a(t)$  in classical chromodynamics we write

$$\begin{aligned} \mathcal{J}_0^a(t, \vec{x}) &= q^a(t) \delta^{(3)}(\vec{x} - \vec{X}(t)) \\ \vec{\mathcal{J}}^a(t, \vec{x}) &= q^a(t) \vec{v}(t) \delta^{(3)}(\vec{x} - \vec{X}(t)) \end{aligned} \quad (63)$$

which satisfies the equation

$$\partial_\mu \mathcal{J}^{\mu a}(t, \vec{x}) = \frac{dq^a(t)}{dt} \delta^{(3)}(\vec{x} - \vec{X}(t)). \quad (64)$$

In a covariant formulation we find from eq. (63)

$$\mathcal{J}^{\mu a}(x) = \int d\tau q^a(\tau) u^\mu(\tau) \delta^{(4)}(x - X(\tau)). \quad (65)$$

Similarly in covariant formulation we find from eq. (64)

$$\partial_\mu \mathcal{J}^{\mu a}(x) = \int d\tau \frac{dq^a(\tau)}{d\tau} \delta^{(4)}(x - X(\tau)) \quad (66)$$

which is not zero at  $x^\mu = X^\mu(\tau)$  which confirms that the color charge  $q^a(\tau)$  is not constant.

The solution of the inhomogeneous wave equation

$$\partial^\nu \partial_\nu \mathcal{A}^{\mu a}(x) = \mathcal{J}^{\mu a}(x) \quad (67)$$

is given by

$$\mathcal{A}^{\mu a}(x) = \int d^4 x' D_r(x - x') \mathcal{J}^{\mu a}(x') \quad (68)$$

where  $D_r(x - x')$  is the retarded Greens function [11]. When  $\mathcal{A}^{\mu a}(x)$  satisfies the Lorentz gauge condition

$$\partial_\mu \mathcal{A}^{\mu a}(x) = 0, \quad (69)$$

then eq. (67) reduces to the inhomogeneous Maxwell-like equation

$$\partial_\nu \mathcal{F}^{\nu \mu a}(x) = \mathcal{J}^{\mu a}(x) \quad (70)$$

where

$$\mathcal{F}^{\mu \nu a}(x) = \partial^\mu \mathcal{A}^{\nu a}(x) - \partial^\nu \mathcal{A}^{\mu a}(x). \quad (71)$$

Hence we find that if  $\mathcal{A}^{\mu a}(x)$  obtained from eq. (67) satisfies Lorentz gauge condition eq. (69) then it satisfies Maxwell-like eq. (70) where  $\mathcal{F}^{\mu \nu a}(x)$  is given by eq. (71).

Using eq. (65) in (68) we find that

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0) \frac{u^\mu(\tau_0)}{u(\tau_0) \cdot (x - X(\tau_0))} \quad (72)$$

where  $\tau_0$  is determined from the solution of the retarded condition as given by eq. (27).

From eqs. (72) and (27) we find

$$\begin{aligned}\partial_\nu \mathcal{A}^{\mu a}(x) &= q^a(\tau_0) \frac{(x - X(\tau_0))_\nu \dot{u}^\mu(\tau_0)}{[u(\tau_0) \cdot (x - X(\tau_0))]^2} - q^a(\tau_0) \frac{u^\mu(\tau_0)}{[u(\tau_0) \cdot (x - X(\tau_0))]^2} \\ &\quad \left[ \frac{[\dot{u}(\tau_0) \cdot (x - X(\tau_0)) - c^2](x - X(\tau_0))_\nu}{u(\tau_0) \cdot (x - X(\tau_0))} + u_\nu(\tau_0) \right] + [\partial_\nu q^a(\tau_0)] \frac{u^\mu(\tau_0)}{u(\tau_0) \cdot (x - X(\tau_0))}\end{aligned}\tag{73}$$

which gives

$$\partial^\nu \partial_\nu \mathcal{A}^{\mu a}(x) = 0\tag{74}$$

where

$$\dot{u}^\mu(\tau_0) = \frac{du^\mu(\tau)}{d\tau}|_{\tau=\tau_0}.\tag{75}$$

Note that a quark has non-zero mass (even if the mass of the light quark is very small) which implies that a quark can not travel exactly at speed of light  $v = c$ . As discussed in section II [see the paragraph after eq. (29)], at a common time  $t = \frac{x_0}{c} = \frac{X_0(\tau)}{c}$  since  $\vec{x} \neq \vec{X}(\tau)$  one finds that eq. (74) is expected because eq. (72) is obtained from eq. (67) where  $\mathcal{J}^{\mu a}(x)$  is given by eq. (65).

From eq. (73) we find

$$\partial_\mu \mathcal{A}^{\mu a}(x) = \frac{\dot{q}^a(\tau_0)}{u(\tau_0) \cdot (x - X(\tau_0))}, \quad \dot{q}^a(\tau_0) = \frac{dq^a(\tau)}{d\tau}|_{\tau=\tau_0}.\tag{76}$$

When the quark in uniform motion is at its highest speed (which is arbitrarily close to speed of light  $v \sim c$ ) we find from eq. (72) that  $\mathcal{A}^{\mu a}(x)$  is given by

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0) \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))}, \quad \beta_{\sim c}^\mu = (1, \vec{\beta}_{\sim c}), \quad \vec{\beta}_{\sim c}^2 = \frac{v^2}{c^2} \sim 1.\tag{77}$$

From eq. (76) we find that at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) the  $\mathcal{A}^{\mu a}(x)$  in eq. (77) satisfies (approximate) Lorentz gauge condition

$$\partial_\mu \mathcal{A}^{\mu a}(x) \sim 0,\tag{78}$$

see section 4.2 of [10] for details. From eqs. (74), (78) and (71) we find that at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) the  $\mathcal{A}^{\mu a}(x)$  in eq. (77) satisfies the equation

$$\partial_\nu \mathcal{F}^{\nu \mu a}(x) \sim 0.\tag{79}$$

As discussed above at a common time  $t = \frac{x_0}{c} = \frac{X_0(\tau)}{c}$  since  $\vec{x} \neq \vec{X}(\tau)$ , we find that eq. (79) (approximately) satisfies Maxwell-like equation (70) where  $\mathcal{J}^{\mu a}(x)$  is given by eq. (65).

From eqs. (72) and (27) we find

$$\begin{aligned} \partial^\mu \mathcal{A}^{\nu a}(x) - \partial^\nu \mathcal{A}^{\mu a}(x) &= q^a(\tau_0) \frac{(x - X(\tau_0))^\mu \dot{u}^\nu(\tau_0) - (x - X(\tau_0))^\nu \dot{u}^\mu(\tau_0)}{[u(\tau_0) \cdot (x - X(\tau_0))]^2} \\ &- q^a(\tau_0) \left[ \frac{[\dot{u}(\tau_0) \cdot (x - X(\tau_0)) - c^2][(x - X(\tau_0))^\mu u^\nu(\tau_0) - (x - X(\tau_0))^\nu u^\mu(\tau_0)]}{[u(\tau_0) \cdot (x - X(\tau_0))]^3} \right] \\ &+ \dot{q}^a(\tau_0) \frac{(x - X(\tau_0))^\mu u^\nu(\tau_0) - (x - X(\tau_0))^\nu u^\mu(\tau_0)}{[u(\tau_0) \cdot (x - X(\tau_0))]^2} \end{aligned} \quad (80)$$

which gives for uniform motion

$$\begin{aligned} \partial^\mu \mathcal{A}^{\nu a}(x) - \partial^\nu \mathcal{A}^{\mu a}(x) &= c^2 q^a(\tau_0) \frac{(x - X(\tau_0))^\mu u^\nu - (x - X(\tau_0))^\nu u^\mu}{[u \cdot (x - X(\tau_0))]^3} \\ &+ \dot{q}^a(\tau_0) \frac{(x - X(\tau_0))^\mu u^\nu - (x - X(\tau_0))^\nu u^\mu}{[u \cdot (x - X(\tau_0))]^2} \end{aligned} \quad (81)$$

where  $\dot{q}^a(\tau_0)$  is given by eq. (76). When the quark in uniform motion is at its highest speed (which is arbitrarily close to the speed of light  $v \sim c$ ) we find from eq. (81) that at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) the  $\mathcal{A}^{\mu a}(x)$  in eq. (77) gives

$$\mathcal{F}^{\mu\nu a}(x) \sim 0, \quad (82)$$

where  $\mathcal{F}^{\mu\nu a}(x)$  is given by eq. (71), see section 4.2 of [10] for details.

From eqs. (77), (78), (79) and (82) we find that at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) the

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0) \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))} \quad (83)$$

satisfies

$$\partial_\mu \mathcal{A}^{\mu a}(x) \sim 0, \quad (84)$$

$$\partial_\mu \mathcal{F}^{\mu\nu a}(x) \sim 0 \quad (85)$$

and

$$\mathcal{F}^{\mu\nu a}(x) \sim 0 \quad (86)$$

where  $\mathcal{F}^{\mu\nu a}(x)$  is given by eq. (71).

From eqs. (83), (86) and (71) we find that at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) the eq. (83) can be written as

$$\mathcal{A}^{\mu a} = q^a(\tau_0) \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))} \sim \partial^\mu \omega^a(x), \quad \omega^a(x) = \int dl_c \frac{q^a(\tau_0)}{l_c}, \quad l_c = \beta_{\sim c} \cdot (x - X(\tau_0)) \quad (87)$$

where  $\int dl_c$  is an indefinite integration. Hence we find that the general expression of the abelian-like non-abelian (approximate) pure gauge color potential at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) produced by the quark in uniform motion at its highest speed (which is arbitrarily close to speed of light  $v \sim c$ ) with time dependent color charge  $q^a(\tau)$  is given by eq. (87). Note that the expression of the abelian (approximate) pure gauge potential at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the electron at the time of closest approach) produced by the electron in uniform motion at its highest speed (which is arbitrarily close to speed of light  $v \sim c$ ) with constant electric charge  $e$  is given by eq. (33).

We call eq. (87) as abelian-like non-abelian (approximate) pure gauge color potential because a quark has non-zero mass (even if the mass of the light quark is very small) and hence it can not travel exactly at speed of light  $\beta_c^\mu = (1, \vec{\beta}_c)$  or  $\vec{\beta}_c^2 = \frac{v^2}{c^2} = 1$  to produce the exact abelian-like non-abelian pure gauge color potential

$$\mathcal{A}^{\mu a}(x) = q^a(\tau_0) \frac{\beta_c^\mu}{\beta_c \cdot (x - X(\tau_0))} = \partial^\mu \omega^a(x), \quad \omega^a(x) = \int dl_c \frac{q^a(\tau_0)}{l_c}, \quad l_c = \beta_c \cdot (x - X(\tau_0)). \quad (88)$$

Note that in Maxwell theory the abelian pure gauge potential produced by the electron is given by

$$A^\mu(x) = \frac{1}{ie} [\partial^\mu U(x)] U^{-1}(x) = \partial^\mu \omega(x), \quad U(x) = e^{ie\omega(x)} \quad (89)$$

where (see eq. (36))

$$\omega(x) \propto e. \quad (90)$$

In Yang-Mills theory the SU(3) non-abelian pure gauge potential  $A^{\mu a}(x)$  in

$$T^a A^{\mu a}(x) = \frac{1}{ig} [\partial^\mu U(x)] U^{-1}(x), \quad U(x) = e^{ig T^a \omega^a(x)} \quad (91)$$

contains infinite number of terms upto infinite powers of  $g$  and/or infinite powers of  $\omega^a(x)$  where

$$\omega^a(x) \propto g. \quad (92)$$

The first term in the expansion in eq. (91) is the abelian-like non-abelian pure gauge color potential given by

$$\mathcal{A}^{\mu a}(x) = \partial^\mu \omega^a(x) \quad (93)$$

where

$$\omega^a(x) \propto q^a(\tau_0) \propto g, \quad (94)$$

see eqs. (87) and (60).

From eqs. (87) and (91) we find that the general expression of the SU(3) (approximate) pure gauge potential at all the time-space points  $x^\mu$  (except at the spatial position  $\vec{x}$  transverse to the motion of the quark at the time of closest approach) produced by the the quark in uniform motion at its highest speed (which is arbitrarily close to the speed of light  $v \sim c$ ) in Yang-Mills theory is given by [10]

$$A^{\mu a}(x) = \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))} q^b(\tau_0) \left[ \frac{e^{g \int dl_c \frac{Q(\tau_0)}{l_c}} - 1}{g \int dl_c \frac{Q(\tau_0)}{l_c}} \right]_{ab} \sim [\partial^\mu \omega^b(x)] \left[ \frac{e^{gM(x)} - 1}{gM(x)} \right]_{ab} \quad (95)$$

where  $\int dl_c$  is an indefinite integration and

$$Q^{ab}(\tau_0) = f^{abd} q^d(\tau_0), \quad l_c = \beta_{\sim c} \cdot (x - X(\tau_0)), \quad M_{ab}(x) = f^{abd} \omega^d(x), \quad \omega^a(x) = \int dl_c \frac{q^a(\tau_0)}{l_c}. \quad (96)$$

By making analogy with Maxwell theory we find from eq. (95) that [see section 4.3 of [10] for details] the general expression of the Yang-Mills potential (color potential) at all the time-space points  $x^\mu$  produced by the color charges  $q^a(\tau)$  of the quark in uniform motion at its highest speed (which is arbitrarily close to the speed of light  $v \sim c$ ) in Yang-Mills theory is given by

$$A^{\mu a}(x) = \frac{\beta_{\sim c}^\mu}{\beta_{\sim c} \cdot (x - X(\tau_0))} q^b(\tau_0) \left[ \frac{e^{g \int dl_c \frac{Q(\tau_0)}{l_c}} - 1}{g \int dl_c \frac{Q(\tau_0)}{l_c}} \right]_{ab}, \quad Q^{ab}(\tau_0) = f^{abd} q^d(\tau_0), \quad l_c = \beta_{\sim c} \cdot (x - X(\tau_0)). \quad (97)$$

From eq. (97) we find that [see section 5 of [10] for details] the general form of the Yang-Mills potential (color potential)  $A^{\mu a}(x)$  produced by the color charges  $q^a(\tau)$  of the quark moving with arbitrary four-velocity  $u^\mu(\tau) = \frac{dX^\mu(\tau)}{d\tau}$  is given by [10]

$$A^{\mu a}(x) = \frac{u^\mu(\tau_0)}{u(\tau_0) \cdot (x - X(\tau_0))} q^b(\tau_0) \left[ \frac{e^{g \int dl \frac{Q(\tau_0)}{l}} - 1}{g \int dl \frac{Q(\tau_0)}{l}} \right]_{ab} \quad (98)$$

where  $\int dl$  is an indefinite integration,

$$Q^{ab}(\tau_0) = f^{abd} q^d(\tau_0), \quad l = u(\tau_0) \cdot (x - X(\tau_0)), \quad (99)$$

$\tau_0$  is obtained from the solution of the retarded equation given by eq. (27) and the repeated color indices  $b, d = 1, 2, \dots, 8$  are summed.

From eqs. (98), (47), (50), (52) and (48) we find that the non-abelian Yang-Mills color current density  $j^{\mu a}(x)$  of the quark which satisfies the equation

$$\begin{aligned} j^{\mu a}(x) &= D_\nu[A] F^{\nu \mu a}(x) = \\ &\partial_\nu [\partial^\nu A^{\mu a}(x) - \partial^\mu A^{\nu a}(x) + g f^{abc} A^{\nu b}(x) A^{\mu c}(x)] + g f^{abc} A_\nu^b [\partial^\nu A^{\mu c}(x) - \partial^\mu A^{\nu c}(x) + g f^{chd} A^{\nu h}(x) A^{\mu d}(x)] \end{aligned} \quad (100)$$

contains infinite powers of  $g$  or infinite powers of  $q^a(\tau)$ . Note that we have used the curly notations  $\mathcal{A}^{\mu a}(x)$ ,  $\mathcal{J}^{\mu a}(x)$  when  $\mathcal{A}^{\mu a}(x)$ ,  $\mathcal{J}^{\mu a}(x)$  are linearly proportional to  $q^a(\tau)$  [see eqs. (87) and (65)] but we have used the usual notations  $A^{\mu a}(x)$ ,  $j^{\mu a}(x)$  for the (full) non-abelian Yang-Mills theory where  $A^{\mu a}(x)$ ,  $j^{\mu a}(x)$  contain infinite powers of  $q^a(\tau)$  [see eqs. (98) and (100)]. Since the  $\mathcal{J}^{\mu a}(x)$  in eq. (63) is linearly proportional to  $g$  (see eq. (94)) and the non-abelian Yang-Mills color current density  $j^{\mu a}(x)$  of the quark in eq. (100) contains infinite powers of  $g$ , we find that  $\int d^3 \vec{x} \mathcal{J}_0^a(t, \vec{x})$  in eq. (63) is different from  $\int d^3 \vec{x} j_0^a(t, \vec{x})$  in eq. (100) in Yang-Mills theory. Note that they are same for electron in Maxwell theory because  $A^\mu(x)$  and  $F^{\mu\nu}(x)$  are linearly proportional to the electric charge  $e$  of the electron in Maxwell theory.

From eqs. (50), (100) and (98) we find that the non-abelian Yang-Mills color current density

$$j^{\mu a}(x) = g \bar{\psi}_i(x) \gamma^\mu T_{ij}^a \psi_j(x) \quad (101)$$

of the quark in Yang-Mills theory contains infinite powers of  $g$ . Note that in Maxwell theory the electromagnetic current density  $j^\mu(x) = e \bar{\psi}(x) \gamma^\mu \psi(x)$  of the electron is linearly proportional to  $e$ .

Hence we find that, unlike Maxwell theory where  $\int d^3\vec{x} \psi^\dagger(t, \vec{x})\psi(t, \vec{x}) = 1$  is independent of  $e$ , we find from eqs. (100) and (101) that in Yang-Mills theory  $\int d^3\vec{x} \psi_i^\dagger(t, \vec{x})T_{ij}^a\psi_j(t, \vec{x})$  contains infinite powers of  $g$  even if the wave function of the quark is normalized, *i.e.*, even if

$$\sum_{i=1}^3 \int d^3\vec{x} \psi_i^\dagger(t, \vec{x})\psi_i(t, \vec{x}) = 1. \quad (102)$$

## VII. GENERAL FORM OF THE YANG-MILLS POTENTIAL (COLOR POTENTIAL) PRODUCED BY THE QUARK AT REST

From the expression of the proper time

$$\tau = \int \frac{dX_0(\tau)}{c\gamma(X_0(\tau))} \quad (103)$$

we find that when  $\vec{\beta} = \frac{\vec{v}}{c} = 0$

$$\tau_0 = \frac{X_0(\tau_0)}{c}. \quad (104)$$

From the retarded condition from eq. (27) we find

$$x_0 - X_0(\tau_0) = ct - X_0(\tau_0) = r \quad (105)$$

where

$$r = |\vec{x} - \vec{X}(\tau_0)|. \quad (106)$$

From eqs. (104) and (105) we find

$$\tau_0 = t - \frac{r}{c}. \quad (107)$$

Using eq. (107) in (98) we find that the general form of the Yang-Mills potential (color potential) produced by the color charges of the quark at rest is given by [10]

$$\Phi^a(x) = A_0^a(t, \vec{x}) = \frac{q^b(t - \frac{r}{c})}{r} \left[ \frac{\exp[g \int dr \frac{Q(t - \frac{r}{c})}{r}] - 1}{g \int dr \frac{Q(t - \frac{r}{c})}{r}} \right]_{ab} \quad (108)$$

where  $dr$  integration is an indefinite integration,

$$r = |\vec{x} - \vec{X}(\tau_0)|, \quad Q_{ab}(\tau_0) = f^{abd}q^d(\tau_0), \quad \tau_0 = t - \frac{r}{c}, \quad (109)$$

$\vec{X}(\tau_0)$  is the spatial position of the quark at rest at the retarded time and the repeated color indices  $b, d(=1,2,\dots,8)$  are summed.

We find from eq. (108) that, unlike Coulomb potential  $A_0(t, \vec{x}) = \frac{e}{|\vec{x} - \vec{X}|}$  produced by the electric charge of the electron at rest in Maxwell theory which is independent of time  $t$ , the color potential  $A_0^a(t, \vec{x})$  produced by the color charges of the quark in Yang-Mills theory depends on the retarded time  $\tau_0 = t - \frac{r}{c}$  even if the quark is at rest [10]. This is a consequence of time dependent color charges of the quark in Yang-Mills theory. The color potential (Yang-Mills potential)  $A_0^a(t, \vec{x})$  at time  $t$  produced by the quark at rest depends on color charges  $q^a(\tau_0)$  of the quark at the retarded time  $\tau_0 = t - \frac{r}{c}$ .

In other words the color potential (Yang-Mills potential)  $A_0^a(t, \vec{x})$  produced by the color charges  $q^a(\tau_0)$  of the quark at rest from the spatial position  $\vec{X}(\tau_0)$  not only depends on the distance  $r = |\vec{x} - \vec{X}(\tau_0)|$  but also depends on the retarded time  $\tau_0 = t - \frac{r}{c} = t - \frac{|\vec{x} - \vec{X}(\tau_0)|}{c}$ , see eq. (108).

Note that when the color charge  $q^a$  is constant we find from eq. (108)

$$\Phi^a(t, \vec{x}) = \frac{q^a}{|\vec{x} - \vec{X}|} \quad (110)$$

which reproduces the Coulomb-like potential (abelian-like potential), similar to Maxwell theory where the constant electric charge  $e$  produces Coulomb potential  $\Phi(t, \vec{x}) = A_0(t, \vec{x}) = \frac{e}{|\vec{x} - \vec{X}|}$ . Hence from eq. (108) we find that the general form of the Yang-Mills potential (color potential)  $A_0^a(t, \vec{x})$  produced by the color charges of the quark at rest is not abelian-like potential.

Hence one finds that the static systems in the Yang-Mills theory are, in general, not abelian-like.

### VIII. YANG-MILLS COLOR CURRENT DENSITY OF THE QUARK AT REST

From eq. (98) we find that the vector component of the Yang-Mills potential (color potential)  $\vec{A}^a(t, \vec{x}) = 0$  when the quark is at rest. However, since the zero ( $\mu = 0$ ) component of the Yang-Mills potential (color potential)  $A_0^a(t, \vec{x})$  in eq. (108) produced by the quark at rest depends on the retarded time  $\tau_0 = t - \frac{r}{c}$ , the Yang-Mills theory has some additional features which are absent in Maxwell theory. For the quark at rest in Yang-Mills theory we find from eq. (100) that the zero ( $\mu = 0$ ) component of the Yang-Mills color current density

$j_0^a(x)$  of the quark obeys the equation

$$\partial_0 j_0^a(x) = \partial_0 [D_\mu[A] F^{\mu 0a}(x)] = \partial_0 \partial_i \partial^i A_0^a(t, \vec{x}) \neq 0, \quad i = 1, 2, 3 \quad (111)$$

because  $A_0^a(t, \vec{x})$  in eq. (108) depends on the retarded time  $\tau_0 = t - \frac{r}{c}$  even if the quark is at rest, *i.e.*,

$$\partial_0 j_0^a(x) \neq 0, \quad \text{for quark at rest.} \quad (112)$$

On the other hand for electron at rest in Maxwell theory we find that the zero ( $\mu = 0$ ) component of the electromagnetic current density  $j_0(x)$  of the electron obeys the equation [see eq. (16)]

$$\partial_0 j_0(x) = 0, \quad \text{for electron at rest,} \quad (113)$$

which is consistent with the fact that the electric charge  $e$  of the electron is constant. The eq. (112) is consistent with the fact that in Yang-Mills theory the color charge  $q^a(\tau)$  of the quark is time dependent and hence the color potential (Yang-Mills potential)  $A_0^a(t, \vec{x})$  at time  $t$  produced by the quark at rest depends on the retarded time  $\tau_0 = t - \frac{r}{c}$ , see eq. (108).

For the quark at rest in Yang-Mills theory we find from eq. (100) that the vector component of the Yang-Mills color current density  $\vec{j}^a(x)$  of the quark is given by

$$j^{ia}(x) = D_\mu[A] F^{\mu ia}(x) = -\partial_0 \partial^i A_0^a(t, \vec{x}) - g f^{abc} A_0^b(t, \vec{x}) \partial^i A_0^c(t, \vec{x}) \neq 0, \quad i = 1, 2, 3 \quad (114)$$

even if the quark is at rest, *i.e.*,

$$\vec{j}^a(x) \neq 0, \quad \text{for quark at rest,} \quad (115)$$

even if  $\vec{\mathcal{J}}^a(x) = 0$  from eq. (63). On the other hand for electron at rest in Maxwell theory we find that the vector component of the electromagnetic current density  $\vec{j}(x)$  of the electron is given by [see eq. (14)]

$$\vec{j}(x) = 0, \quad \text{for electron at rest.} \quad (116)$$

Hence one finds that the static systems in the Yang-Mills theory are, in general, not abelian-like.

## IX. FORM OF FUNDAMENTAL CHARGE FROM DIRAC WAVE FUNCTION IN MAXWELL THEORY AND IN YANG-MILLS THEORY

Note that since the color current density  $j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T_{ij}^a\psi_j(x)$  of the quark in Yang-Mills theory is not gauge invariant, the color charge  $q^a(t)$  of the quark is not gauge invariant and is not a physically measurable quantity but its magnitude  $\vec{q}^2(t) = \sum_{a=1}^8 q^a(t)q^a(t) = g^2$  is gauge invariant and is a physically measurable quantity. As mentioned in section V, the time dependent color charge vector  $\vec{q}(t)$  of a fermion in SU(N) Yang-Mills theory is a  $N^2 - 1$  dimensional (real) vector. The time dependent gauge transformation of the color charge  $q^a(t)$  of the quark in the adjoint representation of SU(3) is given by

$$q'^a(t) = R_{ab}(t)q^b(t), \quad R_{ab}(t) = [e^{S(t)}]_{ab}, \quad S_{ab}(t) = f^{abc}\beta^c(t) \quad (117)$$

where  $\beta^a(t)$  are time dependent parameters,  $f^{abc}$  are the antisymmetric structure constants of the SU(3) group and  $a, b, c = 1, 2, \dots, 8$ . Similarly, the time dependent gauge transformation of the color charge  $q_i(t)$  of a fermion in the adjoint representation of SU(2) is given by

$$q'_i(t) = R_{ij}(t)q_j(t), \quad R_{ij}(t) = [e^{S(t)}]_{ij}, \quad S_{ij}(t) = \epsilon_{ijk}\beta_k(t) \quad (118)$$

where  $\epsilon_{ijk}$  are the antisymmetric structure constants of the SU(2) group and  $i, j, k = 1, 2, 3$ . One may recall that the three generators of the SO(3) group are proportional to the three  $3 \otimes 3$  matrices  $G_{jk}^i = \epsilon_{ijk}$  respectively, which implies that the gauge transformation in the adjoint representation of SU(2) corresponds to a rotation in SO(3), see for example [13]. Hence we find that the time dependent gauge transformation of the color charge (real) three-vector  $\vec{q}(t)$  of a fermion in the adjoint representation of SU(2) in eq. (118) corresponds to a time dependent rotation in SO(3). The gauge transformation in SU(2) may be described in terms of three Euler angles by reparameterising the gauge transformation, see for example [14]. Similarly the gauge transformation in SU(3) may be described in terms of eight Euler angles by reparameterising the gauge transformation. The geometry of SU(3) is described in terms of 8 Euler angles in [15] and the 64 components of the  $8 \otimes 8$  arbitrary matrix in the adjoint representation of SU(3) is obtained in terms of 8 Euler angles in [16]. It is useful to remember that, although the gauge transformation in the adjoint representation of SU(2) corresponds to a rotation in SO(3), the gauge transformation in the adjoint representation of SU(3) does not correspond to a general rotation in SO(8) because a general rotation in

$\text{SO}(8)$  is not described by 8 real parameters but a general rotation in  $\text{SO}(8)$  is described by 28 real parameters, see for example [17]. Hence one finds that the time dependent gauge transformation in the adjoint representation of  $\text{SU}(3)$  in eq. (117) does not correspond to a time dependent general rotation in  $\text{SO}(8)$  although the time dependent gauge transformation in the adjoint representation of  $\text{SU}(2)$  in eq. (118) corresponds to a time dependent rotation in  $\text{SO}(3)$ . Hence one expects that in the Yang-Mills theory the general form of three time dependent color charges  $q_i(t)$  of a fermion in  $\text{SU}(2)$  and the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in  $\text{SU}(3)$  may reflect the fact that  $\text{SU}(2)$  and  $\text{SO}(3)$  are locally isomorphic, while  $\text{SU}(3)$  and  $\text{SO}(8)$  are not [see section XV for more discussion on this]. Since color charge  $q^a(t)$  of the quark is not gauge invariant it does not mean that we can not obtain the form of the color charge  $q^a(t)$  of the quark in Yang-Mills theory. For example, we know that the Maxwell potential  $A^\mu(x)$  (the Lienard-Weichert potential) is not gauge invariant (only  $F^{\mu\nu}(x)$  is gauge invariant, see eq. (29)), but that does not mean that we can not derive the form of the Maxwell potential (Lienard-Weichert potential) in Maxwell theory by solving Maxwell equation using Green's function technique, see [11]. Similarly, one finds that one can obtain the form of the color charge  $q^a(t)$  of the quark in Yang-Mills theory even if the color charge  $q^a(t)$  of the quark is not gauge invariant. Any gauge invariant quantities which are related to physically measurable quantities remain unchanged under the gauge transformation of the fundamental color charge  $q^a(t)$  of the quark.

In Maxwell theory the electric current density  $j^\mu(x)$  of the electron and the fundamental electric charge  $e$  of the electron are gauge invariants. The form of the fundamental electric charge  $e$  of the electron can be obtained from the zero component  $j_0(x)$  of the electric current density  $j^\mu(x)$  of the electron by using the Dirac wave function  $\psi(x)$  of the electron (see below). In Yang-Mills theory the color current density  $j^{\mu a}(x)$  of the quark and the fundamental color charge  $q^a(t)$  of the quark are not gauge invariants. Hence in analogy to Maxwell theory one can obtain the form of gauge non-invariant fundamental color charge  $q^a(t)$  of the quark from the zero component  $j_0^a(x)$  of the gauge non-invariant color current density  $j^{\mu a}(x)$  of the quark by using Dirac wave function  $\psi_i(x)$  of the quark (see below). As mentioned in section III these Dirac wave functions  $\psi(x)$  of the electron and  $\psi_i(x)$  of the quark are yet to be quantized in the sense of second quantization (field quantization), see [2].

Note that in Maxwell theory even if scalar ( $\mu = 0$ ) component

$$j_0(x) = e\bar{\psi}(x)\gamma_0\psi(x) = e\psi^\dagger(x)\psi(x) \quad (119)$$

of the electric current density  $j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x)$  of the electron is proportional to  $\psi^\dagger(x)\psi(x)$ , the number density of the electron, it does not naturally give constant electric charge  $e$  of the electron after integrating over full spatially volume available for the electron unless we implement the continuity equation as given by eq. (1). This is because if the continuity equation is not satisfied, *i.e.*, if

$$\partial_\mu j^\mu(x) \neq 0 \quad (120)$$

then the electric charge  $e$  of the electron becomes time dependent instead of being constant.

Hence in order to determine the form of the fundamental color charge  $q^a(t)$  of the quark from the Dirac wave function  $\psi_i(x)$  of the quark in Yang-Mills theory we make analogy with the Maxwell theory. First of all we notice that the Yang-Mills theory was developed in analogy to the procedure in electromagnetic theory (Maxwell theory) by extending the U(1) gauge group to the SU(3) gauge group appropriately, see [1]. For example in analogy to  $D_\mu\psi = (\partial_\mu - ieA_\mu)\psi$  in electromagnetic theory (Maxwell theory) all derivatives of  $\psi$  in Yang-Mills theory appear in the combination  $D_\mu\psi = (\partial_\mu - i\epsilon B_\mu)\psi$  where  $B_\mu = T^a A_\mu^a$ , see [1]. Similarly, in analogy to commutator equation  $[D_\mu, D_\nu] = -ieF_{\mu\nu}$  in electromagnetic theory (Maxwell theory) which predicts the relation between  $F_{\mu\nu}(x)$  and  $A_\mu(x)$  as given by eq. (29) one finds that in Yang-Mills theory the commutator equation  $[D_\mu, D_\nu] = -igT^a F_{\mu\nu}^a$  predicts the relation between  $F_{\mu\nu}^a(x)$  and  $A_\mu^a(x)$  as given by eq. (48), see [1, 2]. Similarly, in analogy to the electromagnetic field Lagrangian density  $\mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$  in electromagnetic theory (Maxwell theory) where  $F_{\mu\nu}(x)$  is given by eq. (29) one writes down the Yang-Mills field Lagrangian density  $\mathcal{L}_{\text{YM}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}$  in Yang-Mills theory where  $F_{\mu\nu}^a(x)$  is given by eq. (48), see [1]. Similarly, in analogy to electromagnetic theory (Maxwell theory) where the form of the physical electromagnetic potential  $A^\mu(x)$  (Lienard-Wiechert potential or Coulomb potential) produced by the electric charge  $e$  of the electron can be obtained from the form of U(1) pure gauge potential produced by the electron, one finds that the form of the color potential (Yang-Mills potential)  $A^{\mu a}(x)$  produced by the color charges  $q^a(\tau)$  of the quark in Yang-Mills theory can be obtained from the form of SU(3) pure gauge potential produced by the quark, see [10]. Similarly, in order to determine the form of

the fundamental color charge  $q^a(t)$  of the quark from the Dirac wave function  $\psi_i(x)$  of the quark we make analogy with the Maxwell theory. First of all we notice from eqs. (119), (42) and (1) that in Maxwell theory the form of the fundamental electric charge  $e$  of the electron can be determined from the Dirac wave function  $\psi(x)$  of the electron by using, 1) the continuity equation  $\partial_\mu j^\mu(x) = 0$  of the electric current density  $j^\mu(x)$  of the electron, 2) scalar ( $\mu = 0$ ) component of the electric current density  $j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x)$  of the electron and 3) number density  $n_e(x) = \psi^\dagger(x)\psi(x)$  of the electron by using the normalization condition  $\int d^3x\psi^\dagger(x)\psi(x) = 1$ . Hence in analogy to Maxwell theory, one finds in Yang-Mills theory that the form of the fundamental color charge  $q^a(t)$  of the quark can be determined from the Dirac wave function  $\psi_i(x)$  of the quark by using, 1) the equation  $D_\mu[A]j^{\mu a}(x) = 0$  of the color current density  $j^{\mu a}(x)$  of the quark, 2) scalar ( $\mu = 0$ ) component of the color current density  $j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T_{ij}^a\psi_j(x)$  of the quark and 3) number density  $n_q(x) = \psi_i^\dagger(x)\psi_i(x)$  of the quark by using the normalization condition  $\int d^3x\psi_i^\dagger(x)\psi_i(x) = 1$  (see below).

## X. FERMION COLOR CURRENT DENSITY, FERMION WAVE FUNCTION AND PAULI MATRICES IN YANG-MILLS THEORY IN SU(2)

In SU(2) Yang-Mills theory the Yang-Mills color current density of a fermion is given by

$$j^{\mu i}(x) = g\bar{\psi}_k(x)\gamma^\mu\tau_{kn}^i\psi_n(x); \quad i = 1, 2, 3; \quad k, n = 1, 2; \quad (121)$$

where three generators  $\tau^i$  of the SU(2) group are given by

$$\tau_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (122)$$

which are related to Pauli matrices  $\sigma^i$  via the relation  $\tau^i = \frac{\sigma^i}{2}$ .

If there is only one fermion in the entire (physically) allowed volume  $V = \int d^3\vec{x}$  then the total probability of finding that fermion in the entire volume is 1. This implies that the normalized wave functions  $\psi_k(x)$  of the fermion, where  $k = 1, 2$  are the color indices of the fermion wave function in SU(2) Yang-Mills theory, satisfy the normalization condition

$$\int d^3\vec{x} [\psi_1^\dagger(x)\psi_1(x) + \psi_2^\dagger(x)\psi_2(x)] = 1. \quad (123)$$

From eqs. (121) and (122) we find

$$\int d^3\vec{x} j_0^1(x) = \frac{g}{2} \int d^3\vec{x} \psi_1^\dagger(x)\psi_2(x) + \frac{g}{2} \int d^3\vec{x} \psi_2^\dagger(x)\psi_1(x),$$

$$\begin{aligned}\int d^3\vec{x} j_0^2(x) &= -\frac{ig}{2} \int d^3\vec{x} \psi_1^\dagger(x) \psi_2(x) + \frac{ig}{2} \int d^3\vec{x} \psi_2^\dagger(x) \psi_1(x), \\ \int d^3\vec{x} j_0^3(x) &= \frac{g}{2} \int d^3\vec{x} \psi_1^\dagger(x) \psi_1(x) - \frac{g}{2} \int d^3\vec{x} \psi_2^\dagger(x) \psi_2(x).\end{aligned}\quad (124)$$

As we have seen above, since  $\int d^3\vec{x} \psi_i^\dagger(t, \vec{x}) \tau_{ij}^a \psi_j(t, \vec{x})$  in Yang-Mills theory is non-linear function of  $g$  we write

$$d_{11}(t, g) = \int d^3\vec{x} \psi_1^\dagger(x) \psi_1(x), \quad d_{22}(t, g) = \int d^3\vec{x} \psi_2^\dagger(x) \psi_2(x), \quad d_{12}(t, g) = \int d^3\vec{x} \psi_1^\dagger(x) \psi_2(x) \quad (125)$$

where  $d_{11}(t, g)$ ,  $d_{22}(t, g)$  are  $t$  and  $g$  dependent real positive functions and  $d_{12}(t, g)$  is  $t$  and  $g$  dependent complex function.

From eqs. (124) and (125) we find

$$\begin{aligned}\int d^3\vec{x} j_0^1(x) &= \frac{g}{2} \times [d_{12}(t, g) + d_{12}^*(t, g)] \\ \int d^3\vec{x} j_0^2(x) &= -\frac{ig}{2} \times [d_{12}(t, g) - d_{12}^*(t, g)] \\ \int d^3\vec{x} j_0^3(x) &= \frac{g}{2} \times [d_{11}(t, g) - d_{22}(t, g)].\end{aligned}\quad (126)$$

From the normalization condition, see eq. (123), we find

$$d_{11}(t, g) + d_{22}(t, g) = 1 \quad (127)$$

where we have used eq. (125).

Since  $d_{11}(t, g)$  and  $d_{22}(t, g)$  are  $t$  and  $g$  dependent real positive functions (see eq. (125)) we can write eq. (127) as

$$d_{11}(t, g) = \cos^2 \Theta(t, g), \quad d_{22}(t, g) = \sin^2 \Theta(t, g) \quad (128)$$

where

$$0 \leq \Theta(t, g) \leq 2\pi. \quad (129)$$

Using eq. (128) in (126) we find

$$\begin{aligned}\int d^3\vec{x} j_0^1(x) &= \frac{g}{2} \times [d_{12}(t, g) + d_{12}^*(t, g)] \\ \int d^3\vec{x} j_0^2(x) &= -\frac{ig}{2} \times [d_{12}(t, g) - d_{12}^*(t, g)] \\ \int d^3\vec{x} j_0^3(x) &= \frac{g}{2} \times \cos[2\Theta(t, g)].\end{aligned}\quad (130)$$

## XI. GENERAL FORM OF FUNDAMENTAL COLOR CHARGE OF A FERMION IN YANG-MILLS THEORY IN SU(2)

Since the fundamental color charge vector  $\vec{q}(t)$  is linearly proportional to  $g$  (see eqs. (60) and (94)) we find from eq. (130) that the color charge  $q_i(t)$  of a fermion in Yang-Mills theory in SU(2) takes the form

$$\begin{aligned} q_1(t) &= \frac{g}{2} \times [d_{12}(t) + d_{12}^*(t)] \\ q_2(t) &= -\frac{ig}{2} \times [d_{12}(t) - d_{12}^*(t)] \\ q_3(t) &= \frac{g}{2} \times \cos[2\Theta(t)] \end{aligned} \quad (131)$$

where the complex function  $d_{12}(t)$  and the real phase factor  $\Theta(t)$  depend on time  $t$  but are independent of  $g$ . Since  $d_{12}(t)$  is a complex function we can write eq. (131) as

$$\begin{aligned} q_1(t) &= g \times |d_{12}(t)| \times \cos\phi(t) \\ q_2(t) &= g \times |d_{12}(t)| \times \sin\phi(t) \\ q_3(t) &= \frac{g}{2} \times \cos[2\Theta(t)] \end{aligned} \quad (132)$$

where the principal argument

$$\text{Arg}(d_{12}(t)) = \phi(t) = \tan^{-1} \left[ \frac{\text{Im}[d_{12}(t)]}{\text{Re}[d_{12}(t)]} \right]. \quad (133)$$

of the complex function  $d_{12}(t)$  lying in the range

$$-\pi < \phi(t) \leq \pi. \quad (134)$$

It is important to remember that the real phases  $\Theta(t)$  and  $\phi(t)$  in eq. (132) are not independent of time  $t$ . This is because if the real phases  $\Theta(t)$  and  $\phi(t)$  are independent of time  $t$  then the non-abelian Yang-Mills potential  $A^{\mu a}(x)$  in eq. (98) becomes Maxwell-like potential  $A^\mu(x)$ .

From eqs. (62) and (132) we find

$$|d_{12}(t)|^2 + \frac{1}{4}\cos^2[2\Theta(t)] = 1. \quad (135)$$

From eq. (129) we find that the maximum allowed range of  $\Theta(t)$  is

$$0 \leq \Theta(t) \leq 2\pi. \quad (136)$$

From eqs. (135) and (136) we find

$$\frac{\sqrt{3}}{2} \leq |d_{12}(t)| \leq 1. \quad (137)$$

We write

$$\frac{1}{4} \cos^2[2\Theta(t)] = \cos^2\theta(t) \quad (138)$$

where from eqs. (136) and (138) we find

$$0 \leq \cos^2\theta(t) \leq \frac{1}{4}. \quad (139)$$

Since  $|d_{12}(t)|$  is positive we find from eqs. (135), (137), (138) and (139) that we can write

$$|d_{12}(t)| = \sin\theta(t) \quad (140)$$

where

$$\frac{\pi}{3} \leq \theta(t) \leq \frac{2\pi}{3}. \quad (141)$$

Hence from eqs. (138), (140), (141), (134) and (132) we find that the general form of three time dependent fundamental color charges of a fermion in Yang-Mills theory in SU(2) is given by

$$\begin{aligned} q_1(t) &= g \times \sin\theta(t) \times \cos\phi(t), \\ q_2(t) &= g \times \sin\theta(t) \times \sin\phi(t), \\ q_3(t) &= g \times \cos\theta(t) \end{aligned} \quad (142)$$

which reproduces eq. (5) where

$$\frac{\pi}{3} \leq \theta(t) \leq \frac{2\pi}{3}, \quad -\pi < \phi(t) \leq \pi. \quad (143)$$

Note that if these two phases become constants then the three color charges  $q_i$  of the fermion become constants in which case the Yang-Mills potential  $A^{\mu i}(x)$  reduces to Maxwell-like (abelian-like) potential [10]. It should be remembered that the static systems in the Yang-Mills theory are, in general, not abelian-like. For example, unlike Maxwell theory where the electron at rest produces (abelian) Coulomb potential, the fermion at rest in Yang-Mills theory does not produce abelian-like potential [10].

The time dependence must be retained in order not to end up in a quasi-abelian setting. In view of the fact that the whole derivation boils down to a gauge transformation of a given spatially integrated charge configuration this is also no surprise. The only aspect that cannot be captured in a quasi-abelian representation is the time dependence of the non-abelian gauge transformation.

## XII. YANG-MILLS COLOR CURRENT DENSITY OF QUARK, THE WAVE FUNCTION OF QUARK AND GELL-MANN MATRICES IN YANG-MILLS THEORY IN SU(3)

Eight generators  $T^a$  of the SU(3) group in the Yang-Mills theory are given by

$$\begin{aligned} T_1 &= \frac{1}{2} \begin{pmatrix} 0, & 1, & 0 \\ 1, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}, & T_2 &= \frac{1}{2} \begin{pmatrix} 0, & -i, & 0 \\ i, & 0, & 0 \\ 0, & 0, & 0 \end{pmatrix}, & T_3 &= \frac{1}{2} \begin{pmatrix} 1, & 0, & 0 \\ 0, & -1, & 0 \\ 0, & 0, & 0 \end{pmatrix}, & T_4 &= \frac{1}{2} \begin{pmatrix} 0, & 0, & 1 \\ 0, & 0, & 0 \\ 1, & 0, & 0 \end{pmatrix}, \\ T_5 &= \frac{1}{2} \begin{pmatrix} 0, & 0, & -i \\ 0, & 0, & 0 \\ i, & 0, & 0 \end{pmatrix}, & T_6 &= \frac{1}{2} \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 1, & 0 \end{pmatrix}, & T_7 &= \frac{1}{2} \begin{pmatrix} 0, & 0, & 0 \\ 0, & 0, & -i \\ 0, & i, & 0 \end{pmatrix}, & T_8 &= \frac{1}{2\sqrt{3}} \begin{pmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & -2 \end{pmatrix} \end{aligned} \tag{144}$$

which are related to Gell-Mann matrices  $\lambda^a$  via the relation  $T^a = \frac{\lambda^a}{2}$ . In Yang-Mills theory in SU(3) the Yang-Mills color current density of the quark is given by

$$j^{\mu a}(x) = g\bar{\psi}_i(x)\gamma^\mu T_{ij}^a\psi_j(x); \quad a = 1, 2, \dots, 8; \quad i, j = 1, 2, 3. \tag{145}$$

The normalized wave functions  $\psi_i(x)$  of the quark obey the equation

$$\int d^3\vec{x} \psi_1^\dagger(x)\psi_1(x) + \int d^3\vec{x} \psi_2^\dagger(x)\psi_2(x) + \int d^3\vec{x} \psi_3^\dagger(x)\psi_3(x) = 1. \tag{146}$$

From eqs. (145) and (144) we find

$$\begin{aligned} \int d^3\vec{x} j_0^1(x) &= \frac{g}{2}[h_{12}(t, g) + h_{12}^*(t, g)], \\ \int d^3\vec{x} j_0^2(x) &= -\frac{ig}{2}[h_{12}(t, g) - h_{12}^*(t, g)], \\ \int d^3\vec{x} j_0^3(x) &= \frac{g}{2}[h_{11}(t, g) - h_{22}(t, g)], \\ \int d^3\vec{x} j_0^4(x) &= \frac{g}{2}[h_{13}(t, g) + h_{13}^*(t, g)], \end{aligned}$$

$$\begin{aligned}
\int d^3 \vec{x} j_0^5(x) &= -\frac{ig}{2} [h_{13}(t, g) - h_{13}^*(t, g)], \\
\int d^3 \vec{x} j_0^6(x) &= \frac{g}{2} [h_{23}(t, g) + h_{23}^*(t, g)], \\
\int d^3 \vec{x} j_0^7(x) &= -\frac{ig}{2} [h_{23}(t, g) - h_{23}^*(t, g)], \\
\int d^3 \vec{x} j_0^8(x) &= \frac{g}{2\sqrt{3}} [h_{11}(t, g) + h_{22}(t, g) - 2h_{33}(t, g)]
\end{aligned} \tag{147}$$

where

$$\begin{aligned}
h_{11}(t, g) &= \int d^3 \vec{x} \psi_1^\dagger(x) \psi_1(x), & h_{22}(t, g) &= \int d^3 \vec{x} \psi_2^\dagger(x) \psi_2(x), & h_{33}(t, g) &= \int d^3 \vec{x} \psi_3^\dagger(x) \psi_3(x), \\
h_{12}(t, g) &= \int d^3 \vec{x} \psi_1^\dagger(x) \psi_2(x), & h_{13}(t, g) &= \int d^3 \vec{x} \psi_1^\dagger(x) \psi_3(x), & h_{23}(t, g) &= \int d^3 \vec{x} \psi_2^\dagger(x) \psi_3(x)
\end{aligned} \tag{148}$$

and from eq. (146) we find

$$h_{11}(t, g) + h_{22}(t, g) + h_{33}(t, g) = 1. \tag{149}$$

From eq. (148) we find that  $h_{11}(t, g)$ ,  $h_{22}(t, g)$ ,  $h_{33}(t, g)$  are  $t$  and  $g$  dependent real positive functions and  $h_{12}(t, g)$ ,  $h_{13}(t, g)$  and  $h_{23}(t, g)$  are  $t$  and  $g$  dependent complex functions.

From now onwards we can proceed exactly in the way similar to SU(2).

Since  $h_{11}(t, g)$ ,  $h_{22}(t, g)$  and  $h_{33}(t, g)$  are  $t$  and  $g$  dependent real positive functions (see eq. (148)) we can write eq. (149) as

$$h_{11}(t, g) = \sin^2 \Theta(t, g) \times \cos^2 \Phi(t, g), \quad h_{22}(t, g) = \sin^2 \Theta(t, g) \times \sin^2 \Phi(t, g), \quad h_{33}(t, g) = \cos^2 \Theta(t, g) \tag{150}$$

where

$$0 \leq \Theta(t, g) \leq \pi, \quad 0 \leq \Phi(t, g) < 2\pi. \tag{151}$$

Using eq. (150) in (147) we find

$$\begin{aligned}
\int d^3 \vec{x} j_0^1(x) &= \frac{g}{2} \times [h_{12}(t, g) + h_{12}^*(t, g)], \\
\int d^3 \vec{x} j_0^2(x) &= -\frac{ig}{2} \times [h_{12}(t, g) - h_{12}^*(t, g)], \\
\int d^3 \vec{x} j_0^3(x) &= \frac{g}{2} \times \sin^2 \Theta(t, g) \times \cos[2\Phi(t, g)], \\
\int d^3 \vec{x} j_0^4(x) &= \frac{g}{2} \times [h_{13}(t, g) + h_{13}^*(t, g)],
\end{aligned}$$

$$\begin{aligned}
\int d^3 \vec{x} j_0^5(x) &= -\frac{ig}{2} \times [h_{13}(t, g) - h_{13}^*(t, g)], \\
\int d^3 \vec{x} j_0^6(x) &= \frac{g}{2} \times [h_{23}(t, g) + h_{23}^*(t, g)], \\
\int d^3 \vec{x} j_0^7(x) &= -\frac{ig}{2} \times [h_{23}(t, g) - h_{23}^*(t, g)], \\
\int d^3 \vec{x} j_0^8(x) &= \frac{g}{2\sqrt{3}} \times [1 - 3 \cos^2 \Theta(t, g)]. \tag{152}
\end{aligned}$$

### XIII. GENERAL FORM OF FUNDAMENTAL COLOR CHARGE OF THE QUARK IN YANG-MILLS THEORY IN SU(3)

Since the fundamental color charge vector  $\vec{q}(t)$  is linearly proportional to  $g$  (see eqs. (60) and (94)) we find from eq. (152) that the color charge  $q^a(t)$  of a quark in Yang-Mills theory in SU(3) takes the form

$$\begin{aligned}
q_1(t) &= \frac{g}{2} \times [h_{12}(t) + h_{12}^*(t)], \\
q_2(t) &= -\frac{ig}{2} \times [h_{12}(t) - h_{12}^*(t)], \\
q_3(t) &= \frac{g}{2} \times \sin^2 \Theta(t) \times \cos[2\Phi(t)] \\
q_4(t) &= \frac{g}{2} \times [h_{13}(t) + h_{13}^*(t)], \\
q_5(t) &= -\frac{ig}{2} \times [h_{13}(t) - h_{13}^*(t)], \\
q_6(t) &= \frac{g}{2} \times [h_{23}(t) + h_{23}^*(t)], \\
q_7(t) &= -\frac{ig}{2} \times [h_{23}(t) - h_{23}^*(t)], \\
q_8(t) &= \frac{g}{2\sqrt{3}} \times [1 - 3 \cos^2 \Theta(t)]. \tag{153}
\end{aligned}$$

where the complex functions  $h_{12}(t)$ ,  $h_{13}(t)$ ,  $h_{23}(t)$  and the real phases  $\Theta(t)$ ,  $\Phi(t)$  depend on time  $t$  but are independent of  $g$ . Since  $h_{12}(t)$ ,  $h_{13}(t)$ ,  $h_{23}(t)$  are complex functions we can write eq. (153) as

$$\begin{aligned}
q_1(t) &= g \times |h_{12}(t)| \times \cos \phi_{12}(t) \\
q_2(t) &= g \times |h_{12}(t)| \times \sin \phi_{12}(t), \\
q_3(t) &= \frac{g}{2} \times \sin^2 \Theta(t) \times \cos[2\Phi(t)], \\
q_4(t) &= g \times |h_{13}(t)| \times \cos \phi_{13}(t), \\
q_5(t) &= g \times |h_{13}(t)| \times \sin \phi_{13}(t),
\end{aligned}$$

$$\begin{aligned}
q_6(t) &= g \times |h_{23}(t)| \times \cos\phi_{23}(t), \\
q_7(t) &= g \times |h_{23}(t)| \times \sin\phi_{23}(t), \\
q_8(t) &= \frac{g}{2\sqrt{3}} \times [1 - 3 \cos^2\Theta(t)]
\end{aligned} \tag{154}$$

where the principal arguments

$$\begin{aligned}
\text{Arg}(h_{12}(t)) = \phi_{12}(t) &= \tan^{-1} \left[ \frac{\text{Im}[h_{12}(t)]}{\text{Re}[h_{12}(t)]} \right], & \text{Arg}(h_{13}(t)) = \phi_{13}(t) &= \tan^{-1} \left[ \frac{\text{Im}[h_{13}(t)]}{\text{Re}[h_{13}(t)]} \right], \\
\text{Arg}(h_{23}(t)) = \phi_{23}(t) &= \tan^{-1} \left[ \frac{\text{Im}[h_{23}(t)]}{\text{Re}[h_{23}(t)]} \right],
\end{aligned} \tag{155}$$

of the complex functions  $h_{12}(t)$ ,  $h_{13}(t)$  and  $h_{23}(t)$  lying in the range

$$-\pi < \phi_{12}(t), \quad \phi_{13}(t), \quad \phi_{23}(t) \leq \pi. \tag{156}$$

From eqs. (61) and (154) we find

$$|h_{12}(t)|^2 + |h_{13}(t)|^2 + |h_{23}(t)|^2 + \left[ \frac{1}{2} \times \sin^2\Theta(t) \times \cos[2\Phi(t)] \right]^2 + \left[ \frac{1}{2\sqrt{3}} \times [1 - 3 \cos^2\Theta(t)] \right]^2 = 1. \tag{157}$$

From eq. (151) we find that the maximum allowed range of  $\Theta(t)$ ,  $\Phi(t)$  are

$$0 \leq \Theta(t) \leq \pi, \quad 0 \leq \Phi(t) < 2\pi. \tag{158}$$

In order to proceed further we need to find the maximum and minimum values of

$$\begin{aligned}
H &= \left[ \sin^2\Theta(t) \times \cos[2\Phi(t)] \right]^2 + \left[ \frac{1}{\sqrt{3}} \times [1 - 3 \cos^2\Theta(t)] \right]^2 \\
&= [\cos^2\Theta(t) - 1]^2 \times \cos^2[2\Phi(t)] + \frac{1}{3} + [3 \cos^4\Theta(t) - 2 \cos^2\Theta(t)]
\end{aligned} \tag{159}$$

in the range of  $\Theta(t)$ ,  $\Phi(t)$  given by eq. (158). Taking the first derivative with respect to  $\Phi(t)$  we find

$$\frac{dH}{d\Phi(t)} = -2 \sin[4\Phi(t)] \times [\cos^2\Theta(t) - 1]^2. \tag{160}$$

Hence we find

$$\frac{dH}{d\Phi(t)} = 0 \tag{161}$$

when

$$\begin{aligned} 1) \cos^2\Theta(t) &= 1, & \text{for any } \Phi(t); \\ 2) \Phi(t) &= 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, & \text{for any } \Theta(t). \end{aligned} \quad (162)$$

Taking the first derivative with respect to  $\Theta(t)$  we find

$$\frac{dH}{d\Theta(t)} = -2 \sin[2\Theta(t)] \times [ [\cos^2\Theta(t) - 1] \times \cos^2[2\Phi(t)] + 3 \cos^2\Theta(t) - 1 ]. \quad (163)$$

Hence we find

$$\frac{dH}{d\Theta(t)} = 0 = \frac{dH}{d\Phi(t)} \quad (164)$$

when

$$\begin{aligned} 1) \cos^2\Theta(t) &= 1, & \text{for any } \Phi(t); \\ 2) \Phi(t) &= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, & \text{and } \cos^2\Theta(t) = \frac{1}{2} \\ 3) \Phi(t) &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, & \text{and } \cos^2\Theta(t) = \frac{1}{3} \\ 4) \Phi(t) &= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, & \text{and } \Theta(t) = \frac{\pi}{2} \\ 5) \Phi(t) &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, & \text{and } \Theta(t) = \frac{\pi}{2}. \end{aligned} \quad (165)$$

By taking the second derivative we find

$$\begin{aligned} \frac{d^2H}{d\Theta^2(t)} &= -4 \cos[2\Theta(t)] \times [ [\cos^2\Theta(t) - 1] \times \cos^2[2\Phi(t)] + 3 \cos^2\Theta(t) - 1 ] \\ &+ 2 \sin^2[2\Theta(t)] \times [\cos^2[2\Phi(t)] + 3] \end{aligned} \quad (166)$$

and

$$\frac{d^2H}{d\Phi^2(t)} = -8 \cos[4\Phi(t)] \times [\cos^2\Theta(t) - 1]^2 \quad (167)$$

and

$$\frac{d^2H}{d\Theta(t) d\Phi(t)} = 4 \sin[4\Phi(t)] \times \sin[2\Theta(t)] \times [\cos^2\Theta(t) - 1]. \quad (168)$$

We write

$$D = \frac{d^2H}{d\Theta^2(t)} \times \frac{d^2H}{d\Phi^2(t)} - \left[ \frac{d^2H}{d\Theta(t) d\Phi(t)} \right]^2. \quad (169)$$

When

$$\cos^2\Theta(t) = 1, \quad \text{for any } \Phi(t) \quad (170)$$

we find

$$D = 0 \quad (171)$$

which implies that the second derivative test is inconclusive.

When

$$\Phi(t) = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \quad \text{and} \quad \cos^2\Theta(t) = \frac{1}{2} \quad (172)$$

we find

$$D = -16 \sin^2[2\Theta(t)] < 0 \quad (173)$$

which implies that it is saddle point.

When

$$\Phi(t) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \quad \text{and} \quad \Theta(t) = \frac{\pi}{2} \quad (174)$$

we find

$$D = -32 < 0 \quad (175)$$

which implies that it is saddle point.

When

$$\Phi(t) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \quad \text{and} \quad \cos^2\Theta(t) = \frac{1}{3} \quad (176)$$

we find

$$D = \frac{64}{3} \sin^2[2\Theta(t)] > 0, \quad \frac{d^2H}{d\Theta^2(t)} = 6 \sin^2[2\Theta(t)] > 0 \quad (177)$$

which gives the minimum value

$$H_{\min} = 0. \quad (178)$$

When

$$\Phi(t) = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \quad \text{and} \quad \Theta(t) = \frac{\pi}{2} \quad (179)$$

we find

$$D = 64 > 0, \quad \frac{d^2H}{d\Theta^2(t)} = -8 < 0 \quad (180)$$

which gives the maximum value

$$H_{\max} = \frac{4}{3}. \quad (181)$$

Hence we find from eqs. (159), (178) and (181) that

$$0 \leq \left[ \frac{1}{2} \times \sin^2 \Theta(t) \times \cos[2\Phi(t)] \right]^2 + \left[ \frac{1}{2\sqrt{3}} \times [1 - 3 \cos^2 \Theta(t)] \right]^2 \leq \frac{1}{3}. \quad (182)$$

Using eq. (182) in eq. (157) we find

$$\frac{2}{3} \leq |h_{12}(t)|^2 + |h_{13}(t)|^2 + |h_{23}(t)|^2 \leq 1. \quad (183)$$

Since each individual square terms in eq. (157) are positive we find from eqs. (157), (182) and (183) that we can write

$$|h_{12}(t)|^2 + |h_{13}(t)|^2 + |h_{23}(t)|^2 = \sin^2 \theta(t) \quad (184)$$

and

$$\left[ \frac{1}{2} \times \sin^2 \Theta(t) \times \cos[2\Phi(t)] \right]^2 + \left[ \frac{1}{2\sqrt{3}} \times [1 - 3 \cos^2 \Theta(t)] \right]^2 = \cos^2 \theta(t) \quad (185)$$

where

$$\frac{2}{3} \leq \sin^2 \theta(t) \leq 1. \quad (186)$$

Since  $|h_{12}(t)|$ ,  $|h_{13}(t)|$  and  $|h_{23}(t)|$  are positive we write eq. (184) as

$$\begin{aligned} |h_{12}(t)| &= \sin \theta(t) \times \sin \sigma(t) \times \cos \eta(t) \\ |h_{13}(t)| &= \sin \theta(t) \times \sin \sigma(t) \times \sin \eta(t) \\ |h_{23}(t)| &= \sin \theta(t) \times \cos \sigma(t) \end{aligned} \quad (187)$$

where

$$\sin^{-1}(\sqrt{\frac{2}{3}}) \leq \theta(t) \leq \pi - \sin^{-1}(\sqrt{\frac{2}{3}}), \quad 0 \leq \sigma(t), \eta(t) \leq \frac{\pi}{2}. \quad (188)$$

We write eqs. (185) as

$$\begin{aligned} \frac{1}{2} \times \sin^2 \Theta(t) \times \cos[2\Phi(t)] &= \cos\theta(t) \times \sin\phi(t), \\ \frac{1}{2\sqrt{3}} \times [1 - 3 \cos^2 \Theta(t)] &= \cos\theta(t) \times \cos\phi(t) \end{aligned} \quad (189)$$

where

$$0 \leq \phi(t) \leq 2\pi. \quad (190)$$

From eqs. (187), (189), (154), (156), (188) and (190) we find that the general form of eight time dependent fundamental color charges of the quark in Yang-Mills theory in SU(3) is given by

$$\begin{aligned} q_1(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \cos\phi_{12}(t), \\ q_2(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \cos\eta(t) \times \sin\phi_{12}(t), \\ q_3(t) &= g \times \cos\theta(t) \times \sin\phi(t) \\ q_4(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \cos\phi_{13}(t), \\ q_5(t) &= g \times \sin\theta(t) \times \sin\sigma(t) \times \sin\eta(t) \times \sin\phi_{13}(t), \\ q_6(t) &= g \times \sin\theta(t) \times \cos\sigma(t) \times \cos\phi_{23}(t), \\ q_7(t) &= g \times \sin\theta(t) \times \cos\sigma(t) \times \sin\phi_{23}(t), \\ q_8(t) &= g \times \cos\theta(t) \times \cos\phi(t) \end{aligned} \quad (191)$$

which reproduces eq. (7) where

$$\begin{aligned} \sin^{-1}(\sqrt{\frac{2}{3}}) \leq \theta(t) \leq \pi - \sin^{-1}(\sqrt{\frac{2}{3}}), \quad 0 \leq \sigma(t), \eta(t) \leq \frac{\pi}{2}, \\ 0 \leq \phi(t) \leq 2\pi, \quad -\pi < \phi_{12}(t), \phi_{13}(t), \phi_{23}(t) \leq \pi. \end{aligned} \quad (192)$$

Note that, as mentioned in the introduction, if all of these seven phases become constants then all the eight color charges  $q^a$  become constants in which case the Yang-Mills potential  $A^{\mu a}(x)$  reduces to Maxwell-like (abelian-like) potential (see eqs. (98) and (108) or [10]). Since the abelian-like potential can not explain confinement of quarks inside (stable) proton one finds that all the seven real phases can not be constants. Hence one finds that

the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark which we have derived in eq. (191) may provide an insight to the question why quarks are confined inside a (stable) proton once the exact form of these time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  are found out [see section XVI for more discussion about this]. It should be remembered that the static systems in the Yang-Mills theory are, in general, not abelian-like. For example, unlike Maxwell theory where the electron at rest produces (abelian) Coulomb potential, the quark at rest in Yang-Mills theory does not produce abelian-like potential (see eq. (108) and sections VII, VIII or see [10]).

#### XIV. COMPARISON WITH WONG'S EQUATION

In Maxwell theory the electromagnetic current density  $j^\mu(x)$  of the electron is linearly proportional to  $e$ . However, in Yang-Mills theory the situation is different. The Yang-Mills potential  $A^{\mu a}(x)$  in eq. (98) contains infinite powers of  $g$  which implies that the Yang-Mills color current density  $j^{\mu a}(x)$  of the quark in Yang-Mills theory in eq. (100) [or in eq. (101)] contains infinite powers of  $g$ .

The Wong's equation [18]

$$\frac{dI^a(\tau)}{d\tau} = g f^{abc} A_\mu^c \frac{d\xi^\mu(\tau)}{d\tau} I^b(\tau) \quad (193)$$

of  $I^a(\tau)$  with the world line  $\xi^\mu(\tau)$  can be derived by using

$$j^{\mu a}(x) = g \int d\tau I^a(\tau) \frac{d\xi^\mu(\tau)}{d\tau} \delta^{(4)}(x - \xi(\tau)) \quad (194)$$

in the equation

$$D_\nu[A] F^{\nu\mu a}(x) = j^{\mu a}(x) \quad (195)$$

where  $D_\mu^{ab}[A]$  is given by eq. (52) and  $F^{\mu\nu a}(x)$  is given by eq. (48), see [18–20].

The Wong's equation [see eq. (193)] describes the precession of  $I^a(\tau)$  in the background  $A^{\mu a}$ , which naturally contains all orders in  $g$  (unless  $A^{\mu a}$  is of order  $\frac{1}{g}$ ) because the left hand side of eq. (193) is  $\frac{dI^a(\tau)}{d\tau}$  and the right hand side of eq. (193) contains  $gI^b(\tau)$ . As mentioned above, the Yang-Mills potential (color potential)  $A^{\mu a}(x)$  produced by the quark contains infinite powers of  $g$  (see eq. (98) or [10]) which implies that the Yang-Mills color current density  $j^{\mu a}(x)$  of the quark in Yang-Mills theory in eq. (100) [or in eq. (101)] contains infinite powers of  $g$ . Hence from eqs. (195) and (194) one finds that  $I^a(t)$  contains infinite

powers of  $g$  which is consistent with eq. (193) because the left hand side of eq. (193) is  $\frac{dI^a(\tau)}{d\tau}$  and the right hand side of eq. (193) contains  $gI^b(\tau)$ . Since the definition of the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (191) is linearly proportional to  $g$  and the  $I^a(t)$  in eq. (194) [or in eq. (193)] contains infinite powers of  $g$ , we find that the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (191) is different from the  $I^a(t)$  used in the Wong's equation in [18].

In our study we have followed the Yang-Mills paper [1] where the Yang-Mills color current density  $j^{\mu a}(x)$  of the quark as given by eqs. (100) and (101) does not obey the continuity equation ( $\partial_\mu j^{\mu a}(x) \neq 0$ ) [but obeys the equation  $D_\mu[A]j^{\mu a}(x) = 0$ , see eq. (2), which implies that the color charge  $q^a(t)$  of the quark is time dependent] but the isotopic spin current density, say  $\mathcal{I}^{\mu a}(x)$ , of the system where

$$\mathcal{I}^{\mu a}(x) = j^{\mu a}(x) + g f^{abc} A_\nu^b(x) F^{\mu\nu c}(x) \quad (196)$$

(as defined in the Yang-Mills paper [1]) obeys the continuity equation

$$\partial_\mu \mathcal{I}^{\mu a}(x) = 0, \quad (197)$$

(see eqs. (15) and (16) of [1]). Note that, as mentioned earlier, the Yang-Mills theory was developed by making analogy with the corresponding procedure in Maxwell theory. Hence, similar to the procedure in the Yang-Mills paper, we have followed the analogy with the Maxwell theory to determine the general form of the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (191). In Maxwell theory the electron is a fundamental particle of the nature. Hence in analogy to Maxwell theory one finds that the quark in Yang-Mills theory is a fundamental particle of the nature. In Maxwell theory the electric charge  $e$  of the electron is a fundamental charge of the nature and hence it is independent of the Maxwell potential  $A^\mu(x)$ . Hence in analogy to Maxwell theory one finds that since the color charge  $q^a(t)$  of the quark in Yang-Mills theory is a fundamental charge of the nature it is independent of the Yang-Mills potential  $A^{\mu a}(x)$ . This can also be seen as follows.

Since the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (191) is linearly proportional to  $g$  and the Yang-Mills potential  $A^{\mu a}(x)$  in eq. (98) [see also [10]] contains infinite powers of  $g$  we find that the definition of the fundamental time dependent color charge  $q^a(t)$  of the quark in eq. (191) is independent of the Yang-Mills potential  $A^{\mu a}(x)$ .

## XV. SECOND (CUBIC) CASIMIR INVARIANT OF SU(3) AND GENERAL FORM OF COLOR CHARGES OF THE QUARK

Note that there is one Casimir invariant (quadratic Casimir invariant)

$$C = q_i(t)q_i(t), \quad i = 1, 2, 3 \quad (198)$$

of SU(2) which is gauge invariant with respect to the gauge transformation given by eq. (118). We have fixed the exact value of the Casimir invariant  $q_i(t)q_i(t) = g^2$  of SU(2) [see eqs. (135) and (62)] in the derivation of the general form of three time dependent color charges  $q_i(t)$  of a fermion in SU(2) in eq. (142) where  $i = 1, 2, 3$ . Since we have fixed the exact value of the Casimir invariant  $q_i(t)q_i(t) = g^2$  of SU(2) [see eqs. (135) and (62)] we find that the general form of three time dependent color charges  $q_1(t)$ ,  $q_2(t)$ ,  $q_3(t)$  of a fermion in SU(2) in eq. (142) depend on  $g$  and two time dependent phases  $\theta(t)$ ,  $\phi(t)$ . It can be observed that the relation between three time dependent color charges  $q_1(t)$ ,  $q_2(t)$ ,  $q_3(t)$  of a fermion in SU(2) and  $g$ ,  $\theta(t)$ ,  $\phi(t)$  which we have found in eq. (142) is similar to the relation between cartesian coordinates  $x_1, x_2, x_3$  and spherical polar coordinates  $r, \theta_1, \theta_2$  in three dimensions where

$$\begin{aligned} x_1 &= r \times \sin\theta_1 \times \cos\theta_2, \\ x_2 &= r \times \sin\theta_1 \times \sin\theta_2, \\ x_3 &= r \times \cos\theta_1, \\ 0 \leq \theta_1 &\leq \pi, \quad 0 \leq \theta_2 < 2\pi, \quad r^2 = x_i x_i \end{aligned} \quad (199)$$

[except that the ranges of time dependent phases in eq. (142) are different, see eq. (143), which is due the fact that  $\tau^i = \frac{\sigma^i}{2}$  in SU(2) in eq. (122)]. The time dependent gauge transformation of the color charge  $q_i(t)$  of a fermion in the adjoint representation of SU(2) in eq. (118) corresponds to a time dependent rotation in SO(3) [see section IX for more discussion on this]. The general form of three time dependent color charges  $q_i(t)$  of a fermion in SU(2) in eq. (142) is consistent with the fact that the SU(2) and SO(3) are locally isomorphic.

The situation is different in SU(3) as we will see below.

In SU(3) there are two independent Casimir invariants

$$C_1 = q^a(t)q^a(t), \quad a = 1, 2, \dots, 8 \quad (200)$$

and

$$C_2 = d_{abc}q^a(t)q^b(t)q^c(t), \quad a, b, c = 1, 2, \dots, 8 \quad (201)$$

where  $d_{abc}$  are the symmetric structure constants of the SU(3) group. Both the Casimir invariants of SU(3) in eqs. (200) and (201) are gauge invariant with respect to gauge transformation of the color charge  $q^a(t)$  of the quark as given by eq. (117). As mentioned in section IX, the gauge transformation in the adjoint representation of SU(3) in eq. (117) which is described by 8 real parameters does not correspond to a time dependent general rotation in SO(8) because a general rotation in SO(8) is not described by 8 real parameters but a general rotation in SO(8) is described by 28 real parameters, see for example [17]. Hence one finds that for a general vector  $w^a$ , since all the non-zero values of  $d_{abc}$  are not same for different values of  $a, b, c$  where  $a, b, c = 1, 2, \dots, 8$ , the  $d_{abc}w^aw^bw^c$  is not rotationally invariant with respect to the general rotation in SO(8) although  $w^aw^a$  is rotationally invariant with respect to the general rotation in SO(8).

Similarly, it can also be easily verified that since all the non-zero values of  $d_{abc}$  are not same for different values of  $a, b, c$  where  $a, b, c = 1, 2, \dots, 8$  one finds that once the exact value of the first Casimir invariant (quadratic Casimir invariant)

$$q^a(t)q^a(t) = g^2 \quad (202)$$

of SU(3) is fixed, the second Casimir invariant (cubic Casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) satisfies the range [6–8, 21]

$$-\frac{g^3}{\sqrt{3}} \leq d_{abc}q^a(t)q^b(t)q^c(t) \leq \frac{g^3}{\sqrt{3}}. \quad (203)$$

Note that in the derivation of the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (191) we have fixed the exact value of the first casimir invariant  $C_1 = q^a(t)q^a(t) = g^2$  of SU(3) [see eqs. (157) and (61)] but we have not fixed the exact value of the second casimir invariant (cubic casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) because it satisfies the range given by eq. (203). Since we have only fixed the exact value of the first casimir invariant  $C_1 = q^a(t)q^a(t) = g^2$  of SU(3) [see eqs. (157) and (61)], the general form of eight fundamental time dependent color charges  $q_1(t), q_2(t), q_3(t), q_4(t), q_5(t), q_6(t), q_7(t), q_8(t)$  of the quark in eq. (191) depend on  $g$  and seven time dependent phases  $\theta(t), \sigma(t), \eta(t), \phi(t), \phi_{12}(t), \phi_{13}(t), \phi_{23}(t)$ .

However, unlike SU(2) case in eq. (142) which is similar to spherical polar coordinates in three dimensions [see eq. (199)], the relation between eight time dependent fundamental color charges  $q_1(t), q_2(t), q_3(t), q_4(t), q_5(t), q_6(t), q_7(t), q_8(t)$  of the quark in SU(3) and  $g, \theta(t), \sigma(t), \eta(t), \phi(t), \phi_{12}(t), \phi_{13}(t), \phi_{23}(t)$  which we have found in eq. (191) is not similar to the relation between cartesian coordinates  $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$  and spherical polar coordinates  $r, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$  in eight dimensions where

$$\begin{aligned}
x_8 &= r \times \sin\theta_1 \times \sin\theta_2 \times \sin\theta_3 \times \sin\theta_4 \times \sin\theta_5 \times \sin\theta_6 \times \sin\theta_7, \\
x_7 &= r \times \sin\theta_1 \times \sin\theta_2 \times \sin\theta_3 \times \sin\theta_4 \times \sin\theta_5 \times \sin\theta_6 \times \cos\theta_7, \\
x_6 &= r \times \sin\theta_1 \times \sin\theta_2 \times \sin\theta_3 \times \sin\theta_4 \times \sin\theta_5 \times \cos\theta_6, \\
x_5 &= r \times \sin\theta_1 \times \sin\theta_2 \times \sin\theta_3 \times \sin\theta_4 \times \cos\theta_5, \\
x_4 &= r \times \sin\theta_1 \times \sin\theta_2 \times \sin\theta_3 \times \cos\theta_4, \\
x_3 &= r \times \sin\theta_1 \times \sin\theta_2 \times \cos\theta_3, \\
x_2 &= r \times \sin\theta_1 \times \cos\theta_2, \\
x_1 &= r \times \cos\theta_1, \\
0 \leq \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6 &\leq \pi, \quad 0 \leq \theta_7 < 2\pi, \quad r^2 = x_a x_a
\end{aligned} \tag{204}$$

even if we have only fixed the exact value of the first casimir invariant  $C_1 = q^a(t)q^a(t) = g^2$  of SU(3) [see eqs. (157) and (61)] to derive eq. (191). Hence the general form of eight time dependent color charges  $q^a(t)$  of the quark which we have found in eq. (191) reflect the fact that there is a second casimir invariant (cubic casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) whose range is given by eq. (203). One may wonder that since we have not fixed the exact value of the second casimir invariant (cubic casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) in the derivation of eq. (191) how can the the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (191) reflect the fact that there is a second casimir invariant (cubic casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3). The answer to this question is that we have used the exact expressions of eight generators  $T^a$  of SU(3) by using exact expressions of eight Gell-Mann matrices from eq. (144) in the color current density  $j^{\mu a}(x)$  of the quark in eq. (145) to derive eq. (191) which makes it implicit that there is a second casimir invariant (cubic casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) even if we have not fixed its exact value because it satisfies the range given by eq. (203). One way to see this is to observe the corresponding situation in SU(2). In SU(2) we have

used exact expressions of three generators  $\tau^i$  of  $SU(2)$  by using exact expressions of three Pauli matrices from eq. (122) in the color current density  $j^{\mu i}(x)$  of a fermion in  $SU(2)$  in eq. (121) to derive eq. (142) which makes it implicit that the cubic casimir invariant is absent in  $SU(2)$  and hence the general form of three time dependent color charges  $q_1(t)$ ,  $q_2(t)$ ,  $q_3(t)$  of a fermion in  $SU(2)$  in eq. (142) is found to be similar to spherical polar coordinates in three dimensions. Hence we find that the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (191) is consistent with the fact that there is a second casimir invariant (cubic casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of  $SU(3)$  but its exact value is not fixed because it satisfies the range given by eq. (203). As mentioned above the time dependent gauge transformation of the color charge  $q^a(t)$  of the quark in the adjoint representation of  $SU(3)$  in eq. (117) does not correspond to a time dependent general rotation in  $SO(8)$ . Hence we find that in the Yang-Mills theory the general form of three time dependent color charges  $q_i(t)$  of a fermion in  $SU(2)$  in eq. (142) and the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in  $SU(3)$  in eq. (191) are consistent with the fact that  $SU(2)$  and  $SO(3)$  are locally isomorphic, while  $SU(3)$  and  $SO(8)$  are not.

In order to provide an expression for how the second Casimir invariant (cubic Casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of  $SU(3)$  depends on seven time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  of the color charges  $q^a(t)$  of the quark [see eq. (191)] we proceed as follows. From eq. (144) we find

$$T^a q^a(t) = \frac{1}{2} \begin{pmatrix} q_3(t) + \frac{q_8(t)}{\sqrt{3}}, & q_1(t) - iq_2(t), & q_4(t) - iq_5(t) \\ q_1(t) + iq_2(t), & -q_3(t) + \frac{q_8(t)}{\sqrt{3}}, & q_6(t) - iq_7(t) \\ q_4(t) + iq_5(t), & q_6(t) + iq_7(t), & -\frac{2q_8(t)}{\sqrt{3}} \end{pmatrix} \quad (205)$$

which gives

$$\begin{aligned} \text{Det}[T^a q^a(t)] &= \frac{q_8(t)}{8\sqrt{3}} [3q_3^2(t) + \frac{q_8^2(t)}{3} - g^2] \\ &+ [q_1^2(t) + q_2^2(t)] \frac{3q_8(t)}{8\sqrt{3}} - \frac{q_3(t)}{8} [q_6^2(t) + q_7^2(t)] + \frac{q_3(t)}{8} [q_4^2(t) + q_5^2(t)] \\ &+ \frac{q_1(t)}{4} [q_4(t)q_6(t) + q_5(t)q_7(t)] + \frac{q_2(t)}{4} [q_5(t)q_6(t) - q_4(t)q_7(t)] \end{aligned} \quad (206)$$

where  $g^2$  is given by eq. (61). Since [7]

$$\text{Det}[T^a q^a(t)] = \frac{1}{12} d_{abc} q^a(t) q^b(t) q^c(t) \quad (207)$$

we find from eqs. (207) and (206) that

$$\begin{aligned}
d_{abc}q^a(t)q^b(t)q^c(t) = & \frac{3q_8(t)}{2\sqrt{3}}[3q_3^2(t) + \frac{q_8^2(t)}{3} - g^2] + \frac{3\sqrt{3}q_8(t)}{2}[q_1^2(t) + q_2^2(t)] \\
& + \frac{3q_3(t)}{2}[q_4^2(t) + q_5^2(t)] - \frac{3q_3(t)}{2}[q_6^2(t) + q_7^2(t)] \\
& + 3q_1(t)[q_4(t)q_6(t) + q_5(t)q_7(t)] + 3q_2(t)[q_5(t)q_6(t) - q_4(t)q_7(t)]. \tag{208}
\end{aligned}$$

The eq. (208) can also be verified by directly using the non-zero symmetric structure constants  $d_{abc}$  of SU(3) group

<u><math>abc</math></u>	118	146	157	228	247	256	338	344	355	366	377	448	558	668	778	888
<u><math>d_{abc}</math></u>	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$

(209)

which gives

$$\begin{aligned}
d_{abc}q^a(t)q^b(t)q^c(t) = & 3[d_{118}q_1(t)q_1(t)q_8(t) + d_{228}q_2(t)q_2(t)q_8(t) + d_{338}q_3(t)q_3(t)q_8(t) \\
& + d_{344}q_3(t)q_4(t)q_4(t) + d_{355}q_3(t)q_5(t)q_5(t) + d_{366}q_3(t)q_6(t)q_6(t) + d_{377}q_3(t)q_7(t)q_7(t) \\
& + d_{448}q_4(t)q_4(t)q_8(t) + d_{558}q_5(t)q_5(t)q_8(t) + d_{668}q_6(t)q_6(t)q_8(t) + d_{778}q_7(t)q_7(t)q_8(t)] \\
& + 6[d_{146}q_1(t)q_4(t)q_6(t) + d_{157}q_1(t)q_5(t)q_7(t) + d_{247}q_2(t)q_4(t)q_7(t) + d_{256}q_2(t)q_5(t)q_6(t)] \\
& + d_{888}q_8(t)q_8(t)q_8(t). \tag{210}
\end{aligned}$$

Using the non-zero values of symmetric structure constants  $d_{abc}$  of SU(3) from eq. (209) in eq. (210) we find

$$\begin{aligned}
d_{abc}q^a(t)q^b(t)q^c(t) = & \frac{q_8^3(t)}{2\sqrt{3}} + \frac{9}{2\sqrt{3}}q_3^2(t)q_8(t) + \frac{9}{2\sqrt{3}}q_8(t)[q_1^2(t) + q_2^2(t)] \\
& + 3q_1(t)[q_4(t)q_6(t) + q_5(t)q_7(t)] + 3q_2(t)[q_5(t)q_6(t) - q_4(t)q_7(t)] \\
& + \frac{3q_3(t)}{2}[q_4^2(t) + q_5^2(t)] - \frac{3q_3(t)}{2}[q_6^2(t) + q_7^2(t)] - \frac{3q_8(t)}{2\sqrt{3}}g^2 \tag{211}
\end{aligned}$$

which reproduces eq. (208) where  $g^2$  is given by eq. (61).

From eqs. (191) and (208) we find that the expression of the second Casimir invariant (cubic Casimir invariant)  $C_2 = d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) in terms of seven time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  of the color charges  $q^a(t)$  of the quark [see eq. (191)] is given by

$$d_{abc}q^a(t)q^b(t)q^c(t) = \frac{3g^3}{2}\sin^3\theta(t)\sin^2\sigma(t)\cos\sigma(t)\sin2\eta(t)\cos[\phi_{12}(t) - \phi_{13}(t) + \phi_{23}(t)]$$

$$\begin{aligned}
& + \frac{\sqrt{3}g^3}{2} \cos\theta(t) \cos\phi(t) [3 \sin^2\theta(t) \sin^2\sigma(t) \cos^2\eta(t) + 3 \cos^2\theta(t) \sin^2\phi(t) + \frac{\cos^2\theta(t) \cos^2\phi(t)}{3} - 1] \\
& + \frac{3g^3}{2} \cos\theta(t) \sin\phi(t) \sin^2\theta(t) [\sin^2\sigma(t) \sin^2\eta(t) - \cos^2\sigma(t)]. \tag{212}
\end{aligned}$$

Note that the universal coupling  $g$  (which is the physical observable, a fundamental quantity of the nature) is the only parameter (apart from the mass  $m$  of the quark) that appears in the classical Yang-Mills lagrangian density [1, 2] (see eq. (47) above). As mentioned above, from eqs. (61) and (157) one finds that the first Casimir (quadratic Casimir) invariant  $C_1 = q^a(t)q^a(t) = g^2$  of SU(3) is fixed to be  $g^2$  which is a physical observable. Hence we find that the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in Yang-Mills theory in SU(3) in eq. (191) depend on  $g$  and seven time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  where the ranges of these seven time dependent phases are given by eq. (192). Since the first Casimir (quadratic Casimir) invariant  $q^a(t)q^a(t)$  and the second Casimir (cubic Casimir) invariant  $d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) are two independent Casimir invariants, one expects that if the second Casimir (cubic Casimir) invariant  $d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) corresponds to any physical observable then that physical observable should be experimentally measured and that physical observable should be different from  $g$  because the first Casimir (quadratic Casimir) invariant  $q^a(t)q^a(t)$  of SU(3) is fixed to be  $g^2$ , see eqs. (61) and (157). If such a physical observable exists in the nature and is fixed to be, say  $C_3$ , [for example by experiments] where the fixed  $C_3$  is given by eq. (10) then one finds from eqs. (212) and (10) that

$$\begin{aligned}
\phi_{13}(t) &= \phi_{12}(t) + \phi_{23}(t) + \cos^{-1}\left[\frac{1}{\sin^3\theta(t) \sin^2\sigma(t) \cos\sigma(t) \sin 2\eta(t)} \times \left[\frac{2C_3}{3g^3}\right.\right. \\
&\quad \left.\left. - \sin^2\theta(t) \cos\theta(t) \sin\phi(t) [\sin^2\sigma(t) \sin^2\eta(t) - \cos^2\sigma(t)]\right.\right. \\
&\quad \left.\left. - \frac{1}{\sqrt{3}} \cos\theta(t) \cos\phi(t) [3 \sin^2\theta(t) \sin^2\sigma(t) \cos^2\eta(t) + 3 \cos^2\theta(t) \sin^2\phi(t) + \frac{\cos^2\theta(t) \cos^2\phi(t)}{3} - 1]\right]\right] \tag{213}
\end{aligned}$$

in which case the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (191) depend on  $g$ ,  $C_3$  and six time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{23}(t)$  where  $\phi_{13}(t)$  is given by eq. (213). However, note that even if all the physical observables are gauge invariant but not all the gauge invariants are physical observables. Hence if there exists no physical observable in the nature which is related to the fixed value  $C_3$  as given by eq. (10) [for example if one can not find any

such observable from the experiments] then the second Casimir invariant (cubic Casimir invariant)  $d_{abc}q^a(t)q^b(t)q^c(t)$  of SU(3) satisfies the range  $-\frac{g^3}{\sqrt{3}} \leq d_{abc}q^a(t)q^b(t)q^c(t) \leq \frac{g^3}{\sqrt{3}}$  [see eq. (203)] in which case the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in eq. (191) depend on  $g$  and seven time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  where the ranges of these seven time dependent phases are given by eq. (192). Hence we find that the general form of eight time dependent fundamental color charges of the quark in Yang-Mills theory in SU(3) is given by eq. (191) where  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  are real time dependent phases.

## XVI. ADVANTAGE OF TIME DEPENDENT PHASES IN THE COLOR CHARGE OF THE QUARK

One of the main difference between Maxwell theory and Yang-Mills theory is that while Maxwell potential  $A^\mu(x)$ , Maxwell field tensor  $F^{\mu\nu}(x)$ , Maxwell electric current density  $j^\mu(x)$  of the electron are linearly proportional to the electron charge  $e$ ; the non-abelian Yang-Mills potential  $A^{\mu a}(x)$ , the non-abelian Yang-Mills field tensor  $F^{\mu\nu a}(x)$ , the non-abelian Yang-Mills color current density  $j^{\mu a}(x)$  of the quark contain infinite powers of  $g$  (see eqs. (98), (48), and (100)).

Hence one finds that the classical Maxwell theory is in the perturbative regime of the quantum electrodynamics (QED), whereas the classical Yang-Mills theory is in the non-perturbative regime of the quantum chromodynamics (QCD). Hence it may not be surprising if the Yang-Mills potential  $A^{\mu a}(x)$  produced by the quark may provide an insight to the question why quarks are confined inside a (stable) proton, similar to Coulomb potential  $A_0(x)$  which explains (stable) hydrogen atom in Bohr's atomic model.

The general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark derived in this paper depend on the strong-coupling constant  $g$  and time dependent phases  $\theta(t)$ ,  $\sigma(t)$ ,  $\eta(t)$ ,  $\phi(t)$ ,  $\phi_{12}(t)$ ,  $\phi_{13}(t)$ ,  $\phi_{23}(t)$  [see eq. (191)]. When all these phases become constants then the eight color charges  $q^a$  become constants in which case the Yang-Mills potential  $A^{\mu a}(x)$  in eq. (98) reduces to Maxwell-like (abelian-like) potential (see also eq. (108) and sections VII, VIII or [10]). Since abelian-like potential can not explain confinement of quarks inside (stable) proton, one finds that all these phases can not be constants. The information about confinement of quarks inside (stable) proton can be obtained from the

exact form of the gauge invariant  $F_{\mu\nu}^a(x)F^{\mu\nu a}(x)$ . However, the exact form of the gauge invariant  $F_{\mu\nu}^a(x)F^{\mu\nu a}(x)$  can not be obtained from the gauge invariant  $q^a(t)q^a(t) = \bar{q}^2(t) = g^2$ , but the exact form of the gauge invariant  $F_{\mu\nu}^a(x)F^{\mu\nu a}(x)$  can be obtained [see eq. (48)] from the exact form of Yang-Mills potential  $A^{\mu a}(x)$  which can be obtained from the exact form of the color charges  $q^a(t)$  of the quark, see eq. (98). Hence in order to get an insight about confinement of quarks inside (stable) proton it is not enough to know the gauge invariant  $q^a(t)q^a(t) = \bar{q}^2(t) = g^2$  but it is necessary to know the exact form of the color charges  $q^a(t)$  of the quark which means we need to find out the exact form of these time dependent phases  $\theta(t), \sigma(t), \eta(t), \phi(t), \phi_{12}(t), \phi_{13}(t), \phi_{23}(t)$ . Hence the advantage of the general form of the color charge  $q^a(t)$  of the quark derived in this paper is that it may provide an insight to the question why quarks are confined inside a (stable) proton once the exact form of these time dependent phases  $\theta(t), \sigma(t), \eta(t), \phi(t), \phi_{12}(t), \phi_{13}(t), \phi_{23}(t)$  are found out.

## XVII. CONCLUSION

In Maxwell theory the constant electric charge  $e$  of the electron is consistent with the continuity equation  $\partial_\mu j^\mu(x) = 0$  where  $j^\mu(x)$  is the current density of the electron. However, in Yang-Mills theory the Yang-Mills color current density  $j^{\mu a}(x)$  of the quark satisfies the equation  $D_\mu[A]j^{\mu a}(x) = 0$  which is not a continuity equation ( $\partial_\mu j^{\mu a}(x) \neq 0$ ) which implies that the color charge of the quark is not constant where  $a = 1, 2, \dots, 8$  are the color indices. Since the charge of a point particle is obtained from the zero ( $\mu = 0$ ) component of a corresponding current density by integrating over the entire (physically) allowed volume, the color charge  $q^a(t)$  of the quark in Yang-Mills theory is time dependent. In this paper we have derived the general form of eight time dependent fundamental color charges  $q^a(t)$  of the quark in Yang-Mills theory in SU(3) where  $a = 1, 2, \dots, 8$ .

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