Drafts for Project-Particle Methods for Vortex Problems

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We begin with Poisson's equation in continuous form:

$$\nabla^2 \psi(\mathbf{x}) = -\rho(\mathbf{x})$$

The solution using the Green's function is given by:

$$\psi(\mathbf{x}) = \int_{\mathbb{R}^2} G(\mathbf{x} - \mathbf{x}')\, \rho(\mathbf{x}')\, d\mathbf{x}'$$

Solving Poisson's equation numerically on a discrete grid, means the domain, potential, source, and Green's function are sampled at grid points. To discretize this:

- Let the domain be sampled on a uniform 2D grid with spacing h
- Let $i = (i_x, i_y)$ index the grid points
- Let $\mathbf{x}_i = h \cdot i$ be the physical coordinates
- Define $\rho_j = \rho(\mathbf{x}_j)$ Here, $G(i - j)$ is the Green's function for the Laplacian in 2D (solution to Poisson's equation)

Then, we approximate the integral using a Riemann sum and denote $\omega_j^g = \rho_j$.

$$\psi_i = \sum_{j \in \mathbb{Z}^2} G(i - j)\, \omega_j^g$$

where ω_j represents the value of ρ at the grid point j, and the sum runs over all grid points $j \in \mathbb{Z}^2$. The function $G(i - j)$ gives the influence of the source at point j on the potential at point i.

The potential on the grid is given by ψ_i, and the source on the grid is given by ω_j^g.

Hockney's algorithm utilizes that The Fourier transform of a convolution equals the product of the Fourier transforms.

$$\mathcal{F}\left(f * g\right) = \mathcal{F}(f) \cdot \mathcal{F}(g)$$
$$f * g = \mathcal{F}^{-1}\left[\mathcal{F}(f) \cdot \mathcal{F}(g)\right]$$