Isogeometric Analysis of Elasto-static Contact

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1 Introduction

Compared to traditional Lagrangian finite element, Isogeometric analysis has several advantages. The primary one is that the exact geometry of the contact surface is maintained, thus a high level of accuracy is guaranteed. Another is that the smoothness of the contact surface is maintained as well, which may avoid numerical oscillations in contact forces and difficulties that commonly occurs in the nonlinear solution algorithms. In Isogeometry analysis, contact surface can be extracted from the body directly, therefore relive us from the labor of defining contact surfaces.

As an extension of NURBS based linear elasticity Isogeometry analysis code, contact module for 2-D problem was implemented in this project. The non-penetration constraint between two bodies was enforced by penalty method. So called "knot to segment" method was adopted for contact surface discretization.

Some numerical simulations were carried out to test the functionality of the contact module. The accuracy of the algorithm was examined by patch test. The influence of penalty parameter to the result was investigated as well.

2 Geometry of normal contact[2]

Consider two elastic bodies that come into contact at some point of time during their deformation(Fig 1). Let $\boldsymbol{X}^{(i)}$ denote the reference position of a material point in the body i. Each body is associated with a motion $\boldsymbol{x}^{(i)} = \boldsymbol{\varphi}^{(i)}(\boldsymbol{X}^{(i)})$, that bring a material point $\boldsymbol{X}^{(}\boldsymbol{i})$ to its position $\boldsymbol{x}^{(i)}$ in the current configuration $\Omega_t^{(i)}$. The motions are subjected to the usual smoothness and invertibility requirement. The boundaries $\Gamma^{(i)} = \partial\Omega_0^{(i)}$ in the reference configurations are mapped to the current boundaries $\gamma_t^{(i)} = \boldsymbol{x}(\Gamma^{(i)_0}, t)$. Assume that at some time t, the bodies engage a contact through a surface $c = \gamma^{(1)} \cap \gamma^{(2)}$.

For a point $\boldsymbol{x}^{(i)} \in \gamma^{(1)}$, its distance to the surface $\gamma^(2)$ can be defined by the minimization problem (see fig)

$$g(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = min_{\boldsymbol{x}^{(2)} \in \gamma^{(2)}} \|\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}\| \tag{1}$$

The minimizer in eqn1 is called the *closet projection point* of $\boldsymbol{x}^{(1)}$ and is denoted by $\boldsymbol{x}_p^{(2)}$. Once $\boldsymbol{x}_p^{(2)}$ is determined, the gap function between the two surfaces

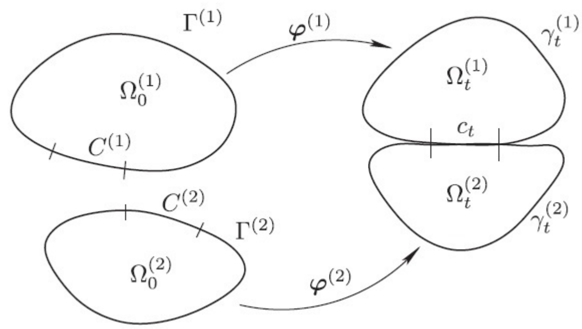

Figure 1: Contact Configuration [2]

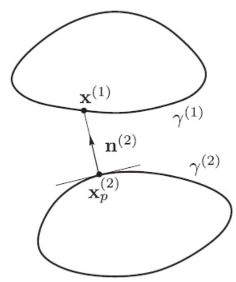

Figure 2: Illustration of the closet point projection in 2D [2]

can be defined as

$$g_N = (\boldsymbol{x}^{(1)} - \boldsymbol{x}_p^{(2)}) \cdot \boldsymbol{n}^{(2)} \tag{2}$$

where $\boldsymbol{n}^{(2)}$ is the unit outward normal of $\gamma^{(2)}$ at $\boldsymbol{x}_p^{(2)}$.

Kuhn-Tucker conditions for impenetrability constraints on c are

$$g_N \geq 0, \quad p_N \geq 0, \quad g_N p_N = 0 \tag{3}$$

3 Isogeometry contact formulation

3.1 Penalty method

Penalty method is one of the most commonly used methods to enforce the contact constraints. Many of the first contributions to Isogeometry contact [2][3] are within in this category. In this method, potential energy relating to the contact is formulated as

$$\Pi_c = \frac{1}{2} \int_c \epsilon < -g_N >^2 \, da \tag{4}$$

3.2 Knot-to-segment discretization

The class node-to-segment(NTS) algorithm of computational contact mechanics cannot be directly employed with NURBS, because the control points are note interpolatory. The straightforward extension of NTS to the isogeometry setting corresponds to a knot-to-segment(KTS) algorithm[3].

Consider two segments(surfaces) involving the contact, one is denoted as master segment, while the other is denoted as slave segment. There is no preference about which one is master or slave. All quadrature points on the slave segment will be checked to see if they are contact with master segment. A quadrature point that is actually contact with the master segment, and its corresponding closest point on the master segment is called an active set. Then, the contact potential energy is represent by

$$\Pi_c = \frac{1}{2} \sum_{\text{Active set}} \epsilon < -g_N >^2 \tag{5}$$

3.3 Contact search (Active set search) algorithm

Given a slave point and a master segment, we can use two steps to determine whether they are in contact. The contact search is composed of two steps. The first step is to find the closest point, which can be achieved by closest point projection method. In figure (3), $\vec{x}$ is the slave point, and $\vec{\rho}(\xi)$ is master segment parameterized in ξ. Out normal of the master segment is $\vec{n}$, while the tangent is $\vec{t}$, which can be computed by

$$\vec{t} = \frac{\partial \vec{\rho}}{\partial \xi} \tag{6}$$

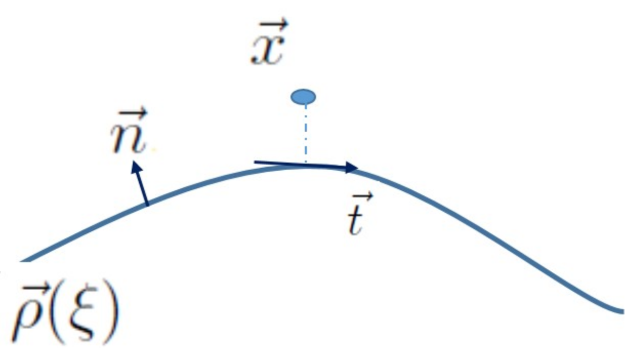

Figure 3: Closest point projection

On master segment, the closest point to slave point satisfies that

$$(\vec{x} - \rho(\xi)) \cdot \vec{t} = (\vec{x} - \rho(\xi)) \cdot \frac{\partial \vec{\rho}}{\partial \xi} = 0 \tag{7}$$

The closet point $\rho(\vec{\xi}_c)$ can be obtained by solving equation (7) using Newton iteration. Increment of Newton iteration is given as,

$$\Delta\xi = -\frac{(\vec{x} - \vec{\rho}(\xi)) \cdot \frac{\partial\vec{\rho}(\xi)}{\partial\xi}}{-\frac{\partial\vec{\rho}(\xi)}{\partial\xi} \cdot \frac{\partial\vec{\rho}(\xi)}{\partial\xi} + (\vec{x} - \vec{\rho}(\xi)) \cdot \frac{\partial\vec{\rho}(\xi)}{\partial^2\xi^2}} \tag{8}$$

NURBS curve and its derivatives are given as

$$\vec{\rho}(\xi) = \sum_a R_a(\xi))\vec{P}_a \tag{9}$$

$$\frac{\partial\vec{\rho}(\xi)}{\partial\xi} = \sum_a \frac{\partial R_a(\xi)}{\partial\xi}\vec{P}_a \tag{10}$$

$$\frac{\partial^2\vec{\rho}(\xi)}{\partial\xi^2} = \sum_a \frac{\partial^2 R_a(\xi)}{\partial\xi^2}\vec{P}_a \tag{11}$$

where R_a is the NURBS basis function, $\vec{P}_a$ is the control points, and ξ is the knot coordinate. Then, the contact is active if

$$(\vec{x} - \rho(\vec{\xi}_c)) \cdot \vec{n} > 0 \tag{12}$$

Otherwise, the contact is inactive.

3.4 System assembling

Adding all active set, we can get contact potential energy according to equation (5). And the contact stiffness matrix is obtained by taking second derivative of the contact potential with respect to displacement $\vec{u}$.

$$K_c = \frac{\partial^2\Pi_c}{\partial u^2} \tag{13}$$

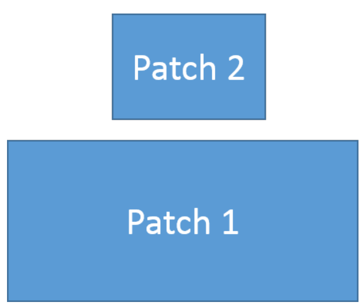

Figure 4: Closest point projection

There are two elastic blocks which are going to have contact with each other in figure (4). The finite element stiffness matrix of the two patches are computed by the code we implemented in homework set 4, and are K_1 and K_2 respectively. Load vectors are obtained to be f_1 and f_2. The whole matrix system is grouped as

$$\left\{\begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} + K_c\right\}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{14}$$

The Dirichlet and Newman boundary are formulated into K_1, K_2, f_1 and f_2. Then we can solve the linear system for response.

4 Numerical examples

4.1 Contact of two blocks

Two blocks in contact are shown in fig(5). Block 1 is on the top, while block 2 is under block 1. The bottom of block 2 is fixed, and the top of block 1 is prescribed with a negative displacement on the vertical direction.

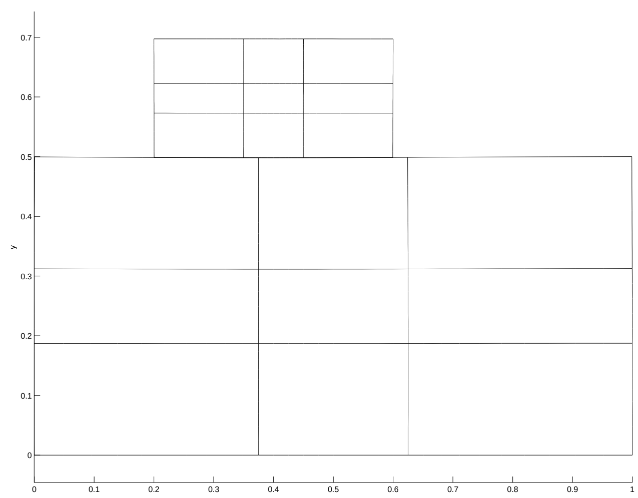

Figure 5: Contact of two blocks: undeformed mesh

Deformed mesh of this problem set is showed in figure (6). We can find that slave segment and master segment (contact surfaces) matches pretty good.

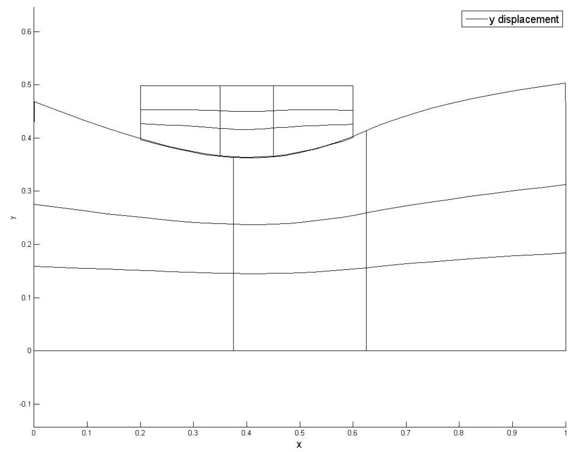

Figure 6: Contact of two blocks: deformed mesh

Normal stress σ_{yy} of the deformed bodies are shown in figure (7)

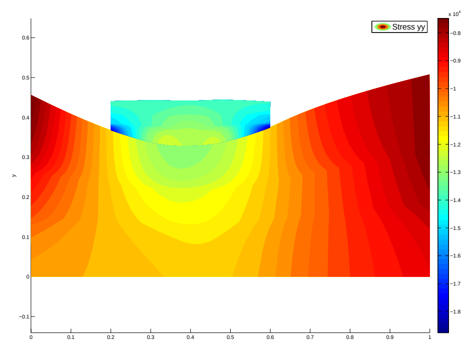

Figure 7: Contact of two blocks: left

Two similar simulations are carried out by move block 1 to right. The results are showed in figure (8) and (9)

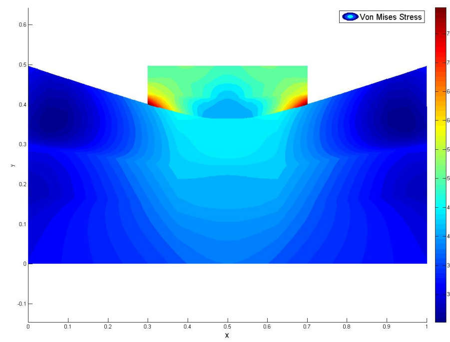

Figure 8: Contact of two blocks: middle

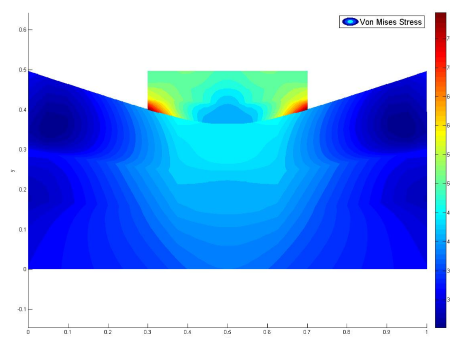

Figure 9: Contact of two blocks: middle

4.2 Patch test: uniform pressure load

The functionality of the contact module is tested in the previous example. We need to investigate the accuracy of the algorithm in addition. A path test commonly used is to apply uniform pressure ($p = -10^4 Pa$) on the top of block 1, and check if the total residual of the normal stress on the contact surfaces is the same as the load applied.

Normal stress contour of the test is shown in the figure below

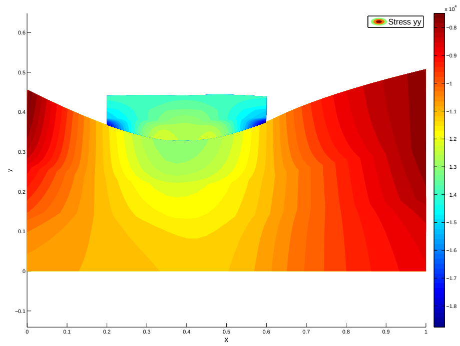

Figure 10: Normal stress of uniform pressure patch test

Normal stress on the bottom of block 1 is plotted in figure (11). The maximum stress occurs in the two end of the contact surface of the block 1. The average value of the normal stress is approximately equal to the pressure we applied, with the deviation of about 4%.

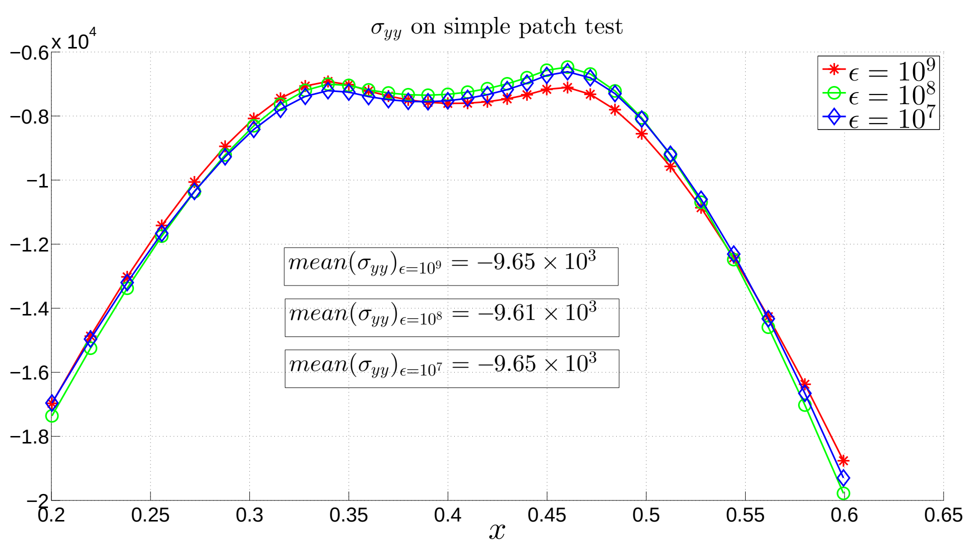

Figure 11: Normal stress on the bottom of block 1

4.3 Patch test: linear pressure load

Instead of uniform load, we apply linear load in this case. The deformed mesh of this problem is shown in figure (13).

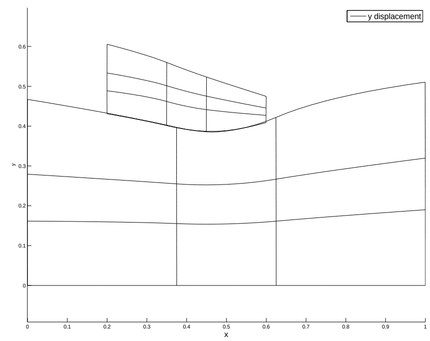

Figure 12: Deformed mesh of linear patch test

Normal stress contour of the test is shown in the figure below

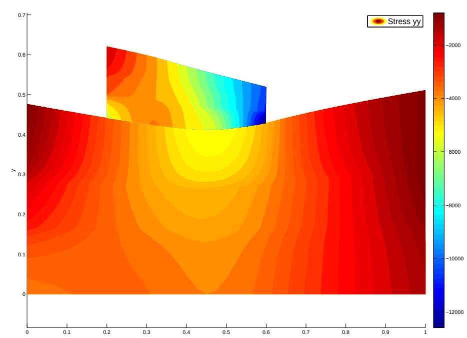

Figure 13: Deformed mesh of linear patch test

Normal stress on the bottom of block 1 is plotted in figure (11). The maximum stress occurs in the two end of the contact surface of the block 1. The average value of the normal stress is approximately equal to half of the pressure we applied, with the deviation of about 4% as well.

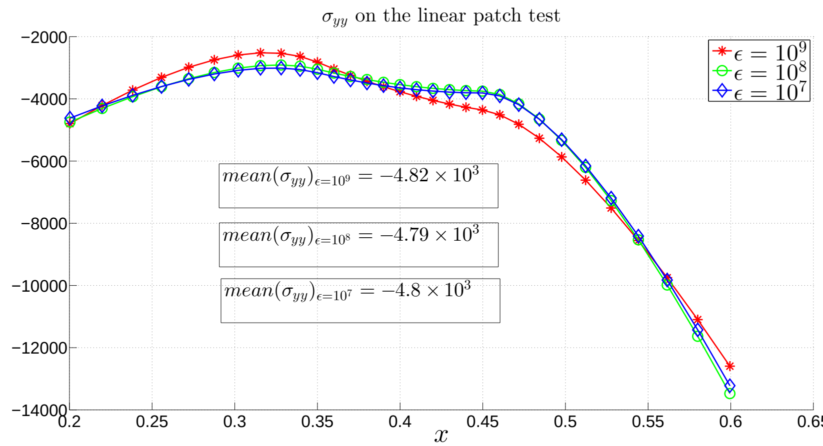

Figure 14: Problem 1 simple test

4.4 Wheel contact

The previous three problem is about two straight contact surfaces(lines in 2-D). To test the robustness of the code, a wheel contact simulation is carried out. Mesh of the undeformed bodies are shown below. On the top is a quarter of a circular wheel. Pressure acting on the wheel makes it move downward to contact with the block.

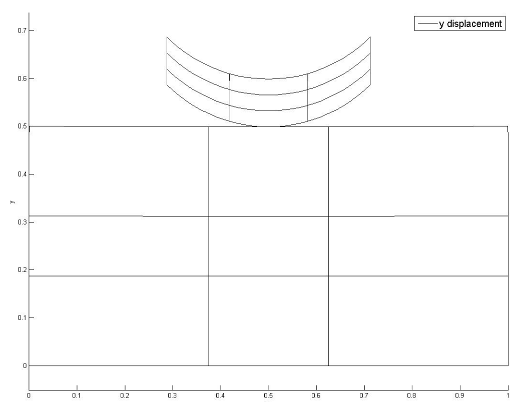

Figure 15: Undeformed mesh of wheel contact problem

Deformed mesh of this problem is shown in figure (16). we can find that the contact surfaces match good.

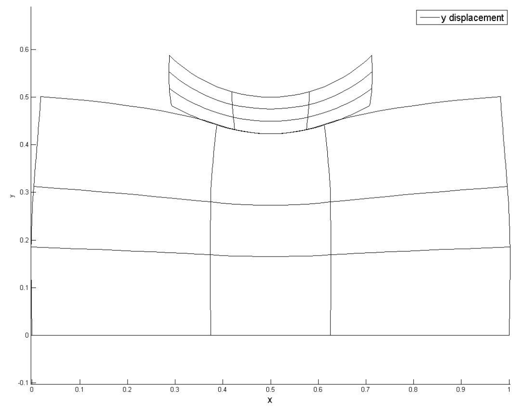

Figure 16: Deformed mesh of wheel contact problem

Contour of y-displacement is shown in figure 17.

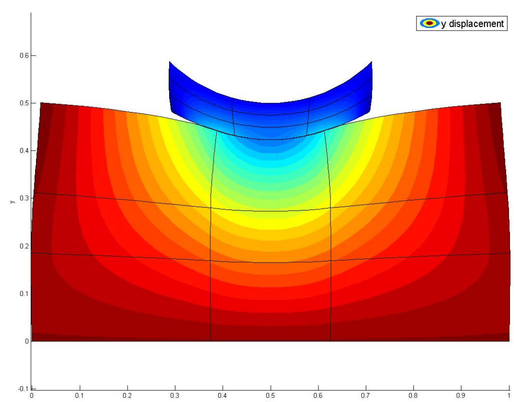

Figure 17: Contour of y displacement of wheel contact problem

5 Conclusion

A contact module for 2-D elasto-static Isogeometry analysis is implemented in this project. Numerical examples shows that the functionality of the code is pretty good, i.e, the contact surfaces matches good with each other. The patch test shows deviation of 4% between the pressure applied and the contact stress.

As suggested in many literature, mortar method can pass the patch test. Thus in future, the knot-to-segment method can be substituted by mortar method to improve the performance of the contact module.

This project is only designed for frictionless contact. Frictional contact should be the next goal of the new versions of the module.

References

[1] Vladislav A.Yastrebov, Numerical Methods in Contact Mechanics, ISTE Ltd 2013.

[2] Jia Lu, Isogeometric contact analysis: Geometric basis and formulation for frictionless contact, Computer Methods in Applied Mechanics and Engineering, 200(2011)726-741.

[3] i. Temizer, P. Wriggers, T.J.R. Hughes, Contact treatment in isogeometric analysis with NURBS, Computer Methods in Applied Mechanics and Engineering, 200(2011)1100-1112.