Moment-angle complexes, toric manifolds, and twisted cohomology

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GENERALIZED MOMENT-ANGLE COMPLEXES

- Let (X, A) be a pair of topological spaces
- Let K be a simplicial complex on vertex set $[m]$.
- Corresponding *generalized moment-angle complex*:

$$\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma \subset X^{\times m}$$

  where $(X, A)^\sigma = \{x \in X^{\times m} \mid x_i \in A \text{ if } i \notin \sigma\}$.
- Construction interpolates between $A^{\times m}$ and $X^{\times m}$.
- Homotopy invariance:
  $(X, A) \simeq (X', A') \implies \mathcal{Z}_K(X, A) \simeq \mathcal{Z}_K(X', A')$.
- Converts simplicial joins to direct products:
  $\mathcal{Z}_{K*L}(X, A) \cong \mathcal{Z}_K(X, A) \times \mathcal{Z}_L(X, A)$.
- Takes a cellular pair (X, A) to a cellular subcomplex of $X^{\times m}$.

Usual moment-angle complexes:

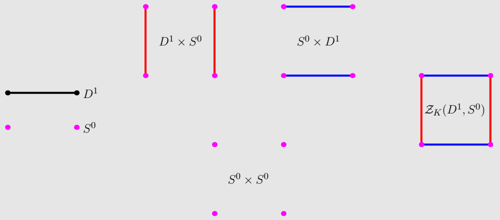
- Complex moment-angle complex, $\mathcal{Z}_K(D^2, S^1)$.
  - $\pi_1 = \pi_2 = \{1\}$.
- Real moment-angle complex, $\mathcal{Z}_K(D^1, S^0)$.
  - $\pi_1 = W_K'$, the derived subgroup of W_K, the right-angled Coxeter group associated to $K^{(1)}$.

EXAMPLE

Let $K =$ two points. Then:

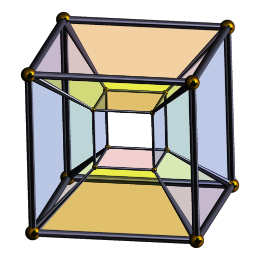
$$\mathcal{Z}_K(D^2, S^1) = D^2 \times S^1 \cup S^1 \times D^2 = S^3$$

$$\mathcal{Z}_K(D^1, S^0) = D^1 \times S^0 \cup S^0 \times D^1 = S^1$$



EXAMPLE

Let K be a circuit on 4 vertices. Then:



$$\mathcal{Z}_K(D^2, S^1) = S^3 \times S^3$$
$$\mathcal{Z}_K(D^1, S^0) = S^1 \times S^1$$

EXAMPLE

More generally, let K be an n-gon. Then:

$$\mathcal{Z}_K(D^2, S^1) = \#_{r=1}^{n-3}\, r \cdot \binom{n-2}{r+1} S^{r+2} \times S^{n-r}$$
$$\mathcal{Z}_K(D^1, S^0) = \text{an orientable surface of genus } 1 + 2^{n-3}(n-4)$$

The second equality was proved by H.S.M. Coxeter in 1937.

- If $(M, \partial M)$ is a compact manifold of dim d, and K is a PL-triangulation of S^m on n vertices, then $\mathcal{Z}_K(M, \partial M)$ is a compact manifold of dim $(d-1)n + m + 1$.
- (Bosio–Meersseman) If K is a *polytopal* triangulation of S^m, then
  - $\mathcal{Z}_K(D^2, S^1)$ if $n + m + 1$ is even, or
  - $\mathcal{Z}_K(D^2, S^1) \times S^1$ if $n + m + 1$ is odd

  is a complex manifold.
- This construction generalizes the classical constructions of complex structures on $S^{2p-1} \times S^1$ (Hopf) and $S^{2p-1} \times S^{2q-1}$ (Calabi–Eckmann).
- In general, the resulting complex manifolds are *not* symplectic, thus, not Kähler. In fact, they may even be non-formal (Denham–Suciu).

- The GMAC construction enjoys nice functoriality properties in both arguments. E.g:
  - Let $f\colon (X,A) \to (Y,B)$ be a (cellular) map. Then $f^{\times n}\colon X^{\times n} \to Y^{\times n}$ restricts to a (cellular) map $\mathcal{Z}_K(f)\colon \mathcal{Z}_K(X,A) \to \mathcal{Z}_K(Y,B)$.

- Much is known about the fundamental group and the asphericity problem for $\mathcal{Z}_K(X) = \mathcal{Z}_K(X,*)$ (work of Davis et al). E.g.:
  - $\pi_1(\mathcal{Z}_K(X,*))$ is the graph product of $G_v = \pi_1(X,*)$ along the graph $\Gamma = K^{(1)} = (\mathsf{V},\mathsf{E})$, where

    $$\mathrm{Prod}_\Gamma(G_v) = \mathop{\ast}_{v\in \mathsf{V}} G_v / \{[g_v, g_w] = 1 \text{ if } \{v,w\} \in \mathsf{E},\, g_v \in G_v,\, g_w \in G_w\}.$$

  - Suppose X is aspherical. Then: $\mathcal{Z}_K(X,*)$ is aspherical iff K is a flag complex.

- Also: $\mathcal{Z}_K(D^2, S^1) \simeq \mathcal{Z}_K(\mathbb{C},\mathbb{C}^*) = \mathbb{C}^m \backslash \bigcup_{\sigma\notin K} H_\sigma$, where $H_\sigma = \{x \in \mathbb{C}^n \mid x_{i_1} = \cdots = x_{i_p} = 0\}$ if $\sigma = \{i_1, \dots, i_p\}$.

GENERALIZED DAVIS–JANUSZKIEWICZ SPACES

- G abelian topological group $G \quad \rightsquigarrow \quad$ GDJ space $\mathcal{Z}_K(BG)$.
- $G = S^1$: Usual Davis–Januszkiewicz space, $\mathcal{Z}_K(\mathbb{CP}^\infty)$.
  - $\pi_1 = \{1\}$.
  - $H^*(\mathcal{Z}_K(\mathbb{CP}^\infty), \mathbb{Z}) = S/I_K$, where $S = \mathbb{Z}[x_1, \dots, x_m]$, $\deg x_i = 2$.
- $G = \mathbb{Z}_2$: Real Davis–Januszkiewicz space, $\mathcal{Z}_K(\mathbb{RP}^\infty)$.
  - $\pi_1 = W_K$: right-angled Coxeter group associated to $K^{(1)} = (\mathsf{V}, \mathsf{E})$.
  - $W_K = \langle v \in \mathsf{V} \mid v^2 = 1,\ vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle$.
  - $H^*(\mathcal{Z}_K(\mathbb{RP}^\infty), \mathbb{Z}_2) = R/I_K$, where $R = \mathbb{Z}_2[x_1, \dots, x_m]$, $\deg x_i = 1$.
- $G = \mathbb{Z}$: Toric complex, $\mathcal{Z}_K(S^1)$.
  - $\pi_1 = G_K$: right-angled Artin group associated to $K^{(1)}$.
  - $G_K = \langle v \in \mathsf{V} \mid vw = wv \text{ if } \{v, w\} \in \mathsf{E} \rangle$.
  - $H^*(\mathcal{Z}_K(S^1), \mathbb{Z}) = E/J_K$, where $E = \bigwedge[e_1, \dots, e_m]$, $\deg e_i = 1$.

- (Denham–Suciu) Let $p\colon (E,E') \to (B,B')$ be a map of pairs, such that both $p\colon E \to B$ and $p|_{E'}\colon E' \to B'$ are fibrations, with fibers F and F', respectively. Suppose that either $F = F'$ or $B = B'$. Then the product fibration, $p^{\times n}\colon E^{\times n} \to B^{\times n}$, restricts to a fibration

$$\mathcal{Z}_K(F,F') \longrightarrow \mathcal{Z}_K(E,E') \xrightarrow{\mathcal{Z}_K(p)} \mathcal{Z}_K(B,B')\ .$$

- Let $G \to EG \to BG$ be universal G-bundle. Applying the above lemma to the relative G-bundle $(G,G) \to (EG,G) \to (BG,*)$, we obtain a bundle

$$G^m \to \mathcal{Z}_K(EG,G) \to \mathcal{Z}_K(BG).$$

- If G is a finitely generated (discrete) abelian group, then $\pi_1(\mathcal{Z}_K(BG))_{\mathrm{ab}} = G^m$, and thus $\mathcal{Z}_K(EG,G)$ is the universal abelian cover of $\mathcal{Z}_K(BG)$.
- In particular, $\mathcal{Z}_K(\mathbb{RP}^\infty)^{\mathrm{ab}} \simeq \mathcal{Z}_K(D^1,S^0)$.

- (Bahri, Bendersky, Cohen, Gitler) Let K a simplicial complex on m vertices. There is a natural homotopy equivalence

$$\Sigma(\mathcal{Z}_K(X,A)) \simeq \Sigma\Big(\bigvee_{I\subset[m]} \widehat{\mathcal{Z}}_{K_I}(X,A)\Big),$$

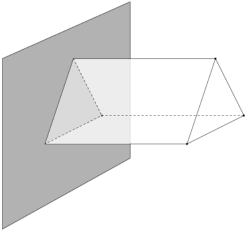
where K_I is the induced subcomplex of K on the subset $I \subset [m]$.

- In particular, if X is contractible and A is a discrete subspace consisting of p points, then

$$H_k(\mathcal{Z}_K(X,A);R) \cong \bigoplus_{I\subset[m]} \bigoplus_{1}^{(p-1)^{|I|}} \widetilde{H}_{k-1}(K_I;R).$$

TORIC MANIFOLDS AND SMALL COVERS

- Let P be an n-dimensional convex polytope; facets $F_1, \dots, F_m$.
- Assume P is *simple* (each vertex is the intersection of n facets).
- Then P determines a dual simplicial complex, $K = K_{\partial P}$, of dimension $n-1$:
  - Vertex set $[m] = \{1, \dots, m\}$.
  - Add a simplex $\sigma = (i_1, \dots, i_k)$ whenever $F_{i_1}, \dots, F_{i_k}$ intersect.



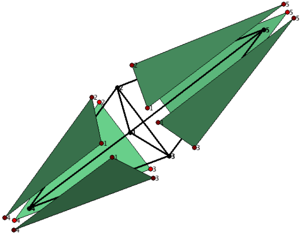


FIGURE: A prism P and its dual simplicial complex K

- Let χ be an n-by-m matrix with coefficients in $G = \mathbb{Z}$ or $\mathbb{Z}_2$.
- χ is *characteristic* for P if, for each vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$, the n-by-n minor given by the columns $i_1, \dots, i_n$ of χ is unimodular.
- Let $\mathbb{T} = S^1$ if $G = \mathbb{Z}$, and $\mathbb{T} = S^0 = \{\pm 1\}$ if $G = \mathbb{Z}_2$.
- Given $q \in P$, let $F(q) = F_{j_1} \cap \cdots \cap F_{j_k}$ be the maximal face so that $q \in F(q)^\circ$. The map χ yields a k-dimensional subtorus

$$T_{F(q)} = T_{F_{j_1}} \cap \cdots \cap T_{F_{j_k}} \subset \mathbb{T}^n.$$

- Here, if F is a face, and $\chi_F \colon G \to G^n$ is the corresponding column vector, then $T_F = \ker(\widehat{\chi_F} \colon \mathbb{T}^n \to \mathbb{T}) \cong \mathbb{T}^{n-1}$.

- To the pair (P, χ), M. Davis and T. Januszkiewicz associate the *(quasi-) toric manifold*

$$X = T^n \times P/\sim$$

where $(t, p) \sim (u, q)$ if $p = q$ and $t \cdot u^{-1} \in \mathbb{T}_{F(q)}$.

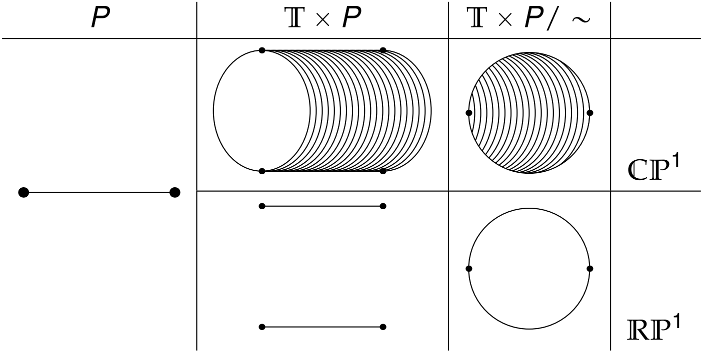
- The projection map $X \to P$ has fibers
  - $\mathbb{T}^n$ over points in the interior of P,
  - $\mathbb{T}^{n-1} = T_F$ over points on a face F, etc.
- For $G = \mathbb{Z}$, the space X is a *complex* toric manifold, denoted $M_P(\chi)$. It is a closed, orientable manifold of dimension $2n$.
- For $G = \mathbb{Z}_2$, the space X is a *real* toric manifold (or, *small cover*), denoted $N_P(\chi)$. It is a closed, not necessarily orientable manifold of dimension n.

EXAMPLE (TORIC MANIFOLDS OVER THE n-SIMPLEX)

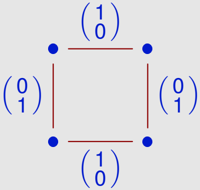
Let $P = \Delta^n$ be the n-simplex, and χ the $n \times (n+1)$ matrix $\begin{pmatrix} 1 & \cdots & 0 & 1 \\ & \ddots & & \vdots \\ 0 & \cdots & 1 & 1 \end{pmatrix}$.

Then

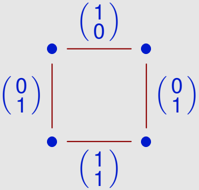
$$M_P(\chi) = \mathbb{CP}^n \quad \text{and} \quad N_P(\chi) = \mathbb{RP}^n.$$



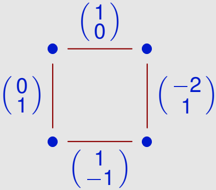
EXAMPLE (TORIC MANIFOLDS OVER THE SQUARE)



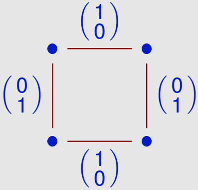
$\mathbb{CP}^1 \times \mathbb{CP}^1$



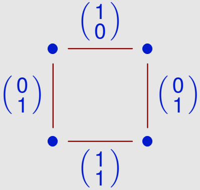
$\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$



$\mathbb{CP}^2 \# \mathbb{CP}^2$



$S^1 \times S^1$



$\mathbb{RP}^2 \# \mathbb{RP}^2$

- If X is a smooth, projective toric variety, then $X(\mathbb{C}) = M_P(\chi)$, for some P and χ, and $X(\mathbb{R}) = N_P(\chi \bmod 2\mathbb{Z})$.

On the other hand:

- $M = \mathbb{CP}^2 \sharp \mathbb{CP}^2$ is a toric manifold over the square, but it does not admit any (almost) complex structure. Thus, $M \ncong X(\mathbb{C})$.
- If P is a 3-dim polytope with no triangular or quadrangular faces, then, by a theorem of Andreev, $N_P(\chi)$ is a hyperbolic 3-manifold. Hence, by a theorem of Delaunay, $N_P(\chi) \ncong X(\mathbb{R})$.
- Concrete example: $P =$ dodecahedron. (Characteristic matrices χ do exist for P, by work of Garrison and Scott.)

Davis and Januszkiewicz showed that:

- $M_P(\chi)$ admits a perfect Morse function with only critical points of even index.
- Moreover,
$$\operatorname{rank} H_{2i}(M_P(\chi), \mathbb{Z}) = h_i(P),$$
where $(h_0(P), \dots, h_n(P))$ is the h-vector of P, which depends only on the number of i-faces of P ($0 \leqslant i \leqslant n$).
- $N_P(\chi)$ admits a perfect Morse function over $\mathbb{Z}_2$.
- Moreover,
$$\dim_{\mathbb{Z}_2} H_i(N_P(\chi), \mathbb{Z}_2) = h_i(P).$$

- They also gave presentations for the cohomology rings $H^*(M_P(\chi), \mathbb{Z})$ and $H^*(N_P(\chi), \mathbb{Z}_2)$, similar to the ones given by Danilov and Jurkiewicz for toric varieties.

- In work with A. Trevisan, we compute $H^*(N_P(\chi), \mathbb{Q})$, both additively and multiplicatively.
- The (rational) Betti numbers of $N_P(\chi)$ no longer depend just on the h-vector of P, but also on the characteristic matrix χ.

EXAMPLE

Recall there are precisely two small covers over the square P:

- The torus $T^2 = N_P(\chi)$, with $\chi = \left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}\right)$.
- The Klein bottle $K\ell = N_P(\chi')$, with $\chi' = \left(\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right)$.

Then $b_1(T^2) = 2$, yet $b_1(K\ell) = 1$.

- Idea: use finite covers involving (up to homotopy) certain generalized moment-angle complexes:

$$\mathbb{Z}_2^{m-n} \longrightarrow \mathcal{Z}_K(D^1, S^0) \longrightarrow N_P(\chi) ,$$

$$\mathbb{Z}_2^{n} \longrightarrow N_P(\chi) \longrightarrow \mathcal{Z}_K(\mathbb{RP}^\infty, *) .$$

FINITE ABELIAN COVERS

- Let X be a connected, finite-type CW-complex, $\pi = \pi_1(X, x_0)$.
- Let $p\colon Y \to X$ a (connected) regular cover, with group of deck transformations Γ. We then have a short exact sequence

$$1 \longrightarrow \pi_1(Y, y_0) \xrightarrow{p_\sharp} \pi_1(X, x_0) \xrightarrow{\nu} \Gamma \longrightarrow 1 .$$

- Conversely, every epimorphism $\nu\colon \pi \twoheadrightarrow \Gamma$ defines a regular cover $X^\nu \to X$ (unique up to equivalence), with $\pi_1(X^\nu) = \ker(\nu)$.
- If Γ is abelian, then $\nu = \chi \circ \mathrm{ab}$ factors through the abelianization, while $X^\nu = X^\chi$ is covered by the universal abelian cover of X:

$$\begin{tikzcd} X^{\mathrm{ab}} \arrow[r] \arrow[dr] & X^\nu \arrow[d, "p"] \\ & X \end{tikzcd} \qquad \longleftrightarrow \qquad \begin{tikzcd} \pi_1(X) \arrow[r, "\mathrm{ab}"] \arrow[dr, "\nu"'] & \pi_1(X)_{\mathrm{ab}} \arrow[d, "\chi"] \\ & \Gamma \end{tikzcd}$$

- Let $C_q(X^\nu;\mathbb{k})$ be the group of cellular q-chains on X^ν, with coefficients in a field $\mathbb{k}$. We then have natural isomorphisms

$$C_q(X^\nu;\mathbb{k}) \cong C_q(X;\mathbb{k}\Gamma) \cong C_q(\widetilde{X}) \otimes_{\mathbb{k}\pi} \mathbb{k}\Gamma.$$

- Now suppose Γ is finite abelian, $\mathbb{k} = \overline{\mathbb{k}}$, and $\operatorname{char}\mathbb{k} = 0$. Then, all $\mathbb{k}$-irreps of Γ are 1-dimensional, and so

$$C_q(X^\nu;\mathbb{k}) \cong \bigoplus_{\rho\in\operatorname{Hom}(\Gamma,\mathbb{k}^\times)} C_q(X;\mathbb{k}_{\rho\circ\nu}),$$

  where $\mathbb{k}_{\rho\circ\nu}$ denotes the field $\mathbb{k}$, viewed as a $\mathbb{k}\pi$-module via the character $\rho\circ\nu\colon \pi\to\mathbb{k}^\times$.

- Thus, $H_q(X^\nu;\mathbb{k}) \cong \bigoplus_{\rho\in\operatorname{Hom}(\Gamma,\mathbb{k}^\times)} H_q(X;\mathbb{k}_{\rho\circ\nu})$.

- Now let P be an n-dimensional, simple polytope with m facets, and let $K = K_{\partial P}$ be the simplicial complex dual to ∂P.
- Let $\chi\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a characteristic matrix for P.
- Then $\ker(\chi) \cong \mathbb{Z}_2^{m-n}$ acts freely on $\mathcal{Z}_K(D^1, S^0)$, with quotient the real toric manifold $N_P(\chi)$.
- $N_P(\chi)$ comes equipped with an action of $\mathbb{Z}_2^m / \ker(\chi) \cong \mathbb{Z}_2^n$; the orbit space is P.
- Furthermore, $\mathcal{Z}_K(D^1, S^0)$ is homotopy equivalent to the maximal abelian cover of $\mathcal{Z}_K(\mathbb{RP}^\infty)$, corresponding to the sequence

$$1 \longrightarrow W_K' \longrightarrow W_K \xrightarrow{\text{ab}} \mathbb{Z}_2^m \longrightarrow 1 .$$

- Thus, $N_P(\chi)$ is, up to homotopy, a regular $\mathbb{Z}_2^n$-cover of $\mathcal{Z}_K(\mathbb{RP}^\infty)$, corresponding to the sequence

$$1 \longrightarrow \pi_1(N_P(\chi)) \longrightarrow W_K \xrightarrow{\chi\circ\text{ab}} \mathbb{Z}_2^n \longrightarrow 1 .$$

To sum up, we have a diagram

$$\begin{tikzcd}
\mathcal{Z}_K(\mathbb{RP}^\infty)^{\mathrm{ab}} \simeq \mathcal{Z}_K(D^1, S^0) \arrow[d, "/\mathbb{Z}_2^{m-n}"] & \\
\mathcal{Z}_K(\mathbb{RP}^\infty)^{\chi\circ\mathrm{ab}} \simeq N_P(\chi) \arrow[r, "/\mathbb{Z}_2^m"] \arrow[d, "/\mathbb{Z}_2^n"] & P \\
\mathcal{Z}_K(\mathbb{RP}^\infty) &
\end{tikzcd}$$

with vertical arrows regular covers, and horizontal arrow the “stratified" (small) cover defining $N_P(\chi)$.

THE HOMOLOGY OF ABELIAN COVERS OF GDJ SPACES

- Let K be a simplicial complex on m vertices.
- Identify $\pi_1(\mathcal{Z}_K(B\mathbb{Z}_p))_{\mathrm{ab}} = \mathbb{Z}_p^m$, with generators $x_1, \dots, x_m$.
- Let $\lambda\colon \mathbb{Z}_p^m \to \Bbbk^\times$ be a character; $\operatorname{supp}(\lambda) := \{i \in [m] \mid \lambda(x_i) \neq 1\}$.
- Let K_λ be the induced subcomplex on vertex set $\operatorname{supp}(\lambda)$.

LEMMA (SUCIU–TREVISAN)

$$H_q(\mathcal{Z}_K(B\mathbb{Z}_p); \Bbbk_\lambda) \cong \widetilde{H}_{q-1}(K_\lambda; \Bbbk).$$

Sketch of proof:

- The inclusion $(S^1,*) \hookrightarrow (B\mathbb{Z}_p,*)$ induces a cellular inclusion

$$T_K = \mathcal{Z}_K(S^1) \hookrightarrow \mathcal{Z}_K(B\mathbb{Z}_p).$$

- The inclusion $\phi\colon K_\lambda \hookrightarrow K$ induces a cellular inclusion

$$T_{K_\lambda} \hookrightarrow T_K.$$

- Let $\bar{\lambda}\colon \mathbb{Z}^m \twoheadrightarrow \mathbb{Z}_p^m \xrightarrow{\lambda} \Bbbk^\times$. We then get (chain) retractions

$$\begin{tikzcd} & C_q(T_K;\Bbbk_{\bar{\lambda}}) \arrow[d] & \\ C_q(\mathcal{Z}_K(B\mathbb{Z}_p);\Bbbk_\lambda) \arrow[r] \arrow[ur, dashed] & C_q(T_{K_\lambda};\Bbbk_{\bar{\lambda}}) \arrow[r, "\cong"] & \widetilde{C}_{q-1}(K_\lambda;\Bbbk) \end{tikzcd}$$

- Hence: $\dim_\Bbbk H_q(\mathcal{Z}_K(B\mathbb{Z}_p);\Bbbk_\lambda) \geqslant \dim_\Bbbk \widetilde{H}_{q-1}(K_\lambda;\Bbbk)$.

For the reverse inequality, we use [BBCG], which, in this case, says

$$H_q(\mathcal{Z}_K(E\mathbb{Z}_p,\mathbb{Z}_p);\Bbbk) \cong \bigoplus_{I\subset[m]} \bigoplus_{1}^{(p-1)^{|I|}} \widetilde{H}_{q-1}(K_I;\Bbbk),$$

and the fact that $\mathcal{Z}_K(E\mathbb{Z}_p,\mathbb{Z}_p) \simeq (\mathcal{Z}_K(B\mathbb{Z}_p))^{\mathrm{ab}}$, which gives

$$H_q(\mathcal{Z}_K(E\mathbb{Z}_p,\mathbb{Z}_p);\Bbbk) \cong \bigoplus_{\rho\in\mathrm{Hom}(\mathbb{Z}_p^m,\Bbbk^\times)} H_q(\mathcal{Z}_K(B\mathbb{Z}_p);\Bbbk_\rho).$$

THEOREM (S–T)

Let $\mathcal{Z}_K(B\mathbb{Z}_p)^\chi$ be the abelian cover defined by an epimorphism $\chi\colon (\mathbb{Z}_p)^m \twoheadrightarrow \Gamma$. Then

$$H_q(\mathcal{Z}_K(B\mathbb{Z}_p)^\chi;\Bbbk) \cong \bigoplus_{\rho\in\mathrm{Hom}(\Gamma;\Bbbk^\times)} \widetilde{H}_{q-1}(K_{\rho\circ\chi};\Bbbk),$$

where $K_{\rho\circ\chi}$ is the induced subcomplex of K on vertex set $\mathrm{supp}(\rho\circ\chi)$.

THE $\mathbb{Q}$-HOMOLOGY OF REAL TORIC MANIFOLDS

- Let again P be a simple polytope, and set $K = K_{\partial P}$.
- Let $\chi\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ be a characteristic matrix for P.
- For each subset S of $[n] = \{1, \dots, n\}$:
  - Compute $\chi_S = \sum_{i \in S} \chi_i$, where χ_i is the i-th row of χ.
  - Find the induced subcomplex $K_{\chi,S}$ of K on vertex set

    $$\operatorname{supp}(\chi_S) = \{j \in [m] \mid \text{ the } j\text{-th entry of } \chi_S \text{ is non-zero}\}.$$

  - Compute the reduced simplicial Betti numbers

    $$\tilde{b}_q(K_{\chi,S}) = \dim_{\mathbb{Q}} \widetilde{H}_q(K_{\chi,S}; \mathbb{Q}).$$

THEOREM (S–T)

The Betti numbers of the real toric manifold $N_P(\chi)$ are given by

$$b_q(N_P(\chi)) = \sum_{S \subseteq [n]} \tilde{b}_{q-1}(K_{\chi,S}).$$

As an application, we recover a result of Nakayama and Nishimura.

COROLLARY

A real, n-dimensional toric manifold $N_P(\chi)$ is orientable if and only if there is a subset $S \subseteq [n]$ such that $K_{\chi,S} = K$.

Reason: $N_P(\chi)$ is orientable iff $b_n(N_P(\chi)) = 1$

EXAMPLE

- Again, let P be the square, $K = K_{\partial P}$ the 4-cycle.
- Let $T^2 = N_P(\chi)$, $\chi = \left(\begin{smallmatrix} 1&0&1&0\\ 0&1&0&1 \end{smallmatrix}\right)$, and $K\ell = N_P(\chi')$, $\chi' = \left(\begin{smallmatrix} 1&0&1&0\\ 0&1&1&1 \end{smallmatrix}\right)$.

S	$\varnothing$	$\{1\}$	$\{2\}$	$\{1,2\}$
χ_S	$(0\ 0\ 0\ 0)$	$(1\ 0\ 1\ 0)$	$(0\ 1\ 0\ 1)$	$(1\ 1\ 1\ 1)$
$K_{\chi,S}$	$\varnothing$	$\{\{1\},\{3\}\}$	$\{\{2\},\{4\}\}$	K
χ'_S	$(0\ 0\ 0\ 0)$	$(1\ 0\ 1\ 0)$	$(0\ 1\ 1\ 1)$	$(1\ 1\ 0\ 1)$
$K_{\chi',S}$	$\varnothing$	$\{\{1\},\{3\}\}$	$\{\{2,3\},\{3,4\}\}$	$\{\{1,2\},\{1,4\}\}$

Hence:

$b_0(T^2) = \tilde{b}_{-1}(\varnothing) = 1$

$b_1(T^2) = \tilde{b}_0(K_{\chi,\{1\}}) + \tilde{b}_0(K_{\chi,\{2\}}) = 2$

$b_2(T^2) = \tilde{b}_1(K_{\chi,\{1,2\}}) = 1$

$b_0(K\ell) = \tilde{b}_{-1}(\varnothing) = 1$

$b_1(K\ell) = \tilde{b}_0(K_{\chi',\{1\}}) + \tilde{b}_0(K_{\chi',\{2\}}) = 1$

$b_2(K\ell) = \tilde{b}_1(K_{\chi',\{1,2\}}) = 0$

THE HESSENBERG MANIFOLDS

- Every Weyl group W determines a smooth, complex projective toric variety $\mathcal{T}_W$.
  - Fan given by the reflecting hyperplanes of W.
  - Polytope P_W is the convex hull of a regular orbit $W \cdot x_0$.
  - $\dim_{\mathbb{C}} \mathcal{T}_W = \operatorname{rank} W$.
- $\mathcal{T}_n = \mathcal{T}_{S_n}$ is the Hessenberg variety, of cx dim $n-1$; polytope is the permutahedron P_n (the iterated truncation of the simplex Δ_{n-1}).
- $\mathcal{T}_n$ is isomorphic to the De Concini–Procesi wonderful model $\overline{Y_{\mathcal{G}}}$, where $\mathcal{G}$ is the building set in $(\mathbb{C}^n)^*$ which consists of all subspaces spanned by $\{x_i \mid i \in I\}$, where $\emptyset \neq I \subseteq [n]$.
- Thus, $\mathcal{T}_n$ can be obtained by iterated blow-ups:
  1. Blow up $\mathbb{CP}^{n-1}$ at the n coordinate points.
  2. Blow up along the proper transforms of the $\binom{n}{2}$ coordinate lines.
  3. Blow up along the proper transforms of the $\binom{n}{3}$ coordinate planes...

- Remark: There is another De Concini–Procesi model, $\overline{Y_{\mathcal{H}}}$, isomorphic to the moduli space $\overline{\mathcal{M}_{0,n+2}}$, and a surjective, S_n-equivariant birational morphism $\overline{\mathcal{M}_{0,n+2}} \twoheadrightarrow \mathcal{T}_n$.
- The real locus of $\mathcal{T}_W$, denoted $\mathcal{T}_W(\mathbb{R})$, is a smooth, connected, compact real toric variety of dimension equal to the rank of W.
- $\mathcal{T}_n(\mathbb{R})$ is a smooth, real toric variety of dim $n-1$, with associated polytope the permutahedron P_n.

THEOREM (HENDERSON 2010)

$$b_i(\mathcal{T}_n(\mathbb{R})) = A_{2i}\binom{n}{2i},$$

where A_{2i} is the Euler secant number, defined as the coefficient of $x^{2i}/(2i)!$ in the Maclaurin expansion of $\sec(x)$, We may recover Henderson's computation, using our general approach. To start with, note that:

- P_n has $2^n - 2$ facets: each subset $\varnothing \neq Q \subset [n]$ determines a facet F^Q with vertices in which all coordinates in positions in Q are smaller than all coordinates in positions not in Q.
- The corresponding column vectors of the characteristic matrix $\chi\colon \mathbb{Z}_2^{2^n-2} \to \mathbb{Z}_2^{n-1}$ are given by:

$$\chi^i = i\text{-th standard basis vector of } \mathbb{R}^{n-1} \ (1 \leqslant i < n),$$

$$\chi^n = \textstyle\sum_{i<n} \chi^i, \qquad \chi^Q = \textstyle\sum_{i \in Q} \chi^i.$$

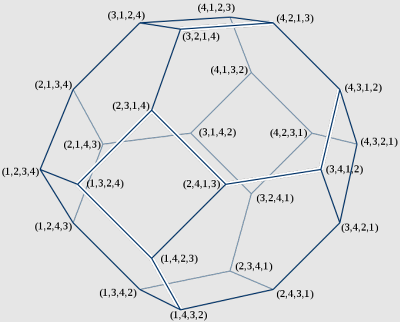
EXAMPLE

- P_3 is a truncated triangle, that is, a hexagon.
- Characteristic matrix

$$\chi = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

- $\mathcal{T}_3(\mathbb{R})$ is obtained from $\mathbb{RP}^2$ by blowing up 3 points.

EXAMPLE



P_4 is a truncated octahedron; it has 14 facets (6 squares and 8 hexagons). Characteristic matrix:

$$\chi = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

- The dual simplicial complex, $K_n = K_{\partial P_n}$, is the barycentric subdivision of the boundary of the $(n-1)$-simplex.
- Given a subset $S \subseteq [n-1]$, the induced subcomplex on vertex set $\operatorname{supp}(\chi_S)$ depends only on $r := |S|$, so denote it by $K_{n,r}$.
- $K_{n,r}$ is the order complex associated to a rank-selected poset of a certain subposet of the Boolean lattice B_n. Thus, $K_{n,r}$ is Cohen–Macaulay; in fact,

$$K_{n,2r-1} \simeq K_{n,2r} \simeq \bigvee^{A_{2r}} S^{r-1}.$$

- Hence:

$$\begin{aligned} b_i(\mathcal{T}_n(\mathbb{R})) &= \sum_{S \subset [n-1]} \tilde{b}_{i-1}((K_n)_{\chi,S}) = \sum_{r=1}^{n-1} \binom{n-1}{r} \tilde{b}_{i-1}(K_{n,r}) \\ &= \left(\binom{n-1}{2i-1} + \binom{n-1}{2i} \right) A_{2i} = \binom{n}{2i} A_{2i}. \end{aligned}$$

CUP PRODUCTS IN ABELIAN COVERS OF GDJ-SPACES

As before, let $X^{\nu} \to X$ be a regular, finite abelian cover, corresponding to an epimorphism $\nu\colon \pi_1(X) \twoheadrightarrow \Gamma$, and let $\Bbbk = \mathbb{C}$. The cellular cochains on X^{ν} decompose as

$$C^q(X^{\nu};\Bbbk) \cong \bigoplus_{\rho\in \operatorname{Hom}(\Gamma,\Bbbk^{\times})} C^q(X;\Bbbk_{\rho\circ\nu}),$$

The cup product map, $C^p(X^{\nu},\Bbbk)\otimes_{\Bbbk} C^q(X^{\nu},\Bbbk) \xrightarrow{\smile} C^{p+q}(X^{\nu},\Bbbk)$, restricts to those pieces, as follows:

$$\begin{tikzcd}
C^p(X;\Bbbk_{\rho\circ\nu})\otimes_{\Bbbk} C^q(X;\Bbbk_{\rho'\circ\nu}) \arrow[r, "\smile"] \arrow[d, "\cong"] & C^{p+q}(X;\Bbbk_{(\rho\cdot\rho')\circ\nu}) \\
C^{p+q}(X\times X;\Bbbk_{\rho\circ\nu}\otimes_{\Bbbk}\Bbbk_{\rho'\circ\nu}) \arrow[r, "\mu^*"] & C^{p+q}(X\times X;\Bbbk_{(\rho\otimes\rho')\circ\nu}) \arrow[u, "\Delta^*"']
\end{tikzcd}$$

where μ^* is induced by the multiplication map on coefficients, and Δ^* is induced by a cellular approximation to the diagonal $\Delta\colon X \to X\times X$.

PROPOSITION (S–T)

Let $\mathcal{Z}_K(B\mathbb{Z}_p)^\nu$ be a regular abelian cover, with characteristic homomorphism $\chi\colon \mathbb{Z}_p^m \to \Gamma$. The cup product in

$$H^*(\mathcal{Z}_K(BG)^\nu;\Bbbk) \cong \bigoplus_{q=0}^{\infty}\left(\bigoplus_{\rho\in\mathrm{Hom}(\Gamma;\Bbbk^\times)} \widetilde{H}^{q-1}(K_{\rho\circ\chi};\Bbbk)\right)$$

is induced by the following maps on simplicial cochains:

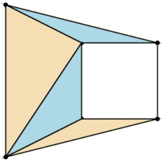
$$\begin{aligned} \widetilde{C}^{p-1}(K_{\rho\circ\chi};\Bbbk^\times)\otimes \widetilde{C}^{q-1}(K_{\rho'\circ\chi};\Bbbk^\times) &\to \widetilde{C}^{p+q-1}(K_{(\rho\otimes\rho')\circ\chi};\Bbbk^\times)\\ \hat{\sigma}\otimes\hat{\tau} &\mapsto \begin{cases} \pm\widehat{\sigma\sqcup\tau} & \text{if } \sigma\cap\tau=\varnothing,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where $\sigma\sqcup\tau$ is the simplex with vertex set the union of the vertex sets of σ and τ, and $\hat{\sigma}$ is the Kronecker dual of σ.

FORMALITY PROPERTIES

- A finite-type CW-complex X is *formal* if its Sullivan minimal model is quasi-isomorphic to $(H^*(X, \mathbb{Q}), 0)$—roughly speaking, $H^*(X, \mathbb{Q})$ determines the rational homotopy type of X.
- (Notbohm–Ray) If X is formal, then $\mathcal{Z}_K(X)$ is formal.
- In particular, toric complexes $T_K = \mathcal{Z}_K(S^1)$ and generalized Davis–Januszkiewicz spaces $\mathcal{Z}_K(BG)$ are always formal.
- (Félix, Tanré) More generally, if both X and A are formal, and the inclusion $i \colon A \hookrightarrow X$ induces a surjection $i^* \colon H^*(X, \mathbb{Q}) \to H^*(A, \mathbb{Q})$, then $\mathcal{Z}_K(X, A)$ is formal.

- (Baskakov, Denham–A.S.) Moment angle complexes $\mathcal{Z}_K(D^2, S^1)$ are not always formal: they can have non-trivial triple Massey products. For instance, $K =$



- (Denham–A.S.) There exist polytopes P and dual triangulations $K = K_{\partial P}$ for which $\mathcal{Z}_K(D^2, S^1)$ is not formal.
- Thus, there are real moment-angle complexes (even manifolds) $\mathcal{Z}_L(D^1, S^0)$ which are not formal.
- (Panov–Ray) Complex toric manifolds $M_P(\chi)$ are always formal.
- Question: are the real toric manifolds $N_P(\chi)$ always formal?

ABELIAN DUALITY & PROPAGATION OF RESONANCE

- Let X be a connected, finite-type CW-complex, with $G = \pi_1(X)$.
- In the background for much of these computations lie the jump loci for cohomology with coefficients in rank 1 local systems,
$$\mathcal{V}^i(X) = \{\rho \in \operatorname{Hom}(G, \mathbb{C}^\times) \mid H^i(X, \mathbb{C}_\rho) \neq 0\}.$$
- Also, the closely related “resonance varieties",
$$\mathcal{R}^i(X) = \{a \in H^1(X, \mathbb{C}) \mid H^i(H^*(X, \mathbb{C}), \cdot a) \neq 0\}.$$
- Question: How do the duality properties of a space X affect the nature of its cohomology jump loci?
- Recall that X is a *duality space* of dimension n if $H^p(X, \mathbb{Z}G) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G) \neq 0$ and torsion-free.
- By analogy, we say X is an *abelian duality space* of dimension n if $H^p(X, \mathbb{Z}G^{\mathrm{ab}}) = 0$ for $p \neq n$ and $H^n(X, \mathbb{Z}G^{\mathrm{ab}}) \neq 0$ and torsion-free.

THEOREM (DENHAM–SUCIU–YUZVINSKY)

Let X be an abelian duality space of dim n. For any character $\rho\colon G \to \mathbb{C}^*$, if $H^p(X, \mathbb{C}_\rho) \neq 0$, then $H^q(X, \mathbb{C}_\rho) \neq 0$ for all $p \leqslant q \leqslant n$. Thus, the characteristic varieties of X "propagate":

$$\mathcal{V}^1(X) \subseteq \mathcal{V}^2(X) \subseteq \cdots \subseteq \mathcal{V}^n(X).$$

COROLLARY

If X admits a minimal cell structure, and X is an abelian duality space of dim n, then resonance propagates:

$$\mathcal{R}^1(X) \subseteq \mathcal{R}^2(X) \subseteq \cdots \subseteq \mathcal{R}^n(X).$$

REMARK

Propagation of $\mathcal{V}^i$'s does not imply propagation of $\mathcal{R}^i$'s.
Eg, let $M = H_{\mathbb{R}}/H_{\mathbb{Z}}$ be the 3-dim Heisenberg manifold. Then $\mathcal{V}^1 = \mathcal{V}^2 = \mathcal{V}^3 = \{1\}$, but $\mathcal{R}^1 = \mathcal{R}^2 = \mathbb{C}^2$, and $\mathcal{R}^3 = \{0\}$.

TORIC COMPLEXES

- Let K be a simplicial complex of dimension d, on vertex set V, and let $T_K = \mathcal{Z}_K(S^1, *)$ be the respective toric complex.
- T_K is a connected, minimal CW-complex, with $\dim T_K = d + 1$.
- $\pi_1(T_K) = G_\Gamma$ is the RAAG associated to the graph $\Gamma = K^{(1)}$.
- $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$, where Δ_Γ is the flag complex of Γ.

THEOREM (PAPADIMA–A.S.)

$$\mathcal{V}^i(T_K) = \bigcup_{\mathsf{W}} (\mathbb{C}^\times)^{\mathsf{W}} \quad \textit{and} \quad \mathcal{R}^i(T_K) = \bigcup_{\mathsf{W}} \mathbb{C}^{\mathsf{W}}$$

where the union is taken over all $\mathsf{W} \subseteq \mathsf{V}$ *for which there is a simplex* $\sigma \in L_{\mathsf{V} \setminus \mathsf{W}}$ *and an index* $j \leqslant i$ *such that* $\widetilde{H}_{j-1-|\sigma|}(\mathrm{lk}_{L_{\mathsf{W}}}(\sigma), \mathbb{C}) \neq 0$.

- K is *Cohen–Macaulay* if for each simplex $\sigma \in K$, the cohomology $\widetilde{H}^*(\mathrm{lk}(\sigma), \mathbb{Z})$ is concentrated in degree $n - |\sigma|$ and is torsion-free.

THEOREM (BRADY–MEIER, JENSEN–MEIER)

G_Γ is a duality group if and only if Δ_Γ is Cohen–Macaulay. Moreover, G_Γ is a Poincaré duality group if and only if Γ is a complete graph.

THEOREM (DSY)

T_K is an abelian duality space (of dimension $d+1$) if and only if K is Cohen–Macaulay, in which case both $\mathcal{V}^i(T_K)$ and $\mathcal{R}^i(T_K)$ propagate.

EXAMPLE (PS)

Let $\Gamma = \circ\!\!-\!\!\!-\!\!\circ \quad \circ\!\!-\!\!\!-\!\!\circ$. Then resonance does not propagate:

$$\mathcal{R}^1(G_\Gamma) = \mathbb{C}^4, \quad \text{but} \quad \mathcal{R}^2(G_\Gamma) = \mathbb{C}^2 \times \{0\} \cup \{0\} \times \mathbb{C}^2.$$

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