"I certify that all solutions are entirely in my own words and that I have not looked at another student's solutions. I have given credit to all external sources I consulted."

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Homework 0 Problem 1

Solution:

1. If a matrix A is PSD, for any non-zero vector $v \in R^n$, $v^T A v \geq 0$.

$$\begin{aligned} v^T E[(Z-\mu)(Z-\mu)^T]v &= \sum_{i=1}^{n}\sum_{j=1}^{n} E[(Z_i-\mu_i)(Z_j-\mu_j)]v_i v_j \qquad \text{Linearity of expectation} \\ &= E[v^T(Z-\mu)(Z-\mu)^T v] \\ &= E[((Z-\mu)^T v)^2] \\ &\geq 0 \end{aligned}$$

2. Denote the probability that an archer hits her target when it is windy as $P(H|W) = 0.4$, the probability that an archer hits her target when it is not windy as $P(H|NW) = 0.7$, and the probability of a gust of wind as $P(W) = 0.3$.

(i) $P_1 = P(W)P(H|W) = 0.3 \times 0.4 = 0.12$

(ii) $P_2 = P(W)P(H|W) + (1-P(W))P(H|NW) = 0.3 \times 0.4 + (1-0.3) \times 0.7 = 0.12 + 0.49 = 0.61$

(iii) $P_3 = \binom{2}{1} P_2(1-P_2) = 2 \times 0.61 \times 0.39 = 0.4758$

(iv) $P_4 = P(W^c|H^c) = \frac{P(H^c|W^c)P(W^c)}{P(H^c)} = \frac{0.3\times0.7}{0.39} = 0.5385$

3.

$$\begin{aligned} \text{Expected score} &= 4\int_0^{\frac{1}{\sqrt{3}}} \frac{2}{\pi(1+x^2)}\,dx \; + 3\int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{\pi(1+x^2)}\,dx \; + 2\int_1^{\sqrt{3}} \frac{2}{\pi(1+x^2)}\,dx \\ &= \frac{2}{\pi}[4\{arctan(\frac{1}{\sqrt{3}}) - arctan(0)\} + 3\{arctan(1) - arctan(\frac{1}{\sqrt{3}})\} + \\ &2\{arctan(\sqrt{3}) - arctan(1)\}] \\ &= \frac{13}{6} \end{aligned}$$

4.

$$\begin{aligned} P(X=k|X+Y=n) &= \frac{P(X=k \cap X+Y=n)}{P(X+Y=n)} \\ &= \frac{P(X=k \cap Y=n-k)}{P(X+Y=n)} \end{aligned}$$

Since $X \perp Y$ and $X \sim Pois(\lambda)$, $Y \sim Pois(\mu)$,

$$\begin{aligned} P(X=k \cap Y=n-k) &= \frac{e^{-\lambda}\lambda^k}{k!}\frac{e^{-\mu}\mu^{n-k}}{(n-k)!} \\ &= \frac{e^{-(\lambda+\mu)}}{n!}\binom{n}{k}\lambda^k \mu^{n-k} \end{aligned}$$

$$
\begin{aligned}
P(X+Y=n) &= \sum_{k=0}^{n} P(X=k \cap Y=n-k) \\
&= \sum_{k=0}^{n} \frac{e^{-(\lambda+\mu)}}{n!}\binom{n}{k}\lambda^k \mu^{n-k} \\
&= \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n
\end{aligned}
$$

Therefore,

$$
P(X=k \cap X+Y=n) = \binom{n}{k}\frac{\lambda^k \mu^{n-k}}{(\lambda+\mu)^n}
$$

This is a binomial distribution with parameters n and $p = \frac{\lambda}{\lambda+\mu}$. ■