6DOF Calibration by Illumination Angles in LED-Array-Microscope Phase Recovery

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1 Introduction

In Fourier ptychography, we aim to recover high-resolution image by capturing a series of low-resolution images captured under different **illumination angles**, which determines the numerical apertures (NA) of the lens. Theoretically, each angle corresponds to certain shift in the frequency spectrum of the object transmission function. In real scenario, however, we observed that the relative positions from the LED source to the sample may vary due to translational or rotational effect of the light source - LED board. If we model the LED array as rigid body, then the offsets have 6 degrees of freedom (**6DOF**) resulted from 3D motion. It is also noted that as the distance from LED source to the board center increases, the measurement suffers more severe distortion. Therefore, the estimated NA is deviated from practical NA, which should be calibrated before the phase recovery procedure.

2 Algorithm Formulation

We formulated an optimization problem to minimize the NA distortions. The aim is to minimize the error between measured and estimated illumination angles.

In our setting, the LED array and the sample are assumed as two equal-sized square plates, upon which we construct two coordinate systems O and L separately. And we denote the centers of the plates as the origins of the coordinate systems ($_O$ and $_L$). We use $_P$ to represent the position of a LED source on the array. The translational and rotational offsets can be denoted by $_O\mathbf{r}_{OL}$ (vector pointing from O to L) and Φ_{OL}, where $\Phi_{OL} \in SO(3)$ denotes the orientation of the coordinate system L w.r.t. coordinate system O.

The relationship of xy coordinate of point p w.r.t. two systems O and L is

$$_O\mathbf{r}_{OP} = {_O\mathbf{r}_{OL}} + \Phi_{OL}({_L\mathbf{r}_{LP}}) \tag{1}$$

where ${}_{O}\mathbf{r}_{OP} = [x_{mn}\, y_{mn}\, z_{mn}]^T$. Then, the objective function of this optimization problem can be written as

$$\min_{{}_{O}\mathbf{r}_{OL}, \Phi_{OL}} F = \frac{1}{2}\sum_{m}\sum_{n}[(k_x^{mn} - \frac{x_{mn}}{\|{}_{O}\mathbf{r}_{OP}\|})^2 + (k_y^{mn} - \frac{y_{mn}}{\|{}_{O}\mathbf{r}_{OP}\|})^2] \tag{2}$$

where $k^{mn} = [k_x^{mn}\, k_y^{mn}]$ are the measured illumination angles, and $(\frac{x_{mn}}{\|{}_{O}\mathbf{r}_{OP}\|}, \frac{y_{mn}}{\|{}_{O}\mathbf{r}_{OP}\|})$ are the estimated angles. (m,n) are the order of points on the LED array.

3 Implementation

We utilized **Gauss-Newton method** to find the minimum of F in (2), which is actually a non-linear least-square problem. The initial value is set assuming no translation and rotational offsets exist, and the condition to reach convergence is when the change of 6DOF decision variable $||\Delta x||$ is smaller than the tolerance parameter (we used 10^{-4}). As shown in Figure 1, the program converged in 12 iterations.

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Method: Gauss-Newton
Iteration           OrOL                        phiOL (axis;angle)              Step-size (norm of dx
    1       [3.081e-05,7.910e-03,7.045e-02]       [0.291 -0.021 0.957 0.300]                  3e-01
    2       [-4.255e-03,5.609e-03,6.581e-02]       [0.235 0.151 0.960 0.439]                  2e-01
    3       [-3.384e-03,4.130e-03,6.290e-02]       [0.203 0.179 0.963 0.495]                  6e-02
    4       [-2.496e-03,3.624e-03,6.145e-02]       [0.195 0.186 0.963 0.521]                  3e-02
    5       [-1.910e-03,3.386e-03,6.072e-02]       [0.193 0.190 0.963 0.533]                  1e-02
    6       [-1.541e-03,3.244e-03,6.036e-02]       [0.192 0.191 0.963 0.538]                  6e-03
    7       [-1.315e-03,3.152e-03,6.018e-02]       [0.192 0.192 0.962 0.541]                  3e-03
    8       [-1.180e-03,3.093e-03,6.009e-02]       [0.192 0.192 0.962 0.543]                  1e-03
    9       [-1.101e-03,3.055e-03,6.004e-02]       [0.192 0.192 0.962 0.543]                  7e-04
   10       [-1.056e-03,3.032e-03,6.002e-02]       [0.192 0.192 0.962 0.544]                  4e-04
   11       [-1.031e-03,3.018e-03,6.001e-02]       [0.192 0.192 0.962 0.544]                  2e-04
   12       [-1.017e-03,3.010e-03,6.001e-02]       [0.192 0.192 0.962 0.544]                  9e-05

Gauss-Newton method converged with tolerance 1e-04!
The ideal OrOL is [000 000 7.000e-02].
The practical OrOL is [-1.000e-03 3.000e-03 6.000e-02].
The estimated OrOL is [-1.017e-03 3.010e-03 6.001e-02].
The ideal phiOL is [0.000 0.000 1.000 0.000](axis;angle).
The practical phiOL is [0.192 0.192 0.962 0.544](axis;angle).
The estimated phiOL is [0.192 0.192 0.962 0.544](axis;angle).
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Figure 1: Results of Gauss-Newton Method

4 Appendix

4.1 Gradient derivation of cost function

We first define $\hat{k}_x^{mn} = \frac{x_{mn}}{\|{}_{O}\mathbf{r}_{OP}\|}$, and $\hat{k}_y^{mn} = \frac{y_{mn}}{\|{}_{O}\mathbf{r}_{OP}\|}$. $\hat{k}^{mn} = [\hat{k}_x^{mn}\, \hat{k}_y^{mn}]^T$. Then,

$$\frac{\partial F}{\partial \hat{k}^{mn}} = [\sum_m \sum_n (\hat{k}_x^{mn} - k_x^{mn}) \quad \sum_m \sum_n (\hat{k}_y^{mn} - k_y^{mn})] \tag{3}$$

The gradient of $\hat{k}_{mn}$ w.r.t. vector $\vec{OP}$ in coordinate system O is

$$\frac{\partial \hat{k}^{mn}}{\partial _O\mathbf{r}_{OP}} = (x_{mn}^2 + y_{mn}^2 + z_{mn}^2)^{-1.5} \begin{bmatrix} y_{mn}^2 + z_{mn}^2 & -x_{mn}y_{mn} & -x_{mn}z_{mn} \\ -x_{mn}y_{mn} & x_{mn}^2 + z_{mn}^2 & -y_{mn}z_{mn} \end{bmatrix} \tag{4}$$

The gradient of $_O\mathbf{r}_{OP}$ w.r.t $[_O\mathbf{r}_{OL}, \Phi_{OL}]^T$ is

$$\frac{\partial _O\mathbf{r}_{OP}}{\partial [_O\mathbf{r}_{OL}, \Phi_{OL}]^T} = [\mathbf{I}_{3\times 3} \ \ -(\Phi_{OL}(_L\mathbf{r}_{LP}))^{\times}] \tag{5}$$

4.2 Derivation of Gauss-Newton Method

We want to write the function in least-square form min $\frac{1}{2}||f(x)||^2$. So $f(x)$ should be set as

$$f(x) = \begin{bmatrix} \hat{k}_x^1 - k_x^1 \\ \vdots \\ \hat{k}_x^{mn} - k_x^{mn} \\ \hat{k}_y^1 - k_y^1 \\ \vdots \\ \hat{k}_y^{1mn} - k_y^{mn} \end{bmatrix}$$

where x = $[_O\mathbf{r}_{OL}^T \ \Phi_{OL}^T]^T \in \mathbb{R}^6$ (here we mean Φ_{OL} is in vector form). Then we follow the standard procedure of Gauss-Newton method to derive the Jacobian matrix $J(x)$ and Δx. It is noted that for the orientation Φ_{OL}, we performed a box-plus operation instead of direct addition when updating x.

$$\Phi_{OL} := exp(\Delta\Phi^{\times}) \cdot \Phi_{OL} \tag{6}$$

where $\Phi_{OL} \in \mathbb{R}^3$ is in rotation vector form, and $\Delta\Phi^{\times}$ is the skew-symmetric matrix of vector $\Delta\Phi$.

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1 Algorithm formulation

1.1 Forward model

Fourier ptychography aims to capture images illuminated under different angles. For the imaging process of our experiment setting, we consider a sample with transmission function $o(\mathbf{r})$, where $\mathbf{r} = (x, y)$ indicates the spatial coordinates in the 2D sample plane. Assume that the distance from LED array to the sample is sufficiently large, then the sample will be illuminated plane waves of different angles emitted by distinct LED sources, with the complex field defined as $exp(i2\pi\mathbf{u}_l\cdot\mathbf{r})$, where $\mathbf{u}_l$ is the spatial frequency corresponding to the l-th LED, with $l = 1, ..., N_{img}$. The exit wave through the sample is the product of the sample and illumination complex fields, $o(\mathbf{r})exp(i2\pi\mathbf{u}_l\cdot\mathbf{r})$. In the frequency domain, the Fourier transform of this exit wave is a shifted version of the Fourier transform of the object, $O(\mathbf{u}-\mathbf{u}_l)$, where $O(\mathbf{u}) = \mathcal{F}\{o(\mathbf{r})\}$ and $\mathcal{F}$ denotes the 2D Fourier transform. After this exit wave passes through the objective lens, it is low-pass filtered by the pupil function $P(\mathbf{u})$ in the Fourier domain. Eventually, the intensity of the received image at spatial domain can be written as (with $\mathcal{F}^{-1}$ as the 2D inverse Fourier transform)

$$I_l(\mathbf{r}) = |\mathcal{F}^{-1}\{P(\mathbf{u})O(\mathbf{u}-\mathbf{u}_l)\}|^2 \tag{1}$$

1.2 Problem Formulation

The recovery process of FPM can be formulated into an optimization problem, which minimizes the difference between measured and estimated intensity (or amplitude). Here we consider minimizing differences of intensity, and the cost function is defined as follows.

$$\underset{O(\mathbf{u})}{min}\, f(O(\mathbf{u})) = \frac{1}{2}\underset{O(\mathbf{u})}{min}\sum_l\sum_{\mathbf{r}} |I_l(\mathbf{r}) - |\mathcal{F}^{-1}\{P(\mathbf{u})O(\mathbf{u}-\mathbf{u}_l)\}|^2|^2 \tag{2}$$

Since solving the optimization problem requires gradient calculation, it is convenient to reformulate it into vector form. First, the images are all flattened into column vectors. The captured images $\mathbf{I}_l(\mathbf{r})$ transformed from $m \times m$ pixels to $m^2 \times 1$ vector. The estimated object $\mathbf{O}(\mathbf{u})$, having $n \times n$ pixels, is also compressed into $n^2 \times 1$ vector. Then, we define a down-sampling matrix Q_l, which converts a $n^2 \times 1$ vector to a $m^2 \times 1$ vector by simply selecting values out of the original vector, since in the down-sampling process we actually crop the images to remove high-frequency information. Last, the pupil function $\mathbf{P}(\mathbf{u})$ is also a vector with size $m^2 \times 1$. And we put all the entries of $\mathbf{P}$ into the diagonal in order to perform multiplication operation. We can then write the estimated intensity as

$$\hat{\mathbf{I}}_l = |\mathbf{g}_l|^2 = |\mathbf{F}^{-1} diag(\mathbf{P})\mathbf{Q}_l\mathbf{O}|^2 \tag{3}$$

The intensity-based cost function becomes

$$\underset{\mathbf{O}}{min}\, f(\mathbf{O}) = \underset{\mathbf{O}}{min} \sum_l (\mathbf{I}_l - |\mathbf{g}_l|^2)^H (\mathbf{I}_l - |\mathbf{g}_l|^2) \tag{4}$$

where H denotes a Hermitian conjugate. To calculate the gradient, we can rewrite $|\mathbf{g}_l|^2$ as

$$|\mathbf{g}_l|^2 = diag(\bar{\mathbf{g}}_l)\mathbf{F}^{-1} diag(\mathbf{P})\mathbf{Q}_l\mathbf{O} \tag{5}$$

where $\bar{\mathbf{g}}_l$ means the complex conjugate of $\mathbf{g}_l$. Define the error term for the l^{th} image as:

$$\mathbf{f}_l = \mathbf{I}_l - |\mathbf{g}_l|^2 \tag{6}$$

$$\mathbf{F}_l = \frac{1}{2}\mathbf{f}_l^H \mathbf{f}_l \tag{7}$$

The gradient of $f(\mathbf{O})$ is calculated as follows:

$$\frac{\partial \mathbf{F}_l}{\partial \mathbf{O}} = \frac{\partial \mathbf{F}_l}{\partial \mathbf{f}_l} \frac{\partial \mathbf{f}_l}{\partial |\mathbf{g}_l|^2} \frac{|\partial \mathbf{g}_l|^2}{\partial \mathbf{O}} = -\mathbf{f}_l diag(\bar{\mathbf{g}}_l)\mathbf{F}^{-1} diag(\mathbf{P})\mathbf{Q}_l \tag{8}$$

$$\nabla_{\mathbf{O}} f(\mathbf{O}) = \sum_l \left[\frac{\partial \mathbf{F}_l}{\partial \mathbf{O}}\right]^H = -\sum_l \mathbf{Q}_l^H diag(\bar{\mathbf{P}})\mathbf{F} diag(\mathbf{g}_l)(\mathbf{I}_l - |\mathbf{g}_l|^2) \tag{9}$$

where $\bar{\mathbf{P}}$ denotes the complex conjugate of $\mathbf{P}$.