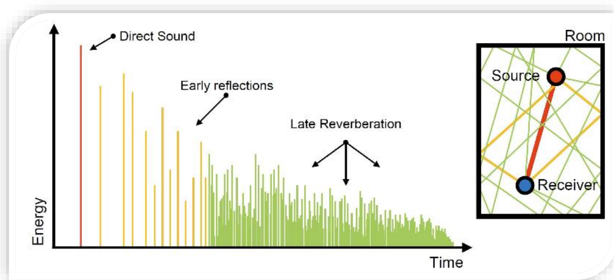
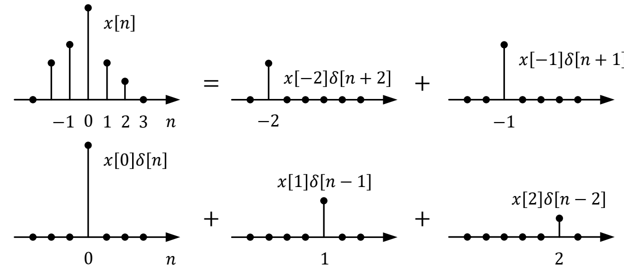
# Chapter 2 Linear Time-Invariant Systems




## §2.1 The Convolution Sum for Discrete-Time LTI Systems

1. Representation of Discrete-Time Signals in terms of $\delta[n]$




$$\boxed{x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]}$$

2. Definition of Convolution Sum

① The unit impulse response of discrete-time LTI systems

Consider a discrete-time LTI system has zero initial-state.

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

$$\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k]$$

For arbitrary input $x[n]$

$$\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

The unit impulse response $h[n]$ of a discrete-time LTI system is the zero-state output of the system when $\delta[n]$ is the input.

The zero-state output of the system with arbitrary input is given by the convolution sum.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y_f[n] = h[n] * x[n]$$

The convolution sum is calculated by

$$h[n] * x[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$\overset{m=n-k}{\Longleftrightarrow} h[n] * x[n] = \sum_{m=-\infty}^{+\infty} x[n-m]h[m] = x[n] * h[n]$$

② The Unit Step Response of Discrete-Time LTI Systems

The unit step response $s[n]$ of a discrete-time LTI system is the zero-state output of the system when $u[n]$ is the input.

$$u[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow s[n]$$

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

$$\begin{cases} \delta[n] = u[n] - u[n-1] \\ u[n] = \sum_{k=-\infty}^{n} \delta[k] \end{cases} \qquad \begin{cases} h[n] = s[n] - s[n-1] \\ s[n] = \sum_{k=-\infty}^{n} h[k] \end{cases}$$

③ The Associative Property:

$h_2[n] * \{h_1[n] * x[n]\} = \{h_1[n] * h_2[n]\} * x[n]$

$$x[n] \longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{h_1[n] * h_2[n]} \longrightarrow y[n]$$

Proof:

$$\sum_{m=-\infty}^{+\infty} h_2[m] \left\{ \sum_{k=-\infty}^{+\infty} x[k] h_1[(n-m)-k] \right\}$$

$$= \sum_{k=-\infty}^{+\infty} x[k] \left\{ \sum_{m=-\infty}^{+\infty} h_2[m] h_1[(n-k)-m] \right\}$$

3. Properties of Convolution Sum

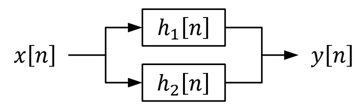
① The Commutative Property:

$h[n] * x[n] = x[n] * h[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad h[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]$$

② The Distributive Property:

$\{h_1[n] + h_2[n]\} * x[n] = h_1[n] * x[n] + h_2[n] * x[n]$




$$x[n] \longrightarrow \boxed{h_1[n] + h_2[n]} \longrightarrow y[n]$$

4. The Convolution Sum of $\delta[n]$

① $x[n] * \delta[n] = x[n]$

② $x[n] * \delta[n-n_0] = x[n-n_0]$

③ $x[n-n_1] * \delta[n-n_2] = x[n-n_1-n_2]$

④ When $x[n] = x_1[n] * x_2[n]$,

$x[n-n_1-n_2] = x_1[n-n_1] * x_2[n-n_2]$

Hint: Consider the output of an LTI system whose impulse response is given by $x[n]$ and input is given by $\delta[n]$.

Discussion 2.1:

Justify the following statements.

a. $a^n x[n] * a^n h[n] = a^n\{x[n] * h[n]\}$

b. $x[n] * \{h[n]g[n]\} = \{x[n] * h[n]\}g[n]$

c. If $y[n] = x[n] * h[n]$, then $y[2n] = 2x[2n] * h[2n]$

Hints:

b. $\delta[n-1] * \{\delta[n]\delta[n-1]\} \neq \{\delta[n-1] * \delta[n]\}\delta[n-1]$

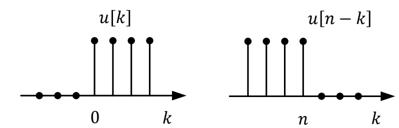
c. $\delta[2n] \neq 2\delta[2n] * \delta[2n]$

Example 2.1: Consider an LTI system with the impulse response $h[n] = u[n]$ and input $x[n] = a^n u[n]$ $(0 < a < 1)$. Suppose that this system has zero-initial state. Determine the output $y[n]$ of this system.

Solution:

$$y[n] = y_f[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} a^k u[k] \cdot u[n-k]$$




$$u[k]u[n-k] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \; and \; 0 \leq k \leq n \end{cases}$$

$$y[n] = \begin{cases} 0 & n < 0 \\ \displaystyle\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} & n \geq 0 \; and \; 0 < a < 1 \end{cases}$$

$$u[n] * a^n u[n] = \left(\frac{1-a^{n+1}}{1-a}\right)u[n] \;\; (0 < a < 1)$$

① Determine $a^n u[n] * a^n u[n]$

$$= \sum_{k=-\infty}^{+\infty} a^k u[k] \cdot a^{n-k} u[n-k] = a^n\{u[n] * u[n]\}$$

② Determine $u[n] * u[n]$

Solution 1:

$$u[n] * a^n u[n] = \left(\sum_{k=0}^{n} a^k\right)u[n]$$

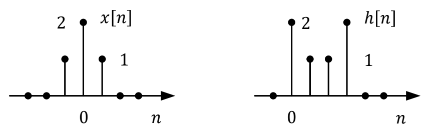
$$u[n] * u[n] = \left(\sum_{k=0}^{n} 1^k\right)u[n] = (n+1)u[n]$$

Solution 2:

$$u[n] * u[n] = \left[\lim_{a \to 1}\left(\frac{1-a^{n+1}}{1-a}\right)\right]u[n]$$

$$= \left[\lim_{a \to 1}\frac{-(n+1)a^n}{-1}\right]u[n] = (n+1)u[n]$$

Example 2.2: Consider an LTI system whose input $x[n]$ and impulse response $h[n]$ are depicted as follows. Suppose that the initial state of this system is zero. Determine the output $y[n]$ of this system.

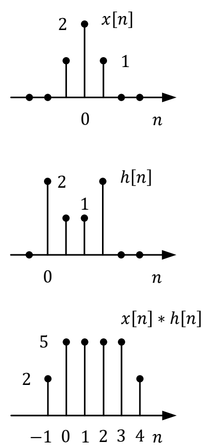


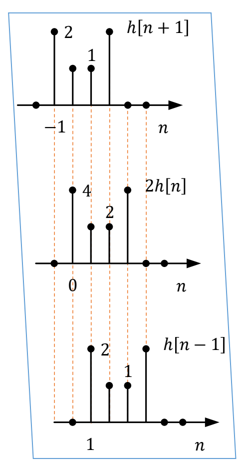

Solution:

$$
\begin{aligned}
x[n] &= \{1, 2, 1\} \quad n = -1, 0, 1 \\
&= \delta[n+1] + 2\delta[n] + \delta[n-1] \\
h[n] &= \{2, 1, 1, 2\} \quad n = 0, 1, 2, 3 \\
&= 2\delta[n] + \delta[n-1] + \delta[n-2] + 2\delta[n-3]
\end{aligned}
$$

$$
\begin{aligned}
x[n] &= \delta[n+1] + 2\delta[n] + \delta[n-1] \\
h[n] &= 2\delta[n] + \delta[n-1] + \delta[n-2] + 2\delta[n-3]
\end{aligned}
$$

$$
\begin{aligned}
y[n] = y_f[n] = h[n] * x[n] &= \{\delta[n+1] + 2\delta[n] + \delta[n-1]\} * \\
&\quad \{2\delta[n] + \delta[n-1] + \delta[n-2] + 2\delta[n-3]\} \\
&= 2\delta[n+1] + \delta[n] + \delta[n-1] + 2\delta[n-2] + \\
&\quad 4\delta[n] + 2\delta[n-1] + 2\delta[n-2] + 4\delta[n-3] + \\
&\quad 2\delta[n-1] + \delta[n-2] + \delta[n-3] + 2\delta[n-4] \\
&= 2\delta[n+1] + 5\delta[n] + 5\delta[n-1] + \\
&\quad 5\delta[n-2] + 5\delta[n-3] + 2\delta[n-4] \\
&= \{2, 5, 5, 5, 5, 2\} \ (n = -1, 0, 1, 2, 3, 4)
\end{aligned}
$$




5. The Convolution Sum of the Sequence with Finite Length

$$
x_1[n] = \begin{cases} x_1[n] & n_1 \le n \le n_2 \\ 0 & otherwise \end{cases} \qquad x_2[n] = \begin{cases} x_2[n] & \hat{n}_1 \le n \le \hat{n}_2 \\ 0 & otherwise \end{cases}
$$

① The length of $x_1[n]$: $N_1 = n_2 - n_1 + 1$

② The length of $x_2[n]$: $N_2 = \hat{n}_2 - \hat{n}_1 + 1$

$$
\begin{aligned}
x[n] &= x_1[n] * x_2[n] \\
&= \{x_1[n_1]\delta[n-n_1] + \cdots + x_1[n_2]\delta[n-n_2]\} * \\
&\quad \{x_2[\hat{n}_1]\delta[n-\hat{n}_1] + \cdots + x_2[\hat{n}_2]\delta[n-\hat{n}_2]\} \\
&= x_1[n_1]\, x_2[\hat{n}_1]\delta[n-\hat{n}_1-n_1] + \cdots + \\
&\quad x_1[n_2] x_2[\hat{n}_2]\delta[n-n_2-\hat{n}_2] \\
&= \begin{cases} x[n] & \hat{n}_1 + n_1 \le n \le n_2 + \hat{n}_2 \\ 0 & otherwise \end{cases}
\end{aligned}
$$

The length of $x[n]$:

$$N = (n_2 + \hat{n}_2) - (\hat{n}_1 + n_1) + 1$$
$$= (n_2 - n_1) + (\hat{n}_2 - \hat{n}_1) + 1$$
$$= N_1 + N_2 - 1$$

$$\left(\sum_{N_1} x_1[n]\right) \times \left(\sum_{N_2} x_2[n]\right) = \sum_{N_1+N_2-1} x[n]$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $h[n]$ | | | 2 | 1 | 1 | 2 |
| $x[n]$ | | | | 1 | 2 | 1 |
| $h[n-1]$ | | | 2 | 1 | 1 | 2 |
| $2h[n]$ | | 4 | 2 | 2 | 4 | |
| $h[n+1]$ | 2 | 1 | 1 | 2 | | |
| $x[n] * h[n]$ | 2 | 5 | 5 | 5 | 5 | 2 |

Example 2.3: Let $x[n] = \{4, 2\}$ $(n = 0, 1)$ and $y[n] = h[n] * x[n] = \{12, 10, 14, 6\}$ $(n = 0, 1, 2, 3)$. Determine $h[n]$.

Solution:

$\because$ The length of $h[n]$: $4 - 2 + 1 = 3$

$\therefore$ Let $h[n] = \{h_0, h_1, h_2\}$ $(n = 0, 1, 2)$

$$\begin{cases} 4h_0 \\ 2h_0 + 4h_1 = 10 \\ 2h_1 + 4h_2 = 14 \\ 2h_1 + 4h_2 = 14 \\ 2h_2 = 6 \end{cases} \Rightarrow \begin{cases} h_0 = 3 \\ h_1 = 1 \\ h_2 = 3 \end{cases}$$

$\therefore$ $h[n] = \{3,1,3\}$ $(n = 0, 1, 2)$

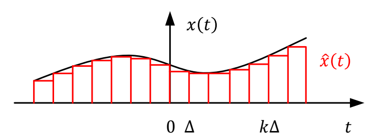
| | | | |
|---|---|---|---|
| | $h_0$ | $h_1$ | $h_2$ |
| | | 4 | 2 |
| | $2h_0$ | $2h_1$ | $2h_2$ |
| $4h_0$ | $4h_1$ | $4h_2$ | |
| 12 | 10 | 14 | 6 |

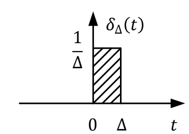
Discussion 2.2:

Find the unit impulse response $h[n]$, which makes $x[n] * h[n] = \delta[n]$ and $x[n] = \delta[n+1] + \frac{5}{2}\delta[n] + \delta[n-1]$.

**§2.2 The Convolution Integral for Continuous-Time Systems**

1. Representation of Continuous-Time Signals in terms of $\delta(t)$




$$\hat{x}(t) = \cdots + x(0)\delta_\Delta(t)\Delta + x(\Delta)\delta_\Delta(t-\Delta)\Delta + \cdots$$
$$+ x(k\Delta)\delta_\Delta(t-k\Delta)\Delta + \cdots$$
$$= \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$x(t) = \lim_{\Delta\to 0} \hat{x}(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta(t-k\Delta)\Delta$$
$$= \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{+\infty} \delta(t-\tau)d\tau$$
$$\xLeftrightarrow{t-\tau=\theta} u(t) = \int_{t}^{-\infty} \delta(\theta)d(-\theta) = \int_{-\infty}^{t} \delta(\theta)d\theta$$
$$\xLeftrightarrow{\theta=\tau} u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

2. Definition of Convolution Integral

① The unit impulse response of continuous-time LTI systems

Consider a continuous-time LTI system has zero initial-state.

$$\delta(t) \rightarrow \boxed{\text{LTI}} \rightarrow h(t)$$
$$\delta(t-\tau) \rightarrow \boxed{\text{LTI}} \rightarrow h(t-\tau)$$

For arbitrary input $x(t)$

$$\sum_{k=-\infty}^{+\infty} x(k\Delta)\delta(t-k\Delta)\Delta \rightarrow \boxed{\text{LTI}} \rightarrow \sum_{k=-\infty}^{+\infty} x(k\Delta)h(t-k\Delta)\Delta$$
$$\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau \quad \lim_{\Delta\to 0} \quad \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

The unit impulse response $h(t)$ of a continuous-time LTI system is the zero-state output of the system when $\delta(t)$ is the input.

The zero-state output of the system with arbitrary input is given by the convolution integral.

$$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y_f(t) = h(t) * x(t)$$

The convolution integral is calculated by

$$h(t) * x(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$
$$\xLeftrightarrow{\theta=t-\tau} h(t) * x(t) = \int_{-\infty}^{+\infty} x(t-\theta)h(\theta)d\theta = x(t) * h(t)$$

② The Unit Step Response of Continuous-Time LTI Systems

The unit step response $s(t)$ of a continuous-time LTI system is the zero-state output of the system when $u(t)$ is the input.

$$u(t) \rightarrow \boxed{\text{LTI}} \rightarrow s(t)$$
$$\delta(t) \rightarrow \boxed{\text{LTI}} \rightarrow h(t)$$

$$\begin{cases} \delta(t) = \dfrac{d}{dt}u(t) \\ u(t) = \displaystyle\int_{-\infty}^{t} \delta(\tau)d\tau \end{cases} \qquad \begin{cases} h(t) = \dfrac{d}{dt}s(t) \\ s(t) = \displaystyle\int_{-\infty}^{t} h(\tau)d\tau \end{cases}$$

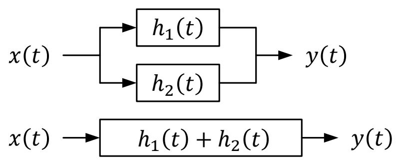
3. Properties of the Convolution Integral

① The Commutative Property:

$h(t) * x(t) = x(t) * h(t)$


② The Distributive Property:

$\{h_1(t) + h_2(t)\} * x(t) = h_1(t) * x(t) + h_2(t) * x(t)$




③ The Associative Property:

$h_2(t) * [h_1(t) * x(t)] = [h_1(t) * h_2(t)] * x(t)$