Northwestern University
Deptartment of EECS

EE 495: Game Theory for Networked Systems Spring 2022

Midterm Exam

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Problem 1 (35 points) For each one of the statements below, state whether it is true or false. If the answer is true, prove it. If the answer is false, give a counterexample. Explanations and counterexamples are required for full credit. (7 points each)

(1) If a static game has a unique strictly dominant strategy equilibrium, then this game *must* also have a unique Nash equilibrium.

Sol: True:

For a game $G = (R, \{S_r\}, \{\Pi_r\})$,

An outcome $S^* = (S_1^* \cdots S_R^*)$

is strictly dominant if $\Pi_r(S_r^*, \bar{S}_{-r}) > \Pi_r(S_r, \bar{S}_{-r})$ $\forall S_r \in S_r$, $S_r \neq S_r^*$, $\forall \bar{S}_r \in \bar{S}_{-r}$

Assume a set of NE $\bar{S}' = (s_1' \cdots s_R')$ $\forall r \in R, \forall S_r' \in S_r$, $\Pi_r(S_r', \bar{S}_{-r}') \underset{\Uparrow}{\geqslant} \Pi_r(S_r, \bar{S}_{-r}')$

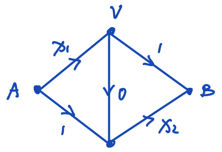
$U(S_r^*, S_{-r}') > U(S_r', \bar{S}_{-r}')$ exist since S^* is strictly dominant, which contradicts the assumption above

So, if a static game has a unique strictly dominant strategy equilibrium, it must have a unique NE.

(2) In a Wardrop equilibrium, each player is infinitesimally small, hence their strategic behavior cannot steer system towards inefficiency and all Wardrop Equilibria maximize social welfare.

Sol: False :

Counter example:



For infinite number of players $P_r \in [0,1]$, from $A \to B$.

WE will be achieved when $x_1 = x_2 = 1$ with total cost $= 2$

Social optimal is achieved when $x_1 = x_2 = \frac{1}{2}$, which has cost $\frac{3}{2}$ < equilibrium total cost

And PoA is always larger than 1, suggests WE does not maximize the social walfare.

(3) In the following simultaneous static prisoner's dilemma game, there exists a correlated equilibrium where players play (C, C) with probability 1.

P2

		D	C
	D	$-2, -2$	$0, -3$
P1	C	$-3, 0$	$-1, -1$

Sol: False.

when play (C,C) with P = 1

Assume P_1 get recommendation C, player2 will also play C. But P_1 can deviate to make a higher payoff from -1 to 0. This mean C is not the best respond.

Therefore, there is no correlated equilibrium when play (C, C) with probability 1.

(4) Let G be any two-player supermodular game and let (x_1, x_2) and $(\tilde{x}_1, \tilde{x}_2)$ be two distinct Nash equilibria of this game, where in each case the first component indicates the action of player 1 and the second component indicates the action of player 2. Furthermore, assume starting from any of these equilibria, unilateral deviation will strictly decrease that player's payoff. If $x_1 > \tilde{x}_1$, it must be that $x_2 \geq \tilde{x}_2$.

Sol: True.

Let's assume $x_1 > \hat{x}_1$, $x_2 < \tilde{x}_2$

Since (x_1, x_2) & $(\hat{x}_1, \tilde{x}_2)$ are two NE points

we have $U(x_1, x_2) \geqslant U(\tilde{x}_1, x_2)$, $U(\hat{x}_1, \tilde{x}_2) \geqslant U(x_1, \tilde{x}_2)$

So that , $U(x_1, x_2) - U(\hat{x}_1, \tilde{x}_2) \geqslant U(x_1, \tilde{x}_2) - U(\tilde{x}_1, \tilde{x}_2) \cdots ①$

which is contradict to the definition of a supermodular game.

$U(x_1, x_2) - U(\hat{x}_1, x_2) \leqslant U(x_1, \tilde{x}_2) - U(\hat{x}_1, \hat{x}_2) \cdots ②$

Therefore, we can approve that if $x_1 > \tilde{x}_1$, it must be that $x_2 \geqslant \tilde{x}_2$.

(5) For any finite game, a weakly dominated action cannot be used with positive probability in a correlated equilibrium.

Sol: False:

By constructing a game that has 4 strategies with the same payoffs, which means every strategy in this game is a weakly dominated action.

In correlated equilibrium, we assign a probability to each strategies. Here, we can assign arbitrary probabilities to every action, and the payoff will not change, which is contradict to the above assumption.

A counter example is:

②

		L	R
①	U	(1,1)	(1,1)
	D	(1,1)	(1,1)

$$\Rightarrow \begin{cases} (U,L) = P_1 \\ (U,R) = P_2 \\ (D,L) = P_3 \\ (D,R) = P_4 \end{cases}$$

to be CE:

$$\begin{cases} P_1 + P_2 \geqslant P_1 + P_2 \\ P_3 + P_4 \geqslant P_3 + P_4 \\ P_1 + P_3 \geqslant P_1 + P_3 \\ P_2 + P_4 \geqslant P_2 + P_4 \end{cases}$$

Obviously, P_i can be any positive probability.

Therefore, a weakly dominated action can be used with positive probability in a CE.

Problem 2 (10 points)*(Mixed Strategy Equilibrium)* Consider a variant of the meeting up for lunch game discussed in class, where the payoff matrix is given as below. Find all (mixed and pure strategy) Nash equilibria.

		Alice	
		TE	Sargent
Bob	TE	(2,1)	(0,0)
	Sargent	(0,0)	(3,4)

Sol:

- Pure strategy:

① Assume Alice choose TE, Bob will choose TE as well, Alice will have no incentive to change unilaterally, because his payoff will decrease from 2 ⇒ 0, which means (TE, TE) is a pure NE point.

② Assume Alice choose Sargent, Bob will choose Sargent as well, Alice will have no incentive to change unilaterally, because his payoff will decrease from 3 ⇒ 0, which means (Sargent, Sargent) is a pure NE point.

- Mixed strategy:

Assume, Alice $\begin{cases} q \Rightarrow \text{TE} \\ 1-q \Rightarrow \text{Sargent} \end{cases}$ Bob $\begin{cases} P \Rightarrow \text{TE} \\ 1-P \Rightarrow \text{Sargent} \end{cases}$

we have $\begin{cases} 2q = 3(1-q) \\ P = 4(1-P) \end{cases} \Rightarrow \begin{cases} q = \frac{3}{5} \\ P = \frac{4}{5} \end{cases}$ ⇒ Mixed strategy

Problem 3 (15 points) *(Potential Game)*

1. Consider the following game

		A	
		L	R
B	U	(1,-2)	(10,5)
	D	(3,100)	(4,99)

   Show that is is an exact potential game by constructing the potential function.

2. Consider the parameterized version of the game

		A	
		L	R
B	U	(a,b)	(c,d)
	D	(e,f)	(g,h)

   Write down the relations between the parameters, such that this game and the one in the previous part would share the same exact potential potential function.

1. Sol:

		A	
		L	R
B	U	(1,-2)	(10,5)
	D	(3,100)	(4,99)

To be a potential Games, it should satisfy:

$U_r(x, s_{-r}) - U_r(z, s_{-r}) = \phi(x, s_{-r}) - \phi(z, s_{-r})$

we first assume $\phi(D,R)=0$

$\phi(U,R) - \phi(D,R) = U_B(U,R) - U_B(D,R) = 10-4 = 6$

$\phi(D,L) - \phi(D,R) = U_A(D,L) - U_A(D,R) = 100-99 = 1$

$\phi(U,L) - \phi(D,L) = U_B(U,L) - U_B(D,L) = 1-3 = -2$

$\phi(U,L) - \phi(U,R) = U_A(U,L) - U_A(U,R) = (-2)-5 = -7$

$\Rightarrow$

		A	
		L	R
B	U	-1	6
	D	1	0

The values of potential function

2. Sol:

we first assume $\phi(D,R)=0$

$\phi(U,R) - \phi(D,R) = u_B(U,R) - u_B(D,R) = c-g = 6$

$\phi(D,L) - \phi(D,R) = u_A(D,L) - u_A(D,R) = f-h = 1$

$\phi(U,L) - \phi(D,L) = u_B(U,L) - u_B(D,L) = a-e = -2$

$\phi(U,L) - \phi(U,R) = u_A(U,L) - u_A(U,R) = b-d = -7$

we also have: $f-h+a-e = c-g+d-b$

$\Rightarrow$

		A	
		L	R
B	U	$f-h+a-e$	$c-g$
	D	$f-h$	0

∴ The value of the potential function

Problem 4 (25 points) *(Patent Race for a New Market)*

Consider a patent race game, where the players are 2 firms: Alps and Bees, which we denote by A and B, respectively. Both firms simultaneously choose a spending budget on research $x_i \geq 0$ ($i = A, B$). Innovation occurs at time $T(x_i)$, which is a function of the spending, where the derivative of $T(x)$ satisfies $T'(x) < 0$ (i.e., more budget leads to faster innovation). The first firm to develop the innovation can file a patent for it worth V dollars (assume no discounting), while any firm developing the innovation later gets no value. The total cost spent by firm i to develop the innovation is x_i, and if both players innovate simultaneously they share its value equally.

1. Formulate the situation as a static game by specifying the payoff functions π_i for firms $i = A, B$.
2. Show that in all pure strategy Nash equilibria one of the players does not invest.
3. Does this game have a pure strategy equilibrium when there are three players, each choosing $x_i \geq 0$, $i = A, B, C$. If there is a tie, the value of the patent is split equally among those who tie.

1. Sol: Spend budget x_A, x_B if $x_A > x_B$ $T(x_A) < T(x_B)$

payoff of $\pi_A = \begin{cases} V - x_A & x_A > x_B \\ \frac{V}{2} - x_A & x_A = x_B \\ -x_A & x_A < x_B \end{cases}$ $\pi_B = \begin{cases} -x_B & x_A > x_B \\ \frac{V}{2} - x_B & x_A = x_B \\ V - x_B & x_A < x_B \end{cases}$

2. Sol:

		Bees		
		$x_A > x_B$	$x_A = x_B$	$x_A < x_B$
Alps	$x_A > x_B$	$(V - x_A, -x_B)$	X	X
	$x_A = x_B$	X	$(\frac{V}{2} - x_A, \frac{V}{2} - x_B)$	X
	$x_A < x_B$	X	X	$(-x_A, V - x_B)$

There exist 2 pure NE strategies:

① Assume Alp's budget $x_A > x_B$, Bees will set its budget $x_B = 0$ to maximize $\pi_B = -x_B$. Alps will have no incentive to change unilaterally, because its payoff will possible decrease to $\frac{V}{2} - x_A$, $-x_A < V - x_A$

Therefore, a pure NE exist when A invest $x_A > 0$ and B invest 0.

① Assume Bee's budget $x_B > x_A$, Alps will set its budget $x_A = 0$ to maximize $\pi_A = -x_A$, Bees will have no incentive to change unilaterally, because its payoff will possible decrease to $\frac{V}{2} - x_B$, $-x_B < V - x_B$

Therefore, a pure NE exist when B invest $x_B > 0$ and A invest 0.

③ Assume $x_A = x_B$, Any one of the firms can deviate to get a higher payoff. Therefore, this is no pure NE

Therefore, in all pure strategy NE, one player does not invest $(x_i = 0)$

3. Sol: Yes, it does. There are three NE points. occurs at

$$\begin{cases} x_A > x_B, x_C = 0 & \Rightarrow \text{payoff } (V - x_A , -x_B , -x_C) \\ x_B > x_A, x_C = 0 & \Rightarrow \text{payoff } (V - x_B , -x_A , -x_C) \\ x_C > x_B, x_A = 0 & \Rightarrow \text{payoff } (V - x_C , -x_B , -x_A) \end{cases}$$

Problem 5 (15 points) *(Iterated Elimination of Strictly Dominated Strategies in Cournot Competition)*

Consider a market in which the price charged for quantity Q (total quantity in the market) of some good is given by $P(Q) = 1 - Q$. Assume that the cost of producing a unit of this good is 0. In this problem, we use subscript to index the firm and superscript to count the number of rounds of eliminations of strictly dominated strategies.

1. Assume that there are two firms in the market. Starting from the original strategy space $S_i^0 = [0, \infty]$, each firm carries out iterated elimination of the strictly dominated strategies. Argue that for $i = 1, 2$, after one round of elimination, we have the set S_i^1 surviving the elimination process can be written as $S_i^1 = [0, 1/2]$.
2. Still for two firms, construct the sets of strategies S_1^k, S_2^k for any $k \geq 1$ and conclude that S_i^∞ is a singleton, i.e. it has only one element. How many Nash Equilibria does this game have?
3. Now assume that there are three firms. Show that S_1^∞ is not a singleton.

①. Sol: payoff ⇒ For firm 1: $P(q_1, q_2) = Q_1(1 - Q_1 - Q_2)$ $u_1' = 0$ ⇒ $Q_1 = \frac{1-Q_2}{2}$

Q_1's best response ⇒ $\frac{1-Q_2}{2}$ as $Q_2 \geqslant 0$ $\frac{1-Q_2}{2} \in [0, \frac{1}{2}]$

$\therefore$ $S_i^1 = [0, \frac{1}{2}]$

②. Sol:

$u_i(Q_i, Q_{-i}) = Q_i P(Q_1, Q_2) = Q_i(1 - Q_1 - Q_2)$

Let $Q_i \rightarrow Q_i + \Delta$ for $\Delta > 0$ then,

$u_i(Q_i + \Delta, Q_{-i}) = (Q_i + \Delta)(1 - Q_1 - Q_2 - \Delta)$

$= (Q_i + \Delta) - (Q_i + \Delta)^2 - (Q_i + \Delta)Q_2$

Let $\begin{cases} S_1 = q_1 \\ S_2 = -q_2 \end{cases}$ the difference between the $u_i(Q_i + \Delta, Q_{-i})$ and $u_i(Q_i, Q_{-i})$ will increase as Q_{-i} increase.

$\therefore$ the game is now a supermodular game, which means:

S^∞ will has $\underline{s}$ and $\bar{s}$, which are the least and greatest pure NE point ⋯ ①

For each iteration: $Q_1 = \frac{1-Q_2}{2}$, Q_i has the same strategy space.

Therefore, it can be formulated as: $Q_i^{k+1} = \frac{1-Q_i^k}{2}$ at the k^{th} iters.

iteration2: $Q_i^2 = \frac{1-\frac{1-Q_i^1}{2}}{2} = \frac{1}{2} - \frac{1}{4} + \frac{Q_i^1}{4}$ iteration2: $Q_i^3 = \frac{1-\frac{1-\frac{1-Q_i^2}{2}}{2}}{2} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{Q_i^2}{8}$

Therefore, iteration k: $Q_i^k = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} \cdots + (-\frac{1}{2})^K Q_i^{K-1}$ as $K \to +\infty$ $S^\infty = Q_i^\infty = -\sum_{k=1}^{\infty} (-\frac{1}{2})^K = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}$

as $k \to \infty$ $Q_i^{\infty+1} = Q_i^\infty \Rightarrow Q_i^\infty = \frac{1}{3}$ which is a singleton. ⋯②

Based on ① and ②, we can know, $\bar{S} = \underline{S} \Rightarrow S_i^\infty = \frac{1}{3}$, and it is the unique NE

3. Sol: For 3 firms, they respetively have quantity Q_1, Q_2, Q_3.

- iteration1:

$P(q_1, q_2) = Q_1(1-Q_1-Q_2-Q_3)$ $u_i' = 0 \Rightarrow Q_1 = \frac{1-Q_2-Q_3}{2}$ as $Q_i \geqslant 0$

$\therefore S_i^1 = [0, \frac{1}{2}]$

- iteration2: Since $S_i^1 = [0, \frac{1}{2}]$

$Q_i^2 = \frac{1-Q_2^1-Q_3^1}{2} \Rightarrow (Q_2+Q_3) \in [0,1]$ which means $Q_i^2 \in [0, \frac{1}{2}]$

thus $S_i^2 = [0, \frac{1}{2}] = S_i^1$, in a similar way, we can conclude as iteration $k \to \infty$, we will always have $Q_i^k \in [0, \frac{1}{2}]$

$\therefore S_i^\infty = [0, \frac{1}{2}]$, which is not a singleton.