ECE495 Game Theory and Networked Systems

Lecture 14: BNE and PBE

1 Bayesian Nash Equilibria

Recall from last class that a Bayesian game (or game with incomplete information) G consists of the following components:

1. R : finite set of players
2. S_r : strategy set for each player r in R
3. Θ_r: type set for each player r in R
4. $F : \Pi_{r=1}^{R} \Theta_r \to [0, 1]$: joint probability distribution over type space
5. $U_r : \Pi_{r=1}^{R} S_r \times \Pi_{r=1}^{R} \Theta_r \to \mathbb{R}$ payoff function for each player r in R

The distribution over the type space does not need to be independent across the players, as we saw in the market of lemons example (seller's type v determines buyer's type αv).

A strategy in this setting is a mapping from type space to an action. Different types of a player can take different actions. The best response against strategy a_{-r} (which consists of $a_{-r}(\theta_{-r})$ for all possible realizations of θ_{-r} for type θ_r player is defined by

$$B_r(a_{-r}|\theta_r) = \arg\max_{s_r \in S_r} \mathbb{E}_{\theta_{-r}}[U_r(s_r, a_{-r}(\theta_{-r}), \theta)|\theta_r].$$

The above definition specifies The unconditioned version of best response of type θ_r player. We next specify the unconditioned best response strategy a_r for player r, which describes the complete strategic mapping from all possible types, Θ_r, to S_r, i.e.,

$$B_r(a_{-r}) = \arg\max_{a_r : \Theta_r \to S_r} \mathbb{E}_{\theta}[U_r(a_r(\theta_r), a_{-r}(\theta_{-r}), \theta)].$$

The following example illustrates information asymmetry, correlated types and how these can make market inefficient.

Example 1. (The market for lemons)

A buyer wants to purchase a single unit of a good that the seller values at v and the buyer at αv, for $\alpha > 1$. The seller's value v is a uniform random variable on $[0, 1]$ and is observed by the seller and not the buyer. A price p in $[0, 1]$ for the good is set exogenously in a market. In an efficient market, trade would take place for $v < p < \alpha v$, since it improves the payoff of the system from a total of v to αv.

To model this as a Bayesian game, we have the players to be the buyer b and the seller s. The seller has two actions: "sell" or "not sell"; the buyer also has two actions: "buy" or "not buy". The transaction happens when seller chooses to sell and buyer chooses to buy. We can view the seller as having a type v that corresponds to her value, while the buyer has a random type of αv. Both types are determined by v. The seller's payoff is given by

$$U_s(s_s, s_b, v) = \begin{cases} p & \text{if } s_s = \textit{sell} \text{ and } s_b = \textit{buy}, \\ v & \text{otherwise.} \end{cases}$$

This payoff reflects how much value the seller goes home with. An alternative setup is to have the first part being $p - v$ and the second part being 0, only the relative ordering matters here.

The buyer's payoff is

$$U_b(s_s, s_b, v) = \begin{cases} \alpha v - p & \text{if } s_s = \textit{sell} \text{ and } s_b = \textit{buy}, \\ 0 & \text{otherwise.} \end{cases}$$

The seller will sell the item only if $p > v$, and v is uniform on $[0, 1]$, so conditioned the seller chooses to sell and the knowing the market price p, the expected value of the good is given by

$$\mathbb{E}\left[v \mid v < p\right] = \frac{p}{2}.$$

Therefore the expected valuation for the buyer will be $\frac{\alpha p}{2}$, and the buyer will buy when $\frac{\alpha p}{2} - p > 0$. This means that the buyer will buy the item only if $\alpha > 2$.

As we can see in this example for $1 < \alpha \leq 2$, no transaction happens which means that the market is inefficient. This phenomenon is aligned with our intuition that the fact the seller chooses to sell means the good is not very good. This model is first used in the market of used cars, where cars with bad quality are called "lemons" and hence the name "market for lemons".

The next example is first price auction where the agents have independent private information.

Example 2. (First price auction with incomplete information)

An object is be assigned to one of two agents via a first price auction. In this mechanism, each player submits a non-negative bid b_r; the object is awarded to the agent with the highest bid. The winning bidder must pay her bid, while the losing bidder pays nothing. In the case of a tie, assume no one is awarded the object, nor pays anything. Each agent r has a private value for the object of θ_r. We assume an independent uniform prior on the valuations; that is, we assume that the values are generated independently chosen according to the uniform distribution on $[0, 1]$.

This situation can be readily modeled as a Bayesian game, where the strategy set for each player r corresponds to the choice of possible bids and the type of each player corresponds to their private value for the object. Thus, the type set Θ_r of each player is simply the interval $[0, 1]$, and the common prior is simply the product of two uniform measures on $[0, 1] \times [0, 1]$. The payoff function of a user in this game is then given by

$$U_r(b_r, b_{-r}, \theta_r) = \begin{cases} \theta_r - b_r & \text{if } b_r > b_{-r}, \\ 0 & \text{otherwise.} \end{cases}$$

A Bayesian Nash equilibrium for this game is for each player to bid half their value, i.e., $b_r = \theta_r/2$. To see that this is an equilibrium, consider agent 1's best response given that agent 2 is bidding half her value. Player 1 will then win the auction if her bid b_1 is larger than $\theta_2/2$. From the assumed uniform prior distribution, the probability that this occurs is given by

$$\mathbb{P}(b_1 > \theta_2/2) = \mathbb{P}(\theta_2 < 2b_1) = \min(2b_1, 1),$$

in which case player 1 receives a payoff of $\theta_1 - b_1$. It follows that agent 1's best response given type θ_1 is to bid value of $b_1 \geq 0$ which maximizes

$$\mathbb{E}_{\theta_2}\left[U_1\left(b_1, \frac{\theta_2}{2}, \theta_1\right) \mid \theta_1\right] = (\theta_1 - b_1)\min(2b_1, 1).$$

When $2b_1 \leq 1$, we are maximizing a concave function, and the unique maximizer is $b_1 = \theta_1/2$. Since $\theta_1 \leq 1$, we have that $2b_1 \leq 1$ for all θ_1 and thus the unique maximizer is $b_1 = \theta_1/2$. In other words, agent 1's best response is to also bid half her value. Hence, from symmetry it follows that this must be a Bayesian Nash equilibrium. Note that bidders in a first price auction are incentivized to shade their bids downward, so that they make a positive profit when they win the auction. This may not be the best outcome for the auctioneer. Other mechanisms, such as VCG (Vickrey Clarke Groves) and reserves may be used to enhance the payoff.

So far we have been constructing BNEs for specific examples. The existence of BNEs are guaranteed by the following theorem, similar to Nash equilibrium existence theorem.

Theorem 3. *Consider a Bayesian game in which the strategy spaces and type sets are compact (continuous) subsets of $\mathbb{R}$ and the payoff functions are continuous in all player's strategies and concave in a player's own strategy. Then a pure strategy Bayesian–Nash equilibrium exists.*

The proof relies on the same tools as Nash existence theorem, except now the utility functions are calculated to reflect the expected value. By the linearity of expectation, all the proof steps of Nash existence theorem carries over.

Just like in the matrix games, we can also consider mixed strategy equilibrium, where the players of different types can use a mixed strategy. We can generalize the notion of BNE to also include mixed strategies and we have the following results on existence of mixed BNE.

Theorem 4. *Mixed Bayesian-Nash equilibria exist for all finite games.*

This is a direct result of the previous theorem, as the mixed strategies come from a probability simplex which is compact.

1.1 Discussion on BNE

The requirement of common prior is quite strong. The players no longer need knowledge of other players' payoff functions (as in Nash Equilibra); instead, they need to know set of possible types and the common prior in order to determine their expected utility. This common knowledge assumption may be strong in certain settings, for instance, when the players have inconsistent beliefs about what is common knowledge, or how to agree upon the common knowledge. One modification to address this concern is by having different types of players, informed vs uninformed for example. This additional level of uncertainty can be added on iteratively and the complexity of the game grows quickly. There is a delicate tradeoff between realism and tractability of the model.

In addition to different types of users, one may also modify other uncertainties, such as the number of participants may be unknown aprior. One way to fix this is by introducing an additional strategy of "not participate" and making this strategy dominant for certain types of players. This also comes at the cost of increasing computation cost.

2 Perfect Bayesian Equilibria

So far we have talked about incomplete information in static games, we next shift our focus to extensive form games, which models games with incomplete information and sequential moves. The definition of a game is now more complex. A dynamic game of incomplete information consists of all the elements from a Bayesian game, as well as sequence of histories and information sets (what happened in the past and what is observable).

We next define Perfect Bayesian Equilibria (PBE), which combines the criterion from Bayesian Nash Equilibria and Subgame Perfect Equilibria. One significant difference between SPE and PBE is the use of "belief system". Like in BNE, due to the uncertainty in types, players form beliefs about their opponents, and to be part of an equilibrium, the beliefs need to be consistent according to Bayes rule, i.e., conditioned on observations, the beliefs do not cause inconsistency.

Definition 5. *A PBE in a dynamic game of incomplete information is a strategy profile s and a belief system μ, such that*

- *The strategy profile s is sequentially rational given μ: each player best responds given the beliefs, history of the game, information set (SPE like).*
- *The belief system μ is consistent given s: for every information set reached with positive probability given the strategy profile s, the probability assigned to each history in the belief system μ is given by Bayes' rule.*

We consider the following example.

Example 2.1 (Lawsuit signaling game)**.** *Here we use model a civil lawsuit as a game. Consider the case where the plaintiff Π knows a bit more information, in particular he knows if he will win the case (we use W to denote the winning type, and L to denote the losing type). The defendant Δ does not have this information but knows that Π knows. The prior belief Δ has is that with probability $1/3$ Π will win the lawsuit. If this lawsuit goes to the court, and if Π wins, the payoff is $(3, -4)$ (for Π, Δ respectively) representing the scenario where Δ pays 3 units to Π and also has to pay 1 unit for the court expenses. If Δ wins, then the payoff is $(-1, 0)$, where there is no payment and only the loser has to pay for 1 unit of court expense.*

Before going to the court, Π will give Δ a chance to settle outside the court. This is done by Π asking for a price of settlement m. This settlement could be either low 1 or high 2. If the defendant chooses to accept, then the payoff is $(m, -m)$, or to reject, in which case they go to court and the payoff is specified as above. We will find the PBE in this game.

We first represent this game in a game tree as in Fig. 1. Note that in all payoffs, we write the plaintiff's payoff first, since he is the first mover in this game (by sending a signal).

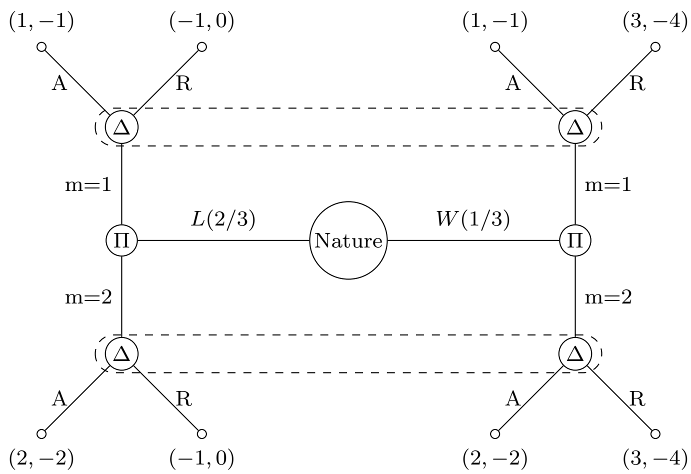

Figure 1: Game tree representation for lawsuit game

There are four different possible signaling possibilities:

1. L and W both send m=1;

2. L and W both send m=2;

3. L send m=1, W send m=2;

4. L send m=2, W send m =1;

The first two cases are also referred to as "pooling" strategy, where both types send the same signal. The last two are referred to as "separating" strategy, where based on signal, one can tell the different types of player Π.

We first analyze pooling strategy. Suppose that both types will send $m = 1$ (case 1). We use $\mu(T|m = i)$ to denote the belief of P2 that P1 is type T given signal m=i. Here the belief system according to Bayes rule is given by

$$\mu(W|m = 1) = \frac{1}{3}.$$

The expected payoffs for Δ associated with to accept is then

$$E[U_\Delta(A, 1, 1|m = 1)] = -1,$$

and associated with to reject is

$$E[U_\Delta(R, 1, 1|m = 1)] = 1/3(-4) + 2/3(0) = -4/3 < -1.$$

Hence Δ will choose to accept. Note that W type is getting a payoff of 1 and he can deviate to sending m=2 and get at least 2. Hence this is not a PBE.

We next consider case 2, where both types send $m = 2$. Here the belief system according to Bayes rule is given by

$$\mu(W|m = 2) = \frac{1}{3}.$$

The expected payoffs for Δ associated with to accept is then

$$E[U_\Delta(A, 2, 2|m = 2)] = -2,$$

and associated with to reject is

$$E[U_\Delta(R, 2, 2|m = 2)] = 1/3(-4) + 2/3(0) = -4/3 > -2.$$

Player Δ chooses to reject. W type player has no incentive to deviate, since he is getting the highest possible payoff. L type player is currently getting a payoff of -1. We have not specified Δ's strategy if m=1 was sent. We need Δ to choose to reject after seeing m=1 to make sure that L type Π will not deviate. The payoff for Δ for choosing A if m=1 is -1 and R is $-4\mu(W|m = 1) + 0(1 - \mu(W|m = 1))$, therefore we need

$$-4\mu(W|m = 1) + 0(1 - \mu(W|m = 1)) \geq 1.$$

This is equivalent to $\mu(W|m = 1) \leq \frac{1}{4}$.

We next analyze separating strategy (cases 3,4). Consider a separating strategy where W sends m=2 and L sends m=1.

$$\mu(W|m = 1) = 0, \quad \mu(W|m = 2) = 1,$$

then P2 will reject if $m = 1$ and accept if $m = 2$. L type Π can then unilaterally deviate and send $m = 1$ to improve payoff. Similarly, if W sends m=1, and L sends m=2. then P2 will accept if $m = 1$ and reject if $m = 2$. However, in this case the W type is getting the least possible payoff and will deviate to send m =2 to improve payoff. Moreover, the L type will also deviate to send $m = 1$ instead to improve payoff from -1 to 1. Hence this is also not an equilibrium.

To summarize, we found a unique equilibrium where W and L both send signal m=2 and defendant rejects when seeing either m =1 or m=2. The associated beliefs are

$$\mu(W|m = 1) \leq \frac{1}{4}, \quad \mu(W|m = 2) = \frac{1}{3}.$$

We note that the second belief is given by equality, as this event $m = 1$ is reached at equilibrium (called along equilibrium path), and therefore its belief is defined by conditional probability. On the other hand, the first inequality is defined for conditioning on m=1, which is not reached at equilibrium (called off equilibrium path). Since this information set m=1 is not reached at equilibrium, there is no constraint on the beliefs and we can define the belief to be anything in [0,1] (Bayes rule at this point involves dividing by 0, so any probability can work), and thus the equilibrium exists. This is often a criticism against PBE, since this process of defining range of $\mu(W|m = 1)$ seems arbitrary.