ECE495 Game Theory and Networked Systems

Lecture 8: Potential Game and Correlated Equilibrium

1 Supermodular Game

From the last class: Recall the definition of supermodular games, and we have

$$\pi_r(s'_r, s_{-r}) - \pi_r(s_r, s_{-r}) \leq \pi_r(s'_r, s'_{-r}) - \pi_r(s_r, s'_{-r}) \tag{1}$$

Theorem 1. *Assume a game, $G = (R, \{S_r\}, \{U_r\})$, is a supermodular game. Let the best response mapping be*

$$B_r(s_{-r}) = \arg\max_{s_r \in S_r} U_r(s_r, s_{-r}). \tag{2}$$

Then,

1. *$B_r(s_{-r})$ has greatest and least elements, denoted by $\bar{B}_r$, and $\underline{B}_r$, respectively.*
2. *If $s'_{-r} \geq s_{-r}$, then $\bar{B}_r(s'_{-r}) \geq \bar{B}_r(s_{-r})$ and $\underline{B}_r(s'_{-r}) \geq \underline{B}_r(s_{-r})$.*

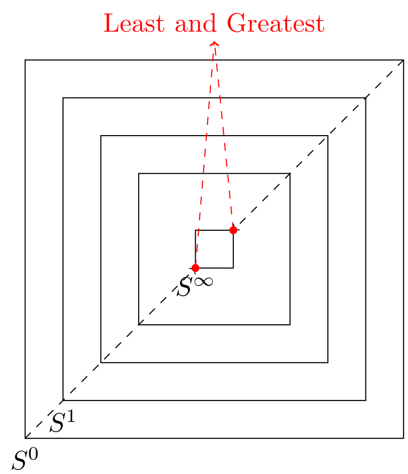

Figure 1: A pictorial description of Theorem 4.

Theorem 2. *For a supermodular game with continuous payoff functions, let S^∞ be the set survived after iterated elimination of strictly dominated strategies. S^∞ has $\underline{s}$ and $\bar{s}$, the least and greatest elements, which are the least and greatest pure strategy Nash Equilibria, respectively.*

Proof. Suppose we have a supermodular game. Let S^k denote the set after the k^{th} iterated elimination process. Then s^0 is the largest element in S^0. Then, we define $s_i^1 = \bar{B}(s^0_{-i})$
The proof follows as $\forall s_i > s_i^1$ and $\forall s_{-i} \in S_{-i}$;

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) &\leq u_i(s_i, s^0_{-i}) - u_i(s_i^1, s^0_{-i}) \\ &< 0 \end{aligned}$$

For the equation above, the first inequality comes from the definition of supermodular games, and the second inequality comes from the fact that $s_i^1 = \bar{B}(\bar{s}_{-i}^0)$, i.e.,$s_i$ is not a best response, so $u_i(s_i^1, \bar{s}_{-i}^0) > u_i(s_i, \bar{s}_{-i}^0)$. We can then eliminate $\forall s_i, s_i > s_i^1$. We can iterate this process, and at each iteration we have for $s_i > s_i^{k+1}$,

$$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k). \tag{3}$$

This inequality holds for all $s_i > s_i^{k+1}$ including those eliminated in earlier rounds, since they were strictly dominated by s_i^t for $t \leq k$. We can iteratively apply this inequality and get that the strategy is dominated by s_i^{k+1}.

We also note that we have $s_i^{k+1} \leq s_i^k$, since we started at the greatest element. Hence, we have a monotone sequence s^k in a compact set and therefore the sequence converges and $\lim_{k\to\infty} s_i^k = s_i^\infty$. We can take limit of Eq. (3) and have $\lim_{k\to\infty} u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k)$. By continuity of u_i function, it follows that,

$$u_i(s_i^\infty, s_{-i}^\infty) \geq u_i(s_i, s_{-i}^\infty).$$

This shows that s^∞ is a NE. A similar argument can be made starting from the smallest element.

2 Potential Games

We introduce a new class of games called potential games, where the existence of a pure strategy Nash equilibrium is guaranteed. For each of these potential games, there is a potential function mapping from outcome space to real numbers, ϕ: $\Pi_r S_r \to \mathbb{R}$. There are two types of potential games: *Ordinal and Exact potential games.* We define these below.

Definition 3 (Exact Potential Game)**.** *A strategic game $G = (R, \{S_r\}_{r\in R}, \{U_r\}_{r\in R})$ is an exact potential game if there exists a function $\phi : \Pi_r S_r \to \mathbb{R}$, such that for all $r \in R$, for all $s_{-r} \in S_{-r}$ and for all $x, z \in S_r$, we have,*

$$U_r(x, s_{-r}) - U_r(z, s_{-r}) = \phi(x, s_{-r}) - \phi(z, s_{-r}).$$

Definition 4 (Ordinal Potential Game)**.** *A strategic game $G = (R, \{S_r\}_{r\in R}, \{U_r\}_{r\in R})$ is an ordinal potential game if there exists a function $\phi : \Pi_r S_r \to \mathbb{R}$ such that for all $r \in R$, for all $s_{-r} \in S_{-r}$ and for all $x, z \in S_r$, we have, $U_r(x, s_{-r}) - U_r(z, s_{-r}) > 0$ if and only if, $\phi(x, s_{-r}) - \phi(z, s_{-r}) > 0$.*

Example 2.1 (Peering Game)**.** Recall the peering game (hot potato example) from Lecture 3. For this game the payoffs for each agent can be represented as Table 1:

		P_2	
		N	F
P_1	N	(-4,-4)	(-1,-5)
	F	(-5,-1)	(-2,-2)

Table 1: The payoffs for the peering game example

We note that the initial value is arbitrary, as only the difference between payoffs matter. To calculate the potential function, we first assume that $\phi(F, F) = 0$. Then we go around this table as shown in the Figure 2. According to the definition of potential function, each time we among the columns, we add the difference of payoffs of the second player. For a row move, we add the difference of payoffs of the first player. For example when we're moving from the point (N, F) to (F, F), which is a horizontal move, we must have, $\phi(N, F) - \phi(F, F) = U_1(N, F) - U_1(F, F)$.

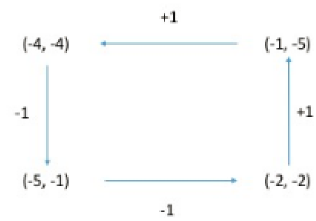

Figure 2: The cycle for the peering game example.

		P_2	
		N	F
P_1	N	2	1
	F	1	0

Table 2: The values of the potential function derived from the cycle above.

Note that, when we go around the cycle of four above, the sum of each edge is zero. From the figure above, we get the potential function assignment as shown in Table 2, i.e., $\phi(F,F) = 0$, $\phi(N,F) = 1$, $\phi(N,N) = 2$, and $\phi(F,N) = 1$.

Theorem 5. *If G is a potential game with potential function ϕ and ϕ attains a maximum over S, then G has a Nash equilibrium.*

A pure strategy Nash equilibrium is at the strategy where the potential function achieves its maximum.

Example 2.2 (Resource Allocation)**.** Allocation of a congestible resource (sharing a link). We define the pay off function for each player r as:

$$U_r(x_r, x_{-r}) = P_r(x_r) - l\left(\sum_{r=1}^{R} x_r\right),$$

where X_r is the strategy space for all players. The first term reflects utility of consumption and the second term represents congestion experienced by all users, which is a function of actions of all players. Then we find the potential function to be,

$$\phi(x) = \sum_{r=1}^{R} P_r(x_r) - l\left(\sum_{r=1}^{R} x_r\right).$$

Example 2.3 (Cournot Game)**.** Now we calculate the potential function for a two player Cournot Game with the price function,

$$P(q_1, q_2) = 1 - q_1 - q_2.$$

The payoff for each player r is given by,

$$U_r(q_r, q_{-r}) = q_r(1 - q_r - q_{-r}) - c_r q_r.$$

Then if player r produces δ more goods, we have $q_r \to q_r + \delta$. Thus, we get,

$$U_r(q_r + \delta, q_{-r}) = (q_r + \delta) - (q_r + \delta)^2 - (q_r + \delta)(q_{-r}) - c_r(q_r + \delta).$$

By analyzing the difference, we find the potential function to be,

$$\phi(q_1, q_2) = q_1 + q_2 - q_1^2 - q_2^2 - q_1 q_2 - c_1 q_1 - c_2 q_2.$$

So far we have concluded that a two player Cournot game is both supermodular and potential.

References

[1] R. Berry and R. Johari, "Economic Modeling in Networking: A Primer", in Foundations and Trends in Networking, 6.3, 2013, pp. 165-286.

[2] MIT, "Game Theory with Engineering Applications Lecture 9: Computation of NE in finite games, Marh 4, 2010