§9.4 The Inverse Laplace Transform

$X(s) = X(\sigma + j\omega) = \mathcal{F}[x(t)e^{-\sigma t}]$

$x(t)e^{-\sigma t} = \mathcal{F}^{-1}[X(\sigma + j\omega)] = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$

$x(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega$

$s = \sigma + j\omega \Rightarrow ds = jd\omega$

$x(t) = \mathcal{L}^{-1}[X(s)] = \dfrac{1}{2\pi j}\displaystyle\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \quad (\sigma \in ROC)$

$\quad = \displaystyle\sum_{all\ poles\ of\ X(s)} Res[X(s)e^{st}]$

Residue Theorem:
http://www.staff.city.ac.uk/~george1/laplace_residue.pdf

Example 9.13 Compute the inverse Laplace transform of

$$X(s) = \frac{1}{s^2 + 3s + 2} \quad (ROC\colon Re\{s\} > -1)$$

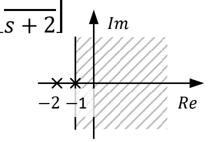
Solution:

$X(s) = \dfrac{1}{s^2 + 3s + 2} = \dfrac{1}{(s+1)(s+2)} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$

$(ROC\colon Re\{s\} > -1)$

$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\dfrac{1}{s+2}\right]$

$\quad = [e^{-t} - e^{-2t}]u(t)$



Example 9.14 Compute the inverse Laplace transform of

$$X(s) = \frac{s+4}{s^3 + 3s^2 + 2s} \quad (ROC\colon -1 < Re\{s\} < 0)$$

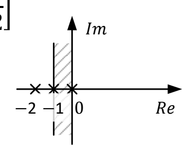
Solution:

$X(s) = \dfrac{s+4}{s^3 + 3s^2 + 2s} = \dfrac{s+4}{s(s+1)(s+2)}$

$\quad = \dfrac{2}{s} + \dfrac{-3}{s+1} + \dfrac{1}{s+2} \quad (ROC\colon -1 < Re\{s\} < 0)$

$x(t) = \mathcal{L}^{-1}\left[\dfrac{2}{s}\right] + \mathcal{L}^{-1}\left[\dfrac{-3}{s+1}\right] + \mathcal{L}^{-1}\left[\dfrac{1}{s+2}\right]$

$\quad = -2u(-t) - 3e^{-t}u(t) + e^{-2t}u(t)$



Discussion 9.3 Compute the inverse Laplace transform of

$$X(s) = \frac{2s+4}{s^2 + 7s + 12} \quad (ROC\colon Re\{s\} > -3)$$

Solution:

$X(s) = \dfrac{4}{s+4} + \dfrac{-2}{s+3} \quad (ROC\colon Re\{s\} > -3)$

$x(t) = 4e^{-4t}u(t) - 2e^{-3t}u(t)$

Discussion 9.4 Compute the inverse Laplace transform of

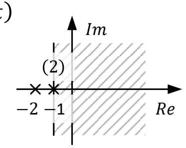
$$X(s)=\frac{s+3}{(s+1)^2(s+2)}\quad (ROC\colon\ Re\{s\}>-1)$$

Solution:

$$X(s)=\frac{s+3}{(s+1)^2(s+2)}=\frac{-1}{s+1}+\frac{2}{(s+1)^2}+\frac{1}{s+2}$$

$(ROC\colon\ Re\{s\}>-1)$

$$\begin{aligned}x(t)&=\mathcal{L}^{-1}[X(s)]\\&=-e^{-t}\,u(t)+2te^{-t}\,u(t)+e^{-2t}\,u(t)\\&=[(2t-1)e^{-t}+e^{-2t}]u(t)\end{aligned}$$



§9.5 Analysis and Characterization of LTI Systems using the Laplace Transform

$y(t)=h(t)*x(t)$

$$x(t)\longrightarrow\boxed{h(t)}\longrightarrow y(t)$$

$Y(s)=H(s)X(s)$

$$X(s)\longrightarrow\boxed{H(s)}\longrightarrow Y(s)$$

$H(s)$ is the system/transfer function of the LTI system

$$H(s)=\mathcal{L}[h(t)]=\int_{-\infty}^{\infty}h(t)e^{-st}dt\quad (s\in ROC)$$

$H(s)|_{s=j\omega}=H(j\omega)$ is the frequency response of the LTI system

1. Causality

①The ROC associated with the system function for a causal system is a right-half plane.

Example 9.16 Consider a system with impulse response $h(t)=e^{-t}u(t)$. Determine the causality of this system.

Solution:

Since $h(t)=0\ \ (t<0)$, this system is a causal system.

$$H(s)=\frac{1}{s+1}\quad (ROC\colon\ Re\{s\}>-1)$$

The ROC is a right-half plane.

Example 9.17 Consider a system function

$$H(s)=\frac{e^s}{s+1}\quad (ROC\colon\ Re\{s\}>-1)$$

Determine the causality of this system.

Solution:

The ROC is a right-half plane.

However,

$$e^{-t}u(t)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s+1}\quad (ROC\colon\ Re\{s\}>-1)$$

$$e^{-(t+1)}u(t+1)\overset{\mathcal{L}}{\longleftrightarrow}\frac{e^s}{s+1}\quad (ROC\colon\ Re\{s\}>-1)$$

$$h(t)=\mathcal{L}^{-1}[H(s)]=e^{-(t+1)}u(t+1)$$

Since $h(t)\neq 0\ \ (t<0)$, this system is not a causal system.

② For a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.

2. Stability

① An LTI system is stable if and only the ROC of its system function $H(s)$ includes the $j\omega$ axis.

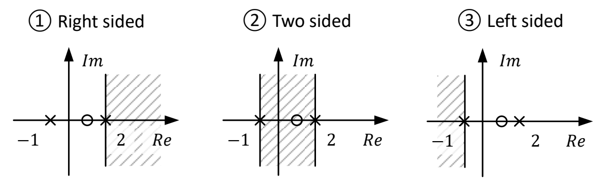
Example 9.18 Consider an LTI system with the system function

$$H(s) = \frac{s-1}{(s+1)(s-2)}$$

Determine all possible ROCs and impulse responses $h(t)$ of this system.

Solution:

$$H(s) = \frac{2}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2}$$



Case (ROC: $Re\{s\} < -1$): Anticausal stable system

$$h(t) = -(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(-t)$$

Case (ROC: $-1 < Re\{s\} < 2$): Noncausal stable system

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$$

Case (ROC: $Re\{s\} > 2$): Causal unstable system

$$h(t) = (\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t})u(t)$$

② A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane, *i.e.* all of the poles have negative real parts.