ECE495 Game Theory and Networked Systems

Lecture 9 Computation of Equilibria

1 Computation of Correlated Equilibrium

In a Nash equilibrium, players choose strategies (or randomize over strategies) independently. In some settings, one may want to allow for some form of communication prior to the play of the game to achieve higher social payoff [2]. We demonstrate this concept with the following example.
Consider the following payoff table.

		P_2	
		L	R
P_1	U	(5,1)	(0,0)
	D	(4,4)	(1,5)

Table 1: The payoffs for the correlated equilibrium example.

There are two pure strategy Nash Equilibria in this game given by, (U, L), and (D, R). To find the mixed strategy Nash equilibria, assume player 1 plays U with probability p and player 2 plays L with probability q. Using the mixed equilibrium characterization, we have $5q = 4q + (1 - q)$, and $p + 4(1 - p) = 5(1 - p)$. Thus $p = q = \frac{1}{2}$. Then first player will have probability $\frac{1}{2}$ to choose U and $\frac{1}{2}$ to choose D, and the second player will have probability $\frac{1}{2}$ to choose L and $\frac{1}{2}$ to choose R. Thus there is a unique mixed strategy equilibrium with expected payoffs $(\frac{5}{2}, \frac{5}{2})$ [2].

Now we notice that neither players will get anything by playing (U, R). However this happens in the mixed equilibrium since the players choose strategy independently. To avoid that state, we introduce a *trusted mediator* or a observable third party (such as traffic light) which is a random variable distributed according to the following probabilities in Table 2:

Outcome	Probability
(U, L)	$\frac{1}{3}$
(D, L)	$\frac{1}{3}$
(D, R)	$\frac{1}{3}$

Table 2: Probabilities of the signals sent by the trusted mediator.

The mediator recommends a strategy according to the random variable. For instance, if the random variable gives (U, L), then player 1 receives the recommendation of U and player 2 receives the recommendation of L. Now we show that no player has an incentive to deviate from the "recommendations" of the mediator.

- If P_1 gets the recommendation U, he will know that P_2 is gets L. P1 has no incentive to change unilaterally because if he does, his pay off will decrease by 1 (from 5 to 4). Thus P_1's best response is to choose U. We can also see this by observing that (U, L) is a NE.

- If P_1 gets D, he will know that P_2 is assigned with either L or R with probability $\frac{1}{2}$, i.e.,

$$\mathbb{P}((D,L)|D) = \mathbb{P}((D,R)|D) = \frac{1}{2}.$$

  With this probability distribution, the expected payoff of playing D is 2.5 and playing U is also 2.5 and thus has no incentive to change (cannot strictly improve payoff by unilaterally deviating). His best response is to play D.

- If P_2 gets the recommendation L, he believes P_1 will play U or D with equal probability, so playing L and R both give the same payoff and playing L is a best response.

- If P_2 gets the recommendation R, he believes P_1 will play D, so his best response is to play R with payoff of 5 (instead of 4 if he were to deviate and play L).

Thus it is in the best interest of both of the the players to follow the mediator's recommendations. With the trusted mediator involved, the expected payoffs are $(\frac{10}{3}, \frac{10}{3})$, which is strictly higher than what the players could get by randomizing between Nash equilibria.
The preceding examples lead us to the notions of correlated strategies and **correlated equilibrium**. We remark that, any convex combination of Nash Equilibria is a Correlated Equilibria, i.e., any payoff in the convex hull of Nash Equilibrium can be obtained via a correlated equilibrium. In the above example, this would correspond to choosing $(U.L)$ with probability γ, and choosing (D,R) with probability $1-\gamma$. Finally we note that Correlated Equilibrium is a more general solution concept than Nash Equilibrium and may achieve strictly higher payoff.

		$Player_2$	
		L	R
$Player_1$	U	(5,1)	(0,0)
	D	(4,4)	(1,5)

Table 3: Correlated Equilibrium Example

In this section, we write out the characteristics of a correlated equilibrium and show that it can be computed efficiently. Recall the correlated equilibrium computation, where we introduce a trusted mediator. Consider the case where (U, L), (U, R), (D, L) and (D,R) are played with P_1, P_2, P_3, P_4 respectively. Given that player 1 received signal to play U, the probability of player 2 playing L is P_1 and R is P_2. We need Player$_1$ to have no incentive choosing D, which means the expected pay off of choosing U is greater or equal to the pay off of choosing D, i.e.,

$$\frac{P_1}{P_1+P_2} \times 5 + 0 \geq \frac{P_1}{P_1+P_2} \times 4 + \frac{P_2}{P_1+P_2} \times 1.$$

We can cancel out the denominator and have solving for the inequality we get:

$$P_1 \times 5 + 0 \geq P_1 \times 4 + P_2 \times 1.$$

We have that for P_1, P_2 to be a correlated equilibrium, they must satisfy a collection of linear inequalities. We can derive three other similar constraints for the cases where Player$_1$ is recommended to play U, Player$_2$ is recommended to play L and R. In general, any correlated equilibrium can be specified by a collection of linear inequalities and hence can be treated as a linear program, which has efficient solvers.

2 Computation of Nash Equilibrium

We now consider the computation of NE for finite games. We first note that we can represent a two player game by expressing strategy space as two matrices. For the example in Table 3, both players have 2 strategies and their payoffs can be represented by two 2×2 matrices. We call these two matrices A and B respectively.

A	$L : y_1$	$R : y_2$
$U : x_1$	5	0
$D : x_2$	4	1

Table 4: Player1 payoff matrix

B	$L : y_1$	$R : y_2$
$U : x_1$	1	0
$D : x_2$	4	5

Table 5: Player2 payoff matrix

We use x, y to denote the probability distribution over the strategies, i.e., $x = [x_1, x_2]'$, where x_1 is the probability the first strategy is played and X and Y to denote the probability space for players 1 and 2 respectively, i.e., $X = \{x | x_1 + x_2 = 1\}$. We have that given player 2 is playing y, the expected payoff for player1 for playing U is given by $5y_1 + 0y_2$ and for playing D is $4y_1 + 1y_2$. The overall expected payoff is $x_1(5y_1 + 0y_2) + x_2(4y_1 + 1y_2)$. This can be represented by $x^T Ay$. The same analysis applies to player 2. In order for a mixed strategy (x^*, y^*) to be a NE, for any strategies $x \in X$ and $y \in Y$, we have that unilateral deviation should not strictly improve expected payoff, i.e.,

$$(x^*)^T Ay^* \geq x^T Ay^*, \quad (x^*)^T By^* \geq (x^*)^T By.$$

We see that this is in bilinear form, which has no known polynomial time algorithm to compute.

An alternative way of computing a mixed strategy NE is by using the definition, where any strategies played with positive probability have the same payoff and that is no less than the strategies played with 0 probability. Now assume we are in a more general case with M strategies for each player, and the set $\phi = \{i | 1 \leq i \leq M, x_i > 0\}$, then for all i in ϕ,

$$u_1(i) = [Ay^*]_i = [Ay^*]_j = u_1(j),$$

and for any k in $\{1, \ldots, M\}$ and not in ϕ,

$$u_1(i) = [Ay^*]_i \geq [Ay^*]_k = u_1(k).$$

While each of these conditions can be represented by a system of linear inequalities. We have to search this on all possibilities of ϕ, which is the number of subsets of $\{1, \ldots, M\}$. We need the same analysis for the opponent, resulting in a total of 2^{2M} possibilities, which is exponential search space. The hardness in finding NE lies in identifying the support set of the strategies.

For more reference on the complexity of NE computation can be found in this paper: Daskalakis, Constantinos, Paul W. Goldberg, and Christos H. Papadimitriou. "The complexity of computing a Nash equilibrium." SIAM Journal on Computing 39.1 (2009): 195-259.

2.1 Zero Sum Game

We next analyze a special class of two player finite game: zero sum games, where we have $A = -B$, and show that we can find equilibria of this class of games efficiently. From the previous section, we have that if (x^*, y^*) is an equilibrium, then for any x in X and y in Y, we have

$$(x^*)^T Ay^* \geq x^T Ay^*, \quad (x^*)^T By^* \geq (x^*)^T By.$$

We can now use the fact that $A = -B$ and rewrite the second inequality as $(x^*)^T Ay^* \leq (x^*)^T Ay$. We can then combine the two inequalities to obtain

$$x^T Ay^* \leq (x^*)^T Ay^* \leq (x^*)^T Ay.$$

Then (x^*, y^*) is the saddle point of $x^T Ay$.

We can also show that for x in X, y in Y,

$$\inf_y \sup_x (x^T Ay) \leq \sup_x (x^T Ay^*) = (x^*)^T Ay^* = \inf_y ((x^*)^T Ay) \leq \sup_x \inf_y (x^T Ay). \tag{1}$$

Moreover, we know for any function f, the minimax inequality holds:

$$\sup_x \inf_y f(x, y) \leq \inf_y \sup_x f(x, y).$$

Therefore the equality holds in Eq. (1) and (x^*, y^*) is NE if and only if:

$$\inf_y \sup_x (x^T Ay) = (x^*)^T Ay^* = \sup_x \inf_y (x^T Ay).$$

The task of finding a mixed strategy NE can be equivalently expressed as finding the $\inf_y \sup_x (x^T Ay)$, since we are searching in a compact probability simplex for x and y, the feasible space is compact and the objective function is continuous, therefore we can use maximization and minimization instead of sup and inf. We note that

$$\max_{x \in X} x^T Ay = \max_{i=1,\ldots,n} \{[Ay]_i\}$$

Then:

$$\min_{y \in Y} \max_{x \in X} x^T Ay = \min_{y \in Y} \max_{i=1,\ldots,n} \{[Ay]_i\} = \min_{\{y \in Y, ve \geq Ay\}} v,$$

where e is the vector of all 1s. Hence finding NE of a two player zero sum game can be represented as a linear programming, which can be solved efficiently.

References

[1] R. Berry and R. Johari, "Economic Modeling in Networking: A Primer", in Foundations and Trends in Networking, 6.3, 2013, pp. 165-286.

[2] MIT, "Game Theory with Engineering Applications Lecture 9: Computation of NE in finite games, Marh 4, 2010