ECE495 Game Theory and Networked Systems

Lecture 4 Cournot competition and mixed strategies

1 Cournot Competition

1.1 Cournot Competition

We introduce Cournot Competition here. This is a competition where two firms are producing the same divisible good, and competing for customers.

- 2 firms producing some good.
- Each firm chooses a quantity to produce $q_i \geq 0$ at a cost per unit c_i.
- Both firms sell their goods at market clearing price $p(q_1 + q_2)$. We may assume this function is decreasing and concave.

The payoffs for each firm i is its profit, given by revenue minus cost:

$$\Pi(q_1, q_2) = p(q_1 + q_2)q_i - c_i q_i$$

For this game it is not immediate if a Nash equilibrium exists and if so how to find it. Each of the firms tries to find $q_i \in [0, \infty)$ to maximize its own payoff function.

For now let's assume linear relationship for market clearing price, say $p(q_1 + q_2) = 1 - q_1 - q_2$. Given that firm 1 already knows the quantity that firm 2 is going to produce q_2, it seeks to solve following maximization problem:

$$\max_{q_1 \geq 0} \; q_1(1 - q_1 - q_2) - c_1 q_1. \tag{1}$$

We define *best response* of agent 1 to agent 2 as $B_1(q_2) \triangleq \arg\max_{q_1 \geq 0} \; q_1(1-q_1-q_2)-c_1 q_1$. Since the objective function is concave in q_1, we can write first order condition of optimality and consider the nonnegativity constraint on the produced quantity to find the best response of agent 1 to agent 2 as

$$B_1(q_2) = \max(\frac{1 - q_2 - c_1}{2}, 0).$$

Correspondingly, we can define best response of agent 2 to agent 1 as

$$B_2(q_1) = \max(\frac{1 - q_1 - c_2}{2}, 0).$$

Using above results and defining q_1^* and q_2^* as the Nash equilibrium quantities to produce for firm 1 and 2 respectively, we have $q_1^* = B_1(q_2^*)$ and $q_2^* = B_2(q_1^*)$ by definition of Nash equilibrium (the point is best response for both players). In other words, finding Nash equilibrium in this setting boils down to solve a system of best response equations (see Figure 1). One can solve the linear system of equations (Assuming both q_i are greater than zero) for our example and arrive at following solutions:

$$q_1^* = \frac{1 - 2c_1 + c_2}{3} \tag{2}$$

$$q_2^* = \frac{1 - 2c_2 + c_1}{3} \tag{3}$$

Note that when $c_1 \to \infty$ and $c_2 \to \infty$, $NE \to (0, 0)$. Also, when $c_1 = c_2 = 0$ we have $NE = (\frac{1}{3}, \frac{1}{3})$.

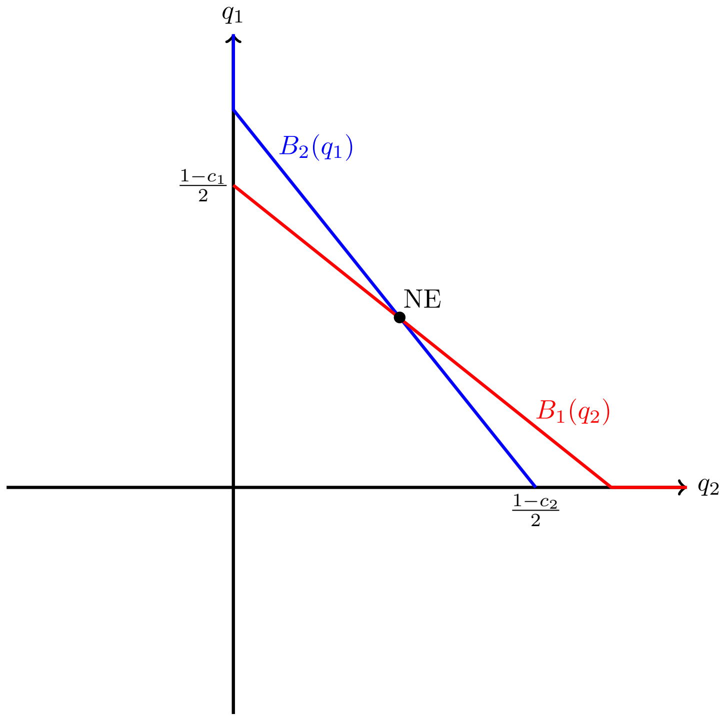

Figure 1: Solving system of equations results in NE

2 General Game and Nash Equilibrium

In a general game $g = (R, \{S_r\}, \{\Pi_r\})$, given $\bar{s}_{-r}$, agent r's best response is given by solving:

$$\max_{s_r \in S_r} \; \Pi_r(s_r, \bar{s}_{-r}). \tag{4}$$

In a general game, multiple solutions might exist. To accommodate that we define the set-valued function $B_r(\bar{s}_{-r})$ as:

$$B_r(\bar{s}_{-r}) \triangleq \arg\max_{s_r \in S_r} \; \Pi_r(s_r, \bar{s}_{-r}). \tag{5}$$

and call it *best response correspondence*. For each r, we can augment the variable and have the best response as a function of entire outcome vector $\bar{s}$ by ignoring the component of s_r. This gives a mapping of $B_r(\bar{s})$. The vector of best response correspondences is then called the joint best response correspondence and is defined as:

$$B(\bar{s}) = \begin{pmatrix} B_1(\bar{s}_{-1}) \\ B_2(\bar{s}_{-2}) \\ . \\ . \\ . \\ B_R(\bar{s}_{-R}) \end{pmatrix}. \tag{6}$$

$B(\bar{s})$ maps action outcomes to action outcomes. Using this, we can define Nash equilibrium again:

Definition 1. *$\bar{s}^*$ is a NE if and only if $\bar{s}^* \in B(\bar{s}^*)$, i.e each agent plays his best response to other players.*

Using definition 1, finding NE is reduced to the well-known fixed point problem. Two fixed point theorems are usually used in game theory, the Brouwer fixed-point theorem and The Kakutani fixed-point theorem. The first theorem is restricted to functions, while the latter is more general and support set-valued functions too. Below we state the Brouwer theorem.

Theorem 2. *[Brouwer theorem]: Let V be a compact convex set. Then any continuous function $f : V \to V$ has a fixed point.*

Here compact set means in Euclidean space, the set is closed and bounded. One implication of this theorem is that a function mapping set [0,1] to [0,1] has to cross the line $y = x$ somewhere in the interval.

3 Mixed Strategy

Example 3. The table below illustrates an example where both Player 1 and Player 2 can choose between L and R.

		Goalie	
		L	R
Penalty taker	L	(-1, 1)	(1, -1)
	R	(1, -1)	(-1, 1)

In this example, no pure strategy NE exists. Therefore, we introduce the idea of mixed strategy.

Definition 4. *A mixed strategy for player r with strategy set S_r is a probability distribution σ_r on S_r.*

3.1 Game model

- Let $\Delta(S_r)$ be a set of probability distributions over S_r.
- Agents in a game $G = (R, \{S_r\}_{r\in R}, \{\Pi_r\}_{r\in R})$ that play mixed strategies can be viewed as playing pure strategies in the game $G' = (R, \{\Delta(S_r)\}_{r\in R}, \{\tilde{\Pi}_r\}_{r\in R})$.
- Payoffs $\tilde{\Pi}_r$ can be interpreted as expectations over the joint distribution on strategies. The payoff of each player r is given by

$$\begin{aligned}\tilde{\Pi}_r(\sigma_r, \bar{\sigma}_{-r}) &= \sum_{\bar{s}\in S} \Pi_r(\bar{s}) \prod_{q\in R} \sigma_q(s_q) \qquad \text{(for discrete strategy cases)}\\ &= \int_{\bar{s}} \Pi_r(\bar{s})\, \mathrm{d}\bar{\sigma}(\bar{s}) \qquad \text{(for continuous strategy cases)}\end{aligned}$$

Definition 5. *Suppose S_r is finite, $\bar{\sigma}$ is a mixed strategy Nash Equilibrium if*

$$\tilde{\Pi}_r(\sigma_r, \bar{\sigma}_{-r}) \geq \tilde{\Pi}_r(\sigma'_r, \bar{\sigma}_{-r}),\ \forall \sigma'_r \in \Delta(S_r),\ \forall r \in R,$$

i.e., σ_r assigns positive probability only to pure strategies that are best response to $\bar{\sigma}_{-r}$.

Existence of mixed strategy Nash Equilibrium is guaranteed by the following theorem.

Theorem 6. *Suppose that $G = (R, \{S_r\}_{r\in R}, \{\Pi_r\}_{r\in R})$ is a finite game, where R and $\{S_r\}_{r\in R}$ are finite. Then there exists a mixed Nash Equilibrium.*

We note that pure strategy NE is a special case of mixed strategy NE where all probability is assigned to one outcome.

3.2 Examples

Consider the following example where Player 1 can choose among T, M and B, and Player 2 can choose between L and R.

Example 7.

		Player 2		
		T	M	B
Player 1	L	(-1, 2)	(1, -1)	(2, -2)
	R	(2, -1)	(-1, 2)	(1, -1)

Clearly, strategy B is a dominated strategy for Player 2. So we can eliminate strategy B from the strategy set S_2. Suppose that Player 1 chooses L (resp. R) with probability p (resp. $1-p$), and Player 2 chooses T (resp. M) with probability q (resp. $1-q$). Note that in a mixed NE, player r must assign positive probability only to strategies that are a best response to $\bar{\sigma}_{-r}$, thus each strategy would yield the same payoff. It follows that a necessary and sufficient condition for a mixed NE to exist is

$$\tilde{\Pi}_1(L) = -1 \times q + 1 \times (1-q) = 2 \times q + (-1) \times (1-q) = \tilde{\Pi}_1(R),$$

$$\tilde{\Pi}_2(T) = 2 \times p + (-1) \times (1-p) = (-1) \times p + 2 \times (1-p) = \tilde{\Pi}_2(M).$$

Solving for p, q gives $p = \frac{1}{2},\ q = \frac{2}{5}$.

References

[1] Berry, Randall A., and Ramesh Johari., "Economic modeling in networking: A primer.," *Foundations and Trends in Networking*, 6.3 (2013): 165-286.

[2] Rosen, J. (1965). Existence and Uniqueness of Equilibrium Points for Concave N-Person Games. Econometrica, 33(3), 520-534.