Chapter 3 Fourier Series Representation of Periodic Signals

§3.1 The Response of LTI Systems to Complex Exponentials

1. Continuous-time LTI Systems

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st} \qquad H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

$H(s)$ — The system's eigenvalue

e^{st} — The system's eigenfunction

2. Discrete-time LTI Systems

$$z^{n} \longrightarrow \boxed{h[n]} \longrightarrow H(z)z^{n} \qquad H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

$H(z)$— The system's eigenvalue

z^{n} — The system's eigenfunction

3. Analysis and Synthesis

$$x(t) = \sum_{k} a_k e^{s_k t} \qquad e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$$

$$y(t) = \sum_{k} a_k H(s_k) e^{s_k t}$$

Continuous-time Fourier series analysis considers $s_k = jk\omega_0$ that is purely imaginary.

$$x[n] = \sum_{k} a_k z_k^{n} \qquad z^{n} \longrightarrow \boxed{h[n]} \longrightarrow H(z)z^{n}$$

$$y[n] = \sum_{k} a_k H(z_k) z_k^{n}$$

Discrete-time Fourier series analysis considers $z_k = e^{jk\omega_0}$ that has unit magnitude.

§3.2 Fourier Series Representation of Continuous-Time Periodic Signals

1. Linear Combinations of Harmonically Related Complex Exponentials

The set of harmonically related complex exponentials:

$$\emptyset_k(t) = e^{jk\omega_0 t} = e^{jk(\frac{2\pi}{T})t} \quad (k \in Z\,)$$

The fundamental frequency of $\emptyset_k(t)$ is a multiple of ω_0.

The fundamental period of $\emptyset_k(t)$ is a fraction of T.

$x(t)$ is a period signal, if $x(t) = x(t+T)$ $(\forall t \in R)$.

$\min\{T|T>0\} = T_0$ determines the fundamental period.

$\omega_0 = \frac{2\pi}{T}$ is called the fundamental frequency.

Fourier Series: a linear combination of harmonically related complex exponentials have the common period T, which is written as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, \emptyset_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

a_k is called the **coefficients of Fourier series**.

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \overset{k=-m}{\Longleftrightarrow} \sum_{m=-\infty}^{\infty} a_{-m}^* e^{jm\omega_0 t}$$

$$\overset{m=k}{\Longleftrightarrow} \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

Conjugate Symmetry
When $x(t) = x^*(t)$, $a_k = a_{-k}^*$, or alternatively $a_k^* = a_{-k}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty}\left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right)$$

$$= a_0 + \sum_{k=1}^{\infty}\left(a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t}\right) = a_0 + \sum_{k=1}^{\infty} 2Re\left\{a_k e^{jk\omega_0 t}\right\}$$

a. Let $a_k = A_k e^{j\theta_k}$

$$x(t) = a_0 + 2\sum_{1}^{\infty} A_k cos(k\omega_0 t + \theta_k)$$

b. Let $a_k = B_k + jC_k$

$$x(t) = a_0 + 2\sum_{1}^{\infty} [B_k cos(k\omega_0 t) - C_k sin(k\omega_0 t)]$$

2. Determination of the Fourier Series Representation of a Continuous-time Period Signal

$$\boxed{\int_{\tau}^{\tau+T} e^{j(k-n)\omega_0 t} dt = T\delta[k-n] = \begin{cases} T & (k=n) \\ 0 & (k \neq n) \end{cases}}$$

Let $x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$.

$$\int_{\tau}^{\tau+T} x(t)e^{-jn\omega_0 t} dt = \int_{\tau}^{\tau+T} \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{\tau}^{\tau+T} e^{j(k-n)\omega_0 t} dt = Ta_n$$

$$\therefore a_n = \frac{1}{T}\int_{\tau}^{\tau+T} x(t)e^{-jn\omega_0 t} dt$$

$$\begin{cases} x(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \\ a_k = \displaystyle\frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_0 t} dt \end{cases}$$

a_k is also called the **spectral coefficients** of $x(t)$.

Example 3.1 Determine the spectral coefficients of
$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos 2\omega_0 t$.

Euler's Relation: $\begin{cases} \cos\omega_0 t = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) \\ \sin\omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) \end{cases}$

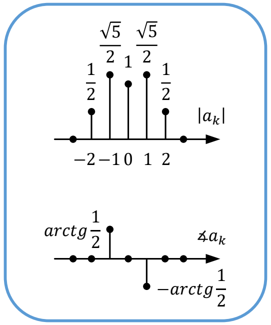
Solution:

$$\sin\omega_0 t + 2\cos\omega_0 t = \left(1 - j\frac{1}{2}\right)e^{j\omega_0 t} + \left(1 + j\frac{1}{2}\right)e^{-j\omega_0 t}$$

$$\cos 2\omega_0 t = \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t}$$

$$x(t) = \sum_{k=-2}^{2} a_k e^{jk\omega_0 t}$$

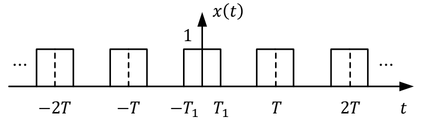
$$\begin{cases} a_0 = 1 \\ a_1 = a_{-1}^* = 1 - j\frac{1}{2} \\ a_2 = a_{-2}^* = \frac{1}{2} \\ a_k = 0 \ \ (|k| > 2) \end{cases}$$



Example 3.2 The periodic square wave sketched as follows is defined over one period as

$$x(t) = \begin{cases} 1 \ \ (|t| < T_1) \\ 0 \ \ (T_1 < |t| < T/2) \end{cases}$$

Determine the Fourier series coefficients for $x(t)$.



Solution:

$$a_0 = \frac{1}{T}\int_{<T>} x(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

$$= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = \frac{1}{T}\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{jk\omega_0}$$

$$= \frac{2j\,\sin k\omega_0 T_1}{jk\omega_0 T} = \frac{\sin k\omega_0 T_1}{\pi k} \ \ (k \neq 0)$$

When $T_1 = \frac{1}{4}T$,

$$\begin{cases} a_0 = \frac{1}{2} \\ a_k = \frac{\sin(\frac{k\pi}{2})}{\pi k} \ (k \neq 0) \end{cases}$$