Example 4.12 Compute $\cos(t) * e^{-t}u(t)$.

Solution:

$$F[\cos(t)] = \pi[\delta(\omega-1)+\delta(\omega+1)]$$

$$F[e^{-t}u(t)] = \frac{1}{1+j\omega}$$

$$F[\cos(t)]F[e^{-t}u(t)] = \pi\left[\frac{\delta(\omega-1)}{1+j}+\frac{\delta(\omega+1)}{1-j}\right]$$

$$= \frac{\pi}{2}[\delta(\omega-1)-j\delta(\omega-1)+\delta(\omega+1)+j\delta(\omega+1)]$$

$$\cos(t) * e^{-t}u(t) = \frac{1}{2}[\cos(t)+\sin(t)] = \frac{\sqrt{2}}{2}\cos\left(t-\frac{\pi}{4}\right)$$

Example 4.12 Compute $\cos(t) * e^{-t}u(t)$.

Solution:

$$h(t) = e^{-t}u(t)$$

$$H(j\omega) = \frac{1}{1+j\omega}$$

$$\omega_0 = 1$$

$$|H(j\omega_0)| = \frac{\sqrt{2}}{2}$$

$$\measuredangle H(j\omega_0) = -\frac{\pi}{4}$$

$$\cos(t) * e^{-t}u(t) = \frac{\sqrt{2}}{2}\cos\left(t-\frac{\pi}{4}\right)$$

Example 4.13 Consider the response of an LTI system with impulse response $h(t) = e^{-at}u(t)$ $(a \in R^+)$ to the input signal $x(t) = e^{-bt}u(t)$ $(b \in R^+)$. Determine the output of this system $y(t)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Solution:

$$h(t) = e^{-at}u(t) \overset{F}{\longleftrightarrow} H(j\omega) = \frac{1}{a+j\omega}$$

$$x(t) = e^{-bt}u(t) \overset{F}{\longleftrightarrow} X(j\omega) = \frac{1}{b+j\omega}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)}$$

① For $a \neq b$

$$Y(j\omega) = \frac{1}{b-a}\left(\frac{1}{a+j\omega}-\frac{1}{b+j\omega}\right)$$

$$y(t) = \frac{1}{b-a}[e^{-at}u(t)-e^{-bt}u(t)]$$

② For $a = b$

$$Y(j\omega) = \frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

$$-jte^{-at}u(t) \overset{F}{\longleftrightarrow} \frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right)$$

$$y(t) = \mathcal{F}^{-1}\left[\frac{1}{(a+j\omega)^2}\right] = te^{-at}u(t)$$

Partial-Fraction Expansion

$$H(v)=\frac{\beta_m v^m+\beta_{m-1}v^{m-1}+\cdots+\beta_1 v+\beta_0}{\alpha_n v^n+\alpha_{n-1}v^{n-1}+\cdots+\alpha_1 v+\alpha_0}$$

For $m<n$, $H(v)$ is called a strictly proper rational function

For $m\geq n$,

$$H(v)=c_{m-n}v^{m-n}+c_{m-n-1}v^{m-n-1}+\cdots+c_1 v+c_0$$
$$+\frac{b_{n-1}v^{n-1}+b_{n-2}v^{n-2}+\cdots+b_1 v+b_0}{v^n+a_{n-1}v^{n-1}+\cdots+a_1 v+a_0}$$

Standard form:

$$H(v)=\frac{b_{n-1}v^{n-1}+b_{n-2}v^{n-2}+\cdots+b_1 v+b_0}{v^n+a_{n-1}v^{n-1}+\cdots+a_1 v+a_0}=\frac{N(v)}{D(v)}$$

$\{p_i\}_{i=1,\cdots,n}$ are roots of the denumerator polynomial $D(v)$

① $\{p_i\}_{i=1,\cdots,n}$ are distinct

$$G(v)=N(v)\prod_{i=1}^{n}\frac{1}{v-p_{\mathrm{i}}}=\sum_{i=1}^{n}\frac{A_i}{v-p_{\mathrm{i}}}$$

$$A_i=G(v)(v-p_i)\Big|_{v=p_i}$$

② $p_1=p_2=\cdots=p_r$ and $\{p_i\}_{i=r+1,\cdots,n}$ are distinct

$$G(v)=\frac{N(v)}{(v-p_1)^r}\prod_{i=r+1}^{n}\frac{1}{v-p_{\mathrm{i}}}=\sum_{i=1}^{r}\frac{A_i}{(v-p_1)^i}+\sum_{i=r+1}^{n}\frac{A_i}{v-p_i}$$

$$A_i=\frac{1}{(r-1)!}\frac{d^{r-1}}{dv^{r-1}}[G(v)(v-p_1)^r]\Bigg|_{v=p_1}\quad(i<r)$$

$$A_r=G(v)(v-p_1)^r\Big|_{v=p_1}\quad A_i=G(v)(v-p_i)\Big|_{v=p_i}\quad(i>r)$$

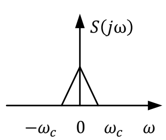
§4.5 The Multiplication Property

$$x(t)\overset{F}{\longleftrightarrow}X(j\omega)\text{ and }y(t)\overset{F}{\longleftrightarrow}Y(j\omega)$$

$$x(t)y(t)\overset{F}{\longleftrightarrow}\frac{1}{2\pi}X(j\omega)*Y(j\omega)$$

(Hint: Duality Property)

Example 4.14 Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is shown here. Sketch the spectrum of $r(t)=s(t)\cos\omega_0 t$.

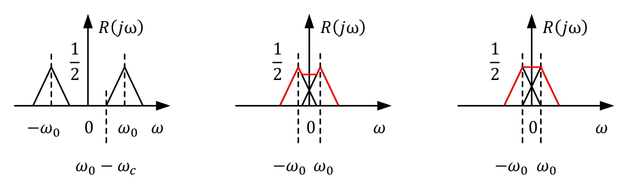


Solution:

$p(t)=\cos\omega_0 t$

$P(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

$$R(j\omega)=\frac{1}{2\pi}S(j\omega)*P(j\omega)$$
$$=\frac{1}{2}S[j(\omega-\omega_0)]+\frac{1}{2}S[j(\omega+\omega_0)]$$



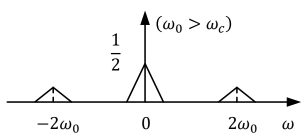
Example 4.15 Consider $r(t) = s(t)\cos\omega_0 t$ as determined in Example 4.14 and let $g(t) = r(t)\cos\omega_0 t$. Determine and sketch the spectrum $G(j\omega)$.

Solution:

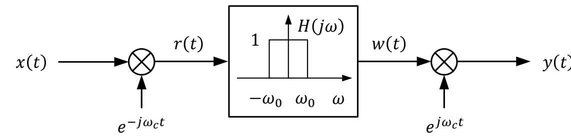
$$R(j\omega) = \frac{1}{2}S[j(\omega - \omega_0)] + \frac{1}{2}S[j(\omega + \omega_0)]$$

$$G(j\omega) = \frac{1}{2}R[j(\omega - \omega_0)] + \frac{1}{2}R[j(\omega + \omega_0)]$$

$$G(j\omega) = \frac{1}{4}S[j(\omega - 2\omega_0)] + \frac{1}{4}S[j(\omega + 2\omega_0)] + \frac{1}{2}S(j\omega)$$



Example 4.16 Analyze the function of the given system.



Solution:

$$R(j\omega) = X(j\omega) * \delta(\omega + \omega_c) = X[j(\omega + \omega_c)]$$

$$W(j\omega) = H(j\omega)X[j(\omega + \omega_c)]$$

$$Y(j\omega) = W(j\omega) * \delta(\omega - \omega_c) = H[j(\omega - \omega_c)]X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = H[j(\omega - \omega_c)]$$

This system is a band-pass filter, of which the center frequency is adjusted by ω_c.

§4.6 Systems Characterized by Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Applying Fourier Transform to both sides

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$H(j\omega) = \sum_{k=0}^{N} b_k (j\omega)^k \Big/ \sum_{k=0}^{M} a_k (j\omega)^k$$

$$h(t) = F^{-1}[\,H(j\omega)]$$

Example 4.17 Determine the unit impulse response $h(t)$ of the LTI system of a stable LTI system characterized by the differential equation

$$\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) + 3y(t) = \frac{d}{dt}x(t) + 2x(t)$$

Solution:

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}$$

$$= \frac{1/2}{j\omega + 1} + \frac{1/2}{j\omega + 3}$$

$$h(t) = F^{-1}[\,H(j\omega)] = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Example 4.18 When the input to the system in Example 4.17 is $x(t) = e^{-t}u(t)$, determine the output $y(t)$.

$$x(t) \longrightarrow \boxed{h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)} \longrightarrow y(t)$$

Solution:

$$X(j\omega) = \frac{1}{j\omega + 1}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}$$

$$= \frac{1/4}{j\omega + 1} + \frac{1/2}{(j\omega + 1)^2} - \frac{1/4}{j\omega + 3}$$

$$y(t) = \frac{1}{2}te^{-t}u(t) + \frac{1}{4}e^{-t}u(t) - \frac{1}{4}e^{-3t}u(t)$$