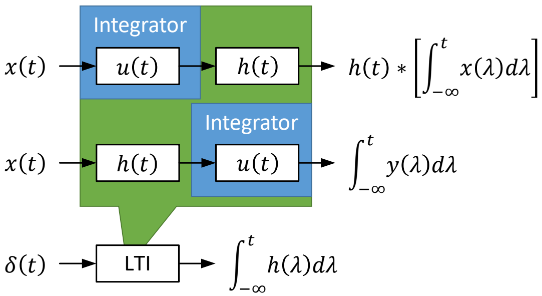
④ The Integral Property

Given $y(t) = h(t) * x(t)$,

$$\int_{-\infty}^{t} y(\lambda)d\lambda = \left[\int_{-\infty}^{t} h(\lambda)d\lambda\right] * x(t) = h(t) * \left[\int_{-\infty}^{t} x(\lambda)d\lambda\right]$$



⑤ The Differentiable Property

Given $y(t) = h(t) * x(t)$,

$$\frac{d}{dt}y(t) = \left[\frac{d}{dt}h(t)\right] * x(t) = h(t) * \left[\frac{d}{dt}x(t)\right]$$

Proof:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$$

$$\frac{d}{dt}y(t) = \int_{-\infty}^{+\infty} x(\tau)\frac{d}{dt}h(t-\tau)\,d\tau = \left[\frac{d}{dt}h(t)\right] * x(t)$$

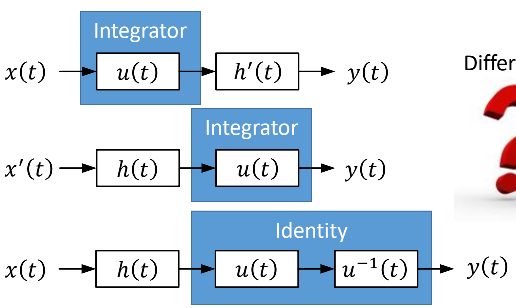
$$y(t) = x\,(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$$

$$\frac{d}{dt}y(t) = \int_{-\infty}^{+\infty} h(\tau)\frac{d}{dt}x(t-\tau)\,d\tau = h(t) * \left[\frac{d}{dt}x(t)\right]$$

⑥ The Differentiable and Integral Property

Given $y(t) = h(t) * x(t)$,

$$y(t) = \left[\frac{d}{dt}h(t)\right] * \left[\int_{-\infty}^{t} x(\lambda)d\lambda\right] = \left[\int_{-\infty}^{t} h(\lambda)d\lambda\right] * \left[\frac{d}{dt}x(t)\right]$$



Differentiator

4. The Convolution Integral of $\delta(t)$

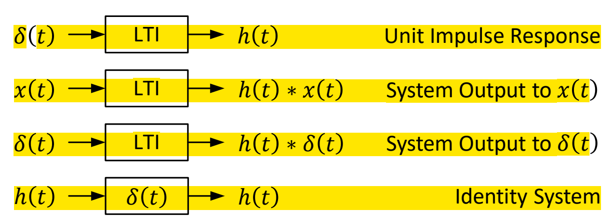
① $x(t) * \delta(t) = x(t)$

② $x(t) * \delta(t - t_0) = x(t - t_0)$

③ $x(t - t_1) * \delta(t - t_2) = x(t - t_1 - t_2)$

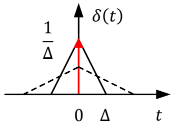
④ When $x(t) = x_1(t) * x_2(t)$,

$$x(t - t_1 - t_2) = x_1(t - t_1) * x_2(t - t_2)$$



5. Unit Doublets and Other Singularity Functions

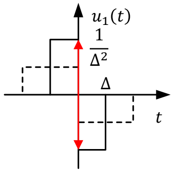
① Definition:



$$u_1(t) \triangleq \frac{d}{dt}\delta(t)$$

$$u_2(t) \triangleq \frac{d^2}{dt^2}\delta(t) = \frac{d}{dt}u_1(t)$$

$$u_k(t) \triangleq \frac{d^k}{dt^k}\delta(t) = \frac{d}{dt}u_{k-1}(t)$$



Consider:

$$\frac{d}{dt}x(t) = x(t) * \left[\frac{d}{dt}\delta(t)\right] = x(t) * u_1(t)$$

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left[\frac{d}{dt}x(t)\right] = \left[\frac{d}{dt}x(t)\right] * \left[\frac{d}{dt}\delta(t)\right]$$

$$\frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left[\frac{d}{dt}x(t)\right] = \left[\frac{d}{dt}x(t)\right] * \left[\frac{d}{dt}\delta(t)\right]$$
$$= x(t) * \left[\frac{d}{dt}\delta(t)\right] * \left[\frac{d}{dt}\delta(t)\right]$$
$$= x(t) * u_1(t) * u_1(t)$$

$$\frac{d^2}{dt^2}x(t) = x(t) * \left[\frac{d^2}{dt^2}\delta(t)\right] = x(t) * u_2(t)$$

$$u_2(t) = u_1(t) * u_1(t) \qquad x(t) \longrightarrow \boxed{u_1(t)} \longrightarrow \frac{d}{dt}x(t)$$

$$u_k(t) = u_{k-1}(t) * u_1(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{k}$$

② Properties of $u_1(t)$

a. $\int_{-\infty}^{+\infty} u_1(t)dt = 0$

Proof:

$\frac{d}{dt}x(t) = x(t) * u_1(t) = \int_{-\infty}^{+\infty} u_1(\tau)x(t-\tau)d\tau$

Let $x(t) = 1$, $\int_{-\infty}^{+\infty} u_1(\tau)d\tau = \frac{d}{dt}(1) = 0$

b. $\int_{-\infty}^{+\infty} g(t)u_1(t)dt = -g'(0)$

Proof:

$g(-t) * u_1(t) = \frac{d}{d(-t)}g(-t) = -g'(-t)$

$g(-t) * u_1(t) = \int_{-\infty}^{+\infty} u_1(\tau)g(\tau-t)d\tau$

Let $t = 0$, $\int_{-\infty}^{+\infty} g(\tau)u_1(\tau)d\tau = -g'(0)$

c. $f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$

Proof:

$\frac{d}{dt}[f(t)\delta(t)] = f(t)\frac{d}{dt}\delta(t) + \delta(t)\frac{d}{dt}f(t)$

$\frac{d}{dt}[f(t)\delta(t)] = \frac{d}{dt}[f(0)\delta(t)] = f(0)u_1(t)$

$\delta(t)\frac{d}{dt}f(t) = f'(0)\delta(t)$

d. $u_1(-t) = -u_1(t) \quad u_2(-t) = u_2(t)$

$u_3(-t) = -u_3(t) \quad u_k(-t) = (-1)^k u_k(t)$

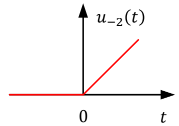
Proof:

$u_1(-t) = \frac{d}{d(-t)}\delta(-t) = \frac{d}{d(-t)}\delta(t) = -\frac{d}{dt}\delta(t) = -u_1(t)$

$u_2(-t) = \frac{d}{d(-t)}u_1(-t) = -\frac{d}{d(-t)}u_1(t) = \frac{d}{dt}u_1(t) = u_2(t)$

③ The Unit Ramp Function

$u_{-2}(t) = tu(t)$



Consider:

$u_{-1}(t) = u(t)$

$$x(t) \longrightarrow \boxed{u_{-1}(t)} \longrightarrow \int_{-\infty}^{t} u(\tau)d\tau$$

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)d\tau = \left(\int_{0}^{t} d\tau\right) u(t) = tu(t)$$

$$u_{-3}(t) = u_{-2}(t) * u(t) = u(t) * u(t) * u(t)$$

$$u_{-k}(t) = u_{-k+1}(t) * u(t) = \underbrace{u(t) * u(t) * \cdots * u(t)}_{k}$$

$$u_0(t) = u_1(t) * u_{-1}(t) = \delta(t)$$

$$u_{k+r}(t) = u_k(t) * u_r(t)$$

Discussion 2.3:

Show by induction that

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!} u(t) \quad (k \in N)$$

Example 2.4: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$ where $x(t) = e^{-at}u(t)$ and $h(t) = u(t)$. Suppose that the initial-state of this system is zero, determine the output $y(t)$.

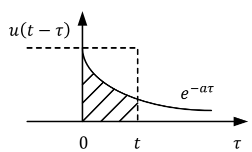
Solution:

$$y(t) = y_f(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

$$= \int_{-\infty}^{t} e^{-a\tau}u(\tau)\, d\tau$$

$$= \left[\int_{0}^{t} e^{-a\tau}\, d\tau\right] u(t) = \frac{1}{a}(1 - e^{-at})u(t)$$



Example 2.5: Let $x_1(t) = u(t+0.5) - u(t-0.5)$ and $x_2(t) = e^{-t}u(t)$. Compute $x(t) = x_1(t) * x_2(t)$.

Solution:

$$\hat{x}(t) = u(t) * x_2(t) = u(t) * e^{-t}u(t)$$

$$= \int_{-\infty}^{t} e^{-\tau}u(\tau)\, d\tau = (1 - e^{-t})u(t)$$

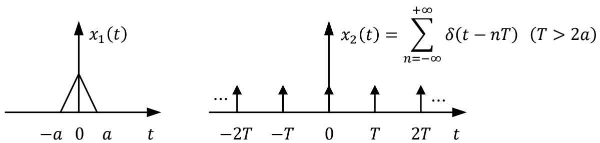
$$x(t) = x_1(t) * x_2(t) = [u(t+0.5) - u(t-0.5)] * x_2(t)$$

$$= u(t+0.5) * x_2(t) - u(t-0.5) * x_2(t)$$

$$= \hat{x}(t+0.5) - \hat{x}(t-0.5)$$

$$= \left[1 - e^{-(t+0.5)}\right]u(t+0.5) - \left[1 - e^{-(t-0.5)}\right]u(t-0.5)$$

Example 2.6: Let $x_1(t)$ be the triangular pulse and $x_2(t)$ be the impulse train as depicted below. Determine and sketch $x(t) = x_1(t) * x_2(t)$.



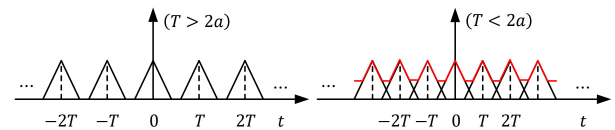
Solution:

$$x(t) = x_1(t) * x_2(t) = x_1(t) * \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

$$x(t) = x_1(t) * \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{n=-\infty}^{+\infty} [x_1(t) * \delta(t - nT)]$$

$$= \sum_{n=-\infty}^{+\infty} [x_1(t - nT)]$$

Periodic Copies of $x_1(t)$:



Example 2.7: Let $x_1(t) = \sin(t)u(t)$ and $x_2(t) = \delta'(t) + u(t)$. Compute $x(t) = x_1(t) * x_2(t)$.

Solution:

$$x(t) = x_1(t) * x_2(t) = x_1(t) * \delta'(t) + x_1(t) * u(t)$$

$$= \frac{d}{dt}[x_1(t) * \delta(t)] + x_1(t) * u(t)$$

$$= \frac{d}{dt}x_1(t) + x_1(t) * u(t)$$

$$\frac{d}{dt}x_1(t) = \cos(t)\, u(t) + \sin(t)\, \delta(t) = \cos(t)\, u(t)$$

$$x_1(t) * u(t) = \int_{-\infty}^{t} \sin(\tau)u(\tau)d\tau = [1 - \cos(t)]u(t)$$

$$x(t) = \cos(t)\, u(t) + [1 - \cos(t)]u(t) = u(t)$$

Example 2.8: Consider an LTI system whose zero-state response is $y_f(\mathrm{t}) = sin\omega_0 t$, when the input is $x(t) = e^{-at}u(t)$. Determine the unit impulse response of this system $h(t)$.

Solution:

$$y_f(t) = h(t) * x(t)$$

$$\frac{d}{dt}x(t) = e^{-at}\delta(t) - ae^{-at}u(t) = \delta(t) - ax(t)$$

$$\frac{d}{dt}y_f(t) = h(t) * \frac{d}{dt}x(t)$$

$$= h(t) * \ \delta(t) - h(t) * [ax(t)] = h(t) - ay_f(t)$$

$$h(t) = ay_f(t) + \frac{d}{dt}y_f(t) = a\sin\omega_0 t + \omega_0 \cos\omega_0 t$$