ECE495 Game Theory and Networked Systems

Lecture 6: Second Price Auction and Wardrop Equilibrium

1 Second Price Auction

- An auction with a single object and n players $\{1, 2, ...n\}$;
- Each player values the object v_i, we order the players such that $v_1 \geq v_2 \geq \cdots \geq v_n \geq 0$ without loss of generality;
- Players simultaneously submit bids $b_i \in [0, \infty)$;
- Player with the highest bid wins the object, and pays the value of the second highest bid. The rest of the players pay nothing and get nothing;
- In the event of a tie, the item goes to one of the player randomly (if players 2 and 3 tie, the object goes to each with probability $\frac{1}{2}$);

1.1 The Game Model

- players $R = \{1, 2, \ldots, n\}$;
- strategy set: $[0, \infty)$;
- payoff:

$$\Pi_r(s_r, \bar{s}_{-r}) = \begin{cases} v_r - b_j & \text{if } r \text{ wins} \\ 0 & \text{if } r \text{ loses} \end{cases}$$

  where b_j is the second highest bid.

1.2 Nash Equilibria

1. Truthful bidding: $b_i = v_i, \ \forall i \in R$, where every player bids their own value. Then player 1 gets payoff $\Pi_1 = v_1 - v_2$, while the rest of the players get payoff of 0. As no players can deviate and improve the payoff, this is a NE.
2. $b_1 = v_1$, and $b_i = 0, \text{ for } i = 2, \ldots, n$. Then player 1 gets payoff $\Pi_1 = v_1$, while the rest of the players get payoff of 0. This is also a NE.

In this game, there are multiple Nash Equilibria, but all ended up giving the good to player 1. The outcome of all players bid 0 is not a NE, since any player may deviate to a positive bid to get the item while still paying 0.

We note that the bid $b_i = v_i$ is a weakly dominating strategy (no worse than any other strategy no matter what the opponents do) for every user i. We can see this by analyzing three different bidding strategies

1. $b_i = v_i$ (blue)
2. $b_i < v_i$ (red)
3. $b_1 > v_i$ (green)

in the following figure, where the horizontal axis is the highest bid other than i.

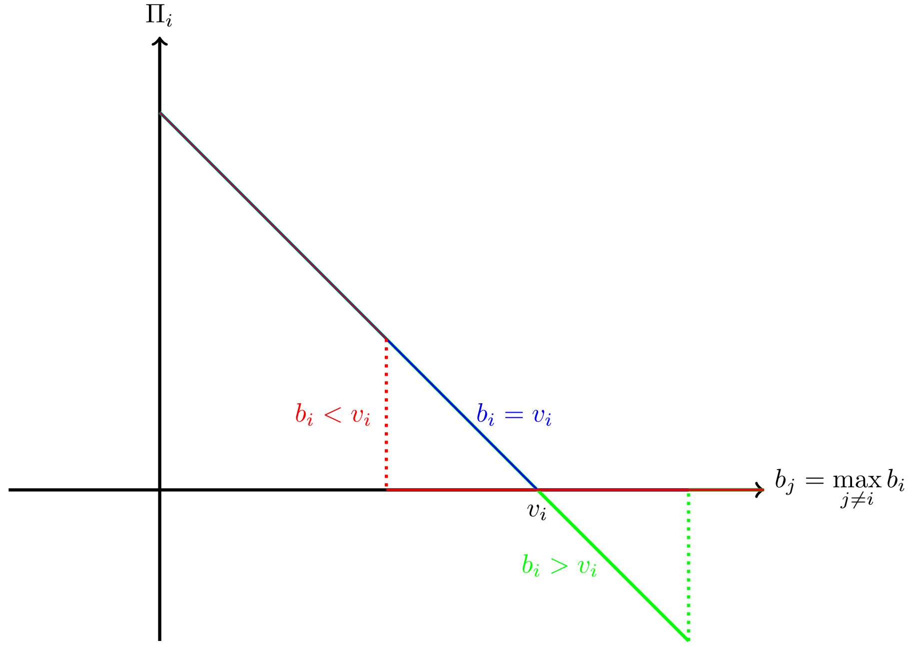

Figure 1: Payoff versus others' highest bidding

2 Weakly Dominant Strategy

Definition 1. *Weakly dominant strategy: Strategy s_r^* is weakly dominant if*

$$\Pi_r(s_r^*, \bar{s}_{-r}) \geq \Pi_r(s_r, \bar{s}_{-r}),\ \forall s_r \in S_r,\ \bar{s}_{-r} \in \bar{S}_{-r}$$

2.1 Examples

Example 2. The table below illustrates an example where both Player 1 and Player 2 can choose B vs C.

		Player 2	
		B	C
Player 1	B	(1, 1)	(0, 0)
	C	(0, 0)	(0, 0)

In this example, both (B, B) and (C, C) are Nash Equilibria. But only (B, B) is a weakly dominant strategy equilibrium. This example illustrates that we cannot eliminate weakly dominated strategies (C) when searching for the set of all Nash Equilibria, and even weakly dominated strategy can be part of a NE.

3 Drawbacks and Criticisms of Mixed Strategies

A mixed strategy equilibrium implies that players use randomization as a fundamental part of their strategy. However, when modeling economic agents such as people or firms, this randomization causes problems. For example behavioral studies have shown that people rarely randomize. The game rock-paper-scissors is a prime example of this. People tend to choose rock, paper or scissors not randomly but based on their previous moves/habits. Moreover if agents meet and play a game once (a one-shot game), a mixed strategy predicts multiple outcomes, but there's only one outcome. Finally, in a mixed strategy equilibrium, computability of the outcomes is an issue because how one plays depends on how the opponent plays (strategy/probability of opponents).

4 Justification of Mixed Strategies

One justification is to consider a setting with a large number of potential agents who may play the game. Each agent has a pure strategy; the randomization reflects the distribution of strategies across the population. Thus mixed strategies makes sense in a one-shot game with a lot of players, especially with engineering protocols. An example of this is N competitors transmitting a signal, considered in the previous lecture. Another justification is known as a purification approach (see here for more detail), which is to view any randomization as a modeling tool that arises due to small imperfections in our model of the "true game."

5 Infinite Number of Players and the Wardrop Equilibrium

We will start with an example to demonstrate a game with an infinite number of players and how the Nash equilibrium could be generalized in this case.

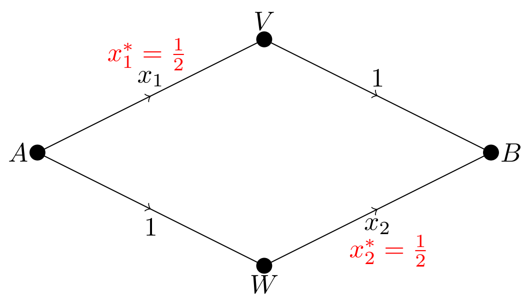

Figure 2: A network modeling a transportation setting for a game with infinite players.

Consider the network in Figure 2. There are an infinite number of players, $P_r \in [0, 1]$, all trying to go from point A to point B. Each player can either choose route V or W. Equilibrium in this game is reached if:

- All used routes have cost less or equal to the cost of other unused routes, experienced by one user.
- All used routes have the same cost.

We call this, **Wardrop Equilibrium**.
The Wardrop Equilibrium for this game is realized when $x_1^* = x_2^* = 1/2$, thus total cost is

$$\frac{1}{2}(1 + \frac{1}{2}) + \frac{1}{2}(1 + \frac{1}{2}) = 3/2.$$

5.1 Wardrop Equilibrium

We give a description of the Wardrop Equilibrium. Wardrop articulated principles that have been widely used to simplify the mathematics associated with routing in traffic models: *Under equilibrium conditions, traffic arranges itself in congested networks in such a way that no individual trip maker can reduce his path costs by switching routes.* [3]

In some cases the socially-optimal pattern of trip-making differs from the pattern that occurs when each individual driver acts according to his or her own self-interest.

5.2 Braess' Paradox

Consider Figure 3, and note that there's an additional route (V,W) added to the previous example with 0 cost. For a Wardrop equilibrium to be established, it must be that $x_1 = x_2 = 1$. Then the total cost is 2. This is a paradox because even though we have an additional road with zero cost, the social welfare is reduced. Hence the addition of an intuitively helpful route negatively impacted the network users.

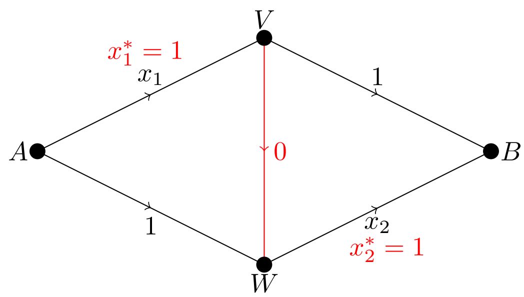

Figure 3: The network in Figure 2 with an added route for demonstrating Braess' Paradox.

6 Price of Anarchy

Now, we introduce a new concept.

Definition 3. *Price of Anarchy (PoA):* $\frac{\textit{Equilibrium total cost}}{\textit{Socially optimal cost}}$.

The Price of Anarchy (PoA) defined by the total equilibrium cost divided by the socially optimal cost quantifies the inefficiency of the equilibrium. We note that PoA is always greater than or equal to 1. If there are multiple equilibrium costs, we choose the equilibrium cost that maximizes the PoA to reflect the worst case scenario.

In the Wardrop Equilibrium example given in the previous lecture, the Price of Anarchy is given by,

$$\text{PoA} = \frac{2}{3/2} = 4/3.$$

Example 6.1 (Peering Game)**.** Recall the peering game (hot potato example) in Lecture 3. For this game the payoffs for each agent can be represented as follows:

		P_1	
		s_1	s_2
P_2	s_1	(-4,-4)	(-1,-5)
	s_2	(-5,-1)	(-2,-2)

Table 1: The payoffs for the peering game example from Lecture 3.

For the Peering Game, the Nash Equilibrium is at $(-4, -4)$. However, both players would be better of, if they picked the point $(-2, -2)$. Therefore, The Price of Anarchy is, PoA $= \frac{8}{4} = 2$. We note that here the goal is to maximize payoff (instead of minimizing cost), so we should multiply by a negative sign to the numerator and denominator. PoA is usually defined for cost minimization.

References

[1] A. Ozdaglar, "6.254 : Game Theory with Engineering Applications Lecture 7: Supermodular Games", MIT, 2010.

[2] R. Berry and R. Johari, "Economic Modeling in Networking: A Primer", in Foundations and Trends in Networking, 6.3, 2013, pp. 165-286.

[3] Ortúzar, J. d. D. and Willumsen, L. G. (2011) Introduction, in Modelling Transport, Fourth Edition, John Wiley & Sons, Ltd, Chichester, UK. doi: 10.1002/9781119993308.ch1