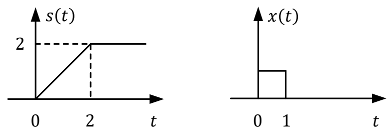
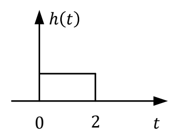
Example 2.9: Let $s(t)$ and $x(t)$ shown as follows be the unit step response and input of an LTI system, respectively. Suppose that the system has zero-initial state. Determine and sketch the output $y(t)$ of the system.

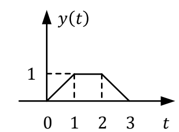


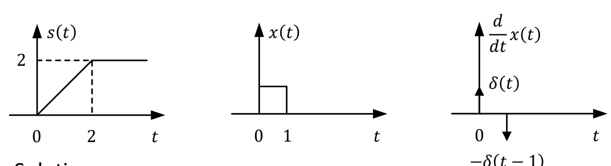
Solution:

$$h(t) = \frac{d}{dt}s(t)$$
$$= u(t) - u(t-2)$$
$$x(t) = u(t) - u(t-1)$$



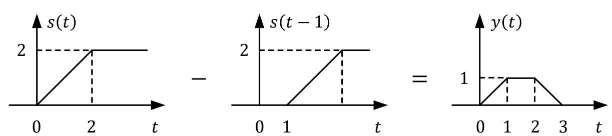
$$y(t) = y_f(t) = [u(t) - u(t-2)] * [u(t) - u(t-1)]$$
$$= u(t) * u(t) - u(t-2) * u(t) - u(t) * u(t-1)$$
$$+ u(t-2) * u(t-1)$$
$$= tu(t) - (t-1)u(t-1) - (t-2)u(t-2)$$
$$+ (t-3)u(t-3)$$
$$= \begin{cases} t < 0: & y(t) = 0 \\ 0 < t < 1: & y(t) = t \\ 1 < t < 2: & y(t) = t - (t-1) = 1 \\ 2 < t < 3: & y(t) = t - (t-1) - (t-2) = 3 - t \\ t > 3: & y(t) = 0 \end{cases}$$





Solution:

$$y(t) = y_f(t) = h(t) * x(t)$$
$$= \left[\frac{d}{dt}s(t)\right] * x(t) = s(t) * \left[\frac{d}{dt}x(t)\right]$$
$$= s(t) * [\delta(t) - \delta(t-1)] = s(t) - s(t-1)$$



§2.3 Properties of Linear Time-Invariant Systems

1. LTI Systems with and without Memory

① The discrete-time LTI system

$$y[n] = kx[n]$$

Memoryless condition:

$$h[n] = k\delta[n]$$

② The continuous-time LTI system

$$y(t) = kx(t)$$

Memoryless condition:

$$h(t) = k\delta(t)$$

2. Invertibility of LTI systems

① The discrete-time LTI system

$$x[n] \rightarrow \boxed{h[n]} \rightarrow \boxed{h^{-1}[n]} \rightarrow x[n]$$

$$x[n] \rightarrow \boxed{\hat{h}[n] = \delta[n]} \rightarrow x[n]$$

$\hat{h}[n] = h[n] * h^{-1}[n] = \delta[n]$

② The continuous-time LTI system

$$x(t) \rightarrow \boxed{h(t)} \rightarrow \boxed{h^{-1}(t)} \rightarrow x(t)$$

$$x(t) \rightarrow \boxed{\hat{h}(t) = \delta(t)} \rightarrow x(t)$$

$\hat{h}(t) = h(t) * h^{-1}(t) = \delta(t)$

3. Causality for LTI systems

① The discrete-time LTI system

$$y[n] = h[n] * x[k] = \sum_{k=-\infty}^{+\infty} x[n-k]h[k]$$

$$= \sum_{k=-\infty}^{-1} x[n-k]h[k] + \sum_{k=0}^{+\infty} x[n-k]h[k]$$

The LTI system is a casual system, when $h[n] = 0\ (n < 0)$.

② The continuous-time LTI system

Similarly, the causal LTI system requires $h(t) = h(t)u(t)$.

$$y(t) = \int_{-\infty}^{0} h(\tau)x(t-\tau)d\tau + \int_{0}^{+\infty} h(\tau)x(t-\tau)d\tau$$

4. Stability for LTI systems

① The discrete-time LTI system

The system is stable if and only if its unit impulse response is absolutely summable, i.e.

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

② The continuous-time LTI system

The system is stable if and only if its unit impulse response is absolutely integable, i.e.

$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty$$

§2.4 Causal LTI Systems described by Differential and Difference Equations

1. Linear Constant-Coefficient Differential Equations

$$y^{(n)}(t) + a_{n-1}y^{(n-1)} + \cdots + a_1 y'(t) + a_0 y(t)$$
$$= b_m x^{(m)}(t) + \cdots + b_1 x'(t) + b_0 x(t)$$

Solution: $y(t) = y_c(t) + y_p(t)$

Homogeneous solution (Natural response): $y_c(t)$

$$y_c{}^{(n)}(t) + a_{n-1}y_c{}^{(n-1)}(t) + \cdots + a_1 y_c{}'(t) + a_0 y_c(t) = 0$$

$$y_c(t) = \sum_{i=1}^{r} c_i\, t^{r-i} e^{\lambda_1 t} + \sum_{i=r+1}^{N} c_i\, e^{\lambda_i t}$$

Particular solution: $y_p(t)$

Example 2.10: Consider a causal LTI system, whose input and output relation is described by a first-order differential equation

$$y'(t) + 2y(t) = x(t)$$

Determine the output signal when the input signal is

$$x(t) = 5e^{3t}u(t)$$

Solution:

$y(t) = y_c(t) + y_p(t)$

$y'(t) + 2y(t) = 0 \implies y_c(t) = Ae^{-2t}$

Let $y_p(t) = Be^{3t}$ for $t > 0$

$3Be^{3t} + 2Be^{3t} = Ke^{3t} \implies B = 1$

$y(t) = Ae^{-2t}+e^{3t}$ for $t > 0$

$y(0^-) = A + 1 = 0 \implies A = -1$

$y(t) = (e^{3t} - e^{-2t})u(t)$

2. Linear Constant Coefficient Difference Equations

$$a_0y[n] + a_1y[n-1] + \cdots + a_Ny[n-N] = b_0x[n] + b_1x[n-1] + \cdots + b_mx[n-M]$$

Solution: $y[n] = y_c[n] + y_p[n]$

Homogeneous solution (Natural response): $y_c[n]$

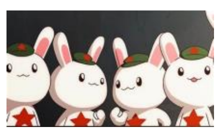
$a_0y[n] + a_1y[n-1] + \cdots + a_Ny[n-N] = 0$

$\{r_i\}$ are the roots of $a_0r^N + a_1r^{N-1} + \cdots + a_N = 0$

$$y_c[n] = \sum_{i=1}^{s} c_i n^{s-i} r_1^n + \sum_{i=s+1}^{N} c_j r_j^n$$

Particular solution: $y_p[n]$

Example 2.11: Derive the generalized equation of the famous Fibonacci sequence.



$f[n] = 0,1,1,2,3,5,8,13,21,34,55,89,\ldots$

Solution:

$f[n] = f[n-1] + f[n-2]$

$y[n] - y[n-1] - y[n-2] = x[n]$

Let $x[n] = 0$, the output will be the Fibonacci sequence

$y[n] = y_c[n] + y_p[n]$ and $y_p[n] = 0$ in this case

$y_c[n] = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$y[0] = 0, y[1] = 1 \implies A = -B = \frac{1}{\sqrt{5}}$

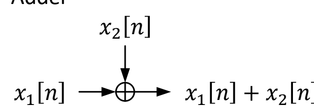
$y[n] = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ for $n > 0$

the Golden Ratio

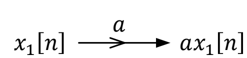
3. Block Diagram Representations of LTI Systems Described by Differential and Difference Equations

① The Discrete-Time LTI System

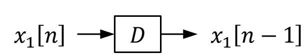
Adder



Multiplication by a coefficient

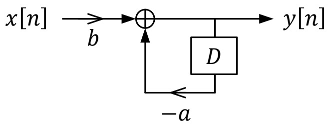


Unit delay



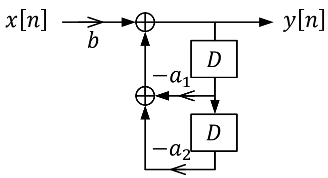
First-Order System:

$y[n] + ay[n-1] = bx[n]$



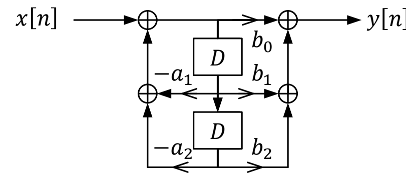
Second-Order System

$y[n] + a_1 y[n-1] + a_2 y[n-2] = bx[n]$



Second-Order System

$$y[n] + a_1 y[n-1] + a_2 y[n-2]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

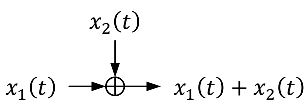


Arbitrary-Order System

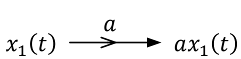
$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_m x[n-M]$$

②The Continuous-Time LTI System

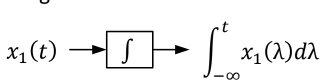
Adder



Multiplication by a coefficient

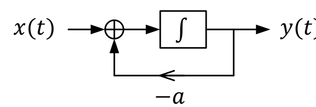


Integrator



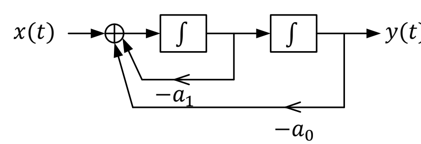
First-Order System:

$$\frac{d}{dt}y(t) + ay(t) = x(t)$$



Second-Order System

$$\frac{d^2}{dt^2}y(t) + a_1\frac{d}{dt}y(t) + a_0 y(t) = x(t)$$



Example 2.12: Consider an LTI system that has the output $y_1(t) = \delta(t) + e^{-t}u(t)$ when the input is $x_1(t) = \delta(t)$. The same LTI system has the output $y_2(t) = 3e^{-t}u(t)$ when the input is $x_2(t) = u(t)$. Determine the impulse response $h(t)$ of this system.

Solution:

Assume the zero-input response of this system is $y_x(t)$.

$$h(t) * \delta(t) + y_x(t) = \delta(t) + e^{-t}u(t)$$

$$h(t) * u(t) + y_x(t) = 3e^{-t}u(t)$$

$$\frac{d}{dt}y_x(t) - y_x(t) = 2\delta(t) - 4e^{-t}u(t)$$

$$y_x(t) = 2e^{-t}u(t)$$

$$h(t) = \delta(t) - e^{-t}u(t)$$

Homework

2.19	**2.23**	**2.40**	
2.5	2.7	2.10	2.11
2.12	2.20	2.46	2.47

① Do not wait until the last minute

② Express your own idea and original opinion

③ Keep in mind the zero-tolerance policy on plagiarism