ECE495 Game Theory and Networked Systems

Infinitely Repeated Games and Games with Incomplete Information

1 Infinitely Repeated Games

Recall the Prisoner's Dilemma with the following payoff matrix:

		P2	
		L	R
P1	L	(1,1)	(5,0)
	R	(0,5)	(4,4)

Table 1: Payoff Matrix for Prisoner's Dilemma

Now let us consider an infinitely repeated game i.e, the players play the game repeatedly at times =0,1,2,... We can view this as either a case where the game is played repeatedly forever or as a setting where the player do not know when the last period is. They always believe there is some chance the game will continue to the next period.

When we study infinitely repeated games we want to derive the payoff that a player receives in each of infinitely many periods. A common assumption is that the player values the present more than the future. Therefore, the future payoffs are *discounted* and are less valuable. The overall payoff for the player is the normalized sum of discounted payoffs at each stage and is given below by the function:

$$U(s) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(s_i^t, s{-}i^t) \tag{1}$$

where $\delta \in [0,1)$ is the discount factor and (1-δ) in front of the equation is for normalization to keep the payoff bounded even when $\delta \to 1$. The normalization also has the effect that if a player receives the same payoff u in each stage game then its normalized discounted payoff will also be u so it is then easy to compare the normalized discounted pay-off to the payoff in a stage game. We also assume that the discount factor for all players is the same.

In such an infinitely repeated game, a strategy for a given player is a rule that specifies for each game t, a action to choose as a function of the history of play in all previous games.

Trigger Strategy: As an example of a strategy in an infinitely repeated games we examine trigger strategies. A trigger strategy basically threatens players with a "punishment" action if they deviate from an agreed upon action.

Lets look at the repeated Prisoner's Dilemma again where we have the following trigger strategy:

1. Play R in first stage;
2. In tth stage, if the outcome in all prior stages is (R,R), play R, otherwise play L.

In the one-shot Prisoners' Dilemma the only equilibrium is for players to choose their dominant strategy (L, L). Even when this game is finitely repeated as shown above — because the stage game has a unique NE —the unique subgame-perfect equilibrium has both players play (L, L) in every period. However, when the players play for long enough and have a large enough discount factor, we will show next that the trigger strategy we can sustain (R, R) in every period as a SPE of the infinitely repeated game.

Suppose Player 2 adopts the trigger strategy, we want to argue that player 1's best response is to also choose a trigger strategy.

First lets consider Player 1's payoff from playing R in every game. In this case, since player 2 is using the trigger strategy, it will also play R in every game, giving the normalized discounted payoff of:

$$(1-\delta)[4+4\delta+4\delta^2+\dots] = 4. \tag{2}$$

Next, consider Player 1's payoff from deviating from the trigger strategy and playing L in some stage, τ. It is clear that since player 2 is following a trigger strategy, player 1 should continue to play L in any subsequent game (since player 2, will play L from game $\tau+1$ onwards). This will give the players each a payoff of 1 in every game from $\tau+1$ onwards. Hence, the only deviations we need to consider are strategies in which player 1 plays R for $\tau-1$ stages and then plays L for the remaining stages.

The normalized discounted payoff for doing this is then given by:

$$(1-\delta)[4+4\delta+4\delta^2+\dots+4\delta^{\tau-1}+5\delta^{\tau}+1\delta^{\tau+1}+1\delta^{\tau+2}+\dots \tag{3}$$

$$= 4(1-\delta^{\tau})+\delta^{\tau+1}+5(\delta^{\tau}-\delta^{\tau+1}) \tag{4}$$

$$= 4-(4\delta-1)\delta^{\tau}. \tag{5}$$

If $\delta \geq 1/4$ then this is increasing in τ and always less than the payoff of 4 obtained by never playing L, showing that for large enough δ player one has no incentive to deviate and therefore, Player 1's best response is to follow the trigger strategy. This shows that both players player the trigger strategy is an equilibrium. An alternative approach is to observe that if it is profitable to deviate, then player 1 will deviate in the first period due to future discounting and this will simplify calculations to also conclude no player will deviate when $\delta \geq 1/4$.

The previous example is a special case of a result known as a Folk Theorem in repeated games. A formal statement of this result is given next:

Theorem 1. *(Folk Theorem) Let G be a static game and let $(e_1,\dots,e_n)$ be a payoff from a Nash Equilibrium for G, and let $(x_1,\dots,x_n)$ be any other feasible payoff from G. If $x_i > e_i$ for each player i, then for δ sufficiently close to 1, there is a sub-game perfect Nash Equilibrium that achieves $(x_1,\dots,x_n)$.*

Here anything feasible can be viewed as convex hull of the payoffs in the game. Essentially, the proof of this theorem is to use traigger strategies as in the previous example , where the Nash equilibrium $(e_1,\dots,e_n)$ is used as the "threat" is users deviate from the desired actions.

Stronger version of this type of results (that require more elaborate strategies) are also known. For example it is possible to replace the Nash equilibrium payoff in the previous game with each players *min-max payoff* defined by

$$v_i = \min_{s_{-i}}(\max_{s_i}\Pi(s_i, s_{-i})).$$

Example 1.1 (Repeated Cournot)**.** Recall the Cournot Competition described below:

1. Demand Curve for Firms : P(Q) = 1 -Q
2. Cost : 0
3. Unique NE : (q1 , q2) = (1/3 , 1/3)
4. Payoffs: (1/9 , 1/9)

If there was no competition, a monopolist could gets a payoff of 1/4 at NE. According to the Folk Theorem, if δ is high enough and the Cournot game is played repeatedly, the two firms can achieve the maximum collusive outcome of 1/4 because it is in the Folk Region. This is an example of *implicit collusion* as the two firms do not need to explicitly collude to obtain this outcome, but rather do this through their strategic behavior in the repeated game.

2 Games with incomplete information

2.1 Introduction

In game theory, complete information refers to any setting where the structure and rules of the game, the payoff functions of the players, and the rationality of all players, are all common knowledge. All the models we have studied thus far in this class assume complete information. This is a strong assumption in many cases and it may often be more reasonable to assume that agents only have partial information about the game. For example, when multiple agents are competing for a common resource, an agent may know its own value of the resource, but not have exact knowledge of the other agents' valuations. Bayesian games or games with incomplete information provide a framework for modeling such situations. Incomplete information refers to a setting where some element of the structure of the game (typically the payoffs of other players) are not common knowledge.
A significant challenge then arises: how should agents reason about each other? The economist John Harsanyi had a key insight in addressing this issue: uncertainty can be modeled by introducing a set of possible "types" for each player, which in turn influence the players' payoffs. Players know their own type, but only have a probabilistic belief about the types of the other players. In this way, players can reason about the game through a structured model of their uncertainty about other players.

2.2 BNE

Formally Bayesian games, or "incomplete information games" are defined as follows.

Definition 2. *A Bayesian game consists of*

1. *A finite set of players R;*
2. *A set of strategies for each player r: S_r;*
3. *A type set for each player r: Θ_r;*
4. *A joint probability distribution over the type space: $F : \Pi_{r=1}^{R}\Theta_r \longmapsto [0,1]$;*
5. *A payoff function for each player r: $U_s = \Pi_{s=1}^{R} S_s \times \Pi_{s=1}^{R}\Theta_s \longmapsto \mathbb{R}$.*

The most common solution concept used for Bayesian games is Bayesian Nash equilibrium (BNE), which naturally generalizes Nash equilibrium to the Bayesian setting. Once again we define equilibria in terms of best response correspondences. However, now in our definition we need to account for the incomplete information in the game. Player r's best response correspondence conditioned on her type is defined as follows:

$$\mathcal{B}_r(a_{-r}(\theta_{-r}) \mid \theta_r) = \underset{s_r \in S_r}{\arg\max}\, \mathbb{E}_{\theta_{-r}} \left[U_r\left(s_r, a_{-r}(\theta_{-r}), \theta\right) \mid \theta_r \right],$$

where θ is a vector include types of all players. For players without various types, we can remove the conditioning (since they have no type) and likewise represent the (unconditioned) best response strategies a_r of player r as:

$$\mathcal{B}_r(a_{-r}(\theta_{-r})) = \underset{a_r:\Theta_r \longmapsto S_r}{\arg\max}\ \mathbb{E}_{\theta} \left[U_r\left(a_r(\theta_r), a_{-r}(\theta_{-r}), \theta\right) \right].$$

Here the best response is a complete mapping from type space to the strategy space. A BNE is defined when the players strategies are best responses to each other, or a fixed point of the best response functions.

Definition 3. *A strategy profile $a^e(\theta_r)$ is a Bayesian-Nash equilibrium (BNE) of a game if and only if $a^e_r \in \mathcal{B}_r(a^e_{-r})$ for all r.*

An alternate definition of BNE is when no player can unilaterally deviate and improve expected payoff (given the joint probability distribution).

2.3 Example of BNE

Example 4. (Coordination game)
Recall the coordination game with complete information where two individuals with different preferences decided between Tech Express and Sargent hall for meeting up for lunch and they both wanted to meet each other (see payoff matrix below). In this game there are two pure strategy Nash equilibria (one of them better for player 1 and the other one better for player 2).

	T	S
T	$(2,1)$	$(0,0)$
S	$(0,0)$	$(1,2)$

Now imagine that player 1 does not know whether player 2 wishes to meet or wishes to avoid. Therefore, this is a situation of incomplete information. We represent this by thinking of player 2 having two different types, one type that wishes to meet player 1 and the other wishes to avoid him. More explicitly, suppose that these two types have probability 3/5 and 2/5. Then the game has two realizations of forms A or B with probability 3/5 and 2/5 respectively (payoff matrix below).

A

	T	S
T	$(2,1)$	$(0,0)$
S	$(0,0)$	$(1,2)$

B

	T	S
T	$(2,0)$	$(0,2)$
S	$(0,1)$	$(1,0)$

Consider the following strategy profile $(T,(T,S))$, which means that player 1 plays T and while in state A player 2 will also play T and in state B player 2 will play S. Clearly given that player 1 plays T, the strategy of player 2 is a best response. We next check if player 1 is best responding. Now consider player 1, the expected payoff is

$$\mathbb{E}[U_1(T,(T,S))] = 2 \times \frac{3}{5} + 0 = \frac{6}{5}.$$

If instead, player 1 deviates and play S, his expected payoff is

$$\mathbb{E}[U_1(S,(T,S))] = \frac{3}{5} \times 0 + \frac{2}{5} \times 1 = \frac{2}{5}.$$

Since the expected payoff is higher for playing T, P1 has no incentive to unilaterally deviate and therefore the profile $(T,(T,S))$ is a BNE.

Now consider another candidate for BNE, $(S,(S,T))$, P2 is best responding to P1. For P1, the expected payoff for player 1 is

$$\mathbb{E}[U_1(S,(S,T))] = 1 \times \frac{3}{5} + 0 = \frac{3}{5}.$$

If he deviates and play T, his expected payoff will be

$$\mathbb{E}[U_1(T,(S,T))] = 0 + 2 \times \frac{2}{5} = \frac{4}{5}.$$

Therefor, player 1 wants to deviate and $(S, (S, T))$ is Not a BNE.

Before we had different types, P2 would prefer (S,S) outcome, however, as we can see in this example, although player 2 has more information, the equilibrium payoff is now worse, (S,S) does not happen for type A player 2. More information is not always helpful.

References

[1] R. Berry and R. Johari, "Economic Modeling in Networking: A Primer", in Foundations and Trends in Networking, 6.3, 2013, pp. 165-286.