**ECE495 Game Theory and Networked Systems**

# Lecture 11: Sequential Games

Recall that we are studying the sequential games with complete information and *perfect monitoring*. By this we mean that agents move one at a time in *stages*, with the ability to view the entire history of moves. Examples of such games include chess and checkers. By contrast, games with *imperfect monitoring* may not reveal all of an opponent's moves, such as when players move simultaneously instead of one at a time, or in games like Battleship.

The concept we consider here is subgame perfect Nash Equilibrium, where we check for one stage deviation at both along equilibrium and off equilibrium path.

## 1 Example: Matching Pennies

We will show that in case of a sequential Cournot game, there may be first mover advantage, but not all games exhibit a first mover advantage. For instance, if we return to the example of matching pennies, we find that making the game sequential causes the second player to always win. This is often referred to as *second mover advantage*.

## 2 Example: Stackelberg Competition

Two competing firms choose quantities of goods to produce, as in a Cournot competition; however, one firm is the leader and the other is the follower, making this a sequential rather than a static game. The firms choose quantities $q_1$ and $q_2$, with Firm 1 acting during the first stage and Firm 2 acting during the second stage. We can solve this problem using backwards induction. This backward induction process works on continuous (the case here, since $q_1$ and $q_2$ are continuous) as well as discrete (as in the previous example) strategy spaces. In the second stage, we have that given player 1's choice $q_1$, player 2 chooses to best respond by solving the following problem:

$$q_2(q_1) = \arg\max_{q_2 \geq 0} (1 - q_1 - q_2)q_2 - cq_2.$$

We solve this and have that

$$q_2(q_1) = \max\left\{0, \frac{1 - q_1 - c}{2}\right\}.$$

Once we have calculated $q_2(q_1)$, we can calculate Firm 1's best response, by solving

$$q_1 = \arg\max_{q_1 \geq 0} (1 - q_1 - q_2(q_1))q_1 - cq_1.$$

We therefore have at subgame perfect equilibrium,

$$q_1^* = \frac{1-c}{2}, \quad q_2^* = \frac{1-c}{4}.$$

for $c \leq 1$.

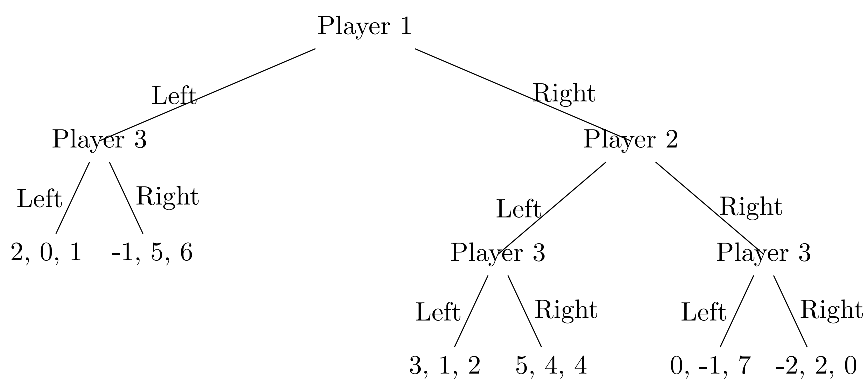
This type of game structure where one player is the leader, and the other one is follower is referred to as *Stackelberg Competition*. We recall that in a simultaneous Cournot competition, the equilibrium outcome is $q_1 = q_2 = \frac{1-c}{3}$. From this, we get a couple of interesting results in comparison to the outcome of a simultaneous Cournot competition.

- A larger total quantity is produced. Here, the total quantity is $\frac{3(1-c)}{4}$, compared to before of $\frac{2(1-c)}{3}$, so consumers are better off.
- Firm 1 is better off (profit went from $\frac{1-c}{9}$ to $\frac{1-c}{8}$.
- Firm 2 is worse off (profit went from $\frac{1-c}{9}$ to $\frac{1-c}{16}$.

This is typically referred to as *first mover advantage*. This is surprising since Firm 2 is worse off with more information. By announcing their production quantity publicly firm 1 controls how firm 2 responds and forces firm 2 to a worse outcome. Firm 2 would really rather that neither firm knew anything about the other's production amounts, which would yield a Cournot instead of a Stackelberg competition. Not all games have a clear first or second mover advantage. Consider the n tiger and 1 meat game, where a tiger would prefer to eat meat than not, but once a tiger eats the meat, he becomes the meat the next time period. A tiger prefers to not eat than be eaten. Therefore, if the game has 1 tiger, he would choose to eat. If the game has 2 tigers, neither would choose to eat. Then we can play induction and conclude the game outcome depends on the even/odd property of n.

# 3 Example: A Three-Player Game

In a straightforward way, we can apply the same ideas of backward induction to games with more than two players and/or more than two stage. This example illustrates a game with 3 players.




In this example, we can use backward induction. In the case where player 1 chooses L, player 3 would prefer 6 over 1 and choose. In case player 1 chooses R and player 2 chooses L, player 3 would prefer 4 over 2 and choose R. In case player 1 chooses R and player 2 chooses R, player 3 would prefer 7 over 0 and choose L. Now back to player 2, if they choose L, then the payoff at next stage is going to be 4 (since player 3 will choose R), compared to -1 when choose R (player 3 chooses L). Therefore, player 2 chooses L. Player 1 is then to choose between -1 (L) and 5 (R) and will choose R. Hence the outcome at SPNE is (R, L, R).

Every finite game of perfect information has a subgame-perfect equilibrium that we can find by backward induction. Furthermore, if each player's payoffs across all terminal nodes are unique, then the equilibrium is unique.

# 4 Example: 3 Period Sequential Bargaining

Consider two players bargaining over a dollar, with three rounds.

- In the first round, Player 1 makes an offer $(S_1, 1 - S_1)$, where Player 1 keeps $S_1$ and Player 2 gets $1 - S_1$. Player 2 may then choose to accept or reject this offer.

- In the second round, which occurs only if Player 2 rejected Player 1's offer in the first round, the process is reversed, with Player 2 offering $(1 - S_2, S_2)$ and Player 1 choosing to accept or reject.
- In the third round, which occurs only if Player 1 rejected Player 2's offer in the second round, the dollar is split 50-50.

If the players are willing to wait forever, they can just move to Round 3; however, often we have to model impatience, as money obtained today can be invested and grow more than money obtained tomorrow. We do this with a discount factor $\delta$ between one and zero, which the dollar is multiplied with each time a new round begins after the first. Hence the payoff at the second stage is $(\delta(1-S_2), \delta S_2)$ and the last stage payoff is $(0.5\delta^2, 0.5\delta^2)$.

To solve this game backwards, we note that in the second stage, player 1 is choosing between $\delta(1 - S_2)$ and $0.5\delta^2$, and will choose to accept if

$$\delta(1 - S_2) \geq 0.5\delta^2.$$

To maximize player 2's payoff, he will maximize $S_2$ and set $\delta(1 - S_2) = 0.5\delta^2$ with

$$S_2 = 1 - 0.5\delta \geq 0.5\delta,$$

where the last inequality follows the fact that $\delta \leq 1$. This way P2 can obtain a payoff of $\delta S_2$, which is larger than $0.5\delta^2$ (the payoff at next stage). To summarize, in the second stage, P2 will offer $1 - 0.5\delta$ and P1 will accept.

Going back to the first stage, P2 is choosing between $1 - S_1$ and $\delta(1 - 0.5\delta)$ (from next stage) and will accept if

$$1 - S_1 \geq \delta S_2 = \delta(1 - 0.5\delta).$$

To maximize P1's payoff, P1 will set $S_1 = 1 - \delta + 0.5\delta^2$. P2 will accept. P1's payoff is $S_1 \leq 1 - \delta + 0.5\delta^2$, which is higher than $0.5\delta^2$ (payoff if reach the second stage), as $\delta \leq 1$. Hence we have, P2 will accept in the first stage at the equilibrium. Hence, from backward induction, we have found that Player 1 should offer

$$S_1^* = 1 - \delta + 0.5\delta^2$$

in the first period and player 2 should accept this.

Note that as $\delta$ goes to zero, Player 2 will accept a greedy initial offer from Player 1, while as $\delta$ goes to one, the players will compromise at 50-50. The players will be able to calculate each other's behavior and will never need to go past a single step of negotiations if they know $\delta$. In equilibrium outcome, we never will reach the second or third stage, but we need to analyze the behavior in these periods as they support of the overall game.

# References

[1] R. Berry and R. Johari. *Economic modeling in networking: A primer.* Foundations and Trends in Networking 6.3 (2013): 165-286.