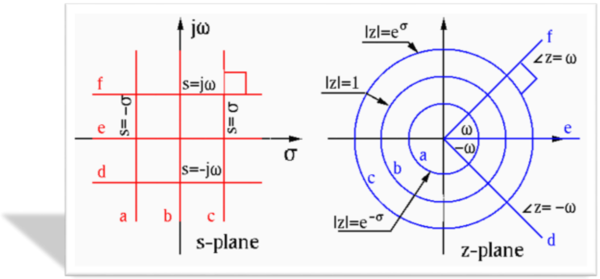
Chapter 10 The Z Transform



$$X(z)|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n}$$
$$= \mathcal{F}\{x[n]r^{-n}\}$$
$$X(z)|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\} = X(e^{j\omega})$$

The range of values z for which $X[z]$ exists is referred to as the Region Of Convergence (ROC)

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty$$

If the ROC includes the unit circle, the discrete-time Fourier transform also convergences.

§10.1 The Z Transform

1. Definition of the Bilateral Z Transform

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z)z^n$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)z^n$$

$$H(z) \triangleq \mathcal{Z}\{h[n]\} = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

Z-transform pairs:

$$x[n] \overset{Z}{\longleftrightarrow} X[z] = \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \quad (z \in ROC)$$

Example 10.1 Determine the z-transforms of

$$x_1[n] = a^n u[n] \ \ (a > 0)$$
$$x_2[n] = -a^n u[-n-1] \ \ (a > 0)$$

Solution:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$
$$= \frac{1}{1-az^{-1}} \quad (ROC:\ |z| > a\,)$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x_2[n]z^{-n} = \sum_{n=-\infty}^{\infty} -a^n u[-n-1]z^{-n}$$
$$= \sum_{n=-\infty}^{-1} -(az^{-1})^n = \frac{1}{1-az^{-1}} \quad (ROC:\ |z| < a\,)$$

2. Pole-Zero Plot

The representation of $X(z)$ through its poles and zeros in the z-plane is referred to as the pole-zero plot of $X(z)$.

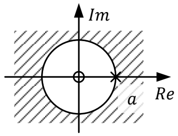
$$X(z) = \frac{N(z)}{D(z)}$$

Zeros of $X(z)$: $N(z_i) = 0$

Poles of $X(z)$: $D\left(z_j\right) = 0$

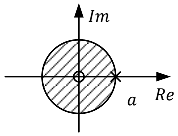
$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

$(ROC\colon\ |z| > a\ )$



$$-a^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}$$

$(ROC\colon\ |z| < a\ )$



§10.2 The Region of Convergence for the Z Transform

Property 1: The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.

Property 2: The ROC does not contain any poles.

Property 3: If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possibly $\mathrm{z} = 0$ and/or $\mathrm{z} = \infty$.

$$\delta[n] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1\ \ (ROC\colon\ z\text{–plane})$$

$$\delta[n-1] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-1}\ \ (ROC\colon\ z \neq 0)$$

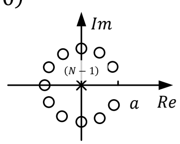
$$\delta[n+1] \overset{z}{\longleftrightarrow} \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z\ \ (ROC\colon\ z \neq \infty)$$

Example 10.2 Determine the z-transform of $x[n] = \begin{cases} a^n & (0 \le n < N,\ a > 0) \\ 0 & (otherwise) \end{cases}$ and sketch its pole-zero plot.

Solution:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1 - (az^{-1})^N}{1 - az^{-1}}$$

$$= \frac{1}{z^{N-1}} \left(\frac{z^N - a^N}{z - a} \right)\ \ (ROC\colon\ |z| > 0)$$



There is an $N-1$ order pole at the origin.

There are $N-1$ zeros.

$$z^N - a^N = 0 \Rightarrow z_k = ae^{j\frac{2\pi}{N}k}\ \ (k = 1, 2, \cdots, N-1)$$

Property 4: If $x[n]$ is a right sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

Property 5: If $x[n]$ is a left sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all non-zero values of z for which $|z| < r_0$ will also be in the ROC.

Property 6: If $x[n]$ is two sided, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z| = r_0$.

Example 10.3 Determine the z-transform of $x[n] = b^{|n|}\ (b > 0)$.

Solution:

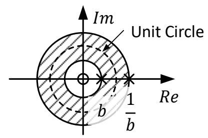
$$x[n] = b^{|n|} = b^n u[n] + b^{-n} u[-n-1]$$

$$b^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - bz^{-1}} \ \ (ROC\text{: } |z| > b)$$

$$b^{-n} u[-n-1] \overset{z}{\longleftrightarrow} \frac{-1}{1 - b^{-1}z^{-1}} \ \ \left(ROC\text{: } |z| < \frac{1}{b}\right)$$

$$X(z) = \mathcal{Z}\{b^n u[n]\} + \mathcal{Z}\{b^{-n} u[-n-1]\}$$

$$= \frac{b^2 - 1}{b} \frac{z}{(z-b)(z-b^{-1})}$$



If $b > 1$, $X(z)$ doesn't exist.

If $b < 1$, the ROC is $b < |z| < \frac{1}{b}$.

Property 7: If the z-transform $X(z)$ of $x[n]$ is rational, the its ROC is bounded by poles or extends to infinity.

Property 8: If the z-transform $X(z)$ of $x[n]$ is rational, then if $x[n]$ is right sided, the ROC is the region in the z-plane outside the outmost pole. Furthermore, if $x[n]$ is causal, the ROC also includes $z = \infty$.

Property 9: If the z-transform $X(z)$ of $x[n]$ is rational, then If $x[n]$ is left sided, the ROC is the region in the z-plane inside the innermost nonzero pole. In particular, if $x[n]$ is anticausal, the ROC also includes $z = 0$.

§10.3 Properties of the Z Transform

$x[n] \overset{z}{\longleftrightarrow} X(z)\ \ (z \in R_1)$ and $y[n] \overset{z}{\longleftrightarrow} Y(z)\ \ (z \in R_2)$

1. Linearity

$$Ax[n] + By[n] \overset{z}{\longleftrightarrow} AX(z) + BY(z)\ \ (ROC \supseteq R_1 \cap R_2)$$

2. Time Shifting

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0} X(z)\ \ (ROC \cong R_1)$$

Caution!
$z = 0$
$z = \infty$

3. Scaling in the z-domain

$$z_0{}^n x[n] \overset{z}{\longleftrightarrow} X(z/z_0)\ \ (ROC = |z_0| R_1)$$

4. Time Reversal

$$x[-n] \overset{z}{\longleftrightarrow} X\left(\frac{1}{z}\right)\ \ \left(ROC = \frac{1}{R_1}\right)$$

5. Conjugation

$$x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*)\ \ (ROC = R_1)$$

6. Convolution Property

$$x[n] * y[n] \overset{z}{\longleftrightarrow} X(z)Y(z)\ \ (ROC \supseteq R_1 \cap R_2)$$

7. First Order Difference

$$x[n] - x[n-1] \overset{z}{\longleftrightarrow} (1 - z^{-1})X(z)\ \ (ROC \cong R_1)$$

The ROC equals to R_1, with possible addition of $z = 0$ and/or deletion of $z = 1$

8. Differentiation in the z-Domain

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}\ \ (ROC = R_1)$$

Example 10.4 Giving $X(z)=\ln(1-az^{-1})$ $(ROC:\ |z|>|a|)$, determine $x[n]$.

Solution:

$$x[n]\xleftrightarrow{z}X(z)$$

$$nx[n]\xleftrightarrow{z}-z\frac{dX(z)}{dz}=\frac{-az^{-1}}{1-az^{-1}}$$

$$a^n u[n]\xleftrightarrow{z}\frac{1}{1-az^{-1}}\quad (ROC:\ |z|>|a|)$$

$$-aa^{n-1}u[n-1]\xleftrightarrow{z}\frac{-az^{-1}}{1-az^{-1}}\quad (ROC:\ |z|>|a|)$$

$$nx[n]=-aa^{n-1}u[n-1]=-a^n u[n-1]$$

$$x[n]=\frac{-a^n}{n}u[n-1]$$

Example 10.5 Compute the z-transforms of

$$x_1[n]=(n+1)a^n u[n]$$
$$x_2[n]=(a^n\cos\omega n)u[n]$$

Solution:

$$a^n u[\mathrm{n}]\xleftrightarrow{z}\frac{1}{1-az^{-1}}\quad (ROC:|z|>|a|\,)$$

$$na^n u[n]\xleftrightarrow{z}-z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)=\frac{az^{-1}}{(1-az^{-1})^2}\quad (ROC:|z|>|a|\,)$$

$$x_1[n]\xleftrightarrow{z}\frac{az^{-1}}{(1-az^{-1})^2}+\frac{1}{1-az^{-1}}=\frac{1}{(1-az^{-1})^2}\quad (ROC:|z|>|a|\,)$$

$$x_1[n]=a^n u[\mathrm{n}]*a^n u[\mathrm{n}]\xleftrightarrow{z}\frac{1}{(1-az^{-1})^2}\quad (ROC:|z|>|a|\,)$$

$$(a^n\cos\omega n)u[n]=\frac{1}{2}e^{j\omega_0 n}a^n u[n]+\frac{1}{2}e^{-j\omega_0 n}a^n u[n]$$

$$e^{j\omega_0 n}a^n u[n]\xleftrightarrow{z}\frac{1}{1-e^{j\omega_0}az^{-1}}\quad (ROC:|z|>|a|\,)$$

$$e^{-j\omega_0 n}a^n u[n]\xleftrightarrow{z}\frac{1}{1-e^{-j\omega_0}az^{-1}}\quad (ROC:|z|>|a|\,)$$

$$x_2[n]\xleftrightarrow{z}\frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}az^{-1}}+\frac{1}{1-e^{-j\omega_0}az^{-1}}\right]$$
$$=\frac{1-(a\cos\omega_0)z^{-1}}{1-(2\,a\cos\omega_0)z^{-1}+a^2z^{-2}}\quad (ROC:|z|>|a|\,)$$

$$\boxed{\begin{aligned}a^n\cos(\omega_0 n)u[n]&\xleftrightarrow{z}\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}\ (ROC:|z|>|a|\,)\\ a^n\sin(\omega_0 n)u[n]&\xleftrightarrow{z}\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}\ (ROC:|z|>|a|\,)\end{aligned}}$$

10. The Initial and Final-Value Theorems

① If $x[n]=x[n]u[n]$,

$$x[0]=\lim_{z\to\infty}X(z)$$

When $x[0]$ is finite, $X(\infty)$ is finite. With $X(z)$ expressed as a ratio of polynomials in z,

a. the order of the numerator polynomial cannot be greater than the order of the denominator polynomial
b. the number of finite zeros cannot be greater than the number of finite poles

② If $x[n]=x[n]u[n]$,

$$x[\infty]=\lim_{z\to 1}(z-1)X(z)$$