ECE495 Game Theory and Networked Systems

Lecture 7: Supermodular Game

1 Supermodular Games

Supermodular games are also referred as games with strategic complementarities. Informally, this means that the marginal utility of increasing a player's strategy raises with increases in the other players' strategies.

Definition 1. *A game, $G = (R, \{S_r\}, \{\Pi_r\})$ is a supermodular game, if following statements hold for each player $r \in R$:*

1. *Assume ${s'}_r \geq s_r$ and ${s'}_{-r} \geq s_{-r}$, then:*

$$\pi_r({s'}_r, s_{-r}) - \pi_r(s_r, s_{-r}) \leq \pi_r({s'}_r, {s'}_{-r}) - \pi_r(s_r, {s'}_{-r}) \tag{1}$$

   This property is called the increasing differences of utility.

2. *S_r is a compact subset of $\mathbb{R}$.*
3. *Π_r is upper-semicontinuous in (s_r, s_{-r}).*

Definition 2. *Let $f : \mathbb{R} \to \mathbb{R}$, then f is upper-semicontinuous iff for any $y \in \mathbb{R}$, the set $\{x|f(x) \geq y\}$ is closed. Informally, a function is upper-semicontinuous if it is either continuous or the top part is closed whenever there is a jump.*

For supermodular games, we need to define an ordering on the strategies. For the ordering part, hen we work with strategies in $\mathbb{R}$, we typically adopt ordering on real numbers (see Example 2.1 for an exception), other times we may need to impose an order on the strategies for the increasing differences properties (see Example 2.2). Defining ordering on vector of s_{-r} could mean a dictionary ordering or other kinds of ordering, as long as we get a payoff function with increasing differences. Note that the order on strategies can be arbitrary, while the utility/payoff function evaluations are given by the order on real numbers.

Example 1.1 (Network Effect)**.** Suppose a cell phone market is composed of only two categories of cell phones: "iPhone" and "Android", denoted by I and A, respectively. For this game, a player i can only have two strategies: $s_i = I$ or $s_i = A$. Let $I(S)$ be the total number of users using "iPhone", and $A(S)$ be the total number of users using "Android". Then, we define a benefit function for player i, as $B_i(J, K)$; where K is the choice of player (I or A), J is the total number of users using same cell phone with the player. Assume that $J < J'$, then the benefit function satisfies following property:

$$B_i(J, K) \leq B_i(J', K) \tag{2}$$

This means that the more users use the same product with player i, the more increase in the player's benefit function. After defining the benefit function, we can define the utility function of player i, as:

$$u_i(s_i, s-i) = \begin{cases} B_i(I(S), I) & \text{if } s_i = I \\ B_i(A(S), A) & \text{if } s_i = A \end{cases} \tag{3}$$

In order to check for supermodularity, we need to assign an order. Let's assume an arbitrary order, $A \geq I$, then assume $s'_{-i} \geq s_{-i}$ meaning that the strategy set S'_{-i} has more or equal users using item A than S_{-i}, and less or equal users using item I than S_{-i}. This could be a result from adopting component-wise ordering of assuming $A \geq I$. From Eq. (2), $u_i(A, s_{-i}) \leq u_i(A, s'_{-i})$ and $u_i(I, s_{-i}) \geq u_i(I, s'_{-i})$. Therefore, for this game we have $u_i(A, s_{-i}) - u_i(I, s_{-i}) \leq u_i(A, s'_{-i}) - u_i(I, s'_{-i})$. We can conclude that the network effect is an example of a supermodular game.

To verify a game is supermodular, we need to find a consistent ordering. Here we found one on the first trial, so we don't have to verify other possibilities. But there could be more than one way of ordering, for instance, we could have chosen $I \geq A$ instead and would got another supermodular game. We could also assign different ordering on different users as the next example illustrates.

Example 1.2 (Cournot Game)**.** Consider a Cournot duopoly model, where two firms choose the quantity they produce $q_i \in [0, \infty)$. [2] Then, the price function can be defined as $P(q_1, q_2) = 1 - q_1 - q_2$. Let $c \geq 0$ denote the cost, the payoff function of each firm will be in the form

$$\begin{aligned} u_i(q_i, q_{-i}) &= q_i P(q_1, q_2) - cq_i \\ &= q_i(1 - q_1 - q_2) - cq_i \end{aligned}$$

Let $q_i \to q_i + \Delta$ for $\Delta > 0$, then,

$$\begin{aligned} u_i(q_i + \Delta, q_{-i}) &= (q_i + \Delta)(1 - q_1 - q_2 - \Delta) - c(q_i + \Delta) \\ &= (q_i + \Delta) - (q_i + \Delta)^2 - (q_i + \Delta)q_2 - c(q_i + \Delta) \end{aligned}$$

Intuitively, we need to verify that the difference between the previous two inequalities increase as q_{-i} increase. The term $-(q_i + \Delta)q_2$ prevents our function from having increasing differences property. As we have $q_2' > q_2$, equivalently $s_{-i}' > s_{-i}$, it will have a negative impact on the function, and the function will decrease, instead of increase (could reach a similar conclusion via twice differentiating the utility function with respect to q_i, q_j). Therefore, it is not supermodular. If we transform the game as

$s_1 = q_1$ (Quantity of Firm 1)

$s_2 = -q_2$ (Negative quantity of Firm 2)

After flipping the sign of s_2, the game is now a supermodular game.

The flipping sign trick only works for two players, as there will not be a consistent way to assign signs for multiple players. The following theorems are based on lattice theory and Tarski's Fixed Point Theorem `https://en.wikipedia.org/wiki/Knaster-Tarski_theorem`.

Theorem 3. *Assume a game, $G = (R, \{S_r\}, \{U_r\})$, is a supermodular game. Let the best response mapping be*

$$B_r(s_{-r}) = \arg\max_{s_r \in S_r} U_r(s_r, s_{-r}). \tag{4}$$

Then,

1. *$B_r(s_{-r})$ has greatest and least elements, denoted by $\bar{B}_r$, and $\underline{B}_r$, respectively.*
2. *If $s_{-r}' \geq s_{-r}$, then $\bar{B}_r(s_{-r}') \geq \bar{B}_r(s_{-r})$ and $\underline{B}_r(s_{-r}') \geq \underline{B}_r(s_{-r})$.*

We note that the argmax could return a set and the greatest and least elements are defined with respect to the ordering we used to derive supermodularity.

Theorem 4. *For a supermodular game with continuous payoff functions, let S^∞ be the set survived after iterated elimination of strictly dominated strategies. S^∞ has $\underline{s}$ and $\bar{s}$, the least and greatest elements, which are the least and greatest pure strategy Nash Equilibria, respectively.*

Proof. Suppose we have a supermodular game. Let S^i denote the set after the i^{th} iterated elimination process. Then $\bar{s}^0$ is the largest element in S^0. Then, we define $s_i^1 = \bar{B}(\bar{s}_{-i}^0)$
The proof follows as $\forall s_i > s_i^1$ and $\forall s_{-i} \in S_{-i}$;

$$\begin{aligned} u_i(s_i, s_{-i}) - u_i(s_i^1, s_{-i}) &\leq u_i(s_i, \bar{s}_{-i}^0) - u_i(s_i^1, \bar{s}_{-i}^0) \\ &< 0 \end{aligned}$$

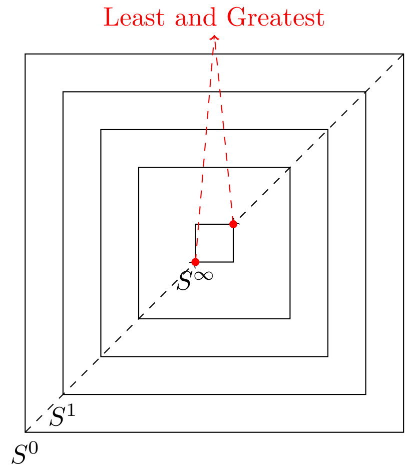

Figure 1: A pictorial description of Theorem 4.

For the equation above, the first inequality comes from the definition of supermodular games, and the second inequality comes from the fact that $s_i^1 = \bar{B}(\bar{s}_{-i}^0)$, i.e.,$s_i$ is not a best response, so $u_i(s_i^1, \bar{s}_{-i}^0) > u_i(s_i, \bar{s}_{-i}^0)$. We can then eliminate $\forall s_i, s_i > s_i^1$. We can iterate this process, and at each iteration we have for $s_i > s_i^{k+1}$,

$$u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k). \tag{5}$$

This inequality holds for all $s_i > s_i^{k+1}$ including those eliminated in earlier rounds, since they were strictly dominated by s_i^t for $t \leq k$. We can iteratively apply this inequality and get that the strategy is dominated by s_i^{k+1}.

We also note that we have $s_i^{k+1} \leq s_i^k$, since we started at the greatest element. Hence, we have a monotone sequence s^k in a compact set and therefore the sequence converges and $\lim_{k\to\infty} s_i^k = s_i^\infty$. We can take limit of Eq. (5) and have $\lim_{k\to\infty} u_i(s_i^{k+1}, s_{-i}^k) \geq u_i(s_i, s_{-i}^k)$. By continuity of payoff functions, it follows that,

$$u_i(s_i^\infty, s_{-i}^\infty) \geq u_i(s_i, s_{-i}^\infty).$$

This shows that s^∞ is a NE. A similar argument can be made starting from the smallest element.

References

[1] R. Berry and R. Johari, “Economic Modeling in Networking: A Primer”, in Foundations and Trends in Networking, 6.3, 2013, pp. 165-286.

[2] A. Ozdaglar, ”6.254 : Game Theory with Engineering Applications Lecture 4: Strategic Form Games-Solution Concepts”, MIT, 2010.