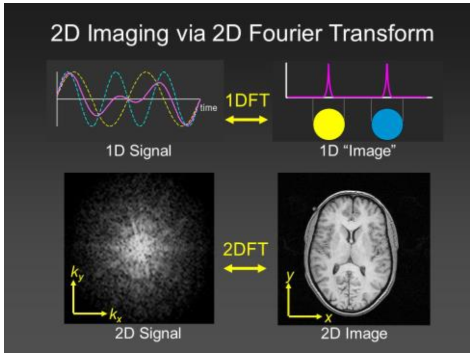
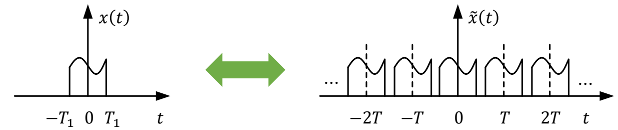
Chapter 4 The Continuous-Time Fourier Transform



§4.1 Representation of Aperiodic Signals: the Continuous-Time Fourier Transform

1. Development of the Fourier Transform Representation

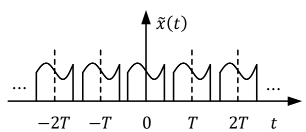


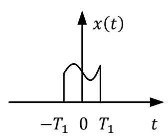
$$\tilde{x}(t) = x(t) * \sum_{n=-\infty}^{+\infty} \delta(t - nT) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

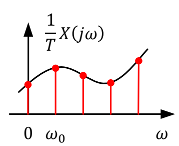
$$x(t) = \lim_{T\to\infty} \tilde{x}(t) = \lim_{\omega_0\to 0} \tilde{x}(t) \quad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_{<T>} \tilde{x}(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T}\int_{-T_1}^{T_1} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt \\ &= \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0} \end{aligned}$$

$$X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-j\omega t} dt$$





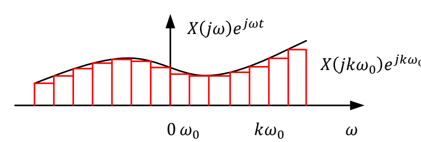


$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0 \quad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$



Fourier Transform Pair

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega)$$

Fourier Transform or Fourier Integral of $x(t)$

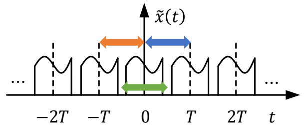
$$X(j\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$X(j\omega)$ is also known as the spectrum of $x(t)$

Inverse Fourier Transform of $X(j\omega)$

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$a_k = \frac{1}{T}X(j\omega)\bigg|_{\omega=k\omega_0}$$

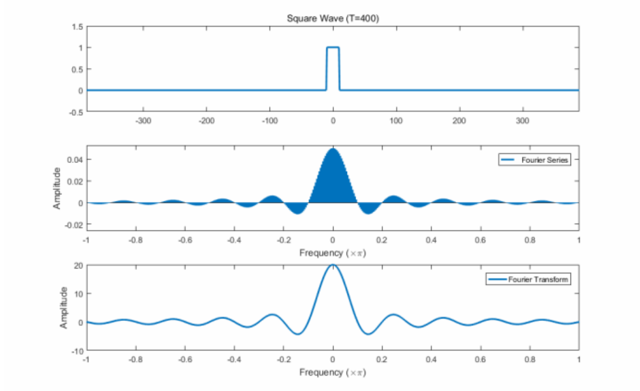


$\{a_k\}$ of a periodic signal $\tilde{x}(t)$ can be expressed in terms of equally spaced samples of the Fourier Transform of any one period of $\tilde{x}(t)$

The set of $\{a_k\}$ is unique, but the Fourier Transforms can be different since there are infinite choices of one period of $\tilde{x}(t)$

$$a_k = \frac{1}{T}X_1(j\omega)\bigg|_{\omega=k\omega_0} = \frac{1}{T}X_2(j\omega)\bigg|_{\omega=k\omega_0} = \frac{1}{T}X_3(j\omega)\bigg|_{\omega=k\omega_0}$$

Conversion from Fourier Series to Fourier Transform



2. Convergence of the Fourier Transform

① $\int_{-\infty}^{\infty}|x(t)|^2dt < \infty$

② Dirichlet conditions:

Condition 1: $\int_{-\infty}^{\infty}|x(t)|dt < \infty$

Condition 2: There are a finite number of maxima and minima within any finite interval of time.

Condition 3: In any finite interval of time, there are only a finite number of finite discontinuities.

Oppenheim: $x(t)$ is absolutely integrable and "well behaved".

3. Examples of Continuous-Time Fourier Transform

① The Unilateral Exponential Signal

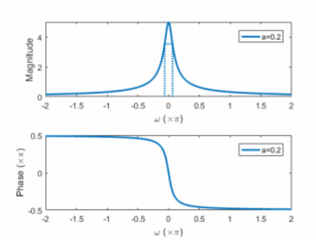
$x(t) = e^{-at}u(t)\ \ (Re\{a\} > 0)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{a + j\omega}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

$$\measuredangle X(j\omega) = -\arctan\left(\frac{\omega}{a}\right)$$



② The Bilateral Exponential Signal

$x(t) = e^{-a|t|}\ \ (Re\{a\} > 0)$

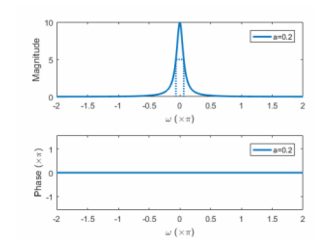
$$X(j\omega) = \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{+\infty} e^{-(a+j\omega)t}dt$$

$$= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$

$$|X(j\omega)| = \frac{2a}{a^2 + \omega^2}$$

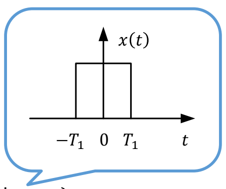
$$\measuredangle X(j\omega) = 0$$



③ The Unit Impulse Signal

$x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = 1$$



④ The Rectangular Pulse Signal

$$x(t) = u(t + T_1) - u(t - T_1) = \begin{cases} 1 & (|t| < T_1) \\ 0 & (|t| > T_1) \end{cases}$$

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}dt = \frac{1}{j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right) = \frac{2\sin(\omega T_1)}{\omega}$$

$$= 2T_1\frac{\sin(\omega T_1)}{\omega T_1} = 2T_1 Sa(\omega T_1)$$

$$Sa(x) = \frac{\sin(x)}{x}$$

⑤ The Ideal Low-Pass Filter

$$X(j\omega) = u(\omega + \omega_c) - u(\omega - \omega_c) = \begin{cases} 1 & (|\omega| < \omega_c) \\ 0 & (|\omega| > \omega_c) \end{cases}$$

$$x(t) = \mathcal{F}^{-1}[X(j\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega = \frac{1}{2\pi}\frac{1}{jt}\left(e^{j\omega_c t} - e^{-j\omega_c t}\right)$$

$$= \frac{\sin(\omega_c t)}{\pi t} = \frac{\omega_c}{\pi}\frac{\sin(\omega_c t)}{\omega_c t} = \frac{\omega_c}{\pi}Sa(\omega_c t)$$

$$= \frac{\omega_c}{\pi}sinc\left(\frac{\omega_c}{\pi}t\right)$$

$$sinc(x) = Sa(\pi x) = \frac{\sin(\pi x)}{\pi x}$$

$$\begin{cases} sinc(x) = \dfrac{\sin(\pi x)}{\pi x} = Sa(\pi x) \\ Sa(x) = \dfrac{\sin(x)}{x} = sinc\left(\dfrac{x}{\pi}\right) \end{cases}$$

Properties of $sinc(x)$

a. $sinc(x) = sinc(-x)$

b. $sinc(x) = 0$ for $x \in Z$ and $x \neq 0$

c. $\int_{-\infty}^{\infty} sinc(x)dx = 1$

Consider the ideal low-pass filter:

$$F(j\omega) = \int_{-\infty}^{\infty} \frac{\omega_c}{\pi} sinc\left(\frac{\omega_c}{\pi} t\right) e^{-j\omega t} dt$$

$$F(0) = \int_{-\infty}^{\infty} \frac{\omega_c}{\pi} sinc\left(\frac{\omega_c}{\pi} t\right) dt = 1$$

Narrowing a Pulse to the Impulse