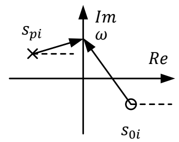
3. Geometric Evaluation of the Frequency Response of LTI Systems from the Pole-Zero Plot of $H(s)$

$$\frac{H(s)}{H_0}=\frac{(s-s_{01})(s-s_{02})\cdots(s-s_{0m})}{(s-s_{p1})(s-s_{p2})\cdots(s-s_{pn})}=\frac{\prod_{i=1}^{m}(s-s_{0i})}{\prod_{i=1}^{n}(s-s_{pi})}$$

$$H(j\omega)=H(s)|_{s=j\omega}=H_0\frac{\prod_{i=1}^{m}(j\omega-s_{0i})}{\prod_{i=1}^{n}(j\omega-s_{pi})}$$



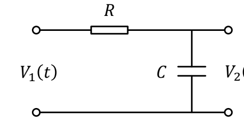
Zero Vector: $\overrightarrow{M_i}=M_ie^{j\theta_{0i}}=j\omega-s_{0i}$

Pole Vector: $\overrightarrow{N_i}=N_ie^{j\theta_{pi}}=j\omega-s_{pi}$

$$H(j\omega)=H_0\frac{\prod_{i=1}^{m}M_ie^{j\theta_{0i}}}{\prod_{i=1}^{n}N_ie^{j\theta_{pi}}}=H_0\frac{\prod_{i=1}^{m}M_i}{\prod_{i=1}^{n}N_i}e^{j(\sum_{i=1}^{m}\theta_{0i}-\sum_{i=1}^{n}\theta_{pi})}$$

$$|H(j\omega)|=H_0\frac{\prod_{i=1}^{m}M_i}{\prod_{i=1}^{n}N_i}\qquad \measuredangle H(j\omega)=\sum_{i=1}^{m}\theta_{0i}-\sum_{i=1}^{n}\theta_{pi}$$

① First-order Systems

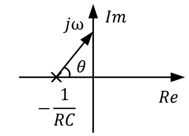


$$RC\frac{d}{dt}V_2(t)+V_2(t)=V_1(t)$$

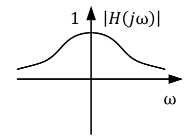
$$\frac{d}{dt}V_2(t)+\frac{1}{RC}V_2(t)=\frac{1}{RC}V_1(t)$$

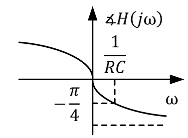
$$sV_2(s)+\frac{1}{RC}V_2(s)=\frac{1}{RC}V_1(s)$$

$$H(s)=\frac{V_2(s)}{V_1(s)}=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$



$$H(j\omega)=\frac{\frac{1}{RC}}{j\omega+\frac{1}{RC}}$$

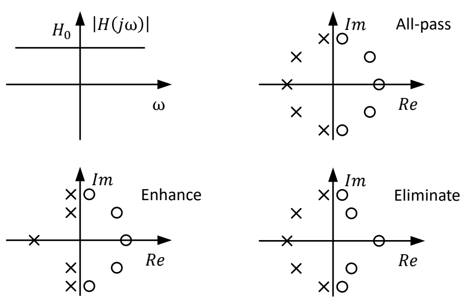




② All-pass Systems

Definition: The magnitude of the frequency response is constant and independent of frequency.

$M_i=N_i$



§9.6 LTI Systems Characterized by Linear Constant-Coefficient Differential Equations

$$y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1y'(t)+a_0y(t)$$
$$=b_mx^{(m)}(t)+\cdots+b_1x'(t)+b_0x(t)$$

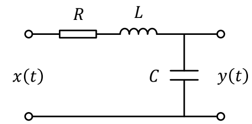
Taking the Laplace transform of both sides of the equation

$$(s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0)Y(s)$$
$$=(b_ms^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0)X(s)$$

So that

$$H(s)=\frac{Y(s)}{X(s)}=\frac{b_ms^m+b_{m-1}s^{m-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$
$$=\frac{\sum_{k=0}^{m}b_ks^k}{\sum_{k=0}^{n}a_ks^k}\ \ (a_n=1)$$

Example 9.19 Determine the system function and causality of the depicted RLC circuit.



Solution:

$$RC\frac{d}{dt}y(t)+L\frac{d}{dt}\left[C\frac{d}{dt}y(t)\right]+y(t)=x(t)$$

$$\frac{d^2}{dt^2}y(t)+\frac{R}{L}\frac{d}{dt}y(t)+\frac{1}{LC}y(t)=\frac{1}{LC}x(t)$$

$$H(s)=\frac{\frac{1}{LC}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$

Since R, L and C are positive, this system must be stable.

Example 9.20 Consider a casual LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2}{dt^2}y(t)+7\frac{d}{dt}y(t)+12y(t)=x(t)$$

a. Determine the system function $H(s)$, sketch the pole-zero plot of $H(s)$ and indicate the ROC of $H(s)$.

b. Compute the impulse response $h(t)$.

c. Determine the output of the system when $x(t)=e^{-t}u(t)$.

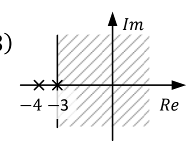
d. Determine the output of the system when $x(t)=e^{2t}\ (-\infty<t<\infty)$.

Solution:

$$H(s)=\frac{1}{s^2+7s+12}=\frac{1}{(s+3)(s+4)}\ (ROC\colon\ Re\{s\}>-3)$$

$$H(s)=\frac{1}{s+3}-\frac{1}{s+4}\ (ROC\colon\ Re\{s\}>-3)$$

$$h(t)=(e^{-3t}-e^{-4t})u(t)$$



When $x(t)=e^{-t}u(t)$,

$$X(s)=\frac{1}{s+1}\ (ROC\colon\ Re\{s\}>-1)$$

$$Y(s)=H(s)X(s)=\frac{1}{(s+1)(s+3)(s+4)}$$

$$Y(s)=\frac{\frac{1}{6}}{s+1}+\frac{-\frac{1}{2}}{s+3}+\frac{\frac{1}{3}}{s+4}\ (ROC\colon\ Re\{s\}>-1)$$

$$y(t)=\left(\frac{1}{6}e^{-t}-\frac{1}{2}e^{-3t}+\frac{1}{3}e^{-4t}\right)u(t)$$

When $x(t)=e^{2t}$,

$x(t)=e^{2t}=e^{2t}u(t)+e^{2t}u(-t)$

$X(s)=L[x(t)]$

$Y(s)=H(s)X(s)$

$y(t)=L^{-1}[Y(s)]$

This is very complicated.

$$x(t)=e^{st}\longrightarrow\boxed{h(t)}\longrightarrow y(t)=H(s)e^{st}$$

When $x(t)=e^{2t}$,

$$y(t)=h(t)*x(t)=H(s_0)e^{s_0t}|_{s_0=2}$$

$$=\frac{1}{30}e^{2t}\ (-\infty<t<\infty)$$

Example 9.21 Suppose that we are given the following information about an LTI system:

a. The system is causal.

b. The system function is rational and has only two poles at $s=-2$ and $s=-4$.

c. When $x(t)=1$, $y(t)=0$.

d. The value of the impulse response at $t=0^+$ is 4.

Compute the system function $H(s)$

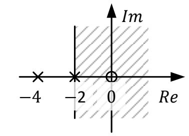
Solution:

$$H(s)=\frac{N(s)}{(s+2)(s+4)}=\frac{N(s)}{s^2+6s+8}\quad (ROC:\ Re\{s\}>-2)$$

$\because x(t)=1=e^{0t}$

$\therefore H(s)|_{s=0}=0$

$\therefore N(s)=s\hat{N}(s)$



Using the initial value theorem,

$$h(0^+)=4=\lim_{s\to\infty} sH(s)=\lim_{s\to\infty}\frac{s^2\hat{N}(s)}{s^2+6s+8}$$

$\therefore \hat{N}(s)=4$

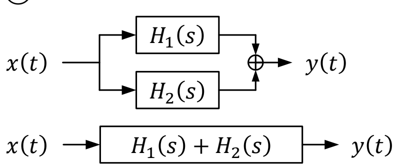
Therefore,

$$H(s)=\frac{4s}{s^2+6s+8}\quad (ROC:\ Re\{s\}>-2)$$

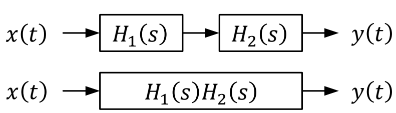
§9.7 System Function Algebra and Block Diagram Representations

1.System Functions for Interconnections of LTI Systems

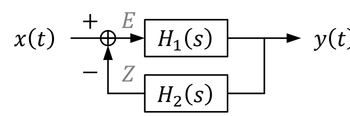
① The Parallel Interconnection



② The Series Interconnection



③ The Feedback interconnection



$$E(s)=X(s)-Z(s)=X(s)-H_2(s)Y(s)$$

$$Y(s)=H_1(s)E(s)=H_1(s)X(s)-H_1(s)H_2(s)Y(s)$$

$$Y(s)=\frac{H_1(s)}{1+H_1(s)H_2(s)}X(s)$$

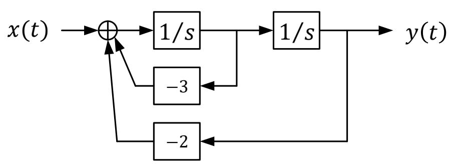
$$H(s)=\frac{H_1(s)}{1+H_1(s)H_2(s)}$$

2. Block Diagram Representations for Causal LTI Systems Described by Differential Equations and Rational System Functions

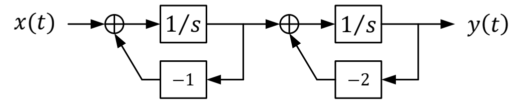
$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

$$H(s) = \underset{①}{\frac{1}{s^2+3s+2}} = \underset{②}{\frac{1}{(s+1)(s+2)}} = \underset{③}{\frac{1}{s+1} - \frac{1}{s+2}}$$

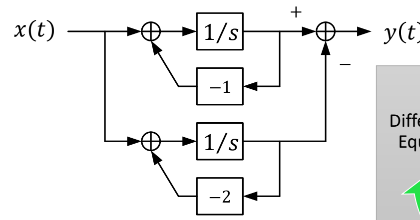
①Direct Form

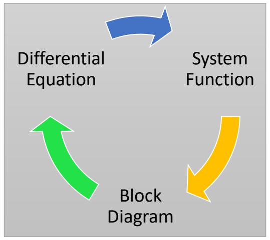


② Cascade Form



③ Parallel Form





§9.8 The Unilateral Laplace Transform

1. Definition

$$\mathcal{X}(s) = \mathcal{UL}[x(t)] \triangleq \int_{0^-}^{\infty} x(t)e^{-st}\,dt \;\; (ROC\colon\, Re\{s\} > \sigma_0)$$

$$x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)$$

2. Properties of Unilateral Laplace Transform

① Differentiation in the time domain

$$\frac{d}{dt}x(t) \overset{\mathcal{UL}}{\longleftrightarrow} s\mathcal{X}(s) - x(0^-)$$

$$\frac{d^2}{dt^2}x(t) \overset{\mathcal{UL}}{\longleftrightarrow} s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

$$\frac{d^n}{dt^n}x(t) \overset{\mathcal{UL}}{\longleftrightarrow} s^n\mathcal{X}(s) - \sum_{i=1}^{n} s^{n-i}x^{(i-1)}(0^-)$$

② Integration in the time domain

$$\int_{0^-}^{t} x(\tau)\,d\tau \overset{\mathcal{UL}}{\longleftrightarrow} \frac{1}{s}\mathcal{X}(s)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{\mathcal{UL}}{\longleftrightarrow} \frac{1}{s}\mathcal{X}(s) + \int_{-\infty}^{0^-} x(t)\,dt$$

③ Initial and Final-value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$,

$$x(0^+) = \lim_{s\to\infty} s\mathcal{X}(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\mathcal{X}(s)$$

Compare the difference between the lateral and unilateral Laplace Transforms:

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+a)^n} \quad (ROC:\ Re\{s\} > -a)$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \overset{\mathcal{UL}}{\longleftrightarrow} \frac{1}{(s+a)^n}$$

$$e^{-a(t+1)}u(t+1) \overset{\mathcal{L}}{\longleftrightarrow} \frac{e^s}{s+a} \quad (ROC:\ Re\{s\} > -a)$$

$$e^{-a(t+1)}u(t+1) \overset{\mathcal{UL}}{\longleftrightarrow} \frac{e^{-a}}{s+a}$$

$$\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\text{Zero input response}} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\text{Zero state response}}$$

For example, $\alpha = 2,\ \beta = 3$ and $\gamma = -5$

$$\mathcal{Y}(s) = \frac{3s^2+4s+2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{1}{s+1} + \frac{3}{s+2}$$

$$y(t) = [1 - e^{-t} + 3e^{-2t}]u(t) \ \ (t > 0)$$

Example 9.22 Consider a casual LTI system described by the differential equation

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 2y(t) = x(t)$$

with initial condition $y(0^-) = \beta$ and $y'(0^-) = \gamma$.

Let $x(t) = \alpha u(t)$, determine $y(t)$.

Solution:

Applying the unilateral Laplace transform to both sides of the differential equation

$$[s^2\mathcal{Y}(s) - sy(0^-) - y'(0^-)] + 3[s\mathcal{Y}(s) - y(0^-)] + 2\mathcal{Y}(s) = \mathcal{X}(s)$$

$$s^2\mathcal{Y}(s) - \beta s - \gamma + 3s\mathcal{Y}(s) - 3\beta + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

UOG Homework

9.13	**9.28**	**9.34**	**9.35**
9.2	9.5	9.7	9.8
9.9	9.21(a)(b)(i)(j)	9.22(a)(b)(c)	9.31
9.32	9.33		

1. Do not wait until the last minute
2. Express your own idea and original opinion
3. Keep in mind the zero-tolerance policy on plagiarism