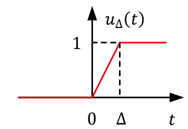
Continuous-Time Complex Exponential Signal	Discrete-Time Complex Exponential Signal
$x(t) = e^{j\omega_0 t}$	$x[n] = e^{j\omega_0 n}$
$t \in R$	$n \in Z$
	Only when $\omega_0/2\pi \in Q$
$x(t) = x(t + kT)$ $(\exists T \neq 0, \forall k \in Z)$	$x[n] = x[n + mN]$ $(\exists N \neq 0, \forall m \in Z)$
	Let $\omega_0/2\pi = p/q$
$T_0 = \dfrac{2\pi}{\lvert\omega_0\rvert}$ $(\omega_0 \neq 0)$	$N_0 = \dfrac{q}{\gcd(p, q)}$ $(\omega_0 \neq 0)$
	$e^{j(\omega_0 + 2\pi k)n} = e^{j\omega_0 n}$ $(\forall k \in Z)$

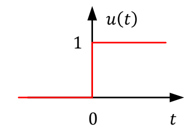
5. Singular Signals and Common Signals

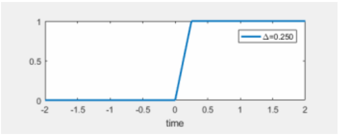
① Continuous-Time Singular Signals

a. The Unit Step Function

$$u(t) \triangleq \lim_{\Delta \to 0} u_\Delta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



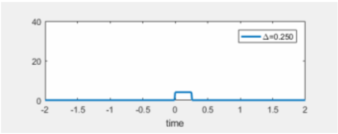


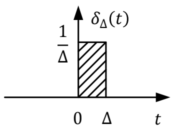


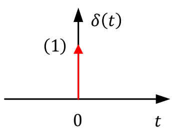
b. The Unit Impulse Function

$$\delta(t) \triangleq \lim_{\Delta \to 0} \delta_\Delta(t) = \begin{cases} \text{"}\infty\text{"} & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1$$



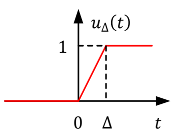


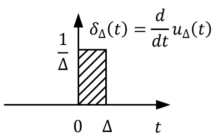


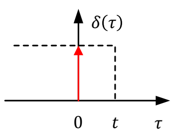
c. Relationship between $u(t)$ and $\delta(t)$

$$\delta(t) \triangleq \lim_{\Delta \to 0} \delta_\Delta(t) = \lim_{\Delta \to 0} \frac{d}{dt} u_\Delta(t) = \frac{d}{dt} \lim_{\Delta \to 0} u_\Delta(t) = \frac{d}{dt} u(t)$$

$$\begin{cases} \delta(t) = \dfrac{d}{dt} u(t) \\ u(t) = \displaystyle\int_{-\infty}^{t} \delta(\tau)d\tau = \int_{0}^{\infty} \delta(t - \tau)d\tau \end{cases}$$







d. Properties of $\delta(t)$

I. Sifting Property

When $x(t)$ is a continuous function at $t=0$

$$\int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0)$$

$$\int_{-\infty}^{+\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t-t_0)dt = 1 \ \ (\varepsilon > 0)$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} x(t)\delta(t-t_0)dt = x(t_0)$$

Proof:

$\because x(t)\delta(t) = x(0)\delta(t)$

$$\therefore \int_{-\infty}^{+\infty} x(t)\delta(t)dt = \int_{-\infty}^{+\infty} x(0)\delta(t)dt = x(0)\int_{-\infty}^{+\infty}\delta(t)dt$$

$$\therefore \int_{-\infty}^{+\infty} x(t)\delta(t)dt = x(0)$$

II. Time scaling

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta[a(t-t_0)] = \frac{1}{|a|}\delta(t-t_0)$$

Proof:

$$\because \int_{-\infty}^{+\infty}\delta(at)dt = \begin{cases} \int_{-\infty}^{\infty}\delta(\tau)d\left(\frac{\tau}{a}\right) = \frac{1}{a} & a>0 \\ \int_{\infty}^{-\infty}\delta(\tau)d\left(\frac{\tau}{a}\right) = -\frac{1}{a} & a<0 \end{cases} = \frac{1}{|a|}$$

$\because \ \delta(at) = 0 \quad \text{when} \quad t \neq 0$

$$\therefore \delta(at) = \frac{1}{|a|}\delta(t)$$

Example 1.2:

a.

$$\int_{-\infty}^{+\infty} \delta\left(t-\frac{1}{2}\right)\sin\pi t dt$$

b.

$$\int_{0^-}^{3} e^{-2t}\sum_{k=-\infty}^{+\infty}\delta(t-2k)\,dt$$

c.

$$\int_{0^+}^{3} e^{-2t}\sum_{k=-\infty}^{+\infty}\delta(t-2k)\,dt$$

d.

$$\int_{-\infty}^{+\infty}\sqrt{t^2+4t-1}\delta(2t-2)dt$$

III. Odd and Even

$\delta(t) = \delta(-t)$

Proof:

$\because \ \delta(at) = \frac{1}{|a|}\delta(t)$

Let $a = -1$

$\therefore \delta(-t) = \frac{1}{|-1|}\delta(t) = \delta(t)$

Moreover,

$$\frac{d}{dt}\delta(t) = -\frac{d}{dt}\delta(-t)$$

$$\frac{d^n}{dt^n}\delta(t) = (-1)^n \frac{d^n}{dt^n}\delta(-t)$$

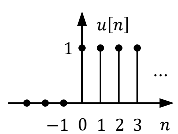
- Mathematically, the delta function is not a function, because it is too singular.
- Instead, it is said to be a distribution.
- It is a generalized idea of functions, but can be used only inside integrals.
- In fact, $\int_{-\infty}^{+\infty} x(t)\delta(t)dt$ can be regarded as an "operator" which pulls the value of a function at zero.
- But as long as it is understood that the delta function is eventually integrated, we can use it as if it is a function.

② Discrete-Time Singular Signals

a. The Unit Step Sequence

$$u[n] \triangleq \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

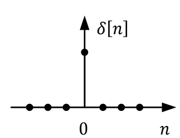
$u[0] = 1$



b. The Unit Impulse Sequence

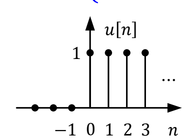
$$\delta[n] \triangleq \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

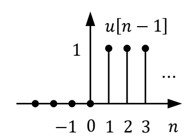
$$\sum_{n=-\infty}^{\infty} \delta[n] = 1$$

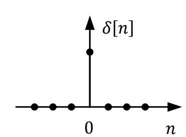


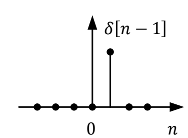
c. Relationship between $u[n]$ and $\delta[n]$

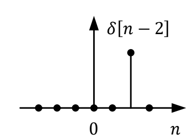
$$\begin{cases} \delta[n] = u[n] - u[n-1] \\ u[n] = \sum_{k=0}^{\infty} \delta[n-k] \overset{m=n-k}{\Longleftrightarrow} u[n] = \sum_{m=-\infty}^{n} \delta[m] \end{cases}$$











d. Properties of $\delta[n]$

I. Sifting Property

$$x[k]\delta[n-k] = x[n]\delta[n-k]$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

II. Time Scaling

$$\delta[kn] = \delta[n]$$

$$\delta[k(n-n_0)] = \delta[n-n_0]$$

III. Odd and Even

$$\delta[n] = \delta[-n]$$

$$\delta[n-n_0] = \delta[n_0-n]$$

Q1.2 Consider the discrete-time signal

$$x[n] = 1 - \sum_{k=3}^{\infty} \delta[n-1-k]$$

Determine the values of the integers M and n_0 so that $x[n]$ may be expressed as

$$x[n] = u[Mn - n_0]$$

6. Signal Energy and Power

① Energy:

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E_\infty = \lim_{N\to\infty} \sum_{-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

② Power:

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N} \sum_{-N}^{N} |x[n]|^2$$

Q1.3 Consider the continuous-time signal

$$x(t) = \delta(t+2) - \delta(t-2)$$

Calculate the value of E_∞ of the signal

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$