§10.6 LTI Systems Characterized by Linear Constant-Coefficient Difference Equations

A general form of the difference equation is written as

$$\sum_{r=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_k x[n-k]$$

Taking the z-transform of both sides of the equation

$$\sum_{r=0}^{N} a_k z^{-k} Y(z) = \sum_{r=0}^{M} b_k z^{-k} X(z)$$

So that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{r=0}^{M} b_k X(z)}{\sum_{r=0}^{N} a_k Y(z)}$$

Example 10.8 Consider a stable discrete-time LTI System described by the following difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = -\frac{3}{2}x[n-1]$$

a. Determine the system function $H(z)$ of the system, sketch the pole-zero plot of $H(z)$ and indicate the ROC of $H(z)$.

b. Compute the impulse response $h[n]$.

c. For $x[n] = (-1)^n \ (-\infty < n < \infty)$, determine the output $y[n]$ of the system.

d. For $x[n] = \cos \pi n \ (-\infty < n < \infty)$, determine the output $y[n]$ of the system.

Solution:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{3}{2}z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

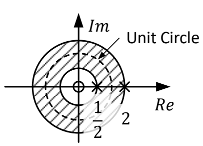
Because this system is stable, the ROC of $H(z)$ must contain the unit circle.

$$ROC\colon \frac{1}{2} < |z| < 2$$

$$H(z) = \frac{-\frac{3}{2}z}{\left(z-\frac{1}{2}\right)(z-2)}$$

First order zero: $z = 0$

First order poles: $z = \frac{1}{2}$ and $z = 2$



$$H(z) = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{-1}{1-2z^{-1}}$$

$$\left(ROC\colon \frac{1}{2} < |z| < 2\right)$$

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[-n-1]$$

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = H(z)z^n$$

$$z_0 = -1 \in ROC$$

$$y[n] = H(z_0){z_0}^n\big|_{z_0=-1} = H(-1)(-1)^n = \frac{1}{3}(-1)^n$$

Example 10.9 Suppose that we are given the following information about an LTI system.

If the input to the system is $x_1[n] = (\frac{1}{6})^n u[n]$, then output is $y_1[n] = \left[A\left(\frac{1}{2}\right)^n + 10\left(\frac{1}{3}\right)^n\right]u[n]$, where A is a real number.

If $x_2[n] = (-1)^n$, then the output is $y_2[n] = \frac{7}{4}(-1)^n(-\infty <$

Solution:

$$X_1(z) = \frac{1}{1-\frac{1}{6}z^{-1}} \quad \left(ROC:\ |z| > \frac{1}{6}\right)$$

$$Y_1(z) = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{10}{1-\frac{1}{3}z^{-1}} = \frac{(A+10)-(5+\frac{A}{3})z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}$$

$$\left(ROC:\ |z| > \frac{1}{2}\right)$$

$$H(z) = \frac{Y_1(z)}{X_1(z)} = \frac{\left[(A+10)-\left(5+\frac{A}{3}\right)z^{-1}\right]\left[1-\frac{1}{6}z^{-1}\right]}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$\left(ROC:\ |z| > \frac{1}{2}\right)$$

$$H(-1) = \frac{\left[(A+10)+\left(5+\frac{A}{3}\right)\right]\times\frac{7}{6}}{\frac{3}{2}\times\frac{4}{3}} = \frac{7}{4}$$

$A = -9$

$$H(z) = \frac{1-\frac{13}{6}z^{-1}+\frac{1}{3}z^{-2}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} \quad \left(ROC:\ |z| > \frac{1}{2}\right)$$

The order of the numerator is not greater than the order of the denominator. Therefore, this system is causal.

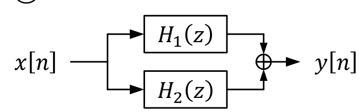
The difference equation describing the system is written as

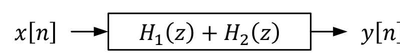
$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{13}{6}x[n-1] + \frac{1}{3}x[n-2]$$

§10.7 System Function Algebra and Block Diagram Representations

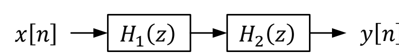
1.System Functions for Interconnections of LTI Systems

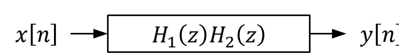
① The Parallel Interconnection



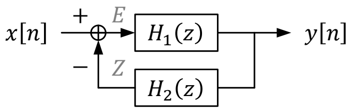


② The Series Interconnection





③ The Feedback interconnection



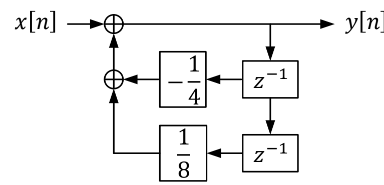
$$E(z) = X(z) - Z(z) = X(z) - H_2(z)Y(z)$$
$$Y(z) = H_1(z)E(z) = H_1(z)X(z) - H_1(z)H_2(z)Y(z)$$
$$Y(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}X(z)$$
$$H(z) = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

2. Block Diagram Representations for Causal LTI Systems Described by Difference Equations and Rational System Functions

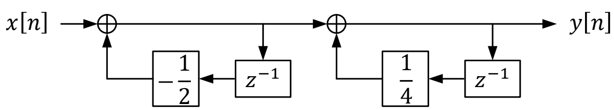
$$y[n] + \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$$

$$H(z) = \underbrace{\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}}_{①} = \underbrace{\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}}_{②} = \underbrace{\frac{\frac{2}{3}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}}_{③}$$

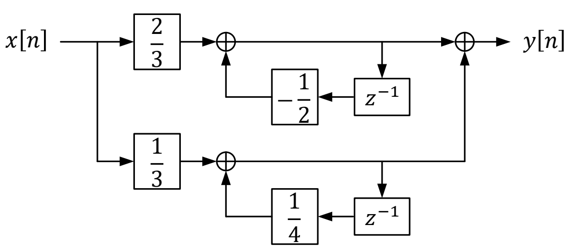
① Direct Form



② Cascade Form



③ Parallel Form



§10.8 The Unilateral Z Transform

1. Definition

$$\mathcal{X}(z) = \mathcal{UZ}\{x[n]\} \triangleq \sum_{n=0}^{\infty} x[n]z^{-n} \quad (ROC: |z| > r)$$

$$\sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]u[n]\}z^{-n} = \mathcal{Z}\{x[n]u[n]\}$$

$$x[n] \xleftrightarrow{\mathcal{UZ}} \mathcal{X}(z)$$

2. Properties of Unilateral Z Transform

① Initial-Value Theorem

$$x[0] = \lim_{z\to\infty} \mathcal{X}(z)$$

② Time Shifting ($n_0 > 0$)

$$x[n+n_0] \overset{UZ}{\longleftrightarrow} z^{n_0}\left\{\mathcal{X}(z) - \sum_{m=0}^{n_0-1} x[m]z^{-m}\right\}$$

$$x[n-n_0] \overset{UZ}{\longleftrightarrow} z^{-n_0}\left\{\mathcal{X}(z) + \sum_{m=-n_0}^{-1} x[m]z^{-m}\right\}$$

Particularly,

$$x[n+1] \overset{UZ}{\longleftrightarrow} z\mathcal{X}(z) - zx[0]$$
$$x[n-1] \overset{UZ}{\longleftrightarrow} z^{-1}\mathcal{X}(z) + x[-1]$$
$$x[n-2] \overset{UZ}{\longleftrightarrow} z^{-2}\mathcal{X}(z) + z^{-1}x[-1] + x[-2]$$

3. Solving Difference Equations Using the Unilateral Z-Transform

Example 10.10 Consider a casual LTI system described by the differential equation

$$y[n] + 3y[n-1] = x[n]$$

with initial condition $y[-1] = \beta$.

Letting $x[n] = \alpha u[n]$, determine $y[n]$.

Solution:

Applying the unilateral z-transform to both sides of the difference equation

$$\mathcal{Y}(z) + 3\{z^{-1}\mathcal{Y}(z) + y[-1]\} = \frac{\alpha}{1-z^{-1}}$$

$$\mathcal{Y}(z) = \underbrace{-\frac{3\beta}{1+3z^{-1}}}_{\text{Zero input response}} + \underbrace{\frac{\alpha}{(1+3z^{-1})(1-z^{-1})}}_{\text{Zero state response}}$$

For example, $\alpha = 8$ and $\beta = 1$

$$\mathcal{Y}(z) = \frac{3}{1+3z^{-1}} + \frac{2}{1-z^{-1}}$$

$$y[n] = 3(-3)^n u[n] + 2u[n] + (n \geq 0)$$

UOG Homework			
10.24	**10.31**	**10.47**	
10.2	10.3	10.6	10.10(a)

① Do not wait until the last minute
② Express your own idea and original opinion
③ Keep in mind the zero-tolerance policy on plagiarism