⑥ The Sign Signal

$$\mathrm{sgn}(t) = \begin{cases} 1 & (t > 0) \\ -1 & (t < 0) \end{cases}$$

$$F[\mathrm{sgn}(t)] = \lim_{a \to 0^+} \{ F[e^{-at}u(t)] - F[e^{at}u(-t)]\}$$

$$= \lim_{a \to 0^+} \left\{ \frac{1}{a + j\omega} - \frac{1}{a - j\omega} \right\} = \frac{2}{j\omega}$$

⑦ The Unit Step Signal

$$\mathcal{F}[u(t)] = \mathcal{F}\left[\frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$

⑧ The DC Signal

$$\mathcal{F}^{-1}[2\pi\delta(\omega)] = \int_{-\infty}^{\infty} \delta(j\omega)e^{j\omega t}d\omega = 1$$

§4.2 The Fourier Transform for Periodic Signals

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\boxed{e^{j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)}$$

Consider:

$$\mathcal{F}^{-1}[2\pi\delta(\omega - \omega_0)\,] = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$\tilde{X}(j\omega) = \sum_{k=-\infty}^{\infty} a_k \mathcal{F}\left[e^{jk\omega_0 t}\right] = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example 4.1 Determine the Fourier Transform of $\cos\omega_0 t$ and $\sin\omega_0 t$.

Solution:

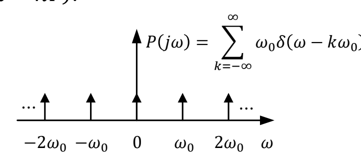
$$e^{j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$

$$e^{-j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega + \omega_0)$$

$$\cos\omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \overset{F}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \overset{F}{\longleftrightarrow} \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

Example 4.2 Determine the Fourier Transform of the impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$.



Solution:

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}F[\delta(t)]\Big|_{\omega = k\omega_0} = \frac{1}{T} \quad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \omega_0 \delta(\omega - k\omega_0)$$

§4.3 Properties of the Continuous-Time Fourier Transform

$$\boxed{x(t)\xleftrightarrow{F}X(j\omega) \text{ and } y(t)\xleftrightarrow{F}Y(j\omega)}$$

1. Linearity

$$Ax(t)+By(t)\xleftrightarrow{F}AX(j\omega)+BY(j\omega)$$

2. Time Shifting

$$x(t-t_0)\xleftrightarrow{F}e^{-j\omega t_0}X(j\omega)$$

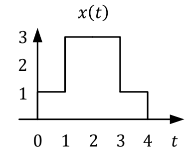
3. Time Reversal

$$x(-t)\xleftrightarrow{F}X(-j\omega)$$

4. Time and Frequency Scaling

$$x(at)\xleftrightarrow{F}\frac{1}{|a|}X\left(\frac{j\omega}{a}\right)$$

Example 4.3 Compute the Fourier Transform $X(j\omega)$ of the signal $x(t)$ shown below.



Solution:

$x_1(t)=u(t+1)-u(t-1)$

$x_2(t)=u(t+2)-u(t-2)$

$x(t)=2x_1(t-2)+x_2(t-2)$

$$\begin{aligned}X(j\omega)&=2X_1(j\omega)e^{-j2\omega}+X_2(j\omega)e^{-j2\omega}\\&=[4Sa(\omega)+4Sa(2\omega)]e^{-j2\omega}\\&=\left[4\frac{\sin(\omega)}{\omega}+2\frac{\sin(2\omega)}{\omega}\right]e^{-j2\omega}\end{aligned}$$

5. Conjugation and Conjugate Symmetry

$$x^*(t)\xleftrightarrow{F}X^*(-j\omega)$$

Proof:

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$X(-j\omega)=\int_{-\infty}^{\infty}x(t)e^{j\omega t}dt$$

$$X^*(-j\omega)=\int_{-\infty}^{\infty}x^*(t)e^{-j\omega t}dt=F[x^*(t)]$$

When $x(t)=x^*(t)$,

$$X(j\omega)=X^*(-j\omega)$$
$$X^*(j\omega)=X(-j\omega)$$

When $x(t)=x^*(t)$,

① $X(j\omega)=Re[X(j\omega)]+jIm[X(j\omega)]\Longrightarrow$

$$\begin{cases}Re[X(j\omega)]=Re[X(-j\omega)]\\Im[X(j\omega)]=-Im[X(-j\omega)]\end{cases}$$

② $X(j\omega)=|X(j\omega)|e^{j\measuredangle X(j\omega)}\Longrightarrow\begin{cases}|X(j\omega)|=|X(-j\omega)|\\\measuredangle X(j\omega)=-\measuredangle X(-j\omega)\end{cases}$

③ $x(t)=x_e(t)+x_o(t)\Longrightarrow\begin{cases}x_e(t)\xleftrightarrow{F}Re[X(j\omega)]\\x_o(t)\xleftrightarrow{F}Im[X(j\omega)]\end{cases}$

④ $x(t)=x(-t)\Longrightarrow\begin{cases}X(j\omega)=X^*\ (j\omega)\\X(j\omega)=X(-j\omega)\end{cases}$

⑤ $x(t)=-x(-t)\Longrightarrow\begin{cases}X(j\omega)=j|X(j\omega)|\\X(-j\omega)=-X(j\omega)\end{cases}$

Example 4.4 Compute the Fourier Transform of the bilateral exponential signal $x(t) = e^{-a|t|}$ $(Re\{a\} \geq 0)$.

Solution:

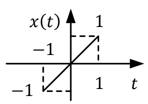
$$e^{-at}u(t) \overset{F}{\longleftrightarrow} \frac{1}{a+j\omega}$$

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2Ev[e^{-at}u(t)]$$

$$Ev[e^{-at}u(t)] \overset{F}{\longleftrightarrow} Re\left[\frac{1}{a+j\omega}\right] = \frac{a}{a^2+\omega^2}$$

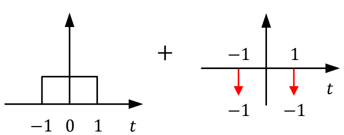
$$X(j\omega) = \mathcal{F}[x(t)] = \frac{2a}{a^2+\omega^2}$$

Example 4.5 Calculate the Fourier Transform $X(j\omega)$ for the signal $x(t)$ as shown.



Solution:

$$g(t) = \frac{d}{dt}x(t)$$



$$G(j\omega) = 2Sa(\omega) - e^{-j\omega} - e^{j\omega} = \frac{2sin\omega - 2\omega cos\omega}{\omega}$$

$$X(j\omega) = \pi G(0)\delta(\omega) + \frac{1}{j\omega}G(j\omega) = \frac{2sin\omega - 2\omega cos\omega}{j\omega^2}$$

$(\because G(0) = 0)$

6. Differentiation

$$\frac{d}{dt}x(t) \overset{F}{\longleftrightarrow} j\omega X(j\omega) \qquad u_1(t) = \frac{d}{dt}\delta(t) \overset{F}{\longleftrightarrow} j\omega$$

7. Integration

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F}{\longleftrightarrow} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega) \qquad u(t)$$

Proof:

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left[\frac{d}{dt}e^{j\omega t}\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(j\omega)]e^{j\omega t}d\omega$$

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)d\tau\right] = \mathcal{F}[u(t)]\mathcal{F}[x(t)] = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right]X(j\omega)$$