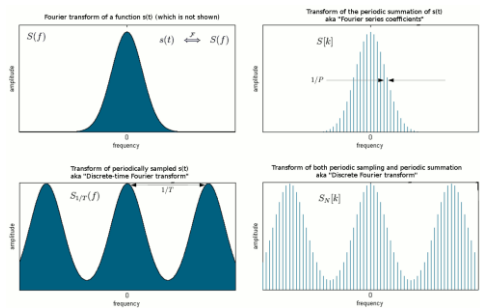


Chapter 5 The Discrete-Time Fourier Transform



$$\begin{aligned}\tilde{x}[n] &= \sum_{k=\langle N \rangle} a_k e^{j\frac{2\pi}{N}kn} \\ a_k &= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk\frac{2\pi}{N}n}\end{aligned}$$

Defining the envelope $X(e^{j\omega})$ as

$$\begin{aligned}X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ a_k &= \frac{1}{N} X(e^{jk\omega_0}) \Big|_{\omega_0 = \frac{2\pi}{N}}\end{aligned}$$

§5.1 Representation of Aperiodic Signals — the Discrete-Time Fourier Transform

1. Development of the Discrete-Time Fourier Transform

$$x[n] = \begin{cases} x[n] & -N_1 \leq n \leq N_2 \\ 0 & \text{otherwise} \end{cases}$$

The plot shows a discrete signal $x[n]$ as a series of vertical lines (impulses) on a horizontal axis labeled n . The signal is non-zero only between $-N_1$ and N_2 , with a peak at $n=0$.

$$\tilde{x}[n]$$

The plot shows the periodic extension $\tilde{x}[n]$ of the signal $x[n]$. It consists of repeated copies of the original signal $x[n]$ shifted by integer multiples of N (where $N = N_1 + N_2$), creating a periodic train of impulses.

$$\tilde{x}[n] = \begin{cases} x[n] & -N_1 \leq n \leq N_2 \\ \tilde{x}[n + N] & \text{always} \end{cases}$$

$$\begin{aligned}\tilde{x}[n] &= \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} \\ &= \frac{1}{2\pi} \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0\end{aligned}$$

The plot shows the periodic extension $\tilde{x}[n]$ on a frequency axis. The horizontal axis is labeled with $-\infty \leftarrow -N$, $-N_1$, 0 , N_2 , and $N \rightarrow \infty$. The signal is periodic with period N .

$$\begin{aligned}x[n] &= \lim_{N \rightarrow \infty} \tilde{x}[n] = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \\ &= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\omega}) e^{j\omega n} d\omega\end{aligned}$$

The **Discrete-Time Fourier Transform (DTFT) Pair** is defined as

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$
$$\begin{cases} x[n] = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\omega}) e^{j\omega n} d\omega \\ X(e^{j\omega}) = F[x[n]] = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \end{cases}$$

$X(e^{j\omega})$ is called the Discrete-Time Fourier Transform of $x[n]$,
a.k.a. the spectrum of $x[n]$.

Periodicity of the Discrete-Time Fourier Transform:

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

2. Examples of the Discrete-Time Fourier Transforms (DTFT)

Example 5.1 Determine the DTFT of $x[n] = a^n u[n]$ ($|a| <$

Example 5.2 Determine the DTFT of $x[n] = a^{|n|}$ ($|a| < 1$).

Solution:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} \\ &= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ \because |ae^{-j\omega}| &= |ae^{j\omega}| = |a| < 1 \\ X(e^{j\omega}) &= \frac{ae^{j\omega}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2} \end{aligned}$$

Example 5.3 Determine the DTFT of $x[n] = e^{j\omega_0 n}$.

Solution:

Presume that

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} 2\pi \delta[\omega - \omega_0 - 2\pi n] \\ F^{-1}[X(e^{j\omega})] &= \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\omega}) e^{j\omega n} d\omega \\ &= \frac{1}{2\pi} \int_{<2\pi>} \sum_{n=-\infty}^{\infty} 2\pi \delta[\omega - \omega_0 - 2\pi n] e^{j\omega n} d\omega \\ &= e^{j\omega_0 n} = x[n] \end{aligned}$$

3. Convergence Issues Associated with the Discrete-Time Fourier Transform

The discrete-time Fourier transform converges when the sequence is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

or the sequence has finite energy

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

The inverse discrete-time Fourier transform generally has no convergence issue because the interval of integration is finite, similar to the discrete-time Fourier series.

Example 5.4 Show that the DTFT of $x[n] = \delta[n]$ is $X(e^{j\omega}) = 1$.

Furthermore, an approximation of $x[n]$ is written as

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-w}^w X(e^{j\omega}) e^{j\omega n} d\omega$$

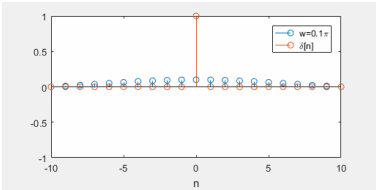
Discuss which value of w ensures $x[n] = \hat{x}[n]$.

Hint:

$$\hat{x}[n] = \frac{\sin wn}{\pi n}$$

When $w = \pi$,

$$\hat{x}[n] = \frac{\sin \pi n}{\pi n} = \delta[n]$$



Homework			

- ① Do not wait until the last minute

② Express your own idea and original opinion

③ Keep in mind the zero-tolerance policy on plagiarism