ECE495 Game Theory and Networked Systems

Lecture 16: Games and Information Spreading over Social Networks

In this class, we study interaction over a social network. We first assume the agents are rational and update according to Bayes rule and then relax the rationality assumption to look at more realistic setup with myopic agents, who will follow their neighbors in some sense.

1 Social Learning and Herding

In a Social Learning setting, an item is equally likely to be "good" or "bad." This 'state of the world' can be reflected by G, B. Furthermore, agents get I.I.D signals S_i with $\Pr(S_i = \theta)$=$p > 0.5$ (aligns with the true state of the world). Using these signals, the agents sequentially decide whether to buy or not buy item, such that $A_i = Yes, No$, where Yes corresponds to the belief that the state is G and No for B. Additionally, each agent can sees the previous agent's decision, so using this information coupled with her signal, the agent decides her agent A_i.

1st Agent: Follows it signal. This can be shown with the following using Bayes rule:

$$\begin{aligned} Pr(\theta = G|S_i = G) &= \frac{Pr(S_i = G|\theta = G)Pr(\theta = G)}{Pr(S_i = G|\theta = G)Pr(\theta = G) + Pr(S_i = \theta|\theta = B)Pr(\theta = B)} \\ &= \frac{0.5p}{0.5p + 0.5(1-p)} \\ &= p > 0.5. \end{aligned}$$

A similar calculation can be made for $Pr(\theta = B|S_i = B) = p > 0.5$. Given the first player's signal, his/her best response is to simply follow his/her signal.

2nd Agent: If its signal matches prior action, then the agent should also follow its signal (i.e., $A_2 = Y$ if $S_2 = G$ or $A_2 = N$ if $S_2 = B$.). We can see this decision from the following probability by again using Bayes rule:

$$\begin{aligned} Pr(\theta = G|A_i = Y, S_2 = G) &= \frac{Pr(A_1 = Y, S_2 = G|\theta = G)Pr(\theta = G)}{Pr(A_1 = Y, S_2 = G|\theta = G)Pr(\theta = G) + Pr(A_1 = Y, S_2 = B|\theta = B)Pr(\theta = B)} \\ &= \frac{0.5p^2}{0.5p^2 + 0.5(1-p)^2} = \frac{p^2}{p^2 + (1-p)^2} > 0.5. \end{aligned}$$

However, what if the agent receives a signal that doesn't match with prior action, it follows that the agent could also follow her signal.

$$\begin{aligned} Pr(\theta = G|A_1 = Y, S_2 = B) &= \frac{Pr(A_1 = Y, S_2 = B|\theta = G)Pr(\theta = G)}{Pr(A_1 = Y, S_2 = B)} \\ &= \frac{Pr(\theta = G)Pr(A_1 = Y|\theta = G)Pr(S_2 = B|\theta = G)}{Pr(A_1 = Y, S_2 = B, \theta = G) + Pr(A_1 = Y, S_2 = B, \theta = B)} \\ &= \frac{0.5p(1-p)}{0.5p(1-p) + 0.5(1-p)p} \\ &= 0.5. \end{aligned}$$

Here we assumed conditional independence of the signals S_1 and S_2 to have $Pr(A_1 = Y, S_2 = B|\theta = G) = Pr(A_1 = Y|\theta = G)Pr(S_2 = B|\theta = G)$.

The answer, 0.5, means that the agent is indifferent between buying and not buying. This agent can follow her signal or flip a coin and decide.

Now consider the third agent. The third agent's action will always follow the majority. Consider an example in which $A_1 = Y, A_2 = Y, S_3 = B$. The majority taken action is to buy, which follows from our probability:

$$\begin{aligned}
Pr(\theta = G|A_1 = Y, A_2 = Y, S_3 = B) &= \frac{Pr(A_1 = Y, A_2 = Y, S_3 = B|\theta = G)Pr(\theta = G)}{Pr(A_1 = Y, A_2 = Y, S_3 = B)} \\
&= \frac{0.5p^2(1-p)}{0.5p^2(1-p) + 0.5(1-p)^2 p} \\
&= \frac{p}{p + (1-p)} = p > 0.5
\end{aligned}$$

Thus, given the observations of two Y, the third agent will conclude that the state of the world is more likely to be G and hence will take action Y, even if his own signal says B. The third agent will therefore ignore his own signal and simply follow the majority rule.

But what about the 4th Agent? We only need to consider the case where $A_1 = Y, A_2 = Y, A_3 = Y, S_4 = B$ (if $S_4 = G$ then the 4th agent will choose Y). We can repeat the above calculation and conclude that the 4th agent will also follow majority rule and may ignore his own signal.

Given that the first two agents, as above, follows their Bayesian optimal action to do, which is to follow their signals. If the first two agents receive the same signal of G, then the third agent's action will always be Y. This illustrates the concept of herding: these agents will follow the "crowd" rather than solely using their information/signal. Thus, when calculating the Bayesian optimal action for the 4th agent, we can effectively remove the action of the 3rd player (because no matter what his signal was, he participated in herding behavior). Thus, given $A_1 = Y, A_2 = Y, S_4 = B$, we can repeat the calculation done for the third agent and conclude that the 4th Agent will also follow the majority opinion, and buy the object.

As more agents join the model, eventually hearding will happen with probability 1 (meaning all agents will carry out the same outcome). From a normative standpoint, are herds good or bad? A "bad" herd could be described as agents following wrong signals, and will happen with probability $(1-p)^2$ (when the first two agents happen to receive bad signals and carry out the wrong actions). This is not very efficient in the social learning context.

2 Network dynamics: Bass Model

The previous section studies a social network where each agent arrives in the system in a linear fashion and are Bayesian. This time let's look at a general model of a social network and are myopic (simple minded).

We will study how fast learning/technology adoption/disease (epidemic) spreads/information spreads in the system. The intuition is that the new technology will get adopted via interaction among the agents.

We first study the baseline model: Bass model 1969. Let $F(t)$ denote the fraction of agents adopted new technology at time t. Then we use the following dynamic to describe the state transition in the system.

$$F(t) = F(t-1) + p(1 - F(t-1)) + q(1 - F(t-1))F(t-1).$$

The first term reflects the state of the previous time slot. The second term captures adoption of new technology of the agents who have not done so yet, p is some parameter of the system. The last time models the social network interaction. The factor q is the imitation factor. The product of $(1 - F(t-1))F(t-1)$ is the interaction of agents who have not adopted the technology with the rest of the population (early adopters). We can gather the terms and have

$$F(t) - F(t-1) = (p + qF(t-1))(1 - F(t-1)).$$

We can then take a continuous time limit and approximate $F(t-1) \approx F(t)$ to derive a ode as

$$\frac{dF(t)}{dt} = (p + qF(t))(1 - F(t)).$$

We can assume $p > 0$ and $F(0) = 0$ as boundary condition to solve the above ode and have

$$F(t) = \frac{1 - e^{-(p+q)t}}{1 + \frac{q}{p} e^{-(p+q)t}}.$$

This evolution scales with time by a factor of $p + q$, meaning the larger the number $p + q$ the faster the dynamic will happen. Also the constant $\frac{q}{p}$ evaluates the relative scale of secondary (q) vs spontaneous (p) effects. When $p >> q$, the spontaneous adoption dominates and we have $F(t) \approx 1 - e^{-pt}$. This is a concavely increasing function approaching 1. In the beginning, the increase is very fast and then as less people are potentially switching, the speed of increase slows down. When $q = 2p$, we have that $F(t)$ is first convex and then concave, i.e., there is a phase transition. The adoption is slow in the beginning and when enough agents in the population adopts the technology then it takes off quickly, and then it will saturate.

This is the base model and there are many variations. For instance, we list the following three.

1. We can have two competing technologies, such as apple vs android adoption. The dynamic can be modeled by two ODEs. The interaction term can be complicated to model complement/competitive effects of the two. The parameters p and q.

2. The parameters p and q can be time varying. Maybe as time goes on, the factor p diminishes, since these are the stubborn agents who refused to adopt in the beginning. The early adopters and technology haters can be described by different groups and may have distinct intrinsic properties.

3. There could also be healing effects. This is more often used to model disease, when an agent can be infected and then go back to recovered state. The baseline model can be easily modeled to reflect this by
$$\frac{dF(t)}{dt} = (p + qF(t))(1 - F(t)) - gF(t).$$
The last term shows that it is possible to revert back to the old state.

3 Social Network Linear Threshold Model

The previous section models a large population/continuous agents. We can also model the network topology explicitly and view networks as individual agents with states. The state can reflect infected or susceptible or recovered in the disease spreading model. The dynamics happens via some local neighborhood rules, such as pairwise information spreading, majority of the neighborhood, probabilistic rules or Bayesian update based on some reward systems.

A few famous models include the SI model: susceptible-infected model. One infected node can infect its neighbors, the process stops when the disease reaches the entire network.

A more realistic model is SIR model: susceptible-infected-recovered model. Here an agent may recover and become immune to the disease.

A another variation is called SIS model: susceptible-infected-susceptible. The agents may get infected, and then recover to become susceptible again.

The question in these setting is typically about properties of the networks to lead to a giant component (covering a large part of the network).

Linear threshold model is another useful model. Here each agent has a threshold θ_i in $[0, 1]$. It assigns weights to its neighbors w_{ij}. The node will switch state to $s_i = 1$ from $s_i = 0$ when

$$\sum_j w_{ij} s_j \geq \theta_i.$$

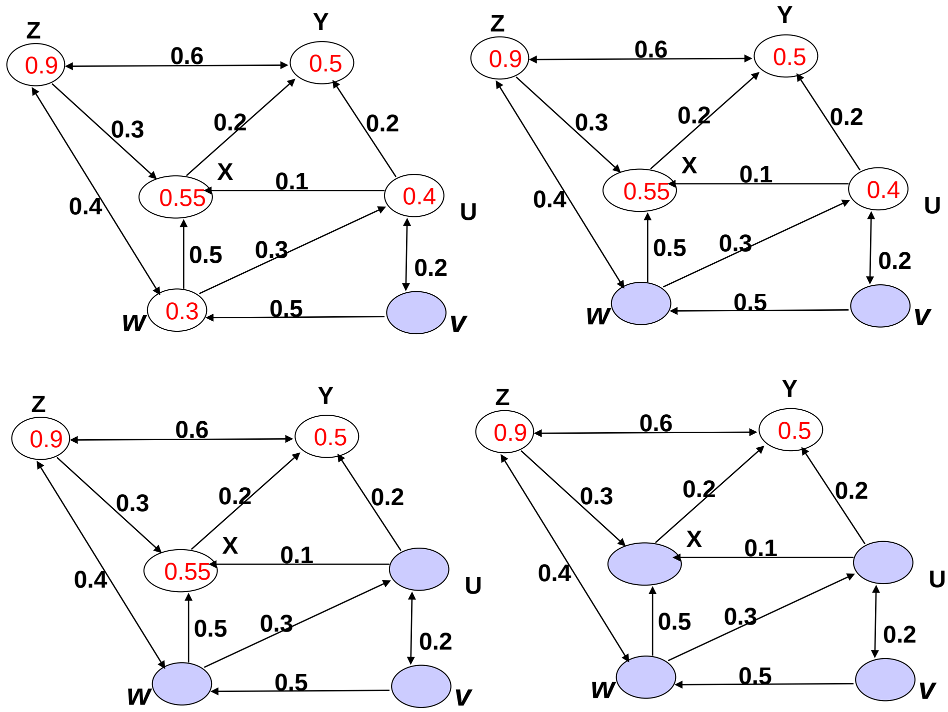

Figure 1: Linear Threshold Model Illustration

The weights may or may not be normalized to 1. The analysis for these kind of problems often does not exhibit closed form solutions. Instead, we will use numerical tools. For instance, observe the network in Fig. 1, where the numbers on the links represent weights and the red numbers inside each node represents their threshold. Starting from only node v with $s_v = 1$, we have after one period, node W switches to state 1 (0.5¿0.3). In the next period, U also switches (0.2+0.3¿0.4). The next period, X also switches. The process then ends at this point, since no more nodes will switch state at this point (or any of the future points).

This dynamic is deterministic. A variant of it could use the incoming weights to reflect the probability of switching and thus leading to a probabilistic model (not linear threshold model), which could continue to evolve after 3 periods.

Let $f(S)$ denote the set of nodes which has state 1, starting from only the set of nodes in S begin in state 1. One can show that this function f is submodular, i.e.,

$$f(S+v) - f(S) \geq f(T+v) - f(T),$$

for some $S \subset T$ and $v \notin T$. This says that the additional initial unit has more contribution for a smaller set than a larger ones. This notion is the opposite of supermodlarity and is related to diminishing return of scale for concavity.

References

[1] Berry, Randall A., and Ramesh Johari. "Economic modeling in networking: A primer." *Foundations and Trends® in Networking*, 6.3 (2013): 165-286.