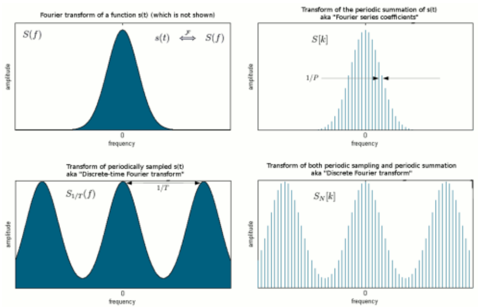
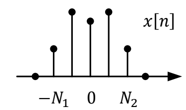
Chapter 5 The Discrete-Time Fourier Transform

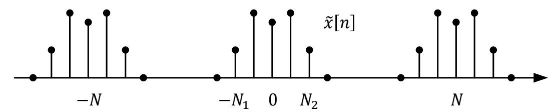


§5.1 Representation of Aperiodic Signals — the Discrete-Time Fourier Transform

1. Development of the Discrete-Time Fourier Transform

$$x[n] = \begin{cases} x[n] & -N_1 \le n \le N_2 \\ 0 & otherwise \end{cases}$$





$$\tilde{x}[n] = \begin{cases} x[n] & -N_1 \le n \le N_2 \\ \tilde{x}[n+N] & always \end{cases}$$

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} \tilde{x}[n]e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]e^{-jk\frac{2\pi}{N}n}$$

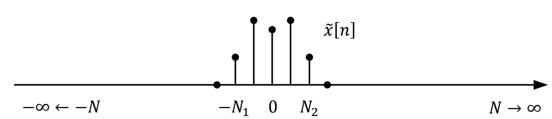
Defining the envelope $X(e^{j\omega})$ as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$a_k = \frac{1}{N}X(e^{jk\omega_0})\bigg|_{\omega_0=\frac{2\pi}{N}}$$

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}$$

$$= \frac{1}{2\pi}\sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n}\,\omega_0$$



$$x[n] = \lim_{N\to\infty} \tilde{x}[n] = \lim_{N\to\infty} \frac{1}{2\pi}\sum_{k=<N>} \frac{1}{N}X(e^{jk\omega_0})\, e^{jk\omega_0 n}\,\omega_0$$

$$= \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}\, d\omega$$

The Discrete-Time Fourier Transform (DTFT) Pair is defined as

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

$$\begin{cases} x[n] = F^{-1}[X(e^{j\omega})] = \dfrac{1}{2\pi}\displaystyle\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega \\ X(e^{j\omega}) = F[x(t)] = \displaystyle\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \end{cases}$$

$X(e^{j\omega})$ is called the Discrete-Time Fourier Transform of $x[n]$, a.k.a. the spectrum of $x[n]$.

Periodicity of the Discrete-Time Fourier Transform:

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$$

2. Examples of the Discrete-Time Fourier Transforms (DTFT)

Example 5.1 Determine the DTFT of $x[n] = a^n u[n]$ $(|a| <$

Example 5.2 Determine the DTFT of $x[n] = a^{|n|}$ $(|a| < 1)$.

Solution:

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^{|n|}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} a^{n}e^{-j\omega n}$$

$$= \sum_{m=1}^{\infty} a^{m}e^{j\omega m} + \sum_{n=0}^{\infty} a^{n}e^{-j\omega n}$$

$$\because \left|ae^{-j\omega}\right| = \left|ae^{j\omega}\right| = |a| < 1$$

$$X(e^{j\omega}) = \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

Example 5.3 Determine the DTFT of $x[n] = e^{j\omega_0 n}$.

Solution:

Presume that

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} 2\pi\delta[\omega-\omega_0-2\pi n]$$

$$F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{<2\pi>} \sum_{n=-\infty}^{\infty} 2\pi\delta[\omega-\omega_0-2\pi n]\, e^{j\omega n}d\omega$$

$$= e^{j\omega_0 n} = x[n]$$

3. Convergence Issues Associated with the Discrete-Time Fourier Transform

The discrete-time Fourier transform converges when the sequence is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

or the sequence has finite energy

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

The inverse discrete-time Fourier transform generally has no convergence issue because the interval of integration is finite, similar to the discrete-time Fourier series.

Homework			

① Do not wait until the last minute
② Express your own idea and original opinion
③ Keep in mind the zero-tolerance policy on plagiarism

Example 5.4 Show that the DTFT of $x[n] = \delta[n]$ is $X\left(e^{j\omega}\right) = 1$.

Furthermore, an approximation of $x[n]$ is written as

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-w}^{w} X\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

Discuss which value of w ensures $x[n] = \hat{x}[n]$.

Hint:

$$\hat{x}[n] = \frac{\sin wn}{\pi n}$$

When $w = \pi$,

$$\hat{x}[n] = \frac{\sin \pi n}{\pi n} = \delta[n]$$