§10.4 The Inverse Z Transform

$X(z) = X(re^{j\omega}) = \mathcal{F}\{r^{-n}x[n]\}$

$r^{-n}x[n] = \mathcal{F}^{-1}[X(re^{j\omega})] = \frac{1}{2\pi}\int_{<2\pi>} X(re^{j\omega})\, e^{j\omega n} d\omega$

$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(re^{j\omega})\,(re^{j\omega})^n d\omega = \frac{1}{2\pi}\int_{<2\pi>} X(z)\, z^n d\omega$

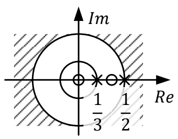
$dz = jre^{j\omega}d\omega = jzd\omega \Rightarrow d\omega = \frac{1}{jz}dz$

$x(t) = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$

1. Partial-Fraction Expansion
2. Power-Series Expansion

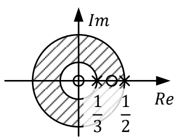
① $ROC\text{: } |z| > \frac{1}{2}$

$x[n] = \left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right]u[n]$



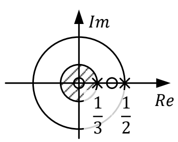
② $ROC\text{: } \frac{1}{3} < |z| < \frac{1}{2}$

$x[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[-n-1]$



③ $ROC\text{: } |z| < \frac{1}{3}$

$x[n] = -\left[\left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n\right]u[-n-1]$



Example 10.6 Determine the inverse z-transform $x[n]$ of $X(z)$

$$X(z) = \frac{2 - \frac{5}{6}z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

using the partial-fraction expansion for

① $ROC\text{: } |z| > \frac{1}{2}$ ② $ROC\text{: } \frac{1}{3} < |z| < \frac{1}{2}$ ③ $ROC\text{: } |z| < \frac{1}{3}$

Solution:

$$X(z) = \frac{2 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{3}z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{1}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Consider the z-transform

$$X(z) = 4z^2 + 2 + 3z^{-1} \quad (0 < |z| < \infty)$$

From the power-series definition of the z-transform,

$X[z] = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$

$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$

The inverse z-transform is obvious as

$$\begin{cases} x[-2] = 4 \\ x[0] = 2 \\ x[1] = 3 \end{cases}$$

That is,

$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$

Example 10.7 Determine the inverse z-transform $x[n]$ of $X(z)$

$$X(z) = \ln(1 - az^{-1}) \quad (ROC\colon\ |z| > |a|)$$

using the power-series expansion.

Solution:

$$\ln(1+v) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} v^n \quad (|v| < 1)$$

$$\ln(1 - az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-az^{-1})^n \quad (|az^{-1}| < 1)$$

$$= \sum_{n=1}^{\infty} -\frac{a^n}{n} z^{-n} = \sum_{n=-\infty}^{\infty} \left\{ -\frac{a^n}{n} u[n-1] \right\} z^{-n}$$

$$x[n] = -\frac{a^n}{n} u[n-1]$$

Example 10.8 Determine the inverse z-transform $x[n]$ of $X(z)$

$$X(z) = \frac{1}{1 - az^{-1}} \quad (ROC\colon\ |z| > |a|)$$

using the power-series expansion.

Solution:

$$X(z) = 1 + \frac{az^{-1}}{1 - az^{-1}} = 1 + az^{-1} + \frac{(az^{-1})^2}{1 - az^{-1}}$$

$$= 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=-\infty}^{\infty} \{a^n u[n]\} z^{-n}$$

$$x[n] = a^n u[n]$$

Now consider the inverse z-transform $x[n]$ of $X(z)$

$$X(z) = \frac{1}{1 - az^{-1}} \quad (ROC\colon\ |z| < |a|)$$

using the power-series expansion.

$$X(z) = -a^{-1}z + \frac{a^{-1}z}{1 - az^{-1}} = -a^{-1}z - a^{-2}z^2 + \frac{(a^{-1}z)^2}{1 - az^{-1}}$$

$$= -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 + \cdots$$

$$= \sum_{n=1}^{\infty} -a^{-n} z^n = \sum_{n=-1}^{-\infty} -a^n z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1]\} z^{-n}$$

$$x[n] = -a^n u[-n-1]$$

§10.5 Analysis and Characterization of LTI Systems using the Z Transform

$y[n] = h[n] * x[n]$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$Y(z) = H(z)X(z)$

$$X(z) \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

$H(z)$ is the system/transfer function of the LTI system

$$H(z) = \mathcal{Z}\big[h[n]\big] = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} \quad (z \in ROC)$$

$H(z)|_{z=e^{j\omega}} = H\left(e^{j\omega}\right)$ is the frequency response of the system

1. Causality

① The ROC associated with the system function for a causal system is the exterior of a circle including the infinity

② A discrete-time LTI system with rational system function $H(z)$ is causal if and only if the following two statements are fulfilled:

a. the ROC is the exterior of a circle outside the outermost pole

b. with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator cannot be greater than the order of the denominator

Example 10.9 Consider a system with the system function

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$

Determine the causality of this system.

Solution:

$$H(z) = z + \frac{-\frac{9}{4}z^2 + \frac{7}{8}z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$

The ROC does not include the infinity.

Therefore, this system is not causal.

The order of the numerator is greater than the order of the denominator in terms of z.

2. Stability

① An LTI system is stable if and only the ROC of its system function $H(z)$ includes the unit circle.

② A LTI system is not only causal but also stable if and only if all of the poles of $H(z)$ lie inside the unit circle, *i.e.* they must all have magnitude smaller than 1.

3. Geometric Evaluation of the Frequency Response of LTI Systems from the Pole-Zero Plot of $H(z)$

$$\frac{H(z)}{H_0} = \frac{(z-c_1)(z-c_2)\cdots(z-c_M)}{(z-d_1)(z-d_2)\cdots(z-d_N)} = \frac{\prod_{i=1}^{M}(z-c_i)}{\prod_{i=1}^{N}(z-d_i)}$$

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = H_0\frac{\prod_{i=1}^{M}(e^{j\omega}-c_i)}{\prod_{i=1}^{N}(e^{j\omega}-d_i)}$$

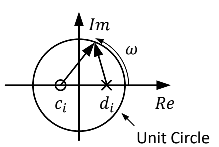
Zero Vector: $\overrightarrow{C_i} = C_i e^{j\alpha_i} = e^{j\omega} - c_i$

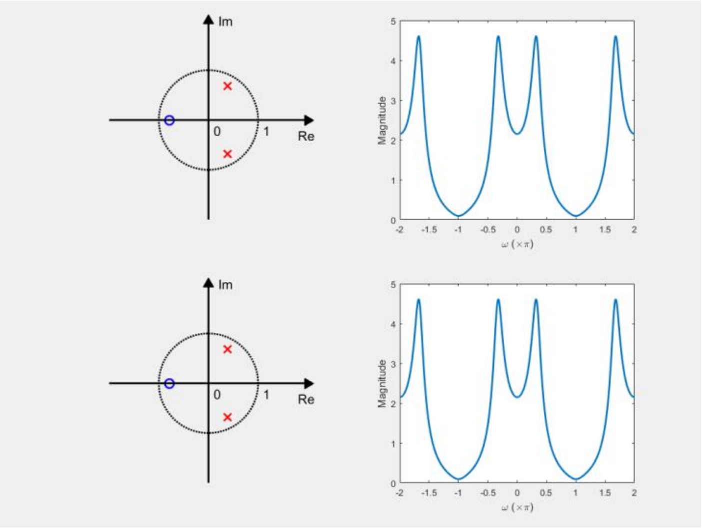
Pole Vector: $\overrightarrow{D_i} = D_i e^{j\beta_i} = e^{j\omega} - d_i$

$$H(e^{j\omega}) = H_0\frac{\prod_{i=1}^{M}C_i}{\prod_{i=1}^{N}D_i}e^{j\left(\sum_{i=1}^{M}\alpha_i - \sum_{i=1}^{N}\beta_i\right)}$$

$$\left|H(e^{j\omega})\right| = H_0\frac{\prod_{i=1}^{M}C_i}{\prod_{i=1}^{N}D_i}$$

$$\measuredangle H(j\omega) = \left(\sum_{i=1}^{M}\alpha_i - \sum_{i=1}^{N}\beta_i\right)$$





[Spring 2018] Assuming $x[n] = \delta[n+1] + \frac{5}{2}\delta[n] + \delta[n-1]$, determine the unit impulse response $h[n]$ of a stable LTI system which makes $h[n] * x[n] = \delta[n]$

Solution:

$$H(z) = \frac{1}{z + \frac{5}{2} + z^{-1}} = \frac{z}{(z+2)\left(z+\frac{1}{2}\right)}$$

$$ROC: \frac{1}{2} < |z| < 2$$

$$H(z) = \frac{z^{-1}}{(z^{-1}+2)\left(z^{-1}+\frac{1}{2}\right)} = \frac{\frac{4}{3}}{z^{-1}+2} + \frac{-\frac{1}{3}}{z^{-1}+\frac{1}{2}}$$

$$h[n] = \frac{2}{3}\left(-\frac{1}{2}\right)^n u[n] + \frac{2}{3}(-2)^n u[-n-1]$$