ECE495 Game Theory and Networked Systems

Lecture 15: PBE and Social Learning

1 Perfect Bayesian Equilibria

We continue to analyze the lawsuit signaling game.

Example 1.1 (Lawsuit signaling game)**.** *Here we use model a civil lawsuit as a game. Consider the case where the plaintiff* Π *knows a bit more information, in particular he knows if he will win the case (we use* W *to denote the winning type, and* L *to denote the losing type). The defendant* Δ *does not have this information but knows that* Π *knows. The prior belief* Δ *has is that with probability* $1/3$ Π *will win the lawsuit. If this lawsuit goes to the court, and if* Π *wins, the payoff is* $(3, -4)$ *(for* Π, Δ *respectively) representing the scenario where* Δ *pays 3 units to* Π *and also has to pay 1 unit for the court expenses. If* Δ *wins, then the payoff is* $(-1, 0)$*, where there is no payment and only the loser has to pay for 1 unit of court expense.*

Before going to the court, Π *will give* Δ *a chance to settle outside the court. This is done by* Π *asking for a price of settlement* m*. This settlement could be either low 1 or high 2. If the defendant chooses to accept, then the payoff is* $(m, -m)$*, or to reject, in which case they go to court and the payoff is specified as above. We will find the PBE in this game.*

We first represent this game in a game tree as in Fig. 1. Note that in all payoffs, we write the plaintiff's payoff first, since he is the first mover in this game (by sending a signal).

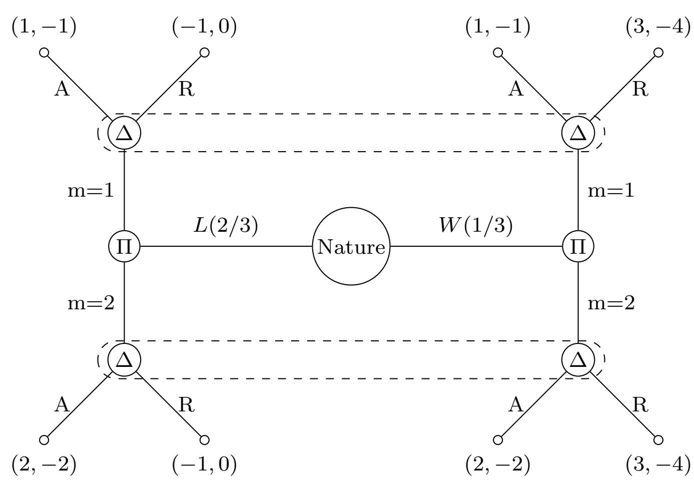

Figure 1: Game tree representation for lawsuit game

There are four different possible signaling possibilities:

1. L and W both send m=1;
2. L and W both send m=2;
3. L send m=1, W send m=2;
4. L send m=2, W send m =1;

The first two cases are also referred to as "pooling" strategy, where both types send the same signal. The last two are referred to as "separating" strategy, where based on signal, one can tell the different types of player Π.

We first analyze pooling strategy. Suppose that both types will send $m = 1$ (case 1). We use $\mu(T|m = i)$ to denote the belief of P2 that P1 is type T given signal m=i. Here the belief system according to Bayes rule is given by

$$\mu(W|m = 1) = \frac{1}{3}.$$

The expected payoffs for Δ associated with to accept is then

$$E[U_\Delta(A, 1, 1|m = 1)] = -1,$$

and associated with to reject is

$$E[U_\Delta(R, 1, 1|m = 1)] = 1/3(-4) + 2/3(0) = -4/3 < -1.$$

Hence Δ will choose to accept. Note that W type is getting a payoff of 1 and he can deviate to sending m=2 and get at least 2. Hence this is not a PBE.

We next consider case 2, where both types send $m = 2$. Here the belief system according to Bayes rule is given by

$$\mu(W|m = 2) = \frac{1}{3}.$$

The expected payoffs for Δ associated with to accept is then

$$E[U_\Delta(A, 2, 2|m = 2)] = -2,$$

and associated with to reject is

$$E[U_\Delta(R, 2, 2|m = 2)] = 1/3(-4) + 2/3(0) = -4/3 > -2.$$

Player Δ chooses to reject. W type player has no incentive to deviate, since he is getting the highest possible payoff. L type player is currently getting a payoff of -1. We have not specified Δ's strategy if m=1 was sent. We need Δ to choose to reject after seeing m=1 to make sure that L type Π will not deviate. The payoff for Δ for choosing A if m=1 is -1 and R is $-4\mu(W|m = 1) + 0(1 - \mu(W|m = 1))$, therefore we need

$$-4\mu(W|m = 1) + 0(1 - \mu(W|m = 1)) \geq 1.$$

This is equivalent to $\mu(W|m = 1) \leq \frac{1}{4}$.

We next analyze separating strategy (cases 3,4). Consider a separating strategy where W sends m=2 and L sends m=1.

$$\mu(W|m = 1) = 0, \quad \mu(W|m = 2) = 1,$$

then P2 will reject if $m = 1$ and accept if $m = 2$. L type Π can then unilaterally deviate and send $m = 1$ to improve payoff. Similarly, if W sends m=1, and L sends m=2. then P2 will accept if $m = 1$ and reject if $m = 2$. However, in this case the W type is getting the least possible payoff and will deviate to send m =2 to improve payoff. Moreover, the L type will also deviate to send $m = 1$ instead to improve payoff from -1 to 1. Hence this is also not an equilibrium.

To summarize, we found a unique equilibrium where W and L both send signal m=2 and defendant rejects when seeing either m =1 or m=2. The associated beliefs are

$$\mu(W|m = 1) \leq \frac{1}{4}, \quad \mu(W|m = 2) = \frac{1}{3}.$$

We note that the second belief is given by equality, as this event $m = 1$ is reached at equilibrium (called along equilibrium path), and therefore its belief is defined by conditional probability. On the other hand, the first inequality is defined for conditioning on m=1, which is not reached at equilibrium (called off equilibrium path). Since this information set m=1 is not reached at equilibrium, there is no constraint on the beliefs and we can define the belief to be anything in [0,1] (Bayes rule at this point involves dividing by 0, so any probability can work), and thus the equilibrium exists. This is often a criticism against PBE, since this process of defining range of $\mu(W|m=1)$ seems arbitrary.

We now recall the definition of Perfect Bayesian Equilibria (PBE), which combines the criterion from Bayesian Nash Equilibria and Subgame Perfect Equilibria. One significant difference between SPE and PBE is the use of "belief system". Like in BNE, due to the uncertainty in types, players form beliefs about their opponents, and to be part of an equilibrium, the beliefs need to be consistent according to Bayes rule, i.e., conditioned on observations, the beliefs do not cause inconsistency.

Definition 1. *A PBE in a dynamic game of incomplete information is a strategy profile s and a belief system μ, such that*

- *The strategy profile s is sequentially rational given μ: each player best responds given the beliefs, history of the game, information set (SPE like).*
- *The belief system μ is consistent given s: for every information set reached with positive probability given the strategy profile s, the probability assigned to each history in the belief system μ is given by Bayes' rule.*

We next analyze a game, where there is no pure strategy PBE.

Example 1.2 (Poker)**.** *We consider a very simple poker game with two players, player 1 (P1) and player 2 (P2). In the beginning, each player puts down* \$1 *in a jar. Then cards are dealt to P1. The cards can either be of type High (good) or Low (bad), each with probability 1/2. After seeing her cards, P1 makes a move to either "See" or "Raise". If P1 chooses to see, then P2 sees the cards and if the cards are of the type high, then P1 wins and takes the* \$2 *from the jar, otherwise, P2 wins and takes the* \$2*. If P1 chooses to raise, then she adds an additional* \$1 *to the jar and P2 gets to choose between "Pass" and "Meet". If P2 chooses to pass, then P1 takes all the money in the jar (*\$3*), otherwise, P2 puts down an additional* \$1 *(now there are* \$4 *in the jar) and P1 reveals her card. If it's High P1 wins and takes all the money, otherwise P2 wins and takes all the money.*

We want to analyze the equilibrium of this game, where we model the payoff of the player by the net change in their wealth. For instance, if P1 chooses to see and she has High type of cards, then P1 gets a payoff of 1 and P2 -1, since among the \$2 *P1 collects in the end, she originally put down* \$1*, the net increase is* \$1*. We represent this game by a game tree as Fig. 2, where the information set indicates that P2 does not know the type of P1.*

We first check that there does not exist a pure strategy equilibrium. If P2 plays a pure strategy P whenever it is his turn to move, then a Low type P1 would play Raise (to get payoff of 1 instead of -1). P1's strategy could be (R,R) or (S,R), where the first letter indicates strategy of a High (H) type P1 and second for a Low (L) type. If P1 chooses to play (S,R), then P2 would choose to Meet whenever he plays. If P1 chooses to play (R,R), then conditioned on P2 plays, the probability of each type is 1/2, and P2 would choose M to get an expected payoff of 0 over -1 associated with P. Now assume P2 chooses M, then P1's strategy would be (R,S) and P2's best response to that would be P. Therefore at any pure strategy profile, at least one player has incentive to unilaterally deviate and improve payoff and hence there is no pure strategy equilibrium.

Now we consider possible mixed strategies, where both players play at least two strategies with positive probability. We note that in a Bayesian game setup, each player's strategy is a mapping from its type space to strategy space. We note that H type player has no incentive to play S, since expected payoff associated with R where P2 plays M with positive probability is higher than 1, which is the payoff associated with S. Therefore, if there is an equilibrium, P1 would play (R,S) and (R,R) with positive probability, which we use q and $1-q$ to denote. We use p and $1-p$ to denote the probability Pass and Meet are played by P2. We use

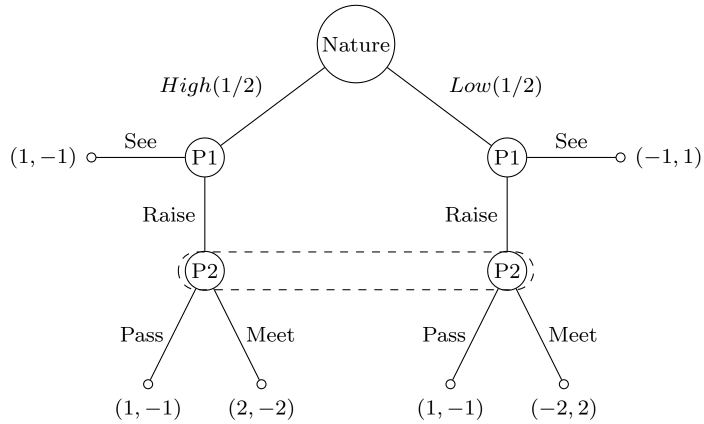

Figure 2: Extensive form Representation for Poker Game

the characterization of a mixed strategy equilibrium, where the strategies with positive probability should have the same expected payoff. We have that the expected payoff for a Low type player 1 associated with S should be equal to R. We have

$$E(u_1((R,S),p|L)) = -1 = p - 2(1-p) = E(u_1((R,R),p|L)),$$

and

$$p = 1/3.$$

Similarly, the expected value for P2 should be equal for P and M against P1 (specified by q).

$$E(u_2(P,q)) = -\frac{1}{2} - \frac{1}{2}(1-q) + \frac{1}{2}q = -\frac{2}{2} + \frac{1}{2}q + \frac{1}{2}(1-q)2 = E(u_2(M,q)),$$

and

$$q = 2/3.$$

This forms a mixed strategy perfect Bayesian Nash equilibrium, where we used conditional probability according to Bayes' rule. We note that when P2 moves, his belief about P1 being type H would be $\frac{1/2}{1/2+1/2(1-q)} = 3/5$. At this belief, mixing between P (with probability 1/3) and M is a best response by the previous equality (consistent with Bayes rule).

At this equilibrium, the expected payoff for player 2 is $-1/3$, and a High type of P1's expected payoff is $1/3 + 2 \times 2/3 = 5/3$ and a Low type's expected payoff is -1. The Low type player bluffs (play Raise) with probability 1/3. Note in this game there is only one subgame, and the perfection in PBE is about aligning strategies with beliefs.