ECE495 Game Theory and Networked Systems

Static One-Shot Game in Strategic Form

1 Static One-Shot Game in Strategic Form

Definition 1. *A Static One-Shot Game in Strategic Form has the following components:*

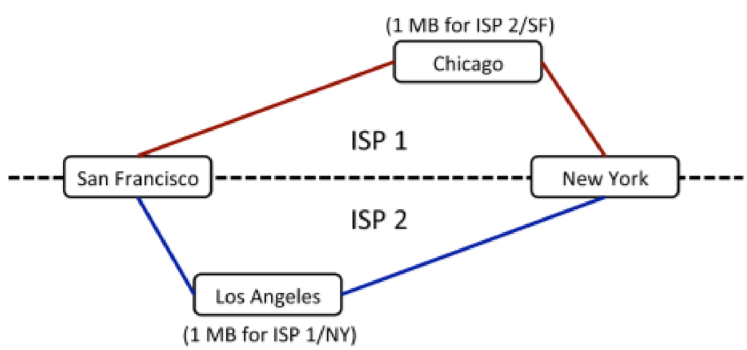
1. *R: a finite set of players;*
2. *A strategy set S_r, for all $r \in R$;*
3. *A payoff function $\Pi_r : \prod_{s=1}^{R} S_s \to \mathbb{R}$, for all $r \in R$.*[1]

The payoff function reflects the utility of player r. This maps actions/strategies of all player's to a real number, since the player r's payoff depends on its own action as well as other players'. Formally: a game is given as

$$G = (R, \{S_r\}, \{\Pi_r\}).$$

1.1 Example 1: Peering

In the figure below, we have a map showing the connection between 2 ISPs. They have a reciprocal agreement to carry each-others data free of charge, while they incur a cost for transmitting data within their own network (ISP1 owns the red lines, and ISP2 owns the blue lines). We assume that each ISP incurs a cost proportional to the length of the route it sends the data across, and that each ISP tries to minimize its own cost.



Notice that if ISP 1 wants to send data from Chicago to San Francisco, it has two options. The first is to send the data directly to San Francisco and incur the cost of the long route. The second option is to send the data to New York, and then pass the rest of the cost off to ISP2 to transport the data from New York to Los Angeles, and then to San Francisco. Notice that if ISP 2 wants to send data from Los Angeles to New York, it has the same options and payoffs. We assume each short route (Los Angeles to San Francisco, Chicago to New York) cost the corresponding owner ISP 1 unit and the long route (Chicago to San Francisco, New York to Los Angeles) cost the owner ISP 2 units. For instance, if ISP 1 chooses far and ISP 2 chooses near,

[1] Here, $\prod_{s=1}^{R} S_s$ represents the Cartesian product of the sets S_s - if you are not familiar with Cartesian products, there is a wikipedia article that explains them well.

then ISP incurs cost of 2 for its own traffic (Chicago to San Francisco) and cost of 3 from traffic of ISP 2 (San Francisco to Chicago to New York), total cost 5. ISP only incurs cost of 1 for transmitting its traffic from Los Angeles to San Francisco, and then pass the traffic to ISP 1, total cost 1. We will represent this game in a matrix below:

1.1.1 The Game Model

1. Players = $\{ISP1, ISP2\}$;
2. Strategy set = $\{Near, Far\}$;
3. Payoffs shown in the Bi-Matrix below:

		ISP 2	
		Near	Far
ISP 1	Near	(-4,-4)	(-1,-5)
	Far	(-5,-1)	(-2,-2)

In each cell of this matrix, the first number represents payoff of the column player (ISP 1), and second element for the row player (ISP 2).

To predict outcome of the game, we use the following reasoning: If ISP 2 is choosing Near (first column), then ISP 1 will get a greater payoff by choosing Near. If ISP 2 chooses Far (second column), ISP 1 will still get a greater payoff by choosing Near. So regardless of what ISP 2 chooses, Near is the better option for ISP 1. Similarly, Near is the better option for ISP 2. When this dynamic plays out in real life, it results in "hot potato" routing, as ISPs try to send the data to another ISP as quickly as possible. This game is also known as the Prisonsers' Dilemma.

1.2 Dominant Strategy

Definition 2. *Strictly dominant strategy: Strategy s_r^* is strictly dominant if*

$$\Pi_r(s_r^*, \bar{s}_{-r}) > \Pi_r(s_r, \bar{s}_{-r}), \quad \forall s_r \in S_r, s_r \neq s_r^*, \quad \forall \bar{s}_{-r} \in \bar{S}_{-r}$$

If the above inequality is weak, i.e., $\geq$, then s_r^* is Weakly Dominant.

Note that from the definition of dominant strategy, each player can have at most one (may not exist) dominant strategy. When each player has a dominant strategy, then we have a dominant strategy equilibrium.

Definition 3. *Dominant Strategy Equilibrium: a set of strategies, where each player plays its dominant strategy.*

Notice that in the Peering example, Near is a dominant strategy and we have a Dominant Strategy Equilibrium, (Near, Near).

1.2.1 Example 2

The table below illustrates an example where Player 1 chooses Up vs Down and Player 2 chooses Left vs Right.

		Player 2	
		Left	Right
Player 1	Up	(1,0)	(1,2)
	Down	(0,3)	(0,1)

In this example, Player 2 does not have a dominant strategy, but Player 1 always chooses Up because it always gets a payoff of 1 for Up and a payoff of 0 for Down. Hence, Up is a dominant strategy for Player 1 and Down is a dominated strategy (no matter what the other player does, Up is better than Down) that should never be played by a rational player. Noticing this, Player 2 assumes Player 1 will always choose Up, in which case Player 2 must choose Right to get the better payoff.

Definition 4. *Iterated Elimination of Dominated Strategies: Each player eliminates their dominated strategies iteratively, until no further strategy can be eliminated.*

Note that the iterated elimination of dominated strategies may not always be possible or converge to a singleton, see the example below.

1.2.2 Example 3: Lunch

In this example, Alice and Bob are each deciding to eat lunch at Sargent or Tech Express. Each has a preferred eating location, but neither wants to eat without the other. The payoffs for their decisions are shown in the table below:

		Alice	
		TE	Sargent
Bob	TE	(2,1)	(0,0)
	Sargent	(0,0)	(1,2)

In this example, neither player has a dominant strategy or dominated strategy.

2 Nash Equilibria

Definition 5. *Given a game* $G = (R, \{S_r\}, \{\Pi_r\})$*, an outcome* $\bar{s}^* = (s_1^*, s_2^*, s_3^* ..., s_R^*)$ *is a Nash Equilibrium if* $\forall r \in R, \forall s_r \in S_r, \Pi_r(s_r^*, \bar{s}_{-r}^*) \geq \Pi_r(s_r, \bar{s}_{-r}^*)$

No one person can do better by herself (unilateral deviation) if all other players keep their strategy fixed. It is different from dominant strategy because it assumes the other players' strategies are fixed. We should note: dominant strategy equilibria is a subset of Nash equilibria.

Recalling Example 3, we notice that there are 2 Nash equilibria; one where both Alice and Bob eat at Tech Express and one where they both eat at Sargent, since at these outcomes, no player can unilateral deviate and improve her payoff. Nash equilibria are not necessarily unique. Even if the payoff for both eating at Sargent were (9,10), there would still be a Nash Equilibrium at which Alice and Bob both ate at Tech Express.

2.1 Example 4: Matching Pennies

In this example, two players pair off and show heads or tails on a penny. If both show the same side, player 1 gets to keep the other's penny. If they show different sides, player 2 keeps the penny. This is illustrated with the payoffs table below:

		Player 1	
		Heads	Tails
Player 2	Heads	(1,-1)	(-1,1)
	Tails	(-1,1)	(1,-1)

There is no (pure strategy) Nash Equilibrium in this example because for each outcome, one player could increase their utility by changing their strategy if the other player's strategy is constant.

2.2 Example 5: First Price Auction

- An auction with a single object and n players $\{1, 2, ...n\}$;
- Each player values the object v_i, we order the players such that $v_1 \geq v_2 \geq ... \geq v_n \geq 0$ (ordering is without loss of generality);
- Players simultaneously submit bids b_i with $0 \leq b_i < \infty$;
- The player with the highest bid pays the value of their bid and obtains the object. The rest of the players pay nothing and get nothing;
- In the event of a tie, the item goes to the lowest-indexed player (if players 2 and 3 tie, the object goes to player 2 and only player 2 pays);

2.2.1 The Game Model

- players $= \{1, 2, ...n\}$;
- strategy set: $(0, \infty)$;
- payoff: $\Pi_r(b_r, \bar{b}_r) = \begin{cases} v_r - b_r & \text{if r wins} \\ 0 & \text{if r loses} \end{cases}$

For this example, Nash equilibrium occurs when $b_1 = v_2$ and $\forall i > 1, b_i = v_i$. The outcome where every player bids their own value is not a NE, since player 1 has the incentive to lower his bid until $b_1 = b_2$ to obtain the item, but pay less. The outcome of $b_i = v_2$ for all i is also an NE, as player 1 still gets the item and no other players can deviate and improve payoff. In this game, there are multiple NE but all end up giving the good to player 1.

References

[1] Berry, Randall A., and Ramesh Johari. "Economic modeling in networking: A primer." *Foundations and Trends® in Networking*, 6.3 (2013): 165-286.