Example 9.4 Determine the Laplace transform of $x(t) = \begin{cases} e^{-at} & (0 < t < T) \\ 0 & (otherwise) \end{cases}$ and sketch the pole-zero plot of $X(s)$.

Solution:

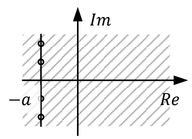
$$\mathcal{L}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{0}^{T} e^{-(a+s)t}dt = \frac{1 - e^{-(a+s)T}}{s+a}$$

Since $x(t)$ is of finite duration, the ROC of $X(s)$ is the entire s-plane.

There is no poles.

$$X(-a) = \lim_{s \to -a} X(s) = T$$

There are a lot of zeros.



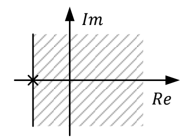
$$e^{-(a+s)T} = 1 \Rightarrow s_k = -a + j\frac{2\pi}{T}k \;\; (k = \pm 1, \pm 2, \cdots)$$

Property 4: If $x(t)$ is right sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all value of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.

Definition: If $x(t) = x(t)u(t - t_0)$, the $x(t)$ is referred to as a right sided signal.

$$\int_{T_1}^{+\infty} |x(t)|e^{-\sigma_1 t}dt = \int_{T_1}^{+\infty} |x(t)|e^{-\sigma_0 t}e^{-(\sigma_1-\sigma_0)t}dt$$

$$\leq e^{-(\sigma_1-\sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)|e^{-\sigma_0 t}dt < \infty$$

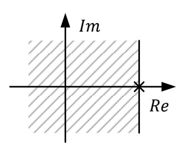


Property 5: If $x(t)$ is left sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all value of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.

Definition: If $x(t) = x(t)u(t_0 - t)$, the $x(t)$ is referred to as a left sided signal.

$$\int_{-\infty}^{T_1} |x(t)|e^{-\sigma_1 t}dt = \int_{-\infty}^{T_1} |x(t)||x(t)|e^{-\sigma_0 t}e^{(\sigma_0-\sigma_1)t}dt$$

$$\leq e^{(\sigma_0-\sigma_1)T_1} \int_{-\infty}^{T_0} |x(t)|e^{-\sigma_1 t}dt < \infty$$



Property 6: If $x(t)$ is two sided, and if the line $Re\{s\}=\sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the line $Re\{s\} = \sigma_0$.

Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, the its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, the if $x(t)$ is right sided, the ROC is the region in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

Example 9.5 Determine the Laplace transform of $x(t) = e^{-b|t|}$ and sketch the pole-zero plot of $X(s)$.

Solution:

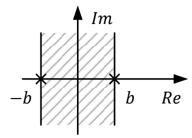
$$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$$

$$e^{-bt}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+b} \quad (ROC\colon Re\{s\} > -b)$$

$$e^{bt}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{-1}{s-b} \quad (ROC\colon Re\{s\} < b)$$

$$X(s) = \mathcal{L}[e^{-bt}u(t)] + \mathcal{L}[e^{bt}u(-t)] = \begin{cases} \dfrac{-2b}{s^2 - b^2} & (b > 0) \\ N.A. & (b < 0) \end{cases}$$

$$X(s)|_{s=j\omega} = \frac{2b}{\omega^2 + b^2} \quad (b > 0)$$



Example 9.6 Sketch the pole-zero plot of

$$X(s) = \frac{1}{(s+1)(s+2)}$$

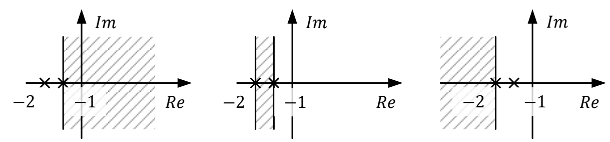
and determine the possible ROCs.

Solution:

The pole of $X(s)$ is at $s = -1$ and $s = -2$.

There are three possible ROCs.

① Right sided ② Two sided ③ Left sided



§9.3 Properties of the Laplace Transform

$$\boxed{x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \ \ (s \in R_1) \text{ and } y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \ \ (s \in R_2)}$$

1. Linearity

$$Ax(t) + By(t) \overset{\mathcal{L}}{\longleftrightarrow} AX(s) + BY(s) \ \ (ROC \supseteq R_1 \cap R_2)$$

2. Time Shifting

$$x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s) \ \ (ROC = R_1)$$

3. Shifting in the s-domain

$$e^{s_0 t}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s - s_0) \ \ (ROC = R_1 + Re\{s_0\})$$

4. Time Scaling

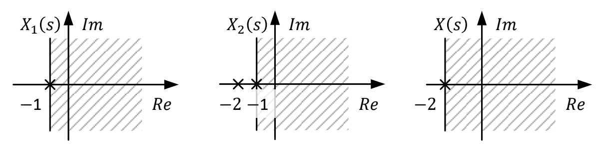
$$x(at) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \ \ (ROC = aR_1)$$

Example 9.7 Determine the Laplace transform of $x(t) = x_1(t) - x_2(t)$, giving that

$$\begin{cases} x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{s+1} & (ROC\colon Re\{s\} > -1) \\ x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \dfrac{1}{(s+1)(s+2)} & (ROC\colon Re\{s\} > -1) \end{cases}$$

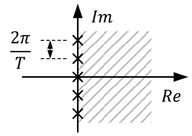
Solution:

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{1}{s+2} \quad (ROC\colon Re\{s\} > -2)$$



Example 9.8 Determine the Laplace transform of

$$x(t) = \sum_{n=0}^{\infty} \delta(t - nT)$$



Solution:

$\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1$ (ROC: entire $s-$plane)

$\delta(t - nT) \overset{\mathcal{L}}{\longleftrightarrow} e^{-nTs}$ (ROC: entire $s-$plane)

$$X(s) = \mathcal{L}[x(t)] = \mathcal{L}\left[\sum_{n=0}^{\infty} \delta(t - nT)\right] = \sum_{n=0}^{\infty} \mathcal{L}[\delta(t - nT)]$$

$$= \sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}} \quad (ROC: Re\{s\} > 0)$$

Example 9.9 Determine the Laplace transforms of

$x(t) = \cos \omega_0 t\, u(t)$

Solution:

$$x(t) = \frac{1}{2} e^{j\omega_0 t} u(t) + \frac{1}{2} e^{-j\omega_0 t} u(t)$$

$$\mathcal{L}[x(t)] = \frac{1}{2}\left(\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right)$$

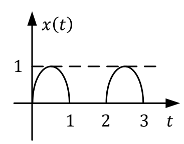
$$= \frac{s}{s^2 + \omega_0^2} \quad (ROC:\ Re\{s\} > 0)$$

$$\cos \omega_0 t\, e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s + a}{(s + a)^2 + \omega_0^2} \quad (ROC:\ Re\{s\} > -a)$$

$$\sin \omega_0 t\, e^{-at} u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{\omega_0}{(s + a)^2 + \omega_0^2} \quad (ROC:\ Re\{s\} > -a)$$

Example 9.10 Consider the outputs of half-wave and full-wave rectifiers when the input is $\sin(\pi t)$. Determine their Laplace transforms.

Solution:



$$x(t) = \sum_{n=0}^{\infty} x_1(t - 2n)$$

$$x_1(t) = \sin(\pi t)\,[u(t) - u(t - 1)\,]$$

$$= \sin(\pi t)\, u(t) + \sin \pi(t - 1)\, u(t - 1)$$

$$\mathcal{L}[x(t)] = \mathcal{L}\left[\sum_{n=0}^{\infty} x_1(t - 2n)\right] = \sum_{n=0}^{\infty} \mathcal{L}[x_1(t - 2n)]$$

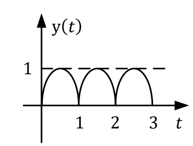
$$= \sum_{n=0}^{\infty} e^{-2sn} \mathcal{L}[x_1(t)] = \mathcal{L}[x_1(t)]\left(\sum_{n=0}^{\infty} e^{-2sn}\right)$$

$$\mathcal{L}[x_1(t)] = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2} \quad (ROC:\ Re\{s\} > 0)$$

$$\sum_{n=0}^{\infty} e^{-2sn} = \frac{1}{1 - e^{-2s}} \quad (ROC:\ Re\{s\} > 0)$$

$$\mathcal{L}[x(t)] = \frac{\pi}{(s^2 + \pi^2)(1 - e^{-s})} \quad (ROC:\ Re\{s\} > 0)$$

$$\mathcal{L}[y(t)] = \mathcal{L}[x_1(t)]\left(\sum_{n=0}^{\infty} e^{-sn}\right)$$



$$\sum_{n=0}^{\infty} e^{-sn} = \frac{1}{1 - e^{-s}} \quad (ROC:\ Re\{s\} > 0)$$

$$L[y(t)] = \frac{\pi(1 + e^{-s})}{(s^2 + \pi^2)(1 - e^{-s})} \quad (ROC:\ Re\{s\} > 0)$$

5. Conjugation

$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*) \ \ (ROC = R_1)$$

6. Convolution Property

$$x(t) * y(t) \xleftrightarrow{\mathcal{L}} X(s)Y(s) \ \ (ROC \supseteq R_1 \cap R_2)$$

7. Differentiation in the Time Domain

$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s) \ \ (ROC \supseteq R_1)$$

8. Differentiation in the s-Domain

$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds} \ \ (ROC = R_1)$$

9. Integration in the Time Domain

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}X(s) \ \ (ROC \supseteq R_1 \cap \{Re\{s\} > 0\})$$

Example 9.11 Determine the Laplace transform of
$x(t) = te^{-at}u(t)$

Solution:

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \ \ (ROC\colon Re\{s\} > -a)$$

$$-te^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}\left(\frac{1}{s+a}\right) = \frac{-1}{(s+a)^2} \ (ROC\colon Re\{s\} > -a)$$

$$te^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^2} \ \ (ROC\colon Re\{s\} > -a)$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n} \ \ (ROC\colon Re\{s\} > -a)$$

Example 9.12 Determine the Laplace transform of
$x(t) = te^{-t}u(t-t_0)$

Solution:

$$x(t) = (t - t_0 + t_0)e^{-(t-t_0+t_0)}u(t-t_0)$$

$$e^{t_0}x(t) = (t-t_0)e^{-(t-t_0)}u(t-t_0) + t_0e^{-(t-t_0)}u(t-t_0)$$

$$\mathcal{L}[e^{t_0}x(t)] = e^{-t_0 s}\left[\frac{1}{(s+1)^2} + \frac{t_0}{s+1}\right] \ \ (ROC\colon Re\{s\} > -1)$$

$$\mathcal{L}[x(t)] = \left[\frac{1 + t_0(s+1)}{(s+1)^2}\right]e^{-t_0 s - t_0} \ \ (ROC\colon Re\{s\} > -1)$$

10. The Initial and Final-Value Theorems

①If $x(t) = x(t)u(t)$ and $x(t)$ contains no impulse or higher order singularities at $t = 0$,

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

②If $x(t) = x(t)u(t)$ and all poles of $sX(s)$ are on left half s-plane,

$$x(\infty) = \lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$