ECE495 Game Theory and Networked Systems

Lecture 2: Utility, Fairness and Equilibrium

1 Formalizing Utility

Assuming that people are rational, we can represent their preferences in terms of Utility, a function which takes in options or choices and assigns them values which are real and satisfy transitivity.
We assume the following about the Utility function $U_r(x_r)$, $\forall x_r \in X$ where X is the set of all possible resource allocations.

Definition 1. *Monotonicity: The more of an item a person has, the higher their happiness, or*

$$\forall x_i \geq x_j, U_r(x_i) \geq U_r(x_j). \tag{1}$$

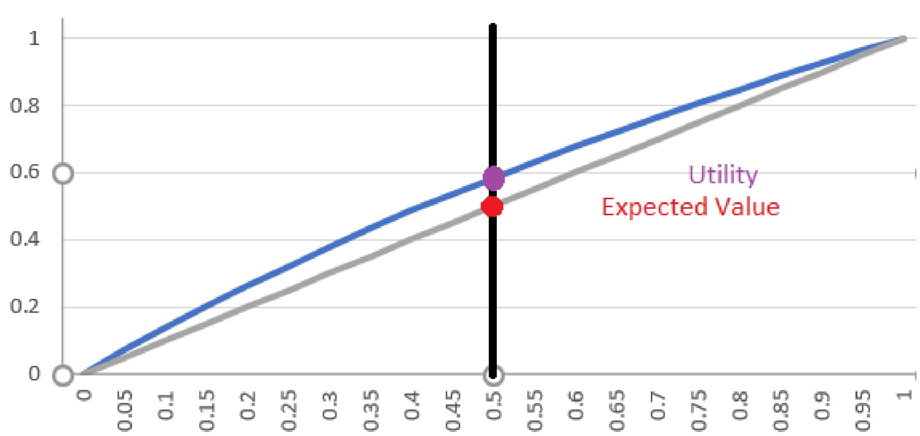
Definition 2. *Concavity: the law of diminishing returns; the more of an item a person has, the less their increase in happiness with each additional item, or in terms of the utility function: for any α in $[0, 1]$ and any allocations $x, y \in X$,*

$$U((1-\alpha)x + \alpha y) \geq (1-\alpha)U(x) + \alpha U(y). \tag{2}$$

Definition 3. *Risk Aversion: an outcome which is certain is preferable to outcomes which are uncertain. Mathematically, we have for all r,*

$$\mathrm{E}[U_r(X)] \leq U_r(\mathrm{E}[x_r]). \tag{3}$$

Risk Aversion follows from Concavity:



Definition 4. *Elasticity: the amount of an item a person wants is highly responsive to the change in the item's price.*

1.1 Prospect Theory

When faced with uncertainty, we usually assume that the players are rational/risk neutral and optimize against expected value. Another related branch of economics called *Prospect Theory*, builds on a different approach. Prospect theory stems from observations of human behaviors and suggests that people make decisions based on perceived losses or gains and are not risk neutral.

Here are some interesting examples illustrating prospect theory.

1. Uncertainty: Option A provides a guaranteed win of \$100 while option B provides the possibility of winning \$200, with a 70% chance of winning and 30% chance of losing. Most people will choose option A since it provides a guaranteed win, even though it offers a lower return compared to B.

2. Small probability discount: People tend to discount very small probabilities even if there is a possibility of losing all their wealth. By discounting the small probabilities, people end up choosing higher-risk options with higher probabilities.

3. Relative positioning: if everybody in the office gets a 20% raise, no individual will feel better off. However, if the person gets a 10% raise, and other people fail to get a raise, that person will feel better off and richer than everyone else.

4. Loss aversion: For example, if a person makes \$200 in profits and \$100 in losses, the person will focus on the loss even though they emerged with a \$\$100 net gain.

1.2 Example: sharing data stream

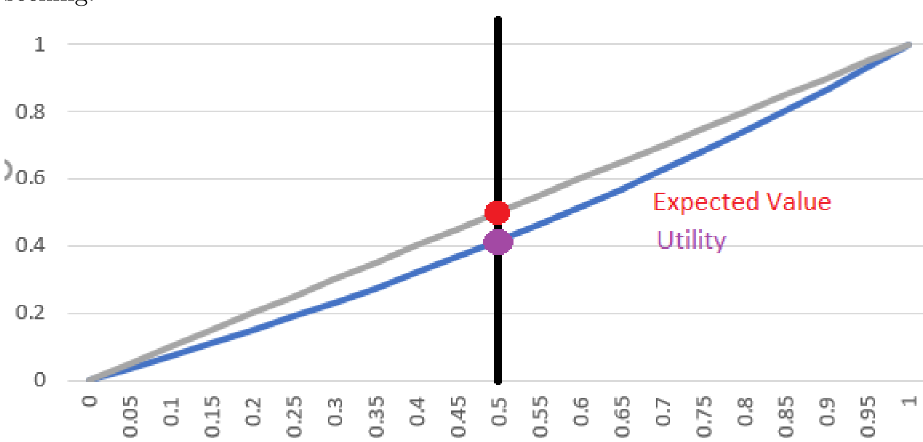
Assume that 2 people are sharing a data stream. We examine the following two schemes hoping to maximize the sum of the people's utility.

1.2.1 Scheme 1

We randomly select one person and give them the whole stream.

This maximizes utility if both parties are streaming video - half the stream results in a slow upload and time spent watching the buffering bar for both users, which is not much better than no stream at all. Giving one of them the whole stream means a smooth video for one of the users which they prefer a lot more than a slow buffer.

Therefore, the utility both users has is convex, and because $\mathrm{E}_x[U_r(x_r)] \geq U_r(\mathrm{E}_x[x_r])$, the users are risk-seeking.



1.2.2 Scheme 2

We give each person half of the stream.

This maximizes utility if both users are doing something which requires a weaker connection, like checking their email. Neither needs the whole stream, so giving it to them would increase their happiness less than taking it away from the other would decrease their happiness. So the users' utility is concave and risk-averse.

2 Optimality

What does it mean to be a good allocation? In economics, efficiency of allocation is achieved through the notion of Pareto optimality.

Definition 5. *Pareto Optimality: x pareto dominates y if* $\forall r$, $U_r(x_r) \geq U_r(y_r)$ *and* $\exists s$, $U_s(x_s) > U_s(y_s)$.

For single resourse allocation, if $U_r(x_r)$ is strictly increasing, and we have a constraint: $\Sigma_r x_r \leq c$ for $x_r \geq 0$ for all r, then we have x is a pareto optimum if $\Sigma_r x_r = c$. However, this claim doesn't holds if there are more than one goods to allocate.

2.1 Example: sharing two links

Suppose there are two users sharing two links. Each of the links has capacity 1. Denote x_{ij} as the amount of bandwidth of link i allocated to user j. We have the utility functions of user 1 and 2 as:

$$U_1(x_{11}, x_{21}) = 2x_{11} + x_{12},$$
$$U_2(x_{12}, x_{22}) = x_{21} + 2x_{22}.$$

Then the two allocations which satisfy $x_{11} + x_{21} = 1$ and $x_{12} + x_{22} = 1$ are:
$x_{11} = 1, x_{22} = 1$, in which case $U_1 = 2, U_2 = 2$, and
$x_{12} = 1, x_{21} = 1$, in which case $U_1 = 1, U_2 = 1$.
We can see that the former one pareto dominates the latter one.

3 Fairness

As illustrated in the previous section, there might be many Pareto optimal allocations. How do we choose among them? In this section, we will focus on fairness considerations in resource allocation. There are many competing notions of fairness providing guidance for choosing between multiple possible Pareto optimal allocations.

3.1 Methodology

We focus on a particular class of fairness criteria that are defined as follows. Let $f : R \to R$ be a strictly concave, strictly increasing function. We solve the following optimization problem to achieve different types of fairness:

$$max_x \ \sum_r f(U_r(x_r)),$$
$$s.t. \ \ x \in X,$$

3.2 Utilitarian Fairness

In this case, Pareto optimal point that maximizes the total utility of the system is chosen. We simply maximize $\sum_r U_r(x_r)$.

3.3 Proportional Fairness

If we assume $U_r(x_r) = x_r$, and x^* is proportional fairness optimal allocation, then for any other allocation x, the inequality below holds:

$$\sum_r \frac{x_r - x_r^*}{x_r^*} \leq 0 \tag{4}$$

In other words, the sum of proportional changes in the allocation across users cannot be positive. By letting $f(x) = \log(x)$, we can achieve proportional fairness.
The reason is that $\forall x \in X$, $(x - x^*)' \nabla f(x^*) \leq 0$.

3.4 α-fairness

Consider the following function $f(U)$:

$$f(U) = \begin{cases} \dfrac{U^{1-\alpha}}{1-\alpha} & \text{if } \alpha \neq 1, \\ \\ \log(U) & \text{if } \alpha = 1. \end{cases}$$

The resulting family of fairness notions, parametrized by α, is called α-fairness. If we assume $U_r(x_r) = x_r$, and x^* is proportional $\alpha-$fairness optimal allocation, then for any other allocation x, the inequality below holds:

$$\sum_r \frac{x_r - x_r^*}{(x_r^*)^\alpha} \leq 0 \tag{5}$$

This family includes several special cases as a result:
When $\alpha = 0$, we get utilitarian fairness;
When $\alpha = 1$, we obtain proportional fairness;
When $\alpha = 2$, we get what is implemented in TCP;

3.5 Max-min fairness

When $\alpha \to \infty$, we have Max-min fairness which solves the problem of

$$\max \min_{x_r} \quad U_r(x_r),$$

where we maximize the amount allocated to the individual who has the least utility.

3.6 Example

To illustrate the differences in the fairness notions above, we consider the following example.

We consider a communication network resource allocation problem with three users and two resources, each of unit capacity. User 1 sends data only through resource 1. User 2 sends data only through resource 2. User 3 sends data through both resources 1 and 2. (See Fig.1) Letting x_r be the rate allocation to user r. In this example, we assume that $U_r(x_r) = x_r$, and the constraints are $x_1 + x_3 \leq 1$ and $x_2 + x_3 \leq 1$.

- The utilitarian allocation is $x_1 = x_2 = 1, x_3 = 0$.
- The proportionally fair allocation is $x_1 = x_2 = \frac{2}{3}, x_3 = \frac{1}{3}$.
- The max-min fair allocation is $x_1 = x_2 = x_3 = \frac{1}{2}$.

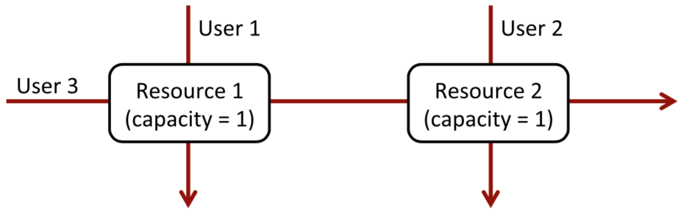


Figure 1: An example illustrating different types of fairness.

4 Quasilinear Environments: The Role of Currency

In this section we introduce the idea of currency. We assume that users' utilities are measured in units of currency. In particular, we consider a quasilinear model where each user r's utility can be written as:

$$U_r(x_r) + w_r,$$

where $U_r(x_r)$ is the utility of consumption and w_r is the allocation of currency. This is called quasilinear because it is linear in one of the goods (currency). This will become our payoff function, which is what each player aims to optimize.

5 Static One-Shot Game in Strategic Form

Definition 6. *A Static One-Shot Game in Strategic Form has the following components:*

1. *R: a finite set of players;*
2. *A strategy set S_r, for all $r \in R$;*
3. *A payoff function $\Pi_r : \prod_{s=1}^{R} S_s \to \mathbb{R}$, for all $r \in R$.*[1]

The payoff function reflects the utility of player r. This maps actions/strategies of all player's to a real number, since the player r's payoff depends on its own action as well as other players'. Formally: a game is given as

$$G = (R, \{S_r\}, \{\Pi_r\}).$$

We will use the following notation to denote the outcome of actions for all players, $\bar{s} = (s_1, s_2, s_3, ..., s_R)$. Then the payoff for player$r$ is written: $\Pi_r(\bar{s}) = \Pi_r(s_r, \bar{s}_{-r})$, where $-r$ denotes the set of all opponents (players other than r). In other words, a player's payoff is a function of their individual choice and the other players' choices.

References

[1] Berry, Randall A., and Ramesh Johari. "Economic modeling in networking: A primer." *Foundations and Trends® in Networking*, 6.3 (2013): 165-286.

[1] Here, $\prod_{s=1}^{R} S_s$ represents the Cartesian product of the sets S_s - if you are not familiar with Cartesian products, there is a wikipedia article that explains them well.