CompSci 351 : Introduction to Computer Graphics

3D Geometric Operations

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3-D Operators for Points and/or Vectors (con'td):

What is the distance between a pair of 3D points? What is the length of a 3D vector? How can we set the magnitude or length of a vector to 1.0 without changing its direction? Well, let's try it:

--Write two 4-element vectors E and F as columns: $[ex\ ey\ ez\ 0]^T$ and $[fx\ fy\ fz\ 0]^T$.

Length of a vector E: $||\mathbf{E}|| = \text{sqrt}(ex^2 + ey^2 + ez^2)$

--JARGON: '**Normalize**' a vector == Scale the vector to make its length =1.0:

norm(E) = (ex/||E||, ey/||E||, ez/||E||)

--JARGON: a '**unit vector**' is a vector whose length is 1.0. Normalizing always creates a unit vector, (length or magnitude of 1.0) but normalizing never changes the direction of the vector.

--Puzzle: does it make any sense geometrically to 'normalize' points?

(**ANS:** No, it doesn't. By definition, a point has neither 'length' nor 'magnitude': it's a location!)

--Puzzle: What should happen to 'w' in a 4-tuple when you normalize it?

(**ANS:** nothing! if you normalize a vector, w==0 before and after. Normalize a point?!? don't!!)

--Puzzle: Suppose you define a 'unit cube' centered at the origin whose 8 corner points have (x,y,z,w) coordinates given by (+/-1, +/-1, +/-1, 1). What is the distance between opposite corners of the cube?

ANS: We can choose any 2 opposing corner points, but I choose
P1 = (1,1,1,1) and P0 = (-1,-1,-1,1). The vector that reaches from P0 to P1 is given by
V01 = (P1 – P0) = (1,1,1,1) – (-1,-1,-1,1) = (1,1,1,1) + (1,1,1,-1) = (2,2,2,0).
The length of this vector is $\text{sqrt}(2^2 + 2^2 + 2^2 + 0^2) = \text{sqrt}(3 \cdot 4) = 2 \cdot \text{sqrt}(3)$.

--Puzzle: Suppose you define a cube whose 8 corner points have (x,y,z,w) coordinates given by (+/- L, +/- L, +/- L, 1). We could then write 8 vectors that reach from the origin to each cube corner as (+/- L, +/- L, +/- L, 0). What is the length of each vector? ANS: length = $L \cdot \text{sqrt}(3 \cdot L^2) = L \cdot \text{sqrt}(3)$.
What distance separates cube corners that share an edge? ANS: 2L: (separation of adjacent corners)

--Puzzle: Suppose we chose fixed A, B, C values to define 8 vectors (+/-A, +/-B, +/-C, 0). We use these to define the corners of a rectangular solid by adding these 8 vectors to the origin point (0,0,0,1).
If we 'normalize' these 8 vectors, will it change the shape of the solid? Does it yield a cube?
ANS: no; all 8 vectors have the same length (length = $\text{sqrt}(A^2 + B^2 + C^2 + 0^2)$), and thus normalizing them scales each of their magnitudes by 1/length, all magnitudes==1, but the directions of these vectors don't change. This centered rectangular solid keeps its shape; non-cubes do not become cubes.

--Puzzle: For a unit cube centered at point (a,b,c,1) what 8 vectors reach from the origin to the corners of the cube? ANS: $[a +/-1,\ b +/-1,\ c +/-1,\ 0]^T$.

--Puzzle: If we normalize those 8 vectors, what happens to the shape they describe? Can any vector have zero length (can't be normalized)? ANS: shape depends strongly on a, b and c values; if all three are +/-1, then the cube must have one of it's vertices at the origin (can you prove that?), where its distance from the origin can't be normalized – we can't divide by zero length! In general, normalizing the vectors will move each corner towards or away from the origin to a distance of 1.0 (moving the point onto the surface of a sphere of radius 1 centered at the origin) without changing the vector direction. Sketch it on paper, or in WebGL software: what happens when a,b,c are all near +/-1? Or all very large?

The vector 'add' and 'subtract' operators are obvious,
but how can we 'multiply' two vectors? Math provides (at least) 2 ways:

1) Dot product

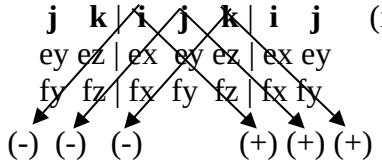
of vectors E and F (written E · F, spoken as "E dot F"), finds a scalar value:

--the 3D inner product, a *projection:*
it means "find how much of E is in the F direction; multiply that by the length of F"
-- More formally: **E · F** = ex·fx + ey·fy + ez·fz, and therefore it commutes: **E·F = F·E**
-- **E · F** = ||E|| ||F|| cos θ, where θ is the angle between the vectors (if connected at one vertex)
-- Dot products can find angles, too, if we make ||E|| = ||F|| = 1 by normalizing **E** and **F** first:
if ||E|| ||F|| = 1, then **E·F** = cos θ; θ= arccos(**E · F**)
if E is perpendicular to F, **E·F = ?** **ans: 0**, because none of vector E is in the F direction.
if E is parallel to F, **E·F = ?** **ans: ||E|| ||F||**, because all of E is in the F direction.
if E is perpendicular to the plane Π,
E · F > 0 if E and F aim towards the *same side* of the plane;
E · F < 0 if E and F aim towards opposite plane sides, and =0 if F is parallel to plane.
--Want more? Want Pictures? Search online – e.g. https://en.wikipedia.org/wiki/Dot_product

2) Cross product

of vectors E and F (written E x F, or "E cross F") finds a vector value,

--also known as the 3D 'outer product';
--finds the vector '**N**' that is perpendicular to E and perpendicular to F:
N = E x F
--The N vector points in the direction perpendicular to the plane formed by E and F.
--Within that plane, E and F define 2 sides of a parallelogram.
The area of that parallelogram is the length of the vector **N**.
--How do we compute a cross product?
E x F = [ey·fz – ez·fy, ez·fx – ex·fz, ex·fy – ey·fx]
Nobody remembers that! I find it easier to remember as this 3x3 matrix determinant:



(i,j,k are unit vectors in the x,y,z directions)

--**ANTI**-COMMUTATIVE! **E x F = -(F x E);** vector of same magnitude, but opposite direction.
--Like dot products, cross-products can also find angles between 3D vectors **E** and **F**:
|| **E x F** || = ||**E**|| ||**F**|| sin θ; where θ is the angle between the vectors (if connected at one vertex)
-- If ||E|| ||F|| = 1 (how do I get that? just normalize **E** and **F** first) then we get θ = arcsin(**E x F**).
NOTE: cross-product result is signed, and it measures angle *from* E *to* F – you can visualize this easily if you imagine both the E and F vectors emerging from the origin to define a 3D plane. Now rest your *right* hand (*!never your left hand!*) on that plane at the origin, and curl your fingers to move from the E vector to the F vector – your thumb will point in the vector direction **E x F**, and your fingers have curled by θ degrees move from the E vector to the F vector on the **EF** plane.
-- If **E** is perpendicular to **F**, ||**E x F**|| = ||**E**|| · ||**F**||, the area of a rectangle with **E**,**F** as edges.
-- If **E** is parallel to **F**, ||**E x F**|| = **0** (a 'degenerate' parallelogram: all edges parallel).

--Puzzle: if we can 'multiply' vectors in two ways, can you find a way to reverse those processes two ways, by defining two kinds of 'vector-divide' operators?

--Puzzle: What procedure would you devise to undo the result of a dot-product? Would it require you to know one or both of the unknown vectors' lengths? Or their directions?

--Puzzle: What procedure would you devise to undo the result of a cross-product? Would it require you to know one or both of the unknown vectors' lengths? Or their directions?

For other concise yet thorough reviews of vector math, geometry, and linear algebra, complete with many helpful solved problems, see:

- Short summaries such as "Vector Analysis (any edition)" – Murray R. Spiegel, Schaum's Outline Series in Mathematics, McGraw-Hill Book Co., or
- Good textbooks such as "An Introduction to Linear Algebra" by Gilbert Strang (and look for his youtube videos or MIT OpenCourseware), or
- Superb youtube videos by 3Blue1Brown on many topics in math and geometry (I especially like the exhilarating presentations on quaternions (a topic in Project B)…
- And of course, our optional textbook I recommend so highly: "*Mathematics for 3D Game Programming and Computer Graphics*" Eric Lengyel, (2013). Chap 2,3,4.

APPENDIX: Column-Vector Notation and Microsoft DirectX

This course, WebGL, OpenGL, and most books on graphics use column-vector notation. However, the authors of the Windows' DirectX library chose to use row-vector notation instead.

This simple linear algebra theorem converts between column-style (WebGL) and row-style (DirectX) vectors and matrices. Recall that if we multiply a column vector v_0 by matrix M we will get a new column vector v_1; and we can write it as $v_1 = Mv_0$.

Recall that a row-vector is just the transpose of a column vector:

$$v_0 == \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad v_0^T == [x \ y \ z \ w] .$$

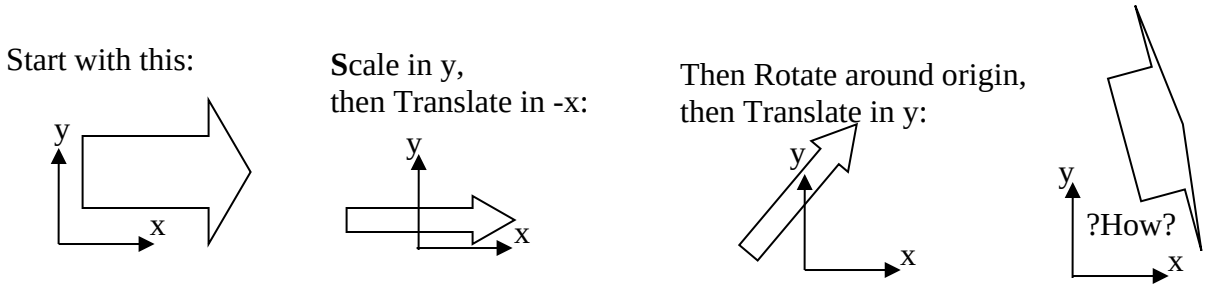
Theorem: if $v_1 = Mv_0$, then $v_1^T = v_0^T M^T$

Why? Remember, 'vectors' are just skinny matrices; use

Theorem: For any matrices A and B, $(AB)^T = B^T A^T$

3D TRANSFORMATIONS

Puzzle: How can you translate (slide up/down/left/right), rotate, zoom, and shear a line-drawing?
Puzzle: How does order of applying these matrix operations affect the geometric results?
Puzzle: Given all the vertices of the 2D shape on the left, how can you make these shapes?



-Define the 'canvas' as part of a plane in 3D, measured in x,y,z (for convenience, think z=0 for now).
-Define a 3D 'vertex' as a point chosen on that x,y,z plane;
write it as a 4-element column vector $[x,y,z,1]^T$.
I will soon explain more about that last '1' (after you see how it enables us to write translation matrices).

CLAIM:

Almost all useful 3D manipulations for all rigid 3D parts defined by 3D vertices consist of a sequence of 'rigid body' transformations; three basic, adjustable operations arranged in some useful sequence.

--Rigid-Body transformations:

Translate (shift position of the shape by some desired direction and magnitude)
Rotate (turn the object around some specified axis by some specified angle)
Scale (shrink or enlarge the shape, scaling all coordinates by the same 'scale factor')

--We can write each one of these **T**, **R**, or **S** transformations as matrix,
--We can use ordinary matrix multiplication to combine or 'concatenate' them; and
We can combine ***ANY*** transformation sequence, no matter how complex, into *just one matrix*!

JUSTIFICATION: 4x4 Matrices

If you write a vertex location point (x_0, y_0, z_0, 1) as a 4-element column vector v_0 and multiply one or more of the matrices below, you can make a new, transformed vertex position v_1:

Scale Matrix:

$$\mathbf{S} = \begin{bmatrix} sx & 0 & 0 & 0 \\ 0 & sy & 0 & 0 \\ 0 & 0 & sz & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(action: x *= sx; y *= sy; z *= sz; and w coordinate does not change)

Translate Matrix :

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & tx \\ 0 & 1 & 0 & ty \\ 0 & 0 & 1 & tz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(action: x += tx; y += ty; z+= tz; and w coordinate does not change)

Rotate on X Matrix: [1 0 0 0] Note: 1st column & row of identity matrix
[0 c(θ) –s(θ) 0] (action: rotate CCW around the x axis;
ROTX = [0 s(θ) c(θ) 0] and w, x coordinates do not change.
[0 0 0 1] c, s are the sines, cosines of angle theta)

Rotate on Y Matrix: [c(θ) 0 –s(θ) 0] Note: 2nd column and row of identity matrix
ROTY = [0 1 0 0] (action: rotate CCW around the y axis;
[s(θ) 0 c(θ) 0] and w, y coordinates do not change.
[0 0 0 1] c, s are the sines, cosines of angle theta)

Rotate on Z Matrix: [c(θ) –s(θ) 0 0] Note 3rd column and row of identity matrix
ROTZ = [s(θ) c(θ) 0 0] (action: rotate CCW around the z axis;
[0 0 1 0] and w, z coordinates do not change.
[0 0 0 1] c, s are the sines, cosines of angle theta)

Graphics conventions:

--We write vectors in lower case, matrices in upper case; such as a point v, and a matrix M.
--We always write 3D vertices, 3D points, and 3D vectors as 4-element 'column' vectors;
--when applied to a vector, matrices *precede* vectors: $v_1 = \mathbf{M}v_0$
v' = **TRS**v changes vector v into vector v' by applying **S**, then **R**, then **T**.
--Theorem: matrix multiplies are associative, but ***not*** commutative; **TRS**v != **SRT**v:
--Theorem: commutation requires careful use of transposes. For matrices A,B: $\mathbf{(AB)^T = B^T A^T}$

Sanity Check: **MUST we transform every point on every line or edge of every drawing? NO.**
IF we transform ONLY the vertices, then use LERPs to 'connect the dots', will we get the same drawing? YES!

--In OpenGL/WebGL we specify all drawing primitives as a sequence of vertices:
edges as point-pairs, line-strip (or 'paths' in PhotoShop) and triangles as an ordered list of vertices, and meshes as a set of triangles. OpenGL/WebGL can draw 3D triangles in any sequence: the same set of triangles drawn in a different order yields the same drawing.
--In OpenGL/WebGL, we transform ONLY the vertices,
then 'rasterizing' hardware draws points, lines or surfaces between those *transformed* vertices.

PUZZLE: What guarantees that 'vertex-only' transforms work correctly for all shapes?
Answer: By 'linear invariance' (and later, by 'projective invariance'): every point in every original edge specifies a unique point in the transformed edge, and any transformed straight-line edge always gives a straight-line result.
You can prove this yourself – try it!
Express an edge as a LERP, let parameter t vary from 0 to 1: 0<=t<=1. Given two vertices v0, v1, every point p along the edge between v0 and v1 is p(t) = v0 + (v1-v0)*t. What happens when you linearly transform p(t) with a 4x4 matrix? **Answer**: lines transform to lines for any 4x4 matrix).

Geometrically, the order we apply these T,R, and S matrices is crucial!
Results from translate-then-rotate are dramatically different from rotate-then-translate!
How do we know which order to use?

Demo: How to 'Compose' a sequence of 2D transforms:
https://www.youtube.com/watch?v=CmgSye6-Ack (from Eric Haines short-course)
How can we do this mathematically? What is the ordering of the matrices?

Geometric Duality:
Matrix multiplies have only one meaning mathematically, but TWO meanings geometrically. This is a true geometric duality: both are valid, consistent, provably correct description of the process, yet they are not equivalent. You need to know BOTH, as WebGL/OpenGL uses both, but in different ways.

Confusing the two or swapping them indiscriminately is a sure path to grief. Remember the two as:

Method 1--a **transformation of vertices** that
---keeps the drawing axes unchanged;
(Same coordinate system: same origin, same coordinate vectors **i, j, k** both *before* and *after* the transformation), but
---moves the vertices; it changes their numbers, their xyz coordinate values:
(Different x,y,z,w values before and after transformation)

Method 2--a **transformation of drawing axes** that
---moves the drawing axes; changes the coordinate system:
(Different origin, different coordinate vectors **i, j, k** *after* the transformation), but
---keeps the vertices unchanged;
(Same coordinate values, same x,y,z,w values before and after transformation).

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SURPRISE!
The OpenGL /WebGL specifications use *Method 2*
to describe all transformations:
gl.translate(), gl.rotate(), gl.scale(), etc.
(…and for a very good reason…)
We will examine this 'duality' very carefully
and thoroughly. Our textbook ignores it, mostly,
as do many introductory computer graphics books.

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If you don't gain complete mastery of this 'duality',
then you'll find that OpenGL/WebGL
"just never seems to works right" for you.
Your robot's legs may walk away from its torso:
your scaled parts get skewed diagonally,
your camera aiming controls work 'backwards',
and your lights never stay where you put them:
tl:dr;
"You're Gonna Have a Bad Time!"