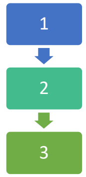
8. Duality

$$x(t)=F^{-1}[X(j\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega$$

$$2\pi x(-t)=\int_{-\infty}^{\infty}X(j\omega)e^{-j\omega t}d\omega$$

$$2\pi x(-j\omega)=\int_{-\infty}^{\infty}X(t)e^{-j\omega t}dt$$



9. Frequency Shifting

$$e^{j\omega_0 t}x(t)\overset{F}{\longleftrightarrow}X[j(\omega-\omega_0)]$$

10. Differentiation in Frequency Domain

$$-jtx(t)\overset{F}{\longleftrightarrow}\frac{d}{d\omega}X(j\omega)$$

Example 4.6 Find the Fourier transform of $\frac{2}{1+t^2}$.

Solution:

$$e^{-|t|}\overset{F}{\longleftrightarrow}\frac{2}{1+\omega^2}$$

→ 找有相近形式的x(t)

$$e^{-|t|}=\mathcal{F}^{-1}\left[\frac{2}{1+\omega^2}\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}d\omega$$

$$2\pi e^{-|t|}=\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}d\omega$$

$$2\pi e^{-|\omega|}=\int_{-\infty}^{\infty}\left(\frac{2}{1+t^2}\right)e^{-j\omega t}dt=\int_{-\infty}^{\infty}g(t)e^{-j\omega t}dt$$

$$\mathcal{F}\left[\frac{2}{1+t^2}\right]=2\pi e^{-|\omega|}$$

Discussion 4.4 Find the Fourier transform $\boxed{\text{of } \frac{1}{\pi t}.}$

Solution:

90°相移,

important!

$$\mathrm{sgn}(t)\overset{F}{\longleftrightarrow}\frac{2}{j\omega}$$

$$\mathrm{sgn}(t)=\mathcal{F}^{-1}\left[\frac{2}{j\omega}\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{j\omega}\right)e^{j\omega t}d\omega$$

$$2\pi\mathrm{sgn}(-t)=-2\pi\mathrm{sgn}(t)=\int_{-\infty}^{\infty}\left(\frac{2}{j\omega}\right)e^{-j\omega t}d\omega$$

$$-2\pi\mathrm{sgn}(\omega)=\int_{-\infty}^{\infty}\left(\frac{2}{jt}\right)e^{-j\omega t}dt=\int_{-\infty}^{\infty}\frac{2\pi}{j}\left(\frac{1}{\pi t}\right)e^{-j\omega t}dt$$

$$\mathcal{F}\left[\frac{1}{\pi t}\right]=j\mathrm{sgn}(\omega)$$

11. Parseval's Relation

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

$$\int_{-\infty}^{+\infty}x(t)y(t)dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)Y(-j\omega)d\omega$$

Proof:

$$\int_{-\infty}^{+\infty}x(t)x^*(t)dt=\int_{-\infty}^{+\infty}x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)\left[\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\right]d\omega$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X^*(j\omega)X(j\omega)d\omega$$

Example 4.7 Calculate the following integral ($a = a^* > 0$).

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+\omega^2)^2} d\omega$$

Solution:

$$e^{-a|t|} \overset{F}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

Parseval's Relation yields

$$\int_{-\infty}^{\infty} \left|e^{-a|t|}\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{2a}{a^2+\omega^2}\right|^2 d\omega$$

$$\int_{-\infty}^{\infty} \frac{1}{(a^2+\omega^2)^2} d\omega = \frac{\pi}{2a^2}\int_{-\infty}^{\infty}\left|e^{-a|t|}\right|^2 dt$$

$$= \frac{\pi}{a^2}\int_{0}^{\infty} e^{-2at} dt = \frac{\pi}{2a^3}$$

Example 4.8 Calculate the following integral ($a = a^* > 0$).

$$\int_{-\infty}^{\infty} \frac{sin^2\, a\omega}{\omega^2} d\omega$$

Solution:

$$u(t+a) - u(t-a) \overset{F}{\longleftrightarrow} \frac{2sina\omega}{\omega}$$

Parseval's Relation yields

$$\int_{-\infty}^{\infty} |u(t+a)-u(t-a)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{2sina\omega}{\omega}\right|^2 d\omega$$

$$\int_{-\infty}^{\infty} \frac{sin^2\, a\omega}{\omega^2} d\omega = \frac{\pi}{2}\int_{-\infty}^{\infty} |u(t+a)-u(t-a)|^2 dt$$

$$= \frac{\pi}{2}\int_{-a}^{a} 1 dt = a\pi$$

§4.4 The Convolution Property

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega) \text{ and } y(t) \overset{F}{\longleftrightarrow} Y(j\omega)$$

$$x(t) * y(t) \overset{F}{\longleftrightarrow} X(j\omega)Y(j\omega)$$

Proof:

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x_1(\tau)x_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\left[\int_{-\infty}^{+\infty} x_2(t-\tau)e^{-j\omega(t-\tau)}dt\right]e^{-j\omega\tau}d\tau$$

$$= X_2(j\omega)\int_{-\infty}^{+\infty} x_1(\tau)e^{-j\omega\tau}d\tau = X_1(j\omega)X_2(j\omega)$$

$$y(t) = h(t) * x(t)$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

$H(j\omega)$ is known as the frequency response of an LTI system

$$x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$y(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} H(jk\omega_0)X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

① Cascade interconnection

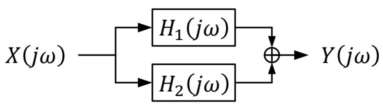
$H_2(j\omega)[H_1(j\omega)X(j\omega)] = [H_1(j\omega)H_2(j\omega)]X(j\omega)$

$$X(j\omega) \to \boxed{H_1(j\omega)} \to \boxed{H_2(j\omega)} \to Y(j\omega)$$

$$X(j\omega) \to \boxed{H_1(j\omega)H_2(j\omega)} \to Y(j\omega)$$

② Paralleled interconnection

$H_1(j\omega)X(j\omega) + H_2(j\omega)X(j\omega) = [H_1(j\omega) + H_2(j\omega)]X(j\omega)$



$$X(j\omega) \to \boxed{H_1(j\omega) + H_2(j\omega)} \to Y(j\omega)$$

Example 4.9 Determine the frequency responses of the differentiator and integrator.

Solution:

$$x(t) \to \boxed{u_1(t)} \to \frac{d}{dt}x(t)$$

Differentiator $h_1(t) = \frac{d}{dt}\delta(t) = u_1(t)$

$$H_1(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega X(j\omega)}{X(j\omega)} = j\omega$$

$$x(t) \to \boxed{u_{-1}(t)} \to \int_{-\infty}^{t} u(\tau)d\tau$$

Integrator $h_2(t) = u(t) = u_{-1}(t)$

$$H_2(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

Example 4.10 Determine the response of an ideal low-pass filter to an input signal $x(t) = \frac{sin\omega_i t}{\pi t}$, when the unit impulse response of the ideal low-pass filter is given by $h(t) = \frac{sin\omega_c t}{\pi t}$.

Solution:

$$X(j\omega) = \mathcal{F}\left[\frac{sin\omega_i t}{\pi t}\right] = \begin{cases} 1 \ \ (|\omega| < \omega_i) \\ 0 \ \ (|\omega| > \omega_i) \end{cases}$$

$$H(j\omega) = \mathcal{F}\left[\frac{sin\omega_c t}{\pi t}\right] = \begin{cases} 1 \ \ (|\omega| < \omega_c) \\ 0 \ \ (|\omega| > \omega_c) \end{cases}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \begin{cases} 1 \ \ (|\omega| < \min\{\omega_i\,, \omega_c\}) \\ 0 \ \ (|\omega| > \min\{\omega_i\,, \omega_c\}) \end{cases}$$

$$y(t) = \begin{cases} h(t) \ \ (\omega_c < \omega_i) \\ x(t) \ \ (\omega_c > \omega_i) \end{cases}$$

Example 4.11 Determine the response of an LTI real system to a sinusoidal input signal $x(t) = \cos(\omega_0 t + \theta)$.

$$X(j\omega) \to \boxed{H(j\omega) = |H(j\omega)|e^{j\measuredangle H(j\omega)}} \to Y(j\omega)$$

Solution:

$$x(t) = \cos(\omega_0 t + \theta) = \frac{1}{2}e^{j\theta}e^{j\omega_0 t} + \frac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

$$X(j\omega) = \pi e^{j\theta}\delta(\omega - \omega_0) + \pi e^{-j\theta}\delta(\omega + \omega_0)$$

$$Y(j\omega) = \pi e^{j\theta}H(j\omega)\delta(\omega - \omega_0) + \pi e^{-j\theta}H(j\omega)\delta(\omega + \omega_0)$$

$$\begin{aligned} y(t) &= \frac{1}{2}e^{j\theta}H(j\omega_0)e^{j\omega_0 t} + \frac{1}{2}e^{-j\theta}H(-j\omega_0)e^{-j\omega_0 t} \\ &= Re\{|H(j\omega_0)|e^{j[\omega_0 t + \theta + \measuredangle H(j\omega_0)]}\} \\ &= |H(j\omega_0)|\cos[\omega_0 t + \theta + \measuredangle H(j\omega_0)] \end{aligned}$$