ECE495 Game Theory and Networked Systems

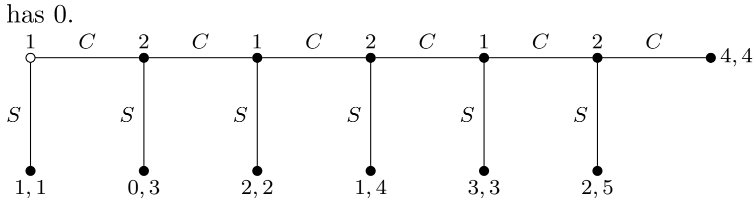
Lecture 12 Dynamic Games and Repeated Games

1 Issues with the bargaining model

There are a few issues with modeling sequential games this way and solving them for subgame-perfect Nash Equilibriums. The first is computational depth: many games, such as Chess, have perfect information but cannot be solved due to sheer complexity of computation. In addition, humans rarely induct backwards more than two steps, so using this model for issues of human behavior may be problematic. Finally, in some cases the predicted equilibrium do not match "common' behavior as illustrated in the following example:

1.1 Example: The Centipede Game

(This game is named after its game tree which looks a bit like a centipede). In the centipede game, each player starts with one dollar. At each subsequent stage, one of the players chooses whether to continue the game or not, with players alternating turns. If a player chooses to continue, he gives one dollar to the other player and the other player gets one dollar from the bank, for a total of two dollars increase for the other player. If he chooses to stop, the game ends and both players keeps the money they have. Played for one hundred rounds, the game will give both players one hundred dollars; however, the only subgame-perfect Nash Equilibrium for either player is to stop at the first turn, where one player has one dollar and the other one has 0.



2 Dynamic Games with Imperfect Information

Previously, we looked at dynamic games with complete and perfect information. Perfect information implies that at any one time only one player is choosing an action and this players knowss the choices made during all previous times by both itself and any other player. This is reasonable for some games like Chess where one can observe an opponent's actions in every single move. However, in games like Battleship or Bridge and poker, we may want to model players needing to act with partial or no knowledge of actions taken by others, or sometimes even themselves.

In order to model this, we introduce the concept of information sets.

Definition 1. *Given a game in extensive form, an information set is a set of nodes that a player can not distinguish among at a given stage of the game. All nodes in a given information set must have the same set of actions available.*

An *imperfect information* game is a dynamic game in which there is at least one information set that contains more than one node (i.e., is not a singleton). In a game with perfect information, all information sets are singletons.

Example 2.1 (Entry Deterrence Game)**.** In this game there are two players. Player 1, the entrant, can choose to enter the market or stay out. Player 2, the incumbent, is already present in the market. If player 1 chooses to enter, both players simultaneously decide to fight(F) or accommodate(A).

This game is depicted in Fig. 1 below.

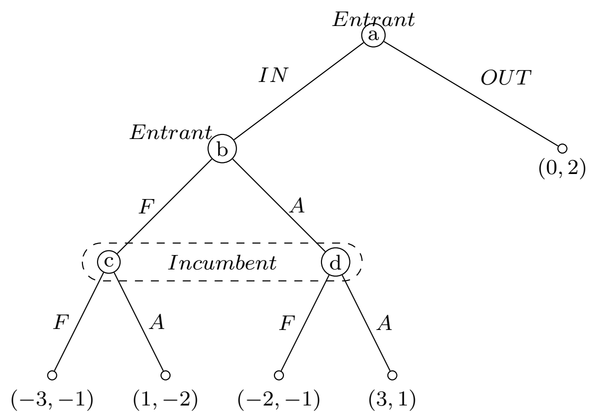


Figure 1: Extensive form Representation for Entry Deterrence Game

If P1 chooses not to enter, the game ends and P1 (Entrant) gets a payoff of 0 and P2 (Incumbent) gets a payoff of 2. However, if P1 chooses to enter and both players decided to fight, then incumbent gets -1 and entrant gets -3. The rest of the payoff is shown in the tree.

- Entrant has two information sets:
  - $\{a\}$
  - $\{b\}$
- Incumbent has one information set:
  - $\{c, d\}$

  Incumbent does not know the entrant's strategy F or A, thus can not distinguish between the two nodes.

Note: A perfect information dynamic game is a special case in which all information sets have just one node.

Finding Sub-Game Perfect Nash Equilibrium:

Now that we have defined the model for a dynamic game with imperfect information, how do we find a SPNE in such a game? In order to answer this question, let's first define what a sub-game would be in such a game.

Definition 2. *A sub-game is a sub-tree rooted at a single information set with the following two properties:*

1. *It includes all subsequent nodes in the game tree;*
2. *it does not cross any information set.*

For example for the game in Figure 1, there is one sub-game rooted at node b.

Using the idea of sub-games we can then generalize the concepts of SPNEs and backward induction introduced in the last class for games with perfect information. Specifically, a strategy profile is a SPNE, if when restricted to any sub-game, it gives a Nash equilibrium for that sub-game. Such strategies can be found by doing backward induction on sub-games.

We illustrate this for the game in Figure 1. This game has one (proper) sub-game as identified above. First, we consider the Nash equilibrium of this sub-game. The normal form of this game is given in Table 1.

		Incumbent	
		F	A
Entrant	F	(-3,-1)	(1,-2)
	A	(-2,-1)	(3,1)

Table 1: The payoffs for the sub-game with root node b

For this sub-game, the Nash Equilibrium is (A,A) with payoffs (3,1). Neither player has incentive to deviate from this pair of strategies. Now that we have a NE for the sub-game, we can replace the sub-game with this equilibrium pay-off as shown in Fig. 2.

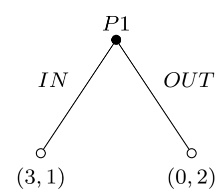


Figure 2: SPNE for Entry Deterrence Game

We then move "backward" and consider the equilibrium for this new game, which just involves the choice of P1. From Figure 2 it is clear that $P1$ would chose IN giving an overall pay-off of $(3, 1)$ in this SPNE.

Note: In some sub-games there will not be a unique Nash Equilibrium. In such cases, we just pick one and derive the SPNE based on that. Different choices will lead to different SPNE's.

3 Repeated Games

Most interactions between players in the real world occur more than once and the same players may behave very differently with these repeated interactions. So we now take a look at a class of games where players repeatedly engage in the same strategic game, called the stage game. At each stage, the player obtains a payoff (as opposed to only in the end like the sequential games). These repetitions can be finite or infinite.

3.1 Finitely Repeated Games

Recall the Prisoner's Dilemma with the following payoff matrix:
Recall that the strategy profile (L,L) is the unique Nash Equilibrium. A question we might ask is if we can make the players play (R,R) and be better off by repeatedly playing the game?

Repeat Twice: Suppose the player's play this game twice. Pay-offs for each strategic action profile are the sum of payoffs in each stage. We can again solve this via backward induction. The second stage game

	P2	
	L	R
P1 L	(1,1)	(5,0)
P1 R	(0,5)	(4,4)

Table 2: Payoff Matrix for Prisoner's Dilemma

is a sub-game that is equivalent to the one-shot version of Prisoner's Dilemma. Hence the only NE of the sub-game is (L, L) giving a pay-off of $(1, 1)$. Using this, we can then view the pay-offs in the first stage as being given by the following pay-off matrix (where we have simply added $(1, 1)$ to the one-stage payoffs.

	P2	
	L	R
P1 L	(2,2)	(6,1)
P1 R	(1,6)	(5,5)

Table 3: Payoff Matrix for Prisoner's Dilemma repeated twice

As we can observe the strategy profile, (L,L) is still the NE. So the only SPNE of the two-stage game is for both player to play L at each stage.

This leads us to the following theorem:

Theorem 3. *(Equilibria of Finitely Repeated Games) Given a game, G , let $G(T)$ be the stage game repeated T times. If G has a unique Nash Equilibrium, then $G(T)$ also has a unique sub-game perfect Nash Equilibrium where the NE for G is played in each game.*

Multiple Equilibria: If we add another action, O, in the Prisoners Dilemma, we will have more than one NE as shown below:

	P2		
	L	R	O
P1 L	(1,1)	(5,0)	(0,0)
P1 R	(0,5)	(4,4)	(0,0)
P1 O	(0,0)	(0,0)	(3,3)

Table 4: Payoff Matrix for modified Prisoner's Dilemma

The modified Prisoner's Dilemma has two NE now: (L,L) and (O,O).

Suppose that this new game is again repeated twice. Consider the following strategy for this repeated game:

1. Play (R,R) in the first period;
2. Play (O,O) if played (R,R) in the first period;
3. Play (L,L) otherwise,

At the second stage, neither players have incentive to unilaterally deviate as the two strategies (O,O) and (L,L) are Nash equilibria.

Back to the first stage, one player can choose to deviate to L to receive a higher payoff of 5, while the other player follows the strategy of playing R. The overall payoff after deviation is then $5 + 1 = 6$, as in second stage, they will play (L,L) (highest payoff given the other player follows the above strategy). The payoff without deviation is $4 + 3 = 7$ and thus no player has incentive to unilaterally deviate.

The *key takeaway* is that if a dynamic game has multiple NE, then there are many sub-game perfect equilibria of the finitely repeated game. Some of these may involve strategies that are collectively more profitable for players than any NE of the one-shot stage game. The equilibrium created by repeating stage game's NE is still SPE, as no player has incentive to unilaterally deviate.

4 Infinitely Repeated Games

Recall the Prisoner's Dilemma with the following payoff matrix:

		P2	
		L	R
P1	L	(1,1)	(5,0)
	R	(0,5)	(4,4)

Table 5: Payoff Matrix for Prisoner's Dilemma

Now let us consider an infinitely repeated game i.e, the players play the game repeatedly at times =0,1,2,... We can view this as either a case where the game is played repeatedly forever or as a setting where the player do not know when the last period is. They always believe there is some chance the game will continue to the next period.

When we study infinitely repeated games we want to derive the payoff that a player receives in each of infinitely many periods. A common assumption is that the player values the present more than the future. Therefore, the future payoffs are *discounted* and are less valuable. The overall payoff for the player is the normalized sum of discounted payoffs at each stage and is given below by the function:

$$U(s) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_i(s_i^t, s{-}i^t) \tag{1}$$

where $\delta \in [0, 1)$ is the discount factor and (1-δ) in front of the equation is for normalization to keep the payoff bounded even when $\delta \to 1$. The normalization also has the effect that if a player receives the same payoff u in each stage game then its normalized discounted payoff will also be u so it is then easy to compare the normalized discounted pay-off to the payoff in a stage game. We also assume that the discount factor for all players is the same.

In such an infinitely repeated game, a strategy for a given player is a rule that specifies for each game t, a action to choose as a function of the history of play in all previous games.

Trigger Strategy: As an example of a strategy in an infinitely repeated games we examine trigger strategies. A trigger strategy basically threatens players with a "punishment" action if they deviate from an agreed upon action.

Lets look at the repeated Prisoner's Dilemma again where we have the following trigger strategy:

1. Play R in first stage;

2. In tth stage, if the outcome in all prior stages is (R, R), play R, otherwise play L.

References

[1] R. Berry and R. Johari. *Economic modeling in networking: A primer.* Foundations and Trends in Networking 6.3 (2013): 165-286.