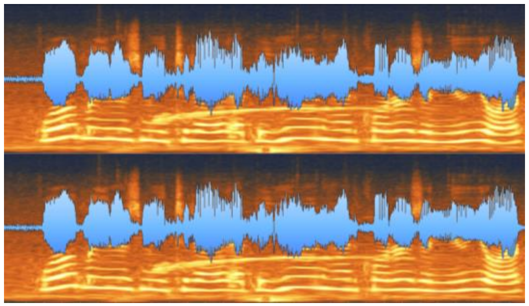
Chapter 6 Time and Frequency Characterization



"In analyzing LTI systems, it is often particularly convenient to utilize the frequency domain because differential and difference equations and convolution operations in the time domain become algebraic operations in the frequency domain."

§6.1 Magnitude-Phase Representation of the Fourier Transform

1. Continuous-Time Fourier Transforms

$$X(j\omega) = |X(j\omega)|e^{j\measuredangle X(j\omega)}$$

2. Discrete-Time Fourier Transforms

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\measuredangle X(e^{j\omega})}$$

§6.2 Magnitude-Phase Representation of the Frequency Response of LTI system

1. Continuous-Time LTI Systems

$$Y(j\omega) = |Y(j\omega)|e^{j\measuredangle Y(j\omega)}$$
$$= H(j\omega)X(j\omega) = |H(j\omega)||X(j\omega)|e^{j[\measuredangle X(j\omega)+\measuredangle H(j\omega)]}$$
$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$
$$\measuredangle Y(j\omega) = \measuredangle X(j\omega) + \measuredangle H(j\omega)$$

$|H(j\omega)|$ is called the gain of the system.

$\measuredangle H(j\omega)$ is called the phase shift of the system.

If $|H(j\omega)|$ and $\measuredangle H(j\omega)$ are not desired, then their effects are referred to as magnitude and phase distortions.

Discussion 6.1 Determine the response of an LTI real system to a sinusoidal input signal $x(t) = \cos(\omega_0 t + \theta)$.

$$X(j\omega) \longrightarrow \boxed{H(j\omega) = |H(j\omega)|e^{j\measuredangle H(j\omega)}} \longrightarrow Y(j\omega)$$

Solution:

$$X(j\omega) = X_1(j\omega) + X_2(j\omega) \qquad X_1(j\omega) = \pi e^{j\theta}\delta(\omega-\omega_0)$$
$$X_2(j\omega) = \pi e^{-j\theta}\delta(\omega+\omega_0)$$
$$|Y_1(j\omega)| = |H(j\omega_0)||X_1(j\omega)|$$
$$\measuredangle Y_1(j\omega) = \measuredangle X_1(j\omega) + \measuredangle H(j\omega_0) = \theta + \measuredangle H(j\omega_0)$$
$$|Y_2(j\omega)| = |H(-j\omega_0)||X_2(j\omega)| = |H(j\omega_0)||X_2(j\omega)|$$
$$\measuredangle Y_2(j\omega) = \measuredangle X_2(j\omega) + \measuredangle H(-j\omega_0) = -\theta - \measuredangle H(j\omega_0)$$
$$\therefore Y_1(j\omega) = Y_2^*(j\omega)$$
$$y(t) = |H(j\omega_0)|\cos[\omega_0 t + \theta + \measuredangle H(j\omega_0)]$$

Example 6.1 Consider $x(t) = \cos t$ is input to three LTI systems with the impulse responses of $h_1(t) = u(t)$, $h_2(t) = -2\delta(t) + 5e^{-2t}u(t)$ and $h_3(t) = 2te^{-t}u(t)$. Please give the expressions of output signals $y_1(t)$, $y_2(t)$ and $y_3(t)$.

Solution:

$$H_1(j\omega)\Big|_{\omega=1} = \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]\Bigg|_{\omega=1} = \frac{1}{j} = -j$$

$$H_2(j\omega)\Big|_{\omega=1} = \left(-2 + \frac{5}{2+j\omega}\right)\Bigg|_{\omega=1} = (-j\omega)\Big|_{\omega=1} = -j$$

$$H_3(j\omega)\Big|_{\omega=1} = \left[\frac{2}{(1+j\omega)^2}\right]\Bigg|_{\omega=1} = \frac{2}{2j} = -j$$

$$y_1(t) = y_2(t) = y_3(t) = \cos\left(t - \frac{\pi}{2}\right) = \sin t$$

Example 6.2 If $H(j\omega) = \frac{1-j\omega}{1+j\omega}$ and $x(t) = \cos\left(\frac{t}{\sqrt{3}}\right) + \cos t + \cos(\sqrt{3}t)$, please give the expression of the output signal $y(t)$ and explain the effect of the system.

Solution:

$$|H(j\omega)| = \left|\frac{1-j\omega}{1+j\omega}\right| = 1$$

$$\measuredangle H(j\omega) = \measuredangle(1-j\omega) - \measuredangle(1+j\omega)$$
$$= \arctan(-\omega) - \arctan(\omega) = -2\arctan(\omega)$$

$$y(t) = \cos\left[\frac{t}{\sqrt{3}} - 2\arctan\left(\frac{1}{\sqrt{3}}\right)\right] + \cos[t - 2\arctan(1)]$$
$$+\cos\left[\sqrt{3}t - 2\arctan(\sqrt{3})\right]$$
$$= \cos\left(\frac{t}{\sqrt{3}} - \frac{\pi}{3}\right) + \cos\left(t - \frac{\pi}{2}\right) + \cos\left(\sqrt{3}t - \frac{2\pi}{3}\right)$$

2. Linear Phase

Example 6.3 When $x(t)$ is an input to an LTI system with the frequency response of $H(j\omega) = e^{j\omega t_0}$, please give the expression of the output signal $y(t)$ and explain the effect of the system.

Solution:

According to the time shifting property of the Fourier Transform

$\because Y(j\omega) = e^{j\omega t_0}X(j\omega)$

$\therefore y(t) = x(t - t_0)$ Time Shift

Moreover,

$|H(j\omega)| \equiv 1$

$\measuredangle H(j\omega) = -t_0\omega$ Linear Phase

3. Group Delay

Consider a narrowband input signal $x(t)$ with the spectrum $X(j\omega) = X(j\omega)[u(\omega - \omega_0 + \Delta) - u(\omega - \omega_0 - \Delta)]$.

We can accurately approximate the phase of the system in the narrowband with the linear approximation

$\measuredangle H(j\omega) \approx -\emptyset - \alpha\omega$

So that,

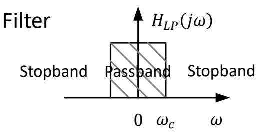
$Y(j\omega) = H(j\omega)\,X(j\omega) \approx |H(j\omega)|X(j\omega)e^{-j\emptyset}e^{-j\alpha\omega}$

α is the group delay at $\omega = \omega_0$

Group delay is defined as $\tau(\omega) = -\frac{d}{d\omega}\measuredangle H(j\omega)$

§6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

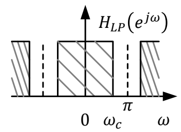
1. Continuous-Time Ideal Low-Pass Filter



$$H_{LP}(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$h_{lp}(t) = F^{-1}[H_{LP}(j\omega)] = \frac{\sin \omega_c t}{\pi t}$$

2. Discrete-Time Ideal Low-Pass Filter



$$H_{LP}(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c < \pi \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

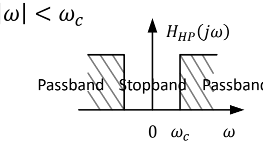
$$h_{lp}[n] = \mathcal{F}^{-1}[H_{LP}(e^{j\omega})] = \frac{1}{2\pi}\int_{<2\pi>} H(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{1}{2\pi}\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} = \frac{\sin \omega_c n}{\pi n}$$

3. Continuous-Time Ideal High-Pass Filter

$$H_{HP}(j\omega) = 1 - H_{LP}(j\omega) = \begin{cases} 0 & |\omega| > \omega_c \\ 1 & |\omega| < \omega_c \end{cases}$$

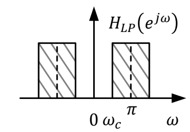
$$h_{hp}(t) = \delta(t) - \frac{\sin \omega_c t}{\pi t}$$



4. Discrete-Time Ideal High-Pass Filter

$$H_{HP}(e^{j\omega}) = 1 - H_{LP}(e^{j\omega})$$

$$h_{hp}[n] = \delta[n] - \frac{\sin \omega_c n}{\pi n}$$

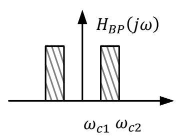


5. Continuous-Time Ideal Band-Pass Filter

$$H_{BP}(j\omega) = H_{LP1}(j\omega) - H_{LP2}(j\omega)$$

$$= \begin{cases} 1 & \omega_{c2} < |\omega| < \omega_{c1} \\ 0 & elsewhere \end{cases}$$

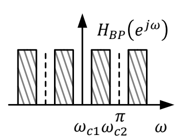
$$h_{bp}(t) = \frac{\sin \omega_{c1} t}{\pi t} - \frac{\sin \omega_{c2} t}{\pi t}$$



6. Discrete-Time Ideal Band-Pass Filter

$$H_{BP}(e^{j\omega}) = H_{LP1}(e^{j\omega}) - H_{LP2}(e^{j\omega})$$

$$h_{bp}[n] = \frac{\sin \omega_{c1} n}{\pi n} - \frac{\sin \omega_{c2} n}{\pi n}$$

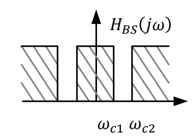


7. Continuous-Time Ideal Band-Stop Filter

$$H_{BS}(j\omega) = 1 - H_{BP}(j\omega) = H_{LP}(j\omega) + H_{HP}(j\omega)$$

$$= \begin{cases} 0 & \omega_{c2} < |\omega| < \omega_{c1} \\ 1 & elsewhere \end{cases}$$

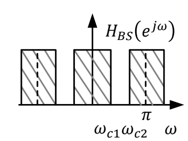
$$h_{bs}(t) = \delta(t) - \frac{\sin \omega_{c1} t}{\pi t} + \frac{\sin \omega_{c2} t}{\pi t}$$



8. Discrete-Time Ideal Band-Stop Filter

$$H_{BS}(e^{j\omega}) = 1 - H_{BP}(e^{j\omega})$$

$$h_{bp}[n] = \delta[n] - \frac{\sin \omega_{c1} n}{\pi n} + \frac{\sin \omega_{c2} n}{\pi n}$$



Homework			
6.27			
6.5	6.23		

① Do not wait until the last minute
② Express your own idea and original opinion
③ Keep in mind the zero-tolerance policy on plagiarism