ECE495 Game Theory and Networked Systems

Lecture 5 Second price auction and mixed strategy

1 Mixed Strategy

Definition 1. *A mixed strategy for player r with strategy set S_r is a probability distribution σ_r on S_r.*

1.1 Game model

- Let $\Delta(S_r)$ be a set of probability distributions over S_r.
- Agents in a game $G = (R, \{S_r\}_{r\in R}, \{\Pi_r\}_{r\in R})$ that play mixed strategies can be viewed as playing pure strategies in the game $G' = (R, \{\Delta(S_r)\}_{r\in R}, \{\tilde{\Pi}_r\}_{r\in R})$.
- Payoffs $\tilde{\Pi}_r$ can be interpreted as expectations over the joint distribution on strategies. The payoff of each player r is given by

$$\begin{aligned}\tilde{\Pi}_r(\sigma_r, \bar{\sigma}_{-r}) &= \sum_{\bar{s}\in S} \Pi_r(\bar{s}) \prod_{q\in R} \sigma_q(s_q) \qquad \text{(for discrete strategy cases)}\\ &= \int_{\bar{s}} \Pi_r(\bar{s})\, \mathrm{d}\bar{\sigma}(\bar{s}) \qquad \text{(for continuous strategy cases)}\end{aligned}$$

Definition 2. *Suppose S_r is finite, $\bar{\sigma}$ is a mixed strategy Nash Equilibrium if*

$$\tilde{\Pi}_r(\sigma_r, \bar{\sigma}_{-r}) \geq \tilde{\Pi}_r(\sigma_r', \bar{\sigma}_{-r}),\ \forall \sigma_r' \in \Delta(S_r),\ \forall r \in R,$$

i.e., σ_r assigns positive probability only to pure strategies that are best response to $\bar{\sigma}_{-r}$.

For mixed strategies, we have the following theorem due to the fact that the mixed extension of a finite game results in payoff that is linear in the strategy (due to linearity of expectation), which is a concave game.

Theorem 3. *Suppose that $G = (R, \{S_r\}_{r\in R}, \{\Pi_r\}_{r\in R})$ is a finite game, where R and $\{S_r\}_{r\in R}$ are finite. Then there exists a mixed Nash Equilibrium.*

We note that pure strategy NE is a special case of mixed strategy NE where all probability is assigned to one outcome. See here for details of the proof based on the construction of a better response and improvement function.

1.2 Examples

Example 4. Suppose that R users are accessing a shared medium. Each user has two strategies: transmit (T) and not transmit (NT). Each transmit incurs a cost of $-c$. If only one user transmits, then the transmission is successful, and the user receives payoff of 1-c. Otherwise, if there are more than one user transmit, then a collision occurs, and each user who transmits gets a payoff of $-c$. If no user transmits, then the payoff is 0 for every user.
Clearly, the game is symmetric among the users. Therefore, the game has a mixed NE where each user transmits independently with probability p (i.e., $\sigma_r(T) = p,\ \sigma_r(NT) = 1-p,\ \forall r$). Then it follows that

$$0 = \tilde{\Pi}_r(NT) = \tilde{\Pi}_r(T) = (1-c)(1-p)^{R-1} - c((1-(1-p)^{R-1})).$$

Solving for p gives $p = 1 - c^{\frac{1}{R-1}}$.

2 Concave Game

Definition 5. *A game $g = (R, \{S_r\}, \{\Pi_r\})$ is concave if for each player $r \in R$ we have:*
1) S_r is a nonempty, compact, convex set of $\mathbb{R}^n$,
2) payoff function $\Pi_r(s_r, \bar{s}_{-r})$ is continuous in $\bar{s}$ for all feasible strategies,
3) $\Pi_r(s_r, \bar{s}_{-r})$ is a concave function of s_r for all $\bar{s}_{-r}$.

We note that any finite game can be viewed as a concave game when considering its mixed strategies.
Example 6. Cournot competition in the previous lecture with payoff functions $\pi_i(q_1, q_2) = (1-q_1-q_2)q_i - c_i q_i$ for $i = 1, 2$ is concave. Note that although we set $q_i \geq 0$, but we can restrict our attention to the set of $q_i \in [0, 1]$, since outside the set, the player's payoff is negative and will not be played (can play $q_i = 0$ instead). Hence the set of possible actions for both agents are essentially closed and bounded, and therefore, compact.

Theorem 7. *Every concave game has at least one (pure strategy) Nash equilibrium.*

Proof We only sketch the proof here. Note that Brouwer Fixed Point Theorem is not directly applicable to the joint best response correspondence $B(\bar{s})$ since it might not be a function (rather it might be a set-valued function). Instead, we can use an improvement function. Let define $\phi_r(\hat{s})$ as:

$$\phi_r(\hat{s}) \triangleq \arg\max_{s_r \in S_r} \ \Pi_r(s_r, \hat{s}_{-r}) - \|s_r - \hat{s}_r\|^2. \tag{1}$$

One can view $\hat{s}$ as a profile we are currently at and we want to know if we can improve the profile. Clearly if $\hat{s}$ is a NE no improvement is possible as the norm expression will penalize deviation from $\hat{s}_r$, and we can not improve the first term. Function $\Pi_r(s_r, \hat{s}_{-r}) - \|s_r - \hat{s}_r\|^2$ is strictly concave and continuous in $\hat{s}$. The maximizer is hence uniquely defined. We can also show that the function ϕ_r is continuous. We can now apply the Brouwer fixed point theorem. Define

$$\phi(\bar{s}) = \begin{pmatrix} \phi_1(\bar{s}) \\ \phi_2(\bar{s}) \\ . \\ . \\ . \\ \phi_R(\bar{s}) \end{pmatrix} \tag{2}$$

$\phi(.)$ is a continuous mapping $\prod_{r=1}^{R} s_r \to \prod_{r=1}^{R} s_r$, since each of the element is continuous. Since the Cartesian product of compact convex sets is compact convex we satisfy conditions of Brouwer Fixed Point Theorem. So a fixed point must exist. By concavity of the payoff functions, this fixed point is our NE. To see why, we note that if Π is differentiable, then $\hat{s}$ satisfies $\nabla_r \Pi(\hat{s}) = 0$ for each r and therefore is component-wise best response for each player. □

Theorem 7 is extendable to quasiconcave games which are defined the same as definition 5 with the difference that payoff function $\Pi_r(s_r, \bar{s}_{-r})$ is a quasiconcave function of s_r for all feasible $\bar{s}_{-r}$.
Conditions also exists which guarantee the uniqueness of a Nash equilibrium for concave game - see [2].

3 Justifying a NE

We know that action of an agent in a Nash equilibria is a rational response to the equilibrium profile of the other users, but how do we coordinate Nash equilibrium with other players if we have just met. In other words, How do we know that other agents will play this profile and why they chose this profile to play? There are couple of justifications that can be used based on the problem we are solving. Here we talk about some possible ones:

Nash equilibrium as a self-enforcing outcome. This first justification works for so-called *one-shot* games that players just meet and play a game just once. In this setting, what will lead players to a NE? One possible approach of justification is a non-binding agreement between players. If this non-binding agreement is a NE, none of these players have any incentive to break the rule and deviate from it. In this sense, each player enforce himself to follow the agreement. Note that this justification assumes rationality of players.

Nash equilibrium as the outcome of long-run learning. One other idea of justification NE comes as a result of learning process of players when they have the chance to play one game many times. We assume that players can experiment with different actions to seek possible actions to improve their payoff functions. Such a process might not reach a NE necessarily, but if it reaches a *steady state* where players can't improve their actions given what others are playing, then this steady-state is necessarily a NE. In this justification, we can assume that each player does not have full information about payoff function and rationality of other players and may learn enough about them by playing the game repeatedly and reaching a NE. One possible downside of this justification is that players might deviate from their learning in order to fool other players.

Nash equilibrium as a result of lots of thinking. Nash equilibrium can also be justified when players put a lot of effort to compute how other people might play the game before actually starting the game.

Note that although these justifications might work in some certain settings, justification becomes more sophisticated when there are more than one NE, where equilibrium selection is needed.

References

[1] Berry, Randall A., and Ramesh Johari., "Economic modeling in networking: A primer.," *Foundations and Trends in Networking*, 6.3 (2013): 165-286.

[2] Rosen, J. (1965). Existence and Uniqueness of Equilibrium Points for Concave N-Person Games. Econometrica, 33(3), 520-534.