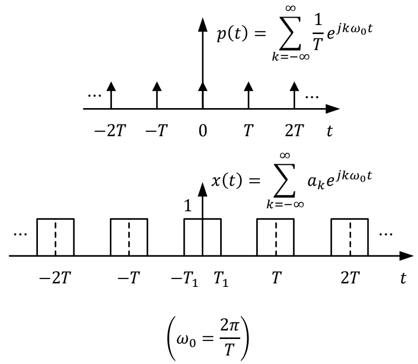
Discussion 3.3: Determine the spectral coefficients of a square wave based on the spectral coefficients of an impulse train.



Solution:

$$p(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{jk\omega_0 t}$$

$$\frac{d}{dt}x(t) = \sum_{k=-\infty}^{\infty} jk\omega_0 a_k e^{jk\omega_0 t} = p(t+T_1) - p(t-T_1)$$

For $k \neq 0$

$$a_k = \frac{\frac{1}{T}e^{jk\omega_0 T_1} - \frac{1}{T}e^{-jk\omega_0 T_1}}{jk\omega_0} = \frac{2j\sin k\omega_0 T_1}{j2k\pi} = \frac{\sin k\omega_0 T_1}{k\pi}$$

For $k = 0$

$$a_0 = \frac{1 \times 2T_1}{T} = \frac{2T_1}{T}$$

§3.5 Fourier Series Representation of Discrete-time Periodic Signals

1. Linear Combinations of Harmonically Related Complex Exponentials

The set of harmonically related complex exponentials:

$$\emptyset_k[n] = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n} \quad (k \in Z)$$
$$\emptyset_k[n] = \emptyset_{k+rN}[n]$$

Fourier Series:

$$x[n] = \sum_{k=\langle N\rangle} a_k \emptyset_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

a_k is called the **discrete-time Fourier series coefficients**.

2. Determination of the Fourier Series Representation of a Discrete-Time Period Signal

$$\boxed{\sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}n} = N\delta[k-r] = \begin{cases} N\,(k-r = 0, \pm N, \cdots) \\ 0\,(k-r \neq 0, \pm N, \cdots) \end{cases}}$$

$$\sum_{n=\langle N\rangle} x[n]e^{-jr\frac{2\pi}{N}n} = \sum_{n=\langle N\rangle}\left(\sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}\right)e^{-jr\frac{2\pi}{N}n}$$
$$= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)\frac{2\pi}{N}n} = Na_r$$

$$\therefore a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr\frac{2\pi}{N}n}$$

The discrete-time Fourier Series Pair:

$$\begin{cases} x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \\ a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\frac{2\pi}{N}n} \end{cases}$$

a_k is also called the spectral coefficients of $x[n]$.
$a_k = a_{k+N}$ has the period N.

§3.6 Properties of Discrete-Time Fourier Series

$x[n] \overset{FS}{\longleftrightarrow} a_k$ and $y[n] \overset{FS}{\longleftrightarrow} b_k$ have the same period T.

1. Multiplication

$$x[n]y[n] \overset{FS}{\longleftrightarrow} c_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

Proof:

$$\begin{aligned} c_k &= \frac{1}{N}\sum_{n=\langle N\rangle} x[n]y[n]e^{-jk\omega_0 n} \\ &= \frac{1}{N}\sum_{n=\langle N\rangle}\left(\sum_{l=\langle N\rangle} a_l e^{jl\omega_0 n}\right) y[n]e^{-jk\omega_0 n} \\ &= \sum_{l=\langle N\rangle} a_l \left\{\frac{1}{N}\sum_{n=\langle N\rangle} y[n]e^{-j(k-l)\omega_0 n}\right\} \\ &= \sum_{l=\langle N\rangle} a_l b_{k-l} \end{aligned}$$

2. Time Shifting

$$x[n-n_0] \overset{FS}{\longleftrightarrow} e^{-jk\frac{2\pi}{N}n_0} a_k$$

Proof:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$\begin{aligned} x[n-n_0] &= \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 (n-n_0)} = \sum_{k=\langle N\rangle} a_k e^{-jk\omega_0 n_0} e^{jk\omega_0 n} \\ &= \sum_{k=\langle N\rangle} \left(e^{-jk\omega_0 n_0} a_k\right) e^{jk\omega_0 n} \end{aligned}$$

Note that $\left(\omega_0 = \frac{2\pi}{N}\right)$

3. First Difference

$$x[n] - x[n-1] \overset{FS}{\longleftrightarrow} \left(1 - e^{-jk\frac{2\pi}{N}}\right) a_k$$

Proof:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}$$

$$x[n-1] = \sum_{k=\langle N\rangle} \left(e^{-jk\frac{2\pi}{N}} a_k\right) e^{jk\omega_0 n}$$

$$x[n] - x[n-1] = \sum_{k=\langle N\rangle} \left(1 - e^{-jk\frac{2\pi}{N}}\right) a_k e^{jk\omega_0 n}$$

4. Parseval's Relation for Discrete-Time Period Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2=\sum_{k=\langle N\rangle}|a_k|^2$$

Proof:

$$\begin{aligned}\frac{1}{N}\sum_{n=\langle N\rangle}|x[n]|^2&=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]x^*[n]\\&=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]\left(\sum_{k=\langle N\rangle}a_k e^{jk\omega_0 n}\right)^*\\&=\sum_{k=\langle N\rangle}a_k^*\left\{\frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk\omega_0 n}\right\}\\&=\sum_{k=\langle N\rangle}a_k^*\,a_k=\sum_{k=\langle N\rangle}|a_k|^2\end{aligned}$$

§3.7 Fourier Series and LTI Systems

1. Continuous-Time LTI systems

$$x(t)=\sum_{k=-\infty}^{\infty}a_k\,e^{jk\omega_0 t}\longrightarrow\boxed{h(t)}\longrightarrow y(t)=\sum_{k=-\infty}^{\infty}b_k e^{jk\omega_0 t}$$

① System Function:

$$H(s)=\int_{-\infty}^{+\infty}h(\tau)e^{-s\tau}d\tau$$

② Frequency Response:

$$H(j\omega)=\int_{-\infty}^{+\infty}h(t)e^{-j\omega t}dt$$

③ Input-Output Relation:

$$b_k=a_kH(jk\omega_0)$$

2. Discrete-Time LTI systems

$$x[n]=\sum_{k=\langle N\rangle}a_k e^{jk\omega_0 n}\longrightarrow\boxed{h[n]}\longrightarrow y[n]=\sum_{k=\langle N\rangle}b_k e^{jk\omega_0 n}$$

① System Function:

$$H(z)=\sum_{k=-\infty}^{+\infty}h[k]z^{-k}$$

② Frequency Response:

$$H\left(e^{j\omega}\right)=\sum_{n=-\infty}^{+\infty}h[n]e^{-j\omega n}$$

③ Input-output Relation:

$$b_k=a_kH\left(e^{jk\omega_0}\right)$$

§3.8 Filtering

Change the relative amplitudes of the frequency components or eliminate some frequency components in a signal

1. Frequency-Shaping Filters

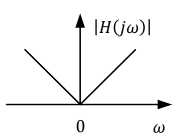
LTI systems that change the shape of the spectrum

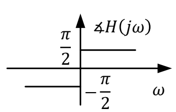
$$x(t)\longrightarrow\boxed{h(t)}\longrightarrow\frac{d}{dt}x(t)$$

$$H(j\omega)=\int_{-\infty}^{+\infty}u_1(t)e^{-j\omega t}dt=j\omega$$

$$|H(j\omega)|=|\omega|$$

$$\measuredangle H(j\omega)=\begin{cases}\frac{\pi}{2} & (\omega>0)\\-\frac{\pi}{2} & (\omega<0)\end{cases}=\frac{\pi}{2}sgn(\omega)$$





2. Frequency-Selective Filters

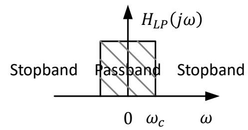
LTI systems that are designed to pass some frequencies

Undistorted and attenuate or eliminate others

① Continuous-Time LTI Systems

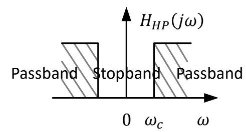
a. Ideal Low-Pass Filter

$$H_{Lp}(j\omega)=\begin{cases}1 & (|\omega|<\omega_c)\\ 0 & (|\omega|>\omega_c)\end{cases}$$



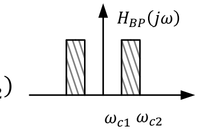
b. Ideal High-Pass Filter

$$H_{Hp}(j\omega)=1-H_{Lp}(j\omega)$$
$$=\begin{cases}0 & (|\omega|<\omega_c)\\ 1 & (|\omega|>\omega_c)\end{cases}$$



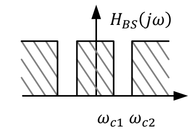
c. Ideal Band-Pass Filter

$$H_{BP}(j\omega)=\begin{cases}1 & (\omega_{c1}<|\omega|<\omega_{c2})\\ 0 & (|\omega|<\omega_{c1})\cup(|\omega|>\omega_{c2})\end{cases}$$



d. Ideal Band-Stop Filter

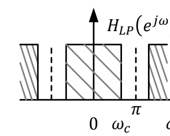
$$H_{BS}(j\omega)=1-H_{Bp}(j\omega)$$
$$=\begin{cases}0 & (\omega_{c1}<|\omega|<\omega_{c2})\\ 1 & (|\omega|<\omega_{c1})\cup(|\omega|>\omega_{c2})\end{cases}$$



② Discrete-Time LTI Systems

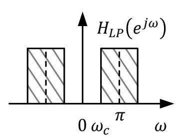
a. Ideal Low-Pass Filter

$$H_{Lp}(e^{jk\omega})=\begin{cases}1 & (|\omega|<\omega_c)\\ 0 & (\omega_c<|\omega|<\pi)\end{cases}$$



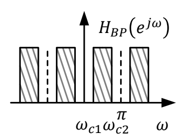
b. Ideal High-Pass Filter

$$H_{Lp}(e^{jk\omega})=\begin{cases}0 & (|\omega|<\omega_c)\\ 1 & (\omega_c<|\omega|<\pi)\end{cases}$$



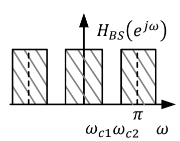
c. Ideal Band-Pass Filter

$$H_{BP}(e^{jk\omega})=\begin{cases}1 & (\omega_{c1}<|\omega|<\omega_{c2})\\ 0 & (|\omega|<\omega_{c1})\\ 0 & (\omega_{c2}<|\omega|<\pi)\end{cases}$$



d. Ideal Band-Stop Filter

$$H_{BS}(e^{jk\omega})=\begin{cases}0 & (\omega_{c1}<|\omega|<\omega_{c2})\\ 1 & (|\omega|<\omega_{c1})\\ 1 & (\omega_{c2}<|\omega|<\pi)\end{cases}$$

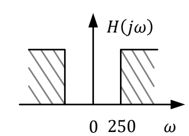


Example 3.4 Consider a continuous-time LTI system of which the frequency response is $H(j\omega)=\begin{cases}1 & (|\omega|>250)\\ 0 & (|\omega|<250)\end{cases}$.

When the input to this system is a signal $x(t)$ with the fundamental period $T=\frac{\pi}{7}$ and Fourier series coefficient a_k, the output $y(t)$ is Identical to $x(t)$. Determine the values of k that lead to $a_k=0$.

Solution:

$$\omega_0=\frac{2\pi}{T}=14$$



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk14t} \qquad y(t) = \sum_{k=-\infty}^{+\infty} H(j14k) a_k e^{j14kt}$$

$\because x(t) = y(t)$

$\therefore a_k = a_k\, H(j14k)$

$$\because H(j14k) = \begin{cases} 1 & \left(|k| \geq \left\lceil \frac{250}{14} \right\rceil = 18\right) \\ 0 & \left(|k| \leq \left\lfloor \frac{250}{14} \right\rfloor = 17\right) \end{cases}$$

$\therefore a_k = 0\ \ (|k| \leq 17)$

Discussion 3.4: The frequency response of a system is written as

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 2, & 2\omega_0 < |\omega| < 4\omega_0 \\ 0, & otherwise \end{cases}$$

(a) Compute the output of the system $y(t)$, when the input signal is given by

$$x(t) = \frac{1}{3} + \frac{1}{2} e^{j\frac{\omega_0}{2}t} - \frac{5}{6} e^{-j3\omega_0 t} + \cos 6\omega_0 t$$

(b) Express the output $y(t)$ with the Fourier series. Is the output a real signal?

Solution:

$$x(t) = \frac{1}{3} + \frac{1}{2} e^{j\frac{\omega_0}{2}t} - \frac{5}{6} e^{-j3\omega_0 t} + \cos 6\omega_0 t = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{\omega_0}{2}t}$$

$$a_0 = \frac{1}{3}, \qquad a_1 = \frac{1}{2}, \qquad a_{-6} = -\frac{5}{6}, \qquad a_{\pm 12} = \frac{1}{2}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{\omega_0}{2}t}$$

$$b_k = a_k H\left(jk\frac{\omega_0}{2}\right)$$

$$b_0 = \frac{1}{3}, \qquad b_1 = \frac{1}{2}, \qquad b_{-6} = -\frac{5}{3}, \qquad b_{\pm 12} = 0$$

$y(t) = \frac{1}{3} + \frac{1}{2} e^{j\frac{\omega_0}{2}t} - \frac{5}{3} e^{-j3\omega_0 t}$ is not a real signal

Homework

3.13	**3.43**		
3.1	3.15	3.34	3.35

① Do not wait until the last minute
② Express your own idea and original opinion
③ Keep in mind the zero-tolerance policy on plagiarism