

HWRS 505: Vadose Zone Hydrology

Lecture 9
9/24/2024

Today: Richards' Equation
Reading: Chapter 11 (Pinder & Celia, 2006).

Review of Lecture 8

❖ Derivation of governing equations for two-phase flow in porous media

- Mass conservation for phase α ($\alpha = w$ or nw)

$$\frac{\partial}{\partial t}(\rho_\alpha \phi s_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{q}_\alpha) = 0$$

- Flux law: Extended Darcy's law for two-phase flow in porous media

$$\mathbf{q}_\alpha = -\frac{k_{r,\alpha} \mathbf{k}}{\mu_\alpha} \nabla(p_\alpha + \rho_\alpha g z)$$

Recall Darcy's law for single-phase (saturated flow): $\mathbf{q}_w = -\frac{\mathbf{k}}{\mu_w} \nabla(p_w + \rho_w g z)$

❖ Richards' assumptions

- $\mu_a \ll \mu_w$ and $\rho_a \ll \rho_w$. $\rho_a \ll \rho_w$ helps to simplify, but it is NOT required. Why?
- Reduces the general two-phase flow equation to only one equation for water flow (i.e., Richards' Equation)

Richards' Equation

Convert water pressure to water pressure head as $\psi_w = p_w/(\rho_w g)$

$$\phi \frac{\partial S_w}{\partial t} - \nabla \cdot \left[k_{rw} \frac{\mathbf{k} \rho_w g}{\mu_w} (\nabla \psi_w + \nabla z) \right] = 0$$

$\mathbf{K}_{sat}^w \equiv \frac{\mathbf{k} \rho_w g}{\mu_w}$, then

$$\phi \frac{\partial S_w}{\partial t} - \nabla \cdot [k_{rw} \mathbf{K}_{sat}^w (\nabla \psi_w + \nabla z)] = 0$$

$\theta_w = \phi S_w$, then

$$\frac{\partial \theta_w}{\partial t} - \nabla \cdot [\mathbf{K}_{sat}^w k_{rw} (\nabla \psi_w + \nabla z)] = 0$$

3D Richards' Equation written in a “mixed” form, i.e., both moisture content and pressure head appear explicitly in the equation.

Richards' Equation

$$\frac{\partial \theta_w}{\partial t} = \frac{d\theta_w}{d\psi_w} \frac{\partial \psi_w}{\partial t} = C(\psi_w) \frac{\partial \psi_w}{\partial t}, \quad C(\psi_w) \equiv \frac{d\theta_w}{d\psi_w} \quad \leftarrow \text{Specific moisture capacity}$$

$$\Rightarrow C(\psi_w) \frac{\partial \psi_w}{\partial t} - \nabla \cdot [k_{rw} \mathbf{K}_{sat}^w (\nabla \psi_w + \nabla z)] = 0 \quad \text{"}\psi\text{-based" or "pressure-based" form}$$

$$\nabla \psi_w = \frac{d\psi_w}{d\theta_w} \nabla \theta_w, \quad \mathbf{D}(\theta_w) \equiv \frac{k_{rw} \mathbf{K}_{sat}^w}{d\theta_w / d\psi_w} \quad \leftarrow \text{Soil moisture diffusivity}$$

$$\Rightarrow \frac{\partial \theta_w}{\partial t} - \nabla \cdot \left[k_{rw} \mathbf{K}_{sat}^w \left(\frac{d\psi_w}{d\theta_w} \nabla \theta_w + \nabla z \right) \right] = 0$$

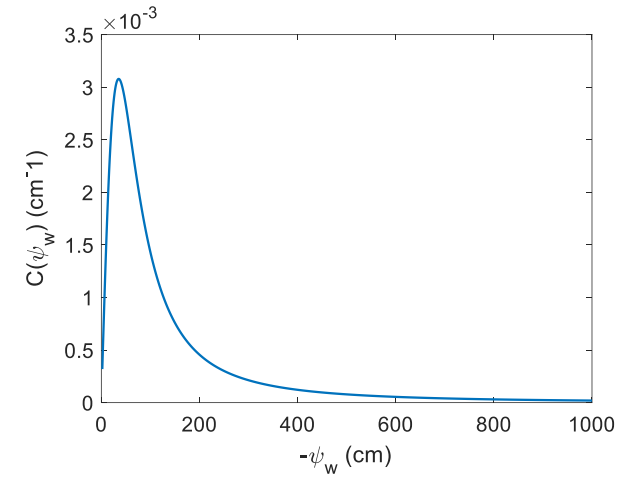
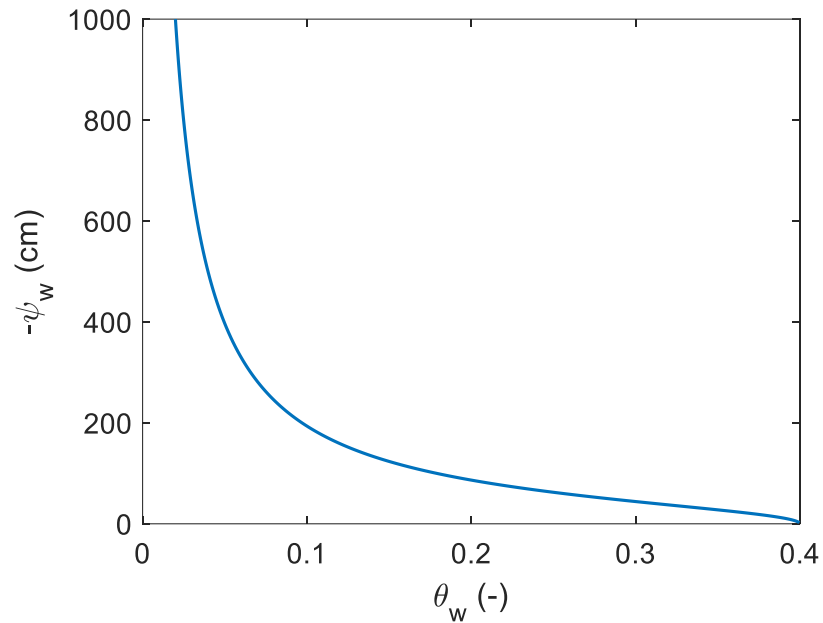
$$\Rightarrow \frac{\partial \theta_w}{\partial t} - \nabla \cdot \left[\frac{k_{rw} \mathbf{K}_{sat}^w}{d\theta_w / d\psi_w} \nabla \theta_w + k_{rw} \mathbf{K}_{sat}^w \nabla z \right] = 0$$

$$\Rightarrow \frac{\partial \theta_w}{\partial t} - \nabla \cdot [\mathbf{D}(\theta_w) \nabla \theta_w] - \nabla \cdot [k_{rw} \mathbf{K}_{sat}^w \nabla z] = 0 \quad \text{"}\theta\text{-based" or "moisture content-based" form}$$

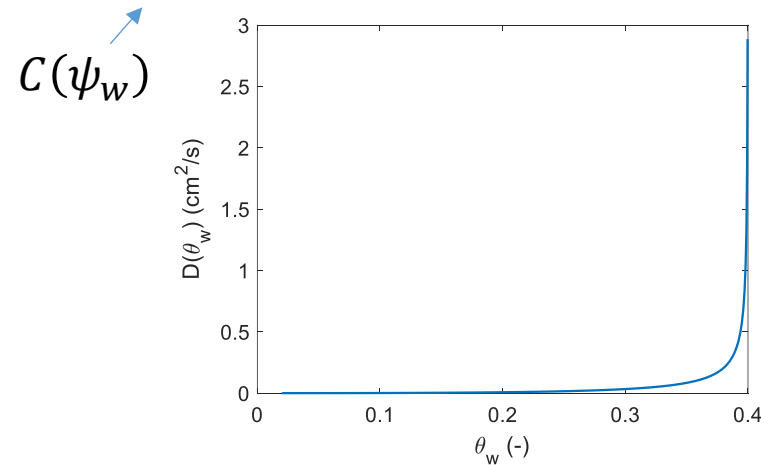
This term represents moisture movement driven by capillary force, which acts like a diffusive process, i.e., water flux is driven by the gradient of moisture where $\mathbf{D}(\theta_w)$ is the (nonlinear) diffusion coefficient.

$C(\psi_w)$ and $D(\theta_w)$

$$C(\psi_w) \equiv \frac{d\theta_w}{d\psi_w} \leftarrow \text{Specific moisture capacity}$$



$$D(\theta_w) \equiv \frac{K_{sat}^w k_{rw}}{d\theta_w/d\psi_w} \leftarrow \text{Soil moisture diffusivity}$$



Some Remarks on Richards' Equation

- θ -based form is appropriate for unsaturated systems, but not for systems that include saturated zones.

(1) $\mathbf{D}(\theta_w) \equiv \frac{k_{rw} \mathbf{K}_{sat}^w}{d\theta_w/d\psi_w}$ becomes unbounded as $\frac{d\theta_w}{d\psi_w} \rightarrow 0$ under saturated condition.

(2) In saturated heterogeneous porous media, $\nabla\theta_w$ does not represent a driving force.

- When saturated conditions occur, the mixed form or the pressure head-based form should be used.

- Soil and fluid compressibility are neglected. They can be included to Richards' equation by relaxing these assumptions.

$$\frac{\partial}{\partial t}(\rho_w \phi s_w) + \nabla \cdot (\rho_w \mathbf{q}_w) = 0.$$

$$\nabla \cdot (\rho_w \mathbf{q}_w) = \rho_w \nabla \cdot \mathbf{q}_w + \mathbf{q}_w \nabla \rho_w \approx \rho_w \nabla \cdot \mathbf{q}_w$$

$\mathbf{q}_w \nabla \rho_w \ll \rho_w \nabla \cdot \mathbf{q}_w$
Water is only slightly compressible.

$$\frac{\partial}{\partial t}(\rho_w \phi s_w) = \rho_w \phi \frac{\partial s_w}{\partial t} + \rho_w s_w \frac{\partial \phi}{\partial t} + \phi s_w \frac{\partial \rho_w}{\partial t} = \rho_w \phi \frac{\partial s_w}{\partial t} + \left(\rho_w s_w \frac{\partial \phi}{\partial p_w} + \phi s_w \frac{\partial \rho_w}{\partial p_w} \right) \frac{\partial p_w}{\partial t}$$

$$\frac{\partial \phi}{\partial p_w} \frac{\partial p_w}{\partial t} \quad \frac{\partial \rho_w}{\partial p_w} \frac{\partial p_w}{\partial t}$$

Note: Contribution to the water storage term by compressibility becomes more important under saturated conditions.

Examples of Air-Water Flow (1D)

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BC at the bottom: fixed pressures for air and water

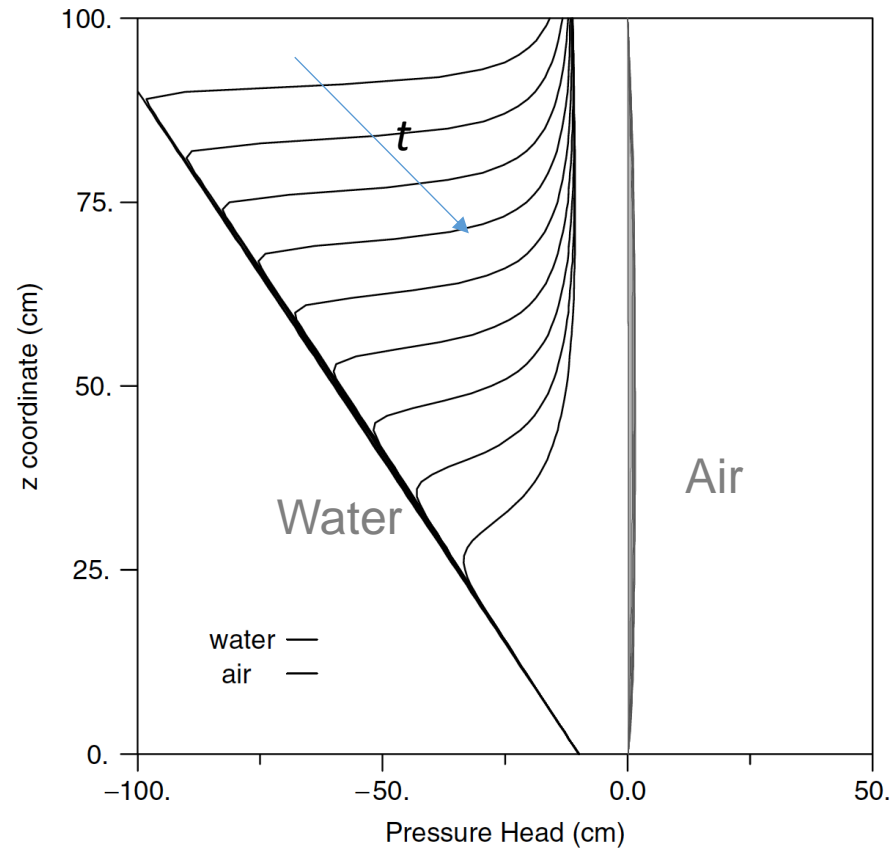


FIGURE 11.6. Plot of both water and air pressure heads as a function of depth, for different values of time. The water pressure head changes over a range of about 100 cm of water, while the air pressure head changes by only a small fraction of that amount (from [15]).

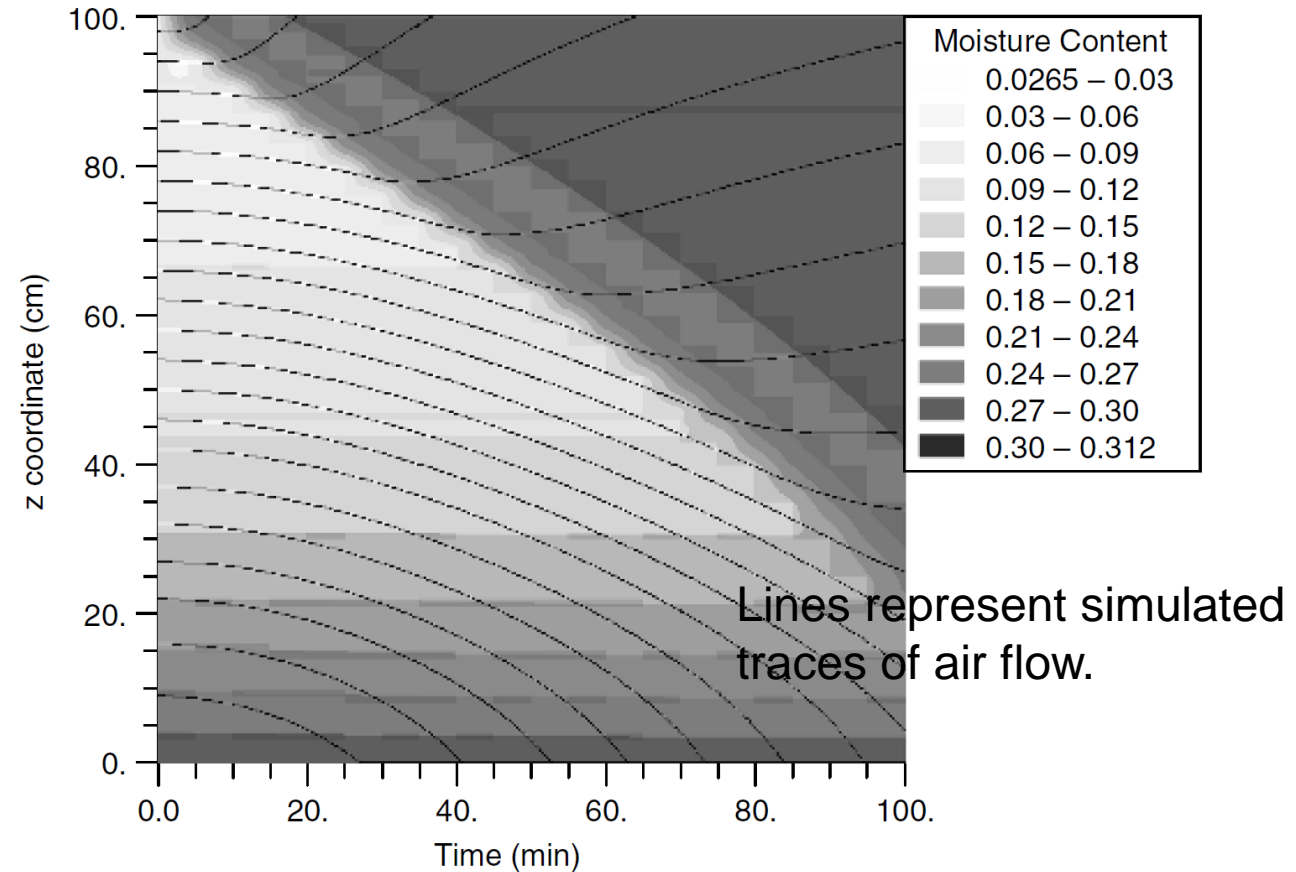


FIGURE 11.7. Water infiltration and air movement (from [15]).

- Air pressure buildup due to water infiltration is very little, but there is clear dynamic air flow

Examples of Air-Water Flow (1D)

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BC at the bottom: impervious for fluids

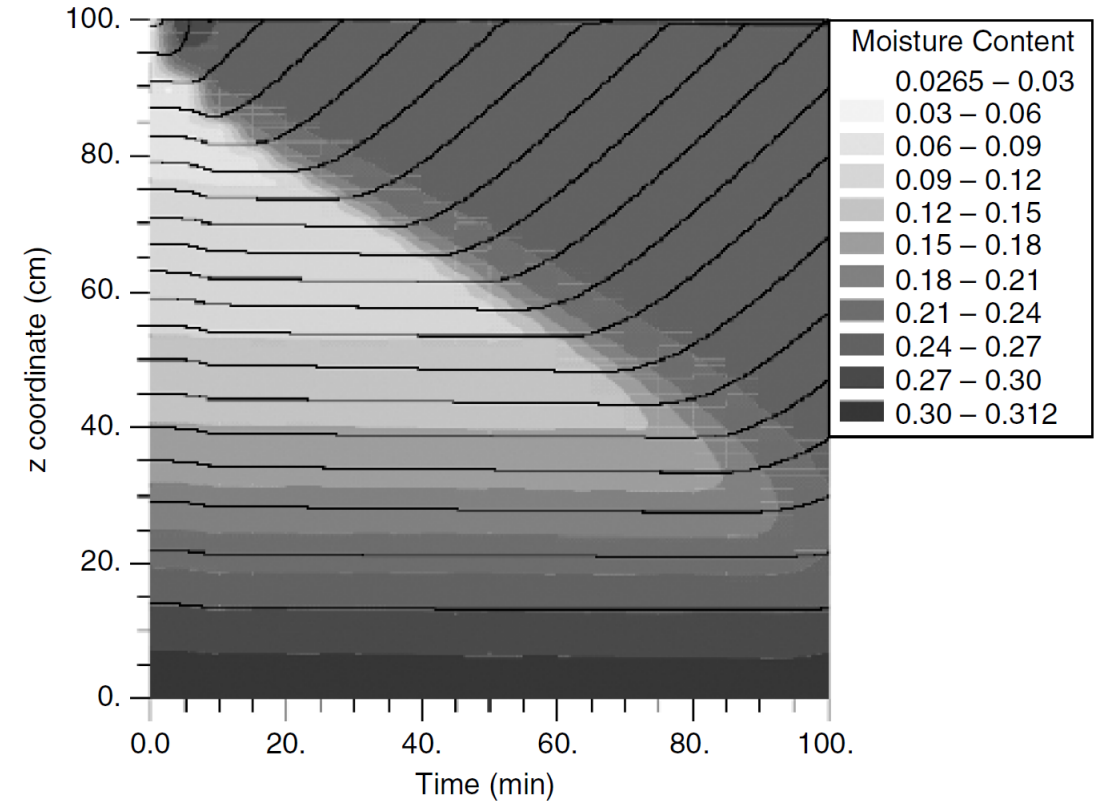
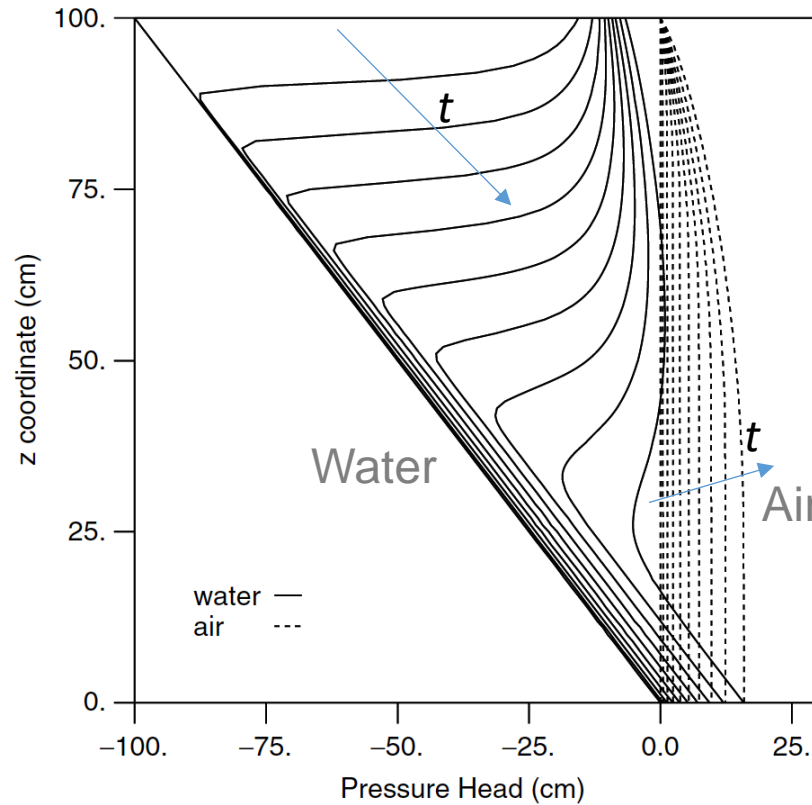


FIGURE 11.8. Plot of water and air pressure heads as a function of depth, for different times, for a system with impervious bottom boundary (from [15]).

FIGURE 11.9. Plot showing water saturation as a function of depth and time (from [15]).

- Notable air pressure buildup due to water infiltration. There is also clear dynamic air flow.
- Richards' equation would be invalid.

Examples of Air-Water Flow (2D)

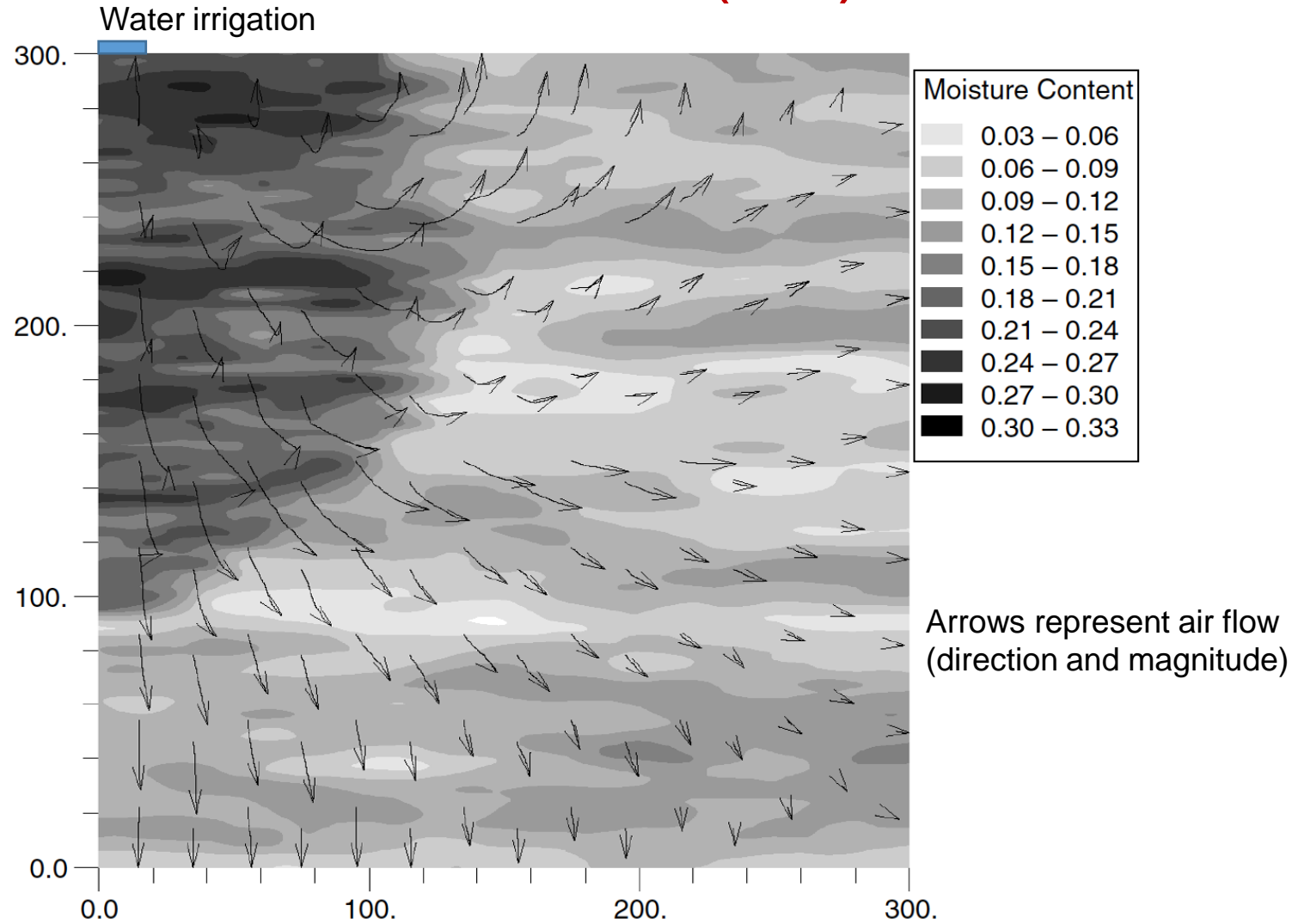


FIGURE 11.10. Plot of moisture content (shaded scale) at a given time, with air pathways superimposed, for the case of infiltration into a heterogeneous two-dimensional system and air pathway traces (from [17]).

1D Richards' Equation

The three forms of Richards' Equation in the vertical dimension. $K \equiv K_{sat}^w k_{rw}$

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial z} \left(K \frac{\partial \psi_w}{\partial z} \right) - \frac{\partial K}{\partial z} = 0 \quad \text{"Mixed" form}$$

$$C(\psi_w) \frac{\partial \psi_w}{\partial t} - \frac{\partial}{\partial z} \left(K \frac{\partial \psi_w}{\partial z} \right) - \frac{\partial K}{\partial z} = 0 \quad \text{"}\psi\text{-based" or "pressure-based" form}$$

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial z} \left(D \frac{\partial \theta_w}{\partial z} \right) - \frac{\partial K}{\partial z} = 0 \quad \text{"}\theta\text{-based" or "moisture content-based" form}$$

The three forms of Richards' Equation in the horizontal dimension.

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial x} \left(K \frac{\partial \psi_w}{\partial x} \right) = 0 \quad \text{"Mixed" form}$$

$$C(\psi_w) \frac{\partial \psi_w}{\partial t} - \frac{\partial}{\partial x} \left(K \frac{\partial \psi_w}{\partial x} \right) = 0 \quad \text{"}\psi\text{-based" or "pressure-based" form}$$

$$\frac{\partial \theta_w}{\partial t} - \frac{\partial}{\partial x} \left(D \frac{\partial \theta_w}{\partial x} \right) = 0 \quad \text{"}\theta\text{-based" or "moisture content-based" form}$$

Steady-State 1D Richards' Equation

In the vertical dimension.

$$\frac{d}{dz} \left(K \frac{d\psi_w}{dz} \right) + \frac{dK}{dz} = 0 \quad \text{"}\psi\text{-based"} \text{ or "pressure-based" form}$$

$$\frac{d}{dz} \left(D \frac{d\theta_w}{dz} \right) + \frac{dK}{dz} = 0 \quad \text{"}\theta\text{-based"} \text{ or "moisture content-based" form}$$

In the horizontal dimension.

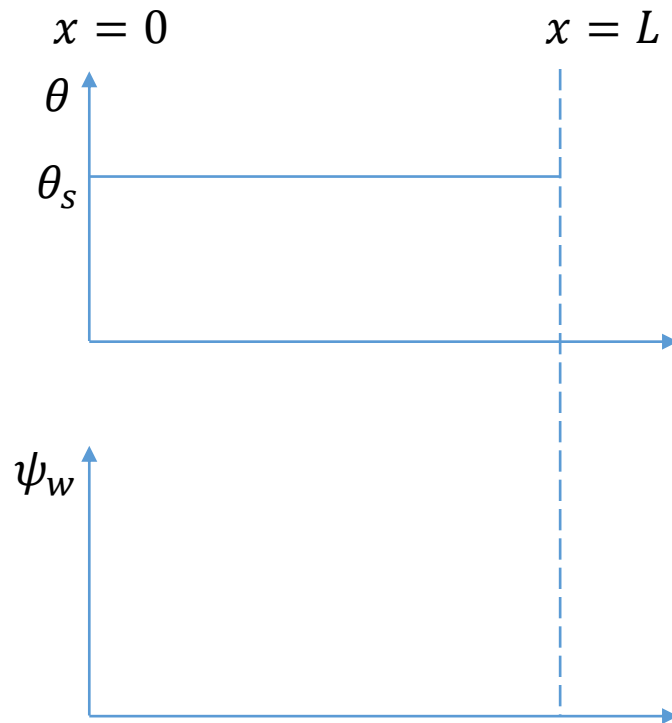
$$\frac{d}{dx} \left(K \frac{d\psi_w}{dx} \right) = 0 \quad \text{"}\psi\text{-based"} \text{ or "pressure-based" form}$$

$$\frac{d}{dx} \left(D \frac{d\theta_w}{dx} \right) = 0 \quad \text{"}\theta\text{-based"} \text{ or "moisture content-based" form}$$

Steady-State Unsaturated Flow

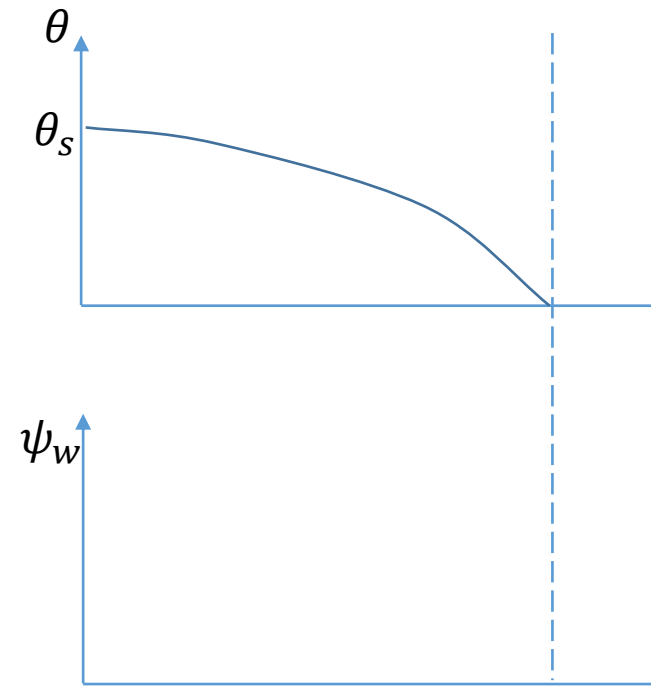
Horizontal flow (Homogeneous column)

Saturated



$\psi_w = 0$ at $x = L$

Unsaturated



$\psi_w = 0$ at $x = 0$