HWRS 505: Vadose Zone Hydrology

Lecture 15

10/15/2024

Today: Numerical solution for 1D transient unsaturated flow
Reading: Celia et al (1990); Ch11 of Pinder & Celia (2006).

Review of Lecture 14

Infiltration models

- Green-Ampt model
- Philip's analytical solutions of the θ-based Richards' equation
- For both the Green-Ampt model and Philip's analytical solutions:
  - ✓ Vertical infiltration: early time vs. late time
  - ✓ Horizontal infiltration is equivalent to the early-time solution of vertical infiltration

Numerical Solution for Transient Richards Eqn

❑ Different forms of the Richards Equation

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) - \frac{\partial K}{\partial z} = 0$$

"Mixed" form

$$C(h)\frac{\partial h}{\partial t} - \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) - \frac{\partial K}{\partial z} = 0$$

"h-based" or "pressure-based" form. Note that h and ψ are both commonly used for denoting pressure head.

$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z}\left(D\frac{\partial \theta}{\partial z}\right) - \frac{\partial K}{\partial z} = 0$$

"θ-based" or "moisture content-based" form

- The three forms are mathematically equivalent, but not so physically nor numerically.
  - ✓ θ-based form cannot be used when saturated condition occurs.
  - ✓ h-based forms may not conserve mass when solved numerically.

Numerical Solution for Transient Richards Eqn

*Cited by **2000+** times on google scholar.*

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A General Mass-Conservative Numerical Solution for the Unsaturated Flow Equation

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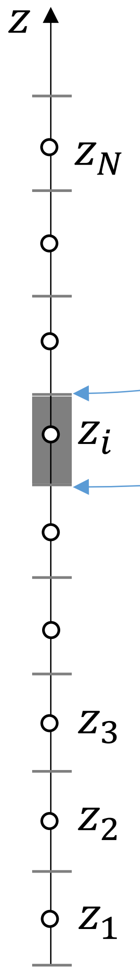
Numerical Solution for Transient Richards Eqn

$$\delta z_i = z_{i+1/2} - z_{i-1/2}$$
$$\nabla z_i = z_i - z_{i-1}$$
$$\Delta z_i = z_{i+1} - z_i$$



$$\frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z}\left[K\left(\frac{\partial h}{\partial z} + 1\right)\right] = 0$$

<u>Spatial discretization</u>

$$\frac{d\theta_i}{dt} - \frac{\left.\left(K(h)\left(\frac{\partial h}{\partial z} + 1\right)\right)\right|_{i+\frac{1}{2}} - \left.\left(K(h)\left(\frac{\partial h}{\partial z} + 1\right)\right)\right|_{i-\frac{1}{2}}}{\delta z_i} = 0$$

$$\frac{d\theta_i}{dt} - \frac{K(h)|_{i+\frac{1}{2}}\left(\frac{h_{i+1} - h_i}{\Delta z_i} + 1\right) - K(h)|_{i-\frac{1}{2}}\left(\frac{h_i - h_{i-1}}{\nabla z_i} + 1\right)}{\delta z_i} = 0$$

$$\Rightarrow \delta z_i \frac{d\theta_i}{dt} - K(h)\Big|_{i+\frac{1}{2}}\left(\frac{h_{i+1} - h_i}{\Delta z_i} + 1\right) + K(h)\Big|_{i-\frac{1}{2}}\left(\frac{h_i - h_{i-1}}{\nabla z_i} + 1\right) = 0$$

<u>Questions</u>:

- How to do temporal discretization? i.e., approximate $\frac{d\theta_i}{dt}$.
- How to deal with the nonlinearities in $K(h)$, and $\theta(h)$?

Numerical Solution for Transient Richards Eqn

<u>Temporal discretization</u>

$$\delta z_i \frac{\theta_i^{n+1,m+1} - \theta_i^n}{\Delta t} - K(h)^{n+1,m}\Big|_{i+\frac{1}{2}} \left(\frac{h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}}{\Delta z_i} + 1 \right) + K(h)^{n+1,m}\Big|_{i-\frac{1}{2}} \left(\frac{h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1}}{\nabla z_i} + 1 \right) = 0$$

- This is the Picard iteration applied to the mixed form of the Richards equation.
- We still have θ_i, but our primary variable is h_i. How can we deal with $\delta z_i \frac{\theta_i^{n+1,m+1} - \theta_i^n}{\Delta t}$?
- How is the Picard iteration different from the Newton-Raphson iteration?

Numerical Solution for Transient Richards Eqn

Time derivative term

$$\delta h = h^{n+1,m+1} - h^{n+1,m}$$

$$\delta z_i \frac{\theta_i^{n+1,m+1} - \theta_i^n}{\Delta t} = \delta z_i \frac{\theta_i^{n+1,m} + \frac{d\theta}{dh}\Big|^{n+1,m}\left(h_i^{n+1,m+1} - h_i^{n+1,m}\right) + O((\delta h)^2) - \theta_i^n}{\Delta t}$$

$$\approx \delta z_i \frac{\theta_i^{n+1,m} + C(h)^{n+1,m}\left(h_i^{n+1,m+1} - h_i^{n+1,m}\right) - \theta_i^n}{\Delta t}$$

$$= \delta z_i \frac{\theta_i^{n+1,m} - \theta_i^n}{\Delta t} + \delta z_i \frac{C(h)^{n+1,m}\left(h_i^{n+1,m+1} - h_i^{n+1,m}\right)}{\Delta t}$$

Substitute back to the full equation

$$\delta z_i \frac{\theta_i^{n+1,m} - \theta_i^n}{\Delta t} + \delta z_i \frac{C(h)^{n+1,m}\left(h_i^{n+1,m+1} - h_i^{n+1,m}\right)}{\Delta t} - K(h)^{n+1,m}\Big|_{i+\frac{1}{2}}\left(\frac{h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}}{\Delta z_i} + 1\right)$$
$$+ K(h)^{n+1,m}\Big|_{i-\frac{1}{2}}\left(\frac{h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1}}{\nabla z_i} + 1\right) = 0$$

Supplemented with initial and boundary conditions, we can then solve for $h_1, h_2, \dots, h_i, \dots, h_N$

Numerical Solution for Transient Richards Eqn

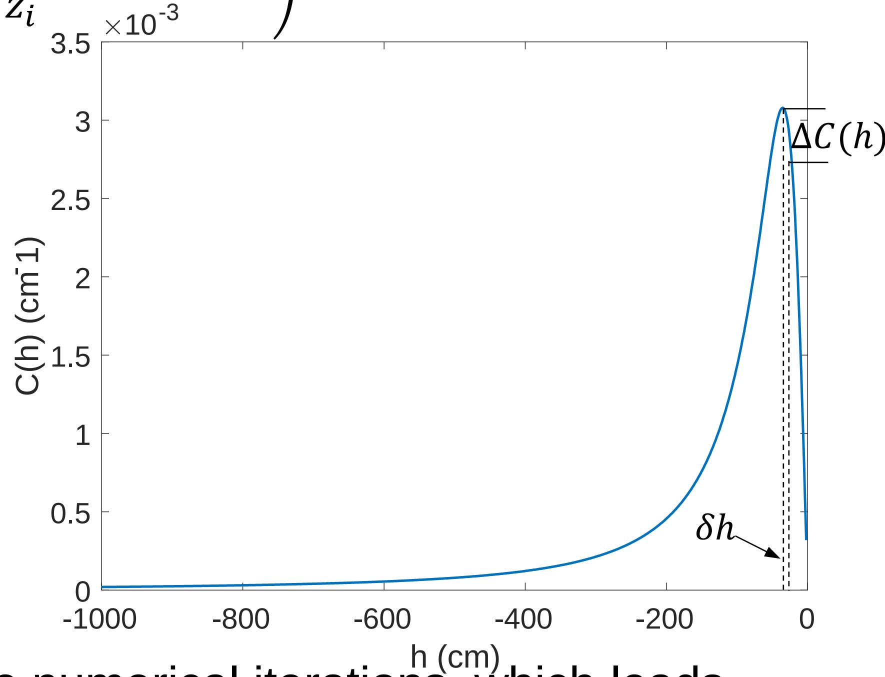
Comparison between the discretized equations for the mixed and h-based form

Mixed form

$$\delta z_i \frac{\theta_i^{n+1,m} - \theta_i^n}{\Delta t} + \delta z_i \frac{C(h)^{n+1,m}\left(h_i^{n+1,m+1} - h_i^{n+1,m}\right)}{\Delta t} - K(h)^{n+1,m}\Big|_{i+\frac{1}{2}}\left(\frac{h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}}{\Delta z_i} + 1\right) + K(h)^{n+1,m}\Big|_{i-\frac{1}{2}}\left(\frac{h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1}}{\nabla z_i} + 1\right) = 0$$

h-based form

$$\delta z_i C(h)^{n+1,m} \frac{\left(h_i^{n+1,m+1} - h_i^n\right)}{\Delta t} - K(h)^{n+1,m}\Big|_{i+\frac{1}{2}}\left(\frac{h_{i+1}^{n+1,m+1} - h_i^{n+1,m+1}}{\Delta z_i} + 1\right) + K(h)^{n+1,m}\Big|_{i-\frac{1}{2}}\left(\frac{h_i^{n+1,m+1} - h_{i-1}^{n+1,m+1}}{\nabla z_i} + 1\right) = 0$$



- Due to strong nonlinearity, $C(h)^{n+1,m}$ can have large errors during the numerical iterations, which leads to mass conservation errors in the h-based form.
- In the mixed form, the error is controlled because $C(h)^{n+1,m}$ is multiplied by a very small number δh.

Numerical Solution for Transient Richards Eqn

Comments from Celia et al (1990)

proximations, it can be seen that the reason for poor mass balance resides in the time derivative term. While $\partial\theta/\partial t$ and $C(\partial h/\partial t)$ are mathematically equivalent in the continuous partial differential equation, their discrete analogs are not equivalent. This inequality in the discrete forms is exacerbated by the highly nonlinear nature of the specific capacity term $C(h)$. This leads to significant mass balance errors in the h-based formulations because the change in mass in the system is calculated using discrete values of $\partial\theta/\partial t$ while the approximating equations use the expansion $C(h)\ (\partial h/\partial t)$. In addition, the spatially distributed form that arises in the finite element approximation appears to further exacerbate the problems inherent in this term.

One of the advantages of the θ-based equation is that discrete approximations to it, such as the finite element and finite difference methods used above, can be formulated so that they are perfectly mass conservative. However, because this form of the Richards equation degenerates in fully saturated media, and because material discontinuities produce discontinuous θ profiles, the θ-based equation is usually not used for general groundwater hydrology problems. Upon examination of the h-based equation and its numerical ap-