HWRS 505: Vadose Zone Hydrology

Lecture 2

8/29/2024

Today:

1. Review: Steady-state saturated flow
2. Derive permeability from Hagen-Poiseuille flow

Steady-state saturated flow

Review of Lecture 1

❖ Vadose zone (Overview)

- Conceptual picture
- Societal impacts
- It's role in the global hydrological and carbon cycles, and the global surface energy balances

❖ Steady-state saturated flow

- Energy potential; hydraulic head
- Darcy's law; saturated hydraulic conductivity; permeability

Assumptions:
1. Volume of the fluid parcel does not change
2. Kinetic energy is neglected:
3. Isothermal conditions $v \approx 0 \Rightarrow \frac{1}{2}(v_1^2 - v_0^2) \ll g(z_1 - z_0) + \psi_1 - \psi_0$

change of kinetic energy is much smaller than the change of gravity and pressure potential

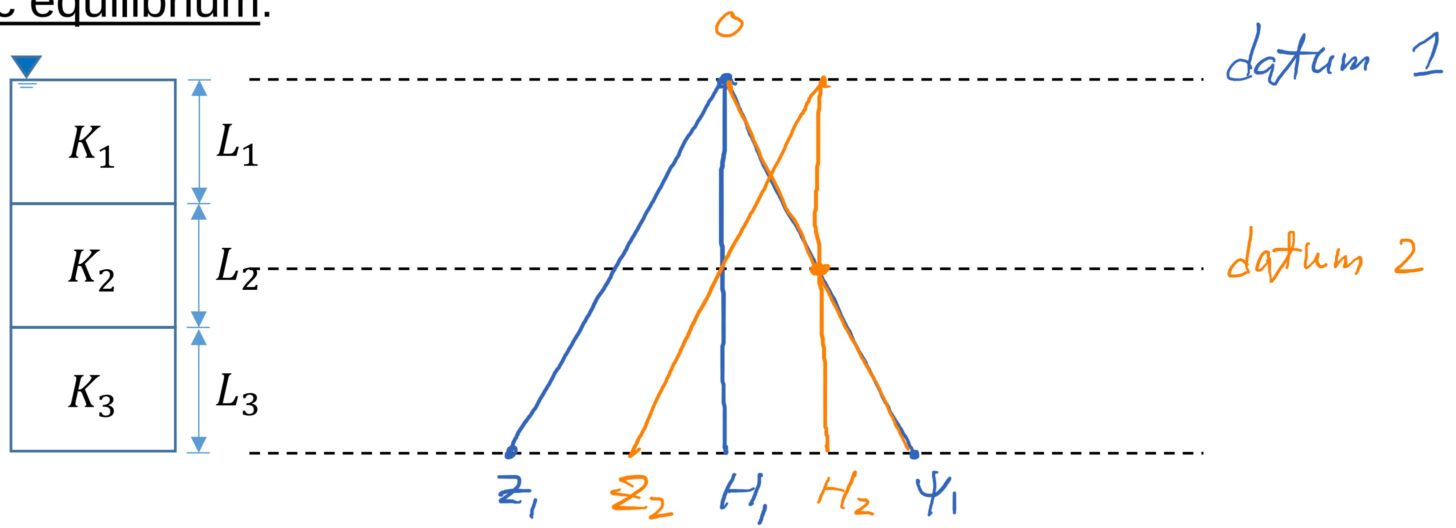
$$q = -K \frac{H_2 - H_1}{L}$$

$$q = -K \frac{dH}{dL} \text{ (differential form in 1D)}$$

$$\underset{\sim}{q} = -\underset{\approx}{K} \nabla \underset{\sim}{H} \text{ (differential form in 3D)}$$

Steady-state saturated flow

Static equilibrium:



$H_1 = z_1 + \psi_1$

$H_2 = z_2 + \psi_2$

Static equilibrium

$q = -K \frac{dH}{dz} = 0$

$\Rightarrow \frac{dH}{dz} = 0$

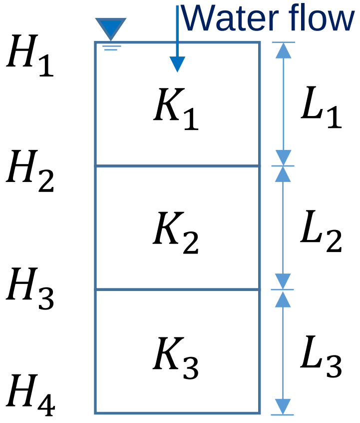
$\Rightarrow H$ is constant.

NOTE:

1. Hydraulic head is constant in space (No flow)
2. water pressure head remains the same for different datum.
3. The solution is independent of K (K_1, K_2, K_3)

Steady-state saturated flow

Steady-state flow:



(1) Homogeneous: $K_1 = K_2 = K_3$
(2) Heterogeneous: $K_1 \neq K_2 \neq K_3$

$$q = -\underline{K} \frac{dH}{dz}$$

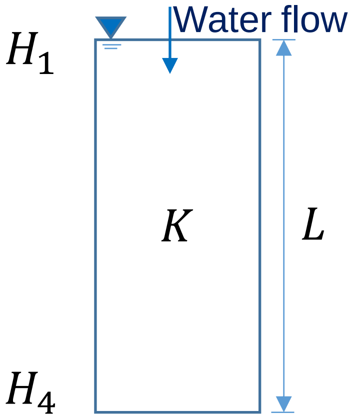
(1) Homogeneous : $\underline{K} = K_1 = K_2 = K_3$

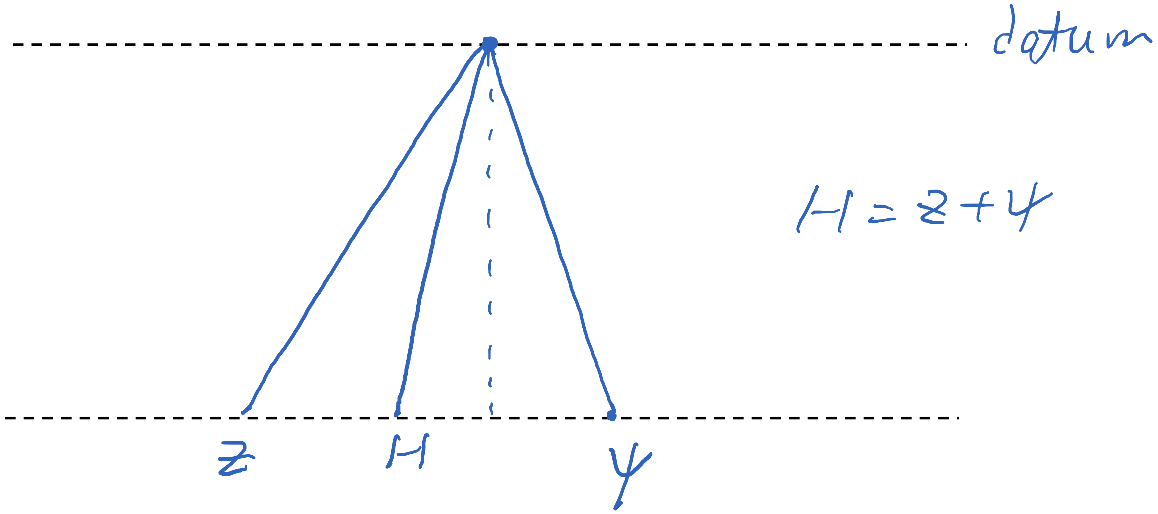
$\Rightarrow \frac{dH}{dz} = -\frac{q}{\underline{K}}$ is constant

$\Rightarrow H$ is a linear function of z.

$\Rightarrow H$ decreases linearly along the flow direction.

(1) Homogeneous: $K_1 = K_2 = K_3$

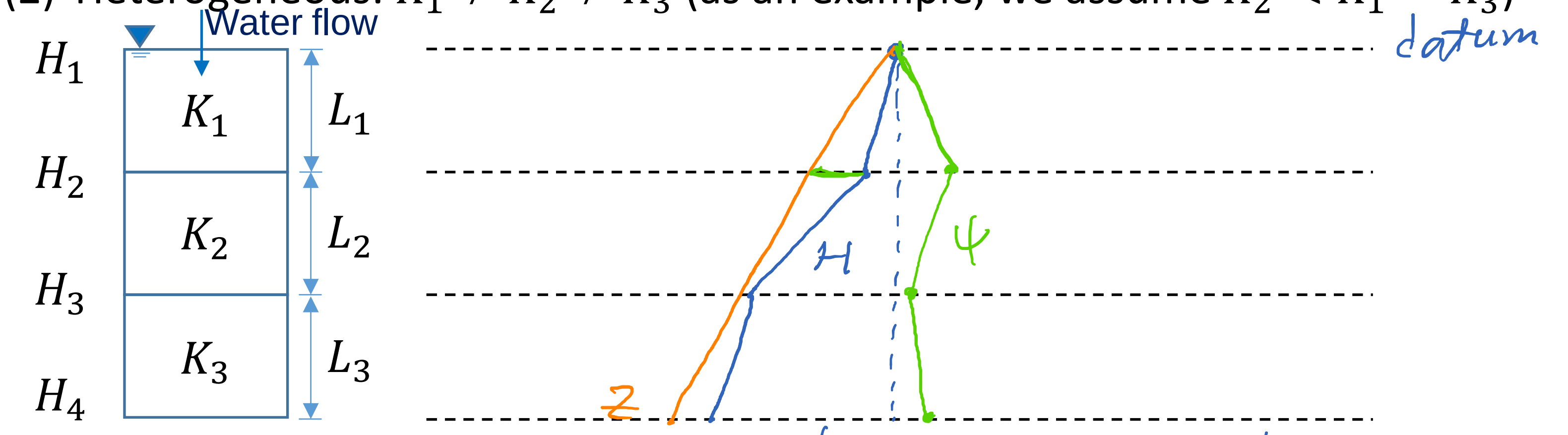




$H = z + \psi$

Steady-state saturated flow

(2) Heterogeneous: $K_1 \neq K_2 \neq K_3$ (as an example, we assume $K_2 < K_1 = K_3$)



$$Q = -K_1 A \left.\frac{dH}{dz}\right|_{(z_1,z_2)} = -K_2 A \left.\frac{dH}{dz}\right|_{(z_2,z_3)} = -K_3 A \left.\frac{dH}{dz}\right|_{(z_3,z_4)} \Rightarrow \left|\frac{dH}{dz}\right|_{(z_1,z_2)} = \left|\frac{dH}{dz}\right|_{(z_3,z_4)} = \frac{Q}{KA}$$

$$K_2 < K_1 = K_3 \qquad < \left|\frac{dH}{dz}\right|_{(z_2,z_3)} = \frac{Q}{K_2 A}$$

$\Rightarrow H$ decreases linealy along the flow direction, but it decreases faster in layer 2.

$$Q = -K_{eq} \frac{H_4 - H_1}{L_1 + L_2 + L_3}$$

It can be shown that $K_{eq} = \dfrac{L_1 + L_2 + L_3}{\frac{L_1}{K_1} + \frac{L_2}{K_2} + \frac{L_3}{K_3}} = \dfrac{\sum_i L_i}{\sum_i \frac{L_i}{K_i}}$

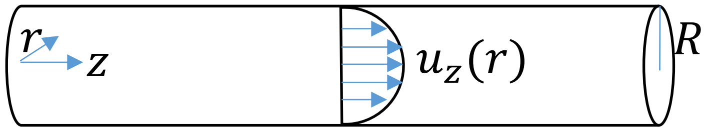
Steady-state saturated flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Permeability $[L^2]$

"Hagen-Poiseuille" flow



Note for the Laplacian: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

Notes for Darcy's Law:

$$q = -K\nabla H = -K\nabla\psi$$
$$= -\frac{k\rho g}{\mu}\nabla\psi$$
$$= -\frac{k}{\mu}\nabla p$$

$K = \frac{k\rho g}{\mu}$

$p = \rho g \psi$

Navier-Stokes Equation

$$\rho\cancel{\frac{Du_z}{Dt}}^{\,0} = -\frac{\partial p}{\partial z} + \cancel{f_z} + \mu\nabla^2 u_z$$

[steady state Low R_e #] [Neglect body forces]

$$\Rightarrow \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{\partial p}{\partial z}$$
$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{r}{\mu}\frac{\partial p}{\partial z}$$
$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\mu}\frac{\partial p}{\partial z} + C_1$$
$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu}\frac{\partial p}{\partial z} + \frac{1}{r}C_1$$

$$\Rightarrow u_z = \frac{r^2}{4\mu}\frac{\partial p}{\partial z} + C_1\ln r + C_2$$

BC:

$$\frac{\partial u_z}{\partial r}\Big|_{r=0} = 0 \Rightarrow C_1 = 0$$
$$u_z\big|_{r=R} = 0 \Rightarrow C_2 = -\frac{R^2}{4\mu}\frac{\partial p}{\partial z}$$

$$\Rightarrow u_z = \frac{1}{4\mu}\frac{\partial p}{\partial z}(r^2 - R^2)$$

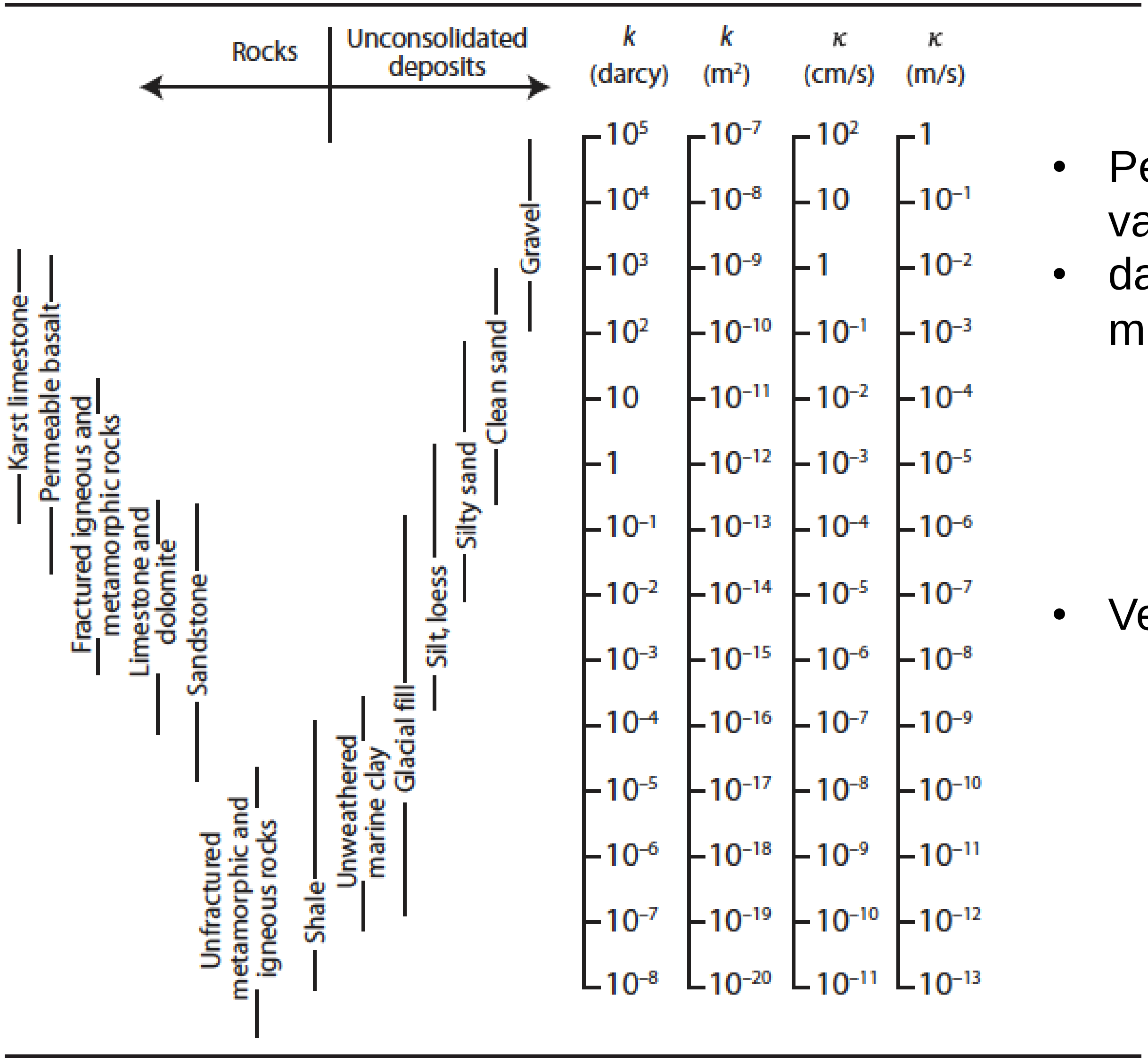
$$Q = \int_0^R 2\pi r u_z\, \mathrm{d}r = \int_0^R \frac{\pi}{2\mu}\frac{\partial p}{\partial z}(r^3 - rR^2)\, \mathrm{d}r$$

$$\Rightarrow q = \frac{Q}{A} = -\frac{R^2/8}{\mu}\frac{\partial p}{\partial z}$$

$$\Rightarrow k = R^2/8 \qquad [L^2]$$

=> The permeability of the tube is $k = R^2/8$

Steady-state saturated flow



(Freeze and Cherry, 1979)

- Permeability and conductivity values for various soil and rock media
- darcy (d) = $9.869233 \times 10^{-13}\ \mathrm{m}^2 \approx 1\ \mu\mathrm{m}^2$. millidarcy (md) = 0.001 darcy.

  A porous medium with a permeability of 1 darcy permits a flow of 1 cm^3/s of a fluid with viscosity 1 cP (1 mPa·s) under a pressure gradient of 1 atm/cm acting across an area of 1 cm^2.

  9.869233=1/1.013250. 1 atm = 1.013250 x 10^5 Pa
- Velocity of groundwater flow $\sim 1\ \mathrm{m/day}$