HWRS 505: Vadose Zone Hydrology

Lecture 8

9/19/2024

Today:

1. Mathematical Representation of Two-Phase Flow
2. Unsaturated Flow and the Richards' Equation

Reading: Chapter 11 (Pinder & Celia, 2006).

Review of Lecture 7

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- Physical meaning of the SWC models (Brooks-Corey vs. van Genuchten)
- Specific yield and drainable porosity in the vadose zone
- Leverett J function/scaling and Miller-Miller scaling
- Unsaturated permeability (physical meaning) and mathematical description
- Extended Darcy's Law for unsaturated flow (Buckingham, 1907)

$$\boldsymbol{q}_w = -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w}\boldsymbol{\nabla}(p_w + \rho_w g z)$$

- How to derive unsaturated permeability from SWC?

Steady-state saturated flow

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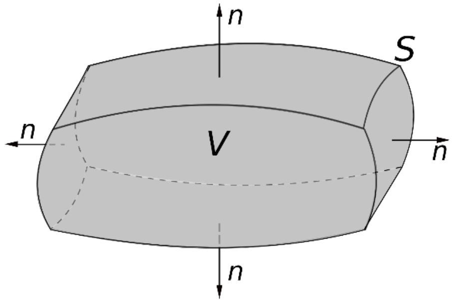
Derive the flow equation:

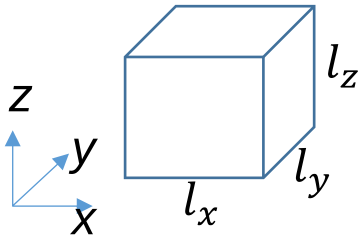


Recall: Divergence theorem

$$\int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}s = \int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{q}\,\mathrm{d}V$$

=> Divergence theorem converts a surface integral to a volume integral.

Mass conservation: Change of mass storage = mass in – mass out.



Rate of mass change: $$\frac{d}{dt}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z} \rho(x,y,z,t)\,\phi(x,y,z,t)\,dx\,dy\,dz = \frac{d}{dt}\int_\Omega \rho\phi\,dV$$

Net fluxes: $$\int_0^{l_y}\int_0^{l_z} q_x\big|_{x=0}\,dy\,dz - \int_0^{l_y}\int_0^{l_z} q_x\big|_{x=l_x}\,dy\,dz$$

$$+ \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=0}\,dx\,dz - \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=l_y}\,dx\,dz$$

$$+ \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=0}\,dx\,dy - \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=l_z}\,dx\,dy$$

Steady-state saturated flow

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$$\Rightarrow -\int_{\partial\Omega} \underset{\sim}{q}\cdot\underset{\sim}{n}\, ds = -\int_{\Omega} \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\, dv$$ divergence theorem

Thus, $$\frac{d}{dt}\int_{\Omega} \rho\phi\, dv = -\int_{\Omega} \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\, dv$$ $\Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{d}{dt}(\rho\phi)\, dv = -\int_{\Omega} \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\, dv$$

$$\Rightarrow \int_{\Omega} \left[\frac{d}{dt}(\rho\phi) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\right] dv = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\right] dv = 0$$

Integral $=0$, Ω is arbitrary $\Rightarrow$ Integrand $=0$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla}\cdot\underset{\sim}{q} = 0$$

Darcy's law: $\underset{\sim}{q} = -\rho \underset{=}{K} \underset{\sim}{\nabla} H$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho\phi) + \underset{\sim}{\nabla}\cdot\left(-\rho \underset{=}{K} \underset{\sim}{\nabla} H\right) = 0}$$

General 3D flow equation for a water-saturated porous medium.

Two-Phase Flow

Flux Law for two-phase flow (Muskat and Meres, 1936)

$$\boldsymbol{q}_w = -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w}\boldsymbol{\nabla}(p_w + \rho_w g z)$$

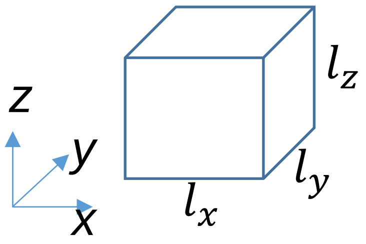
$$\boldsymbol{q}_{nw} = -\frac{k_{r,nw}(S_w)\mathbf{k}}{\mu_{nw}}\boldsymbol{\nabla}(p_{nw} + \rho_{nw} g z)$$

$$\boldsymbol{q}_\alpha = -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha}\boldsymbol{\nabla}(p_\alpha + \rho_\alpha g z), \qquad \alpha = nw \text{ or } w$$

$$= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha}(\boldsymbol{\nabla} p_\alpha - \rho_\alpha \boldsymbol{g})$$

$$\boldsymbol{\nabla}(\rho_\alpha g z) = -\rho_\alpha \boldsymbol{g}$$

The derivation of the governing equations for two-phase flow shares the same procedure as that used for sing-phase flow. The only difference is that a governing equation is needed for each fluid phase.



Two-Phase Flow

Two-Phase Flow

The governing equations for the two fluid phases:

Wetting phase $$\frac{\partial}{\partial t}(\rho_w \phi S_w) - \nabla \cdot \left[\rho_w \frac{k_{r,w}\mathbf{k}}{\mu_w}\nabla(p_w + \rho_w g z)\right] = 0 \quad (1)$$

Nonwetting phase $$\frac{\partial}{\partial t}(\rho_{nw} \phi S_{nw}) - \nabla \cdot \left[\rho_{nw} \frac{k_{r,nw}\mathbf{k}}{\mu_{nw}}\nabla(p_{nw} + \rho_{nw} g z)\right] = 0 \quad (2)$$

Unknowns: $p_w, p_{nw}, S_w, S_{nw}, k_{r,w}, k_{r,nw}$ (Assuming $\rho_\alpha, \mu_\alpha, \phi, \mathbf{k}$ are known.)

=> 6 unknowns, 2 equations. We need 4 more equations to have a mathematically closed system.

$$S_w + S_{nw} = 1 \quad (3)$$

Constitutive equations (e.g., VG or B-C models)

$$\begin{cases} p_{nw} = p_w + p_c(S_w) & (4) \\ k_{r,w} = k_{r,w}(S_w) & (5) \\ k_{r,nw} = k_{r,nw}(S_w) & (6) \end{cases}$$

- 6 equations, 6 unknowns. The system of equations is mathematically closed.
- Substituting Eqs. (3-6) to Eqs. (1-2) => 2 equations and 2 unknowns.
- Several options for the primary variables. Two common options:
  - ✓ One phase pressure + one phase saturation (e.g., p_w, S_{nw})
  - ✓ Two phase pressures (e.g., p_w, p_{nw})

Unsaturated Flow: Richards' Equation

Let's now think about the air-water system in the vadose zone.

At a common condition: 1 atm, 20 °C

$$\left.\begin{aligned} \rho_w &\approx 1000\ \mathrm{kg/m^3} \\ \rho_a &\approx 1\ \mathrm{kg/m^3} \end{aligned}\right\} \rho_a \ll \rho_w \qquad \left.\begin{aligned} \mu_w &\approx 10^{-3}\ \mathrm{Pa\cdot s} \\ \mu_a &\approx 1.8\times 10^{-5}\ \mathrm{Pa\cdot s} \end{aligned}\right\} \mu_a \ll \mu_w$$

Air flux: $\boldsymbol{q}_a = -\dfrac{k_{r,a}\mathbf{k}}{\mu_a}\boldsymbol{\nabla}(p_a + \rho_a g z)$

Richards' Equation

Governing equations for air-water flow in porous media:

$$\frac{\partial}{\partial t}(\rho_a \phi s_a) - \boldsymbol{\nabla} \cdot \left[\frac{\rho_a k_{r,a} \mathbf{k}}{\mu_a} \boldsymbol{\nabla}(p_a + \rho_a g z) = 0\right] \tag{1}$$

$$\frac{\partial}{\partial t}(\rho_w \phi s_w) - \boldsymbol{\nabla} \cdot \left[\frac{\rho_w k_{r,w} \mathbf{k}}{\mu_w} \boldsymbol{\nabla}(p_w + \rho_w g z) = 0\right] \tag{2}$$

$$s_a + s_w = 1 \tag{3}$$

$$k_{r,a} = k_{r,a}(s_w) \tag{4}$$

$$k_{r,w} = k_{r,w}(s_w) \tag{5}$$

$$p_a - p_w = p_c(s_w) \tag{6}$$