HWRS 505: Vadose Zone Hydrology

Lecture 8

9/19/2024

Today:
    1. Mathematical Representation of Two-Phase Flow
    2. Unsaturated Flow and the Richards' Equation
Reading: Chapter 11 (Pinder & Celia, 2006).

Review of Lecture 7

Review of Lecture 7

❖ Physical meaning of the SWC models (Brooks-Corey vs. van Genuchten)

❖ Specific yield and drainable porosity in the vadose zone

❖ Leverett J function/scaling and Miller-Miller scaling

❖ Unsaturated permeability (physical meaning) and mathematical description

❖ Extended Darcy's Law for unsaturated flow (Buckingham, 1907)

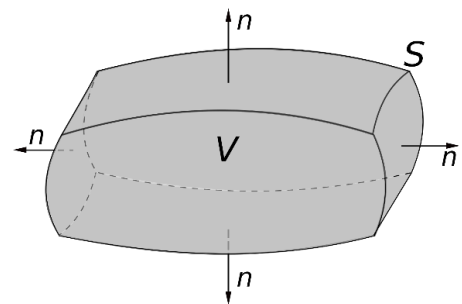$$\boldsymbol{q}_w = -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w}\boldsymbol{\nabla}(p_w + \rho_w g z)$$

❖ How to derive unsaturated permeability from SWC?
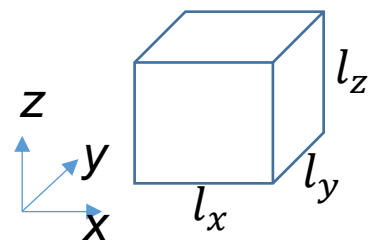
Steady-state saturated flow

Derive the flow equation:

Recall: Divergence theorem

$$\int_{\partial\Omega} \boldsymbol{q} \cdot \boldsymbol{n}\, \mathrm{d}s = \int_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{q}\, \mathrm{d}V$$

=> Divergence theorem converts a surface integral to a volume integral.

Mass conservation: Change of mass storage = mass in – mass out.

Rate of mass change:
$$\frac{d}{dt}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z} \rho(x,y,z)\,\phi(x,y,z)\,dx\,dy\,dz = \frac{d}{dt}\int_\Omega \rho\phi\,dv$$

Net fluxes:
$$\int_0^{l_y}\int_0^{l_z} q_x\big|_{x=0}\,dy\,dz - \int_0^{l_y}\int_0^{l_z} q_x\big|_{x=l_x}\,dy\,dz$$

$$+ \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=0}\,dx\,dz - \int_0^{l_x}\int_0^{l_z} q_y\big|_{y=l_y}\,dx\,dz$$

$$+ \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=0}\,dx\,dy - \int_0^{l_x}\int_0^{l_y} q_z\big|_{z=l_z}\,dx\,dy$$

Steady-state saturated flow

$$\Rightarrow -\int_{\partial\Omega} \underset{\sim}{q}\cdot\underset{\sim}{n}\,dS = -\int_{\Omega} \nabla\cdot\underset{\sim}{q}\,dv \qquad \text{divergence theorem}$$

Thus,
$$\frac{d}{dt}\int_{\Omega} \rho\phi\,dv = -\int_{\Omega} \nabla\cdot\underset{\sim}{q}\,dv \Big\} \Omega \neq \Omega(t)$$

$$\Rightarrow \int_{\Omega} \frac{d}{dt}(\rho\phi)\,dv = -\int_{\Omega} \nabla\cdot\underset{\sim}{q}\,dv$$

$$\Rightarrow \int_{\Omega} \left[\frac{d}{dt}(\rho\phi) + \nabla\cdot\underset{\sim}{q}\right] dv = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot\underset{\sim}{q}\right] dv = 0$$

Integral $= 0$, Ω is arbitrary $\Rightarrow$ Integrand $= 0$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot\underset{\sim}{q} = 0$$

Darcy's law: $\underset{\sim}{q} = -\rho\underset{=}{K}\nabla H$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot\left(-\rho\underset{=}{K}\nabla H\right) = 0}$$

General 3D flow equation for a water-saturated porous medium.

Two-Phase Flow

Flux Law for two-phase flow (Muskat and Meres, 1936)

$$\boldsymbol{q}_w = -\frac{k_{r,w}(S_w)\mathbf{k}}{\mu_w}\boldsymbol{\nabla}(p_w + \rho_w gz)$$

$$\boldsymbol{q}_{nw} = -\frac{k_{r,nw}(S_w)\mathbf{k}}{\mu_{nw}}\boldsymbol{\nabla}(p_{nw} + \rho_{nw} gz)$$

$$\boldsymbol{q}_\alpha = -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha}\boldsymbol{\nabla}(p_\alpha + \rho_\alpha gz), \qquad \alpha = nw \text{ or } w$$

$$= -\frac{k_{r,\alpha}(S_w)\mathbf{k}}{\mu_\alpha}(\boldsymbol{\nabla}p_\alpha - \rho_\alpha \boldsymbol{g})$$

$$\boldsymbol{\nabla}(\rho_\alpha gz) = -\rho_\alpha \boldsymbol{g}$$

The derivation of the governing equations for two-phase flow shares the same procedure as that used for sing-phase flow. The only difference is that a governing equation is needed for each fluid phase.
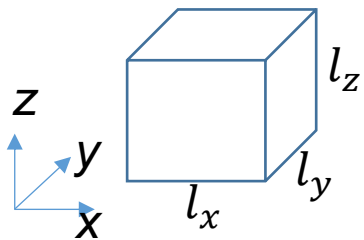
Rate of mass change:

$$\frac{\partial}{\partial t}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z} \rho_\alpha(x,y,z,t)\,\phi(x,y,z,t)\,S_\alpha(x,y,z,t)\,dx\,dy\,dz = \frac{\partial}{\partial t}\int_\Omega \rho_\alpha \phi S_\alpha\,dv$$

Net fluxes:

$$\int_0^{l_y}\int_0^{l_z} \rho_\alpha\, q_x\Big|_{x=0}\, l_y l_z - \int_0^{l_y}\int_0^{l_z} \rho_\alpha\, q_x\Big|_{x=l_x}\, l_y l_z$$

$$\int_0^{l_x}\int_0^{l_z} \rho_\alpha\, q_y\Big|_{y=0}\, l_x l_z - \int_0^{l_x}\int_0^{l_z} \rho_\alpha\, q_y\Big|_{y=l_y}\, l_x l_z \Bigg\} \Rightarrow -\int_{\partial\Omega} \rho_\alpha\, \underset{\sim}{q_\alpha}\cdot\underset{\sim}{n}\, ds$$

$$\int_0^{l_x}\int_0^{l_y} \rho_\alpha\, q_z\Big|_{z=0}\, l_x l_y - \int_0^{l_x}\int_0^{l_y} \rho_\alpha\, q_z\Big|_{z=l_z}\, l_x l_y$$

$$= -\int_\Omega \boldsymbol{\nabla}\cdot(\rho_\alpha \underset{\sim}{q_\alpha})\,dv$$

Divergence theorem

$$\Rightarrow \frac{\partial}{\partial t} \int_{\Omega} \rho_\alpha \phi S_\alpha \, dv = -\int_{\Omega} \nabla \cdot (\rho_\alpha \underset{\sim}{q}_\alpha) \, dv$$

$$\swarrow \Omega \neq \Omega(t)$$

$$\Rightarrow \int_{\Omega} \frac{\partial}{\partial t} (\rho_\alpha \phi S_\alpha) \, dv = -\int_{\Omega} \nabla \cdot (\rho_\alpha \underset{\sim}{q}_\alpha) \, dv$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t} (\rho_\alpha \phi S_\alpha) + \nabla \cdot (\rho_\alpha \underset{\sim}{q}_\alpha) \right] dv = 0$$

$$\swarrow \Omega \text{ is arbitrary}$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho_\alpha \phi S_\alpha) + \nabla \cdot (\rho_\alpha \underset{\sim}{q}_\alpha) = 0 \left. \right\} \Rightarrow$$

$$\text{Flux law:} \quad \underset{\sim}{q}_\alpha = - \frac{k_{r,\alpha} \, \underline{\underline{k}}}{\mu_\alpha} \nabla (P_\alpha + \rho_\alpha g z)$$

$$\boxed{ \frac{\partial}{\partial t} (\rho_\alpha \phi S_\alpha) - \nabla \cdot \left[\rho_\alpha \frac{k_{r,\alpha} \, \underline{\underline{k}}}{\mu_\alpha} \nabla (P_\alpha + \rho_\alpha g z) \right] = 0 }$$

General governing equation for two-phase flow.

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Two-Phase Flow

The governing equations for the two fluid phases:

Wetting phase

$$\frac{\partial}{\partial t}(\rho_w \phi S_w) - \boldsymbol{\nabla} \cdot \left[\rho_w \frac{k_{r,w}\mathbf{k}}{\mu_w}\boldsymbol{\nabla}(p_w + \rho_w gz)\right] = 0 \qquad (1)$$

Nonwetting phase

$$\frac{\partial}{\partial t}(\rho_{nw} \phi S_{nw}) - \boldsymbol{\nabla} \cdot \left[\rho_{nw} \frac{k_{r,nw}\mathbf{k}}{\mu_{nw}}\boldsymbol{\nabla}(p_{nw} + \rho_{nw} gz)\right] = 0 \qquad (2)$$

Unknowns: $p_w, p_{nw}, S_w, S_{nw}, k_{r,w}, k_{r,nw}$ (Assuming $\rho_\alpha, \mu_\alpha, \phi, \mathbf{k}$ are known.)

=> 6 unknowns, 2 equations. We need 4 more equations to have a mathematically closed system.

$$S_w + S_{nw} = 1 \qquad (3)$$

Constitutive equations
(e.g., VG or B-C models)

$$\begin{cases} p_{nw} = p_w + p_c(S_w) & (4) \\ k_{r,w} = k_{r,w}(S_w) & (5) \\ k_{r,nw} = k_{r,nw}(S_w) & (6) \end{cases}$$

- 6 equations, 6 unknowns. The system of equations is mathematically closed.
- Substituting Eqs. (3-6) to Eqs. (1-2) => 2 equations and 2 unknowns.
- Several options for the primary variables. Two common options:
  - ✓ One phase pressure + one phase saturation (e.g., p_w, S_{nw})
  - ✓ Two phase pressures (e.g., p_w, p_{nw})

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Unsaturated Flow: Richards' Equation

Let's now think about the air-water system in the vadose zone.

At a common condition: 1 atm, 20 °C

$$\rho_w \approx 1000 \text{ kg/m}^3$$

$$\rho_a \approx 1 \text{ kg/m}^3$$

$$\rho_a \ll \rho_w$$

$$\mu_w \approx 10^{-3} \text{ Pa} \cdot \text{s}$$

$$\mu_a \approx 1.8 \times 10^{-5} \text{ Pa} \cdot \text{s}$$

$$\mu_a \ll \mu_w$$

Air flux: $\quad \boldsymbol{q}_a = -\dfrac{k_{r,a}\mathbf{k}}{\mu_a}\boldsymbol{\nabla}(p_a + \rho_a g z)$

Richards' assumptions

$\mu_a \ll \mu_w$.

Air is essentially inviscid. In that sense, almost no gradient of energy potential is required to drive air flow.

$\Rightarrow \nabla(p_a + \rho_a g z) \approx 0 \Rightarrow p_a + \rho_a g z \approx$ const

At the ground surface ($z \approx 0$), $p_a = 0$

$p_a(z) = 0 - \rho_a g z$

$p_a \approx 1 \text{ kg/m}^2$

$\Rightarrow p_a \approx 0$ in the unsaturated zone.

$\rho_a \quad g \quad z \quad \ll \text{ atm}$
$1 \times 10 \times 100$
$\text{kg/m}^3 \quad \text{m/s}^2 \quad \text{m}$

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Richards' Equation

Governing equations for air-water flow in porous media:

Pa = 0. Thus, no need to solve equation (1)

$$\frac{\partial}{\partial t}(\rho_a \phi s_a) - \boldsymbol{\nabla} \cdot \left[\frac{\rho_a k_{r,a} \mathbf{k}}{\mu_a} \boldsymbol{\nabla}(p_a + \rho_a g z) = 0 \right] \qquad (1)$$

Pw(Sw)

$$\frac{\partial}{\partial t}(\rho_w \phi s_w) - \boldsymbol{\nabla} \cdot \left[\frac{\rho_w k_{r,w} \mathbf{k}}{\mu_w} \boldsymbol{\nabla}(p_w + \rho_w g z) = 0 \right] \qquad (2)$$

now, this is the only eqn We need to solve. It's essentially the Richards' eqn, but in a more general form.

$$s_a + s_w = 1 \qquad (3)$$

$$k_{r,a} = k_{r,a}(s_w) \qquad (4)$$

$$k_{r,w} = k_{r,w}(s_w) \qquad (5)$$

$$p_a - p_w = p_c(s_w) \qquad (6)$$

If we simply by assuming constant density and porosity:

$$\phi \frac{\partial s_w}{\partial t} - \boldsymbol{\nabla} \cdot \left[\frac{k_{r,w} \underline{\mathbf{k}}}{\mu_w} \boldsymbol{\nabla}(p_w + \rho_w g z) \right] = 0$$

Richards' Eqn.