HWRS 505: Vadose Zone Hydrology

Lecture 4

9/5/2024

Today:

1. Wrap up the review of solute transport under saturated flow
2. Air-water system in capillary tubes

Solute transport under saturated flow

Review of Lecture 3

- ❖ Derivation of 3D transient groundwater flow.
- ❖ Solute transport under saturated flow.
    - Advection ($\boldsymbol{v} = \boldsymbol{q}/\phi$)
    - Molecular diffusion
    - Mechanical dispersion

“The dispersion coefficient is a lumped fitting parameter that adequately describes relatively large-scale observations.”

$$\left\{ \int_\Omega \left[\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot \underset{\sim}{q} \right] dV = 0 \right.$$

Ω arbitrary

$$\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot \underset{\sim}{q} = 0$$

$\int_0^L f(x)\,dx = 0$ $[0, L]$ arbitrary

Suppose $f(x^*) \neq 0$

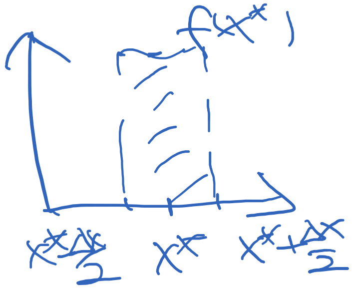
$$\lim_{\Delta x\to 0} \int_{x^*-\Delta x/2}^{x^*+\Delta x/2} f(x)\,dx = f(x^*)\,\Delta x = 0$$

$f(x^*) = 0$

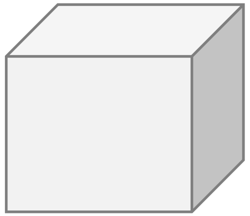
$\Rightarrow f(x^*) = 0$ contradicts $f(x^*) \neq 0$

$\Rightarrow f(x) = 0$ everywhere

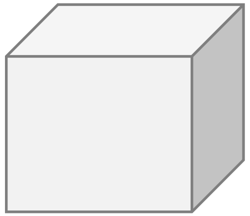
$$\boldsymbol{q}_c = \phi(\boldsymbol{v}C - \boldsymbol{D}\nabla C)$$
$$= \boldsymbol{q}C - \phi\boldsymbol{D}\nabla C$$



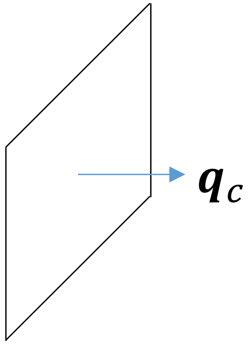
Solute transport under saturated flow

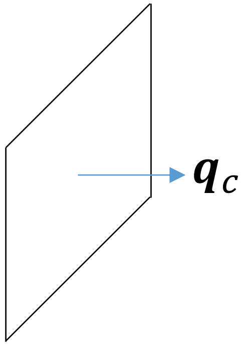


Saturated
porous medium



Fluid only
(e.g., free water)



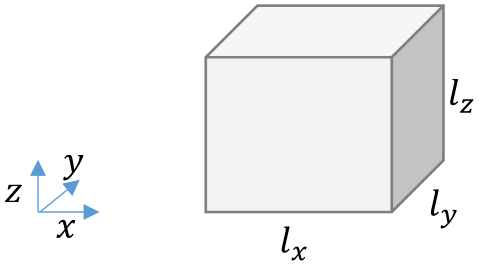


$$\begin{aligned}\boldsymbol{q}_c &= \phi(\boldsymbol{v}C - \boldsymbol{D}\nabla \mathrm{C}) \\ &= \boldsymbol{q}C - \phi\boldsymbol{D}\nabla \mathrm{C}\end{aligned}$$

$$\boldsymbol{D} = \alpha_T|\boldsymbol{v}|\mathbf{I} + (\alpha_L - \alpha_T)\frac{\boldsymbol{v}\boldsymbol{v}}{|\boldsymbol{v}|} + wD_0\mathbf{I}$$

$$\boldsymbol{q}_c = \boldsymbol{v}C - D_0\nabla \mathrm{C}$$

Solute transport under saturated flow



$$C(x,y,z,t) = \frac{\text{mass of solute}}{\text{unit fluid volume}}$$

Mass conservation: Change of mass storage = mass in – mass out.

Rate of mass change:

$$\frac{d}{dt}\int_0^{l_x}\int_0^{l_y}\int_0^{l_z}\phi C(x,y,z,t)\,dx\,dy\,dz = \frac{d}{dt}\int_\Omega \phi C(\underset{\sim}{x},t)\,dV$$

Net fluxes:

$$\int_0^{l_x}\int_0^{l_z} q_{c,x}\Big|_{x=0}\,dy\,dz - \int_0^{l_y}\int_0^{l_z} q_{c,x}\Big|_{x=l_x}\,dy\,dz$$

$$+\int_0^{l_x}\int_0^{l_z} q_{c,y}\Big|_{y=0}\,dx\,dz - \int_0^{l_x}\int_0^{l_z} q_{c,y}\Big|_{y=l_y}\,dx\,dz$$

$$+\int_0^{l_x}\int_0^{l_y} q_{c,z}\Big|_{z=0}\,dx\,dy - \int_0^{l_x}\int_0^{l_y} q_{c,z}\Big|_{z=l_z}\,dx\,dy$$

Solute transport under saturated flow

$$\Rightarrow -\int_{\partial\Omega} \underset{\sim}{q} \cdot \underset{\sim}{n}\, dS = -\int_{\Omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{q_c}\, dV$$ divergence theorem

$$\Rightarrow \frac{\partial}{\partial t}\int_{\Omega} \phi c\, dV = -\int_{\Omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{q_c}\, dV$$

$\downarrow$ $\Omega \neq \Omega(t)$

$$\int_{\Omega} \frac{\partial}{\partial t}\phi c\, dV = -\int_{\Omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{q_c}\, dV$$

$$\int_{\Omega} \left(\frac{\partial}{\partial t}\phi c + \underset{\sim}{\nabla} \cdot \underset{\sim}{q_c}\right) dV = 0$$

$\downarrow$ Ω is arbitrary

$$\Rightarrow \left.\begin{aligned} &\frac{\partial}{\partial t}\phi c + \underset{\sim}{\nabla} \cdot \underset{\sim}{q_c} = 0 \\ &\underset{\sim}{q_c} = \underset{\sim}{q} c - \phi \underset{\sim}{D} \underset{\sim}{\nabla} c \end{aligned}\right\}$$

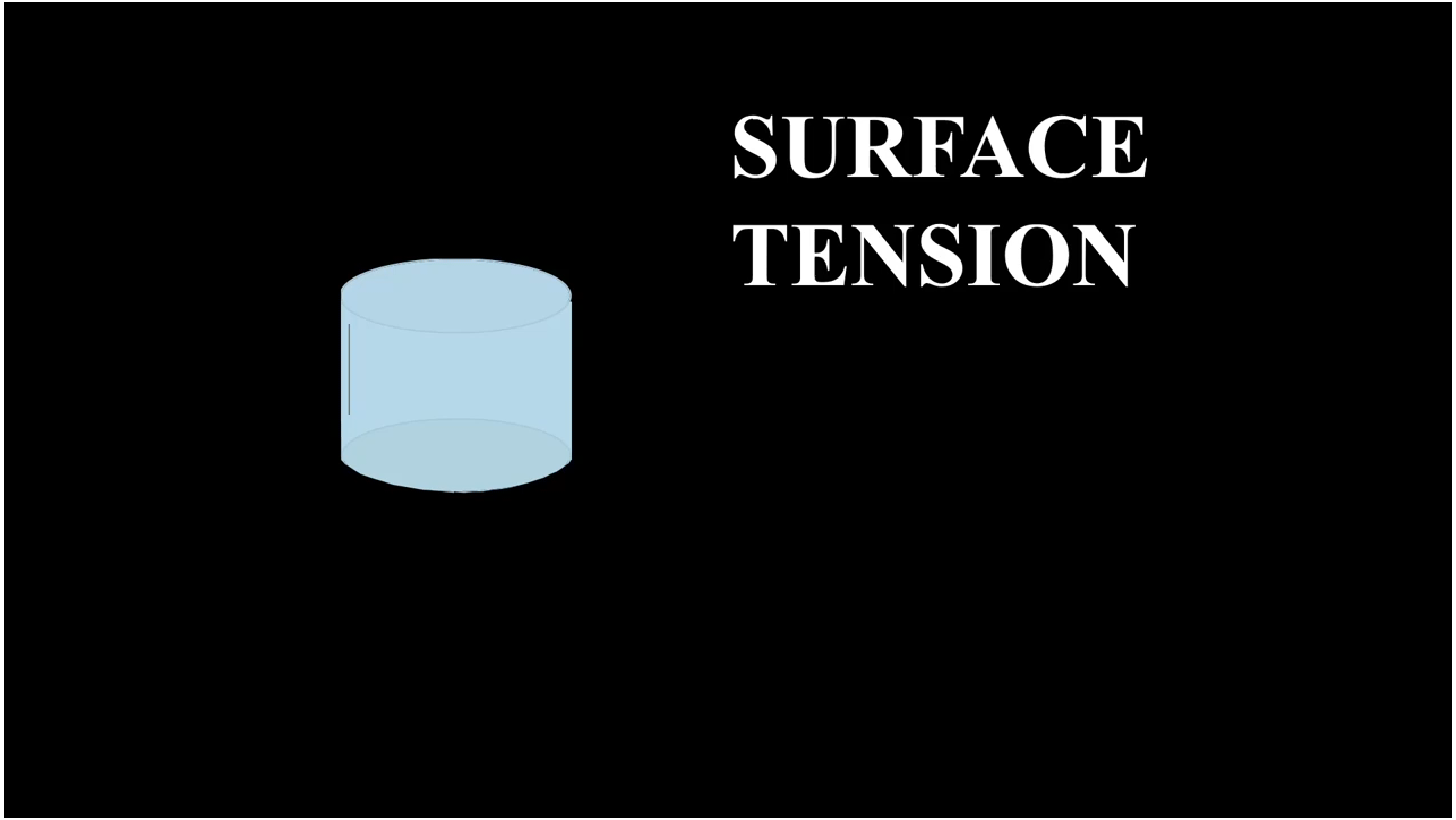
$$\boxed{\frac{\partial}{\partial t}(\phi c) + \underset{\sim}{\nabla} \cdot (\underset{\sim}{q} c) - \underset{\sim}{\nabla} \cdot (\phi \underset{\sim}{D} \underset{\sim}{\nabla} c) = 0}$$

General 3D governing equation for solute transport in porous media

Air-water system in capillary tubes

1. Why does the water try to hold together?
2. Why does the water not wet the surface?

Air-water system in capillary tubes

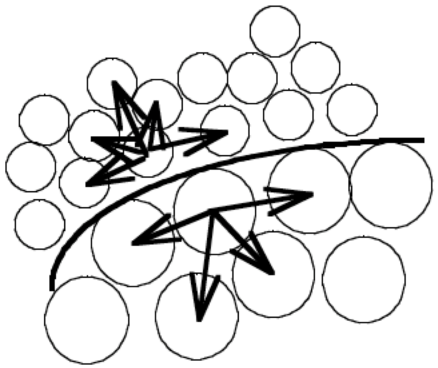


Link to the video: https://youtu.be/zMzqiAuOSz0

Air-water system in capillary tubes

- **Two and three phase systems: water, oil, air**

- *Interfacial tension (<u>cohesive</u> forces between fluid molecules)*

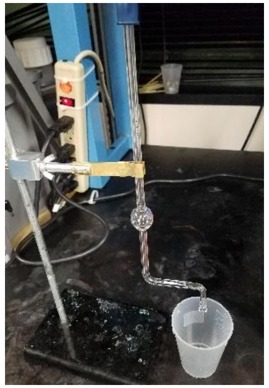


<u>Typical values of surface tension</u>:

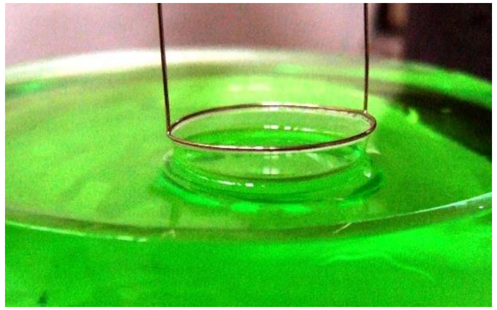
air-water	0.072 N/m
oil-water	0.20 N/m
oil-water w/ soap	0.0001 N/m

How to measure interfacial tension?

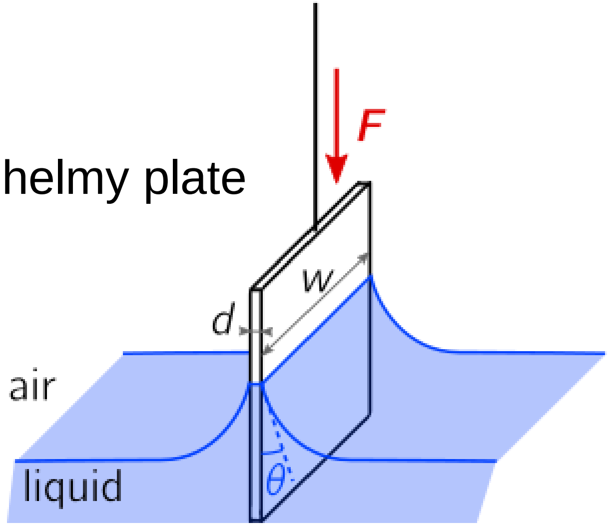
Drop weight method



ring method

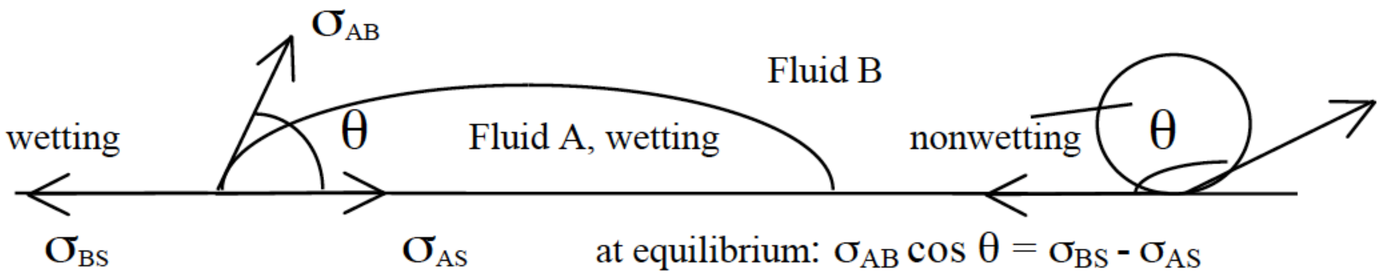


Wilhelmy plate



Air-water system in capillary tubes

- *Wettability (<u>adhesive</u> forces between the fluid and solid surface)*



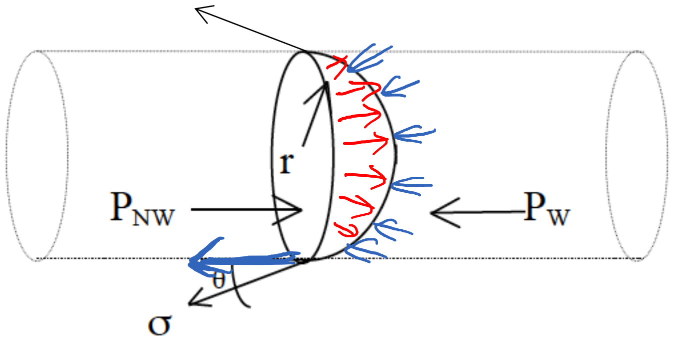
$\theta < 90^o$: fluid A is wetting with respect to fluid B on the solid S
$\theta > 90^o$: fluid A is nonwetting with respect to fluid B on the solid S

Wettability is a function of the fluid properties, soil properties, and history of contact. For most soils, the relative wettabilities are: water > oil > air

Recommended video for the concepts of *viscosity, cohesive and adhesive forces, surface tension, and capillary action* https://www.youtube.com/watch?v=P_jQ1B9UwpU

Air-water system in capillary tubes

Capillary pressure (difference between the nonwetting and wetting phase pressures)



Force balance at equilibrium:

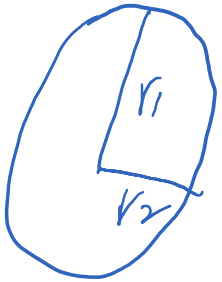
$$\pi r^2 P_{nw} - \pi r^2 P_w = 2\pi r \sigma \cos\theta$$

$$P_{nw} - P_w = \frac{2\sigma\cos\theta}{r}$$

$$\boxed{P_c = \frac{2\sigma\cos\theta}{r}}$$

Young-Laplace Equation

1. more general equation for any nw-w interface



$$P_c = \sigma\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

For the capillary tube: $r_1 = r_2 = \frac{r}{\cos\theta}$

2. For a perfectly wetting fluid

$$\theta = 0, \quad P_c = \frac{2\sigma}{r}$$