HWRS 505: Vadose Zone Hydrology

Lecture 10

9/25/2024

Today: Richards' Equation and steady-state unsaturated flow
Reading: Chapter 11 (Pinder & Celia, 2006) and Ferre Lecture Notes

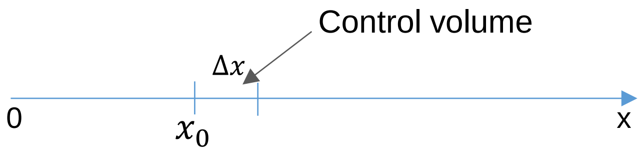
Comments on Homework #1

- Problems #1 and #2

Vertical $q_z = -K\dfrac{\mathrm{d}H}{\mathrm{d}z}$ $\quad H = \psi + z$

Horizontal $q_x = -K\dfrac{\mathrm{d}H}{\mathrm{d}x}$ $\quad H = \psi\ (z = 0)$

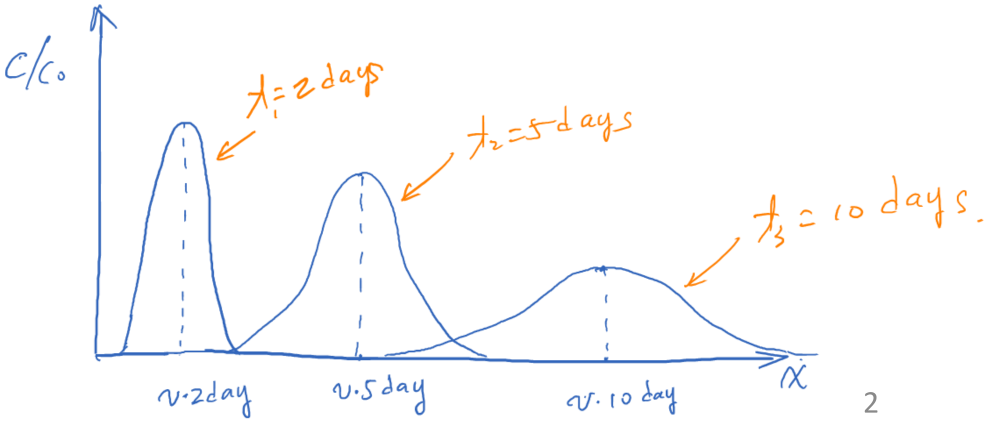
- Problems #3



Net flux

$$q_x\Big|_{x=x_0} - q_x\Big|_{x=x_0+\Delta x} = -\int_{x=x_0}^{x=x_0+\Delta x}\frac{\mathrm{d}q}{\mathrm{d}x}\,\mathrm{d}x$$

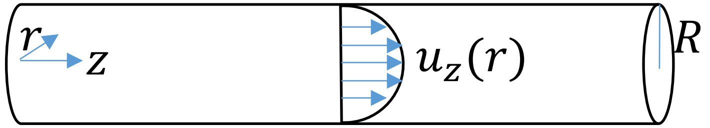
Draw schematics of the solutions of solute concentrations ($c(x,t)$)) at three times, $t = 2, 5, 10$ days.



Solute transport under saturated flow

Taylor-Aris Dispersion: Dispersion in a capillary tube [G.I. Taylor (1953) and R. Aris (1956)]

Solute transport in "Hagen-Poiseuille" flow



Governing equation for solute transport in the tube:

$$\frac{\partial C}{\partial t} + 2u\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{\partial C}{\partial x} - D_0\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) = 0$$

At sufficiently long times (derivation via the perturbation method)

$$\frac{\partial \bar{C}}{\partial t} + \bar{u}\frac{\partial \bar{C}}{\partial x} - D_L\frac{\partial^2 \bar{C}}{\partial x^2} = 0, \qquad D_L = D_0 + \frac{R^2\bar{u}^2}{48 D_0}$$

$\bar{u}$: mean velocity in the tube.
C_m: mean concentration in the tube.

$$C_m = \frac{\int_0^{2\pi}\int_0^R C(r,x)\, r\, \mathrm{d}r\, \mathrm{d}\theta}{\int_0^{2\pi}\int_0^R r\, \mathrm{d}r\, \mathrm{d}\theta} = \frac{2}{R^2}\int_0^R C\, r\, \mathrm{d}r$$

Insights: The average solute concentration spreads out by a dispersion process (radial diffusion and axial advection) and follows Fickian diffusion. The effective diffusivity is not the molecular diffusivity D_0. Rather, it is D_0 + a quadratic function of the mean velocity.

Review of Lecture 9

- Three forms of Richards' equation
  - Mixed form
  - Pressure head-based form
    - ✓ Specific moisture capacity
  - Water content-based form
    - ✓ Soil moisture diffusivity (What does the equation have to do with "diffusion"?)
    - ✓ Cannot be used if the domain involves saturated water flow
  - How to include soil and fluid compressibility?
- Richards' assumptions
  - Air pressure remains almost zero everywhere, but air does move.
  - Does "air movement" make the Richards' equation invalid? No, as long as air pressure remains almost zero everywhere.

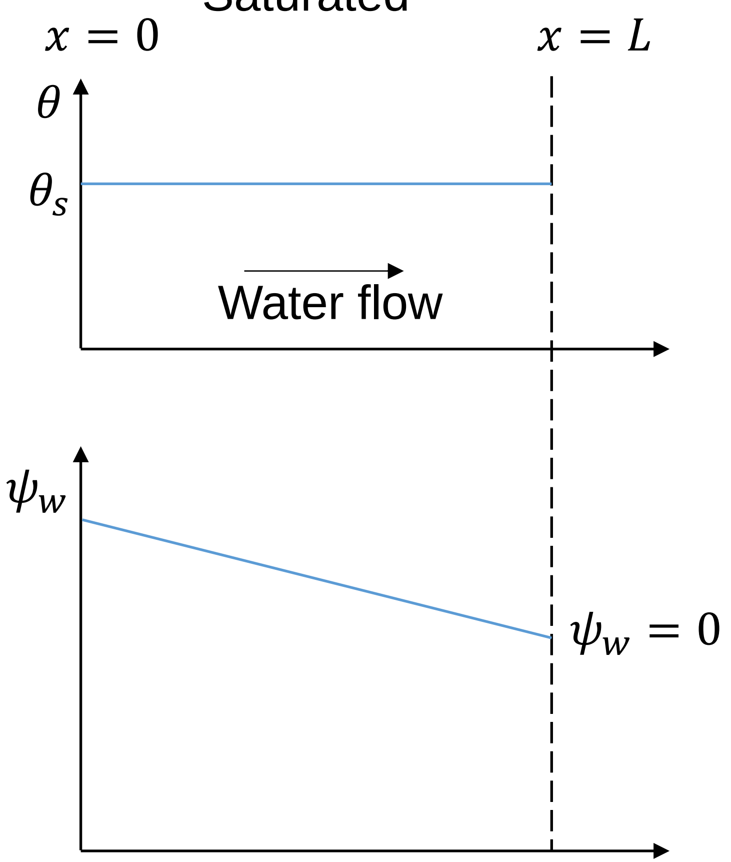
$\boxed{\text{solute diffusion} \quad \text{In analogy: } q_c \sim -D\frac{\partial c}{\partial x}}$

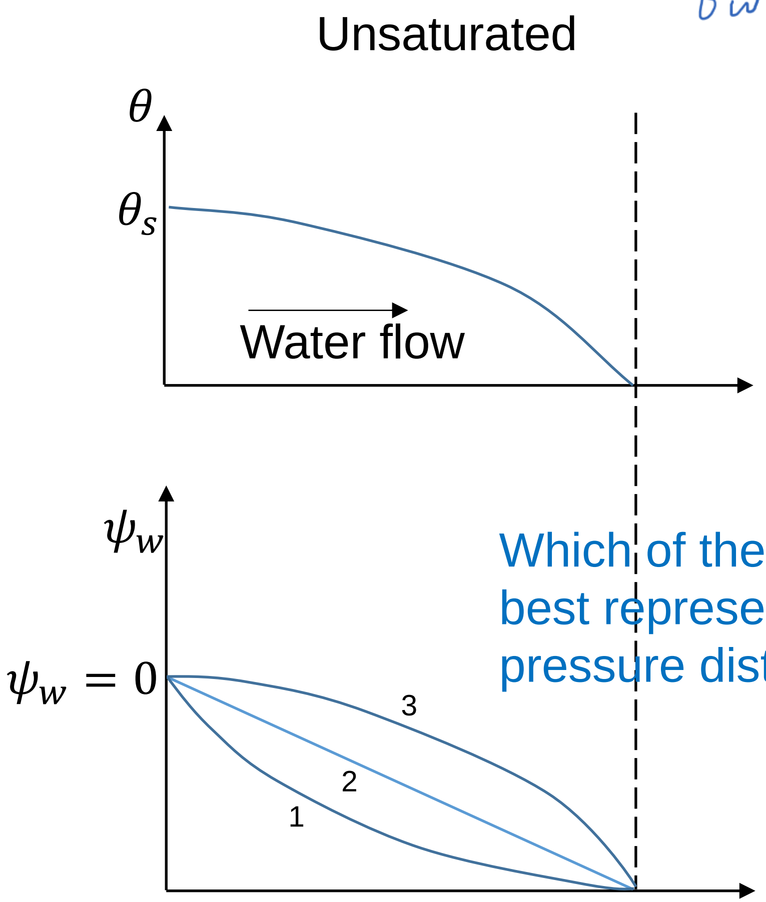
$q_w \sim -\frac{\partial P_w}{\partial x} \sim \frac{\partial P_c}{\partial x} \sim \frac{\partial P_c}{\partial \theta}\frac{\partial \theta}{\partial x}$

Soil moisture diffussivity

Steady-State Unsaturated Flow

Horizontal flow (Homogeneous column)





$q_w = -K_{sat} k_{r,w} \frac{d\psi_w}{dx}$

- $k_{r,w}$ decreases from left to right
- $\left|\frac{d\psi_w}{dx}\right|$ increases from left to right,

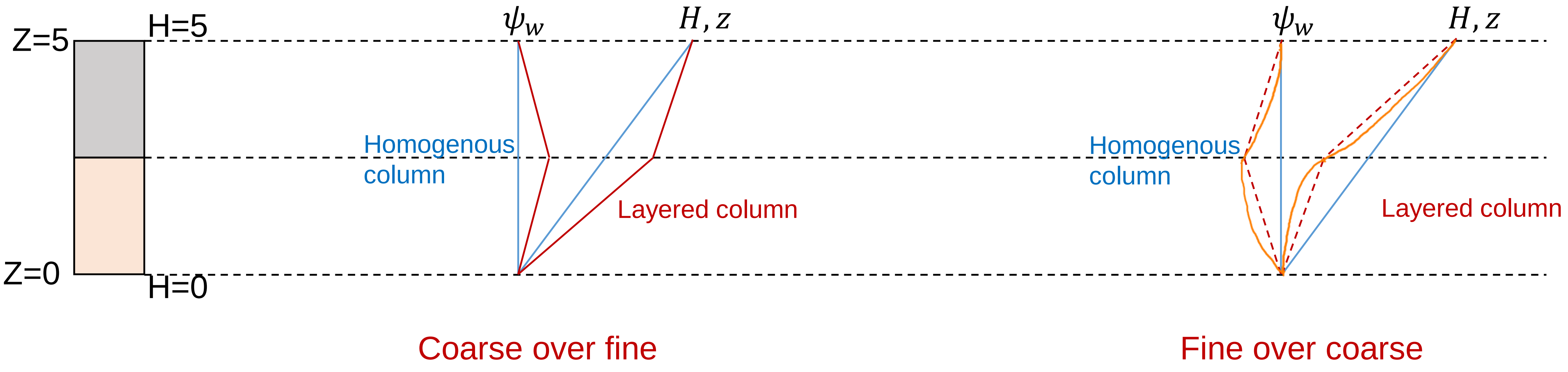
Which of the curves (1, 2, or 3) best represents the water pressure distribution?

Answer: 3

- Unsaturated flow involves nonlinearities that make their behaviors differ from that of the saturated flow

Steady-State Unsaturated Flow

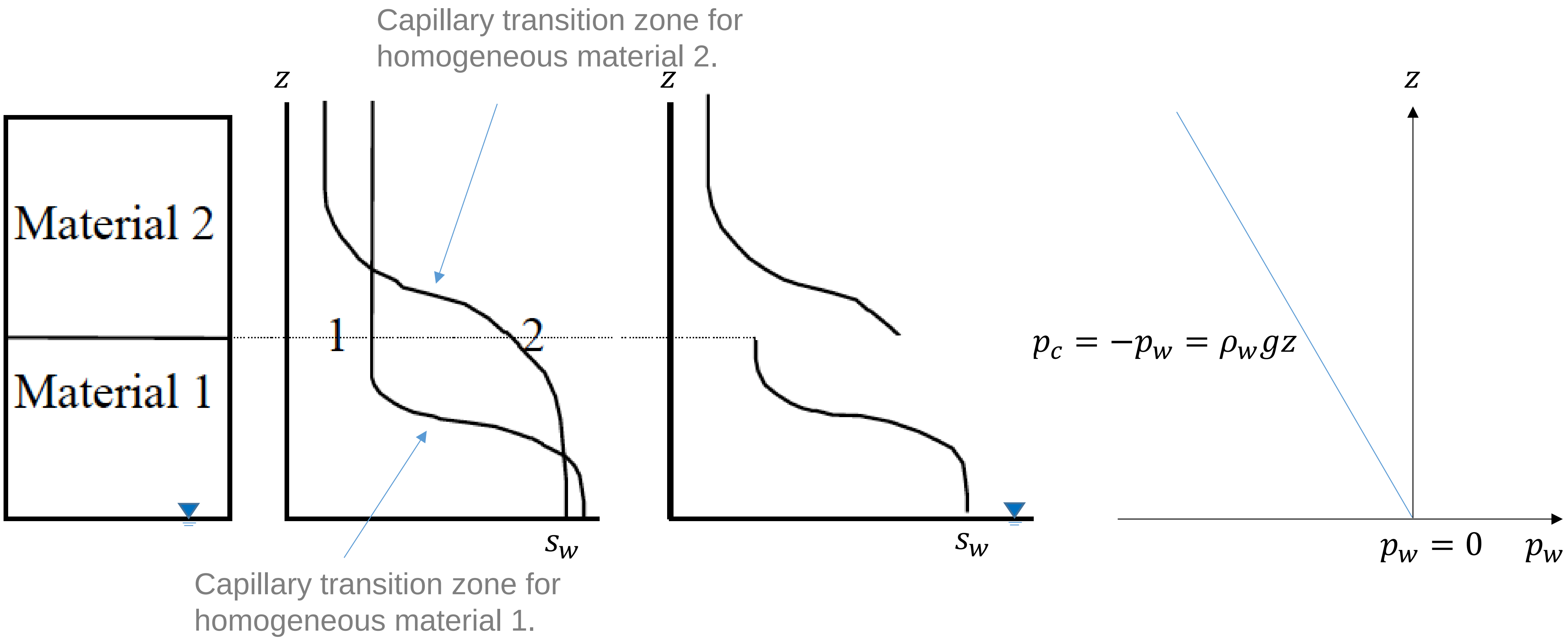
Vertical flow (in layered columns)



- Hydraulic head and pressure head are both continuous in space.
- Is water saturation continuous?

Steady-State Unsaturated Flow

Hydrostatic unsaturated layered systems

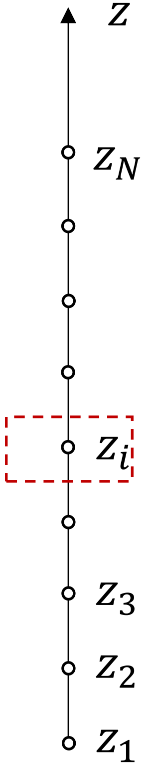


- S_w is discontinuous at the material interface, but the p_c and p_w or ψ_w are continuous.

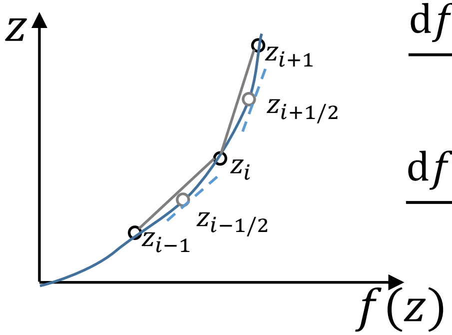
Steady-State Unsaturated Flow: Numerical Soln.

$$\cancelto{0}{\frac{\partial \theta_w}{\partial t}} - \frac{\partial}{\partial z}\left(K\frac{\partial \psi_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0 \xrightarrow{\text{Steady-state}} \underbrace{\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)}_{1} + \underbrace{\frac{\mathrm{d}K}{\mathrm{d}z}}_{2} = 0 \qquad (1)$$

- Equation (1) is a second-order ordinary differential equation in 1D.
- It is nonlinear because $K = K(\psi_w)$ is a nonlinear function.
- To solve it, we need two boundary conditions and we need to do it in an iterative procedure.



- How to solve this 1D **nonlinear ordinary differential equation**?
- Key idea: Divide the domain into many boxes and convert the **differential equation** to a system of nonlinear **algebraic equations**.
- Technique: Use finite difference to approximate derivatives

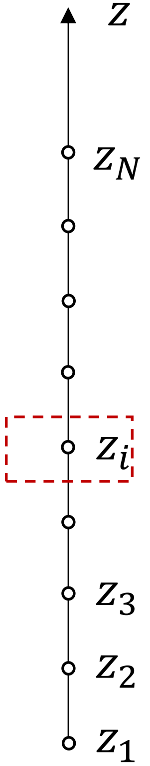


$$\frac{\mathrm{d}f(z_{i+1/2})}{\mathrm{d}z} \approx \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}$$

$$\frac{\mathrm{d}f(z_{i-1/2})}{\mathrm{d}z} \approx \frac{f(z_i) - f(z_{i-1})}{z_i - z_{i-1}}$$

Steady-State Unsaturated Flow: Numerical Soln.

$$\cancelto{0}{\frac{\partial \theta_w}{\partial t}} - \frac{\partial}{\partial z}\left(K\frac{\partial \psi_w}{\partial z}\right) - \frac{\partial K}{\partial z} = 0 \quad \xrightarrow{\text{Steady-state}} \quad \underbrace{\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)}_{1} + \underbrace{\frac{\mathrm{d}K}{\mathrm{d}z}}_{2} = 0 \qquad (1)$$



$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}z}\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)}_{1}\Bigg|_{z_i} \approx \frac{\left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)_{i+1/2} - \left(K\frac{\mathrm{d}\psi_w}{\mathrm{d}z}\right)_{i-1/2}}{\Delta z} \approx \frac{K_{i+1/2}\frac{\psi_{w,i+1} - \psi_{w,i}}{\Delta z} - K_{i-1/2}\frac{\psi_{w,i} - \psi_{w,i-1}}{\Delta z}}{\Delta z}$$

$$= \frac{K_{i+1/2}(\psi_{w,i+1} - \psi_{w,i}) - K_{i-1/2}(\psi_{w,i} - \psi_{w,i-1})}{\Delta z^2}$$

$$\underbrace{\frac{\mathrm{d}K}{\mathrm{d}z}}_{2}\Bigg|_{z_i} \approx \frac{K_{i+1/2} - K_{i-1/2}}{\Delta z}$$

$$\Rightarrow \frac{K_{i+1/2}}{\Delta z^2}(\psi_{w,i+1} - \psi_{w,i}) - \frac{K_{i-1/2}}{\Delta z^2}(\psi_{w,i} - \psi_{w,i-1}) + K_{i+1/2}\frac{1}{\Delta z} - K_{i-1/2}\frac{1}{\Delta z} = 0$$

$$\Rightarrow \frac{K_{i+1/2}}{\Delta z^2}\psi_{w,i+1} - \left(\frac{K_{i+1/2}}{\Delta z^2} + \frac{K_{i-1/2}}{\Delta z^2}\right)\psi_{w,i} + \frac{K_{i-1/2}}{\Delta z^2}\psi_{w,i-1} + K_{i+1/2}\frac{1}{\Delta z} - K_{i-1/2}\frac{1}{\Delta z} = 0$$

This is an algebraic equation with 3 unknowns. We can write such an algebraic equation for each node or box and we can get N algebraic equations for the N unknowns $(\psi_{w,1}, \psi_{w,2}, \dots, \psi_{w,N})$

Steady-State Unsaturated Flow: Numerical Soln.

In matrix form:

$$
\begin{pmatrix}
1 & 0 & 0 & & & & \\
\frac{K_{3/2}}{\Delta z^2} & -\frac{(K_{3/2}+K_{5/2})}{\Delta z^2} & \frac{K_{5/2}}{\Delta z^2} & & & & \\
& \frac{K_{5/2}}{\Delta z^2} & -\frac{(K_{5/2}+K_{7/2})}{\Delta z^2} & \frac{K_{7/2}}{\Delta z^2} & & & \\
& & \ddots & \ddots & \ddots & & \\
& & \frac{K_{i-1/2}}{\Delta z^2} & -\frac{(K_{i-1/2}+K_{i+1/2})}{\Delta z^2} & \frac{K_{i+1/2}}{\Delta z^2} & & \\
& & & \ddots & \ddots & \ddots & \\
& & & & \frac{K_{n-1-1/2}}{\Delta z^2} & -\frac{(K_{n-1-1/2}+K_{n-1+1/2})}{\Delta z^2} & \frac{K_{n-1+1/2}}{\Delta z^2} \\
& & & & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\psi_{w,1} \\ \psi_{w,2} \\ \psi_{w,3} \\ | \\ \psi_{w,i} \\ | \\ \psi_{w,n-1} \\ \psi_{w,n}
\end{pmatrix}
=
\begin{pmatrix}
B.C. \\ \frac{K_{3/2}-K_{5/2}}{\Delta z} \\ \frac{K_{5/2}-K_{7/2}}{\Delta z} \\ | \\ \frac{K_{i-1/2}-K_{i+1/2}}{\Delta z} \\ | \\ \frac{K_{n-1-1/2}-K_{n-1+1/2}}{\Delta z} \\ B.C.
\end{pmatrix}
$$

$\Rightarrow \boldsymbol{F}(\boldsymbol{\psi}_w)\cdot\boldsymbol{\psi}_w = \boldsymbol{R}$ How to solve this nonlinear system of algebraic equations?

=> Employ an iterative methods