

HWRS 505: Vadose Zone Hydrology

Lecture 3

9/3/2024

Today:

1. Derive the 3D groundwater flow equation
2. Review solute transport under saturated flow

Steady-state saturated flow

Review of Lecture 2

❖ Steady-state saturated flow

- Boundary conditions -> Distribution of hydraulic head and pressure head.
 - ✓ The definition of total hydraulic head ($H = z + \psi$)
 - ✓ Darcy's law ($q = -K \frac{dH}{dz}$)
- For a 1D system, steady state => constant Darcy flux and porewater velocity in space.
- Derivation of permeability for a tube based on the Hagen-Poiseuille flow (simplified N-S equation).
 - ✓ Permeability $\sim R^2$
 - ✓ The origin of Darcy's law is N-S equation
 - ✓ Permeability for a bundle of tubes of uniform size?

Approximate as a "tube"

$n \text{ tubes}$
 k_{ST}
 ϕ

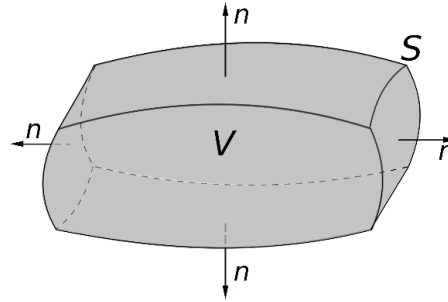
$$\bar{q} = \frac{nq\pi R^2}{n\pi R^2/\phi} = \phi q = -\phi \frac{k_{ST}}{\mu} \frac{\partial p}{\partial x}$$
$$\bar{q} = -\frac{k_{BoT}}{\mu} \frac{\partial p}{\partial x}$$
$$k_{BoT} = \phi k_{ST}$$

Steady-state saturated flow

Derive the flow equation:

Recall: Divergence theorem

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, ds = \int_{\Omega} \nabla \cdot \mathbf{q} \, dV$$



=> Divergence theorem converts a surface integral to a volume integral.

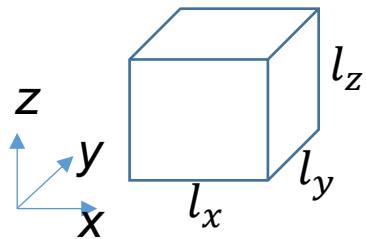
Mass conservation: Change of mass storage = mass in – mass out.

Rate of mass change:

$$\frac{d}{dt} \int_0^{l_x} \int_0^{l_y} \int_0^{l_z} \rho(x,y,z) \phi(x,y,z) \, dx \, dy \, dz = \frac{d}{dt} \int_V \rho \phi \, dV$$

Net fluxes:

$$\begin{aligned} & \int_0^{l_y} \int_0^{l_z} q_x|_{x=0} \, dy \, dz - \int_0^{l_y} \int_0^{l_z} q_x|_{x=l_x} \, dy \, dz \\ & + \int_0^{l_x} \int_0^{l_z} q_y|_{y=0} \, dx \, dz - \int_0^{l_x} \int_0^{l_z} q_y|_{y=l_y} \, dx \, dz \\ & + \int_0^{l_x} \int_0^{l_y} q_z|_{z=0} \, dx \, dy - \int_0^{l_x} \int_0^{l_y} q_z|_{z=l_z} \, dx \, dy \end{aligned}$$



Steady-state saturated flow

$$\Rightarrow - \int_{\partial\Omega} \underline{q} \cdot \underline{n} \, dS = - \int_{\Omega} \nabla \cdot \underline{q} \, dV \quad \text{Divergence theorem.}$$

$$\text{thus, } \frac{d}{dt} \int_{\Omega} \rho \phi \, dV = - \int_{\Omega} \nabla \cdot \underline{q} \, dV$$

$\downarrow \Omega \neq \Omega(t)$

$$\Rightarrow \int_{\Omega} \frac{d}{dt} (\rho \phi) \, dV = - \int_{\Omega} \nabla \cdot \underline{q} \, dV$$

$$\Rightarrow \int_{\Omega} \left[\frac{d}{dt} (\rho \phi) + \nabla \cdot \underline{q} \right] dV = 0$$

$$\Rightarrow \int_{\Omega} \left[\frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot \underline{q} \right] dV = 0$$

$$\Rightarrow \frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot \underline{q} = 0$$

Darcy's law: $\underline{q} = -\rho \underline{k} \nabla H$

$\int_{\Omega} \dots = 0, \Omega \text{ is arbitrary} \Rightarrow \text{Integrand} = 0$

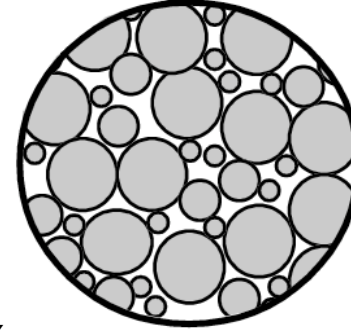
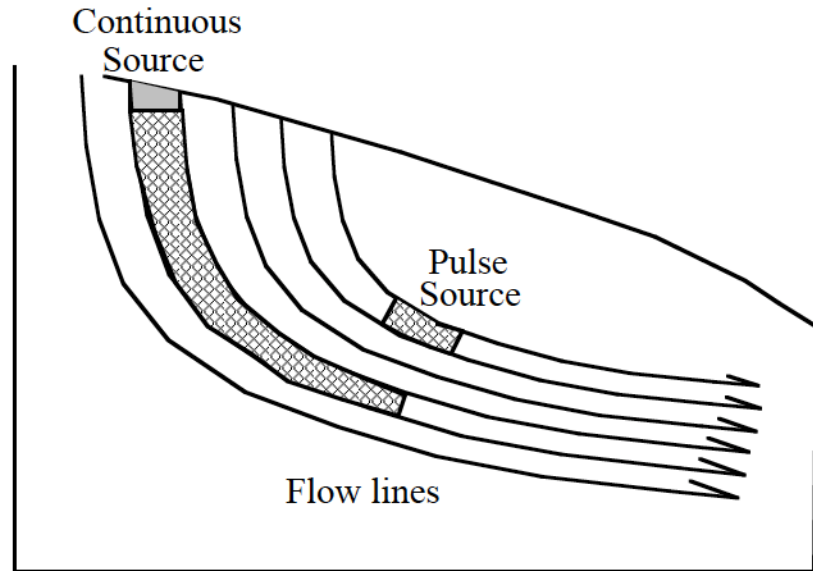
$$\Rightarrow \boxed{\frac{\partial}{\partial t} (\rho \phi) + \nabla \cdot (-\rho \underline{k} \nabla H) = 0}$$

General 3D flow equation for a water-saturated porous medium.

Solute transport under saturated flow

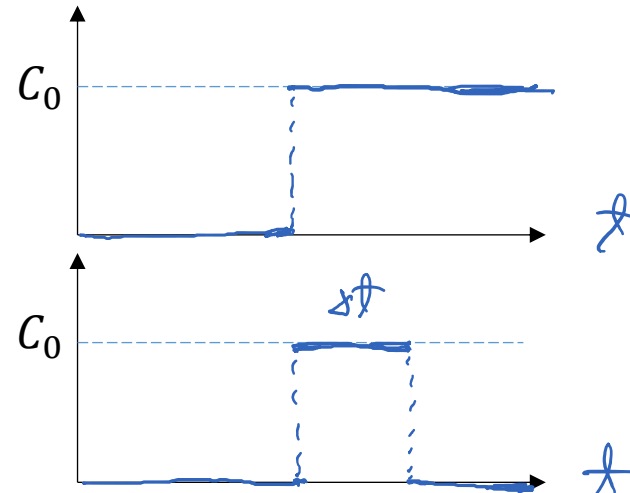
Advection: The solute particles move along streamlines with a velocity equal to the groundwater velocity

$$q = Q/A = -\underline{K} \nabla H$$
$$v = q/\phi$$



Porewater velocity is greater than the volumetric Darcy flux

Breakthrough curve (BTC)



Continuous source:

Pulse source:

Solute transport under saturated flow

Molecular diffusion: Solute particles move due to random molecular motion.

Free water: Recall Fick's Law

$$\mathbf{q}_c = -D_0 \nabla C.$$

Porous medium:

$$\mathbf{q}_c = -\phi D_d \nabla C.$$

$D_d = D_0 w$ is the effective diffusion coefficient, and w ($0 < w < 1$) is the tortuosity factor. (Note: tortuosity vs. tortuosity factor)

Mechanical dispersion: Mixing due to local-scale variations in groundwater velocity

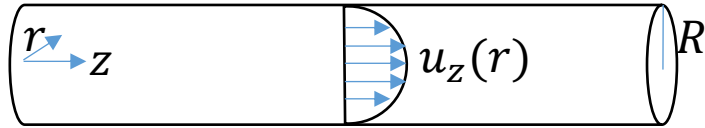


Example: Dispersion in a capillary tube [G.I. Taylor (1953) and R. Aris (1956)]

Solute transport under saturated flow

Taylor-Aris Dispersion: Dispersion in a capillary tube [G.I. Taylor (1953) and R. Aris (1956)]

Solute transport in “Hagen-Poiseuille” flow



Governing equation for solute transport in the tube:

$$\frac{\partial C}{\partial t} + 2u \left[1 - \left(\frac{r}{R} \right)^2 \right] \frac{\partial C}{\partial x} - D_0 \left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r} \frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2} \right) = 0$$

At sufficiently long times (derivation via the perturbation method)

$$\frac{\partial \bar{C}}{\partial t} + \bar{u} \frac{\partial \bar{C}}{\partial x} - D_L \frac{\partial^2 \bar{C}}{\partial x^2} = 0, \quad D_L = D_0 + \frac{R^2 \bar{u}^2}{48 D_0}$$

\bar{u} : mean velocity in the tube.

C_m : mean concentration in the tube.

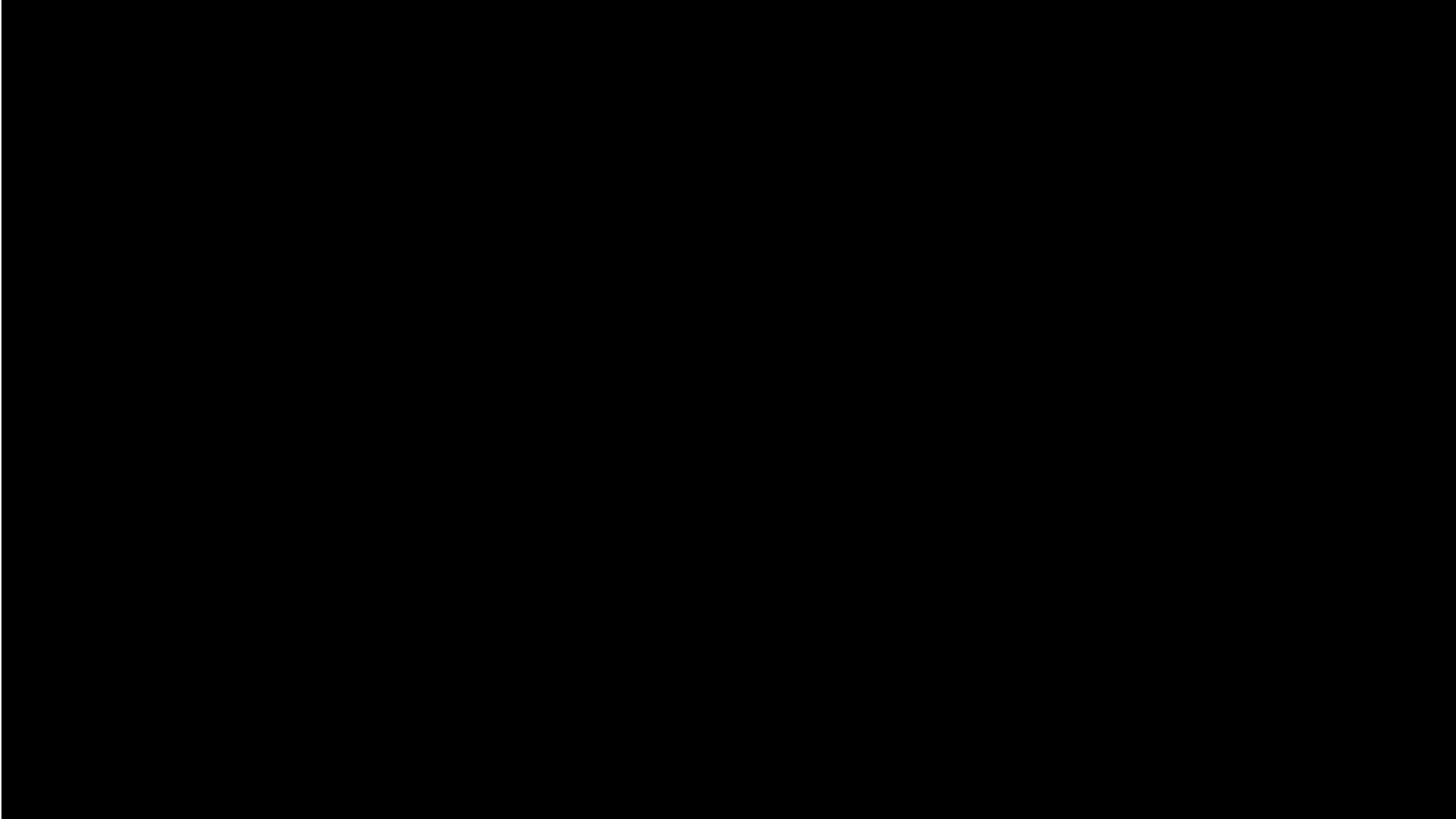
$$C_m = \frac{\int_0^{2\pi} \int_0^R C(r, x) r \, dr \, d\theta}{\int_0^{2\pi} \int_0^R r \, dr \, d\theta} = \frac{2}{R^2} \int_0^R C r \, dr$$

Insights: The average solute concentration spreads out by a dispersion process (radial diffusion and axial advection) and follows Fickian diffusion. The effective diffusivity is not the molecular diffusivity D_0 . Rather, it is D_0 + a quadratic function of the mean velocity.

Solute transport under saturated flow

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Illustrative numerical simulations of Taylor-Aris dispersion (https://youtu.be/toC4RM_aUS4)

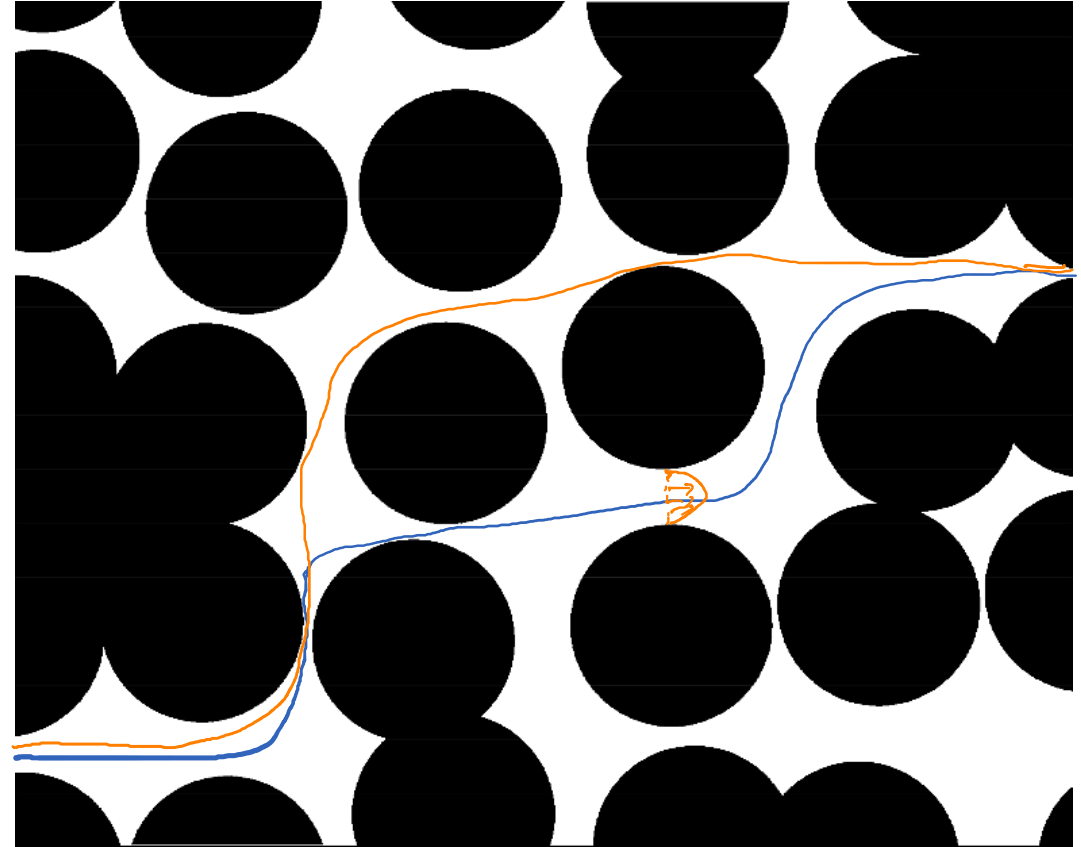


Solute transport under saturated flow

Taylor-Aris Dispersion: Spatial variation of velocity leads to dispersion of solutes

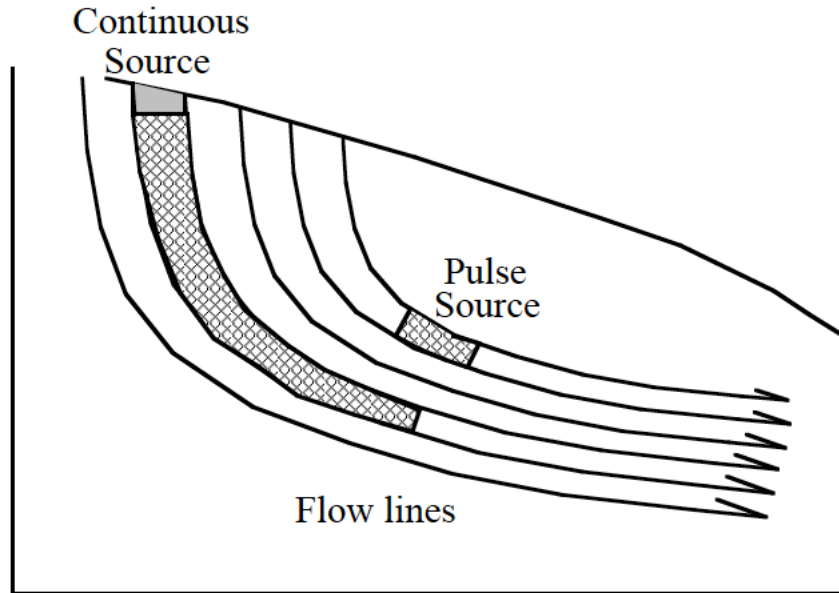
Spatial variation of velocity in a porous medium:

- (1) Velocity profile through a pore throat (related to T-A dispersion)
- (2) Velocity variation among pore throats
- (3) Tortuosity

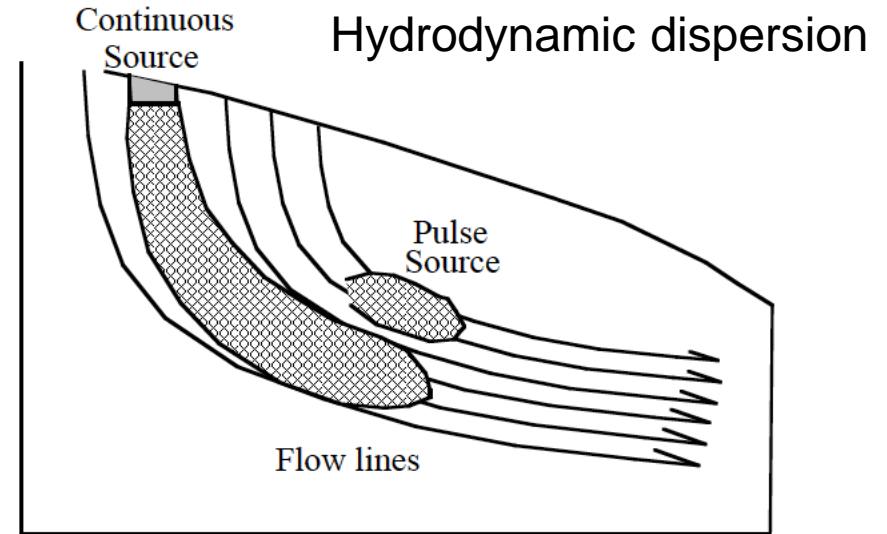


Solute transport under saturated flow

Pure advection



Advection + Mechanical dispersion + Molecular diffusion



A widely used theory for describing solute flux in saturated porous media [J. Bear (1961)]

$$\mathbf{q}_c = \phi(\mathbf{v}C - \mathbf{D}\nabla C),$$

- \mathbf{q}_c is the solute flux vector.
- \mathbf{v} is the velocity field.
- \mathbf{D} is a tensor that depends on the velocity field and molecular diffusion in the porous medium.

For a special case in 1D flow:

$$D_{xx} = \alpha_L v_x + D_d$$

$$D_{yy} = D_{zz} = \alpha_T v_x + D_d$$

- α_L and α_T are longitudinal and transverse dispersivities, respectively.