Homework I

Q-01

An isosceles triangle OAB rotates around an axis through point O at a constant angular velocity ω, as shown in the Figure 1, while Point M moves along the AB side at a constant relative speed. When the triangle finishes one round, Point M happens to move from A to B. If AB = b, please derive the absolute speed and absolute acceleration for Point M when it is at A.

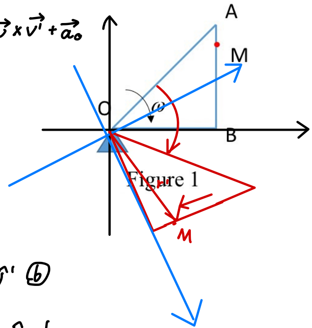

Figure 1

$T = \frac{2\pi}{\omega}$

$|\vec{v}'| = b \div \frac{2\pi}{\omega} = \frac{b\omega}{2\pi}$

$\vec{r}' = b\hat{\imath} + (b - \frac{b\omega t}{2\pi})\hat{\jmath}$

$\vec{v}' = \frac{\tilde{d}}{dt}\vec{r}' = -\frac{b\omega}{2\pi}\hat{\jmath}'$ ①

here the rotation is clockwise.

Therefore. $\vec{\omega} = -\omega\hat{k}$

$\because \vec{V} = \vec{v}' + \vec{\omega}\times\vec{r}'$

$$\vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & 0 & -\omega \\ b & b-\frac{b\omega t}{2\pi} & 0 \end{vmatrix}$$

$= b\omega(\hat{\imath}'(1-\frac{\omega t}{2\pi}) - \hat{\jmath}')$ ②

①+② = $\vec{V} = b\omega(\hat{\imath}'(1-\frac{\omega t}{2\pi}) + \hat{\jmath}'(-1-\frac{1}{2\pi}))$

$|\vec{V}| = b\omega\sqrt{(1-\frac{\omega t}{2\pi})^2 + (1+\frac{1}{2\pi})^2}$

$\boxed{= b\omega\sqrt{2 - \frac{\omega t}{\pi} + \frac{\omega^2 t^2}{4\pi^2} + \frac{1}{\pi} + \frac{1}{4\pi^2}}}$

Here $\because \vec{a} = \vec{a}' + \dot{\omega}\times\vec{r}' - \omega^2\vec{r}' + 2\vec{\omega}\times\vec{v}' + \vec{a}_0$

$\vec{a}' = \frac{\tilde{d}}{dt}\vec{v}' = 0$ ③

$\because$ no translational motion.

$\therefore \vec{a}_0 = 0$ ④

$\because \dot{\omega} = 0$

$\therefore \dot{\omega}\times\vec{r} = 0$ ⑤

$-\omega^2\vec{r}' = -\omega^2(b\hat{\imath}' + (b-\frac{b\omega t}{2\pi})\hat{\jmath}')$

$= -\omega^2 b\,\hat{\imath}' - \omega^2(b-\frac{b\omega t}{2\pi})\hat{\jmath}'$ ⑥

$$2\vec{\omega}\times\vec{v}' = 2\begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & 0 & -k \\ 0 & -\frac{b\omega}{2\pi} & 0 \end{vmatrix} = \hat{\imath}'(-\frac{b\omega k}{\pi})$$ ⑦

③+④+⑤+⑥+⑦:

$\vec{a} = \hat{\imath}'(-\omega^2 b - \frac{b\omega k}{\pi}) + \hat{\jmath}'(\frac{b\omega^3 t}{2\pi} - \omega^2 b)$

When M at point A, t=0:

$\boxed{\vec{a}(0) = \hat{\imath}'(-\omega^2 b - \frac{b\omega k}{\pi}) + \hat{\jmath}'(-\omega^2 b)}$

Q-02

A straight line OX rotates around an axis through Point O at a constant angular velocity ω, as shown in the Figure 2, while Point M moves along the line. When t=0, θ=0 and Point M locates at Point O. If the magnitude of the absolute velocity for Point M is a constant, how should Point M move along the line OX? What is the trajectory and acceleration of Point M?

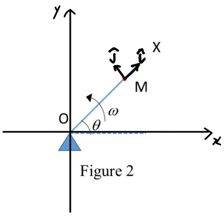

Figure 2

$$\vec{r} = r\hat{i}$$

$$\vec{v} = \dot{r}\hat{i} + r\dot{\theta}\hat{j}$$

$$= \dot{r}\hat{i} + r\omega\hat{j}$$

$$|\vec{v}| = \sqrt{(\dot{r})^2 + (r\omega)^2} = V_0$$

$$\left(\frac{dr}{dt}\right)^2 + r^2\omega^2 = V_0^2$$

$$\frac{dr}{dt} = \sqrt{V_0^2 - r^2\omega^2}$$

$$\frac{dr}{\sqrt{V_0^2 - \omega^2 r^2}} = dt$$

$$\frac{dr}{\omega\sqrt{\frac{V_0^2}{\omega^2} - r^2}} = dt$$

$$\frac{1}{\omega}\arcsin\left(\frac{\omega}{V_0} r\right) = t$$

$$\sin(\omega t) = \frac{\omega}{V_0} r$$

$$\boxed{r = \frac{V_0}{\omega}\sin(\omega t)}$$

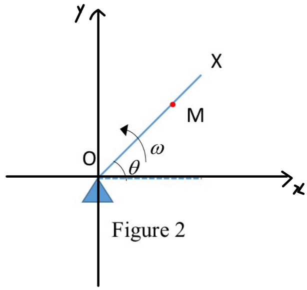

Figure 2

Therefore the trajectory:

$$\boxed{\begin{cases} r = \frac{V_0}{\omega}\sin(\omega t) \\ \theta = \omega t \end{cases}}$$

$$\dot{r} = V_0 \cdot \cos(\omega t)$$

$$\ddot{r} = -V_0\omega\sin(\omega t)$$

$$\dot{\theta} = \omega \qquad \ddot{\theta} = 0$$

$$\vec{a} = \left(-V_0\omega\sin(\omega t) - \frac{V_0}{\omega}\sin(\omega t)\cdot\omega^2\right)\hat{i}$$

$$+ \left(2\cdot V_0\cos(\omega t)\cdot\omega\right)\hat{j}$$

$$\boxed{= \hat{i}\left(-2V_0\omega\sin(\omega t)\right) + \hat{j}\left(2\omega V_0\cos(\omega t)\right)}$$

Q-03

A straight line AB moves on surface of a circle (radius r) at a constant velocity *c*, as shown in the Figure 3. The intersection of the line and the circle is Point M. What is the speed and acceleration of Point M?

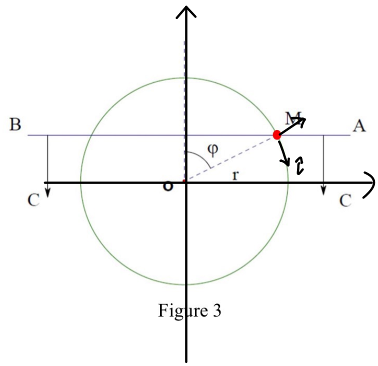

Figure 3

$$y = r\cos\varphi$$

$$y' = -c = -r\sin\varphi\,\varphi'$$

$$\omega = \varphi' = \frac{c}{r\sin\varphi}$$

r is constant

$$\dot{r} = 0$$

$$\vec{V} = r\cdot\omega\,\hat{\jmath} = \frac{c}{\sin\varphi}\,\hat{\jmath}$$

$$\boxed{|\vec{v}| = \left|\frac{c}{\sin\varphi}\,\hat{\jmath}\right| = \frac{c}{\sin\varphi}}$$

$$\dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\varphi}\cdot\frac{d\varphi}{dt}$$

$$= -\frac{c}{r}\,\frac{\cos(\varphi)}{\sin^2(\varphi)}\cdot\frac{c}{r\sin\varphi}$$

$$= -\frac{c^2}{r^2}\,\frac{\cos(\varphi)}{\sin^3(\varphi)}$$

$$\vec{a} = (\cancel{\ddot{r}} - r\dot{\theta}^2)\hat{\imath} + (r\ddot{\theta} + \cancel{2\dot{r}\dot{\theta}})\hat{\jmath}$$

$$= \left(-r\cdot\left(\frac{c}{r\sin\varphi}\right)^2\right)\hat{\imath} + r\cdot\left(-\frac{c^2}{r^2}\,\frac{\cos(\varphi)}{\sin^3(\varphi)}\right)\hat{\jmath}$$

$$= \left(-\frac{c}{r\sin^2\varphi}\right)\hat{\imath} + \left(-\frac{c^2\cos(\varphi)}{r\sin^3(\varphi)}\right)\hat{\jmath}$$