Tutorial #2

1. Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

as z tends zero does not exist.

2. Verify the Cauchy–Riemann equations for $f(z) = z^3$.

3. Determine where $f(z) = xy^2 + ix^2y$ is differentiable.

4. Let

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{when} \quad z \neq 0 \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then

$$\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \begin{cases} 1 & \text{when} \quad \Delta z = \Delta x, \\ -1 & \text{when} \quad \Delta z = \Delta y + i\Delta y. \end{cases}$$

Thus $f(z)$ is not differentiable at $z = 0$. Verify that the Cauchy-Riemann equations hold at $z = 0$. Hence C-R equations hold true is only necessary but not sufficient condition for $f(z)$ to be differentiable at a point.

5. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

6. Recall that

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}.$$

By *formally* applying the chain rule to a function $F(x, y)$, derive the expression

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

Determine the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

suggested by the above formula, to show that if the real and imaginary parts of $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

which is the *complex form* of the Cauchy–Riemann equations. Using this to show that $f(z) = |z|^2$ is not differentiable for all $z \neq 0$.

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