Recap

Improper Integrals : 2 Types

How are the 2 types of integrals related to proper integrals ?

Limit of proper (Riemann) integrals

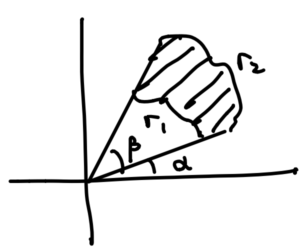
I. $\int_a^{\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$

II $\int_a^{c} f(x)\,dx = \lim_{b \to c} \int_a^b f(x)\,dx$

How to establish convergence? (Type I)

- Comparison test (Recall series)
- Limit Comparison Test .
- Absolute Convergence .

· Polar coordinates ; Simplify certain types of calculation.



Area : $\frac{1}{2} \int_{\alpha}^{\beta} (r_2(\theta)^2 - r_1(\theta)^2)$

Prob 1

- $\displaystyle\int_0^{+\infty} \frac{x\ln x}{(1+x^2)^2}\,dx$

Observe $\frac{x\ln x}{(1+x^2)^2}$ is not defined at $x=0$

$\Rightarrow$ Type I + Type II Improper integral

We do the integration first. (Newton—Leibniz)

$$\int \frac{2x\ln x}{2(1+x^2)^2}\,dx = -\frac{1}{2}\int \ln x \; d\,\frac{1}{1+x^2}$$

By Parts

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int \frac{dx}{x(1+x^2)}$$

$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{4}\ln\frac{x^2}{1+x^2} + C$$

Using N-L Formula

$$I = \lim_{\substack{a\to 0\\ b\to+\infty}} \left[-\frac{\ln x}{2(1+x^2)} + \frac{1}{4}\ln\frac{x^2}{1+x^2} \right]_a^b$$

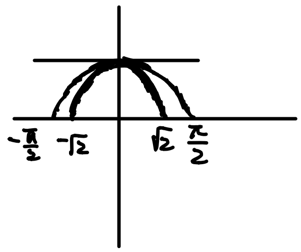
$$= \lim_{a\to 0}\left[\frac{\ln a}{-2(1+a^2)} + \frac{1}{4}\ln\frac{a^2}{1+a^2}\right] + \lim_{b\to\infty}\left[\frac{\ln b}{-2(1+b^2)} + \frac{1}{4}\ln\frac{b^2}{1+b^2}\right]$$

Cancel Term

$$= \lim_{a\to 0}\left(-\frac{1}{4}\ln(1+a^2)\right) + \lim_{b\to\infty}\left(\frac{1}{4}\ln\frac{b^2}{1+b^2}\right)$$

$$= 0$$

- $\lim_{x \to 0} x \int_x^1 \frac{\cos t}{t^2} \, dt$



<u>Claim</u> : $1 - \frac{x^2}{2} \leq \cos x$

Take derivative of $f(x) = \cos x - 1 + \frac{x^2}{2}$

$f'(x) = -\sin x + x > 0$ when $x \in (0, 1]$

Hence true.

$$\therefore \int_x^1 \frac{1 - (t^2/2)}{t^2} dt \leq \int_x^1 \frac{\cos t}{t^2} \, dt \leq \int_x^1 \frac{1}{t^2} \, dt$$

$$\Rightarrow \quad -\frac{3}{2} + \frac{x}{2} + \frac{1}{x} \leq I \leq -1 + \frac{1}{x}$$

Since $\lim_{x \to 0} x\left(-\frac{3}{2} + \frac{x}{2} + \frac{1}{x}\right) = 1$

$\lim_{x \to 0} x\left(-1 + \frac{1}{x}\right) = 1$

By Squeeze Theorem,

$\lim_{x \to 0} x \cdot I = 1.$

<u>Prob 2</u>

Investigate the convergence.

- $\int_1^{+\infty} \frac{1}{\sqrt{x^3 - e^{-2x} + \ln x + 1}}$

Since $-e^{-2x} > -1$ $\ln x > 0$

$\therefore \sqrt{x^3 - e^{-2x} + \ln x + 1} > x^{3/2}$

$\therefore \int_1^{+\infty} \frac{1}{\sqrt{x^3 - e^{-2x} + \ln x + 1}} < \int_1^{+\infty} x^{-3/2} = 2$

- $\int_0^{1/2} \frac{1}{\sqrt{x^2(1-x)}}$

$I \geq \int_0^{1/2} \frac{1}{x} = \lim_{b \to 0} \ln x \Big|_b^{1/2} \to +\infty$

$\therefore I$ diverges.

Prob 3

- $\int_0^{+\infty} \frac{2x\,dx}{x^2+1}$ diverges

$\int_0^{+\infty} \ln(1+x^2)$

$= \lim_{b \to +\infty} \ln(1+x^2) \Big|_0^b$

Obviously, diverges.

- $\lim_{b \to +\infty} \int_{-b}^{b} \frac{2x\,dx}{x^2+1} = 0$

Note that $\frac{2x}{x^2+1}$ is odd.

Integrating an odd function over a symmetric interval yields 0.

Remark. It is tempting to say

$$\int_{-\infty}^{\infty} x \, dx = 0$$

But we've seen it's false by the previous example.

The problem lies in infinity is not a "fixed" number

$$\lim_{b \to \infty} \int_{-b}^{b^2} x \, dx \quad \& \quad \lim_{b \to \infty} \int_{-b}^{\log b} x \, dx$$

can both be regarded as $\int_{-\infty}^{\infty} x \, dx$. yet they do not converge to a fixed #.

As opposed to this

$$\int_{-\infty}^{\infty} f(x) \, dx = c \quad \text{implies}$$

$$\lim_{t \to \infty} \int_{l(t)}^{u(t)} f(x) \, dx = c$$

where $u(t)$, $l(t)$ are arbitrary s.t.

$u(t) \to +\infty$, $l(t) \to -\infty$ as $t \to \infty$.

Prob 4

For what values of a does

$$\int_1^\infty \left(\frac{ax}{x^2+1} - \frac{1}{2x} \right) dx$$

converge? Evaluate the corresponding integrals.

$$I = \int_1^\infty \left(\frac{ax}{x^2+1} \right) dx - \int_1^\infty \frac{1}{2x} \, dx$$

$$= \frac{a}{2} \ln(x^2+1) \Big|_1^\infty - \frac{1}{2} \ln x \Big|_1^\infty$$

$$= \frac{1}{2} \ln \frac{(x^2+1)^a}{x} \Big|_1^\infty$$

We observe when $a = \frac{1}{2}$

$$I = \frac{1}{2} \ln 1 - \frac{1}{2} \ln \sqrt{2} = -\frac{1}{4} \ln 2$$

When $a > \frac{1}{2}$

$$\lim_{x \to +\infty} \frac{(x^2+1)^a}{x} = +\infty$$

When $a < \frac{1}{2}$

$$\lim_{x \to +\infty} \frac{(x^2+1)^a}{x} = 0$$

Both cases, divergent.