- One sided derivatives
  - Compare w/ one sided limits
  - Diff'ble condition
- Th'm of Eq. 1-Sided Derivatives
- Diff'ble $\overset{\Rightarrow}{\not\Leftarrow}$ Cts
- Higher-Order derivatives

Notation: $f^{(n)}(x)$, $y^{(n)}$, $\frac{d^n y}{dx^n}$, $\frac{d^n}{dx^n}[f(x)]$ Differential operator

- Differentials: $\Delta y \approx f'(x)\Delta x$
- 1^{st}-order approximation (linear ~)

(App'n : Gradient Descent [Opt & ML])
Newton's Method [Alg'm]

- Local Extreme : Critical Pt → $f'(c)=0$
  $f'(c)$ X exist

Global Extreme : Boundaries / Critical Pts

- Rolle's Th'm : $f(a)=f(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c)=0$
- MVT : $\exists c$ in (a,b) s.t. $f'(c) = \frac{f(a)-f(b)}{a-b}$
- Monotonicity ↔ 1^{st} Derivative

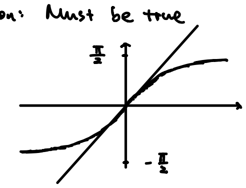
(- Concavity ↔ 2^{nd} Derivative)

Prob 1

Prove $|\arctan a - \arctan b| \le |a-b|$

Intuition: Must be true

Graph:



- Derivative max: at $x=0$, $f'(x)=1$ elsewhere "flatter" than $y=x$.
- $y=x$ is a tangent.

Proof (MVT)

Let $f(x)=\arctan x$

$\forall a<b \in \mathbb{R}$, $\exists c \in (a,b)$

$\frac{1}{1+c^2} = f'(c) = \frac{f(b)-f(a)}{b-a}$

$\because 0 < \frac{1}{1+c^2} \le 1$

$\therefore 0 < f(a)-f(b) \le a-b$

$\therefore |\arctan a - \arctan b| \le |a-b|$

Q: How do we prove using monotonicity
(Hint: Use twice, different occasions)

Problem 2

Show that $f(x)=\sqrt{x}+\sqrt{1+x}-4$ has exactly 1 zero on $(0,+\infty)$

Intuition:

$\sqrt{x} \nearrow$, $\sqrt{1+x} \nearrow$, $f(x)$ Monotonic

At most cross the x-axis once.

$f(1)<0$ $f(4)>0$.

At least cross t x-axis once.

$\Rightarrow$ Cross once. i.e. 1 zero.

Q1: If f has more than 1 zero, how does it conflict with Rolle's Thm?

Q2: What technical detail should we add in the intuition part to add rigor?

Continuity

Q3: If instead, we establish the monotonicity by calculating $f'(x)$. Do we need the detail? why?

Problem 3

If $|f(w)-f(x)| \le |w-x|$ $\forall w,x$
f diff'ble, show that $-1 \le f'(x) \le 1$ $\forall x$.

Proof 1 (Def'n)

Since $f'(x) = \lim_{w\to x} \frac{f(w)-f(x)}{w-x}$

$|f'(x)| = \left|\lim_{w\to x} \frac{f(w)-f(x)}{w-x}\right| \overset{①}{=} \lim_{w\to x}\left|\frac{f(w)-f(x)}{w-x}\right| \overset{②}{\le} 1$

How to justify

①: Let $g(x)=|x|$: Cts, interchange ✓

②: Order limit Th'm *** Very Important

Attempt 2 (MVT)

$\forall x$, $\exists y<x<z$
s.t. $f'(x) = \frac{f(y)-f(z)}{y-z}$

... Hold the horses!

Is this true?

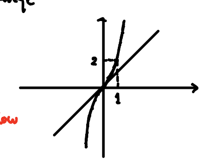
If not, give a counterexample.

- $f(x)=x^3+x$ Take $x=0$
  $f'(0)=1$ But $\forall y<0<z$.
  $\frac{f(y)-f(z)}{y-z} = y^2+yz+z^2+1 \ge 1+\frac{3}{4}z^2 > 1$

- This does not affect the validity of the MVT though.
  Still consider the case $f(x)=x^3$
  $y=-1$, $z=1$, $\exists x = \frac{1}{3}$ s.t.
  $f'(x) = 1 = \frac{f(y)-f(z)}{y-z}$
- Caveat: The converse fails when $f''(x)=0$, or less abstractly, when concavity change

Our Example

Right side of x
y on top of tg
left side y below
tg.



Problem 4.

Prove that

$f(x) = \arctan\left(\frac{1+x}{1-x}\right) - \arctan x = \begin{cases} \frac{\pi}{4}, & x<1 \\ -\frac{3\pi}{4}, & x>1 \end{cases}$

Observations:

- $f(x)$ def'd on $\mathbb{R}\setminus\{1\}$
- Piecewise constant
  ↳ Describing a constant function
  ↳ $f'(x)=0$
- Recall identity
  $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$

Fun properties combined with rational transformations

Proof.

Let $h(x)=\arctan x$ $g(x)=\frac{1+x}{1-x}$

$h'(x)=\frac{1}{1+x^2}$ $g'(x)=\frac{2}{(1-x)^2}$

$f'(x) = h'(g(x))\,g'(x) - h'(x)$
$= 0$ (Verify it) $(x \ne 1)$

$\because \lim_{x\to 1^-} f(x) = ?$

$\lim_{x\to 1^+} f(x) = ?$

$\therefore f(x) = ?$

Q: Why do we introduce one-side limits in the proof? Why can it show the behavior of the function?

Remark:

Expand $\tan(x+\frac{\pi}{4})$, the problem will be demystified.

$\tan(x+\frac{\pi}{4}) = \frac{\sin(x+\frac{\pi}{4})}{\cos(x+\frac{\pi}{4})} = \frac{\sin x + \cos x}{\cos x - \sin x}$
$= \frac{\tan x + 1}{1-\tan x}$

Now, apply the arctan function to both sides, what do you get?