Computations of Definite Integral

Recall Computation techniques last time:

0) Algebraic Properties

i) Riemann Sums: Definition

ii) FT of calculus (II)

More efficient ways of computing definite integrals:

- Change of Var
- Integration by Parts

Change of var.

Recall FT of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

We can view x as a fxn of u: $x = g(u)$, then if $\begin{cases} a = u(i) \\ b = u(j) \end{cases}$

$$\int_{g(i)}^{g(j)} f(x)dx = \int_a^b f(x)\,dx = \int_i^j f(g(u))\,dg(u) = \int_i^j f(g(u))\,g'(u)\,du$$

Integration by Parts

; "Reverse" of Product Rule

Since $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$

By FT of calculus

$$\int_a^b [f'(x)g(x) + f(x)g'(x)]\,dx = f(x)g(x)\Big|_a^b$$

$$\Rightarrow \quad \int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

Discontinuities

A. Removable:

If $\tilde{f}$ & f differ at finitely many pts. then

$$\int_a^b \tilde{f}(x)\,dx = \int_a^b f(x)\,dx$$

B Jump

Suppose f has a jump contiunity at $c \in [a,b]$ and everywhere else is cts, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)\,dx$$

Note: Can be generalized to finite case. How? Try to formulate.

<u>Numerical Integration</u>:

1 Midpoint Approximation

Evenly partition $[a,b]$ into n subintervals

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x)\,dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(c_k)\cdot h \ , \quad c_k = \frac{x_{k-1}+x_k}{2}$$

<u>Error Bound</u>:

By Taylor Exp'n: if $f(x)$, $f'(x)$ cts, $f''(x)$ diff'ble

$\exists\ \xi$ s.t. $\quad f(x) = f(c_k) + f'(c_k)(x-c_k) + \frac{f''(\xi)}{2}(x-c_k)^2$

If $|f''(\xi)|$ bdd by M, then

$$\left|\int_{x_{k-1}}^{x_k} f(x)\,dx - f(c_k)\cdot h\right| = \left|0 + \int_{x_{k-1}}^{x_k} \frac{f''(\xi)}{2}(x-c_k)^2\,dx\right|$$

$$\leq \int_{x_{k-1}}^{x_k} \left|\frac{M}{2}(x-c_k)^2\right| dx$$

$$= \frac{M}{24}\frac{(b-a)^3}{n^3}$$

$$|E_M| \leq \left(\frac{M}{24}\frac{(b-a)^3}{n^3}\right)\cdot n = \frac{M}{24}\frac{(b-a)^3}{n^2}$$

2 Trapezoidal Approx $|E_T| \le \frac{M(b-a)^3}{12n^2}$

3 Parabola Approx $|E_S| \le \frac{M(b-a)^5}{180n^4}$
(Simpson's Method)

Remark

- From the denominator, we observe:
  the finer the partition ($n \uparrow$), the better the approximation ($\downarrow$)
- When $n \to \infty$, they are basically equivalent with Riemann sums.

- 1 2 are using (piecewise)-linear approximation of f.
  3 is using (piecewise)-quadratic approximation of f.
- 1 2 has slower rate of convergence: $O(\frac{1}{n^2})$
  3 has faster rate of convergence: $O(\frac{1}{n^4})$
- 1 2 has weaker requirements on the derivatives
  3 requires much more.

Prob 2

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}}\,dt$$

Observation: x is a function of y

It is tempting for us to use FT.

$$\frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}}$$

We aim to establish y as a fxn of x. So we take the reciprocal.

$$\frac{dy}{dx} = \sqrt{1+4y^2} \quad \cdots \quad (1)$$

Caveat: y on the RHS is still implicit.

Note: Equations in form (2) is what we usually call <u>Ordinary Differential Equation</u> (ODE), in which we are interested in find $y = f(x)$.

For our purpose now, we only want to find y''

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 8y \cdot y' \cdot \frac{1}{2\sqrt{1+4y^2}}$$ Plug in (1)

$$= 8y \cdot \sqrt{1+4y^2} \cdot \frac{1}{2\sqrt{1+4y^2}}$$

$$= 4y$$

$\therefore$ The constant of proportionality is 4.

Note that in the whole solution, we didn't find the explicit formula of y.

Problem 3 (Change of Var)

- $\int_0^{\pi/2} \sin^3 x \cos^4 x \, dx$

Observation: $\sin x = (-\cos x)' = -d\cos x$

$\sin^2 x = 1 - \cos^2 x$

$$\Rightarrow \int_0^{\pi/2} \sin x \cdot \sin^2 x \cos^4 x \, dx$$

$$= \int_0^{\pi/2} (1-\cos^2 x) \cos^4 x \, d\cos x$$

$$= -\int_1^0 (1-t^2)\, t^4 \, dt$$

$$= \left(\frac{1}{5}t^5 - \frac{1}{7}t^7\right)\Big|_0^1$$

$$= \frac{2}{35}$$

$$\int_1^2 \frac{1}{x(1+x^4)}\,dx$$

How do we deal with this big thing?

One attempt: Factorize $(1+x^4)$ into quadratic terms.

Too much computation,

Observation: $\frac{1}{x}dx = \frac{x^3}{x^4}dx = \frac{1}{x^4}\,d\frac{1}{4}x^4 = \frac{1}{4x^4}\,dx^4$

$$\therefore \int_1^2 \frac{1}{x(1+x^4)}\,dx = \int_1^2 \frac{1}{4x^4(1+x^4)}\,dx^4$$

$$= \frac{1}{4}\int \frac{1}{u(1+u)}\,du$$

$$= \frac{1}{4}\left[\ln u - \ln(u+1)\right]_1^{16}$$

$$= \frac{1}{4}\ln\frac{32}{17}$$

Prob 4 : Find $f(4)$

a) $\int_0^{x^2} f(t)\,dt = x\cos\pi x$ (FALSE PROBLEM)

Whenever seeing variable at the upper/lower limit of an integral
A natural thought is FT of Calculus

Take the derivatives wrt x in both sides:

$$\frac{d}{dx}\int_0^{x^2} f(t)\,dt = \frac{d}{dx}(x\cos\pi x)$$

$$2x\cdot f(x^2) = \cos\pi x - \pi x\sin\pi x$$

- We want $f(4)$, so we plug in $x=2$ (-2 yields the same result?)

$4\cdot f(2^2) = 1$

b)

$$\int_0^{f(x)} t^2\,dt = x\cos \pi x$$

Note that in this part, we can evaluate $\int t^2 dt$ explicitly

This is done by the other part of FT of Calculus: N-L formula.

$$\left.\frac{t^3}{3}\right|_0^{f(x)} = x\cos \pi x$$

$$[f(x)]^3 = 3x\cos \pi x$$

$$f(x) = \sqrt[3]{3x\cos \pi x}$$

Plug in $x = 4$, $f(4) = \sqrt[3]{12}$