Recap.

Definite Integral : Finding the area under a curve

Idea : Using "THIN" enough rectangles to approximate "TRAPEZOIDS"
and sum them up

↓ Take limit of summation

Definite Integral.

P : partition of $[a,b]$: $\{x_0, \dots, x_n\}$
View as a vector in a simplex
Take infinity norm.

If limit of Riemann sums exists

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(c_k)\, \Delta x_k = J$$

norm

Then f is (Riemann) integrable → "Lebesque Integrable"

Note J is independent of c_k & P.

Suff Cond'ns:

Continuous → Integrable

Bdd, Finite discontinuities → Integrable

MVT for Def. Integrals : f cts

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Generalized Version: : f cts, g integrable, nonneg.

$$\int_a^b f(x) g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

Reduces to the original by setting $g \equiv 1$

FT of Calculus

I) f continuous on $[a,b]$

Def $F : [a,b] \to \mathbb{R}$ by

$F(x) := \int_a^x f(t)\, dt$, then F diff'ble on (a,b) w/ $F'(x) = f(x)$

II) f cts on $[a,b]$, let F be antiderivative of f on $[a,b]$

Then $\int_a^b f(x)\,dx = F(b) - F(a)$

<u>Integration:</u>

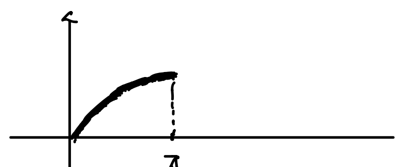
0) Algebraic Properties

i) Riemann Sums: Definition

ii) FT of calculus (II)

Problem 1

$$\int_0^{\frac{\pi}{2}} \sin x \, dx$$



$$S = \sum_{k=1}^{n} \Delta x \, f(x_k)$$

where $x_k = \frac{\pi}{2} \cdot \frac{k}{n}$ $\quad \Delta x = \frac{\pi}{2n}$

$$= \frac{\pi}{2n} \sum_{k=0}^{n-1} \sin\left(\frac{k\pi}{2n}\right)$$

$\downarrow$ Algebraic manipulation: $\sin a \cdot \sin b = \cos\frac{a-b}{2} - \cos\frac{a+b}{2}$

$$= \frac{\pi}{2n} \sum_{k=0}^{n-1} \frac{1}{2\sin\frac{\pi}{4n}} \left(\cos\left(\frac{2k-1}{2} \cdot \frac{\pi}{2n}\right) - \cos\left(\frac{2k+1}{2} \cdot \frac{\pi}{2n}\right)\right)$$

$$= \frac{\pi/2n}{2\sin \pi/4n} \cdot \left(\cos\frac{\pi}{4n} - \cos\frac{(2n-1)\pi}{4n} \right)$$

$$I = \lim_{n\to\infty} S = 1 \cdot 1 = 1$$

Problem 2

Compare the values

- $I = \int_0^{\frac{\pi}{2}} \sin^{10} x \, dx$ & $J = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$

$$J - I = \int_0^{\frac{\pi}{2}} \underbrace{\sin^2 x \,(1 - \sin^8 x)}_{\geq 0 \ , \ (> 0 \text{ when } x \neq 0 \text{ and } \frac{\pi}{2})} \, dx$$

> 0

- $I = \int_0^1 e^{-x} \, dx$ & $J = \int_0^1 e^{-x^2}$

$$J - I = \int \underbrace{e^{-x^2}}_{>0} \,(\underbrace{1 - e^{x^2 - x}}_{\text{Since } x^2 - x \leq 0})$$

$e^{x^2 - x} \leq 1$

Integrand > 0 when $x \neq 0$ and 1

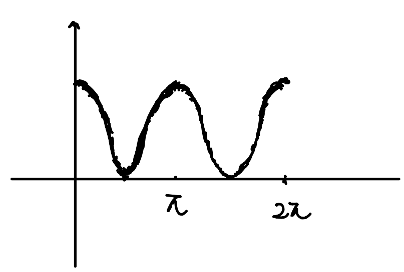
> 0

- $I = \int_0^{\pi} e^{-x^2} \cos^2 x \, dx$ & $J = \int_{\pi}^{2\pi} e^{-x^2} \cos^2 x \, dx$

Observe the symmetry of $\cos^2 x$:

- Periodic with period π

$\cos^2(x + \pi) = \cos^2(x)$



Also note

$$J = \int_{\pi}^{2\pi} e^{-x^2} \cos^2 x \, dx$$

$$= \int_0^{\pi} e^{-(x+\pi)^2} \cos^2(x+\pi) \, dx$$

$$= \int_0^{\pi} e^{-x^2} \cos^2 x \cdot e^{-2x\pi - \pi^2} \, dx$$

$$\therefore I - J = \int_0^{\pi} \underbrace{e^{-x^2} \cos^2 x}_{>0} \left(\underbrace{1 - e^{-2x\pi - \pi^2}}_{>0} dx \right)$$

> 0

Prob 3

f cts on $[a,b]$, $\int_a^b [f(x)]^2 \, dx = 0$

Prove $f \equiv 0$ on $[a,b]$

Composition

Let $g(x) = [f(x)]^2$ $\therefore$ $g(x)$ cts, nonneg.

By FT of calculus

$G'(x) = g(x) \geq 0$

$\because \int_a^b g(x) \, dx = G(b) - G(a) = 0 \Rightarrow G(a) = G(b)$

$\because$ G is nondecreasing on $[a,b]$ (by $g(x) \geq 0$)

$\therefore \forall c \in (a,b)$, $G(a) \leq G(c) \leq G(b) \Rightarrow G(c) = G(a)$

$\therefore G(x) = k$ where k is a constant.

$\therefore g(x) = G'(x) = 0$ on $[a,b]$

$\Rightarrow f(x) = 0$ on $[a,b]$

What if we remove the cts condition?

Does it have to be zero fxn?

Actually no. One can add finitely many removable discontinuities.

Problem 4.

Find the derivative of $F = \int_{\sin x}^{\cos x} \cos(\pi t^2)\, dt$

We want to use the FT of Calculus, but things look a little messy here.
We rewrite F in following form:

$$F = \int_0^{\cos x} \cos(\pi t^2)\, dt - \int_0^{\sin x} \cos(\pi t^2)\, dt$$

We want:

$$f = \frac{d}{dx} \int_0^{\cos x} \cos(\pi t^2)\, dt - \int_0^{\sin x} \cos(\pi t^2)\, dt$$

$$= \frac{d\cos x}{dx} \frac{d}{d\cos x} \int_0^{\cos x} \cos(\pi t^2)\, dt$$

$$- \frac{d\sin x}{dx} \frac{d}{d\sin x} \int_0^{\sin x} \cos(\pi t^2)\, dt$$

$$= -\sin x \cdot \cos(\pi \cos^2 x) - \cos x \cos(\pi \sin^2 x)$$

$$= -\sin x \cos(\pi - \pi\cos^2 x) - \cos x \cos(\pi \sin^2 x)$$

$$= (\sin x - \cos x) \cos(\pi \sin^2 x)$$