Tutorial #1

1. Find an example such that $\operatorname{Arg}(z \cdot w) \neq \operatorname{Arg} z + \operatorname{Arg} w$. And show that if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$ then $\operatorname{Arg}(z \cdot w) = \operatorname{Arg} z + \operatorname{Arg} w$.

2. Prove that

$$\left| \frac{z-w}{1-\overline{z}w} \right| < 1$$

if $|z| < 1$ and $|w| < 1$.

3. Prove the Cauchy-Schwarz inequality

$$\left| \sum_{k=1}^{n} z_k w_k \right|^2 \leq \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |w_k|^2.$$

4. Use de Moivre's formula to derive

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin\theta^2 \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

5. Sketch the following sets and determine which are domains

(a) $|z-2+i| \leq 1$ $\quad$ (b) $|2z+3| > 4$ $\quad$ (c) $\operatorname{Im} z < 2$ $\quad$ (d) $|z-4| \leq |z|$

6. Some topology. A point z_0 is an *interior point* of a set S if there is a neighbourhood of z_0 contained in S. A point z_0 is an *exterior point* of a set S if there is a neighbourhood of z_0 contained in the complement of S. If z_0 is neither an interior nor an exterior point of S, it is called a *boundary* point. A set S is *open* if it doesnot contain any boundary points, or equivalently, each point of S is an interior point. A set S is *closed* if it contains all its boundary points; hence its complement is open.

A point z_0 is an *accumulation point* or *limit point* of a set S if each deleted neighbourhood of z_0 contains at least one point of S. A set is closed if and only if it contains all of its accumulation points.

A set is *bounded* if it is contained in a disk $|z| < R$ for some positive R, otherwise it is *unbounded*.

Bolzano-Weierstrass theorem says that every bounded infinite set has at least one accumulation point.

A set S (as a topological space) is *sequentially compact* if every sequence of points in S has a convergent subsequence converging to a point in S. Assume S is a subset of $\mathbb{C}$, (or more general, of $\mathbb{R}^n$). If S is sequentially compact, then S is bounded and closed.

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