Indefinite Integral : Reverse operation of differentiation

$$\int f(x)\,dx = F(x) + C$$

Continuous $\overset{\Rightarrow}{\not\Leftarrow}$ Integrable

- Finding derivative is relatively easier than the reverse.
- Operator p.o.v. : Similar to Diff'n, it is linear

$$\int \alpha f + \beta g = \alpha \int f + \beta \int g$$

- Change of variable

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \quad \text{by letting } u = g(x)$$

- Typical cases
  1) $g(x)$ Affine: $g(x) = ax + b$ $\quad \int f(Ax+B)\,dx = \frac{1}{A}F(Ax+B)$
  2) $f(x) = \frac{1}{x}$: $\int \frac{g'(x)}{g(x)} = \ln|g(x)| + C$

- Integrate by parts

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)$$

- When seeing e^x, trig, do not hesitate to use this!

- Trig substitution :

Observe $\quad a^2 - x^2, \quad x^2 - a^2, \quad a^2 + x^2$

$$\begin{array}{ccc} a^2 - x^2 & x^2 - a^2 & a^2 + x^2 \\ \downarrow & \downarrow & \downarrow \\ x = a\sin\theta & x = a\sec\theta & x = a\tan\theta \\ \downarrow & \downarrow & \downarrow \\ a^2\cos^2\theta & a^2\tan^2\theta & a^2\sec^2\theta \end{array}$$

- Partial fraction :

$$\frac{P(x)}{Q(x)} = \sum_i \sum_{k=1}^{r_i} \frac{C_i}{(A_i x + B_i)^k} + \sum_i \sum_{k=1}^{r_i} \frac{G_i + H_i}{(D_i x^2 + E_i x + F_i)^k}$$

By linearity. Integrate term-by-term.

Prob 1.

- $\int \frac{dx}{x\sqrt{x^2+1}}$

$$I = \int \frac{dx}{x \cdot x\sqrt{1+\frac{1}{x^2}}}$$

$$= -\int \frac{d(\frac{1}{x})}{\sqrt{1+(\frac{1}{x})^2}}$$

Let $\frac{1}{x} = \tan\theta$

$\Rightarrow I = -\int \sec\theta \, d\theta$

$$\int \frac{1}{\cos\theta} d\theta = \int \frac{1}{1-\sin^2\theta} d\sin\theta$$

$$= \frac{-1}{2}\int \frac{1}{\sin\theta - 1} - \frac{1}{\sin\theta+1} \, d\sin\theta$$

A first thought would be using identities such as

$\sec\theta = \frac{1}{\cos\theta}$, but watch this

$$I = -\int \boxed{\sec\theta \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}} d\theta$$

$\sec^2\theta + \sec\theta\tan\theta$

$= (\tan\theta + \sec\theta)'$

$$= -\ln|\sec\theta + \tan\theta| + C$$

$$= -\ln\left|\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right| + C$$

- Any problems ?

- Sign of x $\sqrt{x^2+1} \neq x\sqrt{1+\frac{1}{x^2}}$ when $x<0$.

- How to remedy ?

- What is $\left(\frac{1}{|x|}\right)'$? $-\frac{1}{x|x|}$

- Please complete the rest of the problem using the hints,

Another attempt

$x = \sqrt{u^2-1}$

But notice the range of x.

$\frac{1}{\sqrt{u^2-1}\cdot u}\, d\sqrt{u^2-1}$

$= \ldots$

$= \ldots$

- $\int \frac{dx}{\sqrt{1+e^x}}$

Let $\sqrt{1+e^x} = t$, $x = \ln(t^2-1)$, $dx = \frac{2t}{t^2-1}\,dt$

$$I = 2\int \frac{dt}{t^2-1}$$

$$= \ln\left(\frac{t-1}{t+1}\right) + C$$

$$= \ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + C$$

- $\int \frac{x\,dx}{(x^2-1)^{\frac{3}{2}}}$

Observe that $\left(\frac{x^2-1}{2}\right)' = x$

$$I = \frac{1}{2}\int \frac{d(x^2-1)}{(x^2-1)^{3/2}}$$

Observe that $\left(-\frac{1}{\sqrt{x}}\right)' = \frac{1}{2x^{3/2}}$

$\therefore I = -\frac{1}{\sqrt{x^2-1}} + C$

Prob 2

$$\int \frac{dx}{\sqrt{x(1+x)}}$$

- What is the domain of $\frac{1}{\sqrt{x(1+x)}}$

Want $x(1+x) > 0 \Rightarrow x > 0$ or $(1+x) < 0$

- Separate into 2 situations

When $x > 0$

$$I = \int \frac{dx}{\sqrt{x(1+x)}} = \int \frac{dx}{\sqrt{x} \cdot \sqrt{1+x}}$$

$$= 2\int \frac{d\sqrt{x}}{\sqrt{1+(\sqrt{x})^2}}$$

Recall $\int \frac{dx}{\sqrt{1+x^2}} = \ln|x+\sqrt{1+x^2}| + C$

$$\Rightarrow I = 2\ln|\sqrt{x}+\sqrt{1+x}| + C$$

- What if when $\boxed{x < -1}$?

- Analogous, we've just write $\sqrt{1+x} = \sqrt{1+(\sqrt{x})^2}$ when we substitute put x in the differential operator.

- Now observe $\sqrt{x(1+x)} = \sqrt{(-x)(-1-x)} = \sqrt{-x} \cdot \sqrt{-1-x}$

- $\sqrt{-x} = \sqrt{1+(\sqrt{-1-x})^2}$

The rest is left as an exercise

<u>Prob 3</u> $\int \sqrt{a^2-x^2}\, dx$

- Remark: The toughest part of the problems with parameters is to discuss the signs. When $a>0$, it would be a lot less annoying.

$$I = \int \sqrt{a^2-x^2}\, dx$$

$$= \int |a| \sqrt{1-\left(\frac{x}{|a|}\right)^2}\, dx$$

$$= a^2 \int \sqrt{1-\left(\frac{x}{|a|}\right)^2}\, d\left(\frac{x}{|a|}\right)$$

Let $\frac{x}{|a|} = \sin\theta$

$$= a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \int \frac{1}{2} + \frac{\cos 2\theta}{2}\, d\theta$$

$$= \frac{a^2\theta}{2} + \frac{a^2 \sin 2\theta}{4} + C$$

$$\theta = \arcsin\left(\frac{x}{|a|}\right)$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{x}{|a|}\cdot\sqrt{1-\left(\frac{x}{|a|}\right)^2}$$

$$\therefore\ I = \frac{1}{2}a^2\cdot\arcsin\left(\frac{x}{|a|}\right) + \frac{1}{2}x\sqrt{a^2-x^2} + C$$

Note that the ref sol. provides another solution using integral by parts.

Prob 4

- $$\int \frac{dx}{x \ln x \ \ln(\ln x)}$$

$$I = \int \frac{d \ln x}{\ln x \ \ln(\ln x)}$$

$$= \int \frac{d \ln(\ln(x))}{\ln(\ln(x))}$$

If not obvious, write $u = \ln x$.

$$= \ln \left| \ln(\ln(x)) \right|$$

- $$\int \frac{x^{\frac{n}{2}}}{\sqrt{1+x^{n+2}}} \, dx$$

When $n = -2$: left as exercise.

When $n \neq -2$. observe $\left(\frac{2}{n+2} x^{\frac{n+2}{2}}\right)' = x^n$

$$I = \frac{2}{n+2} \int \frac{d x^{\frac{n+2}{2}}}{\sqrt{1+x^{n+2}}}$$

The rest follows automatically: $u = x^{\frac{n+2}{2}}$

Using previous result :

$$I = \frac{2}{n+2} \ln \left| x^{\frac{n+2}{2}} + \sqrt{1+x^{n+2}} \right| + C$$

- $$\int \frac{x^2}{(1-x)^{100}} \, dx$$

Observe : $x = x - 1 + 1$

$$x^2 = [(x-1) + 1]^2$$

$$\Rightarrow I = \int \frac{1}{(x-1)^{98}} + \frac{2}{(x-1)^{99}} + \frac{1}{(x-1)^{100}} \, dx$$

$$= -\frac{1}{97(x-1)^{97}} - \frac{1}{49(x-1)^{98}} - \frac{1}{99(x-1)^{99}} + C$$