Tutorial #2 Solutions

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February 2020

1. Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}}\right)^2$$

as z tends zero does not exist.

The following solution is natural when considering the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$.

Solution 1. Consider $z = x + i \cdot 0$, then

$$\lim_{x\to 0} f(x + i \cdot 0) = \lim_{x\to 0} \left(\frac{x + i \cdot 0}{x + i \cdot 0}\right)^2 = 1.$$

Consider $z = y + i \cdot y$, then

$$\lim_{y\to 0} f(y + i \cdot y) = \lim_{y\to 0} \left(\frac{y + i \cdot y}{y - i \cdot y}\right)^2 = -1.$$

When approaching 0 from different paths, the limit of the function along the path varies. Hence the limit does not exist. $\square$

Recall the following theorem:

Theorem 1. *If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$, then*

$$\lim_{z\to z_0} f(z) = w_0$$

if and only if

$$\lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

We have the second solution:

Solution 2. For $z = x + iy$, the function $f(z)$ can be written as

$$f(z) = f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} + i \cdot \frac{2xy}{x^2 + y^2}.$$

Denote $\frac{x^2-y^2}{x^2+y^2}$ by $u(x, y)$ and $\frac{2xy}{x^2+y^2}$ by $v(x, y)$. We know that the limit of $f(z)$ as $z \to 0$ exists if and only if the limits of $u(x, y)$ and $v(x, y)$ both exists as $x, y \to 0$. Clearly this is not the case. $\square$

Exercise 1. Finish the proof by arguing

$$\lim_{x,y\to 0} v(x,y)$$

does not exist.

2. Verify the Cauchy–Riemann equations for $f(z) = z^3$.

We note that this function is extremely easy to represent using polar form, giving us our first solution:

Solution 1. Let $z = re^{i\theta}$, and the function can be written as

$$f(z) = f(re^{i\theta}) = r^3 \cdot e^{i3\theta} = r^3 \cdot \cos(3\theta) + i \cdot r^3 \sin(3\theta).$$

Denote $r^3 \cdot \cos(3\theta)$ by $u(r,\theta)$ and $r^3 \cdot \sin(3\theta)$ by $v(r,\theta)$. Taking partial derivatives with respect to r, θ respectively, we have

$$\begin{aligned} u_r &= 3r^2 \cos(3\theta), & v_r &= 3r^2 \sin(3\theta). \\ u_\theta &= -3r^3 \sin(3\theta), & v_\theta &= 3r^3 \cos(3\theta). \end{aligned}$$

Compare the terms, we have

$$\begin{aligned} u_r = 3r^2 \cos(3\theta) &= \frac{1}{r} v_\theta, \\ \frac{1}{r} u_\theta = -3r^2 \sin(3\theta) &= -v_r, \end{aligned}$$

which is exactly the *C-R equations* in polar form. □

Exercise 2. Is this verification complete? If not, what is missing?

Remark. Since the C-R equations in polar form only applies to the region $r > 0$, it still remains to show they hold at $z = 0$.

Exercise 3. Verify the C-R equations in the standard form.

3. Determine where $f(z) = xy^2 + ix^2y$ is differentiable.

Satisfying Cauchy-Riemann equations is necessary for a function to be differentiable at a point, and it suffices if the function satisfies Cauchy-Riemann equations alongside with the partial derivatives being continuous at that point. However, note that these are not *if and only if* conditions.

Solution. Let $u(x,y) = xy^2$, $v(x,y) = x^2y$. Taking partial derivatives with respect to x, y, we get

$$u_x = y^2, \qquad v_x = 2xy.$$
$$u_y = 2xy, \qquad v_y = x^2.$$

It is clear that the partial derivatives are continuous for all $x, y \in \mathbb{R}$. To satisfy the C-R equations, we require:

$$x^2 = y^2, \quad 2xy = -2xy,$$

which reduces to $x = y = 0$. Hence we conclude that the function is only differentiable at $x = y = 0$, i.e. $z = 0$. $\square$

4. Let

$$f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{when} \quad z \neq 0 \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Show that if $z = 0$, then

$$\lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z} = \begin{cases} 1 & \text{when} \quad \Delta z = \Delta x, \\ -1 & \text{when} \quad \Delta z = \Delta y + i\Delta y. \end{cases}$$

Thus $f(z)$ is not differentiable at $z = 0$. Verify that the Cauchy-Riemann equations hold at $z = 0$. Hence C-R equations hold true is only accessory but not sufficient condition for $f(z)$ to be differentiable at a point.

Solution. We first verify the non-differentiability of $f(z)$ at $z = 0$. When $\triangle z = \triangle x$,

$$\lim_{\triangle x \to 0} \frac{f(\triangle x)}{\triangle x} = \lim_{\triangle x \to 0} \frac{(\triangle x)^2}{(\triangle x)^2} = 1,$$

When $\triangle z = \triangle y + i\triangle y$,

$$\lim_{\triangle y \to 0} \frac{f(\triangle y + i\triangle y)}{\triangle y + i\triangle y} = \lim_{\triangle y \to 0} \frac{(\triangle y - i\triangle y)^2}{\triangle y + i\triangle y)^2} = -1.$$

Thus the limit of $\frac{f(z)}{z}$ as $z \to 0$ does not exist, implying that $f(x)$ is not differentiable at $z = 0$.

Now we check the Cauchy-Riemann equations.

$$f(z) = f(x,y) = \frac{x^3 - 3xy^2}{x^2 + y^2} + i \cdot \frac{-3x^2y + y^3}{x^2 + y^2}.$$

Let $\frac{x^3-3xy^2}{x^2+y^2}$ be $u(x,y)$ and $\frac{-3x^2y+y^3}{x^2+y^2}$ be $v(x,y)$. Taking the partial derivatives at $z = 0$, i.e. $x = y = 0$, we get

$$u_x|_{(0,0)} = \lim_{x \to 0} \frac{x^2}{x^2} = 1, \qquad v_x|_{(0,0)} = \lim_{x \to 0} \frac{0}{x^3} = 0,$$
$$u_y|_{(0,0)} = \lim_{y \to 0} \frac{0}{y^3} = 0, \qquad v_x|_{(0,0)} = \lim_{x \to 0} \frac{y^2}{y^2} = 1.$$

Obviously, the C-R equations hold at $z = 0$. $\square$

Exercise 4. Let

$$f(z) = \begin{cases} |z|^2 \sin\left(\frac{1}{|z|}\right) & \text{when} \quad z \neq 0 \\ 0 & \text{when} \quad z = 0. \end{cases}$$

Verify that $f(z)$ is differentiable at $z = 0$. Verify that the C-R equations hold at $z = 0$. Show that the partial derivatives, however, are discontinuous at $z = 0$. Reach to the conclusion that satisfying C-R equations and having continuous partial derivatives are only *sufficient* but not *necessary* conditions for $f(z)$ to be differentiable at a point.

Remark. This is very analogous to the example $f(x) = x^2 \sin(\frac{1}{x})$ that we have seen in calculus.

5. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

In calculus, this is the well-known *L'Hôpital's Rule*. We aim to show this rule still holds for functions of a complex variable.

Solution. Expand $f'(z_0)$ and $g'(z_0)$ according to definition:

$$f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z} = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z)}{\triangle z}$$

$$g'(z_0) = \lim_{\triangle z \to 0} \frac{g(z_0 + \triangle z) - g(z_0)}{\triangle z} = \lim_{\triangle z \to 0} \frac{g(z_0 + \triangle z)}{\triangle z}$$

The limits on the right hand side are well-defined because of the existence of the derivatives. Now

$$\frac{f'(z_0)}{g'(z_0)} = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z)}{g(z_0 + \triangle z)} = \lim_{z \to z_0} \frac{f(z)}{g(z)}.$$

$\square$

6. Recall that

$$x = \frac{z + \bar{z}}{2}, \qquad y = \frac{z - \bar{z}}{2i}.$$

By *formally* applying the chain rule to a function $F(x, y)$, derive the expression

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

Determine the operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

suggested by the above formula, to show that if the real and imaginary parts of $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy–Riemann equations, then

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

which is the *complex form* of the Cauchy–Riemann equations. Using this to show that $f(z) = |z|^2$ is not differentiable for all $z \neq 0$.

Solution. We first regard x, y as functions of two variables: $z, \bar{z}$. Differentiate x, y with respect to $\bar{z}$, we have:

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \qquad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}.$$

Applying the chain rule,

$$\frac{\partial F}{\partial \bar{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right) \tag{1}$$

For a function f satisfying the Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2}$$

Since

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{3}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}. \tag{4}$$

Plug (3),(4) into (1),

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right),$$

then plug in the Cauchy-Riemann equations (3), we arrive to the result that

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

□

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