Tutorial #1 Solutions

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1. Find an example such that $\operatorname{Arg}(z \cdot w) \neq \operatorname{Arg} z + \operatorname{Arg} w$. And show that if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$ then $\operatorname{Arg}(z \cdot w) = \operatorname{Arg} z + \operatorname{Arg} w$.

Remark. Recall that the argument, denoted by arg, is not unique. Basically, for arguments, the relation

$$\arg(z \cdot w) = \arg z + \arg w \tag{1}$$

holds. This can be proved by the exponential form of complex numbers. However, the notation Arg denotes the principal value of the argument, which is restricted to $-\pi < \operatorname{Arg} z \leq \pi$. The restriction makes the equality (1) unable to be inherited to principal values.

Solution. Take

$$z = w = e^{i\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}(-1 + i).$$

Now $\operatorname{Arg} z = \operatorname{Arg} w = \frac{3\pi}{4}$, and $\operatorname{Arg} z + \operatorname{Arg} w = \frac{3\pi}{4}$.
Now we compute $z \cdot w$.

$$z \cdot w = \left(e^{i\frac{3\pi}{4}}\right)^2 = e^{i\frac{-\pi}{2}} = -i,$$

and it is obvious from the exponential form that $\operatorname{Arg}(z \cdot w) = \frac{-\pi}{2}$.
Still, we observe that although

$$\operatorname{Arg}(z \cdot w) \neq \operatorname{Arg} z + \operatorname{Arg} w,$$

two sides of the inequality differ by an integer multiple of 2π. This follows readily from the definition.
For the second part, note that if $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, we have

$$\frac{-\pi}{2} < \operatorname{Arg} z,\ \operatorname{Arg} w < \frac{\pi}{2},$$

or more compactly,

$$|\operatorname{Arg} z|, \ |\operatorname{Arg} w| < \frac{\pi}{2}.$$

By triangle inequality for real numbers,

$$|\operatorname{Arg} z + \operatorname{Arg} w| \leq |\operatorname{Arg} z| + |\operatorname{Arg} w| < \pi.$$

Since $\operatorname{Arg}(z \cdot w) = \operatorname{Arg} z + \operatorname{Arg} w + 2n\pi$, $n \in \mathbb{N}$, and $-\pi < \operatorname{Arg} z + \operatorname{Arg} w < \pi$. We can say that $n = 0$, which proves our claim.

2. Prove that

$$\left|\frac{z - w}{1 - \overline{z}w}\right| < 1$$

if $|z| < 1$ and $|w| < 1$.

Remark. One property of complex arithmetic is that the modulus of the product/quotient equals to the product/quotient of the modulus (Easily shown by exponential form).

Solution 1. Using the property stated in the remark, we have

$$\left|\frac{z - w}{1 - \overline{z}w}\right| < 1 \iff |z - w| < |1 - \overline{z}w| \iff |z - w|^2 < |1 - \overline{z}w|^2.$$

Expand both sides,

$$\begin{aligned}
|z - w|^2 &= (z - w)\overline{(z - w)} \\
&= (z - w)(\overline{z} - \overline{w}) \\
&= |z|^2 - 2\mathrm{Re}\,(z\overline{w}) + |w|^2, \\
|1 - \overline{z}w|^2 &= (1 - \overline{z}w)\overline{(1 - \overline{z}w)} \\
&= (1 - \overline{z}w)(1 - z\overline{w}) \\
&= 1 - 2\mathrm{Re}\,(z\overline{w}) + |z|^2|w|^2.
\end{aligned}$$

Note that when $|z| < 1$ and $|w| < 1$,

$$(|z| - 1)(|w| - 1) = |z|^2|w|^2 - |z|^2 - |w|^2 + 1 > 0.$$

This essentially says that $|z - w|^2 < |1 - \overline{z}w|^2$, which finishes our proof.

Solution 2. Note that we can write $z - w$ and $\overline{z}w$ in the following form:

$$\begin{aligned} z - w &= |z|e^{i\theta_z} - |w|e^{i\theta_w} \\ &= e^{i\theta_z} \cdot \left(|z| - |w|e^{i(\theta_w - \theta_z)}\right), \\ \overline{z}w &= |z|e^{-i\theta_z} \cdot |w|e^{i\theta_w} \end{aligned}$$
$$|z||w|e^{i(\theta_w-\theta_z)}.$$

Hence,

$$\begin{aligned} \left|\frac{z-w}{1-\overline{z}w}\right| &= \left|\frac{e^{i\theta_z} \cdot \left(|z| - |w|e^{i(\theta_w - \theta_z)}\right)}{1 - |z||w|e^{i(\theta_w-\theta_z)}}\right| \\ &= \left|\frac{|z| - |w|e^{i(\theta_w-\theta_z)}}{1 - |z||w|e^{i(\theta_w-\theta_z)}}\right| \end{aligned}$$

The whole expression does not depend on the choice of θ_z, but rather the difference $\theta_w - \theta_z$. W.l.o.g, we can assume that z is real.

Exercise 1. Finish the proof with the assumption that z is real.

Exercise 2. If $\overline{z}w \neq 1$, prove that

$$\left|\frac{z-w}{1-\overline{z}w}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

Exercise 3. Consider the mapping F_w

$$F_w : z \mapsto \frac{w - z}{1 - \overline{w}z}.$$

This family of mappings, sometimes called **Blaschke factors**, has various applications in complex analysis. It satisfies the following conditions:

(i) F_w maps $\mathbb{D}$ to $\mathbb{D}$, and is holomorphic.

(ii) F_w interchanges 0 and w, namely $F_w(0) = w$ and $F_w(w) = 0$.

(iii) $|F_w(z)| = 1$ if $|z| = 1$.

(iv) F_w is a involution, i.e. $F_w \circ F_w = \imath$. This implies F_w is bijective.

(v) Construct a bijective, holomorphic function that interchanges two given complex numbers z and w.

Solution. By Exercise 3, it is direct that F_w maps $\mathbb{D}$ to $\mathbb{D}$ and $|F_w(z)| = 1$ if $|z| = 1$. The holomorphicity follows from that the function is the quotient of two holomorphic (linear) functions. Note that the denominator is never 0 when $z \in \mathbb{D}$ because $|\overline{w}z| = |w||z| < 1$.
For (ii), this is obtained by direct calculation.
For (iv), we compute $(F_w \circ F_w)(z)$:

$$(F_w \circ F_w)(z) = \frac{w - \frac{w-z}{1-\overline{w}z}}{1 - \overline{w} \cdot \frac{w-z}{1-\overline{w}z}} = \frac{w - |w|^2 z - w + z}{1 - \overline{w}z - |w|^2 + \overline{w}z} = \frac{z - z|w|^2}{1 - |w|^2} = z.$$

Hence $F_w \circ F_w = \imath$, otherwise put, $F_w = {F_w}^{-1}$. Since bijectivity and invertibility are equivalent, F_w is bijective.
For (v), by our previous argument, we essentially need to find a complex number a such that F_ζ maps z to w and maps w to z (readily follows by F_ζ is an involution.) Hence we have:

$$w = F_\zeta(z) = \frac{\zeta - z}{1 - \overline{\zeta} z}.$$

Simplify, we get

$$\zeta = z + w - \overline{\zeta} z w.$$

Equating the real and imaginary parts respectively, we eventually get

$$\begin{aligned} \operatorname{Re}\zeta &= \frac{bd - a(1-c)}{d^2 + c^2 + 1} \\ \operatorname{Im}\zeta &= \frac{ad - b(1+c)}{d^2 + c^2 + 1}, \end{aligned}$$

where

$$a = \operatorname{Re}(z+w),\ b = \operatorname{Im}(z+w),\ c = \operatorname{Re}(zw),\ d = \operatorname{Im}(zw).$$

3. Prove the Cauchy-Schwarz inequality

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \le \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |w_k|^2.$$

Solution. As you might have noticed, the form is not very similar to *Cauchy-Schwarz inequality* that you have learnt in Linear Algebra at first sight. However, a direct use of triangle inequality yields

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \le \left(\sum_{k=1}^{n} |z_k w_k|\right)^2 \le \sum_{k=1}^{n} |z_k|^2 \sum_{k=1}^{n} |w_k|^2.$$

The right hand part of this inequality is the classical *Cauchy-Schwarz inequality*, always written as $(x^\top y)^2 \le \|x\|^2 \|y\|^2$ in a Linear Algebra context.

Remark. Here we give a delicate proof of the Cauchy-Schwarz inequality. We claim that

$$(\sum_i x_i y_i)^2 = (\sum_i x_i^2)(\sum_i y_i^2) - \frac{1}{2}\sum_i \sum_j (x_i y_j - x_j y_i)^2.$$

Note that the Cauchy-Schwarz inequality is the direct consequence of our claim since the last term is always negative.
We expand the L.H.S. as

$$\text{L.H.S.} = (\sum_i x_i y_i)^2 = (\sum_i x_i y_i)(\sum_j x_j y_j) = \sum_i \sum_j x_i y_i x_j y_j.$$

While the R.H.S can be written as

$$\begin{aligned}
\text{R.H.S.} &= \frac{1}{2}(\sum_i x_i^2)(\sum_j y_j^2) + \frac{1}{2}(\sum_j x_j^2)(\sum_i y_i^2) - \frac{1}{2}\sum_i \sum_j (x_i y_j - x_j y_i)^2 \\
&= \frac{1}{2}\sum_i \sum_j \left(x_i^2 y_j^2 + x_j^2 y_i^2 - x_i^2 y_j^2 - x_j^2 y_i^2 + 2x_i y_i x_j y_j\right) \\
&= \sum_i \sum_j x_i y_i x_j y_j.
\end{aligned}$$

The proof is complete.

4. Use de Moivre's formula to derive

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin\theta^2 \qquad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Solution. We use de Moivre's formula together with the binomial formula.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin\theta^2 - i\sin^3\theta \\
&= \cos 3\theta + i\sin 3\theta.
\end{aligned}$$

By equating the real part and the imaginary part of both sides, the two identities are readily established.

5. Sketch the following sets and determine which are domains

(*a*) $|z-2+i| \leq 1$ (*b*) $|2z+3| > 4$ (*c*) $\operatorname{Im} z < 2$ (*d*) $|z-4| \leq |z|$

Remark. It is useful to remind yourself that $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$, and modulus is "isomorphic" to the Euclidean distance.

Solution.

(b). By the analysis in the remark, it is not hard to find that the set is simply $\mathbb{C} \setminus \{z : |z - (-3/2)| \leq 2\}$. It is the complement of a closed disc, so it is open and connected. By definition, it is a domain.

(d). This characterizes the halfplane $\{z : \operatorname{Re} z \geq 2\}$. This is a closed set, so it's not a domain.

6. Some topology. A point z_0 is an *interior point* of a set S if there is a neighbourhood of z_0 contained in S. A point z_0 is an *exterior point* of a set S if there is a neighbourhood of z_0 contained in the complement of S. If z_0 is neither an interior nor an exterior point of S, it is called a *boundary* point. A set S is *open* if it doesnot contain any boundary points, or equivalently, each point of S is an interior point. A set S is *closed* if it contains all its boundary points; hence its complement is open.

A point z_0 is an *accumulation point* or *limit point* of a set S if each deleted neighbourhood of z_0 contains at least one point of S. A set is closed if and only if it contains all of its accumulation points.

A set is *bounded* if it is contained in a disk $|z| < R$ for some positive R, otherwise it is *unbounded.*

Bolzano-Weierstrass theorem says that every bounded infinite set has at least one accumulation point.

A set S (as a topological space) is *sequentially compact* if every sequence of points in S has a convergent subsequence converging to a point in S. Assume S is a subset of $\mathbb{C}$, (or more general, of $\mathbb{R}^n$). If S is sequentially compact, then S is bounded and closed.

Exercise 4. A set Ω is said to be **pathwise connected** if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

(a) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0, 1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t^* = \sup_{0 \leq t \leq 1} \{t : z(s) \in \Omega_1, \quad \forall\, 0 \leq s < t\}$$

Arrive at a contradiction by considering the point $z(t^*)$.

(b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

The proof actually shows that the regularity and type of curves we used to define pathwise connectedness can be relaxed without changing the equivalence between the two definitions when Ω is open. For instance, we may take all curves to be continuous, or simply polygonal lines.

Solution.
(a) By the way we select t^*, $z(t^*)$ is a limit point of Ω_1. But $z(t^*) \notin \Omega_1$. Consider the part of the curve γ parametrized by $z(t)$ where $t \in [t^*, 1]$. If $z(t^*) \in \Omega_1$, by openness of Ω_1, there exists a $\epsilon > 0$ where $z(t) \in \Omega_1$ whenever $t \in [t^*, t^* + \epsilon)$. This contradicts with the selection of t^*. Hence $z(t^*) \in \Omega_2$ since $\Omega = \Omega_1 \cup \Omega_2$. But by openness of Ω_2, there exists a δ-neighborhood $V_\delta(z(t^*))$ that is entirely contained in Ω_2. But since $z(t^*)$ is a limit point of Ω_1, $V_\delta(z(t^*)) \cap \Omega_1 \neq \emptyset$, which implies $V_1 \cap V_2 \neq \emptyset$. This contradicts with the assumption that Ω_1 and Ω_2 are disjoint.
(b) The disjointness of the two sets are straightforward from the definitions. It is also direct that $\Omega = \Omega_1 \cup \Omega_2$. Now consider an arbitrary element $x \in \Omega_1$, by openness of Ω, there exists $V_\delta(z(t^*)) \subset \Omega$. It is obvious that any point x' in a open disc $V_\delta(z(t^*))$ can be connected by a smooth curve to the center x. By adjoining the smooth curve from x to x' to the curve from w to x, we find a piecewise smooth curve from w to x', proving that $x' \in \Omega_1$, i.e. Ω_1 is open. Likewise, for any $y \in \Omega_2$, assume for contradiction that for every $\delta > 0$ such that $V_\delta(z(t^*)) \subset \Omega$, we can find a $y' \in \Omega_1$. Using similar argument, pick one such δ, there exists a piecewise smooth curve from w to y, contradicting with our selection of y. Hence Ω_2 must be open as well.
Still by openness of Ω, there exists a $V_\delta(w) \subset \Omega$. Therefore Ω_1 is not empty since $V_\delta(w) \subset \Omega_1$. By definition of connectedness, we conclude that $\Omega_2 = \emptyset$.

— End —