1.

Inverse function:

$f: A \to B$, bijection

$f^{-1}: B \to A$, $f^{-1}(y) = x$ where $f(x) = y$

Verification:

Piecewise defined:

1° when $x \in [0,1] \cap \mathbb{Q}$

$f(x) = x$

$f(f(x)) = f(x) = x$

2° when $x \notin [0,1] \cap \mathbb{Q}$ (or $x \in [0,1] \setminus \mathbb{Q}$]

Set subtraction

$f(x) = 1 - x$

$f(f(x)) = 1 - (1-x) = x$

$\Downarrow$

Thus for all $x \in [0,1]$

$f(f(x)) = x \iff f(f(x)) = i(x)$

a) Any flaws in the proof? What should be further clarified?

We should state the irrationality of $(1-x)$ when $x \notin \mathbb{Q}$.

b) In general, if the function $f(x)$ $[0,1]$ is defined as

$\llcorner$ as the whole set.

$$f(x) = \begin{cases} x & x \in A \\ 1-x & x \in A^c \end{cases}$$

and we still have $f(f(x)) = i(x)$

What should they satisfy?

We note that this function can be rewritten as

$$f(x) = \begin{cases} i(x) & x \in A \\ 1-x & x \in A^c. \end{cases}$$

Note: the range of $i(x)$ is identical to its domain.

If we want f to be invertible (i.e. having an inverse)

f should be <u>bijective</u>.

We denote the range of $f(x)$, $x \in A^c$ as $f(A^c)$

Injectivity: $f(A^c) \subset A^c$

Surjectivity: $A^c \subset f(A^c)$

$\Rightarrow$ <u>$f(A^c) = A^c$</u>

We have already shown $1-x$ is a self-inverse function

i.e. $1-(1-x) = x$.

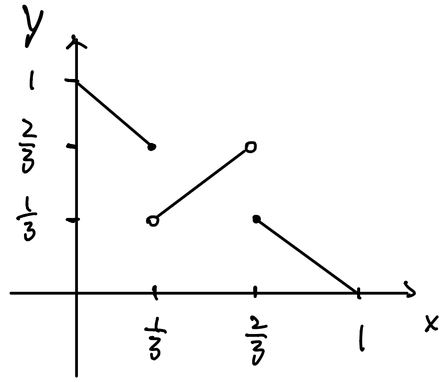
So we only need $f(A^c) = A^c$

We give an example: $A = (\frac{1}{3}, \frac{2}{3})$

$A^c = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

<u>Easy to verify</u>: if $x \in A^c$, $1-x \in A^c$

the graph of the function:



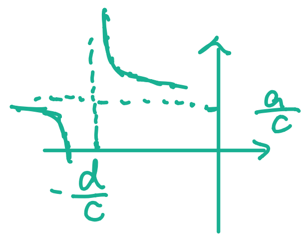
c) What is so special about Self-inverse functions?

What does its graph look like?

Wikipedia: Involution.

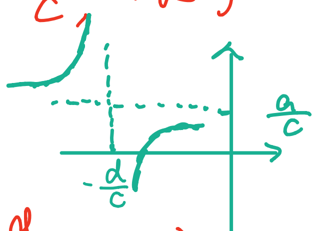
2. $f(x) = \dfrac{ax+b}{cx+d} = \dfrac{\frac{a}{c}(cx+d) + (b-\frac{ad}{c})}{(cx+d)}$ $(c>0)$ (Shifted) Reciprocal fxn (Mult. inv fxn)

$= \dfrac{a}{c} + \dfrac{b-\frac{ad}{c}}{cx+d}$ — Constant — Linear



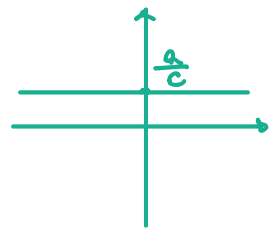
① If: $b - \frac{ad}{c} > 0$

Decreasing on $(-\infty, -\frac{d}{c})$ and $(-\frac{d}{c}, +\infty)$



② If $b - \frac{ad}{c} < 0$

Increasing on $(-\infty, -\frac{d}{c})$ and $(-\frac{d}{c}, +\infty)$



③ If $b - \frac{ad}{c} = 0 \iff bc = ad \iff \frac{a}{c} = \frac{b}{d}$

This ratio could be simplified to be $\boxed{\frac{a}{c}}$, a constant

Find the inverse of f

$y = \dfrac{ax+b}{cx+d} \Rightarrow cxy + dy = ax + b$

$\Rightarrow cxy - ax = b - dy$

$\Rightarrow x = \dfrac{-dy+b}{cy-a}$

$\therefore f^{-1}(x) = \dfrac{-dx+b}{cx-a} \Rightarrow a = -d$

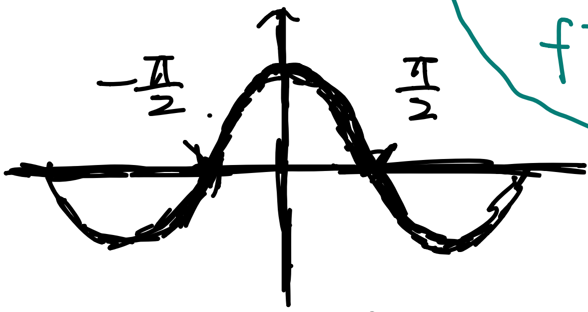
Situation ③
Bijection ✗
↳ No inverse.

Situation ① & ②
Bijection ✓
$f: \mathbb{R} \setminus \{-\frac{d}{c}\} \to \mathbb{R} \setminus \{\frac{a}{c}\}$

$f^{-1}: \mathbb{R}\setminus\{\frac{a}{c}\} \to \mathbb{R}\setminus\{-\frac{d}{c}\}$

3

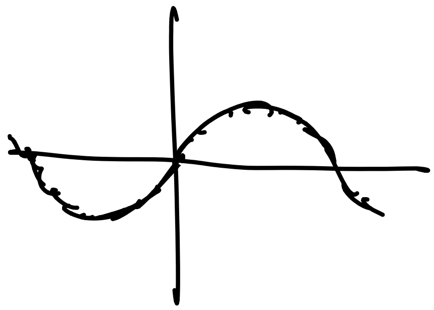
1) $\cos x^2 \geq 0$



$x^2 \in [0, \frac{\pi}{2}] \cup [2k\pi - \frac{\pi}{2}, 2k\pi + \frac{\pi}{2}]$, $k \in \mathbb{N}_+$

$x \in [-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}] \cup [\sqrt{\frac{(4k-1)\pi}{2}}, \sqrt{\frac{(4k+1)\pi}{2}}] \cup [-\sqrt{\frac{(4k+1)\pi}{2}}, -\sqrt{\frac{(4k-1)\pi}{2}}]$

2) $\sin\frac{\pi}{x} > 0$



$\frac{\pi}{x} \in (2k\pi, (2k+1)\pi)$, $k \in \mathbb{Z}$.

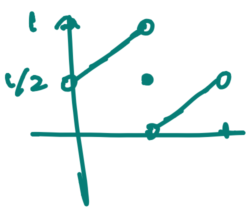
$\Rightarrow \frac{1}{x} \in (2k, 2k+1)$, $k \in \mathbb{Z}$.

$\Rightarrow \frac{1}{x} \in (2k, 2k+1) \cup (-2k, -2k+1) \cup (0,1)$, $k \in \mathbb{N}_+$

$\Rightarrow x \in (\frac{1}{2k+1}, \frac{1}{2k}) \cup (\frac{-1}{2k+1}, \frac{-1}{2k}) \cup (1, +\infty)$, $k \in \mathbb{N}_+$

4. $D: (0,1)$

Construct ?



Bijective (1-1 Correspondence) $\Longleftrightarrow$ Invertible

We use contradiction to prove this.

Assume: $A^c \not\subset f(A^c)$, i.e. $\exists z \in A^c$, $z \notin f(A^c)$

Recall <u>Surjectivity</u>: for every $y \in [0,1]$, $\exists\, x$ s.t. $f(x) = y$

1) if $y \in A$ $f(y) = y$ $\checkmark$

2) if $y \in A^c$:

Since $z \in A^c$, $z \notin f(A^c)$

$\therefore \forall x \in [0,1]$, $f(x) \neq z$.

We have a contradiction

$\therefore A^c \subset f(A^c)$