Recap

- ## Continuity Def'n

  1) ε-δ
  2) $\underbrace{\lim_{x\to c} f(x) = f(c)}$ — Equ. by lim def'n

  Provided c is an interior point of D

  Q: Why?
  A: To ensure one-sided limits well-def'd.

- ## One-sided continuity

- ## Properties:

  $f+g$, $f-g$, $f\cdot g$, f/g all continuous ($g(c)\neq 0$)

- ## Limits of continuous fxns

  If $\lim_{x\to c} f(x) = b$, g continuous at b

  $$\lim_{x\to c} g(f(x)) = g(b) = g\left(\lim_{x\to c} f(x)\right)$$

Remark

1° We can substitute $\lim_{x\to c}$ to $\lim_{x\to c^+}$ or $\lim_{x\to c^-}$

2° Compositions of continuous functions are continuous.

- ## Inverses of cts fxns are cts.

- ## Technique:

  If $\lim_{x\to c} \frac{f(x)}{g(x)} = 1$, $\Rightarrow \lim_{x\to c} f(x)h(x) = \lim_{x\to c} g(x)h(x)$

- Discontinuities

1. Removable : $\lim_{x\to c}$ exists
2. Jump : $\lim_{x\to c^+} \neq \lim_{x\to c^-}$
3. Essential : $\lim_{x\to c^+}$ or $\lim_{x\to c^-}$ does not exist.

- Cts fxns & Bddness

  - If f is a continuous fxn def'd on $[a,b]$. then f is bdd on $[a,b]$

If interested, check Heine-Borel Theorem

Remark

1° Th'm false in general if function not cts.

What if function with only (finite) removable/jump discontinuities?

Can we discard this?

2° Th'm false if interval not closed/unbdd.

Example?

Global Extrema

f cts on $[a,b]$, then f has a global max/min on $[a,b]$

IVT

If f cts on $[a,b]$. $\forall y_0 \in [f(a), f(b)]$

$\exists c \in [a,b]$, $f(c) = y_0$

Problem 1 (Computation Technique)

$$\cdot \lim_{x\to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$= \lim_{x\to 0} \frac{\tan x\,(1-\cos x)}{\sin^3 x}$$

$$= \lim_{x\to 0} \frac{\tan x}{x}\cdot\left(\frac{x}{\sin x}\right)^3 \frac{1-\cos x}{\frac{1}{2}x^2}\cdot\frac{\frac{1}{2}x^3}{x^3}$$

$$= \frac{1}{2}$$

- Remember the discussion ?

Why can't we do the following ?

$$\ldots = \lim_{x\to 0} \frac{x\cdot\frac{\tan x}{x} - x\,\frac{\sin x}{x}}{\frac{\sin^3 x}{x^3}\cdot x^3}$$

$$= \frac{\lim\limits_{x\to 0} \frac{\tan x}{x} - \lim\limits_{x\to 0} \frac{\sin x}{x}}{\lim\limits_{x\to 0} \frac{\sin^3 x}{x^3}\, x^2}$$

$$= \frac{1 - 1}{\lim\limits_{x\to 0} x^3}$$

$$= \frac{0}{\lim\limits_{x\to 0} x^3}$$

$$= 0$$

Why we can substitute $\tan x$ to x in products but not sums?

Because when $x \to 0$, $\tan x = x + o_1(x)$, where $o(x)$ is a function where $\lim\limits_{x\to 0} \frac{o_1(x)}{x} = 0$

Similarly, when $x \to 0$, $1-\cos x = \frac{1}{2}x^2 + o_2(x^2)$, $\sin(x) = x + o_3(x)$

$$\lim_{x\to 0} \frac{o_2(x^2)}{x^2} = 0 \quad , \quad \lim_{x\to 0} \frac{o_3(x)}{x} = 0$$

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} = \lim_{x\to 0} \frac{o_1(x) - o_3(x)}{x^3}$$

→ We don't know the relationship between $o_1(x)$, $o_3(x)$ and x^3.

$$\parallel$$

$$\lim_{x\to 0} \frac{\tan x(1-\cos x)}{x^3} = \lim_{x\to 0} \frac{[x+o_1(x)][\frac{1}{2}x^2 + o_2(x^2)]}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2}x^3 + \boxed{x \cdot o_2(x^2)} + \boxed{\frac{1}{2}x^2 \cdot o_1(x)} + \boxed{o_1(x) \cdot o_2(x^2)}}{x^3}$$

<u>Think</u>: Why these three terms are $o(x^3)$?

$$o(f(x)) : \lim_{x\to 0} \frac{o(f(x))}{f(x)} = 0$$

$$= \frac{1}{2}$$

- $$\lim_{x\to 0} \frac{\tan x - \sin x}{(\sqrt[3]{1+x^2}-1)(\sqrt{1+\sin x}-1)}$$

Lemma 1

$$\lim_{x\to 0} \frac{(1+g(x))^{\frac{1}{n}}-1}{\frac{1}{n}[g(x)]} = 1 \quad \text{if} \quad \lim_{x\to 0} g(x) = 0$$

$(n\in\mathbb{Z})$

Finite

By $(a^n-1) = (a-1)(a^{n-1}+\dots+1)$

$$(1+g(x))^{\frac{1}{n}}-1 = \frac{\overbrace{[1+g(x)]-1}^{g(x)}}{\underbrace{[1+g(x)]^{\frac{n-1}{n}}+\dots+1}_{\sum_{i=0}^{n-1}[1+g(x)]^{\frac{i}{n}}}}$$

$\because \lim_{x\to 0} g(x) = 0 \Rightarrow \lim_{x\to 0} 1+g(x) = 1$

$\Rightarrow \lim_{x\to 0} [1+g(x)]^{\frac{i}{n}} = 1$

$$\lim_{x\to 0} \frac{[1+g(x)]^{\frac{1}{n}}-1}{\frac{1}{n}\cdot g(x)} = \lim_{x\to 0} \frac{\frac{1}{\sum_{i=0}^{n-1}[1+g(x)]^{\frac{i}{n}}}}{\frac{1}{n}} = 1$$

Lemma 2

$$\lim_{x\to 0} \frac{(1+g(x))^\alpha - 1}{\alpha \cdot g(x)} = 1 \quad \text{if} \quad \lim_{x\to 0} g(x) = 0$$

If interested, check Taylor expansion

Pf

By Newton's Binomial Theorem (Proof too advanced)

$$(1+g(x))^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} [g(x)]^k$$

where $\binom{\alpha}{k}$ is def'd by $\dfrac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$

$$\therefore \lim_{x\to 0} \frac{(1+g(x))^\alpha - 1}{\alpha \cdot g(x)} = \lim_{x\to 0} \frac{\alpha \cdot g(x) + \frac{\alpha(\alpha-1)}{2}[g(x)]^2 + \dots}{\alpha \cdot g(x)}$$

$$= \lim_{x\to 0}\left[1 + \sum_{k=2}^{\infty} \binom{\alpha}{k} \underbrace{(g(x))^{k-1}}_{g(x)\to 0}\right]$$

$$= 1$$

Using the lemma

$$\lim_{x\to 0} \frac{\tan x - \sin x}{(\sqrt[3]{1+x^2} - 1)(\sqrt{1+\sin x} - 1)} = \lim_{x\to 0} \frac{\frac{1}{2}x^3}{\frac{1}{3}x^2 \cdot \frac{1}{2}\sin x}$$

$$= 3$$

Problem 2

By Composition of cts fxns, this is obvious.

$h(x) = \sqrt{x}$ $\quad g(x) = x - x^2$: Cts

$f = h \circ g$

- Sketch of Proof:

$\forall x_0 \in (0,1)$, $|x - x_0| < \delta$

$$f(x) - f(x_0) = \sqrt{x - x^2} - \sqrt{x_0 - x_0^2} = \frac{(x - x^2) - (x_0 - x_0^2)}{\sqrt{x - x^2} + \sqrt{x_0 - x_0^2}}$$

$$f(x) - f(x_0) < \frac{3|x - x_0|}{\sqrt{x_0 - x_0^2}} < \frac{3\delta}{\sqrt{x_0 - x_0^2}}$$

$|x - x_0| + |x^2 - x_0^2|$

$= |(x - x_0)(x + x_0)|$

$< |2(x - x_0)|$

Pf

$\forall x_0 \in (0,1)$, $\forall \varepsilon > 0$

Pick $\delta = \min\left(x_0,\ 1 - x_0,\ \frac{\sqrt{x_0 - x_0^2} \cdot \varepsilon}{3}\right)$

why?

$\forall x$ s.t. $|x - x_0| < \delta$

$$|f(x) - f(x_0)| = \left|\sqrt{x - x^2} - \sqrt{x_0 - x_0^2}\right|$$

$$\leq \frac{|x - x_0| + |x^2 - x_0^2|}{\sqrt{x - x^2} + \sqrt{x_0 - x_0^2}}$$

$$\leq \frac{3|x - x_0|}{\sqrt{x_0 - x_0^2}}$$

$$\leq \varepsilon$$

Any steps left?

* ENDPOINTS!

If $x_0 = 0$ $\quad \forall \varepsilon > 0$

Pick $\delta = \varepsilon^2$

$\forall\ 0 < x < \delta$

$|f(x) - f(0)| = \sqrt{x(1-x)} < \sqrt{x} < \sqrt{\delta} = \varepsilon$

By symmetry (Any symmetry you see?)

Cts at 1.

Problem 3

- Find $\lim_{x \to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right)$

$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\downarrow$

Cts by ? Inv of cts still cts

$$\lim_{x \to c} g(f(x)) = g\left[\lim_{x \to c}(f(x))\right]$$

Valid. $\begin{cases} \lim_{x \to c} f(x) \text{ exists: } \lim_{x \to 1} \frac{1-x}{1-x^2} = \frac{1}{2} \\ g \text{ cts} \end{cases}$

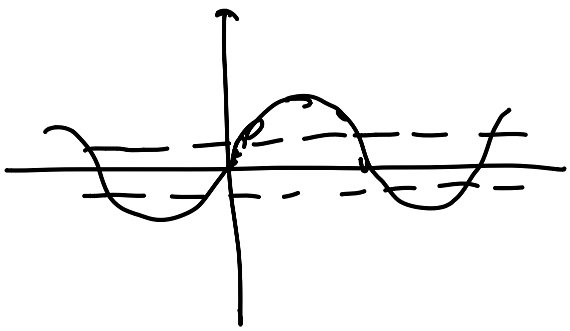
$\therefore \lim_{x \to 1} \sin^{-1}\left(\frac{1-x}{1-x^2}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Problem 4

Find all discontinuities

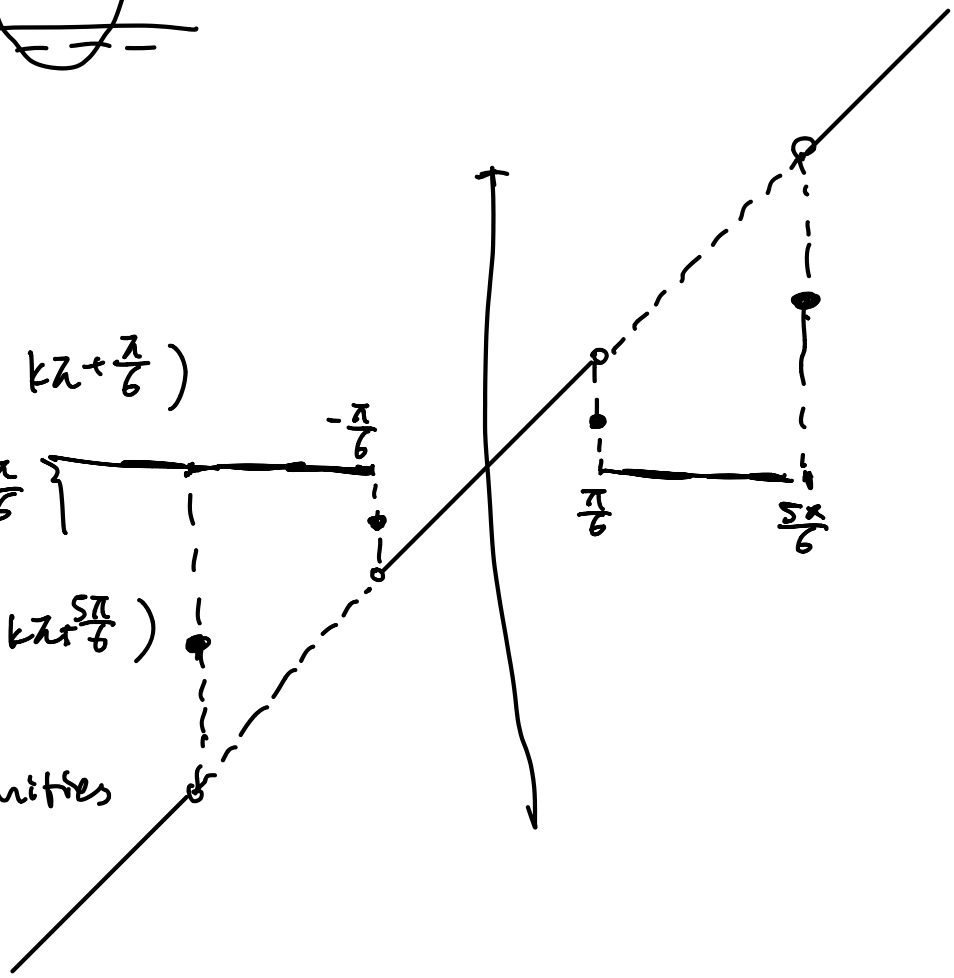
$$\cdot\ y = \lim_{n\to\infty} \frac{x}{1+(2\sin x)^{2n}} = \frac{x}{1+\lim\limits_{n\to\infty}(2\sin x)^{2n}}$$

Note that $$(2\sin x)^{2n} = \begin{cases} 0 & x \in \left(k\pi-\frac{\pi}{6},\ k\pi+\frac{\pi}{6}\right) \\ 1 & x \in \left\{k\pi\pm\frac{\pi}{6}\right\} \\ \infty & x \in \left(k\pi+\frac{\pi}{6},\ k\pi+\frac{5\pi}{6}\right) \end{cases}$$



A Quick Plot.

$$y = \begin{cases} x & x \in \left(k\pi-\frac{\pi}{6},\ k\pi+\frac{\pi}{6}\right) \\ \frac{x}{2} & x \in \left\{k\pi\pm\frac{\pi}{6}\right\} \\ 0 & x \in \left(k\pi+\frac{\pi}{6},\ k\pi+\frac{5\pi}{6}\right) \end{cases}$$



$\therefore$ All the discontinuities are at $\left\{k\pi\pm\frac{\pi}{6}\right\}$

At $\tilde{x} = k\pi+\frac{\pi}{6}$,

$$\lim_{x\to\tilde{x}^-} y = y, \quad \lim_{x\to\tilde{x}^+} y = 0$$

$\tilde{x} = k\pi-\frac{\pi}{6}$

$$\lim_{x\to\tilde{x}^-} y = 0, \quad \lim_{x\to\tilde{x}^+} y = y$$

- $y = \frac{1}{1-e^{\frac{x}{1-x}}}$

First Consider

$f(x) = 1 - e^{\frac{x}{1-x}}$; $x \to 1^-$, $e^{\frac{x}{1-x}} \to \infty$, $1 - e^{\frac{x}{1-x}} \to -\infty$

$x \to 1^+$, $e^{\frac{x}{1-x}} \to 0$, $1 - e^{\frac{x}{1-x}} \to 1$

$\therefore \lim_{x \to 1^-} y = 0$, $\lim_{x = 1^+} y = 1$

Then consider when $1 - e^{\frac{x}{1-x}} = 0 \Leftrightarrow x = 0$

$x \to 0^-$, $e^{\frac{x}{1-x}} \to 1^-$, $1 - e^{\frac{x}{1-x}} \to 0^+$. $y \to \infty$

$x \to 0^+$, $e^{\frac{x}{1-x}} \to 1^+$, $1 - e^{\frac{x}{1-x}} \to 0^-$, $y \to -\infty$