Errata: • All $\frac{\partial y}{\partial x}$ in today's notes should be $\frac{dy}{dx}$

Differentiation (at $x=c$)

↳ Find the derivative at $x=c$

↳ Define $g_c(x) = \frac{f(x)-f(c)}{x-c}$ → find its limit at c

* If limit exist
Then this number is
$f'(c)$

- $g_c(x) \neq f'(x)$ But $\lim_{x\to c} g_c(x) = f'(c)$
- $g_c(x)$ has a "discontinuity" at $x=c$ (Not quite, actually not def'd)
  if removable, then $f(x)$ dif'ble at c.
  - Example: Jump ? $f(x) = |x|$
  - Example: Essential ? $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x = 0 \end{cases}$
    ↳ Discontinuous

  What else: $f(x) = x^{\frac{1}{3}}$ (Vertical tg)

- Derivative discontinuous ?

  Consider $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

- If $x=0$

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin(1/x)$$

$$-x \leq x \sin(1/x) \leq x$$

- Limit-order theorem

$$\lim_{x\to 0} x \sin 1/x = 0 \implies f'(0) = 0$$

- If $x \neq 0$

$$f'(x) = 2x \cdot \sin(1/x) + x^2 \cdot \left(-\frac{1}{x^2}\right) \cos(1/x)$$

$$= 2x \sin(1/x) - \cos(1/x)$$

- Recall def'n of continuity

$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall |x-c| < \delta,\ |f(x) - f(c)| < \varepsilon$

Let $\varepsilon = \frac{1}{2}$ $\quad \forall \delta > 0 \quad \exists n \in \mathbb{N}$ s.t. $n > \frac{1}{2\delta\pi}$

i.e. $\frac{1}{2n\pi} < \delta$

$$\left|f'\left(\frac{1}{2n\pi}\right)\right| = |0 - 1| = 1 > \frac{1}{2}$$

$\therefore$ Not cts.

- At a pt $x = c$

Continuity $\underset{\not\Rightarrow}{\Leftarrow}$ Differentiability $\underset{\not\Rightarrow}{\Leftarrow}$ Continuously Differentiable. ($f'(x)$ cts)

Example (at $= 0$)

$f(x)$	Cts ?	Diff'ble ?	Cts'ly Diff'ble?
$\sin\frac{1}{x}$; $f(0)=0$	×	×	×
$x\sin\frac{1}{x}$; $f(0)=0$	✓	×	×
$x^2\sin\frac{1}{x}$; $f(0)=0$	✓	✓	×
$x^k\sin\frac{1}{x}$; $f(0)=0$; $k \geq 3$	?	?	?

- Implicit function differentiation

$$F(x,y) = 0$$

Since y dependent on x as well.

$$\Rightarrow \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = 0$$

A contour, might not be a curve

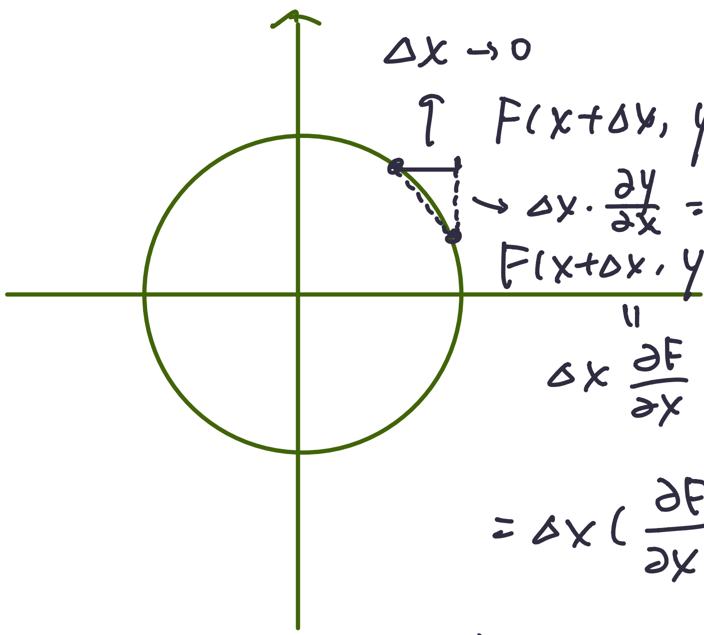
Change in F due to change in (unit) x with y fixed

Change in F due to change in (unit) y with x fixed

Change in y due to change in (unit) x

- "∂" pronounced partial

$F(x,y) = x^2+y^2-25 = 0$ Circle w/ radius 5



$\Delta x \to 0$

$F(x+\Delta x, y) = \Delta x \cdot \frac{\partial F}{\partial x}$

$\to \Delta x \cdot \frac{\partial y}{\partial x} = \Delta y$

$$F(x+\Delta x, y+\Delta y) = 0$$

$$\Delta x \frac{\partial F}{\partial x} + \Delta y \cdot \frac{\partial F}{\partial y}$$

$$= \Delta x \left(\frac{\partial F}{\partial x} + \frac{\partial y}{\partial x}\frac{\partial F}{\partial y}\right) = 0$$

$$\Delta x \to 0 \Rightarrow \Delta x \neq 0$$

$$\Rightarrow \boxtimes$$

Explicit fxn (Validation)

$y = f(x) \Leftrightarrow F(x,y) = y - f(x) = 0$

$\big\downarrow$ Implicit Differentiation

$-f'(x) + 1 \cdot \frac{\partial y}{\partial x} = 0 \Rightarrow \frac{\partial y}{\partial x} = f'(x)$

Example $y = x^{\sqrt{x}}$

Let $F(x,y) = y^{(1/\sqrt{x})} - x = 0$

$0 = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial x}$

$= \left(\frac{1}{\sqrt{x}} y^{(1/\sqrt{x})-1}\right)\frac{\partial y}{\partial x} + \left(-\frac{1}{2}x^{-\frac{3}{2}} \ln y \cdot y^{1/\sqrt{x}} - 1\right)$

$= \frac{1}{\sqrt{x}} \frac{x}{y} \frac{\partial y}{\partial x} + \left(-\frac{1}{2}x^{-\frac{3}{2}} \sqrt{x} \ln x \cdot x - 1\right)$

$= \frac{\sqrt{x}}{x^{\sqrt{x}}} \frac{\partial y}{\partial x} - \frac{1}{2} \ln x - 1$

$\Rightarrow y' = \frac{\partial y}{\partial x} = x^{\sqrt{x} - \frac{1}{2}} \left(\frac{1}{2} \ln x + 1\right)$

Direct consequence: Differentiation of Inverse fcns

If $f: D \to X$ has a inverse $f^{-1}: X \to D$

if f diff'ble, $y = f^{-1}(x)$, then

$$(f^{-1})'(x) = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

$F(x,y) = y - f(x) = 0$

$$\begin{cases} \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = 0 \\ \frac{\partial F}{\partial y} - \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial y} = 0 \end{cases} \Rightarrow \begin{aligned} \frac{\partial y}{\partial x} &= \frac{-\partial F / \partial x}{\partial F / \partial y} \\ \frac{\partial x}{\partial y} &= \frac{-\partial F / \partial y}{\partial F / \partial x} \end{aligned}$$

$$\therefore \frac{\partial y}{\partial x} = 1 / \frac{\partial x}{\partial y}$$

(Natural, even seems trivial when you think of the fraction form)

Interpretation: if change in 1 unit of x changes a unit(s) of y, changing 1 unit of y will change $\frac{1}{a}$ unit(s) of x.

Geometric view: Inverse function: Sym wrt $y=x$

Derivative $\longleftrightarrow$ Slope of tg

Any tg line $y = kx \xrightarrow{\text{Sym wrt } y=x} y = \frac{1}{k}x$.

Problem 2

$f(x) = x^2 D(x)$, $D(x)$: Dirichlet : $f(x) = \begin{cases} 1 , & \mathbb{Q} \\ 0 , & \mathbb{I} \end{cases}$

Prove that $f(x)$ differentiable at only 1 point, $x=0$

We will prove 1) Diff'ble at $x=0$

2) NOT diff'ble at $x \neq 0$

1) If $x=0$

$$f'(x) = \lim_{x \to 0} \left(\frac{x^2 D(x) - 0}{x} \right)$$

$$= \lim_{x \to 0} x D(x)$$

$\because$ $0 \le x D(x) \le x$

Limit - order

$$0 = \lim_{x \to 0} 0 \le \lim_{x \to 0} x D(x) \le \lim_{x \to 0} x = 0$$

$\therefore$ $\lim_{x \to 0} x D(x) = 0 \Rightarrow f'(0) = 0$

2) If $x \neq 0$

a) if $x \in \mathbb{Q}$, aim to show $\lim_{y \to x} \frac{f(y) - f(x)}{y - x}$ doesn't exist

$\because$ $f(x) \neq 0$, suffices to find an irrational seq cvgs to x

(why?)

$f(y)=0$ if $y \in \mathbb{I}$

$\Rightarrow$ Then $\lim\limits_{y \to x} \frac{0-f(x)}{y-x}$ doesn't exist.

$\because \forall \delta > 0, \quad \exists n \in \mathbb{N}$ s.t $\frac{1}{n} < \delta$

(Archimedean Property)

Consider $y_n = \left(\frac{\pi}{4n} + x\right) \in \mathbb{I}$ $\qquad y_n \to x$

$$\Rightarrow \lim_{n \to \infty} \frac{f(y_n) - f(x)}{y-x}$$

$$= \lim_{n \to \infty} -\left(\frac{4n}{\pi}\right) x^2$$

Unbdd.

= NOT exist.

By fxn & seq limits

fxn limit at c exists if every seq cvgs to c cvgs

b) $x \notin \mathbb{Q}$

Similar argument with a)

Suffices to find a rational sequence cvgs to x.

Consider the decimal expansion of x.

$$x = x_0 . x_1 x_2 \ldots = \sum_{i=0}^{\infty} x_i 10^{-i}$$

Let $\{y_n\} = \sum_{i=0}^{n} x_i 10^{-i}$

$$|y_n - x| < 10^{-n+1} \qquad y_n \to x$$

$$\Leftrightarrow \lim_{n\to\infty} \frac{f(y_n) - f(x)}{y_n - x}$$

$$> \lim_{n\to\infty} y_n^2 \cdot 10^{n-1}$$

$$= x^2 \lim_{n\to\infty} 10^{n-1} \nearrow \text{Unbdd}$$

$$= \text{NOT exist.}$$

$\therefore$ $f(x)$ only has a limit at $x=0$

Exercise

1) Show that $g(x) = x D(x)$ is continuous only at $x=0$

2) Show that $R(x) : \begin{cases} \frac{1}{q}, & x = p/q, \ p<q, \ \gcd(p,q)=1 \\ 0, & x \in \mathbb{I}. \end{cases}$

is not diff'ble. (Hint 1: Only need to consider $x \in \mathbb{I}$)
(Hint 2: Decimal expansion)

Remark

(At $x=0$)

$f(x)$	Cts ?	Diff'ble ?	Cts'ly Diff'ble ?
$D(x)$; $f(0)=0$	×	×	×
$xD(x)$; $f(0)=0$	✓	×	×
$x^2 D(x)$; $f(0)=0$	✓	✓	×

Cts at only 1 pt

Diff'ble at only 1 pt