# Recap:

- Sequence limit, $N$-$\varepsilon$ definition
- Properties of Sequence limits
  (Preserved under arithmetic operations)
- Bounded Sequence, Thm of Bdd Seq's.
- Squeeze Theorem
- Monotone Convergence Thm. (MCT)
- Important limits
  - $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$
  - $\lim_{n\to\infty} \sqrt[n]{n} = 1$

- Prove $\lim_{n\to\infty} \frac{2n}{n+1} = 2$

- Prove $\lim_{n\to\infty} \frac{n^2+1}{4n-4n} = \frac{1}{4}$

( $N-\varepsilon$ def'n )

(a)

- Want to show :

  $\exists$ some $N$. when $n > N$ .

  $$\left|\frac{2n}{n+1} - 2\right| = \frac{2}{n+1} < \varepsilon$$

- Want to find $N$.

  $\because \frac{2}{n+1} < \frac{2}{n}$

  Pick $N = \lceil \frac{2}{\varepsilon} \rceil$

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Pf For arbitrary $\varepsilon > 0$

Pick $N = \lceil \frac{2}{\varepsilon} \rceil$ $\quad n > N \geq \frac{2}{\varepsilon}$

when $n > N$

$$\left|\frac{2n}{n+1} - 2\right| = \frac{2}{n+1} < \frac{2}{n} < \frac{2}{2/\varepsilon} = \varepsilon$$

Where $\lceil a \rceil$ is the ceil fxn : least integer that is g/eq to a ; $\lfloor a \rfloor$ is the floor fxn: greatest int that is l/eq to a.

(b)

- Want to show

$\exists$ some $N$, when $n > N$

$$\left| \frac{n^2+1}{4n-4n} - \frac{1}{4} \right| = \frac{n+1}{4n^2-4n} < \varepsilon$$

- Want to find $N$

$$\because \frac{n+1}{4n^2-4n} \leq \frac{2n-2}{4n^2-4n} = \frac{1}{2n} \quad \text{if } n \geq 3$$

Note: $n$ goes to infinity, we can make certain assumptions to make life easier

pick $N = \max\left(\left\lceil \frac{1}{2\varepsilon} \right\rceil, 3\right)$, $n > N$

If $\left\lceil \frac{1}{2\varepsilon} \right\rceil < 3$

$$\frac{1}{2\varepsilon} \leq 2 \Rightarrow \varepsilon \geq \frac{1}{4}$$

$$\frac{n+1}{4n^2-4n} \leq \frac{1}{2n} < \frac{1}{6} < \frac{1}{4} \leq \varepsilon$$

(By $n > N \geq 3$)

If $\lceil \frac{1}{2\varepsilon} \rceil \geq 3$

$$\frac{n+1}{4n^2-4n} < \frac{1}{2n} < \frac{1}{2 \cdot \frac{1}{2\varepsilon}} = \varepsilon$$

Formal Proof: Left as an exercise

Remark $1^\circ$ Proving using $N-\varepsilon$ is just about properly chosing an $N$.

Remark $2^\circ$ The choice of "$N$" is not unique. It is related to how you bound the absolute difference between $x_n$ and $\lim_{n\to\infty} x_n$.

Q2 Find the limits :

(Squeeze Thm)

- $a_n = \frac{n}{2^n}$
- $b_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n}$

(a) By observation, we guess $\lim_{n\to\infty} a_n = 0$

Aim: find $\{x_n\}, \{y_n\}$ s.t.

$$x_k \le a_k \le y_k \quad \forall k \ge N$$

and $\lim_{n\to\infty} x_n = 0$, $\lim_{n\to\infty} y_n = 0$

Left : Immediate: $x_n = 0$

Right : Give a upper bound of $a_n$. (not so loose!)

Why? Criteria?

$\Updownarrow$

Upper bounding $\frac{n}{2^n}$

$\Updownarrow$

Lower bounding $2^n$

Attempt #1 : Binomial Exp'n.

$$2^n = (1+1)^n$$
$$= \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} + \binom{n}{n}$$

When $n \geq 5$

Strict ineq?

$$2^n > \binom{n}{1} + \binom{n}{2} + \binom{n}{n-2} = n^2$$

$$\therefore \frac{n}{2^n} < \frac{n}{n^2} = \frac{1}{n}, \quad \lim_{n\to\infty} \frac{1}{n} = 0$$

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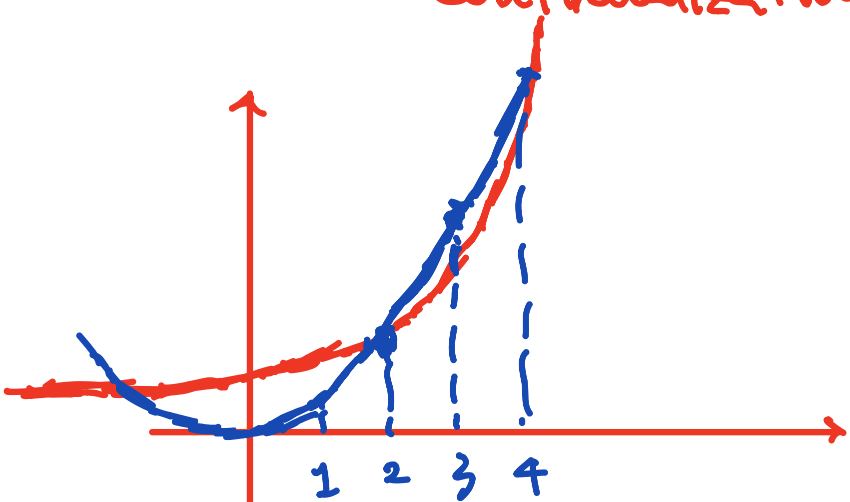
Attempt #2

Recall <u>Sequence</u> : $f: \mathbb{N} \to \mathbb{R}$

$\downarrow$ Expand its domain

$\mathbb{R}$

"Continualization"




$$\boxed{2^n \geq n^2 \text{ when } n \geq 4}$$

Increase Rate: (Asymptotically $(n\to\infty)$) $(\alpha > 1)$

$$c < \log_\alpha n < \sqrt[\alpha]{n} < n < p(n) < \alpha^n < n! < n^n$$

**Remark** The ratio of any term in the (asymptotic) inequality and its subsequent terms goes to 0 when n goes to $\infty$

**Exercise** Prove

1. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

2. $\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = 0$

Hint:

1. $\frac{n!}{n^n} = \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n-1}{n} < \frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n}$
$= \frac{1}{n}$ (Squeeze)

2. Use $\frac{n}{2^n} \to 0$ when $n \to \infty$

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# Some other method: Optional

$$\sum_{n=1}^{N} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \cdots + \frac{N-1}{2^{N-1}} + \frac{N}{2^N}$$

$$2\sum_{n=1}^{N} \frac{n}{2^n} = 1 + \frac{2}{2} + \frac{3}{4} + \cdots + \frac{N}{2^{N-1}}$$

$$\Downarrow$$

$$\sum_{n=1}^{N} \frac{n}{2^n} = 2\sum_{n=1}^{N} \frac{n}{2^n} - \sum_{n=1}^{N} \frac{n}{2^n} = \sum_{n=0}^{N-1} \frac{1}{2^n} - \frac{N}{2^N}$$

Consider $\{c_n\}$, $c_n = \sum_{k=1}^{n} \frac{k}{2^k}$

- An upper bound of $c_n$ is 2 (Geometric)
- $c_n$ is monotonically $\nearrow$

- By MCT. $\{c_n\}$ is conv.

Actually. $\lim_{n\to\infty} c_n = 2$

$\exists N$. s.t. $n > N$ $\forall \varepsilon > 0$

$|c_n - c| < \varepsilon$ (Def)

$\because c_n < c_{n+1} \le c$ (Monotonicity)

$\therefore |a_{n+1} - 0| = a_{n+1} = |c_{n+1} - c_n| < \varepsilon$

<u>Exercise</u> :

- Assume $\lim_{n\to\infty} c_n = c$

Use this result and <u>Def</u> of concergence,

prove $\lim_{n\to\infty} \frac{n}{2^n} = 0$

<u>Note</u> Sometimes what we need is not the limit itself. but merely its existence.

(By MCT)

b) Construct a helper sequence $\{c_n\}$

$$c_n = \frac{2}{3} \cdot \frac{4}{5} \cdot \dots \cdot \frac{2n}{2n+1}$$

Now we check :

- $b_n, c_n$ convergent : (MCT) ☑
- $b_n < c_n$ : $\because \frac{1}{2} < \frac{2}{3}, \frac{3}{4} < \frac{4}{5}, \dots, \frac{2n-1}{2n} < \frac{2n}{2n+1}$

$$\therefore \prod_{i=1}^{n} \frac{2i-1}{2i} < \prod_{i=1}^{n} \frac{2i}{2i+1} \quad ☑$$

$$b_n \cdot c_n = \frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{2n-1}{2n} \cdot \frac{2n}{2n+1} = \frac{1}{2n+1} \quad ☑$$

$$b_n \cdot b_n < b_n \cdot c_n \Rightarrow b_n < \sqrt{b_n c_n} \quad \text{(Pos seq)}$$
$$= \sqrt{\frac{1}{2n+1}}$$

By Squeeze Th'm

$0 < b_n < \sqrt{\frac{1}{2n+1}}$ . $\lim_{n\to\infty} 0 = 0$ , $\lim_{n\to\infty} \sqrt{\frac{1}{2n+1}} = 0$

(Please Verify it!)

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Q3 Find

- $\lim_{n\to\infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$

- $\lim_{n\to\infty} n(\sqrt{n^2+1} - \sqrt{n^2-1})$

(a)

- Divide by $3^{n+1}$ in both the numerator and denominator :

$$\ldots = \frac{-\frac{1}{2}\cdot(-\frac{2}{3})^{n+1} + \frac{1}{3}}{(-\frac{2}{3})^{n+1} + 1}$$

- Verify they are convergent

$$\lim \frac{\cdots}{\cdots} = \frac{\lim \cdots}{\lim \cdots} = \frac{1}{3}$$

Exercise

- $$\lim_{n\to\infty} \frac{2^n + (-3)^n}{2^{n+1} + 3^{n+1}}$$

- $$\lim_{n\to\infty} \frac{3^n + 4^n}{2^n + 5^n}$$

Does these limits exist? What pattern do you discover?

Remark The greatest base dominates.

(b)

- Note that $(\sqrt{n^2+1} - \sqrt{n^2-1})(\sqrt{n^2+1} + \sqrt{n^2-1}) = 2$

$$\text{``} = \frac{2 \cdot n}{\sqrt{n^2+1} + \sqrt{n^2-1}}$$

- Divide by $n$ in both the num. & denom.

$$\text{``} = \frac{2}{\sqrt{1+\frac{1}{n^2}} + \sqrt{1-\frac{1}{n^2}}}$$

Conv? 1

$$\lim \frac{a_n}{b_n + c_n} = \frac{\lim a_n}{\lim b_n + \lim c_n}$$

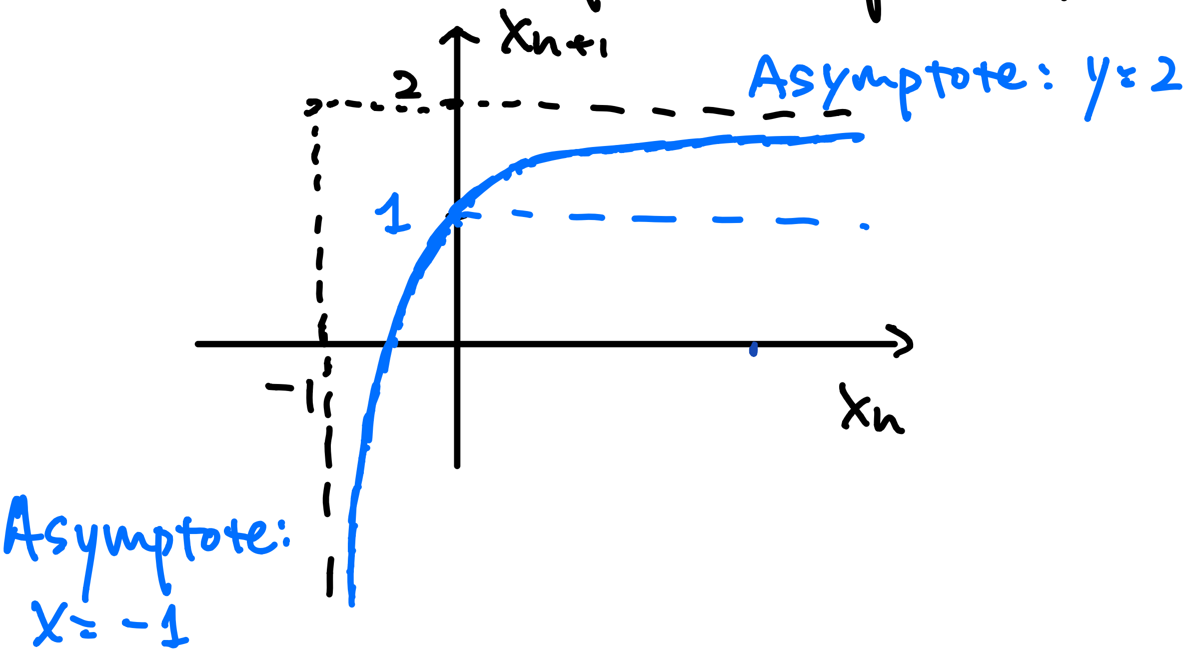
Q4 Unconventional : Recursively Def'd
Not explicit.

- Some intuition :
  - View $x_{n+1}$ as a fxn of $x_n$
  - Continualization

Remark We sometimes also do vice versa, i.e. discretize continuous fxns.

- $x_{n+1} = 2 - \frac{1}{1+x_n} \Rightarrow x_{n+1} = f(x_n)$

(Last tut, "Shifted" reciprocal fxn)




Observations

1° $x_n > 0 \Rightarrow x_{n+1} \in (1,2)$ $n \in \mathbb{N}$ (Graph)

Alternatively: $\frac{1}{1+x_n} \in (0,1)$ $2-\frac{1}{1+x_n} \in (1,2)$

2° Monotonically increasing on $(0, +\infty)$

$\Leftrightarrow$ (If $a > b \Rightarrow f(a) > f(b)$ )

3° $\exists$ a point $\gamma$ s.t

if $x_k = \gamma \Rightarrow x_{k+1} = \gamma, \dots, x_{k+m} = \gamma$

$m \in \mathbb{N}_+$

. By the recursive relationship
Calculate the $\gamma$

$$\gamma = 2 - \frac{1}{1+\gamma} \Rightarrow \gamma = \frac{1+\sqrt{5}}{2} \text{ (+ve)}$$

GOLDEN RATIO

$\Downarrow$ $x_1 > \gamma \Rightarrow x_n > \gamma$
$x_1 < \gamma \Rightarrow x_n < \gamma$ $\forall n$ $\boxed{\text{WHY ?}}$

$x_1 > \gamma$

$f(x_1) > \gamma$

$f(f(x_1)) > \gamma$ ....

Informal Proof (Sketch):

1) Consider $x_1 = \gamma \Rightarrow x_n = \gamma \quad \forall n$ (3°)

Constant sequence: $\lim_{n\to\infty} \{x_n\} = \gamma$

2) Consider $x_1 < \gamma \xRightarrow{(2°, 3°)} x_n < \gamma$ (U.B)

$$x_{n+1} - x_n = \frac{1 + x_n - x_n^2}{1 + x_n} > 0 \qquad (\nearrow)$$

By MCT, convergent.

(3) $x_1 > \gamma$ (Exercise)

- By $$x_{n+1} = \frac{x_n}{1+x_n} + 1$$

$$\Rightarrow \lim x_{n+1} = \lim\left(\frac{x_n}{1+x_n} + 1\right)$$

$$\Leftrightarrow \lim x_n = \frac{\lim x_n}{1 + \lim x_n} + 1$$

$$\Leftrightarrow (\lim x_n)^2 - \lim x_n - 1 = 0$$

$$\Leftrightarrow \lim x_n = \gamma = \frac{1+\sqrt{5}}{2}$$

- A more elegant way to show monotonicity

$$x_{n+1} = 1 + \frac{x_n}{1+x_n} \qquad x_n = 1 + \frac{x_{n-1}}{1+x_{n-1}}$$

$$x_{n+1} - x_n = \frac{x_n}{1+x_n} - \frac{x_{n-1}}{1+x_{n-1}}$$

$$= \frac{x_n - x_{n-1}}{(1+x_n)(1+x_{n-1})}$$

Since $(1+x_n)(1+x_{n-1}) > 0$

$\therefore$ $x_{n+1} - x_n$ and $x_n - x_{n-1}$ have the same sign.

<u>Remark</u> The following statements are equivalent regarding monotonicity of sequence $\{a_n\}$

(a) $\{a_n\}$ is a strictly monotonic sequence

(b) $\forall k \in N_+ \quad (a_{k+2} - a_{k+1})(a_{k+1} - a_k) > 0$