Recap.

- Function Limits & Properties < Thm of Bdd Fxns / Squeeze / Limit prsv order
  - Analogy: Seq is special fxn
- One sided limit $\lim = L$ iff $\lim^+ = \lim^- = L$
- Function limits & Sequence limits

$$\lim_{x\to c} f = L \Rightarrow \lim_{n\to\infty} f(x_n) = L \text{ for all } x_n \to c \text{ when } n\to\infty$$

- Two important limits
  - $\lim_{x\to 0} \frac{\sin x}{x} = 1$ Consequence of Squeeze
  - $\lim_{x\to 0} (1+x)^{1/x} = e$ Consequence of Thm of Fxn & Sequence Limits
- Limits involving infinity
  - Asymptotes < horizontal / vertical

Problem 1

Prove that $\lim_{x\to 2} \frac{x^2-x-2}{2x^2-3x-2} = \frac{3}{5}$ by using the definition of function limits.

$$\frac{x^2-x-2}{2x^2-3x-2} = \frac{(x+1)(x-2)}{(2x+1)(x-2)}$$

$x \to 2$: x lies in a punctured/deleted neighborhood of 2.

$0 < |x-2| < \varepsilon$

$$\left|\frac{x+1}{2x+1} - \frac{3}{5}\right| = \left|\frac{5x+5-6x-3}{10x+5}\right|$$

$$= \left|\frac{2-x}{10x+5}\right|$$

Sequence limit proof : Find N suf. large → $|f(x)-a|<\varepsilon$

Function limit proof : Find δ suf. small

$\frac{2-x}{10x+5}$: Is this function bdd?

How do we bd it in $(-\varepsilon, \varepsilon)$?

Recall last time: (Seq. limit)

We are interested in the limiting behavior when $n \to \infty$ $\Rightarrow$ We give constraints to n to easily bound a_n

$n \geq 4$ $2^n \geq n^2$ $\frac{n}{2^n} \leq \frac{n}{n^2} = \frac{1}{n}$

Can we give constraints to x this time?

How? Which particular part are we interested in?

What do we want to find: (δ) ?

We let $|x-2|<1 \Rightarrow x \in (1,3) \Rightarrow |10x+5| \geq 15$

When $|x-2| \leq 15\varepsilon$ $\quad \frac{|x-2|}{|10x+5|} \leq \frac{15\varepsilon}{15} = \varepsilon$

Problem 2

$\therefore$ Pick $\delta = \min(15\varepsilon, 1)$

Find the following limits.

- $\lim_{x\to 0} \frac{a^{x^2} - b^{x^2}}{(a^x - b^x)^2}$ (a>0, b>0),
- $\lim_{x\to \frac{\pi}{3}} \frac{\sin(x-\frac{\pi}{3})}{1-2\cos x}$.

lemma 2

Limits of Continuous Functions

Theorem

Let f and g be functions, such that $g \circ f$ is defined. Suppose that g is continuous at a point b and $\lim_{x\to c} f(x) = b$. Then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

Proof Let $\epsilon > 0$ be given. Since g is continuous at b, there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \epsilon \quad \text{whenever} \quad |y - b| < \delta_1.$$

Since $\lim_{x\to c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \epsilon$ whenever $|x - c| < \delta$. From the definition of limit, this proves that $\lim_{x\to c} g(f(x)) = g(b)$. ■

lemma 1

- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

let $y = a^x - 1$ $\quad x\to 0, y \to 0$

$x = \log_a(1+y)$

$x = \frac{\ln(1+y)}{\ln a}$

$\lim_{x\to 0} \frac{y}{x} = \lim_{y\to 0} \frac{\ln a}{\frac{1}{y}\ln(1+y)}$

$= \frac{\ln a}{\lim_{y\to 0} \ln(1+y)^{\frac{1}{y}}}$

$= \frac{\ln a}{\ln \lim_{y\to 0} (1+y)^{\frac{1}{y}}}$

$= \frac{\ln a}{\ln e}$

$= \ln a$

$\frac{a^x - b^x}{x} = \ln a - \ln b$

Why ?

$\frac{a^{x^2} - b^{x^2}}{x^2} = ?$

Hint

$\frac{a^{x^2} - b^{x^2}}{(a^x - b^x)^2} = \frac{a^{x^2} - b^{x^2}}{x^2} \left(\frac{x}{a^x - b^x}\right)^2$

Trig identity

$\lim_{y\to 0} \frac{\sin y}{1 - 2\cos(y + \frac{\pi}{3})}$

$\frac{\sin y}{1 - 2\cos y \cos\frac{\pi}{3} + 2\sin y \sin\frac{\pi}{3}}$

$= \frac{\sin y}{1 - \cos y + \sqrt{3}\sin y}$

$= \frac{1}{\frac{1-\cos y}{\sin y} + \sqrt{3}}$

$= \frac{1}{\frac{2\sin^2 y/2}{\sin y} + \sqrt{3}}$

$$\frac{\frac{2\sin^2 y/2}{(y/2)^2}}{\frac{\sin y}{(y/2)^2}} = 2 \cdot 4y = 8y \to 0$$

when $y \to 0$

Problem 3

Prove that

$$\lim_{x\to 0} \cos\frac{1}{x}$$

does not exist by using the theorem of function and sequence limits.

Theorem (Theorem of Function and Sequence Limits, I)

Let f be a function defined on some open interval D containing some fixed number c, except possibly at c, and let $L \in \mathbb{R}$. Then the following are equivalent:

(a) $\lim_{x\to c} f(x) = L$.

(b) *For every sequence (x_n) in $D \setminus \{c\}$ converging to c, we have* $\lim_{n\to\infty} f(x_n) = L$.

Note: $\cos\frac{1}{x}$ Bdd.

a) $\Leftrightarrow$ (b)

$\neg(a) \Leftrightarrow \neg(b)$

- The negation of (b)?

$\forall \to \exists$

$\exists$ 2 seqs (x_n), (y_n) converging to 0, $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$

let $x_n = \frac{1}{2n\pi}$ $\quad y_n = \frac{1}{(2n+1)\pi}$

LIMIT NOT EXIST (AS A FINITE NUMBER)

- UNBOUNDED
- TWO SIDED LIMIT $\neq$
- OSCILLATING

Problem 4

Prove that

$$\lim_{x\to 0} \frac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1}$$

does not exist by using the theorem of equaled one-sided limits.

Note

$(2^{\frac{1}{x}}) \cdot 2^x \neq 2^2$!

Let $\frac{1}{x_0} = y$

$$\lim_{x\to 0^+} \frac{2^{\frac{1}{x}}+1}{2^{\frac{1}{x}}-1} = \lim_{y\to +\infty} \frac{2^y+1}{2^y-1}$$

$$= 1 + \lim_{y\to +\infty} \frac{2}{2^y-1}$$

$$= 1$$

$$\lim_{x\to 0^-} \frac{2^y+1}{2^y-1} = 1 + \lim_{y\to -\infty} \frac{2}{2^y-1}$$

$$= 1 + (-2)$$

$$= -1$$

- Show that $\lim_{x\to x_0} [f(x)g(x)]$ may exist even though neither $\lim_{x\to x_0} f(x)$ nor $\lim_{x\to x_0} g(x)$ exists.

$f(x) : \sin\frac{1}{x}$

$g(x) : \frac{1}{\sin\frac{1}{x}}$

$x_0 = 0$

- Hw 2 Hint $\lim_{x \to \infty} x^{\frac{1}{x}}$

  Take logarithm, change variable

- Hw 5 Hint $\lim_{x \to 0^+} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$

  Why is the problem using different thm compared to Hw 4? What's their difference.

  What sequences will you pick?