- One sided derivatives
  - Compare w/ one sided limits
  - Diff'ble condition
- Th'm of Eq. 1-Sided Derivatives
- Diff'ble $\overset{\Rightarrow}{\cancel{\Leftarrow}}$ Cts
- Higher-Order derivatives

Notation: $f^{(n)}(x)$, $y^{(n)}$,

$\frac{d^n y}{dx^n}$, $\frac{d^n}{dx^n}[f(x)]$ Differential operator

- Differentials: $\Delta y \approx f'(x)\Delta x$
  - 1^{st}-order approximation (linear ~)

[App'n: Gradient Descent [Opt & ML])
Newton's Method [Alg'm]

- Local Extreme: Critical Pt $\to$ $f'(c)=0$ / $\nexists f'(c)$ $\times$ exist

Global Extreme: Boundaries / Critical Pts

- Rolle's Th'm: $f(a)=f(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c)=0$
- MVT : $\exists c$ in (a,b) s.t. $f'(c)=\frac{f(a)-f(b)}{a-b}$
- Monotonicity $\leftrightarrow$ 1^{st} Derivative

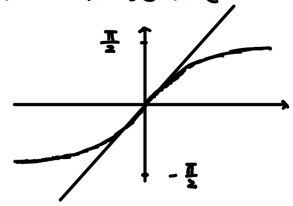
(- Concavity $\leftrightarrow$ 2^{nd} Derivative)

Prob 1

Prove $|\arctan a - \arctan b| \le |a-b|$

Intuition: Must be true

Graph:



- Derivative max: at $x=0$, $f'(x)>1$ elsewhere "flatter" than $y=x$.
- $y=x$ is a tangent.

Proof (MVT)

Let $f(x)=\arctan x$

$\forall a<b \in \mathbb{R}$, $\exists c \in (a,b)$

$\frac{1}{1+c^2}=f'(c)=\frac{f(b)-f(a)}{b-a}$

$\because 0<\frac{1}{1+c^2}\le 1$

$\therefore 0<f(a)-f(b)\le a-b$

$\therefore |\arctan a - \arctan b| \le |a-b|$

Q: How do we prove using monotonicity
(Hint: Use twice, different occasions)

Problem 2

Show that $f(x)=\sqrt{x}+\sqrt{1+x}-4$
has exactly 1 zero on $(0,+\infty)$

Intuition:

$\sqrt{x}\nearrow$, $\sqrt{1+x}\nearrow$, $f(x)$ Monotonic

At most cross the x-axis once.

$f(1)<0$ $f(4)>0$.

At least cross t x-axis once.

$\Rightarrow$ Cross once. i.e. 1 zero.

Q1: If f has more than 1 zero, how does it conflict with Rolle's Thm?

Q2: What technical detail should we add in the intuition part to add rigor?

Continuity

Q3: If instead, we establish the monotonicity by calculating $f'(x)$. Do we need the detail? why?

Problem 3

If $|f(w)-f(x)|\le |w-x|$ $\forall w,x$

f diff'ble, show that $-1\le f'(x)\le 1$ $\forall x$.

Proof 1 (Def'n)

Since $f'(x)=\lim_{w\to x_0}\frac{f(w)-f(x)}{w-x}$

$|f'(x)|=\left|\lim_{w\to x_0}\frac{f(w)-f(x)}{w-x}\right| \overset{①}{=} \lim_{w\to x_0}\left|\frac{f(w)-f(x)}{w-x}\right| \overset{②}{\le} 1$

How to justify

①: Let $g(x)=|x|$: Cts, Interchange ✓

②: Order limit Th'm *** Very Important

Attempt 2 (MVT)

$\forall x$. $\exists y<x<z$

s.t $f'(x)=\frac{f(y)-f(z)}{y-z}$

... Hold the horses!

Is this true?

If not, give a counterexample.

- $f(x)=x^3+x$ Take $x=0$

$f'(0)=1$ But $\forall y<0<z$.

$\frac{f(y)-f(z)}{y-z}=y^2+yz+z^2+1 \ge 1+\frac{3}{4}z^2>1$

- This does not affect the validity of the MVT though.

Still consider the case $f(x)=x^3$

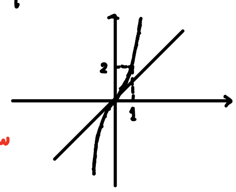
$y=-1$, $z=1$, $\exists x=\frac{1}{3}$ s.t.

$f'(x)=1=\frac{f(y)-f(z)}{y-z}$

- Caveat: The converse fails when $f''(x)=0$, or less abstractly, when concavity change

Our Example

Right side of x
y on top of tg
left side y below
tg.



Problem 4.

Prove that

$f(x)=\arctan\left(\frac{1+x}{1-x}\right)-\arctan x = \begin{cases}\frac{\pi}{4}, & x<1\\ -\frac{3\pi}{4}, & x>1\end{cases}$

Observations:

- $f(x)$ def'd on $\mathbb{R}\setminus\{1\}$
- Piecewise constant
  - ↳ Describing a constant function
    - ↳ $f'(x)=0$
- Recall identity

$\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}$

Fun properties combined with rational transformations

Proof.

Let $h(x)=\arctan x$ $g(x)=\frac{1+x}{1-x}$

$h'(x)=\frac{1}{1+x^2}$ $g'(x)=\frac{2}{(1-x)^2}$

$f'(x)=h'(g(x))g'(x)-h'(x)$

$=0$ (Verify it) $(x\ne 1)$

$\because \lim_{x\to 1^-}f(x)=?$

$\lim_{x\to 1^+}f(x)=?$

$\therefore f(x)=?$

Q: Why do we introduce one-side limits in the proof? Why can it show the behavior of the function?

Remark:

Expand $\tan(x+\frac{\pi}{4})$, the problem will be demystified.

$\tan(x+\frac{\pi}{4})=\frac{\sin(x+\frac{\pi}{4})}{\cos(x+\frac{\pi}{4})}=\frac{\sin x+\cos x}{\cos x-\sin x}$

$=\frac{\tan x+1}{1-\tan x}$

Now, apply the arctan function to both sides, what do you get?