or, equivalently, as

$$\begin{array}{ll} \text{minimize} & \|y\| \\ \text{subject to} & I_C(x) \le 0 \\ & x_0 - x = y \end{array}$$

where the variables are $x$ and $y$. The dual function of this problem is

$$\begin{aligned} g(z,\lambda) &= \inf_{x,y} \left(\|y\| + \lambda I_C(x) + z^T(x_0 - x - y)\right) \\ &= \begin{cases} z^T x_0 + \inf_x \left(-z^T x + I_C(x)\right) & \|z\|_* \le 1, \quad \lambda \ge 0 \\ -\infty & \text{otherwise} \end{cases} \\ &= \begin{cases} z^T x_0 - S_C(z) & \|z\|_* \le 1, \quad \lambda \ge 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

so we obtain the dual problem

$$\begin{array}{ll} \text{maximize} & z^T x_0 - S_C(z) \\ \text{subject to} & \|z\|_* \le 1. \end{array}$$

If $z$ is dual optimal with a positive objective value, then $z^T x_0 > z^T x$ for all $x \in C$, *i.e.*, $z$ defines a separating hyperplane.

## 8.2 Distance between sets

The distance between two sets $C$ and $D$, in a norm $\|\cdot\|$, is defined as

$$\mathbf{dist}(C,D) = \inf\{\|x - y\| \mid x \in C,\ y \in D\}.$$

The two sets $C$ and $D$ do not intersect if $\mathbf{dist}(C,D) > 0$. They intersect if $\mathbf{dist}(C,D) = 0$ and the infimum in the definition is attained (which is the case, for example, if the sets are closed and one of the sets is bounded).

The distance between sets can be expressed in terms of the distance between a point and a set,

$$\mathbf{dist}(C,D) = \mathbf{dist}(0, D - C),$$

so the results of the previous section can be applied. In this section, however, we derive results specifically for problems involving distance between sets. This allows us to exploit the structure of the set $C - D$, and makes the interpretation easier.

### 8.2.1 Computing the distance between convex sets

Suppose $C$ and $D$ are described by two sets of convex inequalities

$$C = \{x \mid f_i(x) \le 0,\ i = 1,\dots,m\}, \qquad D = \{x \mid g_i(x) \le 0,\ i = 1,\dots,p\}.$$

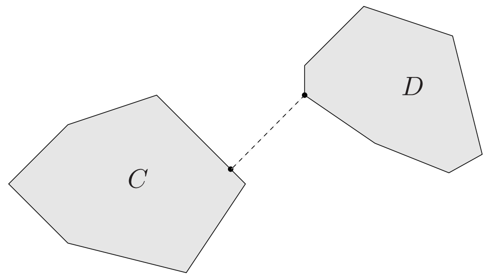


**Figure 8.2** Euclidean distance between polyhedra $C$ and $D$. The dashed line connects the two points in $C$ and $D$, respectively, that are closest to each other in Euclidean norm. These points can be found by solving a QP.

(We can include linear equalities, but exclude them here for simplicity.) We can find $\mathbf{dist}(C, D)$ by solving the convex optimization problem

$$\begin{array}{ll} \text{minimize} & \|x-y\| \\ \text{subject to} & f_i(x) \le 0, \quad i=1,\ldots,m \\ & g_i(y) \le 0, \quad i=1,\ldots,p. \end{array} \tag{8.3}$$

### Euclidean distance between polyhedra

Let $C$ and $D$ be two polyhedra described by the sets of linear inequalities $A_1 x \preceq b_1$ and $A_2 x \preceq b_2$, respectively. The distance between $C$ and $D$ is the distance between the closest pair of points, one in $C$ and the other in $D$, as illustrated in figure 8.2. The distance between them is the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & \|x-y\|_2 \\ \text{subject to} & A_1 x \preceq b_1 \\ & A_2 y \preceq b_2. \end{array} \tag{8.4}$$

We can square the objective to obtain an equivalent QP.

## 8.2.2 Separating convex sets

The dual of the problem (8.3) of finding the distance between two convex sets has an interesting geometric interpretation in terms of separating hyperplanes between the sets. We first express the problem in the following equivalent form:

$$\begin{array}{ll} \text{minimize} & \|w\| \\ \text{subject to} & f_i(x) \le 0, \quad i=1,\ldots,m \\ & g_i(y) \le 0, \quad i=1,\ldots,p \\ & x-y=w. \end{array} \tag{8.5}$$

The dual function is

$$g(\lambda, z, \mu) = \inf_{x,y,w} \left( \|w\| + \sum_{i=1}^m \lambda_i f_i(x) + \sum_{i=1}^p \mu_i g_i(y) + z^T(x-y-w) \right)$$

$$= \begin{cases} \inf_x \left(\sum_{i=1}^m \lambda_i f_i(x) + z^T x\right) + \inf_y \left(\sum_{i=1}^p \mu_i g_i(y) - z^T y\right) & \|z\|_* \leq 1 \\ -\infty & \text{otherwise,} \end{cases}$$

which results in the dual problem

$$\begin{array}{ll} \text{maximize} & \inf_x \left(\sum_{i=1}^m \lambda_i f_i(x) + z^T x\right) + \inf_y \left(\sum_{i=1}^p \mu_i g_i(y) - z^T y\right) \\ \text{subject to} & \|z\|_* \leq 1 \\ & \lambda \succeq 0, \quad \mu \succeq 0. \end{array} \tag{8.6}$$

We can interpret this geometrically as follows. If $\lambda$, $\mu$ are dual feasible with a positive objective value, then

$$\sum_{i=1}^m \lambda_i f_i(x) + z^T x + \sum_{i=1}^p \mu_i g_i(y) - z^T y > 0$$

for all $x$ and $y$. In particular, for $x \in C$ and $y \in D$, we have $z^T x - z^T y > 0$, so we see that $z$ defines a hyperplane that strictly separates $C$ and $D$.

Therefore, if strong duality holds between the two problems (8.5) and (8.6) (which is the case when (8.5) is strictly feasible), we can make the following conclusion. If the distance between the two sets is positive, then they can be strictly separated by a hyperplane.

#### Separating polyhedra

Applying these duality results to sets defined by linear inequalities $A_1 x \preceq b_1$ and $A_2 x \preceq b_2$, we find the dual problem

$$\begin{array}{ll} \text{maximize} & -b_1^T \lambda - b_2^T \mu \\ \text{subject to} & A_1^T \lambda + z = 0 \\ & A_2^T \mu - z = 0 \\ & \|z\|_* \leq 1 \\ & \lambda \succeq 0, \quad \mu \succeq 0. \end{array}$$

If $\lambda$, $\mu$, and $z$ are dual feasible, then for all $x \in C$, $y \in D$,

$$z^T x = -\lambda^T A_1 x \geq -\lambda^T b_1, \qquad z^T y = \mu^T A_2 x \leq \mu^T b_2,$$

and, if the dual objective value is positive,

$$z^T x - z^T y \geq -\lambda^T b_1 - \mu^T b_2 > 0,$$

*i.e.*, $z$ defines a separating hyperplane.

### 8.2.3 Distance and separation via indicator and support functions

The ideas described above in §8.2.1 and §8.2.2 can be expressed in a compact form using indicator and support functions. The problem of finding the distance between two convex sets can be posed as the convex problem

$$\begin{array}{ll} \text{minimize} & \|x - y\| \\ \text{subject to} & I_C(x) \leq 0 \\ & I_D(y) \leq 0, \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & \|w\| \\ \text{subject to} & I_C(x) \leq 0 \\ & I_D(y) \leq 0 \\ & x - y = w. \end{array}$$

The dual of this problem is

$$\begin{array}{ll} \text{maximize} & -S_C(-z) - S_D(z) \\ \text{subject to} & \|z\|_* \leq 1. \end{array}$$

If $z$ is dual feasible with a positive objective value, then $S_D(z) < -S_C(-z)$, *i.e.*,

$$\sup_{x \in D} z^T x < \inf_{x \in C} z^T x.$$

In other words, $z$ defines a hyperplane that strictly separates $C$ and $D$.

## 8.3 Euclidean distance and angle problems

Suppose $a_1, \dots, a_n$ is a set of vectors in $\mathbf{R}^n$, which we assume (for now) have known Euclidean lengths

$$l_1 = \|a_1\|_2, \quad \dots, \quad l_n = \|a_n\|_2.$$

We will refer to the set of vectors as a *configuration*, or, when they are independent, a *basis*. In this section we consider optimization problems involving various geometric properties of the configuration, such as the Euclidean distances between pairs of the vectors, the angles between pairs of the vectors, and various geometric measures of the conditioning of the basis.

### 8.3.1 Gram matrix and realizability

The lengths, distances, and angles can be expressed in terms of the *Gram matrix* associated with the vectors $a_1, \dots, a_n$, given by

$$G = A^T A, \qquad A = \left[ \begin{array}{ccc} a_1 & \cdots & a_n \end{array} \right],$$

so that $G_{ij} = a_i^T a_j$. The diagonal entries of $G$ are given by

$$G_{ii} = l_i^2, \quad i = 1, \dots, n,$$

which (for now) we assume are known and fixed. The distance $d_{ij}$ between $a_i$ and $a_j$ is

$$\begin{array}{rcl} d_{ij} & = & \|a_i - a_j\|_2 \\ & = & (l_i^2 + l_j^2 - 2a_i^T a_j)^{1/2} \\ & = & (l_i^2 + l_j^2 - 2G_{ij})^{1/2}. \end{array}$$