


# Bayesian Linear Regression

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# Reference

## Mathematics for Machine Learning:

<https://mml-book.com>

### Chapter 9

# Probabilistic Models

**Probabilistic model:** Model of the data is a probability distribution.

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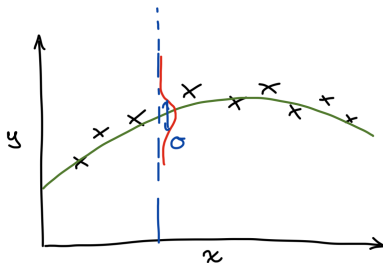
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# Probabilistic Models

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$$y_n = f(\mathbf{x}_n; \theta) + \epsilon_n \quad \epsilon_n \sim \mathcal{N}(0, \sigma^2)$$

We can now also estimate the **unpredictability** of our problem:



$$(\theta^*, \sigma^{2*}) = \operatorname{argmax}_{\theta, \sigma^2} \log p(\mathbf{y} | \theta, \sigma^2, X) \quad (1)$$

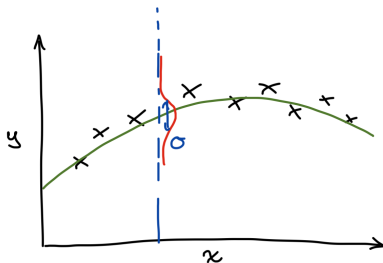
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Goes by many names... e.g. **aleatoric uncertainty**.

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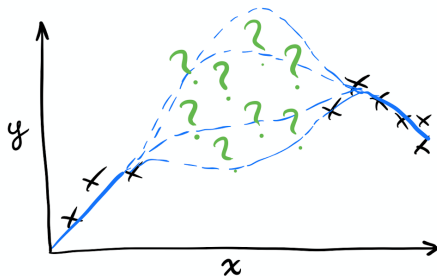


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# Uncertainty in Parameters/Function

Aren't we also uncertain when we have a lack of data?



This is uncertainty in the parameters that define the function!  
Also goes by many names... e.g. **epistemic uncertainty**.

# Quantifying Uncertainty with Bayesian Inference

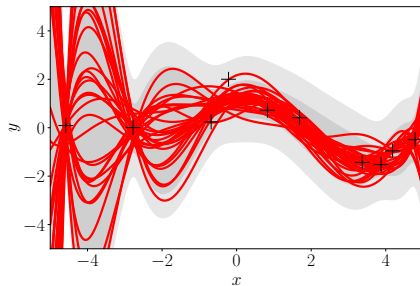
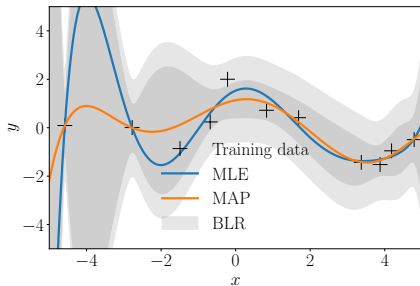
If we knew that for a series of problems, our parameters  $\theta$  were sampled from  $p(\theta)$ , then Bayes' rule would give us the probability distribution after observing our data  $\mathbf{y}$ :

$$\underbrace{p(\theta|\mathbf{y})}_{\text{posterior}} = \frac{\overbrace{p(\mathbf{y}|\theta)}^{\text{likelihood}} \overbrace{p(\theta)}^{\text{prior}}}{\underbrace{p(\mathbf{y})}_{\text{evidence}}} \quad (2)$$

- ▶ Allows us to quantify uncertainty in parameters  $\theta$ .
- ▶ Bayesian inference makes a leap of faith: Choose a prior and assume this is the correct one.
- ▶ Choosing priors is important  $\implies$  Probabilistic Inference (Spring).

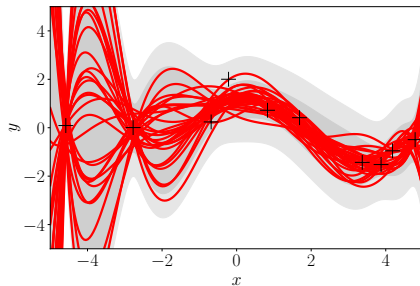
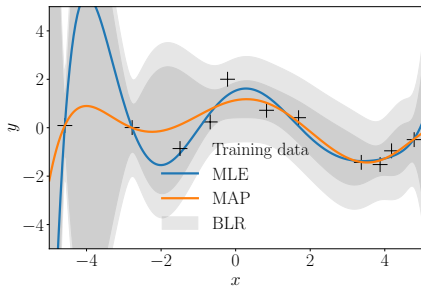


# Example



- ▶ Light-gray: uncertainty due to noise  
(aleatoric uncertainty / unpredictability)
- ▶ Dark-gray: uncertainty due to parameter uncertainty  
(epistemic uncertainty)

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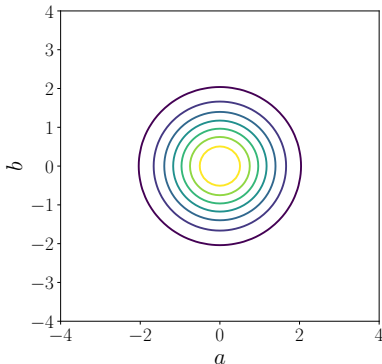


- ▶ Light-gray: uncertainty due to noise (aleatoric uncertainty / unpredictability)
- ▶ Dark-gray: uncertainty due to parameter uncertainty (epistemic uncertainty)
- ▶ Right: Plausible functions under the parameter distribution (every single parameter setting describes one function)

# Distribution over Functions

Consider a linear regression setting

$$y = f(x) + \epsilon = a + bx + \epsilon, \quad \epsilon \sim \mathcal{N}(0, \sigma_n^2)$$
$$p(a, b) = \mathcal{N}(\mathbf{0}, \mathbf{I})$$



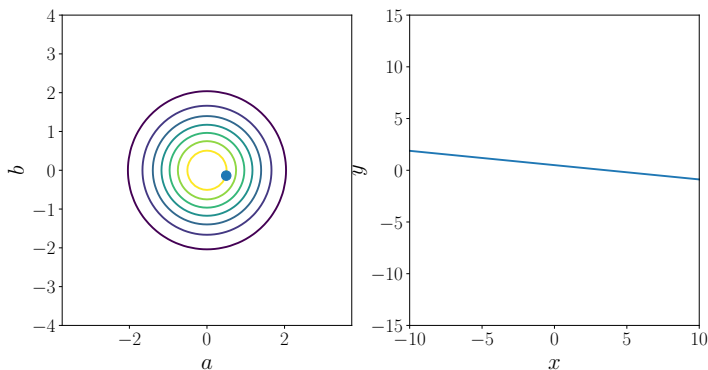
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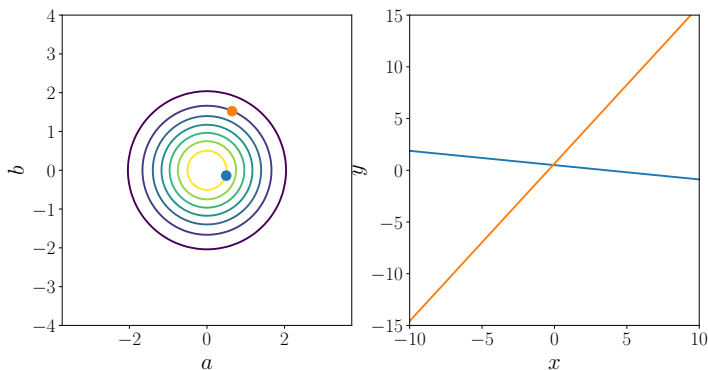
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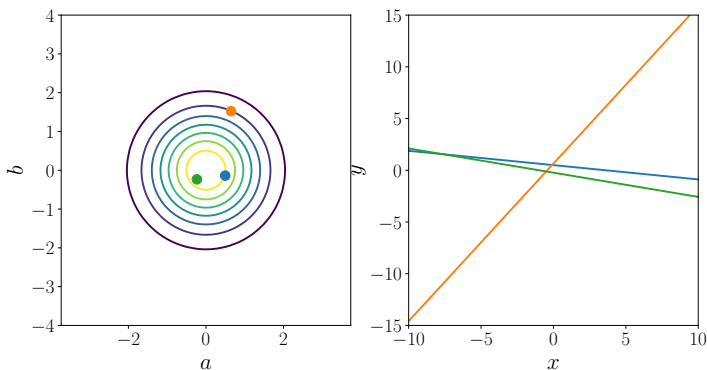
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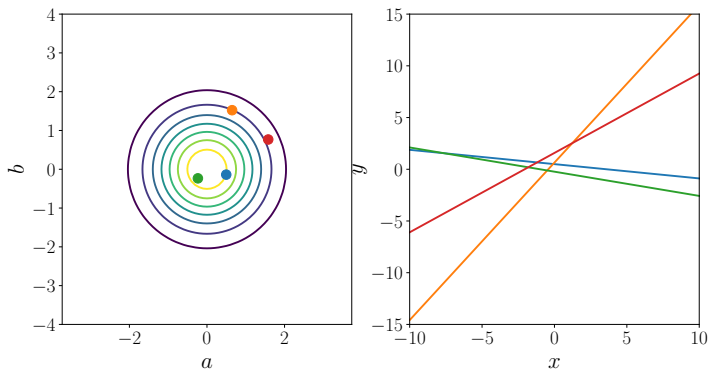
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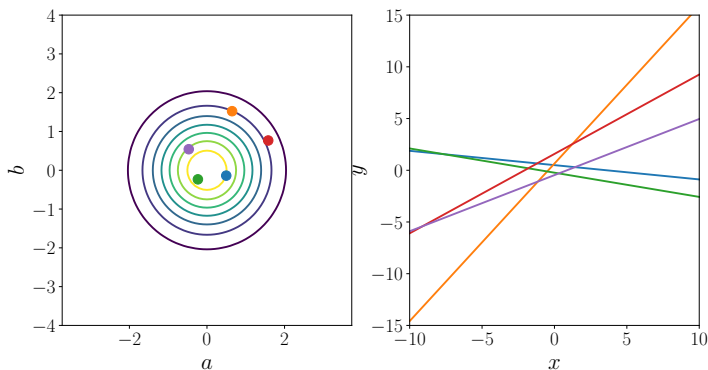
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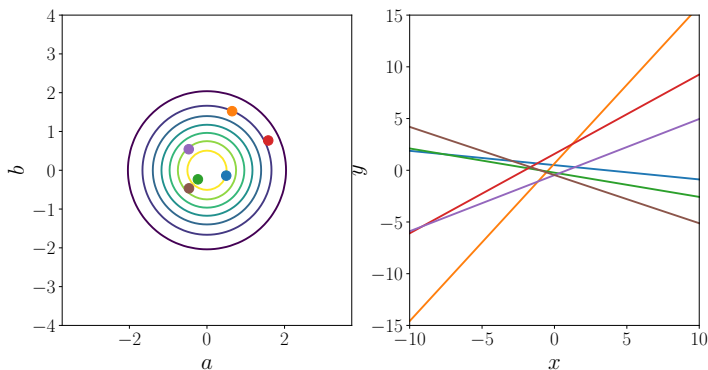
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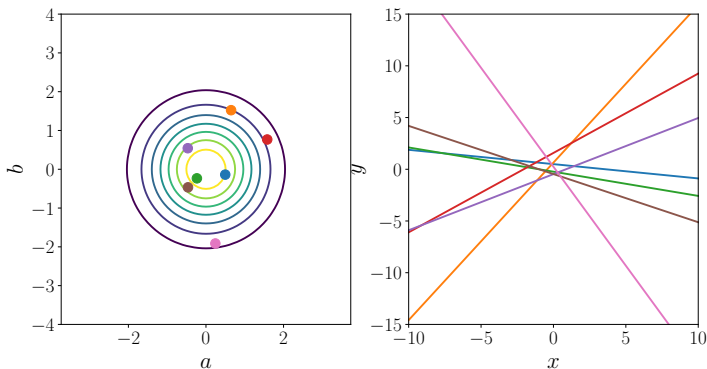
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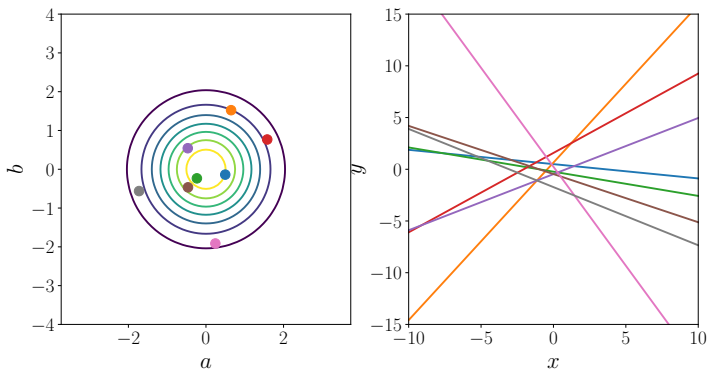
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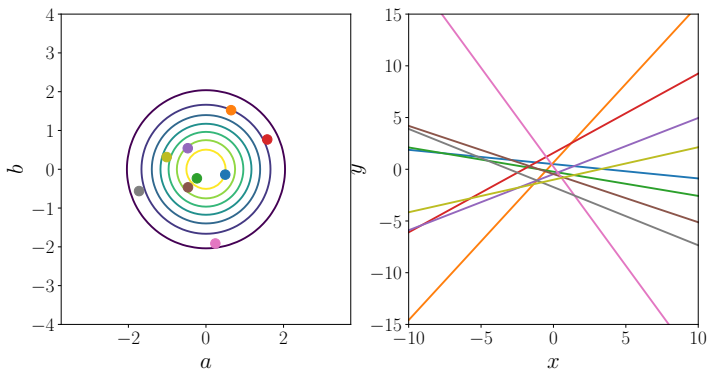
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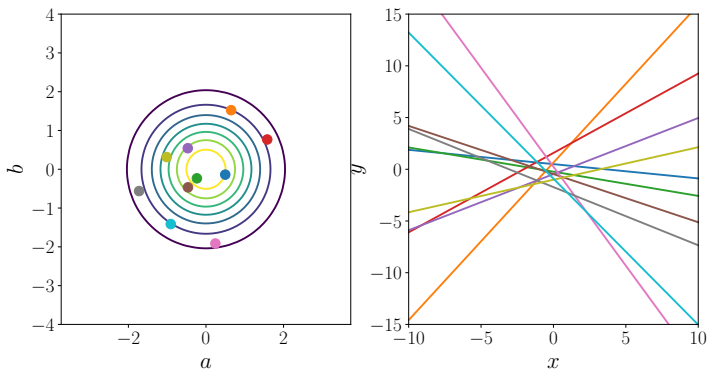
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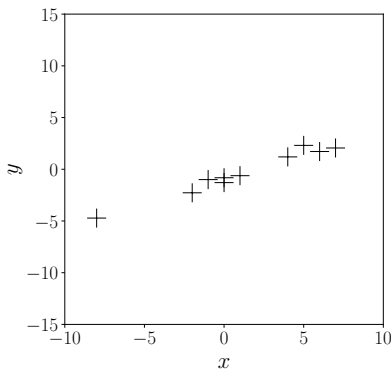
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$$\mathbf{X} = [x_1, \dots, x_N], \quad \mathbf{y} = [y_1, \dots, y_N] \quad \text{Training inputs/targets}$$



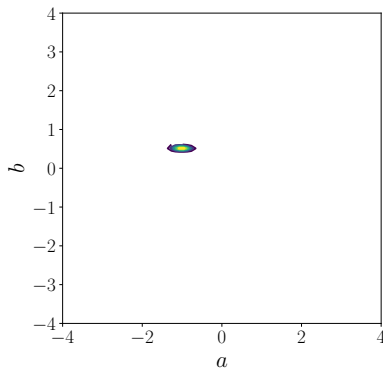
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$$p(a, b | \mathbf{X}, \mathbf{y}) = \mathcal{N}(\mathbf{m}_N, \mathbf{S}_N) \quad \text{Posterior}$$



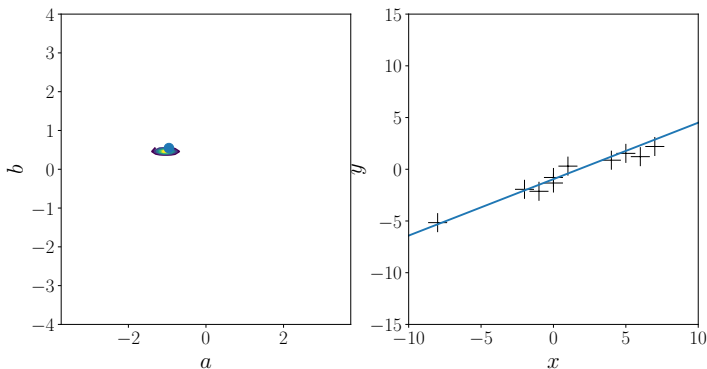
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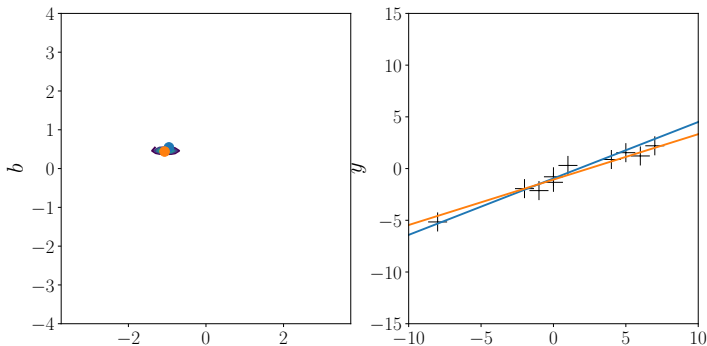
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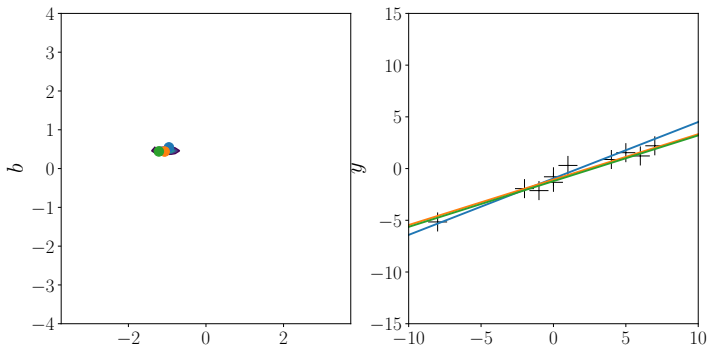
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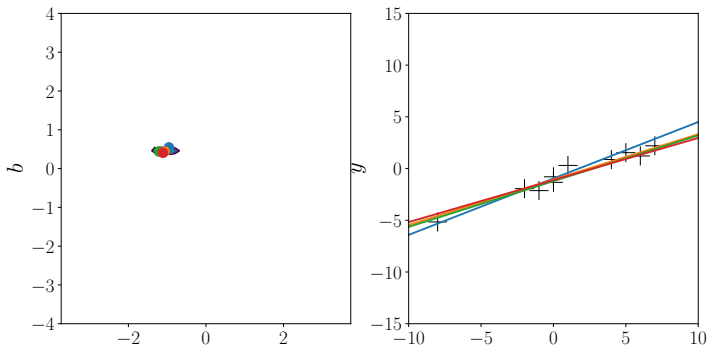
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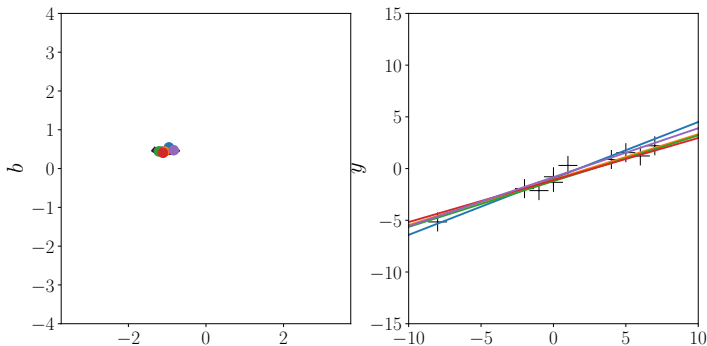
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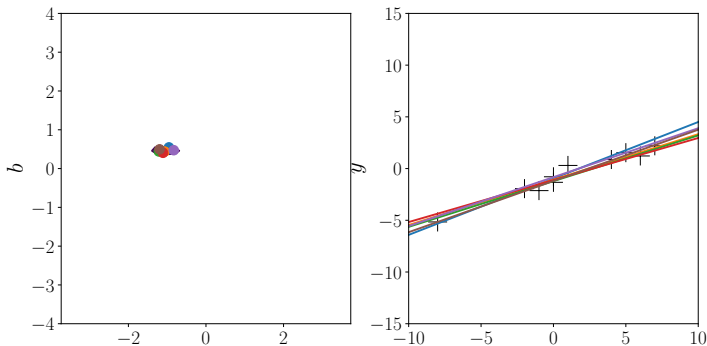
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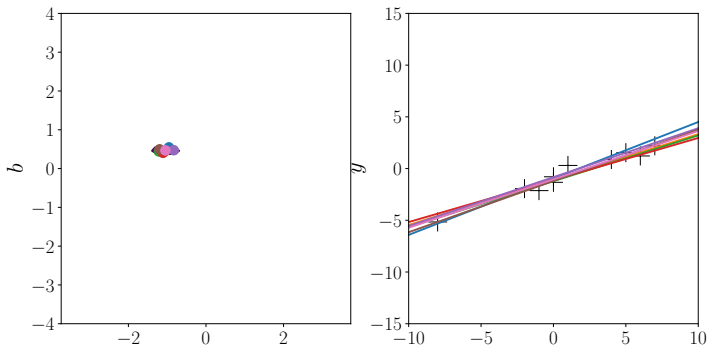
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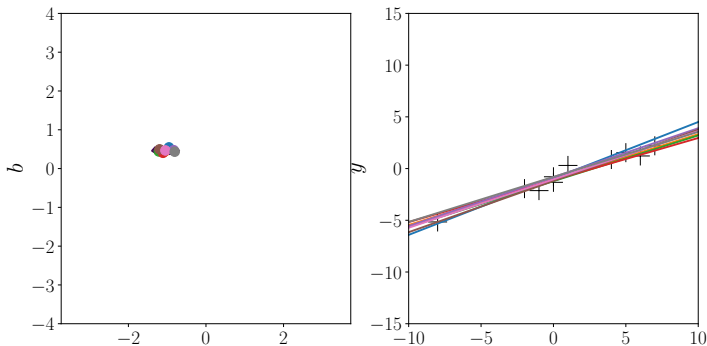
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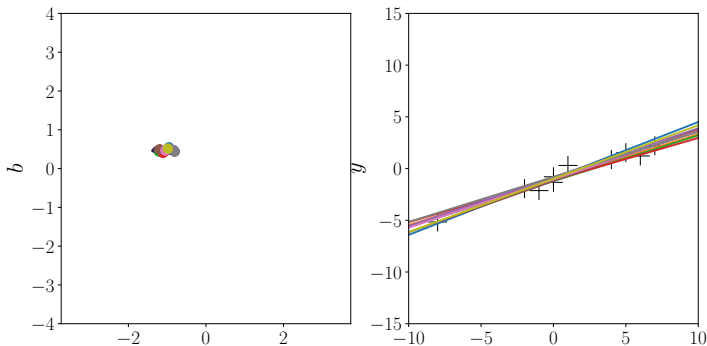
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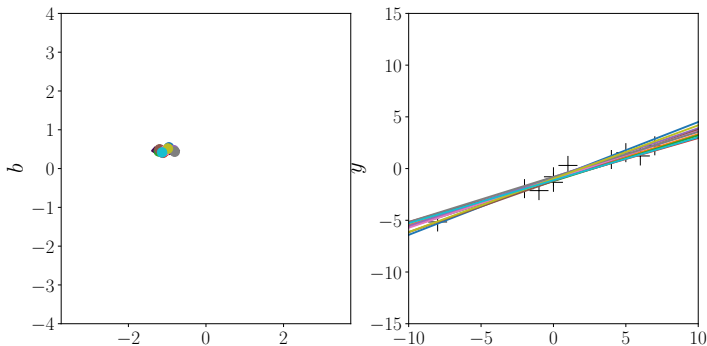
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# Model: Bayesian Linear Regression

We never put a distribution on any  $\mathbf{x}_n$ , so we drop from conditioning.

$$p(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta}; 0, \mathbf{I}_M) \quad (3)$$

$$p(y_n | \boldsymbol{\theta}) = \mathcal{N}(y_n; \boldsymbol{\phi}(\mathbf{x}_n)^\top \boldsymbol{\theta}, \sigma^2) \quad (4)$$

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Two goals:

- ▶ Find posterior over parameters  $p(\boldsymbol{\theta} | \mathbf{y})$
- ▶ Find predictive posterior  $p(\mathbf{y}^* | \mathbf{y})$

# Posterior over Parameters

Board:

- ▶ Equating coefficients (tests your matrix algebra skills!)
- ▶ Joint Gaussian
- ▶ Woodbury identity

## Method 1: Crunching densities

$$\log p(\mathbf{y}|\boldsymbol{\theta}) = \log p(\mathbf{y}|\boldsymbol{\theta}) + \log p(\boldsymbol{\theta}) \quad (6)$$

$$= c - \frac{1}{2\sigma^2}(\mathbf{y} - \Phi(\mathbf{X})\boldsymbol{\theta})^\top(\mathbf{y} - \Phi(\mathbf{X})\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{\theta}^\top\boldsymbol{\theta} \quad (7)$$

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- ▶ Find maximum to find mean  $\text{board}$
- ▶ Find Hessian to find covariance  $\text{board}$

$$p(\boldsymbol{\theta}|\mathbf{y}) = \mathcal{N}\left(\boldsymbol{\theta}; \left[\frac{1}{\sigma^2}\Phi(\mathbf{X})^\top\Phi(\mathbf{X}) + \mathbf{I}_M\right]^{-1} \frac{1}{\sigma^2}\Phi(\mathbf{X})^\top\mathbf{y} \quad (8)$$

$$\left[\frac{1}{\sigma^2}\Phi(\mathbf{X})^\top\Phi(\mathbf{X}) + \mathbf{I}_M\right]^{-1}) \quad (9)$$

## Method 2: Joint Gaussian

Find

$$p(\boldsymbol{\theta}, \mathbf{y}) = \mathcal{N}\left(\begin{bmatrix} \boldsymbol{\theta} \\ \mathbf{y} \end{bmatrix}; \begin{bmatrix} \mathbb{E}_{\boldsymbol{\theta}}[\boldsymbol{\theta}] \\ \mathbb{E}_{\mathbf{y}}[\mathbf{y}] \end{bmatrix}, \begin{bmatrix} \mathbb{V}[\boldsymbol{\theta}] & \mathbb{C}[\boldsymbol{\theta}, \mathbf{y}] \\ \mathbb{C}[\mathbf{y}, \boldsymbol{\theta}] & \mathbb{V}[\mathbf{y}] \end{bmatrix}\right) \quad (10)$$

board

$$p(\boldsymbol{\theta}|\mathbf{y}) = \mathcal{N}\left(\boldsymbol{\theta}; \Phi(X)^{\top} [\Phi(X)\Phi(X)^{\top} + \sigma^2 \mathbf{I}_N]^{-1} \mathbf{y}, \right. \\ \left. \mathbf{I}_M - \Phi(X)^{\top} [\Phi(X)\Phi(X)^{\top} + \sigma^2 \mathbf{I}_N]^{-1} \Phi(X)\right) \quad (11)$$

# Computational Considerations

# Woodbury identity



# Predictive posterior

Board:

- ▶ Crunching densities
- ▶ Equating coefficients (tests your matrix algebra skills!)
- ▶ Joint Gaussian
- ▶ Woodbury identity

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Next do the integrals / equating coefficients.  $\implies$  I recommend other method.

# Conclusion