Bayesian Linear Regression

Mark van der Wilk

Department of Computing
Imperial College London

@markvanderwilk
m.vdwilk@imperial.ac.uk

November 16, 2021

Last time

- Discussed the principle of Bayesian inference
- Skill: Compute posteriors
  - Example: Coin flipping (continuous latent, discrete obs)
  - Example: Noisy communication (discrete latent, continuous obs)
- Introduced: Noisy communication with Gaussian prior on signal.
- Skill: Gaussian conditioning.

Last time

- Discussed the principle of Bayesian inference
- Skill: Compute posteriors
  - Example: Coin flipping (continuous latent, discrete obs)
  - Example: Noisy communication (discrete latent, continuous obs)
- Introduced: Noisy communication with Gaussian prior on signal.
- Skill: Gaussian conditioning.

To help with clarity, I made a notebook of figures to clarify last Friday's lecture (`l10-bayes-voltage.ipynb`). Do carefully look at it.

Gaussian Conditioning

Many important models are **Linear Gaussian Models**.

- Bayesian Linear Regression
- Factor Analysis / PCA / Linear Autoencoders
- Kalman filters (time series)

Gaussian Conditioning

Many important models are **Linear Gaussian Models**.

- Bayesian Linear Regression
- Factor Analysis / PCA / Linear Autoencoders
- Kalman filters (time series)

Special because:

- You can compute posteriors in closed form

  (**closed form** means that you can express some solution in terms of typical functions log, sin, etc...)

- This is rare in Bayesian inference!

Gaussian Conditioning

Many important models are **Linear Gaussian Models**.

- Bayesian Linear Regression
- Factor Analysis / PCA / Linear Autoencoders
- Kalman filters (time series)

Special because:

- You can compute posteriors in closed form

  (**closed form** means that you can express some solution in terms of typical functions log, sin, etc...)

- This is rare in Bayesian inference!

Key skill: Conditioning in Gaussian distributions.

Example: Noisy analogue communication

Similar set-up to before.

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{1}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{2}$$

Example: Noisy analogue communication

Similar set-up to before.

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{1}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{2}$$

We are interested in our posterior belief on S:

$$p(s|v) = \frac{p(v|s)p(v)}{p(v)} \tag{3}$$

$$= \frac{p(v,s)}{\int p(v,s)\mathrm{d}s} \qquad \text{Alternative formulation (sum/prod rules)} \tag{4}$$

Two formulations, two methods of solving

Both share the same goal:

1. Find an expression of the posterior density that you can directly implement (closed-form)
2. Express posterior as a known standard type of distribution

Method 1: Crunching through densities

One method always works: Crunching through the densities.

We only care about terms that depend on s, since we know that a distribution has to normalise to 1!

$$p(s|v) \propto p(v|s)p(s) \tag{5}$$

$$= \mathcal{N}(v; s, \sigma^2)\mathcal{N}(s; 0, 1) \tag{6}$$

$$\propto \exp\left(-\frac{(v-s)^2}{2\sigma^2}\right)\exp\left(-\frac{s^2}{2}\right) \tag{7}$$

$$= \exp\left(-\frac{v^2}{2\sigma^2} + \frac{sv}{\sigma^2} - \frac{s^2}{2\sigma^2} - \frac{s^2}{2}\right) \tag{8}$$

$$\propto \exp\left(-\frac{1+\sigma^2}{2\sigma^2}s^2 + \frac{v}{\sigma^2}s\right) \tag{9}$$

Method 1: Equating coefficients

$$\mathcal{N}(s; a, b) = c \cdot \exp\left(-\frac{1}{2b}x^2 + \frac{a}{b}x\right) \tag{10}$$

$$\implies b = \frac{\sigma^2}{1+\sigma^2} \tag{11}$$

$$\implies a = \frac{1}{1+\sigma^2}v \tag{12}$$

$$\implies p(s|v) = \mathcal{N}\left(s; \frac{1}{1+\sigma^2}v, \frac{\sigma^2}{1+\sigma^2}\right) \tag{13}$$

Method 1: Equating coefficients

$$\mathcal{N}(s; a, b) = c \cdot \exp\left(-\frac{1}{2b}x^2 + \frac{a}{b}x\right) \tag{10}$$

$$\implies b = \frac{\sigma^2}{1+\sigma^2} \tag{11}$$

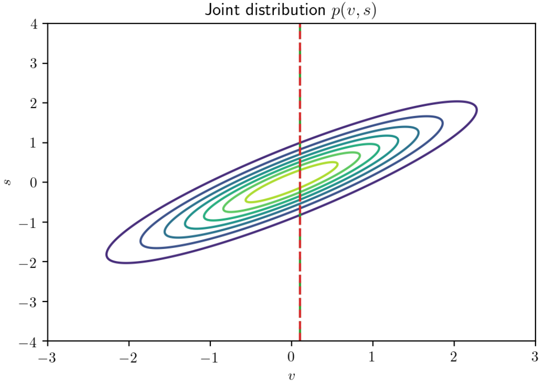
$$\implies a = \frac{1}{1+\sigma^2}v \tag{12}$$

$$\implies p(s|v) = \mathcal{N}\left(s; \frac{1}{1+\sigma^2}v, \frac{\sigma^2}{1+\sigma^2}\right) \tag{13}$$

- Best guess (mode/mean) is more conservative than MaxLik (biased towards the mean of the prior)
- Variance tends to zero as $\sigma^2 \to 0$

Method 2: Joint and Conditioning Formula

Alter: we could find the joint first, and then evaluate along a line



Method 2: Gaussian Conditioning Formula

For a joint Gaussian density

$$p(x, y) = \mathcal{N}\left(\begin{bmatrix} x \\ y \end{bmatrix}; \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} A & B \\ B & C \end{bmatrix}\right) \tag{14}$$

we have the conditional

$$p(y|x) = \mathcal{N}\left(y; \frac{B}{A}(x - a) + b, C - \frac{B^2}{A}\right) \tag{15}$$

Linear Gaussian Model

Linear-Gaussian model:

Linear Gaussian Model

Linear-Gaussian model:

- All RVs are conditionally Gaussian.

Linear Gaussian Model

Linear-Gaussian model:

- All RVs are conditionally Gaussian.
- Conditional means can depend **linearly** on other RVs.

Linear Gaussian Model

Linear-Gaussian model:

- All RVs are conditionally Gaussian.
- Conditional means can depend **linearly** on other RVs.
- Can always rewrite as linear combination of independent Gaussian RVs.

Linear Gaussian Model

Linear-Gaussian model:

- All RVs are conditionally Gaussian.
- Conditional means can depend **linearly** on other RVs.
- Can always rewrite as linear combination of independent Gaussian RVs.
- Linear transformations of Gaussians are still Gaussian (earlier lectures / exercise)

Linear Gaussian Model

Linear-Gaussian model:

- All RVs are conditionally Gaussian.
- Conditional means can depend **linearly** on other RVs.
- Can always rewrite as linear combination of independent Gaussian RVs.
- Linear transformations of Gaussians are still Gaussian (earlier lectures / exercise)
- This gives an easy way of finding the joint.

Method 2: Example

Set-up:

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{16}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{17}$$

Method 2: Example

Set-up:

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{16}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{17}$$

Write as linear transform, use expectation identities to find means and covariances:

$$\begin{bmatrix} s \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ \epsilon \end{bmatrix} \tag{18}$$

(19)

(20)

(21)

(22)

Method 2: Example

Set-up:

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{16}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{17}$$

Write as linear transform, use expectation identities to find means and covariances:

$$\begin{bmatrix} s \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ \epsilon \end{bmatrix} \tag{18}$$

$$\mathbb{E}_{s,\epsilon}[v] = \mathbb{E}_{s,\epsilon}[s + \epsilon] = \mathbb{E}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 0 \tag{19}$$

(20)

(21)

(22)

Method 2: Example

Set-up:

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{16}$$
$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{17}$$

Write as linear transform, use expectation identities to find means and covariances:

$$\begin{bmatrix} s \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ \epsilon \end{bmatrix} \tag{18}$$
$$\mathbb{E}_{s,\epsilon}[v] = \mathbb{E}_{s,\epsilon}[s + \epsilon] = \mathbb{E}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 0 \tag{19}$$
$$\mathbb{V}_{s,\epsilon}[v] = \mathbb{V}_{s,\epsilon}[s + \epsilon] = \mathbb{V}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 1 + \sigma^2 \qquad \text{Indep.} \tag{20}$$
$$\tag{21}$$
$$\tag{22}$$

Method 2: Example

Set-up:

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{16}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{17}$$

Write as linear transform, use expectation identities to find means and covariances:

$$\begin{bmatrix} s \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ \epsilon \end{bmatrix} \tag{18}$$

$$\mathbb{E}_{s,\epsilon}[v] = \mathbb{E}_{s,\epsilon}[s + \epsilon] = \mathbb{E}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 0 \tag{19}$$

$$\mathbb{V}_{s,\epsilon}[v] = \mathbb{V}_{s,\epsilon}[s + \epsilon] = \mathbb{V}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 1 + \sigma^2 \qquad \text{Indep.} \tag{20}$$

$$\mathbb{C}_{s,\epsilon}[s, v] = \mathbb{E}_{s,\epsilon}[(s + \epsilon)s] = \mathbb{E}_s[s]\mathbb{E}_\epsilon[\epsilon] + \mathbb{E}_\epsilon\left[s^2\right] \qquad \text{Indep.} \tag{21}$$

$$\tag{22}$$

Method 2: Example

Set-up:

$$p(s) = \mathcal{N}(s; 0, 1) \qquad \text{Assume a prior on source voltage.} \tag{16}$$

$$p(v|s) = \mathcal{N}\left(v; s, \sigma^2\right) \qquad \text{Assumptions about the channel.} \tag{17}$$

Write as linear transform, use expectation identities to find means and covariances:

$$\begin{bmatrix} s \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ \epsilon \end{bmatrix} \tag{18}$$

$$\mathbb{E}_{s,\epsilon}[v] = \mathbb{E}_{s,\epsilon}[s + \epsilon] = \mathbb{E}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 0 \tag{19}$$

$$\mathbb{V}_{s,\epsilon}[v] = \mathbb{V}_{s,\epsilon}[s + \epsilon] = \mathbb{V}_s[s] + \mathbb{E}_\epsilon[\epsilon] = 1 + \sigma^2 \qquad \text{Indep.} \tag{20}$$

$$\mathbb{C}_{s,\epsilon}[s, v] = \mathbb{E}_{s,\epsilon}[(s + \epsilon)s] = \mathbb{E}_s[s]\mathbb{E}_\epsilon[\epsilon] + \mathbb{E}_\epsilon\left[s^2\right] \qquad \text{Indep.} \tag{21}$$

$$= 0 + \mathbb{V}_s[s] = 1 \qquad \mathbb{E}_\epsilon[\epsilon] = 0 \tag{22}$$

Method 2: Example continued

Given the means, variances and covariances we have computed, we can fill in the full joint:

$$p(s, v) = \mathcal{N}\left(\begin{bmatrix} v \\ s \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 + \sigma^2 & 1 \\ 1 & 1 \end{bmatrix}\right) \tag{23}$$

Method 2: Example continued

Given the means, variances and covariances we have computed, we can fill in the full joint:

$$p(s, v) = \mathcal{N}\left(\begin{bmatrix} v \\ s \end{bmatrix}; \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 + \sigma^2 & 1 \\ 1 & 1 \end{bmatrix}\right) \tag{23}$$

We can now apply the Gaussian conditioning formula (make sure s, v are the right way round):

$$p(s|v) = \mathcal{N}\left(s; \frac{\sigma^2}{1+\sigma^2} v, 1 - \frac{1}{1+\sigma^2}\right) \tag{24}$$

$$= \mathcal{N}\left(s; \frac{\sigma^2}{1+\sigma^2} v, \frac{\sigma^2}{1+\sigma^2}\right) \tag{25}$$

Conclusion

Gaussian conditioning:

- Method 1: Equating coefficients
- Method 2: Finding joint, using conditioning formula