Bayesian Inference

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Coin lottery

Trick question (i.e. many correct answers):

- I pick a coin out of my pocket.
- I flip it 4 times.
- I observe Heads 4 times.
- What do you think the chance is of Heads on the next round?

Coin lottery

Trick question (i.e. many correct answers):

- I pick a coin out of my pocket.
- I flip it 10 times.
- I observe Heads 10 times.
- What do you think the chance is of Heads on the next round?

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$$X_i \sim \text{Bernoulli}(p_h) \tag{1}$$

$$P(X_i = 1) = p_h \qquad\qquad P(X_i = 0) = (1 - p_h) \tag{2}$$

$$\implies P_{X_i}(x) = p_h^x (1 - p_h)^{1-x} \tag{3}$$

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- How do we find p_h?
- Maximum likelihood?

Coin: Maximum Likelihood

$$p(x_1, x_2, x_3, \ldots | p_h) = p(\mathbf{x}|p_h) = \prod_{n=1}^{N} P_{X_i}(x_i) \tag{4}$$

(5)

(6)

(7)

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$$p(x_1, x_2, x_3, \dots | p_h) = p(\mathbf{x}|p_h) = \prod_{n=1}^{N} P_{X_i}(x_i) \tag{4}$$

$$\log p(\mathbf{x}|p_h) = \underbrace{\left(\sum_{n=1}^{N} x\right)}_{N_1} \log p_h + \underbrace{\left(\sum_{n=1}^{N} (1-x)\right)}_{N_0} \log(1-p_h) \tag{5}$$

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- After $N_1 = 4, N_0 = 0$, would you bet all your savings on heads?
- Maximum likelihood tells you that you should...

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- Consider the unknown parameter **unobserved**
- Data is drawn conditional on parameter
- Find probability of parameter given the data
- Use conditional probability (Bayes rule) to quantify your uncertainty!

Bayes

$$P(\text{hidden}|\text{data}) = \frac{P(\text{data}|\text{hidden})p(\text{hidden})}{p(\text{data})} \tag{8}$$

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Coin flipping example.