Character variety

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1 Introduction

This is a project of M1S1 in Paris 13 under the instruction of Tamiozzo. The theme will focus on affine geometrc invariance theory and their applications in Painleve VI equations.

2 Affine GIT

2.1 Linear algebraic groups

In this section, we would like to explain the most notion and results about linear algebraic groups we will use after.

Definition 2.1. *A linear algebraic group G over k is an affine variety(not necessarily irreducible here) together with a group structure such that the multiplication $m : G \times G \to G$ and the inversion $i : G \to G$ are morphisms of varieties.*

Naturally, there will be a coalgebra structure on the ring $A(G)$ of G. We call "comultiplication" that $m^* : A(G) \to A(G) \otimes A(G)$ and "coinversion" $i^* : A(G) \to A(G)$.

Example 2.2 (Additive group)**.** *$\mathbb{G}_a :=$ Spec $k[t]$ is an algebraic group whose underlying variety is the affine line $\mathbb{A}_k^1$ and the co-group structure is given by:*

$$\begin{aligned} m^*(t) &= t \otimes 1 + 1 \otimes t, \\ i^*(t) &= -t. \end{aligned}$$

Example 2.3 (Multiplicative group)**.** *$\mathbb{G}_m :=$ Spec $k[t, t^{-1}]$ is an algebraic group whose underlying variety is $\mathbb{A}_k^1 - \{0\}$ and the co-group structure is given by:*

$$\begin{aligned} m^*(t) &= t \otimes t, \\ i^*(t) &= t^{-1}. \end{aligned}$$

Example 2.4 (General linear group)**.** *The general linear group $GL_n(k)$ over k is an open subvariety of $\mathbb{A}^{n^2}$ which is birational to a hyperplane in $\mathbb{A}^{n^2+1}$. It is an affine variety with coordinate ring $k[x_{i,j}, \Delta^{-1}]$. The co-group structure is defined by:*

$$\begin{aligned} m^*(x_{i,j}) &= \sum_{s=1}^{n} x_{i,s} \otimes x_{s,j}, \\ i^*(x_{i,j}) &= (x_{i,j})^{-1}_{i,j}. \end{aligned}$$

Example 2.5 (Finite group)**.** *All finite groups all linear algebraic groups. Consider the emmbedding $G \to \mathcal{S}_n \to GL_n(k)$ as closed subgroups.*

One important kind of linear algebraic groups are tori.

Definition 2.6. *A torus T is a linear algebraic group over k which is isomorphic to $\mathbb{G}_m^n$ for some n. We call the commutative group $X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m)$ character group and $X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T)$ cocharacter group.*

Definition 2.7. *A morphism of algebraic groups $G \to GL(V)$ is called a rational representation of G in V, where V is a finite dimensional vector space over k.*

An important fact about tori is that the rational representations of tori are completely reducible. We show the decomposition below and we will prove a version a little stronger, see Theorem 2.15.

Theorem 2.8. *For a finite demensional rational representation of a torus $\rho : T \to GL(V)$, there is a weight space decomposition*

$$V \simeq \bigoplus_{\chi \in X^*(T)} V^{\chi}.$$

Remark 2.9. *Actually, there is an equivalence between the category of linear representations of T and $X^*(T)-$graded k-vector spaces.*

There are several important results about the structure of linear algebraic groups over an algebraically closed field k. From then on, all field k will be algebraically closed unless mentioned.

Definition 2.10. *Let G be a linear algebraic group over k. An element g is semisimple(resp. unipotent) if there is a faithful linear representation $\rho : G \to GL_n$ such that $\rho(g)$ is diagonalizable(unipotent).*

Remark 2.11 (Jordan decomposition)**.** *Let G be a linear algebraic group over k. For every $g \in G$, there exists a unique semisimple element g_{ss} and a unique unipotent element g_u such that*

$$g = g_{ss} g_u = g_u g_{ss}.$$

Furthermore, this decomposition is functorial with respect to morphisms of linear algebraic subgroup of G.

Definition 2.12. *(1) Let $g \in G$, then g is called semisimple(resp. unipotent) if $g = g_s$(resp. $g = g_u$).*
(2) We denote by G_s (resp. G_u) the set of semisimple(resp. unipotent) elements in G.
(3) The group G is called unipotent if $G = G_u$.
(4) The group G is called diagonalisable if there exists a faithful representation $G \to GL(V)$ such that the image of G is contained in the subgroup of diagonal matrices.

Obviously, tori are diagonalizable groups. Below is a useful lemma from group theory which describes the characters.

Lemma 2.13 (Dedekind's Lemma)**.** *Let G be any group. $\mathbb{X}(G) = \operatorname{Hom}_{Groups}(G, \mathbb{G}_m)$ is a linearly independent subset of k^G the set of all functions on G.*

Proof. If there is a relation between the elements in $\mathbb{X}(G)$, let us choose such a relation with minimal length i.e. n minimal such that there is a relation $\sum_{i=1}^n a_i \chi_i = 0$ with $a_i \neq 0$ and $\chi_i \in \mathbb{X}(G)$. Let $g, h \in G$, we have

$$\sum_{i=1}^n a_i \chi_i(g) \chi_i(h) = \sum_{i=1}^n a_i \chi_i(gh) = 0,$$
$$\chi_1(g) \sum_{i=1}^n a_i \chi_i(h) = \sum_{i=1}^n a_i \chi_i(h) = 0.$$

Taking the difference, we get $\sum_{i=2}^n a_i \left(\chi_i(g) - \chi_1(g)\right) \chi_i(h) = 0$ and thus the relation

$$\sum_{i=2}^n a_i \left(\chi_i(g) - \chi_1(g)\right) \chi_i = 0$$

Because $\chi_2 \neq \chi_1$, there exists $g \in G$ such that $\chi_2(g) - \chi_1(g) \neq 0$ thus we have a smaller relation. Contradiction. □

Corollary 2.14. *The subset $X^*(G)$ is linearly independent in $k[G]$.*

Now it's the structure of diagonalisable groups.

Theorem 2.15 (Structure theorem of diagonalisable groups)**.** *Let G be an algebraic group. The following properties are equivalent:*
(1) The group G is commutative and $G = G_s$.
(2) The group G is diagonalisable.
(3) The group $X^*(G)$ is abelian of finite type and forms a base of $k[G]$.
(4) Any representation V of G is a direct sum of representations of dimension 1.

Proof. (1) $\Rightarrow$ (2) : it suffices to prove there axists a base of V such that all matrices of the elements in G are upper triangular matrices in this base. This can be showed by induction that the kernel is stable for commutativity. Furthermore, for semisimple elements, the base can be chosen such that all the elements have a diagonal matrix in that base.

Let us prove the implication (2) $\Rightarrow$ (3). If G is diagonalisable, then G is closed subgroup of rmD_n thus we have a surjective map $k\left[T_1^{\pm}, \cdot, T_n^{\pm}\right] = k\left[\mathrm{D}_n\right] \to k[G]$. Furthermore, the T_i are characters of D_n. By restriction, we have that $T_i|_G$ is also a character of G thus the characters of G span $k[G]$. Furthermore, we have a surjective map $X^*\left(\mathrm{D}_n\right) \to X^*(G)$ thus the former is of finite type.

Let us prove the implication (ml) $\Rightarrow$ (iv). Let $\phi : G \to \mathrm{GL}(V)$ be a representation. This can be seen as a map to $\mathfrak{gl}(V) \simeq k^{n^2}$ therefore, we have $\phi(g)_{i,j} \in k[G]$ and we may write $\phi(g)_{i,j} = \sum_\chi a(i,j)_\chi \chi$. Thus we have

$$\phi = \sum_\chi \chi A_\chi$$

for some linear map $A_\chi \in \mathrm{GL}(V)$. Note that only finitely many A_χ are non zero. Now the equality $\phi(gg') = \phi(g)\phi(g')$ yields the equality

$$\sum_\chi \chi(g)\chi(g') A_\chi = \sum_{\chi'} \sum_{\chi''} \chi'(g)\chi''(g') A_{\chi'} A_{\chi''}$$

which by Dedekin's Lemma applied twice gives $A_{\chi'} A_{\chi''} = \delta_{\chi',\chi''} A_{\chi'}$. Note that by evaluating at e, we have $\sum_\chi A_\chi = \mathrm{Id}$. In particular, if we set $V_\chi = \mathrm{im}\, A_\chi$, we have $V = \oplus_\chi V_\chi$. Furthermore, any $g \in G$ acts on V_χ as $\chi(g) \cdot \mathrm{Id}$. This proves the implication.

The implication (1v) $\Rightarrow$ (11) is obvious because if we embed G in $\mathrm{GL}(V)$, then G will be contained in the diagonal matrices given by the base coming from (iv). □

Corollary 2.16. *Let G be a diagonalisable group, then $X^*(G)$ is an abelian group of finite type without p-torsion if $p = char(k)$. The algebra $k[G]$ is isomorphic to the group algebra of $X^*(G)$.*

Proof. We have already seen that $X^*(G)$ is an abelian group of finite type. Furthermore, if $X^*(G)$ has p-torsion, then there exists a character $\chi : G \to \mathbb{G}_m$ such that $\chi^p = 1$. This gives the relation $(\chi - 1)^p = \chi^p - 1 = 0$ in $k[G]$ which would not be reduced. A contradiction.

Recall that the group algebra of $X^*(G)$ has a base $(e(\chi))_{\chi \in X^*(G)}$ and the multiplication rule $e(\chi)e(\chi') = e(\chi\chi')$. But by the previous Theorem $k[G]$ has a base indexed by $X^*(G)$ with the same multiplication rule.

Conversely, let M be any abelian group of finite type without p-torsion if $p = \mathrm{char}(k)$. We define $k[M]$ to be its groups algebra. □

Corollary 2.17. *Let G be a diagonalised algebraic group, then the following are equivalent.*

(1) G is a torus.
(2) G is connected.
(3) The group $X^*(G)$ is a free abelian group.

Proof. Obviously (1) implies (2) and we have seen that (2) implies (3). Furthermore if (3) holds, then $X^*(G) \simeq \mathbb{Z}^r$, thus $G \simeq G(X^*(G)) \simeq G(\mathbb{Z})^r \simeq \mathbb{G}_M^r$. □

Corollary 2.18. *A diagonalisable algebraic group is a product of a torus and a finite abelian group of ordre primes to $p = char(k)$.*

2.2 Group action

Definition 2.19. *Let X be a variety, an action of an algebraic group G on X is given by a morphism of varieties $\varphi : G \times X \to X$. We call X is a G-variety or G-space.*

If X is an affine variety over k and $A(X)$ denotes its algebra of regular functions, then an action of G on X gives rise to a coaction of k-algebras;

$$\begin{aligned} \varphi^* : A(X) &\to A(G) \otimes A(X) \\ f &\mapsto \sum h_i \otimes f_i. \end{aligned}$$

This gives rise to an action $G \to \mathrm{Aut}(A(X))$ by

$$f \mapsto \sum h_i(g) f_i \in A(X).$$

These action are called rational actions.

Lemma 2.20. *For any $f \in A(X)$, the linear space spanned by the translates $g \cdot f$ for $g \in G$ is finite dimensional.*

Proof. Fix $f \in A(X)$. Because $g \cdot f = \sum_{i=1}^n h_i(g) f_i \in A(X)$, so f_i are a basis. □

As in the group theory, there are two important notion of group action.

Definition 2.21. *Let G be an affine algebraic group acting on a variety X.*
(1) We define the orbit $G\cdot x$ of x to be the image of the morphism: $G\cdot x=\bigcup_g \varphi(g,x)$.
(2) We define the stabiliser G_x of x to be the fibre of φ over x, which is a closed subgroup of G.

Remark 2.22. *Chevalley theorem on constructible sets proved that the orbit $G\cdot x$ is a locally closed subvariety of X. The proof can be found in Hartshorne[**?**].*

Now we prove two important closeness lemma.

Lemma 2.23. *Let X be a G-variety.*
(1) If Y and Z are subvarieties of X such that Z is closed, then $\{g\in G: gY\subset Z\}$ is closed.
(2) For any subgroup $H\subset G$, the fixed point locus $X^H=\{x\in X: H\cdot x=x\}$ is closed.

Proof. For the first statement,

$$\{g\in G: gY\subset Z\}=\bigcap_{y\in Y}\varphi_y^{-1}(Z)$$

is closed.

Secondly, fix a $h\in H$, consider the graph $\Gamma_h: X\to X\times X$ given by $x\mapsto (x,\varphi(h,x))$. So

$$X^H=\bigcap_{h\in H}\Gamma_h^{-1}(\Delta_X)$$

, where Δ_X is closed because affine morphisms are separated. $\square$

They we apply them to orbits.

Proposition 2.24. *Let X be a G-variety. Then*
(1) The orbit $G\cdot x$ is open in $\overline{G\cdot x}$.
(2) The boundary $\overline{G\cdot x}-G\cdot x$ is a union of orbits of strictly smaller dimension.
(3) There exists at least one closed orbit in X.

Proof. (1) By Chavalley's theorem, the orbit $G\cdot x$ contains an open subset U which is open and dense in the closure $\overline{G\cdot x}$. And $G\cdot x=\bigcup_{g\in G} gU$ which is open.

(2) Because $\overline{G\cdot x}-G\cdot x$ is invariant under G so it's a union of orbits and is closed by (1). The dimensions of these orbits must be strictly smaller the $G\cdot x$.

(3) The orbits of minimum dimension must be closed by (2). $\square$

As we can see, because the obits are not closed generally, it's not easy to get a good enough quotient space arbitrarily.While we really hope we can get an affine variety after quotient, but it's not true for general case. There are some conterexample.

Example 2.25. *Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $\varphi(t,(x,y))=(tx,t^{-1}y)$. The orbits of this action are*
(1) Conics: $xy=a$ for $a\in k^*$,
(2) Punctured x-axis,
(3) Punctured y-axis,
(4) The origin $(0,0)$.
The origin and the conic orbits are closed while the others both contain the origin in their orbit closures.

We introduce a new kind of quotient which works.

Definition 2.26. *We definie a categorical quotient for the action of G on X is a G-invariant morphism $\pi: X\to S$ of varieties which is universal; that is, for every other G-invariant morphism $f: X\to Y$, there is a factors uniquely through π, which means there exists a unique morphism $g: S\to Y$ such that $f=g\circ\pi$.*

Because the projection is contunuous, it must be constant on a orbit and its closure. So the categorical quotient will be the orbit space only if all the orbits are closed.

Example 2.27. *Compared to the conterexample 2.25, the categorical quotient of the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ will be the morphism: $\pi:\mathbb{A}^2\to\mathbb{A}$ such that $\pi(x,y)=xy$. Because are the case (2) (3) (4) will be in one closed orbit.*

To construct such a quotient, we may need to consider about the stucture ring. Let X be a G-variety. The action of G on X induce an action on $A(X)$ given by

$$g \cdot f(x) = f(g^{-1}x).$$

Definition 2.28. *As the conditon above, we denote the subalgebra of invariant functions by*

$$O_X(X)^G := \{f \in O_X(X) : g \cdot f = f \text{ for all } g \in G\}.$$

Similarly, if $U \subset X$ is a subset which is invariant under the action of G, then G will act on $O_X(U)$ and we denote $O_X(U)^G$ for the subalgebra of invariant functions.

We want our quotient to have nice geometric properties and so we introduce a notion of good quotient:

Definition 2.29. *We call a morphism $\pi : X \to S$ is a good quotient for the action of G on X if*
(1) π is surjective and constant on orbits.
(2) If $U \subset S$ is an open subset, then the morphism $\pi^\sharp : O_S(U) \to O_X(\pi^{-1}(U))$ induce an isomorphism to rings of invariant functions $O_S(U) \cong O_X(\pi^{-1}(U))^G$.
(3) The image of G-invariant closed subset of X is closed, and if V_1 and V_2 are two disjoint G-invariant closed subsets, then $\pi(V_1) \cap \pi(V_2) = \emptyset$.
(4) π is affine (i.e. the preimage of every affine open is affine).
If moreover, we say π is a geometric quotient, if
(5) the preimage of each point is a single orbit.

Remark 2.30. *The definition of good and geometric quotients are local: if $\pi : X \to Y$ is a good(resp. geometric) quotient, then for any open set $U \subset Y$, $\pi|_{\pi^{-1}(U)}$ is also a good(resp. geometric) quotient. Inversely, if a G-invariant map which restricted on preimages of a cover of Y are all good (respectively geometric) quotients, then so is the map.*

Proposition 2.31. *Let X be a G-variety, then a good quotient $\pi : X \to S$ is a categorical quotient.*

Corollary 2.32. *Let $\pi : X \to S$ be a good quotient, then:*
(1) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$ if and only if $\pi(x_1) = \pi(x_2)$.
(2) For each $y \in Y$, the preimage $\pi^{-1}(y)$ contains a unique closed orbit. In particular, if the action is closed (i.e. all orbits are closed), then π is a geometric quotient.

Proof. The fact is given by definition of good quotient (1) and (3). ☐

2.3 Reductive groups

An interesting problem first considered by Hilbert was the following:

Conjecture 2.33 (Hilbert's 14th problem.)**.** *Given a rational action of an affine algebraic group G on a finitely generated k-algebra A, is the algebra of G-invariants A^G finitely generated?*

Unfortunately, the answer to this problem is FALSE. But for a very large class of groups(known as reductive groups), the answer is yes.

Definition 2.34. *Let G be a linear algebraic group over k.*
(1) The unipotent radical of G, denoted $\mathcal{R}_u(G)$ is the maximal closed connected unipotent normal linear algebraic subgroup of G.
(2) The radical of G, denoted $\mathcal{R}(G)$ is the maximal closed connected solvable normal linear algebraic subgroup of G.
(3) G is semisimple if it has trivial radical $\mathcal{R}(G) = 1$.
(4) G is reductive if it has trivial unipotent radical $\mathcal{R}_u(G) = \{1\}$.

Theorem 2.35 (Lie–Kolchin)**.** *Every unipotent linear algebraic subgroup is solvable.*

Example 2.36. *The general linear group GL_n has radical consisting of the scalar matrices $\mathcal{R}(\mathrm{GL}_n) \cong \mathbb{G}_m$ and trivial unipotent radical. In particular, this is reductive but not semisimple. The special linear group and projective linear group are both semisimple (and thus reductive).*

There are several close notion of reductive:

Definition 2.37. *A linear algebraic group G is*

(1) linearly reductive if for every finite dimensional linear representation $\rho : G \to \mathrm{GL}_n(k)$ decomposes as a direct sum of irreducible representations.

(2) geometrically reductive if for every representation $\rho : G \to \mathrm{GL}_n(k)$ and every non-zero G-invariant point $v \in \mathbb{A}^n$, there is a G-invariant non-constant homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ such that $f(v) \neq 0$.

Remark 2.38. *It is clear that a linearly reductive group is also geometrically reductive. Nagata showed that every geometrically reductive group is reductive. In characteristic zero we have that all three notions coincide as a theorem of Weyl shows that every reductive group is linearly reductive. In positive characteristic, the different notions of reductivity are related as follows:*

$$\textit{linearly reductive} \implies \textit{geometrically reductive} \iff \textit{reductive}$$

Example 2.39. *Every torus $(\mathbb{G}_m)^r$ and finite group is reductive. Also the groups $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$ and $\mathrm{PGL}(n, k)$ are all reductive. The additive group $\mathbb{G}_a$ of k under addition is not reductive. In positive characteristic, the groups $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$ and $\mathrm{PGL}_n(k)$ are not linearly reductive for $n > 1$.*

A good property for geometrically reductive groups is that we will have enough polynomials to distinguish every closed point.

Lemma 2.40. *Suppose G is geometrically reductive and acts on an affine variety X. If W_1 and W_2 are disjoint G-invariant closed subsets of X, then there is an invariant function $f \in A(X)^G$ which separates these sets i.e.*

$$f(W_1) = 0 \quad \textit{and} \quad f(W_2) = 1$$

Proof. As W_i are disjoint and closed

$$(1) = I(\phi) = I(W_1 \cap W_2) = I(W_1) + I(W_2)$$

and so we can write $1 = f_1 + f_2$. Then $f_1(W_1) = 0$ and $f_1(W_2) = 1$. The linear subspace V of $A(X)$ spanned by $g \cdot f_1$ is finite dimensional by Lemma 2.23 and so we can choose a basis $h_1, \ldots, h_n$. This basis defines a morphism $h : X \to k^n$ by

$$h(x) = (h_1(x), \ldots, h_n(x))$$

For each i, we have that $h_i = \sum a_i(g) g \cdot f_i$ and so $h_i(x) = \sum_g a_i(g) f_1(g^{-1} \cdot x)$. As W_i are G-invariant, we have $h(W_1) = 0$ and $h(W_2) = v \neq 0$. The functions $g \cdot h_i$ also belong to V and so we can write them in terms of our given basis as

$$g \cdot h_i = \sum_j a_{ij}(g) h_j$$

This defines a representation $G \to \mathrm{GL}_n(k)$ given by $g \mapsto (a_{ij}(g))$. We note that $h : X \to \mathbb{A}^n$ is then G-equivariant with respect to the action of G on X and $\mathrm{GL}_n(k)$ on $\mathbb{A}^n$; therefore $v = h(W_2)$ is a G-invariant point. As G is geometrically reductive, there is a non-constant homogeneous polynomial $f_0 \in k[x_1, \ldots, x_n]^G$ such that $f_0(v) \neq 0$ and $f_0(0) = 0$. Then $f = cf_0 \circ h$ is the desired invariant function where $c = 1/f_0(v)$. ☐

Also the structure of reductive groups is interesting:

Theorem 2.41 (Structure of reductive groups)**.** *Let G be a reductive group and let T be a maximal torus. Let $W(G,T)$ be the Weyl group of G and $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h}$ be the Lie algebra of T and R the set of non trivial characters of T appearing in $\mathfrak{g}$. Then we have $\mathfrak{g}^T = \mathfrak{h}$ and a decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

such that

For any $\alpha \in R$, we have dim $\mathfrak{g}_\alpha = 1$ and there exists a unique closed connected unipotent subgroup U_α, normalised by T such that $L(U_\alpha) = \mathfrak{g}_\alpha$. And the group G is spanned by T and the U_α for $\alpha \in R$.

2.4 Nagata's theorem

We recall that an action of G on a k-algebra A is rational if $A = A(X)$ for some affine variety X and this action comes from an (algebraic) action of G on X.

Theorem 2.42 (Nagata)**.** *Let G be a geometrically reductive group acting rationally on a finitely generated k-algebra A. Then the G-invariant subalgebra A^G is finitely generated.*

As every reductive group is geometrically reductive, we can use Nagata's theorem for reductive groups. In the following section, we will prove this result for linearly reductive groups using Reynolds operators.

Given a linearly reductive group G, for any finite dimensional linear representation $\rho : G \to \mathrm{GL}(V)$, we can write $V = V^G \oplus W$ where W is the direct sum of all non-trivial irreducible sub-representations. In particular, there is a projection $p : V \to V^G$. This motivates the following definition:

Definition 2.43. *For a group G acting rationally on a k-algebra A. A linear map $R_A : A \to A^G$ is called a Reynolds operator if if it a projection onto A^G and for $a \in A^G$ and $b \in A$ we have $R_A(ab) = aR_A(b)$.*

Proposition 2.44. *Let G be a linearly reductive group acting rationally on a k-algebra A; then there exists a Reynolds operator and A^G is finitely generated.*

Proof. As the action is rational, there is an affine variety X such that $A(X) = A$ and the action comes from an algebraic action on X. There is a natural homogeneous grading of $A(X)$ and so we write $A = \oplus_{r \geq 0} A_r$ with $A_0 = k$. As each graded piece A_n is a finite dimensional vector space and G is linearly reductive, there is a Reynold's operator $R_r : A_n \to A_n^G$. This allows us to define a Reynold's operator R_A on A. In particular, $A^G = \oplus_{n \geq 0} A_n^G$ is a graded k-algebra: if we take $f \in A^G$ and write $f = f_0 + \cdots + f_d$ with f_i homogeneous of degree m, then

$$f = R_A(f) = R_A(f_0 + \cdots + f_d) = R_0(f_0) + \cdots + R_d(f_d)$$

with $R_i(f_i) \in A_i^G$.

By Hilbert's basis theorem, A is a Noetherian ring and so the ideal $I = \oplus_{r>0} A_r^G$ is finitely generated; that is, we have $I =< f_1, \ldots, f_m >$ for f_i of degree d_i. We claim that the elements $f_1, \ldots, f_m$ generate A^G as a k-algebra. The proof is by induction on degree. The degree zero case is trivial as we have $A_0^G = 0$. We fix $d > 0$ and assume that all elements of degree strictly less than d can be written as a polynomial in the f_i with coefficients in k. Now take $f \in A_d^G$; then we can write

$$f = a_0 f_0 + \cdots + a_m f_m$$

for $a_i \in A$. If we replace a_i by its homogeneous piece of degree $d - d_i$ then the same equation still holds and so we can assume each a_i is homogeneous of degree $d - d_i$. We apply the Reynolds operator R_A to the above expression for f to obtain

$$f = R_A(f) = R_A(a_0) f_0 + \cdots + R_A(a_m) f_m$$

as a_i is homogeneous of degree $d - d_i$, we have that $R_A(a_i) \in A^G$ is also homogeneous of degree $d - d_i$. Hence each $R_A(a_i)$ is homogeneous of degree strictly less than d and so we deduce by induction that they are polynomials in the f_j with coefficients in k. In particular, f is also a polynomial in the f_i with coefficients in k. □

Now we can construct the affine GIT quotient. We have seen that this induces an action of G on the affine coordinate ring $A(X)$ which is a finitely generated k-algebra without zero-divisors. By Nagata's Theorem the subalgebra of invariants $A(X)^G$ is finitely generated.

Definition 2.45. *Let G be a reductive group acting on an affine variety X. The affine GIT quotient is the morphism $\varphi : X \to X /\!\!/ G := \operatorname{Spec} A(X)^G$ of varieties associated to the inclusion $\varphi^* : A(X)^G \hookrightarrow A(X)$.*

Theorem 2.46. *Let G be a reductive group acting on an affine variety X. Then the affine GIT quotient $\varphi : X \to Y := X /\!\!/ G$ is a good quotient and in particular Y is an affine variety. Moreover if the action of G is closed on X, then it is a geometric quotient.*

Proof. As G is reductive and so also geometrically reductive, it follows from Nagata's Theorem that the algebra of G-invariant regular functions on X is a finitely generated reduced k-algebra. Hence $Y := \operatorname{Spec} A(X)^G$ is an affine variety. The affine GIT quotient is defined by the inclusion $A(X)^G \hookrightarrow A(X)$ and so is G-invariant and affine.

Latter, we take $y \in Y$ and want to construct $x \in X$ whose image under $\varphi : X \to Y$ is y. Let $\mathfrak{m}_y$ be the maximal idea in $A(Y) = A(X)^G$ corresponding to the point y. We choose generators $f_1, \ldots, f_m$ of $\mathfrak{m}_y$ and, as G is reductive, we claim that

$$\sum_{i=1}^{m} f_i A(X) \neq A(X)$$

It suffices to prove that $(\sum_{i=1}^{m} f_i A(X)) \cap A(X)^G = \sum f_i A(X)^G$ as the ideal $\mathfrak{m}_y$ in $A(X)^G$ generated by the f_i is a proper maximal ideal. The right hand side of this expression is contained in the left hand side and to prove the opposite containment we use the Reynold's operator $R_A : A(X) \to A(X)^G$ (which exists as G is linearly reductive). We write $f = \sum f_i a_i \in A(X)^G$ with $a_i \in A(X)$ and apply the Reynold's operator; then

$$f = R_A(f) = \sum R_A\left(a_i\right) f_i$$

with $R_A\left(a_i\right) \in A(X)^G$ which proves the result. Then, as $\sum_{i=1}^{m} f_i A(X)$ is not equal to $A(X)$, it is contained in some maximal idea $\mathfrak{m}_x \subset A(X)$ corresponding to a closed point $x \in X$. In particular, we have that $f_i(x) = 0$ for $i = 1, \ldots, m$ and so $\varphi(x) = y$ as required.

As the open sets of the form $U = Y_f := \{y \in Y : f(y) \neq 0\}$ for non-zero $f \in A(X)^G$ form a basis for the open sets of Y it suffices to verify iii) for these open sets. If $U = Y_f$, then $\mathcal{O}_Y(U) = \left(A(X)^G\right)_f$ is the localisation of $A(X)^G$ with respect to f and

$$\mathcal{O}_X\left(\varphi^{-1}(U)\right)^G = \mathcal{O}_X\left(X_f\right)^G = \left(A(X)_f\right)^G = \left(A(X)^G\right)_f = \mathcal{O}_Y(U)$$

as localisation commutes with taking G-invariants. Hence the image of the inclusion homomorphism $\mathcal{O}_Y(U) = \left(A(X)^G\right)_f \to \mathcal{O}_X\left(\varphi^{-1}(U)\right) = A(X)_f$ is $\mathcal{O}_X\left(\varphi^{-1}(U)\right)^G = \left(A(X)_f\right)^G$ and this homomorphism is an isomorphism onto its image.

Finally, we use the fact that G is geometrically reductive: by Lemma for any two disjoint closed subsets W_1 and W_2 in X there is a function $f \in A(X)^G$ such that f is zero on W_1 and equal to 1 on W_2. We may view f as a regular function on Y and as $f\left(\varphi\left(W_1\right)\right) = 0$ and $f\left(\varphi\left(W_2\right)\right) = 1$, we must have

$$\overline{\varphi\left(W_1\right)} \cap \overline{\varphi\left(W_2\right)} = \phi$$

The final statement follows immediately from Corollary 2.32. □

Example 2.47. *Consider the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$ by $t \cdot (x, y) = \left(tx, t^{-1}y\right)$. In this case $A(X) = k[x, y]$ and $A(X)^G = k[xy] \cong k[z]$ so that $Y = \mathbb{A}^1$ and the GIT quotient $\varphi : X \to Y$ is given by $(x, y) \mapsto xy$. The three orbits consisting of the punctured axes and the origin are all identified and so the quotient is not a geometric quotient.*

Example 2.48. *Consider the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ by $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. Then $A(X) = k\left[x_1, \ldots, x_n\right]$ and $A(X)^G = k$ so that $Y = \operatorname{Spec} k$ is a point and the GIT quotient $\varphi : X \to Y = \operatorname{Spec} k$ is given by the structure morphism. In this case all orbits are identified and so this good quotient is not a geometric quotient.*

Remark 2.49. *We note that the fact that G is reductive was used several times in the proof, not just to show the ring of invariants is finitely generated. In particular, there are non-reductive group actions which have finitely generated invariant rings but for which other properties listed in the definition of good quotient fail. For example, consider the additive group $\mathbb{G}_a$ acting on $X = \mathbb{A}^4$ by the linear representation $\rho : \mathbb{G}_a \to \mathrm{GL}_4$*

$$s \mapsto \begin{pmatrix} 1 & s & & \\ & 1 & & \\ & & 1 & s \\ & & & 1 \end{pmatrix}$$

Even though $\mathbb{G}_a$ is non-reductive, the invariant ring is finitely generated:

$$\mathbb{C}\left[x_1, x_2, x_3, x_4\right]^{\mathbb{G}_a} = \mathbb{C}\left[x_2, x_4, x_1 x_4 - x_2 x_3\right].$$

However the GIT 'quotient' map $X \to X /\!\!/ \mathbb{G}_a = \mathbb{A}^3$ is not surjective; its image misses the punctured line $\{(0, 0, \lambda) : \lambda \in k^*\} \subset \mathbb{A}^3$.

2.5 Stable criterium

Now back to geometric quotients. As we saw above, when a reductive group G acts on an affine variety X in general a geometric quotient (i.e. orbit space) does not exist because in general the action is not closed. For finite groups G, every good quotient is a geometric quotient as the action of a finite group is always closed (every orbit is a finite number of points which is a closed subset). For general G, we ask if there is an open subset of X for which there is a geometric quotient.

Definition 2.50. *We say $x \in X$ is stable if its orbit is closed in X and $\dim G_x = 0$ (or equivalently, $\dim G \cdot x = \dim G$). We let X^s denote the set of stable points.*

Proposition 2.51. *Suppose a reductive group G acts on an affine variety X and let $\varphi : X \to Y$ be the associated good quotient. Then $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$ is also open. Moreover, $\varphi : X^s \to Y^s$ is a geometric quotient.*

Proof. We first show that X^s is open by showing for every $x \in X^s$ there is an open neighbourhood of x in X^s. The set $X_+ := \{x \in X : \dim G_x > 0\}$ of points with positive dimensional stabilisers is a closed subset of X. If $x \in X^s$, then there is a function $f \in A(X)^G$ such that

$$f(X_+) = 0, \quad f(G \cdot x) = 1$$

It is clear that x belongs to the open subset X_f, but in fact we claim that $X_f \subset X^s$ so it is an open neighbourhood of x. It is clear that all points in X_f must have stabilisers of dimension zero but we must also check that their orbits are closed. Suppose $z \in X_f$ has a non closed orbit so $w \notin G \cdot z$ belongs to the orbit closure of z; then $w \in X_f$ too as f is G-invariant and so w must have stabiliser of dimension zero. However, the boundary of the orbit $G \cdot z$ is a union of orbits of strictly lower dimension and so the orbit of w must be of dimension strictly less than that of z which contradicts the orbit stabiliser theorem. Hence X^s is open and is covered by sets of the form X_f.

Since $\varphi(X_f) = Y_f$ is also open in Y and $X_f = \varphi^{-1}(Y_f)$, we see that Y^s is open and also $X^s = \varphi^{-1}(\varphi(X^s))$. Then $\varphi : X^s \to Y^s$ is a good quotient and the action of G on X^s is closed; thus $\varphi : X^s \to Y^s$ is a geometric quotient by Corollary 2.32. □

Example 2.52. *We can now calculate the stable set for the action of $G = \mathbb{G}_m$ on $X = \mathbb{A}^2$. The closed orbits are the conics $\{xy = a\}$ for $a \in k^*$ and the origin, however the origin has a positive dimensional stabiliser and so*

$$X^s = \left\{(x, y) \in \mathbb{A}^2 : xy \neq 0\right\} = X_{xy}$$

In this example, it is clear why we need to insist that $\dim G_x = 0$ in the definition of stability: so that the stable set is open.

Example 2.53. *We may also consider which points are stable for the action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ as in Examples 2.48. In this case the only closed orbit is the origin whose stabiliser is positive dimensional and so $X^s = \phi$. In particular, this example shows that the stable set may be empty.*

Example 2.54. *Consider $G = \mathrm{GL}_2(k)$ acting on the space of 2×2 matrices $M_{2\times 2}(k)$ by conjugation. The characteristic polynomial of a matrix A is given by*

$$\mathrm{char}_A(t) = \det(xI - A) = x^2 + c_1(A)x + c_2(A)$$

where $c_1(A) = -\operatorname{Tr}(A)$ and $c_2(A) = \det(A)$ and is well defined on the conjugacy class of a matrix. The Jordan canonical form of a matrix is obtained by conjugation and so lies in the same orbit of the matrix. The theory of Jordan canonical forms says there are three types of orbits:

(1) matrices with characteristic polynomial with distinct roots α, β. These matrices are diagonalisable with Jordan canonical form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

These orbits are closed and have dimension 2 - the stabiliser of the above matrix is the subgroup of diagonal matrices which is 2 dimensional.

(2) matrices with characteristic polynomial with repeated root α for which the minimum polynomial is equal to the characteristic polynomial. These matrices are not diagonalisable - their Jordan canonical form is

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}.$$

These orbits are also 2 dimensional but are not closed - for example

$$\lim_{t\to 0}\left(\begin{array}{cc} t & 0 \\ 0 & t^{-1} \end{array}\right)\left(\begin{array}{cc} \alpha & 1 \\ 0 & \alpha \end{array}\right)\left(\begin{array}{cc} t^{-1} & 0 \\ 0 & t \end{array}\right)=\left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha \end{array}\right)$$

(3) matrices with characteristic polynomial with repeated root α for which the minimum polynomial is $x-\alpha$. These matrices have Jordan canonical form

$$\left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha \end{array}\right)$$

and as scalar multiples of the identity commute with everything, their stabilisers are full dimensional. Hence these orbits are closed and have dimension zero.

We note that every orbit of the second type contains an orbit of the third type and so will be identified in the quotient. There are only two G-invariant functions: the trace and determinant which define the GIT quotient

$$\varphi=(\operatorname{Tr}, \operatorname{det}): M_{2\times 2}(k)\rightarrow \mathbb{A}^2 .$$

Theorem 2.55 (Hilbert-Mumford)**.** *Let G be a reductive group acting on an affine variety X and let x be a point of X. Let O_x be the unique closed orbit adherent to Gx. Then there exists a 1-parameter subgroup $\lambda: \mathbb{G}_m \rightarrow G$ such that $\lim_{t\to 0}\lambda(t)x$ exists and belong to O_x.*

In particular, a point is stable if and only if for any non-trivial 1-parameter subgroup $\lambda: \mathbb{G}_m\rightarrow G$, the limit $\lim_{t\mapsto 0}\lambda(t)x$ does not exist.

We will prove a stronger criterion for projective GIT in appendix.

3 Character variety

Since we have obtained many powerful tools mentioned above, we can talk about what we really care about.

Definition 3.1. *Let k be a field of characteristic 0, Γ be a finitely generated group and G be a reductive affine algebraic group. Then the character variety $X(\Gamma, G)$ is defined as the algebraic quotient*

$$X(\Gamma, G)=\operatorname{Hom}(\Gamma, G) /\!\!/ G$$

where $\operatorname{Hom}(\Gamma, G)$ is the space of homomorphisms $\rho:\Gamma\rightarrow G$ and the action of G is by conjugation: $g.\rho=g\rho g^{-1}$.

The questions we need to answer are the following three:
(1) How to describe the algebra of regular functions by generators and relations?
(2) How far is from the point on character variety to the set of conjugacy classes of representations?
(3) What's the tangent space of $X(\Gamma, G)$ at a point $\pi(\rho)$? When is it smooth?

3.1 Generators and relations

Before the general result, we consider an example first.

Example 3.2. *Consider the special case $\Gamma=\mathbb{Z}$ and $G=\mathrm{GL}_n(k)$. We clearly have $\operatorname{Hom}(\mathbb{Z}, \mathrm{GL}_n(k))=\mathrm{GL}_n(k)$ by the map $\rho\mapsto\rho(1)$. The algebra of regular functions on $\mathrm{GL}_n(k)$ is simply*

$$k\left[\mathrm{GL}_n\right]=k\left[X_{ij}, \Delta^{-1}\right] \text{ where } i, j=1, \ldots, n \text{ and } \Delta=\operatorname{det}\left(X_{ij}\right).$$

By definition the character variety $X(\mathbb{Z}, \mathrm{GL}_n(k))=\mathrm{GL}_n(k) /\!\!/ \mathrm{GL}_n(k)$ is the spectrum of the subalgebra of invariants

$$k\left[\mathrm{GL}_n /\!\!/ \mathrm{GL}_n\right]=k\left[\mathrm{GL}_n\right]^{\mathrm{GL}_n}.$$

We can show that $k\left[\mathrm{GL}_n\right]^{\mathrm{GL}_n}\cong k\left[c_1, \ldots, c_n^{\pm 1}\right]$, where $c_0=1, c_1=\operatorname{Tr}(X), \ldots, c_n=\Delta$ are the coefficients of the characteristic polynomial of $X=\left(X_{ij}\right)_{i,j=1,\ldots,n}$ given by:

$$\operatorname{det}(\lambda \operatorname{Id}-X)=\sum_{i=0}^n c_{n-i}(X)(-\lambda)^i .$$

We can alternatively replace the generators $c_1, \ldots, c_n$ by $t_i = \operatorname{Tr} X^i$ for $i \in \mathbb{Z}$. And the elements $t_1, \ldots, t_{n+1} \in k\left[\mathrm{GL}_n\right]^{\mathrm{GL}_n}$ satisfy the Frobenius formula:

$$\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \prod_{k=1}^{n+1} t_k^{l_k(\sigma)} = 0,$$

where $l_k(\sigma)$ denote the number of cycles of order k in σ.

Remark 3.3. *Consider now the general case of $X(\Gamma, \mathrm{GL}_n(k))$. By assumption, Γ is finitely generated so that we can take a set of generators $\gamma_1, \ldots, \gamma_r$. Denoting by F_r the free group of rank r, then there is a presentation of Γ in the form*

$$1 \longrightarrow R \longrightarrow F_r \longrightarrow \Gamma \longrightarrow 1.$$

Consider the algebra A of functions on $\operatorname{Hom}(\Gamma, \mathrm{GL}_n(k))$. First, there are polynomials generated by (X_{ij}^l) where $1 \leq l \leq r, 1 \leq i, j \leq n$. And the matrix $X_l = (X_{ij}^l)$ is the image of generators of Γ. Then $\det(X_l)$ should be invertible. And all the word in R should give an ideal to do quotient.

Before the theorem, we will mention a standard theorem of representation theory first.

Lemma 3.4 (Schur-Weyl duality)**.** *Let V be a k-vector space of dimension n and fix an integer m. Recall that we defined a representation $\rho : S_m \to \mathrm{GL}\left(V^{\otimes m}\right)$. We set $A = \operatorname{Span}\left\{\rho(\sigma), \sigma \in S_m\right\} \subset \operatorname{End}\left(V^{\otimes m}\right)$. We also set $B = \operatorname{Span}\{g \otimes \cdots \otimes g, g \in \mathrm{GL}_n(k)\} \subset \operatorname{End}\left(V^{\otimes m}\right)$.*

Then A is the centralizer of B and B is the centralizer of A. In formulas

$$A = \operatorname{End}_B\left(V^{\otimes m}\right) \text{ and } B = \operatorname{End}_A\left(V^{\otimes m}\right).$$

Now we can give an answer to the first question.

Theorem 3.5 (Generators)**.** *Denote $\rho_n : \Gamma \to \mathrm{GL}_n(k)$ which maps γ_l to X_l, and $t_\gamma = \operatorname{Tr}(\rho_n)(\gamma)$ The elements t_γ for $\gamma \in \Gamma$ and Δ_γ^{-1} generate $k\left[X\left(\Gamma, \mathrm{GL}_n(k)\right)\right]$.*

Proof. Because the map $k\left[\mathrm{M}_n(k)\right] \to k\left[\mathrm{GL}_n(k)\right]$ is the localization of the determinant. Hence it is sufficient to show that $k\left[\mathrm{M}_n(k)^r\right]^{\mathrm{GL}_n(k)}$ is generated by the t_γ 's. ◻

As for the case in example, the trace functions t_γ are not algebraically independent. The second fundamental theorem of invariants gives generators for the ideal of invariants, also called syzygies.

Lemma 3.6 (Frobenius formula)**.** *Given $\rho : \Gamma \to \mathrm{GL}_n(k)$ and $\gamma_0, \ldots, \gamma_n \in \Gamma$, we have*

$$\operatorname{Tr}\left(P_{n+1} \circ \left(\rho\left(\gamma_0\right) \otimes \cdots \otimes \rho\left(\gamma_n\right)\right)\right) = 0.$$

Proof. Let $V = k^n$ and consider the action of $X \in \operatorname{End}(V)$ on $V^{\otimes(n+1)}$ given by $X\left(v_0 \otimes \cdots \otimes v_n\right) = \left(X v_0\right) \otimes \cdots \otimes \left(X v_n\right)$. We also define the representation $\rho : S_{n+1} \to \mathrm{GL}\left(V^{\otimes(n+1)}\right)$ by setting

$$\rho(\sigma)\left(v_0 \otimes \cdots \otimes v_n\right) = v_{\sigma^{-1}(0)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

and set $P_{n+1} = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \rho(\sigma)$. As P_{n+1} takes its values in $\Lambda^{n+1} V = 0$, it vanishes identically, giving

$$\operatorname{Tr}\left(P_{n+1} \circ X\right) = \sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \operatorname{Tr}(\rho(\sigma) \circ X) = 0$$

◻

Theorem 3.7 (Relations)**.** *The ideal of relations among the trace functions $t_\gamma \in k\left[X\left(\Gamma, \mathrm{GL}_n\right)\right]$ is generated by $t_1 - n$ and the elements*

$$\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) T^\sigma\left(\gamma_0, \ldots, \gamma_n\right)$$

where $T^{(i_1, \ldots, i_k)} = t_{\gamma_{i_1} \cdots \gamma_{i_k}}$ and $T^\sigma = \prod_{j=1}^l T^{\sigma_j}$ where $\sigma = \sigma_1 \cdots \sigma_l$ is the decomposition of σ into cycles (including the trivial ones).

Example 3.8. *For instance, if $n = 2$ we get that all relations are consequences of*

$$t_\alpha t_\beta t_\gamma - t_\alpha t_{\beta\gamma} - t_\beta t_{\alpha\gamma} - t_\gamma t_{\alpha\beta} + t_{\alpha\beta\gamma} + t_{\alpha\gamma\beta} = 0, \quad \alpha, \beta, \gamma \in \Gamma$$

It is a good exercise to show these relations in the case of $X(\Gamma, \mathrm{SL}_2(k))$ reduce to the relations $t_1 - 2$ and the famous trace relation

$$t_{\alpha\beta} + t_{\alpha\beta^{-1}} = t_\alpha t_\beta \text{ for all } \alpha, \beta \in \Gamma.$$

Corollary 3.9. *Let S be a closed oriented surface of genus g. For any isotopy class of subvariety $\gamma \subset S$ without homotopically trivial component, put $t_\gamma = \prod t_{\gamma_i}$ where $\gamma_1, \ldots, \gamma_r$ are the connected components of γ. Then the trace functions t_γ form a linear basis of $k[X(\pi_1(S), \mathrm{SL}_2)]$.*

Proof. By Theorem 1, the trace functions t_γ generate $k[X(\pi_1(S), \mathrm{SL}_2)]$ as an algebra. Using the trace equation, one can replace products by sums hence the trace functions generate $k[X(\pi_1(S), \mathrm{SL}_2)]$ linearly.

Take now $\gamma \in \pi_1(S)$ and represent it by a curve on S with a minimal number of intersection points. Applying the trace relation allows to reduce this number inductively to 0 . This corresponds to the generators of the corollary, the second theorem of invariants show that they are linearly independent. ☐

3.2 Irreducible representations

Let us apply the tools of the preceding section to the character variety $X(\Gamma, \mathrm{GL}_n(k))$. The main result is the following theorem showing that any fiber of the quotient map $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(k)) \to X(\Gamma, \mathrm{GL}_n(k))$ contains a unique conjugacy class of completely reducible representations. Moreover, the fiber reduces to one orbit of representations if and only if this representation is irreducible.

Theorem 3.10. *Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation. Then*

1. *$\mathrm{GL}(V)\rho$ is closed if and only if ρ is completely reducible.*
2. *ρ is stable for the action of $\mathrm{PGL}(V)$ if and only if ρ is irreducible.*

Proof. Up to conjugation, any 1-parameter subgroup can be written in the form

$$\lambda(t) = \begin{pmatrix} t^{a_1} & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & t^{a_n} \end{pmatrix} \quad a_1 \geq \ldots \geq a_n \in \mathbb{Z}$$

We write $\{a_1, \ldots, a_n\} = \{w_1, \ldots, w_r\}$ where $w_1 > \cdots > w_r$ and call the w_i 's the weights of λ. We compute that $\left(\lambda(t)\rho(\gamma)\lambda(t)^{-1}\right)_{ij} = t^{a_i - a_j}\rho(\gamma)_{ij}$. This shows that $\lambda(t).\rho$ converges when $t \to 0$ if and only if $\rho(\gamma)_{ij} = 0$ whenever $a_i < a_j$. This condition is equivalent to the fact that ρ preserves the flag $F = (F_1, \ldots, F_r)$ where F_l is generated by the first $w_1 + \cdots + w_l$ vectors.

Hence, by the above description, ρ is stable if and only if it does not preserve a non-trivial flag, that is if and only if it is irreducible, proving the second point.

Let us show that GL $(V)\rho$ closed implies ρ completely reducible. Let W be a ρ-invariant subspace and take λ a 1-parameter subgroup which acts by t on W and fixes a complement W'. Then $\lambda(t).\rho$ converges to a representation ρ' which stabilizes W and W' As the orbit of ρ is closed, ρ' is conjugated to ρ, and the conjugating matrix sends W' to a ρ-invariant complement of W, hence ρ is completely reducible.

Suppose now that ρ is completely reducible. Applying criterium, it suffices to show that for any 1-parameter subgroup λ such that $\lambda(t).\rho$ converges to ρ' when $t \to 0, \rho'$ is conjugated to ρ. To simplify the proof, suppose that λ has only two weights corresponding to a flag $W \subset V$ so that one has

$$\rho = \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}, \quad \rho' = \lim_{t \to 0} \lambda(t).\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix}.$$

This shows that there exists W' stabilized by ρ' such that $V = W \oplus W'$. As ρ is completely reducible and W is G-stable, there exists a G-stable complement W'', hence an element $g \in \mathrm{GL}(V)$ acting trivially on W and V/W and mapping W to W''. Conjugating ρ with g shows that ρ' and ρ are conjugate, ending the proof.

Let us give the example of $X(\mathbb{Z}^r, \mathrm{GL}_n(k))$. Any r-tuple of commuting matrices can be simultaneously trigonalized: said differently, any representation $\rho : \mathbb{Z}^r \to \mathrm{GL}_n(k)$ can be conjugated to take its values in the standard Borel subgroup of upper-triangular matrices. Denote by T the sub-torus of diagonal matrices. As shown above, the representation ρ is in the same orbit than the representation $\rho' : \mathbb{Z}^r \to T$ consisting in taking out the diagonal part of ρ.

This shows that the map $X(\mathbb{Z}^r, T) \to X(\mathbb{Z}^r, \mathrm{GL}_n(k))$ is surjective. Moreover, this map is invariant by the action of the Weyl group $S_n = N_{\mathrm{GL}_n}(T)/T$ yielding an isomorphism

$$X(\mathbb{Z}^r, \mathrm{GL}_n) = (T^r) /\!\!/ S_n$$

□

There is a generalization of this result replacing $\mathrm{GL}_n(k)$ with any reductive group G. We introduce the notions of irreducibility and completely reducibility so that it works in the same way.

Definition 3.11. *A Borel subgroup of G is a maximal connected solvable linear algebraic subgroup of G.*

A parabolic subgroup P of G is a closed subgorup such that G/P is a complete variety.

Definition 3.12. *A subgroup $P \subset G$ is called parabolic if G/P it contains a Borel subgroup (maximal closed connected solvable subgroup of G). It is equivalent to the fact that G/P is a projective variety.*

A Levi subgroup of P is a connected subgroup L such that $P = R_u(P) \rtimes L$ where $R_u(P)$ is the unipotent radical of P (maximal normal unipotent subgroup). Such subgroups always exist and are reductive.

When $G = \mathrm{GL}(V)$, parabolic subgroups are the stabilizers of flags. Levi subgroup of the parabolic subgroup stabilizing $F = (F_1, \dots, F_r)$ correspond to the stabilizer of a decomposition $E = E_1 \oplus \cdots E_r$ such that $F_l = E_1 \oplus \cdots E_l$ for all $l \leq r$.

Definition 3.13. *A subgroup $H \subset G$ will be said*

1. *irreducible if it is not contained in any proper parabolic subgroup.*
2. *completely reducible if whenever $H \subset P$, there exists a Levi subgroup $L \subset P$ such that $H \subset L$.*

We observe that in this definition, H can be safely replaced with its Zariski closure.

The somewhat technical definition of complete reducibility makes sense due to the following characterization:

Theorem 3.14. *A subgroup $H \subset G$ is completely reducible if and only if its closure $\overline{H}$ is reductive.*

Any Lie group has an adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. It is instructive to compare the properties of ρ and $\mathrm{Ad} \circ \rho$.

Theorem 3.15. *Let $\rho : \Gamma \to G$ be a homomorphism.*

1. *$G\rho$ is closed if and only if ρ is completely reducible.*
2. *ρ is stable for the action of $G/Z(G)$ if and only if it is irreducible.*

Proof. When G is semi-simple then $Z(G)$ is finite and the adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ has finite kernel. A representation $\rho : \Gamma \to G$ is then Ad-irreducible if and only if $\rho(\Gamma)$ is Zariski-dense in G. Indeed, the Lie algebra of $\mathrm{Ad}\,\overline{\rho(\Gamma)}$ is an Ad $\circ\rho$-invariant subspace of $\mathfrak{g}$. Here is the generalization of Theorem 8 in the context of general reductive groups. We refer to [6] for the proof. □

Consider the character variety $X(\Gamma, \mathrm{GL}_n(k))$ where k is no longer algebraically closed (but still has zero characteristic). A representation $\rho : \Gamma \to \mathrm{GL}_n(k)$ will be said absolutely irreducible if it is irreducible when extended to $\mathrm{GL}_n(\bar{k})$ where $\bar{k}$ is an algebraic closure of k. Let us start with a useful lemma.

Lemma 3.16. *The subset $X^{\mathrm{irr}}(\Gamma, \mathrm{GL}_n(k))$ of characters of irreducible representations is Zariski-open in $X(\Gamma, \mathrm{GL}_n(k))$.*

Proof. This is true for any reductive group, but we give here a simple proof, based on Burnside's theorem on matrix algebras. This claims that:

$$\rho : \Gamma \to \mathrm{GL}_n(k) \text{ is absolutely irreducible } \iff \mathrm{Span}\{\rho(\gamma), \gamma \in \Gamma\} = \mathrm{M}_n(k)$$

For any subset $I \subset \Gamma$ of cardinality n^2, the determinant $\Delta_I = \det\left(\mathrm{Tr}\left(\rho\left(\gamma_i \gamma_j\right)\right)\right)_{i,j \in I}$ is non zero iff $\left(\rho\left(\gamma_i\right)\right)_{i \in I}$ is a basis of $\mathrm{M}_n(k)$. As $\Delta_I \in k\left[X\left(\Gamma, \mathrm{GL}_n\right)\right]$, the locus $\bigcup_{I \subset \Gamma} \{\Delta_I \neq 0\}$ is open and equal to $X^{\mathrm{irr}}(\Gamma, \mathrm{GL}_n(k))$, proving the lemma. □

3.3 Tangent space

Let's recall the definition of cohomology with twisted coefficients. Let V be a representation of Γ : we write $g.v$ for the action of g on v. We set $C^n(\Gamma, V) = \{f : \Gamma^n \to V\}$ and define $d : C^n(\Gamma, V) \to C^{n+1}(\Gamma, V)$ by the formula

$$(df)(\gamma_0, \ldots, \gamma_n) = \gamma_0 \cdot f(\gamma_1, \ldots, \gamma_n) + \sum_{i=1}^{n} (-1)^i f(\gamma_0, \ldots, \gamma_{i-1}\gamma_i, \ldots, \gamma_n) \\ + (-1)^{n+1} f(\gamma_0, \ldots, \gamma_{n-1})$$

We denote by $H^*(\Gamma, V)$ the cohomology of this complex. We easily check that $H^0(\Gamma, V) = V^\Gamma$, the space of invariants and $H^1(\Gamma, V) = Z^1(\Gamma, V)/B^1(\Gamma, V)$ where $B^1(\Gamma, V)$ is the space of maps $f : \Gamma \to V$ of the form $f(\gamma) = \gamma \cdot v - v$ for some $v \in V$ and

$$Z^1(\Gamma, V) = \{f : \Gamma \to V, f(\gamma_0\gamma_1) = f(\gamma_0) + \gamma_0.f(\gamma_1)\}.$$

Let $\rho : \Gamma \to G$ be a homomorphism. Corresponding dually to $\rho \in \mathrm{Hom}(\Gamma, G)$, there is an algebra morphism $\phi_\rho : k[\mathrm{Hom}(\Gamma, G)] \to k$. This follows from the universal property that we recall now. For any k-algebra B, there is a group structure on $G(B) = \mathrm{Hom}_{\mathrm{alg}}(k[G], B)$ such that

$$\mathrm{Hom}_{\mathrm{alg}}(k[\mathrm{Hom}(\Gamma, G)], B) = \mathrm{Hom}_{\mathrm{gr}}(\Gamma, G(B))$$

A Zariski tangent vector at ρ is an algebra morphism $\phi_\epsilon : k[\mathrm{Hom}(\Gamma, G)] \to k[\epsilon]/\left(\epsilon^2\right)$ which reduces to ϕ_ρ modulo ϵ.

This shows that $T_\rho \mathrm{Hom}(\Gamma, G) = \left\{\rho_\epsilon : \Gamma \to G\left(k[\epsilon]/\left(\epsilon^2\right)\right), \rho_\epsilon = \rho \bmod \epsilon\right\}$. Setting $f : \Gamma \to \mathfrak{g}$ so that $\rho_\epsilon(\gamma) = (1 + \epsilon f(\gamma))\rho(\gamma)$. We denote by $\mathrm{Ad} \circ \rho$ the vector space $\mathfrak{g}$ viewed as a Γ module by the formula $\gamma \cdot v = \mathrm{Ad}(\rho(\gamma)).v$.

A direct computation, shows that ρ_ϵ is a group homomorphism if and only if f satisfies the cocycle condition defining $Z^1(\Gamma, \mathrm{Ad} \circ \rho)$, hence the following proposition holds.

Proposition 3.17. *For any $\rho : \Gamma \to G$ there is a natural isomorphism*

$$T_\rho \mathrm{Hom}(\Gamma, k) \simeq Z^1(\Gamma, \mathrm{Ad} \circ \rho).$$

In general, the differential $D_\rho \pi : T_\rho \mathrm{Hom}(\Gamma, G) \to X(\Gamma, G)$ induces a linear map $Z^1(\Gamma, \mathrm{Ad} \circ \rho) \to T_{\pi(\rho)} X(\Gamma, G)$.

Define $C_\rho : G \to \mathrm{Hom}(\Gamma, G)$ by $C_\rho(g) = g\rho g^{-1}$: almost by definition of the linear representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, one induces a linear map

$$\Phi_\rho : H^1(\Gamma, \mathrm{Ad} \circ \rho) \to T_{[\rho]} X(\Gamma, G)$$

Example 3.18. *Recall that $X(F_2, \mathrm{SL}_2(\mathbb{C})) = \mathbb{C}^3$. Show that at the trivial representation, the map Φ_1 below vanishes:*

$$H^1(F_2, \mathrm{Ad} \circ \rho) = \mathrm{sl}_2(\mathbb{C})^2 \xrightarrow{\Phi_1} T_1 X(F_2, \mathrm{SL}_2(\mathbb{C})) = \mathbb{C}^3$$

Corollary 3.19. *The tangent space of $X(\Gamma, \mathrm{SL}_2(k))$ at the trivial character is given by functions $f : \Gamma \to k$ satisfying the parallelogram identity for all $\alpha, \beta \in \Gamma$*

$$f(\alpha\beta) + f\left(\alpha\beta^{-1}\right) = 2f(\alpha) + 2f(\beta).$$

Proof. Put $t_\gamma = 2 + \epsilon f(\gamma) \in k[\epsilon]/\left(\epsilon^2\right)$ and the trace equation becomes the parallelogram identity. □

Theorem 3.20. *Let $\rho : \Gamma \to G$ be a completely reducible homomorphism and denote by $Z(\rho)$ the centralizer of ρ.*

1. If $Z(\rho) = Z(G)$ then Φ_ρ is an isomorphism.

2. If ρ is a smooth point of $\mathrm{Hom}(\Gamma, G)$ *then*

$$T_0\left(H^1(\Gamma, \mathrm{Ad} \circ \rho)//Z(\rho)\right) \simeq T_{[\rho]} X(\Gamma, G)$$

Proof. We prove 1. Both statements follow from Luna's Slice theorem which applies anytime we have a closed orbit in an affine variety. Replacing G by $G/Z(G)$ we can suppose that G acts freely at ρ. Luna's theorem says that there exists a subvariety $S \subset \mathrm{Hom}(\Gamma, G)$ containing ρ such that the map $\Phi : G \times S \to \mathrm{Hom}(\Gamma, G)$ sending (g, ρ) to $g.\rho$ is etale, as the map $\pi|_S : S \to X(\Gamma, G)$.

This implies that $D\Phi$ and $D\pi|_S$ are isomorphisms. Hence $T_\rho S$ is a complement of $B^1(\Gamma, \mathrm{Ad} \circ \rho)$ in $Z^1(\Gamma, \mathrm{Ad} \circ \rho)$, its projection on $H^1(\Gamma, \mathrm{Ad} \circ \rho)$ induces the isomorphism Φ_ρ, proving the theorem. Suppose that $\Gamma = F_r$ so that $X(F_r, G) = G^r /\!\!/ G$. As $F_r = \pi_1(B_r)$ where B_r is a bouquet of r circles which is aspherical, one has $H^*(F_r, \mathrm{Ad} \circ \rho) = H^*(B_r, \mathrm{Ad} \circ \rho)$. In particular, these cohomology groups vanish in degree distinct from 0,1 and computing the twisted Euler characteristic gives

$$\dim H^0(B_r, \mathrm{Ad} \circ \rho) - \dim H^1(B_r, \mathrm{Ad} \circ \rho) = \chi(B_r) \dim \mathfrak{g} = (1 - r) \dim \mathfrak{g}.$$

Definition 3.21. *We say that a representation $\rho : \Gamma \to G$ is good if it is irreducible and $Z(\rho) = Z(G)$.*

If ρ is good, we then have $\dim H^1(F_r, \mathrm{Ad} \circ \rho) - \dim z(G) = \dim T_{[\rho]} X(F_r, G) = (r-1)\dim \mathfrak{g}$. We sum up this discussion in the following proposition:

Proposition 3.22. *The open set $X^{good}(F_r, G)$ is smooth of dimension $(r-1)\dim \mathfrak{g} + \dim z(G)$.*

Of course, this also follows from the fact that $X^{\text{good}}(F_r, G)$ is the quotient of an open subset of G^r by a free and proper action of $G/Z(G)$. One can show using Reidemeister torsion that there is an algebraic volume form on $X^{\text{good}}(F_r, G)$, natural in the sense that it is invariant by the action of $\mathrm{Out}(F_r)$ given by $[\phi] \cdot [\rho] = [\rho \circ \phi^{-1}]$.

Remark 3.23. *Let $\phi : \mathbb{Z}^2(\Gamma, \mathbb{R}) \to \mathbb{R}$ be any continuous linear function that vanishes on $B^2(\Gamma, \mathbb{R})$. Then, $\phi \circ \omega$ is a closed 2-form on $\mathrm{Hom}(\Gamma, G)$, where $\omega : Z^1(\Gamma, \mathbb{R}) \times Z^1(\Gamma, \mathbb{R}) \to Z^2(\Gamma, \mathbb{R})$.*

Hence there is a natural symplectic structure on so-called relative character varieties.

3.4 Applications to Painleve VI

The sixth Painlevé equation $P_{VI} = P_{VI}(\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$ is the second order non linear ordinary differential equation

$$P_{VI} : \begin{cases} \dfrac{d^2q}{dt^2} = \dfrac{1}{2}\left(\dfrac{1}{q} + \dfrac{1}{q-1} + \dfrac{1}{q-t}\right)\left(\dfrac{dq}{dt}\right)^2 - \left(\dfrac{1}{t} + \dfrac{1}{t-1} + \dfrac{1}{q-t}\right)\left(\dfrac{dq}{dt}\right) \\ \qquad + \dfrac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\dfrac{(\theta_\delta - 1)^2}{2} - \dfrac{\theta_\alpha^2}{2}\dfrac{t}{q^2} + \dfrac{\theta_\beta^2}{2}\dfrac{t-1}{(q-1)^2} + \dfrac{1-\theta_\gamma^2}{2}\dfrac{t(t-1)}{(q-t)^2}\right) \end{cases}$$

the coefficients of which depend on complex parameters

$$\theta = (\theta_\alpha, \theta_\beta, \theta_\gamma, \theta_\delta)$$

Remark 3.24. *The main property of this equation is the absence of movable singular points, the so-called Painlevé property: all essential singularities of all solutions $q(t)$ of the equation only appear when $t \in \{0, 1, \infty\}$; in other words, any solution $q(t)$ extends analytically as a meromorphic function on the universal cover of $\mathbb{P}^1(\mathbf{C}) \backslash \{0, 1, \infty\}$.*

Let $\mathbb{S}_4^2$ be the four punctured sphere. Its fundamental group is isomorphic to a free group of rank 3:

$$\pi_1\left(\mathbb{S}_4^2\right) = \langle \alpha, \beta, \gamma, \delta \mid \alpha\beta\gamma\delta = 1 \rangle$$

Let $\mathrm{Rep}\left(\mathbb{S}_4^2\right)$ be the set of representations of $\pi_1\left(\mathbb{S}_4^2\right)$ into $\mathrm{SL}(2, \mathbf{C})$. Such a representation ρ is uniquely determined by the 3 matrices $\rho(\alpha), \rho(\beta)$, and $\rho(\gamma)$, so that $\mathrm{Rep}\left(\mathbb{S}_4^2\right)$ can be identified with the affine algebraic variety $(\mathrm{SL}(2, \mathbf{C}))^3$. Let us associate the 7 following traces to any element ρ of $\mathrm{Rep}\left(\mathbb{S}_4^2\right)$:

$$\begin{array}{ccccccc} a = \mathrm{tr}(\rho(\alpha)) & ; & b = \mathrm{tr}(\rho(\beta)) & ; & c = \mathrm{tr}(\rho(\gamma)) & ; & d = \mathrm{tr}(\rho(\delta)) \\ x = \mathrm{tr}(\rho(\alpha\beta)) & ; & y = \mathrm{tr}(\rho(\beta\gamma)) & ; & z = \mathrm{tr}(\rho(\gamma\alpha)). & & \end{array}$$

Remark 3.25. *The polynomial map $\chi : \mathrm{Rep}\left(\mathbb{S}_4^2\right) \to \mathbf{C}^7$ defined by*

$$\chi(\rho) = (a, b, c, d, x, y, z) \tag{4.1}$$

is invariant under conjugation, by which we mean that $\chi(\rho') = \chi(\rho)$ if ρ' is conjugate to ρ by an element of $\mathrm{SL}(2, \mathbf{C})$. Moreover,

(1) the algebra of polynomial functions on $\mathrm{Rep}\left(\mathbb{S}_4^2\right)$ which are invariant under conjugation is generated by the components of χ;

(2) the components of χ satisfy the quartic equation

$$x^2 + y^2 + z^2 + xyz = Ax + By + Cz + D \tag{4.2}$$

in which the variables $A, B, C,$ and D are given by

$$A = ab + cd, \quad B = bc + ad, \quad C = ac + bd, \quad D = 4 - a^2 - b^2 - c^2 - d^2 - abcd. \tag{4.3}$$

(3) the algebraic quotient $\mathrm{Rep}\left(\mathbb{S}_4^2\right) /\!\!/ \mathrm{SL}(2, \mathbf{C})$ of $\mathrm{Rep}\left(\mathbb{S}_4^2\right)$ by the action of $\mathrm{SL}(2, \mathbf{C})$ by conjugation is isomorphic to the six-dimensional quartic hypersurface of $\mathbf{C}^7$ defined by equation (4.2).

Definition 3.26. *The affine algebraic variety* $\mathrm{Rep}\left(\mathbb{S}_4^2\right) /\!\!/ \mathrm{SL}(2,\mathbf{C})$ *will be denoted* $\chi\left(\mathbb{S}_4^2\right)$ *and called the character variety of* $\mathbb{S}_4^2$*.*

For each choice of four complex parameters A, B, C, and D, we will denote by $S_{(A,B,C,D)}$ the cubic surface of $\mathbb{C}^3$ defined by the equation (4.2). The family of these surfaces $S_{(A,B,C,D)}$ will be denoted *Fam*.

Singular points of the surface $S_{(A,B,C,D)}$ arise from semi-stable points of $\mathrm{Rep}\left(\mathbb{S}_4^2\right)$, that is to say either from reducible representations, or from those representations for which one of the matrices $\rho(\alpha), \rho(\beta), \rho(\gamma)$ or $\rho(\delta)$ is in the center $\pm I$ of $\mathrm{SL}(2,\mathbf{C})$. The smooth part of the surface $S_{(A,B,C,D)}$ is equipped with the holomorphic volume form

$$\Omega = \frac{dx \wedge dy}{2z + xy - C} = \frac{dy \wedge dz}{2x + yz - A} = \frac{dz \wedge dx}{2y + zx - B} \tag{4.4}$$

After minimal resolution of the singular points by blowing-up, the 2 -form extends as a global holomorphic volume form on the smooth surface.

Theorem 3.27. *Let* $A, B, C,$ *and* D *be four complex numbers. Let* M *be an element of* $\Gamma_2^{\pm}$*, and* f_M *be the automorphism of* $S_{(A,B,C,D)}$ *which is determined by* M*. The topological entropy of* $f_M : S_{(A,B,C,D)}(\mathbb{C}) \to S_{(A,B,C,D)}(\mathbb{C})$ *is equal to the logarithm of the spectral radius of* M*.*

Definition 3.28. *A specific choice of the parameters will play a central role, for* $(A, B, C, D) = (0, 0, 0, 4)$*. The surface* $S_{(0,0,0,4)}$ *is the unique surface in our family having four singularities. We call it the Cayley cubic.*

The main result that we shall prove concerns the classification of parameters (A, B, C, D) for which $S_{(A,B,C,D)}$ admits a $\Gamma_2^{\pm}$-invariant holomorphic geometric structure.

Theorem 3.29. *The group* $\Gamma_2^{\pm}$ *does not preserve any holomorphic curve of finite type, any (singular) holomorphic foliation, or any (singular) holomorphic web. The group* $\Gamma_2^{\pm}$ *does not preserve any meromorphic affine structure, except in the case of the Cayley cubic, i.e. when* $(A, B, C, D) = (0, 0, 0, 4)$*, or equivalently when*

$$(a, b, c, d) = (0, 0, 0, 0) \text{ or } (2, 2, 2, -2)$$

up to multiplication by -1 and permutation of the parameters.

Corollary 3.30. *The sixth Painlevé equation is irreducible in the sense of Malgrange and Casale except when* $(A, B, C, D) = (0, 0, 0, 4)$*, i.e. except in one of the following cases:*
(1) $\theta_\omega \in \frac{1}{2} + \mathbf{Z}, \forall \omega = \alpha, \beta, \gamma, \delta,$
(2) $\theta_\omega \in \mathbf{Z}, \forall \omega = \alpha, \beta, \gamma, \delta,$ *and* $\sum_\omega \theta_\omega$ *is even.*

At last, we would like to mention the Riemann-Hilbert correpondence.

Theorem 3.31. *Suppose* Σ *is a smooth compact complex algebraic curve. The following categories are equivalent:*
(1) Algebraic connections on algebraic vector bundles on Σ*,*
(2) Holomorphic connections on holomorphic vector bundles on Σ*,*
(3) Flat C^∞ *connections on* C^∞ *complex vector bundles on* Σ*,*
(4) Local systems of finite dimensional complex vector spaces on Σ*,*
(5) For any fixed basepoint $b \in \Sigma$*, the category of finite dimensional complex* $\pi_1(\Sigma, b)$ *representations.*

A Projective GIT and Kempf–Ness

Definition A.1. *An action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$ is said to be linear if G acts via a homomorphism $G \to \mathrm{GL}_{n+1}$.*

If we have a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, then G acts on the affine cones $\mathbb{A}^{n+1}$ and $\tilde{X}$ over $\mathbb{P}^n$ and X. In particular G acts on $R := A(\tilde{X})$ and preserves the graded pieces so that $R^G = A(\tilde{X})^G$ is a homogeneous graded subalgebra of R. By Nagata's theorem this is also finitely generated and so we can consider the associated projective variety $\operatorname{Proj}\left(R^G\right)$.

Definition A.2. *For a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$, we let $X /\!\!/ G$ denote the projective variety $\operatorname{Proj}\left(R^G\right)$ associated to the finitely generated graded k-algebra R^G of G-invariant functions where $R = R(X)$ is the homogeneous coordinate ring of X. The inclusion $R^G \hookrightarrow R$ defines a rational map*

$$\varphi : X \to X /\!\!/ G$$

which is undefined on the null cone

$$N_{R^G}(X) := \left\{x \in X : f(x) = 0 \forall f \in R_+^G\right\}.$$

We define the semistable locus $X^{ss} := X - N_{R^G}(X)$ to be the complement to the nullcone. Then the projective GIT quotient for the linear action of G on $X \subset \mathbb{P}^n$ is the morphism $\varphi : X^{ss} \to X /\!\!/ G$.

Proposition A.3. *The projective GIT quotient for a linear action of a reductive group G on a projective variety $X \subset \mathbb{P}^n$ is a good quotient of the action of G on X^{ss}.*

We can now ask if there is an open subset X^s of X^{ss} on which this quotient becomes a geometric quotient. For this we want the action to be closed on X^s, or at least the action is closed on some affine open G-invariant subsets which cover X^s. This motivates the definition of stability:

Definition A.4. *Consider a linear action of a reductive group G on a closed subvariety $X \subset \mathbb{P}^n$. Then a point $x \in X$ is*

(1) semistable if there is a G-invariant homogeneous polynomial $f \in R(X)_+^G$ such that $f(x) \neq 0$.

(2) stable if $\dim G_x = 0$ and there is a G-invariant homogeneous polynomial $f \in R(X)_+^G$ such that $x \in X_f$ and the action of G on X_f is closed.

(3) unstable if it is not semistable.

We denote the set of stable points by X^s and the set of semistable points by X^{ss}.

Remark A.5. *The semistable set X^{ss} is the complement of the null cone $N_{R^G}(X)$ and so is open in X. The stable locus X^s is open in X (and also in X^{ss}): let $X_c := \cup X_f$ where the union is taken over $f \in R(X)_+^G$ such that the action of G on X_f is closed; then X_c is open in X and it remains to show X^s is open in X_c. By Proposition, the function $x \mapsto \dim G_x$ is an upper semi-continuous function on X and so the set of points with zero dimensional stabiliser is open. Therefore, we have open inclusions $X^s \subset X_c \subset X$.*

Theorem A.6. *For a linear action of a reductive group G on a closed subvariety $X \subset \mathbb{P}^n$, we have: i) The GIT quotient $\varphi : X^{ss} \to Y := X//G$ is a good quotient and a categorical quotient. Moreover, Y is a projective variety. ii) $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \cap X^{ss} \neq 0$ if and only if $\varphi(x_1) = \varphi(x_2)$. iii) There is an open subset $Y^s \subset Y$ such that $\varphi^{-1}(Y^s) = X^s$ and $\varphi : X^s \to Y^s$ is a geometric quotient.*

Remark A.7. *Different linearizations will give different sets of (semi-)stable points.*

Theorem A.8 (Kempf–Ness)**.** *If the complex vector space is given a norm that is invariant under a maximal compact subgroup of the reductive group, then the Kempf–Ness theorem states that a vector is stable if and only if the norm attains a minimum value on the orbit of the vector.*

Corollary A.9. *If X is a complex smooth projective variety and if G is a reductive complex Lie group, then $X//G$ (the GIT quotient of X by G) is homeomorphic to the symplectic quotient of X by a maximal compact subgroup of G.*