Artin L- function

0. ramification

1. Artin L- function (non-ramified)

   (from Galois representations)

2. automorphic (cuspidal) representation

3. Langlands functorially, naive form

4. 吹牛.

Dedekind domain B/A $p \in A$

$$\underline{pB = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_g^{e_g}} \qquad e_i \geq 1$$

def: p 是分歧的 (ramified in B) if. 存在 $e_i > 1$

注: 其中 e_i 为 ramification index. 记 $\mathfrak{P} | p$.

$f_i(\mathfrak{P}/p)$: degree $[B/\mathfrak{P} : A/p]$ residue class degree.

称 p 是分裂的. 指 $e_i = f_i = 1$ 对 all i

称 p 是 inert pB 还是素理想.

称 $\mathfrak{P}$ 是分歧的 如果 $\mathfrak{P}$ 所"<u>对应的</u>素理想 $p \subset A$"

是分歧的.

$$\underline{p = \mathfrak{P} \cap K}$$

Theorem: 如果 A 是 Dedekind domain, 其分式域记作 K

L/K 的有限扩张. 记 A 在 L 中整闭包为 B.

如果 $[L:K] = m$.

$$\sum_{i=1}^{g} e_i f_i = m$$

L/K 是 Galois extension, 那么.

e_i 相同

$\Rightarrow \quad efg = m$

milne. ANT.

Only finitely many prime ideals ramify.

1. Artin L- function.

$\boxed{E/F}$ 上的 finite Galois extention.

let $\underline{\rho:}$ $\mathrm{Gal}(E/F) \longrightarrow GL(n, \mathbb{C})$

$L(s, \rho)$.

Artin L(1930).
Brauer (1941)

关心 non-ramified primes. E/F

如果 p 是 F 上的素理想, 无分歧 over E.

记 $\underline{\mathfrak{P}|p}$, $\sigma_{\mathfrak{P}|p} \in \mathrm{Gal}(E/F)$.

$$\sigma_{\mathfrak{P}|\mathfrak{p}}(x) \equiv x^{N(\mathfrak{p})} \mod \mathfrak{P}.$$

其中 $\boxed{N(\mathfrak{p}) = |F/\mathfrak{p}|}$

引入 decomposition group $D_{\mathfrak{P}} < \mathrm{Gal}(E/F)$.

$$D_{\mathfrak{P}} := \{\sigma \in \mathrm{Gal}(E/F) \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}.$$

($\mathfrak{P}$ | $\mathfrak{p}$).

$$D_{\mathfrak{P}} \xrightarrow{\phi} \mathrm{Gal}\left((B/\mathfrak{P})/(A/\mathfrak{p})\right)$$

ϕ 是满射.

unramified $\Rightarrow$ ϕ 是同构.

Bump: E/F Abelian $\Rightarrow$ $\sigma_{\mathfrak{P}|\mathfrak{p}} = \sigma_{\mathfrak{p}}$

$\mathfrak{p} = \mathfrak{P}_1 \mathfrak{P}_2 \cdots \mathfrak{P}_g$ $\quad \sigma_{\mathfrak{P}_i|\mathfrak{p}}$ 是共轭的.

$\mathfrak{p}$: $\quad L_{\mathfrak{p}}(s,\rho) = \det\left(I_n - \rho(\sigma_{\mathfrak{p}}) N(\mathfrak{p})^{-s}\right)^{-1}$

如果 $\rho(\sigma_{\mathfrak{p}})$ 的特征值为

$$\alpha_1(\mathfrak{p}), \quad \cdots, \quad \alpha_n(\mathfrak{p})$$

$$L_{\mathfrak{p}}(s,\rho) = \prod_{i=1}^{n} (1 - \alpha_i(\mathfrak{p}) N(\mathfrak{p})^{-s})^{-1}$$

$$L(s,\rho) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(s,\rho).$$

如果 ρ 是分段的.

$$L_p(s,\rho) = \det\left(I - \rho(p) \,|\, V^{I_p} N(p)^{-s} \right)^{-1}$$

$E = \mathbb{Q}$. $\qquad \mathrm{Gal}(\bar{E}/\mathbb{Q})$.

2. automorphic cuspidal representation

$GL(n, F)$, F 是一个数域

$GL(n, \mathbb{A})$ $\quad \underline{\mathbb{A}} = \mathbb{A}_F$ 是 F 的 adele ring.

$\mathbb{Q}$. $\quad \mathbb{A}_{\mathbb{Q}}$: $\quad \prod (\mathbb{Q}_v = \mathbb{Z}_v)$.

$\mathbb{Q}_p$, $\mathbb{Z}_p$, $\qquad \mathbb{Q}$. $\qquad \infty$. $\quad \mathbb{R}$.

$\qquad\qquad\qquad p$. $\quad \mathbb{Q}_p$

$(a_i) \in \mathbb{A}_{\mathbb{Q}}$, $\quad a_i \in \mathbb{Q}_v \qquad v \in \{p, \infty\}$.

$a_i \notin \mathbb{Z}_v$ 只有有限多个.

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{R} \times \prod \mathbb{Z}_p)$$

$$\frac{a}{b} \otimes_{\mathbb{Z}} (r, \mathbb{Z}_p)$$

$$= a \otimes_{\mathbb{Z}} \left(\frac{r}{b}, \boxed{\frac{\mathbb{Z}_p}{b}} \right).$$

$$b = \underline{p_1^{\alpha_1}} \cdots \underline{p_n^{\alpha_n}} \qquad v_p(b) = 0$$

GL_n over F. $\quad \mathbb{A}_F\ (\mathbb{A})$.

$GL(n, F)$ 是 $GL(n, \mathbb{A})$ 的离散子群.

$$a \longrightarrow (a)$$

$\underline{Z_{\mathbb{A}}}$: scalar matrices. entries $\in \mathbb{A}^\times$

idele group.

$$\underline{GL(n,\mathbb{A})} \backslash \underline{(\boxed{Z_{\mathbb{A}}}\ GL(n,F))}$$

取 ω 为 $\mathbb{A}^\times / F^\times$ 的一个特征

$|\omega| = 1$

$$GL(n,\mathbb{A}) / GL(n,F) \longrightarrow \underline{L^2(GL(n,F)\backslash GL(n,\mathbb{A}),\ \omega)}$$

① $\displaystyle\int_{\underline{Z_{\mathbb{A}} GL(n,F)\backslash GL(n,\mathbb{A})}} |f(g)|^2 \underline{dg} < \infty.$

② $\underline{f\left(\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} g\right)} = \underline{\omega(z)} f(g).$ $\quad \in GL(n,\mathbb{A})$

cusp form:

③ $\displaystyle\int_{(\mathbb{A}/F)^{r(n-r)}} f\left(\begin{pmatrix} I_r & \boxed{x} \\ & I_{n-r} \end{pmatrix} g\right) dx = 0$

$1 \le r < n.$

可以证明. $\rho : GL(n,\mathbb{A}) \longrightarrow \mathrm{End}(L^2)$. $\underline{\text{right translation}}$

$$(\rho(g)f)(g') = f(g'g).$$

满足 ③ L^2 的子空间. $\boxed{L_0^2}$ cusp form

$$\Rightarrow \quad L(s,\pi) = \prod_{p} \left(1 - \alpha_1(p) N(p)^{-s}\right)^{-1} \cdots \left(1 - \alpha_n(p) N(p)^{-s}\right)^{-1}$$

Conjecture 1.8.1. Langlands.

$$\rho : \mathrm{Gal}(E/F) \longrightarrow GL(n, \mathbb{C})$$

存在一个自守表示

$$GL(n, F) \longrightarrow L^2$$

使得 Artin L-function 相同.

Conjecture 1.8.2.

Operations on Artin L-function correspond to operations on automorphic forms.

(a). Galois representation

$$\begin{tikzcd} \mathrm{Gal}(E/F) \arrow[r] \arrow[d] & GL(n, \mathbb{C}) \arrow[r] & GL(m, \mathbb{C}) \\ {} & & \end{tikzcd}$$

取 $n=2$. $m = r+1$

$$\underline{GL(2, \mathbb{C})} \longrightarrow \underline{GL(r+1, \mathbb{C})}$$

$$\underline{V \longrightarrow \vee^r V} \quad \dim = C_{r+1}^{1} = r+1.$$

twist $(\det^{-\alpha} \otimes \vee^r V.)$ 后谈..

$$\det{}^{-1} \otimes \vee^2 V.$$

$$\alpha_1, \alpha_2 \longrightarrow \alpha_1^r, \alpha_1^{r-1}\alpha_2, \cdots, \alpha_2^r$$

$$\underline{L(s,\rho)} = \prod_p (1-\alpha_1(p) N(p)^{-s})^{-1} (1-\alpha_2(p) N(p)^{-s})^{-1} \quad (1)$$

$$\Rightarrow \quad L(s,\rho) = \prod_p (1-\alpha_1^r(p) N(p)^{-s})^{-1} \cdots\cdots (1-\alpha_2(p)^r N(p)^{-s})^{-1} \quad (2)$$

Prediction

如果 π 是 $GL(2)$ 的自守表示

$L(s,\pi) =$ (1).

存在一个自守表示 $V^r\pi$ 满足

$L(s, V^r\pi) =$ (2)

(2). $GL(n_1, \mathbb{C}) \times \cdots GL(n_r, \mathbb{C}) \longrightarrow GL(n, \mathbb{C})$.

$r=2$ 时.

$\rho_1 : \quad Gal(E_1/F) \longrightarrow GL(n, \mathbb{C})$

$\rho_2 : \quad Gal(E_2/F) \longrightarrow GL(m, \mathbb{C})$

$E = \langle E_1, E_2 \rangle$

$$\begin{tikzcd} & E \arrow[dl] \arrow[dr, hook] & \\ E_1 \arrow[dr] & & E_2 \arrow[dl] \\ & F & \end{tikzcd}$$

$\rho \qquad Gal(E, F)$.

$E_1 = E_2 = E$

$\underline{\rho} = \rho_1 \otimes \rho_2$

$$\alpha_i(\rho) = \alpha_{ij}(\rho_1)\,\alpha_{ik}(\rho_2).$$

$$L_p(s, \rho_1) = \prod_{i=1}^{n} (1 - \underline{\alpha_i(p)}\, N(p)^{-s})^{-1}$$

$$L_p(s, \rho_2) = \prod_{i=1}^{m} (1 - \beta_i(p)\, N(p)^{-s})^{-1}$$

$$\Rightarrow L_p(s, \rho) = \prod_{i,j} (1 - \alpha_i(p)\, \beta_j(p)\, N(p)^{-s})^{-1}$$

prediction

π_1, π_2 是 $GL(n)$, $GL(m)$ automorphic representations.

则存在 $GL(nm)$ ______.

满足 上述 的 Artin L-function.

(3). Galois group restricted to subgroup.

$E \supset K/F$ Galois. $Gal(K/F) < Gal(E/F)$

$$\Rightarrow. \quad \rho = Gal(E/F) \longrightarrow GL(n, \mathbb{C}).$$

$$\downarrow$$

$$\rho' = Gal(E/K) \longrightarrow GL(n, \mathbb{C}).$$

$\Downarrow$.

$$\pi: GL(n, F) \longrightarrow GL(n, K).$$

base change.

$$|K/F| = 2.$$

Hecke character over F

$$\chi(p) = \begin{cases} 1 & \text{if } p \text{ split} \\ -1 & \text{if } p \text{ remained.} \\ 0 & \text{if } p \text{ ramified.} \end{cases}$$

K/F

p 是 F 上 co prime

$$L_p(s, \rho) = \prod_{i=1}^{n} (1 - \alpha_i(p) N(p)^{-s})^{-1}$$

① split. $\quad p O_K = \mathfrak{P}_1 \mathfrak{P}_2$

$$\sigma_p = \sigma_{\mathfrak{P}_1} = \sigma_{\mathfrak{P}_2}$$

$$L_{\mathfrak{P}_1}(s, \rho') = L_{\mathfrak{P}_2}(s, \rho') = L_p(s, p)$$

$$\Rightarrow \quad L_{\mathfrak{P}_1} L_{\mathfrak{P}_2} = \prod_{i=1}^{n} (1 - \alpha_i(p) N(p)^{-s})^{-2}$$

② p 是 inert. $\quad p O_K = \mathfrak{P}.$

$$\Rightarrow \quad \sigma_{\mathfrak{P}} = \sigma_p^2$$

$$efg = m = 2 \Rightarrow ef = 1 \Rightarrow f = 2.$$

$$[B/\mathfrak{P} : A/p] = 2.$$

$$N(\mathfrak{P}) = N(p)^2$$

$$\Rightarrow \quad L_{\mathfrak{P}}(s, \rho') = \prod_{i=1}^{n} (1 - \alpha_i(p)^2 N(p)^{-2s})^{-1}$$

$$= \prod_{i=1}^{n} (1 - \alpha_i(p) N(p)^{-s})^{-1} (1 + \alpha_i(p) N(p)^{-s})^{-1}$$

$$\Rightarrow \quad \underline{L_p(s, \rho') = L_p(s,\rho)\, L_p(s, \underline{\chi\otimes\rho})}.$$

其中 $$L(s, \chi\otimes\rho) = \prod_{i=1}^{n} \left(1 - \chi(p)\, \alpha_i(p)\, N(p)^{-s}\right)^{-1}$$

Prediction $\qquad L(s, \pi, \chi)$

(4). Galois subgroup induced.

K/F

$$\begin{array}{ccc} \mathrm{Gal}(E/K) & \longrightarrow & GL(n, \mathbb{C}) \\ \downarrow & & \\ \mathrm{Gal}(E/F) & \longrightarrow & GL(nr, \mathbb{C}) \end{array}$$

其中 $r = [K:F]$

$$L(s, \tau') = L(s, \tau).$$

$r=2$ 时.

$$\tau'(\sigma) = \begin{cases} \begin{pmatrix} \tau(\sigma) & \\ & \tau(r\sigma r^{-1}) \end{pmatrix} & \text{if } \sigma \in \mathrm{Gal}(E/K) \\ \begin{pmatrix} & \tau(\sigma r^{-1}) \\ \tau(r\sigma) & \end{pmatrix} & \text{otherwise.} \end{cases}$$

① split: $\qquad p\mathcal{O}_K = \mathfrak{P}_1 \mathfrak{P}_2$

$\sigma_{p}, \quad \underline{\sigma_{\mathfrak{P}_1}}, \quad \underline{\sigma_{\mathfrak{P}_2}}$ 在 $\mathrm{Gal}(E/F)$ 中共轭.

$p \qquad \sigma_p$ 在 $\mathrm{Gal}(E/K)$ 中. 共轭中 $\sigma_{\mathfrak{P}_1}$

$\gamma \sigma_p \gamma^{-1}$ 共轭于 $\sigma_{\mathfrak{P}_2}$

$$\tau'(\sigma_p) = \begin{pmatrix} \underline{\tau(\sigma_p)} & \\ & \underline{\tau(\gamma \sigma_p \gamma^{-1})} \end{pmatrix}$$

$$\Rightarrow L_p(s, \tau') = L_{\mathfrak{P}_1}(s, \tau) L_{\mathfrak{P}_2}(s, \tau).$$

② inert: $\sigma_p^{2} = \boxed{\sigma_{\mathfrak{P}}}$ $\sigma_p \notin \mathrm{Gal}(F/K)$.

$$\underline{\sqrt{\alpha_1}, -\sqrt{\alpha_1}, \cdots, \sqrt{\alpha_n}, -\sqrt{\alpha_n}.}$$

$$\Rightarrow L_p(s, \tau') = \prod_{i=1}^{n} (1 - \sqrt{\alpha_i} N(p)^{-s})^{-1} (1 + \sqrt{\alpha_i} N(p)^{-s})^{-1}$$

$$= \prod_{i=1}^{n} (1 - \alpha_i N(p)^{-2s})^{-1} = L_{\mathfrak{P}}(s, \tau).$$

Prediction Op.
1.3-4.

quadratic base change for $GL(2)$.