Introduction à la théorie analytique et probabiliste des nombres

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It's a brief note about the lesson given by Xianchang Meng and obviously, it hasn't been finished yet. And perhaps it's too much to collect them all. If you need hand-write file, email me.

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1 Basic analytic number theory

1.1 Some number-theory function

Mobius Function

$$\mu(n) = \begin{cases} (-1)^k & , \quad n = p_1 p_2 ... p_k \\ 0 & , \quad \text{else} \end{cases}$$

and we have Mobius identity:

$$\sum_{d|n} \mu(d) = [\frac{1}{n}] = \begin{cases} 1 & , \quad n = 1 \\ 0 & , \quad \text{else} \end{cases}$$

Euler function

$$\phi(n) = \sum_{\substack{m \leq n \\ (m,n)=1}} 1$$

Liouville's function

$$\lambda(n) = (-1)^{a_1 + ... + a_k}, \qquad n = p_1^{a_1} ... p_k^{a_k}$$

1.2 Coprime convolution

$$\sum_{\substack{n \\ (n,k)=1}} f(n) = \sum_n f(n) \sum_{d|(n,k)} \mu(d) = \sum_{d|k} \mu(d) \sum_n f(n) = \sum_{d|k} \mu(d) \sum_m f(dm) \tag{1}$$

Apply: Calculate Euler function with mobius function

$$\phi(n) = \sum_{\substack{m \leq n \\ (m,n)=1}} 1 = \sum_{m \leq n} \sum_{d|(n,m)} \mu(d) = \sum_{d|n} \mu(d) \sum_{\substack{m \leq n \\ d|m}} 1 = \sum_{d|n} \mu(d) [\frac{n}{d}]$$

1.3 Dirichlet convolution

$$f * g(n) = \sum_{d|n} f(d)g(\frac{n}{d}) = \sum_{d|n} f(\frac{n}{d})g(d) = \sum_{ab=n} f(a)g(b) \tag{2}$$

And if we let

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

Then

$$F(s) \cdot G(s) = \sum_{n=1}^{\infty} \frac{A(n)}{n^s}$$

where

$$A(n) = f * g(n)$$

Apply: Represent with zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (\sum_{n=1}^{\infty} \frac{1}{p^{sn}}) = \prod_p (1 - \frac{1}{p^s})^{-1}$$

If

$$F(s) = \sum_{n=1}^{\infty} \frac{\mu(s)}{n^s}$$

And considering the mobius identity:

$$\mu(s) * 1 = \sum_{d|s} \mu(d) = 0$$

for $s \neq 1$. Which means

$$F(s)\zeta(s) = 1$$

and the conclusion is:

$$\sum_{n=1}^{\infty} \frac{\mu(s)}{n^s} = \prod_p (1 - \frac{1}{p^s}) = \frac{1}{\zeta(s)} \tag{3}$$

1.4 Multiplication of function

If f(n) is multiplicative, we have:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(s)}{n^s} = \prod_p (1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + ...)$$

And $F(s)$ converges absolutely iff

$$\sum_{p,m} |\frac{f(p^{ms})}{p^{ms}}| < \infty$$

And if f(n) is completely multiplicative, then

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p (\sum_{n=0}^{\infty}) \frac{f^n(p)}{p^{sn}} = \prod_p (1 - \frac{f(p)}{p^s})^{-1}$$

Apply: Dirichlet series of λ function

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1 + \frac{1}{p^s})^{-1} = \frac{\zeta(2s)}{\zeta(s)} \tag{4}$$

1.5 Summation by parts

$a(n)$ is a complex sequence, and f has continuous derivative. Let

$$A(t) := \sum_{n \le t} a_n, \qquad t > 0$$

Then, we have:

$$\sum_{n \le x} a(n) f(n) = A(x) f(x) - \int_1^x A(t) f(t) dt \tag{5}$$

Proof:

$$\begin{aligned}
I &= \int_1^x A(t)f'(t)dt \\
&= \int_1^x (\sum_{n\leq t})f'(t)dt \\
&= \int_1^x \sum_{n\leq x} a(n)\cdot 1_{t\geq n} f'(t)dt \\
&= \sum_{n\leq x} a(n)\int_1^x 1_{t\geq n} f'(t)dt \\
&= \sum_{n\leq x} a(n)\int_n^x f'(t)dt = A(x)[f(x)-f(n)]
\end{aligned}$$

Apply: Calculate the order of $\sum_{n=1}^{\infty}\frac{1}{n}$

Let $a(n)=1, f(x)=\frac{1}{x}$ then

$$\begin{aligned}
A(x) &= \sum_{n\leq x} 1 = [x] = x-\{x\} \\
\sum_{n=1}^{\infty}\frac{1}{n} &= A(x)f(x) - \int_1^x A(t)f'(t)dt \\
&= (x-\{x\})\frac{1}{x} - \int_1^x (t-\{t\})(-\frac{1}{t^2})dt \\
&= 1-\frac{\{x\}}{x} + \int_1^x \frac{1}{t}dt - \int_1^x \frac{\{t\}}{t^2}dt \\
&= logx + 1 - \int_1^{\infty}\frac{\{t\}}{t^2}dt + \int_x^{\infty}\frac{\{t\}}{t^2}dt + O(\frac{1}{x}) \\
&= logx + \gamma + O(\frac{1}{x})
\end{aligned}$$

where the Euler constant:

$$\gamma = 1 - \int_1^{\infty}\frac{\{t\}}{t^2}dt \to 0.5772...$$

Conclusion:

$$\sum_{n=1}^{\infty}\frac{1}{n} = \log x + \gamma + O(\frac{1}{x}) \tag{6}$$

1.6 Riemann zeta function (1)

Fix s, $Re(s) > 0$, let

$$s(x) = \sum_{n \le x} \frac{1}{n^s}$$

Use the summation by parts:

$$a(n) = 1, f(x) = \frac{1}{x^s}$$

$$\begin{aligned}
s(x) &= [x]\frac{1}{x^s} - \int_1^x [t]f'(t)dt \\
&= x^{1-s} - \frac{\{x\}}{x^s} - \int_1^x tf'(t)dt + \int_1^x \{t\}f'(t)dt \\
&= x^{1-s} - \frac{x}{x^s} - tf(t)|_1^x + \int_1^x f(t)dt + \int_1^x \{t\}f'(t)dt \\
&= x^{1-s} - \frac{\{x\}}{x^s} + f(1) + \frac{1}{1-s}t^{1-s}|_1^x - s\int_1^x \{t\}t^{-s-1}dt \\
&= 1 + \frac{1}{s-1} - \frac{\{x\}}{x^s} + \frac{x^{1-s}}{1-s} - s\int_1^x \{t\}t^{-s-t}dt
\end{aligned}$$

Then, if we let the x limit to infinity, we get an important conclusion

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt \tag{7}$$

and the last term absolutely convergent for $Re(s) > 0$.

1.7 Estimate the Euler function

Considering that:

$$\phi(n) = \sum_{d|n} 1 = \sum_{ab=n} 1 = 1 * 1$$

Then

$$\sum_{n=1}^\infty \frac{\phi(n)}{n^s} = \zeta(s) \cdot \zeta(s) \tag{8}$$

Actually, we can calculate directly:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\phi(n)}{n^s} &= \prod_p(1+\frac{\phi(p)}{p^s}+\frac{\phi(p^2)}{p^{2s}}+...)\\ &= \prod_p(1+\frac{2}{p^s}+\frac{3}{p^s}+...)\\ &= \prod_p(1+\frac{1}{p^s}+...)^2\end{aligned}$$

And

$$\sum_{n\le x}\phi(n)=\sum_{\substack{n\le x\\ ab=n}}1=(\sum_{\substack{a\le\sqrt{x}\\ b\le\frac{x}{a}}}+\sum_{\substack{a\le\sqrt{x}\\ a\le\frac{x}{a}}}-\sum_{\substack{b\le\sqrt{x}\\ a\le\sqrt{x}}})1=I_1+I_2-I_3$$

where

$$I_1=I_2=\sum_{a\le\sqrt{x}}(\frac{x}{a}-\{\frac{x}{a}\})$$

And

$$\begin{aligned}I_1 &= x\sum_{a\le\sqrt{x}}\frac{1}{a}+O(\sum_{a\le\sqrt{x}}O(1))\\ &= x(log\sqrt{x}+\gamma+O(\frac{1}{\sqrt{x}}))+O(\sqrt{x})\\ &= \frac{1}{2}xlogx+\gamma x+O(\sqrt{x})\end{aligned}$$

$$I_3=([\sqrt{x}])^2=(\sqrt{x}+O(1))^2=x+(\sqrt{x})$$

Conclusion:

$$\sum_{n\le x}\phi(n)=xlogx+(2\gamma-1)x+O(\sqrt{x}) \tag{9}$$

1.8 Estimate the Mobius function

$$|\sum_{n\le x}\frac{\mu(n)}{x}|\le 1 \tag{10}$$

Proof:

Denote:

$$e(n)=\sum_{d|n}\mu(d)$$

Then

$$\begin{aligned}
S(N)&=\sum_{n\le N}e(n)\\
&=\sum_{n\le N}\sum_{d|n}\mu(d)\\
&=\sum_{d\le N}\mu(d)(\sum_{\substack{d|n\\ n\le N}}1)\\
&=\sum_{d\le N}\mu(d)[\frac{N}{d}]\\
&=N\sum_{d\le N}\frac{\mu(d)}{d}-\sum_{d\le N}\mu(d)\{\frac{N}{d}\}
\end{aligned}$$

So

$$\begin{aligned}
|N\sum_{d\le N}\frac{\mu(d)}{d}|&\le S(N)+|\sum_{d\le N-1}1|\\
&=1+N-1=N\\
&\Rightarrow|\sum_{n\le x}\frac{\mu(n)}{n}|\le 1
\end{aligned}$$

Now the question is: What's true order of $\sum_{n\le x}\mu(n)$?

The *Merten's conjecture*

$$|\sum_{n\leq x}\mu(n)| \leq \sqrt{x}$$

But it's false. There are infinitely many x s.t.

$$|\sum_{n\leq x}\mu(n)| \geq 1.06\sqrt{x}$$

And the RH(suppose it's true) tell us:

$$\sum_{n\leq x}\mu(n) \leq Cx^{\frac{1}{2}+\epsilon} \tag{11}$$

The elementary method can also estimate that:

$$\sum_{n=1}^{\infty}\frac{u^2(n)}{n^s} = \prod_p(1+\frac{1}{p^s}) = \frac{\zeta(s)}{\zeta(2s)} = (\sum_{n=1}^{\infty}\frac{1}{n^s})(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2s}}) \tag{12}$$

If we consider that:

$$\mu^2(n) = \sum_{ab^2=n} 1\cdot\mu(b)$$

$$\begin{aligned}
\sum_{n\leq x}\mu^2(n) &= \sum_{n\leq x}\sum_{ab^2=n} 1\cdot\mu(b) \\
&= \sum_{b^2\leq x}\mu(b)(\sum_{a\leq\frac{x}{b^2}} 1) \\
&= \sum_{b\leq\sqrt{x}}\mu(b)(\frac{x}{b^2}+O(1)) \\
&= x\sum_{b\leq\sqrt{x}}\frac{\mu(b)}{b^2} + (\sum_{b\leq\sqrt{x}} 1) \\
&= x\sum_{b=1}^{\infty}\frac{\mu(b)}{b^2} - x\sum_{b>\sqrt{x}}\frac{\mu(b)}{b^2} + O(\sqrt{x}) \\
&= \frac{x}{\zeta(2)} + O(\sqrt{x})
\end{aligned}$$

Conclusion:

$$\sum_{n\le x}\mu^2(n)=\frac{6}{\pi^2}x+O(\sqrt{x}) \tag{13}$$

1.9 Lambda function and its application

$$\Lambda(n)=\begin{cases}\log p, & n=p^k\\ 0, & \text{else}\end{cases}$$

And it's easy to verify that:

$$\sum_{d|n}\Lambda(d)=\log n$$

Suppose that

$$\sum_{n\le x}\Lambda(n)=O(x)$$

Then

$$\sum_{n\le x}\sum_{d|n}\Lambda(d)=\sum_{n\le x}\log n=xlogx+O(x)$$

Also

$$\begin{aligned}
&=\sum_{d\le n}\Lambda(d)(\sum_{d'\le\frac{x}{d}}1)\\
&=\sum_{d\le x}\lambda(d)(\frac{x}{d}-\{\frac{x}{d}\})\\
&=x\sum_{d\le x}\frac{\Lambda(d)}{d}+O(\sum_{d\le x}\Lambda(d))\\
&=x\sum_{d\le x}\frac{\Lambda(d)}{d}+O(x)
\end{aligned}$$

Which means

$$\sum_{d\leq x}\frac{\Lambda(d)}{d}=\log x+O(1) \tag{14}$$

For the logarithmic function of prime, we can use the property of gamma function:

$$\sum_{p\leq x}\frac{\log p}{p}=\sum_{n\leq x}\frac{\Lambda(n)}{n}+O(1)=\sum_{p}(\frac{\log p}{p}+...)=\log x+O(1) \tag{15}$$

Also, for the inverse of prime number, we can combine the partial summation:

$$\sum_{p\leq x}\frac{1}{p}=\sum_{p\leq x}\frac{\log p}{p}\frac{1}{\log p}$$

If we let:

$$f(x)=\frac{1}{\log x}$$

$$a_n=\begin{cases}\frac{\log p}{p}, & n=p\\ 0, & \text{else}\end{cases}$$

and the sum of the sequence:

$$A(t)=\log t+O(1)$$

(denote $O(1):=R(t)$)

$$\begin{aligned}\sum_{n\leq x}a_nf(n)&=(\sum_{p\leq x}\frac{\log p}{p})\frac{1}{\log x}+\int_2^x A(t)(\frac{dt}{t\log^2 t})\\&=1+\frac{R(x)}{\log x}+\int_2^x\frac{dt}{t\log t}+\int_2^x\frac{R(t)}{t\log^2 t}dt\\&=\log\log x-\log\log 2+1+O(\frac{1}{\log x})+\int_2^\infty\frac{R(t)}{t\log^2 t}dt-\int_x^\infty\frac{R(t)}{t\log^2 t}dt\end{aligned}$$

Hence:

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+A+O(\frac{1}{\log x}) \tag{16}$$

1.10 Mellin Transform

Given that:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad M(x) = \sum_{n \le x} f(n)$$

Then suppose $F(s)$ converges for some s with $Re(s) > 0$, the Mellin transform has:

$$F(s) = s \int_1^{\infty} \frac{M(x)}{x^{s+1}} dx \tag{17}$$

Let's use mellin transform to see some proposition of the order of mobius function. As we say before:

$$M(x) = \sum_{n \le x} \mu(n)$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$$

Suppose $M(x) = O(x^{\theta})$, and the integral $\int_1^{\infty} \frac{M(s)}{x^{s+1}} dx$ convergent for $Re(s) > \theta$, so if $\theta < \frac{1}{2}$, $\frac{1}{\zeta(s)}$ is absolutely convergent for $Re(s) \ge \frac{1}{2}$. But $\zeta(s)$ has zeros on $Re(s) = \frac{1}{2}$, which is impossible.

1.11 Perron's formula

$$F(s) = \sum_{n} \frac{f(n)}{n^s}$$

absolutely convergent for $Re(s) > \delta_a$. Then for any $c > \max\{0, \delta_a\}$, we have

$$M(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds \tag{18}$$

Proof: Given $T > 0$,

$$M(x) = \frac{1}{2\pi i} \int_{C-iT}^{C+iT} F(s) \frac{x^s}{s} ds + R(T)$$

What we will do first is to calculate the remainder term $R(T)$. And we claim that:

$$|R(T)| \leq \frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^c|\log\frac{x}{n}|}$$

And the other equation's proof is complex, wait when I'm free.

1.12 Riemann zeta function (2)

As we get in before:

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty}\frac{\{x\}}{x^{s+1}}dt$$

and we let:

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt \tag{19}$$

Then consider the Dirichlet series of gamma function:

$$\frac{\Gamma(s)}{n^s} = \int_0^{\infty} e^{-t}(\frac{t}{n})^{s-1}d(\frac{t}{n}) = \int_0^{\infty} e^{-nt}t^{s-1}dt$$

Thus, summation equals

$$\Gamma(s)\zeta(s) = \sum_{n\geq 1}\int_0^{\infty} e^{-nt}s^{s-1}dt = \int_0^{\infty}(\sum_{n\geq 1}e^{-nt})t^{s-1}dt = \int_0^{\infty}\frac{t^{s-1}}{e^t-1}dt$$

So, naturally, let's consider a complex-plane function

$$I(s) = \int_{\partial l_p}\frac{z^{s-1}}{e^z-1}dz$$

and the integral path is a contour(I don't have ability to draw it with Latex) composed with a circle whose radius are ρ lack a little and two lines from the gap, of which one right forward to infinity but the others inverse, and the function is an entire function of s. Let's calculate it:

First, on $|z| = \rho \leq \pi$

$$|\int_{|z|=\rho}\frac{z^{s-1}}{e^z-1}dz| \leq \int_{|z|=\rho}\frac{|z|^{Re(s)-1}}{e^{|z|}-1}|dz| \leq \int_0^{2\pi}\rho^{Re(s)-2}\rho d\theta \leq 2\pi\rho^{Re(s)-1} \to 0$$

as $\rho \to 0$, for $Re(s) > 1$. Which means, under the condition above, the function of $I(s)$ is dependent only on the real function $\Gamma(s)\zeta(s)$. Consider the direction of the path, we have:

$$I(s) = (e^{2\pi is} - 1)\Gamma(s)\zeta(s) \tag{20}$$

and for the gamma function we have reflection formula:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{21}$$

so we can combine these to reflect function:

$$\zeta(s) = \frac{e^{-i\pi s}}{2\pi i}\Gamma(1-s)I(s) \tag{22}$$

which is valid for $Re(s) \leq 0$.

Theorem: functional equation

for $s \neq 1$,

$$\zeta(s) = 2^s \pi^{s-1} \sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s) \tag{23}$$

(denote $\Phi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$, then the theorem is: $\Phi(s) = \Phi(1-s)$)

Proof:

1.13 Prime Number Theory

Under RH, $\zeta(s)$ has no zero in the stripe $\frac{1}{2} < Re(s) < 1$. And we will present some conclusion. Denote $s = \sigma + it$ for $Re(s) > \frac{1}{2}$

$$\zeta(s) = O(t^{\epsilon})$$
$$\frac{1}{\zeta(s)} = O(t^{\epsilon})$$

Let

$$F(s) = \sum_{p} \frac{1}{p^s}$$

and

$$\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}$$

then

$$\log \zeta(s) = -\sum_p \log(1 - \frac{1}{p^s}) = \sum_p (\frac{1}{p^s} + \frac{1}{2p^{2s}} + \frac{1}{3p^{3s}} + ...)$$

So we can find that

$$F(s) = \log \zeta(s) - \frac{1}{2} \log \zeta(2s) + G(s)$$

where, $G(s)$ is absolutely convergent for $Re(s) \geq 0.34$. And we treat the $\pi(x)$ with Perron's formula:

$$\pi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \frac{x^s}{s} ds$$

Choose $c = 1 + \frac{1}{\log x}$

2 Chebychev's bias(h)

It's just a slight glance at the chebychev's bias about primes module 3 as the homework of course "probabilistic number theory and its applications".

Define that

$$\chi_0(n) = 1, n \neq 3$$

$$\chi_1(n) = \begin{cases} 1, \ n \equiv 1(mod 3) \\ \\ -1, n \equiv 2(mod 3) \end{cases}$$

Obviously, the function we defined are both completely multiplitive. Then we can define the corresponding Dirichlet series:

$$L(s, \chi_0) = \sum_{n=1}^{\infty} \frac{\chi_0(n)}{n^s} = \prod_{p \neq 3} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + ...) = \prod_{p \neq 3} (1 - \frac{1}{p^s})^{-1} = \zeta(s)(1 - \frac{1}{3^s})$$
$$L(s, \chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s} = \prod_{p \equiv 1(mod 3)} (1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + ...) \prod_{p \equiv 2(mod 3)} (1 - \frac{1}{p^s} + \frac{1}{p^{2s}} + ...) = \prod_{p \neq 3} (1 - \frac{\chi_(p)}{p^s})^{-1}$$

The former is similarly to the zeta function and have a pole at $s = 1$. But the $L(s, \chi_1)$ convergent at that point. Don't be worry, we both have the similar zero-free region as zeta function:

$$\sigma \geq 1 - \frac{c}{\log(|t| + 1)}$$

for some c if $s = \sigma + it$.

To detect the prime number we need to define these two function too:

$$F(s, \chi_0) = \sum_{p} \frac{\chi_0(p)}{p^s} = (\sum_{p \neq 3} \frac{1}{p^s})$$
$$F(s, \chi_1) = \sum_{p} \frac{\chi_1(p)}{p^s}$$

The main tool we will use is Perron's formula, and we can easily get:

$$\pi(x; 3; 1) = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{1}{2} (F(s, \chi_0) + F(s, \chi_1)) \frac{x^s}{s} ds$$

Calculate separately, for the $F(s,\chi_0)$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s,\chi_0)\frac{x^s}{s}ds=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\sum_p\frac{1}{p^s}-\frac{1}{3^s})\frac{x^s}{s}ds=\sum_{p\neq 3}1=\pi(x)-1\sim\frac{x}{\log x}$$

For the another, considering that:

$$\log L(s,\chi_1)(s)=-\sum_p\log(1-\frac{\chi_1(p)}{p^s})=\sum_p\frac{\chi_1(p)}{p^s}-\frac{1}{2}\sum_p\frac{\chi_1^2(p)}{p^{2s}}+...$$

$$\Rightarrow F(s,\chi_1)=\log L(s,\chi_1)+G(s)$$

where G(s) is absolutely convergent for $Re(s)>\frac{1}{2}$. And on this region, the function $L(s,\chi_1)$ is entire for the changes between plus and minus. Hence, this term will not contribute to the main term.

In conclusion:

$$\pi(x;3,1)=\frac{1}{2}\frac{x}{\log x}+\text{error term}$$
$$\pi(x;3,2)=\frac{1}{2}\frac{x}{\log x}+\text{error term}$$

The question now is how large of the error term and the chebychev's bias for them?

What we want to detect is:

$$\Delta(x;3,2,1):=\pi(x;3,2)-\pi(x;3,1)$$

Actually, it equals to:

$$\Delta(x;3,2,1)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(x,\chi)\frac{x^s}{s}ds$$

And the idea come from that, what we neglect before are the therms that have non-pole in the region that $\sigma\geq\frac{1}{2}$, and how about we add some terms which consider a larger non-zero region and that indicate the main term of bias?

$$F(s,\chi_1)=\log L(s,\chi_1)-\frac{1}{2}\log L(2s,\chi_1^2)+G'(s)$$

where $G'(s)$ absolutely convergent for $\sigma > \frac{1}{3}$.

$$\Delta(x;3,2,1) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\log L(s,\chi_1) - \frac{1}{2}\log L(2s,\chi_1^2))\frac{x^s}{s}ds + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}G'(s)\frac{x^s}{s}ds$$

The latter are the error term and the size are about $O(x^{\frac{1}{2}+\epsilon})$. And we can observe that:

$$\begin{aligned}\log L(2s,\chi_1^2) = \log L(2s,\chi_0) &= \log\zeta(2s) + \log(1-\frac{1}{3^{2s}})\\ &= \sum_p \frac{1}{p^{2s}} + G''(s)\end{aligned}$$

where G(s) absolutely convergent in $\sigma > \frac{1}{4}$. These are all the error terms.

Hence,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{2}\log L(2s,\chi_1^2)\frac{x^s}{s}ds = \frac{\sqrt{x}}{\log x} + \text{error}$$

But the question is when cross the line $\sigma = \frac{1}{2}$, there will exist poles in the first term.

And at each pole, we can calculate roughly:

if $\rho = \frac{1}{2} + i\gamma$

$$-\frac{1}{2\pi i}\int_{H_\rho}\log(s,\chi_1)\frac{x^s}{s}ds = \frac{\sqrt{x}}{\log x}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma} + \text{error}$$

where H_ρ is a path to cross the point of pole to calculate the line $x = c$ by Cauchy's integral theorem. Hence, we can obtain:

$$\Delta(x;3,2,1) = \frac{\sqrt{x}}{\log x}(1 + \sum_{|\gamma|\le T}\frac{x^{i\gamma}}{\frac{1}{2}+i\gamma}) + \text{error}$$

Hence, in most time, the prime number of the form 3p+2 is much more than 3p+1.

3 Basic graph theory

4 L-function

5 Dirichlet character

6 Probabilistic tools

7 Exponential moment method

8 Jason's inequality

9 Markov chains

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