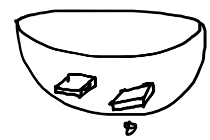
An introduction to decoupling

① $\mathbb{P}^{n-1} = \{ \xi_n = \xi_1^2 + \xi_2^2 + \cdots + \xi_{n-1}^2 ,\ |\xi_i| \le 1 \}$

$N_{R^{-1}}(\mathbb{P}^{n-1}) = \bigsqcup \theta$, each θ : $\underbrace{R^{-1/2} \times \cdots R^{-1/2}}_{n-1} \times R^{-1}$ - cap

R^{-1}-nbh of $\mathbb{P}^{n-1}$



For function f, define $\hat{f}_\theta = 1_\theta \hat{f}$ (which is the Fourier projection to θ)

(Thm) (Bourgain–Demeter) If $\operatorname{spt} \hat{f} \subset N_{R^{-1}}(\mathbb{P}^{n-1})$, then

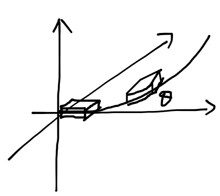
$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon R^\varepsilon (1 + R^{\frac{n-1}{4} - \frac{n+1}{2p}}) (\Sigma \|f_\theta\|^2_{L^p(\mathbb{R}^n)})^{1/2}$$

The critical $p = \frac{2(n+1)}{n-1}$

Moment curve in $\mathbb{R}^n$:

$\Gamma_n := \{ (t, t^2, \cdots t^n) ,\ t \in [0,1] \}$

Fix N. Can cover Γ_n with $N^{-1} \times N^{-2} \times \cdots N^{-n}$ - caps $\{\theta\}$



Define the N- anisotropic nbh of Γ_n to be $\Gamma_n(N) := \cup \theta$

$\supset N_{N^{-n}}(\Gamma_n)$.

(Thm) (Bourgain–Demeter–Guth) If $\operatorname{spt} \hat{f} \subset \Gamma_n(N)$, then

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim_\varepsilon N^\varepsilon (1 + N^{\frac{1}{2} - \frac{n(n+1)}{2p}}) (\Sigma \|f_\theta\|^2_{L^p(\mathbb{R}^n)})^{1/2}$$

The critical $p = n(n+1)$ $\qquad \#\theta = N$

Application to Vinogradov Conjecture :

Consider the system : $s, n, N \in \mathbb{N}$

$$\begin{cases} j_1 + j_2 + \cdots j_s = j_{s+1} + \cdots j_{2s} \\ j_1^2 + \cdots j_s^2 = j_{s+1}^2 + \cdots j_{2s}^2 \\ \vdots \\ j_1^n + \cdots j_s^n = j_{s+1}^n + \cdots j_{2s}^n \end{cases} \qquad j_1, j_2 \cdots j_{2s} \in \{1, 2, \cdots N\}$$

Denote the # of solution to be $J_{s,n}(N)$

(Vinogradov Conj) $J_{s,n}(N) \lesssim_\varepsilon N^\varepsilon (N^s + N^{2s - \frac{n(n+1)}{2}})$

$\gtrsim$ by randomness

pf. $\quad e(\alpha) := e^{2\pi i \alpha}$

Plug in $f(x) = \phi_{B_{N^n}} \sum_{j=1}^{N} e(\frac{j}{N}x_1 + \cdots (\frac{j}{N})^n x_n)$ to decoupling
Note if θ is the cap centered at $(\frac{j}{N}, \cdots (\frac{j}{N})^n)$, then
$f_\theta = \phi_{B_{N^n}} e(\frac{j}{N}x_1 + \cdots (\frac{j}{N})^n x_n)$

So, $$\int_{B_{N^n}} \Big| \sum_{j=1}^{N} e(\frac{j}{N}x_1 + \cdots (\frac{j}{N})^n x_n) \Big|^{2s} dx \lesssim N^{\varepsilon}(1+N^{s-\frac{n(n+1)}{2}}) \Big(\sum_\theta \|\phi_{B_{N^n}}\|_{2s}^2\Big)^s$$
$$\sim N^{\varepsilon}(1+N^{s-\frac{n(n+1)}{2}}) N^s N^{n^2}$$

By change of variable $x_1 \mapsto N x_1, \cdots x_n \mapsto N^n x_n$
$$\text{LHS} = N^{1+2+\cdots+n} \int_{B_{N^{n-1}} \times N^{n-2} \times \cdots \times 1} \Big| \sum_{j=1}^{N} e(j x_1 + \cdots j^n x_n) \Big|^{2s} dx$$
$$\overset{\text{by periodicity}}{=} N^{n^2} \underbrace{\int_{[0,1]^n} \Big| \sum_{j=1}^{N} e(j x_1 + \cdots j^n x_n) \Big|^{2s} dx}$$
$$= \int_{[0,1]^n} \sum_{\substack{1 \le j_1, \cdots, j_s \le N \\ 1 \le j_{s+1}, \cdots, j_{2s} \le N}} e((j_1 + \cdots j_s - j_{s+1} - \cdots j_{2s}) x_1)\, e((j_1^2 + \cdots j_s^2 - j_{s+1}^2 - \cdots j_{2s}^2) x_2) \cdots e((j_1^n + \cdots j_s^n - j_{s+1}^n - \cdots j_{2s}^n) x_n)\, dx_1 \cdots dx_n$$

(Fact : for $j \in \mathbb{N}$, $\int_{[0,1]} e(jx)\, dx = \begin{cases} 1 & j=0 \\ 0 & j \neq 0 \end{cases}$)

$$= J_{s,n}(N)$$

We obtain $\quad N^{n^2} J_{s,n}(N) \lesssim N^{\varepsilon}(1+N^{s-\frac{n(n+1)}{2}}) N^s N^{n^2} \Rightarrow J_{s,n}(N) \lesssim N^{\varepsilon}(N^s + N^{2s-\frac{n(n+1)}{2}})$

□

② Global to local reduction $\quad \operatorname{spt} \hat{f} \subset N_{R^{-1}} \mathbb{P}^{n-1}, \quad \theta : R^{-1/n} \times \cdots R^{-1}$

Global : $\|f\|_{L^p(\mathbb{R}^n)} \lesssim R^{\varepsilon} \big(\sum \|f_\theta\|_{L^p(\mathbb{R}^n)}^2 \big)^{1/2}$

Local : $\|f\|_{L^p(B_R)} \lesssim R^{\varepsilon} \big(\sum \|f_\theta\|_{L^p(B_R)}^2 \big)^{1/2}$

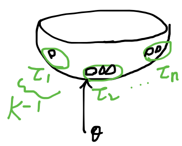
(Lem) Global $\Leftrightarrow$ Local

③ Broad-Narrow reduction (Bourgain-Guth reduction)

Idea: Multilinear version $\Longrightarrow$ linear version

Fix $K \gg 1$ (but $K \ll R$). $\tau_1, \tau_2, \cdots \tau_n \subset P^{n-1}$ are <u>transverse</u> K^{-1}-caps

i.e., $\forall \xi_i \in \tau_i$, $|N(\xi_1) \wedge \cdots \wedge N(\xi_n)| \gtrsim_K 1$



(Linear decoupling in $\mathbb{R}^n$) $\|f\|_{L^p(B_R)} \lessapprox D_n(R)\left(\sum \|f_\theta\|^2_{L^p(B_R)}\right)^{1/2}$

(Multilinear decoupling in $\mathbb{R}^n$) Suppose $\operatorname{spt} \widehat{f_i} \subset N_{R^{-1}}(\tau_i)$, then

$$\left\| \left(\prod_{i=1}^n |f_i|\right)^{1/n} \right\|_{L^p(B_R)} \lessapprox MD_n(R)\left(\prod_{i=1}^n \left(\sum_{\theta \subset \tau_i} \|f_{i,\theta}\|^2_{L^p(B_R)}\right)^{1/2}\right)^{1/n}$$

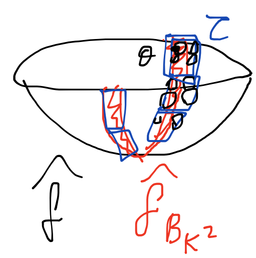
Lem: $D_n(R) \lesssim K^{O(1)} MD_n(R) + D_{n-1}(K^2)\, D_n(R/K^2)$

Sketch pf: $B_R = \bigsqcup B_{K^2} = \bigsqcup_{\text{broad}} B_{K^2} \cup \bigsqcup_{\text{narrow}} B_{K^2}$

$$\|f\|_{L^p(\text{broad})} \lesssim K^{O(1)} \left\| \left(\prod_{i=1}^n |f_i|\right)^{1/n} \right\|_{L^p(B_R)} \lessapprox K^{O(1)} MD_n(R)\left(\sum \|f_\theta\|_p^2\right)^{1/2}$$

$\|f\|^p_{L^p(\text{Narrow})} \lesssim \sum_{B_{K^2}} \|f_{B_{K^2}}\|^p_{L^p(B_{K^2})}$ Here, $f_{B_{K^2}}$ has Fourier spt in $(n-2)$-dim paraboloid

$$\lesssim D_{n-1}(K^2)^p \sum_{B_{K^2}} \left(\sum_{|\tau| = K^{-1}} \|f_\tau\|^2_{L^p(B_{K^2})}\right)^{p/2}$$



$$\lesssim D_{n-1}(K^2)^p D_n(R/K^2)^p \sum_{B_{K^2}} \left(\sum_{|\theta| = R^{-1/2}} \|f_\theta\|^2_{L^p(B_{K^2})}\right)^{p/2}$$

$$\overset{\text{Minkowski}}{\lessapprox} D_{n-1}(K^2)^p D_n(R/K^2)^p \left(\sum \|f_\theta\|^2_{L^p(B_R)}\right)^{1/2}$$

$\square$

4 Multilinear restriction

$\boxed{\text{(Restriction Conj)}}$ For g sptted in $[0,1]^{n-1}$, set $Eg(x) := \int_{[0,1]^{n-1}} g(\xi) e^{i x\cdot(\xi, |\xi|^2)} d\xi$, $x \in \mathbb{R}^n$

$$\|Eg\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^p([0,1]^{n-1})} \quad p > \frac{2n}{n-1}$$

(local): $\quad \|Eg\|_{L^{\frac{2n}{n-1}}(B_R)} \lessapprox R^{\varepsilon} \|g\|_{L^{\frac{2n}{n-1}}}$

(Equivalent form) If $\operatorname{spt}\hat{f} \subset N_{R^{-1}}(\mathbb{P}^{n-1})$, then

$$\|f\|_{L^p(B_R)} \lesssim R^{-1+\frac{1}{p}} \|\hat{f}\|_{L^p} \quad p \geq \frac{2n}{n-1}$$

(Multilinear Restriction) If $\tau_1, \cdots \tau_n$ are transverse caps in $\mathbb{P}^{n-1}$, $\operatorname{spt}\hat{f_i} \subset N_{R^{-1}}(\tau_i)$

then $\left\|\left(\prod_{i=1}^{n} |f_i|\right)^{1/n}\right\|_{L^{\frac{2n}{n-1}}(B_R)} \lessapprox \left(\prod_{i=1}^{n} \|f_i\|_{L^2(B_R)}\right)^{1/n} \quad \left(\lessapprox \prod \|f_i\|_{L^p(B_R)}\right)^{1/n}$.

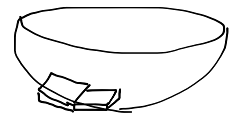
RM: Multilinear restriction $\Longrightarrow$ $MD_{\frac{2n}{n-1}} \lessapprox 1$

Since $\left(\prod_{i=1}^{n} \|f_i\|_{L^2(B_R)}\right)^{1/n} \sim \left(\prod_{i=1}^{n} \left(\sum_{\theta} \|f_{i,\theta}\|^2_{L^2(B_R)}\right)^{1/2}\right)^{1/n}$

$$\overset{\text{Hölder}}{\lesssim} \left(\prod_{i=1}^{n} \left(\sum_{\theta} \|f_{i,\theta}\|^2_{L^{\frac{2n}{n-1}}}\right)^{1/2}\right)^{1/n} \quad \square$$

$$\|f\|_{L^p(X)} := \left(\frac{1}{|X|}\int_X |f|^p\right)^{1/p}$$

$$\operatorname{spt} \hat{f} \subset N_{R^{-1}}(P^{n-1}) = \bigcup \theta \qquad \theta : R^{-1/2} \times \cdots R^{-1/2} \times R^{-1}$$



θ

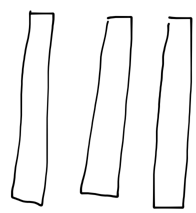
$$f_\theta = \left(1_\theta \hat{f}\right)^\vee$$

$$\| f \|_{L^p(B_R)} \lessapprox \left(\sum \| f_\theta \|^2_{L^p(B_R)} \right)^{1/2}$$

WPD : $f = \sum_\theta f_\theta \qquad f_\theta = \sum_{T \in \mathbb{T}_\theta} f_T \qquad \mathbb{T}_\theta$ is a set of

$R^{1/2} \times R$ - tubes $/\!/ \ \theta^*$

$a_T \in \mathbb{C}$



$\mathbb{T}_\theta$

w_θ

θ

$$f_T(x) \approx a_T e^{i x w_\theta} 1_T$$

Amplitude $\| f_T \|_\infty = a_T$

$$\mathbb{T} = \bigcup_\theta \mathbb{T}_\theta \qquad f = \sum_{T \in \mathbb{T}} f_T \qquad T \xRightarrow{\text{determin}} \theta .$$

$$\mathbb{T}' \subset \mathbb{T} \qquad f_{\mathbb{T}'} = \sum_{T \in \mathbb{T}'} f_T \quad \left(f = f_{\mathbb{T}} \right).$$

$\forall$ dyadic $\lambda > 0$ (amplitude). $\quad \mathbb{T}_{\theta,\lambda} := \{ T \in \mathbb{T}_\theta : |a_T| \sim \lambda \}$.

$$\mathbb{T}_\theta = \bigcup_\lambda \mathbb{T}_{\theta,\lambda} .$$

$\forall$ number $N \geq 1 \qquad \Theta_{\lambda,N} := \{ \theta : \# \mathbb{T}_{\theta,\lambda} \sim N \}$.

WPs with amplitude λ $= \bigcup_\theta \mathbb{T}_{\theta,\lambda} = \bigcup_N \bigcup_{\theta \in \Theta_{\lambda,N}} \mathbb{T}_{\theta,\lambda}$

$$\mathbb{T} = \bigcup_{\theta} \mathbb{T}_\theta = \bigcup_{\lambda} \bigcup_{\theta} \mathbb{T}_{\theta,\lambda} = \bigcup_{\lambda} \bigcup_{N} \boxed{\bigcup_{\theta \in \Theta_{\lambda,N}} \mathbb{T}_{\theta,\lambda}}$$

$$f = f_{\mathbb{T}} = \sum_{\lambda,N} \sum_{\theta \in \Theta_{\lambda,N}} f_{\mathbb{T}_{\theta,\lambda}}$$

$$\exists \lambda, N$$

$$\|f\|_{L^p(B_R)} \lessapprox (\log R)^2 \Big\| \underbrace{\sum_{\theta \in \Theta_{\lambda,N}} f_{\mathbb{T}_{\theta,\lambda}}}_{\triangleq f_{\lambda,N}} \Big\|_{L^p(B_R)}$$

Suffice to prove

$$\|f_{\lambda,N}\|_{L^p} \lessapprox \Big(\sum_{\theta \in \Theta_{\lambda,N}} \|(f_{\lambda,N})_\theta\|_p^2\Big)^{1/2} \qquad (f_{\lambda,N})_\theta = f_{\mathbb{T}_{\theta,\lambda}}$$

$$\| \quad (\#\Theta_{\lambda,N})^{1/2} \, \underbrace{\|(f_{\lambda,N})_\theta\|_p} = (\#\Theta_{\lambda,N})^{1/2} \lambda N^{\frac{1}{p}} R^{\frac{3}{2p}}$$

$$\approx \Big(\int \Big(\sum_{T \in \mathbb{T}_{\theta,\lambda}} f_T\Big)^p\Big)^{1/p}$$

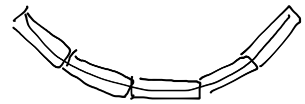
$$\approx \Big(\int \sum |f_T|^p\Big)^{1/p}$$

$$\approx \Big(\int \sum |a_T|^p \mathbb{1}_T\Big)^{1/p}$$

$$= \lambda \Big(\sum |T|\Big)^{1/p} = \lambda \Big(\#\mathbb{T}_{\theta,\lambda} R^{3/2}\Big)^{1/p}$$

$$= \lambda N^{\frac{1}{p}} R^{\frac{3}{2p}}$$

$\mathrm{spt}\,\hat{f} \subset N_{R^{-1}}(\mathbb{P}^1) = \sqcup\,\theta \qquad \theta : R^{-1/2} \times R^{-1}$

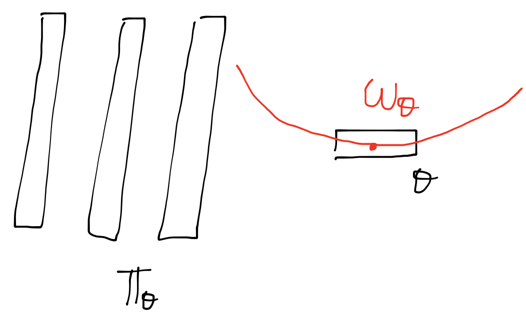


$$f_\theta = (1_\theta \hat{f})^\vee$$

$$\|f\|_{L^6(B_R)} \lessapprox \left(\sum \|f_\theta\|_{L^6}^2\right)^{1/2}$$

WPD $\quad f = \sum_\theta f_\theta \qquad f_\theta = \sum_{T \in \mathbb{T}_\theta} f_T \qquad \mathbb{T}_\theta$ is a set of

$R^{1/2} \times R$ - tubes $\,/\!/\, \theta^*$



$$f_T(x) \approx a_T\, e^{ix\cdot\omega_\theta}\, 1_T$$

amplitude $\quad \|f_T\|_\infty = |a_T|$

$\mathbb{T} = \bigcup_\theta \mathbb{T}_\theta \qquad f = \sum_{T \in \mathbb{T}} f_T$, $\quad$ For $\mathbb{T}' \subset \mathbb{T}$, $\; f_{\mathbb{T}'} := \sum_{T \in \mathbb{T}'} f_T$

$(f_{\mathbb{T}} = f)$.

Dyadic pigeonholing.

$\forall \lambda > 0$ (amplitude) $\qquad \mathbb{T}_{\theta,\lambda} := \{T \in \mathbb{T}_\theta : |a_T| \sim \lambda\}$

$\forall N \geq 1$ (# of $\mathbb{T}_{\theta,\lambda}$) $\qquad \Theta_{\lambda,N} := \{\theta : \#\mathbb{T}_{\theta,\lambda} \sim N\}$

$$\mathbb{T} = \bigsqcup_{\lambda, N} \; \bigcup_{\theta \in \Theta_{\lambda,N}} \mathbb{T}_{\theta,\lambda}$$

$$f = \sum_{\lambda,N} \underbrace{\sum_{\theta \in \Theta_{\lambda,N}} f_{\mathbb{T}_{\theta,\lambda}}}_{\triangleq f_{\lambda,N}} \qquad \sum_{\lambda,N} f_{\lambda,N}$$

$$\|f\|_{L^p} \overset{\exists \lambda, N}{\lesssim} (\log R)^2 \, \|f_{\lambda,N}\|_{L^p}$$

Suffice to prove $\|f_{\lambda,N}\|_{L^p} \lessapprox \left(\sum_{\theta \in \Theta_{\lambda,N}} \|(f_{\lambda,N})_\theta\|_{L^p}^2\right)^{1/2}$

$$\text{RHS} = (\#\Theta_{\lambda,N})^{1/2} \, \|\underbrace{(f_{\lambda,N})_\theta}\|_p \qquad (f_{\lambda,N})_\theta = f_{\mathbb{T}_{\theta,\lambda}}$$

$$= \left(\int \Big(\sum_{T \in \mathbb{T}_{\theta,\lambda}} f_T\Big)^p\right)^{1/p} \qquad |f_T| \sim \lambda \, 1_T$$

$$\sim \left(\int \sum_{\mathbb{T}_{\theta,\lambda}} |f_T|^p\right)^{1/p}$$

$$\sim \lambda \, |T|^{1/p} \, \#\mathbb{T}_{\theta,\lambda} \sim \lambda N R^{\frac{3}{2p}}$$

Dec ineq $\Leftrightarrow$ $\|f_{\lambda,N}\|_{L^6}^6 \lessapprox (\#\Theta_{\lambda,N})^3 \lambda^6 N^6 R^{\frac{3}{2}}$

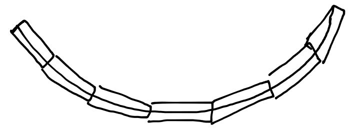
Lecture 2. Decoupling for parabola

Ref: Guth-Maldague-Wang
"Improved decoupling for the parabola"

(Thm) Spt $\hat{f} \subset N_{R^{-1}}(\mathbb{P}^1)$, $\{\theta\}$ are $R^{-1/2} \times R^{-1}$-caps that cover $N_{R^{-1}}(\mathbb{P}^1)$

$$\|f\|_{L^6(B_R)} \lessapprox \Big(\sum_\theta \|f_\theta\|^2_{L^6(B_R)}\Big)^{1/2}$$

$\lesssim (\log R)^{O(1)}$ $\qquad \lessapprox C_\varepsilon R^\varepsilon \quad \forall \varepsilon > 0$



pf: Define level set: for $\alpha > 0$, let $U_\alpha := \{x \in B_R : |f(x)| \sim \alpha\}$

Suffice to show: $\alpha^6 |U_\alpha| \lessapprox \big(\sum_\theta \|f_\theta\|^2_{L^6}\big)^3$ $(*)$

Recall wave packet decomposition: $f = \sum_\theta f_\theta = \sum_\theta \sum_{T \in \mathbb{T}_\theta} f_T$, $f_\theta = \sum_{T \in \mathbb{T}_\theta} f_T$

$\mathbb{T}$ is a set of $R^{1/2} \times R$-tubes, $\mathbb{T}_\theta$ is the set of tubes dual to θ

each f_T is locally constant on T. Each f_T is called a WP of f

By pigeonholing: ① Can assume all the wave packets of f has amplitude 1.

$|f_T| \sim 1_T$. $\|f_T\|_\infty \sim 1$

② Also assume $\#\mathbb{T}_\theta = 0$ or $\sim M$. Denote $\Theta = \{\theta : f_\theta \neq 0\}$

$\forall \theta \in \Theta$ $\|f_\theta\|_\infty \sim 1$

Fact: $\|f_\theta\|^p_{L^p} \sim N R^{3/2}$

pf: $\int |f_\theta|^p = \int \big|\sum_{T \in \mathbb{T}_\theta} f_T\big|^p \overset{\mathbb{T}_\theta \text{ disjoint}}{\sim} \int \sum_{T \in \mathbb{T}_\theta} |f_T|^p \sim \int \sum_{\mathbb{T}_\theta} 1_T \sim M R^{3/2}$

So for any p, q, $\theta, \theta' \in \Theta$,

$\|f_\theta\|^p_{L^p} \sim \|f_\theta\|^q_{L^q} \sim \|f_{\theta'}\|^p_{L^p}$

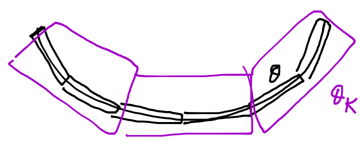
$(*) \iff \alpha^6 |U_\alpha| \lessapprox (\#\Theta)^2 \sum_{\theta \in \Theta} \|f_\theta\|^6_{L^6} \sim (\#\Theta)^2 \sum_{\theta \in \Theta} \|f_\theta\|^2_{L^2}$.

$(\|f_\theta\|_\infty \sim 1)$.

High-low method: Fix small $\varepsilon > 0$, let $N = \varepsilon^{-1}$ (may assume $N \in \mathbb{N}$)

Let $R_K = R^{\varepsilon K}$ ($R_N = R$).

Let $\{\theta_K\}$ be $R_K^{-1/2} \times R_K^{-1}$-caps that cover $N_{R_K^{-1}}(\mathbb{P}^1)$



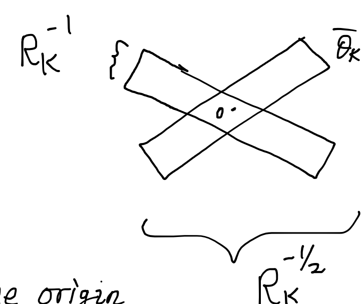
Define the square function $g_K = \sum_{\theta_K} |f_{\theta_K}|^2$ (Here, $f_{\theta_K} = \sum_{\theta \subset \theta_K} f_\theta$)

Q: What is spt $\widehat{g_K}$?

$$\widehat{g_K} = \sum_{\theta_K} \widehat{f_{\theta_K}} * \widehat{\bar{f}_{\theta_K}}$$

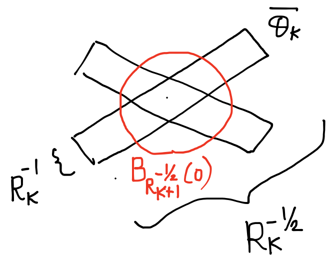
$$\text{spt } \widehat{f_{\theta_K}} * \widehat{\bar{f}_{\theta_K}} \subset \theta_K - \theta_K \triangleq \overline{\theta_K}$$

$\overline{\theta_K}$ is rougly the translation θ_K to the origin



Define $\eta_K(\xi)$ to be a cut off function at $B_{R_{K+1}^{-1/2}}(0)$

$R_K = R^{\varepsilon K}$



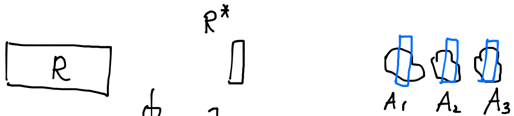
$$\widehat{g_K} = \eta_K \widehat{g_K} + (1-\eta_K)\widehat{g_K}$$

$$\Longleftrightarrow \quad g_K = \underbrace{\check{\eta}_K * g_K}_{\text{low part}} + \underbrace{(1-\check{\eta}_K) * g_K}_{\text{high part}}$$

low part of sq fcn $\lesssim$ Sq fcn at a smaller scale.

[low Lem] $\check{\eta}_K * g_K \lesssim g_{K+1}$

Local orthogonality: If $\operatorname{spt} \hat{f}_i \subset A_i$, R is a rectangle R^* is the dual of R. If $\{A_i + R^*\}_i$ are finite overlap



$\phi_R \sim 1_R$

then $\int_R |\sum_i f_i|^2 \lesssim \int \sum_i |f_i|^2 w_R$

pf: $\text{LHS} \sim \int |\phi_R \sum_i f_i|^2 \overset{\text{Plancherel}}{=} \int |\sum_i \hat{\phi}_R * \hat{f}_i|^2$ $\quad \operatorname{spt} \hat{\phi}_R * \hat{f}_i \subset R^* + A_i$

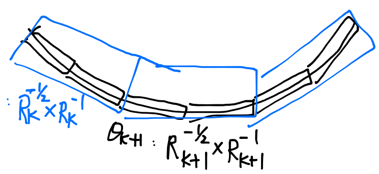
$\sim \int \sum_i |\hat{\phi}_R * \hat{f}_i|^2 = \int \sum_i |\phi_R f_i|^2 \sim \int_R \sum_i |f_i|^2$ $\square$

pf of [low Lem] Recall η_K is a bump function at $B_{R_{K+1}^{-1/2}}(0)$,

So $\check{\eta}_K \sim \frac{1}{|B_{R_{K+1}^{1/2}}|} 1_{B_{R_{K+1}^{1/2}}(0)}$.

$\fint_X f = \frac{1}{|X|}\int f$

$\check{\eta}_K * g_K(x) \sim \sum_{\theta_K} \fint_{B_{R_{K+1}^{1/2}}(x)} |f_{\theta_K}|^2$



Note $f_{\theta_K} = \sum_{\theta_{K+1} \subset \theta_K} f_{\theta_{K+1}}$

Since $(B_{R_{K+1}^{1/2}}(x))^* = B_{R_{K+1}^{-1/2}}(0)$ and $\{\theta_{K+1} + B_{R_{K+1}^{-1/2}}(0)\}_{\theta_{K+1}}$ is finite overlap

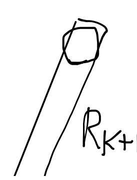
By local orthogonality,

$$\fint_{B_{R_{K+1}^{1/2}}(x)} |f_{\theta_K}|^2 \sim \sum_{\theta_{K+1} \subset \theta_K} \fint_{B_{R_{K+1}^{1/2}}(x)} |f_{\theta_{K+1}}|^2$$

$g_{K+1} = \sum_{\theta_{K+1}} |f_{\theta_{K+1}}|^2$

So, $\check{\eta}_K * g_K(x) \sim \sum_{\theta_{K+1}} \fint_{B_{R_{K+1}^{1/2}}(x)} |f_{\theta_{K+1}}|^2 \overset{\text{local const.}}{\sim} g_{K+1}(x)$

$\theta_{K+1} : R_{K+1}^{-1/2} \times R_{K+1}^{-1}$



$$\int \Big| \sum_{\theta_K} (1-\check{\eta}_K) * |f_{\theta_K}|^2 \Big|^2 \qquad R_{K+1}^{1/2}$$

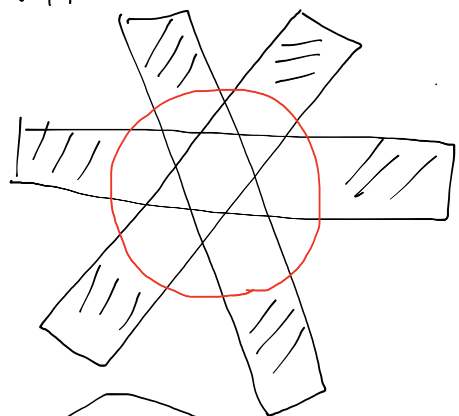
$$\|$$

[High lem] $\displaystyle\int |(1-\check{\eta}_K) * g_K|^2 \lesssim R^{O(\varepsilon)} \sum_{\theta_K} \int |f_{\theta_K}|^4$

pf: $\displaystyle\int \Big| \sum_{\theta_K} (1-\check{\eta}_K) * |f_{\theta_K}|^2 \Big|^2$

$$\lesssim R^{\varepsilon} \sum_{\theta_K} \int \big| (1-\check{\eta}_K) * |f_{\theta_K}|^2 \big|^2$$

$$\lesssim R^{\varepsilon} \sum_{\theta_K} \int |f_{\theta_K}|^4$$



$\widehat{\mathrm{spt}\,(1-\check{\eta}_K) * |f_{\theta_K}|^2}$ are R^{ε}-overlapping

$\subset$ 

By reverse sq est, $\displaystyle\int_{B_R} |f|^4 \lesssim \int_{B_R} \Big(\sum |f_\theta|^2\Big)^2$

$g_N = \sum_{\theta_N} |f_{\theta_N}|^2 = \sum_{\theta} |f_\theta|^2$

$$\Longrightarrow \alpha^4 |U_\alpha| \lesssim \int \Big(\sum |f_\theta|^2\Big)^2 = \int g_N^2$$

$$\lesssim \int |\underbrace{\check{\eta}_N * g_N}_{\text{low}}|^2 + \int |\underbrace{(1-\check{\eta}_N) * g_N}_{\text{high}}|^2$$

High case: $\displaystyle\int |(1-\check{\eta}_N) * g_N|^2 \overset{\text{high lem}}{\lesssim} \sum_\theta \int |f_\theta|^4$

$$\alpha^4 |U_\alpha| \lesssim \sum_\theta \int |f_\theta|^4 \sim \sum_\theta \int |f_\theta|^2$$

Suffice: $\alpha^2 \lesssim (\#\theta)^2 \iff \alpha \lesssim \#\theta$

$$\alpha \sim |f(x)| \le \sum_\theta |f_\theta(x)| \lesssim \sum_\theta \|f_\theta\|_\infty \lesssim \#\theta$$

Goal: $\alpha^6 |U_\alpha| \lesssim (\#\theta)^2 \sum \|f_\theta\|_2^2$

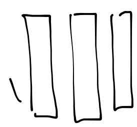
If high part dominate at scale R_k:

$$\alpha^4 |U_\alpha| \lesssim \int |(1-\check{\eta}_k) * g_k|^2 \lesssim \sum_{\theta_k} \int |f_{\theta_k}|^4$$

$$\lesssim \sup_{\theta_k} \|f_{\theta_k}\|_\infty^2 \int \sum_{\theta_k} |f_{\theta_k}|^2 \overset{\text{orth}}{=} \sup_{\theta_k} \|f_{\theta_k}\|_\infty^2 \int \sum_{\theta} |f_\theta|^2$$

Suffice: $\alpha^2 \sup_{\theta_k} \|f_{\theta_k}\|_\infty^2 \lesssim (\#\theta)^2$

i.e., $\sup_{\theta_k} \|f_{\theta_k}\|_\infty \lesssim \frac{\#\theta}{\alpha}$ (RM: $k=N$ ✓ since $\|f_\theta\|_\infty \lesssim 1$, $\alpha \lesssim \#\theta$)

f_{θ_k} 

Pruning process (prune away high-amplitude WPs)

Recall $U_\alpha = \{x \in B_R : |f(x)| \sim \alpha\}$

$1 \le k \le N$ $\quad g_k \le \underbrace{|\check{\eta}_k * g_k|}_{\lesssim g_{k+1}} + \underbrace{|(1-\check{\eta}_k) * g_k|}_{\lesssim g_{k,h}} \lesssim g_{k+1} + g_{k,h}$

$g_k(x) \lesssim g_{k+1}(x)$
or $g_k(x) \lesssim g_{k,h}(x)$

$$U_\alpha^{N-1} = \{x \in U_\alpha : g_{N-1}(x) \lesssim g_{N-1,h}(x)\}$$

$$U_\alpha^{N-2} = \{x \in U_\alpha : g_{N-1}(x) \lesssim g_N,\ g_{N-2} \lesssim g_{N-2,h}(x)\}$$

$\vdots$

$$U_\alpha^{k} = \{x \in U_\alpha : g_l(x) \lesssim g_{l+1}(x),\ k+1 \le l \le N-1,$$

$$g_K(x) \lesssim g_{K,h}(x) \}$$

$$L = U_\alpha \setminus \bigcup_{K=1}^{N-1} U_\alpha^K = \{ x \in U_\alpha : g_1(x) \lesssim g_2(x) \cdots \lesssim g_{N-1}(x) \}$$

Suffice: $\alpha^6 |U_\alpha^K| \lesssim (\#\theta)^2 \sum_\theta \|f_\theta\|_2^2$

$$\alpha^6 |L| \lesssim (\#\theta)^2 \sum_\theta \|f_\theta\|_2^2$$

$$\int_L |f|^6 \lesssim R^{2\varepsilon} \int_L g_1^3 \lesssim R^{2\varepsilon} \int g_{N-1}^3 \leq R^{O(\varepsilon)} \int \left(\sum |f_\theta|^2\right)^3$$

$$\leq (\#\theta)^2 \int \sum |f_\theta|^{6} \sim (\#\theta)^2 \int \sum |f_\theta|^2$$

For $x \in U_\alpha^K$, $g_{K+1}(x) \lesssim g_{K+2}(x) \cdots \lesssim g_{N-1}(x) \leq R^\varepsilon g_N(x)$

Let $r = R^\varepsilon \|g_N\|_\infty \lesssim R^\varepsilon \#\theta$, then $g_{K+1}(x) \leq r$.

$$\frac{\alpha}{2} \leq |f(x)| = \left| \sum_{\theta_{K+1}} f_{\theta_{K+1}}(x) \right| = \left| \sum_{|f_{\theta_{K+1}}(x)| \geq \frac{100r}{\alpha}} f_{\theta_{K+1}}(x) + \sum_{|f_{\theta_{K+1}}(x)| \leq \frac{100r}{\alpha}} f_{\theta_{K+1}}(x) \right|$$

Since $\displaystyle \sum_{|f_{\theta_{K+1}}(x)| \geq \frac{100r}{\alpha}} f_{\theta_{K+1}}(x) \leq \frac{\alpha}{100r} \underbrace{\sum |f_{\theta_{K+1}}(x)|^2}_{= g_{K+1}(x)} \leq \frac{\alpha}{100}$

so, $\displaystyle \left| \sum_{|f_{\theta_{K+1}}(x)| \leq \frac{100r}{\alpha}} f_{\theta_{K+1}}(x) \right| \gtrsim \alpha$

May assume $\displaystyle f = \sum_{\|f_{\theta_{K+1}}\|_\infty \leq \frac{100r}{\alpha}} f_{\theta_{K+1}}$, so $\|f_{\theta_{K+1}}\|_\infty \leq \frac{100r}{\alpha}$

$$\sup_{\theta_k} \|f_{\theta_k}\|_\infty \lesssim \frac{\#\Theta}{\alpha}$$

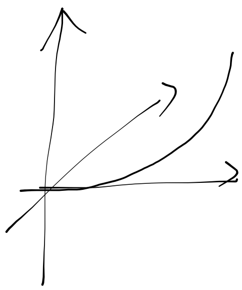
$$r \le R^{O(\varepsilon)} \#\Theta$$

$$r = R^{O(\varepsilon)} \|g_N\|_\infty = R^{O(\varepsilon)} \| \sum |f_\theta|^2 \|_\infty \lesssim R^{O(\varepsilon)} \#\Theta$$

$\square$

$$\sum a_n e(\alpha n^3 + \beta n)$$

$$\begin{cases} n_1 + n_2 = n_3 + n_4 \\ n_1^3 + n_2^3 = n_3^3 + n_4^3 \end{cases}$$



$$(x, x^3)$$

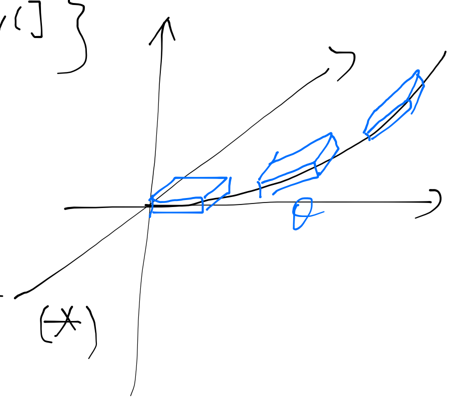
In $\mathbb{R}^3$ $\quad \Gamma_3 = \{ (t, t^2, t^3) \ t \in [0,1] \}$

θ : $R^{-1/3} \times R^{-2/3} \times R^{-1}$

$(\lesssim C_\varepsilon R^\varepsilon \ \forall \varepsilon > 0)$

$$\|f\|_{L^{12}(B_R)} \lessapprox \left(\sum_\theta \|f_\theta\|^2_{L^{12}} \right)^{1/2} \quad (*)$$

$L^{n(n+1)}$ $\quad$ $L^{n(n+1)}$



Pf. $\quad U_\alpha = \{ x \in B_R : |f(x)| \sim \alpha \}$

By pigeonholing $\quad \|f_\theta\|_\infty \sim 1 \quad \#\Pi_\theta \sim N$

$$\|f_\theta\|^p_{L^p} \sim \|f_\theta\|^q_{L^q} \sim \|f_{\theta'}\|^p_{L^p}$$

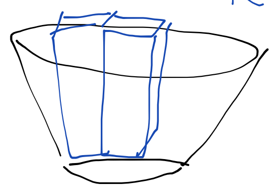
$$(*) \Leftarrow \quad \alpha^{12} |U_\alpha| \lessapprox \left(\sum_\theta \|f_\theta\|^2_{12} \right)^6 \sim (\#\theta)^5 \sum \|f_\theta\|^{12}_{12}$$

$$\sim (\#\theta)^5 \sum_\theta \|f_\theta\|_2^2$$

$n(n+1)$

$\alpha^{n(n+1)}|U_\alpha| \lesssim (\#\theta)^{\frac{}{2}-1}\sum\|f_\theta\|_2^2$

τ : $R^{-1/2}\times R^{-1}\times 1$ - cap



(L^6 - dec for cone in $\mathbb{R}^3$)

$L^{\frac{2n}{n-2}}$ - - . . . $\mathbb{R}^n$

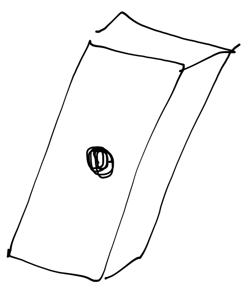
$$\|f\|_{L^6} \lessapprox \left(\sum\|f_\theta\|_{L^6}^2\right)^{1/2}$$

(L^6 - reverse sq in $\mathbb{R}^3$

L^{2n} — . . . . . $\mathbb{R}^n$)

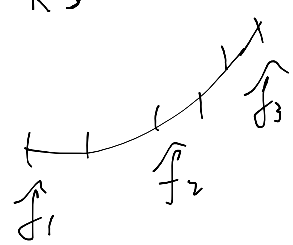
spt $\hat f \subset \tilde N(\Gamma_3) = \bigsqcup\theta$

$$\int_{B_{R^{1/3}}}|f|^6 \lessapprox \int_{B_{R^{1/3}}}\left(\sum|f_\theta|^2\right)^3$$



$R^{1/3}\times R^{2/3}\times R$.

(true when LHS $= \int_{B_{R^{1/3}}}(|f_1||f_2||f_3|)^2$



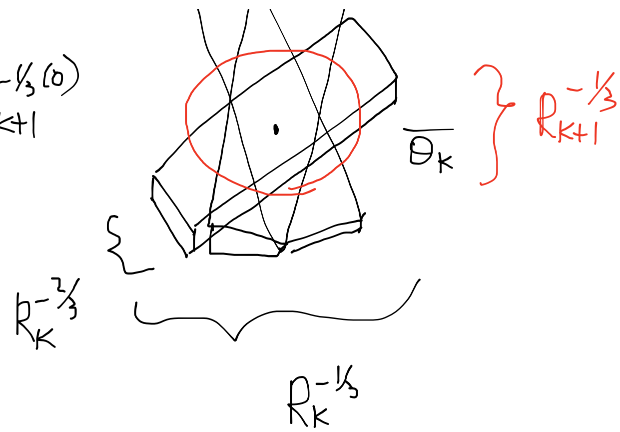
Fix $\varepsilon>0$ $\quad N=\varepsilon^{-1}$ $\quad R_k = R^{\varepsilon k}$ $\quad R_1<R_2\cdots<R_N=R$

Let $\{\theta_k\}$ $\quad R_k^{-1/3}\times R_k^{-2/3}\times R_k^{-1}$

$g_k := \sum_{\theta_k}|f_{\theta_k}|^2$ $\qquad \hat g_k = \sum_{\theta_k}\underbrace{\hat f_{\theta_k} * \hat{\bar f}_{\theta_k}}$

spt $\hat g_k$? $\qquad$ spt $\hat f_{\theta_k} * \hat{\bar f}_{\theta_k} \subset \theta_k - \theta_k \triangleq \bar\theta_k$

Choose $\eta_K(\xi)$ spted in $B_{R_{K+1}^{-1/3}}(0)$



$$g_K = \check{\eta}_K * g_K + (1-\check{\eta}_K) * g_K$$

(low lem) $\check{\eta}_K * g_K \lesssim g_{K+1}$

(high lem) (Recall yesterday: $\int (\sum |f_{\theta_K}|^2) \lesssim \int \sum |f_{\theta_K}|^4$)

$\frac{2n}{n-2}$

$$\int |(1-\check{\eta}_K) * g_K|^6 = \int \Big| \sum_{\theta_K} (1-\check{\eta}_K) * |f_{\theta_K}|^2 \Big|^6$$

$\left\{ \operatorname{spt} \widehat{(1-\check{\eta}_K) * |f_{\theta_K}|^2} \right\}$ form a partition of cone in $\mathbb{R}^3$

L^6-dec for cone

$$\le \Big(\sum_{\theta_K} \| (1-\check{\eta}_K) * |f_{\theta_K}|^2 \|_{L^6}^2 \Big)^3 \le \underline{\Big(\sum_{\theta_K} \| f_{\theta_K} \|_{L^{12}}^4 \Big)^3}$$

$$\Big(\sum_{\theta_K} \| f_{\theta_K} \|^4_{L^{\frac{4n}{n-2}}} \Big)^{\frac{n}{2(n-2)}}$$

$(g_K \gtrsim R^{-\varepsilon^2 K} \# \theta)$

$U_\alpha^N = \{ x \in U_\alpha : g_N(x) \gtrsim \# \theta \}$

$U_\alpha^{N-1} = \{ \cdots : g_N(x) \lesssim \# \theta,\ g_{N-1} \gtrsim \# \theta \}$

$U_\alpha^k = \{ x \in U_\alpha : g_l(x) \lesssim \# \theta \ \ k+1 \le l \le N,\ g_k(x) \gtrsim \# \theta \}$

$L = \{ x \in U_\alpha : \forall l \ \ g_l(x) \lesssim \# \theta \}$

Suffice to show: $\alpha^{12} |U_\alpha^k| \lesssim (\#\theta)^5 \sum \|f_\theta\|_2^2$

$\alpha^{12} |L| \lesssim (\#\theta)^5 \sum \|f_\theta\|_2^2$

$$\overset{\alpha^{2n}}{\alpha^6} |L| \lesssim \int |f|^{\overset{2n}{6}} \overset{L^6\text{- reverse sq}}{\lesssim} \int \left(\sum |f_\theta|^2\right)^{\overset{n}{3}} \leq (\#\theta)^{\overset{n-1}{2}} \int \sum_\theta |f_\theta|^6$$

$$\sim (\#\theta)^2 \sum \|f_\theta\|_2^2$$

Remains to show $\alpha^6 \lesssim (\#\theta)^3 \iff \alpha^2 \lesssim \#\theta$.

$\exists x \in L$

$$\alpha^2 \lesssim |f(x)|^2 \lesssim R^\varepsilon \sum_{\theta_1} |f_{\theta_1}|^2 = R^\varepsilon g_1(x) \lesssim R^\varepsilon \#\theta$$

$$\boxed{|f(x)| = \left|\sum_{\theta_1} f_{\theta_1}(x)\right| \lesssim (\#\theta_1)^{1/2} \left(\sum |f_{\theta_1}|^2\right)^{1/2}} \quad \square$$

$$\alpha^{2n} |L| \lesssim (\#\theta)^{n-1} \int \sum |f_\theta|^2 \quad \Leftarrow \quad \frac{\alpha^{n(n+1)} |U| \lesssim (\#\theta)^{\frac{n(n+1)}{2} - 1} \sum \|f_\theta\|_2^2}{\alpha^{n(n+1)-2n} \lesssim (\#\theta)^{\frac{n(n+1)}{2} - 1 - (n-1)}}$$

$$\alpha^{n(n-1)} \lesssim (\#\theta)^{\frac{n(n-1)}{2}}$$

$\forall x \in U_\alpha^k$, $\#\theta \lesssim g_k(x) = \eta_k * g_k + (1-\eta_k) * g_k$

$$\overset{\text{low lem}}{\leq} g_{k+1} + |(1-\check{\eta}_k) * g_k|$$

$$g_{k+1}(x) \lesssim \#\theta \implies \#\theta \lesssim g_k(x) \lesssim |(1-\check{\eta}_k) * g_k| \quad (**)$$

$$\alpha^6 |U_\alpha^k| \lesssim \int_{U_\alpha^k} |f|^6 \leq \int_{N_{R_k^{1/3}}(U_\alpha^k)} |f|^6$$

$$\overset{L^6\text{- reverse sq}}{\lesssim} \int_{N_{R_k^{1/3}}(U_\alpha^k)} \left(\sum_{\theta_k} |f_{\theta_k}|^2\right)^3 \lesssim \int_{N_{R_k^{1/3}}(U_\alpha^k)} \left(\sum_{\theta_k} |(1-\check{\eta}_k) * |f_{\theta_k}|^2|\right)^3$$

$(1-\check{\eta}_K) * g_K$ has Fourier trans spt in $B_{R_K^{-1/3}}(0)$.

$\Longrightarrow$ Locally const in ball of radius $R_K^{1/3}$

$\cancel{(*)}$ + local const $\Longrightarrow$ $|(1-\check{\eta}_K) * g_K(x)| \gtrsim \#\theta$ $\quad x \in N_{R_K^{1/3}}(U_\alpha^K)$

$$\lesssim \frac{1}{(\#\theta)^3} \int \left| \sum_{\theta_K} (1-\check{\eta}_K) * |f_{\theta_K}|^2 \right|^6$$

$\sum_{\theta_K} \int |f_{\theta_K}|^2 \lesssim \sum_\theta \int |f_\theta|^2$.

$$\lesssim \frac{1}{(\#\theta)^3} \left(\sum_{\theta_K} \left(\int |f_{\theta_K}|^{12} \right)^{1/3} \right)^3$$

$$\lesssim \frac{1}{(\#\theta)^3} \sup_{\theta_K} \|f_{\theta_K}\|_\infty^6 \underbrace{\left(\sum_{\theta_K} \left(\int |f_{\theta_K}|^6 \right)^{1/3} \right)^3}_{= \left(\sum_{\theta_K} \|f_{\theta_K}\|_{L^6}^2 \right)^3}$$

$$\overset{L^6\text{-dec for parabola}}{\lesssim} \left(\sum_{\theta_K} \sum_{\theta \subset \theta_K} \|f_\theta\|_{L^6}^2 \right)^3$$

$$= \left(\sum_\theta \|f_\theta\|_{L^6}^2 \right)^3$$

$$\sim (\#\theta)^2 \sum_\theta \|f_\theta\|_6^6$$

$$\sim (\#\theta)^2 \sum_\theta \|f_\theta\|_2^2$$

$$\alpha^6 |U_\alpha^K| \lesssim \frac{1}{\#\theta} \sup_{\theta_K} \|f_{\theta_K}\|_\infty^6 \sum_\theta \|f_\theta\|_2^2$$

Remain to show $\quad \dfrac{\alpha^6}{\#\theta} \sup_{\theta_K} \|f_{\theta_K}\|_\infty^6 \lesssim (\#\theta)^5$

$\Longleftrightarrow \quad \sup \|f_{\theta} \| \lesssim \dfrac{\#\theta}{}$

θ_K: "ν_{θ_K}"$\|_\infty \sim \alpha$.

WP pruning.

$\forall x \in U_\alpha^K$

$$\frac{\alpha}{2} \le |f(x)| = \left| \sum_{\theta_{K+1}} f_{\theta_{K+1}}(x) \right|$$

$$= \left| \underbrace{\sum_{|f_{\theta_{K+1}}(x)| \ge \frac{100\#\theta}{\alpha}} f_{\theta_{K+1}}(x)} + \sum_{\le \frac{100\#\theta}{\alpha}} f_{\theta_{K+1}}(x) \right|$$

$$\le \frac{\alpha}{100\#\theta} \sum_{\theta_{K+1}} |f_{\theta_{K+1}}(x)|^2 = \frac{\alpha}{100\#\theta} g_{K+1}(x)$$

$$\lesssim \frac{\alpha}{100}$$

So, can assume $\sup_{\theta_{K+1}} \|f_{\theta_{K+1}}\|_\infty \lesssim \frac{\#\theta}{\alpha}$

So, $\sup_{\theta_K} \|f_{\theta_K}\|_\infty \le R^\varepsilon \sup_{\theta_{K+1}} \|f_{\theta_{K+1}}\|_\infty \le R^\varepsilon \frac{\#\theta}{\alpha}$