Varieties

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September 11, 2022

1 Affine varieties

All rings should be thought as commutative with an identity.

1.1 Affine space

Let k be a fixed algebraically closed field. We define affine n-space over k, denoted A_k^n or simply A^n, to be the set of all n-tuples of elements of k. An element $P \in A^n$ with $P = (a_1, ..., a_n), a_i \in k$ will be called a point. In a word, an affine space is a linear space, but there is no distinguished point that serves as an origin.

Let $A = k[x_1, ..., x_n]$ be the polynomial ring in n variables over k. Of course, the elements f in ring can be viewed as functions from the affine n-space to k. What we care about is the zeros. Let

$$Z(f) = \{P \in A^n | f(P) = 0\}$$

Also, consider a subset T of A, we define the zero set of T to be the common zeros of all the elements of T, as

$$Z(T) = \{P \in A^n | f(P) = 0, \text{ for all } f \in T\}$$

Clearly, if $\mathfrak{a}$ is the ideal of A generated by T, then

$$Z(T) = Z(\mathfrak{a})$$

1.2 Noetherian ring

Let Σ be a partially ordered set, then

Ascending Chain Condition(a.c.c.): every increasing sequence $x_1 \leq x_2 \leq ...$ is stationary.

Similarly, we have the Descending Chain Condition(d.c.c.): every increasing sequence $x_1 \geq x_2 \geq ...$ is stationary.

And A module(ring) under the relation $\subseteq$ satisfying the a.c.c. on the submodules(ideals) will be called Noetherian. And the d.c.c will be called the Artinian. What we need to mention is that every Artinian ring is a Noetherian ring.

Proposition 1.1

M is a Noetherian A-module iff every submodule of M is finitely generated.

proof:

"$\Rightarrow$": If M is Noetherian and N is a submodule of M, consider Σ be all the f.g. submodule of N(of course are submodule of M), then it's a partially ordered set under inclusion. Hence, there will existe a maximal element N_0, otherwise, we can construct a non-stationary sequence, which contract the Noetherian condition. We claim that $N_0 = N$, otherwise, there will be a element $x \notin N_0$, then consider the submodule of $N_0 + Ax$ which is bigger than N_0, contradiction.

"$\Leftarrow$": Let $M_1 \subseteq M_2 \subseteq ...$ be an ascending chain of submodule of M, then let $N = \cup M_n$, it's a submodule of M too, hence f.g. say by $x_1, ..., x_n$, then $x_i \in M_{n_i}$, hence there exist a n s.t. $N \subseteq M_n$, hence, the sequence is stationary. $\square$

Proposition 1.2

A-module M is Noetherian, then the submodules of M are also Noetherian.

A is Noetherian, then the finitely generated modules are also Noetherian.

Exercise. $\square$

So, if we consider the Noetherian ring as regular module, we get immediately that every ideals is f.g.

Back to the polynomial ring A, since A is a noetherian ring, any ideal $\mathfrak{a}$ has a finite set of generators $f_1, f_2, ..., f_r$. Thus $Z(T)$ can be expressed as the common zeros of $f_1, f_2, ..., f_r$.

1.3 Algebraic set

A subset Y of A^n is an algebraic set if there exists a subset $T \subseteq A$ s.t. $Y = Z(T)$.

Zariski topology

The set of all algebraic sets is closed under finite union and any intersection. Naturally, we can define the Zariski topology on A^n by the complements of the algebraic sets.

A good exercise for topolygy class. And the proof can be found in textbook. □

Example:

Consider the Zariski topology on the affine line A^1. A=k[x] is a PID, so every ideal of A is principle, which means every algebraic set is the set of zeros of a single polynomial. And since k is algebraically closed, $Z(f) = \{a_1, ..., a_n\}$. Thus the algebraic sets in A^1 are just finite subsets or the whole space.

1.4 Affine algebraic variety

A nonempty subset Y of a topological space X is irreducible(connected) if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y.

An affine algebraic variety is an irreducible closed subset of A^n. An open subset of an affine variety is a quasi-affine variety. To explore the relationship between subsets of A^n and ideals in A, for any subset $Y \subset A^n$, we define the ideal of Y in A by

$$I(Y) - \{f \in A | f(P) = 0, \text{ for all } P \in Y\}$$

And we now have a function Z from subsets of A to algebraic subsets and a function I form subsets of A^n to ideals.

Proposition 1.3 (a) If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

(b)If $Y_1 \subseteq Y_2$ are subsets of A^n, then $I(Y_1) \supseteq I(Y_2)$.

(c)For any two subsets Y_1, Y_2 of A^n, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d)For ant ideal $\mathfrak{a} \subseteq A$, $I(Z(\mathfrak{a})) = r(\mathfrak{a})$.

(e)For any susbet $Y \subseteq A^n$, $Z(I(Y)) = \bar{Y}$.

Supplement

We need to define the radical of ideal and give some propositions:

$$\sqrt{\mathfrak{a}} = r(\mathfrak{a}) = \{f \in A | f^r \in \mathfrak{a} \text{ for some } r > 0\}$$

First, we claim that the radical of ideal is also ideal. Consider the naturally homomorphism:

$$A \to A/\mathfrak{a} \to (A/\mathfrak{a})/\mathfrak{R}$$

where $\mathfrak{R}$ denote all the nilpotent elements of $A/\mathfrak{a}$ which is ideal(denoted as nilradical, which also equals to the intersection of all prime ideals), and then the kernel of composition is $r(\mathfrak{a})$.

proof:

(d) This is the direct consequence of Hilbert's Nullstellensatz(strong form). The whole proof you can found in the page 85 of Atiyah.

(e) Also, we note that $Y \subseteq Z(I(Y))$, and the latter as an algebraic set is closed, hence $\bar{Y} \subseteq Z(I(Y))$. What we need to prove is $ZI(Y) \subseteq \bar{Y}$. We say $\bar{Y} = Z(\mathfrak{a})$ for some ideal. We claim that:

$$\mathfrak{a} \subseteq IZ(\mathfrak{a}) \subseteq I(Y)$$

Because $\bar{Y} = Z(\mathfrak{a}) \supseteq Y$, then (b) tells us $I(Z(\mathfrak{a})) \subseteq I(Y)$, by (d), $\mathfrak{a} \subseteq IZ(\mathfrak{a})$. So use (a) we have

$$\bar{Y} = Z(\mathfrak{a}) \supseteq ZIZ(\mathfrak{a}) \supseteq ZI(Y)$$

Thus $ZI(Y) \subseteq \bar{Y}$. □

1.5 Functoriality

There is a one-to-one inclusion **reversing** correspondence between algebraic sets in affine space A^n and radical ideals in the polnomial ring with n variables over k $A = k[x_1, ..., x_n]$. Given by

$$Y \to I(Y)$$
$$Z(\mathfrak{a}) \leftarrow \mathfrak{a}$$

Corollary: an algebraic set is irreducible iff it's ideal is a prime ideal.

proof:

"$\Rightarrow$": If $fg \in I(Y)$, then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Then we can write $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, each part is closed, but Y is irreducible, then either $Y = Y \cap Z(f)$, or $Y = Y \cap Z(g)$. Hence $f \in I(Y)$ or $g \in I(Y)$.

"$\Leftarrow$": let $\mathfrak{p}$ be a prime ideal, and suppose that $Z(\mathfrak{p}) = Y_1 \cup Y_2$, $\mathfrak{p} = IZ(\mathfrak{p}) = I(Y_1) \cap I(Y_2)$. We claim that $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Otherwise, we can choose $y_i \in I(Y_i)$ but $y_i \notin \mathfrak{p}$, while $y_1 y_2 \in I(Y_1) \cap I(Y_2) = \mathfrak{p}$, this contract the $\mathfrak{p}$ is prime. $\square$

Example(weak form)

A maximal ideal $\mathfrak{m}$ of $A = k[x_1, ..., x_n]$ corresponds to a minimal irreducible closed subset of A^n, which must be a point, say $P = (a_1, ..., a_n)$. This shows that every minimal ideal of A is of form $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$.

1.6 Noetherian toplogical space

A topology space X is called noetherian if it satisfies the descending chain condition for closed subsets.

Hilbert's basis theorem

If A is Noetherian, then the polynomial ring $A[x]$ is Noetherian.

proof:

Let $\mathfrak{a}$ be an ideal in A[x]. The leading coefficients of the polynomials form an ideal $\mathfrak{i}$ in A. Since A is Noetherian, $\mathfrak{i}$ is finitely generated by $a_1, ..., a_n$. We define n polynomial for each i:

$$f_i = a_i x^{r_i} + ...$$

and let $r = \max r_i$, and the ideal generated by them are $\mathfrak{b} \subseteq \mathfrak{a}$.

For any element $f = ax^m + ...$ in $\mathfrak{a}$, we discuss them in two types. If $m \geq r$, we can write $a = \sum u_i a_i$, then by induction let $f' = f - \sum u_i f_i x^{m-r_i}$, we reduce $f = g + h$, where $h \in \mathfrak{b}$ and degree of g is less than r. If $m < r$, we say that it can be put into a finitely generated A-module M which is generated by $1, x, , , x^{r-1}$. And we now have:

$$\mathfrak{a} = (\mathfrak{a} \cap M) + \mathfrak{b}$$

Hence, $\mathfrak{a}$ is finitely generated, and A[x] is Noetherian. $\square$

A^n is a noetherian topological space. Indeed, if $Y_1 \supseteq Y_2 \subseteq ...$ is a descending chain of closed subsets, then $I(Y_1) \subseteq I(Y_2) \subseteq ...$ will be an ascending chain of ideals. Since A is a noetherian ring, the sequence is stationary. And $Y_i = ZI(Y_i)$.

Proposition 1.4

In a noetherian topological space X, every nonempty closed subset Y can be expressed as a finite union $Y = Y_1 \cup ... \cup Y_n$. If we require there is no inclusion between Y_i, then the Y_i are uniquely determined. They are called the irreducible components of Y. Further, in such way, every algebraic set in A^n can be expressed uniquely as a union of varieties.

1.7 Dimension

If X is a topological space, we define the dimension of X to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset ... \subset Z_n$ of distinct irreducible closed subsets of X.

Example

The dimension of A^1 is 1. Indeed, the only irreducible closed subsets of A^n are the whole space or single points.

Also, we will define the height of a prime ideal $\mathfrak{p}$ is the supremum of all integers n such that there exist a chain $\mathfrak{p}_\mathfrak{o} \subset \mathfrak{p}_\mathfrak{1} \subset ... \subset \mathfrak{p}_\mathfrak{n} = \mathfrak{p}$, and the dimension of the ring is the supremum of the hights.

Proposition 1.5

If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y) := A/I(Y)$.

This proposition allows us to apply results of dimension theory of noetherian rings to algebraic geometry!

But because the proof is complex and require us a lot of time and patience to grasp, the proposition below will just be stated without proof, of which you can find the reference in GTM52.

Theorem in rings

Let k be a field, and let B be an integral domain which is a finitely generated k-algebra. Then:

(a) the dimension of B is equal to the transcendence degree of the quotient field K(B) of B over k.

(b) For any prime ideal $\mathfrak{p}$ in N we have:

$$\text{height } \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$$

Corollary in geometry

(a) The dimension of A^n is n.

(b) A variety Y in A^n has dimension n-1 iff it is the zero set Z(f) of a single nonconstant irreducible polynomial in $A = k[x_1, ..., x_n]$.