Case Study: Portfolio Optimization

(Adapted from previous offering by Prof. Michael P. Friedlander)

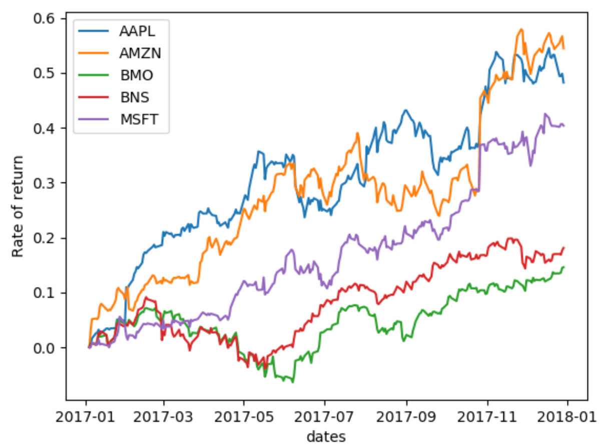

Figure 1: The historical rate of return of five technology stocks from the beginning of 2017.

Modern portfolio theory is based on the Markowitz model for determining a portfolio of stocks with a desired expected rate of return that has the smallest amount of variance. The main idea is that by *diversifying* (investing in a mixture of different stocks), one can guard against large amounts of variance in the rates of return of the individual stocks.

Suppose $p_1, \ldots, p_m$ are the historical prices of a stock over some period of time. We define the *rate of return* at time t, relative to the initial price p_1 by

$$r_t := (p_t - p_1)/p_1, \quad \text{for} \quad t = 1, \ldots, m. \tag{1}$$

The *expected rate of return* is the mean μ of the rates of return, and the *risk* is defined as the *standard deviation* σ of the rates of return:

$$\mu := \frac{1}{m}\sum_{t=1}^{m} r_t \quad \text{and} \quad \sigma := \sqrt{\frac{1}{m}\sum_{t=1}^{m}(r_t - \mu)^2}.$$

Given a collection of n stocks, let r_t^i be the rate of return of stock i at time t. Let r be the $n \times 1$ vector of the expected rates of return of the n stocks. In addition, let Σ be the $n \times n$ *covariance matrix* of the rates of return of the n stocks. Thus, r_i is the mean of the rates of return of stock i, Σ_{ii} is the variance of the rates of return of stock i, and Σ_{ij} is the covariance of the rates of return of stocks i and j:

$$r_i := \frac{1}{m}\sum_{t=1}^{m} r_t^i \quad \text{and} \quad \Sigma_{ij} := \frac{1}{m}\sum_{t=1}^{m}(r_t^i - r_i)(r_t^j - r_j).$$

We let x_i be the fraction of our investment money we put into stock i, for $i = 1, \ldots, n$. For the sake of this study, we assume there is no *short selling* (i.e., holding a stock in negative quantity). Thus, x is a vector of length n that has nonnegative entries that sum to one (i.e., $x \geq 0$ and $\sum_{i=1}^{n} x_i = 1$). The vector x represents our *portfolio* of investments. The expected rate of return and standard deviation of a portfolio x are then given by

$$\mu := r^T x \quad \text{and} \quad \sigma := \sqrt{x^T \Sigma x}.$$

Exercise 1 (*In class 1*)

1. Download financial data (`csv` files) from Yahoo! Canada Finance for the following twenty stocks:

   - Technology: AAPL, BBRY, GOOG, MSFT, YHOO
   - Services: AMZN, COST, EBAY, TGT, WMT
   - Financial: BMO, BNS, HBC, RY, TD
   - Energy: BP, CVX, IMO, TOT, XOM

   Store the `csv` files in a directory called '`data`'.

2. Download and complete the function `load_stocks.m`:

   ```
   [X, dates, names] = load_stocks(dirname, startdate, enddate)
   ```

   This function must read the *adjusted closing prices* of all stocks in the given directory between the start date and end date, and compute the rates of return as in equation (1).

   Use the following start and end dates:

   ```
   startdate = '2017-01-03';  enddate = '2017-12-31';
   ```

   Plot your results using `disp_stocks.m`:

   ```
   disp_stocks(X, dates, names)
   ```

   You may compare your output against the solution `load_stocks_soln.p`.

3. Create a function `meancov.m` that returns the $n \times 1$ vector `r` of means and the $n \times n$ covariance matrix `Sig` of the rates of returns of n stocks given by `X`:

   ```
   [r, Sig] = meancov(X)
   ```

4. Download and complete the function `portfolio_scatter.m`:

   ```
   h = portfolio_scatter(r, Sig, num)
   ```

   This function must generate random portfolios and make a scatter plot of their expected rates of return and standard deviation. Each random portfolio is generated by randomly allocating a fraction of the overall investment among a small set of 5 randomly chosen stocks. Make a scatter plot with `num = 1000` points. This function returns a handle `h` to the figure. You may compare your output against the solution `portfolio_scatter_soln.p`.

Exercise 2 *(Homework)*

1. Use CVX to compute the portfolio with minimum risk. What is the expected rate of return and standard deviation of this portfolio? Plot the rate of return of this portfolio over the entire time period. What is the portfolio with maximum possible expected rate of return? Create a function `return_range.m` that returns `num` linearly spaced rates of return between the rate of return of the portfolio with minimum risk and the maximum possible rate of return:

```
rrange = return_range(r, Sig, num)
```

   You may compare your output against the solution `return_range_soln.p`.

2. Given a desired expected rate of return, we can see from the scatter plot that there are many portfolios that we can choose that have this expected rate of return. However, each of these portfolios have a different level of risk, or standard deviation. Among these, the most *efficient* portfolio is the one giving us the least amount of risk.

   Each expected rate of return determines a different efficient portfolio. Plotting the expected rate of return and standard deviation of each of the efficient portfolios will give us a curve called the *efficient frontier*.

   Download and complete the function `efficient_frontier.m`:

```
[Y, rates, sigs] = efficient_frontier(r, Sig, num)
```

   This function will compute `num` efficient portfolios with linearly spaced rates of return (obtained from `return_range.m`). These portfolios will be stored in the `n`×`num` matrix `Y`, and their corresponding expected rates of return and standard deviation in vectors `rates` and `sigs`. Plot `sigs` and `rates` on the scatter plot, with `num = 12`:

```
h = portfolio_scatter(r, Sig, 1000);
[Y, rates, sigs] = efficient_frontier(r, Sig, num);
figure(h); hold on; plot(sigs, rates, 'ro-');
ylim([0 0.5]); xlim([0 max(sigs)]);
```

   Display your results using `disp_portfolios.m`:

```
h = disp_portfolios(Y, rates, sigs, names)
```

   You may compare your output against the solution `efficient_frontier_soln.p`.

Exercise 3 *(Homework)*

Add a risk-free investment called 'RF' to the collection of stocks with a 3% rate of return. Use your `efficient_frontier.m` code from Exercise 2 to determine the new efficient frontier and plot it on the same plot with the original efficient frontier. You will notice that the new efficient frontier has two pieces: (1) a linear piece, and (2) a nonlinear piece that coincides with the original efficient frontier. What does the linear piece represent? The portfolio where these two pieces join is called the *market portfolio*. Download and complete the function `market_portfolio.m` that computes the market portfolio corresponding to a risk-free rate of return `f`:

```
x = market_portfolio(f, r, Sig)
```

You may compare your output against the solution `market_portfolio_soln.p`. Plot the line that is tangent to the original efficient frontier at the market portfolio. What does the top half of this tangent line represent?