VE203 Assignment 8

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Q1. Suppose the recurrence relation with initial conditions a_0, a_1 is $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, then the characteristic polynomial of this relation is

$$\lambda^2 - c_1\lambda - c_2 = (\lambda - \alpha)^2 = \lambda^2 - 2\lambda\alpha + \alpha^2,$$

which means that $c_1 = 2\alpha, c_2 = -\alpha^2$, i.e. $a_n = 2\alpha a_{n-1} - \alpha^2 a_{n-2}$. Putting $a_n = q_1\alpha^n + q_2 n\alpha^n$ back to the relation above, we have

$$\begin{aligned} a_n &= 2\alpha a_{n-1} - \alpha^2 a_{n-2} \\ q_1\alpha^n + q_2 n\alpha^n &= 2\alpha(q_1\alpha^{n-1} + q_2(n-1)\alpha^{n-1}) - \alpha^2(q_1\alpha^{n-2} + q_2(n-2)\alpha^{n-2}) \\ &= 2q_1\alpha^n + 2q_2(n-1)\alpha^n - q_1\alpha^n - q_2(n-2)\alpha^n \\ &= q_1\alpha^n + q_2 n\alpha^n \end{aligned}$$

Therefore, we know that $a_n = q_1\alpha^n + q_2 n\alpha^n$ satisfies the relation. Next, we need to show that such a sequence can satisfy the prescribed initial conditions. The initial conditions would be satisfied if we could choose q_1, q_2 such that for all $n = 0, 1$, $a_n = q_1\alpha^n + q_2 n\alpha^n$. Letting $A = (a_0, a_1)^T$ and $Q = (q_1, q_2)^T$, this system of equations can be written as $A = MQ$, where

$$M = \begin{pmatrix} 1 & 0 \\ \alpha & \alpha \end{pmatrix}$$

Since the recurrence relation is degree 2, $-\alpha^2 = c_2 \neq 0$, which means $\alpha \neq 0$. Therefore, the determinant of M is $\alpha \neq 0$, which means M is invertible. Then $Q = M^{-1}A$ yields values for q_1, q_2 that ensures that the sequence (a_n) satisfies the prescribed initial conditions.

Q2. The characteristic polynomial of this recurrence relation is

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda + 1)(\lambda - 2).$$

And so, the characteristic equation has three distinct roots: $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 2$. Therefore (a_n) has the form

$$a_n = q_1 + q_2(-1)^n + q_3 2^n.$$

Thus we have

$$\begin{aligned} a_0 &= 3 = q_1 + q_2 + q_3 \\ a_1 &= 6 = q_1 - q_2 + 2q_3 \\ a_2 &= 0 = q_1 + q_2 + 4q_3 \end{aligned}$$

Therefore, $q_1 = 6, q_2 = -2, q_3 = -1$. And so $a_n = 6 - 2(-1)^n - 2^n$.

Q3. The characteristic polynomial of this recurrence relation is

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

And so, the characteristic equation has two distinct roots: $\alpha_1 = 2$, $\alpha_2 = 3$. Since $f'(n) = 2^n + 2n^2 + n$, we consider $a_n = c_n + d_n$, where $c_n = q_1 2^n + q_2 3^n + 2^n$ and $d_n = q_3 2^n + q_4 3^n + 2n^2 + n$. For c_n, we seek a particular sequence (p_n) satisfying the inhomogeneous recurrence relation in the form $p_n = xn2^n$. This requires that

$$xn2^n = 5x(n-1)2^{n-1} - 6x(n-2)2^{n-2} + 2^n$$

So $x = -2$, and (c_n) is in the form,

$$c_n = q_1 2^n + q_2 3^n - 2n2^n$$

Similarly, for d_n, we seek a particular sequence (q_n) satisfying the inhomogeneous recurrence relation in the form $q_n = q_5 n^2 + q_6 n + q_7$. This requires that

$$\begin{aligned} q_5 n^2 + q_6 n + q_7 &= 5(q_5(n-1)^2 + q_6(n-1) + q_7) - 6(q_5(n-2)^2 + q_6(n-2) + q_7) + 2n^2 + n \\ 0 &= (-2q_5 + 2)n^2 + (14q_5 - 2q_6 + 1)n + (5q_5 - 3q_6 + q_7) \end{aligned}$$

So $q_5 = 1, q_6 = \frac{15}{2}, q_7 = \frac{67}{4}$, and (d_n) is in the form,

$$d_n = q_3 2^n + q_4 3^n + n^2 + \frac{15}{2}n + \frac{67}{4}$$

Therefore, $a_n = (q_1 + q_3)2^n + (q_2 + q_4)3^n - 2n2^n + n^2 + \frac{15}{2}n + \frac{67}{4}$. The fact that $a_0 = 0, a_1 = 4$, yields

$$\begin{aligned} 0 &= (q_1 + q_3) + (q_2 + q_4) + \frac{67}{4} \\ 4 &= (q_1 + q_3)2 + (q_2 + q_4)3 - 4 + 1 + \frac{15}{2} + \frac{67}{4} \end{aligned}$$

Therefore, $q_1 + q_3 = -33$, $q_2 + q_4 = \frac{65}{4}$. The sequence (a_n) satisfies

$$a_n = -33 \times 2^n + \frac{65}{4} \times 3^n - n2^{(n+1)} + n^2 + \frac{15}{2}n + \frac{67}{4}$$

Q4. The characteristic polynomial of this recurrence relation is

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda - 2)^2(\lambda - 3).$$

And so, the characteristic equation has two distinct roots: $\alpha_1 = 2$, $\alpha_2 = 3$. It follows that (a_n) is of the form

$$a_n = (q_3 + q_4 n)2^n + q_5 3^n$$

Since $f'(n) = n4^n$, we consider a particular sequence (p_n) satisfying the inhomogeneous recurrence relation in the form $p_n = (q_1 n + q_2)4^n$. This requires that

$$\begin{aligned} (q_1 n + q_2)4^n &= 7(q_1(n-1) + q_2)4^{(n-1)} - 16(q_1(n-2) + q_2)4^{(n-2)} + 12(q_1(n-3) + q_2)4^{(n-3)} + n4^n \\ 0 &= (-4q_1 + 64)n - 20q_1 - 4q_2 \end{aligned}$$

Hence, $q_1 = 16, q_2 = -80$. Therefore, $p_n = (16n - 80)4^n$ and $a_n = (q_3 + q_4 n)2^n + q_5 3^n + (16n - 80)4^n$. Given that $a_0 = -3, a_1 = 2, a_2 = 5$, we have

$$\begin{aligned} -3 &= q_3 + q_5 - 80 \\ 2 &= (q_3 + q_4) \times 2 + q_5 3 + (16 - 80) \times 4 \\ 5 &= (q_3 + q_4 \times 2) \times 2^2 + q_5 3^2 + (16 \times 2 - 80) \times 4^2 \end{aligned}$$

Therefore, $q_3 = 28, q_4 = \frac{55}{2}, q_5 = 49$. Therefore,

$$a_n = (28 + \frac{55}{2}n)2^n + 49 \times 3^n + (16n - 80)4^n$$

Q5. The recurrence relation is $a_n = a_{n-1} + n^4$ with initial condition $a_1 = 1$. The characteristic polynomial of it is $\lambda - 1 = 0$ so that it only has one root 1 with multiplicity 1. Therefore, solutions to the homogeneous recurrence relation are sequences (b_n) with $b_n = q_1$. Since $f'(n) = n^4$ we guess that a particular solution (p_n) can be found with $p_n = q_2 n^5 + q_3 n^4 + q_4 n^3 + q_5 n^2 + q_6 n$. This requires that

$$q_2 n^5 + q_3 n^4 + q_4 n^3 + q_5 n^2 + q_6 n = q_2(n-1)^5 + q_3(n-1)^4 + q_4(n-1)^3 + q_5(n-1)^2 + q_6(n-1) + n^4$$

Therefore, we have $0 = n^4(1-5q_2) + n^3(10q_2 - 4q_3) + n^2(-10q_2 + 6q_3 - 3q_4) + n(5q_2 - 4q_3 + 3q_4 - 3q_5) + (-q_2 + q_3 - q_4 + q_5 - q_6)$. Hence, $q_2 = \frac{1}{5}, q_3 = \frac{1}{2}, q_4 = \frac{1}{3}, q_5 = 0, q_6 = -\frac{1}{30}$. Therefore, (a_n) is of the form

$$a_n = p_1 + \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

The initial conditions yield equations

$$1 = p_1 + \frac{1}{5} + \frac{1}{2} + \frac{1}{3} - \frac{1}{30}$$

Therefore, $p_1 = 0$. So

$$a_n = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

Q6. According to the initial conditions, $a_1 = 3a_0 + 2b_0 = 3 + 4 = 7$. Then rearrange the relations, we get

$$\begin{aligned}
a_n - b_n &= 2a_{n-1} \\
b_n &= a_n - 2a_{n-1} \\
b_{n-1} &= a_{n-1} - 2a_{n-2} \\
a_n &= 3a_{n-1} + 2a_{n-1} - 4a_{n-2} = 5a_{n-1} - 4a_{n-2}
\end{aligned}$$

Therefore, the characteristic polynomial of (a_n) is

$$\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$$

And so, the characteristic equation has two distinct roots: $\alpha_1 = 1, \alpha_2 = 4$. Therefore (a_n) has the form

$$a_n = q_1 + q_2 4^n.$$

Thus we have

$$\begin{aligned}
a_0 &= 1 = q_1 + q_2 \\
a_1 &= 7 = q_1 + 4q_2
\end{aligned}$$

Therefore, $q_1 = -1, q_2 = 2$. And so $a_n = -1 + 2 \times 4^n$. Then, $a_{n-1} = -1 + 2 \times 4^{n-1}$ and so $b_n = a_n - 2a_{n-1} = -1 + 2 \times 4^n - 2(-1 + 2 \times 4^{n-1}) = 1 + 4^n$

Q7. The characteristic polynomial of this recurrence relation is

$$\lambda^2 - 2\lambda + 2 = (\lambda - (1-i))(\lambda - (1+i))$$

And so, the characteristic equation has two distinct roots: $\alpha_1 = 1 - i$, $\alpha_2 = 1 + i$. It follows that (a_n) is of the form

$$a_n = q_1(1-i)^n + q_2(1+i)^n$$

Since $f'(n) = 3^n$, we consider a particular sequence (p_n) satisfying the inhomogeneous recurrence relation in the form $p_n = c3^n$. This requires that

$$\begin{aligned}
c3^n &= 2c \times 3^{n-1} - 2c \times 3^{n-2} + 3^n \\
9c &= 6c - 2c + 9 \\
5c &= 9 \\
c &= \frac{9}{5}
\end{aligned}$$

Therefore, $p_n = \frac{9}{5}3^n$ and $a_n = q_1(1-i)^n + q_2(1+i)^n + \frac{9}{5}3^n$. Given that $a_0 = 1, a_1 = 2$, we have

$$\begin{aligned} 1 &= q_1 + q_2 + \frac{9}{5} \\ 2 &= q_1(1-i) + q_2(1+i) + \frac{9}{5} \times 3 \end{aligned}$$

Therefore, $q_1 = -\frac{13i+4}{10}, q_2 = \frac{13i-4}{10}$. Therefore,

$$a_n = -\frac{13i+4}{10}(1-i)^n + \frac{13i-4}{10}(1+i)^n + \frac{9}{5}3^n$$

Q8.

(i)

$$G(x) = x - 1 + \sum_{n=0}^{\infty} 3^n x^n = 4x + \sum_{n=2}^{\infty} 3^n x^n$$

So the sequence is

$$a_n = \begin{cases} 0 & n = 0 \\ 4 & n = 1 \\ 3^n & n \geq 2 \end{cases}$$

(ii)

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(3x^2)^n - 1 = \sum_{n=1}^{\infty} \frac{1}{n!} 3^n x^{2n}$$

So the sequence is

$$a_n = \begin{cases} 0 & n = 0 \text{ and } n \text{ is odd.} \\ \frac{3^n}{n!} & \text{otherwise} \end{cases}$$

(iii)

$$\begin{aligned} G(x) &= x\frac{1}{1-(-(x+x^2))} \\ &= x\sum_{n=0}^{\infty}(-1)^n(x+x^2)^n \\ &= x\sum_{n=0}^{\infty}(-1)^n x^n(1+x)^n \\ &= x\sum_{n=0}^{\infty}(-1)^n x^n \sum_{k=0}^{n}\binom{n}{k}x^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(-1)^n x^{n+k+1} \end{aligned}$$

So the sequence is

$$a_n = \begin{cases} 0 & n = 0 \\ \sum_{k=0}^{n-1-\lfloor \frac{n}{2} \rfloor} \binom{n-1-k}{k}(-1)^{n-1-k} & \text{otherwise} \end{cases}$$

Q9.

(i) Let both sides multiply with $\frac{g(n+1)Q(n+1)}{f(n)}$, therefore, the relation becomes

$$\begin{aligned} f(n)a_n &= g(n)a_{n-1} + h(n) \\ g(n+1)Q(n+1)a_n &= \frac{g(n+1)Q(n+1)g(n)}{f(n)}a_{n-1} + \frac{g(n+1)Q(n+1)}{f(n)}h(n) \end{aligned}$$

Since we know

$$\frac{Q(n+1)}{Q(n)} = \frac{\frac{f(1)f(2)\cdots f(n)}{g(1)g(2)\cdots g(n+1)}}{\frac{f(1)f(2)\cdots f(n-1)}{g(1)g(2)\cdots g(n)}} = \frac{f(n)}{g(n+1)},$$

we have

$$\begin{aligned} g(n+1)Q(n+1)a_n &= \frac{g(n+1)Q(n+1)g(n)}{f(n)}a_{n-1} + \frac{g(n+1)Q(n+1)}{f(n)}h(n) \\ &= \frac{Q(n)Q(n+1)g(n)}{Q(n+1)}a_{n-1} + \frac{Q(n)Q(n+1)}{Q(n+1)}h(n) \\ &= Q(n)g(n)a_{n-1} + Q(n)h(n) \\ b_n &= b_{n-1} + Q(n)h(n) \end{aligned}$$

(ii)

$$\begin{aligned} b_n - b_{n-1} &= Q(n)h(n) \\ b_{n-1} - b_{n-2} &= Q(n-1)h(n-1) \\ &\cdots \\ b_1 - b_0 &= Q(1)h(1), \end{aligned}$$

where $b_0 = g(1)Q(1)C = g(1)\frac{1}{g(1)}C = C$. Adding the above n equations together, we get

$$\begin{aligned} b_n - b_0 &= \sum_{i=1}^{n} Q(i)h(i) \\ b_n &= C + \sum_{i=1}^{n} Q(i)h(i) \\ g(n+1)Q(n+1)a_n &= C + \sum_{i=1}^{n} Q(i)h(i) \\ a_n &= \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)} \end{aligned}$$

(iii) In this sequence, $f(n) = 1, g(n) = n+3, h(n) = n, C = 1$. Therefore, $Q(n) = \frac{f(1)f(2)\cdots f(n-1)}{g(1)g(2)\cdots g(n)} = \frac{6}{(n+3)!}$.

$$\begin{aligned} a_n &= \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)} \\ &= \frac{1 + \sum_{i=1}^{n} \frac{6i}{(i+3)!}}{(n+4)\frac{6}{(n+4)!}} \\ &= \frac{1 + \sum_{i=1}^{n} \frac{6i}{(i+3)!}}{\frac{6}{(n+3)!}} \\ &= \frac{(n+3)! + \sum_{i=1}^{n} \frac{6i(n+3)!}{(i+3)!}}{6} \end{aligned}$$