Q1. Let $(L, \preceq)$ be a complete lattice and let $f : L \longrightarrow L$ be an order-preserving function.

(i) Let $a, b \in L$ with $a \preceq b$. Define

$$[a, b] = \{x \in L \mid a \preceq x \preceq b\}$$

Show that $([a, b], \preceq)$ is a complete lattice.

(ii) Consider $X = \{x \in L \mid f(x) = x\}$. The Tarski-Knaster Theorem shows that there exists $a \in X$ such that for all $x \in X$, $a \preceq x$. Let $S \subseteq X$. Since $(L, \preceq)$ is a complete lattice, S has a g.l.b.

$$s = \bigwedge S$$

in L. Show that $a \preceq s$.

(iii) By considering f restricted to $[a, s]$, for s and a defined in (ii), show that S has a g.l.b. in X. Note that it is not necessarily the case that the g.l.b. of S in X is the same as the g.l.b. of S in L.

(iv) Show that $(X, \preceq)$ is a complete lattice.

(v) Define an order-preserving function $f : [0, 1] \longrightarrow [0, 1]$ such that the set of fixed points of f endowed with the usual order $(\leq)$ on $\mathbb{R}$ is a linear order with exactly 4 elements.

(8 marks)

Q2. Define $G : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \times \mathbb{N}$ by: for all $(n, m) \in \mathbb{N} \times \mathbb{N}$,

$$G((n, m)) = (n + 1, 2^m)$$

Let $X = \{R \in \mathcal{P}(\mathbb{N} \times \mathbb{N}) \mid (0, 0) \in R\}$. Define $F : X \longrightarrow X$ by: for all $R \in X$, $F(R) = R \cup G``R$.

(i) Show that $(X, \subseteq)$ is a complete lattice.

(ii) Prove that F is an order preserving function on $(X, \subseteq)$.

By the Tarski-Knaster Theorem, we know that F must have a least fixed point in $(X, \subseteq)$. Let f be the least fixed point of F.

(iii) Prove that f is a function with $\mathrm{dom}(f) = \mathbb{N}$.

(iv) Prove that for all $n \in \mathbb{N}$ with $n \geq 2$, $f(n)$ is even.

(v) Prove that f is injective.

(7 marks)

Q3. Prove that

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

(2 marks)

Q4. Prove that for all $n \geq 1$,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$$

(1 mark)

Q5. Prove that for all $m \geq 0$ and for all $n \geq 1$,

$$\sum_{k=0}^{n}\binom{m+k}{m} = \binom{m+n+1}{m+1}$$

(2 marks)

Q6. What is coefficient of $x^{31}y^4$ in the expansion of $((x+y)^8+y)^7$?
(2 mark)

Q7. Find the greatest term in the expansion of $(1+4x)^9$ when $x = \frac{1}{3}$.
(2 marks)

Q8. Find the number solutions to the equation $x_1+x_2+x_3+x_4 \leq 6$ with $x_1, x_2, x_3, x_4 \in \mathbb{N}$.
(2 mark)

Q9*. Use the binomial theorem to evaluate $(1.2)^5$.
(1 marks)

Q10. This question refers to $\mathbb{N}_{\text{def}}$ discussed in lectures.

(i) Prove that if $n \in \mathbb{N}_{\text{def}}$ with $n \neq \emptyset$, then there exists $m \in \mathbb{N}_{\text{def}}$ such that

$$n = S(m)$$

For (ii) you may assume the definition of $+$ on $\mathbb{N}_{\text{def}}$ given in lectures ensures that $\leq$ (defined in the first lecture) is a linear ordering of $\mathbb{N}_{\text{def}}$.

(ii) Prove that the usual $\leq$ order on $\mathbb{N}_{\text{def}}$ is a well order.

(6 marks)