

**Q1.** Let  $m \in \mathbb{N}$  with  $m > 1$ . Prove that if  $ac \equiv bc \pmod{m}$ , then

$$a \equiv b \pmod{\frac{m}{\gcd(c, m)}}$$

(2 marks)

**Q2.** Find a solution to the linear congruence  $36x \equiv 75 \pmod{1309}$ .

(2 marks)

**Q3.** Find all solutions to the system:

$$2x \equiv 4 \pmod{5}$$

$$3x \equiv 5 \pmod{7}$$

$$7x \equiv 2 \pmod{13}$$

(2 marks)

**Q4.** Find all, if any, solutions to the system:

$$x \equiv 5 \pmod{6}$$

$$x \equiv 3 \pmod{10}$$

$$x \equiv 8 \pmod{15}$$

(2 marks)

**Q5.** Find all, if any, solutions to the system:

$$x \equiv 5 \pmod{5}$$

$$x \equiv 3 \pmod{7}$$

$$x \equiv 8 \pmod{11}$$

$$x \equiv 2 \pmod{17}$$

(2 marks)

**Q6.** Show that  $n \log_2(n)$  is not  $O(\log_2(n))$ .

(1 mark)

**Q7.** Show that  $\log_2(n)$ ,  $\log_{10}(n)$  and  $\ln(n)$  all have the same order.

(1 mark)

**Q8.** Show that  $\lfloor x^3 - 4 \rfloor$  is order  $x^3$ .

(1 mark)

**Q9.** Let  $k > 1$ . Do  $n^n$  and  $n^{n-k}$  have the same order?

(2 marks)

**Q10.** For all  $n \in \mathbb{N} \setminus \{0\}$  define

$$H(n) = \sum_{k=0}^{n-1} \frac{1}{n-k}$$

(i) Show that

$$\sum_{j=2}^n \frac{1}{j} < \int_1^n \frac{1}{x} dx$$

(ii) Prove that  $H(n)$  is  $O(\ln(n))$ .

**(4 marks)**

**Q11.** The **Binary Insertion Sort Algorithm** is a variation of the Insertion Sort Algorithm that uses a binary search technique rather than a linear search technique to insert the  $i^{\text{th}}$  element in the correct place among the previously sorted elements.

- (i) Express the Binary Insertion Sort Algorithm in pseudocode.
- (ii) Compare the number of comparisons of elements used by the Insertion Sort Algorithm and the Binary Insertion Sort Algorithm when sorting the list  $(7, 4, 3, 8, 1, 5, 4, 2)$ .
- (iii) Show that the Insertion Sort Algorithm uses  $O(n^2)$  comparisons of elements.
- (iv) Find the complexity of the Binary Insertion Sort Algorithm. Is it significantly faster than Insertion Sort?

**(8 marks)**