Q1.

(i)* How many different anagrams can be made from the word "unnecessarily".

(ii) Prove that the number of ways of distributing n distinguishable objects into k boxes, $A_1, \ldots, A_k$, such that for all $1 \leq i \leq k$, n_i objects appear in box A_i is

$$\frac{n!}{n_1! \cdots n_k!}$$

(iii) How many surjections are there from $[5]$ to $[3]$?

(6 marks)

Q2. Let $(G, \cdot)$ be a group.

(i) Prove that the identity element of $(G, \cdot)$ is unique.

(ii) Prove that for all $x \in G$, x has a unique inverse.

(2 marks)

Q3. Write the following bijections as products of disjoint cycles and state their order in the group S_9:

(i) $f : [9] \longrightarrow [9]$ defined by: for all $n \in [9]$,

$$f(n) = \begin{cases} n+3 & \text{if } n+3 < 9, \\ n+3-9 & \text{if } n+3 \geq 9 \end{cases}$$

(ii) (13)(203)(16)(38)(14)(234)

(iii) (1203)(245)(231)(105)

(iv) (45)(123)(456)(12)

(4 marks)

Q4. Write the following bijections as products of 2-cycles and state whether they are even or odd:

(i) (1256)(12439)

(ii) $f : [9] \longrightarrow [9]$ defined by: for all $n \in [9]$,

$$f(n) = \begin{cases} n+2 & \text{if } n+2 < 9, \\ n+2-9 & \text{if } n+2 \geq 9 \end{cases}$$

(iii) (0124)(2198)(132568)

(iv) (120)(94567)(0427)

(4 marks)

Q5. For the following sets G and binary operations $\star : G \times G \longrightarrow G$ either prove that $(G, \star)$ is a group or show that $(G, \star)$ is not a group:

(i) $G = \{x \in \mathbb{R} \mid x > 0\}$ and $x \star y = \sqrt{xy}$

(ii) $G = \mathbb{R} \backslash \{0\}$ and $x \star y = \frac{x}{y}$

(iii) G is the set of all 2×2 matrices and $\star$ is matrix multiplication

(iv) $G = \{x \in \mathbb{Q} \mid x > 0\}$ and $x \star y = \frac{xy}{2}$

(4 marks)

Q6. Let $n \geq 3$ and consider S_n.

(i) We say that a 2-cycle (pq) is *adjacent* if $p = k$ and $q = k + 1$. Prove that for all $\sigma \in S_n$, if σ can be written as an odd number of 2-cycles, then σ can be written as an odd number of adjacent 2-cycles, and if σ can be written as a product of an even number of 2-cycles, then σ can be written as an even number of adjacent 2-cycles.

(ii) For all $\sigma \in S_n$, define

$$P(\sigma) = |\{(k, l) \in [n] \times [n] \mid (k < l) \wedge (\sigma(l) < \sigma(k))\}|$$

Prove that if (pq) is an adjacent cycle and $\sigma \in S_n$, then $P((pq)\sigma) = P(\sigma) \pm 1$.

(iii) Use (ii) to prove that no $\sigma \in S_n$ is both even and odd.

(iv) The *Alternating Group* on $[n]$, denoted A_n, is the set of all even bijections in S_n. Prove that A_n is a subgroup of S_n.

(v) Prove that $|A_n| = \frac{n!}{2}$.

(12 marks)

Q7. Let $(G, \cdot)$ be a group. Let $x, y \in G$ be such that $xyx^{-1} = y^2$ and $y \neq e$.

(i) Show that $x^5 y x^{-5} = y^{32}$.

(ii) If the order of x is 5, then what is the order of y? Justify your answer.

(6 marks)

Q8. Find a group $(G, \cdot)$, $x, y \in G$ and $n \in \mathbb{N} \backslash \{0, 1\}$ such that

$$(xy)^n \neq x^n y^n$$

(2 marks)

Q9. Let $(G, \cdot)$ be a group. Prove that if for all $x \in G$, $x^2 = e$, then $(G, \cdot)$ is abelian.
(2 marks)

Q10. Find all of the subgroups of D_4.
(4 marks)