VE203 Assignment 3

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Q1.

(i) According to the definition of $[a, b]$, all the elements in this set is also contained in L. Since $(L, \preceq)$ is a complete lattice, $(l, \preceq)$ must be a lattice and therefore also a poset. Therefore, $([a, b], \preceq)$ must also be a poset since all the element containing in it satisfies the prerequisite for a set to be a lattice and a poset. Therefore, for all non-empty set $[c, d] \subseteq [a, b]$, since for $\forall y \in [c, d]$, we have $c \preceq y \preceq d$, which means $c \in [a, b]$ is the g.l.b and $d \in [a, b]$ is the l.u.b of $[c, d]$. Besides, as for the empty set, we first have all $z \in [a, b]$ is both the u.b and l.b of $\emptyset$. And then we know that $\forall z \in [a, b], z \preceq b$ and $a \preceq z$, $a, b \in [a, b]$, which means that b is the g.l.b and a is the l.u.b of $\emptyset$. Therefore, every $X \subseteq [a, b] \subseteq L$ has both a l.u.b. and a g.l.b, which means that $([a, b], \preceq)$ is a complete lattice.

(ii) For $\forall x \in S$, since $S \subseteq X$, $x \in X$. And $a \in X$ is the element that for all $y \in X$, $a \preceq y$. Therefore, we have $a \preceq x$ for all element $x \in S$. Therefore, a is one lower bound of the set S. However, we also know that s is the g.l.b of S. Therefore, we must have $a \preceq s$.

(iii)
- We first need to prove that if $(L, \preceq)$ is a complete lattice, and $f : (L, \preceq) \Rightarrow (L, \preceq)$ is an order-preserving function, then f has a greatest fixed point. Consider

$$Y = \{y \in L | y \preceq f(y)\} \text{ and b} = \bigvee \text{Y}$$

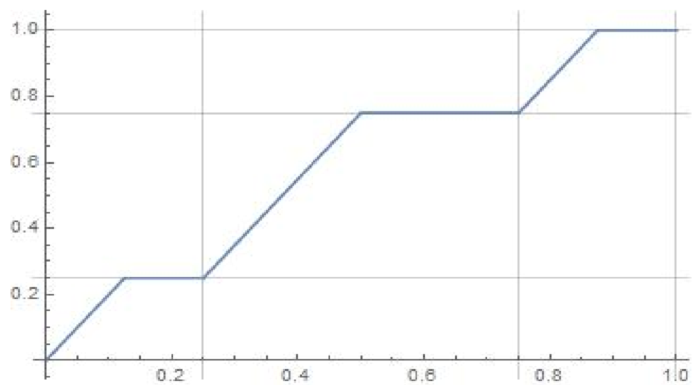
  Claim I: If $y \in Y$, then $f(y) \in Y$. To see this, let $y \in Y$. Therefore, $y \preceq f(y)$. Since f is order preserving, $f(y) \preceq f(f(y))$. This shows that $f(y) \in Y$.
  Claim II: $f(b)$ is an upper bound on Y. To see this, let $y \in Y$. Therefore, $y \preceq b$, since b is an upper bound on Y. Since f is order-preserving, $f(y) \preceq f(b)$. Since $y \preceq f(y)$, it follows that $y \preceq f(b)$.
  It follows from Claim II that $b \preceq f(b)$, because b is the l.u.b of Y. Therefore, $b \in Y$. So, by Claim I, $f(b) \in Y$. Therefore, $f(b) \preceq b$ and $b = f(b)$. So b is a fixed point of f.
  Since for any fixed point x, $f(x) = x$, by the reflexive property, we must have $x \preceq x = f(x)$. Therefore, all the fixed points are in Y. Since b is the upper bound of Y and $b \in Y$, b must be the greatest fixed point because any fixed point $x \preceq b$.
- From the question (i), we know that $([a, s], \preceq)$ is a complete lattice. Then, since f is order-preserving and $s \preceq z, \forall z \in S$, we have $f(s) \preceq f(z), \forall z \in S$, which means $f(s)$ is also a lower bound of S. But s is the g.l.b of S, therefore $s \preceq f(s)$. Also since f is order-preserving and $a \preceq x \preceq s$ for any $x \in [a, s]$, we have $a = f(a) \preceq f(x) \preceq f(s) \preceq s$, which means when the domain of f is $[a, s]$, range of $f \subseteq [a, s]$. Therefore, there must exist a greatest fixed point among all the fixed points in $[a, s]$, and we denote it as s_X. Since $s_X \in [a, s]$, we have $s_X \preceq s$. Since s is the g.l.b of S, for $\forall x \in S$, we have $s \preceq x$. Therefore, we have $\forall x \in S, s_X \preceq x$ due to transitive property. Since for any other fixed point t in $[a, s]$ we have $t \preceq s_X$ and thus s_X must be the g.l.b of S in X.

(iv) First, by Tarski-Knaster Theorem, we denote the least fixed point in X as a and the greatest fixed point n X is b.

- For $\emptyset$, its l.u.b is a since all the fixed points is its upper bound and $a \preceq x, forall x \in X$. Similarly, its g.l.b is b since all the fixed points is its least bound and $\forall x \in X, x \preceq b$.
- For every non-empty subset of X (denote as X_s), it is also a subset of L, therefore, it must have a g.l.b (denoted as $c \in L$) and a l.u.b (denoted as $d \in L$) in L. Therefore, by question (i), we know that $[c, d]$ is also a complete lattice, therefore, there are must exist a greatest fixed point and a least fixed points among all fixed points in $[c, d]$, and the former one is of course the only upper bound (i.e. l.u.b) of X_s and the latter one is the only lower bound (i.e. g.l.b) of X_s.

(v) The expression for f is shown below as well as its graph.

$$f(x) = \begin{cases} 2x & 0 \leq x < \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} \leq x < \frac{1}{4} \\ 2x - \frac{1}{4} & \frac{1}{4} \leq x < \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \leq x < \frac{3}{4} \\ 2x - \frac{3}{4} & \frac{3}{4} \leq x < \frac{7}{8} \\ 1 & \frac{7}{8} \leq x \leq 1 \end{cases}$$



Q2.

(i)
- We first prove that $(X \subseteq)$ is a poset.
  (a) It is reflexive since every set is the subset of itself.
  (b) It is antisymmetry since for any two set A, B if $A \subseteq B$ and $B \subseteq A$, they must be equal.
  (c) It is transitive. For any three set A, B, C, if $A \subseteq B$ and $B \subseteq C$, we must have $A \subseteq C$.
- For any subset $S \subseteq X$, we can always find $\bigwedge S = \bigcup S$ and $\bigvee S = \bigcap S$. Therefore, $(X \subseteq)$ is a complete lattice.

(ii) Suppose $A, B \subseteq X$ and $A \subseteq B$. Since all elements in A must in B but there exists element in B that is not in A, we have $G“A \subseteq G“B$. Therefore, we have $F(A) = A \cup G“A \subseteq F(B) = B \cup G“B$. Therefore, F is an order preserving function on $(X, \subseteq)$.

(iii)
- We first know that $0 \in$ dom(f).
- For any $k, m_k \in \mathbb{N}, k \geq 0$, suppose $(m, k) \in f$ (i.e. $k \in$ dom(f)). Therefore, since f is a fixed point of F, we have $f = F(f) = f \cup G“f$. Since $k \in$ dom(f), $G((k, m_k)) = (k+1, 2^{m_k}) \in G“f$. Therefore, $k+1 \in$ dom(f).
- In this way, we prove that all natural numbers must be in the domain of f.
- Since $f \in \mathcal{P}(\mathbb{N} \times \mathbb{N})$, the domain of f can only take values from $\mathbb{N}$, we have dom(f) $= \mathbb{N}$.

(iv) Since $f(n)$ is the result of the power of 2, where the power is equal to $f(n-1)$, we only need to guarantee that $f(n-1)$ is a positive integer.

- First, we know $G(0, 0) = (1, 1)$, $G(1, 1) = (2, 2)$. Since $G(1, 1) \in G“f$, we have $G(1, 1) \in f$. We can see that $f(1) = 1$ is a positive integer, therefore, $f(2) = 2^1 = 2$ is even.
- Let $n \in \mathbb{N}$ with $n \geq 2$ and assume that $f(n)$ is a positive integer. Therefore, $f(n+1) = 2^{f(n)}$ is also a positive integer and also an even number.
- Therefore, for all $n \in \mathbb{N}, n \geq 2$, f(n) is even.

(v) Suppose that f is not injective, which means there exists $a, b \in \mathbb{N},\ a \neq b$ such that $(a, m) \in f$ and $(b, m) \in f$.

- Suppose a or b is zero, one from (a, m) and (b, m) must be (0,0), therefore, assume $(a, m) = (0, 0)$, consider $f' = f/(b, m)$, then $f' \subseteq f$, which is against the fact that f is the $\subseteq -$least in X.
- Suppose neither a nor b is zero. Since f is $\subseteq -$least, there exists a unique pair $(n, \log_2 m) \in f$. Then apply G, we can only get one from (a, m) and (b, m). Then consider $f' = f/(b, m)$, then $f' \subseteq f$, which is against the fact that f is the $\subseteq -$least in X.

Therefore, f is injecive.

Q3. We first consider choosing elements one by one, when choosing the first element from the n elements, we have n choices. Continue choosing, when choosing the ith element from the rest $n - i + 1$ elements, we have $n - i$ choices. Because we need to choose k elements in total, we end up choosing after finishing choosing the kth element. Therefore, during choosing those elements, we have $n(n-1)(n-2)...(n-k+1) = \frac{n!}{(n-k)!}$. However, since if we choose the k elements out of n at the same time, we don't need to consider the order of them. Therefore, we need to factor out the ways to place k elements in order, which is the number of bijections from $[k]$ to $[k]$, i.e. $k!$.

Therefore, we have

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Q4. Since $(x+y)^n = \sum\limits_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, let $x = 1$, $y = -1$, we have

$$0 = (1 + (-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

Q5.

$$\begin{aligned}
\binom{m+n+1}{m+1} &= \binom{m+n}{m} + \binom{m+n}{m+1} \\
&= \binom{m+n}{m} + \binom{m+n-1}{m} + \binom{m+n-1}{m+1} \\
&= \text{...... (keep making the last term into two terms with m and m + 1 at the bottom)} \\
&= \binom{m+n}{m} + \binom{m+n-1}{m} + ... + \binom{m+1}{m} + \binom{m+1}{m+1} \\
&= \binom{m+n}{m} + \binom{m+n-1}{m} + ... + \binom{m+1}{m} + \binom{m}{m} \\
&= \sum_{k=0}^{n} \binom{m+k}{m}
\end{aligned}$$

Q6.

$$\begin{aligned}
\left((x+y)^8 + y\right)^7 &= \sum_{m=0}^{7} \binom{7}{m} ((x+y)^8)^{7-m} y^m \\
&= \sum_{m=0}^{7} \binom{7}{m} (x+y)^{56-8m} y^m \\
&= \sum_{m=0}^{7} \binom{7}{m} (\sum_{k=0}^{56-8m} \binom{56-8m}{k} x^{56-8m-k} y^k) y^m
\end{aligned}$$

Therefore, to make $56-8m-k=31$ and $k+m=4$, we have $k=1, m=3$. Therefore, the coefficient is $\binom{7}{3} \times \binom{56-24}{1} = 1120$.

Q7.

$$
\begin{aligned}
(1+4\times\frac{1}{3})^9 &= \sum_{k=0}^{9}\binom{9}{k}1^{9-k}(\frac{4}{3})^k \\
&= \sum_{k=0}^{9}\frac{9!}{k!(9-k)!}(\frac{4}{3})^k
\end{aligned}
$$

Suppose the greatest term is the $i+1$th (if we can find an integer solution i, it is not necessary for us to check whether i is the first or the last term) term, $0 \le i \le 9$. Then we must have

$$
\begin{aligned}
\frac{9!}{i!(9-i)!}(\frac{4}{3})^i &\ge \frac{9!}{(i+1)!(8-i)!}(\frac{4}{3})^{(i+1)} \\
\frac{9!}{i!(9-i)!}(\frac{4}{3})^i &\ge \frac{9!}{(i-1)!(10-i)!}(\frac{4}{3})^{(i-1)}
\end{aligned}
$$

Through calculation, we find that $4.7 \le i \le 5.71$. Therefore, $i=5$ and the greatest term is $\frac{9!}{5!(9-5)!}(\frac{4}{3})^5$ =530.96.

Q8. The number of solutions is

$$
\begin{aligned}
\sum_{r=0}^{6}\binom{4+r-1}{r} &= \binom{3}{0}+\binom{4}{1}+\binom{5}{2}+\binom{6}{3}+\binom{7}{4}+\binom{8}{5}+\binom{9}{6} \\
&= 1+4+10+20+35+56+84 \\
&= 210
\end{aligned}
$$

Q9.

$$
\begin{aligned}
1.2^5 = (1+0.2)^n &= \sum_{k=0}^{5}\binom{5}{k}1^{n-k}(0.2)^k \\
&= \sum_{k=0}^{5}\binom{5}{k}(0.2)^k \\
&= 2.48832
\end{aligned}
$$

Q10.

(i) Suppose there exists $n \in \mathbb{N}_{\text{def}}$ with $n \neq \emptyset$ such that there doesn't exist $m \in \mathbb{N}_{\text{def}}$ such that $n = S(m)$, which means $n \in \mathbb{N}_{\text{def}}$ but $n \notin S``\mathbb{N}_{\text{def}}$. Because $S``\mathbb{N}_{\text{def}} \cup \mathbb{N}_{\text{def}} = \mathbb{N}_{\text{def}}$, we have $S``\mathbb{N}_{\text{def}} \subseteq \mathbb{N}_{\text{def}}$. Given that $n \notin S``\mathbb{N}_{\text{def}}$, we also have $S``\mathbb{N}_{\text{def}} \subseteq \mathbb{N}_{\text{def}}/\{n\}$.

Therefore, consider $X = \mathbb{N}_{\text{def}}/\{n\}$. We then have $S``\mathbb{N}_{\text{def}} \subseteq X$, which means $X \cup S``\mathbb{N}_{\text{def}} = X$, which is against the fact that $\mathbb{N}_{\text{def}}$ is the least fixed point for this operation.

(ii) (a) For every non-empty subset $A \subseteq \mathbb{N}_{\text{def}}$ except itself, it is finite. Since it is linear order, it is also well-order, which means we can find a least element in that set.

(b) For $\mathbb{N}_{\text{def}}$ itself, we can see that the element $\emptyset \in \mathbb{N}_{\text{def}}$ can satisfy the quality that for all $x \in \mathbb{N}_{\text{def}}$, if $x \le \emptyset$, then $x = \emptyset$. To see this, we need to prove that for all $x \in \mathbb{N}_{\text{def}}$, if $x \le \emptyset$, then $x = \emptyset$.

If $x \le \emptyset$, then there exists sets A, B and $k \in \mathbb{N}_{\text{def}}$, such that $A \cup B = \emptyset$, $|A| = |x|$ and $|B| = |k|$, and $|A \cup B| = |\emptyset|$. Therefore, $A \cup B = \emptyset$ since if there is a bijection from one set to the empty set, this set must also be the empty set. Therefore, we get $A, B = \emptyset$ and thus $x, k = \emptyset$ and we are done.