Q1. Prove that if (a_n) is a sequence that satisfies a linear homogeneous recurrence relation of degree 2 whose characteristic polynomial has only one real root α, then there exists $q_1, q_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$a_n = q_1\alpha^n + q_2 n\alpha^n$$

(4 marks)

Q2. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$.
(2 marks)

Q3. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 5a_{n-1} - 6a_{n-2} + 2^n + 2n^2 + n$$

with $a_0 = 0$, $a_1 = 4$.
(3 marks)

Q4. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3} + n4^n$$

with $a_0 = -3$, $a_1 = 2$, $a_2 = 5$.
(2 marks)

Q5. For all $n \in \mathbb{N}\backslash\{0\}$, let

$$a_n = \sum_{i=1}^{n} i^4$$

By finding a recurrence relation that (a_n) satisfies and solving that recurrence relation, find an expression for the terms of the sequence (a_n).
(2 marks)

Q6. Solve the simultaneous recurrence relations

$$\begin{aligned} a_n &= 3a_{n-1} + 2b_{n-1} \\ b_n &= a_{n-1} + 2b_{n-1} \end{aligned}$$

with $a_0 = 1$ and $b_0 = 2$.
(3 marks)

Q7. Find an expression for the terms of the sequence (a_n) that satisfy

$$a_n = 2a_{n-1} - 2a_{n-2} + 3^n$$

with $a_0 = 1$, $a_1 = 2$.
(2 marks)

Q8. Find a closed formula for the sequences generated by the following generating functions:

(i) $G(x) = x - 1 + \frac{1}{1-3x}$

(ii) $G(x) = e^{3x^2} - 1$

(iii) $G(x) = \frac{x}{1+x+x^2}$

(3 marks)

Q9.

(i) Consider a recurrence relation in the form

$$f(n)a_n = g(n)a_{n-1} + h(n)$$

for $n \geq 1$, and with $a_0 = C$. Show that this recurrence relation can be reduced to a recurrence relation of the form

$$b_n = b_{n-1} + Q(n)h(n)$$

where

$$b_n = g(n+1)Q(n+1)a_n \text{ and } Q(n) = \frac{f(1)f(2)\cdots f(n-1)}{g(1)g(2)\cdots g(n)}$$

(ii) Use (i) to solve the original recurrence relation and obtain

$$a_n = \frac{C + \sum_{i=1}^{n} Q(i)h(i)}{g(n+1)Q(n+1)}$$

(iii) Find an expression for the sequence (a_n) that satisfies

$$a_n = (n+3)a_{n-1} + n \text{ with } a_0 = 1$$

(6 marks)