VE203 Assignment 3

Name: YIN Guoxin Student ID: 517370910043

Q1.

1. We first prove that if there exists an injective function $f : \mathbb{N} \to A$, then the set A is (Dedekind) infinite.

   - Because $f : \mathbb{N} \to A$ is injective, then $|\mathbb{N}| = |C|$, where $C \subseteq A$ and this means there exists a bijective function that $g : \mathbb{N} \to C$. Suppose $C = \{c_0, c_1, c_2, ...\}$ with c_i related to i in the natural numbers due to the bijection property of g.
   - For all $x \in A$, define a function h such that if $x \notin C$, $h(x) = x$, otherwise, $h(x)$ equals to the next element behind x, which means $h(c_0) = c_1, h(c_1) = c_2, ..., h(c_n) = h(c_{n+1})$...and so on. Since the function f makes C bijective with $\mathbb{N}$, the function h on C is equal to $y(n) = n + 1$ for all natural numbers n, which we know is not surjective since $y(n)$ cannot be zero. Therefore, $h(x)$ cannot be c_0 either.
   - Therefore, we know that the function h is injective (since it works on all $x \in A$) but not surjective (since $h(x)$ cannot be c_0.), which means A is (Dedekind) infinite.

2. We then prove that if A is (Dedekind) infinite, then there exists an injective function $f : \mathbb{N} \to A$.

   - Because A is (Dedekind) infinite, then there exists $g : A \to A$ that is an injection but not a surjection. Suppose $a_0 \in A$ and $a_0 \notin$ ran g, since g is a function from A to A, g works on every element in A, including a_0, and we denote $g(a_0) = a_1 \in A$. Due to the same reason, $g(a_1) = a_2$,..., $g(a_n) = a_{n+1}$ and so on, where n is arbitrary natural numbers and $a_n \in A$.
   - Therefore, we can find a function f such that $f(n) = a_n$. For every $a_n \in A$, we can find n because the process of finding a_n in never stopped. Therefore, f is surgective. In addition, since g is injective, every a_n is distinctive because their ancestor a_{n-1} is distinctive. Therefore, f is injective.
   - Form above, we know that f is bijective. Since all $a_n \in A$, the set of a_n, denoted as B, is a subset of A. Therefore, the function $f : \mathbb{N} \to B$ is bijective, which means $|\mathbb{N}| = |B|$ and $B \subseteq A$, which means there exists an injective function $f : \mathbb{N} \to A$.

From above, we know that a set A is (Dedekind) infinite iff there exists an injective function $f : \mathbb{N} \to A$.

Q2. If the three expressions are equivalent, we need to prove that if one holds, the other two expressions also hold.

(i) If $a \preceq b$ holds,

- then b is the upper bound of $\{a, b\}$ (note that $b \preceq b$ according to reflexive property of a poset) and we only need to prove that b is the least upper bound. Suppose c is the l.u.b of $\{a, b\}$ and $c \neq b$. Because c is l.u.b but b is only one u.b, then $c \preceq b$. However, if c is the l.u.b of $\{a, b\}$, $b \preceq c$. Therefore, by antisymmetry, $c = b$, which is a contradiction. Therefore, b is the least upper bound of $\{a, b\}$, i.e. $a \vee b = b$.
- then a is the lower bound of $\{a, b\}$ (note that $a \preceq a$ according to reflexive property of a poset) and we only need to prove that a is the greatest lower bound. Suppose c is the g.l.b of $\{a, b\}$ and $c \neq a$. Because c is g.l.b but a is only one l.b, then $a \preceq c$. However, if c is the l.u.b of $\{a, b\}$, $c \preceq a$. Therefore, by antisymmetry, $c = a$, which is a contradiction. Therefore, a is the greatest lower bound of $\{a, b\}$, i.e. $a \wedge b = a$.

(ii) If $a \vee b = b$,

- then $a \preceq b$ according to the definition of l.u.b of a set.

- then $a \preceq b$ according to the definition of l.u.b of a set. And then a is the lower bound of $\{a, b\}$ (note that $a \preceq a$ according to reflexive property of a poset) and we only need to prove that a is the greatest lower bound. Suppose c is the g.l.b of $\{a, b\}$ and $c \neq a$. Because c is g.l.b but a is only one l.b, then $a \preceq c$. However, if c is the l.u.b of $\{a, b\}$, $c \preceq a$. Therefore, by antisymmetry, $c = a$, which is a contradiction. Therefore, a is the greatest lower bound of $\{a, b\}$, i.e. $a \wedge b = a$.

(iii) If $a \wedge b = a$,

- then $a \preceq b$ according to the definition of l.u.b of a set.
- then $a \preceq b$ according to the definition of l.u.b of a set. And then b is the upper bound of $\{a, b\}$ (note that $b \preceq b$ according to reflexive property of a poset) and we only need to prove that b is the least upper bound. Suppose c is the l.u.b of $\{a, b\}$ and $c \neq b$. Because c is l.u.b but b is only one u.b, then $c \preceq b$. However, if c is the l.u.b of $\{a, b\}$, $b \preceq c$. Therefore, by antisymmetry, $c = b$, which is a contradiction. Therefore, b is the least upper bound of $\{a, b\}$, i.e. $a \vee b = b$.

Q3. $(a \vee b) \vee c = a \vee (b \vee c)$ is true because the L.H.S=R.H.S=the highest order of $\{a, b, c\}$. The details are shown below and we have already known from **Q2.** that $a \preceq b \iff a \vee b = b$.

1. Suppose $a \preceq b \preceq c$, then $(a \vee b) \vee c = b \vee c = c$ and $a \vee (b \vee c) = a \vee c = c$, so they are equal.
2. Suppose $a \preceq c \preceq b$, then $(a \vee b) \vee c = b \vee c = b$ and $a \vee (b \vee c) = a \vee b = b$, so they are equal.
3. Suppose $b \preceq a \preceq c$, then $(a \vee b) \vee c = a \vee c = c$ and $a \vee (b \vee c) = a \vee c = c$, so they are equal.
4. Suppose $b \preceq c \preceq a$, then $(a \vee b) \vee c = a \vee c = a$ and $a \vee (b \vee c) = a \vee c = a$, so they are equal.
5. Suppose $c \preceq a \preceq b$, then $(a \vee b) \vee c = b \vee c = b$ and $a \vee (b \vee c) = a \vee b = b$, so they are equal.
6. Suppose $c \preceq b \preceq a$, then $(a \vee b) \vee c = a \vee c = a$ and $a \vee (b \vee c) = a \vee b = a$, so they are equal.

Q4. To prove that $\mathbb{N} \times \mathbb{N}$ is countable, we need to prove $|\mathbb{N} \times \mathbb{N}|\ leq|\mathbb{N}|$. Consider the function

$$f(a, b) = 2^a 3^b, \quad a, b \in \mathbb{N},$$

since every natural number has a unique factorization into primes, we know for those natural numbers with only 2 or 3 prime factor, if (a_1, b_1) and (a_2, b_2) are distinct and $a_1, b_1, a_2, b_2 \in \mathbb{N}$, $f(a_1, b_1)$ and $f(a_2, b_2)$ must be different due to the uniqueness of prime factorization. Therefore, the function f is injective from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, which means $|\mathbb{N} \times \mathbb{N}| \ \leq |\mathbb{N}|$, i.e. $\mathbb{N} \times \mathbb{N}$ is countable.

Q5.

(i)

- To prove that S is countable, we need to prove $|S| \leq |\mathbb{N}|$, which means there exists a function $g : S \to \mathbb{N}$ that is an injection.

  From the definition of S, we know that the element in S are those functions with n elements in its domain ($\{0, 1, .., n-1\}$, $n \in \mathbb{N}/\{0\}$ or $\emptyset$) and exactly n natural numbers in its range ($f(0), f(1), ..., f(n-1)$, $n \in \mathbb{N}/\{0\}$ or $\emptyset$).

  Define the function $g : S \to \mathbb{N}$ such that

  $$g(f) = \begin{cases} 0 & dom\ f = \emptyset \\ 2^{(f(0)+1)} 3^{(f(1)+1)} 5^{(f(2)+1)} 7^{(f(3)+1)} \times ... \times i^{(f(n-1)+1)} & otherwise \end{cases},$$

  where i is the nth smallest prime number. Because every natural number has a unique factorization into primes, every $g(f)$ is different if f is different. In this way, we find the function $g : S \to \mathbb{N}$ that is an injection.

- Yes, $|S| = |\mathbb{N}|$. To prove this, because we have already found one injective function $g : S \to \mathbb{N}$, we only need to find another injective function $h : \mathbb{N} \to S$.

For every natural numbers greater than 0, it can be uniquely expressed as the product of prime numbers as below,

$$n = 2^{n_0} 3^{n_1} 5^{n_2} \times ......$$

Therefore, we can define the function $h : \mathbb{N} \to S$ as

$$h(n) = \begin{cases} f \text{ with domain} = \emptyset \text{ andrange} = \emptyset, & n = 0 \\ f \text{ with domain} = \text{range} = \text{a} & otherwise \end{cases},$$

Particularly, when $n \neq 0$, a is the ath prime number such that it is the smallest prime factor that if any prime factor is bigger than it, the exponential of this bigger number is zero. Besides, the function f is also determined by $f(0) = n_0, f(1) = n_1, ..., f(n-1) = n_{a-1}$. Therefore, for any different natural numbers, we can find different functions $f \in S$ that corresponding to them because the prime factorization of one natural number is unique so that the permutation of the exponentials n_i in the prime factorization of one specific natural number is unique. Therefore, the function h is injective.

Therefore, we prove that $|S| = |\mathbb{N}|$.

(ii)

- Yes, it is partial order.

  (a) It is reflexive. For all $f \in S, f : [n] \to \mathbb{N}$, $A = [n] \cap [n] = [n]$ and $[n] \subseteq [n]$ always holds. Since $\forall i \in [n], (f(i) = f(i)) \wedge [n] \subseteq [n]$, we have $f \preceq_1 f$.

  (b) It is antisymmetric. For all $f, g \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}$, if $f \preceq_1 g$ and $g \preceq_1 f$, we have first $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j)))$ or $(\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m])$, and second $(\exists a \in A)(g(a) < f(a) \wedge (\forall j < a)(f(j) = g(j)))$ or $(\forall a \in A)(f(a) = g(a)) \wedge ([m] \subseteq [n])$, $A = [n] \cap [m]$. Suppose $f \neq g$. To satisfy the first condition,
  suppose we have $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j)))$. Because there exists an inequality, we must have $(\exists a \in A)(g(a) < f(a) \wedge (\forall j < a)(f(j) = g(j)))$ to satisfy the second condition. Define $n = \min(i, a)$ (and we suppose $n = a$ here and another condition can be proved similarly), for j from 0 to $n-1$, $f(j) = g(j)$. But the first condition says $f(n) = g(n)$ if $a < i$ and $f(n) < g(n)$ if $a = i$, both are contradicted to what the second condition says, which is $f(n) > g(n)$. So this assumption leads to contradiction.
  Suppose we have $(\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m])$ to satisfy the first condition, because there doesn't exist any inequality, we must have $(\forall a \in A)(f(a) = g(a)) \wedge ([m] \subseteq [n])$, $A = [n] \cap [m]$ to satisfy the second condition. To satisfy both condition, we then must have $[n] = [m]$ to satisfy $[n] \subseteq [m]$ and $[m] \subseteq [n]$. Then the two condition becomes $(\forall i \in A)(f(i) = g(i)) \wedge ([m] = [n])$, which means $f = g$, and this is also a contradiction.
  From above, we know that it is antisymmetry.

  (c) It is transitive. For all $f, g, h \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}, h : [p] \to \mathbb{N}$, if $f \preceq_1 g$ and $g \preceq_1 h$, we have first $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j)))$ or $(\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m])$, and second $(\exists a \in B)(g(a) < h(a) \wedge (\forall j < a)(h(j) = g(j)))$ or $(\forall a \in B)(h(a) = g(a)) \wedge ([m] \subseteq [p])$, $A = [n] \cap [m]$ and $B = [m] \cap [p]$.

    - If $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j)))$ and $(\exists a \in B)(g(a) < h(a) \wedge (\forall j < a)(h(j) = g(j)))$ hold, then $(\exists b = \min(i, a) \in ([n] \cap [p])(f(b) < h(b) \wedge (\forall j < b)(f(j) = g(j) = h(j)))$, which means $f \preceq_1 h$.
    - If $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j)))$ and $(\forall a \in B)(h(a) = g(a)) \wedge ([m] \subseteq [p])$ hold, then we know $\forall i \in [m], h(i) = g(i)$. Therefore, $(\exists i \in ([n] \cap [p])(f(i) < h(i) \wedge (\forall j < i)(f(j) = g(j) = h(j)))$, which means $f \preceq_1 h$.
    - If $(\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m])$ and $(\exists a \in B)(g(a) < h(a) \wedge (\forall j < a)(h(j) = g(j)))$ hold, then we know $\forall i \in [n], h(i) = g(i)$. Therefore, either $(\forall i \in ([n] \cap [p]))(f(i) = h(i)) \wedge ([n] \subseteq [p])$ or $(\exists a \in ([n] \cap [p])(f(a) < h(a) \wedge (\forall j < a)(f(j) = g(j) = h(j)))$, and both of which mean $f \preceq_1 h$.
    - If $(\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m])$ and $(\forall a \in B)(h(a) = g(a)) \wedge ([m] \subseteq [p])$ hold, then we know $\forall i \in [n], h(i) = g(i)$ and $\forall i \in [m], h(i) = g(i)$ and $[n] \subseteq [p]$. Therefore, we have $(\forall i \in ([n] \cap [p]))(f(i) = h(i)) \wedge ([n] \subseteq [p])$, which means $f \preceq_1 h$.

From above, we know that for all $f, g, h \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}, h : [p] \to \mathbb{N}$, if $f \preceq_1 g$ and $g \preceq_1 h$, we have $f \preceq_1 h$.

- Yes, it is linear order. We are going to prove that for all $f, g \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}$, if $f \not\preceq_1 g$, we must have $g \preceq_1 f$. Because if we have equality of $g(i)$ and $f(i)$, the subset relation must be $[m] \subset [n]$ to fail $f \preceq g$. If we have inequality, the $(\forall i \in A)(f(i) = g(i)) \wedge ([n] \subseteq [m])$ has been failed already obviously. Then to fail $(\exists i \in A)(f(i) < g(i) \wedge (\forall j < i)(f(j) = g(j)))$, we can find a smallest number i in A such that $f(i) > g(i) \wedge (\forall j < i)(f(j) = g(j))$.

  Therefore, if $f \not\preceq_1 g$, we have either $(\forall i \in A)(f(i) = g(i)) \wedge ([m] \subset [n])$ or $(\exists i \in A)(f(i) > g(i) \wedge (\forall j < i)(f(j) = g(j)))$.

  The first condition ensures that $(\forall i \in A)(f(i) = g(i)) \wedge ([m] \subseteq [n])$, which means $g \preceq_1 f$.

  The second condition ensures that $(\exists i \in A)(g(i) > f(i) \wedge (\forall j < i)(f(j) = g(j)))$, which means $g \preceq_1 f$.

- No, it isn't chain complete. Since S itself is a chain and it is infinite, we cannot find the l.u.b of S.

- Yes, it is a lattice. For all $f, g \in S$, since $(S, \preceq_1)$ is a linear order, we can assume $g \preceq_1 f$ ($f \preceq_1 g$ can be proved similarly). Then $f \vee g = f$ and $f \wedge g = g$ according to **Q2.**.

- Yes, it is a well-order. Since $(S, \preceq_1)$ is a linear order, every two elements in S are related, and it is similar to $(\mathbb{N}, \leq)$, every non-empty subset A of S except itself is finite, so there must be a least element in A. As for itself, we have the function that applies on $\emptyset$ such that it is the least element in the whole S.

(iii)

- Yes, it is a partial order.

  (a) It is reflexive. For all $f \in S, f : [n] \to \mathbb{N}$, $[n] \subseteq [n]$ always holds. Since $\forall i \in [n], (f(i) = f(i))$, we have $f \preceq_2 f$.

  (b) It is antisymmetric. For all $f, g \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}$, if $f \preceq_2 g$ and $g \preceq_2 f$, we have first $[n] \subseteq [m]$ and $(\forall i \in [n])(f(i) \leq g(i))$, and second $[m] \subseteq [n]$ and $(\forall i \in [m])(f(i) \leq g(i))$. Suppose $f \neq g$. To satisfy both conditions, we must have $[n] = [m]$ and $(\forall i \in [n] = [m])(g(i) = f(i))$, which means $f = g$. Therefore, we know that it is antisymmetry.

  (c) It is transitive. For all $f, g, h \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}, h : [p] \to \mathbb{N}$, if $f \preceq_2 g$ and $g \preceq_2 h$, we have first $[n] \subseteq [m]$ and $(\forall i \in [n])(f(i) \leq g(i))$, and second $[m] \subseteq [p]$ and $(\forall i \in [m])(g(i) \leq h(i))$. To satisfy both conditions, we must have $[n] \subseteq [p]$ and $(\forall i \in [n])(f(i) \leq g(i) \leq h(i))$. Therefore, we know that for all $f, g, h \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}, h : [p] \to \mathbb{N}$, if $f \preceq_2 g$ and $g \preceq_2 h$, we have $f \preceq_2 h$.

- No, it isn't chain complete, since we cannot find a natural number that is greater or equal to any other natural numbers so that we cannot satisfy $[n] \subseteq [m]$.

- Yes, it is a lattice.

  For all $f, g \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}$, we can find a function $h \in S, h : [p] \to \mathbb{N}$ such that $[p] = [n]$ if $[n] \subseteq [m]$ or $[p] = [m]$ if $[m] \subseteq [n]$, and $(\forall i \in [p])(h(i) = \min(h(i), g(i)))$. Therefore, we know that h is the lower bound of $\{f, g\}$. Suppose there exists any other lower bound $y \in S, y : [q] \to \mathbb{N}$ such that $h \preceq_2 y$ and $h \neq y$, which means $[p] \subseteq [q]$ and $(\forall i \in [p])(h(i) \leq y(i))$, and $[q] \subseteq [n]$ and $[q] \subseteq [m]$, and $(\forall i \in [q])(y(i) \leq \min(h(i), g(i)))$. It also means that $[q] = [p]$ and $y(i) = h(i)$, which is contradicted to $y \neq h$. Therefore, h is the g.l.b.

  Similarly, for all $f, g \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}$, we can find a function $z \in S, h : [r] \to \mathbb{N}$ such that $[z] = [m]$ if $[n] \subseteq [m]$ or $[z] = [n]$ if $[m] \subseteq [n]$, and $(\forall i \in [r])(z(i) = \max(h(i), g(i)))$. Therefore, we know that z is the upper bound of $\{f, g\}$. And it is also the l.u.b of $\{f, g\}$ and the proof is the same as the proof of g.l.b.

- No, it is not a linear order. Suppose it is a linear order, then for all $f, g \in S, f : [n] \to \mathbb{N}, g : [m] \to \mathbb{N}$, if $f \not\preceq_2 g$, we must have $g \preceq_2 f$. If $f \not\preceq_2 g$, we have either $[m] \subset [n]$ or, $[n] \subseteq [m]$ but $(\exists i \in [n])(f(i) > g(i))$. Suppose it is the second condition, $[n] \subseteq [m]$ but $(\exists i \in [n])(f(i) > g(i))$. Since we cannot guarantee that $\forall i \in [m], g(i) \leq f(i)$, we still cannot guarantee $g \preceq_2 f$. Therefore, it is not a linear order.

Q6. Denote $a = \frac{1}{2}(x+y)(x+y+1)+y$, then we have,

$$\begin{aligned}
f(x,y,z) =& \frac{1}{2}(a+z)(a+z+1)+z \\
=& \frac{1}{2}(\frac{1}{2}(x+y)(x+y+1)+y+z)(\frac{1}{2}(x+y)(x+y+1)+y+z+1)+z \\
=& \frac{x^4}{8}+\frac{x^3y}{2}+\frac{x^3}{4}+\frac{3x^2y^2}{4}+\frac{5x^2y}{4}+\frac{x^2z}{2}+\frac{3x^2}{8}+\frac{xy^3}{2}+\frac{7xy^2}{4}+xyz \\
&+\frac{5xy}{4}+\frac{xz}{2}+\frac{x}{4}+\frac{y^4}{8}+\frac{3y^3}{4}+\frac{y^2z}{2}+\frac{11y^2}{2}+\frac{3yz}{2}+\frac{3y}{4}+\frac{z^2}{2}+\frac{3z}{2}
\end{aligned}$$

Q7. Every nonzero rational number has a unique representation in the form $\frac{(-1)^k a}{b}$ where $k \in \{0,1\}$ and $a, b \in \mathbb{N}$ with $a, b \neq 0$ and the greatest common divisor of a and b is 1. Moreover, any two distinct sets (k_1, a_1, b_1) and (k_2, a_2, b_2), where $k \in \{0,1\}$ and $a, b \in \mathbb{N}$ with $a, b \neq 0$ and the greatest common divisor of a and b is 1, yield distinct non-zero integers $\frac{(-1)_1^k a_1}{b_1}$ and $\frac{(-1)_2^k a_2}{b_2}$. This means the function

$$f(\frac{(-1)^k a}{b}) = \begin{cases} 0 & a \text{ or b} = 0 \\ (-1)^k \times (\frac{1}{2}(a+b)(a+b+1)+b) & a \neq 0 \text{ and b} \neq 0 \end{cases},$$

is injective, which means $|\mathbb{Q}| \leq |\mathbb{Z}|$. Since $|\mathbb{Z}| = |\mathbb{N}|$, we thus have $|\mathbb{Q}| \leq |\mathbb{N}|$, which means that $\mathbb{Q}$ is countable.

Q8. From the graph of the Cantor's Pairing Function, the ith diagonal has i element, which means there are $\frac{i(i+1)}{2}$ elements in the first i diagonals. Since

$$223 = \frac{21 \times 22}{2} - 8 > \frac{20 \times 21}{2} = 210,$$

we get $(x, y) = (0+7, 21-8) = (7, 13)$.

Q9.

1. First, suppose a function $f : \mathcal{P}(\mathbb{N} \times \mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that all the natural number pairs in the set that is the element of the set $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ are turned into specific natural numbers in the set that is the element of the $\mathcal{P}(\mathbb{N})$ by the function f. How the f works is that

   $$f(\{(a,b),(c,d),...\}) = \{\frac{1}{2}(a+b)(a+b+1)+b, \frac{1}{2}(c+d)(c+d+1)+d, ...\}$$

   Since the Cantor's Pairing Function form $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is bijective (i.e. one specific natural number pairs is only corresponded to one specific natural number), the function f is injective.

2. Second, suppose a function $g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N} \times \mathbb{N})$ such that all the natural numbers in the set that is the element of the $\mathcal{P}(\mathbb{N})$ are turned into specific natural number pairs in the set that is the element of the set $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ by the function g. How the g works is that

   $$\begin{aligned}
   &g(\{a, b, ...\}) = \{(n_1, n_2), (n_3, n_4), ...\}, \\
   &\text{where a} = \frac{1}{2}(\mathrm{n}_1+\mathrm{n}_2)(\mathrm{n}_1+\mathrm{n}_2+1)+\mathrm{n}_2, \mathrm{b} = \frac{1}{2}(\mathrm{n}_3+\mathrm{n}_4)(\mathrm{n}_3+\mathrm{n}_4+1)+\mathrm{n}_4, ... \text{ and so on}
   \end{aligned}$$

   Since the Cantor's Pairing Function form $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is bijective (i.e. one specific natural number is only corresponded to one specific natural number pairs), the function g is injective.

3. Therefore, there exists a bijection function between $|\mathcal{P}(\mathbb{N} \times \mathbb{N})|$ and $|\mathcal{P}(\mathbb{N})|$, which means $|\mathcal{P}(\mathbb{N} \times \mathbb{N})| = |\mathcal{P}(\mathbb{N})|$.

Q10. Consider a function $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ such that for $A \in \mathcal{P}(\mathbb{N})$, $n \in A$, we have,

$$f(A) = \sum_{n \in A} 10^{-n}$$

Therefore, the function f is injective, since 10^{-n} has different digits for different n and thus the summation of various 10^{-n} must be different for different A because they contain different n. Therefore, we have $|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Because we have $|N| < |\mathcal{P}(\mathbb{N})|$, we have $|\mathbb{N}| < |\mathbb{R}|$.