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Hybrid approaches to the solutions of the "Lighthill–Whitham–Richards" model

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Abstract

This paper presents a hybrid "Lighthill–Whitham–Richards" (LWR) model combining both macroscopic and microscopic traffic descriptions. Simple interfaces are defined to translate the boundary conditions when changing the traffic description. They are located in discrete points in space and are based on a generalisation of demand and supply concepts. Simulation results show that the proposed interfaces ensure an accurate wave propagation along two links differently described. Furthermore, the demand and supply formulation of the hybrid model makes it fully compatible with major LWR extensions developed over the past few years (intersections modeling, multi-lanes representation, etc.). Finally, we prove that Newell's optimal velocity car-following model can be derived from the Godunov scheme applied to the LWR model in Lagrangian coordinates. This proposes a microscopic resolution of the LWR model that is fully compatible with the usual macroscopic representation. This microscopic resolution is included in the hybrid model.

Keywords: Kinematic wave (LWR) model; Macroscopic and microscopic resolution; Coupling conditions; Macro/micro and micro/macro interfaces

1. Introduction

Traffic flow models can be described either microscopically or macroscopically. Microscopic models use individual representation of vehicles based on car-following approaches, whereas macroscopic models consider traffic as a continuum stream obeying global rules. Hybrid models are able to combine both descriptions, which makes it possible to treat different links or nodes of a network with different levels of details. Driver behaviour can thus be treated more precisely or more coarsely when needed, e.g. route choice, stochastic behaviour or gap acceptance processes are easier to represent microscopically; macroscopic intersection models (Jin and Zhang, 2003; Lebacque and Koshyaran, 2005) may be a simple and relevant way to represent nodes while links are microscopically described.

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Previous works (Magne et al., 2002; Hennecke et al., 2000; Poshinger et al., 2002; Bourrel and Lesort, 2003; Burghout et al., 2005) considered that the interfaces between the microscopic and the macroscopic models had a spatial extend where both models coexist. This imposes some limitations, e.g. virtual vehicles are necessary and obey fuzzy rules. Additionally, the proposed models are constrained to have the same description for all entering and outgoing links.

The aim of this paper is to overcome these limitations in the particular case of the Lighthill and Whitham (1955) and Richards (1956) (LWR) framework. Section 2 presents a preliminary result of formally deriving a microscopic resolution of the LWR model using the Godunov scheme in Lagrangian coordinates. This formulation, which is proven to be equivalent to Newell's optimal velocity car-following (OVCF) model (Newell, 2002), generalizes the results presented in Daganzo (2006). Point interfaces based on the generalisation of the demand (sending) and the supply (receiving) concept (Daganzo, 1995; Lebacque, 1996) are proposed in Section 3 to couple together a microscopic and a macroscopic resolution of this model. This guarantees the compatibility of the two traffic descriptions. Finally, a brief discussion is included in Section 4.

2. Background and preliminary result

2.1. Macroscopic resolution of the LWR model

The continuum formulation of the LWR model in Eulerian (x,t) coordinates is

$$\partial_t K + \partial_x (KV) = 0 \tag{1}$$

where $K(x,t)$ and $V(x,t)$ are respectively the density and the speed at the space–time point (x,t). The flow $Q = KV$. Speed V is related to K through the fundamental diagram (FD): $V = V_e(K)$.

This equation is usually solved numerically with finite difference methods (Leveque, 1992; Daganzo, 1995; Lebacque, 1996) associated with a fixed Eulerian grid (Fig. 1a). Highway is partitioned into small sections (cells of length Δx) and time into discrete time steps (of duration Δt). In each cell i at time t the density is approximated by a constant value K_i^t and its exit flow $Q_i^{t\to t+\Delta t}$ can be expressed as the minimum between the upstream demand $\Delta(K_i)$ and the downstream supply $\Omega(K_{i+1})$ by applying the Godunov scheme (Godunov, 1959), i.e.

$$\Delta(K) = \begin{cases} KV_e(K) & \text{if } K \leqslant K_c \\ Q_c & \text{if } K > K_c \end{cases} \quad \text{and} \quad \Omega(K) = \begin{cases} Q_c & \text{if } K \leqslant K_c \\ KV_e(K) & \text{if } K > K_c \end{cases} \tag{2}$$

where K_c is the density associated with the capacity Q_c. The density can then be updated:

$$K_i^{t+\Delta t} = K_i^t + \frac{\Delta t}{\Delta x}\left(Q_{i-1}^{t\to t+\Delta t} - Q_i^{t\to t+\Delta t}\right) \tag{3}$$

Scheme (3) requires that the Courant–Friedrichs–Lewy (CFL) stability condition (4) be satisfied.

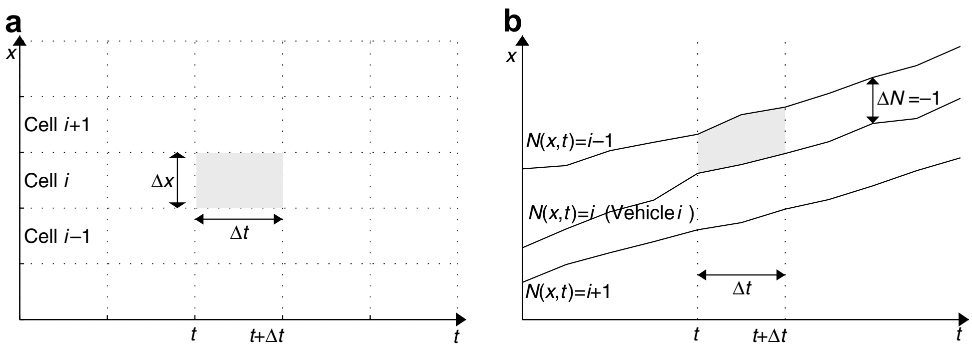

Fig. 1. (a) Eulerian grid and (b) Lagrangian grid.

$$\Delta t \leqslant \Delta x / V_{\mathrm{f}} \quad (V_{\mathrm{f}}: \text{free-flow speed}) \tag{4}$$

2.2. Microscopic resolution of the LWR model (preliminary result)

Let $N(x,t)$ (N curves) be the cumulative number of vehicles that have crossed location x by time t. This function is constant along a vehicle trajectory $x = X(t)$. The flow $Q = \partial_t N$, the density $K = -\partial_x N$ and the speed $V = \partial_t X$ on a vehicle trajectory.

The Eulerian coordinates (x,t) can now be changed into the (N,t) coordinates following (Courant and Friedrichs, 1948) and Rascle's works (Aw et al., 2000). Note that, on opposite to Eulerian coordinates which are fixed in space, (N,t) are called Lagrangian coordinates in fluid dynamics as they are fixed to a given parcel of fluid, but move in space. The partial derivatives of V with respect to N can be reformulated as follows:

$$\partial_N V = \partial_N(\partial_t X) = \partial_{N,t} X = \partial_{t,N} X = \partial_t(\partial_N X) = \partial_t(1/\partial_X N) = -\partial_t(1/K) \Rightarrow \partial_t(1/K) + \partial_N V = 0 \tag{5}$$

Let us introduce the spacing s corresponding to the inverse of the density and the fundamental diagram (FD*) expressed as a function of $s: V_e^*(s)$. The LWR model in the Lagrangian coordinates corresponds to the following non-linear, hyperbolic conservation equation in s:

$$\partial_t s + \partial_N V_e^*(s) = 0 \tag{6}$$

The Godunov method can be used to solve numerically this hyperbolic equation. The chosen Lagrangian grid corresponds to two consecutive vehicles i and $i-1$, observed between t and $t+\Delta t$ (Fig. 1b). Note that vehicle i's trajectory is defined by $N = i$. As vehicle i follows vehicle $i-1$ to be consistent with the definition of N (N decreases in space), the discretization step in N is $\Delta N = -1$. Between two vehicles, the spacing at time t is approximated by a constant value $s_i^t = x_{i-1}^t - x_i^t$ (x_i^t: position of vehicle i at time t). As the flux function $V_e^*(s)$ for Eq. (6) increases in s, the characteristic speed is always non-negative. The Godunov's method reduces to the upwind method:

$$s_i^{t+\Delta t} = s_i^t + \frac{\Delta t}{\Delta N}\left(V_e^*(s_i^t) - V_e^*(s_{i-1}^t)\right) = s_i^t + \Delta t(V_e^*(s_{i-1}^t) - V_e^*(s_i^t)) \tag{7}$$

The CFL condition (8) defines the stability domain of (7). As the Godunov scheme is consistent, this also guarantees that it converges (Leveque, 1992).

$$\Delta N \geqslant \max_s |\partial_s(V_e^*(s))|\Delta t \tag{8}$$

The numerical scheme (7) is equivalent to the following expression in x:

$$\underbrace{x_i^{t+\Delta t} - x_i^t - V_e^*(x_{i-1}^t - x_i^t)\Delta t}_{A_i} = \underbrace{x_{i-1}^{t+\Delta t} - x_{i-1}^t - V_e^*(x_{i-2}^t - x_{i-1}^t)\Delta t}_{A_{i-1}} \tag{9}$$

Eq. (9) is true for all i. As the first downstream vehicle $i = 0$ has a constant speed V_{f}, it comes that $A_0 = 0$. Thus A_i is null for all i. Eq. (9) is therefore equivalent to (10). This is the discrete-time expression of the OVCF model

$$x_i^{t+\Delta t} = x_i^t + V_{\mathrm{e}}^*(x_{i-1}^t - x_i^t)\Delta t \tag{10}$$

Wagner (1987) has proven that the weak solutions of (1) and (6) are the same. Therefore, the Lagrangian resolution of the LWR model gives rise to a class of microscopic models that are equivalent, by definition, to this model. This result has already been highlighted in Daganzo (2006) for the particular case of a triangular FD. The demonstration proposed here does not require any assumption on the shape of the fundamental diagram except for the classical concavity condition. Note also that the time step used by Daganzo for its demonstration corresponds to the limiting case when the CFL condition (8) is satisfied as an equality.

3. Coupling together the macroscopic and the microscopic resolution

This section is dedicated to the elaboration of a hybrid model, connecting the two discretization forms of the LWR model.

3.1. Interfaces definition

To stitch together the macroscopic (M) and the microscopic (m) descriptions, interfaces translating boundary conditions should be defined. The proposed interfaces are located at discrete points (macro/micro M2m interface, Fig. 3a; micro/macro m2M interface, Fig. 2a) and do not require transition areas as in previous works. The proposed interfaces are based on a generalized definition of the demand and the supply. Indeed, Lebacque (1996) proved that the flow Q at any point (x,t) can be expressed in the continuous LWR model as (11). The demand and supply expressions are well-known for a M link. Therefore, one only requires expressing the demand at the exit and the supply at the beginning of a m link. The transformation of vehicles into flow (or vice versa) will be made using reservoirs that stock vehicles when arriving and spread the right value of flow (or vice versa).

$$Q(x,t) = \min(\Delta(x^-,t);\Omega(x^+,t)) \tag{11}$$

The M2m and m2M interfaces will be described in the next two subsections. Note that one assumes that the same time step is used for both resolution modes satisfying both CFL conditions (4) and (8). Condition (8) is more restrictive, as it is possible to adapt the cell length to satisfy (4). However, time steps up to around 1 s satisfy (8) for classical parameters values of the fundamental relationship.

3.2. m2M interface

Suppose that the m2M interface is located at x_0 (m link ends at x_0^- and M link begins at x_0^+, Fig. 2a). The demand at a point x_0^- is given by (2) and requires an estimation of the density at x_0^- between t and $t+\Delta t$. Let us first suppose that a vehicle (called veh_0) is at x_0 at time t (Fig. 2b). It has just left the m link. The following vehicle (veh_1) is considered as the reference vehicle to determine the demand at x_0. The spacing at time t between veh_1 and veh_0 is $s_1^t = x_0^t - x_1^t$. In the Lagrangian Godunov scheme (m link), this spacing is supposed to be constant between t and $t+\Delta t$. The density at x_0^- is then uniform during the same time period and its value is $K(x_0^-, t\to t+\Delta t) = 1/s_1^t$. Therefore, the demand during this period can be expressed as

$$\Delta(x_0^-, t\to t+\Delta t) = \begin{cases} \frac{V_e(1/s_1^t)}{s_1^t} = \frac{V_e^*(s_1^t)}{s_1^t} & \text{if } s_1^t \geqslant s_c \\ Q_c & \text{if } s_1^t < s_c \end{cases} \quad (s_c = 1/K_c : \text{critical spacing}) \tag{12}$$

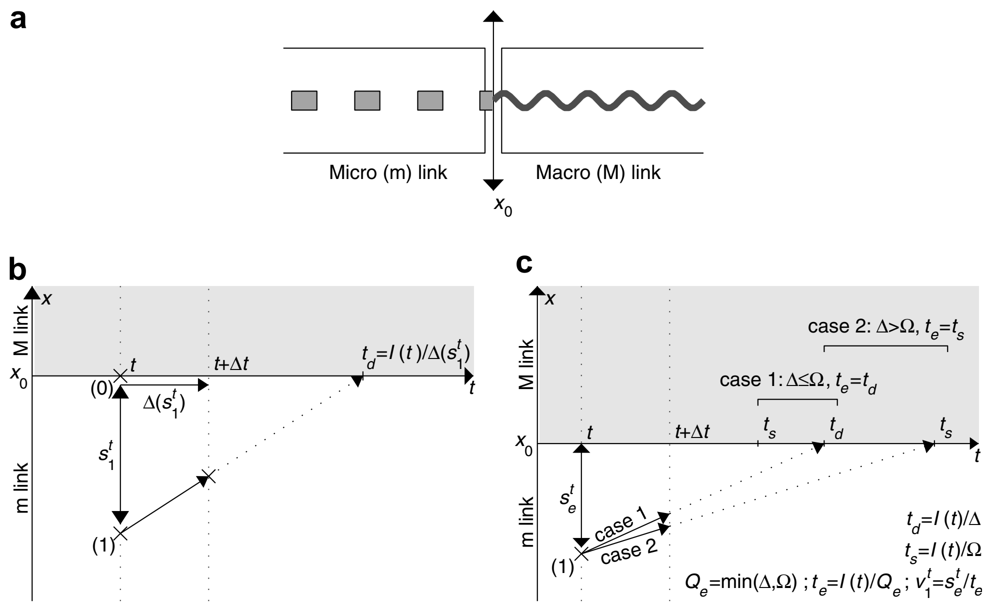

Fig. 2. (a) m2M interface, (b) definition of the demand from a m link, and (c) m2M interface functioning.

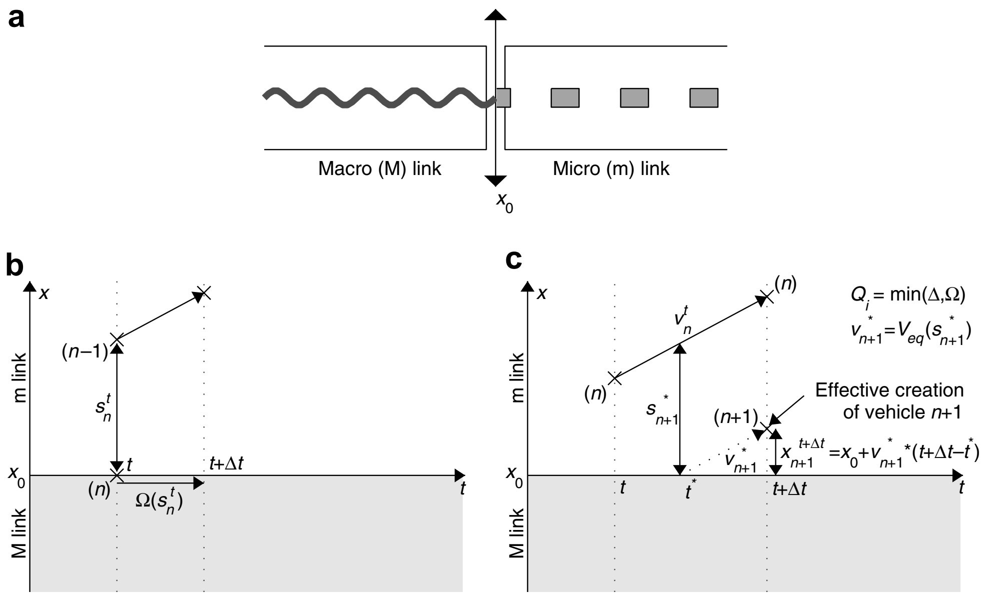

Fig. 3. (a) M2m interface, (b) definition of the supply to a m link, and (c) M2m interface functioning.

In the general case (when veh_0 is left before t), one uses the spacing $s_1^{t^*}$, observed at time t^* when veh_0 reaches x_0, to define the demand at x_0. This reference spacing is then assumed to be constant until veh_1 reaches x_0. In that case, veh_1 becomes veh_0, and veh_2 becomes the reference vehicle for the demand calculation. No virtual vehicle, driving at the beginning of the M link, is needed (as contrary to Bourrel and Lesort, 2003). Moreover, this scheme is more consistent with the Lagrangian description in the upstream link: the demand varies each time a discretization unit (a vehicle) crosses the border x_0.

The exit flow $Q_e(x_0,t \to t+\Delta t)$ can now be calculated as the minimum between the m link demand (12) and the M link supply $\Omega(K_{0^+}^t)$ ($K_{0^+}^t$ is the density in the first cell of the M link at time t). This boundary condition in flow is taken into account by the microscopic model through the vehicles conservation at x_0: the cumulative flow between two vehicles exit must be equal to 1. The speed of the reference vehicle (veh_1) is then adapted to respect this condition (Fig. 2c). This vehicle has no leader in front of it and its trajectory is thus imposed by the boundary condition at x_0.

In practice, a reservoir manages the conservation condition at x_0. Its capacity at time t is $I(t) \in [0,1]$. When veh_0 reaches x_0 at t^*, $I(t^*) = 1$. This capacity will decrease depending on the exit flow Q_e: $I(t+\Delta t) = I(t) - Q_e\Delta t$. The reservoir capacity should be null when veh_1 arrives at x_0 and comes back to 1 when this vehicle has just left. To respect this condition, veh_1 exit time t_e (from time t) is calculated depending on the reservoir state and the exit flow: $t_e = I(t)/Q_e$. The veh_1 speed v_1 is then adapted to fit this exit time: $v_1^t = s_e^t/t_e$ (s_e^t: distance between veh_1 and the m link exit). This operation is repeated every time step (Fig. 2c). This guarantees that the boundary condition in flow is correctly taken into account by the microscopic model.

3.3. M2m interface

Let us now study a M2m interface located at x_0 (the M link ends at x_0^- and the m link begins at x_0^+, Fig. 3a). The supply at x_0^+ is given by (2). When a vehicle (called veh_n) is at x_0 at time t, the density can be estimated from the spacing between this vehicle and its leader veh_$(n-1)$ $s_n^t = x_{n-1}^t - x_n^t = x_{n-1}^t - x_0$ (Fig. 3b). Its value is then $K(x_0^+, t \to t+\Delta t) = 1/s_n^t$. Therefore, the supply between t and $t+\Delta t$ is

$$\Omega(x_0^+, t \to t+\Delta t) = \begin{cases} Q_c & \text{if } s_n^t \geqslant s_c \\ \frac{V_e(1/s_n^t)}{s_n^t} = \frac{V_e^*(s_n^t)}{s_n^t} & \text{if } s_n^t < s_c \end{cases} \quad (s_c : \text{critical spacing}) \tag{13}$$

In the general case (when veh_n is entered before t), one uses the spacing $s_n^{t^*}$, observed at time t^* when veh_n was created, to define the supply at x_0. This reference spacing s_r is then assumed to be constant until the next vehicle veh_$(n+1)$ is created. Thus, the supply varies each time a discretization unit cross the border x_0. Finally, note that the reference spacing is adjusted to avoid input flow oscillations when a shockwave arrives at the M2m interface coming from the m link. Its complete expression is $s_r = \max(s_n^{t^*}; s_{n-1}^{t^*})$, where $s_{n-1}^{t^*}$ is the spacing between veh_$(n-1)$ and veh_$(n-2)$ observed when veh_n enters the m link.

The input flow $Q_i(x_0, t \to t+\Delta t)$ is calculated as the minimum between the m link supply (13) and the M link demand $\Delta(K_{0^-}^t)$ ($K_{0^-}^t$ is the density in the last cell of the M link at time t). As for the m2M interface, this boundary condition is introduced satisfying flow conservation: a vehicle is created when the cumulative flow since the last creation is equal to 1. A reservoir also manages the conservation condition in that case.

When the reservoir capacity I reaches 1 at time t^*, the vehicle $n+1$ should be created at x_0. Its speed v_{n+1}^* is evaluated using the spacing s_{n+1}^* between veh_n and x_0 at time t^* (Fig. 3c). In practice, if t^* occurs between t and $t+\Delta t$, veh_$(n+1)$ is only created at the end of the time step $(t+\Delta t)$. Its position is then $x_{n+1}^{t+\Delta t} = x_0 + (t + \Delta t - t^*)v_{n+1}^*$. The reservoir capacity is reset each time a vehicle is created. This capacity $I(t)$ increases each time step depending on the input flow Q_i: $I(t+\Delta t) = I(t) + Q_i\Delta t$.

3.4. Simulation results

Four typical cases representing a highway that is composed of a m and a M link (cases a and b, respectively a M and a m link, cases c and d) are simulated:

Case a: supply variations just downstream of the m2M interface;
Case b: a shockwave coming from the M link;
Case c: demand variations just upstream of the M2m interface;
Case d: a shockwave coming from the m link.

All of the results are presented in Fig. 4. In each case, the left diagram shows the flow evolution at the interface level ($x_0 = 0$) and the reservoir capacity depending on time. The right diagram presents the vehicles trajectories in the m link and the density in the M link (cases b and d). The flow at the entry of the highway is assumed to be constant and equal to 0.4 veh/s (except for case c where the demand varies). The FD is supposed to be isosceles (capacity $Q_c = 0.5$ veh/s; free-flow speed $V_f = 5$ m/s; jam density $K_x = 0.2$ veh/m). Such an isosceles FD is of course unrealistic in practice but it guaranties that the Eulerian (3) and the Lagrangian (7) Godunov schemes do not introduce numerical viscosity when the CFL conditions (4) and (8) are satisfied as an equality (here $\Delta t = 1$ s) (Daganzo and Laval, 2005; Daganzo, 2006). Differences between the numerical results and the analytical solutions of the LWR model can thus only occur due to the interface treatment. This makes it possible to easily study the accuracy of the proposed treatment.

The four cases demonstrate that the wave's propagation at both interfaces level is correct. Case a shows that the exit flow immediately fits the supply variations. This corresponds to the LWR analytical solution (Whitham, 1974; Daganzo, 1997) one should observe in that case at x_0. Same conclusions can be drawn from case c, as the input flow fits the analytical solution for such demand variations. In cases b and d, the shockwave is generated by a downstream supply equal to 0.2 veh/s at the end of the highway. These cases show that the shockwave properly travels through both interfaces: neither delays nor modifications of the shockwave speed are observed. Crossing the interfaces appears to be a transparent operation for the model solutions.

The four cases also highlight that the m2M and the M2m interfaces never introduce oscillations in the exit flow. This is an advantage comparing to previous works on hybrid models (Magne et al., 2002; Hennecke et al., 2000; Poshinger et al., 2002; Bourrel and Lesort, 2003). In all cases, the reservoir capacity always varies between 0 and 1. This demonstrates that the interfaces work well. Note that the value 0 or 1 is not always reached because the capacity is only evaluated at discrete time steps. The capacity is equal to 0 or 1 only when vehicles cross the interface. This event often occurs inside a time step. Finally, no irregularity is observed for the vehicles trajectories in the vicinity of both interfaces. Note that the trajectories in cases c and d do not

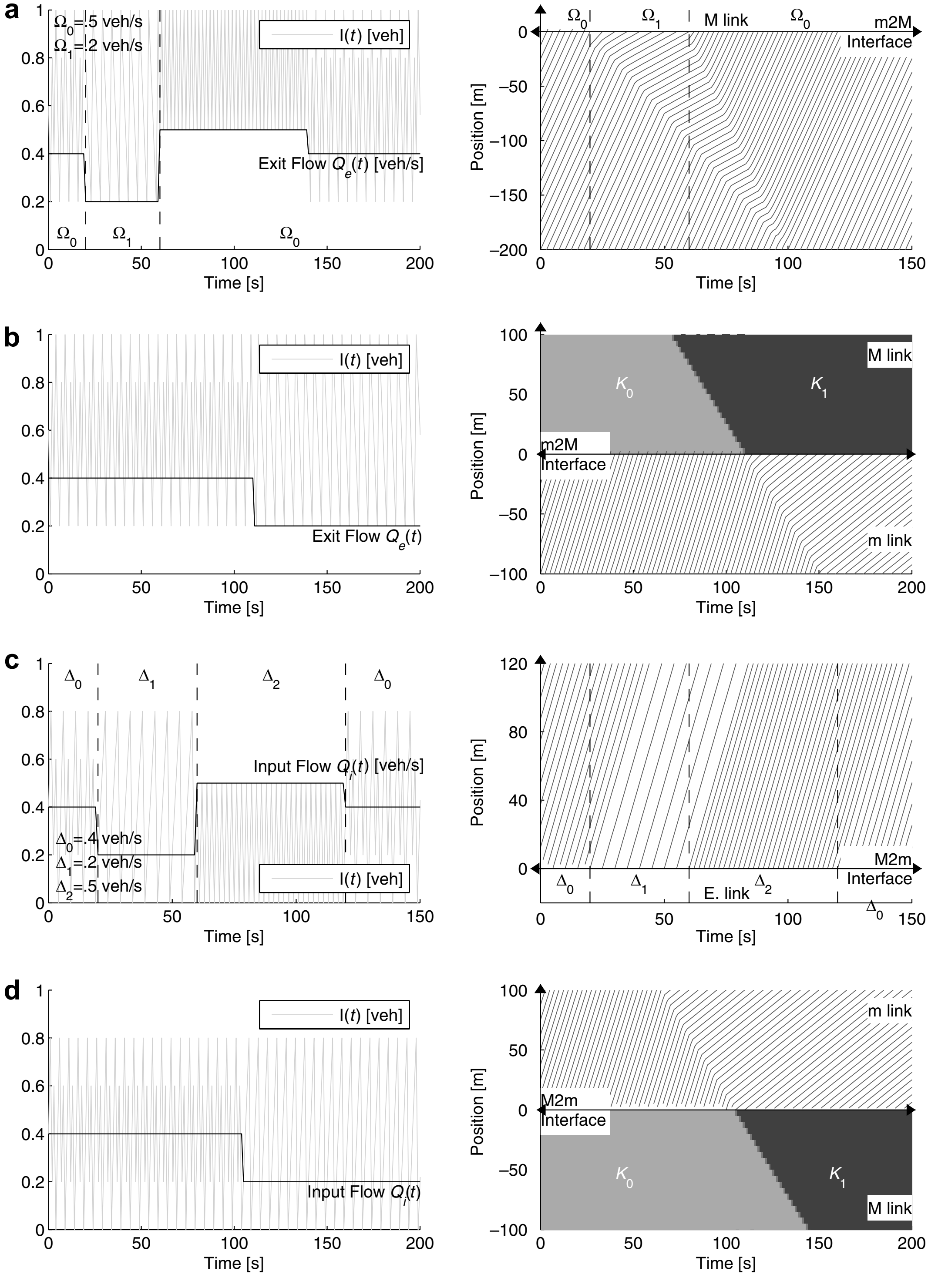

Fig. 4. Simulation results (a) m2M interface – discontinuous variation of the downstream supply, (b) m2M interface – shockwave coming from the M link, (c) M2m interface – discontinuous variation of the upstream demand, and (d) M2m interface – shockwave coming from the m link.

always begin at x_0 because vehicles are only created at the end of a time step and at the position they have reached at this moment.

4. Conclusion

This paper shows that the LWR model can be numerically solved in two different manners: the classical Eulerian scheme and a Lagrangian scheme following the *N* curves. This method leads to the same discrete-time expression as the car-following models based on Newell's optimal velocity approach. Both discretization schemes converge to the same solutions. This highlights that the way traffic is represented does not influence the results. A ''microscopic'' (car-following) and a ''macroscopic'' approach leads to the same solutions in such a case. In fact, the model properties are governed by the behaviour rule (the fundamental diagram) rather than by the traffic representation.

The paper also proposes a hybrid model for coupling together both discretizations. Simple interfaces are defined to translate the boundary conditions when changing the traffic representation (M2m and m2M). These interfaces are based on the generalisation of the demand and the supply concept for a m link. The simulation results show that these interfaces ensure a correct wave's propagation. They do not induce delays, wave speed modification or oscillations when changing the flow representation. Thus they appear to be transparent for the model solutions.

The demand and supply formulation of the hybrid model makes it fully compatible with major LWR extensions developed over the past few years. This enlarges its possible practical applications. The most significant example is convergent, divergent and intersections models which are all formulated using the demand and supply concepts (see Lebacque and Koshyaran, 2005, for a review). Therefore, they can be directly introduced in the hybrid model at the interfaces level. This notably permits to simply representing intersections in the optimal velocity car-following models. Another example is multi-lane representation (Munjal and Pipes, 1971). This can be achieved in the hybrid model by dedicating one interface per lane. Furthermore, Laval and Daganzo (2005) have proposed to introduce lane-changing representation in LWR multi-lanes formulation. Again, the m2M and the M2m interfaces are already compatibles with lane-changing modeling. It would be interesting to look how such an extension can be expressed in the microscopic scheme in order to develop a global multi-lanes hybrid model including lane-changing.

Finally, the proposed hybrid model could help the development of new extensions of the LWR model. Indeed, some phenomena can be more easily described using an individual representation of vehicles (individual route choice, stochastic aspect, etc.). This can be introduced in the microscopic scheme devoted to m link.

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