where $\bar{v}(x, v_{\text{des}}; t)$ is defined as

$$\bar{v}(x, v_{\text{des}}; t) = \frac{\int \mathrm{d}v \; v \; g(x, v, v_{\text{des}}; t)}{f_0(x, v_{\text{des}}; t)}. \tag{9.52}$$

Equation (9.51) is an equation of continuity for each desired speed $v_{\text{des}}$ separately. It is a consequence of the assumption that $\mathrm{d}v_{\text{des}}/\mathrm{d}t = 0$, i.e., no driver changes the desired speed.

Using the Boltzmann-like equation (9.3.2) and the definition (9.50), we can get separate partial differential equations for the moments of $v$, moments of $v_{\text{des}}$, and the mixed moments of $v$ and $v_{\text{des}}$. Unfortunately, these lead to a *hierarchy* of moment equations where each evolution equation for moments of a given order involves also moments of the next higher order. To close this system of equations, one needs to make appropriate justifiable assumptions.

This program can in principle be carried out for arbitrary orders $k$. For $k = 0$, it leads to the continuity equation, i.e., a first-order traffic. In the next order, $k = 1$, the velocity (momentum) equation of the second-order models is obtained. Furthermore, a new macroscopic variable, the product of density and variance, which is interpreted as *traffic pressure*, is introduced. To close the system, assumptions about the traffic pressure are needed, e.g., its density and speed dependence. $k = 2$ leads to an equation for the velocity variance, which is sometimes called *energy equation*. Furthermore, new variables are introduced, which can, e.g., be related to the skewness of the speed distribution.

## 9.4. CAR-FOLLOWING MODELS

In the *car-following theories* [425, 560, 1199], one writes, for each individual vehicle, an equation of motion. This is analogous to the Newtonian description of a classical system of interacting particles. In Newtonian mechanics, the acceleration may be regarded as the *response* of the particle to the *stimulus* it receives in the form of force, which includes both the external force and those arising from its interaction with all the other particles in the system. Therefore, the basic philosophy of the car-following theories [425, 560, 1199] can be summarized by the equation

$$\boxed{[\text{Response}]_n \propto [\text{Stimulus}]_n} \tag{9.53}$$

for the $n$-th vehicle ($n = 1, 2, \ldots$). Each driver can respond to the surrounding traffic conditions only by accelerating or decelerating the vehicle. Different forms of the equations of motion of the vehicles in the different versions of the car-following models arise from the differences in their postulates regarding the nature of the stimulus (i.e., *behavioral force* or a *generalized force* [542]). The stimulus may be composed of the speed of the vehicle, the difference in the speeds of the vehicle under consideration and its lead vehicle, the distance headway, etc., and therefore, in general,

$$\ddot{x}_n = f_{\text{sti}}(v_n, \Delta x_n, \Delta v_n), \tag{9.54}$$

where the function $f_{\text{sti}}$ represents the stimulus received by the $n$-th vehicle. Different versions of the car-following models model the function $f_{\text{sti}}$ differently. In the next two subsections, we discuss two different conceptual frameworks for modeling $f_{\text{sti}}$.

### 9.4.1. Follow-the-Leader Model

In the earliest car-following models [1111, 1179, 1180], the difference in the velocities of the $n$-th and $(n+1)$-th vehicles was assumed to be the stimulus for the $n$-th vehicle.[6] In other words, it was assumed that every driver tends to move with the same speed as that of the corresponding leading vehicle so that

$$\boxed{\ddot{x}_n(t) = \frac{1}{\tau}\left[\dot{x}_{n+1}(t) - \dot{x}_n(t)\right],} \tag{9.55}$$

where $\tau$ is a parameter that sets the time scale of the model. Note that $1/\tau$ in the equation (9.55) can be interpreted as a measure of the sensitivity coefficient $\mathcal{S}$ of the driver; it indicates how strongly the driver responds to unit stimulus. According to such models (and their generalizations proposed in the 1950s and 1960s), the driving strategy is to follow the leader and, therefore, such car-following models are collectively referred to as the *follow-the-leader* model.

Pipes [1111] derived the equation (9.55) by differentiating, with respect to time, both sides of the equation

$$\Delta x_n(t) = x_{n+1}(t) - x_n(t) = (\Delta x)_{\text{safe}} + \tau \dot{x}_n(t), \tag{9.56}$$

which encapsulates his basic assumption that (1) the higher is the speed of the vehicle, the larger should be the distance headway, and (2) in order to avoid collision with the leading vehicle, each driver must maintain a *safe distance* $(\Delta x)_{\text{safe}}$ from the leading vehicle.

It has been argued [187] that for a more realistic description, the strength of the response of a driver at time $t$ should depend on the stimulus received from the other vehicles at time $t - T$ where $T$ is a response time lag. Therefore, generalizing the equation (9.55), one would get [187]

$$\ddot{x}_n(t + T) = \mathcal{S}[\dot{x}_{n+1}(t) - \dot{x}_n(t)] \tag{9.57}$$

where the sensitivity coefficient $\mathcal{S}$ is a constant independent of $n$.

[6] In the following, we label the vehicles in driving direction such that the $(n+1)$-th vehicle is in front of the $n$-th vehicle.

According to the equations (9.55) and (9.57), a vehicle would accelerate or decelerate to acquire the same speed as that of its leading vehicle. This implies that, as if, slower following vehicle are dragged by their faster leading vehicle. In these *linear* dynamical models, the acceleration response of a driver is completely independent of the distance headway. Therefore, this oversimplified equation fails to account for the clustering of the vehicles observed in real traffic. Moreover, since there is no density-dependence in this dynamical equation, the fundamental relation cannot be derived from this dynamics. In order to make the model more realistic, we now assume [428] that the closer is the $n$-th vehicle to the $(n+1)$-th, the higher is the sensitivity of the driver of the $n$-th car. In this case, the dynamical equation (9.57) is further generalized to

$$\ddot{x}_n(t+T) = \frac{\kappa}{[x_{n+1}(t) - x_n(t)]}[\dot{x}_{n+1}(t) - \dot{x}_n(t)], \tag{9.58}$$

where $\kappa$ is a constant. An even further generalization of the model can be achieved [429, 436] by expressing the sensitivity factor for the $n$-th driver as

$$\mathcal{S}_n = \frac{\kappa[v_n(t+\tau)]^m}{[x_{n+1}(t) - x_n(t)]^\ell}, \tag{9.59}$$

where $\ell$ and $m$ are phenomenological parameters to be fixed by comparison with empirical data. These generalized follow-the-leader models lead to coupled *nonlinear* differential equations for $x_n$. Thus, in this microscopic theoretical approach, the problem of traffic flow reduces to problems of nonlinear dynamics.

So far, as the stability analysis is concerned, there are two types of analyses that are usually carried out. The *local stability analysis* gives information on the nature of the response offered by the following vehicle to a fluctuation in the motion of its leading vehicle. On the other hand, the manner in which a fluctuation in the motion of any vehicle is propagated over a long distance through a sequence of vehicles can be obtained from an *asymptotic stability analysis*.

From experience with real traffic, we know that drivers often observe not only the leading vehicle but also a few other vehicles ahead of the leading vehicle. For example, the effect of the leading vehicle can be incorporated in the same spirit as the effect of next-nearest-neighbors in various lattice models in statistical mechanics. A linear dynamical equation, which takes into account this next-nearest-neighbor within the framework of the follow-the-leader model, can be written as [1199]

$$\ddot{x}_n(t+T) = \mathcal{S}^{(1)}[\dot{x}_{n+1}(t) - \dot{x}_n(t)] + \mathcal{S}^{(2)}[\dot{x}_{n+2}(t) - \dot{x}_n(t)], \tag{9.60}$$

where $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are two phenomenological response coefficients.

The weakest point of these theories is that these involve several phenomenological parameters that are determined through calibration, i.e., by fitting some predictions of the theory with corresponding empirical data [744, 1429]. Besides, an extension to multilane traffic is difficult since every driver is satisfied if he or she can attain the desired speed!

### 9.4.2. Optimal Velocity Model and Its Extensions

Some general principles should be considered while formulating the dynamical equations for updating the velocities and positions of vehicles in any microscopic theory:

- In the absence of any disturbance from the road conditions and interactions with other vehicles, a driver tends to drive with a *desired velocity* $V^{\mathrm{des}}$; if the actual current velocity of the vehicle $v(t)$ is smaller (larger) than $V^{\mathrm{des}}$, the vehicle accelerates (decelerates) so as to approach $V^{\mathrm{des}}$.
- In freely-flowing traffic, even when a driver succeeds in attaining the desired velocity $V^{\mathrm{des}}$, the velocity of the vehicle fluctuates around $V^{\mathrm{des}}$ rather than remaining constant in time.
- The interactions between a pair of successive vehicles in a lane cannot be neglected if the gap between them is short in relation to $V^{\mathrm{des}}$; in such situations, the following vehicle must decelerate so as to avoid collision with the leading vehicle.

In the car-following models, such driving strategy is expressed mathematically as[7]

$$\dot{v}_n(t) = \frac{1}{\tau}\left[V_n^{\mathrm{des}}(t) - v_n(t)\right], \tag{9.61}$$

where $V_n^{\mathrm{des}}(t)$ is the desired speed of the $n$-th driver at time $t$. In all follow-the-leader models mentioned above, the driver maintains a safe *distance* from the leading vehicle by choosing the *speed* of the leading vehicle as his or her own desired speed, i.e., $V_n^{\mathrm{des}}(t) = v_{n+1}$.

An alternative possibility has been explored in works based on the car-following approach. This formulation is based on the assumption that $V_n^{\mathrm{des}}$ depends on the distance headway of the $n$-th vehicle, i.e., $V_n^{\mathrm{des}}(t) = V^{\mathrm{opt}}(\Delta x_n(t))$ so that

$$\boxed{\dot{v}_n(t) = \frac{1}{\tau}\left[V^{\mathrm{opt}}(\Delta x_n(t)) - v_n(t)\right],} \tag{9.62}$$

where the so-called optimal-velocity (OV) function $V^{\mathrm{opt}}(\Delta x_n)$ depends on the corresponding instantaneous distance headway $\Delta x_n(t) = x_{n+1}(t) - x_n(t)$. In other words,

[7] Fluctuations can in principle be included by an additional noise term.

according to this alternative driving strategy, the $n$-th vehicle tends to maintain a *safe speed* that depends on the relative position, rather than relative velocity, of the $n$-th vehicle. In general, $V^{\text{opt}}(\Delta x) \to 0$ as $\Delta x \to 0$ and must be bounded for $\Delta x \to \infty$. For explicit calculations, one has to postulate a specific functional form of $V^{\text{opt}}(\Delta x)$. Car-following models along this line of approach have been introduced by Bando et al. [67–69]. For obvious reasons, these models are usually referred to as *OV models*.

Since the equations of motion in the follow-the-leader models involve only the velocities, and not positions, of the vehicles, these can be formulated as essentially *first-order* differential equations (for velocities) with respect to time. In contrast, since the equations of motion in the OV model involve the positions of the vehicles explicitly, the theoretical problems of this model are formulated mathematically in terms of *second-order* differential equations (for the positions of the vehicles) with respect to time [67–69].

Clearly, the reliability of the predictions of the OV model depends on the appropriate choice of the OV function. The simplest choice for $V^{\text{opt}}(\Delta x)$ is [1324, 1325]

$$V^{\text{opt}}(\Delta x) = v_{\max}\Theta(\Delta x - d), \tag{9.63}$$

where $d$ is a constant and $\Theta$ is the Heaviside step function. According this form of $V^{\text{opt}}(\Delta x)$, a vehicle should stop if the corresponding distance headway is less than $d$; otherwise, it can accelerate so as to reach the maximum allowed velocity $v_{\max}$. A somewhat more realistic choice [1031, 1325] is

$$V^{\text{opt}}(\Delta x) = \begin{cases} 0 & \text{for } \Delta x < \Delta x_A, \\ f\,\Delta x & \text{for } \Delta x_A \le \Delta x \le \Delta x_B. \\ v_{\max} & \text{for } \Delta x_B < \Delta x. \end{cases} \tag{9.64}$$

The main advantage of the forms (9.63) and (9.64) of the OV function is that exact analytical calculations, e.g., in the jammed region, are possible [1325]. Although (9.63) and (9.64) may not appear very realistic, they capture several key features of more realistic forms of OV function [68, 69], e.g.,

$$V^{\text{opt}}(\Delta x) = \tanh[\Delta x - \Delta x_c] + \tanh[\Delta x_c] \tag{9.65}$$

for which analytical calculations are very difficult. For the convenience of numerical investigation, the dynamical equation (9.62) for the vehicles in the OV model has been discretized and, then, rewritten as a *difference* equation [1017].

The main question addressed by the OV model is the following: what is the *condition* for the stability of the homogeneous solution?

$$x_n^h = bn + ct, \tag{9.66}$$

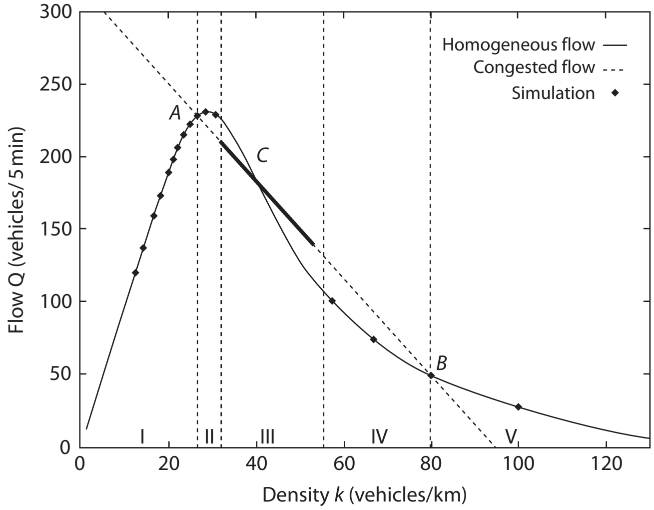


**Figure 9.3** Fundamental diagram of the OV model. The solid line shows the OV function and the dots simulation data. One can distinguish five different density regimes with respect to the stable stationary state (from [1325]).

where $b = (\Delta x)_{\text{av}} = L/N$ is the constant average spacing between the vehicles and $c$ is the constant velocity. It is not difficult to argue that in general, in the OV models, the homogeneous flow becomes unstable when $\frac{\partial V^{\text{opt}}}{\partial \Delta x}\big|_{\Delta x = b} > \frac{2}{\tau}$ [68, 69].

One can distinguish five different density regimes with respect to the stability of microscopic states (Fig. 9.3). At low and high densities, the homogeneous states are stable. For intermediate densities, three regimes with jammed states exist. In region III, the jammed state is stable, whereas in regions II and IV, both homogeneous and jammed states form stable structures. Beyond the formation of jams, also hysteresis effects have been observed. Thus, the OV model is able to reproduce many aspects of experimental findings.

A *modified Korteweg-de Vries (KdV) equation* has been derived from the equation (9.62) in a special regime of the parameters [785], and the relations between its kink solutions and traffic congestion have been elucidated [960]. In order to account for traffic consisting of two different types of vehicles, say, cars and trucks, Mason and Woods [916] generalized the formulation of Bando et al. [68, 69] by replacing the constant $\tau$ by $\tau_n$ so that

$$\dot{v}_n(t) = \frac{1}{\tau_n}[V_n^{\text{opt}}(t) - v_n(t)], \tag{9.67}$$

where $\tau_n$ now depends on whether the $n$-th vehicle is a car or a truck. Since a truck is expected to take longer to respond than a car, we should assign larger $\tau$ to trucks and smaller $\tau$ to cars. Some other mathematically motivated generalizations of the OV model have also been considered [503, 987, 991–993, 1016].

In [542], the OV model was calibrated using empirical car-following data. It was found that the relaxation time is rather short ($\tau \approx 1.2$ s), which leads to unrealistically high accelerations and overshooting of the velocity. Nevertheless the unrealistically large decelerations are not sufficient to avoid accidents in the model [1469]. Therefore, it has been suggested to extend the OV model to the *generalized force model* (see Section 9.4.3).

Another possible solution to these problems is to take into account the driver reaction times through a delay time $t_d$, e.g., by replacing $\dot{v}_n(t)$ in the equation of motion (9.62) by $\dot{v}_n(t + t_d)$ [66]. Davis [262, 263] has used the maximum size of a *safe platoon*, i.e., a platoon of vehicles that avoids collisions, as a measure of stability. He found that safe platoons require small delay time $t_d$ which is much smaller than typical reaction times. As possible solution, he suggested to replace the OV function $V^{\text{opt}}(\Delta x_n(t))$ by $V^{\text{opt}}(\Delta xNn(t - t_d) + t_d \Delta v_n(t - t_d))$, i.e., a kind of anticipation of future gaps. Further related studies have been performed by Wilson et al. [1085, 1469].

As mentioned earlier, drivers often receive stimulus not only from the leading vehicle but also from a few other vehicles ahead of the leading vehicle. One possible way to generalize the OV models for taking into account such *multivehicle* or *multianticipative interactions* [855] is to write the dynamical equations as

$$\dot{v}_n = \sum_{j=1}^{m} \mathcal{S}_j \left[ V^{\text{opt}} \left( \frac{x_{n+j} - x_n}{j} \right) - v_n \right], \qquad (9.68)$$

where $\mathcal{S}_j$ are sensitivity coefficients. One of the commonly used explicit forms of the OV function, e.g. (9.65), can be chosen for that of the function $V^{\text{opt}}$ in (9.68).

Extensions to multilane traffic [264, 995, 1225] and effects of ramps [105, 264] have also been studied. Open boundary conditions have been investigated by Mitarai and Nakanishi [934, 935]. They found an oscillatory solution in the linearly unstable region, which shares some features with synchronized traffic. Computer simulations of open systems were reported in [379]. Another OV model that reproduces certain aspects of synchronized traffic has been proposed in [504]. The parameters in the OV function of this model depend on the local traffic situation. The full velocity difference model (see Section 9.4.3) can also be considered a generalization of the OV model. It has been used to describe the transition from free flow to synchronized flow [675].

Berg et al. [104] have performed a continuum limit of the OV model. In [515, 530], it has been argued that the Payne model introduced in Section 9.2.1 can be considered to be a macroscopic approximation of the OV model. Some ideas of the OV model have been utilized by Mahnke et al. [897, 899–901] in their master equation approach to the study of jam dynamics. This is discussed in Section 9.6.1.

### 9.4.3. Generalized Force Models

The *generalized force model* introduced in [542] is motivated by the *social-force concept* [864], which has also been applied to model pedestrian dynamics (see Section 11.5.2). It assumes that the amount and direction of a behavioral change, which is typically an acceleration in traffic dynamics, is determined by *generalized forces* (*behavioral* or *social forces*), which reflect the motivation of an individual in response to the state of its environment. In general, these forces do not fulfill Newton's laws[8] like *actio* = *reactio*.

In the case of highway traffic, the main motivation for each driver $n$ is to reach a desired velocity $v^{(0)}$, which is given by the typical relaxation term $(v^{(0)} - v_n)/\tau$, and keeping a safe distance from other vehicles. For the latter, mainly the distance $d_n$ to the preceding car $n+1$ is relevant. The general structure of the equations of motion is then given by

$$\dot{v}_n = \frac{v^{(0)} - v_n}{\tau} + f(x_n, v_n; x_{n+1}, v_{n+1}), \tag{9.69}$$

where $f$ is a repulsive interaction force. The choice

$$f^{(\mathrm{opt})}(d_n) = \frac{1}{\tau}\left(V^{\mathrm{opt}}(d_n) - v^{(0)}\right) \tag{9.70}$$

corresponds to the OV model.

To guarantee that drivers brake early and strongly enough for large velocity differences $\Delta v_n = v_{n+1} - v_n$ apart from $f^{(\mathrm{opt})}$, an additional term

$$f^{(\mathrm{brake})}(v_n, v_{n+1}, d_n) = -\lambda(v_n, d_n)\Delta v_n \Theta(-\Delta v_n) \tag{9.71}$$

has to be taken into account so that the interaction force has the form

$$f = f^{(\mathrm{opt})} + f^{(\mathrm{brake})}. \tag{9.72}$$

The Heaviside function $\Theta$ in (9.71) guarantees that the term is only effective if the velocity $v_{n+1}$ of the preceding vehicle is smaller than $v_n$. $\lambda(v_n, d_n)$ is a sensitivity function, which should be chosen such that $f^{(\mathrm{brake})}$ grows with increasing velocity difference $|\Delta v_n|$ and decreasing distance $d_n$ and vanishes for $d_n \to \infty$. In [542], the sensitivity function

$$\lambda(v_n, d_n) = \frac{1}{\tau'} \exp\left(-\frac{d_n - d(v_n)}{R'}\right) \tag{9.73}$$

[8] There are further problems, like the validity of the superposition principle which is nevertheless usually assumed in all models.

was used, which is motivated by the assumption that drivers try to keep a velocity-dependent safe distance

$$d(v_n) = d_n^{(0)} + Tv_n. \tag{9.74}$$

Here, $d^{(0)}$ is the minimal vehicle distance and $T$ is a safe time headway (typically of the order of the reaction time). The braking time $\tau'$ should be smaller than $\tau$ since deceleration capabilities are larger than acceleration capabilities. $R'$ can be interpreted as the range of the braking interaction. The full model can then be written in the form of a *generalized OV model*,

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = \frac{V^*(d_n, v_n, \Delta v_n) - v_n(t)}{\tau^*} \tag{9.75}$$

with

$$V^*(d_n, v_n, \Delta v_n) = \frac{\tau^*}{\tau} V^{\mathrm{opt}}(d_n) + \frac{\tau^*}{\tau''} \Theta(\Delta v_n) v_{n+1} \tag{9.76}$$

and

$$\tau'' = \tau' \exp\left[(d_n - d(v_n))/R'\right] \quad \text{and} \quad \frac{1}{\tau^*} = \frac{1}{\tau} + \frac{\Theta(\Delta v_n)}{\tau''}. \tag{9.77}$$

In [542], it was also suggested to replace the standard OV function by

$$V^{\mathrm{opt}}(d_n) = v^{(0)} \left[1 - \mathrm{e}^{-(d_n - d(v_n))/R}\right], \tag{9.78}$$

where $R$ is the range of the acceleration interaction. A calibration using empirical car-following data then yielded the optimal parameter values, which are shown in Table 9.1.

A different form for $f^{(\mathrm{brake})}$ has been proposed in [673] called the *full velocity difference model*. It uses a braking term of the form

$$f^{(\mathrm{brake})}(v_n, v_{n+1}, d_n) = \lambda(v_n, d_n) \Delta v_n. \tag{9.79}$$

This term takes both positive and negative velocity differences into account and guarantees that a driver will brake also when the preceding car is much faster and the headway is smaller than the safe distance [683].

**Table 9.1** Optimal parameter values [542] for the generalized force model based on a calibration with empirical follow-the-leader data

| **Parameter** | $v^{(0)}$ | $\tau$ | $d^{(0)}$ | $T$ | $\tau'$ | $R$ | $R'$ |
|---|---|---|---|---|---|---|---|
| **Value** | 16.98 m/s | 2.45 s | 1.38 m | 0.74 s | 0.77 s | 5.59 m | 98.78 m |

Different variants of the full velocity difference model have been discussed, e.g., by introducing an asymmetry between positive and negative velocity differences [453], an additional acceleration difference term [1520, 1521], or a form of anticipation by including the velocity difference $\Delta v_{n+1}$ [430]. In [683, 1088], continuum analogs of the full velocity difference model have been derived.

A force model that tries to capture the tendency of vehicles to keep speed and distance has been proposed in [1518]. The first effect is taken into account by an acceleration term

$$a(v) = a_0\left(1 - \frac{v}{v_0}\right), \tag{9.80}$$

where $v_0$ corresponds, e.g., to a speed limit of 15 m/s and $a_0$ is the initial acceleration of the order of 2 m/s$^2$. The second effect is modeled in analogy to molecule dynamics by a Lennard-Jones potential

$$U(d) = \frac{k}{d}\left[-\left(\frac{d_{\mathrm{opt}}}{d}\right)^4 + \left(\frac{d_{\mathrm{opt}}}{d}\right)^2\right], \tag{9.81}$$

where $d$ is the headway and

$$d_{\mathrm{opt}} = d_0 + k_0 v^2 \tag{9.82}$$

is the optimal distance the drivers want to keep. Typical parameter values suggested in [1518] are $d_0 = 9$ m, $k_0 = 0.1$ s$^2$/m. In terms of the effective density $\rho^* = 1/d$, the equation of motion for each vehicle is then given by

$$a(\rho^*, v) = k\rho^*\left[-(\rho^* d_{\mathrm{opt}})^4 + (\rho^* d_{\mathrm{opt}})^2\right] + a_0\left(1 - \frac{v}{v_0}\right). \tag{9.83}$$

A closer analysis shows that problems might occur when the preceding car has velocity $v_{\mathrm{prec}} = 0$. In this case, equation (9.83) is no longer applicable.

### 9.4.4. Intelligent Driver Model

In the *intelligent driver model (IDM)* [1382, 1387], each driver $n$ adjusts speed $v_n$ depending on the velocity difference $\Delta v_n = v_{n+1} - v_n$ and headway $d_n = x_{n+1} - x_n - \ell$ according to

$$\boxed{\dot{v}_n = a\left[1 - \left(\frac{v_n}{v_0}\right)^\delta - \left(\frac{d_n^*}{d_n}\right)^2\right].} \tag{9.84}$$

Here, $a$ is the maximum acceleration, $v_0$ is the desired velocity, $d^*$ is the minimum desired headway, and $\delta$ is an acceleration exponent. The right-hand side interpolates between a free acceleration $a_f = a(1-(v/v_0)^\delta)$ on a free road and braking maneuvers with deceleration $-a(d^*/d)^2$ when the headway becomes too small. The desired headway depends dynamically on the velocity $v_n$ and relative velocity $\Delta v_n$ according to

$$d_n^*(v_n, \Delta v_n) = d_0 + Tv_n - \frac{v_n \Delta v_n}{2\sqrt{ab}}, \qquad (9.85)$$

where $d_0$ is the typical headway in a jam, $T_n$ is a safe time headway, and $b$ is a comfortable deceleration. The third term becomes relevant only in nonstationary traffic. It implements an accident-free ("intelligent") driving strategy, which implies in almost all situations braking decelerations less than the comfortable value $b_n$. In emergency situations, the decelerations can become larger to make the model collision-free [1387], at least for single-lane traffic. Note that the IDM does not take into account the reaction time of the drivers, which would lead to an effective reduction of the time headway.

The IDM has a homogeneous flow solution with headway

$$d_h = \frac{d_0 + Tv_h}{\sqrt{1-(v_h/v_0)^\delta}} \qquad (9.86)$$

corresponding to the global density $\rho = 1/(\ell + d_h)$. The flux in this homogeneous state is $J_h = \rho v_h$. Simple expressions for $v_h$ only result in certain limiting cases [1387].

Table 9.2 shows typical parameter values for the IDM. The acceleration $a$ corresponds to 38 s needed to accelerate from 0 to 100 km/h. The maximum deceleration is approximately 4.5 m/s$^2$, i.e., $g/2$ and much larger than $b$. A proper calibration of the IDM with empirical car-following data based on trajectory sets is described in [744]. The fitted parameters are found to have realistic values.

**Table 9.2** Typical parameter values for the IDM as proposed in [1387]

| **Parameter** | **Typical value** |
|---|---|
| Desired velocity $v_0$ | 120 km/h |
| Safe time headway $T$ | 1.6 s |
| Maximal acceleration $a$ | 0.73 m/s$^2$ |
| Desired deceleration $b$ | 1.67 m/s$^2$ |
| Acceleration exponent $\delta$ | 4 |
| Headway in jam $d_0$ | 2 m |
| Car length $l = 1/\rho_{\max}$ | 5 m |

In [1382], slightly different values for $a$, $b$, and $T$ have been used.

The IDM describes most macroscopic aspects of the spatiotemporal dynamics, especially the various types of congested traffic [1388] including the scattered flow-density data in the synchronized regime [1384, 1387], the behavior in systems with inhomogeneities like ramps [524, 1387], and hysteresis phenomena [259]. The *IDM with memory (IDMM)* [1384] allows also to take into account memory effects in the adaption of drivers to surrounding traffic. However, like most car-following models, it produces unrealistic dynamics and crashes if realistic reaction times (of the order of 1 s) are introduced [1389], although it is crash-free for vanishing reaction times [1382, 1387].

In order to overcome the problems described above, the IDM has been extended to the *human driver model (HDM)* in [1389]. It is a meta-model that incorporates additional aspects into basic car-following models of the type

$$\dot{v}_n = a(d_n, v_n, \Delta v_n)\,. \tag{9.87}$$

For a more realistic description (i) finite reaction times, (ii) estimation errors (for the headway $d_n$ and the relative velocity $\Delta v_n$), (iii) spatial anticipation (interactions with preceding vehicles), and (iv) temporal anticipation (of future headways and velocities) have to be included. Basically this implies that the input parameters in (9.87) have to be replaced by effective or estimated ones.

The HDM includes not only the IDM, but also OV models (Section 9.4.2), the velocity difference model (Section 9.4.3), bounded rational driver models [886, 887] and the Gipps model (Section 9.5.1). Lane-changes have been implemented based on the MOBIL (minimizing overall braking induced by lane changes) concept [745].

### 9.4.5. Kerner–Klenov Model

The Kerner-Klenov (KK) model [731, 732] was introduced to reproduce the basic features of Kerner's three-phase traffic theory. The Kerner–Klenov–Wolf model discussed in Section 8.3.2 is the CA variant of the KK model.

The basic rules of vehicle motion for a vehicle which is at time $t$ located at $x(t)$ and has velocity $v(t)$ and (net) headway $d(t) = x_{\text{prec}}(t) - x(t)$ ($x_{\text{prec}}$ is the position of the preceding car) are given by [727]

$$v(t+\tau) = \max[0, \min\{v_{\max}, v_{\text{des}}(t), v_{\text{safe}}(t)\}]\,, \tag{9.88}$$

$$x(t+\tau) = x(t) + v(t+\tau)\tau\,, \tag{9.89}$$

where $\tau$ is the discrete time step and $v_{\max}$ the maximal velocity in free flow. The *desired speed*

$$v_{\text{des}}(t) = \begin{cases} v(t) + \Delta(t) & \text{for } d(t) \le D(t) \\ v(t) + a(t)\tau & \text{for } d(t) > D(t) \end{cases} \tag{9.90}$$

depends on the headway $d(t)$ and the *synchronization distance* $D(t) = D(v(t), v_{\text{prec}}(t))$ where $v_{\text{prec}}(t)$ is the velocity of the preceding car. $D(t)$ will be specified later. $\Delta(t)$ is given by

$$\Delta(t) = \max\left[-b(t)\tau, \min\{a(t)\tau, \Delta v(t)\}\right] \tag{9.91}$$

with the velocity difference $\Delta v(t) = v_{\text{prec}}(t) - v(t)$ to the preceding car. The acceleration and deceleration functions $a(t) \geq 0$ and $b(t) \geq 0$, respectively, restrict speed changes during a time step. They contain a stochastic component to simulate driver time delays in acceleration and deceleration. The safe speed $v_{\text{safe}}(t)$ is given by

$$v_{\text{safe}}(t) = \min\left[v^{(s)}(t), \frac{1}{\tau}d(t) + v_{\text{anti}}\right] \tag{9.92}$$

where

$$v_{\text{anti}} = \max\left\{0, \min\left\{v_{\text{prec}}^{(s)}(t) - a\tau, v_{\text{prec}}(t) - a\tau, d_{\text{prec}}(t)/\tau\right\}\right\} \tag{9.93}$$

is an anticipated velocity of the preceding vehicle at the next time step and $v^{(s)}(t)$ is the solution of the Gipps equation for a safe velocity (Section 9.5.1).

The precise rules are rather complex and involve a large number of parameters (see chapter 16.3 of Kerner's book [727]). They are designed such that in the deterministic limit, the stationary states cover a two-dimensional region in the flow-density plane (Fig. 9.4). This is realized through the *speed adaption effect* in synchronized flow where all vehicles move at the same constant speed and headway.

Two related but somewhat simpler deterministic models that capture some essential ideas of the KK model have been suggested in [733]. The *acceleration time delay (ATD) model* emphasizes the importance of delays in acceleration and deceleration processes. In the *speed adaption model (SAM)* [733, 1431], speed adaption occurs in synchronized flow depending on the driving conditions. Both models yield congested traffic patterns that are consistent with empirical results.

### 9.4.6. Inertial Car-Following Model

The model proposed by Tomer et al. [1205, 1371] assumes that car acceleration is affected by four factors, namely, (1) keeping a safe time gap, (2) early braking if the preceding car is much slower, (3) obeying speed limits, and (4) random noise. These factors are captured by the acceleration function of vehicle $n$,

$$a_n = A\left(1 - \frac{\Delta x_n^0}{\Delta x_n}\right) - \frac{\Theta^2(-\Delta v_n)}{2(\Delta x_n - D)} - k\Theta(v_n - v_{\max} + \eta), \tag{9.94}$$

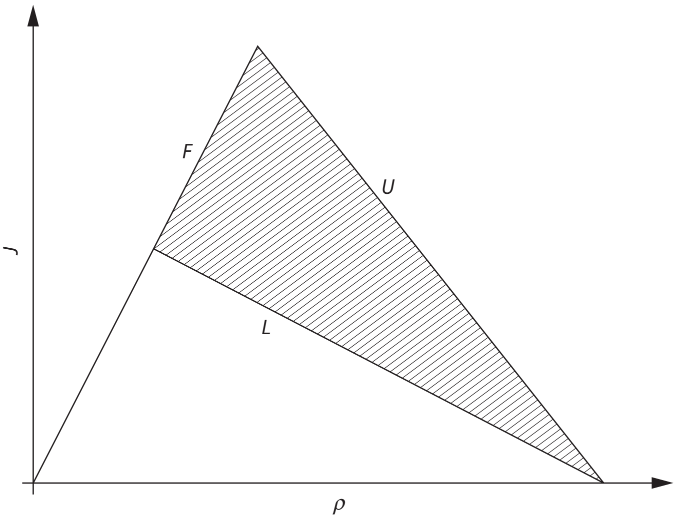


**Figure 9.4** Fundamental diagram of the KK model in the deterministic limit. The boundaries of the allowed region are associated with free flow ($F$), the synchronization gap ($L$), and the safe gap ($U$).

where $\Delta x_n = x_{n+1} - x_n$ is the headway of vehicle $n$ and $\Delta v_n = v_{n+1} - v_n$ is the velocity difference. Furthermore, the safety distance $\Delta x_n^0 = v_n T + D$ has been introduced, which is determined by the safe time gap $T$ and the minimal distance $D$ between cars. Finally, $\Theta(x)$ is the Heaviside function (with $\Theta(0) = 0$), $A$ and $k$ are sensitivity constants, $v_{\max}$ is the maximal velocity, and $\eta$ is a noise term.

The first term in (9.94) becomes important for small velocity differences $\Delta v_n$ and leads to braking (accelerating) if $\Delta x_n < \Delta x_n^0$ ($\Delta x_n > \Delta x_n^0$). The second term is relevant if the preceding car is approached very fast ($v_n \gg v_{n+1}$) and corresponds to the deceleration, which is necessary to reduce the velocity difference $\Delta v_n$ to 0 as the minimal headway $D$ is approached. The third term is dissipative and represents a repulsive force, which reduces the velocity if $v_{\max}$ is exceeded.

The equation of motion is then given by $\dot{x}_n = v_n$ and $\dot{v}_n = a_n$ with $a_n$ as defined in (9.94). Assuming periodic boundary conditions ($x_{N+1} = x_1 + N/\rho,\ v_{N+1} = v_1$) and neglecting the noise term $\eta$, a solution corresponding to homogeneous flow can be found:

$$v_n^{(0)} = \begin{cases} \dfrac{A(1 - D\rho) + k v_{\max}}{a\rho T + k} & \text{for } \rho \leq \rho_c \\[2ex] \dfrac{1 - D\rho}{\rho T} & \text{for } \rho \geq \rho_c \end{cases} \tag{9.95}$$

with $\rho_c = \frac{1}{D + T v_{\max}}$ so that

$$x_n^{(0)} = \frac{n-1}{\rho} + v_n^{(0)} t. \tag{9.96}$$

In computer simulations, one finds in an intermediate density regime $\rho_1 < \rho < \rho_2$ flows which are considerably lower and velocity fluctuations occur. For small values of $A$, the upper critical density can become larger than the maximal density $\rho_{\max} = 1/D$. Therefore, one can distinguish three different flow regimes: (1) free flow for $\rho < \rho_1$, (2) nonhomogeneous congested traffic (NHC) for $\rho_1 < \rho < \rho_2$, and (3) homogeneous congested traffic $\rho > \rho_2$. In the NHC regime, the presence of humps (dense regions) is observed, which can move forward or backward. In the stationary state, the humps are equidistant, and the NHC state is similar to the *recurring humps state* of [844] and the empirical observations in [742].

The NHC is not unique since, depending on the initial conditions, different wavelengths of the recurring humps and corresponding flows can be realized. In [1371], this is interpreted as indication for the existence of many different attractive limit cycles. Some of these cycles are more sensitive to noise than others.

The values of $\rho_1$ and $\rho_2$ can be estimated by an analytical calculation [1371]:

$$\rho_1' = \frac{1}{D + Tv_{\max}}, \qquad \rho_2' = \frac{2}{AT^2}. \tag{9.97}$$

Comparison with simulation data show that $\rho_2 \approx \rho_2'$, but that $\rho_1$ is considerably smaller than $\rho_1'$. In the regime $\rho_1 < \rho < \rho_1'$, both the NHC and the homogeneous flow solution appear to be stable as indicated by hysteresis loops in the density-flow plane.

## 9.5. COUPLED-MAP MODELS

In the car-following models, space is assumed to be a continuum, and time is represented by a continuous variable $t$. Besides, velocity and acceleration of the individual vehicles are also real variables. However, most often, for numerical manipulations of the differential equations of the car-following models, one needs to discretize the continuous variables with appropriately chosen grids. In contrast, in the coupled-map approach [708], one starts with a discrete time variable. The dynamical equations for the individual vehicles are formulated as discrete dynamical maps that relate the state variables at time $t$ with those at time $t+1$, although position, velocity, and acceleration are not restricted to discrete integer values. The unit of time in this scheme (i.e., one time step) may be interpreted as the reaction time of the individual drivers as the velocity of a vehicle at the time step $t$ depends on the traffic conditions at the preceding time step $t-1$. The general form of the dynamical maps in the coupled-map models can be expressed as follows:

$$v_n(t+1) = \mathrm{Map}_n[v_n(t), v_{\mathrm{des}}, \Delta x_n(t)], \tag{9.98}$$

$$x_n(t+1) = v_n(t) + x_n(t), \tag{9.99}$$