Mean-Field Limit of Infinite-range Quantum Spin-1/2 Systems

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Guoyi Ma (Matrikelnummer 5406470), January 25, 2023

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Abstract

The aim of this thesis is to present a derivation of mean-field limits for time-dependent quantum spin systems with N interacting particles. The core of the idea is to present the case of a system with a bounded interaction scaling potential to make each pair of interactions weaker than the mean-field potential. The method we use in this proof is to translate the physical description of a many-particle quantum spin system directly into a mathematical counting algorithm. This method was originally developed for the derivation of the Gross-Pitaevskii equation from a N-particle non-spin quantum mechanical system for typical initial conditions [1].

Dedication

To my parents who always believe in me and give me great support during my studies.

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Contents

1 Introduction **5**

1.1 Quantum Spin Systems . . . 5

1.1.1 Quantum Mechanics of a Single Spin . . . 6

1.1.2 Infinite-range Quantum Spin Systems . . . 9

1.1.3 A Concise Definition of the Mathematical Model . . . 9

1.2 Structure of the Thesis . . . 11

2 Mean-field Theory **12**

2.1 Mean-field Theory in Infinite-range Quantum Spin Systems . . . 13

3 Mean-field Limits of General Quantum Spin-1/2 Systems **16**

3.1 Counting Measure . . . 16

3.1.1 Control of $\alpha_N(t)$ . . . 18

3.2 Construction of the Counting Measure . . . 20

4 Conclusion **30**

Chapter 1

Introduction

The mean-field theory is an approximation. However, it can give the exact solution in the case of an infinite-range model where all possible pairs of sites have interactions. In this work, we present the rigorous derivation of infinite-range quantum spin-1/2 mean-field models starting from a non-relativistic many-particle Schrödinger equation. In this chapter, we introduce the basic principles of many-body quantum spin systems and give an outline of the main part of this thesis including a summary of our main results.

1.1 Quantum Spin Systems

We begin by giving a brief introduction to the mathematical framework for the quantum spin system by following [2, 3, 4].

The term "quantum spin system" often refers to such models of quantum systems with an infinite number of degrees of freedom that each have a finite-dimensional state space, e.g. an n-dimensional Hilbert space associated with each site of a d-dimensional lattice.

1.1.1 Quantum Mechanics of a Single Spin

In nonrelativistic quantum mechanics, we will commonly denote by $\mathcal{H}$ the complex Hilbert space of states of a quantum system. The inner product $< \cdot, \cdot >$ on $\mathcal{H} = \mathbb{C}^n$ is given by

$$\langle u, v \rangle = \sum_{i=1}^{n} \overline{u_i} v_i. \tag{1.1}$$

The norm induced by this inner product is denoted by $\| \cdot \|$.

Spin (or more precisely, spin angular momentum) is a quantum mechanical object described by the spin operator $\boldsymbol{S} = \left(S^{(1)}, S^{(2)}, S^{(3)}\right)$ for which the operators $S^{(1)}, S^{(2)}, S^{(3)}$ are self-adjoint, and satisfy the commutation relations

$$\left[S^{(\alpha)}, S^{(\beta)}\right] = i \sum_{\gamma=1,2,3} \varepsilon_{\alpha\beta\gamma} S^{(\gamma)}, \tag{1.2}$$

for any $\alpha, \beta = 1, 2, 3$, where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol, defined as $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = 1$, and $\varepsilon_{\alpha\beta\gamma} = 0$ for other components.

The spin operators $S^{(1)}, S^{(2)}, S^{(3)}$ are defined by the property that they are Hermitian matrices satisfying the $SU(2)$ commutation relations. Instead of S^1 and S^2, one often works with the spin raising and lowering operators, $S^+$and S^-, defined by the relations $S^1 = (S^+ + S^-)/2$, and $S^2 = (S^+ - S^-)/(2i)$. In terms of these, the $SU(2)$ commutation relations are

$$\left[S^+, S^-\right] = 2S^3, \quad \left[S^3, S^\pm\right] = \pm S^\pm,$$

where we have used the standard notation for the commutator for two elements A and B in an algebra: $[A, B] = AB - BA$. In the standard basis S^3, S^+, and $S^-$are given by the following matrices:

$$S^3 = \begin{pmatrix} S & & & \\ & S-1 & & \\ & & \ddots & \\ & & & -S \end{pmatrix}$$

$S^- = (S^+)^*$, and

$$S^+ = \begin{pmatrix} 0 & c_S & & & \\ & 0 & c_{S-1} & & \\ & & \ddots & \ddots & \\ & & & 0 & c_{-S+1} \\ & & & & 0 \end{pmatrix}$$

where, for $m = -S, -S+1, \ldots, S$,

$$c_m = \sqrt{S(S+1) - m(m-1)}.$$

The spin operators act on the $(2S+1)$ dimensional Hilbert space $\mathcal{H}_0$. A commonly used basis for $\mathcal{H}_0$ is $\{|\psi^s\rangle\}$ with $s = -S, -S+1, \ldots, S-1, S$. The basis states are fully characterized by the properties

$$S^{(3)} |\psi^s\rangle = s\, |\psi^s\rangle\,, \tag{1.3}$$

$$S^{\pm} |\psi^s\rangle = \sqrt{S(S+1) - s(s \pm 1)}\, \left|\psi^{s\pm 1}\right\rangle\,, \tag{1.4}$$

where $S^{\pm} := S^{(1)} \pm iS^{(2)}$, and the normalization condition $\langle \psi^s \mid \psi^s \rangle = 1$. The interpretation is that $\left|\psi^S\right\rangle$ is the state in which the spin is pointing in the positive 3-direction.

For $S = 1/2$ we often write $\left|\psi^{1/2}\right\rangle$ and $\left|\psi^{-1/2}\right\rangle$ as $\left|\psi^{\uparrow}\right\rangle$ and $\left|\psi^{\downarrow}\right\rangle$, (or sometimes as $|\uparrow\rangle$ and $|\downarrow\rangle$), respectively. Then (1.3) and (1.4) read

$$S^{(3)} \left|\psi^{\uparrow}\right\rangle = \frac{1}{2} \left|\psi^{\uparrow}\right\rangle, \quad S^{(3)} \left|\psi^{\downarrow}\right\rangle = -\frac{1}{2} \left|\psi^{\downarrow}\right\rangle$$
$$S^+ \left|\psi^{\uparrow}\right\rangle = 0, \quad S^- \left|\psi^{\uparrow}\right\rangle = \left|\psi^{\downarrow}\right\rangle, \quad \hat{S}^+ \left|\psi^{\downarrow}\right\rangle = \left|\psi^{\uparrow}\right\rangle, \quad S^- \left|\psi^{\downarrow}\right\rangle = 0.$$

It is common to identify the basis state $|\psi^s\rangle$ with a column vector

$$|\psi^s\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

whose $(S - s + 1) - th$ component is 1 and the rest are 0. Then the operators $S^{(1)}$, $S^{(2)}$, and $S^{(3)}$ are identified with $(2S + 1) \times (2S + 1)$ Hermitian matrices whose components can be read off from (1.3) and (1.4).

For example, when $S = 1/2$, we find that

$$S^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S^{(3)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is also common to use the Pauli matrices defined by $\sigma^{(\alpha)} = 2S^{(\alpha)}$ when dealing with $S = 1/2$ systems, i.e.,

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a vector $\boldsymbol{n} = (n_1, n_2, n_3)$ for which $n_1, n_2, n_3 \in \mathbb{R}$, $|\boldsymbol{n}| = 1$, the spin $\boldsymbol{\sigma}$ in the $\boldsymbol{n}$ direction corresponds to the Hermitian matrix

$$\boldsymbol{\sigma_n} = n_1\sigma^{(1)} + n_2\sigma^{(2)} + n_3\sigma^{(3)}.$$

For a $spin - 1/2$ particle, the observable $\frac{1}{2}\boldsymbol{\sigma_n}$ is the spin along the $\boldsymbol{n}$-axis. The spin vector $\boldsymbol{\sigma}$ can be interpreted as the generator of rotations in the sense that there is a unitary operator $U(\theta)$

$$U(\theta) = e^{-\frac{1}{2}i\theta\boldsymbol{\sigma_n}},$$

which represents a rotation around the $\boldsymbol{n}$-axis by an angle θ.

1.1.2 Infinite-range Quantum Spin Systems

Any two quantum systems described by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ can be considered as one composite system. The Hilbert space of the composite system is given by the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $e_1, ..., e_n$ and $f_1, ..., f_m$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\mathcal{H} = \mathcal{H}_1 \bigotimes \mathcal{H}_2$ is defined as the linear span of nm orthonormal vectors denoted $e_i \bigotimes f_j, 1 \leq i \leq n, 1 \leq j \leq m$. The tensor notation is extended by linearity to identify $\phi_1 \bigotimes \phi_2 \in \mathcal{H}_1 \bigotimes \mathcal{H}_2$, for any $\phi_1 \in H_1, \phi_2 \in H_2$.

The combination of spins, meaning, considering a physical context in which both exist, is described as a composite system using the tensor product of the Hilbert spaces of the individual systems.

Consider the lattice Λ to be an arbitrary finite set, in which the elements of Λ are denoted as $x, y, \ldots$, and called sites. Each lattice site represents a single spin. We also fix the spin quantum number S, and assume that each site (i.e., an atom) carries a spin quantum angular momentum with quantum number S.

The Hilbert space of the spin system on Λ is $\mathcal{H} := \bigotimes\limits_{x \in \Lambda} \mathcal{H}_x$, whose dimension is $(2S+1)^{|\Lambda|}$. A natural basis state is

$$\left|\Psi^N\right\rangle := \bigotimes_{x \in \Lambda} |\psi_x\rangle \tag{1.5}$$

with every $|\psi_x\rangle \in \mathcal{H}_x \coloneqq \mathbb{C}^{(2S+1)}$.

The Hilbert space of an N-particle spin-1/2 system is thus

$$\mathcal{H} = (\mathbb{C}^2)^{\bigotimes N}.$$

1.1.3 A Concise Definition of the Mathematical Model

We now state the mathematical description of the model analyzed in this thesis. Consider a quantum spin system with S= 1/2 on a one-dimensional lattice Λ_N of N interacting particles, with random infinite-range exchange interactions. The system is described by a wave function $\Psi_N \in \mathcal{H}^N$, Where

$$\mathcal{H}^N := L^2_+\left(\Omega^N,\ \mathrm{d}\boldsymbol{\sigma_1}\cdots\mathrm{d}\boldsymbol{\sigma_N}\right)$$

is a subspace of $L^2\left(\Omega^N,\ \mathrm{d}\boldsymbol{\sigma_1}\cdots\mathrm{d}\boldsymbol{\sigma_N}\right)$, the wave functions $\Psi_N\left(\boldsymbol{\sigma_1},\ldots,\boldsymbol{\sigma_N}\right)$ are symmetric under permutation of their arguments $\boldsymbol{\sigma_1},\ldots,\boldsymbol{\sigma_N} \in \Omega$.

The time evolution of Ψ_N is described by the N-particle Schrödinger equation

$$i\partial_t \Psi_N(t) = H_N \Psi_N(t) \quad \Psi_N(0) = \Psi_{N,0},$$

where the Hamiltonian has the form

$$H_N = a\sum_{i<j}^{N} J_{ij}\boldsymbol{\sigma_i}\boldsymbol{\sigma_j} + h\sum_{i=1}^{N}\boldsymbol{\sigma_i}.$$

where a is a scaling parameter, i and j label lattice sites of Λ_n and σ is the triplet of Pauli spin matrices $\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}$. Each interaction J_{ij} is supposed to be distributed independently according to a probability distribution $P(J_{ij})$. One often uses the Gaussian model as a typical example of the distribution of $P(J_{ij})$. Their explicit forms are

$$P\left(J_{ij}\right) = \frac{1}{\sqrt{2\pi J^2}}\exp\left\{-\frac{\left(J_{ij}-J_0\right)^2}{2J^2}\right\} \tag{1.6}$$

with mean J_0 and variance J^2, and each interaction J_{ij} is not stronger than J^* which $|J_{ij}| \leq J^*$ for all $i,j \in N$; h is the external potential, and it can in theory be removed. The external potential h does not depend on N, and H_N conserves symmetry, i.e. any symmetric function $\Psi_N(0)$ involves a symmetric function $\Psi_N(t)$.

Lemma 1. *Let $J^* = (lnN)^{\frac{2}{3}}J + J_0$, we have*

$$\lim_{N\to\infty} \mathbb{P}(J_{ij} > J^*) = 0.$$

Proof. Since the interactions J_{ij} is a Gaussian random variable with mean J_0 and variance J, the probability density function of J_{ij} is showed in 1.6, then $X = \frac{J_{ij}-J_0}{J}$ is stardand normal and and the probability of $J_{ij} > J^*$ will be

$$\mathbb{P}\left(J_{ij} > J^*\right) = \mathbb{P}\left(X > x\right) = \int_x^\infty f(u)du < \frac{f(x)}{x}$$

where $f(u)$ is the density function of the standard normal distribution, and $x = \frac{J^* - J_0}{J} = (lnN)^{\frac{2}{3}}$.

Thus we have

$$
\begin{aligned}
\mathbb{P}\left(J_{ij} > J^*\right) &< \frac{e^{-\frac{1}{2}(\ln N)^{\frac{4}{3}}}}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\ln N)^{\frac{2}{3}}} \\
&= \frac{\left(e^{-(\ln N)^{\frac{4}{3}}}\right)^{\frac{1}{2}}}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\ln N)^{\frac{2}{3}}} \\
&= \frac{\left(e^{-\ln N \cdot -(\ln N)^{\frac{1}{3}}}\right)^{\frac{1}{2}}}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\ln N)^{\frac{2}{3}}} \\
&= \frac{1}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\ln N)^{\frac{2}{3}} N^{\frac{1}{2}(lnN)^{\frac{1}{3}}}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{P}\left(J_{ij} > J^*\right) &< \lim_{N\to\infty} \frac{1}{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\ln N)^{\frac{2}{3}} N^{\frac{1}{2}(lnN)^{\frac{1}{3}}}} \\
&= 0.
\end{aligned}
$$

□

1.2 Structure of the Thesis

This thesis deals with the derivation of mean-field limits for quantum spin-1/2 systems with infinite-range interactions. In the previous section, we introduced the system being considered in this thesis. In chapter 2 we outline the basics of mean-field theory, and discuss its application to infinite-range quantum spin systems. In chapter 3 we introduce a counting measure that can help us control the ratio of the particles which are not in the mean-field state at time t introduced by P.Pickl in [1] and translate it into quantum spin-1/2 systems, ultimately demonstrating that the ratio at time t is bounded. We conclude in chapter 4 with a brief summary of the primary result, namely the validity of the mean-field limit of infinite-range quantum spin-1/2 systems.

Chapter 2

Mean-field Theory

Simple, noninteracting systems for which the total Hamiltonian decomposes as the sum of the single-particle Hamiltonians are easily considered. However, there are also lots of interacting systems (e.g. the Ising model), in which the parts of the Hamiltonian corresponding to interactions between particles are difficult to calculate. Thus it can be difficult to find explicit solutions for such many-body systems, and the wave function Ψ will be in general a highly entangled object, which makes it impossible to solve the Schrödinger equation with a large particle number N.

In this case, we will introduce a very useful method in solving many-body interacting systems in statistical physics by finding an approximation of the system, mean-field theory, which was originally introduced by Pierre Curie and Pierre Weiss to describe phase transitions of ferromagnetic properties[5, 6].

With the perspective of the Law of large numbers, the fundamental idea of the mean-field theory is to assume the particles of the system are independent, i.e. the fluctuations of every single pair of interactions will not affect the average behavior of the system. One can transform the many-body interacting problem into a one-body problem, by considering every single particle as experiencing a mean external field, arising from the average behavior of the particles with which it interacts.

Since the mean-field theory is an approximation, there might be problems with its validity. One has to determine how independence between the particles can arise and exist in such interacting systems. This certainly happens only under particular physical conditions and on specific lengths and time scales.

The interacting many-body model is commonly studied in a specific limit, where one or more physical parameters approach infinity (or zero), such as the particle number N or the average density $\rho = N/|V|$ ($|V|$ being the volume of the system), which are well-motivated examples of physical parameters that can be chosen to tend to this limit[7].

The validity of the mean-field theory is also affected by the characteristics of the system. In order to determine the precise valid conditions for the mean-field theory of various interacting systems, one can adjust the mass of the particles, the strength of the interaction, the range of the potential, the appropriate time scale, etc.

Let us emphasize that there are multiple experimental (physical) and mathematical approaches to determine whether mean-field limits in interacting systems are sensible and nontrivial under certain boundaries. The mathematical component of the program involves proofing that under the specified conditions (such as an appropriately scaled Hamiltonian and appropriate initial data), the solution of the many-body Schrödinger equation converges with respect to a meaningful notion of distance to the solution of the associated effective mean-field description.

2.1 Mean-field Theory in Infinite-range Quantum Spin Systems

The mean-field theory can be valid in a many-body interacting system with a large number of particles, in which we can consider the particles of the system as independent and moving through an effective external potential produced by the other particles. By understanding the law of large numbers, this can only be true only if the strength of the interaction is neither too strong nor too weak for the whole system.

Let us assume the weak coupling constant $a = \frac{1}{N}$, such that the interaction term in H corresponds to a scaled sum of N independent random variables. The coupling is chosen such that kinetic and interaction terms are of the same order, as can be argued for by rescaling space and time coordinates, at least for some interactions. Thus the Hamiltonian is defined by

$$H_N = \frac{1}{N}\sum_{i<j}^{N} J_{ij}\boldsymbol{\sigma_i}\boldsymbol{\sigma_j} + h\sum_{i=1}^{N}\boldsymbol{\sigma_i}. \tag{2.1}$$

For the sake of simplicity let us assume the particles in the spin system are initially identically and independently distributed (*i.i.d.*), with the initial N-particle wave function $\Psi_N \in \mathcal{H}^N$ as the exact tensor product of N wave functions of every individual particle

$$\Psi_N(t=0) = \bigotimes_{j=1}^{N}\varphi_j(t=0)$$

where $\varphi_j(t=0) \in L^2(\Omega)$ is the initial wave function of every single particle $\boldsymbol{\sigma}_j$. To prove the validity of the mean-field theory one has to prove that the many-body ground state wave function obeys the assumed product structure at time t, i.e.

$$\Psi_N(t) \approx \bigotimes_{j=1}^{N}\varphi_j(t)$$

for N large. $\varphi_j(t)$ is the solution of the Schrödinger equation under the mean-field Hamiltonian $h^{mf,j}$

$$i\partial_t\varphi_j(t) = h^{mf,j}\varphi_j(t) \tag{2.2}$$

for every single particle $\boldsymbol{\sigma}_j$ at time t with φ_j^0 is the initial wave function as above.

The basic idea of $\Psi_N(t) \approx \bigotimes_{j=1}^{N}\varphi_j(t)$ is: For N large, “most” particles are in the state $\varphi_j(t)$ at time t and the identical and independent character is maintained, while only “few” particles are not and behave in an entangled manner, meaning that the mean-field theory is valid in this system at time t. We will introduce a method to help us measure the ratio of these particles in our system.

Chapter 3

Mean-field Limits of General Quantum Spin-1/2 Systems

3.1 Counting Measure

The goal of this thesis is to develop a rigorous proof of the validity of the mean-field limit in infinite-range quantum spin-1/2 systems. We will use a simple and effective method presented by P.Pickl in[1] for deriving mean-field descriptions in general quantum spin systems.

The strategy of proving the mean-field limit is valid is to control the number of particles not in the product state (as explained in the previous Chapter)

$$\Psi_N(t) = \bigotimes_{j=1}^{N} \varphi_j(t)$$

at time t, in the condensate. To measure the relative number of particles that are not in the state $\varphi_j(t)$, we first give the following definition:

Definition 1. *For any $\varphi_j(t) \in L^2(\Omega)$ with $\|\varphi_j(t)\|_{L^2(\Omega)} = 1$ where $\varphi_j(t)$ is the solution of Schrödinger equation at time of particle $\boldsymbol{\sigma_j}$ at time t, and $1 \leq j \leq N$, we define the time-depended projectors $p_j(t)$: $L^2(\Omega^N) \to L^2(\Omega^N)$ and its orthogonal* *complement* $q_j(t)$*:* $L^2(\Omega^N) \to L^2(\Omega^N)$ *as*

$$p_i(t) := \varphi_j(t) \left\langle \varphi_j(t), \cdot \right\rangle_{L^2(\Omega, \mathrm{d}\boldsymbol{\sigma}_j)} \quad q_j(t) := \mathbf{1} - p_i(t).$$

We shall also use the bra-ket notation $p_j(t) = |\varphi_j(t)\rangle \langle \varphi_j(t)|$*.*

Furthermore we define on $L(\Omega^N)$ *for any* $0 \le k \le N$ *the projector*

$$\begin{aligned} P_{k,N} &\coloneqq (q_1(t) \dots q_k(t) p_{k+1}(t) \dots p_N(t))_{sym} \\ &= \sum_{a_i \in \{0,1\}; \sum_{i=1}^N a_i = k} \prod_{i=1}^N (q_i(t))^{a_i} (p_i(t))^{1-a_i}. \end{aligned}$$

Remark. *From the definition above one can conclude [8]* $P_{k,N}$ *is an orthogonal projector and*

$$\sum_{k=0}^N P_{k,N} = 1. \tag{3.1}$$

Proposition 1. *An orthogonal projection on a Hilbert space* $\mathcal{H}$ *is a linear map* $P : \mathcal{H} \to \mathcal{H}$ *that satisfies*

$$P^2 = P, \quad \langle Px, y \rangle = \langle x, Py \rangle \quad \text{for all } x, y \in \mathcal{H}.$$

An orthogonal projection is necessarily bounded.

Definition 2. *Let* $\Psi_N(t) \in \mathcal{H}^N$ *be symmetric with* $\|\Psi_N(t)\|_{L^2(\Omega^N)} = 1$*,* $\langle \cdot, \cdot \rangle$ *be the scalar product on* $L^2(\Omega^N)$*. We define for any* $N \in \mathbb{N}$ *the functional* $\alpha_N : L^2(\Omega^N) \times L^2(\Omega^N) \to \mathbb{R}_0$ *as*

$$\alpha_N(t) \coloneqq \left\langle \Psi_N(t), \sum_{k=0}^N \frac{k}{N} P_{k,N}(t) \Psi_N(t) \right\rangle_{L^2(\Omega^N)}. \tag{3.2}$$

Remark. *Let us explain a bit more about the physical interpretation of* $\alpha_N(t)$*: Assume the wave function for a* N *particle spin system*

$$\Psi_N(t) = \left(\bigotimes_{j=1}^k \varphi_j^\perp(t) \bigotimes_{j=k+1}^N \varphi_j(t) \right)_{sym}$$

for some $\varphi_j^\perp \perp \varphi_j$*, with* $\varphi_j(t)$ *means the* j*-th particle* σ_j *is in the "good" state (is the solution of the Schrödinger equation under the mean-field Hamiltonian) at time*

t and $\varphi_j^{\perp}(t)$ stands the j-th particle is in "bad" state. $\|\Psi_N(t)\|_{L^2(\Omega^N)} = 1$ *and* $\Psi_N(t)$ *is symmetric, then by definition*

$$\alpha_N(t) = \left\langle \Psi_N(t), \sum_{k=0}^{N} \frac{k}{N} P_{k,N}(t)\Psi_N(t) \right\rangle_{L^2(\Omega^N)}$$
$$= \frac{k}{N}.$$

Hence the quantity $\alpha_N(t)$ is what we defined to count the average relative number of particles outside the condensate in $\Psi_N(t)$.

Remark. *One has to be cautious about the meaning of $\Psi_N(t) \approx \bigotimes_{j=1}^{N} \varphi_j(t)$. In particular, one should not expect $\Psi_N(t)$ converges to $\bigotimes_{j=1}^{N} \varphi_j(t)$ in L^2-sense as $N \to \infty$.*

The principle behind this counting measure is directly translating the mean-field description of a many-particle quantum system into a mathematical algorithm, eliminating the requirement for propagation estimates on $\Psi_N(t)$.

Theorem 2. *Assume that for any $N \in \mathbb{N}$ there exists a solution $\Psi_N(t) \in \mathcal{H}^N$ be the wave function of the N-particle spin system and $\varphi_j(t)$ be the solution of Schrödinger equation of every single particle $\boldsymbol{\sigma_j}$ at time t as we defined above. Then for $N \to \infty$ we have*

$$\alpha_N(t) \leq e^{8J^*t}\alpha_N(0),$$

where $J^* = (lnN)^{\frac{2}{3}}J + J_0$ is defined in Lemma 1.

3.1.1 Control of $\alpha_N(t)$

We defined $\alpha_N(t)$ to help us measure "the relative number of the particles which are in the right state". To prove the mean-field theory is valid in this case is to control the relative number $\alpha_N(t)$ small at time t in this thesis we will show that if the measure $\alpha_N(0)$ was initially small then $\alpha_N(t)$ keeps being small for all times. Controlling the growth of $\alpha_N(t)$ will most likely involve an application of Grönwall's lemma, which we will now state and prove[9]:

Lemma 3. *[Grönwall's inequality] Let f and c denote real-valued functions defined on [0,∞) and let f be differentiable on (0,∞). If f satisfies*

$$\frac{d}{dt}f(t) \leq c(t)f(t)$$

then

$$f(t) \leq e^{\int_0^t c(s)ds} f(0).$$

Proof of Lemma 3. Define a function g by

$$g(t) = e^{\int_0^t c(s)ds}$$

for $t \in [0, \infty)$. Then

$$\frac{d}{dt}g(t) = c(t)g(t).$$

It holds

$$\frac{d}{dt}\frac{f}{g} = \frac{f'g - fg'}{g^2} \leq \frac{cfg - cfg}{g^2} = 0$$

on (0,∞) and therefore

$$\frac{f(t)}{g(t)} \leq \frac{f(0)}{g(0)} = f(0)$$

since $g(0) = 1$ and $g(t) > 1$ for all $t \in (0, \infty)$. □

Lemma 4. *Let f denote a continuous real-valued function on $[0, \infty)$ and let f be differentiable on $(0, \infty)$. If f satisfies for positive constants c_1 and c_2*

$$\frac{d}{dt}f(t) \leq c_1 f(t) + c_2,$$

then

$$f(t) \leq e^{c_1 t} f(0) + \left(e^{c_1 t} - 1\right)\frac{c_2}{c_1}.$$

Proof of Lemma 4. Define

$$g(t) := f(t) + \frac{c_2}{c_1},$$

then

$$\frac{d}{dt}g(t) = \frac{d}{dt}f(t) \leq c_1 f(t) + c_2 = c_1 g(t)$$

and thus by Grönwall's lemma

$$g(t) \leq e^{c_1 t} g(0),$$

which implies

$$f(t) = g(t) - \frac{c_2}{c_1} \leq e^{c_1 t} f(0) + \left(e^{c_1 t} - 1\right) \frac{c_2}{c_1}.$$

□

Lemma 3 and Lemma 4 are sufficient to show that

$$\partial_t \alpha_N(t) \leq C \alpha_N(t) + \mathcal{O}(1).$$

for a constant C, in which the subsequent growth of $\alpha_N(t)$ is bounded by a multiple of its initial value so that the mean-field description above is valid.

3.2 Construction of the Counting Measure

We now apply the counting measure in the case of the general infinite-range 1/2-spin system. An infinite-range spin system is a model where every spin interacts with every other spin, regardless of their relative locations. Since an external force has the same effect on all particles, independently of their distribution, this should not affect the derivation of the equation. We will in the following only consider interactions between exact spins and mean-field dynamics, and this external potential will not appear anymore.

Let us drop the dependencies on t and N in the following for better representation whenever this does not lead to confusion. For the scalar product in $L^2\left(\Omega^N\right)$ we define the shorthand

$$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L^2(\Omega^N)}$$

and for the L^2-norm on Ω^N we use

$$\| \cdot \| := \| \cdot \|_{L^2(\Omega^N)}.$$

We now consider the Hamiltonian of the system as

$$H_N = \frac{1}{N}\sum_{i=1}^{N}\sum_{i<j}^{N-1} J_{ij}\boldsymbol{\sigma_i}\boldsymbol{\sigma_j}. \tag{3.3}$$

Lemma 5. *Let the measure α and p_j, q_j be defined as above, Let Ψ be the solution of the N-particle infinite-range spin system at time t with $\Psi \in \mathcal{H}^N$, $\|\Psi\|_{L^2(\Omega^N)} = 1$ and Ψ is symmetric. We have*

$$\partial_t \alpha := -i\left\langle \Psi, \left[H - h^{mf,1}, q_j\right]\Psi\right\rangle$$

where $H = \frac{1}{N}\sum_{i=1}^{N}\sum_{i<j}^{N-1} J_{ij}\boldsymbol{\sigma_i}\boldsymbol{\sigma_j}$ is the Hamiltonian of the system, and $h^{mf,1}$ is the mean-field Hamiltonian which acts on particle $\boldsymbol{\sigma_1}$.

Proof of Lemma 5. Recalling the definition of α and applying (3.1) we get

$$\begin{aligned}\alpha &:= \left\langle \Psi, \sum_{k=0}^{N}\frac{k}{N}P_{k,N}\Psi\right\rangle \\ &= \left\langle \Psi, \frac{1}{N}\sum_{i=1}^{N} q_i \sum_{k=0}^{N}\frac{k}{N}P_{k,N}\Psi\right\rangle \\ &= \langle \Psi, q_1\Psi\rangle \end{aligned}\tag{3.4}$$

We will use the form $\alpha = \langle \Psi, q_1\Psi\rangle$ in the following calculation as it will be much simpler, we have

$$\alpha : \mathbb{R} \to [0,1],$$
$$t \mapsto \langle \Psi, q_1\Psi\rangle.$$

The image of α is $[0,1]$ since $\|\Psi\| = 1$ and q_j is a orthonormal projection. The functional α is an element of $C^1(\mathbb{R})$.

Recall φ_j is the solution of the Schrödinger equation of the j-th particle at time t

$$\partial_t \varphi_j = -ih^{mf,j}\varphi_j.$$

For the next calculation, we note

$$\begin{aligned}\partial_t\left(\varphi_j\langle\varphi_j,\cdot\rangle\right) &= \left(\partial_t\varphi_j\right)\langle\varphi_j,\cdot\rangle + \varphi_j\left\langle\partial_t\varphi_j,\cdot\right\rangle \\ &= -ih^{mf,j}\varphi_j\langle\varphi_j,\cdot\rangle + i\varphi_j\langle\varphi_j, h^{mf,j}\rangle.\end{aligned}$$

This equation can be written in a more compact form for the operator q_j

$$\partial_t q_j = \left[h^{mf,j}, q_j\right]. \tag{3.5}$$

With the above remarks, we can calculate

$$\begin{aligned}
\partial_t \alpha &= \partial_t \langle \Psi, q_1 \Psi \rangle \\
&= \langle \partial_t \Psi, q_1 \Psi \rangle + \langle \Psi, q_1 \partial_t \Psi \rangle + \langle \Psi, (\partial_t q_1) \Psi \rangle \\
&= i \langle \Psi, H q_1 \Psi \rangle - i \langle \Psi, q_1 H \Psi \rangle - i \left\langle \Psi, [h^{mf,1}, q_1] \Psi \right\rangle \\
&= i \langle \Psi, [H, q_1] \Psi \rangle - i \left\langle \Psi, [h^{mf,1}, q_1] \Psi \right\rangle \\
&= i \left\langle \Psi, [H - h^{mf,1}, q_1] \Psi \right\rangle
\end{aligned}$$

where we used equation (3.5). □

Lemma 6. *Control of the derivative of* α

$$\partial_t \alpha \leq I + II + III + IV,$$

where

$$\begin{aligned}
I &:= \left| 2 \left\langle \Psi, p_1 \left(\frac{1}{N} \sum_{j=1}^{N-1} p_{1+j} J_{1j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} p_{1+j} - h^{mf,1} \right) q_1 \Psi \right\rangle \right| \\
II &:= \left| \frac{2}{N} \left\langle \Psi, p_1 \sum_{j=1}^{N-1} p_{1+j} J_{1j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} q_{1+j} q_1 \Psi \right\rangle \right| \\
III &:= \left| \frac{2}{N} \left\langle \Psi, p_1 \sum_{j=1}^{N-1} q_{1+j} J_{1j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} p_{1+j} q_1 \Psi \right\rangle \right| \\
IV &:= \left| \frac{2}{N} \left\langle \Psi, p_1 \sum_{j=1}^{N-1} q_{1+j} J_{1j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} q_{1+j} q_1 \Psi \right\rangle \right|
\end{aligned}$$

with $h^{mf,1} := \frac{1}{N} \sum_{j=1}^{N-1} J_{1j} \langle \varphi, \boldsymbol{\sigma} \varphi \rangle \boldsymbol{\sigma_1}$.

Proof of Lemma 6. From Lemma 5 we have

$$\partial_t \alpha = i \left\langle \Psi, [H - h^{mf,1}, q_1] \Psi \right\rangle.$$

We find

$$\langle\Psi,[H,q_1]\Psi\rangle = \left\langle \Psi,[\frac{1}{N}\sum_{i=1}^{N}\sum_{i<j}^{N-1} J_{ij}\boldsymbol{\sigma_i}\boldsymbol{\sigma_j},q_1]\Psi \right\rangle$$
$$= \left\langle \Psi,[\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}},q_1]\Psi \right\rangle$$

since all terms which do not depend on the first particle $\boldsymbol{\sigma_1}$ in H will yield zero when commuting with q_1. We now left with

$$\begin{aligned}\partial_t\alpha =& i\left\langle \Psi,[H-h^{mf,1},q_1]\Psi\right\rangle \\ =& i\left(\left\langle \Psi,(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})q_1\Psi\right\rangle\right. \\ &\left.-\left\langle \Psi,q_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})\Psi\right\rangle\right). \qquad (3.6)\end{aligned}$$

To decompose (3.6), let us insert $\mathbf{1}=p_1+q_1$,

$$\begin{aligned}\partial_t\alpha =& i\left(\left\langle \Psi,\underbrace{(p_1+q_1)}_{\mathbf{1}}(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})q_1\Psi\right\rangle\right. \\ &\left.-\left\langle \Psi,q_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})\underbrace{(p_1+q_1)}_{\mathbf{1}}\Psi\right\rangle\right) \\ =& i\left(\langle \Psi,p_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})q_1\Psi\rangle\right. \\ &+\left\langle \Psi,q_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})q_1\Psi\right\rangle \\ &-\left\langle \Psi,q_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})p_1\Psi\right\rangle \\ &\left.-\left\langle \Psi,q_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})q_1\Psi\right\rangle\right) \\ =& i\left(\left\langle \Psi,p_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})q_1\Psi\right\rangle\right. \\ &\left.-\left\langle \Psi,q_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}-h^{mf,1})p_1\Psi\right\rangle\right),\end{aligned}$$

since the projection operators p_1 and q_1 are self-adjoint, we have

$$\partial_t \alpha = 2\,\mathrm{Im}\left\langle \Psi, p_1(\frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} - h^{mf,1})q_1\Psi \right\rangle . \tag{3.7}$$

Last we insert $\mathbf{1} = p_{1+j} + q_{1+j}$ on both sides of $J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}$,

$$\begin{aligned}
\partial_t \alpha =& 2\,\mathrm{Im}\left\langle \Psi, p_1\left(\frac{1}{N}\sum_{j=1}^{N-1} \underbrace{(p_{1+j}+q_{1+j})}_{\mathbf{1}} J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} \underbrace{(p_{1+j}+q_{1+j})}_{\mathbf{1}} - h^{mf,1}\right) q_1\Psi \right\rangle \\
=& 2\left(\mathrm{Im}\left\langle \Psi, p_1(\frac{1}{N}\sum_{j=1}^{N-1} p_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}p_{1+j} - h^{mf,1})q_1\Psi \right\rangle\right. \\
&+ \mathrm{Im}\left\langle \Psi, p_1\frac{1}{N}\sum_{j=1}^{N-1} p_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}q_{1+j}q_1\Psi \right\rangle \\
&+ \mathrm{Im}\left\langle \Psi, p_1\frac{1}{N}\sum_{j=1}^{N-1} q_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}p_{1+j}q_1\Psi \right\rangle \\
&\left.+ \mathrm{Im}\left\langle \Psi, p_1\frac{1}{N}\sum_{j=1}^{N-1} q_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}q_{1+j}q_1\Psi \right\rangle\right) .
\end{aligned}$$

Taking the absolute value of the right side we have

$$\begin{aligned}
\partial_t \alpha \leq & \left| 2\left\langle \Psi, p_1(\frac{1}{N}\sum_{j=1}^{N-1} p_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}p_{1+j} - h^{mf,1})q_1\Psi \right\rangle\right| \\
&+ \left|\frac{2}{N}\left\langle \Psi, p_1\sum_{j=1}^{N-1} p_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}q_{1+j}q_1\Psi \right\rangle\right| \\
&+ \left|\frac{2}{N}\left\langle \Psi, p_1\sum_{j=1}^{N-1} q_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}p_{1+j}q_1\Psi \right\rangle\right| \\
&+ \left|\frac{2}{N}\left\langle \Psi, p_1\sum_{j=1}^{N-1} q_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}q_{1+j}q_1\Psi \right\rangle\right| .
\end{aligned} \tag{3.8}$$

□

Lemma 7.

$$(1) I \leq 2J^*\alpha,$$

$$(2) II \leq 2J^*\alpha,$$

$$(3) III \leq 2J^*\alpha,$$

$$(4) IV \leq 2J^*\alpha.$$

Proof of Lemma 7(1). Here we show the mean-field interaction cancels the full interaction. The proof of this inequality is based on a law of large numbers argument, as one wants to sum over the interactions of the infinite-range system from the statistical mean for large N.

Recalling the notation $p_{j+1} = |\varphi_{j+1}\rangle\langle\varphi_{j+1}|$ and we have the term

$$\begin{aligned} p_{1+j} J_{1j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} p_{1+j} = & \boldsymbol{\sigma_1} |\varphi_{1+j}\rangle\langle\varphi_{1+j}| \boldsymbol{\sigma_{1+j}} |\varphi_{1+j}\rangle\langle\varphi_{1+j}| \\ = & p_{1+j} \langle\varphi_{1+j}| \boldsymbol{\sigma_{1+j}} |\varphi_{1+j}\rangle \boldsymbol{\sigma_1} \end{aligned}$$

Recalling Ψ is symmetric and with $h^{mf,1} := \frac{1}{N}\sum_{j=1}^{N-1} J_{1j}\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1}$, by inserting $\mathbf{1} = p_{1+j} + q_{1+j}$ we have

$$\begin{aligned} I = & \left| 2\left\langle \Psi, p_1\left(\frac{1}{N}\sum_{j=1}^{N-1} p_{1+j} J_{1j} \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j} - h^{mf,1}\right) q_1\Psi \right\rangle \right| \\ = & \left| \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} \left\langle \Psi, p_1 \left(p_{1+j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j} - \underbrace{(p_{1+j} + q_{1+j})}_{\mathbf{1}}\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1} \right) q_1\Psi \right\rangle \right| \\ = & \left| \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} \left\langle \Psi, p_1 \left(p_{1+j}\langle\varphi_{1+j}|\boldsymbol{\sigma_{1+j}}|\varphi_{1+j}\rangle\boldsymbol{\sigma_1} - (p_{1+j}+q_{1+j})\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1}\right) q_1\Psi \right\rangle \right| \\ = & \left| \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} \langle \Psi, p_1 q_{1+j}\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1} q_1\Psi\rangle \right| \\ = & \left| \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} \langle q_{1+j}\Psi, p_1 q_{1+j}\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1} q_1\Psi\rangle \right|. \end{aligned}$$

Using triangle inequality and Cauchy-Schwarz inequality I is bounded by

$$\frac{2}{N}\sum_{j=1}^{N-1} |J_{1j}| \|q_{1+j}\Psi\| \|q_1\Psi\| \|p_1\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1} q_1\|_{op} \tag{3.9}$$

where the operator norm $\|\cdot\|_{op}$ is defined in (3.11) below. Recalling that J_{ij}s are bounded by J^*, also by applying Law of large numbers we obtain

$$\frac{2}{N}\sum_{j=1}^{N-1}|J_{1j}| \leq 2\frac{N-1}{N}J^* \overset{N\to\infty}{\to} 2J^*.$$

By the property of projection operators and recalling Ψ is symmetric, we have

$$\|q_{1+j}\Psi\|\|q_1\Psi\| = \|q_1\Psi\|^2 = \langle q_1\Psi, q_1\Psi\rangle = \langle\Psi, q_1\Psi\rangle = \alpha. \tag{3.10}$$

For all $\psi \in L^2(\mathbb{C}^2)$, the operator norm

$$\|p_1\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1}q_1\|_{op} = \underset{\|\psi\|=1}{sup}\ \|p_1\langle\varphi, \boldsymbol{\sigma}\varphi\rangle\boldsymbol{\sigma_1}q_1\psi\|_{L^2(\mathbb{C}^2)} = 1. \tag{3.11}$$

Thus we conclude that

$$I \leq 2J^*\alpha. \tag{3.12}$$

$\square$

Proof of Lemma 7*(2).* We now estimate II. In this term it is a bit complicated because we don't have enough q_js to get an α.

$$\begin{aligned} II =& \left|\frac{2}{N}\left\langle\Psi, p_1\sum_{j=1}^{N-1}p_{1+j}J_{1j}\boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}}q_{1+j}q_1\Psi\right\rangle\right| \\ =& \left|\left\langle\frac{2}{N}\sum_{j=1}^{N-1}J_{1j}q_{1+j}\boldsymbol{\sigma_{1+j}}\boldsymbol{\sigma_1}p_{1+j}p_1\Psi, q_1\Psi\right\rangle\right| \\ \leq& \|q_1\Psi\|\ \underbrace{\|\frac{2}{N}\sum_{j=1}^{N-1}J_{1j}q_{1+j}\boldsymbol{\sigma_{1+j}}\boldsymbol{\sigma_1}p_{1+j}p_1\Psi\|}_{:=A}, \end{aligned} \tag{3.13}$$

which A^2 can be considered as

$$\begin{aligned} A^2 =& \|\frac{2}{N}\sum_{j=1}^{N-1}J_{1j}q_{1+j}\boldsymbol{\sigma_{1+j}}\boldsymbol{\sigma_1}p_{1+j}p_1\Psi\|^2 \\ =& \left\langle\frac{2}{N}\sum_{j=1}^{N-1}J_{1j}q_{1+j}\boldsymbol{\sigma_{1+j}}\boldsymbol{\sigma_1}p_{1+j}p_1\Psi, \frac{2}{N}\sum_{l=1}^{N-1}J_{1l}q_{1+l}\boldsymbol{\sigma_{1+l}}\boldsymbol{\sigma_1}p_{1+l}p_1\Psi\right\rangle, \end{aligned}$$

we can now consider the inner product into two parts: when the sum indexes are the same and when they are not, thus we have

$$
\begin{aligned}
A^2 =& \left\langle \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi, \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi \right\rangle \\
&+ \left\langle \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi, \frac{2}{N}\sum_{l\neq j}^{N-1} J_{1l} q_{1+l} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1 \Psi \right\rangle \\
=& \frac{4}{N^2}\sum_{j=1}^{N-1} (J_{1j})^2 \left\langle q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi, q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi \right\rangle \\
&+ \frac{2}{N}\sum_{j=1}^{N-1} J_{1j} \frac{2}{N}\sum_{l\neq j}^{N-1} J_{1l} \left\langle \Psi, p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} q_{1+j} q_{1+l} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1 \Psi \right\rangle \\
\leq& \frac{4(N-1)}{N^2}(J^*)^2 \underbrace{\left\langle \Psi, p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi \right\rangle}_{\leq 1} \\
&+ \frac{4(N-1)(N-2)}{N^2}(J^*)^2 \left\langle q_{1+l}\Psi, p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1 q_{1+j} \Psi \right\rangle \\
\overset{N\to\infty}{=}& 4(J^*)^2 \left\langle q_{1+l}\Psi, p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1 q_{1+j} \Psi \right\rangle .
\end{aligned}
$$

Applying Cauchy-Schwarz inequality and the symmetry of Ψ

$$
\begin{aligned}
&\|\frac{2}{N}\sum_{j=1}^{N-1} J_{1j} q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi\|^2 \\
\leq& 4(J^*)^2 \|q_{1+l}\Psi\| \|p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1\|_{op} \|q_{1+j}\Psi\| \\
=& 4(J^*)^2 \|p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1\|_{op} \alpha,
\end{aligned}
$$

in which the operator norm

$$
\|p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1\|_{op} = \underset{\|\psi\|=1}{sup} \|p_1 p_{1+j} \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_{1+l}} \boldsymbol{\sigma_1} p_{1+l} p_1 \psi\|_{L^2(\mathbb{C}^2)} = 1
$$

for all $\psi \in L^2(\mathbb{C}^2)$, thus we have

$$
\|\frac{2}{N}\sum_{j=1}^{N-1} J_{1j} q_{1+j} \boldsymbol{\sigma_{1+j}} \boldsymbol{\sigma_1} p_{1+j} p_1 \Psi\|^2 \leq 4(J^*)^2 \alpha \tag{3.14}
$$

Putting (3.13), (3.14) together yields

$$
II \leq \sqrt{\alpha}\sqrt{4(J^*)^2\alpha}
$$

$$= 2J^*\alpha. \tag{3.15}$$

□

Proof of Lemma 7*(3).* We next estimate III. In this term, we have enough q_js to get an α.

Using (3.10) and the fact that p_i and q_i are self-adjoint operators we have

$$\begin{aligned} III =& \left| \frac{2}{N} \sum_{j=1}^{N-1} J_{1j} \left\langle q_{1+j}\Psi, p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j} q_1 \Psi \right\rangle \right| \\ \leq& \frac{2}{N} \sum_{j=1}^{N-1} |J_{1j}| \left| \left\langle q_{1+j}\Psi, p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j} q_1 \Psi \right\rangle \right| \\ \leq& \frac{2}{N} \sum_{j=1}^{N-1} |J_{1j}| \, \|q_{1+j}\Psi\| \, \|p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j}\|_{op} \|q_1\Psi\| \\ \leq& 2\frac{N-1}{N} J^* \|q_1\Psi\|^2 \|p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j}\|_{op} \\ \overset{N\to\infty}{=}& 2J^*\alpha \end{aligned} \tag{3.16}$$

since the operator norm

$$\|p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j}\|_{op} = \underset{\|\psi\|=1}{sup} \|p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} p_{1+j}\psi\|_{L^2(\mathbb{C}^2)} = 1$$

for all $\psi \in L^2(\mathbb{C}^2)$. □

Proof of Lemma 7*(4).* Using Cauchy-Schwarz's inequality and (3.10) IV is bounded by

$$\begin{aligned} IV =& \left| \frac{2}{N} \sum_{j=1}^{N-1} J_{1j} \left\langle q_{1+j}\Psi, p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} q_{1+j} q_1 \Psi \right\rangle \right| \\ \leq& \frac{2}{N} \sum_{j=1}^{N-1} |J_{1j}| \left| \left\langle q_{1+j}\Psi, p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} q_{1+j} q_1 \Psi \right\rangle \right| \\ \leq& \frac{2}{N} \sum_{j=1}^{N-1} |J_{1j}| \, \|q_{1+j}\Psi\| \|p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} q^t_{1+j}\|_{op} \|q_1\Psi\| \\ \leq& 2\frac{N-1}{N} J^* \|q_1\Psi\|^2 \|p_1 \boldsymbol{\sigma_1}\boldsymbol{\sigma_{1+j}} q_{1+j}\|_{op} \\ \overset{N\to\infty}{=}& 2J^*\alpha \end{aligned} \tag{3.17}$$

with

$$\|p_1 \boldsymbol{\sigma_1} \boldsymbol{\sigma_{1+j}} q_{1+j}\|_{op} = 1.$$

$\square$

By summarizing Lemma 5, Lemma 6, and Lemma 7 together we can conclude that Theorem 2 is proved. We can therefore bound α under time evolution using Lemma 4:

Writing

$$\partial_t \alpha(t) \leq 8J^* \alpha(t),$$

we have

$$\begin{aligned}
\alpha(t) &\leq e^{8J^*t} \alpha(0) \\
&= e^{8[(lnN)^{\frac{2}{3}} J + J_0]t} \alpha(0) \\
&= e^{8J_0 t} \alpha(0) \left(e^{lnN (lnN)^{-\frac{1}{3}}} \right)^{8Jt} \\
&= e^{8J_0 t} \alpha(0) \left(N^{(lnN)^{-\frac{1}{3}}} \right)^{8Jt},
\end{aligned}$$

and thus with the initial value $\alpha(0) = 0$, we find

$$\begin{aligned}
\lim_{N \to \infty} \alpha(t) &\leq \lim_{N \to \infty} e^{8J_0 t} \alpha(0) \left(N^{(lnN)^{-\frac{1}{3}}} \right)^{8Jt} \\
&= \lim_{N \to \infty} e^{8J_0 t} \alpha(0) \\
&= 0.
\end{aligned}$$

which if we have our spin system in an independent and identical initial condition, the mean-field limit exists at time t.

Chapter 4

Conclusion

The purpose of this thesis is to test if we can determine the validity of the mean-field limit of the quantum 1/2-spin system by using the counting method introduced in [1], the introduction of this counting measure made it possible to derive mean-field equations and quantify macroscopic dynamics using microscopic interactions. This was achieved by counting the number of particles that are (not) in the spin state that they are expected to be by the mean-field theory, and verifying the convergence of the relative number where the system has a large particle number N. This counting measure has also proven successful for non-spin quantum mechanical systems and classical mechanical systems, e.g. for the time-dependent Hartree equation and the Vlasov equation[1, 10].

We succeeded in translating the method into the quantum spin system and were able to derive that the mean-field approximation is accurate by choosing a proper scaling parameter of the system and setting up a strong condition of the independent and identical initial state. This way we showed that the mean-field limit remains valid for a quantum many-body system with spin.

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