The Davis-Wielandt Shell

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Review - Numerical Range

$$W(A) = \{\langle Ax, x\rangle : \langle x, x\rangle = 1\}.$$

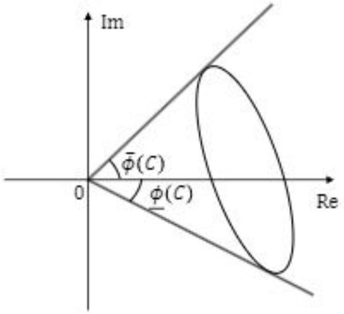


Figure 7.1: Geometric interpretations of $\overline{\phi}(C)$ and $\underline{\phi}(C)$.

Example 1 - Numerical Range

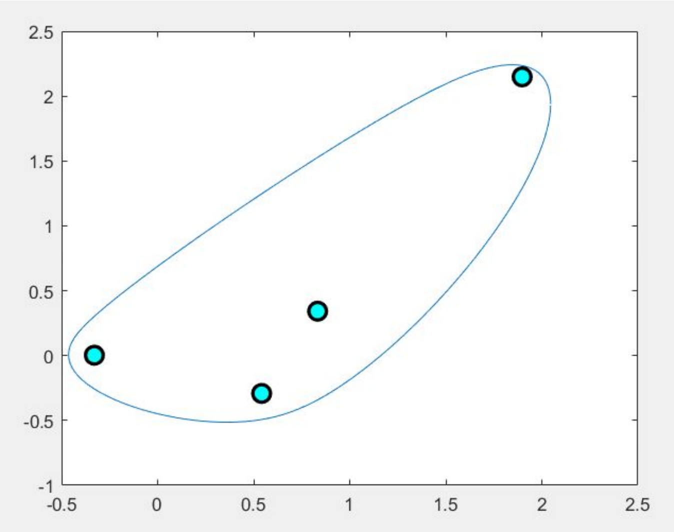
```
C =

   0.6557 + 0.6948i   0.6787 + 0.4387i   0.6555 + 0.1869i   0.2769 + 0.7094i
   0.0357 + 0.3171i   0.7577 + 0.3816i   0.1712 + 0.4898i   0.0462 + 0.7547i
   0.8491 + 0.9502i   0.7431 + 0.7655i   0.7060 + 0.4456i   0.0971 + 0.2760i
   0.9340 + 0.0344i   0.3922 + 0.7952i   0.0318 + 0.6463i   0.8235 + 0.6797i
```

```
>> eig(C)

ans =

   1.8976 + 2.1484i
  -0.3291 + 0.0027i
   0.8329 + 0.3423i
   0.5416 - 0.2917i
```



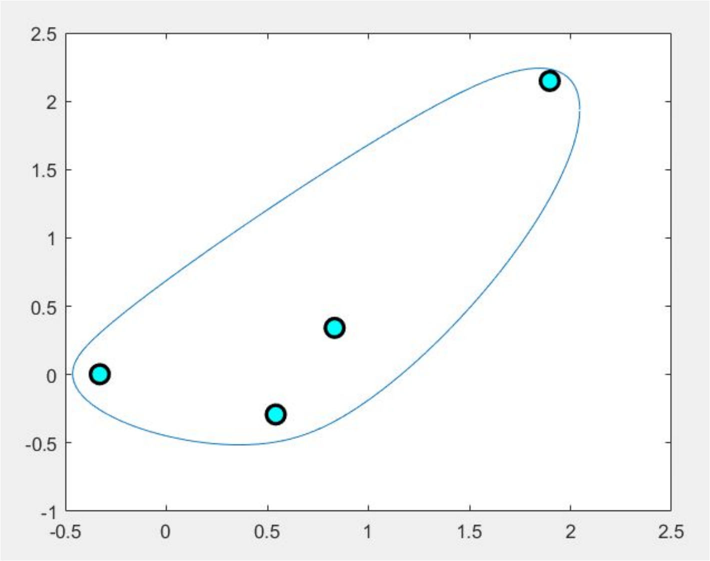
Davis Wielandt Shell

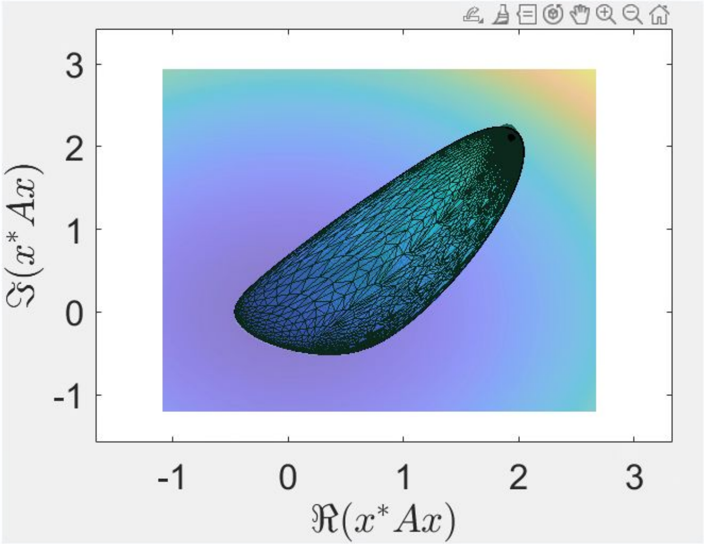
$$DW(A) = \{(\underline{\langle Ax, x\rangle}, \underline{\langle Ax, Ax\rangle}) : \langle x, x\rangle = 1\}.$$

First Term is the Numerical Range: <Ax, x> = x*Ax → Complex Number

Second Term is Gain: <Ax, Ax> = (Ax)*(Ax) = x*A*Ax = $||Ax||^2$ → Non-negative scalar

Example 1 - DW Shell





Proving Convexity of DW Shells

Prove that the DW shell of $n \times n$ matrix is convex with $n \geq 3$

AN EXTENSION OF THE HAUSDORFF-TOEPLITZ THEOREM ON THE NUMERICAL RANGE

YIK-HOI AU-YEUNG AND NAM-KIU TSING

ABSTRACT. Let $\mathcal{H}_n$ be the set of all $n \times n$ hermitian matrices and $\mathcal{U}_n$ the set of all $n \times n$ unitary matrices. For any $c = (c_1, \ldots, c_n) \in \mathbf{R}^n$ and A_1, A_2, $A_3 \in \mathcal{H}_n$, let $W\ (A\ \ , A\ ;, A\ ;)$ denote the set

$$\{(\mathrm{tr}[c]UA_1U^*, \mathrm{tr}[c]UA_2U^*, \mathrm{tr}[c]UA_3U^*) \colon U \in \mathcal{U}_n\},$$

where $[c]$ is the diagonal matrix with $c_1, \ldots, c_n$ as diagonal entries. In this present note, the authors prove that if $n > 2$, then $W_c(A_1, A_2, A_3)$ is always convex. Equivalent statements of this result, in terms of definiteness and inclusion relations, are also given. These results extend the theorems of Hausdorff-Toeplitz, Finsler and Westwick on numerical ranges, respectively.

b<<c, iff b = cQ | Q is a Pinching Matrix

$$\sum_i x_{ij} = \sum_j x_{ij} = 1,$$

An $n \times n$ doubly-stochastic matrix $Q = (q_{ij})$ is called a pinching matrix if for some $1 \leq k < l \leq n$ and $0 \leq \mu \leq 1$,

$$q_{ij} = \begin{cases} \mu, & \text{if } (i,j) = (k,k) \text{ or } (l,l), \\ 1-\mu, & \text{if } (i,j) = (k,l) \text{ or } (l,k), \\ 1, & \text{if } i = j \neq k, l, \\ 0, & \text{otherwise}. \end{cases} \longrightarrow \quad k = 1, l = 2 \quad \begin{cases} b_1 = \mu c_1 + (1-\mu) c_2 \\ b_2 = (1-\mu) c_1 + \mu c_2 \\ b_i = c_i (i \neq 1, 2) \end{cases}$$

The following lemma is well known (for example, see [**8**]).

LEMMA 2. *For any b, $c \in \mathbf{R}^n$, we have $b \ll c$ if and only if $b = cQ$ where Q is a finite product of pinching matrices.*

bases in $\mathbf{C}^n$. Then $W_c(A_1, A_2, A_3)$ is equal to the set

$$\left\{\left(\sum_{i=1}^n c_i x_i A_1 x_i^*, \sum_{i=1}^n c_i x_i A_2 x_i^*, \sum_{i=1}^n c_i x_i A_3 x_i^*\right) : (x_1, \ldots, x_n) \in \Lambda_n\right\}.$$

Let $(r_1, r_2, r_3) \in W_b(A_1, A_2, A_3)$. Then there is $(e_1, \ldots, e_n) \in \Lambda_n$ such that

$$\begin{aligned} r_j &= \sum_{i=1}^n b_i e_i A_j e_i^* \\ &= \tfrac{1}{2}(b_1 + b_2)(e_1 A_j e_1^* + e_2 A_j e_2^*) + \tfrac{1}{2}(b_1 - b_2)(e_1 A_j e_1^* - e_2 A_j e_2^*) + \sum_{i=3}^n b_i e_i A_j e_i^* \end{aligned}$$

for $j = 1, 2, 3$. For any θ, $\phi \in \mathbf{R}$ and $(x_1, \ldots, x_n) \in \Lambda_n$, define $y_1 = \cos\theta x_1 + \sin\theta e^{\phi\sqrt{-1}} x_2$, $y_2 = -\sin\theta x_1 + \cos\theta e^{\phi\sqrt{-1}} x_2$ and $y_i = x_i$ for all $i \geq 3$. Then $(y_1, \ldots, y_n) \in \Lambda_n$ and for $j = 1, 2, 3$,

$$\begin{aligned} \sum_{i=1}^n c_i y_i A_j y_i^* = \tfrac{1}{2}(c_1 + c_2)(x_1 A_j x_1^* + x_2 A_j x_2^*) \\ + \tfrac{1}{2}(c_1 - c_2)[p_j \cos 2\theta + \sin 2\theta(q_j \cos\phi + s_j \sin\phi)] \\ + \sum_{i=3}^n c_i x_i A_j x_i^* \end{aligned}$$

where

$$p_j = x_1 A_j x_1^* - x_2 A_j x_2^*, \quad q_j = 2\,\mathrm{Re}(x_1 A_j x_2^*), \quad s_j = 2\,\mathrm{Im}(x_1 A_j x_2^*)$$

Transformation

Apply the rotation matrix to the vector with the phase shift:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 e^{i\phi} \end{pmatrix} = \begin{pmatrix} \cos(\theta) x_1 + \sin(\theta) x_2 e^{i\phi} \\ -\sin(\theta) x_1 + \cos(\theta) x_2 e^{i\phi} \end{pmatrix}$$

Note: Since the Rotation Matrix is Unitary, and for the phase shift, $||e^{i\phi}||^2 = 1$. Hence, as $||x||^2 = 1$, the transformed vectors y_1 and y_2 are also unit vectors. This results in the transformed vectors:

$$y_1 = x_1 \cos(\theta) + x_2 \sin(\theta) e^{i\phi}$$

$$y_2 = -x_1 \sin(\theta) + x_2 \cos(\theta) e^{i\phi}$$

are real scalars. As θ and ϕ vary in $\mathbf{R}$, the locus of the point

$$\left(\sum_{i=1}^{n} c_i y_i A_1 y_i^*, \sum_{i=1}^{n} c_i y_i A_2 y_i^*, \sum_{i=1}^{n} c_i y_i A_3 y_i^*\right)$$

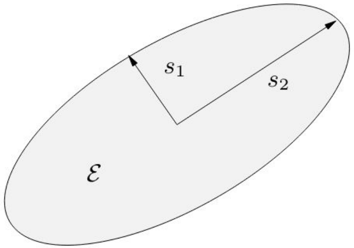
in $\mathbf{R}^3$ is an ellipsoid, which we shall denote by $E_{(x_1,\ldots,x_n)}$ and is degenerate convex) if $c_1 = c_2$ or

$$\det\begin{pmatrix} p_1 & q_1 & s_1 \\ p_2 & q_2 & s_2 \\ p_3 & q_3 & s_3 \end{pmatrix} = 0.$$

if $A = A^{\mathsf{T}} > 0$, the set

$$\mathcal{E} = \{ x \mid x^{\mathsf{T}} A x \leq 1 \}$$

is an *ellipsoid* in $\mathbb{R}^n$, centered at 0



It is clear that $E_{(x_1,\ldots,x_n)} \subset W_c(A_1, A_2, A_3)$ for any $(x_1, \ldots, x_n) \in \Lambda_n$. Since $c_1 + c_2 = b_1 + b_2$, $|c_1 - c_2| \geq |b_1 - b_2|$ and $c_i = b_i$ for all $i \geq 3$, $(r_1, r_2, r_3) \in \operatorname{conv} E_{(e_1,\ldots,e_n)}$ where "conv" denotes the convex hull. If $|c_1 - c_2| = |b_1 - b_2|$ or $E_{(e_1,\ldots,e_n)}$ degenerates, then $(r_1, r_2, r_3) \in E_{(e_1,\ldots,e_n)} \subset W_c(A_1, A_2, A_3)$. So we may assume that $E_{(e_1,\ldots,e_n)}$ does not degenerate and (r_1, r_2, r_3) lies in the interior of the ellipsoid $E_{(e_1,\ldots,e_n)}$. Let $w_1 \in \mathbf{C}^n$ be a unit eigenvector of A_1. Since $n > 2$, we can choose a unit vector $w_2 \in \mathbf{C}^n$ which is orthogonal to both w_1 and $w_1 A_2$. Consider a continuous function $f\colon [0,1] \to \Lambda_n$ such that $f(0) = (e_1, \ldots, e_n)$ and $f(1) = (w_1, \ldots, w_n)$, where $(w_1, \ldots, w_n)$ is any extension of (w_1, w_2) to an orthonormal basis of $\mathbf{C}^n$. Now (r_1, r_2, r_3) lies within the interior of $E_{f(0)}$, and $E_{f(1)}$ degenerates (hence convex) as $w_1 A_j w_2^* = 0$ for $j = 1, 2$. If $(r_1, r_2, r_3) \in E$, then (r_1, r_2, r_3) is automatically in $W_c(A_1, A_2, A_3)$. If $(r_1, r_2, r_3) \notin E_{f(1)}$, then, by continuity, there exists $\alpha \in (0,1)$ such that $(r_1, r_2, r_3) \in E_{f(\alpha)}$. Thus $(r_1, r_2, r_3) \in W_c(A_1, A_2, A_3)$. $\square$