Contents

Harmonic Oscillator.

Oscillatory Motion is a very important chapter for engineers. One should be thorough with the basic equations of SHM. Though this chapter contains Forced oscillations & damped oscillations for your syllabus but I have included the basic SHM in case you have forgotten the basic equations. Generally questions are asked from damped oscillations & resonance in Forced oscillations. Be clear about 3 different $\omega \rightarrow \omega_0, \omega, \& \omega_d$. And relations between them. Your basic understanding of Simple Harmonic motion is sometimes very important in solving the problems of damped oscillations.

I have consulted following books to make this note.

(1) Kleppner – I differ from Kleppner in Resonance part in Forced oscillations with damped oscillator.

(2) Engineering Mechanics – Manoj Harbola (IITK)

(3) Intermediate Dynamics – Patrick Hamill.

(4) Mechanics, Part-2, DC Pandey (For SHM only)

(5) Internet Resources.

If you find any error somewhere please point it out to me. So that I send you a corrected Version.

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17 Nov, 2016.

The Harmonic Oscillator.

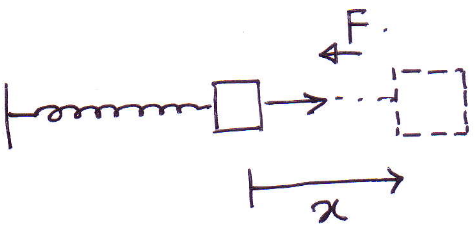
This chapter is divided into 3 parts

(1) Simple Harmonic Oscillator / SHM.
(2) Damped Harmonic Motion
(3) Forced Oscillations & Resonance.

Out of these three sections you have already studied in quite a good detail about the first section that is Simple Harmonic Motion. Here I will give a brief review of those formulas so that in later years if you ~~have~~ forget ~~forgotten~~ the formulas then you can see this section.

Simple Harmonic Motion.

Here we study about simple periodic motion where a body oscillates about a mean position. One simple exam is of a block which is attached to a spring. When it is stretched & released it executes SHM about a mean position.



As the spring is stretched, a restoring force acts on the body proportional to its displacement from its mean position towards the origin. So the force equation is

$$F_x = -kx$$

x component of displacement of the body from mean position

x component of force
Restoring Force.

Spring constant

The differential Equation for SHM.

$$F_x = -kx$$

$$m\frac{d^2x}{dt^2} + kx = 0.$$

$$\frac{d^2x}{dt^2} + \left(\frac{k}{m}\right)x = 0.$$

$\frac{k}{m} \rightarrow \omega_0^2$

Take $\frac{k}{m} = \omega_0^2 \Rightarrow \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

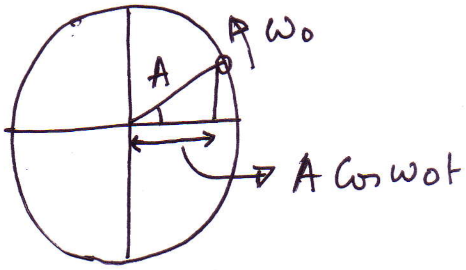
$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$$

Both $\sin x$ & $\cos x$ OR $\cos\omega_0 t$ and $\sin\omega_0 t$ are Solutions of the above equation.

$x = C\cos\omega_0 t + D\sin\omega_0 t.$

OR $x = A\sin(\omega_0 t + \phi)$ OR $\boxed{A\cos(\omega_0 t + \phi)}$ are Solutions

- Now what is Significance of ω_0? (V. Imp)



If a particle is rotating in a circle with angular velocity ω_0, the projection of this particle on the x axis is simple harmonic motion. $\Rightarrow \omega_0 = 2\pi n$

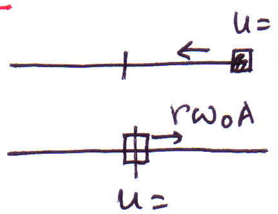
- What is initial phase ϕ?

It denotes the initial position of the particle wrt to mean position.— like for $t=0$, $A\cos\phi$ will give the x displacement.

- What is the difference between ($\Rightarrow$ Difference in Their position at $t=0$)

$x = A\cos(\omega t)$

$x = A\sin(\omega t)$



→ position of particle at $t=0$

position of particle at $t=0$.

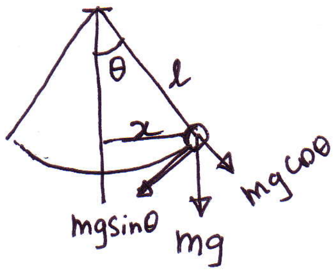
A table showing displacement, velocity, acceleration, KE, PE and Total Energy in terms of (as a function of) time and displacement.

	Time		Displacement
Disp	$x = A\cos(\omega t + \phi)$		
Vel	$v = -A\omega\sin(\omega t + \phi)$	$\rightarrow$	$v = \pm\omega\sqrt{(A^2 - x^2)}$
Acceleration	$a = -A\omega^2\cos(\omega t + \phi)$	$\rightarrow$	$a = -\omega^2 x$
KE	$KE = \frac{1}{2}mv^2$	$\rightarrow$	$KE = \frac{1}{2}m\omega^2(A^2 - x^2)$
PE	$PE = \frac{1}{2}kx^2 = \frac{1}{2}(m\omega_0^2)(A\cos(\omega t + \phi))^2$	$\rightarrow$	$PE = \frac{1}{2}m\omega^2 x^2$
Total E	$TE = \frac{1}{2}m\omega^2 A^2$	$\rightarrow$	$E = \frac{1}{2}m\omega^2 A^2$.

I have taken this idea above from the book of DC Pandey.

Some Common Examples of SHM.

(a) Simple Pendulum.



Linear Approach.

θ quite Small.

$F_R = -mg\sin\theta$

$a = -g\sin\theta = -g\theta$

Now $\theta = \frac{x}{l}$

$a = -g\frac{x}{l}$.

$a = -\omega^2 x$

$T = 2\pi\sqrt{\frac{l}{g}} \Leftarrow \omega = \frac{g}{l}$

Angular Approach.

$a_T = l\alpha$

$l\alpha = -g\theta$

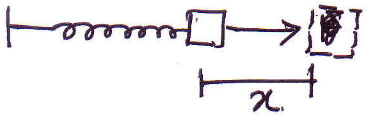
$\alpha = -\left(\frac{g}{l}\right)\theta$

$\ddot{\theta} + \left(\frac{g}{l}\right)\theta = 0$

$\omega_0^2 = \frac{g}{l}$.

$T = 2\pi\sqrt{\frac{l}{g}}$.

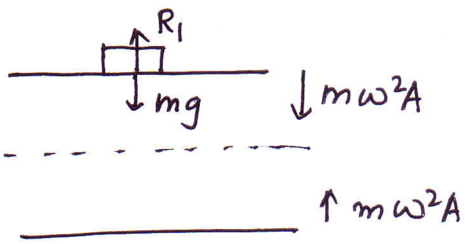
(b) Spring block System.



$$F = -kx$$
$$ma = -kx$$
$$a = -\frac{k}{m}x$$

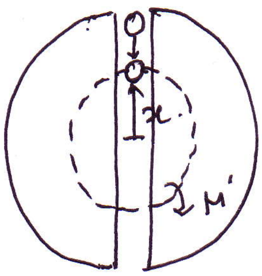
$$\omega_0^2 = \frac{k}{m}$$
$$T = 2\pi\sqrt{\frac{m}{k}}$$

A horizontal platform vibrats up and down with a Simple harmonic motion of amplitude 20 cm. At what frequency will an object kept on the platform just lose contact with the platform.



$$mg - R_1 = m\omega^2 A$$
$$R_1 = 0$$
$$mg = m\omega^2 A$$
$$\omega = \sqrt{\frac{g}{A}} \Rightarrow n = \frac{1}{2\pi}\sqrt{\frac{g}{A}}$$

If you drill a hole directly through the center of earth & drop a mass m — what will be the time period.



When the mass is at dist x from center.

$$F = \frac{GmM'}{x^2}$$
$$= \frac{GmMx^3}{R^3 \times x^2} = \frac{GmMx}{R^3}$$

$$M' = \frac{4}{3}\pi x^3 \times \rho.$$
$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$
$$M' = \cancel{\frac{4}{3}}\cancel{\pi}x^3 \times \frac{M}{\cancel{\frac{4}{3}}\cancel{\pi}R^3}$$
$$= \frac{Mx^3}{R^3}.$$

So $F = -\left(\frac{GmM}{R^3}\right)x$

Restoring force & disp in opp direction

$$\cancel{m}a = -\frac{G\cancel{m}M}{R^3}x = \cancel{-\omega}$$
$$a = -\omega_0^2 x \qquad \omega_0 = \frac{GM}{R^3}$$
$$T = 2\pi\sqrt{\frac{R^3}{GM}}$$

If a SHM is represented by eqn $x = 10\sin(\pi t + \pi/6)$ in SI units, then determine its amplitude, time period & maximum velocity.

$$x = A\sin(\omega t + \phi)$$

$A = 10m.$

$\omega = \pi$ rad/s & $\phi = \pi/6.$

$T = \frac{2\pi}{\omega} \Rightarrow T = 2s.$; $v_{max} = \omega A = 10\pi$ m/s.

A particle executes SHM with a time period of 4s. Find the time taken by the particle to go directly from its mean position to half its amplitude.

$$x = A\sin(\omega t)$$

$\frac{A}{2} = A\sin\omega t \Rightarrow \sin\omega t = \frac{1}{2} \Rightarrow \omega t = \sin^{-1}\left(\frac{1}{2}\right) = \pi/6.$

$t = \pi/6\omega. = \frac{\pi}{6} \times \frac{T}{2\pi} = \frac{T}{12} = \frac{4}{12} = \frac{1}{3}$ sec

The equation of an oscillating particle is given by $x = 2\sin\left(\frac{\pi}{2}t + \pi/4\right)$ cm. Find.

(i) period of oscillation. (ii) Maximum velocity
(iii) max acceleration (iv) Initial displacement.

Compare it $x = A\sin(\omega t + \delta)$

$A = 2$ cm

$\omega = \frac{\pi}{2}$ rad/s.

(i) period of oscillation $\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/2} = 4s.$

(ii) Max Velocity $v_{max} = \omega A = \frac{\pi}{2} \times 2 = \pi$ cm/s

(iii) Max Acceleration $a_{max} = \omega^2 A = (\pi/2)^2 \times 2 = \frac{\pi^2}{2}$ cm/s^2

(iv) Initial disp $\Rightarrow$ $x_0 = 2\sin(\pi/4) = \sqrt{2}$ cm.

Some Simple Practice problems.

Equivalent Force Constants

(You have done these proofs before – just a quick revision)

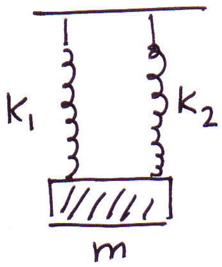
(1)

$$\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}, \quad T = 2\pi\sqrt{\frac{m}{K}}$$

[Proof: $x = x_1 + x_2 \Rightarrow \frac{F}{K} = \frac{F}{K_1} + \frac{F}{K_2}$

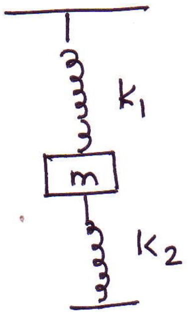
so $\frac{1}{K} = \frac{1}{K_1} + \frac{1}{K_2}$]

(2)



$$K = K_1 + K_2 \quad ; \quad T = 2\pi\sqrt{\frac{m}{K}}$$

(3)

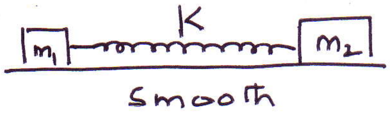


$$K = K_1 + K_2 \qquad T = 2\pi\sqrt{\frac{m}{K}}$$

(4) If the spring has mass m_s and a mass m is suspended from it, then time period is given by

$$T = 2\pi\sqrt{\frac{m + m_s/3}{K}}$$

(5)



If the spring connected with two mass m_1 & m_2 is compressed or elongated by x_0 & then left for oscillations then both blocks execute SHM with same time period but diff amplitudes.

$$T = 2\pi\sqrt{\frac{\mu}{k}}$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Rightarrow \text{reduced mass.}$$

Damped Harmonic Oscillator.

Here we consider an oscillator which is under the influence of a retarding force (like air friction etc) proportional to the velocity.

The equation of motion for an undamped simple Harmonic Oscillator is

$$F = -kx \quad \Rightarrow \quad m \cdot \frac{d^2x}{dt^2} = -kx.$$

Here we add one more term — That is damping force proportional to velocity $= -bv$.
So the differential equation become.

$$m \cdot \frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$ ← damping term.

OR $$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = 0$$

Rewriting $$\boxed{\ddot{x} + \underbrace{\frac{b}{m}}_{\gamma}\dot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2}x = 0}$$ ← V. Imp.

Damping Term. Oscillatory Term.

These two terms that is $\gamma = b/m$ & $\omega_0^2 = k/m$ are very very important to remember.

How to Solve this Differential Equation.

(Detailed Solution of Differential Equation will be taught in Mathematics Course)

Differential Equation for Damping is

$$\boxed{\ddot{x} + \gamma \dot{x} + \omega_0^2 x = 0.}$$

$\gamma = \frac{b}{m}$, $\omega_0^2 = \frac{k}{m}$

In Order to Solve this equation i-e to find the Value of x, we Substitute

$x = e^{\lambda t}$ in the above equation.

$$\lambda^2 e^{\lambda t} + \gamma\lambda e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0.$$

$$\Rightarrow \quad \lambda^2 + \gamma\lambda + \omega_0^2 = 0.$$

This is a quadratic eqn. of form. $ax^2+bx+c=0$

$$\lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

or

$$\boxed{\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}}$$

γ is $= b/m \Rightarrow$ damping term.

$\omega_0^2 = \frac{k}{m} \Rightarrow$ oscillatory term.

So there are two Value of the roots given by λ_1 & λ_2

General Solution will be a linear Combination of both

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

Here the term $\sqrt{\frac{\gamma^2}{4} - \omega_0^2}$ is V. V. Important.

It has three possibilities. depending on whether $\left(\frac{\gamma^2}{4} - \omega_0^2\right)$ it is >0, $=0$ or <0. — We will discuss them in next page. But we will only deal in detail with the Case where the above term is <0.

The Three Possibilities.

1) Heavily damped (Overdamped) (2) Critically damped (3) Lightly damped.

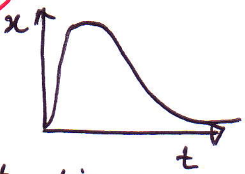
$$\left(\frac{\gamma^2}{4} - \omega_0^2\right)$$

>0

$>0, \ \frac{\gamma^2}{4} > \omega_0^2 \Rightarrow$ Damping (γ) dominats Oscillation (ω_0)

$\Downarrow$

1) Heavily Damped Oscillation



- $x(t) = c_1 e^{-\gamma_1 t} + c_2 e^{-\gamma_2 t}$

$\downarrow$

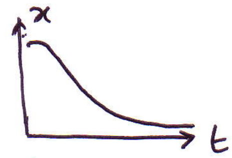
- exponentially decreasing with time.
- Here displacement rapidly decays to 0. (depending on the parameters)

$=0$

$=0, \ \frac{\gamma^2}{4} = \omega_0^2$ Damping (γ) dominats Oscillation (ω_0).

$\Downarrow$

2) Critically damped Oscillation



- $x(t) = c_1 e^{-\gamma_c t} + c_2 t e^{-\gamma_c t}$

$\downarrow$

- exponentially decreasing with time.
- Value of x – displacement decays quickly to

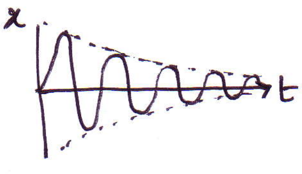
<0

$<0, \ \frac{\gamma^2}{4} < \omega_0^2$ Oscillation (ω_0) dominats damping (γ)

$\Downarrow$

3.) Lightly damped Oscillation (under damped Oscillation)

We will study in detail about this third case only. $\Leftarrow$



$$x(t) = e^{-\gamma t}\left(c_1 e^{i\omega_1 t} + c_2 e^{-i\omega_1 t}\right)$$

OR

$$x(t) = A e^{-\gamma t} \cos(\omega_1 t + \phi), \quad \omega_1 \Rightarrow \text{next}$$

Lightly Damped | Underdamped Oscillator.

$\omega_0^2 > \gamma^2/4$.

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \gamma^2/4} \rightarrow \text{Say} = \omega_1$$

$$= -\frac{\gamma}{2} \pm i\omega_1$$

$$x(t) = C_1 e^{(-\frac{\gamma}{2}+i\omega_1)t} + C_2 e^{(-\frac{\gamma}{2}-i\omega_1)t}$$

$$= e^{-\frac{\gamma}{2}t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

Which can be finally written as

$$\boxed{x(t) = Ae^{-\frac{\gamma}{2}t}\left[\cos(\omega_1 t + \phi)\right]} \rightarrow \omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}.$$

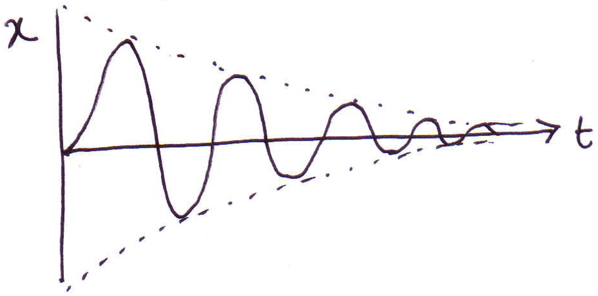
OR $\quad x(t) = Ae^{-\frac{\gamma}{2}t}\left[A\cos\omega_1 t + B\sin\omega_1 t\right]$

ω_1 and ω_0.

Since $\omega_1^2 = \omega_0^2 - \frac{\gamma^2}{4}$, $\quad \omega_0^2 = \frac{k}{m}$, $\quad \gamma = \frac{b}{m}$

So $\omega_1 < \omega_0$.

Angular frequency in case of Damped oscillator is smaller than the natural frequency ω_0 – Damped System oscillats more slowly than Undamped system.



So the displacement in case of damped oscillation is given by

$$x(t) = \underbrace{A e^{-\frac{\gamma t}{2}}}_{\text{Amplitude.}} \cos(\omega_1 t + \phi)$$

$$A(t) = A e^{-\frac{\gamma}{2}t}$$

Here motion is similar to undamped case, only the amplitude decays exponentially.

This can also be written as.

$$A(t) = A e^{-bt/2m} \cos(\omega_1 t + \phi)$$

$$= A e^{-t/\tau} \cos(\omega_1 t + \phi) \quad , \quad \boxed{\tau = \frac{2m}{b}}$$

τ is damping time const or the mean lifetime of the oscillation. When $t = \tau$

$$A(t) = \frac{1}{e} A$$

So mathematically τ is the time necessary for the amplitude to drop to $1/e$ of its initial value.

Energy $E(t)$

Here $E(t)$ decreases with time because the friction force continually dissipates energy

$$E(t) = K(t) + U(t) = \text{Kinetic energy} + PE.$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

From the above expressions one can find v & treating damping to be small $\omega_1^2 \sim \omega_0^2$

$$\boxed{E(t) = \frac{1}{2} k A^2 e^{-\gamma t}} = E_0 e^{-\gamma t}$$

$E_0 = \frac{1}{2} k A^2$, energy at time $t = 0$.

The decay can be characterized by the time τ required for the energy to drop to $1/e$ of its initial value.

$$E(\tau) = E_0 e^{-\gamma\tau} = e^{-1} E_0 .$$

$$\gamma\tau = 1 \quad \Rightarrow \quad \tau = \frac{1}{\gamma} = \frac{m}{b} .$$

τ is often called the damping time.

The Q of an Oscillator.

The degree of damping of an oscillator is often specified by a dimensionless parameter Q, the quality factor.

$$\boxed{Q = \frac{\text{Energy stored in the oscillator}}{\text{Energy dissipated per radian}}}$$

Energy dissipated

$$\frac{dE}{dt} = -\gamma E_0 e^{-\gamma t} = -\gamma E$$

The energy dissipated in time Δt

$$\Delta E \simeq \left(\frac{dE}{dt}\right) \Delta t = \gamma E \, \Delta t$$

Now 2π radian takes time T.

1 " " $\qquad \frac{T}{2\pi} = \frac{1}{\omega_1} = \Delta t$

so $\Delta E = \gamma E / \omega_1$

$$\boxed{Q = \frac{E}{\gamma E/\omega_1} = \frac{\omega_1}{\gamma} \approx \frac{\omega_0}{\gamma}}$$

A lighly damped oscillator has $Q >> 1$

A heavily " " has low Q.

A undamped oscillator has infinite Q.

\# A musician's tuning fork rings at A above middle c, 440 Hz. A sound level meter indicats That the sound intensity decreases by a factor of 5 in 4 s. What is the Q of the tuning fork?

⇒ The sound intensity from the tuning fork is proportional to the energy of oscillation.

We can find γ by taking the ratio of the energy at $t=0$ to that at $t = 4\,s$.

$$5 = \frac{E(0)\, e^{0}}{E(0)\, e^{-4\gamma}} = e^{4\gamma}.$$

Hence $4\gamma = \ln 5 = 1.6.$

$\gamma = 0.4\ s^{-1}.$

$$Q = \frac{\omega_1}{\gamma} = \frac{2\pi(440)}{0.4} \simeq 700.$$

\# In one experiment, a paperweight suspended from a hefty rubber band had a period of 1.2 s and the amplitude of oscillation decreased by a factor of 2 after 3 periods. What is the estimated Q of this system.

⇒ Amplitude is given by $Ae^{(-\gamma/2)t}$. The ratio of amplitude at $t=0$ to that at $t = 3(1.2) = 3.6\ s$ is

$$2 = \frac{Ae^{0}}{Ae^{-3.6\gamma/2}}$$

Hence $1.8\gamma = \ln 2$

$\Rightarrow \gamma = 0.39\ s^{-1}$

$$Q \approx \frac{\omega_1}{\gamma} = \frac{2\pi/T}{0.39} = \frac{2\pi/1.2}{0.39} = \underline{\underline{13}}$$

A mass attached to a spring is set in vibration by displacing it from equilibrium at $t=0$. After 10 sec the amplitude has decreased by 75% and the damped period is 2 sec. Write the differential equation of motion from these two observations.

$\Rightarrow$

$$x = A e^{-\left(\frac{\gamma}{2}\right)t} \cos(\omega_1 t + \phi)$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4}$$

$$\frac{3A}{4} = A e^{-(\gamma/2)\,10} \Rightarrow \underline{\gamma \text{ is } 0.057}$$

$$T = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{r^2}{4}}} = 2 \Rightarrow \underline{\omega_0 \text{ is } 20}$$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0. \qquad \gamma = 0.057$$

$$\omega_0 = 20.$$

An underdampled harmonic oscillator has $k = 2\,N/m$, $m = 1\,kg$ and $b = 0.1\,kg/s$. How many oscillations does the system make before the amplitude decreases to $\frac{1}{e}$ of its initial value

$\Rightarrow$ We need to find out total angle $(\omega_1 t)$, then I can find out n that is $2\pi \times (n) = \omega_1 t$. ; we need to calculate t, γ

1. $A(t) = A e^{-\frac{\gamma}{2}t}$, Amplitude becomes $\frac{1}{e}$ at $\frac{r}{2}t = 1 \Rightarrow t = \frac{2}{\gamma}$.
2. $\gamma = b/m = \frac{0.1}{1}\,kg/s = 0.1\,s^{-1} \Rightarrow \underline{t = 20 \text{ sec}}$
3. $\omega_1 t = \sqrt{\omega_0^2 - \gamma^2/4}\; t = \underline{28.26} \text{ rad.}$ $[\underline{\omega_0^2 = k/m}]$
4. Number of oscillations $= \frac{28.26}{2\pi} = \underline{\underline{4.5}}$ (Ans)

Forced Oscillations.

Three Cases
- Simplest Case - Oscillator Subjected to Const Force
  - Ex: Mass is attached to a Vertical Spring.
- Undamped Oscillator subject to Periodic Force
- Damped Oscillator subject to Periodic Force.

(a) Simplest Case - Oscillator Subjected to a Const Force.

Example: A mass is attached to a Vertical Spring.

$$\left.\begin{aligned} m\ddot{x} + kx &= F \\ m\ddot{x} + kx - F &= 0 \\ m\ddot{x} + k\left(x - \frac{F}{k}\right) &= 0 \quad -(1) \end{aligned}\right\} \text{ Undamped Oscillator.}$$

$$\left.\begin{aligned} m\ddot{x} + b\dot{x} + k(x - F/m) &= 0 \\ \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}(x - F/m) &= 0 \quad (2) \end{aligned}\right\} \text{ Damped Oscillator.}$$

Solution: put $y = x - F/k$.

$\dot{x} = \dot{y}$, $\ddot{x} = \ddot{y}$

Undamped case changes as. (eq 1).

$$m\ddot{y} + ky = 0$$

Sol $y = A\cos(\omega_0 t + \phi)$

$$x - F/k = A\cos(\omega_0 t + \phi)$$

$$x = A\cos(\omega_0 t + \phi) + F/k$$

Therefore mass oscillats about F/k.

Similarly we can Write the equation for damped case.

(b) Undamped Oscillator subject to Periodic Force –

$F(t) \Rightarrow$ time dependent force.

$F(t)$ can be decomposed into its Fourier Components

$$F(t) = \Sigma F_n \cos(n\omega t)$$

Let us take $F(t) = F_0 \cos \omega_d t$

ω_d = angular frequency of time dependent force = driving frequency

So

$$m\ddot{x} + kx = F_0 \cos \omega_d t$$

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m} \cos \omega_d t$$

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega_d t$$

The general Solution is the Combination of the solution of homogeneous part of the equation and a particular solution $x_p(t)$.

$$x = x_H + x_p = \boxed{A \cos(\omega_0 t + \phi) + \frac{F_0/m \cos \omega_d t}{(\omega_0^2 - \omega_d^2)}}$$

Resonance (in case of Undamped Osc).

This expression indicats that as the driving frequency ω_d gets closer and closer to ω_0, the natural frequency of oscillation, the amplitude of oscillation gets larger & larger. If $\omega_d = \omega_0$, the amplitude is infinite. This phenomena is known as resonance. In real physical system – damped oscillator it does not actually become infinite.

(c) Damped Oscillator Subject to Periodic Force.

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos W_d t$$ ← Driving Force.

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos W_d t.$$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos W_d t.}$$

The general solution is sum of homogeneous & parti Part

$$x(t) = e^{-\frac{\gamma}{2}t}\cos(\omega_1 t + \phi) + x_{part}(t).$$

As the time progresses, the $e^{-\frac{\gamma}{2}t}$ term will make the homogeneous Solution die down and finally the only solution

$$x(t) = x_{particular}(t)$$

This is a steady state solution and not for a short time.

Substitute $x(t) = A\cos W_d t + B\sin W_d t.$

Finally $\boxed{x(t) = A\cos(W_d t + \phi)}$

with

$$\boxed{\begin{aligned} A &= \frac{F_0/m}{\sqrt{(\omega_0^2 - W_d^2)^2 + \gamma^2 W_d^2}} \\ \tan\phi &= \frac{\gamma\omega}{\sqrt{W_d^2 - W_0^2}} \end{aligned}}$$

Resonance (in Damped Case)

It is interesting to note That the amplitude of $x(t)$ is not maximized at $W_d = W_0$ but rather at a nearby frequency called The resonance frequency, which you can determine by evaluating the derivative of the denominator and setting it equal to zero. Thus

$$\frac{d}{d\omega_d}\left[(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2\right]=0$$

$$2(\omega_0^2-\omega_d^2)(-2\omega_d)+2\gamma^2\omega_d=0.$$

$$\Rightarrow \boxed{\omega_d^2=\omega_0^2-\frac{\gamma^2}{2}=\omega^2_{resonance}}$$

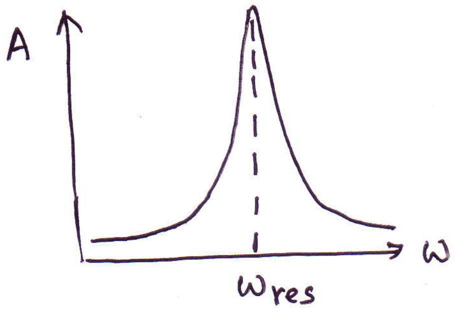
This is neither freq of undamped oscillator ω_0 nor the freq of damped oscillator ω_1.

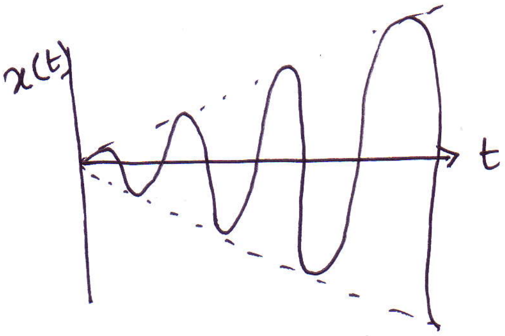
But if γ is very small which is true for many light damping case Then in That Case

$$\boxed{\omega_d \approx \omega_0 = \omega_{res}}$$ If γ is very small.

The resonance phenomenon is often associated with transfer of energy. The average energy of a harmonic oscillator is proportional to the amplitude Squared, so an important & useful parameter in studying resonance is A^2

$$A^2=\frac{(F_0/m)^2}{(\omega_0^2-\omega_d^2)^2+\gamma^2\omega_d^2}$$





At Resonance.

Energy.

The average value of Energy is given by

$$\langle E \rangle = \frac{1}{4} m A^2 (\omega_d^2 + \omega_0^2)$$

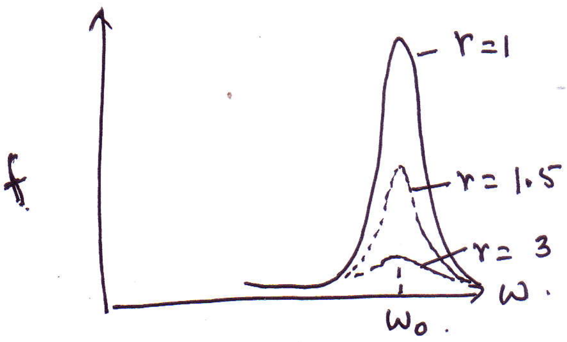
Simplifying for A.

$$\langle E(\omega_d) \rangle = \frac{1}{4} \frac{F_0^2}{m} \frac{(\omega_d^2 + \omega_0^2)}{[(\omega_0^2 - \omega_d^2)^2 + (\omega_d \gamma)^2]}$$

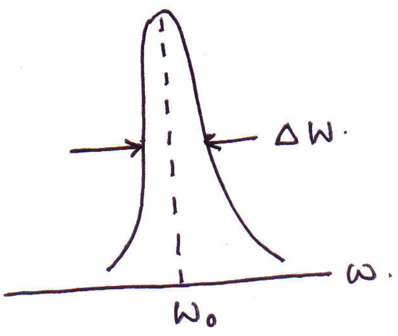
It can be further simplified for $\gamma << \omega_0$ light damping

$$\langle E(\omega_d) \rangle = \frac{1}{8} \frac{F_0^2}{m} \frac{1}{(\omega_d - \omega_0)^2 + (\gamma/2)^2}$$

The plot of the function $[(\omega_d - \omega_0)^2 + (\gamma/2)^2]^{-1}$ as a function of ω is called a resonance curve.



Resonance Width.



The full width of the curve at half maximum value is often called resonance width.

$$\omega_+ - \omega_- = 2\left(\frac{\gamma}{2}\right) = \gamma.$$

$$\Delta\omega = \gamma.$$

Q

Q for lightly damped system ($\omega_1 \sim \omega_0$)

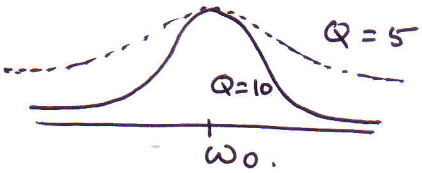
$$Q = \frac{\omega_0}{\gamma}.$$

Now.

$$\frac{\text{Resonance freq}}{\text{freq width of Resonance Curve}} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{\gamma} = Q.$$

So Q is often defined as

$$Q = \frac{\text{Resonance frequency}}{\text{frequency width of Resonance Curve}}$$



Appendix 1: You may read about particular solution & homogeneous sol (here complementary sol) here.

Forced (Driven) simple harmonic motion

Undamped forced oscillations: As we have seen in the last chapter that due to the resistance oscillations eventually die down. To maintain the oscillations one needs a driving force. First we shall study the forced oscillations without the damping term. So the basic equation of motion in this case is

$$m\ddot{x} + kx = F(t) \tag{1}$$
$$\ddot{x} + \omega_0^2 x = F(t)/m. \tag{2}$$

The general solution of the equation (2) is given as sum of two parts. First part is known as *particular solution*, say $P(t)$, which satisfies the equation (2). The second part, known as *complementary function*, say $C(t)$, is the solution of the equation (2) with right hand side set to zero (i.e. solution of ordinary SHM). Writing in terms of equations we have,

$$\frac{d^2P(t)}{dt^2} + \omega_0^2 P(t) = F(t)/m, \tag{3}$$
$$\frac{d^2C(t)}{dt^2} + \omega_0^2 C(t) = 0, \tag{4}$$

Adding equations (3) and (4) we have,

$$\frac{d^2(P(t)+C(t))}{dt^2} + \omega_0^2(P(t)+C(t)) = F(t)/m. \tag{5}$$

So $x(t) = P(t) + C(t)$ gives the general solution of the equation (2) with two arbitrary constants coming form the complementary function and determined by initial conditions.

We shall now assume a sinusoidal time dependence ($F(t) = F_0 \sin \omega t$) of forcing. The angular frequency *omega* appearing in the driving force is called the *driving frequency*. Why we would like to study the sinusoidal forcing will become clear as we proceed. So the equation we are interested in solving, is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \sin \omega t = f_0 \sin \omega t, \tag{6}$$

where, $F_0/m = f_0$. Since we already know the complementary function we look for a particular solution. We try a solution of the type $P(t) = A \sin \omega t$. Substituting this $P(t)$ in the equation (6), we get

$$-A\omega^2 \sin \omega t + A\omega_0^2 \sin \omega t = f_0 \sin \omega t. \tag{7}$$

Above equation (7) finds the amplitude,

$$A = \frac{f_0}{(\omega_0^2 - \omega^2)}. \tag{8}$$

The general solution can now be written as by adding the complementary function

$$x(t) = \frac{f_0}{\omega_0^2 - \omega^2} \sin \omega t + B \cos \omega_0 t + C \sin \omega_0 t. \tag{9}$$

We find that as the driving frequency ω approaches the natural frequency ω_0 from below the phenomenon of resonance occurs and the amplitude A tends to ∞. Once it crosses ω_0 the amplitude tends to $-\infty$. Now since we always consider amplitude as a positive quantity we define the amplitude as $|A|$ and compensate the $-$ve sign of amplitude for $\omega > \omega_0$ including a phase in the argument of the sine function of particular solution. For this we are left with two choices in hand $\sin(\omega t - \pi)$ and $\sin(\omega t + \pi)$. We cannot *a priori* decide whether the oscillations lead or lag the driving force. We take the hint from damped forced oscillations (which is to be done in the next section) and settle for $\sin(\omega t - \pi)$ for $\omega > \omega_0$. So in this case there is an abrupt change of $-\pi$ radians in the phase. So the first term of(9) is written as.

$$x(t) = \frac{f_0}{\omega^2 - \omega_0^2} \sin(\omega t - \pi), \qquad (\omega > \omega_0) \tag{10}$$

The amplitude and the phase are plotted against the frequency below.

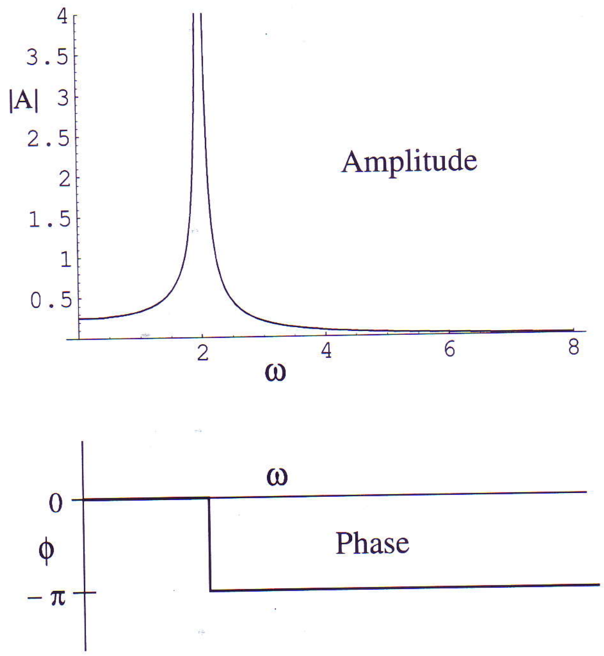


Figure 1: Amplitude and phase of undamped driven oscillator as a function of driving frequency ω.

We fix B and C using some initial conditions. Let us choose $x(t=0) = \dot{x}(t=0) = 0$. The condition $x(0) = 0$ fixes $B = 0$ and $\dot{x}(0)$ finds $C = -\frac{f_0 \omega}{\omega_0(\omega_0^2 - \omega^2)} = -A\omega/\omega_0$.