Rotational Dynamics - 1.

Moment of Inertia - Rotational analogue of Mass.

Here we learn to calculate moment of inertia of solid bodies - mostly symmetric bodies.

$$I = \int r^2 \, dm.$$

Earlier we have learnt to Calculate Center of Mass.

$$\bar{R}_{cm} = \frac{1}{M} \int \bar{r} \, dm.$$

We must be Careful about the meaning of r in both these expressions. In the Calculation of moment of Inertia, r is the ~~d~~ perpendicular distance from the axis of rotation where as in the Center of Mass Calculation $\bar{r}$ is the position vector of the mass dm from the Origin.

In MI Calculation The final answer Contain mass M. Where as in Center of mass calculation the final answer is a Coordinate, the position vector of a point - a Coordinate.

Few Important pts to remember for MI Calculation.

1. Identify the axis about which MI is to be Calculated.
2. Identify dm → if dm is a infinitesimally small mass Then apply the above formula by knowing r That is ~~d~~ perp dist from axis of rotation.

3. If it is a solid body like a disk, cylinder, hollow sphere, solid sphere, solid cone one needs to be careful.

Suppose you want to calculate MI of solid sphere and you take a hollow sphere at a dist- r and thickness dr as your elemental mass - small mass which you want to integrate, then it is always useful to introduce a small step that is you write that MI of hollow elemental sphere as dI.

$$dI = \frac{2}{3} r^2 dm. \qquad \text{Then } I = \int dI.$$

Otherwise it is possible to make mistake.

- If you find any error please let me know.
- These notes are for internal circulation only.
- I consulted mostly H.C. Verma and other internet sites while preparing the notes of this chapter. I acknowledge them.

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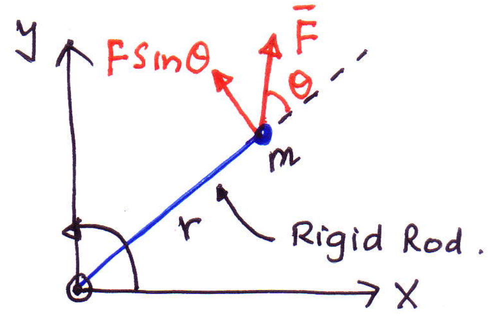
(AMIT NEOGI)
7 Sept, 2016.

In this chapter most of the material You have studied in your earlier classes. I have one question for you. Why do we define $I = mr^2$. Do you remember?
If not just see the explanation:



Suppose there is a particle of mass m attached to a rigid rod of negligible mass that rotats in the x-y plane.

The tangential Comp of Force which acts on the particle & rotats is $= F\sin\theta$.

$$F\sin\theta = m a_T = m r \alpha \qquad (\alpha = \text{angular acc})$$

Multiply both sids by r, the left hand side $rF\sin\theta$ is the torque which is responsible to rotate the body as well as the particle. $[rF\sin\theta = mr^2\alpha]$

$$rF\sin\theta = \tau_z = \tau \quad (\text{let us write } \tau \text{ only})$$

$$\tau = mr^2\alpha$$

$$F = m a$$

If we compare both equations mr^2 in rotational motion plays similar role of mass in linear motion.

So $I = mr^2$ &

$$\tau = I\alpha.$$

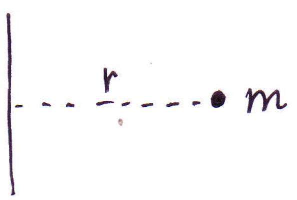
Moment of Inertia.

(1) The role of moment of Inertia in the study of rotational motion is analogous to that of mass in translational motion.

(2) It is a property of an object that is related to its mass distribution.

(3) Moment of Inertia gives a measurement of the resistance of a body to a change in its rotational motion.

MI of a Single Particle.



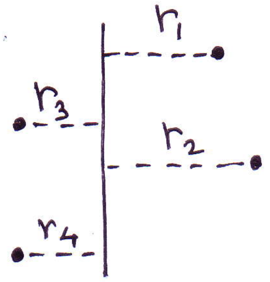
$$I = mr^2$$

m is the mass of the particle.

r is the perp distance from the axis under consideration.

(Here r is not the distance from Origin)

MI of a System of Particles.

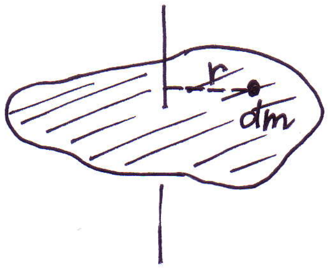


$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots$$

Here r_i is the perpendicular distance from the axis to the ith particle which has a mass m_i

Moment of Inertia of Rigid Bodies.

(Continuous Mass distribution)



$$\boxed{I = \int r^2\, dm}$$

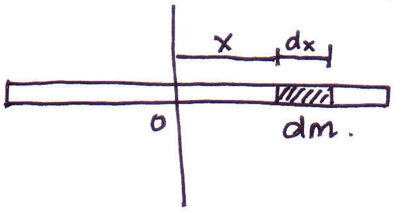
(Please find out from your class XII book why you get $r^2 dm$).

(A) Uniform rod about a perpendicular bisector.

Mass = M & Length = ℓ

Step 1 : $I = \int r^2\, dm$

Step 2 : Mark dm in the figure & find its value.



$dm = \left(\frac{M}{L}\right) dx.$

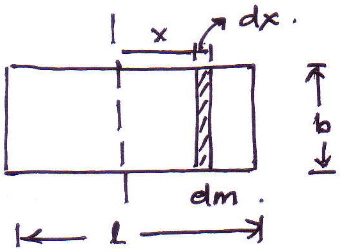
Step 3 : $I = \int r^2\, dm = \int_{-\ell/2}^{+\ell/2} x^2 \left(\frac{M}{L}\right) dx = \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\ell/2}^{+\ell/2}$

$= \frac{M\ell^2}{12}$

(B) MI of a rectangular Plate about an axis passing through Center parallel to an edge.

Step 1 : $I = \int r^2 dm$

Step 2 : Mark dm & its value.



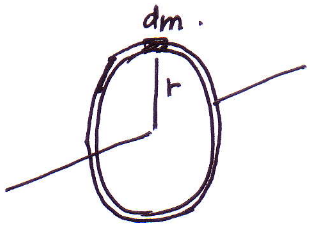
$dm = \left(\frac{M}{b\ell}\right)(b\,dx) = \frac{M}{\ell} dx.$

Step 3 : $I = \int r^2 dm = \int_{-\ell/2}^{\ell/2} x^2 \frac{M}{L} dx = \frac{M}{L}\int_{-\ell/2}^{\ell/2} x^2 dx = \frac{M\ell^2}{12}.$

(c) Moment of Inertia of a Circular ring about its axis.

(1) Step 1 : $I = \int r^2 dm$

(2) step2 : Mark dm & its Value.
We can keep dm as dm.



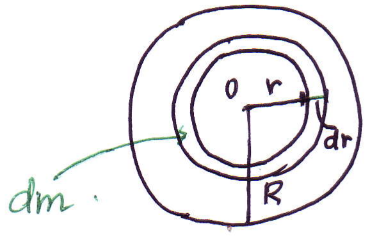
(3) Step3 : $I = \int r^2 dm = \int R^2 dm$
$= R^2 \int dm = MR^2$

(D) MI of a Uniform Circular Plate about its axis.
Mass = M , Radius = R.

(1) Step 1 : $I = \int r^2 dm$

(2) Step2 : Mark dm & its Value.

Consider a ring at a distance r of Thickness dr.



$$dm = \left(\frac{M}{\pi R^2}\right) 2\pi r\, dr = \frac{2Mr\,dr}{R^2}$$

One can Write $dI = \_$ but it is not necessary.

(3) Step 3. : $I = \int r^2 dm = \int_0^R r^2 \left(\frac{2Mr\,dr}{R^2}\right) = \frac{2M}{R^2}\int_0^R r^3 dr$

$$= \frac{MR^2}{2}$$

(E) MI of Hollow Cylinder about its axis.

Step1 : $I = \int r^2 dm$.

Step 2 : dm ⇒ a stack of rings & all of Them at the same distance.

Step3 : $I = \int r^2 dm = R^2 \int dm = MR^2$.

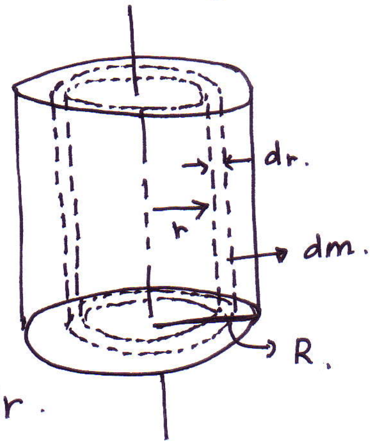
(F) MI of a Uniform Solid Cylinder about its axis.

Mass = M Radius = R. length = l.

Step 1: $I = \int r^2 \, dm$.

Step 2: Mark the dm & its value.

Mark a hollow cylinder of radius r inside the main Cylinder.



$$\rho = \frac{M}{\pi R^2 l}.$$

$$dm = \left(\frac{M}{\pi R^2 l}\right)(2\pi r \, dr \, l) = \frac{2M}{R^2} r \, dr.$$

$\uparrow$ density $\uparrow$ vol. of dm element

Step 3: $I = \int r^2 \, dm = \int_0^R r^2 \left(\frac{2M}{R^2} r \, dr\right) = \frac{MR^2}{2}$

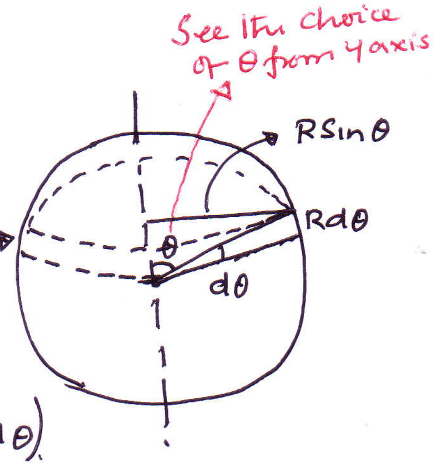
* dI step is not necessary because for ring $dI = r^2 dm$ only.

(G) MI of a Uniform hollow Sphere about a diameter.

Step 1: $I = \int r^2 \, dm$

Step 2: Mark dm & its value.

Consider a ring as dm which extends an angle $d\theta$ at the Center.



Area of the ring = $(2\pi R \sin\theta)(R \, d\theta)$

Mass per Unit area of Sphere = $\frac{M}{4\pi R^2}$

$$dM = \frac{M}{4\pi R^2}(2\pi R \sin\theta)(R \, d\theta) = \frac{M}{2} \sin\theta \, d\theta.$$

Step 3: $I = \int r^2 \, dm = \int_0^{\pi} (R \sin\theta)^2 \left(\frac{M}{2} \sin\theta \, d\theta\right) = \frac{2}{3} MR^2$

Integration of $0 - \pi$ will Cover the whole Space.

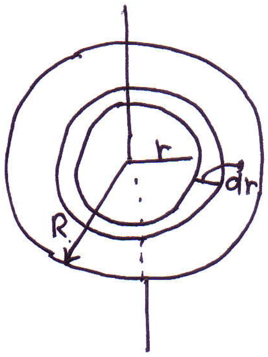
Count θ from ~~negative~~ +ve y axis to -ve y axis to cover whole Space. $\left[\int \sin^3\theta = -\cos\theta + \frac{1}{3}\cos^3\theta + c\right]$

(H) M.I of a Uniform Solid sphere about a diameter.

Step 1: $I = \int r^2 dm$

Step 2: Mark dm & find its value.

Consider a thin Spherical Shell of thickness dr at a distance r.



$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}.$$

It has surface area $= 4\pi r^2$ & Thickness dr.

volume $= 4\pi r^2 dr$.

$$dm = \left(\frac{3M}{4\pi R^3}\right) \cdot \left(4\pi r^2 dr\right) = \frac{3M}{R^3} . r^2 dr.$$

Step 3:

$$I = \int r^2 dm = \int_0^R r^2 \left(\frac{3M}{R^3}\right) r^2 dr)$$

$$= \frac{3}{5} MR^2$$ X The answer is Wrong.

Where did we go wrong. ? ⇒ V. Imp. Think!

We had calculated earlier the case of hollow sphere.

The moment of Inertia of a hollow Sphere is not MR^2 but $\frac{2}{3} MR^2$

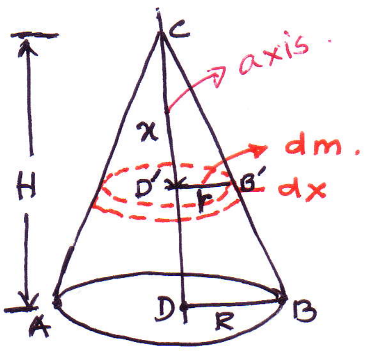
⇒ $dI = \frac{2}{3} r^2 dm.$ $I = \int dI$ | V. Imp.

$$\text{So } I = \frac{2}{3} \int_0^R r^2 \left(\frac{3M}{R^3} r^2 dr\right) = \frac{2}{5} MR^2.$$

Be Careful: The meaning of r is different in case of COM calculation & Moment of Inertia Calculation.

Moment of Inertia of a Solid Cone. (about an axis through COM as shown)

(Mass = M, Height = H, Radius = R)



We can imagine the cone to be made up of large number of discs of small thickness.

Let us consider one such disk at a dist x from the vertex and thickness dx.

Why not from base.

The vertical symmetrical axis which is passing through COM is shown - CD in the figure.

$$dm = \left(\frac{M}{\frac{1}{3}\pi R^2 H}\right) \cdot \pi r^2 dx.$$

vol of the cone.

⇒ We have two variabls r and x here. We need to convert one in terms of other.

Consider the similar triangles. $CD'B'$ and CDB.

$$\frac{r}{x} = \frac{R}{H} \Rightarrow r = \frac{R}{H} x$$

This step is simpler because x is taken from vertex & not from base.

$$dm = \frac{3M}{\pi R^2 H} \pi \left(\frac{R^2}{H^2} x^2\right) dx$$

MI of elemental disk $= dI = \frac{1}{2} dm\, r^2$ | Imp.

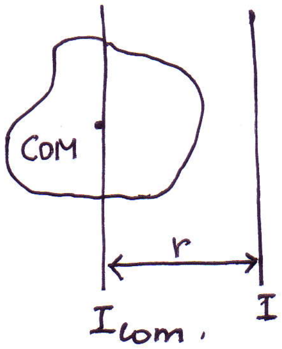
$$= \frac{1}{2}\left(\frac{3M}{H^3} \cdot x^2 dx\right) \frac{R^2}{H^2} \cdot x^2.$$

$$I = \int dI = \frac{3}{2} \frac{M}{H^5} \cdot R^2 \int_0^H x^4 dx = \frac{3}{10} MR^2.$$

Theorems on Moment of Inertia.

Theorem of Parallel Axes.

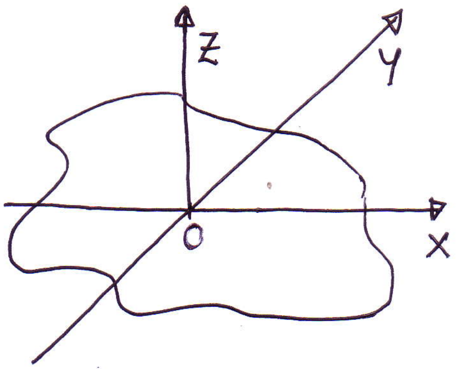
This theorem relats the moment of inertia of a rigid body about two parallel axes, one of which passes through the Centre of mass.



$$\boxed{I = I_{COM} + Mr^2}$$

M is the mass of the body.

Theorem of Perpendicular Axes.



Consider a plane body (2 Dim) of mass M. Let X and Y axes be two mutually perpendicular lines in the plane of the body. The axes intersect at origin O.

I_x = Moment of inertia of the body about X axis

I_y = " " " about Y axis.

The moment of inertia of the body about Z axis (passing through O and perpendicular to the plane of the body) is given by

$$\boxed{I_z = I_x + I_y}$$

Few More Questions on MI. (Both these problems are taken from D.C. Pandey).

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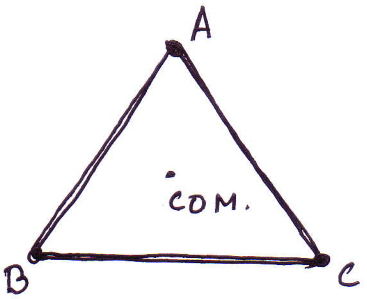
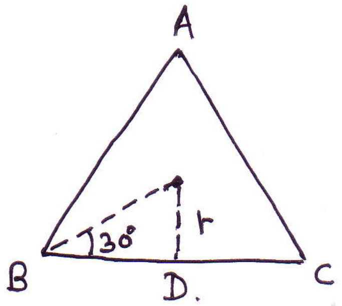


- Three rods are connected to form an equilateral triangle.

Find MI about an axis passing through COM & perpendicular to the plane of the triangle.



MI of rod BC $\perp$ to plane of ABC & passing through mid point of rod BC is

$$I_1 = \frac{ml^2}{12}$$

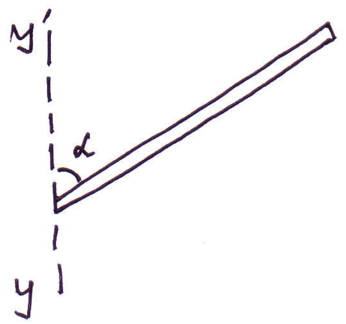
$$r = BD \tan 30^\circ = \frac{l}{2\sqrt{3}}.$$

From Parallel axes theorem MI of this rod about the axis through COM is

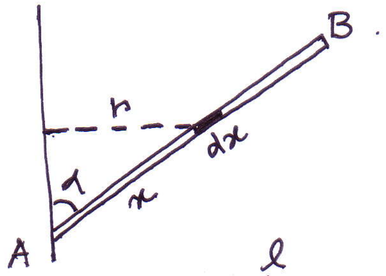
$$I_2 = I_1 + mr^2 = \frac{ml^2}{12} + m\left(\frac{l}{2\sqrt{3}}\right)^2 = \frac{ml^2}{6}.$$

Moment of Inertia of all three rods $= 3\left(\frac{ml^2}{6}\right) = \frac{ml^2}{2}.$

\# Find MI of the rod AB about an axis YY as shown.



$\Rightarrow$



$$I = \int r^2 dm.$$

$$r = x \sin\alpha$$

$$dm = \left(\frac{m}{l}\right) dx$$

$$I = \int_0^l (x \sin\alpha)^2 \left(\frac{m}{l} dx\right)$$

If $\alpha = 0$ $I = 0$
$\alpha = \frac{\pi}{2}$ $I = \frac{ml^2}{3}$ $\Leftarrow$

$$= \frac{ml^2}{3} \sin^2\alpha$$