Central Force Motion.

The chapter on Central Force is basically divided into two parts — (1) Basic Theory (pg 1 - 11) & then applications. Mostly you get your questions from applications part. But theory part is necessary for your understanding. Concepts of Central Force is used to explain Orbit motion. of Satellits.

I have used following resources to prepare this note

(1) Kleppner

(2) Intermediate Physics - Patrick Hamill.

(3) Internet Resources — including Wikipedia.

If you find any mistake please let me know.

Aneogi
(AMIT NEOGI)
15 NOV, 2016.

NOTE: You must be thorough with the theory of the graph
1) that is plotted in page 12 & 13.
2) Kleppner Problem solved in page 19.

Important For Exam.

Contents

pages.

Central Force Motion

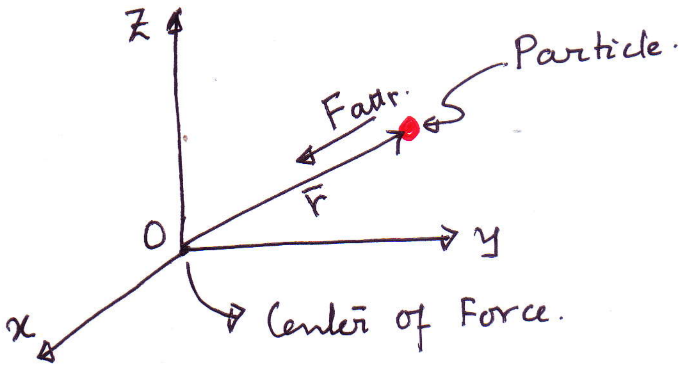
What is a Central Force Problem ?

Suppose a force acting on a particle of mass m has the property –

(1) The force is always directed from m toward or away from a fixed point O.

(2) The magnitude of force only depends on the dist r from O.

Forces having the above properties are called Central Forces.



$$\boxed{\bar{F} = \underbrace{f(r)}_{\text{pt (2) above}}\ \underbrace{\hat{r}}_{\text{pt (1) above}}} = m\ddot{\vec{r}}$$

Here $\bar{r}$ is the position vector of the particle

Force is attractive towards the Origin O.

If $f(r) < 0 \Rightarrow$ Force is attractive towards O

$f(r) > 0 \Rightarrow$ Force is repulsive from O.

Applications & Examples.

Central forces are V. Imp in Physics & Engineering. Some examples are

(1) Gravitational force of attraction between two pt mass.

Mathematically $F_{12} = -G \frac{m_1 m_2}{r^2} \hat{r}$

(2) The Coulomb force of attraction & repulsion between Charged particles $\Rightarrow$ $F_{12} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r^2} \hat{r}$.

(3) Linear Restoring Force $F_{12} = -k r \hat{r}$.

(4) Motion of Planets in Solar System.

Some more complicated problems in Physics such as two body problem with forces along the line connecting the two bodies can be reduced to a Central Force motion.

Properties of a Particle Moving Under the Influence of Central Force.

1) The path of the particle must be a plane Curve. i.e it must lie on a plane.

2) The angular momentum of the particle is Conserved (Const with time). Both angular momentum & Energy are Constants of motion.

3) The particle moves in such a way that the position Vector (from point) Sweeps out equal areas in equal time. The time rate of change in area is const. This is referred to as Law of Areas.

Potential Energy.

A Central force is always a conservative force. So $F(r)$ can be written in terms of U as.

$$F(r) = -\frac{dU}{dr}$$

$$W = \int_{r_1}^{r_2} \bar{F}\cdot d\bar{r} = \int_{r_1}^{r_2} F(r)\,\hat{r}\cdot d\bar{r} = \int_{r_1}^{r_2} F\,dr = U(r_1) - U(r_2)$$

$$\bar{\nabla}\times\bar{F} = 0.$$

Uniform Circular Motion.

Every Central Force can produce circular motion provided that initial radius r and speed v satisfy the equation for the Centripetal force.

$$\boxed{\frac{mv^2}{r} = F(r).}$$

If this equation is satisfied at the initial moment it will be satisfied at all later times. The particle will continue to move in a circle of radius r & speed v for ever.

Relation to the classical two-body Problem.

The Central-force problem Concerns an ideal situation ("one-body problem") in which a single particle is attracted or repelled from an immovable point O, the center of force. However, physical forces are generally between two bodies and by Newton's third Law; If the first body applies a force on the second, the second body applies an equal and opposite force on the first. Therefore both bodies are accelerated if a force is present between them; There is no perfectly immovable ~~const~~ Center of force. However, if one body is overwhelmingly more massive than other, its acceleration relative to the other may be neglected.

Newton's Law of motion allow ~~any~~ any classical two body problem to be Converted into a Corresponding exact one body problem.

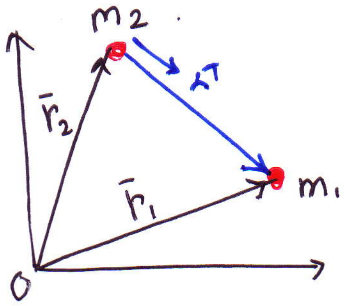
$\Downarrow$

Using Reduced Mass formulation.

$$M = \frac{m_1 m_2}{m_1 + m_2}$$

Discuss later

Two body problem (It will be reduced to one body Prob)



From fig: $\bar{r}_2 + \bar{r} = \bar{r}_1$

$$\bar{r} = \bar{r}_1 - \bar{r}_2$$

Separation between mass.

$\bar{r}_1$ and $\bar{r}_2$ are position vectors of masses m_1 and m_2

The equation of motion are

$$m_1 \ddot{\bar{r}}_1 = f(r)\,\hat{r} \quad -(1)$$

$$m_2 \ddot{\bar{r}}_2 = -f(r)\,\hat{r} \quad -(2)$$

There is –ve sign in eq (2) because direction of force in body 2 is opp in direction to that of body 1.

Confusion: These are highly confusing equations. Double differentiation of $\bar{r}_1$ gives $\hat{r}$. Moreover the origin of $\hat{r}$ is not from O but from m_2.

$\Rightarrow$ Actually these are coupled equations. Soon we will combine them in one equations and then it will be only in terms of $\hat{r}$.

What is our Purpose?

To find the value of r_1 and r_2 as fn of t.

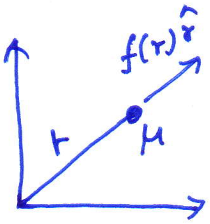
Here we combine them first, solve for r as a fn of t and then get back r_1 and r_2

$$\ddot{\bar{r}}_1 - \ddot{\bar{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\,\hat{r} = \left(\frac{m_1 + m_2}{m_1 m_2}\right) f(r)\,\hat{r}$$

$\frac{1}{\mu}$:= Reduced mass.

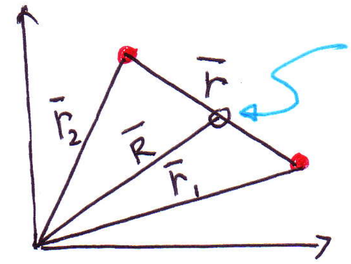
$$\Rightarrow \mu\left(\ddot{\bar{r}}_1 - \ddot{\bar{r}}_2\right) = f(r)\,\hat{r}$$

$$\Rightarrow \boxed{\mu \ddot{\bar{r}} = f(r)\,\hat{r}}$$



It is a quantity which allows the two body Problem to be solved as if it were a one body Problem.

Find out $\bar{r}$ as fn of time from this eqn & then $\bar{r}_1$ and $\bar{r}_2$ can be calculated using the equations below.



$$\bar{r} = \bar{r}_1 - \bar{r}_2 \quad - (3)$$

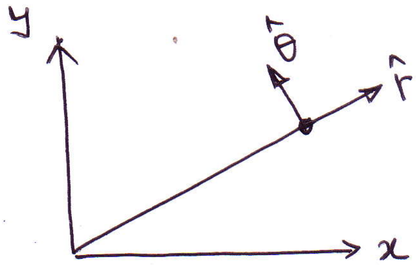
$$\bar{R} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{m_1 + m_2} \quad - (4)$$

Solving (3) & (4)

$$\boxed{\bar{r}_1 = \bar{R} + \left(\frac{m_2}{m_1 + m_2}\right)\bar{r}} \quad ; \quad \boxed{\bar{r}_2 = \bar{R} - \left(\frac{m_1}{m_1 + m_2}\right)\bar{r}}$$

So our purpose of calculating r_1 and r_2 as a fn of t is fulfilled.

Now we will study about the general properties of Central Force Field. For this we will using the formulas of plane polar coordinate again & again. Lets revise.



Position Vector of any Pt $P = r\hat{r}$.

Velocity $= v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

Acceleration $a = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$.

$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$

$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$.

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$$

General Properties in Central Force Field.

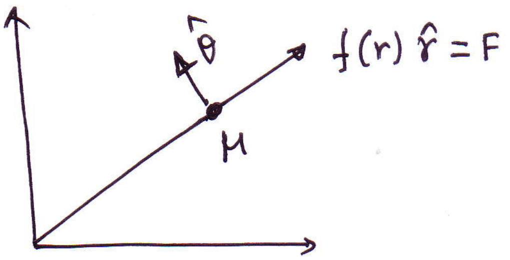
(1) Equation of Motion.

When we write equations of motion for a particular body we write

$$F_x = m a_x$$

$$F_y = M a_y.$$

Similarly here we write equations using plane polar coord.



$$\mu \bar{a} = f(r)\hat{r}$$

$$\mu[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}] = f(r)\hat{r}.$$

Equating $\hat{r}$ and $\hat{\theta}$ component

$$\boxed{\begin{aligned} \mu(\ddot{r} - r\dot{\theta}^2) &= f(r) \\ \mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= 0 \end{aligned}}$$

$\Rightarrow$ Equations of motion in Central Force Field.

One Constant of motion.

$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{\mu}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \frac{\mu}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

So $\boxed{r^2\dot{\theta} = \text{constant} = h}$

There are two other constants of motion in Central Force Field — Angular Momentum & Energy.

(2) Motion is confined to a plane

- The motion of a particle under a Central Force F always remain in the plane defined by its initial position & velocity. Since the position r, velocity v and Force F all lie in the same plane, there is never an acceleration $\perp$ to that plane.
- The Central Force $f(r)\hat{r}$ is along $\hat{r}$ and can exert no torque on the reduced mass μ. Hence angular momentum is constant.

$$\bar{\tau} = \bar{r} \times \bar{F} = r\hat{r} \times f(r)\hat{r} = 0.$$

So $\frac{d\bar{L}}{dt} = 0 \Rightarrow L = \text{const.}$

$$\bar{L} = \bar{r} \times \bar{p} = \bar{r} \times \mu\bar{v}$$

Direction of $\bar{L}$ is const means $\bar{r}$ and $\bar{v}$ are confined to a plane. & $\perp$ to the plane.

(3) Constants of Motion – $\bar{L}$, E

There are two more constants of Central Force Motion

(a) Magnitude of angular Momentum $|\bar{L}| = l$

(b) Total Energy E

They are very useful to solve problems.

I a) Angular Momentum.

$$\bar{L} = \bar{r} \times \mu\bar{v} = r\hat{r} \times \mu(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$
$$= \mu r\dot{r}\,\hat{r}\times\hat{r} + \mu r^2\dot{\theta}\,\hat{r}\times\hat{\theta}$$
$$= 0 + \mu r^2\dot{\theta}\,\hat{k}$$

$\boxed{|\bar{L}| = l = \mu r^2\dot{\theta}}$ = Constant of motion.

(b) Energy.

$$E = \frac{1}{2}\mu v^2 + U(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

Where $U(r)$ is given by

$$U(r) - U(r_a) = -\int_{r_a}^{r} f(r)\,dr$$

- The constant $U(r_a)$ is not Physically Significant so we can leave r_a Unspecified – leaving a const has no effect on the motion.
- Eliminate $\dot{\theta}$ from the last equation.

$$\dot{\theta} = \frac{l}{\mu r^2}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r).$$

This looks like the equation of motion of a particle moving in one dimension. All reference to θ is gone.

We can introduce

Known as true Potential

$$\underline{\underline{U_{eff}(r)}} = \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$$ so that

Effective Potential

Known as Centrifugal Potential

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)}$$

OR $$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{eff})}$$

OR $$\boxed{\int_{r_0}^{r}\frac{dr}{(2/\mu)(E - U_{eff})} = t - t_0}$$

Gives r as a fn of t.

To find θ as a fn of time

$$l = \mu r^2 \dot{\theta}$$

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2}$$

$$\boxed{\theta - \theta_0 = \int_{t_0}^{t} \frac{l}{\mu r^2}\, dt}$$

r is known as a fn of t from previous eqn.

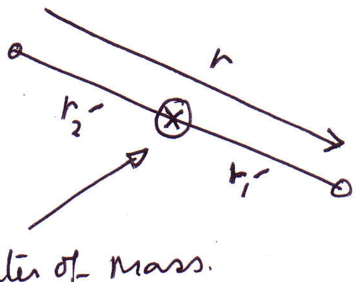
To find θ as a fn of r - path of the particle.

From above equations.

$$\boxed{\frac{d\theta}{dr} = \frac{1}{\mu r^2} \frac{1}{\sqrt{(2/\mu)\,(E - U_{eff})}}}$$

So we can obtain $r(t)$, $\theta(t)$ or $r(\theta)$.

Center of Mass Coordinate.



$$\bar{r}_1' = \frac{m_2}{m_1 + m_2} \bar{r}$$

$$r_2' = - \frac{m_2}{m_1 + m_2} \bar{r}$$

- It is easy to show that L is simply the angular momentum of m_1 and m_2 about the Center of mass L_c
  $\Rightarrow$ $L_c = L$
- Similarly, the total Energy E is the energy of m_1 & m_2 relative to their Center of mass E_c.
  $\Rightarrow$ $E_c = E$

(4) The Law of Equal Areas. – r sweeps out equal areas in equal time.

Suppose that in time Δt the position vector moves from $\mathbf{r}$ to $\mathbf{r}+\Delta\mathbf{r}$. Then the area swept out by the position vector in this time is approximately half the area of a parallelogram with sides $\mathbf{r}$ and $\Delta\mathbf{r}$. We give a proof of this:

$$\begin{aligned}\text{Area of parallelogram} &= \text{height} \times |\mathbf{r}|,\\ &= |\Delta\mathbf{r}|\sin\theta|\mathbf{r}|,\\ &= |\mathbf{r}\times\Delta\mathbf{r}|,\end{aligned}$$

see Fig. 3.

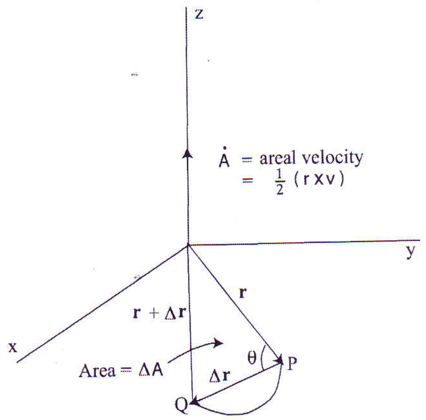


Figure 3:

Hence,

$$\Delta A = \frac{1}{2}|\mathbf{r}\times\Delta\mathbf{r}|.$$

Dividing this expression by Δt, and letting $\Delta t \to 0$, gives:

$$\lim_{\Delta t\to 0}\frac{\Delta A}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{2}\left|\mathbf{r}\times\frac{\Delta\mathbf{r}}{\Delta t}\right| = \frac{1}{2}|\mathbf{r}\times\mathbf{v}|,$$

or

$$\dot{A} = \frac{1}{2}|\mathbf{r}\times\mathbf{v}|.$$

Now we need to evaluate $\mathbf{r}\times\mathbf{v}$. Using $\mathbf{r} = r\mathbf{r}_1$, we have:

$$\mathbf{r}\times\mathbf{v} = \mathbf{r}\times(\dot{r}\mathbf{r}_1 + r\dot{\theta}\boldsymbol{\theta}_1) = \dot{r}(\mathbf{r}\times\mathbf{r}_1) + r\dot{\theta}(\mathbf{r}\times\boldsymbol{\theta}_1) = r^2\dot{\theta}\mathbf{k}.$$

Therefore we have:

$$r^2\dot{\theta} = 2\dot{A} = \text{constant}. \tag{7}$$

The vector:

$$\boxed{\dot{\mathbf{A}} = \dot{A}\mathbf{k} = \frac{1}{2}r^2\dot{\theta}\mathbf{k},}$$

is called the *areal velocity*.

$\dot{\theta} = \frac{l}{\mu r^2}$

So $\dot{A} = \frac{1}{2} r^2 \frac{l}{\mu r^2}\hat{k} = \frac{l}{2\mu}\hat{k}$

$\dot{A} = \frac{1}{2} r^2 \dot{\theta}\hat{k} = \frac{l}{2\mu}\hat{k}$

Applications of Central Force.

Planetary Motion.

One of the major application of Central force is in discussing Planetary Motion. Let us jot down the important formulas that we have read.

1) $l = \mu r^2 \dot{\theta}$ (l = magnitude of angular Momentum)

2) Energy : $E =$ Radial Energy $\rightarrow \frac{1}{2}\mu \dot{r}^2$

Tangential Energy $\rightarrow \frac{1}{2}\mu (r\dot{\theta})^2 = \frac{l^2}{2\mu r^2}$

3) Effective Potential : $U_{eff} = \frac{l^2}{2\mu r^2} + U(r)$

$\frac{l^2}{2\mu r^2}$ → Centrifugal potential.

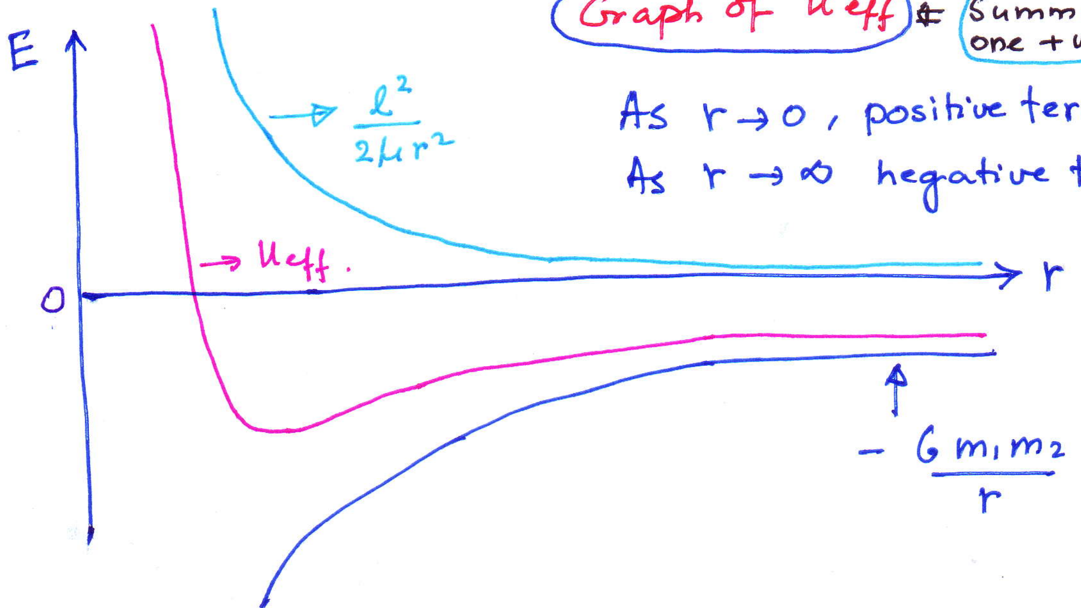
4) Energy : $\boxed{E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)}$

In. Planetary Motion : $f(r) = -\frac{G m_1 m_2}{r^2} \Rightarrow U(r) = -\frac{G m_1 m_2}{r}$

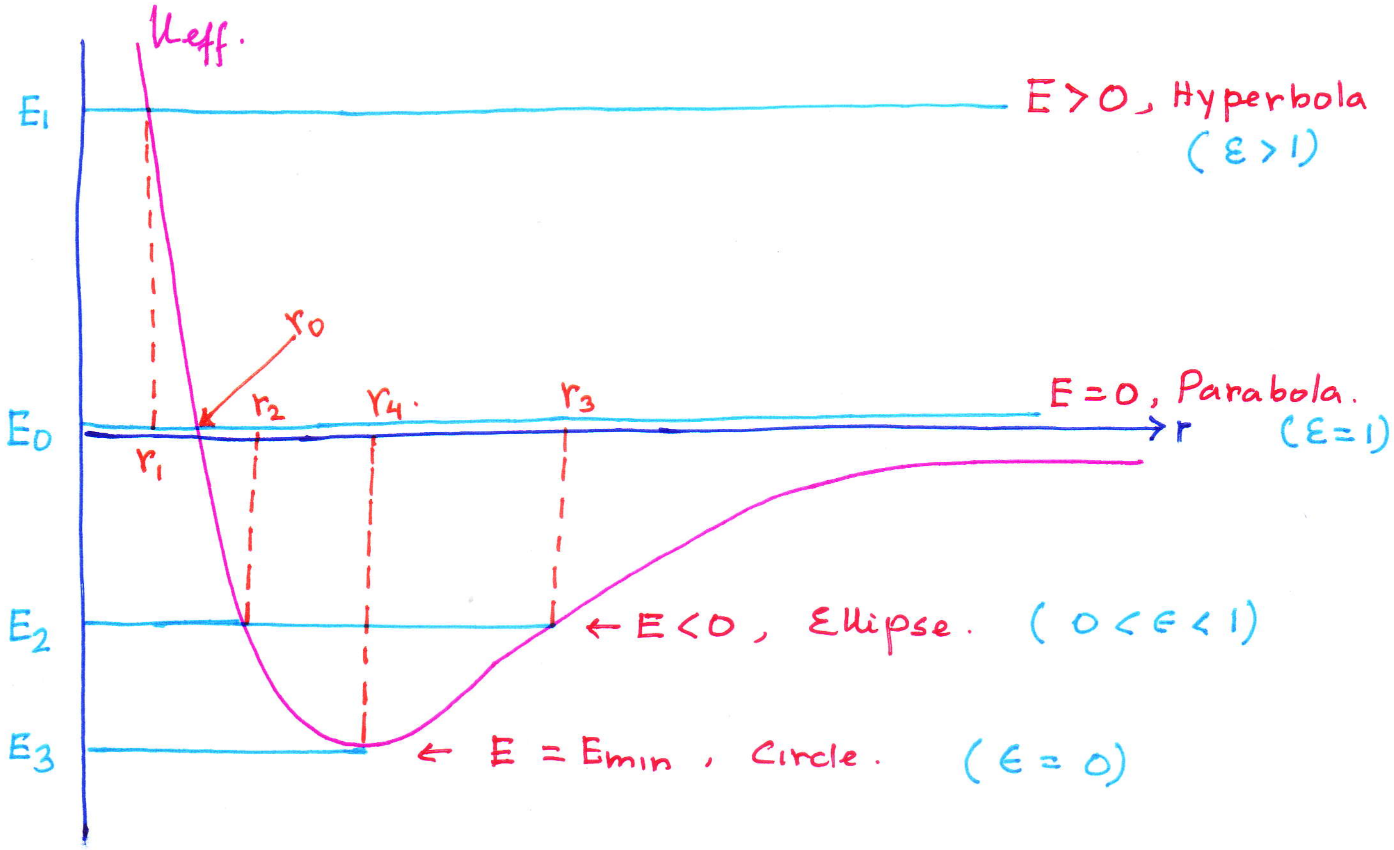
Graph of U_{eff} ⇐ Summation of 2 terms. one +ve & other −ve.

As $r \rightarrow 0$, positive term dominats

As $r \rightarrow \infty$ negative term dominats.



Let us draw only U_{eff} – enlarged view. (V. Imp).



ε = Eccentricity.
↓
We will study later.

$$\boxed{\frac{1}{2}\mu\dot{r}^2 = E - U_{eff}(r)}$$

↓

This radial KE term Cannot become negative.

The particle Cannot be located at values of r for which $\underline{E \leq U_{eff}}$

✓ Case 1 : E > 0. (See figure)

- Suppose a particle starting at r = ∞ Coming closer & closer to the primary Until it reaches r_1 (the turning point) and then moving back to infinity.

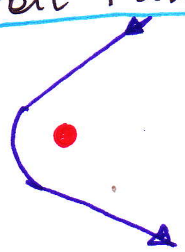
E_1 in the figure.

- This is not a Complete description. It is only a description of the radial motion. Meanwhile the particle is also moving in θ with a Velocity given by $\dot{\theta} = l/mr^2$. The angular velocity increases as the radial distance decreases, in agreement with Keplers second Law.

- At r_1, the value of U_{eff} is E_1, so $\dot{r} = 0$. That is, at the turning point the particle has zero radial velocity but the angular component of the velocity is a maximum because $r\dot{\theta} = (r)(l/mr^2) = l/mr$ is greatest when r is smallest.

- Example: A good example is the case of a Comet coming in from infinity, speeding up as it approachs the Sun, swooping around the Sun, and then moving out to infinity

- Orbit Path: The path of this particle will be a hyperbola. (with Eccentricity $\epsilon > 1$)



Case 2: $E = 0$ (path is Parabola, $\epsilon = 1$)

- This means that positive "radial KE" $\frac{1}{2}m\dot{r}^2$ is equal to $-(U_{eff})$. The particle comes from $r = \infty$ ($\dot{r} \sim 0$) to the turning pt at r_0 ($\dot{r} = 0$) & then goes back to infinity. The main difference from the previous case is here the orbit path is parabola.

E_0 in the figure.

- As the particle comes in from infinity, $\dot{r}$ increases reaching a maximum at r_4 where the value of $(E - U_{eff})$ is greatest. Then slows down to zero at r_0.

Case 3: $E < 0$ (Orbit path – Ellipse, $0 < \epsilon < 1$)

E2 in figure

If the particle has negative total energy, the motion is quite different. There are two turning points r_2 and r_3. It can neither reach $r = 0$ nor move out to $r = \infty$ because $KE < 0$.

As it moves back and forth radially between the two points denoted by r_2 and r_3, it is also moving azimuthally with varying angular velocity $\dot{\theta}$. This combination of radial and angular motion represents a trajectory of ellipse.

Case 4 : $E = E_{min}$. (orbit path – Circle, $\epsilon = 0$).

E3 in the fig.

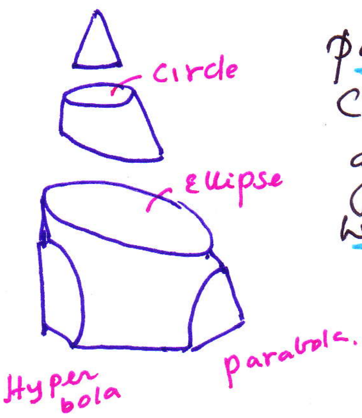
If the particle has energy (E_3 in fig), the minimum possible total energy, the value of $\dot{r}$ is zero at all times and the particle is at a constant radial position $r = r_4$. The path is a circle

Please note $\dot{r} = 0$, the radial velocity is zero, not the angular velocity which is equal to

$$\dot{\theta} = \frac{l}{m r_4^2} = \text{constant}$$

. The particle is therefore moving with constant angular velocity in a circular path.

Conclusion: Depending on the value of energy, the trajectory of the particle is a hyperbola, a parabola, an ellipse, or a circle. These are called conic sections because they can be generated by cutting a cone in various diff ways.



Planetary Motion.

$$\mathcal{U}(r) = -\frac{GMm}{r} = -\frac{c}{r}.$$

$c = GMm.$

$$\cancel{\frac{d\theta}{d\theta}} \quad \frac{d\theta}{dr} = \frac{1}{\mu r^2} \frac{1}{\sqrt{(2/\mu)(E - U_{eff})}}$$

(derived earlier)

$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{c}{r}.$$

$$\theta - \theta_0 = \ell \int \frac{dr}{r(2\mu E r^2 + 2\mu C r - \ell^2)^{1/2}}$$

After Integral

$$\theta - \theta_0 = \sin^{-1}\left(\frac{\mu c r - \ell^2}{r\sqrt{\mu^2 c^2 + 2\mu E \ell^2}}\right)$$

Solving for r gives.

$$r = \frac{\ell^2/\mu c}{1 - \sqrt{1 + (2E\ell^2/\mu c^2)}\sin(\theta - \theta_0)}$$

We introduce parameter

$$\boxed{r_0 = \frac{\ell^2}{\mu c}}$$

$\Rightarrow$ radius of circular orbit corr to given values of ℓ, μ, c.

$$\boxed{\epsilon = \sqrt{1 + \frac{2E\ell^2}{\mu c^2}}}$$

$\Rightarrow$ ϵ = eccentricity $\Rightarrow$ Shape of orbit

Simplified

$$\boxed{r = \frac{r_0}{1 - \epsilon\cos\theta}}$$

Take $\theta_0 = -\pi/2$

← orbit equation. (Try to remember this)

$$E \longrightarrow \epsilon = \sqrt{1 + \frac{2El^2}{\mu c^2}} \longrightarrow r = \frac{r_0}{1 - \epsilon\cos\theta}$$

Energy $\longrightarrow$ Eccentricity $\longrightarrow$ orbit equation.

1) $E > 0 \longrightarrow \epsilon > 1 \longrightarrow$ hyperbola.

2) $E = 0 \longrightarrow \epsilon = 1 \longrightarrow$ Parabola.

3) $-\frac{\mu c^2}{2l^2} \leq E < 0 \longrightarrow 0 \leq \epsilon < 1 \longrightarrow$ Ellipse.

4) $E = E_{min} = -\frac{\mu c^2}{2l^2} \longrightarrow \epsilon = 0 \longrightarrow$ Circle.

* How do you know that they are hyperbola / Parabola / Ellipse.
  — The detailed working of these equations are worked out in page 392 of Kleppner. If interested please see the book.

Elliptical Orbits ($E<0$, $0 \leq \epsilon < 1$)

Often our satellites move in elliptic optics around the earth. They are important & worth looking in detail.

The Orbit Equation

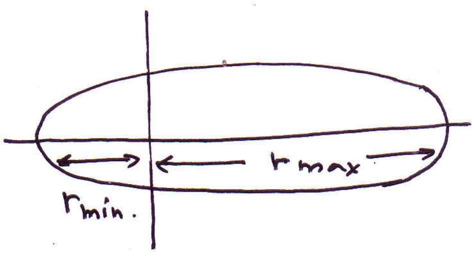
$$\boxed{r = \frac{r_0}{1 - \epsilon \cos\theta}}$$

- The maximum value of r occurs at $\theta = 0$.

$$r_{max} = \frac{r_0}{1-\epsilon}$$

- The minimum value of r occurs at $\theta = \pi$ $\Rightarrow$ $r_{min} = \dfrac{r_0}{1+\epsilon}$.

- The length of the major axis is



$$A = r_{min} + r_{max}$$

$$= r_0 \left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon} \right) = \frac{2r_0}{1-\epsilon^2}$$

- Expressing r_0 & ϵ in terms of E, ℓ, μ, C gives.

$$A = \frac{2r_0}{1-\epsilon^2} = \frac{2\ell^2/(\mu C)}{1-[1+2E\ell^2/(\mu C^2)]} = \frac{C}{-E}$$

$$\boxed{A = \frac{C}{-E}}$$

$C = GMm$.

Total Energy

length of major axis

$$\boxed{\frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon}}$$

$\epsilon \to 1 \Rightarrow$ elongated ellipse

$\epsilon \to 0 \Rightarrow$ ellipse $\to$ Circle.

One Important Problem in Central force on Elliptical orbit – (Solved Example in Kleppner – Page No- 396).

- If perigee (closest approach to the earth) & apogee (farthest dist from earth) is given one can calculate (1) Energy E of satellite
(2) Angular Momentum l
(3) Speed of satellite at perigee & apogee.

A satellite of mass = 2,000 kg is in elliptic orbit about the earth. It is given that at

Perigee ⇒ Its altitude is ~~4100~~ 1100 km. &

apogee ⇒ Its altitude is 4100 km.

(1) What are the satellite's Energy E $\left\{\begin{array}{l}\text{Initial Energy } E_i \text{ before launch}\\ \text{Final Energy in orbit} - E\end{array}\right.$

(2) Angular Momentum.

(3) Speed at perigee & apogee.

⇒ How to go about it? (Major Steps).

(1) Calculate ~~semi~~ length of Major Axis A.

(2) $A = \dfrac{C}{-E}$, Calculate $C = GmM_e = mgR_e^2$

Use this

(3) Calculate E, Since A & C are known.

E_i can also be calculated

(4) Calculate $\epsilon = \dfrac{r_{max} - r_{min}}{r_{max} + r_{min}} = \dfrac{r_{max} - r_{min}}{A}$

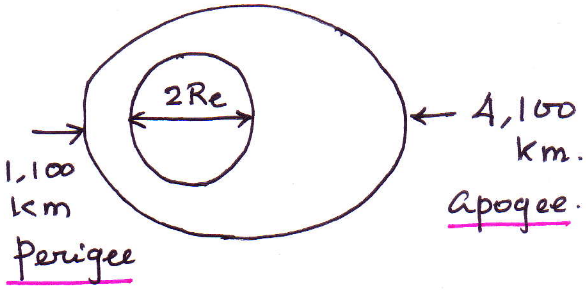
(5) $\epsilon^2 = 1 + \dfrac{2El^2}{mc^2}$ ⇒ calculate l.

(6) Calculate Speed at perigee & apogee.

Detailed Solution:

- Since $m << M_e$, $\mu \approx m$, assume earth is fixed.

- **Major Axis. Calculation.**



$$A = [1{,}100 + 4{,}100 + 2(6{,}400)]\ \text{km}$$
$$= 1.8 \times 10^7\ \text{m}.$$

- **Calculation of E.**

$A = \dfrac{C}{-E}$ OR $E = \dfrac{C}{A}$, C is not known.

- $C = GmM_e = mgR_e^2$ Since $g = GM_e/R_e^2$

  $C = (2\times10^3)(9.8)(6.4\times10^6)^2 = 8.0 \times 10^{17}$ J.m.

So
$$E = -\frac{C}{A} = -4.5\times10^{10}\ \text{J}$$

- Calculation of **Initial Energy E_i**

$$E_i = -\frac{GmM_e}{R_e} = -\frac{C}{R_e} = -12.5\times10^{10}\ \text{J}$$

The energy needed to put the Satellite into Orbit, neglecting losses due to friction is

$$E - E_i = 8\times10^{10}\ \text{J}$$

Calculation of Angular Momentum.

$$r_{min} = \frac{r_0}{1+\epsilon} \quad \text{and} \quad r_{min} = \frac{r_0}{1-\epsilon}.$$

So $\quad (1+\epsilon)\, r_{min} = (1-\epsilon)\, r_{min} \quad$ [Equating r_0]

$$\underline{\underline{\epsilon}} = \frac{r_{max} - r_{min}}{r_{max} + r_{min}} = \frac{r_{max} - r_{min}}{A} = \frac{3 \times 10^3}{1.8 \times 10^4} = \frac{1}{6}$$

From the definition of ϵ.

$$\boxed{\epsilon^2 = 1 + \frac{2 E l^2}{mc^2}}$$ which yields $\underline{\underline{l}} = 1.2 \times 10^{14}$ kg·m²/s.

Calculation of Speed v of the Satellite

$$\boxed{E = \frac{1}{2} m v^2 - \frac{C}{r}}$$

→ At perigee. $r = (1{,}100 + 6{,}400)$ km $= 7.5 \times 10^6$ m
and the speed at perigee is

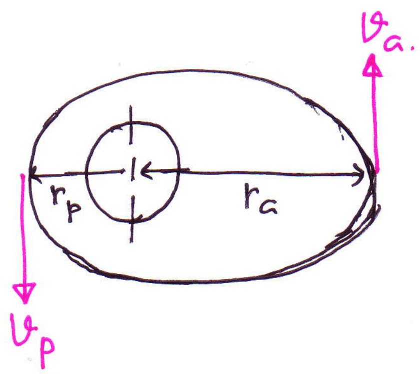
$\underline{\underline{v_p}} = 7{,}900$ m/s.

→ v_a = Speed at apogee

Note that at apogee & perigee the Velocity of the Satellite is purely tangential. Hence by Conservation of angular momentum

$$\mu v_p r_p = \mu v_a r_a$$

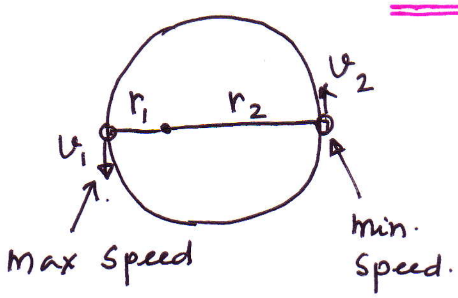
$$\underline{\underline{v_a}} = \frac{v_p r_p}{r_a} = 5{,}600 \text{ m/s}.$$



A particle moves in a circular orbit of radius a. It is attracted by a Central force to a point that is inside the Circle, but is not at the Center of the Circle. The velocity of the particle Varies from a minimum Value v_1 to a maximum Value v_2. Determine the period.

⇒ The angular momentum of the particle is Constant.

$$l = mvr = \text{constant}.$$



$$m v_1 r_1 = m v_2 r_2$$

$$r_1 + r_2 = 2a \qquad [a = \text{radius}]$$

$$l = m v_1 (2a - r_2) = m v_2 r_2$$

$$r_2 = \frac{2 a v_1}{v_1 + v_2}$$

So Angular Momentum Can be Written as.

$$l = m v_2 r_2 = \frac{m\, 2a\, v_1 v_2}{v_1 + v_2}$$

The angular momentum is related to the areal Velocity by

$$\boxed{\frac{l}{2m} = \frac{ds}{dt}} = \text{areal velocity}$$

The period is equal to the area divided by areal velocity

$$\boxed{\tau = \frac{\text{area}}{\text{areal vel}}} = \frac{\pi a^2}{l/2m} = \frac{\pi a^2 (2m)}{\frac{m\, 2a v_1 v_2}{v_1 + v_2}} = \frac{\pi a (v_1 + v_2)}{v_1 v_2}$$

The above Problem is taken from Intermediate ~~Physics~~ Dynamics – Patrick Hamill. [Worked Example 10.3]

Assignment - 4
Q5.

#5

Alternative Forms to the Basic Equations of Motion for a Particle in a Central Force Field. Recall the basic equations of motion as they will be our starting point:

$$m(\ddot{r} - r\dot{\theta}^2) = f(r), \qquad (8)$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \qquad (9)$$

Replace $m \to \mu$.

we derived the following *constant of the motion*:

$$r^2\dot{\theta} = h = \text{constant}. \qquad (10)$$

This constant of the motion will allow you to determine the θ component of motion, provided you know the r component of motion. However, (8) and (9) are coupled (nonlinear) equations for the r and θ components of the motion. How could you solve them without solving for both the r and θ components? This is where alternative forms of the equations of motion are useful.

Let us rewrite (8) in the following form (by dividing through by the mass m):

$$\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{m}. \qquad (11)$$

Now, using (10), (11) can be written entirely in terms of r:

$$\ddot{r} - \frac{h^2}{r^3} = \frac{f(r)}{m}. \qquad (12)$$

We can use (12) to solve for $r(t)$, and the use (10) to solve for $\theta(t)$.

Equation (12) is a *nonlinear* differential equation. There is a useful change of variables, which for certain important central forces, turns the equation into a *linear* differential equation with constant coefficients, and these can always be solved analytically. Here we describe this coordinate transformation.

Let

$$r = \frac{1}{u}.$$

This is part of the coordinate transformation. We will also use θ as a new "time" variable. Coordinate transformation are effected by the chain rule, since this allows us to express derivatives of "old" coordinates in terms of the "new" coordinates. We have:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = \frac{h}{r^2}\frac{dr}{du}\frac{du}{d\theta} = -h\frac{du}{d\theta}, \qquad (13)$$

and

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right) = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = -h^2u^2\frac{d^2u}{d\theta^2}, \qquad (14)$$

where, in both expressions, we have used the relation $r^2\dot{\theta} = h$ at strategic points.

Now

$$r\dot{\theta}^2 = r\frac{h^2}{r^4} = h^2u^3. \qquad (15)$$

Substituting this relation, along with (14) into (8), gives:

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3\right) = f\left(\frac{1}{u}\right),$$

or

$$\frac{d^2u}{d\theta^2} + u = -\frac{f\left(\frac{1}{u}\right)}{mh^2u^2}. \qquad (16)$$

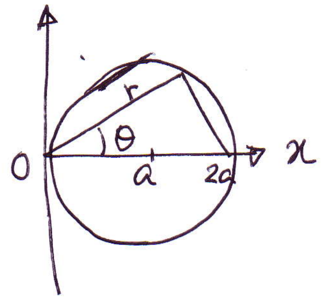
$m \to \mu$.

Now if $f(r) = \frac{K}{r^2}$, where K is some constant, (16) becomes a linear, constant coefficient equation.

Q5) For a particle moving under central force sometime it is easier to solve the problem by introducing the variable $u = \frac{1}{r}$ and the constant $h = r^2\dot{\theta}$ (since magnitude of angular mom is const i.e $\mu r^2\dot{\theta} = \text{constant}$). Under these two variables show the differential eq governing the motion of a particle under central force is

$$\frac{d^2u}{d\theta^2} + u = -f(1/u)/\mu h^2 u^2.$$

(6) In polar coordinates the eqn of a circle of radius a passing through O is



$$r = 2a\cos\theta \quad - (1)$$

Since $$\frac{1}{r} = \frac{\sec\theta}{2a} \quad - (2)$$

$$\therefore \frac{du}{d\theta} = \frac{\sec\theta\tan\theta}{2a} \quad - (3)$$

$$\frac{d^2u}{d\theta^2} = \frac{\sec\theta\sec^2\theta + (\sec\theta\tan\theta)(\tan\theta)}{2a}$$

$$\frac{d^2u}{d\theta^2} = \frac{\sec^3\theta + \sec\theta\tan^2\theta}{2a} \quad - (4)$$

$\therefore$ using the eqn.

$$f\left(\frac{1}{u}\right) = -\mu h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= -\mu h^2u^2\left(\frac{\sec^3\theta + \sec\theta\tan^2\theta + \sec\theta}{2a}\right)$$

$$= -\frac{\mu h^2u^2}{2a}\left\{\sec^3\theta + \sec\theta(1+\tan^2\theta)\right\}$$

$$= -\frac{\mu h^2u^2}{2a}(2\sec^3\theta) = -\frac{\mu h^2u^2}{a}(u.2a)^3$$

Using eq (2)

$$f\left(\frac{1}{u}\right) = -8\mu h^2a^2u^5$$

$$f(r) = -\frac{8mh^2a^2}{r^5}$$