Rigid body Dynamics.

Part-1.

Moment of Inertia Tensor.

This chapter deals with General Rigid body motion when $\bar{L}$ and $\bar{\omega}$ are not parallel to each other. Here they are related by equation

$$\bar{L} = \tilde{I}\,\bar{\omega}$$

Here I is not a scalar like before, when $\bar{L}$ and $\bar{\omega}$ are in same direction, but a tensor known as Moment of Inertia tensor. In case of Principal axis this moment of Inertia tensor reduces only to diagonal form — That means the off diagonal elements are zero. Through different examples we will learn how to calculate the elements of Inertia tensor & then calculate angular momentum.

The Second pt That we need to remember That whenever we are asked to calculate the angular momentum ~~we~~ it is about a point. Either Center of mass or any fixed pt about which the body is rotating. Now about That Origin — we can have two set of axes —

(i) Fixed Coordinate axes.

(ii) A symmetrical axis rotating with the body known as Principal axis.

At one time They Coincide. But the main point is that We are Calculating about a point which will Come out to be same for both. Only the principal axis will make the Calculation easier.

The two major books which I have consulted while making the notes are (1) Kleppner & (2) Engineering Mechanics — Manoj K. Harbola. I have taken materials directly from these books, & I acknowledge them too.

If you find any mistake/error in the notes please notify me so that I can post a corrected one.

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3 october, 2016.

Rigid body Dynamics.

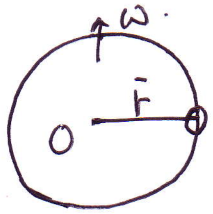
First we have considered the case of a particle rotating in a Circle & calculated its moment of Inertia. Then we studied the case of rigid body whose axis is fixed. Then we have gone through the example of a cylinder rolling down an incline where the axis is moving but the direction of axis is not changing. Here we will deal with general Cass where body is undergoing any kind of motion, the direction of axis is also changing.

In earlier chapters in most of the cass, the angular momentum $\bar{L}$ is $\parallel$ to $\bar{\omega}$ – The case of Symmetric body rotating about symmetric axis. Here we will deal with those Cass where L is not $\parallel$ to ω – We will study about Moment of Inertia tensor by which we can deal with these cases.

Angular Momentum.

We have Calculated Angular momentum for different Cases.

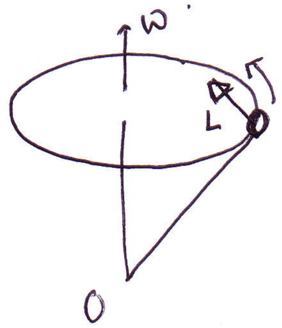
(1) Angular Momentum of a particle rotating about a fixed pt in a circle.



$\bar{L}_O = \bar{r} \times m\bar{v}$ $\qquad$ $\bar{L}_O = I\bar{\omega}$

Here L and ω are ∥ to each other

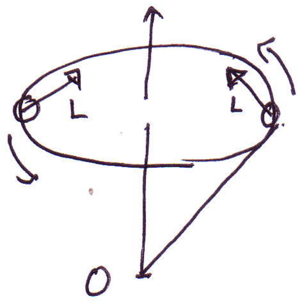
(2)



$L_O \neq I\omega$ $\qquad$ L is not ∥ to ω.

This case is Similar to a non symmetric object.

(3)

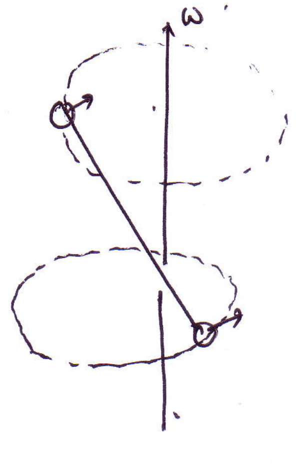


Here we have two particles diametrically opposite to each other. The Component of L which is along xy plane gets Cancelled but along z axis gets added up.

$L = I\omega$ $\qquad$ $\bar{L}$ is ∥ $\bar{\omega}$

This case is similar to Symmetric object rotating about a symmetric axis.

(4)



A case of non Symmetric object

$\bar{L} \neq \bar{\omega}$ not parallel.

We will see now how to Calculate L for this Case — (1) $\bar{r} \times m\bar{v}$

(2) $I\omega_\perp$

(3) Inertia Tensor.

Is θ a vector?

θ is not a vector. We have already studied before.

$$\theta \stackrel{?}{=} \theta_x \hat{i} + \theta_y \hat{j} + \theta_z \hat{k}$$

We can't write this because the order in which we add them affects the results.

$$\theta_x \hat{i} + \theta_y \hat{j} \neq \theta_y \hat{j} + \theta_x \hat{i}.$$

Vector addition should be commutative.

Is ω a vector?

Yes because infinitesimal rotations do commute.

$\bar{\omega} = \lim\limits_{\Delta t \to 0} (\Delta\theta / \Delta t)$ represents a true vector.

Direction of ω is given by right hand rule.

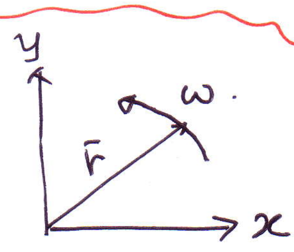
$$\frac{d\bar{r}}{dt} = \bar{v} = \bar{\omega} \times \bar{r}$$

$$(\omega \hat{k}) \qquad (x\hat{i} + y\hat{j}).$$

$$\Downarrow$$

$$(r\cos\theta\, \hat{i} + r\sin\theta\, \hat{j})$$

note this Step.

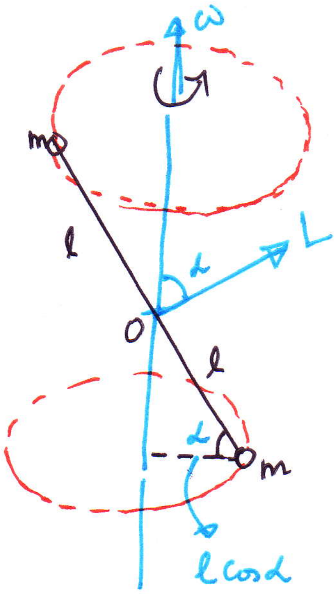


$\bar{v}$ Comes out to be $\omega r \hat{\theta}$. $\Rightarrow$ Try to work out.

$$\Downarrow$$

Example of a body rotating in x-y plane. — Please note how we write vector nature of $\bar{\omega}$, $\bar{r}$, & $\bar{v}$. It will help you solve problems

Rotating Skew Rod : Angular Momentum.



1. Consider a simple Rigid body Consisting of two particles of mass m separated by a massless rod of length $2l$.
2. The midpoint of the rod is attached to a vertical axis which rotates at angular speed ω.
3. The rod is Skewed at angle α as Shown.

There are different methods to Calculate the angular momentum L. Here it is a case of nonsymmetric body. The vector $\bar{L}$ and $\bar{\omega}$ are not parallel to each other.

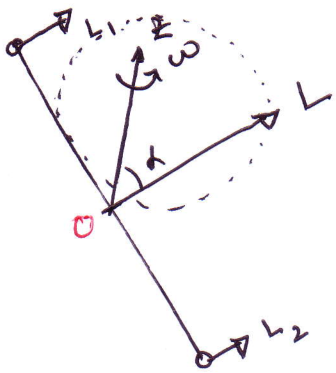
Method - 1.

$$L_0 = \Sigma\, r_i \times (m_i v_i) = \Sigma\, \bar{r}_i \times (m_i \times (\bar{\omega} \times \bar{R}_i))$$

radius of Circle

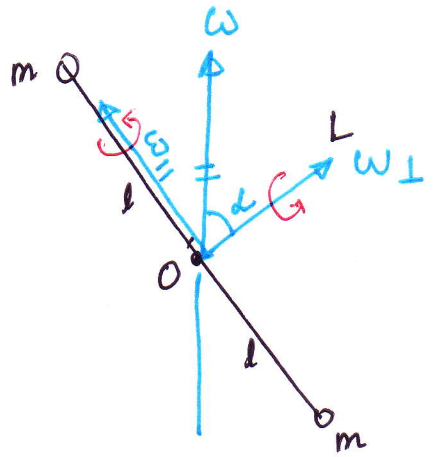
$$|L_0| = 2 m \omega l^2 \cos\alpha$$

L_0 is $\perp$ to the Skew rod and lies in the plane of the rod and the z axis. L turns with the rod & its tip traces out a circle about the z axis.



Method - 2.

Here we resolve ω into two components $\omega_{\parallel}$ which is parallel to the rod and $\omega_{\perp}$ which is perpendicular to the rod. $\omega_{\parallel}$ is perpendicular to the vector $\bar{L}$ and $\omega_{\perp}$ is along $\bar{L}$ vector.



Now.

$\omega_{\parallel}$ does not contribute to L. It is because the direction of $\omega_{\parallel}$ means that the particles are rotating about an axis which is along the rod. Since the rod's angle is fixed, that means that the particles are just spinning to give $\omega_{\parallel}$. Since they are point particles — their r is 0. — So $I = mr^2$ about that axis (along the length of the rod) is 0.

$\omega_{\perp}$ is $\perp$ to the rod. which means that particles are rotating about $\omega_{\perp}$ axis — which means that the particles are rotating with radius $= l$.

$$L = I\omega_{\perp} = 2(ml^2)\,\omega_{\perp} = 2ml^2(\omega\cos\alpha)$$

$$|L_0| = 2ml^2\omega\cos\alpha.$$

Method - 3.

General case when L is not parallel to ω — By moment of Inertia tensor.

Angular Momentum of a Rigid body Rotating with Angular Velocity ω : Moment of Inertia Tensor.

We have seen earlier that we cannot write $\bar{L} = I\bar{\omega}$ as a general case. This is not true.

This is true only for symmetric object rotating about a symmetric axis i.e when $\bar{L} \parallel \bar{\omega}$ only. But there is another way of writing this & that is

$$\boxed{\bar{L} = \tilde{I}\,\bar{\omega}}$$

$\Rightarrow$ General Case even if L is not parallel to ω.

$\bar{L}$ → Vector, $\tilde{I}$ → Tensor, $\bar{\omega}$ → Vector

Tensor of Inertia.

$$\boxed{\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}}$$

We can separately write as.

$$L_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z$$

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Similarly for others.

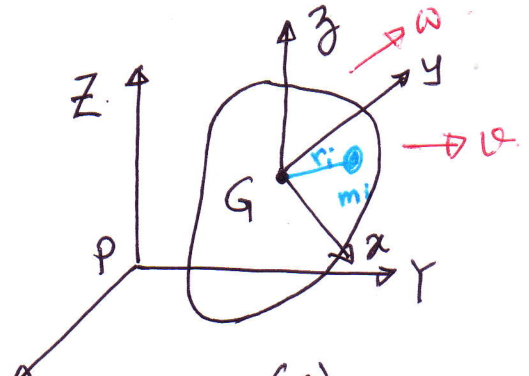
Let us see the mathematical proof of this eqn.

Angular Momentum

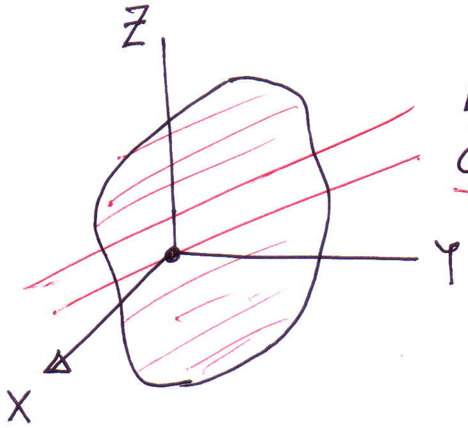
If we want to calculate Angular momentum there can be two cass

(a) A body rotating as well as translating – a general motion.

(b) A body rotating about a fixed point. (a Top)

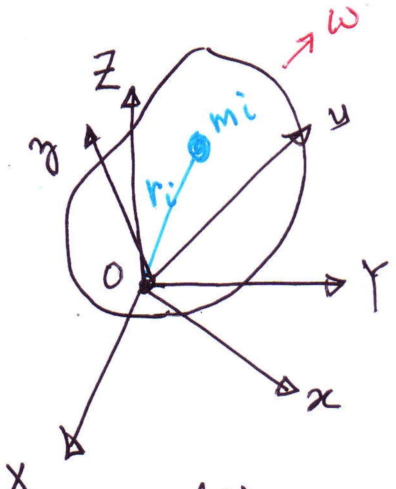


(a)



If we shift the Origin P to G, then

$$\bar{L}_P = \sum \bar{r}_i \times m_i \bar{v}_i$$
$$= \bar{L}_G$$



(b)

(a) Here we fix a coordinate xyz moving with the body whose origin is at center of mass. The XYZ is fixed to the earth.

(b) In this case the body is rotating about a fixed pt O. We fix the xyz frame rotating with the body & the coordinate system XYZ fixed to Lab with center O

$$\bar{L}_0 = \sum \bar{r}_i \times (m_i \times \bar{v}_i)$$

So in both cases we need to calculate the above exp.

Proof :

Let us derive the relationship between the angular momentum of a body rotating in space with one point fixed, & its angular velocity

$$\bar{L} = \sum m_i \bar{r}_i \times \bar{v}_i = \sum m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i)$$

For a body rotating with one point fixed, it is true.

Now.

$$v_i = \bar{\omega} \times \bar{r}_i = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x_i & y_i & z_i \end{vmatrix} = \hat{i}\,(\omega_y z_i - \omega_z y_i) + \hat{j}\,(\omega_z x_i - \omega_x z_i) + \hat{k}\,(\omega_x y_i - \omega_y x_i)$$

$$\bar{L} = \sum m_i \underbrace{(x_i \hat{i} + y_i \hat{j} + z_i \hat{k})}_{r_i} \times \Big\{ \overbrace{(\omega_y z_i - \omega_z y_i)\,\hat{i} + (\omega_z x_i - \omega_x z_i)\,\hat{j} + (\omega_x y_i - \omega_y x_i)\,\hat{k}}^{v_i} \Big\}$$

After multiplying & rearranging terms.

$$\bar{L} = (\;L_x\;)\,\hat{i} + (\;L_y\;)\,\hat{j} + (L_z)\,\hat{k}$$

$$L_x = \underbrace{\sum m_i (y_i^2 + z_i^2)}_{I_{xx}}\,\omega_x \underbrace{-\Big(\sum m_i x_i y_i\Big)}_{I_{xy}}\omega_y \underbrace{-\Big(\sum m_i x_i z_i\Big)}_{I_{xz}}\omega_z$$

$$L_x = I_{xx}\,\omega_x + I_{xy}\,\omega_y + I_{xz}\,\omega_z$$

$$L_y = \underbrace{-\Big(\sum m_i y_i x_i\Big)}_{I_{yx}}\omega_x + \Big(\underbrace{\sum m_i (z_i^2 + x_i^2)}_{I_{yy}}\,\omega_y\Big) \underbrace{-\Big(\sum m_i y_i z_i\Big)}_{I_{yz}}\omega_z$$

$$= I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z$$

$$L_z = -\underbrace{\left(\sum m_i z_i x_i\right)}_{I_{zx}} \omega_x - \underbrace{\left(\sum m_i z_i y_i\right)}_{I_{zy}} \omega_y + \overbrace{\sum m_i (x_i^2 + y_i^2)}^{I_{zz}} \omega_z$$

$$= I_{zx}\,\omega_x + I_{zy}\,\omega_y + I_{zz}\,\omega_z$$

This is usually written in the matrix form.

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Commonly known as
Inertia Tensor / Moment of Inertia tensor.

Diagonal Terms

$I_{xx} = \sum m_i (y_i^2 + z_i^2)$

$I_{yy} = \sum m_i (z_i^2 + x_i^2)$

$I_{zz} = \sum m_i (x_i^2 + y_i^2)$

These diagonal terms are the moment of Inertia about $x, y, \& z$ axes respectively.
As you see for I_{xx} only y & z coordinate is present because it is around x axis & similarly for the other cases.

What are the off diagonal terms ?

These off diagonal terms are present in case of objects which are not symmetric wrt to the axis chosen. [For example I_{xy} may tell me how much an object will be accelerated around the y axis when I apply torque around x axis] – It depends upon the choice of axis – In principal axis all the off diagonal terms will be zero].

The off - diagonal terms are

$$I_{xy} = I_{yx} = -\sum m_i x_i y_i$$

$$I_{xz} = I_{zx} = -\sum m_i x_i z_i$$

$$I_{yz} = I_{zy} = -\sum m_i y_i z_i$$

~~are~~ known as products of Inertia. The Values of moments & products of Inertia depend on the set of axes chosen.

Now the problem of Skew Rod which we solved earlier will be solved using Inertia tensor. We will choose / solve the problem using two diff set of axes. In one set of axes off-diagonal terms are zero. They are known as Principal axes.

Principal Axes.

Here we have to be clear about ~~two~~ few things.

(a) We can choose a set of axes which is symmetric with the body & moving with the body. They are known as Principal axes.

(b) There is a set of axes which is fixed to the earth / lab known as Coordinate axes.

(c) We can reorient our Cartesian Coordinate axes so that they coincide with the principal axes — even instantaneously for a rotating body.

In this case the tensor of Inertia takes a simple diagonal form wrt to Coordinate System.

$$\tilde{\underline{\underline{I}}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

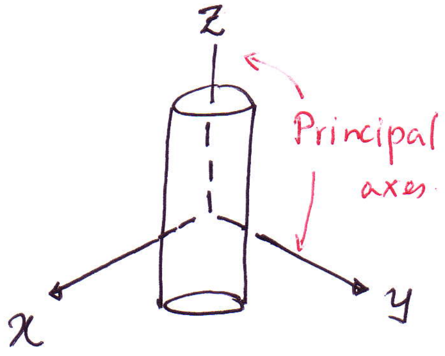
(d) If the body rotats about one of the principal axes then L is $\parallel$ to ω

(e) For a body of any shape & mass distribution it is always possible to find a set of Three orthogonal axes such that the products of inertia vanish i.e principal axis.

(f)

- Uniform Sphere : Any perpendicular axes through the center are principal axes.

- Solid Cylinder : The axis of revolution is principal axis. The other two principal axis are mutually perpendicular & lie in a plane through the center of mass perpendicular to the axis of revolution.



Kinetic Energy of a Rotating Rigid body.

In case of fixed axis rotation where $\bar{L}$ & $\bar{\omega}$ are in the same direction. We know.

$$KE = \frac{1}{2} I \omega^2$$

In terms of angular momentum

$$KE = \frac{L^2}{2I}$$

Work Energy Theorem is expressed as

$$\Delta KE = \int \tau \, d\theta$$

If L is not parallel to ω then. KE will have little ~~general~~ different expression & it cannot be written as $\frac{1}{2} I \omega^2$.

$$T = \frac{1}{2} \sum m_i \, v_i \cdot v_i$$

Now $\bar{v}_i = \bar{\omega} \times \bar{r}_i$, so

$$T = \frac{1}{2} \sum m_i \bar{v}_i \cdot (\bar{\omega} \times \bar{r}_i)$$

$$= \frac{1}{2} \sum m_i \, r_i \cdot (\bar{v}_i \times \omega)$$

$$= \frac{1}{2} \sum m_i \, \bar{\omega} \cdot (\bar{r}_i \times \bar{v}_i)$$

$$\boxed{KE = \frac{1}{2} \bar{\omega} \cdot \bar{L}}$$

In terms of Components in Principal axis

$$\boxed{KE = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2}$$

$I_1, I_2, I_3 \Rightarrow$ moments of Inertia about Pr axis

$\omega_1, \omega_2, \omega_3 \Rightarrow$ Components of ω along same Pr axis.

Tensor of Inertia for a Rotating Skew Rod.

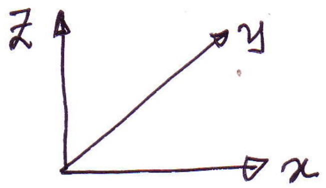
Let us Calculate the moment of Inertia of a skew Rod rotating about the Z axis by two different axes system.

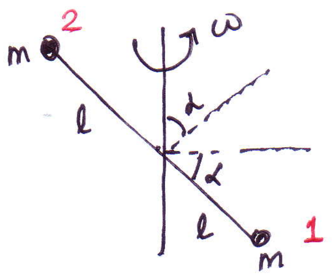
Coordinate Axes Fixed to the earth

Here we will see That Coordinats are function of time.

Principal axes which is fixed to the body & Coordinate axes is aligned with principal axes instantaneously.

Solution: Coordinate Axes.





At $t=0$ the skew rod it lies in $x-z$ plane. At time t

Particle - 1

$x_1 = R\cos\omega t$

$y_1 = R\sin\omega t$

$z_1 = -h.$

Particle - 2

$x_2 = -R\cos\omega t$

$y_2 = -R\sin\omega t$

$z_2 = h.$

Here $R = l\cos\alpha$ and $h = l\sin\alpha$. Now.

$$I_{xx} = m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2)$$

$$= 2m(R^2\sin^2\omega t + h^2)$$

Similarly.

$$I_{yy} = R^2\cos^2\omega t + h^2 \quad ; \quad I_{zz} = R^2$$

Off diagonal terms.

$$I_{xy} = I_{yx}$$
$$= -(m x_1 y_1 + m x_2 y_2)$$
$$= -(m R^2 \cos\omega t \sin\omega t + m R^2 \cos\omega t \sin\omega t)$$
$$= -2mR^2 \cos\omega t \sin\omega t.$$

Similarly other diagonal terms are calculated.

The above steps gives a feel to calculate off diagonal terms.

Finally

$$I = 2m \begin{pmatrix} R^2 \sin^2\omega t + h^2 & -R^2 \sin\omega t \cos\omega t & Rh \cos\omega t \\ -R^2 \sin\omega t \cos\omega t & R^2 \cos^2\omega t + h^2 & Rh \sin\omega t \\ Rh \cos\omega t & Rh \sin\omega t & R^2 \end{pmatrix}$$

The common factor 2m multiplies each term.

Now $\bar{L} = \tilde{I}\bar{\omega}$. So.

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \overset{\tilde{I}}{\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

So

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 2m\omega Rh \cos\omega t$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z = 2m\omega Rh \sin\omega t$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z = 2mR^2\omega$$
$$\bar{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}.$$

One can find out the Principal axis by solving eigen value eqn $\tilde{I}\omega = \lambda\omega$ — not in our syllabus.

One Can Calculate applied torque by diff.

$$\tau_x = -2mRh\,\omega^2 \sin\omega t$$

$$\tau_y = 2mRh\,\omega^2 \cos\omega t$$

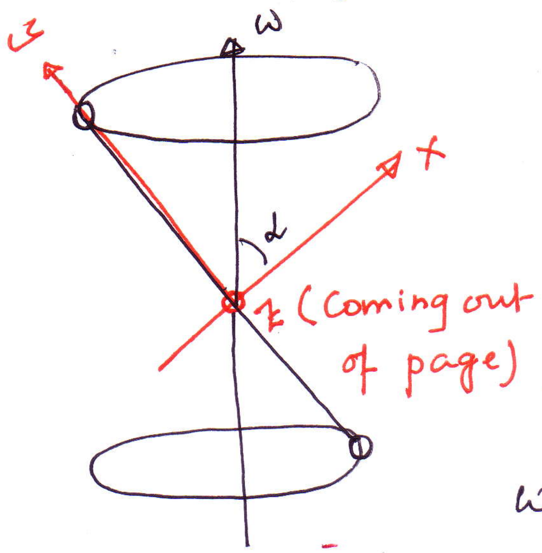
$$\tau_z = 0.$$

The results are identical to that before.

Now the above method is very lengthy & Cumbersome. Let us solve it by principal axis method which will become quite simple.

Method-2 — Principal Axis Method. — Simpler.

Find the axis of Symmetry. — If we choose one principal axis along the skew rod joining two particles then one principal axes is defined. The other two axis can be perpendicular to this axis. See the figure below.



Principal Axis — $\Rightarrow$ Here all the product terms $= 0$.

$$I_{xx} = \sum m_i (y^2 + z^2) = ml^2 + ml^2 = 2ml^2$$

$$I_{yy} = \sum m_i (x^2 + z^2) = 0$$

$$I_{zz} = \sum m_i (x^2 + y^2) = 2ml^2$$

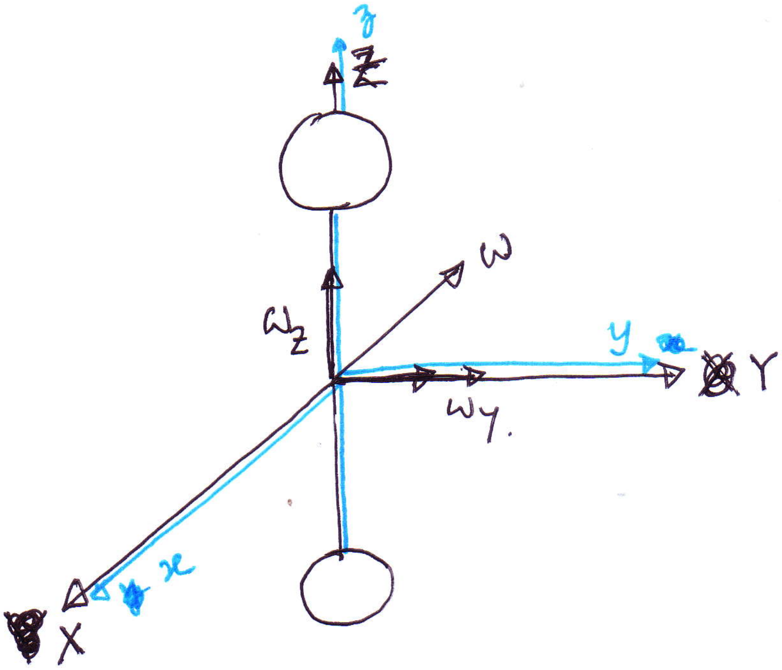
$$\bar{L} = I_{xx}\omega_x \hat{i} + I_{yy}\omega_y \hat{j} + I_{zz}\omega_z \hat{k}$$

$$\omega_x = \omega\cos\alpha, \quad \omega_y = \omega\sin\alpha, \quad \omega_z = 0$$

$$\bar{L} = 2ml^2\omega\cos\alpha\,\hat{i}$$ (Same as before)

Rotating Dumbbell.

A dumbbell made up of two spheres of radius b & mass M separated by a thin rod. The distance between the center is $2l$. The body is rotating about some axis through its center of mass. At a certain instant the rod coincides with the z axis, and ω lies in the yz plane, $\omega = \omega_y \hat{j} + \omega_z \hat{k}$. Find L.



Here I look for a axis which is symmetric to the body. I choose the blue z axis which is symmetric to the body. The other two symmetric axes are orthogonal to z that is x & y.

So xyz is symmetric to the body shown in blue color which coincides instantaneously with the fixed coordinate system XYZ. So the xyz frame is the principal axes of the body which is rotating with the body whose center is fixed at the middle of the rod which is center of mass of the system.

In principal axes the products terms are zero i.e
$I_{xy} = -\sum m_i x_i y_i = 0$ because for every mass m there

is another mass diametrically opposite because of which it is 0.

$$I_{\cancel{xx}\,zz} = 2\left(\frac{2}{5}Mb^2\right) = \frac{4}{5}Mb^2$$

$$I_{yy} = 2\left(\frac{2}{5}Mb^2 + Ml^2\right) = \frac{4}{5}Mb^2 + 2Ml^2$$

$$I_{xx} = I_{yy} = \frac{4}{5}Mb^2 + 2Ml^2$$

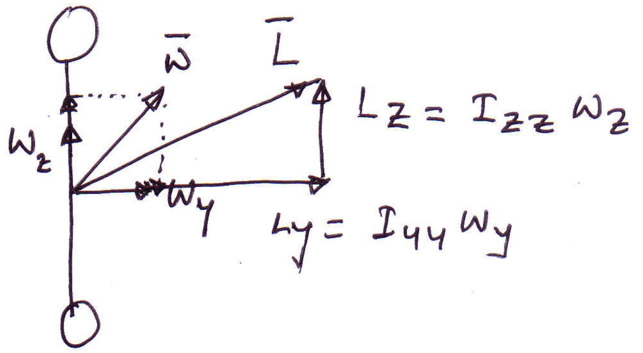
NOW $\bar{\omega} = \omega_y \hat{j} + \omega_z \hat{k}$

$$L_x = I_{xx}\omega_x = 0$$

$$L_y = I_{yy}\omega_y = \left(\frac{4}{5}Mb^2 + 2Ml^2\right)\omega_y.$$

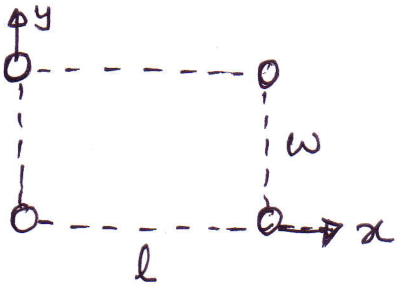
$$L_z = I_{zz}\omega_z = \frac{4}{5}Mb^2\omega_z$$

$$\bar{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$



In this case the fixed axis XYZ at that instant is acting as symmetry axis → principal axis, so choosing xyz axis is not necessary.

Four point particles of equal mass m are kept at the corners of a rectangle of length l and width w in the x-y plane as shown.



(i) Find the moment and products of inertia of the masses.

(ii) Find the principal axis of the system with the same Origin.

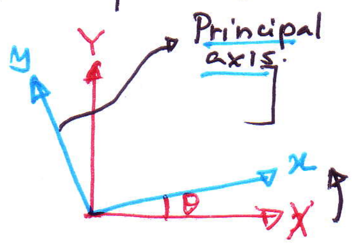
(Ans (i) $I_{xx} = 2mw^2$, $I_{yy} = 2ml^2$, $I_{zz} = 2m(l^2+w^2)$

$I_{xy} = I_{yx} = -mlw$; $I_{yz} = I_{zy} = 0$; $I_{xz} = 0$

(ii) The principal x axis is at an angle θ from the original x axis such that $\tan 2\theta = \dfrac{lw}{l^2-w^2}$

[Hint: One can get this value by applying the formula

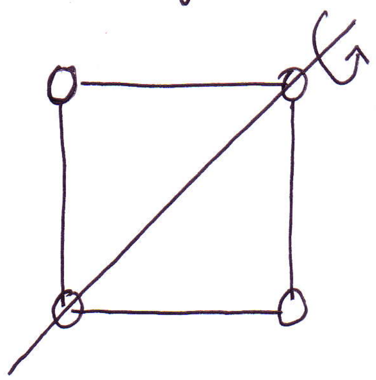
* $$\boxed{\tan 2\theta = \frac{2I_{xy}}{I_{yy}-I_{xx}}} = \frac{lw}{l^2-w^2}$$



You need to go through some advanced book of Mechanics to see the proof of this formula. For example you can go through the book ~~by~~ Engineering Mechanics by Manoj K. Harbola – page NO – 132 – 133. (Cengage Learning)

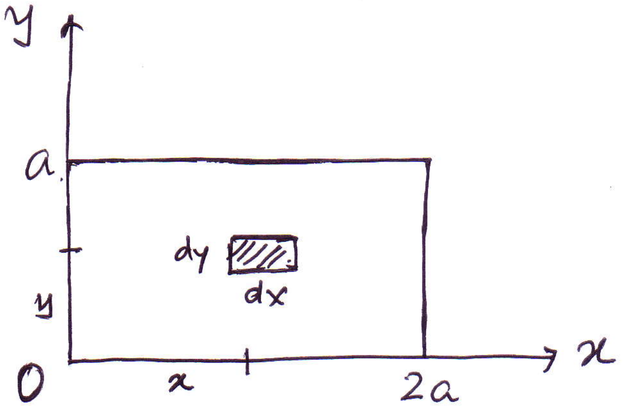
* Please note I_{yy}, I_{xx}, I_{xy} is not of principal axis. but of the axis chosen here. θ is the angle by which this chosen axis is to be rotated to get Pr axis.

Four masses each of mass m, are connected by massless rods to form a rectangle of length a & width b. This rectangle is rotated about one of its diagonals at an angular speed ω. Find the rate of change (both magnitude & direction) of the angular momentum about the CM, when the rectangle is in the plane of the paper.



(Ans: $m\omega^2 ab\left(\dfrac{a^2-b^2}{a^2+b^2}\right)$ into the page).

Find the moment of Inertia tensor I about the Origin of Coordinats O,



Consider a Square mass element of area $dxdy$ at the location $(x, y, 0)$.

$$I_{xx} = \sum m_i (y_i^2 + z_i^2) = \sum m_i y_i^2 = \sigma \int_0^a dx \int_0^a y^2 dy$$

$$\sigma = \frac{m}{2a^2} \qquad = \frac{m}{2a^2} (a) \frac{a^3}{3} = \frac{1}{3} ma^2$$

$$I_{yy} = \sum m_i (x_i^2 + z_i^2) = \sum m_i x_i^2 = \sigma \int_0^{2a} x^2 dx \int_0^a dy$$

$$= \frac{4}{3} ma^2$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2) = I_{xx} + I_{yy} = \frac{5}{3} ma^2$$

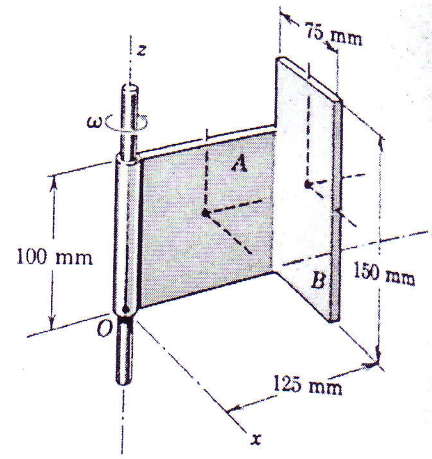
$$I_{xy} = -\sum m_i x_i y_i = \sigma \int_0^{2a} x\, dx \int_0^a y\, dy = -\frac{1}{2} ma^2$$

$$I_{xz} = -\sum m_i x_i z_i = 0 \quad , \quad I_{yz} = 0$$

$$\tilde{I} = \begin{pmatrix} 1/3\, ma^2 & -\frac{1}{2} ma^2 & 0 \\ -1/2\, ma^2 & 4/3\, ma^2 & 0 \\ 0 & 0 & 5/3\, ma^2 \end{pmatrix}$$

Sample Problem 8/6

The bent plate has a mass of 70 kg per square meter of surface area and revolves about the z-axis at the rate $\omega = 30$ rad/s. Determine (a) the angular momentum $\mathbf{H}$ of the plate about point O and (b) the kinetic energy T of the plate. Neglect the mass of the hub at the rotation axis and the thickness of the plate compared with its surface dimensions.



Solution. The moments and products of inertia are written with the aid of Eqs. A3 and A9 in Appendix A by transfer from the parallel ① centroidal axes for each part.

First, the mass of each part is

$$m_A = (0.100)(0.125)(70) = 0.875 \text{ kg}, \quad m_B = (0.075)(0.150)(70) = 0.788 \text{ kg}$$

Part A

$$[I_{xx} = \bar{I}_{xx} + md^2] \quad I_{xx} = \frac{0.875}{12}[(0.100)^2 + (0.125)^2] + 0.875[(0.050)^2 + (0.0625)^2] = 0.007\,47 \text{ kg}\cdot\text{m}^2$$

$$[I_{yy} = \tfrac{1}{3}ml^2] \quad I_{yy} = \frac{0.875}{3}(0.100)^2 = 0.002\,92 \text{ kg}\cdot\text{m}^2$$

$$[I_{zz} = \tfrac{1}{3}ml^2] \quad I_{zz} = \frac{0.875}{3}(0.125)^2 = 0.004\,56 \text{ kg}\cdot\text{m}^2$$

$$\left[I_{xy} = \int xy\,dm,\ I_{xz} = \int xz\,dm\right] \quad I_{xy} = 0, \quad I_{xz} = 0$$

$$[I_{yz} = \bar{I}_{yz} + md_y d_z] \quad I_{yz} = 0 + 0.875(0.0625)(0.050) = 0.002\,73 \text{ kg}\cdot\text{m}^2$$

Part B

$$[I_{xx} = \bar{I}_{xx} + md^2] \quad I_{xx} = \frac{0.788}{12}(0.150)^2 + 0.788[(0.125)^2 + (0.075)^2] = 0.018\,21 \text{ kg}\cdot\text{m}^2$$

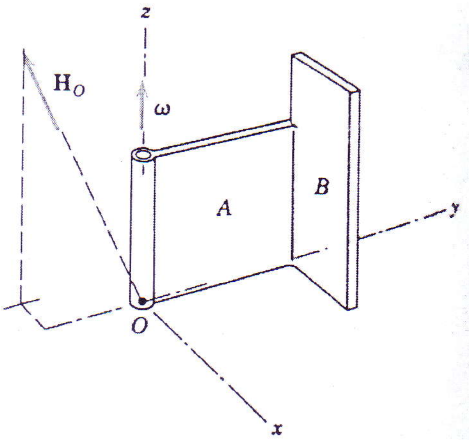
$$[I_{yy} = \bar{I}_{yy} + md^2] \quad I_{yy} = \frac{0.788}{12}[(0.075)^2 + (0.150)^2] + 0.788[(0.0375)^2 + (0.075)^2] = 0.007\,38 \text{ kg}\cdot\text{m}^2$$

$$[I_{zz} = \bar{I}_{zz} + md^2] \quad I_{zz} = \frac{0.788}{12}(0.075)^2 + 0.788[(0.125)^2 + (0.0375)^2] = 0.013\,78 \text{ kg}\cdot\text{m}^2$$

$$[I_{xy} = \bar{I}_{xy} + md_x d_y] \quad I_{xy} = 0 + 0.788(0.0375)(0.125) = 0.003\,69 \text{ kg}\cdot\text{m}^2$$

$$[I_{xz} = \bar{I}_{xz} + md_x d_z] \quad I_{xz} = 0 + 0.788(0.0375)(0.075) = 0.002\,21 \text{ kg}\cdot\text{m}^2$$

$$[I_{yz} = \bar{I}_{yz} + md_y d_z] \quad I_{yz} = 0 + 0.788(0.125)(0.075) = 0.007\,38 \text{ kg}\cdot\text{m}^2$$



The sum of the respective inertia terms gives for the two plates together

$$I_{xx} = 0.0257 \text{ kg}\cdot\text{m}^2 \qquad I_{xy} = 0.003\,69 \text{ kg}\cdot\text{m}^2$$
$$I_{yy} = 0.010\,30 \text{ kg}\cdot\text{m}^2 \qquad I_{xz} = 0.002\,21 \text{ kg}\cdot\text{m}^2$$
$$I_{zz} = 0.018\,34 \text{ kg}\cdot\text{m}^2 \qquad I_{yz} = 0.010\,12 \text{ kg}\cdot\text{m}^2$$

(a) The angular momentum of the body is given by Eq. 8/11 where $\omega_z = 30$ rad/s and ω_x and ω_y are zero. Thus

② $L_O =$ $\mathbf{H}_O = 30(-0.002\,21\mathbf{i} - 0.010\,12\mathbf{j} + 0.018\,34\mathbf{k})$ N · m · s *Ans.*

(b) The kinetic energy from Eq. 8/18 becomes

$$T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_O = \tfrac{1}{2}(30\mathbf{k})\cdot 30(-0.002\,21\mathbf{i} - 0.010\,12\mathbf{j} + 0.018\,34\,\mathbf{k}) = 8.25 \text{ J}$$

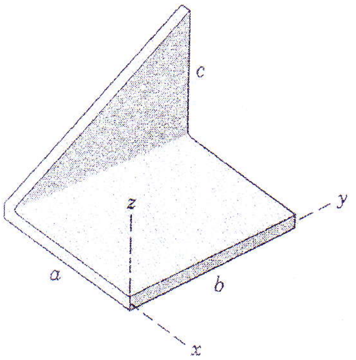
① The parallel-axis theorems for transferring moments and products of inertia from centroidal axes to parallel axes are explained in Appendix A and are most useful relations.

Recall that the units of angular momentum may also be written in the base units as kg · m²/s.

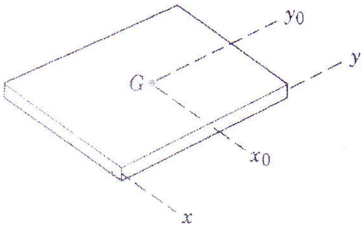
You need not a understand the Solution fully. But try to Understand how if you are asked to calculate the angular momentum of a body of Complicated Shape — How to do it.

Sample Problem A/4

The bent plate has a uniform thickness t which is negligible compared with its other dimensions. The density of the plate material is ρ. Determine the products of inertia of the plate with respect to the axes as chosen.



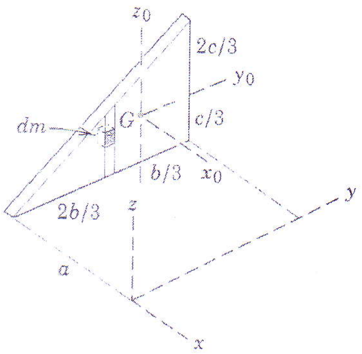
Solution. Each of the two parts is analyzed separately.



Rectangular part. In the separate view of this part we introduce parallel axes x_0-y_0 through the mass center G and use the transfer-of-axis theorem. By symmetry we see that $\bar{I}_{xy} = I_{x_0y_0} = 0$ so that

$$[I_{xy} = \bar{I}_{xy} + md_xd_y] \qquad I_{xy} = 0 + \rho tab\left(-\frac{a}{2}\right)\left(\frac{b}{2}\right) = -\frac{1}{4}\rho ta^2b^2$$

Since the z-coordinate of all elements of the plate is zero, it follows that $I_{xz} = I_{yz} = 0$.



Triangular part. In the separate view of this part we locate the mass center G and construct x_0-, y_0-, and z_0-axes through G. Since the x_0-coordinate of all elements is zero, it follows that $\bar{I}_{xy} = I_{x_0y_0} = 0$ and $\bar{I}_{xz} = I_{x_0z_0} = 0$. The transfer-of-axis theorems then give us

$$[I_{xy} = \bar{I}_{xy} + md_xd_y] \qquad I_{xy} = 0 + \rho t\frac{b}{2}c(-a)\left(\frac{2b}{3}\right) = -\frac{1}{3}\rho tab^2c$$

$$[I_{xz} = \bar{I}_{xz} + md_xd_z] \qquad I_{xz} = 0 + \rho t\frac{b}{2}c(-a)\left(\frac{c}{3}\right) = -\frac{1}{6}\rho tabc^2$$

We obtain I_{yz} by direct integration, noting that the distance a of the plane of the triangle from the y-z plane in no way affects the y- and z-coordinates. With the mass element $dm = \rho t\,dy\,dz$ we have

$$\left[I_{yz} = \int yz\,dm\right] \qquad I_{yz} = \rho t\int_0^b\int_0^{cy/b} yz\,dz\,dy = \rho t\int_0^b y\left[\frac{z^2}{2}\right]_0^{cy/b} dy$$

$$= \frac{\rho tc^2}{2b^2}\int_0^b y^3\,dy = \frac{1}{8}\rho tb^2c^2$$

Adding the expressions for the two parts gives

$$I_{xy} = -\tfrac{1}{4}\rho ta^2b^2 - \tfrac{1}{3}\rho tab^2c = -\tfrac{1}{12}\rho tab^2(3a + 4c) \qquad \textit{Ans.}$$

$$I_{xz} = \quad 0 \quad -\tfrac{1}{6}\rho tabc^2 = -\tfrac{1}{6}\rho tabc^2 \qquad \textit{Ans.}$$

$$I_{yz} = \quad 0 \quad +\tfrac{1}{8}\rho tb^2c^2 = +\tfrac{1}{8}\rho tb^2c^2 \qquad \textit{Ans.}$$

① We must be careful to preserve the same sense of the coordinates. Thus plus x_0 and y_0 must agree with plus x and y.

② We choose to integrate with respect to z first, where the upper limit is the variable height $z = cy/b$. If we were to integrate first with respect to y, the limits of the first integral would be from the variable $y = bz/c$ to b.