Rigid body Dynamics.

Part-2.

Gyroscopes.

This chapter deals with Gyroscopes & gyro-scopic principles. We can explain many phenomena using gyroscopic principle.

Here we have a heavy disk which when rotats with Large spin gives rise to Large angular momentum. Now when a small torque is added that gives rise to small change in angular spin momentum because of which there is a interesting phenomena of precession — that is the angular momentum vector rotats. One has to understand the precession, the direction of precession & how to relate ω_s — spin angular velocity with ω_p — the precession angular velocity. This has large number of applications.

I have mainly consulted Kleppner & Young & Freedman while making these nots. My acknowledgment to them.

Anwg.
(AMIT NEOGI)
3 October, 2016

Gyroscopes & Precession.

Some familiar questions.

- What are Gyroscopes.
- Why should we study them.
- What is precession
- Spin angular frequency, precessional ang freq etc.

A Gyroscope is a spinning wheel or disc in which the axis of rotation is free to assume any orientation by itself. When rotating, the orientation of this axis is unaffected by tilting or rotation of the mounting, according to the conservation of angular momentum. Because of this, gyroscopes are useful for measuring or maintaining orientation.

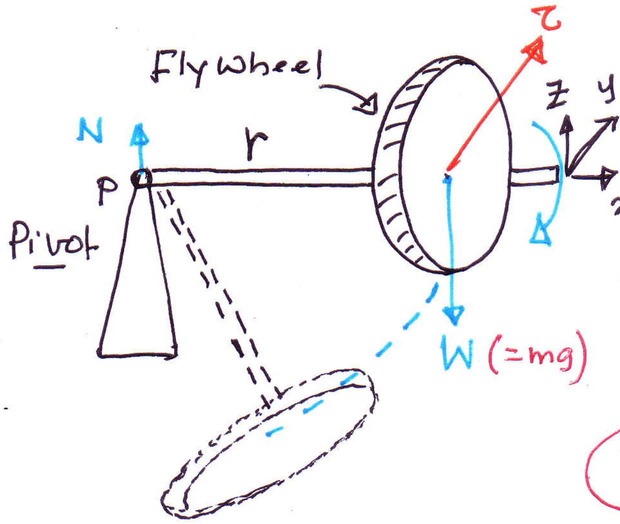
There are different type of gyroscopes with diff operating principles such as electronic, microchip packaged Mems gyroscopes found in consumer electronics devices, Solid state ring lasers, fiber optic gyroscopes, & extremely sensitive Quantum gyroscope.

So there are large number of applications which include inertial navigation systems, stabilization of flying vehicles like radio-controlled helicopters or Unmanned aerial vehicles, recreational boats and commercial ships, Gyro theodolits — maintain direction in tunnel mining, construct Gyrocompasses etc.

Why should we Study Them.

We will Study the principle behind Simple Gyroscope. It will help us to explain many phenomena including the principle of balance in bicycle. If we keep bicycle by itself without a stand it falls but when we ride that it does not fall. We say it is balance. What is the physics behind this? It can be explained by gyroscopic principle. We will see here what is gyroscopic principle and how it can be used to explain many everyday phenomena & applications.

Simple Gyroscope: A Simple gyroscope can be thought of a flywheel connected by a rod to a pivot.



What will happen at rest?

Net Force → $N - W = 0$ $N = W$

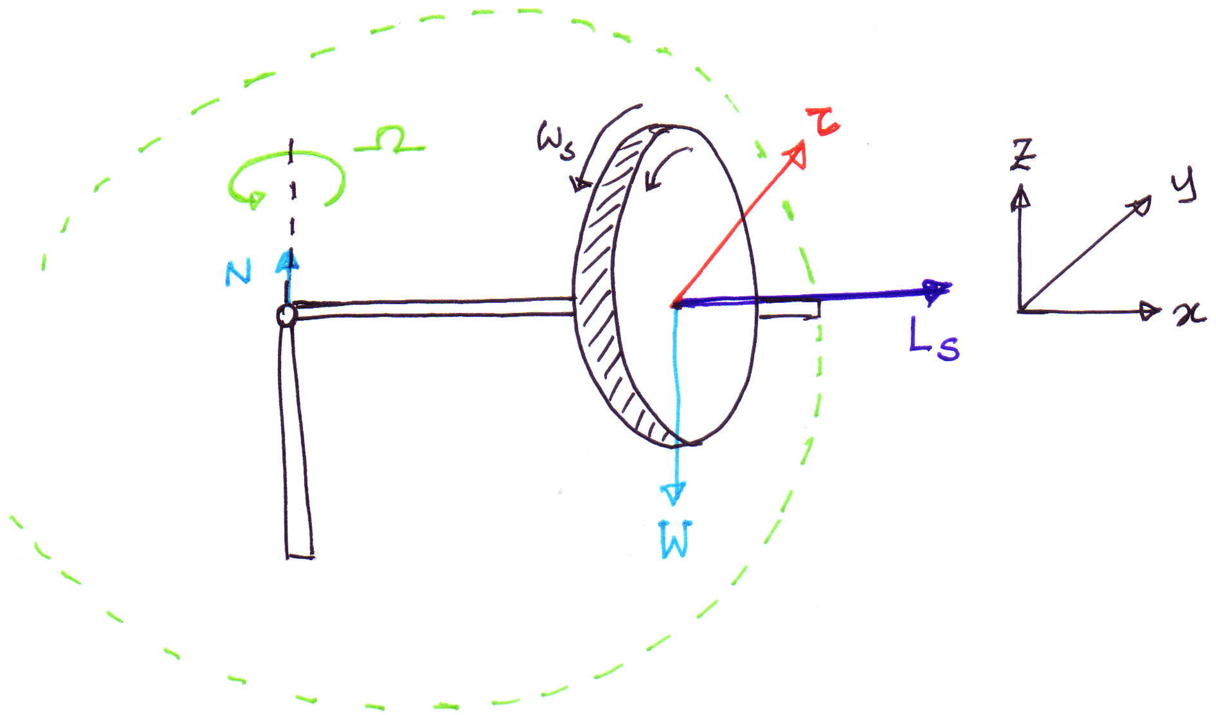
Net Force $= 0$.

Net Torque about P → $\bar{r} \times \bar{W} = \tau_p \hat{j}$ (along y)

Because of this net torque the flywheel will swing like a pendulum & fall down. That is very natural & it should happen like that. (Shown by dotted line)

V. Imp. What will happen when the fly wheel is in motion

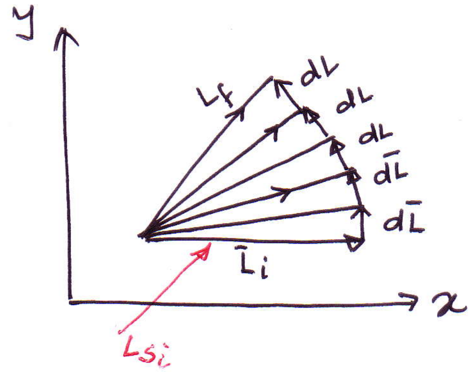
- Suppose it is spinning anticlockwise as shown with some angular frequency ω_s.



Surprisingly when the flywheel is spinning the situation is completely different. It does not fall but the fly wheel starts rotating in a circle - which is known by the name Precession with angular freq/vel. Ω. Why does it happen? This explains many phenomena. Let us see the physics behind it.

Because of this large fly wheel which is spinning it will give rise to a large angular momentum L_s which was not there before. The torque which was there before will give rise to small change in angular momentum given by $\left(\frac{dL}{dt} = \tau, \; dL = \tau dt\right)$

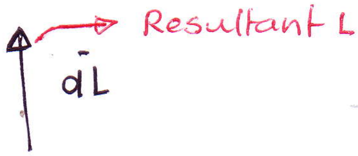
This $d\bar{L}$ vector is in a direction $\perp$ to the $\bar{L}_s$ vector which will constantly change the direction of L_s but not its magnitude. The situation is like this.



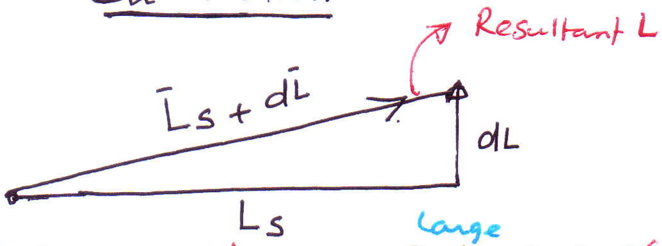
$d\bar{L}$ vector is very small compared to $\bar{L}_s$ vector & acts in a direction $\perp$ to it. When $d\bar{L}$ vector is added to $\bar{L}_s$ vector it starts rotating the L_s vector. At every step the $d\bar{L}$ vector is $\perp$ to $\bar{L}_s$ vector & the flywheel makes a regular rotation with angular freq Ω. about z axis.

Large L_s vector in the second case of flywheel rotating with ω_s makes all the difference. So

At Rest



In Motion



See how the direction of resultant L changes because of large L_s in 2nd case.

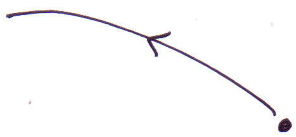
We can understand the situation better if we look at

Linear motion:

Particle at rest & Force is applied.

Particle will move towards the force.

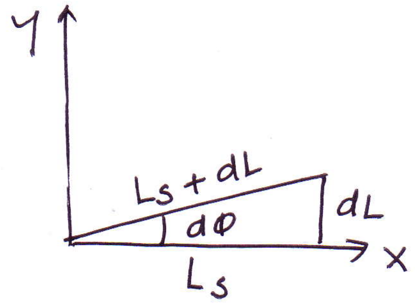
Particle in Motion & same Force is applied.



Here it will be decided by change of momentum by Force & initial momentum. Think of Centripetal Force in Circular Motion.

Relation between Precessional Ω and Spin ω_s.

(also written as ω_p)



For small $d\phi$ we can write

$$d\phi = \frac{dL}{L_s} \qquad \left[\tan\theta \simeq \sin\theta \simeq \theta \text{ for small } \theta\right]$$

$$\omega_p = \Omega$$

Some books write ω_p, Kleppner writes Ω for the precessional angular velocity. Let us write the ω_p notation

ω_p = ω for precession which is along z axis.

ω_s = ω of the spinning flywheel along x axis.

$$\omega_p = \frac{d\phi}{dt} = \frac{(dL)/L_s}{dt} = \left(\frac{dL}{dt}\right)\cdot\frac{1}{L_s} = \frac{\tau}{L_s} = \frac{rMg}{I\omega_s}$$

$$\boxed{\omega_p = \frac{\tau}{L_s} = \frac{rMg}{I\omega_s}}$$

If you increase ω_s, ω_p will decrease. If you increase M, ω_p will increase.

Now with the above knowledge, can you explain the physics of balance when you are riding a bicycle. Why doesn't it fall? What happens in slow cycling race? Why you apply +ve & -ve torque in your handle repeatedly? If you are clear about the answers that means you have understood Gyroscope.

Time Derivative of a rotating Vector of fixed Length.

Suppose there is a vector $\bar{A}$ which is rotating about some point with angular velocity ω; then

$$\boxed{\frac{d\bar{A}}{dt} = \bar{\omega} \times \bar{A}}$$

This is used in many places especially when a vector is precessing about some axis like Ls vector precessing with angular velocity Ω

$$\frac{d\bar{L}}{dt} = \bar{\Omega} \times \bar{L}$$

OR when a body is rotating in a circle, we can take $\bar{A}$ to be the position vector $\bar{r}$ we get the formula

$$\bar{v} = \frac{d\bar{r}}{dt} = \bar{\omega} \times \bar{r}$$

for the linear velocity v of a particle due to its pure rotation of its position vector.

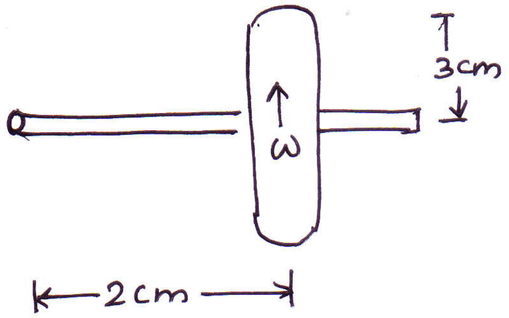
Please note that magnitude of $\bar{r}$ is not changing. Magnitude is fixed. Only direction of the vector is changing.

Please see the proof of this from any book or from internet [www.real-world-physics-problems.com]

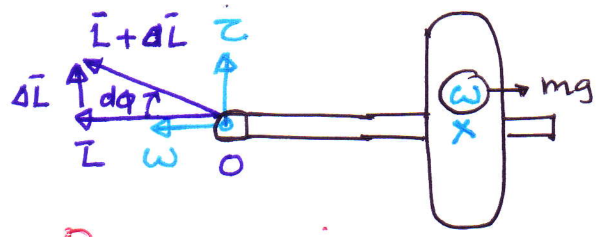
Figure shows a top view of cylindrical gyroscope wheel that has been set spinning by an electric motor. The pivot is at O, and the mass of the axle is negligible

(a) As seen from above, is the precession clockwise or anticlockwise (Draw a vector diagram $\bar{L}, d\bar{L}, \bar{\omega}$)

(b) If the Gyro takes 4.0 s for one revolution of precession, at what angular speed does the wheel spin.



(a) Vector diagram.



Precession is clockwise as shown.

Weight into the page.

(b) $\Omega_{\wedge}$ or ω_p $= (1\,\text{rev})/(4.0\,\text{s}) = (2\pi\,\text{rad})/(4.0\,\text{s}) = 1.57\,\text{rad/s}$

NOTE This is precession Ω / ω_p & not ω_s.

$$\omega_p = \frac{mgr}{I\omega_s} = \frac{\cancel{m}gr}{\frac{\cancel{m}R^2}{2} \times \omega_s} = \frac{2gr}{R^2\omega_s}$$

$$\text{so } \omega_s = \frac{2gr}{R^2 \omega_p} = \frac{2 \times (9.8\,\text{m/s}^2)(2.0 \times 10^{-2}\,\text{m})}{(3 \times 10^{-2}\,\text{m})^2 (1.57\,\text{rad/s})}$$

$$= 280\,\text{rad/s} = 2600\,\text{rev/min}.$$

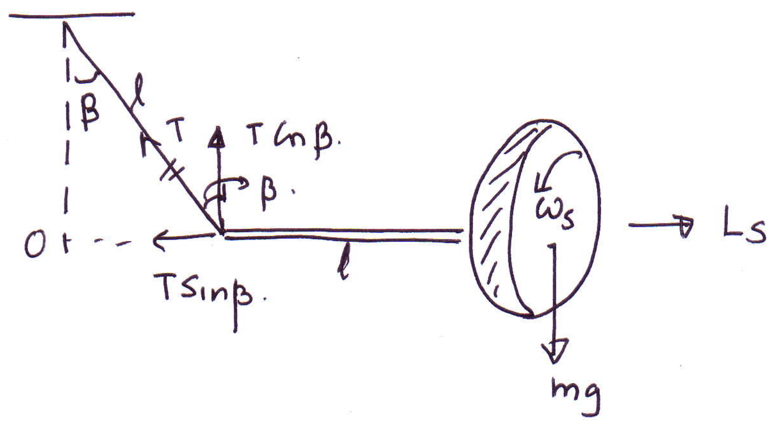
This problem is taken from Mechanics – Young & Freedman.

A gyroscope wheel is at one end of an axle of length l. The other end of the axle is suspended from a string of length l. The wheel is set into motion so that it executes uniform precession in the horizontal plane. The wheel has mass M and moment of Inertia about its center of mass I_0. Its spin angular velocity is ω_s. Neglect the mass of the shaft and of the string.

Find the angle β that the string makes with the vertical. Assume that β is so small that approximations like $\sin\beta \simeq \beta$ are justified.

(Problem 7.3 of Kleppner)

1.



Force : $T\cos\beta = Mg.$

$T\sin\beta = M\Omega^2(l + l\sin\beta)$

Torq ab (0) $\tau = \tau_{Mg} + \tau_T$

$(\bar{r} \times (-\hat{k}) = \hat{\theta}$.

Small angle β.

Ang Mom. about O

$L = L_{cm} + L_{about\ cm}.$

$\bar{r}\times\bar{p} \quad + I_0 W_s \hat{r}$

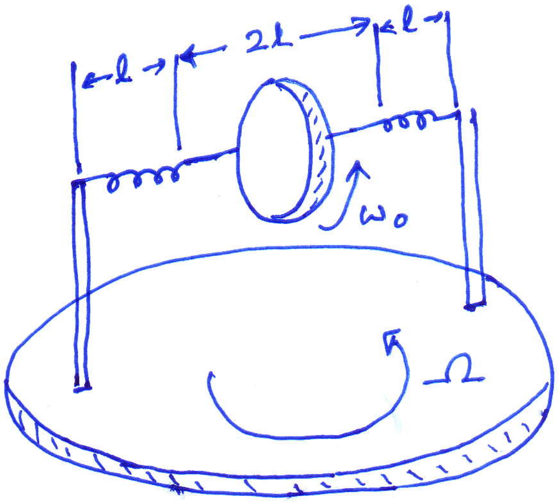
$\frac{dL}{dt} = \tau \Rightarrow \frac{dL}{dt} = 0 + (\quad) \frac{d\hat{r}}{dt}$

$\Downarrow$

$\Omega\hat{\theta}.$

A very popular problem of Kleppner. Try to figure out yourself before going into the solution. - This is an interesting problem. (Prob 7.2 of Kleppner)

A fly wheel of moment of Inertia I_0 rotates with angular velocity ω_0 at the middle of an axle of length $2l$. Each end of the axle is attached to a support by a spring which is stretched to length l & provides tension T. You may assume that T remains const for small displacement of the axle. The supports are fixed to a table which rotates at const angular velocity Ω, where $\Omega \ll \omega_0$. The center of mass of the flywheel is directly over the center of rotation of the table. Neglect gravity & assume that the motion is completely uniform so that nutational effects are absent. The problem is to find the direction of the axle wrt a straight line between the supports



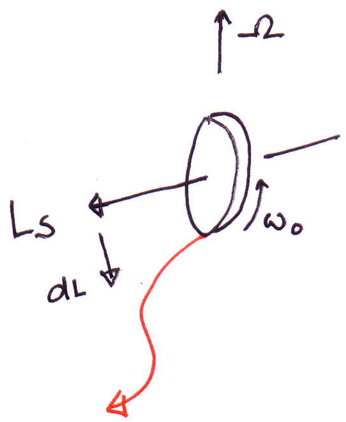
Interesting Problem.

I had shown you one demonstration in the class with the bicycle wheel. When I wanted to rotate the wheel in one direction it is deflected in a perpendicular direction. This problem is designed similar to that demonstration.

You can easily find out the direction of angular momentum vector L_s due to spin. Now the moment ~~It is~~ the table is rotated with angular velocity Ω the constant vector L_s is rotated because of which there will be dL given by.

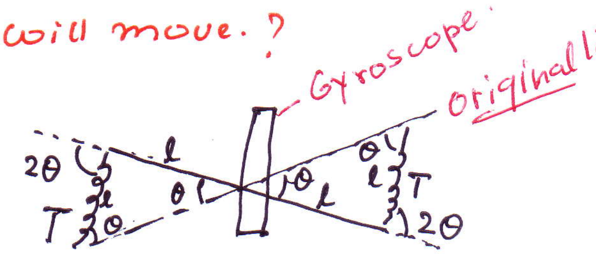
$$\frac{d\bar{L}}{dt} = \bar{\Omega} \times \bar{L}$$
$$= \Omega I_0 \omega_0$$

(We have read this formula before) (Derivative of a rotating vector).



- See the direction of $d\bar{L}$
- There must be some corresponding torque to provide this $d\bar{L}$ vector. How is it possible.?
- The Disc will twist in such a way so as to give a necessary torque which will give the necessary $d\bar{L}$.

In which direction it will move.?



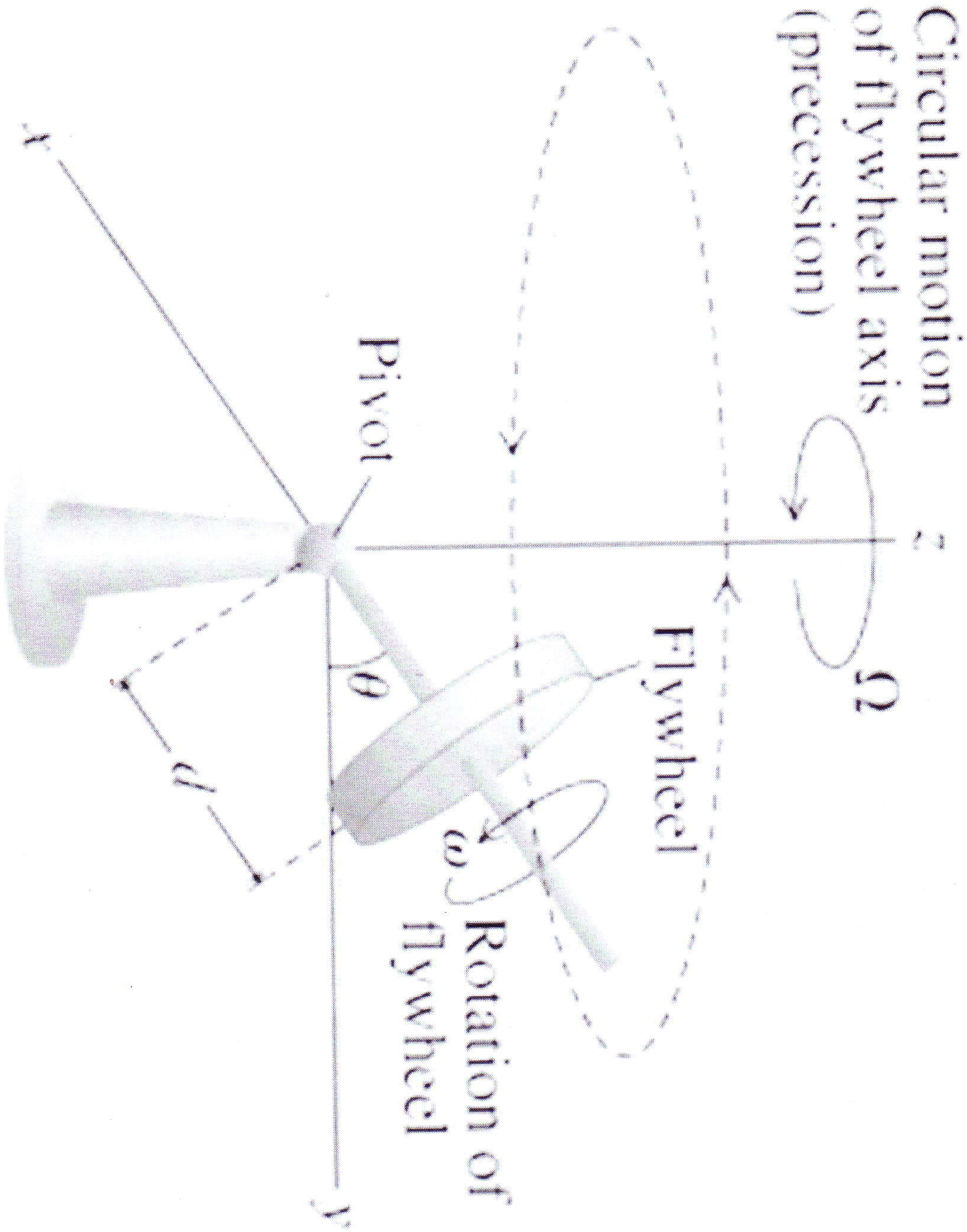
It will tilt upward by an θ from the original.

$$\text{Torque} = Tl \sin 2\theta + Tl \sin 2\theta$$
$$\tau = 4Tl\theta.$$

Now $4Tl\theta = \Omega I_0 \omega_0$

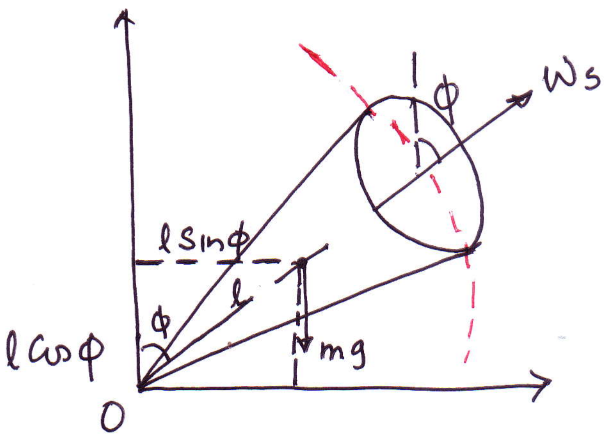
$$\boxed{\theta = \Omega I_0 \omega_0 / 4Tl}$$

A Gyroscope is precessing about Z axis with angular Velocity Ω at an angle θ to the x-y plane. Can you prove that Ω does not depend upon θ — that means for any value of θ Ω remains the same.



A top of mass M spins with angular speed ω_s about its axis as shown below. The moment of Inertia of the top about the Spin axis is I_0 and the Center of mass of the top is a dist l from the point. The axis is inclined at angle ϕ wrt to the Vertical & the top is undergoing Uniform precession. Gravity is directed downward. Find the rate of precession Ω clearly indicating its direction.

(Problem from Kleppner)



$\Rightarrow$ Break in Cylindrical Coord

(r, ϕ, z)

About O.

$$\tau_O = \bar{r} \times \bar{F}$$

$$= (l\sin\phi\,\hat{r} + l\cos\phi\,\hat{k}) \times (-Mg\,\hat{k})$$

$$L_O = \underset{(CM)}{L_{orb}} + \underset{(\text{about CM})}{L_{spin}}$$

$$L_{orb} = (\bar{r} \times m\bar{v}) = m(l\sin\phi\,\hat{r} + l\cos\phi\,\hat{k}) \times (l\sin\phi)\Omega\hat{\phi}$$

$$L_{spin} = I\omega_s = I(\omega_s \sin\phi\,\hat{r} + \omega_s\cos\phi\,\hat{k})$$

$\omega_s >> \Omega$ ignore L_{orb}.

$$\frac{dL_{spin}}{dt} = I\omega_s\sin\phi\,\frac{d\hat{r}}{dt} + 0 = I\omega_s\sin\phi\,\Omega\,\hat{\phi}$$

$$\left[\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}\right]$$

$$\tau = \frac{dL}{dt}$$

$$\boxed{\Omega = \frac{Mgl}{I\omega_s}}$$