Special Theory of Relativity.

My present note has become little lengthy – 43 pages. But I am happy about it. Most of the important topics are clearly written. If you keep it carefully, then even after 10 years you can read it and within a day or two you can revise the whole of STR. The first part deals with relativistic mechanics where as second part Momentum, Energy. I could not revise once the whole note after writing ~~once~~. There may be some errors. Please go through it carefully & if you find some please let me know so that I can send a corrected version. I have consulted the following books while making this note.

1) Resnick Halliday Krane

2) Physics – Serway & Jewett.

3) Modern Physics – Mani & Mehta.

I have not consulted Kleppner for this chapter. Please don't hesitate to ask me something if you don't understand or is not clear. If you have any suggestions please let me know.

(AMIT NEOGI)
4 NOV, 2016.

Contents

Topics

page Numbers.

Special Theory of Relativity.

The kinematics developed by Galileo and the mechanics developed by Newton which form the basis of Classical Physics had many triumphs. However a number of experimental phenomena cannot be understood with these successful classical theories — limitations of classical Physics.
There were problems at different places.

- Troubles with our ideas about Time — decay of unstable particle pion — (For details please go through Resnick Halliday Krane)
- Trouble with our ideas about Length
- Troubles with our ideas about speed.
- Troubles with our ideas about Energy.
- Troubles with our ideas about Light. —

Einstein proposed his special theory of relativity in 1905, based on a thought experiment that he had devised. Einstein, as a 16 year old student had thought about a paradox: If you were to move at the speed of light parallel to a light beam travelling in empty space, you will observe "static" electric and magnetic field patterns. However, Einstein knew that such static electric and magnetic field patterns in empty space violated the theory of electromagnetism. Einstein was faced with two choices to resolve this paradox: either electromagnetic theory was wrong or else the classical kinematics that permits an observer to travel along a light beam was wrong. With the intuition that was perhaps his greatest attribute, Einstein put his faith in electromagnetic theory and sought an alternative to the kinematics of Galileo & Newton. — which forms the basis of Special Relativity.

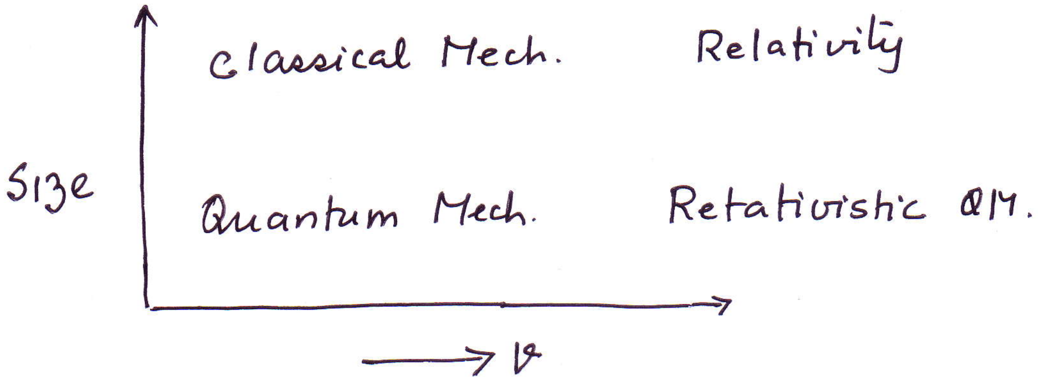
<u>Einstein Relativity</u> $\left\{ \begin{array}{l} \text{STR – Special Theory of Relativity} \\ \text{GTR – General Theory of Relativity.} \end{array} \right.$

<u>STR</u> – Deals with only (<u>Mostly</u>) inertial frames & not accelerated frame. (?) ⇒ this is not true really.
STR deals with inertial frames & accelerated frames differently.

- It applies only to a Special Case – Here Curvature of Spacetime due to gravity is negligible.
- In order to include gravity, Einstein formulated general relativity in 1915

So Galilean relativity is an approximation of Special relativity for low speeds, special relativity is considered an approximation of general relativity that is valid for weak gravitational fields.

A Comparison of Size and speed.



Concept of an Event & Observer. & Frame of Reference

Event: An event which takes place in space is independent of any reference frame.

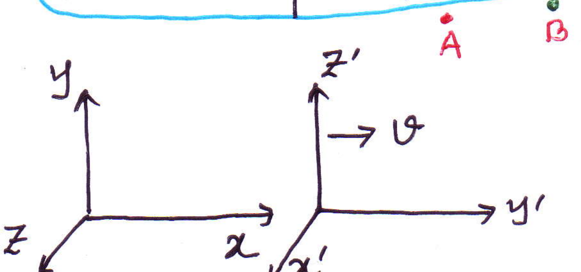
Observer: Different observers in different frames of reference make the measurement. Suppose there are two events A & B. An observer in one frame of reference measures these events as.

In S Frame	In S' Frame
$A(x_1, t_1)$	$A(x_1', t_1')$
$B(x_2, t_2)$	$B(x_2', t_2')$

Here S' frame is moving at a velocity v wrt to S frame.

Frames of Reference: As discussed above the frame of reference is v. Imp to describe any motion.

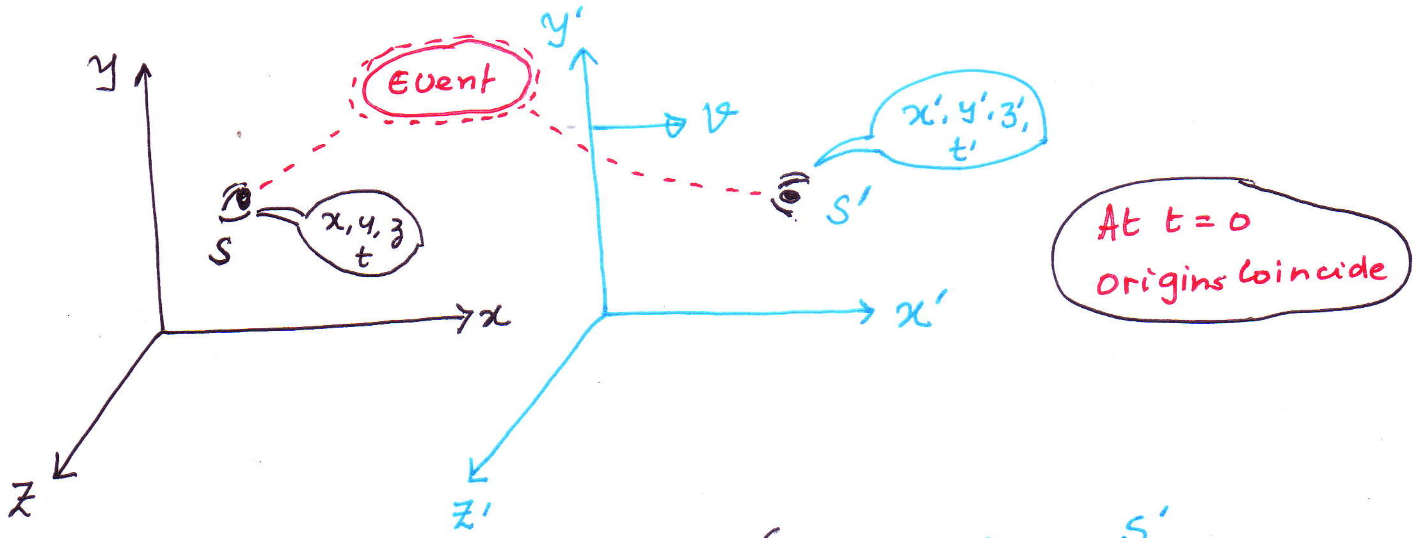
An inertial frame of reference is that frame where Newton First Law of motion holds good.



Galilean Transformation | Galilean Relativity.

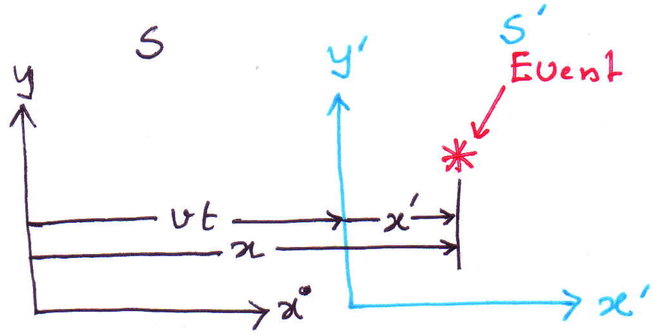
We have studied Galilean transformation equation in the previous chapter – Non – Inertial Frame. For the sake of completeness I include here once again.

An event happens independent of any observer in space. Two observers, whose frames of reference are represented by S and S', observe the same event. S' moves relative to S with velocity $\vec{v}$ along the common xx' direction. S measures the coordinate x, y, z, t of the event, while S' measures the coordinate x', y', z', t' of the same event. [At $t=0$ S' frame coincide with frame S]



$$\boxed{\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}}$$

⇒



These equations are the Galilean Space – time transformation equations. Note time is assumed to be same in both Inertial frames.

Now.

$$\Delta x' = \Delta x - v\,\Delta t$$

$$\Delta t' = \Delta t$$

So $$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\Delta t} - v.$$

In the limit $\Delta t = \Delta t' \to 0$ we can write

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

Velocity Transformation $\boxed{u_x' = u_x - v.}$, $u_y' = u_y$, $u_z' = u_z$

Where u_x and u_x' are the x component of the velocity of the particle measured by observers in S and S' respectively

$\vec{v}$ is the relative velocity of two frames.

This is the Galilean Velocity transformation equation.

It is consistent with our intuitive notion of time & Space (Concept of Relative motion)

Consequences of Galilean Relativity.

We find few important results from Galilean Relativity at that time.

1) Speed of light is not constant but depends upon the reference frame $u \pm c$. Like if you are sitting in a train & the other train crosses you travelling in same direction $(u_1 - u_2)$ or in opposite direction $(u_1 + u_2)$. So Speed of light can be much greater than c.

Concept of Aether drift — Discarded Later

It is worth while to mention here that during that time people used to believe that there is a hypothetical medium pervading the Universe known as 'ether' through which light used to travel. Depending upon the ether drift the Speed of light relative to Earth will increase $(c+v)$ or decrease $(c-v)$ depending on whether ether drift is in the direction of light or opposite to it. But later Michelson Morley experiment showed that ether does not exist.

2) Laws of Mechanics hold good in all inertial frame according to Galilean Relativity. But the question is whether laws of electricity and magnetism also holds good in all inertial frame. This was a very important question in that era. But Maxwell's equations seem to imply that speed of light always has fixed value 3.00×10^8 m/s in all inertial frames, a result in direct contradiction to what is expected based on Galilean velocity transformation equation. According to Galilean relativity, the speed of light should not be the same in all inertial frame. To resolve this contradiction in theories we must conclude that either

⇒ (1) Laws of Electricity & Magnetism are not same in all inertial Frame OR.

⇒ (2) The Galilean Velocity transformation eq is incorrect

3) If we want laws of electricity & magnetism to be same in all inertial frame then we need to change Galilean transformation equation.

Time and length were absolute in Galilean relativity (What does it mean?). Time that is time interval between two events $(t_2 - t_1)$ is constant in any reference frame & similarly length $(x_2 - x_1)$. If there are two event happening in space which is observed by two observers, one is at rest wrt to earth & other observer is moving wrt to first observer at a speed v. Then both observers will measure same time difference and space difference. But now we are forced to abandon the notion of absolute time & absolute length if we want all the laws that is both laws of Mechanics & laws of electromagnetism as well as other laws of Physics to hold good in all ~~se~~ inertial reference frame.

The Michelson-Morley Experiment.

The most famous experiment designed to detect small changes in the speed of light on account of ether wind. in 1881 by A. Michelson & later repeated under various conditions by Michelson & Morley.

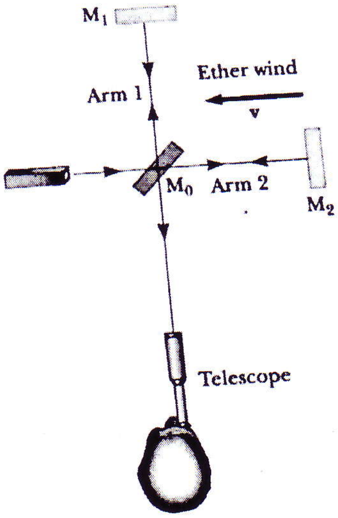
The outcome of the experiment contradicted the ether hypothesis

from Physics by Serway & Jewett.

The experiment was designed to determine the velocity of the Earth relative to that of the hypothetical ether. The experimental tool used was the Michelson interferometer, which was discussed in Section 37.7 and is shown again in Figure 39.4. Arm 2 is aligned along the direction of the Earth's motion through space. The Earth moving through the ether at speed v is equivalent to the ether flowing past the Earth in the opposite direction with speed v. This ether wind blowing in the direction opposite the direction of Earth's motion should cause the speed of light measured in the Earth frame to be $c - v$ as the light approaches mirror M_2 and $c + v$ after reflection, where c is the speed of light in the ether frame.

The two light beams reflect from M_1 and M_2 and recombine, and an interference pattern is formed, as discussed in Section 37.7. The interference pattern is observed while the interferometer is rotated through an angle of 90°. This rotation interchanges the speed of the ether wind between the arms of the interferometer. The rotation should cause the fringe pattern to shift slightly but measurably. Measurements failed, however, to show any change in the interference pattern! The Michelson–Morley experiment was repeated at different times of the year when the ether wind was expected to change direction and magnitude, but the results were always the same: **no fringe shift of the magnitude required was ever observed.**[2]

The negative results of the Michelson–Morley experiment not only contradicted the ether hypothesis but also showed that it was impossible to measure the absolute velocity of the Earth with respect to the ether frame. However, Einstein offered a postulate for his special theory of relativity that places quite a different interpretation on these null results. In later years, when more was known about the nature of light, the idea of an ether that permeates all of space was abandoned. **Light is now understood to be an electromagnetic wave, which requires no medium for its propagation.** As a result, the idea of an ether in which these waves travel became unnecessary.



Active Figure 39.4 According to the ether wind theory, the speed of light should be $c - v$ as the beam approaches mirror M_2 and $c + v$ after reflection.

At the Active Figures link at http://www.pse6.com, you can adjust the speed of the ether wind to see the effect on the light beams if there were an ether.

⇓

Here earth along with the experimental set is assumed to be moving towards the right ⟶

So ether wind wrt to earth is leftward as shown in fig.

The negative result had 2 Consequences.

(1) Ether does not exist and so there is no such thing as "absolute motion" relative to ether. All motion is relative to a specified frame of reference, not to a universal frame.

(2) The speed of light is same for all observers, which is not true of waves that need a material medium in which to occur (as Sound & Water Waves)

Einstein Principle of Relativity : The Postulats.

Einstein proposed a Theory That removed all the existing difficulties as discussed before and altered our notion of space and time. He based his special theory of relativity on two postulats.

1. The Principle of Relativity – The laws of physics must be the same in all inertial reference frames.

2. The Constancy of the Speed of light – The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the Velocity of the observer or the velocity of the source emitting the light.

The laws of physics are absolute, Universal and same for all inertial observers. Laws That hold good for one inertial observer Cannot be violated for any inertial observer. These laws include Mechanics, Electrodynamics, optics, thermodynamics unlike Galilean relativity where it was only for Mechanics. This is the essence of his first postulate.

The second postulate Contradict our "Common Sense" which is firmly grounded in Galilean Kinematics. Here the Speed of light is independent of the relative motion between the Source & the observer. The two postulate taken together have another Consequence: It is impossible to accelerate a particle to a speed greater than c, no matter how much KE We give it.

Experimental Verification of the Postulats of Einstein.

The postulats of Einstein were experimentally verified later

(a) In 1964 an experiment was performed at CERN, the European high-energy particle Physics ~~Laboratory~~ accelerators made possible the study of the motions of particles at speeds close to c.

$$\boxed{\pi^0 \longrightarrow \gamma + \gamma}$$

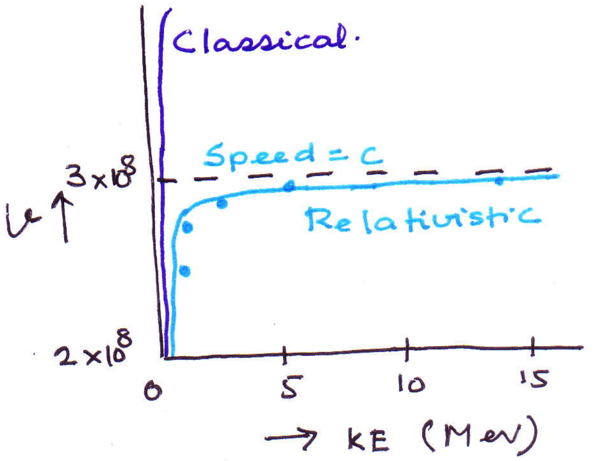
Laboratory speed = 0.99975 c (for π^0)

c, Lab Speed = c. (for each γ)

π^0 = neutral pions decaying to gamma rays.

- According to Galilean Relativity the Speed of Gamma rays Should be = 0.99975c + c = 1.99975 c
- According to Einstein they Should have a speed c
- The measured Speed was 2.9977×10^8 m/s ~ c.

Another Exp in 1964.

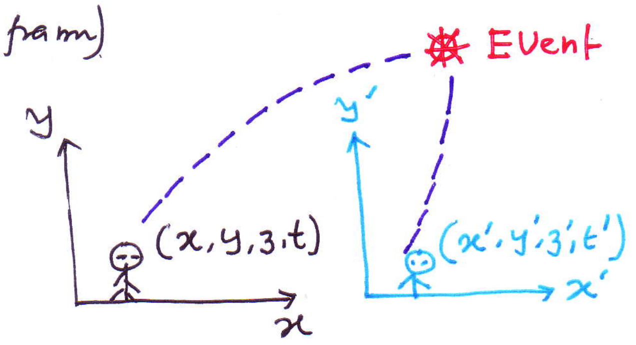


Electrons were accelerated by a large voltage difference & Speed of electrons were directly determined. No matter how much the accelerating voltage is increased, the speed never quite reaches or exceeds c. Once again it Confirms the postulats of Special relativity.

The Lorentz Transformation.

We have already seen the limitations of Galilean transformation. The Galilean transformation is not valid when v approaches the speed of light. So Lorentz transformation equations are relativistic relationship that is when we go to high speed of the order of light. These equations were first proposed by H. A. Lorentz for quite a different reason and who was not fully aware of their implications about the nature of space & time. The equations can be derived directly from Einstein's postulats; if we invoke certain reasonable assumptions about the symmetry and the homogeneity of space & time. Consider once again an event in space & 2 observers like GT.

Observer	Frame	Event
observer 1 →	S frame (Lab frame) →	(x, y, z, t)
observer 2 →	S′ frame (moving with vel $\bar{v}$ wrt to S fram) →	(x', y', z', t')



$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - vx/c^2) \\ \text{where } \gamma &= \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

→ (Lorentz Factor)

Proof of these equations are not given here. It may be included in appendix

Important observations about Lorentz Transformation eq.

(1) In Galilean Case $t' = t$ but in Lorentz case $t' \neq t$

In Lorentz transformation t' assigned to an event by an observer O' in S' frame depends both on time t and on the coordinate x as measured by the observer O in S frame. This is consistent with the notion that an event is characterized by four space-time coordinats (x, y, z, t)

In other words, in relativity, space and time are not separate concepts but rather are closely interwoven with each other.

(2) The Lorentz transformation equations should reduce to Galilean equations in the classical limit that is $v << c$.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

When $v << c$ that is $\frac{v}{c} \to 0$ then $\gamma = 1$

So

$$x' = \gamma (x - vc) \Rightarrow x' = x - vc$$

$$\Downarrow$$

Galilean Case.

Any new Theory should accomodate the results of old Theory. The new transformation equns now accomodate both high speed $(v \sim c)$ & low speed $(v << c)$

Inverse Lorentz transformation for $s' \to s$.

If we wish to transform Coordinats in s' frame to Coordinats in S frame, We Simply replace v by $-v$ and interchange primed & Unprimed Coordinats in the previous L-T equations.

$$\boxed{\begin{aligned} x &= \gamma\,(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}}$$

If you look from the S' frame. That is we make s' frame the rest frame then S frame is moving in $-$ve x direction — So the logic of $-$ to $+$.

Relating Difference in Coordinats of two event.

If There are two events in Space, their difference in Space Coordinate is Δx & That of time Δt in S frame Then in S' they are related by

$$\boxed{\left.\begin{aligned} \Delta x' &= \gamma\,(\Delta x - v\,\Delta t) \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \end{aligned}\right\} S \to S'}$$

$$\left.\begin{aligned} \Delta x &= \gamma\,(\Delta x' + v\,\Delta t') \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \end{aligned}\right\} S' \to S$$

Space - Time Interval

Let us consider two events P_1 and P_2 in S & S' frame.

	Event 1 ⇒ P_1	Event 2 ⇒ P_2
S frame	(x_1, y_1, z_1, t_1)	(x_2, y_2, z_2, t_2)
S' frame	(x_1', y_1', z_1', t_1')	(x_2', y_2', z_2', t_2')

Then it can be shown.

$$\boxed{\begin{aligned} c^2(t_1-t_2)^2 - (x_1-x_2)^2 - (y_1-y_2)^2 - (z_1-z_2)^2 \\ = c^2(t_1'-t_2')^2 - (x_1'-x_2')^2 - (z_1'-z_2')^2 \end{aligned}}$$

OR .

$$\boxed{c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = \text{Invariant in every frame}}$$

This means that above quantity known as space time interval between any two event is the same in any inertial frame.

Proof: Write the L-T equations for both these events, subtract them & then substitute in the above equation. — you can prove that LHS = RHS.

Two events E_1 and E_2 are described in S by Coordinats E_1: $x_1 = 10$ km, $t_1 = 2\times10^{-5}$ sec, $E_2 \Rightarrow x_2 = 100$ km, $t_2 = 10^{-5}$ sec. Is there a frame S' in which E_1 and E_2 are simultaneous & if so, find its Velocity wrt to S [Ans. 10^7 m/s]

Method 1

Use Lorentz transformation.

$$t_1' - t_2' = 0$$

$$t_1' = \gamma(t_1 \;\cdots\quad)$$

$$t_2' = \gamma(t_2 - \;\cdots)$$

$$t_2' - t_1' = \gamma\left[(t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right]$$

$$t_2 - t_1 = \frac{v}{c}\,(x_2 - x_1)$$

Method 2.

Use Space time Interval formula.

$$c^2\Delta t^2 - \Delta x^2 = \text{same for both frame.}$$

If an event A precedes B occuring at the same point in one Inertial frame, will A precede B in all other inertial frame? Will they occur at the same point in any other inertial frame? Will the time interval between the event be the same in any other inertial ref. frame.

[Ans Yes, NO, NO] $\Rightarrow$ Use Space time Interval.

Consequences of Lorentz transformation.

In this topic I would like to discuss three topics

(1) Length Contraction

(2) Time dilation.

(3) Simultaneity & Relativity of Time.

Now all these topics are discussed under the heading 'Consequences of Special Theory of Relativity' in Resnick Halliday, Serway & Jewett, Beiser and many other books. The expressions are derived using the postulats of relativity. That gives real feeling of relativity. Moreover it gives an importance to the postulats of Einstein.

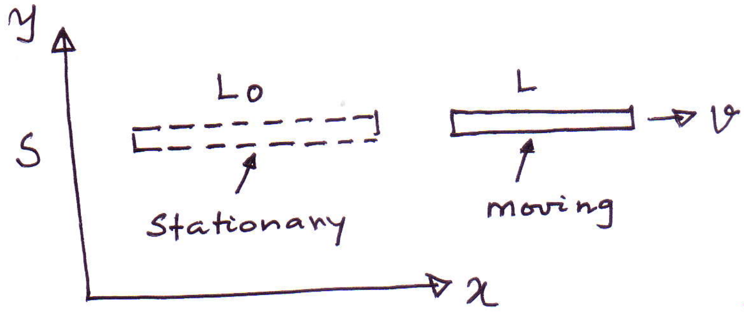
But most of the teachers do it using Lorentz transformation equations because it is short & it gives us a basic practise to use Lorentz transformation eqns.

Sometimes a student gets a doubt in his/her mind —— what is the importance of Einstein two postulats? Why we are never using it? ⇒ We can certainly derive all the three concepts using only Einstein postulats. But here I am doing using Lorentz transformation only.

I hope you understand what I want to convey.

Length Contraction.

Suppose there is a rod whose length is L_0 when it is at rest in frame S. Now the rod starts moving with velocity v wrt to frame S in x direction. It is found that the length of the rod is reduced as measured by an observer in S frame. See the diagram below. This effect is known as length contraction



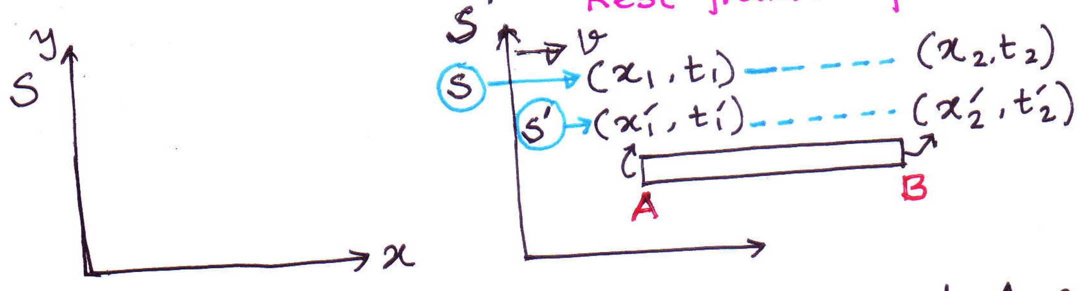
L_0 = Rest length/ Proper length

L_0 is the length measured by an observer in a frame where rod is at rest.

We have to find out a relation between L_0 & L.

⇒ Let us introduce another frame S' which moves with speed v such that in that frame the rod is at rest.

Rest frame of the RoD (Proper length)



Lets take two ends of the rod as event A and B.

	A	B
S	(x_1, t_1)	(x_2, t_2)
S'	(x_1', t_1')	(x_2', t_2')

⇒

$$\boxed{\begin{aligned} x_2' - x_1' &= L_0 \\ t_2 - t_1 &= 0 \\ x_2 - x_1 &= ? \end{aligned}}$$

Why?

According Lorentz transformation.

$$x_1' = \gamma(x_1 - vt_1)$$

$$x_2' = \gamma(x_2 - vt_2)$$

$$x_2' - x_1' = \gamma[(x_2 - x_1) - v\cancel{(t_2 - t_1)}^{\,0}]$$

$$L_0 = \gamma L$$

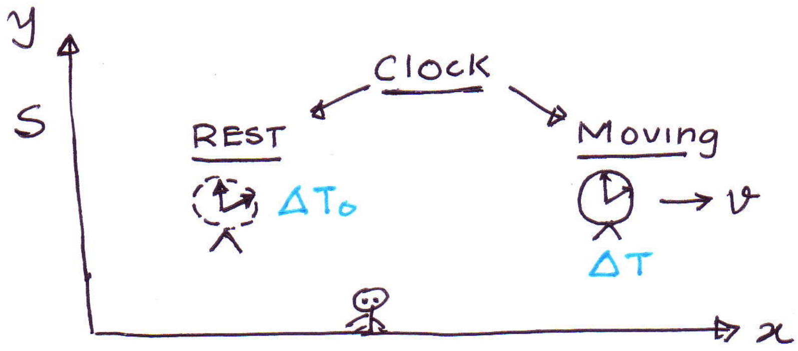
So $\boxed{L = \frac{L_0}{\gamma}}$ $\gamma > 1$, Length is Contracted

L_0 = proper length / rest length = Frame in which the rod is at rest.

- Length Contraction takes place only along the direction of motion.
- Proper length and proper time interval are defined differently. The proper length is measured by an observer for whom the end points of the length remain fixed in Space.
- The proper time interval is measured by someone for whom the two events take place at the same position in Space.

Time dilation.

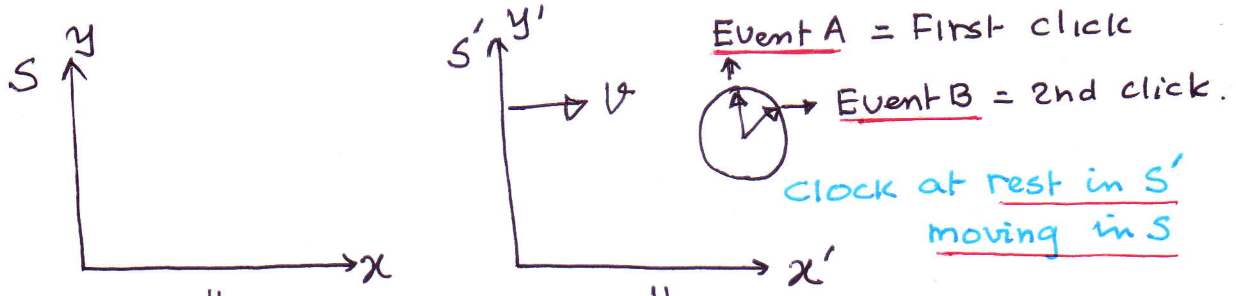
A moving clock ticks more slowly Than a clock at rest.



Suppose There is a clock at rest as observed by a person in S frame. The time interval between two ticks is $\Delta T_0 = 1$ Sec (say). Then the clock start moving with velocity v. It is observed that time interval between two ticks has increased that is $\Delta T > 1$ Sec.

What is the Relation between ΔT_0 and ΔT?

In order to solve the problem let us take help of another frame S' moving with velocity $\bar{v}$ such that the moving clock is at rest in that frame.



$\Downarrow$ Event A (x_1, t_1)

Event B (x_2, t_2)

$\Downarrow$ Event A $= (x_1', t_1')$

Event B $= (x_2', t_2')$

Few Important Points ~~of~~ about ΔT_0 & ΔT.

Students often get confused between ΔT_0 and ΔT. Please read the lines below very very carefully. — V. Imp

1) Differentiate first between event and observer.
Here there are two events — Event A → first click
Event B – Second click.

Event
- Event A – First click
- Event B – 2nd click

Events occur in Space independent of any observer

Observer
- In S frame
- In S′ frame.

Different observer record the same event differently

2) What is ΔT_0? V. V. V. Imp. (often we make mistake)

ΔT_0 is the time interval between two events where both the events are occuring at the same point in space — as observed by an observer in a frame.

- It may be in S′, it may be in S — V. Imp
- In this particular case it is S′. In S′ the clock is at rest & hence both the clicks are occuring at the same point. → ~~It is not~~ That is ΔT_0
- To an observer in S frame the two ticks are not occuring at the same point because clock is moving → That is ΔT

Please read this page atleast three times.

ΔT_0 is also known as Proper Time.

Now to find the relation between ΔT_0 & ΔT.
There are different ways to do it.

Method 1: (Simple way) Using Space time interval.
Space time interval is invariant in any frame.

$$c^2 \Delta t^2 - \Delta x^2 = c^2 \Delta t'^2 - \Delta x'^2$$

Since the clock is at rest in S', $\Delta x' = 0$.

$$c^2 \Delta T^2 - v^2 \Delta T^2 = c^2 \Delta T_0^2$$

$$\Delta T^2 (c^2 - v^2) = c^2 \Delta T_0^2$$

$$\Delta T^2 = \frac{c^2 \Delta T_0^2}{c^2 - v^2} = \frac{\Delta T_0^2}{1 - v^2/c^2}$$

$$\boxed{\Delta T = \gamma \Delta T_0} \qquad \gamma > 1. \qquad \Delta T_0 = \text{Proper time}.$$

Thus 1 sec in S' ($\Delta T_0 = 1$) appears as γ secs in S
In other words moving clock appears to go slow —
time appears dilated — hence time dilation.

Method 2

Things we know: $x_2' - x_1' = 0$; $t_2' - t_1' = \Delta T_0$
Using Lorentz Transf (Inverse). $t_2 - t_1 = \Delta T = ?$

$$t_1 = \gamma (t_1' + v x_1'/c^2)$$

$$t_2 = \gamma (t_2' + v x_2'/c^2)$$

$$t_2 - t_1 = \gamma [(t_2' - t_1') + v/c^2 \underbrace{(x_2' - x_1')}_{0}]$$

$$\boxed{\Delta T = \gamma \Delta T_0} \qquad \gamma > 1, \quad \Delta T_0 = \text{Proper time}.$$

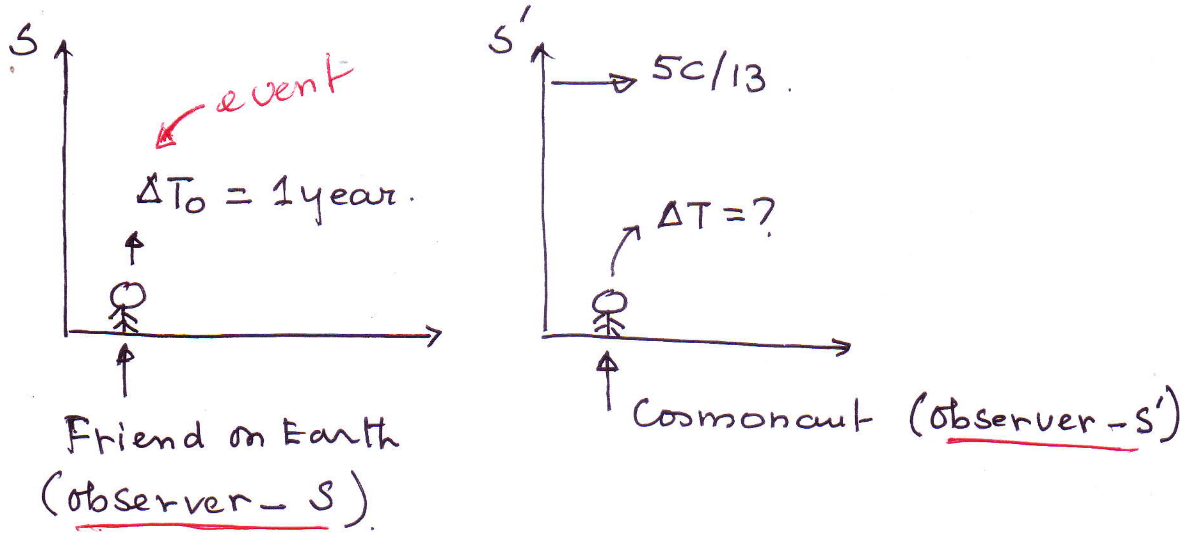
Find the time interval Calculated by a Cosmonaut moving at 5c/13 between Consecutive birthday celebrations of his friend on the earth. What time interval does the friend on earth Calculate between Consecutive birthday celebrations by the cosmonaut.

⇒ Steps to solve the problems of this type.

Step 1: Draw two frames S and S′ which is moving with speed v relative to S.

Step 2: Identify the events and observers.

Step 3: Apply the formula, $L = L_0/\gamma$, $\Delta T = \gamma T_0$ or Lorentz transformation / Inverse whatever is necessary.



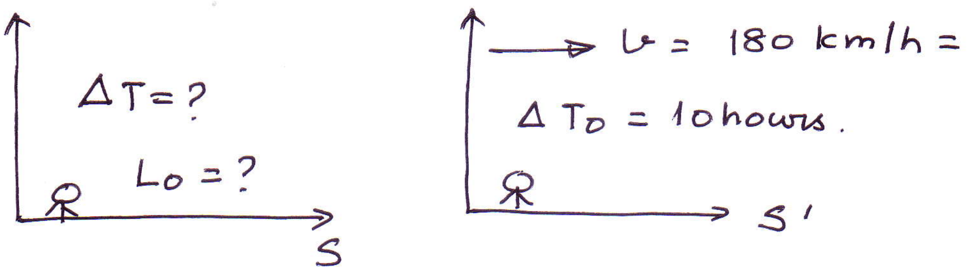
NOTE: Here the friend on earth – Stationary observes his Consequitive birthday → The two events & the time diff between them is 1 year. So the person is at rest ΔT_0 is in S frame. Cosmonaut is moving wrt to the clock of his friend on earth So he will see ΔT. – Apply time dilation

$\Delta T = \gamma T_0$. (13/12 years in both case).

If a person travelling by car at 180 km/h takes exactly 10 hours by his wristwatch to go from station A to B. (a) What is the rest distance between the two stations (b) How much time is taken in the road frame to go by car from A to B.

Step 1 : Draw two reference Frames S & S'.

Step 2 : Identity the observers & events & place. Them in the frames.



(b) $\Delta T = \gamma \Delta T_0$. (Ans: 0.50 ns more than 10 hours)

(a) $L_0 = v \times \Delta T = 25$ nm more than 1800 km.

Quite a Complicated Problem: We have marked two observers but what are the events.

Event 1: Observing / starting Station A.
Event 2: Reaching Station B.

$\Rightarrow \Delta T$, ΔL

ΔT ↓ Time Interval, ΔL ↓ length Int Between events.

Now ΔT, ΔL Can be taken as events.

According to Question → The watch of observer in S' is at rest — ΔT_0 is given — in S'
So find ΔT in S.

- Observer in S observes rest length between stations That is L_0 — but observer in S' observes L which is not asked in question.

Simultaneity of Events – Frame-Dependent

Suppose two events P_1 and P_2 occur simultaneously as seen by an observer in the frame S. Let the coordinates of P_1 and P_2 be (x_1, t_0) and (x_2, t_0) respectively. Then the events as seen by an observer in frame S', will occur at times t_1' and t_2'

$$t_1' = \gamma(t_0 - vx_1/c^2)$$

$$t_2' = \gamma(t_0 - vx_2/c^2)$$

$$t_2' - t_1' = \gamma(0) - \frac{\gamma v}{c^2}(x_2 - x_1) \neq 0.$$

It is clear that if $x_1 \neq x_2$, $t_1' \neq t_2'$

This shows that the two events do not occur simultaneously in the frame S'. Thus, the simultaneity concept is not absolute but frame dependent.

Paradoxes in Special Theory of Relativity

1. Twin Paradox

The story is that one of a pair of twins leaves on a high speed space journey during which he travels at a large fraction of the speed of light while the other remains on the Earth. Because of time dilation, time is running more slowly in the spacecraft as seen by the earthbound twin and the traveling twin will find that the earthbound twin will be older upon return from the journey. The common question: Is this real? Would one twin really be younger?

Many textbooks explain that the traveling twin must be accelerated up to traveling speed, turned around, and decelerated again upon return to Earth. Accelerations are outside the realm of special relativity. But it is not totally correct. Please read through an article in scientific American https://www.scientificamerican.com/article/how-does-relativity-theor/

Despite the experimental difficulties, an experiment on a commercial airline confirms the existence of a time difference between ground observers and a reference frame moving with respect to them.

2. Rocket Warfare
3. Car and Ditch
4. Stick and Plane

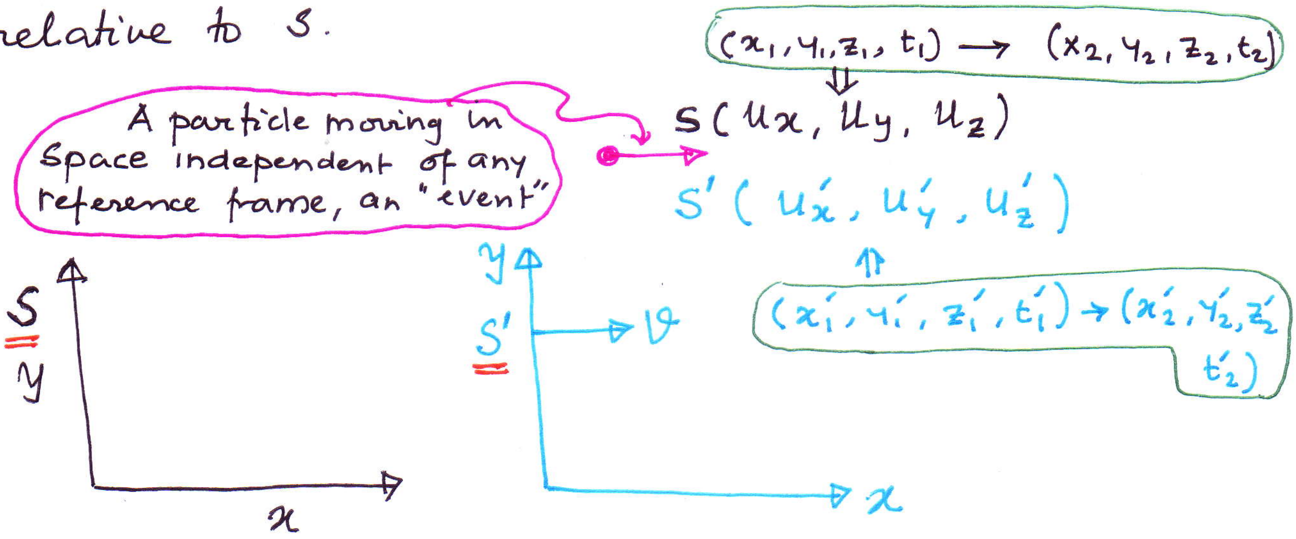
} other Paradoxes in STR.

Frames cannot have velocity greater than that of light.

The factor $\gamma = 1/(1-v^2/c^2)^{1/2}$ plays a crucial role in the Lorentz transformation. The value of γ is 1 when $v=0$ and ∞ when $v=c$. If $v>c$, then $v^2/c^2>1$ and γ is imaginary. Thus if the coordinates (x,t) are real, the coordinates (x',t') become imaginary. Since the Lorentz transformation connects two inertial frames specifying the same event, the relative velocity between the two frames must be less than the speed of light c.

The Relativistic Transformation of Velocities.

Here we use the Lorentz transformation equations to relate the velocity $\bar{u}$ of a particle measured by an observer in the S frame to the velocity $\bar{u}'$ of the same particle measured by an observer in the S' frame. Please note that S' is moving with velocity $\vec{v}$ relative to S.



Observer in S finds particle to move from $(x_1, y_1, z_1, t_1) \longrightarrow (x_2, y_2, z_2, t_2)$

Observer in S' finds the particle to move from $(x_1', y_1', z_1', t_1') \longrightarrow (x_2', y_2', z_2', t_2')$

$$u_x' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - v\Delta x/c^2)} = \frac{(\Delta x/\Delta t) - v}{1 - \frac{v}{c^2}\left(\frac{\Delta x}{\Delta t}\right)}$$

Replacing $\Delta x/\Delta t$ by u_x

$$\boxed{u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}}$$

Similarly $u_y' = \dfrac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$

$$u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

One must be careful what is u & v.

- One must note that in Lorentz transformation eqn. $\Delta y = \Delta y'$ but $u_y' \neq u_y$ because $\Delta t \neq \Delta t'$.
- One must note that denominators of all the three equations include the factor u_x.
- Let us see the Lorentz Velocity transformation eqns ~~in both~~ as well as Inverse velocity transformation.

Velocity transformation		Inverse Velocity Transformation
$u_x' = \dfrac{u_x - v}{1 - \frac{u_x v}{c^2}}$	$\longleftrightarrow$	$u_x = \dfrac{u_x' + v}{1 + \frac{u_x' v}{c^2}}$
$u_y' = \dfrac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$	$\longleftrightarrow$	$u_y = \dfrac{u_y'}{\gamma\left(1 + \frac{u_x' v}{c^2}\right)}$
$u_z' = \dfrac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$	$\longleftrightarrow$	$u_z = \dfrac{u_z'}{\gamma\left(1 + \frac{u_x' v}{c^2}\right)}$

- $(u_x, u_y, u_z) \Rightarrow$ Velocity observed in S frame – Lab frame.
- $(u_x', u_y', u_z') \Rightarrow$ Velocity observed in S' frame
- S' is moving wrt to S frame with velocity v in positive x direction.

- The above transformation equations reduce to Classical Galilean transformation when $v \ll c$. In that case

$$u'_x = u_x - v, \quad u'_y = u_y \quad \text{and} \quad u'_z = u_z.$$

Which are indeed Galilean results.

- Lorentz velocity transformation gives the result demanded by Einstein's second postulate (the Constancy of the speed of light): a speed of c measured by one observer must also be measured to be c by any other observer.

Suppose there is a common event i.e is "Passage of light" is observed by S & S'.

- Observer S measures $u_x = c$, $u_y = u_z = 0$.
- What velocity does observer S' measure?

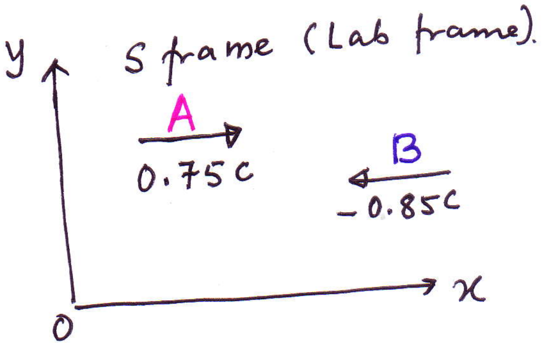
$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} = \frac{c - v}{1 - cv/c^2} = \frac{c - v}{1 - v/c} = \frac{(c-v)c}{(c-v)} = c$$

So $u'_x = c$

The result is independent of the relative speed v between S and S'.

A speed of c measured in one inertial frame leads to a speed of c measured in all frames. Thus the speed of light is indeed the same for all observers.

Two space craft A and B are moving in opposite directions as shown below. An observer on the earth measures the speed of craft A to be $0.750c$ and the speed of craft B to be $0.850c$. Find the velocity of craft B as measured by the crew on craft A.



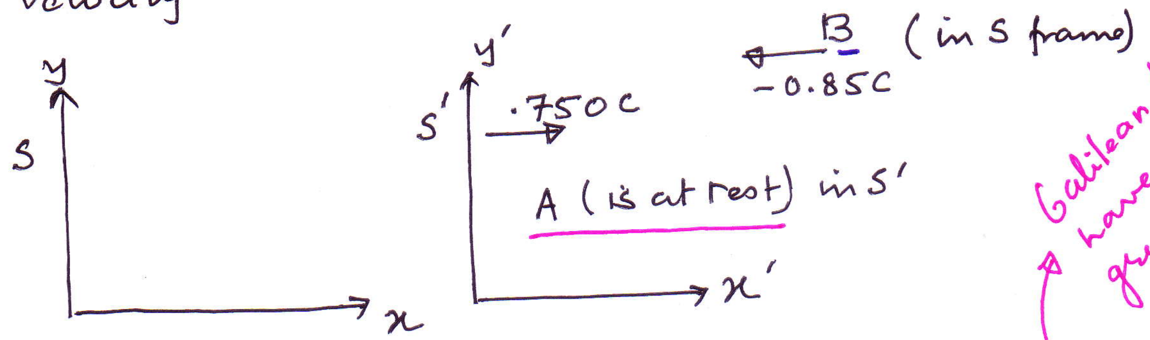
⇒ We have to learn the rules of the game – How to attack the problem in the right manner so that we don't make mistake.

Who are the observers & event here?

The two observers are on Earth and on Space craft A.

The event is the motion of Space craft B.

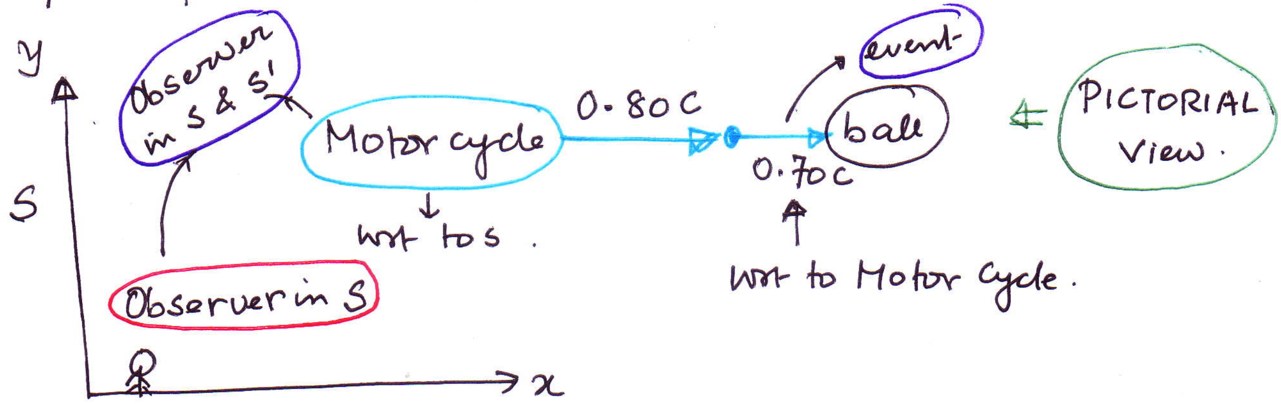
So we choose S' frame which is moving with velocity $0.75c$ so that spacecraft A is at rest.

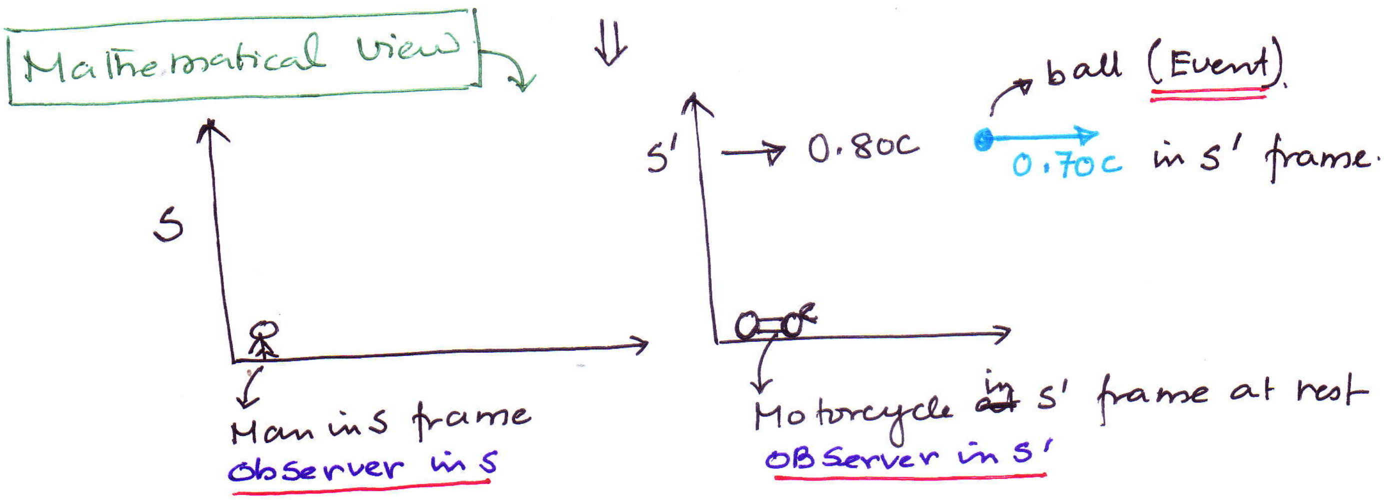


Galilean transf would have given value greater than c

So
$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850c - 0.750c}{1 - \frac{(-0.850c)(0.750c)}{c^2}} = -0.977c$$

Imagine a motor cycle moving with a speed 0.80c past a stationary observer, as shown below. If the rider tosses a ball in the forward direction with a speed of 0.70c relative to himself, what is the speed of the ball relative to stationary observer?



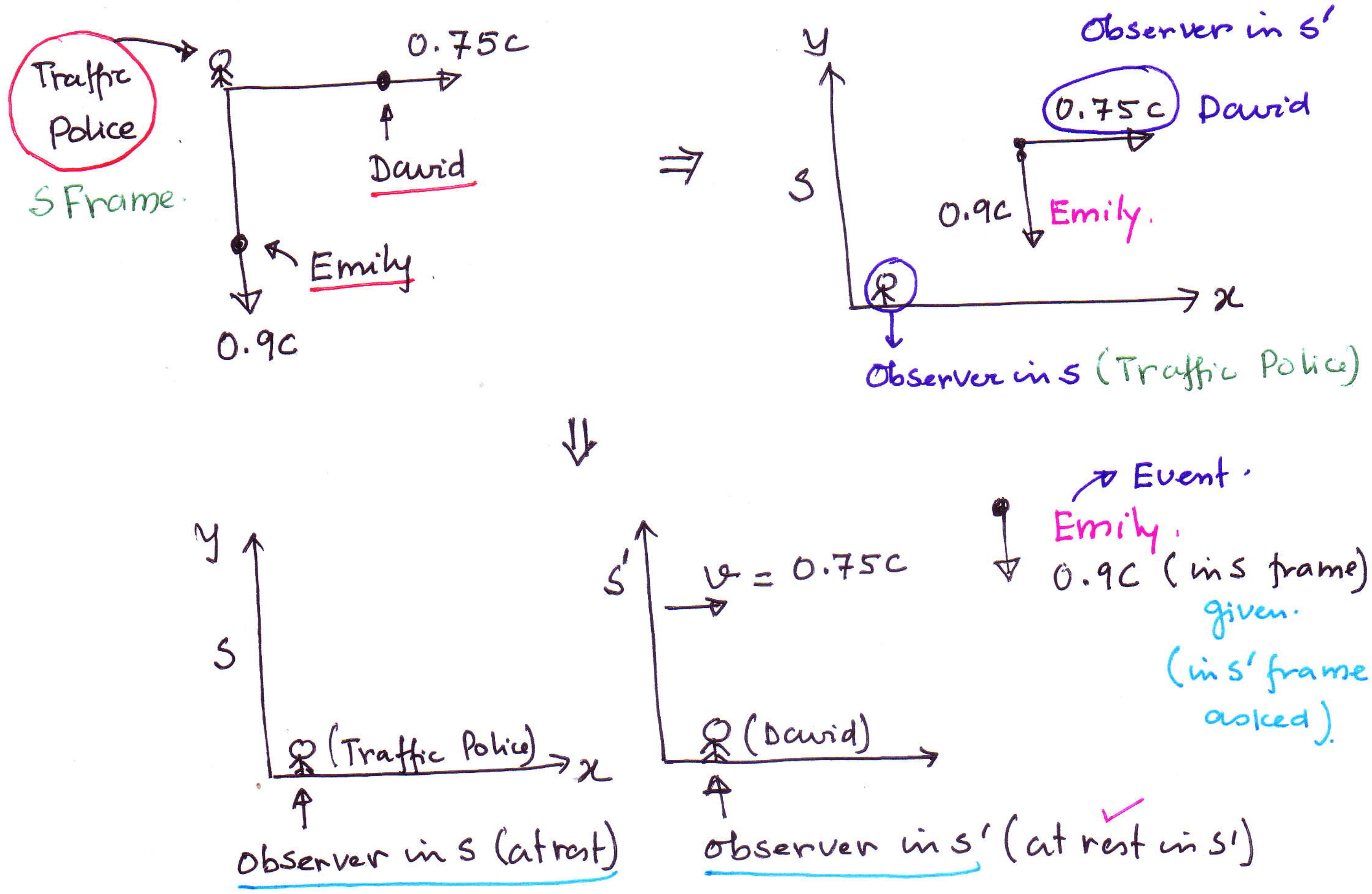


I have to find the velocity of ball (an event) wrt to the stationary observer in S frame That is $\underline{\underline{u_x}}$

Use Inverse Velocity transformation.

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{0.70c + 0.80c}{1 + \frac{(0.70c)(0.80c)}{c^2}} = 0.96c$$

\# Two motor cycle pack leaders named David & Emily are racing at relativistic speeds along perpendicular paths as shown below. How fast does Emily recede as seen by David over his right shoulder.



S frame: $u_y = (-0.9c)$ $\quad u_x = 0.$

S' frame: $u'_y = ?$ $\quad u'_x = ?\ (\neq 0)$

$v = 0.75c$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{\sqrt{1 - \frac{(0.75c)^2}{c^2}}}{\left(1 - \frac{(0)(0.75c)}{c^2}\right)} = -0.60c$$

$$u'_x \neq 0$$

$$= \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{0 - 0.75c}{1 - \frac{(0)(0.75c)}{c^2}} = -0.75c$$

$\Rightarrow u' = \sqrt{(u'_x)^2 + (u'_y)^2} = 0.96c$

Relativistic Momentum & Relativistic Form of Newton's Laws.

With the Lorentz transformation equation it was found that momentum was not conserved when we observe a particular collision in S and S′ frame. Therefore if we are to retain the conservation of momentum as a general law consistent with Einsteins first postulate, we must find a new definition of momentum.

The relativistic formula for the momentum of a particle of mass m moving with velocity $\bar{v}$ is

$$\boxed{\bar{p} = \frac{m\bar{v}}{\sqrt{1-v^2/c^2}} = \gamma m \bar{v}}$$

Here
m = rest Mass.

reduces to old formula $\bar{p} = mv$ in classical limit.

$$p_x = \frac{m v_x}{\sqrt{1-v^2/c^2}} \qquad p_y = \frac{m v_y}{\sqrt{1-v^2/c^2}}$$

Here v is the Speed of the particle as measured in a particular inertial frame. It is not the speed of an inertial frame.

The linear momentum of an isolated system must be conserved in all collisions.

Newton's Second Law:

$$\boxed{\bar{F} = \frac{d\bar{p}}{dt}}$$

"Relativistic Mass." ??

One would alternatively regard the increase in an objects momentum over the classical value as being due to an increase in the objects mass. One may write as

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Then we would call m_0 = rest mass of the object and $m(v)$ its relativistic mass, its mass when moving relative to an observer, so that $p = mv$. This is the view often taken in the past, at one time even by Einstein. However, as Einstein later wrote, the idea of relativistic mass is "not good" because "no clear definition can be given. It is better to introduce no other mass concept than the rest mass "m". In this note the term mass and symbol m will always refer to proper (or rest) mass, which will be considered relativistically invariant.

Now if you see Wikipedia, it says that 60% of the authors are following the idea of ~~relativistic mass~~ invariant mass proposed by Einstein but 40% of the authors are still going through the concept of relativistic mass.— our own book Kleppner is one among them. Where as Resnick Halliday, Beiser, Serway & Jewett are following Einstein that is mass is invariant in all ~~se~~ frames.

Among our instructors two of them are following relativistic mass, other two are invariant mass.

Now you as a student, what should you do? If you get a question about ~~Invariant~~ relativistic mass (which is also there in your assignment) then use the formula

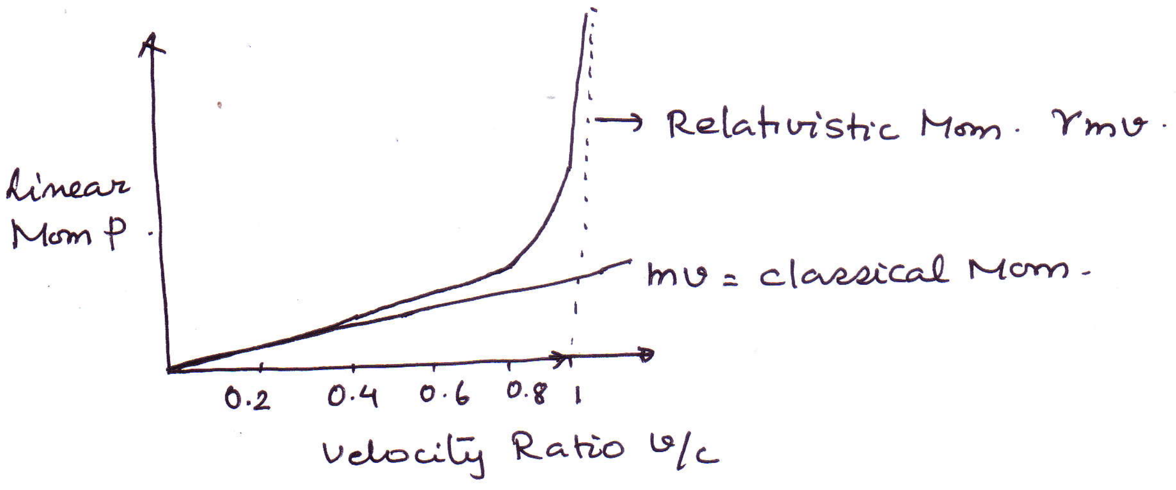
$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

m_0 → rest mass.

$m(v)$ → Relativistic mass

But otherwise in this note we will not follow the above formula. Wherever you find 'm' in this note take it as invariant mass which is rest-mass but 'm_0' will not be written to specify that as rest mass.

Plot of Relativistic Momentum Versus Classical Mom.



The momentum of an object moving at velocity v relative to an observer. The mass m of the object is its value when it is at rest relative to observer. The object's velocity can never reach c because its momentum would then be infinite which is impossible. The relativistic momentum $\gamma m v$ is always correct — the classical momentum mv is valid for velocities much smaller than c.

How to find acceleration of a Particle in Relativity?

Hint: Once relativistic momentum is established, use That for acceleration / Energy or any other place.

#. Find the acceleration of a particle of mass m & velocity $\bar{v}$ when it is acted upon by the Constant Force $\bar{F}$, where $\bar{F}$ is parallel to $\bar{v}$

$$a = \frac{dv}{dt}$$

$$F = \frac{d}{dt}(\gamma m v) = m \frac{d}{dt}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right)$$

$$= m\left[\frac{dv}{dt}\frac{1}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt}\right]$$

$$= m\left[\frac{1}{\sqrt{1-v^2/c^2}} + \frac{v^2/c^2}{(1-v^2/c^2)^{3/2}}\right]\frac{dv}{dt}$$

$$F = \frac{ma}{(1-v^2/c^2)^{3/2}} = \gamma^3 Ma.$$

$$\boxed{a = \frac{F}{m}\frac{1}{\gamma^3}}$$

Two Important Conclusions.

(1) We note That F is equal to $\gamma^3 ma$ & not γma. Merely replacing m by γm in classical Formula does not always give a relativistically Correct Result.

(2) Even Though the force is Constant, the acceleration of the particle decreases as its Velocity increases. As $v \to c$, $a \to 0$, So the particle Can never reach the speed of light – a conclusion We expect.

Energy & Mass in Relativity.

- Just like the case of momentum it was observed that for an elastic collision, energy is conserved ($k_i = k_f$) in S frame but not conserved in S' frame.

- The classical expression for KE ($\frac{1}{2}mv^2$) also violates the second relativity postulate by allowing speeds in excess of the speed of light. There is no limit (in either classical or relativistic dynamics) to the energy we can give to a particle. If we allow the KE to increase without limit, the classical expression $K = \frac{1}{2}mv^2$ implies that velocity must correspondingly increase without limit, thereby violating the second postulate.

Let us derive the expression of KE using work – KE Theorem.

$$KE = \int_0^s F\,ds$$

$\bar{F}$ is in the same direction as $\bar{S}$

In non relativistic physics, the KE of an object of mass m and speed v is $KE = \frac{1}{2}mv^2$. To find the correct relativistic formula for KE we start from relativistic form of the second Law of motion.

$$KE = \int_0^s \frac{d}{dt}(\gamma m v)\,ds = \int_0^{mv} v\,d(\gamma m v) = \int_0^v v\,d\left(\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

Integrating by parts

$$\int x\,dy = xy - \int y\,dx.$$

$$KE = \frac{mv^2}{\sqrt{1-v^2/c^2}} - m\int_0^v \frac{v\,dv}{\sqrt{1-v^2/c^2}}$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + \left[mc^2\sqrt{1-v^2/c^2}\right]_0^v$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

Kinetic Energy $\boxed{KE = \gamma mc^2 - mc^2} = (\gamma-1)mc^2$

This is interpreted as Total Energy E

Then this term is Rest Energy E_0

So $\boxed{E = E_0 + KE}$

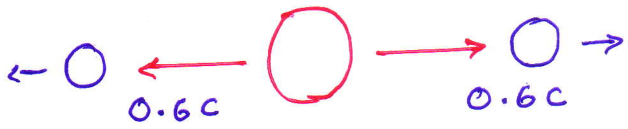
Rest Energy $\Rightarrow$ $E_0 = mc^2$

Total Energy $\Rightarrow$ $E = \gamma mc^2 = \dfrac{mc^2}{\sqrt{1-v^2/c^2}}$

Mass Energy Equivalence: Mass & energy are not independent entities, their separate conservation principles are properly a single one – the principle of conservation of mass energy. Mass can be created or destroyed, but when this happens, an equivalent amount of energy simultaneously vanishes or Comes into being and vice Versa. Mass and Energy are different aspects of the Same thing.

- The rest energy is in effect the total energy of a particle measured in a frame of reference in which the particle is at rest. The rest energy can be regarded as the internal energy of a particle or a system of particles at rest. Whenever we add energy ΔE to a material object that remains at rest, we increase its mass by an amount $= \Delta E/c^2$. We must include rest energy among the kinds of energy that can characterize a system. The sum of all possible kinds of energy that is the total energy must be conserved in any interaction.

\# A Stationary body explodes into two fragments of mass 1.0 kg that move apart at Speeds of 0.6c relative to the Original body. Find the mass of the original body.



The rest energy of the Original body must equal the total energies of the fragments.

$$E_0 = mc^2 = \gamma m_1 c^2 + \gamma m_2 c^2 = \frac{m_1 c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - v_2^2/c^2}}$$

$$m = \frac{E_0}{c^2} = \frac{(2)(1.0\ \text{kg})}{\sqrt{1-(0.60)^2}} = \underline{\underline{2.5}}\ \text{kg}.$$

KE at low speeds.

When the relative speed v is small compared with c, the formula for KE must reduce to the familiar $\frac{1}{2}mv^2$

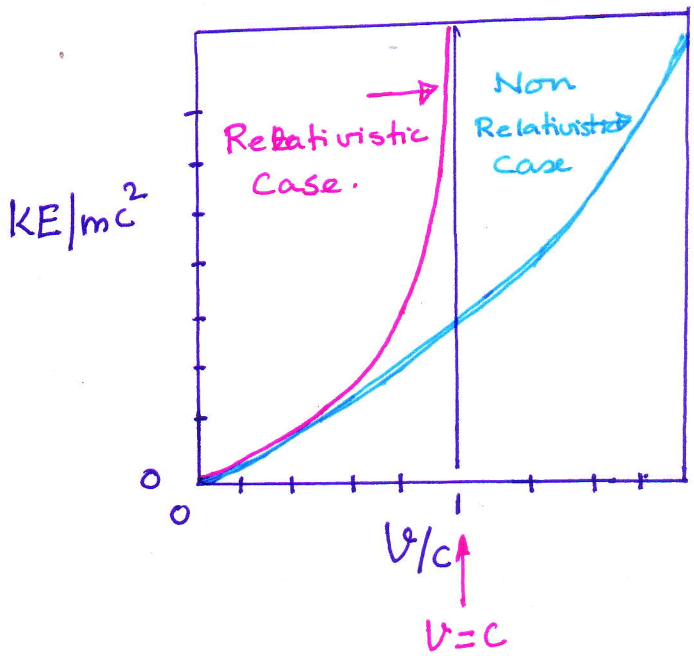
$$KE = \gamma mc^2 - mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

Since $v^2/c^2 \ll 1$ We can use the binomial expansion $(1+x)^n = 1+nx$.

$$\frac{1}{\sqrt{1-v^2/c^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} \qquad v \lll c$$

Thus we have

$$KE = \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)mc^2 - mc^2 \simeq \frac{1}{2}mv^2 \qquad v \lll c$$



- A comparison of relativistic case with non-relativistic case of KE/mc^2 (ratio of KE/rest energy).
- According to Relativistic Mechanics a body would need an infinite KE to travel with speed of light. Whereas in classical Mech it would need only a KE of half its rest energy to have this speed.

Conserved Quantity – Invariant Quantity.

It is important to note the difference between a conserved quantity such as total energy and an invariant quantity such as proper mass. Conservation of E means that in a given reference frame, the total energy of some isolated system remains the same regardless of what events occur in the system. However the total energy may be different as measured from another frame.

On the other hand invariance of m means that m has the same value in all inertial frame.

Any other Invariant Quantity?

- mass m is invariant.
- Space time interval $c^2 \Delta t^2 - \Delta x^2 = \text{Const.}$

Energy & Momentum.

Total energy and momentum are conserved in an isolated system, and the rest energy of a particle is invariant. Let us see how total energy, momentum and rest energy are related.

Total Energy $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \quad -(1)$$

Square it $$E^2 = \frac{m^2c^4}{1 - v^2/c^2} \quad -(2)$$

Momentum $$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \quad -(3)$$

Square $\times c^2$, So $$p^2c^2 = \frac{m^2v^2c^2}{1 - v^2/c^2} \quad -(4).$$

Now Subtract (4) from (2)

$$E^2 - p^2c^2 = \frac{m^2c^4 - m^2v^2c^2}{1 - v^2/c^2} = \frac{m^2c^4(1 - v^2/c^2)}{1 - v^2/c^2} = (mc^2)^2$$

Hence $$\boxed{E^2 = (mc^2)^2 + p^2c^2}$$

Total Energy Rest Energy Momentum.

Important Relation between Total Energy, Rest Energy & Momentum in Relativity.

Now $mc^2 \Rightarrow$ Rest energy is an invariant quantity

So $E^2 - p^2c^2 \Rightarrow$ Also an invariant quantity

$$E^2 - p^2c^2 = (mc^2)^2$$

This should also be invariant.

Rest Mass $\Rightarrow$ Invariant Quantity

The value of an invariant quantity remains same in all reference frame.

Massless Particles.

Can a Massless Particle exist? & exhibits particle like properties Such as energy & momentum

- Classical Mech – NO
- Relativistic Mech – Yes.

In Relativity: Massless Particle

- Not possible $\Rightarrow m=0,\ v<c$
- Possible $\Rightarrow m=0,\ v=c$

$\Downarrow$

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{0}{0}, \quad p = \frac{mv}{\sqrt{1-v^2/c^2}} = \frac{0}{0}$$

If $v=c$, $m=0$

Indeterminate = E & P Can take any Value.

So $E^2 - p^2c^2 = (mc^2)^2 = 0$ (Because $m=0$)

$\Rightarrow E^2 = p^2c^2 \Rightarrow \boxed{E = pc} \Rightarrow$ Photons do exist as massless particles with speed c

Relationship between E & P for $m=0$.

Lorentz Transformation

A reasonable guess about the nature of the correct relationship between x and x' is

$$x' = k(x - vt) \tag{1.33}$$

Here k is a factor that does not depend upon either x or t but may be a function of v. The choice of Eq. (1.33) follows from several considerations:

1 It is linear in x and x', so that a single event in frame S corresponds to a single event in frame S', as it must.
2 It is simple, and a simple solution to a problem should always be explored first.
3 It has the possibility of reducing to Eq. (1.26), which we know to be correct in ordinary mechanics.

Because the equations of physics must have the same form in both S and S', we need only change the sign of v (in order to take into account the difference in the direction of relative motion) to write the corresponding equation for x in terms of x' and t':

$$x = k(x' + vt') \tag{1.34}$$

The factor k must be the same in both frames of reference since there is no difference between S and S' other than in the sign of v.

As in the case of the Galilean transformation, there is nothing to indicate that there might be differences between the corresponding coordinates y, y' and z, z' which are perpendicular to the direction of v. Hence we again take

$$y' = y \tag{1.35}$$

$$z' = z \tag{1.36}$$

The time coordinates t and t', however, are *not* equal. We can see this by substituting the value of x' given by Eq. (1.33) into Eq. (1.34). This gives

$$x = k^2(x - vt) + kvt'$$

from which we find that

$$t' = kt + \left(\frac{1 - k^2}{kv}\right)x \tag{1.37}$$

Equations (1.33) and (1.35) to (1.37) constitute a coordinate transformation that satisfies the first postulate of special relativity.

The second postulate of relativity gives us a way to evaluate k. At the instant $t = 0$, the origins of the two frames of reference S and S' are in the same place, according to our initial conditions, and $t' = 0$ then also. Suppose that a flare is set off at the common origin of S and S' at $t = t' = 0$, and the observers in each system measure the speed with which the flare's light spreads out. Both observers must find the same speed c (Fig. 1.23), which means that in the S frame

$$x = ct \tag{1.38}$$

and in the S' frame

$$x' = ct' \tag{1.39}$$

Substituting for x' and t' in Eq. (1.39) with the help of Eqs. (1.33) and (1.37) gives

$$k(x - vt) = ckt + \left(\frac{1 - k^2}{kv}\right)cx$$

and solving for x,

$$x = \frac{ckt + vkt}{k - \left(\frac{1 - k^2}{kv}\right)c} = ct\left[\frac{k + \frac{v}{c}k}{k - \left(\frac{1 - k^2}{kv}\right)c}\right] = ct\left[\frac{1 + \frac{v}{c}}{1 - \left(\frac{1}{k^2} - 1\right)\frac{c}{v}}\right] \Rightarrow \text{Solving}$$

$$k = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Appendix 1: Proof of Lorentz transformation.