Solution to Gate ST 2023 Q 26

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Question : Consider the following regression model

$$y_k = \alpha_0 + \alpha_1 \log_e k + \epsilon_k, \qquad k = 1, 2, \ldots, n,$$

where ϵ_k's are independent and identically distributed random variables each having probability density function $f(x) = \frac{1}{2}e^{-|x|}, x \in \mathbb{R}$. Then which one of the following statements is true?

(A) The maximum likelihood estimator of α_0 does not exist
(B) The maximum likelihood estimator of α_1 does not exist
(C) The least squares estimator of α_0 exists and is unique
(D) The least squares estimator of α_1 exists, but it is not unique

Solution:

$$f(\epsilon_k) = \frac{1}{2}e^{-|\epsilon_k|} \tag{1}$$

$$\text{Likelihood function} : f(\epsilon_1\epsilon_2....\epsilon_n) = \prod_{k=1}^{n} f(\epsilon_k) \tag{2}$$

$$L = \prod_{k=1}^{n} \frac{1}{2}e^{-|\epsilon_k|} \tag{3}$$

$$L_1 = \ln L = \ln\left(\prod_{k=1}^{n} \frac{1}{2}e^{-|\epsilon_k|}\right) \tag{4}$$

$$= \sum_{k=1}^{n} \ln\left(\frac{1}{2}e^{-|\epsilon_k|}\right) \tag{5}$$

$$= \sum_{k=1}^{n} \left(-\ln 2 - |y_k - \alpha_0 - \alpha_1 \log_e k|\right) \tag{6}$$

$$= -n\ln 2 - \sum_{k=1}^{n} \left(|y_k - \alpha_0 - \alpha_1 \log_e k|\right) \tag{7}$$

$$L_1 = \text{function of } \alpha_0, \alpha_1 \tag{8}$$

1) Maximum likelihood estimator
We need to find the value of α_0 and α_1 which will maximise the value of L_1 i.e. the value of α_0 and

α_1 which will minimise the value of $\sum_{k=1}^{n} |y_k - \alpha_0 - \alpha_1 \log_e k|$

$$|y_k - \alpha_0 - \alpha_1 \log_e k| = \left| \mathbf{y_k} - \begin{pmatrix} 1 & \log_e k \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \right| \tag{9}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{10}$$

$$\mathbf{A} = \begin{pmatrix} 1 & \log_e 1 \\ 1 & \log_e 2 \\ \vdots & \vdots \\ 1 & \log_e n \end{pmatrix} \tag{11}$$

$$\mathbf{x} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \tag{12}$$

$$\begin{aligned} \min_{\alpha_0, \alpha_1} \quad & \mathbf{y} - \mathbf{Ax} \\ \text{s.t.} \quad & \mathbf{y} - \mathbf{Ax} \geq 0 \end{aligned}$$

$$\begin{aligned} \min_{\alpha_0, \alpha_1} \quad & -(\mathbf{y} - \mathbf{Ax}) \\ \text{s.t.} \quad & \mathbf{y} - \mathbf{Ax} \leq 0 \end{aligned}$$