Solution to Q11.16.3.17

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Question: Determine the probability p, for each of the following events.
(a) An odd number appears in a single toss of a fair die.
(b) At least one head appears in two tosses of a fair coin.
(c) A king, 9 of hearts, or 3 of spades appears in drawing a single card from a well shuffled ordinary deck of 52 cards.
(d) The sum of 6 appears in a single toss of a pair of fair dice.

Solution:

parameter	**value**	**description**
Random Variable X	0	Odd number appears in the throw
	1	Even number appears in the throw

TABLE (a)
SINGLE TOSS OF DIE

(a)

$$p = P_X(0) = \frac{1}{2} \tag{1}$$

parameter	**value**	**description**
Random Variable X	0	No head
	1	One head
	2	Two heads

TABLE (b)
TWO COIN TOSS

(b)

$$P_X(1) = \frac{1}{2} \tag{2}$$

$$P_X(2) = \frac{1}{4} \tag{3}$$

$$p = P_X(1) + P_X(2) \tag{4}$$

$$p = \frac{3}{4} \tag{5}$$

parameter	**value**	**description**
Random Variable X	0	King appears in the draw
	1	9 of hearts appears in the draw
	2	3 of spades appears in the draw
	3	Any other card appears in the draw

TABLE (c)
DECK OF CARDS

(c)

$$P_X(0) = \frac{1}{13} \tag{6}$$
$$P_X(1) = \frac{1}{52} \tag{7}$$
$$P_X(2) = \frac{1}{52} \tag{8}$$
$$p = P_X(0) + P_X(1) + P_X(2) \tag{9}$$
$$p = \frac{3}{26} \tag{10}$$

(d) The Z-transform of a die is defined as

$$M_X(z) = z^{-X} = \sum_{k=-\infty}^{\infty} p_X(k) z^{-k} \tag{11}$$

The Z-transform of the first die X_1 is given by

$$M_{X_1}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} = \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})}, |z| > 1 \tag{12}$$

The Z-transform of the second die X_2 is given by

$$M_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} = \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})}, |z| > 1 \tag{13}$$

The Z-transform of X is given as:

$$M_X(z) = \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})} \times \frac{z^{-1}(1-z^{-6})}{6(1-z^{-1})} \tag{14}$$
$$M_X(z) = \frac{1}{36}\left[\frac{z^{-2}(1-2z^{-6}+z^{-12})}{(1-z^{-1})^2}\right] \tag{15}$$

We also know that,

$$p_X(n-k) \overset{Z}{\longleftrightarrow} M_X(z) z^{-k}; \tag{16}$$
$$nu(n) \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \tag{17}$$

Hence, after some algebra, it can be shown that,

$$\frac{1}{36}[(n-1)u(n-1) - 2(n-7)u(n-7) + (n-13)u(n-13)] \quad \overset{Z}{\longleftrightarrow} \quad \frac{1}{36}\left[\frac{z^{-2}1-2z^{-6}+z^{-12}}{(1-z^{-1})^2}\right] \tag{18}$$

where,

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{19}$$

hence,

$$p_X(n) = \frac{1}{36}[(n-1)u(n-1) - 2(n-7)u(n-7) + (n-13)u(n-13)] \tag{20}$$

$$p_X(n) = \begin{cases} 0 & n \leq 1 \\ \frac{n-1}{36} & 2 \leq n \leq 6 \\ \frac{13-n}{36} & 7 \leq n \leq 12 \\ 0 & n \geq 13 \end{cases} \tag{21}$$

Hence, the probability ,

$$p = p_X(6) = \frac{5}{36} \tag{22}$$

The experiment of rolling the dice was simulated using Python for 10000 samples.

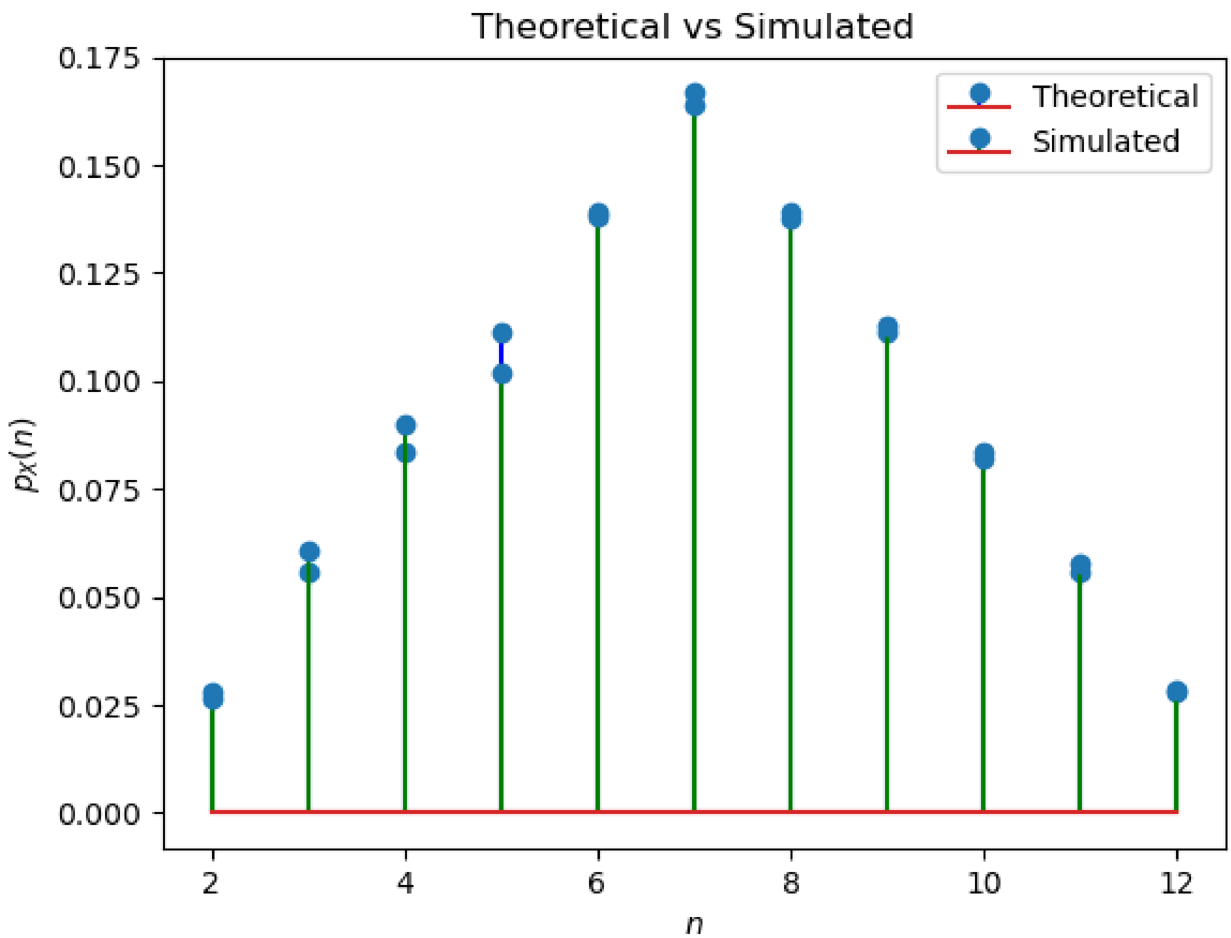

Fig. (d). Plot of $p_X(n)$.Simulations are close to the analysis.