Indian Institute of Information Technology Kota, India
Department of Mathematics

Tutorial Sheet (MAT-101: Mathematics I)

Topics: Differential Equations – Differential equations of first order and first degree - linear form, reducible to linear form, exact form, reducible to exact form. Linear differential equations of higher order with constant coefficients. Second order ordinary differential equations with variables coefficients – Homogeneous, exact form, reducible to exact form, change of dependent variable (normal form), change of independent variable, method of variation of parameters.

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1. Find the integrating factor of the following differential equations:

   (i) $(8ydx + 8xdy) + x^2y^3\,(4ydx + 5xdy) = 0.$ Ans. $xy.$

   (ii) $(x^3 + y^3)\,dx - xy^2dy = 0.$ Ans. $\dfrac{1}{x^4}.$

   (iii) $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0.$ Ans. $\dfrac{1}{3x^2y^2}.$

2. Obtain the value of λ such that given differential equation is exact:
   $(xy^2 + \lambda x^2y)\,dx + x^2\,(x + y)\,dy = 0.$ Ans. $\lambda = 3.$

3. Find the solution of the following differential equations:

   (i) $(2xy\cos x^2 - 2xy + 1)\,dx + (\sin x^2 - x^2 + 3)\,dy = 0.$ Ans. $y\sin x^2 - yx^2 + x + 3y = C.$

   (ii) $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$ Ans. $x\left(y + \dfrac{2}{y^2}\right) + y^2 = C.$

   (iii) $y\,(xy + 2x^2y^2)\,dx + x\,(xy - x^2y^2)\,dy = 0.$ Ans. $-\dfrac{1}{xy} + 2\log x - \log y = C.$

   (iv) $\left(1 + e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0.$ Ans. $x + ye^{x/y} = C.$

   (v) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$ Ans. $xy + 2\dfrac{x}{y^2} + y^2 = c.$

   (vi) $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0.$ Ans. $\dfrac{x}{y} + \log\dfrac{y^3}{x^2} = c.$

   (vii) $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$ Ans. $\dfrac{x^2}{y} = ce^{\frac{1}{xy}}.$

4. Solve the following initial value problems:

   (i) $(x - y\cos x)\,dx - \sin x dy = 0, \quad y(\frac{\pi}{2}) = 1.$ Ans. $x^2 - 2y\sin x = \left(\dfrac{\pi^2}{4}\right) - 2.$

   (ii) $(2xy + e^y)\,dx + (x^2 + xe^y)\,dy = 0, \quad y(1) = 1.$ Ans. $x^2y + xe^y = e + 1.$

5. Solve the differential equation $x\,(1 + y^2)\,dy + y\,(1 + x^2)\,dx = 0$ by finding an integrating factor by inspection. Hint: $\dfrac{xdy + ydx}{xy} + ydy + xdx = 0.$

6. Show that $F(x, y)$ is an integrating factor of $M(x, y)dx + N(x, y)dy = 0$, if and only if

$$\left(M\frac{\partial F}{\partial y} - N\frac{\partial F}{\partial x}\right) + \left(M\frac{\partial M}{\partial y} - N\frac{\partial N}{\partial x}\right) = 0.$$

7. Solve following differential equations:

   (i) $(3 + 2\sin x + \cos x)dy = (1 + 2\sin y + \cos y)dx$.
   Ans. $\tan^{-1}\left(1 + \tan\left(\frac{x}{2}\right)\right) = \frac{1}{2}\log\left(1 + 2\tan\left(\frac{x}{2}\right)\right) + c$.

   (ii) $\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}$. Ans. $\left(x - \frac{1}{5}\right)^2 - \left(x - \frac{1}{5}\right)\left(y - \frac{7}{5}\right) - \left(y - \frac{7}{5}\right)^2 = c$.

8. Find the solution of the differential equation $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$ and show that solution is having asymptotes $x + y = 0, 2x + y + 1 = 0$.
   Ans. $2x^2 + x + 3xy + y^2 + y = c$.

9. If $x^h y^k$ is integrating factor of differential equation $(y^2 + 2x^2 y)dx + (2x^3 - xy)dy = 0$, then find the values of h and k.
   Ans. $h = -\frac{5}{2}, k = -\frac{1}{2}$.

10. Solve the following second order differential equations:

    (a) $D^2 y + y = x^3 + e^x \sin x$. Ans. $y = c_1 \cos x + c_2 \sin x + (x^3 - 6x) - \frac{e^x(2\cos x - \sin x)}{5}$.

    (b) $(D^2 - 2D + 1)y = x \sin x$. Ans. $y = (c_1 + c_2 x)e^x + \frac{(x + 1)\cos x}{2} - \frac{\sin x}{2}$.

    (c) $D^2 y + n^2 y = \sec nx$.
    Ans. $y = c_1 \cos(nx) + c_2 \sin(nx) + \frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\log(\cos(nx))$.

11. Solve following Cauchy-Euler linear equation:

    (i) $x^4 \frac{d^3 y}{dx^3} + 2x^3 \frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1$. Ans. $y = (c_1 + c_2 \log x)x + c_3 x^{-1} + \frac{1}{4}x^{-1}\log x$.

    (ii) $(1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4\cos\log(1 + x)$.
    Ans. $y = c_1 \cos\log(1 + x) + c_2 \sin\log(1 + x) + 2\log(1 + x)\sin\log(1 + x)$.

12. Reduce the differential equation $\frac{d^2 y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x$ into normal form and obtain the solution.
    Ans. $y = (c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x)\sec x + \frac{1}{7}e^x$.

13. Apply the method of variation of parameters to solve the following equations:

    (i) $(1 - x)\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = (1 - x)^2$. Ans. $y = (c_1 e^x + c_2 x) + 1 + x + x^2$.

    (ii) $\frac{d^2 y}{dx^2} + (1 - x)\frac{dy}{dx} - y\cot x = \sin^2 x$.
    Ans. $c_1(\sin x - \cos x) + c_2 e^{-x} - \frac{1}{2}\cos x(\sin x - \cos x) + \frac{1}{20}(3\sin 2x - \cos 2x - 5)$.

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