$f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

1) $m = 1$ (funzione scalare)

2) $n = 1$ $m \geq 1$ curve

3) $n = 2$ $m \geq 2$ superfici

4) $n < m$ varietà $n$-dimensionali in $\mathbb{R}^m$

5) $n = m$

$f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$

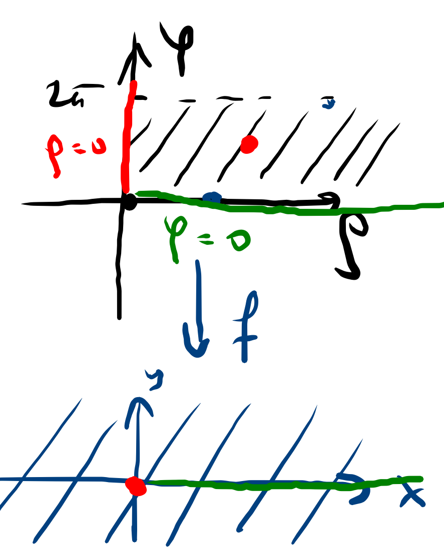
cambio di coordinate

# Esempi

**1°** $\quad f : \underbrace{[0,+\infty)}_{\rho} \times \underbrace{[0,2\pi)}_{\varphi} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

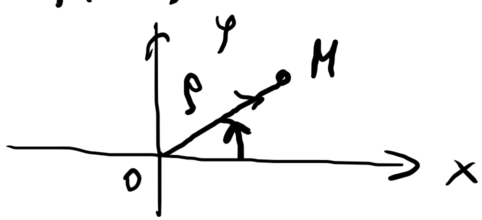
$(n = m = 2)$

$[0,2\pi]$, oppure $\mathbb{R}$ ...me identificando $\varphi$ con $\varphi + 2k\pi$, $k \in \mathbb{Z}$




$$f(\rho,\varphi) = (\rho\cos\varphi, \rho\sin\varphi)$$

$f$ : coordinate polari in $\mathbb{R}^2$ $(\rho,\varphi)$ $\longmapsto$ coordinate cartesiane $(x,y)$




$$f(\rho,\varphi) = (x(\rho,\varphi), y(\rho,\varphi))$$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

$$Df(\rho,\varphi) = \begin{pmatrix} \dfrac{\partial x}{\partial\rho} & \dfrac{\partial x}{\partial\varphi} \\[2ex] \dfrac{\partial y}{\partial\rho} & \dfrac{\partial y}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix}$$

matrice jacobiana

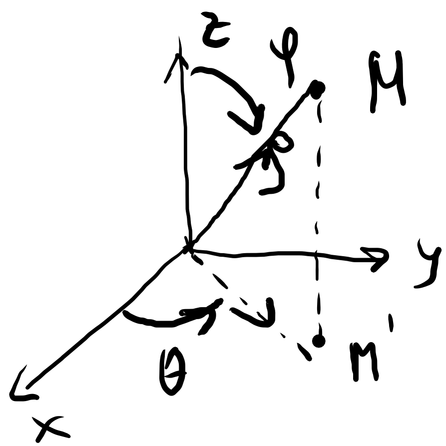
$$\det Df(\rho,\varphi) = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho$$

jacobiano

$\mathbb{R}$

$$2^\circ)\quad f : \underbrace{(\rho,\theta,\varphi)}_{\text{coordinate sferiche}} \in [0,+\infty)\times[0,2\pi)\times[0,\pi] \longmapsto$$

$$\longmapsto \underbrace{\big(\underbrace{\rho\cos\theta\sin\varphi}_{=\,x},\ \underbrace{\rho\sin\theta\sin\varphi}_{=\,y},\ \underbrace{\rho\cos\varphi}_{=\,z}\big)}_{\text{coord. cartesiane}}$$



$$Df(\rho,\varphi,\theta) = \begin{pmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\theta \sin\varphi & -\rho \sin\theta \sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}$$

$$J(\rho,\varphi,\theta) := \det Df(\rho,\varphi,\theta) = \rho^2 \sin\varphi \begin{vmatrix} \cos\theta\sin\varphi & -\sin\theta & \cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \cos\theta & \sin\theta\cos\varphi \\ \cos\varphi & 0 & -\sin\varphi \end{vmatrix}$$

$$= \rho^2\sin\varphi \left( \cos\varphi\left(-\sin^2\theta \underline{\cos\varphi} - \cos^2\theta\, \underline{\cos\varphi}\right) - \sin\varphi\left(\cos^2\theta\, \underline{\sin\varphi} + \sin^2\theta\, \underline{\sin\varphi}\right)\right)$$

$$= \rho^2\sin\varphi\left(\cos^2\varphi\cdot(-1) - \sin^2\varphi\left(\cancelto{1}{\cos^2\theta+\sin^2\theta}\right)\right) =$$

$$= -\rho^2\sin\varphi$$

3°) $f : [0,+\infty) \times \cancel{[0,2\pi)} \times \mathbb{R} \to \mathbb{R}^3$

$\mathbb{R}$

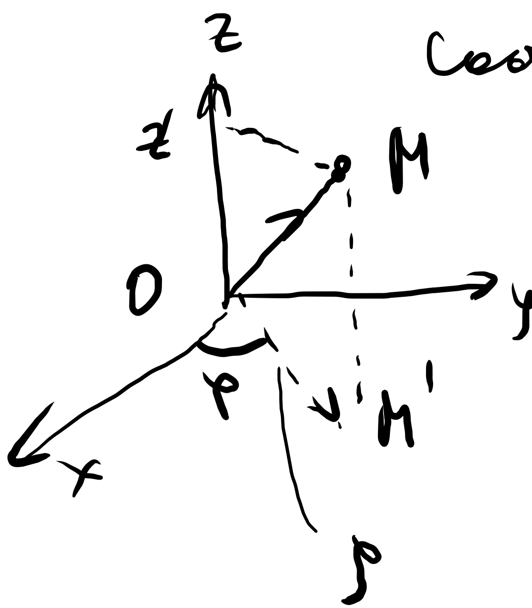
$$\underbrace{(\rho, \varphi, z')}_{\text{coord. cilindriche}} \in [0,+\infty) \times [0,2\pi) \times \mathbb{R} \mapsto$$

$$\mapsto \underbrace{(\underset{x}{\underset{\parallel}{\rho\cos\varphi}}, \underset{y}{\underset{\parallel}{\rho\sin\varphi}}, \underset{z}{\underset{\parallel}{z'}})}_{\text{coordinate cartesiane}}$$

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$z = z'$



$$Df(\rho, \varphi, z) = \left( \begin{array}{cc|c} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ \hline 0 & 0 & 1 \end{array} \right)$$

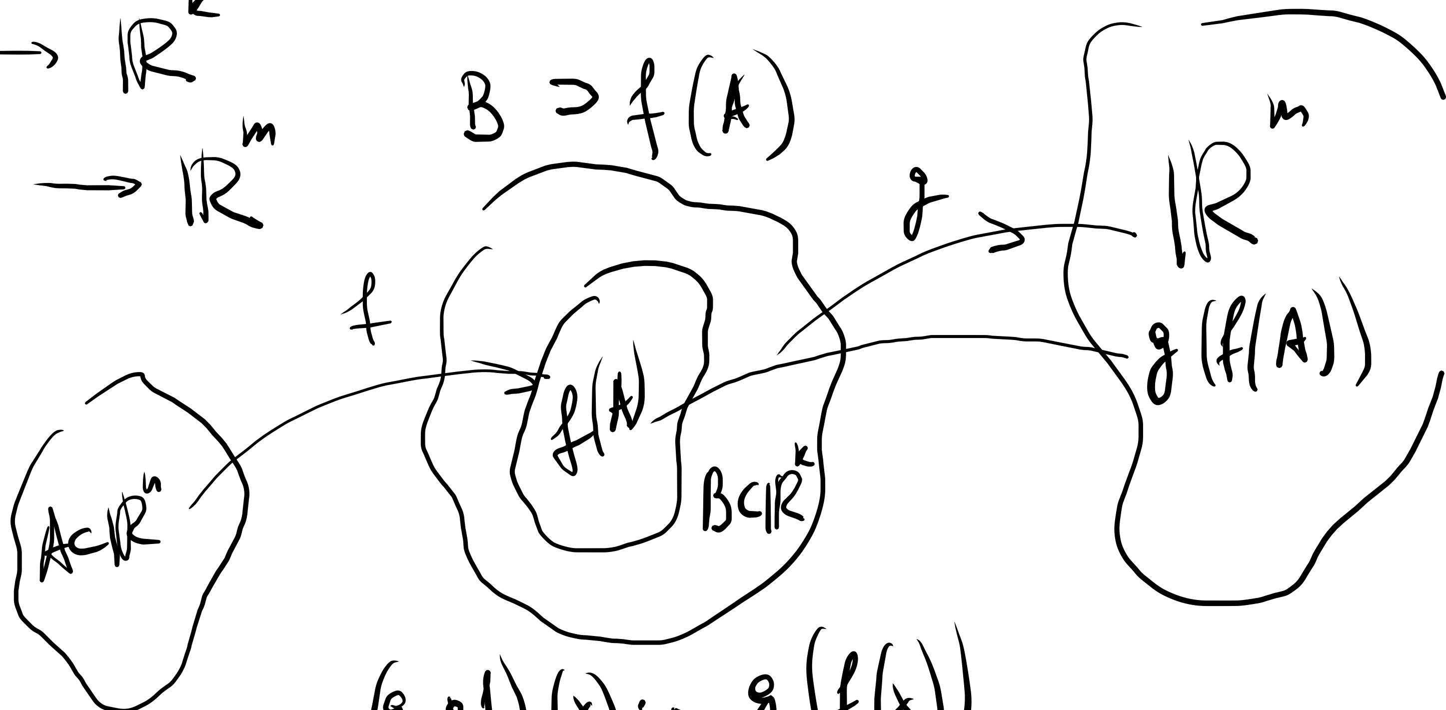
$$J(\rho, \varphi, z) = \det Df(\rho, \varphi, z) = \rho$$

# Differenziale di funzioni Composte

$f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^k$

$g : B \subset \mathbb{R}^k \longrightarrow \mathbb{R}^m$

$B \supset f(A)$




$(g \circ f)(x) := g(f(x))$

$g \circ f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Teorema

Sia $x_0$ punto interno di $A$

$f(x_0)$ —— " —— di $B$

$f$ differenziabile in $x_0$

$g$ —— " —— in $f(x_0)$

Allora, $g \circ f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ è differenziabile in $x_0$ e

$$\underbrace{D(g \circ f)(x_0)}_{m \times n} = \underbrace{(Dg)(f(x_0))}_{m \times k}\,\underbrace{(Df)(x_0)}_{k \times n}$$

## Casi particolari

1) $m = k = n = 1$ $\quad A, B$ intervalli

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$$

2) $m = 1$, $n = 1$, $A$ intervallo

$\gamma : A \subset \mathbb{R} \to \mathbb{R}^k$

$g : B \subset \mathbb{R}^k \to \mathbb{R}$

$B \supset \gamma(A)$

$g \circ \gamma : A \subset \mathbb{R} \to \mathbb{R}$

$(g \circ \gamma)'(t_0) =$

$= \nabla g(\gamma(t_0)) \cdot \dot{\gamma}(t_0)$

$$\nabla g(\gamma(t_0))^T \dot{\gamma}(t_0) = \underbrace{\nabla g(\gamma(t_0))}_{\text{matrice } 1 \times k} \underbrace{\dot{\gamma}(t_0)}_{\text{vettore colonna (ovvero matrice } k \times 1)}$$

$\nabla g(\gamma(t_0))$: vettore (colonna

3) $f: A \subset \mathbb{R}^n \to \mathbb{R}$

$g: \mathbb{R} \to \mathbb{R}$

$x_0 \in A$ interno, $f$ è diff in $x_0$,

$g$ — " — in $f(x_0)$

$$g \circ f : A \subset \mathbb{R}^n \to \mathbb{R}$$

$$\nabla (g \circ f)(x_0) = g'(f(x_0)) \nabla f(x_0)$$

**Esercizi**

1) $h(x,y) = \sin(x^2+y^2)$

$h = g \circ f \qquad g(z) = \sin z$

$\qquad\qquad\quad f(x,y) = x^2+y^2$

$$\nabla g(x,y) = g'(f(x,y))\,\nabla f(x,y) =$$

$$= \cos(x^2+y^2)\,(2x,\ 2y) =$$

$$= (2x\cos(x^2+y^2),\ 2y\cos(x^2+y^2)).$$

2).

$$f(x,y) = x^2+y^2$$

$$\gamma(t) = (a\cos t, b\sin t) \qquad a,b>0$$

$$(f\circ\gamma)(t)' = \nabla f(\gamma(t))\cdot\dot\gamma(t) =$$

$$= (2\gamma_x(t), 2\gamma_y(t))\cdot(\dot\gamma_x(t), \dot\gamma_y(t))$$

$$= (2a\cos t, 2b\sin t)\cdot(-a\sin t, b\cos t)$$

$$= -2a^2\cos t\sin t + 2b^2\sin t\cos t =$$

$$= 2(b^2-a^2)\sin t\cos t$$

Volendo, si può fare anche direttamente

$$(f\circ\gamma)'(t) = (a^2\cos^2 t + b^2\sin^2 t)' =$$

$$= 2(b^2-a^2)\sin t\cos t$$

# Applicazioni

**Def.** $f : A \subset \mathbb{R}^n \to \mathbb{R}$ (scalare)

differenziabile

$x_0$ è punto di $\overset{\text{massimo}}{\text{minimo}}$ locale di $f$

se $\exists\, \rho > 0 \;:\; f(x_0) \overset{\geq}{\leq} f(x)$ $\forall x \in B_\rho(x_0)$

Se inoltre $f(x_0) \overset{>}{<} f(x)$ $\forall x \in B_\rho(x_0)$, $x \neq x_0$,

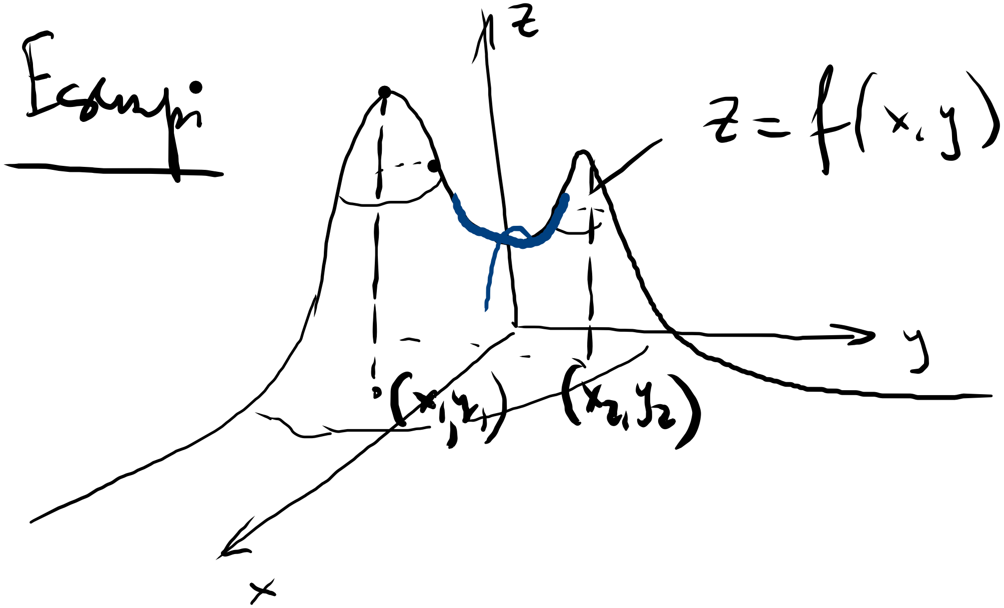
allora $x_0$ si dice di $\overset{\text{massimo}}{\text{minimo}}$ locale stretto

$x_0$ è punto di $\overset{\text{massimo}}{\text{minimo}}$ (globale), se $f(x) \overset{\geq}{\leq} f(x_0)$ $\forall x \in A$

Se inoltre $f(x_0) \overset{>}{<} f(x)$ $\forall x \in A$, $x \neq x_0$,

allora $x_0$ è di $\overset{\text{massimo}}{\text{minimo}}$ (globale) stretto

## Esempi




$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x_1,y_1) > f(x_2,y_2) > 0$

$\lim_{(x,y)\to\infty} f(x,y) = 0$

$f(x,y) > 0$

$(x_1,y_1)$ e $(x_2,y_2)$ sono punti di massimo locale di $f$ (anche stretto)

$(x_1,y_1)$ è anche punto di massimo (globale), anche strett

$(x_2,y_2)$ non è di massimo globale, solo locale

non ci sono minimi globali

Th (Fermat) Se $f: A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$
aperto
ammette minimo (o massimo) locale in $x_0 \in A$, ed è derivabile in direzione di $\overline{n}$ in $x_0$, allora

$$\frac{\partial f}{\partial \overline{n}}(x_0) = 0$$

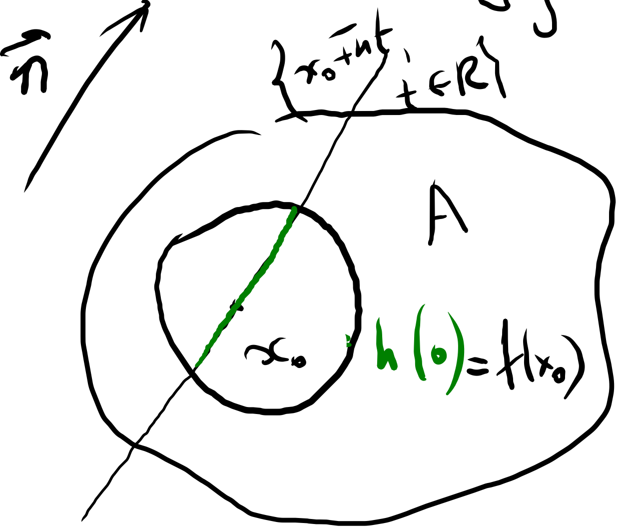
<u>Corollario</u> Sotto ipotesi del th di Fermat è f è derivabile in tutte le direzioni in $x_0$, allora

$$\nabla f(x_0) = 0$$

In particolare questo vale per f differenziabili in $x_0$.

**Dim** $h(t) := f(x_0 + \bar{u}t) \qquad t \in \mathbb{R}$

$\exists \rho > 0 :$ $h$ è definita su $(-\rho, \rho) \subset \mathbb{R}$, ed inoltre $0$ è punto di minimo/massimo locale per $h$

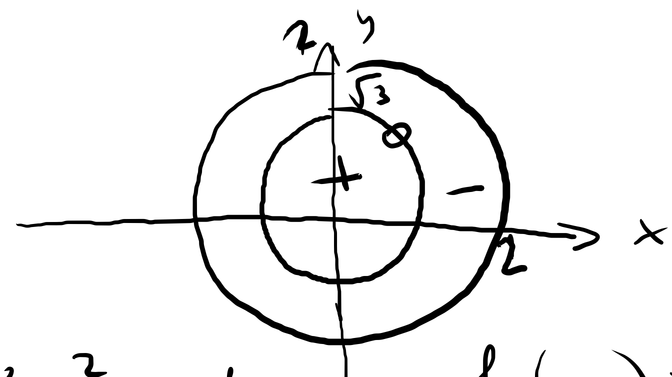



$$\underset{\displaystyle 0}{\underset{\displaystyle \parallel}{h'(0)}} =: \frac{\partial f}{\partial \bar{u}}(x_0), \quad \underline{\text{q.e.d.}}$$

Esempio. $f(x,y) := \log(4-x^2-y^2) \rightarrow \max$

$$D(f) := \{(x,y) \in \mathbb{R}^2 : 4-x^2-y^2 > 0\} =$$

$$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 4\}$$




$$4-x^2-y^2 \geq 1 \iff f(x,y) \geq 0$$

$$\Updownarrow$$

$$x^2+y^2 \leq 3$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{1}{4-x^2-y^2}\, 2x = 0 \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{1}{4-x^2-y^2}\, 2y = 0 \end{cases} \iff (x,y) = 0$$

punto di massimo

Da ricordare : $f : I \subset \mathbb{R} \to \mathbb{R}$

$I$ intervallo aperto

Th. di Fermet

1) $x_0 \in I$ $\exists f'(x_0)$ ; $x_0$ punto di massimo / minimo locale di $f$

$$\Rightarrow f'(x_0) = 0$$

2) $x_0 \in I$, $\exists f'(x_0)$, $f''(x_0)$

$f'(x_0) = 0$, $f''(x_0) < 0 \Rightarrow x_0$ è di massimo locale

$f'(x_0) = 0$, $f''(x_0) > 0 \Rightarrow x_0$ è di min locale

# Derivate di ordine superiore

$$f : A \subset \mathbb{R}^n \to \mathbb{R}, \qquad x_0 \in A$$

$A$ aperto

$$\nabla f(\cancel{x_0}) = \left( f_{x_1}(\cancel{x_0}), \dots, f_{x_n}(\cancel{x_0}) \right)$$

$$\frac{\partial f_{x_j}}{\partial x_k} =: \frac{\partial^2 f}{\partial x_j \partial x_k}$$

$$\frac{\partial f_{x_k}}{\partial x_k} =: \frac{\partial^2 f}{\partial x_k^2}$$

$$\frac{\partial}{\partial x_m}\left( \frac{\partial^2 f}{\partial x_j \partial x_k} \right) =: \frac{\partial^3 f}{\partial x_j \partial x_k \partial x_m}$$

$k = j$: $\quad \dfrac{\partial^3 f}{\partial x_j^2 \partial x_m}$

$k = j = m$: $\quad \dfrac{\partial^3 f}{\partial x_j^3}$

Notazione comoda

multiindice $(\alpha_1, \dots, \alpha_n)$ $\alpha_j \in \mathbb{N}$

$$D^\alpha f = D^{(\alpha_1, \dots, \alpha_n)} f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$$

Esempi

- $n = 3$ $\alpha = (1, 0, 2)$ $|\alpha| = 3$

$$D^\alpha f = \frac{\partial^3 f}{\partial x_1 \partial x_3^2}$$

- $n = 5$ $\alpha = (2, 1, 0, 1, 3)$, $|\alpha| = 7$

$$D^\alpha f = \frac{\partial^7 f}{\partial x_1^2 \partial x_2 \partial x_4 \partial x_5^3}$$

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \qquad f = f(x,y)$$

$$D^2 f = \text{Hess} f := \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

<u>Esempio</u> $f(x,y) = e^{-x^2+3y^2}$

$$\nabla f(x,y) = e^{-x^2+3y^2} \left( -2x, 6y \right) =$$

$$= \left( -2x e^{-x^2+3y^2}, 6y e^{-x^2+3y^2} \right)$$

$$D^2 f(x,y) = \begin{pmatrix} -2e^{-x^2+3y^2} + (2x)^2 e^{-x^2+3y^2} & -2x e^{-x^2+3y^2} \cdot 6y \\ 6y e^{-x^2+3y^2} \cdot (-2x) & 6e^{-x^2+3y^2} + (6y)^2 e^{-x^2+3y^2} \end{pmatrix}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$