·) STAZIONARIETÀ IN SENSO LATO (SSL)

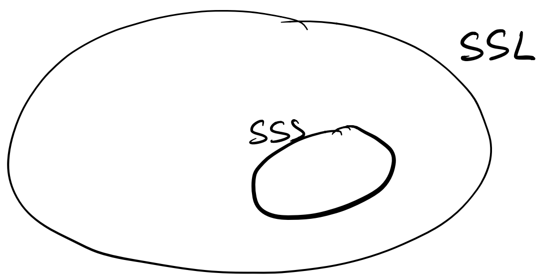
·) Un processo aleatorio è SSL se

1) il suo valor medio è costante $\eta_X(t) = \eta_X$

2) la sua autocorrelazione dipende dalla differenza $t_1 - t_2$ e non separatamente da t_1 e t_2

$$R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$$

$$\tau = t_1 - t_2$$



SSS $\Rightarrow$ SSL

SSL $\not\Rightarrow$ SSS

·) AUTOCOVARIANZA DI UN PROCESSO SSL

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \eta_X^2(t)$$

$$= R_X(t_1 - t_2) - \eta_X^2$$

$$= C_X(\tau) \qquad , \tau = t_1 - t_2$$

·) PROPRIETÀ DELLA AUTOCORRELAZIONE DI UN PROCESSO SSL

1) $R_X(\tau) = R_X(-\tau)$

Dim.

$$R_X(\tau) = E[X(t)\, X(t - \tau)]$$

$t = t_1$

$\tau = t_1 - t_2)$

$$= E\left[X(t'+\tau)\,X(t')\right] \qquad t-\tau = t'$$

$$= E\left[X(t')\,X(t'-(-\tau))\right]$$

$$= R_X(-\tau)$$

2) $R_X(0) = E\left[X(t)\,X(t)\right] = E\left[X^2(t)\right]$

$$= P_X \geq 0$$

3) $R_X(0) \geq |R_X(\tau)|$

Dim

$$E\left[\left\{X(t) \pm X(t-\tau)\right\}^2\right] \geq 0$$

$$\underbrace{E\left[X^2(t)\right]}_{P_X} + \underbrace{E\left[X^2(t-\tau)\right]}_{P_X} \pm 2E\left[X(t)\,X(t-\tau)\right] \geq 0$$

$$\cancel{2}P_X \pm \cancel{2}R_X(\tau) \geq 0$$

$$\begin{cases} P_X \geq -R_X(\tau) \\ P_X \geq R_X(\tau) \end{cases} \Rightarrow P_X \geq |R_X(\tau)|$$

$$\Downarrow$$

$$R_X(0) \geq |R_X(\tau)|$$

4) Se la $R_X(\tau)$ non contiene componenti periodiche

$$\Rightarrow \lim_{\tau \to \infty} R_X(\tau) = \eta_X^2$$

SIGNIFICATO DELLA AUTOCORRELAZIONE

$N(t)$ processo di rumore

$N(t_1)$ $N(t_2)$ 2 campioni del rumore

$\Downarrow$

2 V.A. estratte dal processo di rumore

$\Downarrow$

sono correlate ??

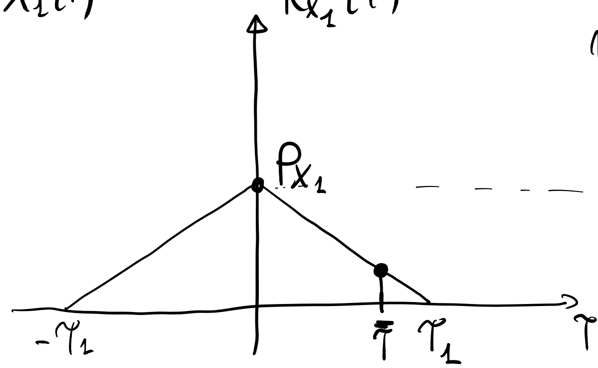
2 V.A. sono correlate se la loro covarianza è non nulla

$$R_X(\tau) = C_X(\tau) + \eta_X^2$$

$\Rightarrow$ spesso η_X nei processi di rumore presenti nei sistemi di comunicazione è nullo

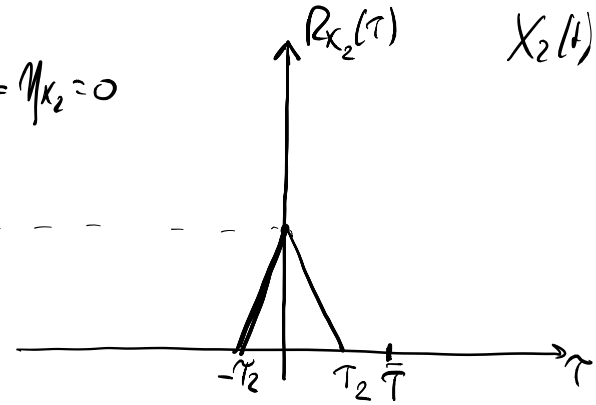
$$\eta_X = 0 \Rightarrow \eta_X^2 = 0 \Rightarrow R_X(\tau) = C_X(\tau)$$

$X_1(t)$



$\eta_{x_1} = \eta_{x_2} = 0$

$X_2(t)$



$\tau_1 > \tau_2$

$\Downarrow$

V.A. incorrelati

$\bar{\tau} = t_1 - t_2$

$X(t_1), X(t_2)$

$t_1 - t_2 = \bar{\tau}$

$\Downarrow$

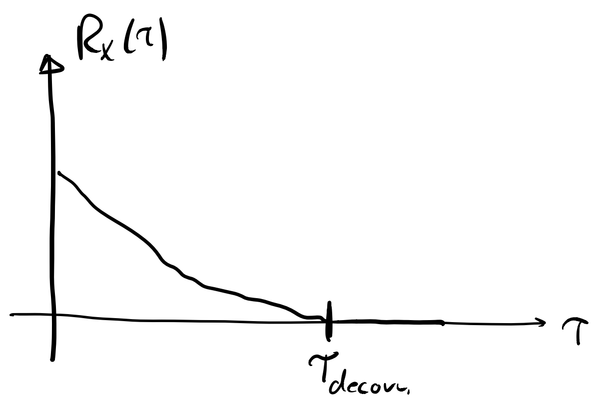
V.A. correlati

TEMPO DI DECORRELAZIONE

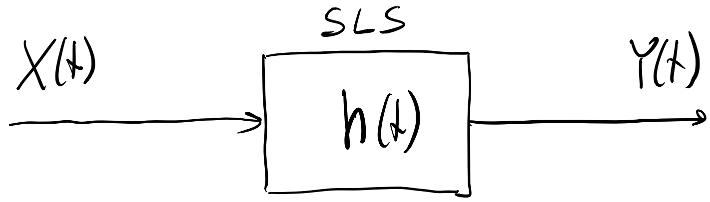
$$\tau_{decorr.} \quad : \quad R_X(\tau) = 0 \qquad \tau \geq \tau_{decorr.}$$

simm.

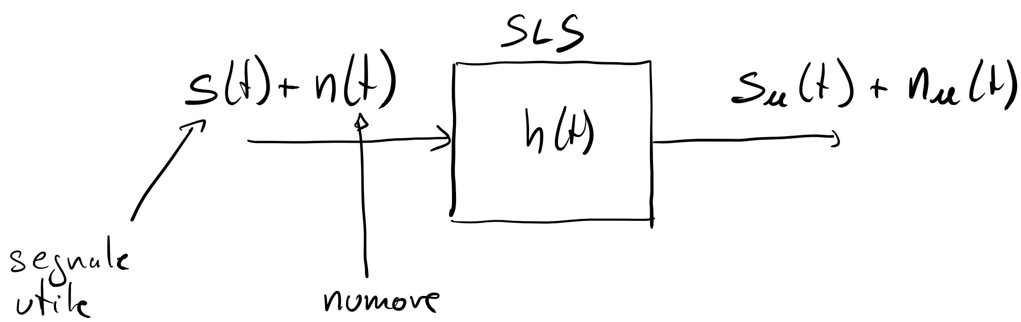
//



·) FILTRAGGIO DI SEGNALI ALEATORI



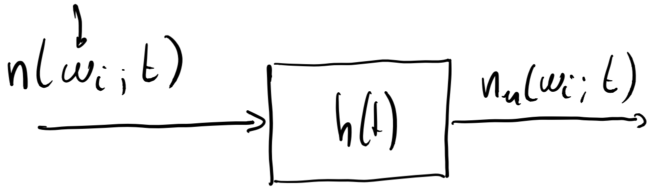
·) nei sistemi di comunicazione sono presenti SLS così come è presente del rumore



$$s_u(t) = s(t) \otimes h(t)$$
$$n_u(t) = n(t) \otimes h(t)$$

per la linearità

ris. di un esp aleatorio $\Rightarrow$ realizzazione di rumore



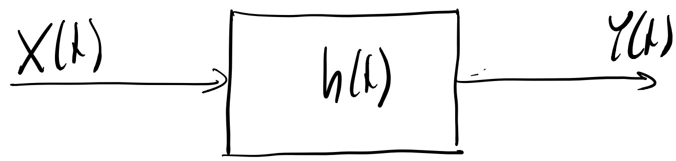
$$n_u(\omega_i ; t) = n(\omega_i ; t) \otimes h(t)$$

formalmente $\Rightarrow$ $\underline{\underline{N_u(t) = N(t) \otimes h(t)}}$

$X(t)$ ⇒ conosco le ddp di ordine N con N arbitrario

$h(t)$ ⇒ conosco

⇒ posso ricavare le ddp di ordine N del processo in uscita $Y(t)$



⇒ Non esiste un modo per risalire alle ddp di ordine N di $Y(t)$

⇒ CALCOLO DEGLI INDICI STATISTICI DI $Y(t)$

·) VALOR MEDIO

$$\eta_Y(t) = E[Y(t)] = \int_{-\infty}^{+\infty} y \, f_Y(y;t) \, dy$$

$f_Y(y;t)$ non è calcolabile nota che sia la $f_X(x;t)$ e la $h(t)$

$$\eta_Y(t) = E[Y(t)] = E[X(t) \otimes h(t)]$$

$$= E\left[\int_{-\infty}^{+\infty} X(\tau) \, h(t-\tau) \, d\tau\right] = \int_{-\infty}^{+\infty} E[X(\tau)] \, h(t-\tau) \, d\tau$$

$$= \int_{-\infty}^{+\infty} \eta_X(\tau)\, h(t-\tau)\, d\tau = \eta_X(t) \otimes h(t)$$

$$\Rightarrow \eta_Y(t) = \eta_X(t) \otimes h(t)$$

INTERPRETAZIONE

·) $\eta_X(t) = 0 \Rightarrow \eta_Y(t) = 0$

·) v.m nullo, v.m. ; v.m. nullo, v.m.

$$X(t) = \underbrace{X_0(t)}_{\text{v.m nullo}} + \underbrace{\eta_X(t)}_{\text{v.m.}} \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = \underbrace{Y_0(t)}_{\text{v.m. nullo}} + \underbrace{\eta_Y(t)}_{\text{v.m.}}$$

$\eta_X(t) = E\left[X(t)\right]$

$X_0(t) \Rightarrow$ V.M. nullo

$$Y_0(t) = X_0(t) \otimes h(t)$$

$$\boxed{\eta_Y(t) = \eta_X(t) \otimes h(t)}$$

$$X_0(t) = X(t) - \eta_X(t)$$

$$E\left[X_0(t)\right] = E\left[X(t) - \eta_X(t)\right] = \underbrace{E\left[X(t)\right]}_{\eta_X(t)} - \eta_X(t) = 0$$

$$\underbrace{\eta_Y(t)}_{\text{determ.}} = \underbrace{\eta_X(t)}_{\text{determ}} \otimes \underbrace{h(t)}_{\text{determ}}$$

$$\Updownarrow \text{TCF} \qquad \text{TCF} \Updownarrow \qquad \Updownarrow \text{TCF}$$

$$\underline{\eta}_Y(f) = \underline{\eta}_X(f)\, H(f)$$

← risp in freq. del filtro

|| Per il calcolo del v.m. in uscita

·) AUTOCORRELAZIONE DEL PROCESSO IN USCITA

$$R_Y(t_1, t_2) = E\left[Y(t_1)\, Y(t_2) \right]$$

$$= E\left[\left(\underset{Y_1(t_1)}{X(t_1) \otimes h(t_1)} \right) \left(\underset{Y(t_2)}{X(t_2) \otimes h(t_2)} \right) \right]$$

$$= E\left[\int_{-\infty}^{+\infty} X(\tau_1)\, h(t_1 - \tau_1)\, d\tau_1 \int_{-\infty}^{+\infty} X(\tau_2)\, h(t_2 - \tau_2)\, d\tau_2 \right]$$

$$= \underset{(\tau_1)}{\int_{-\infty}^{+\infty}} \underset{(\tau_2)}{\int_{-\infty}^{+\infty}} \underbrace{E\left[X(\tau_1)\, X(\tau_2) \right]}_{R_X(\tau_1, \tau_2)} h(t_1 - \tau_1)\, h(t_2 - \tau_2)\, d\tau_1 d\tau_2$$

$$= \underset{\tau_2}{\int_{-\infty}^{+\infty}} \underbrace{\underset{\tau_1}{\int_{-\infty}^{+\infty}} R_X(\tau_1, \tau_2)\; h(t_1 - \tau_1)}\, d\tau_1\; h(t_2 - \tau_2)\, d\tau_2$$

$$= \underset{\tau_2}{\int_{-\infty}^{+\infty}} R_X(t_1, \tau_2) \otimes h(t_1)\;\; h(t_2 - \tau_2)\, d\tau_2$$

$$= \int_{-\infty}^{+\infty} h(t_1) \otimes R_X(t_1, \tau_2)\, h(t_2 - \tau_2)\, d\tau_2$$

$$= h(t_1) \otimes \left[\int_{-\infty}^{+\infty} R_X(t_1, \tau_2)\, h(t_2 - \tau_2)\, d\tau_2 \right]$$

$$= h(t_1) \otimes R_X(t_1, t_2) \otimes h(t_2)$$

$$= \boxed{R_X(t_1, t_2) \otimes h(t_1) \otimes h(t_2) = R_Y(t_1, t_2)}$$

·) FILTRAGGIO DI PROC. ALEATORI SSL

·) VALOR MEDIO

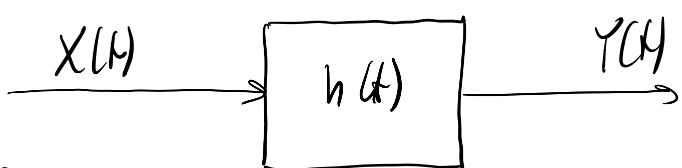
$$\eta_Y(t) = \eta_X(t) \otimes h(t) =$$

$$\int_{-\infty}^{+\infty} \eta_X(\tau)\, h(t-\tau)\, d\tau = \eta_X \int_{-\infty}^{+\infty} h(t-\tau)\, d\tau$$

$$= \eta_X \int_{-\infty}^{+\infty} h(t')\, dt' = \boxed{\eta_X\, H(0) = \eta_Y}$$

ANCHE $\eta_Y(t)$ non dipende dal tempo

$\eta_Y(t) = \eta_Y$



$X(t)$ SSL

·) $\eta_X(t) = \eta_X$

·) $R_X(t_1, t_2) = R_X(\tau)$

$\tau = t_1 - t_2$

$$\int_{-\infty}^{+\infty} h(t)\, dt =$$

$$= \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt \Big|_{f=0}$$

$$= H(f)\Big|_{f=0} = H(0)$$

-) AUTOCORRELAZIONE

$$R_Y(t_1, t_2) \Rightarrow R_Y(t, t-\tau)$$

$t = t_1$
$t-\tau = t_2$

$$R_Y(t, t-\tau) = E\left[Y(t)\, Y(t-\tau) \right]$$

$$= E\left[\left[X(t) \otimes h(t)\right] \left[X(t-\tau) \otimes h(t-\tau)\right] \right]$$

$$= E\left[\int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha \int_{-\infty}^{+\infty} X(\beta)\, h((t-\tau)-\beta)\, d\beta \right]$$

α $\qquad$ β

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E\left[X(\alpha)\, X(\beta)\right] h(t-\alpha)\, h(t-\tau-\beta)\, d\alpha\, d\beta$$

α $\qquad$ β

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\alpha-\beta)\, h(t-\alpha)\, h(t-\tau-\beta)\, d\alpha\, d\beta$$

α $\qquad$ β

$\alpha - \beta = \gamma$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\gamma)\, h((t-\beta)-\gamma)\, d\gamma\, h(t-\tau-\beta)\, d\beta$$

(β) $\qquad$ (γ)

$$= \int_{-\infty}^{+\infty} \left[R_X(t-\beta) \otimes h(t-\beta)\right] h(t-\beta-\tau)\, d\beta$$

(β)

$$= \int_{-\infty}^{+\infty} \Big[\underbrace{R_X(\eta) \otimes h(\eta)}_{g(\eta)}\Big]\, h(\eta - \tau)\, d\eta \qquad t-\beta = \eta$$

(η)

$$= \int_{-\infty}^{+\infty} g(\eta)\, h\big[-(\tau-\eta)\big]\, d\eta = g(\tau) \otimes h(-\tau)$$

$$g(\tau) = R_X(\tau) \otimes h(\tau)$$

$$R_Y(t, t-\tau) = \boxed{R_Y(\tau) = R_X(\tau) \otimes h(\tau) \otimes h(-\tau)}$$

R_Y dipende solo da τ

$\tau = t_1 - t_2$

Se $X(t)$ è SSL

$Y(t) = X(t) \otimes h(t)$

$\Downarrow$

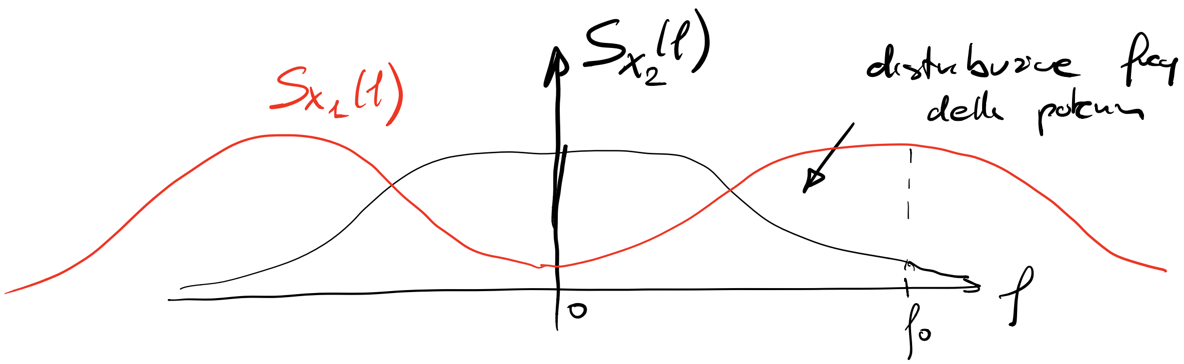
$Y(t)$ è SSL

SSL
- VALOR MEDIO COSTANTE
- AUTOCORRELAZIONE DIP. SOLO DA τ

-> DENSITÀ SPETTRALE DI POTENZA DI UN PROCESSO SSL

I processi di rumore sono di classe energetica
- -> potenza finita
- -> energia infinita

=> DENSITÀ SPETTRALE DI POTENZA



$$S_X(f) = \lim_{T \to \infty} \frac{|X_T(f)|^2}{T}$$

per segnali det non periodici a potenza finita

=> per i processi

$$x(t) \Rightarrow X(t)$$
$$x_T(t) \Rightarrow X_T(t)$$

$$S_X(\omega_i; f) = \lim_{T \to \infty} \frac{|X_T(\omega_i; f)|^2}{T}$$ definizione

$\Rightarrow$ TEOREMA DI WIENER - KHINTCHINE

$$\boxed{S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau)\, e^{-j2\pi f \tau}\, d\tau}$$

$$\boxed{S_X(f) = TCF\left[R_X(\tau)\right]}$$

ESERCIZIO

$$X(t) = 1 + 9\cos\left(2\pi f_0 t + 2\Theta_0\right)$$

Θ_0 è una V.A. $\in \mathcal{U}[-\pi, \pi)$

1) Valor medio

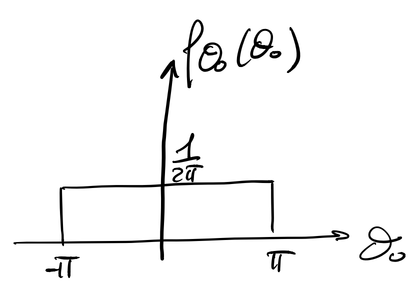
2) DSP

Soluzione

1) $$\eta_X(t) = E\left[X(t)\right] = \int_{-\infty}^{+\infty} x\, f_X(x;t)\, dx$$

$$X(t) = g(\Theta_0; t)$$

$$f_{\Theta_0}(\theta_0) = \frac{1}{2\pi}\,\mathrm{rect}\left(\frac{\theta_0}{2\pi}\right)$$



I^a strada

$$f_{\Theta_0}(\theta_0) \xrightarrow{g(\theta_0;t)} f_X(x;t) \Rightarrow \eta_X(t) = \int_{-\infty}^{+\infty} x\, f_X(x;t)\, dx$$

II^a strada – TEO. DEL VALOR MEDIO

$$E[X(t)] = E[g(\theta_0;t)]$$

$$= \int_{-\infty}^{+\infty} [1 + g\cos(2\pi f_0 t + 2\theta_0)]\, f_{\Theta_0}(\theta_0)\, d\theta_0$$

θ_0

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [1 + g\cos(2\pi f_0 t + 2\theta_0)]\, d\theta_0$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_0 + \frac{g}{2\pi}\underbrace{\int_{-\pi}^{\pi} \cos(2\theta_0 + \varphi_0)\, d\theta_0}_{\substack{= \\ 0 \\ \text{ind. da } \varphi_0}} \qquad \varphi_0 \triangleq 2\pi f_0 t$$

$$= 1 + 0 = 1$$

$$\boxed{\eta_X(t) = \eta_X = 1}$$

2) DSP $\Rightarrow$ $S_X(f) = ?$

$$S_X(f) = TCF\left[R_X(\tau)\right]$$

$$R_X(t_1, t_2) = E\left[X(t_1)\,X(t_2)\right]$$

$$= E\left[\left(1 + 9\cos(2\pi f_0 t_1 + 2\theta_0)\right)\left(1 + 9\cos(2\pi f_0 t_2 + 2\theta_0)\right)\right]$$

$$= E[1] + E\left[9\cos(2\pi f_0 t_2 + 2\theta_0)\right] + E\left[9\cos(2\pi f_0 t_1 + 2\theta_0)\right]$$

$$+ E\left[81\cos(2\pi f_0 t_1 + 2\theta_0)\cos(2\pi f_0 t_2 + 2\theta_0)\right]$$

$$= 1 + \frac{9}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos(2\pi f_0 t_2 + 2\theta_0)\,d\theta_0}_{0} + \frac{9}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos(2\pi f_0 t_1 + 2\theta_0)\,d\theta_0}_{=0}$$

$$+ \frac{81}{2} E\underbrace{\left[\cos\left(4\theta_0 + 2\pi f_0(t_1 + t_2)\right)\right]}_{\frac{1}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos(4\theta_0 + \varphi)\,d\theta_0}_{=0}} +$$

$$+ \frac{81}{2} E\left[\cos\left(2\pi f_0(t_1 - t_2)\right)\right]$$

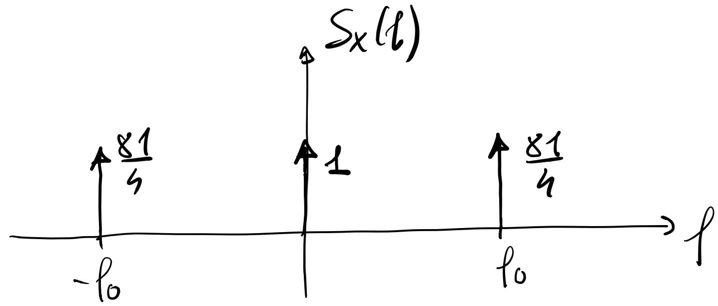
$$= 1 + \frac{81}{2}\cos\left[2\pi f_0(t_1 - t_2)\right]$$

$$= 1 + \frac{81}{2}\cos(2\pi f_0 \tau) \qquad \tau = t_1 - t_2$$

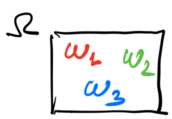
$$= R_X(\tau)$$

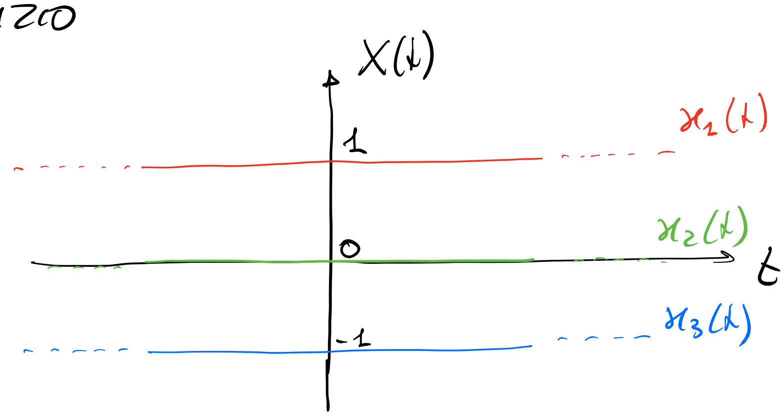
$$\Rightarrow S_X(f) = TCF\left[R_X(\tau)\right] \qquad \text{Teo di W-K}$$

$$S_X(f) = TCF\left[1 + \frac{81}{2}\cos(2\pi f_0 \tau)\right]$$
$$= \delta(f) + \frac{81}{2}\left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right]$$
$$= \delta(f) + \frac{81}{4}\delta(f-f_0) + \frac{81}{4}\delta(f+f_0)$$



ESERCIZIO





Processo aleatorio che ammette 3 realizzazioni equiprobabili

$$P\{\omega_1\} = P\{\omega_2\} = P\{\omega_3\} = \frac{1}{3}$$

Calcolare

1) $\eta_X(t)$

2) $R_X(t_1, t_2)$

3) SSL - ?

Soluzione

$$\eta_X(t) = E\left[X(t)\right] = \int_{-\infty}^{+\infty} x \, f_X(x;t) \, dx$$

$\Rightarrow$ $\underline{X(t) = A}$ processo aleatorio parametrico

A V.A. discreta

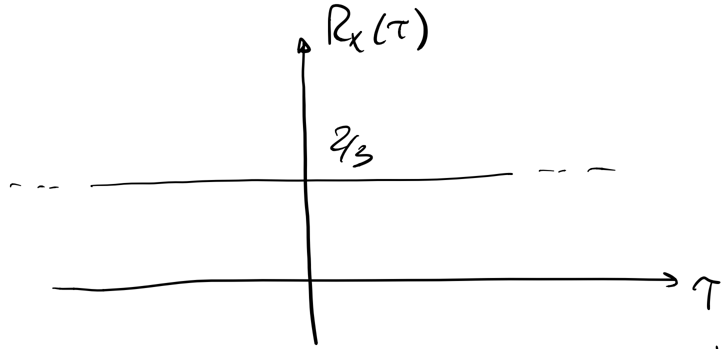
$$E\left[X(t)\right] = E\left[A\right] = \sum_{n=1}^{3} p_n x_n$$

$$= p_1 x_1 + p_2 x_2 + p_3 x_3 = \frac{1}{3}(+1) + \frac{1}{3} \cdot 0 + \frac{1}{3}(-1)$$

$$\eta_X(t) = 0 = \eta_X$$

2) $$R_X(t_1, t_2) = E\left[X(t_1)\, X(t_2)\right] = E\left[A^2\right]$$

$$= \sum_{n=1}^{3} p_n x_n^2 = \frac{1}{3} 1^2 + \frac{1}{3} 0^2 + \frac{1}{3} (-1)^2 = \frac{2}{3}$$



3) SSL $\Rightarrow$ si poiché $\to$ v.m. è costante

$\to$ $R_X(\tau)$

ESERCIZIO

$$X(t) = A \cos(2\pi f_0 t + \varphi)$$

$A \in \mathcal{U}[1, 2)$ $\qquad$ A, φ sono indipendenti

$\varphi \in \mathcal{U}[-\pi, \pi)$

·) Dire se $X(t)$ è SSL

Soluzione

1) $\eta_X(t) = E[X(t)] = E[A \cos(2\pi f_0 t + \varphi)]$

A e φ sono indipendenti

$\Downarrow$

A e $\cos(2\pi f_0 t + \varphi)$ sono indipendenti

$$= E[A]\, E[\cos(2\pi f_0 t + \varphi)]$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A \cos(2\pi f_0 t + \varphi)\, f_{A,\varphi}(A,\varphi)\, dA\, d\varphi$$

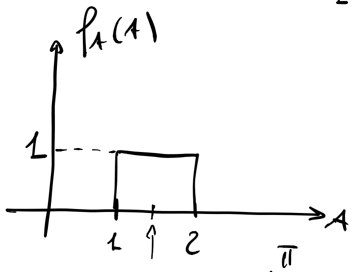
$A \qquad \varphi$

$\Downarrow$ per l'indipendenza

$f_A(A)\, f_\varphi(\varphi)$

$$\underbrace{\int_{-\infty}^{+\infty} A\, f_A(A)\, dA}_{E[A]} \; \underbrace{\int_{-\infty}^{+\infty} \cos(2\pi f_0 t + \varphi)\, f_\varphi(\varphi)\, d\varphi}_{E[\cos(2\pi f_0 t + \varphi)]}$$

$$E[A] = \int_{-\infty}^{+\infty} A\, f_A(A)\, dA = \int_1^2 A\, dA = \left.\frac{A^2}{2}\right|_1^2 = \frac{4-1}{2} = \frac{3}{2}$$



$$E\left[\cos\left(2\pi f_0 t + \varphi\right)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\left(2\pi f_0 t + \varphi\right) d\varphi = 0$$

$$\eta_X(t) = \frac{3}{2}\cdot 0 = 0 \qquad \underline{\underline{\text{costante}}}$$

2) $R_X(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$

$$= E\left[A\cos(2\pi f_0 t_1 + \varphi)\ A\cos(2\pi f_0 t_2 + \varphi)\right]$$

$$= \frac{1}{2}E\left[A^2\cos\left(2\varphi + 2\pi f_0 (t_1 + t_2)\right)\right] + \frac{1}{2}E\left[A^2\cos\left(2\pi f_0 (t_1 - t_2)\right)\right]$$

$$= \frac{1}{2}E\left[A^2\right]E\left[\cos\left(2\varphi + 2\pi f_0 (t_1 + t_2)\right)\right] +$$

$$+ \frac{1}{2}E\left[A^2\right]E\left[\cos\left(2\pi f_0 (t_1 - t_2)\right)\right]$$

$$E\left[A^2\right] = \int_{-\infty}^{+\infty} A^2 f_A(A)\, dA = \int_1^2 A^2\, dA = \left.\frac{A^3}{3}\right|_1^2 = \frac{8-1}{3} = \frac{7}{3}$$

$$E\left[\cos\left(2\varphi + 2\pi f_0 (t_1 + t_2)\right)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\varphi + \theta_0)\, d\varphi = 0$$

$$\theta_0 = 2\pi f_0 (t_1 + t_2)$$

$$E\left[\underbrace{\cos\left(2\pi f_0 (t_1 - t_2)\right)}_{\text{determ.}}\right] = \cos\left[2\pi f_0 (t_1 - t_2)\right]$$

$$R_X(t_1, t_2) = \frac{1}{2} \cdot \frac{7}{3} \cdot 0 + \frac{1}{2} \cdot \frac{7}{3} \cdot \cos\left[2\pi f_0 (t_1 - t_2)\right]$$

$$\boxed{R_X(\tau) = \frac{7}{6} \cos\left(2\pi f_0 \tau\right)}$$

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