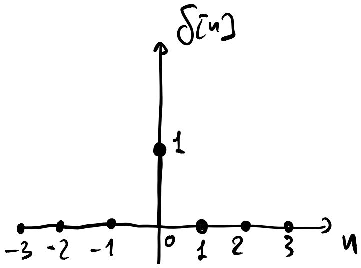
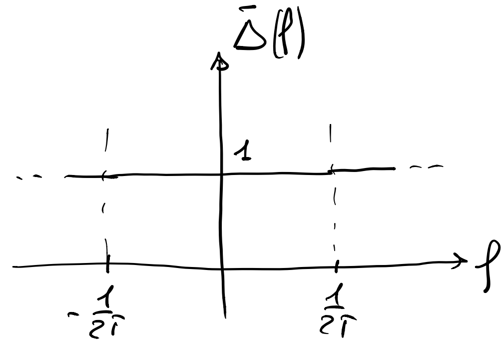
·) TFS $\delta[n]$

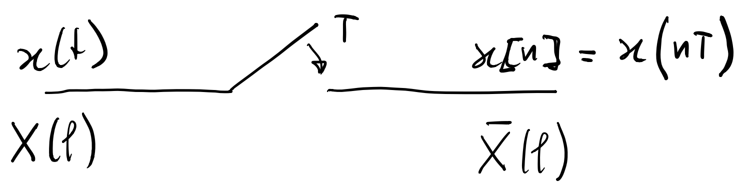
$$\bar{\Delta}(f) = \sum_{n=-\infty}^{+\infty} \delta[n]\, e^{-j2\pi f n T} = 1 \quad \forall f$$



$\overset{TFS}{\Longleftrightarrow}$



·) RELAZIONE TRA LA TFS $(\bar{X}(f))$ E LA TCF $(X(f))$



$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$ $\qquad$ $x[n] \overset{TFS}{\Longleftrightarrow} \bar{X}(f)$

$$\bar{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right)$$

$\downarrow$

periodicizzazione della TCF
con periodo $\frac{1}{T} = f_c$

Dimostrazione

segnale analogico campionato ogni "T"

$$\bar{X}(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f n T} = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi f n T}$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

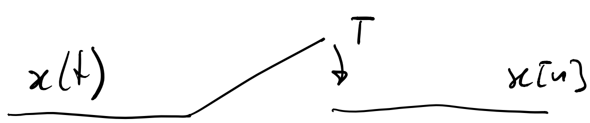
antitrasformata continua di Fourier

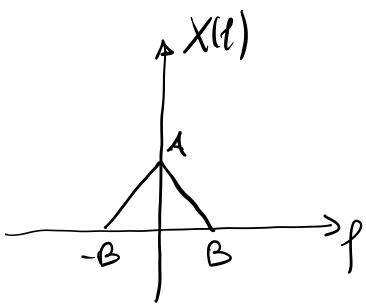
$$x(nT) = \int_{-\infty}^{+\infty} X(f)\, e^{j 2\pi f n T}\, df$$

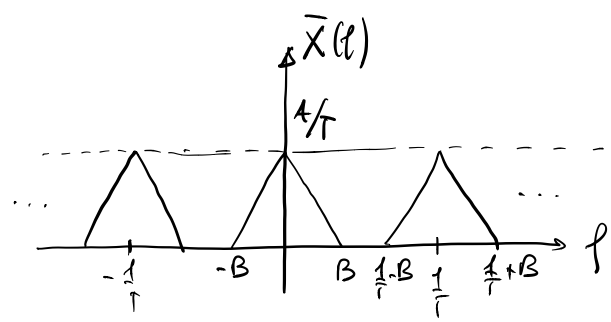
$$\bar{X}(f) = \sum_{n=-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} X(\alpha)\, e^{j 2\pi \alpha nT}\, d\alpha}_{x(nT)}\; e^{-j 2\pi f n T}$$

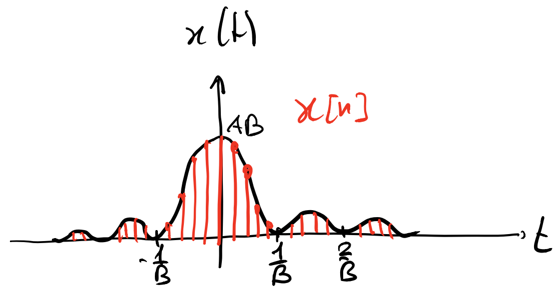
$$= \int_{-\infty}^{+\infty} X(\alpha) \underbrace{\sum_{n=-\infty}^{+\infty} e^{-j 2\pi (f-\alpha) n T}}_{\frac{1}{T}\sum_{n=-\infty}^{+\infty} \delta\left(f-\alpha-\frac{n}{T}\right)} d\alpha$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T} \int_{-\infty}^{+\infty} X(\alpha)\, \delta\left(\alpha - \left(f - \frac{n}{T}\right)\right) d\alpha = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right)$$









ANALISI ENERGETICA DI SEQUENZE

- correlazione tra sequenze

$$R_{xy}[k] = \sum_{n=-\infty}^{+\infty} x[n]\, y^*[n-k]$$

- autocorrelazione

$$R_x[k] = \sum_{n=-\infty}^{+\infty} x[n]\, x^*[n-k]$$

- DENSITÀ SPETTRALE DI ENERGIA

$$\bar{S}_x(f) = TFS\left[R_x[k]\right] = \sum_{k=-\infty}^{+\infty} R_x[k]\, e^{-j2\pi f k T}$$

$$= \sum_{k=-\infty}^{+\infty} \underbrace{\sum_{n=-\infty}^{+\infty} x[n]\, x^*[n-k]}_{R_x[k]}\, e^{-j2\pi f k T}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \sum_{k=-\infty}^{+\infty} x^*[n-k]\, e^{-j2\pi f k T} \qquad n-k = k'$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \sum_{k'=-\infty}^{+\infty} x^*[k']\, e^{-j2\pi f (n-k')T}$$

$$= \underbrace{\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f n T}}_{\bar{X}(f)} \underbrace{\left[\sum_{k'=-\infty}^{+\infty} x[k']\, e^{-j2\pi f k' T}\right]^*}_{\bar{X}^*(f)} = \left|\bar{X}(f)\right|^2$$

$$\boxed{E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = T\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \left|\bar{X}(f)\right|^2 df} \Leftarrow$$

$$E_x = R_x[0] = \sum_{n=-\infty}^{+\infty} x[n]\, x^*[n-k] \Big|_{k=0} = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

·) TEOREMA DI PARSEVAL PER SEQUENZE

$$\sum_{n=-\infty}^{+\infty} x[n]\, y^*[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, \bar{Y}^*(f)\, df$$

Dim

$$\sum_{n=-\infty}^{+\infty} x[n]\, y^*[n] = \sum_{n=-\infty}^{+\infty} \underbrace{T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, e^{j2\pi f nT}\, df}_{x[n]}\; y^*[n]$$

$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f) \underbrace{\sum_{n=-\infty}^{+\infty} y^*[n]\, e^{j2\pi f nT}}_{\bar{Y}^*(f)} df$$

$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, \bar{Y}^*(f)\, df$$

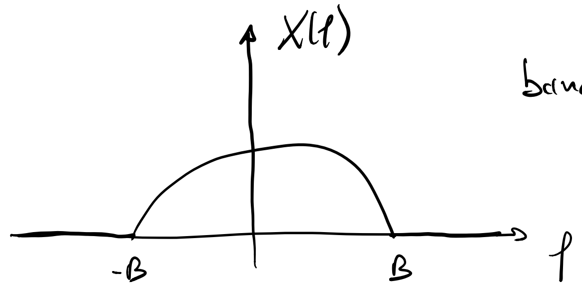
$$x[n] = y[n]$$

$$\Downarrow$$

$$E_x = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |\bar{X}(f)|^2\, df$$

→ CONDIZIONE DI NYQUIST

→ Si applica ai segnali a banda rigorosamente limitata



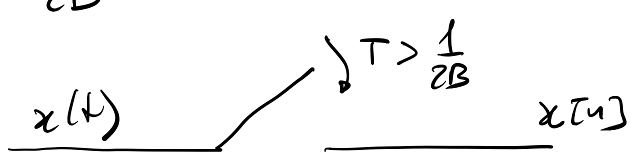
banda rig. lim. B

→ campionando con un intervallo di campionamento

T

$T > \frac{1}{2B}$ $\qquad$ $T \leq \frac{1}{2B}$

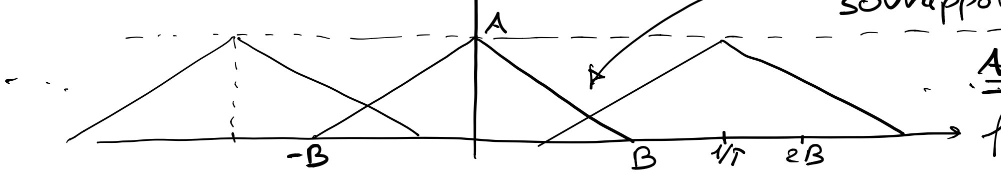
I) $T > \frac{1}{2B}$



$$\bar{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right)$$

$$\frac{1}{T} < 2B$$

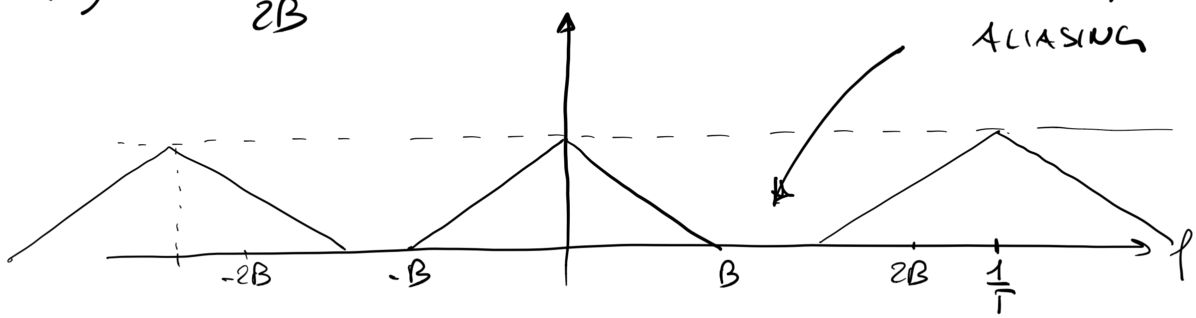
le repliche si sovrapponogono



ALIASING

$T > \frac{1}{2B} \Rightarrow$ ALIASING

II) $T \leq \frac{1}{2B}$



$T \leq \frac{1}{2B} \Rightarrow$ NO ALIASING

<u>CONDIZIONE DI NYQUIST</u>: è una condizione che si applica sull'intervallo di campionamento affinché non si produca ALIASING

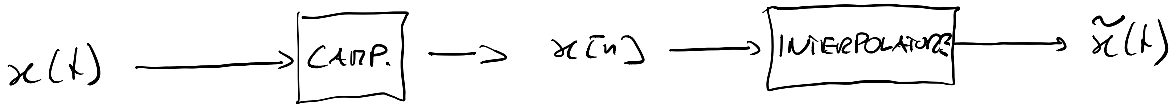
$\Downarrow$

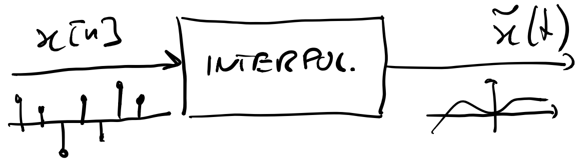
$\boxed{T \leq \frac{1}{2B}}$, B = banda (rig. lim.) del segnale da campionare

$\boxed{f_c \geq 2B}$

$\Downarrow$

bisogna campionare con una freq. di campionamento almeno il doppio della banda del segnale

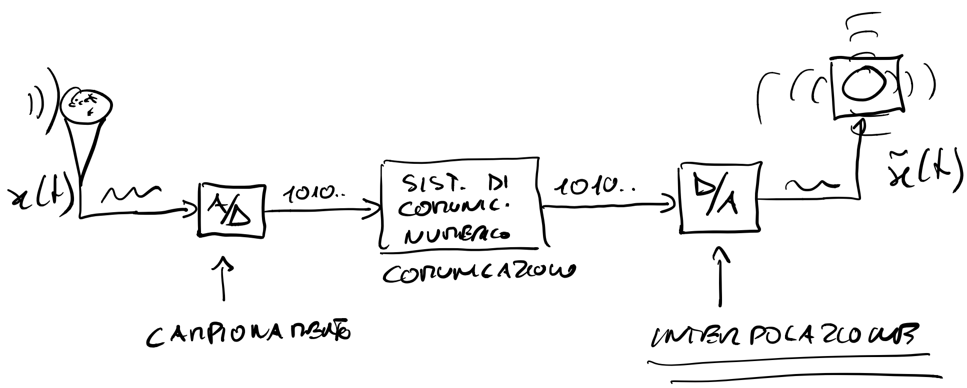




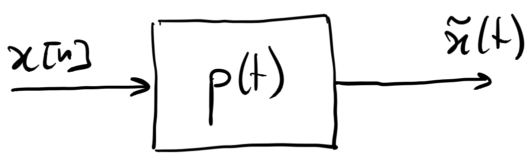
$\tilde{x}(t)$ = segnale interpolato

INTERPOLATORE : è un dispositivo che permette di ottenere un segnale analogico a partire da un segnale tempo-discreto (sequenza)

$\tilde{x}(t) \rightarrow x(t)$ mantenere il più possibile inalterato il contenuto informativo



INTERPOLATORE



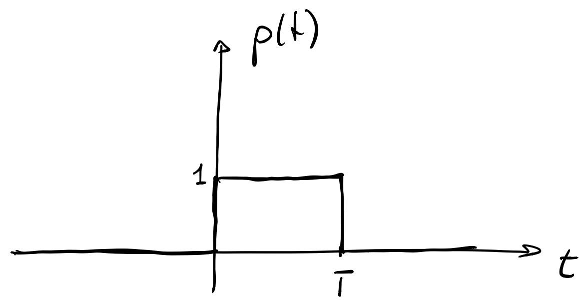
$p(t)$ = funzione interpolatrice

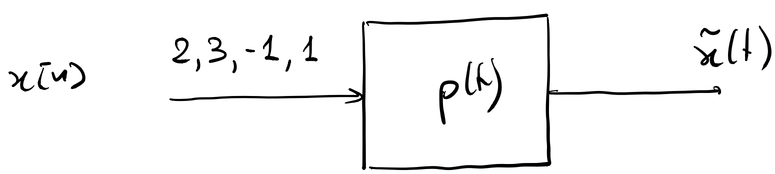
$$\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT)$$

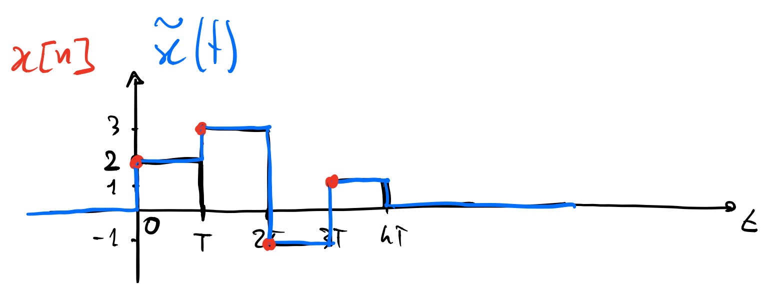
uscita ingresso

Esempio : interpolatore a mantenimento

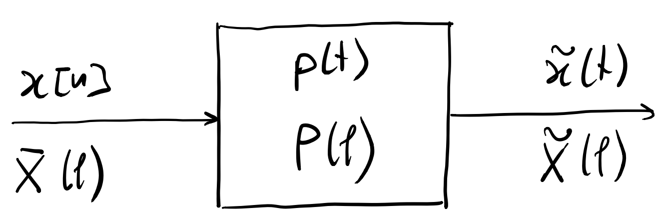
$$p(t) \triangleq \mathrm{rect}\left(\frac{t - T/2}{T}\right)$$







·) INTERPRETAZIONE DELL'INTERPOLATORE NEL DOMINIO DELLA FREQUENZA



$$x[n] \overset{TFS}{\Longleftrightarrow} \bar{X}(f) \;,\quad p(t) \overset{TCF}{\Longleftrightarrow} P(f) \;,\quad \tilde{x}(t) \overset{TCF}{\Longleftrightarrow} \tilde{X}(f)$$

$$\Rightarrow \boxed{\tilde{X}(f) = \bar{X}(f)\,P(f)}$$

Dimostrazione

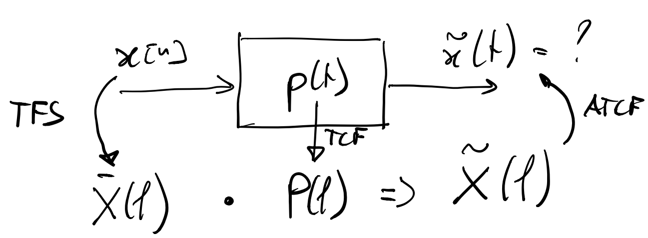
$$\tilde{X}(f) = \text{TCF}\left[\tilde{x}(t)\right] = \int_{-\infty}^{+\infty} \tilde{x}(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \underbrace{\sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT)}_{\tilde{x}(t)}\, e^{-j2\pi f t}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \int_{-\infty}^{+\infty} p(t-nT)\, e^{-j2\pi f t}\, dt \qquad t - nT = t'$$

$$= \sum_{n} x[n] \int_{-\infty}^{+\infty} p(t')\, e^{-j2\pi f (t'+nT)}\, dt'$$

$$= \underbrace{\sum_{n} x[n]\, e^{-j2\pi f nT}}_{\bar{X}(f)} \underbrace{\int_{-\infty}^{+\infty} p(t')\, e^{-j2\pi f t'}\, dt'}_{P(f)} = \bar{X}(f)\, P(f)$$



-) TEOREMA DEL CAMPIONAMENTO

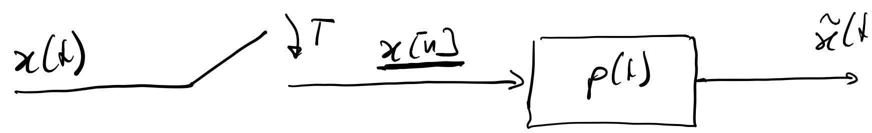
Ip: 1) $x(t)$ segnale a banda rig. limitata B

2) $T \leq \frac{1}{2B}$, con T: intervallo di campionamento

3) $p(t) = 2B \operatorname{sinc}(2Bt)$

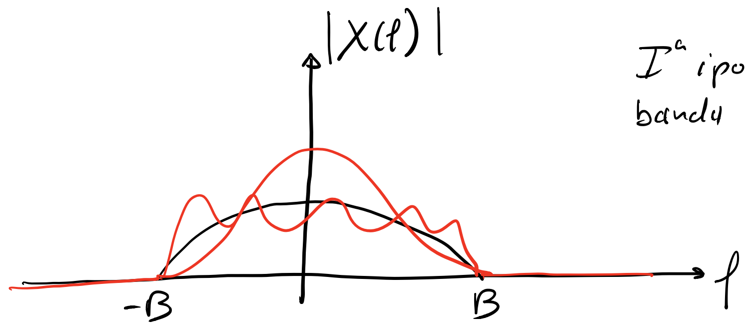
$\Downarrow$

Tesi: posso ricostruire il segnale $x(t)$ perfettamente a partire dai suoi campioni



$\tilde{x}(t)$ replica fedele di $x(t)$

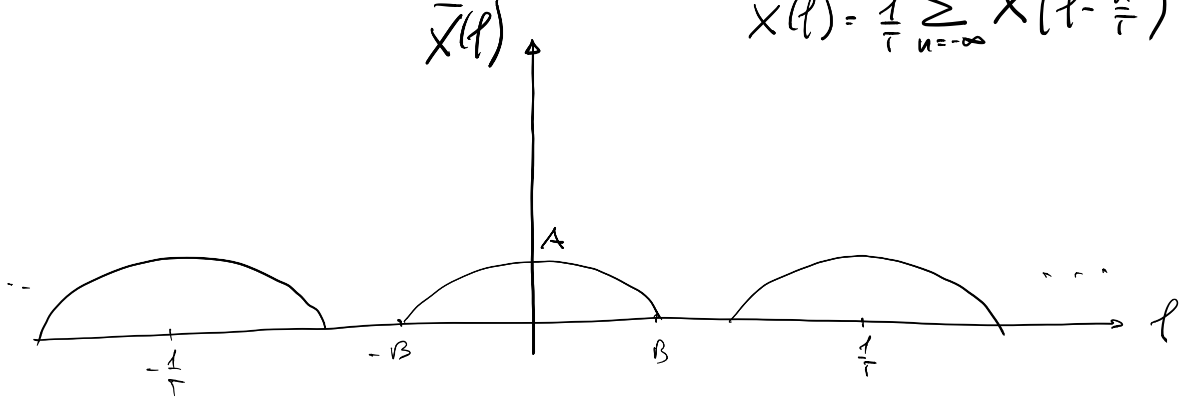
Dimostrazione (x via grafica)



I^a ipotesi
banda rig. lim. B

$\Downarrow$ dopo campionamento con $T \leq \frac{1}{2B}$
II^a ipotesi

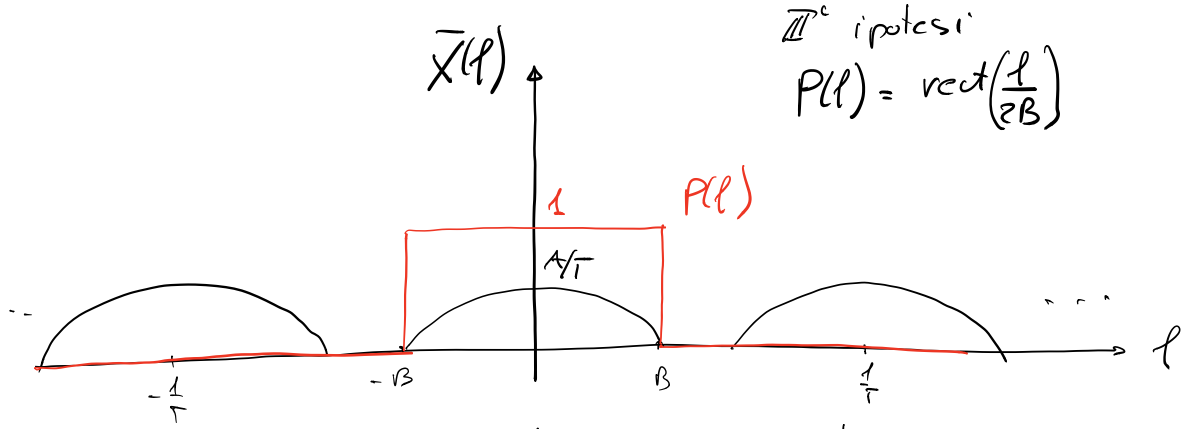
$$\bar{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right)$$



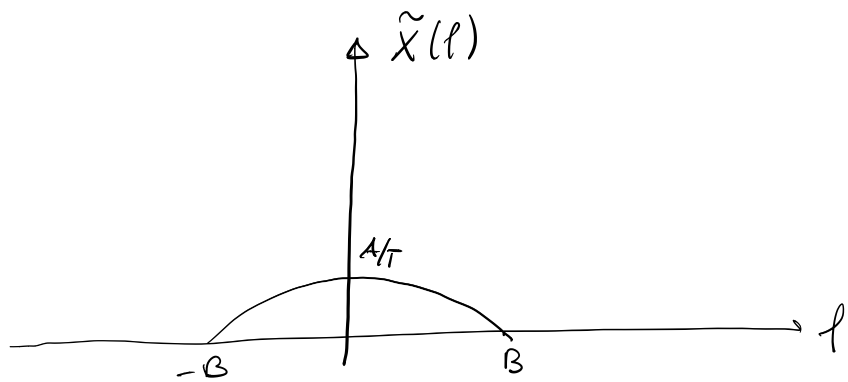
$\Downarrow$ interpolazione

III° ipotesi

$P(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$



$\Downarrow$ dopo interpolazione



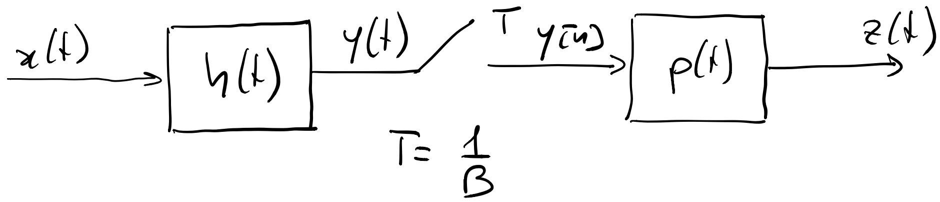
$$\tilde{X}(f) = \frac{1}{T} X(f) \Rightarrow \tilde{x}(t) = \frac{1}{T} x(t)$$

$\Downarrow$

$$p(t) = T \cdot 2B \,\mathrm{sinc}(2Bt)$$

$$P(f) = T\, \mathrm{rect}\left(\frac{f}{2B}\right) \Rightarrow \tilde{X}(f) = X(f)$$

·) ESERCIZIO - I Compitino 2019



$$x(t) = 2AB\,\mathrm{sinc}(2Bt) + AB\,\mathrm{sinc}^2(Bt)$$

$$h(t) = B\,\mathrm{sinc}(Bt)$$

$$p(t) = 2B\,\mathrm{sinc}(2Bt)$$

·) Calcolare $z(t)$

·) Calcolare E_z

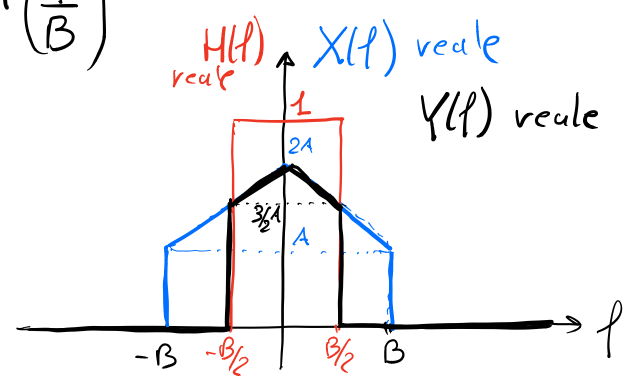
Soluzione

$$y(t) = x(t) \otimes h(t)$$

$$Y(f) = X(f)\,H(f)$$

$$X(f) = A\,\mathrm{rect}\left(\frac{f}{2B}\right) + A\left(1 - \frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$H(f) = \mathrm{rect}\left(\frac{f}{B}\right)$$



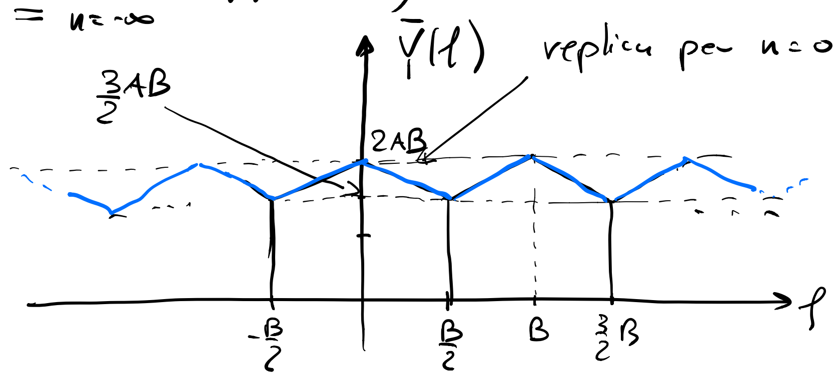
$$Y(f) = \frac{3}{2}A\,\mathrm{rect}\left(\frac{f}{B}\right) + \frac{A}{2}\left(1 - \frac{|f|}{B/2}\right)\mathrm{rect}\left(\frac{f}{B}\right)$$ uscita dal filtro

$\Rightarrow$ campionamento di $y(t)$

$$y[n] = y(nT)$$

$$\bar{Y}(f) = \mathrm{TFS}\left[y[n]\right] = \frac{1}{T}\sum_{n=-\infty}^{+\infty} Y\left(f - \frac{n}{T}\right)$$

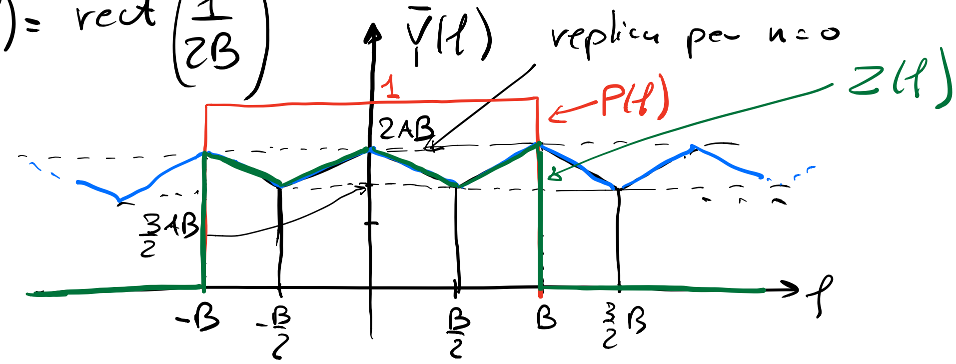
$$= \underline{B}\sum_{n=-\infty}^{+\infty} Y(f - nB)$$

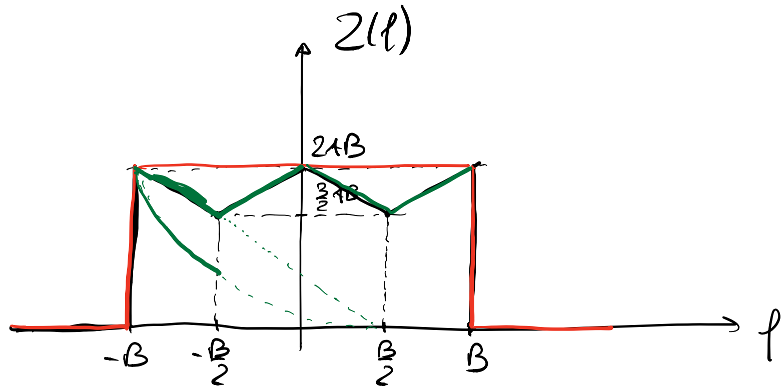


$\Rightarrow$ interpolazione

$$Z(f) = \bar{Y}(f)\,P(f)$$

$$P(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$$





$$Z(f) = \underbrace{2AB\,\mathrm{rect}\left(\frac{f}{2B}\right)} - \frac{AB}{2}\left[\left(1-\frac{|f-\frac{B}{2}|}{B/2}\right)\mathrm{rect}\left(\frac{f-\frac{B}{2}}{B}\right) + \underbrace{\left(1-\frac{|f+B/2|}{B/2}\right)\mathrm{rect}\left(\frac{f+B/2}{B}\right)}\right]$$

$$= 2AB\,\mathrm{rect}\left(\frac{f}{2B}\right) - \frac{AB}{2}\left(1-\frac{|f|}{B/2}\right)\mathrm{rect}\left(\frac{f}{B}\right)\otimes\left[\delta\left(f-\frac{B}{2}\right)+\delta\left(f+\frac{B}{2}\right)\right]$$

$$z(t) = 4AB^2\,\mathrm{sinc}(2Bt) - \frac{AB^2}{4}\mathrm{sinc}^2\left(\frac{B}{2}t\right)\cdot\left[e^{j2\pi\frac{B}{2}t} + e^{-j2\pi\frac{B}{2}t}\right]$$

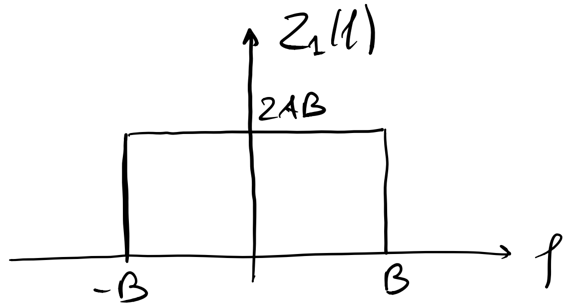
$$= \boxed{4AB^2\,\mathrm{sinc}(2Bt) - \frac{AB^2}{2}\mathrm{sinc}^2\left(\frac{B}{2}t\right)\cos(\pi Bt)}$$

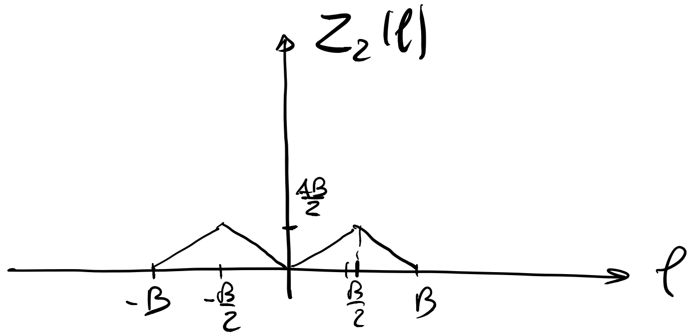
•) $E_z = \int_{-\infty}^{+\infty} |z(t)|^2\,dt = \int_{-\infty}^{+\infty} |Z(f)|^2\,df$

$$Z(f) = Z_1(f) - Z_2(f)$$

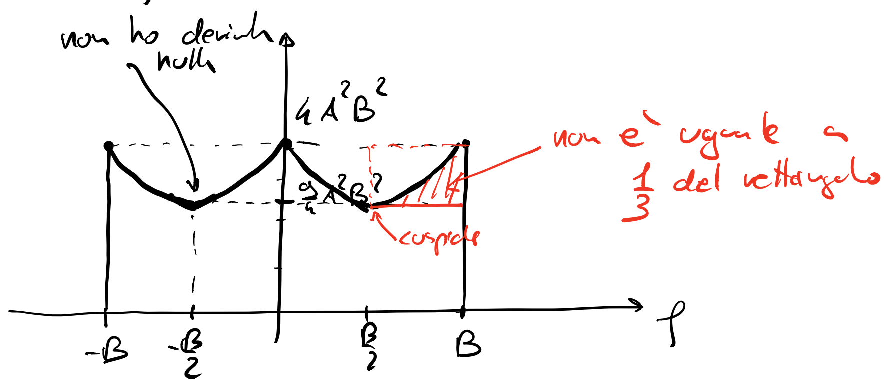
$$Z_1(f) = 2AB \, \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Z_2(f) = \frac{AB}{2}\left[\left(1 - \frac{|f - \frac{B}{2}|}{\frac{B}{2}}\right)\mathrm{rect}\left(\frac{f-\frac{B}{2}}{B}\right) + \left(1 - \frac{|f+\frac{B}{2}|}{\frac{B}{2}}\right)\mathrm{rect}\left(\frac{f+\frac{B}{2}}{B}\right)\right]$$



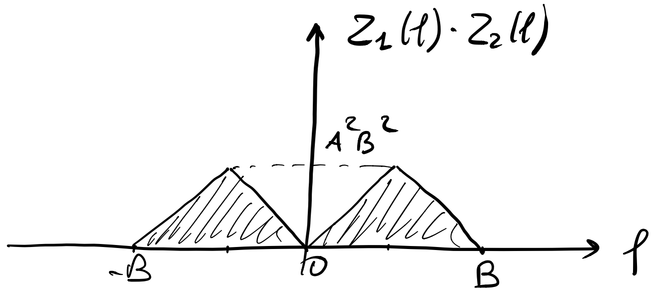


$$|Z(f)|^2 = Z^2(f)$$



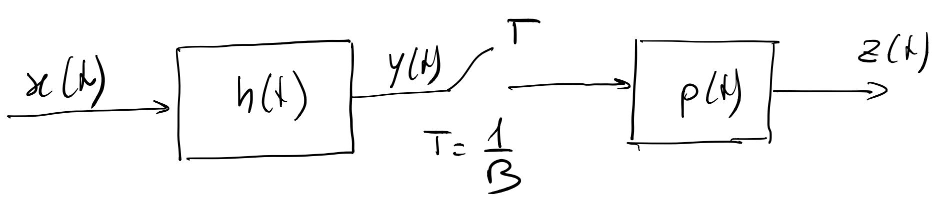
$$Z^2(f) = Z_1^2(f) + Z_2^2(f) - 2\, Z_1(f)\, Z_2(f)$$

$$E_2 = \underbrace{4A^2B^2\cdot 2B}_{E_{Z_1}} + \underbrace{4\cdot\frac{1}{3}\,\frac{B}{2}\cdot\frac{A^2B^2}{4}}_{E_{Z_2}} - 2A^2B^2B$$



$$E_2 = 8A^2B^3 + \frac{1}{6}A^2B^3 - 2A^2B^3 = \left(8+\frac{1}{6}-2\right)A^2B^3 = \boxed{\frac{37}{6}A^2B^3}$$

ESERCIZIO - DAL I COMPITINO DEL 2018



$x(t) = 3B\,\text{sinc}\left(\frac{3B}{2}t\right) - \frac{B}{2}\,\text{sinc}\left(\frac{B}{2}t\right)$

$h(t)$ è un passa-basso ideale di banda B

$p(t) = 2B\,\text{sinc}(2Bt)$

1) $y(t)$
2) verificare se $y[n]$ è ottenuta rispettando Nyquist
3) $z(t)$
4) E_z, P_z

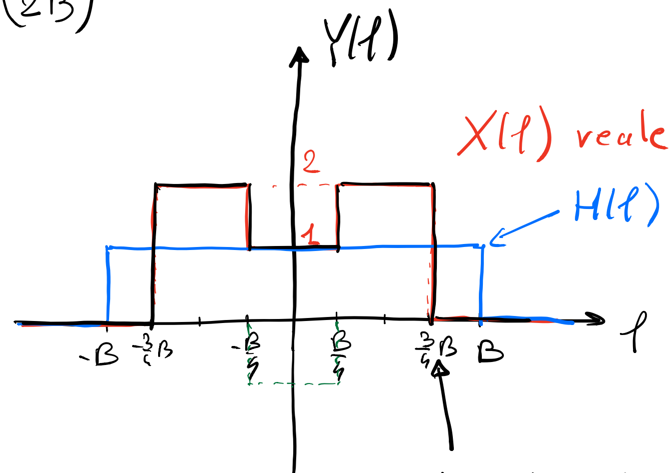
Svolgimento

1) $y(t) = x(t) \otimes h(t)$

$$Y(f) = X(f)\,H(f)$$

$$X(f) = 2\,\mathrm{rect}\left(\frac{f}{\frac{3B}{2}}\right) - \mathrm{rect}\left(\frac{f}{\frac{B}{2}}\right)$$

$$H(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$$



banda di $x(t)$ e quindi di $y(t)$ banda rig. limitata

$$Y(f) = X(f)$$

$$\boxed{y(t) = x(t)}$$

2) $B' = \frac{3}{4}B$

$$\Rightarrow T_{Nyq} \leq \frac{1}{2B'} = \frac{1}{2\cdot\frac{3}{4}B} = \frac{2}{3B}$$

$$\Rightarrow T = \frac{1}{B} > \frac{2}{3B}$$

non è rispettata la condizione di Nyquist !!