ESERCIZIO - PRODOTTO DI SINUSOIDI

$$z(t) = A\cos(2\pi f_0 t + \varphi)\cos(4\pi f_0 t)$$

→ Calcolare e disegnare lo spettro di $z(t)$

Svolgimento

$$Z_n = TSF[z(t)]$$

$$z(t) = \frac{A}{2}\cos(6\pi f_0 t + \varphi) + \frac{A}{2}\cos(-2\pi f_0 t + \varphi)$$

$$\left(\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta) \right.$$

$$= \frac{A}{2}\cos(\underbrace{6\pi f_0}_{3f_0} t + \varphi) + \frac{A}{2}\cos(\underbrace{2\pi f_0}_{f_0} t - \varphi)$$

f_0 = freq. fond

$$z(t - nT_0) = z(t), \quad T_0 = \frac{1}{f_0}, \quad n \in \mathbb{Z}$$

$$z(t - nT_0) = \frac{A}{2}\cos(6\pi f_0(t - nT_0) + \varphi) + \frac{A}{2}\cos(2\pi f_0(t - nT_0) - \varphi)$$

$$= \frac{A}{2}\cos(6\pi f_0 t - \underbrace{6\pi n \cancel{f_0 T_0}}_{} + \varphi) + \frac{A}{2}\cos(2\pi f_0 t - \underbrace{2\pi n f_0 T_0}_{} - \varphi)$$

$$= \frac{A}{2}\cos(6\pi f_0 t + \varphi) + \frac{A}{2}\cos(2\pi f_0 t - \varphi) = z(t)$$

$$Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j(6\pi f_0 t+\varphi)} + e^{-j(6\pi f_0 t+\varphi)}}{2}\, e^{-j2\pi n f_0 t}\, dt$$

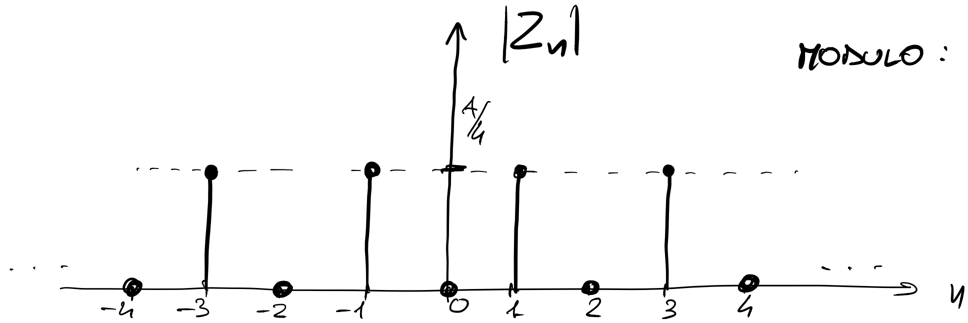
$$+ \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j(2\pi f_0 t-\varphi)} + e^{-j(2\pi f_0 t-\varphi)}}{2}\, e^{-j2\pi n f_0 t}\, dt$$

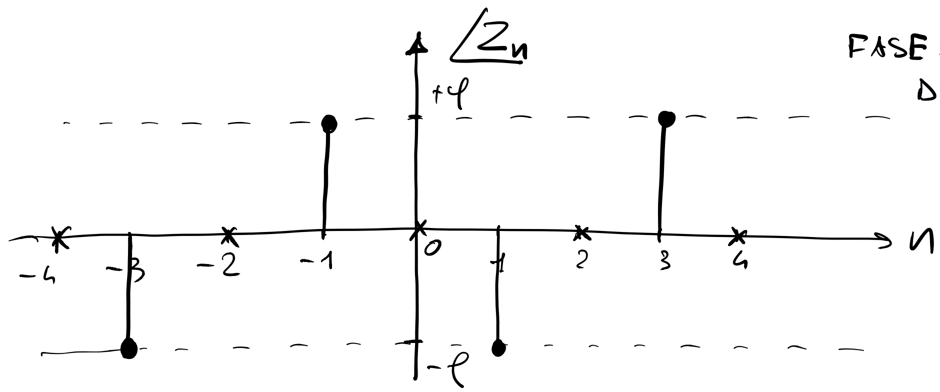
$$= \frac{A e^{j\varphi}}{4T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi \underbrace{(3-n)}_{n=3} f_0 t}\, dt + \frac{A}{4T_0} e^{-j\varphi}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi \underbrace{(3+n)}_{n=-3} f_0 t}\, dt$$

$$+ \frac{A}{4T_0} e^{-j\varphi}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi \underbrace{(1-n)}_{n=1} f_0 t}\, dt + \frac{A}{4T_0} e^{j\varphi}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi \underbrace{(1+n)}_{n=-1} f_0 t}\, dt$$

$$Z_n = \begin{cases} \frac{A}{4\cancel{T_0}} e^{j\varphi} \cancel{T_0} & n = 3 \\ \frac{A}{4\cancel{T_0}} e^{-j\varphi} \cancel{T_0} & n = -3 \\ \frac{A}{4\cancel{T_0}} e^{-j\varphi} \cancel{T_0} & n = 1 \\ \frac{A}{4\cancel{T_0}} e^{j\varphi} \cancel{T_0} & n = -1 \\ 0 & n \neq \pm 1, \pm 3 \end{cases}$$

MODULO: SIMM. PARI





FASE: SIMM. DISPARI

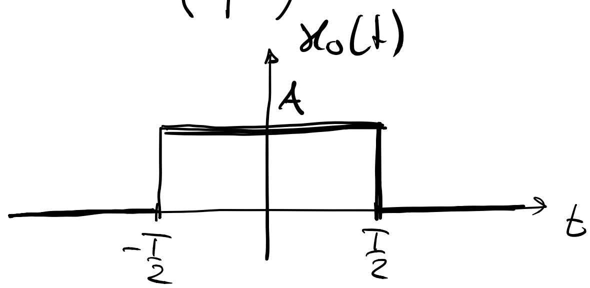
SI NOTA LA SIMM. HERMITIANA

$z(t)$ reale $\Rightarrow$ Z_n Hermitiana

- modulo ha simm. pari
- fase ha simm. disp.

ESERCIZIO - TRENO DI IMPULSI RETTANGOLARI

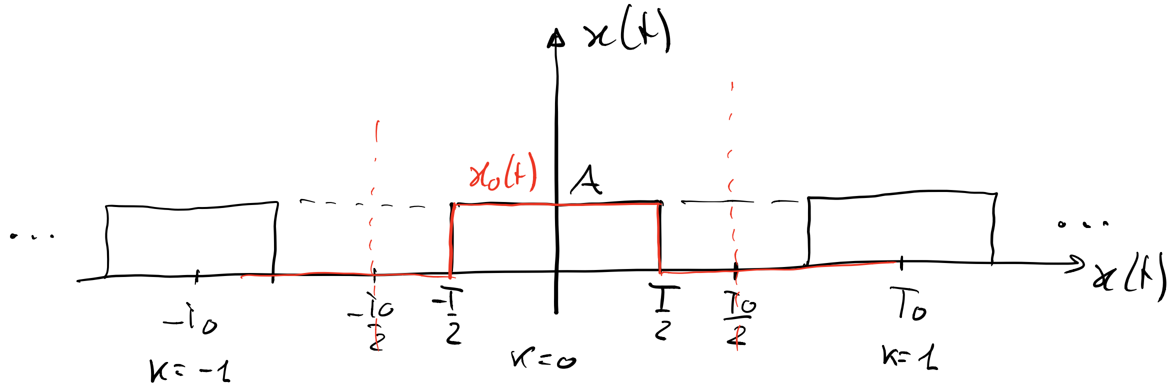
$$x_0(t) = A \, \text{rect}\left(\frac{t}{T}\right)$$



$T < T_0$

$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)}$$

$$= A \sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{t - kT_0}{T}\right)$$



$$x(t-nT_0) = x(t)$$

$$\sum_{k=-\infty}^{+\infty} x(t-nT_0-kT_0) = \sum_{k=-\infty}^{+\infty} x(t-(n+k)T_0)$$

$$n+k = k'$$

$$= \sum_{k'=-\infty}^{+\infty} x(t-k'T_0) = x(t)$$

$$X_n = \,?$$

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_0(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \mathrm{rect}\left(\frac{t}{T}\right) e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{A}{T_0}\int_{-T/2}^{T/2} e^{-j2\pi n f_0 t}\, dt = \frac{A}{\cancel{T_0}}\left(-\frac{1}{j2\pi n \cancel{f_0}}\right) e^{-j2\pi n f_0 t}\Big|_{-T/2}^{T/2}$$

$$= -\frac{A}{j2\pi n}\left(e^{-j2\pi n f_0 \frac{T}{2}} - e^{j2\pi n f_0 \frac{T}{2}}\right)$$

$$= \frac{A}{\pi n}\left(\frac{e^{j\pi n f_0 T} - e^{-j\pi n f_0 T}}{2j}\right)$$

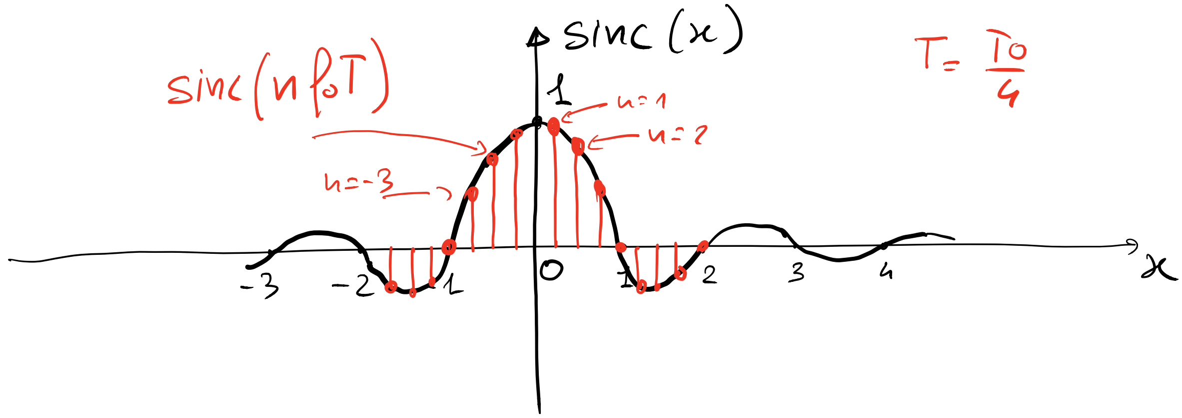
$$= A\frac{\sin(\pi n f_0 T)}{\pi n} = A f_0 T \frac{\sin(\pi n f_0 T)}{\pi n f_0 T} = \boxed{A f_0 T \operatorname{sinc}(n f_0 T)}$$

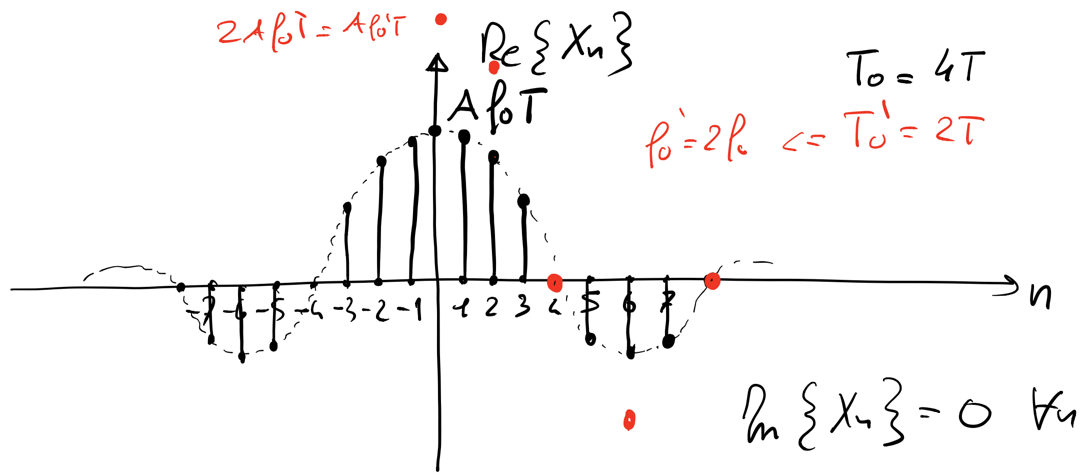
$$\operatorname{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x} \qquad \text{pari}$$

π omesso

$$\boxed{A \sum_{n=-\infty}^{+\infty} \operatorname{rect}\left(\frac{t - nT_0}{T}\right) \overset{TSF}{\Longleftrightarrow} A f_0 T \operatorname{sinc}(n f_0 T)}$$

$\operatorname{sinc}(x)$ $\qquad$ x variabile continua $\in \mathbb{R}$





$x(t)$ è reale e pari $\Rightarrow X_n$ è reale e pari

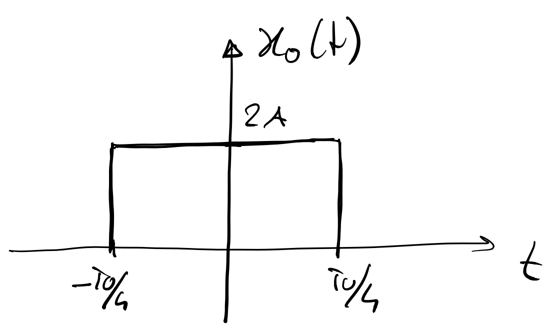
ESERCIZIO

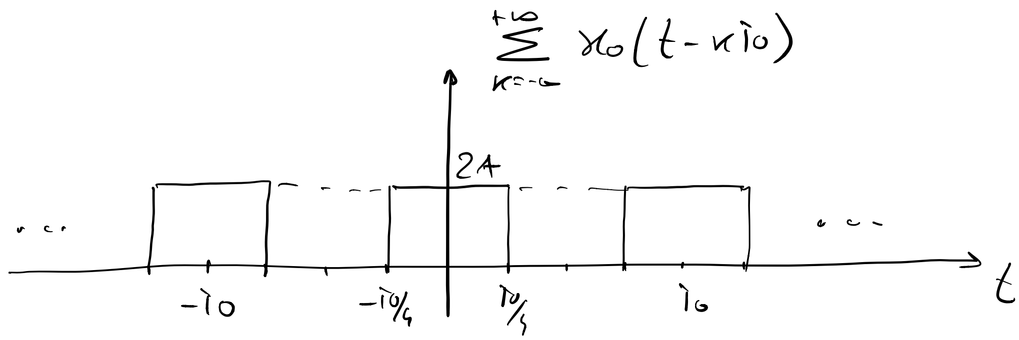
$$x_0(t) = 2A \ \mathrm{rect}\left(\frac{t}{T_0/2}\right)$$

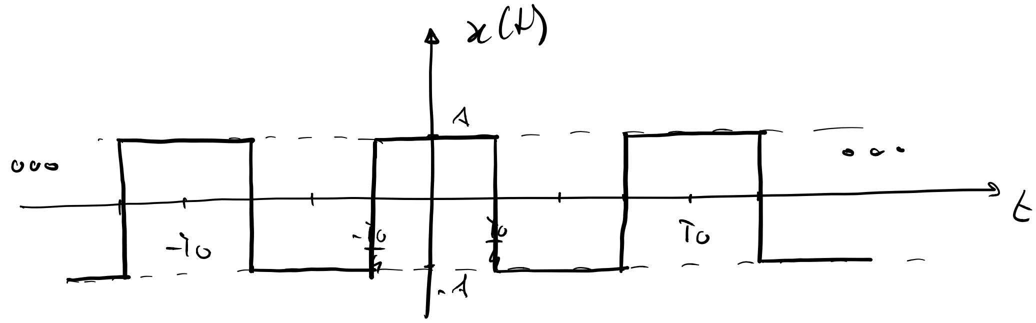
$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0) - A$$

$X_n = ?$

Svolgimento







I^a strada

- reale e pari $\Longrightarrow$ X_n reale e pari
- segnale alternativo

$$\hookrightarrow X_n = \begin{cases} 0 & n \text{ pari} \\ \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt & n \text{ disp} \end{cases}$$

II^a strada

$$x(t) = x_1(t) + x_2(t)$$

$$X_1(t) = \sum_{k=-\infty}^{+\infty} x_0(t-kT_0) = 2A\sum_{k=-\infty}^{+\infty} \text{rect}\left(\frac{t-kT_0}{T_0/2}\right)$$

$$x_2(t) = -A$$

-) Linearità $\Rightarrow$ $X_n = X_{1n} + X_{2n}$

$$X_{1n} = TSF[x_1(t)]$$

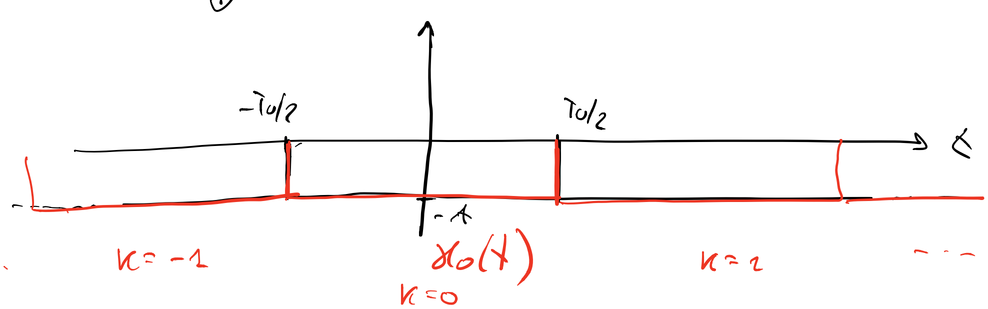
$$X_{2n} = TSF[x_2(t)]$$

$$A\sum_{k=-\infty}^{+\infty} rect\left(\frac{t-kT_0}{T}\right) \Longleftrightarrow A\frac{T}{T_0}\, sinc\left(n\frac{T}{T_0}\right)$$

$$2A\sum_{k=-\infty}^{+\infty} rect\left(\frac{t-kT_0}{T_0/2}\right) \Longleftrightarrow \cancel{2}A\,\frac{T_0/\cancel{2}}{\cancel{T_0}}\, sinc\left(n\frac{T_0/2}{\cancel{T_0}}\right)$$

$$X_{1n} = A\ sinc\left(\frac{n}{2}\right)$$

$$X_{2n} = -A = -A\sum_{k=-\infty}^{+\infty} rect\left(\frac{t-kT_0}{T_0}\right) \Longleftrightarrow -A\frac{T_0}{T_0}\, sinc\left(n\frac{T_0}{T_0}\right)$$



$$X_{2n} = -A\ sinc(n)$$

$$X_n = A\ sinc\left(\frac{n}{2}\right) - A\ sinc(n)$$

I^a strada

$$X_n = \begin{cases} 0 & n \text{ pari} \\ \dfrac{2}{T_0}\displaystyle\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt \end{cases}$$

(dispari)

$$X_n = \frac{2}{T_0}\int_0^{\frac{T_0}{2}} \left[2A\ \mathrm{rect}\left(\frac{t}{T_0/2}\right) - A\right] e^{-j2\pi n f_0 t}\, dt$$

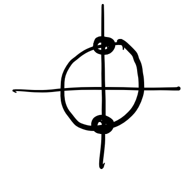
$$= \frac{4A}{T_0}\int_0^{\frac{T_0}{4}} e^{-j2\pi n f_0 t}\, dt - \frac{2A}{T_0}\int_0^{\frac{T_0}{2}} e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{\overset{2}{\cancel{4}}A}{\cancel{T_0}}\left(-\frac{1}{j\cancel{2}\pi n \cancel{f_0}}\right) e^{-j2\pi n f_0 t}\Big|_0^{T_0/4} - \frac{\cancel{2A}}{\cancel{T_0}}\left(-\frac{1}{j2\pi n \cancel{f_0}}\right) e^{-j2\pi n f_0 t}\Big|_0^{T_0/2}$$

$$= -\frac{2A}{j\pi n}\left(e^{-j\frac{\pi n \cancel{f_0 T_0}}{2}} - 1\right) + \frac{A}{j\pi n}\left(e^{-j\pi n f_0 T_0} - 1\right)$$

$$= \frac{2A}{j\pi n}\left(1 - e^{-j\frac{\pi n}{2}}\right) + \frac{A}{j\pi n}\underbrace{\left(e^{-j\pi n} - 1\right)}_{= -2}$$ n dispari

$$= \cancel{\frac{2A}{j\pi n}} - \cancel{\frac{2A}{j\pi n}} - \frac{2A e^{j\frac{\pi}{2}n}}{j\pi n} = 2A\frac{\cancel{j}(-1)^{\frac{n-1}{2}}}{\cancel{j}\pi n}$$



$$= \frac{2A\,(-1)^{\frac{n-1}{2}}}{\pi n} \qquad n \text{ dispari}$$

$$0 \qquad n \text{ pari}$$

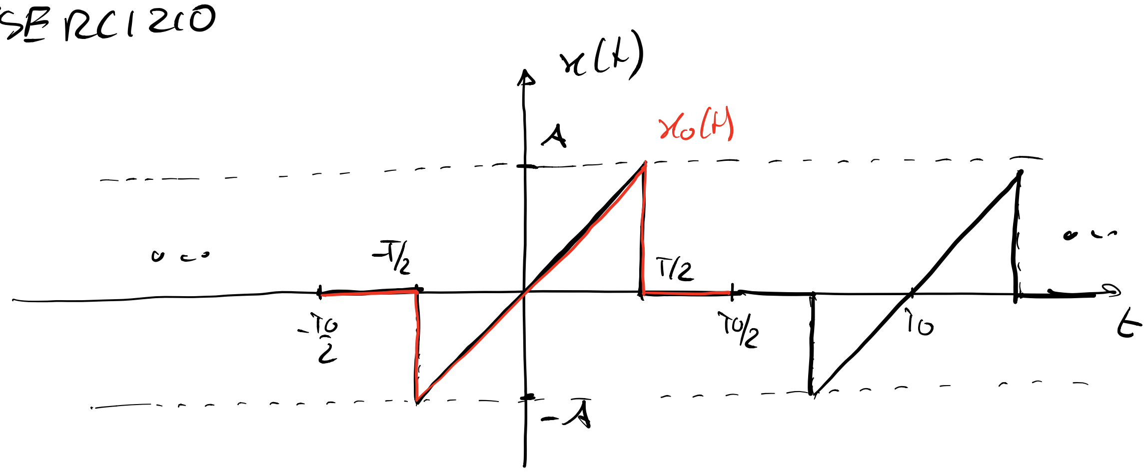
$$X_n = A\,\mathrm{sinc}\left(\frac{n}{2}\right) - A\,\mathrm{sinc}(n)$$

n pari $\Rightarrow$ $X_n = 0$

n dispari $\quad A\,\mathrm{sinc}\left(\frac{n}{2}\right) = A\,\frac{\sin \pi n/2}{\pi n/2}$

$$= \frac{A\,(-1)^{\frac{n-1}{2}}}{\pi n/2} = \frac{2A\,(-1)^{\frac{n-1}{2}}}{\pi n}$$

ESERCIZIO



$X_n = ?$

Ci aspettiamo che X_n sia dispari e imm. pura

Svolgimento

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$T < T_0$$

$$x_0(t) = \frac{2A}{T} t \; \mathrm{rect}\left[\frac{t}{T}\right]$$

$$X_n = \frac{1}{T_0}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A t}{T}\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{2A}{T_0 T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{2A}{T_0 T}\left[t\left(-\frac{1}{j2\pi n f_0}\right) e^{-j2\pi n f_0 t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} + \right.$$

$$\left. - \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi n f_0 t}\, dt \right]$$

$$= \frac{2A}{T_0 T}\left[\left(-\frac{1}{j2\pi n f_0}\right)\left(\frac{T}{2} e^{-j2\pi n f_0 \frac{T}{2}} + \frac{T}{2} e^{j2\pi n f_0 \frac{T}{2}}\right) + \right.$$

$$\left. - \left(-\frac{1}{j2\pi n f_0}\right)^2 e^{-j2\pi n f_0 t}\Big|_{-T/2}^{T/2} \right]$$

$$= \frac{2A}{T_0 T}\left[\frac{jT}{2\pi n f_0}\cos(\pi n f_0 T) + \frac{1}{4\pi^2 n^2 f_0^2}\left(e^{-j\pi n f_0 T} - e^{j\pi n f_0 T}\right)\right]$$

$$= \frac{2A}{T_0 T}\left[\frac{jT}{2\pi n f_0}\cos(\pi n f_0 T) - \frac{j}{2\pi^2 n^2 f_0^2}\sin(\pi n f_0 T)\right]$$

$$= \frac{jA}{\pi n}\cos(\pi n f_0 T) - \frac{jA}{\pi^2 n^2 f_0 T}\sin(\pi n f_0 T)$$

$$= \frac{jA}{\pi n}\cos(\pi n f_0 T) - \frac{jA}{\pi n}\operatorname{sinc}(n f_0 T)$$

$$= \underbrace{\frac{jA}{\pi n}\left[\cos(\pi n f_0 T) - \operatorname{sinc}(n f_0 T)\right]}$$

SEGNALI APERIODICI AD
ENERGIA FINITA

$$x(t) \neq x(t - kT_0) \quad \forall k \qquad \text{aperiodico}$$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty$$

$\Rightarrow$ TRASFORMATA CONTINUA DI FOURIER (TCF)

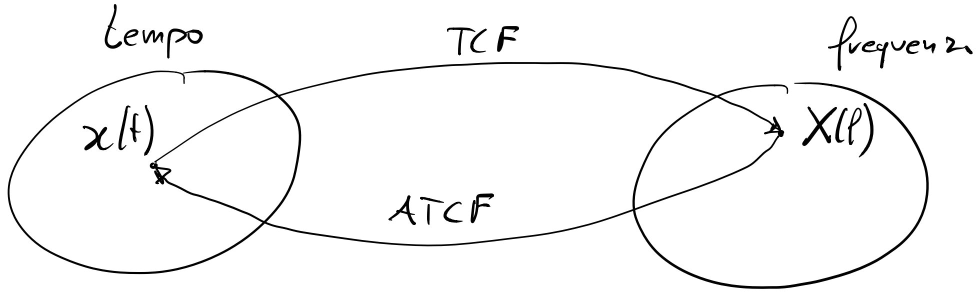
$$\boxed{X(f) \triangleq \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt}$$ Eq. di analisi

$$\boxed{x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, dt}$$ Eq. di sintesi

$n f_0$ TSF

f TCF

$$x(t) \xleftrightarrow{TCF} X(f)$$



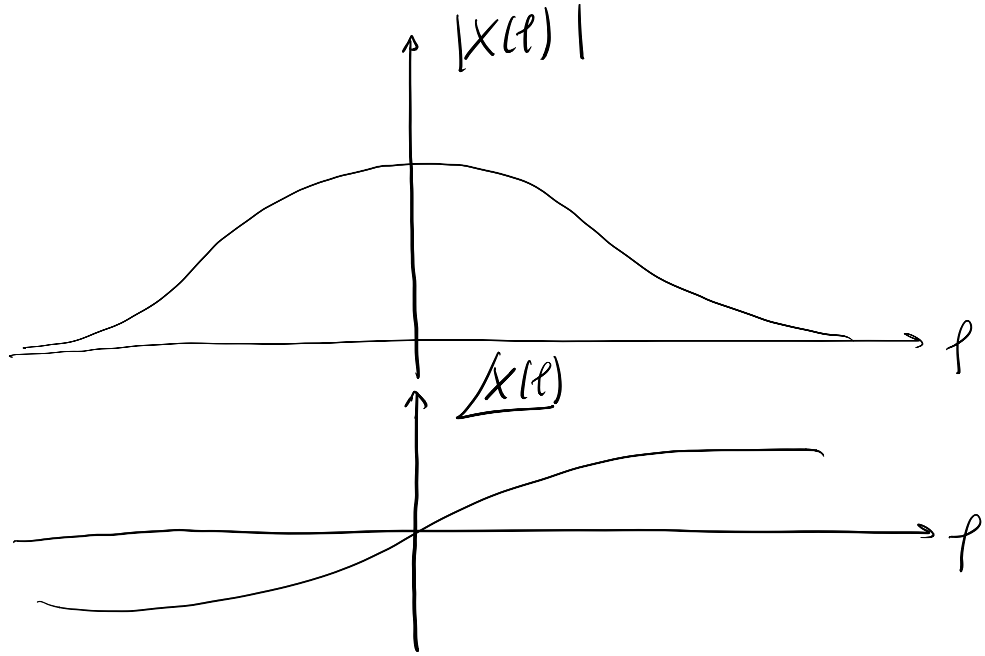
$$ATCF\big[\,TCF[x(t)]\,\big] = x(t)$$

Dimostrazione della biunivocità è rimandata

Rappresentazione dello spettro

ampiezza $|X(f)|$

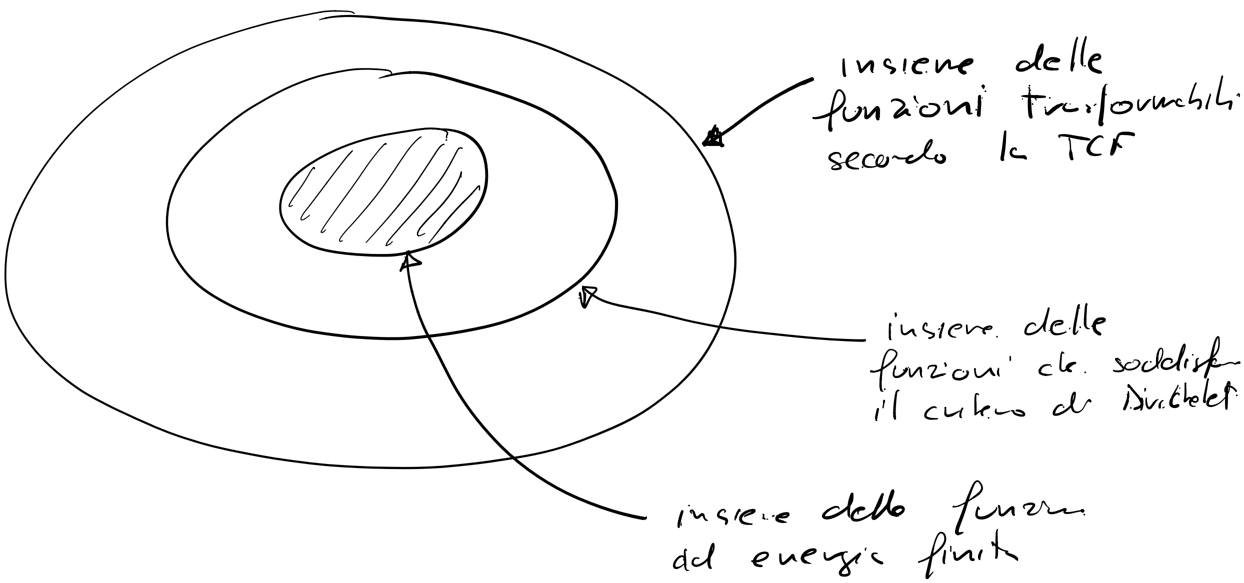
fase $\angle X(f)$



f = frequenza [Hz]

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty \Rightarrow \text{TCF esiste}$$

- Criterio di Dirichlet (sufficiente



SIMMETRIE

- Hermitiana

$x(t)$ è reale $\qquad x(t) = x^*(t)$

$$\Downarrow$$

$X(f)$ è Hermitiana $\Rightarrow X(-f) = X^*(f)$

ampiezza pari $\qquad$ fase dispari

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(-f)t}\, dt$$

$$= \left[\int_{-\infty}^{+\infty} x^*(t)\, e^{-j2\pi f t}\, dt \right]^*$$

$$= \left[\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt}_{X(f)} \right]^* = X^*(f)$$

parità

$$|X(-f)| = |X^*(f)| = |X(f)|$$

$$\angle X(-f) = \angle X^*(f) = -\angle X(f)$$

disparità

SEGNALI REALI E PARI

$x(t)$ reale $\Rightarrow$ $x(t) = x^*(t)$

$x(t)$ pari $\Rightarrow$ $x(t) = x(-t)$

$\Downarrow$

$X(f)$ è reale e pari

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(-f)t}\, dt \qquad t' = -t$$

$$= \int_{-\infty}^{+\infty} x(-t')\, e^{-j2\pi f t'}\, dt'$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt' = X(f)$$

$$\overbrace{X(-f) = X(f)}^{\text{pari}}$$

$$X(-f) = X^*(f) \quad \Rightarrow \quad \underbrace{X(f) = X^*(f)}_{\text{reale}}$$

SEGNALI REALI E DISPARI

$$x(t) \text{ reale} \Rightarrow x(t) = x^*(t)$$

$$x(-t) = -x(t) \quad \text{dispari}$$

$$\Downarrow$$

$X(f)$ è imm e dispari

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{j2\pi f t}\, dt \qquad t' = -t$$

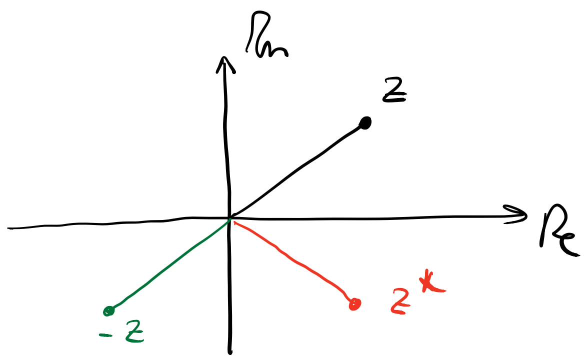
$$= \int_{-\infty}^{+\infty} x(-t')\, e^{-j2\pi f t'}\, dt' = \int_{-\infty}^{+\infty} -x(t')\, e^{-j2\pi f t'}\, dt'$$

$= - X(f)$

$$X(-f) = -X(f) \quad \text{dispari}$$

$$X(-f) = X^*(f)$$

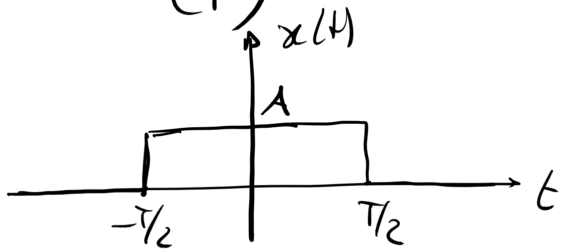
$$X^*(f) = -X(f) \quad \text{imm. puro}$$



$-z = z^*$ solo se $\mathrm{Re}\{z\} = 0$

Esercizio

$$x(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)$$



$X(f) = ?$

Svolgimento

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= A \int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{t}{T}\right) e^{-j2\pi f t}\, dt$$

$$= A \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi f t}\, dt = A \left(-\frac{1}{j2\pi f}\right) e^{-j2\pi f t} \Big|_{-T/2}^{T/2}$$

$$= A \left(-\frac{1}{j2\pi f}\right) \left(e^{-j\pi f T} - e^{j\pi f T}\right)$$

$$= \frac{A}{\pi f} \, \frac{e^{j\pi f T} - e^{-j\pi f T}}{2j} = \frac{AT}{\pi f T} \sin(\pi f T)$$

$$X(f) = AT\, \mathrm{sinc}(fT)$$

reale

pari

$Re\{X(f)\}$, $Im\{X(f)\} = 0$