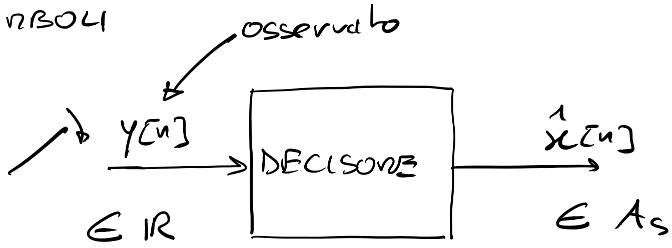
⇒ CRITERIO PER "OTTIMIZZARE" LA DECISIONE DEI SIMBOLI



⇒ CRITERIO A MINIMA PROB. DI ERRORE

$\Updownarrow$

⇒ CRITERIO A MASSIMA PROB. A POSTERIORI

$$P\{x=\alpha_i \mid \underbrace{y}_{\text{osservato}}\}$$

Dim.

$$P_E(M) \triangleq P\{\hat{x} \neq x\} = 1 - P\{\hat{x} = x\} \dots$$

$$\dots = 1 - \sum_{i=1}^{M} \int_{y \in R_i} P\{x=\alpha_i\}\, f_Y(y \mid x=\alpha_i)\, dy \quad \Leftarrow$$

$$= 1 - \sum_{i=1}^{M} P\{x=\alpha_i\} \underbrace{\int_{y\in R_i} f_Y(y \mid x=\alpha_i)\, dy}_{P\{\hat{x}=\alpha_i \mid x=\alpha_i\}}$$

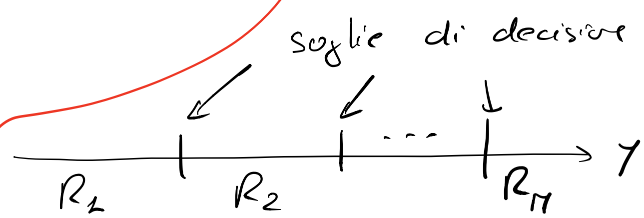
prob. di transizione

$$P_E(M) = 1 - \sum_{i=1}^{M} P\{\hat{x}=\alpha_i \mid x=\alpha_i\}\, P\{x=\alpha_i\}$$

$$P\{x=\alpha_i\}\, f_Y(y|x=\alpha_i) = P\{x=\alpha_i|y\}\, f_Y(y)$$

$$= P_E(M) = 1 - \underbrace{\sum_{i=1}^{M} \int_{y\in R_i} P\{x=\alpha_i|y\}\, f_Y(y)\, dy}_{} \longrightarrow \text{minima}$$

$$\max_{R_i} \sum_{i=1}^{M} \int_{y\in R_i} \underbrace{P\{x=\alpha_i|y\}}_{}\, \underbrace{f_Y(y)}_{\text{non dipende da "i"}}\, dy$$



PROBABILITÀ A POSTERIORI : devono essere massime per massimizzare il funzionale

$\Downarrow$

quando la $P\{x=\alpha_i|y\}$ è massima $\Rightarrow$ decido per il simbolo α_i

$$\left.\begin{matrix} P\{x=\alpha_1|y\} \\ P\{x=\alpha_2|y\} \\ \vdots \\ P\{x=\alpha_M|y\} \end{matrix}\right\} \Rightarrow \max \Rightarrow \alpha_i \;\Big|\; P\{x=\alpha_i|y\} \text{ è } \max$$

CRITERIO MAP

CRITERIO A min P_E $\Rightarrow$ $\underbrace{\text{CRITERIO MAP}}$

$\Rightarrow$ CASO DI $P\{\alpha_i\} = \dfrac{1}{M} \quad \forall i$

simboli sono equiprobabili

$$\hat{x} = \max_{\alpha_1,\dots,\alpha_M} P\{x=\alpha_i \mid y\} =$$

$$= \max_{\alpha_1,\dots,\alpha_n} \frac{P\{x=\alpha_i\}\, f_Y(y \mid x=\alpha_i)}{f_Y(y)}$$

$$= \frac{1}{M}\,\frac{1}{f_Y(y)} \max_{\alpha_1,\dots,\alpha_n}\left[f_Y(y \mid x=\alpha_i)\right]$$

funzione di verosimiglianza

$f_Y(y \mid x=\alpha_i)$ è la ddp della v.a.

$$\underbrace{y|x=\alpha_i = h(0)\,\alpha_i + n_k}$$

si può determinare

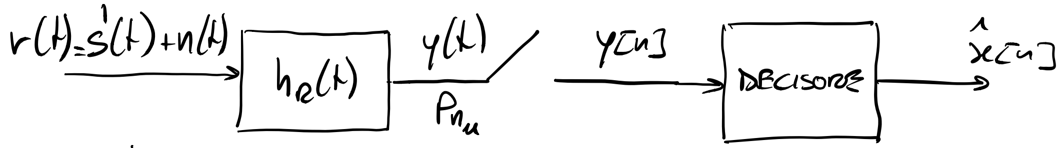
$$= \max_{\alpha_1,\dots,\alpha_n}\left[f_Y(y \mid x=\alpha_i)\right]$$

CRITERIO A MASSIMA VEROSIMIGLIANZA

$\Rightarrow$ $P_E(n)$ minima $\Rightarrow$ MAP $\xRightarrow{\text{simb. equip}}$ MAX VEROSIMIGLIANZA

Maximum A posteriori Probability

⇒ Caso Gaussiano Bianco



$s'(t) = s(t) \otimes c(t)$

In assenza di ISI

$$y[n] = h(0)\,x[n] + n_u$$

è una v.a. Gaussiana con valor medio nullo e varianza $\sigma_{n_u}^2 = P_{n_u}$

$$n_u \in \mathcal{N}(0, \sigma_{n_u}^2) \Rightarrow f_{n_u}(n_u) = \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}}\, e^{-\frac{n_u^2}{2\sigma_{n_u}^2}}$$

$$h(0) = h(t)\Big|_{t=0} = \Big[p(t) \otimes c(t) \otimes h_R(t)\Big]\Big|_{t=0}$$

$$f_Y(y \mid x = \alpha_i)$$

$$y \mid x = \alpha_i = h(0)\alpha_i + \underset{a}{\uparrow} n_u \quad \Leftarrow \text{TRASF. LINEARE DI UNA V.A. GAUSSIANA}$$

$$g(n_u) = a\, n_u + b \quad , \quad a = 1 \quad , \quad b = h(0)\alpha_i$$

$$y = n_u + h(0)\alpha_i \qquad n_u = y - h(0)\alpha_i \qquad \text{1 sol per } g^{-1}(y)$$

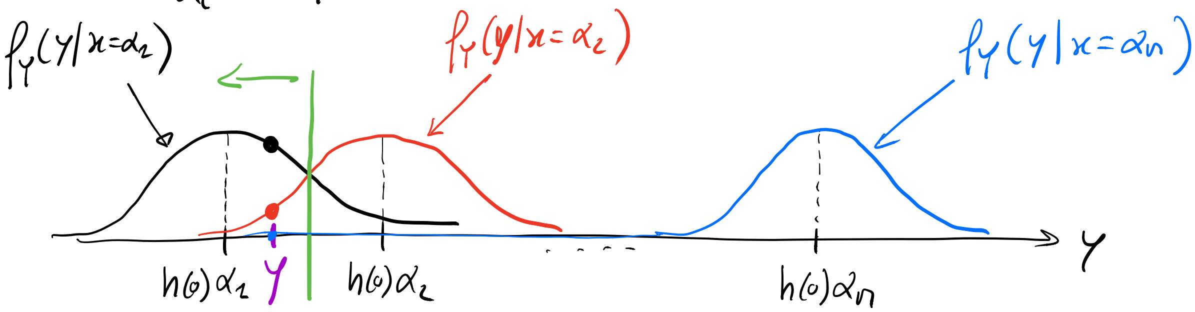
$$f_Y(y \mid x = \alpha_i) = \frac{f_{n_u}(y - h(0)\alpha_i)}{|1|} = f_{n_u}(y - h(0)\alpha_i)$$

$$f_Y(y|x=\alpha_i) = \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{(y-h(0)\alpha_i)^2}{2\sigma_{n_u}^2}} \in \mathcal{N}(h(0)\alpha_i, \sigma_{n_u}^2) \quad \forall i$$

$\Rightarrow$ se i simboli sono equiprob.

min $P_E(\pi)$ $\Rightarrow$ max verosimiglianza

$$\hat{x} = \max_{\alpha_i} f_Y(y|x=\alpha_i)$$ CRITERIO DI DECISIONE



$\Rightarrow$ decido guardando quale valore della funzione di verosimiglianza è massimo

$\Rightarrow$ nell'esempio sopra è $\hat{x} = \alpha_1$

$$\max_{\alpha_i}\left[f_Y(y|x=\alpha_i)\right] = \max_{\alpha_i} \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{(y-h(0)\alpha_i)^2}{2\sigma_{n_u}^2}}$$

non dipende da α_i

$$= \max_{\alpha_i} e^{-\frac{(y-h(0)\alpha_i)^2}{2\sigma_{n_u}^2}} = \min_{\alpha_i} \frac{(y-h(0)\alpha_i)^2}{2\sigma_{n_u}^2}$$

non dipende da α_i

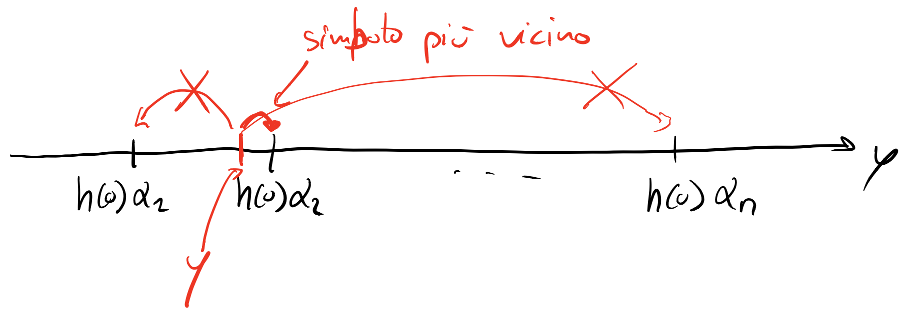
l'esponenziale negativo è max quando l'esponente è minimo

$$= \min_{\alpha_i} (y-h(0)\alpha_i)^2 = \min_{\alpha_i} |y-h(0)\alpha_i|$$

distanza Euclidea

⇒ CRITERIO A MINIMA DISTANZA

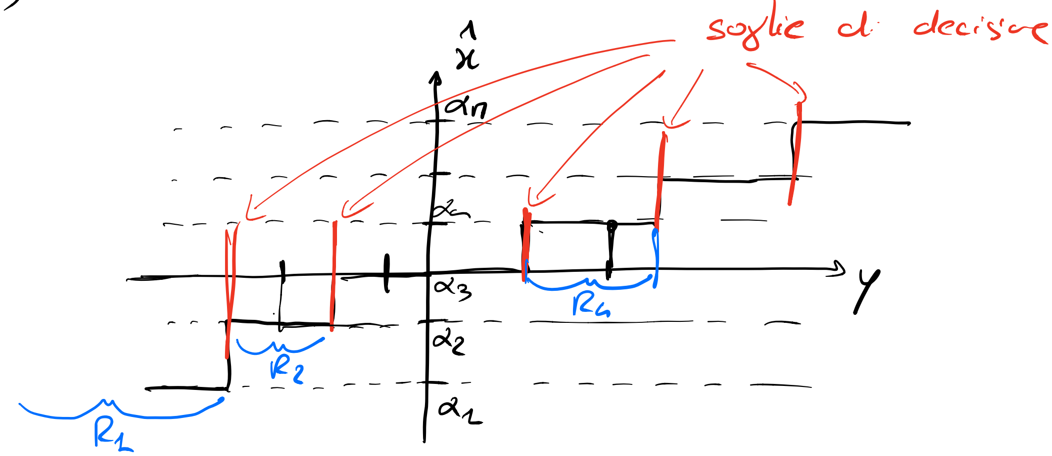
si decide per il simbolo più vicino



minima distanza ⇒ max verosimiglianza ⇒ MAP ⇒ min P_E

Ipotesi

- simboli equiprob
- rumore Gaussiano Bianco ⇒ minima distanza



⇒ il decisore si comporta, o meglio, viene implementato da un quantizzatore

⇒ Nel caso Gaussiano Bianco, anche con simboli non equiprobabili, si arriva ad un decisore a soglie, ma non vale più la regola della minima distanza Euclidea

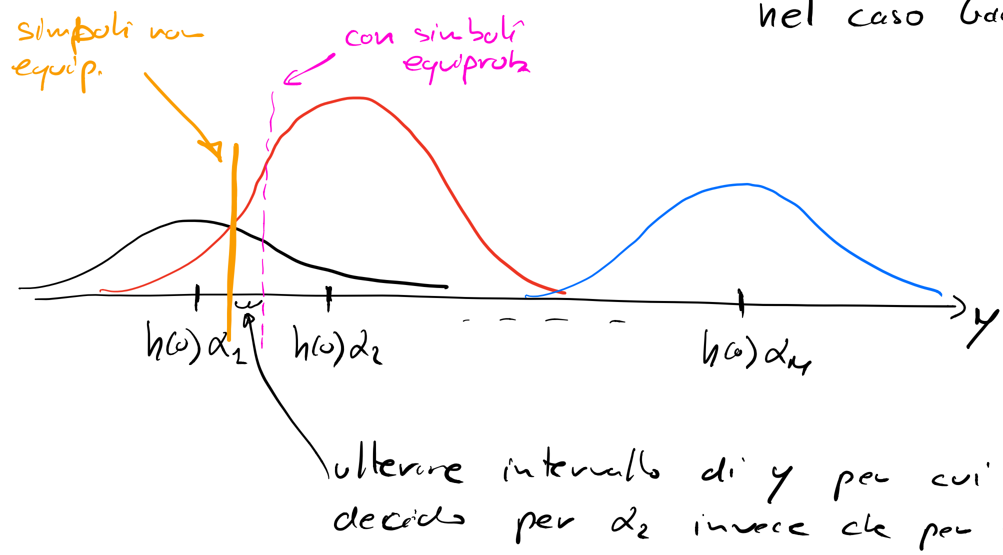
$$\max_{\alpha_i} P\{x=\alpha_i \mid y\} = \max_{\alpha_i} \frac{P\{x=\alpha_i\}\, f_Y(y|x=\alpha_i)}{f_Y(y)}$$

$$= \max_{\alpha_i} P\{x=\alpha_i\}\, f_Y(y|x=\alpha_i)$$

VEROSIMIGLIANZE "PESATE" DALLE PROB. A PRIORI

non solo equiprob

nel caso Gaussiano



⇒ Il decisore è ancora un decisore a soglia

⇒ Noi tratteremo solo rumore Gaussiano bianco

⇓

sempre decisioni a soglia

ANALISI DELLE PERFORMANCE DI SISTEMI DI COMUNICAZIONE IN B.B. CON MODULAZIONE PAM (ANCHE NON STANDARD)

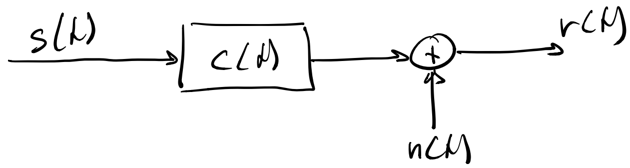
$\hookrightarrow \alpha_i \neq 2i-1-M$

$\Downarrow$

CALCOLARE LA PROB. DI ERRORE

Faremo sempre l'ipotesi di

- rumore Gaussiano bianco
- $M = 2$ (PAM BINARIA)
- modello di canale con $c(t)$ noto e rumore additivo

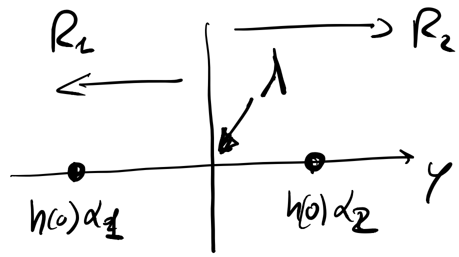


- decisore a soglia con soglia nota

Il testo del problema fornisce sempre

- $s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT_s)$
- $p(t)$ nota
- $x[n] \in A_s$ noto
- $P\{\alpha_i\}$ $i = 1, 2$ note

- $c(t)$ noto
- $n(t)$ Gaussiano bianco con DSP nota
- $h_R(t)$ nota
- T_S noto → in Tx / → al Rx (campionatore)
- soglia di decisione "λ" nota



$\Rightarrow P_E(a) = P_E(b) = ?$

- VERIFICARE L'ASSENZA DI ISI

$$\Downarrow$$

$$y[n] = h(0)\,x[n] + n_u[n]$$

conosciamo questa V.A.

$$y|_{x=\alpha_i} = h(0)\,\alpha_i + n_u$$

questo caso è risolvibile

$\Rightarrow$ in presenza di ISI

non lo sappiamo modellare

$$y[n] = h(0)\,x[n] + \sum_{\substack{k=-\infty \\ k \neq n}}^{+\infty} h[n-k]\,x[k] + n_u[n]$$

$\Rightarrow$ in presenza di ISI la P_E aumenta

$\Rightarrow$ VERIFICA DELL' ASSENZA DI ISI

- $h(t)\Big|_{t=nT_s} = h(0)\,\delta[n]$ Nyquist nel tempo

- $\displaystyle\sum_{n=-\infty}^{+\infty} H\left(f-\frac{n}{T_s}\right) = K$ costante (in freq.)

- CALCOLARE LA $h(0)$
  - $h(t)\Big|_{t=0}$
  - $H(f) \Rightarrow h(t) = \text{ATCF}\left[H(f)\right]\Big|_{t=0}$

$$\Downarrow$$

$$\int_{-\infty}^{+\infty} H(f)\,df = h(0)$$

- modello di $y|\alpha_i$

$$y|\alpha_i = \underset{\Uparrow}{h(0)}\,\alpha_i + n_u$$

$$P_E(b) = P\{\hat{b} \neq b\} = P\{\hat{b} = \alpha_1, b = \alpha_2\} + P\{\hat{b} = \alpha_2, b = \alpha_1\}$$

·) due eventi errore
·) sono disgiunti

$$= P\{\hat{b} = \alpha_1 \mid b = \alpha_2\} P\{b = \alpha_2\} + P\{\hat{b} = \alpha_2 \mid b = \alpha_1\} P\{b = \alpha_1\}$$

$P\{b = \alpha_1\}$, $P\{b = \alpha_2\}$ sono le prob. a priori (note)

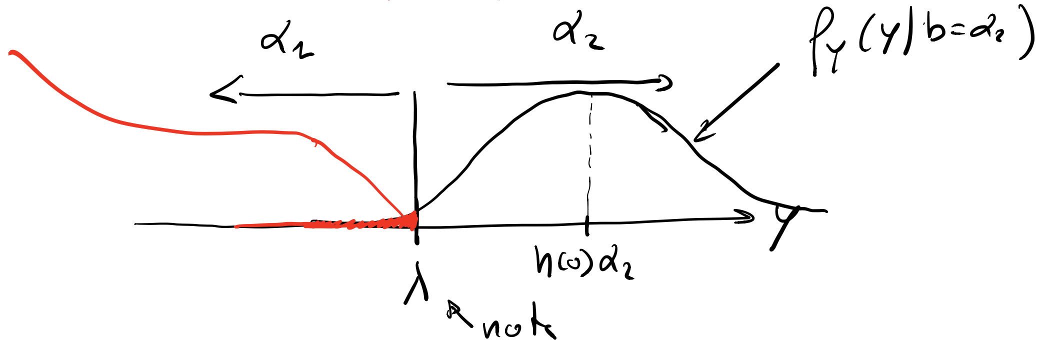
$P\{\alpha_1\}$ $\qquad$ $P\{\alpha_2\}$

·) devo calcolare

$$P\{\hat{b} = \alpha_1 \mid b = \alpha_2\} = ?$$

$$P\{\hat{b} = \alpha_2 \mid b = \alpha_1\} = ?$$

area sottesa $\Rightarrow P\{\hat{b} = \alpha_1 \mid b = \alpha_2\}$



$$P\{\hat{b}=\alpha_1 \mid b=\alpha_2\} = \int_{-\infty\,(R_1)}^{\lambda} f_Y(y \mid b=\alpha_2)\,dy$$

$$y \mid b=\alpha_2 = h(0)\alpha_2 + n_u$$

$$f_Y(y \mid b=\alpha_2) = \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}}\, e^{-\frac{(y-h(0)\alpha_2)^2}{2\sigma_{n_u}^2}}$$

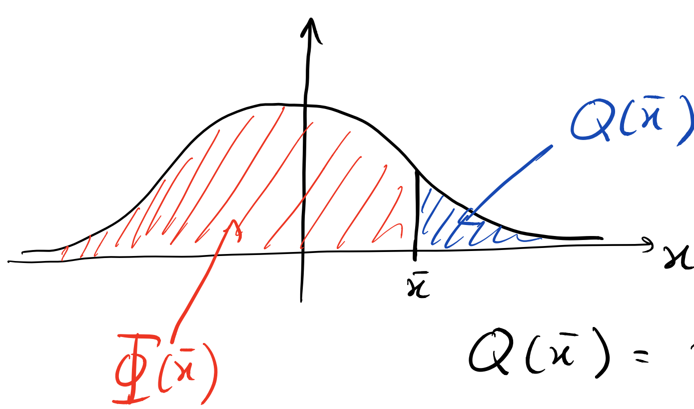
$$\Rightarrow = F_Y(\lambda \mid b=\alpha_2) = \Phi\left(\frac{\lambda - h(0)\alpha_2}{\sigma_{n_u}}\right)$$

$$y \mid b=\alpha_2 \in \mathcal{N}(0, \sigma_{n_u}^2)$$

$$y \mid b=\alpha_2 = n_u + h(0)\alpha_2$$

$$n_u = \frac{y \mid b=\alpha_2 - h(0)\alpha_2}{\sigma_{n_u}}$$

$$Q(\bar{x}) = 1 - \Phi(\bar{x})$$

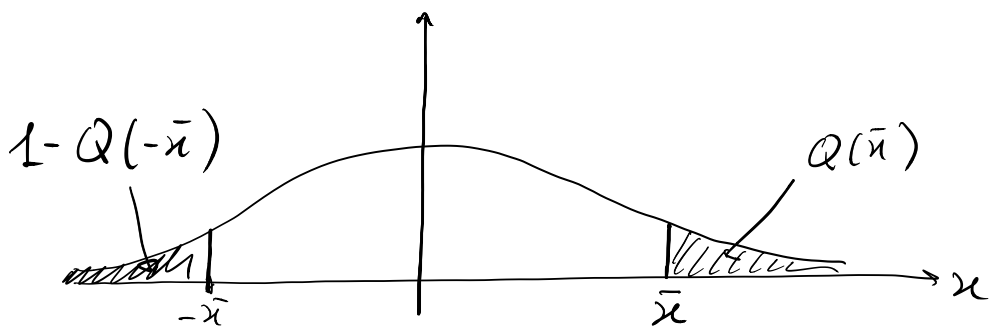


$$Q(\bar{x}) = 1 - \Phi(x)$$

$$P\{\hat{b}=\alpha_1 \mid b=\alpha_2\} = F_Y(\lambda \mid b=\alpha_2) = \Phi\left(\frac{\lambda - h(0)\alpha_2}{\sigma_{n_u}}\right)$$

$$= 1 - Q\left(\frac{\lambda - h(0)\alpha_2}{\sigma_{n_u}}\right)$$

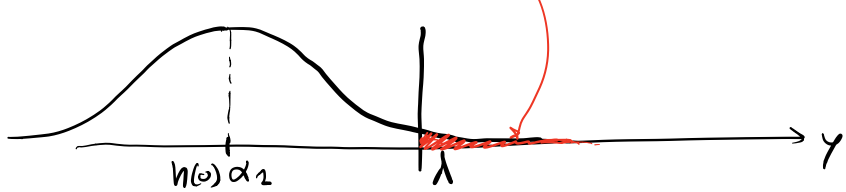
$$= Q\left(-\frac{\lambda - h(0)\alpha_2}{\sigma_{n_u}}\right) = \boxed{Q\left(\frac{h(0)\alpha_2 - \lambda}{\sigma_{n_u}}\right)}$$



$$Q(\bar{x}) = 1 - Q(-\bar{x})$$

$$\Rightarrow \boxed{P\{\hat{b}=\alpha_2 \mid b=\alpha_2\} = Q\left(\frac{h(0)\alpha_2 - \lambda}{\sigma_{n_u}}\right)}$$

$$\Rightarrow \underline{P\{\hat{b}=\alpha_2 \mid b=\alpha_1\}} = \int_{\lambda}^{+\infty} f_Y(y \mid x=\alpha_1)\, dy$$



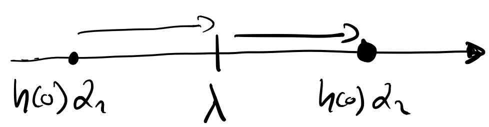
$$= Q\left(\frac{\lambda - h(0)\alpha_1}{\sigma_{n_u}}\right)$$

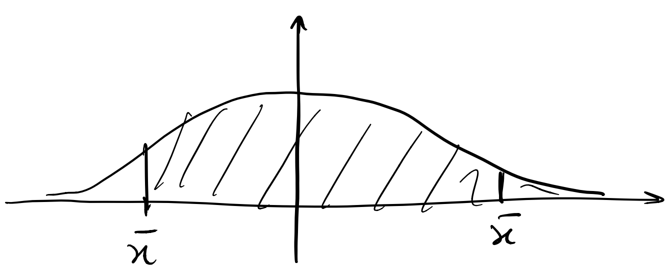
$$\Uparrow$$

$$y|\alpha_1 = h(0)\alpha_2 + n_u$$

$$P\{\hat{b} = \alpha_2 \mid b = \alpha_2\} = Q\left(\frac{\overbrace{h(0)\alpha_2 - \lambda}^{\text{quantità positiva}}}{\sigma_{n_u}}\right)$$

$$P\{\hat{b} = \alpha_2 \mid b = \alpha_1\} = Q\left(\frac{\overbrace{\lambda - h(0)\alpha_1}^{\text{positiva}}}{\sigma_{n_u}}\right)$$





negativo

$\bar{x}$ positivo

$Q(\bar{x}) < \frac{1}{2}$

$$P_E(b) = P\{\alpha_1\}\, P\{\hat{b} = \alpha_2 \mid b = \alpha_1\} +$$
$$P\{\alpha_2\}\, P\{\hat{b} = \alpha_1 \mid b = \alpha_2\}$$

$$= \boxed{P\{\alpha_1\}\, Q\left(\frac{\lambda - h(0)\,\alpha_1}{\sigma_{n_u}}\right) + P\{\alpha_2\}\, Q\left(\frac{h(0)\,\alpha_2 - \lambda}{\sigma_{n_u}}\right)}$$

$\Rightarrow$ $P\{\alpha_1\}$, $P\{\alpha_2\}$ note (le cerco nel testo del problema)

$\Rightarrow$ λ nota la cerco nel testo

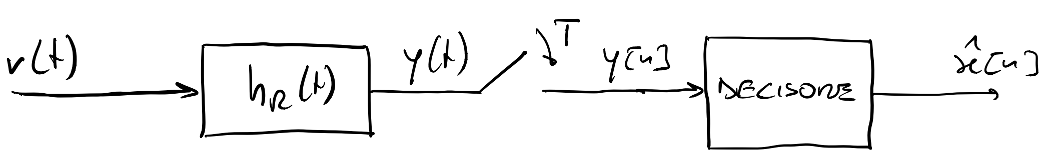
$\Rightarrow$ α_1, α_2 noti $\Rightarrow$ A_s

$\Rightarrow$ $\sigma_{n_u}^2 = P_{n_u}$ $\Rightarrow$ lo devo calcolare all'uscita di $h_R(t)$

$$\boxed{P_{n_u} = \frac{N_0}{2} E_{h_R}}$$

$$\sigma_{n_u} = \sqrt{P_{n_u}} = \sqrt{\frac{N_0}{2} E_{h_R}}$$

ESERCIZIO - 06 Febbraio 2017



$A_s = \{-1, 2\}$ equiprobabili

$w(t)$ Gaussiano bianco con DSP $\frac{N_0}{2}$

$$p(t) = \frac{2}{T}\,\mathrm{sinc}^2\left(\frac{2t}{T}\right)\cos\left(\frac{4\pi t}{T}\right)$$

$$h_R(t) = \frac{4}{T}\,\mathrm{sinc}\left(\frac{4t}{T}\right)$$

$$\lambda = 0$$

$$c(t) = \delta(t)$$ (quando non è specificato nel testo)

4) Verificare l'assenza di ISI

$$h(t) \quad , \quad H(f)$$

$$h(t) = p(t) \otimes c(t) \otimes h_R(t) = p(t) \otimes h_R(t)$$

essendo $c(t) = \delta(t)$

$$H(f) = P(f)\,H_R(f)$$

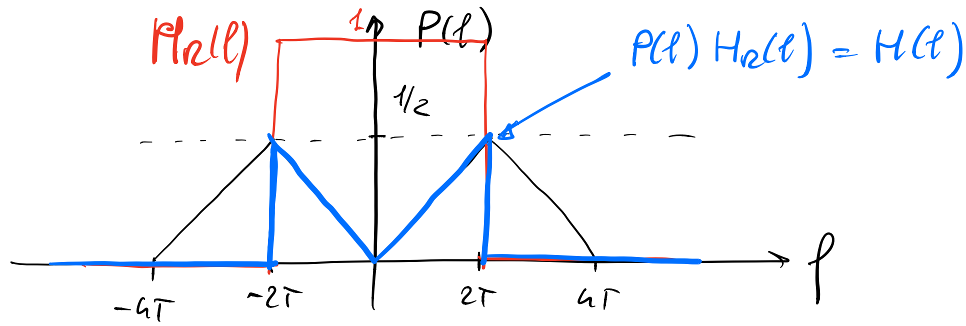
$$p(t) = p_0(t)\cos\left(\frac{4\pi t}{T}\right) \xRightarrow{\text{TCF}} P(f) = \frac{1}{2}P_0\left(f - \frac{2}{T}\right) + \frac{1}{2}P_0\left(f + \frac{2}{T}\right)$$

$$p_0(t) = \frac{2}{T}\,\mathrm{sinc}^2\left(\frac{2t}{T}\right)$$

$$\Downarrow \text{TCF}$$

$$P_0(f) = \left(1 - \frac{|f|}{2/T}\right)\mathrm{rect}\left(\frac{f}{4/T}\right)$$

$$P(f) = \frac{1}{2}\left(1 - \frac{|f - \frac{2}{T}|}{2/T}\right)\mathrm{rect}\left(\frac{f - \frac{2}{T}}{4/T}\right) + \frac{1}{2}\left(1 - \frac{|f + \frac{2}{T}|}{2/T}\right)\mathrm{rect}\left(\frac{f + \frac{2}{T}}{4/T}\right)$$



$$H_R(f) = \text{rect}\left(\frac{f}{4/T}\right)$$

$$H(f) = \frac{1}{2}\,\text{rect}\left(\frac{f}{4/T}\right) - \frac{1}{2}\left(1 - \frac{|f|}{2/T}\right)\text{rect}\left(\frac{f}{4/T}\right)$$

$$h(t) = \frac{1}{2}\,\frac{4}{T}\,\text{sinc}\left(\frac{4t}{T}\right) - \frac{1}{2}\cdot\frac{2}{T}\,\text{sinc}^2\left(\frac{2t}{T}\right)$$

$$= \frac{2}{T}\,\text{sinc}\left(\frac{4t}{T}\right) - \frac{1}{T}\,\text{sinc}^2\left(\frac{2t}{T}\right)$$

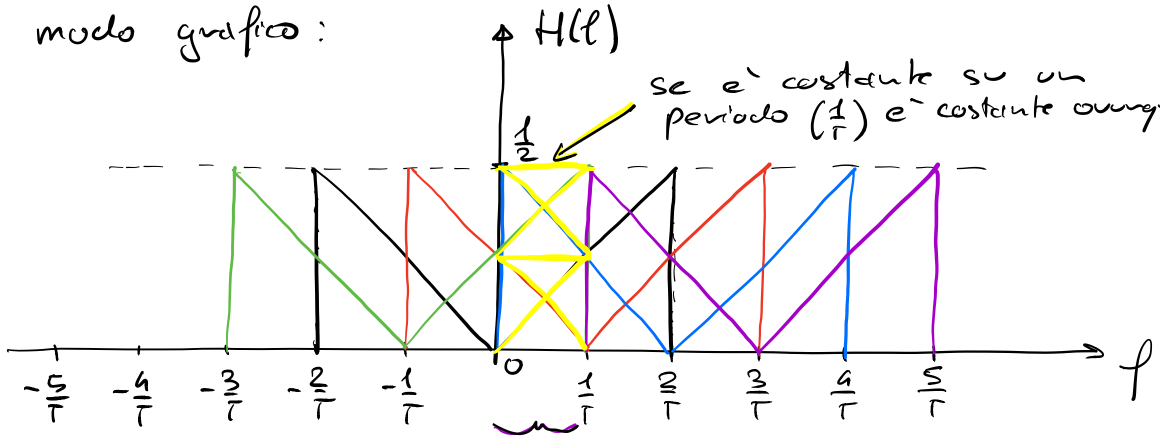
$$\Rightarrow h(t)\Big|_{t=nT} = \frac{2}{T}\,\text{sinc}\left(\frac{4}{\cancel{T}}n\cancel{T}\right) - \frac{1}{T}\,\text{sinc}^2\left(\frac{2}{\cancel{T}}n\cancel{T}\right)$$

$$= \frac{2}{T}\,\text{sinc}(4n) - \frac{1}{T}\,\text{sinc}^2(2n)$$

la sinc(·) si annulla quando l'argomento è un intero, eccetto per $t=0$

$$= \frac{2}{T}\,\delta[n] - \frac{1}{T}\,\delta[n] = \frac{1}{T}\,\delta[n]$$

è verificata la cond. di Nyquist nel tempo in modo grafico:



$$\frac{1}{T}\sum_{n=-\infty}^{+\infty} H\left(f-\frac{n}{T}\right) = \text{funzione periodica di } \frac{1}{T} = \overline{H}(f)$$

$$= \frac{1}{T}\cdot\frac{1}{2} = \text{costante}$$

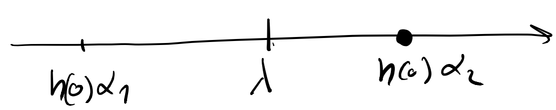
$\Rightarrow$ <u>ISI è assente</u>

5) $P_E(b) = P\{\hat{b}=\alpha_1, b=\alpha_2\} + P\{\hat{b}=\alpha_2, b=\alpha_1\}$

$$= P\{\hat{b}=\alpha_1 \mid b=\alpha_2\}\underbrace{P\{\alpha_2\}}_{\frac{1}{2}} + P\{\hat{b}=\alpha_2 \mid b=\alpha_1\}\underbrace{P\{\alpha_1\}}_{\frac{1}{2}}$$

$\downarrow$ $Q\left(\frac{h(0)\alpha_2 - \overset{=0}{\lambda}}{\sqrt{P_{n_u}}}\right)$, $\sqrt{P_{n_u}} = \sigma_{n_u}$

$\downarrow$ $Q\left(\frac{\overset{=0}{\lambda} - h(0)\alpha_1}{\sqrt{P_{n_u}}}\right)$, $\sqrt{P_{n_u}} = \sigma_{n_u}$



$$h(0) = \frac{1}{T} \qquad \alpha_1 = -1 \quad , \quad \alpha_2 = 2$$

$$P_{n_u} = \frac{N_0}{2} E_{h_R} = \frac{N_0}{2} \cdot \frac{4}{T} = \frac{2N_0}{T}$$

$$P_E(b) = \frac{1}{2} Q\left(\frac{\frac{1}{T} \cdot 2 - 0}{\sqrt{\frac{2N_0}{T}}}\right) + \frac{1}{2} Q\left(\frac{0 - \frac{1}{T}(-1)}{\sqrt{\frac{2N_0}{T}}}\right)$$

$$= \frac{1}{2} Q\left(\frac{2/T}{\sqrt{\frac{2N_0}{T}}}\right) + \frac{1}{2} Q\left(\frac{1/T}{\sqrt{\frac{2N_0}{T}}}\right)$$

$$= \frac{1}{2} Q\left(\sqrt{\frac{4}{T^2}\frac{T}{2N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{1}{T^2}\frac{T}{2N_0}}\right)$$

$$= \boxed{\frac{1}{2} Q\left(\sqrt{\frac{2}{N_0 T}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{1}{2N_0 T}}\right)}$$