

Segnali Analogici

Potenza istantanea

$$P_x(t) \triangleq |x(t)|^2 \quad x \in \mathbb{C}$$

Energia

$$E_x \triangleq \int_{-\infty}^{\infty} P_x(t) dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

Potenza Media

$$P_{x_T} \triangleq \frac{E_{x_T}}{T}$$

$$P_x \triangleq \lim_{T \rightarrow \infty} \frac{E_{x_T}}{T} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

1 $x(t)$: $E_x < \infty \Rightarrow P_x = 0$

Dim:

$$P_x = \lim_{T \rightarrow \infty} \frac{E_{x_T}}{T} \xrightarrow[T \rightarrow \infty]{K < \infty} 0$$

2 $x(t)$: $P_x = K < \infty \Rightarrow E_x = \infty$

Dim: \emptyset

$$x(t) : \emptyset < P_x < \infty$$

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot E_{x_T} = K \neq 0 \Rightarrow \lim_{T \rightarrow \infty} E_{x_T} = \infty$$

$$\lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = E_x = \infty$$

Valore Efficace

$$x_{eff} \triangleq \sqrt{P_x}$$

Valore Medio

$$x_m \triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt$$

Segnali Tipici

Costante

$$x(t) = A + t$$

- $E_x = \infty$
- $P_x = A^2$
- $x_{eff} = |A|$
- $x_m = A$

Sinusoide

$$x(t) = A \cos(2\pi f_0 t + \varphi)$$

$$\begin{aligned} -Ex &= \int_{-\infty}^{\infty} |A \cos(2\pi f_0 t + \varphi)|^2 dt = A^2 \int_{-\infty}^{\infty} \cos^2(2\pi f_0 t + \varphi) dt = A^2 \cdot \int_{-\infty}^{\infty} \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi f_0 t + 2\varphi) \right] dt = \\ &= \underbrace{\frac{A^2}{2} \int_{-\infty}^{\infty} dt}_{L \rightarrow \infty} + \underbrace{\frac{A^2}{2} \int_{-\infty}^{\infty} \cos(4\pi f_0 t + 2\varphi) dt}_{=0} = \infty \end{aligned}$$

$$-Px = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cos^2(2\pi f_0 t + \varphi) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \left[\frac{A^2 T}{2} + \frac{A^2}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(4\pi f_0 t + 2\varphi) dt \right] = \frac{A^2}{2}$$

$$-X_{eff} = \frac{|A|}{\sqrt{2}} \quad \frac{-A}{2\pi f_0} \leq K \leq \frac{A}{2\pi f_0}$$

$$-X_m = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} A \cos(2\pi f_0 t + \varphi) dt = \lim_{T \rightarrow \infty} \frac{K}{T} = \emptyset$$

Gradino

$$x(t) \triangleq \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$-Ex = \int_{-\infty}^{\infty} u^2(t) dt = \int_{-\infty}^{\infty} u(t) dt = \int_{-\infty}^{\infty} 1 dt = \infty$$

$$-Px = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u^2(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{\frac{T}{2}} 1 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \frac{T}{2} = \frac{1}{2}$$

Rettangolo

$$x(t) = A \cdot \text{rect}\left(\frac{t}{T}\right) = \begin{cases} A & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{altrav} \end{cases}$$

$$-Ex = \int_{-\infty}^{+\infty} A^2 \text{rect}^2\left(\frac{t}{T}\right) dt = A^2 \cdot \int_{-\infty}^{\infty} \text{rect}^2\left(\frac{t}{T}\right) dt = A^2 \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 dt = A^2 T$$

$$-Px = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \cdot \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A^2 \text{rect}^2\left(\frac{t}{T_0}\right) dt = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \cdot A^2 T = \emptyset$$

$$-X_{eff} = \emptyset$$

$$-X_m = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \cdot \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A \text{rect}\left(\frac{t}{T_0}\right) dt = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \cdot AT = \emptyset$$

Esponenziale Unilatera (monopolare)

$$x(t) = e^{-t} \cdot u(t)$$

$$-Ex = \int_{-\infty}^{\infty} (e^{-t} u(t))^2 dt = \int_0^{\infty} e^{-2t} dt = \frac{1}{2} e^{-2t} \Big|_0^{\infty} = \frac{1}{2}$$

$$-Px = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2t} u(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{\frac{T}{2}} e^{-2t} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \left(-\frac{1}{2} e^{-2t} \right) \Big|_0^{\frac{T}{2}} = \emptyset$$

$$-xe_{\frac{d}{dt}} = \emptyset$$

$$-xm = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-t} u(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_0^{\frac{T}{2}} e^{-t} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot (-e^{-t}) \Big|_0^{\frac{T}{2}} = \emptyset$$

Esponenziale Bilatera

$$x(t) = e^{-|t|} \quad \text{punti e modulo}$$

$$-Ex = \int_{-\infty}^{\infty} e^{-2|t|} dt = 2 \cdot \int_0^{\infty} e^{-2t} dt = 2 \cdot \left(-\frac{1}{2} e^{-2t} \right) \Big|_0^{\infty} = 1$$

$$-Px = \emptyset$$

$$-xe_{\frac{d}{dt}} = \emptyset$$

$$-xm = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-|t|} dt = 2 \cdot \lim_{T \rightarrow \infty} \int_0^{\frac{T}{2}} e^{-t} dt = \emptyset$$

Segno

$$x(t) \triangleq \operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$$

$$-Ex = \int_{-\infty}^{\infty} \operatorname{sgn}^2(t) dt = \int_{-\infty}^{\infty} 1 dt = \infty$$

$$-Px = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \operatorname{sgn}^2(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot T = 1$$

$$-xe_{\frac{d}{dt}} = 1$$

$$-xm = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \operatorname{sgn}(t) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} -1 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{\frac{T}{2}} 1 dt = \emptyset$$

Dimostrazione $Px = \emptyset \Rightarrow xm = \emptyset$

Sia $x(t)$ qualsiasi. $x(t) = xm + x'(t)$ segnale a media nulla

$$x'(t) = x(t) - xm$$

$$x_m' = \lim_{T \rightarrow \infty} \underbrace{\frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) dt}_{x_m} - \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_m dt = \phi$$

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \stackrel{\text{i.potesi}}{=} \phi$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t) + x_m|^2 dt = \lim_{T \rightarrow \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} (x_m + x'(t))(x_m^* + x'^*(t)) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} [|x_m|^2 + |x'(t)|^2 + \underbrace{x_m \cdot x'(t)^*}_{\varepsilon} + \underbrace{x_m^* \cdot x'(t)}_{\varepsilon^*}] dt$$

$$\begin{aligned} \varepsilon + \varepsilon^* &= 2 \operatorname{Re}\{x_m^*\} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_m|^2 + |x'(t)|^2 + 2 \cdot \operatorname{Re}\{x_m x'^*(t)\} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x_m|^2 dt + \lim_{T \rightarrow \infty} \frac{1}{T} \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} |x'(t)| dt + 2 \operatorname{Re}\left\{ \lim_{T \rightarrow \infty} \frac{1}{T} x_m \int_{-\frac{T}{2}}^{\frac{T}{2}} x'(t)^* dt \right\} \end{aligned}$$

$$P_x = |x_m|^2 + P_{x'} = \phi \Rightarrow x_m = \phi$$

" "
" "

Segnali periodici

$$x(t) = x(t - KT_0) \text{ con } K \in \mathbb{Z} \text{ e } T_0 \in \mathbb{R}^+$$

Energia

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} \int_{-\frac{T_0}{2} + KT_0}^{\frac{T_0}{2} + KT_0} |x(t)|^2 dt = \sum_{K=-\infty}^{\infty} x = \lim_{K \rightarrow \infty} Kx = \infty \Rightarrow \text{Tutti i segnali periodici hanno energia} = \infty$$

Potenza Media

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt \stackrel{\text{disegni:}}{=} \lim_{K \rightarrow \infty} \frac{1}{KT_0} \cdot \int_{-\frac{KT_0}{2}}^{\frac{KT_0}{2}} |x(t)|^2 dt = \lim_{K \rightarrow \infty} \frac{1}{KT_0} \cdot K \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt$$

$$\Rightarrow P_x = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt \quad \text{Per i segnali periodici}$$

Valor Medio

$$x_m = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) dt$$

Analisi di Fourier

$x(t)$ periodico trasformabile ammette TSF

$$x_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-j2\pi n f_0 t} dt \quad \text{Equazione di analisi}$$

CS trasformabilità $x(t) \rightarrow x_n$ (Condizioni di Dirichelet)

Antitrasformata Serie di Fourier

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t} \quad \text{Equazione di sintesi}$$

Dimostrazione Biunivocità

$$\begin{matrix} \xrightarrow{\text{TSF}} \\ x(t) \Leftrightarrow x_n \\ \xleftarrow{\text{ATSF}} \end{matrix}$$

$$\text{TSF} [\text{ATSF} [x_n]] =$$

$$\begin{aligned} &= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n f_0 t} \cdot e^{-j2\pi f_0 m t} dt = \sum_{n=-\infty}^{\infty} x_n \cdot \underbrace{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(n-m)f_0 t} dt}_{e^{j\alpha} = \cos \alpha + j \sin \alpha} \\ &\qquad\qquad\qquad \downarrow \\ &\qquad\qquad\qquad \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos [2\pi(n-m)f_0 t] dt}_{\begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}} + j \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin [2\pi(n-m)f_0 t] dt}_{\begin{cases} \phi & n \neq m \\ T_0 & n=m \end{cases}} \quad \Leftrightarrow = \phi \\ &= \sum_{n=-\infty}^{\infty} x_n \frac{1}{T_0} \cdot T_0 \delta[n-m] = x_n \end{aligned}$$

Linerità TSF

$$z(t) = a x(t) + b y(t); \quad a, b \in \mathbb{C}, \quad x(t) \xrightarrow{\text{TSF}} x_n, \quad y(t) \xrightarrow{\text{TSF}} y_n; \quad x(t), y(t) \text{ periodici di periodo } T_0$$

Tesi:

$$z_n = a x_n + b y_n$$

Dimostrazione:

$$\begin{aligned} z_n &= \text{TSF}[z(t)] = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t) \cdot e^{-j2\pi n f_0 t} dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} [a x(t) + b y(t)] \cdot e^{-j2\pi n f_0 t} dt = \\ &= \frac{a}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi n f_0 t} dt + \frac{b}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) \cdot e^{-j2\pi n f_0 t} dt = a x_n + b y_n \end{aligned}$$

Simmetria Hermitiana

Se $x(t)$ è reale

Tesi:

$$x_{-n} = x_n^*$$

Dimostrazione

$$x_{-n} = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi(-n)\frac{1}{T_0}t} dt = \left\{ \left[\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{j2\pi n \frac{1}{T_0}t} dt \right]^* \right\}^* = \left\{ \left(\frac{1}{T_0} \right)^* \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (x(t))^* e^{-j2\pi n \frac{1}{T_0}t} dt \right\}^* =$$

$$= \left\{ \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi n \frac{1}{T_0}t} dt \right\}^* = x_n^*$$

modulo: $|x_{-n}| = |x_n|$; simmetria pari

x_n^* ↗
base: $\angle x_{-n} = -\angle x_n$; simmetria dispari

$x(t)$ reale e pari

$$\begin{cases} x(t) = x^*(t) \\ x(t) = x(-t) \end{cases}$$

Tesi:

x_n è reale e pari

Dim

$$x_{-n} = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{j2\pi n \frac{1}{T_0}t} dt \stackrel{t' = -t}{=} \frac{1}{T_0} \int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} x(-t') e^{-j2\pi n \frac{1}{T_0}t'} (-dt') = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t') \cdot e^{-j2\pi n \frac{1}{T_0}t'} dt' = x_n$$

\Downarrow
 x_n è pari

$x_n^* = x_{-n} = x_n$

reale

$x(t)$ reale e dispari

$$\begin{cases} x(t) = x^*(t) \\ x(-t) = -x(t) \end{cases}$$

Tesi:

x_n è immaginario e dispari

Dim

$$x_{-n} = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{j2\pi n \frac{1}{T_0}t} dt \stackrel{t' = -t}{=} -\frac{1}{T_0} \int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} x(t') \cdot e^{-j2\pi n \frac{1}{T_0}t'} dt' = -x_n$$

immaginario

$$x_n^* = x_n = -x_n$$

dispari

x(t) periodico alternativo

I_P: $x(t) = -x(t - \frac{T_0}{2})$ con T_0 periodo

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi n f_0 t} dt = \underbrace{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^0 x(t) \cdot e^{-j2\pi n f_0 t} dt}_{\downarrow t' = t + \frac{T_0}{2}} + \frac{1}{T_0} \int_0^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi n f_0 t} dt$$

$$\begin{aligned} &= \frac{1}{T_0} \int_0^{\frac{T_0}{2}} x(t - \frac{T_0}{2}) e^{-j2\pi n f_0 (t - \frac{T_0}{2})} dt' + [\dots] \\ &= -\frac{1}{T_0} \int_0^{\frac{T_0}{2}} x(t') e^{-j2\pi n f_0 t'} dt' \cdot e^{j\pi n f_0 T_0} + [\dots] \end{aligned}$$

uguale

$$= \frac{1}{T_0} \int_0^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi n f_0 t} dt \cdot \left[1 - e^{j\pi n f_0 T_0} \right]$$

$\downarrow \begin{cases} 1 \text{ se } n \text{ pari} \\ -1 \text{ se } n \text{ dispari} \end{cases}$

$$= \frac{1}{T_0} \int_0^{\frac{T_0}{2}} x(t) e^{-j2\pi n f_0 t} dt \cdot (1 - (-1)^n)$$

$$\Rightarrow X_n = \begin{cases} \emptyset & n \text{ pari} \\ \frac{1}{T_0} \int_0^{\frac{T_0}{2}} x(t) \cdot e^{-j2\pi n f_0 t} dt & n \text{ dispari} \end{cases}$$

Sinc

$$\text{sinc}(x), x \in \mathbb{R}: \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Trasformata continua di Fourier

Si applica a segnali aperiodici che rispettano il criterio della TCF (CS di Dirichlet)

CS: Tutti i segnali ad energia finita sono trasformabili

TCF è bimivoca

$$x(j) \triangleq \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi j t} dt \quad \boxed{\text{TCF}}$$

$$x(t) = \int_{-\infty}^{\infty} x(j) \cdot e^{j2\pi j t} dj \quad \boxed{\text{ATCF}}$$

Proprietà

1) Simmetria Hermitiana

$$x(t) \text{ Reale} \Rightarrow x(j) \text{ Hermitiana: } x(-j) = x^*(j) \rightarrow \begin{cases} |x(j)| = |x(-j)| \\ \angle x(j) = -\angle x(-j) \end{cases}$$

2) Parità

$$x(t) \text{ Reale e pari} \Rightarrow x(j) \text{ Reale e pari}$$

3) Disparità

$$x(t) \text{ Reale e dispari} \Rightarrow x(j) \text{ Immaginaria e dispari}$$

4) Linearità

$$\begin{aligned} x(t) &= ax_1(t) + bx_2(t); \quad x_1(t) \xrightarrow{\text{TCF}} x_1(j), \quad x_2(t) \xrightarrow{\text{TCF}} x_2(j); \quad a, b \in \mathbb{C} \\ \Rightarrow x(j) &= ax_1(j) + bx_2(j) \end{aligned}$$

Teorema della dualità

Sia $x(t) \xrightarrow{\text{TCF}} X(j)$

Tesi:
 $X(t) \xrightarrow{\text{TCF}} x(-j)$

Dimostrazione

$$\begin{aligned} x(j) &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi jt} dt \quad t \rightarrow j \\ X(t) &= \int_{-\infty}^{\infty} x(j) \cdot e^{j2\pi jt} dj \quad j \rightarrow -j \\ &= \int_{-\infty}^{\infty} x(-j') \cdot e^{-j2\pi(-j')t} dj' = \int_{-\infty}^{\infty} x(-j') e^{j2\pi j't} dj' = \text{ATCF}[x(-j)] \end{aligned}$$

$$X(t) = \text{ATCF}[x(-j)]$$

$$\stackrel{\text{II}}{\uparrow} \quad \quad \quad X(t) \xrightarrow{\text{TCF}} x(-j)$$

Teorema del ritardo

$$x(t) \xrightarrow{\text{TCF}} X(j), \quad y(t) = x(t-t_0) \text{ con } t_0 \in \mathbb{R}$$

Tesi:

$$Y(j) = \text{TCF}[y(t)] = X(j) e^{-j2\pi jt_0}$$

Dimostrazione

$$Y(j) = \int_{-\infty}^{\infty} y(t) \cdot e^{-j2\pi jt} dt = \int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-j2\pi jt} dt = \int_{-\infty}^{\infty} x(t') \cdot e^{-j2\pi j(t+t_0)} dt' = e^{-j2\pi jt_0} \cdot \int_{-\infty}^{\infty} x(t') \cdot e^{-j2\pi jt'} dt' = x(j) \cdot e^{-j2\pi jt_0}$$

Modulazione di Ampiezza

$$y(t) = x(t) \cdot \cos(2\pi f_0 t), \quad x(t) \xrightarrow{\text{TCF}} X(f)$$

Tesi:

$$Y(f) = \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)$$

Dimostrazione

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} y(t) \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(t) \cos(2\pi f_0 t) \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(t) \cdot \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \cdot e^{-j2\pi ft} dt = \\ &= \frac{1}{2} \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi(f-f_0)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi(f+f_0)t} dt = \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0) \end{aligned}$$

Modulazione con il seno

$$y(t) = x(t) \cdot \sin(2\pi f_0 t) \Rightarrow Y(f) = \frac{1}{2j} X(f - f_0) + \frac{1}{2j} X(f + f_0)$$

Modulazione con cosinuside generica

$$y(t) = x(t) \cdot \cos(2\pi f_0 t + \varphi)$$

Tesi:

$$Y(f) = \frac{e^{j\varphi}}{2} X(f - f_0) + \frac{e^{-j\varphi}}{2} X(f + f_0)$$

Dimostrazione

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} x(t) \cdot \frac{e^{j(2\pi f_0 t + \varphi)} + e^{-j(2\pi f_0 t + \varphi)}}{2} \cdot e^{-j2\pi ft} dt = \\ &= \frac{e^{j\varphi}}{2} \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi(f-f_0)t} dt + \frac{e^{-j\varphi}}{2} \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi(f+f_0)t} dt = \frac{e^{j\varphi}}{2} X(f - f_0) + \frac{e^{-j\varphi}}{2} X(f + f_0) \end{aligned}$$

Modulazione con esponenziale complesso

$$y(t) = x(t) \cdot e^{-j2\pi f_0 t}, \quad x(t) \xrightarrow{\text{TCF}} X(f)$$

Tesi:

$$Y(f) = X(f - f_0)$$

Dimostrazione

$$Y(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi f_0 t} \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi(f-f_0)t} dt = X(f - f_0)$$

Similitudini e Differenze con il Teorema del ritardo

$$\text{RIT: } x(t - t_0) \Leftrightarrow X(f) \cdot e^{-j2\pi f t_0}$$

$$\text{MOD: } x(t) \cdot e^{j2\pi f_0 t_0} \Leftrightarrow X(f - f_0)$$

Teorema della derivazione

$$x(t) \xrightarrow{\text{TCF}} X(j\omega), y(t) = \frac{d}{dt} x(t)$$

Tesi:

$$Y(j\omega) = j\omega \cdot X(j\omega)$$

Dimostrazione

$$\begin{aligned} y(t) &= \frac{d}{dt} x(t) = \frac{d}{dt} \underbrace{\int_{-\infty}^{\infty} x(\tau) e^{j\omega\tau} d\tau}_{\text{ATCF}[x(\tau)]} = \int_{-\infty}^{\infty} x(\tau) \frac{d}{dt} e^{j\omega\tau} d\tau = \int_{-\infty}^{\infty} j\omega \cdot x(\tau) e^{j\omega\tau} d\tau \\ y(t) &= \text{ATCF}[Y(j\omega)] = \int_{-\infty}^{\infty} Y(j\omega) e^{j\omega\tau} d\tau = \int_{-\infty}^{\infty} j\omega \cdot X(j\omega) e^{j\omega\tau} d\tau \Rightarrow Y(j\omega) = X(j\omega) \cdot j\omega \end{aligned}$$

Teorema dell'integrazione

$$x(t) \xrightarrow{\text{TCF}} X(j\omega), y(t) = \int_{-\infty}^t x(\alpha) d\alpha, \int_{-\infty}^{\infty} x(t) dt = \phi \Leftrightarrow X(0) = \phi$$

Tesi:

$$Y(j\omega) = \frac{x(j\omega)}{j\omega}$$

Dimostrazione

Th. deriv.

$$y(t) = \int_{-\infty}^t x(\alpha) d\alpha \Rightarrow x(t) = \frac{d}{dt} y(t) \Rightarrow X(j\omega) = j\omega Y(j\omega) \Rightarrow Y(j\omega) = \frac{x(j\omega)}{j\omega}$$

↑
fa sì che esista una discontinuità nel denominatore generata da

TCF Notevole: triangolo

$$A(1 - \frac{|t|}{T}) \text{rect}(\frac{t}{\pi T}) \xrightarrow{\text{TCF}} A T \text{sinc}^2(\frac{j\omega}{2T})$$

E, per dualità

$$AB \text{sinc}^2(Bt) \xrightarrow{\text{TCF}} A(1 - \frac{|t|}{B}) \cdot \text{rect}(\frac{t}{2B})$$

B???

Teorema della derivazione in frequenza

$$Y(j\omega) = \frac{d}{d\omega} X(j\omega), x(t) \xrightarrow{\text{TCF}} X(j\omega)$$

Tesi:

$$y(t) = -j\omega t x(t)$$

Teorema integrazione in frequenza

$$x(t) \xrightarrow{\text{TCF}} X(f), Y(f) = \int_{-\infty}^{\infty} X(\alpha) d\alpha, \int_{-\infty}^{\infty} Y(\alpha) d\alpha = 0 \Leftrightarrow x(s) = 0$$

Tesi:

$$y(t) = \frac{x(t)}{j2\pi f}$$

Teorema della Convoluzione

$$x(t), y(t) \Rightarrow z(t) = x(t) \otimes y(t) \triangleq \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau) d\tau$$

Ipotesi:

$$x(t) \xrightarrow{\text{TCF}} X(f), y(t) \xrightarrow{\text{TCF}} Y(f), z(t) = x(t) \otimes y(t)$$

Tesi:

$$Z(f) = X(f) \cdot Y(f)$$

Dimostrazione

$$\begin{aligned} Z(f) &= \int_{-\infty}^{\infty} z(t) \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau) d\tau \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(\tau) \cdot \int_{-\infty}^{\infty} y(t-\tau) \cdot e^{-j2\pi ft} dt d\tau = \\ &= \int_{-\infty}^{\infty} x(\tau) \cdot y(f) \cdot e^{-j2\pi f\tau} d\tau = Y(f) \cdot \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j2\pi f\tau} d\tau = X(f) \cdot Y(f) \end{aligned}$$

Proprietà della convoluzione

Commutativa

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

$$\int_{-\infty}^{\infty} x(\tau) y(t-\tau) d\tau \Rightarrow t-\tau = t', \tau = t - t' \Rightarrow \int_{-\infty}^{\infty} y(t') x(t-t') d\tau'$$

Distributiva

$$x(t) \otimes (y(t) + z(t)) = x(t) \otimes y(t) + x(t) \otimes z(t)$$

Associativa

$$(x(t) \otimes y(t)) \otimes z(t) = x(t) \otimes (y(t) \otimes z(t)) = x(t) \otimes y(t) \otimes z(t)$$

Dimostrazione con le TCF

$$(X(f)Y(f))Z(f) = X(f)Y(f)Z(f)$$

Teorema del Prodotto

$$x(t) \xrightarrow{\text{TCF}} X(f), y(t) \xrightarrow{\text{TCF}} Y(f), z(t) = x(t)y(t)$$

Tesi:

$$z(f) = X(f) \otimes Y(f)$$

Dimostrazione

$$\begin{aligned} z(f) &= \int_{-\infty}^{\infty} x(t)y(t) \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} X(\alpha) \cdot e^{j2\pi \alpha t} d\alpha}_{(t)} y(t) \cdot e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} X(\alpha) \int_{-\infty}^{\infty} y(t) \cdot e^{j2\pi f - \alpha t} dt d\alpha = \\ &= \int_{-\infty}^{\infty} X(\alpha) \cdot Y(f - \alpha) d\alpha = X(f) \otimes Y(f) \end{aligned}$$

Trasformate notevoli esponenziali

Monofatera

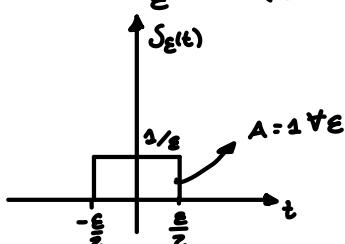
$$e^{-t} u(t) \xrightarrow{\text{TCF}} \frac{1}{1 + j2\pi f} \quad \text{e per dualità} \quad \frac{1}{1 + j2\pi f} \xrightarrow{\text{TCF}} e^{(-f)} u(-f) = e^f u_1(f)$$

Bifatera

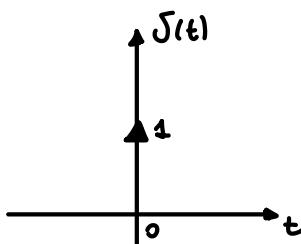
$$e^{-|t|} \xrightarrow{\text{TCF}} \frac{2}{1 + 4\pi^2 f^2} \quad \text{e per dualità} \quad \frac{2}{1 + 4\pi^2 f^2} \xrightarrow{\text{TCF}} e^{-|f|}$$

Delta di Dirac

$$\delta_\epsilon(t) \triangleq \frac{1}{\epsilon} \operatorname{rect}\left(\frac{t}{\epsilon}\right)$$



$$\delta(t) \triangleq \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t)$$



Proprietà

1] $\int_{-\infty}^{\infty} \delta(t) dt = 1$

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \text{rect}\left(\frac{t}{\epsilon}\right) dt = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{\epsilon}\right) dt = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \cdot \epsilon = 1$$

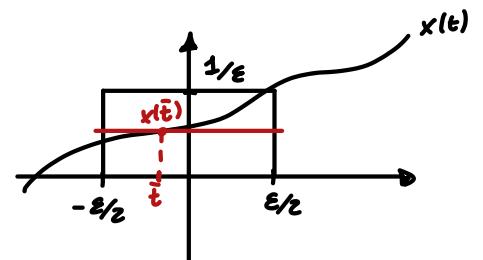
2| Proprietà campionatrice

$$\int_{-\infty}^{\infty} x(t) \delta(t) dt = x(0) \quad \text{Se } x(t) \text{ continuo in } t=0$$

Dimostrazione

$$\int_{-\infty}^{\infty} x(t) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \text{rect}\left(\frac{t}{\epsilon}\right) dt = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} x(t) \text{rect}\left(\frac{t}{\epsilon}\right) dt = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \cdot \epsilon \cdot x(0) = x(0)$$

media temporale



Proprietà integrali discendenti dalla campionatrice

3| Parità

$$\delta(t) = \delta(-t)$$

Dimostrazione

$$\int_{-\infty}^{\infty} x(-t) \delta(t) dt = x(-0) = x(0)$$

4] $\int_{-\infty}^{\infty} x(t) \delta(t-t_0) dt = x(t_0)$

Dimostrazione

$$\int_{-\infty}^{\infty} x(t) \delta(t-t_0) dt = \int_{-\infty}^{\infty} x(t'+t_0) \delta(t') dt' = x(t'+t_0) \Big|_{t'=0} = x(t_0)$$

5] $x(t) \delta(t-t_0) = x(t_0) \delta(t-t_0)$

Dimostrazione

$$\int_{-\infty}^{\infty} x(t_0) \delta(t-t_0) dt = x(t_0) \int_{-\infty}^{\infty} \delta(t-t_0) dt = x(t_0) \int_{-\infty}^{\infty} \delta(t') dt' = x(t_0)$$

6] $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau = \int_{-\infty}^{\infty} x(\tau) \delta(\tau-t) d\tau = x(t) \Rightarrow \delta(t) \text{ è l'elemento neutro della convoluzione}$

6bis| Traslazione Rigida

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau) \delta(t-t_0-\tau) d\tau = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(\tau - \underbrace{(t-t_0)}_{t'}) d\tau = x(t') = x(t-t_0)$$

Energia $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{-\infty}^{\infty} \left[\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \text{rect}\left(\frac{t}{\epsilon}\right) \right]^2 dt = \lim_{\epsilon \rightarrow 0} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} \left(\frac{1}{\epsilon} \right)^2 dt = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\epsilon^2} \rightarrow \infty$$

TCF $\delta(t)$

$$\Delta(f) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-j2\pi f t} dt = 1 \Rightarrow \delta(t) \xleftrightarrow{\text{TCF}} 1 + \frac{1}{2}$$

Inoltre

$$x(t) = A \forall t \Leftrightarrow A \delta(f)$$

$$A \delta(t) \xrightarrow{\text{TCF}} A + \frac{1}{2}$$

TCF di $1/t$

$$x(t) = \frac{1}{t} \xrightarrow{\text{TCF}} -j\pi \text{sgn}(f)$$

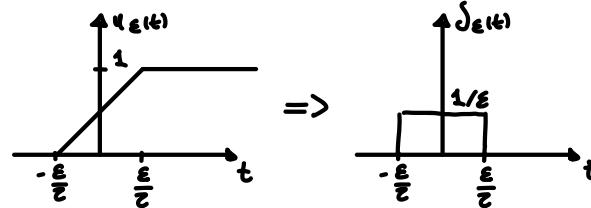
Funzione Gradino

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$u(t) = \int_{-\infty}^t \delta(\alpha) d\alpha ; \quad \delta(t) = \frac{d}{dt} u(t)$$

Dimostrazione

$$\delta_\epsilon(t) = \frac{d}{dt} u_\epsilon(t)$$



$$\delta(t) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t) = \lim_{\epsilon \rightarrow 0} \frac{d}{dt} u_\epsilon(t) = \frac{d}{dt} \lim_{\epsilon \rightarrow 0} u_\epsilon(t) = \frac{d}{dt} u(t)$$

TCF $u(t)$

$$U(f) = \text{TCF}[u(t)] = \int_{-\infty}^{\infty} u(t) \cdot e^{-j2\pi f t} dt = \int_0^{+\infty} e^{-j2\pi f t} dt = \frac{-1}{2j\pi f} \cdot e^{-j2\pi f t} \Big|_0^{+\infty}$$

\hookrightarrow oscilla

Sceviamo allora:

$$u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t)$$

$$U(f) = \text{TCF}[1/2] + \text{TCF}[1/2 \text{sgn}(t)]$$

$$\text{TCF}[1/2] = 1/2 \delta(f)$$

$$\text{TCF}[1/2 \text{sgn}(t)]: \quad \frac{1}{t} \xrightarrow{\text{TCF}} -j\pi \text{sgn}(f) \xrightarrow{\text{Dualità}} -j\pi \text{sgn}(t) \xleftrightarrow{\text{TCF}} -\frac{1}{f} \Rightarrow \frac{1}{2} \text{sgn}(t) \xrightarrow{\text{TCF}} \frac{1}{2j\pi f}$$

$$U(f) = \frac{1}{2} \delta(f) + \frac{1}{2j\pi f}$$

Teorema dell'integrazione completa

Ipotesi:

$$\begin{cases} x(t) \xrightarrow{\text{TCF}} X(j) \\ y(t) = \int_0^t x(\alpha) d\alpha \\ \int_{-\infty}^{\infty} x(t) dt = \phi \end{cases}$$

Non c'è, al contrario del th. dell'integrazione incompleta

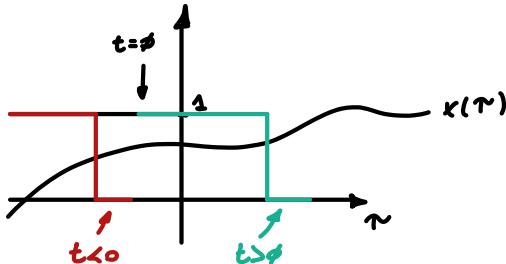
Tesi:

$$y(j) = \text{TCF}[y(t)] = \underbrace{\frac{x(0)}{z}}_{\text{In più rispetto al teorema dell'integrale incompleto}} \mathcal{J}(j) + \underbrace{\frac{x(j)}{j\pi j}}$$

In più rispetto al teorema dell'integrale incompleto

Dimostrazione

$$y(t) = \int_{-\infty}^t x(\alpha) d\alpha = x(t) \otimes u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau) d\tau = \int_{-\infty}^t x(\tau) d\tau$$



$$\Rightarrow y(t) = x(t) \otimes u(t)$$

$$\stackrel{\text{TCF}}{\downarrow} y(j) = x(j) u(j) = x(j) \cdot \left[\frac{1}{2} \mathcal{J}(j) + \frac{1}{j\pi j} \right]$$

$$y(j) = \frac{x(0)}{z} \mathcal{J}(j) + \frac{x(j)}{z j \pi j}$$

TCF Notevoli

$$\mathcal{J}(t-t_0) \xrightarrow{\text{TCF}} e^{-j2\pi j t_0} \quad \text{Dualità} \quad e^{j2\pi t_0} \xrightarrow{\text{TCF}} \mathcal{J}(-j+j_0) = \mathcal{J}(j-j_0)$$

$$\mathcal{J}(t+t_0) \xrightarrow{\text{TCF}} e^{j2\pi j t_0} \quad \Rightarrow \quad e^{-j2\pi t_0} \xrightarrow{\text{TCF}} \mathcal{J}(j+j_0)$$

TCF di un Coseno

$$x(t) = A \cos(2\pi f_0 t)$$

$$\begin{aligned} X(j) &= \int_{-\infty}^{\infty} A \cos(2\pi f_0 t) \cdot e^{-j2\pi j t} dt = \frac{A}{2} \int_{-\infty}^{\infty} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \cdot e^{-j2\pi j t} dt = \\ &= \frac{A}{2} \int_{-\infty}^{\infty} e^{j2\pi f_0 t} \cdot e^{-j2\pi j t} dt + \frac{A}{2} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} \cdot e^{-j2\pi j t} dt = \frac{A}{2} \text{TCF}[e^{j2\pi f_0 t}] + \frac{A}{2} \text{TCF}[e^{-j2\pi f_0 t}] = \end{aligned}$$

$$\frac{A}{2} \delta(j-j_0) + \frac{A}{2} \delta(j+j_0)$$

TCF di un segnale

$$x(t) = A \sin(2\pi f_0 t)$$

$$x(j) = \frac{A}{2j} \int (e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) \cdot e^{-j2\pi jt} dt = \frac{A}{2j} TCF[e^{j2\pi f_0 t}] - \frac{A}{2j} TCF[e^{-j2\pi f_0 t}] = \\ = j \frac{A}{2} \delta(j+f_0) - j \frac{A}{2} \delta(j-f_0)$$

Biunivocità TCF

$$x(t) \xrightarrow{\text{TCF}} x(j)$$

Tesi:

$$x(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\alpha) \cdot e^{-j2\pi f \alpha} d\alpha \cdot e^{j2\pi jt} df$$

Dimostrazione

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\alpha) \cdot e^{-j2\pi f \alpha} d\alpha \cdot e^{j2\pi jt} df = \int_{-\infty}^{\infty} x(\alpha) \int_{-\infty}^{\infty} e^{j2\pi f(t-\alpha)} df d\alpha = \int_{-\infty}^{\infty} x(\alpha) \delta(t-\alpha) d\alpha = x(t) \otimes \delta(t) = x(t)$$

$e^{j2\pi jt} \xrightarrow{\text{TCF}} \delta(j-j')$

Analisi energetica segnali aperiodici

Segnali ad energia finita:

Correlazione tra segnali aperiodici

$$C_{xy}(\tau) \triangleq \int_{-\infty}^{\infty} x(t) y^*(t-\tau) dt$$

Auto correlazione

$$C_x(\tau) \triangleq \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt$$

Proprietà Auto-correlazione

$$\underline{1} C_x(\phi) = \int_{-\infty}^{\infty} x(t) x^*(t-\phi) dt = \int_{-\infty}^{\infty} |x(t)|^2 dt = E_x$$

2] Simmetria Hermitiana: $C_x^*(\tau) = C_x(-\tau)$

Dimostrazione

$$C_x(-\tau) = \int_{-\infty}^{\infty} x(t) x^*(t - (-\tau)) dt = \int_{-\infty}^{\infty} x(t) x^*(t + \tau) dt = \int_{-\infty}^{\infty} x(t') x^*(t' - \tau) dt' = \underbrace{\left[\int_{-\infty}^{\infty} x(t') x^*(t' - \tau) dt' \right]}_{C_x(\tau)}^*$$

3] TCF Autocorrelazione

$$\begin{aligned} S_x(j) &\triangleq \text{TCF}[C_x(\tau)] = \int_{-\infty}^{\infty} C_x(\tau) \cdot e^{j2\pi j\tau} d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) x^*(t - \tau) dt \cdot e^{j2\pi j\tau} d\tau = \\ &= \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} x^*(t - \tau) \cdot e^{-j2\pi j\tau} d\tau dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t') x^*(\tau') \cdot e^{-j2\pi j(t - \tau')} d\tau' dt = \\ &= \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi jt} dt \int_{-\infty}^{\infty} x^*(\tau') \cdot e^{j2\pi j\tau'} d\tau' = x(j) x^*(j) = |x(j)|^2 = S_x(j) \end{aligned}$$

Densità spettrale di energia

Inoltre:

$$\int_{-\infty}^{\infty} S_x(j) dj = \int_{-\infty}^{\infty} \underbrace{S_x(j) \cdot e^{j2\pi j\tau}}_{C_x(\tau) \Big|_{\tau=0}} dj = C_x(0) = E_x \Rightarrow E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(j)|^2 dj$$

Teorema di Parseval

$$\int_{-\infty}^{\infty} x(t) y^*(t) dt = \int_{-\infty}^{\infty} X(j) Y^*(j) dj$$

Relazione tra correlazione e convoluzione

$$C_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t - \tau) dt = x(\tau) \otimes y^*(-\tau) = \int_{-\infty}^{\infty} x(t) y^*(-(\tau - t)) dt = \int_{-\infty}^{\infty} x(t) y^*(t - \tau) dt$$

Periodizzazione di un segnale aperiodico

$x(t)$ aperiodico

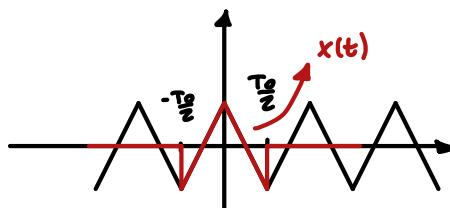
$y(t) = \sum_{n=-\infty}^{\infty} x(t - nT_0)$; $n \in \mathbb{Z}$, $T_0 \in \mathbb{R}^+$ $\Rightarrow y(t)$ è periodico di periodo T_0 , infatti:

$$y(t - kT_0) = \sum_{n=-\infty}^{\infty} x(t - kT_0 - nT_0) = \sum_{n=-\infty}^{\infty} x(t - (n+k)T_0) = \sum_{n'=0}^{\infty} x(t - n'T_0) = y(t)$$

$\xrightarrow{\text{TCF}}$

$$x(j) \Leftrightarrow x(t)$$

$$Y_n = \text{TSF}[y(t)] = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) \cdot e^{-j2\pi n f_0 t} dt$$



Un segnale $y(t)$ periodico si può sempre scrivere come periodizzazione di uno aperiodico.

$$x(t) = \begin{cases} y(t) & -\frac{T_0}{2} \leq t \leq \frac{T_0}{2} \\ \emptyset & \text{altrimenti} \end{cases} \Rightarrow y(t) = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

$$y_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) \cdot e^{-j2\pi n \frac{f}{T_0} t} dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{\infty} x(t-kT_0) \cdot e^{-j2\pi n \frac{f}{T_0} t} dt = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t-kT_0) \cdot e^{-j2\pi n \frac{f}{T_0} t} dt =$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{\infty} \int_{-\frac{T_0}{2}-kT_0}^{\frac{T_0}{2}-kT_0} x(t') \cdot e^{-j2\pi n \frac{f}{T_0} (t'+kT_0)} dt' = \frac{1}{T_0} \int_{-\infty}^{\infty} x(t') \cdot e^{-j2\pi n \frac{f}{T_0} t'} dt' = \frac{1}{T_0} x(\tilde{f}) \text{ con } \tilde{f} = n \frac{f}{T_0}$$

[1]

$$= \frac{1}{T_0} \times \left(\frac{n}{T_0} \right) = Y_n$$

1^a Formula di Poisson

$$\sum_{n=-\infty}^{\infty} x(t-nT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_0}\right) \cdot e^{j2\pi n \frac{f}{T_0} t}$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(t-nT_0) \stackrel{\text{TCF}}{\Leftrightarrow} \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_0}\right)$$

$$\sum_{n=-\infty}^{\infty} x(t-nT_0) = \text{DTFT} \left[\underbrace{\frac{1}{T_0} X\left(\frac{n}{T_0}\right)}_{Y_n} \right] = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} X\left(\frac{n}{T_0}\right) \cdot e^{j2\pi n \frac{f}{T_0} t} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_0}\right) e^{j2\pi n \frac{f}{T_0} t}$$

2^a Formula di Poisson

$$\sum_{n=-\infty}^{\infty} x(nT_0) \cdot e^{-j2\pi n \frac{f}{T_0} t} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X\left(\frac{n}{T_0}\right)$$

Applicazione di Poisson su $\delta(t)$

$$[1] \quad \sum_{n=-\infty}^{\infty} \delta(t-nT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} 1 \cdot e^{-j2\pi n \frac{f}{T_0} t}$$

$$[2] \quad \sum_{n=-\infty}^{\infty} 1 \cdot e^{-j2\pi n \frac{f}{T_0} t} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(\frac{n}{T_0} - f\right)$$

TCF di un traino di delta

2^a Poisson

$$\text{TCF} \left[\sum_{n=-\infty}^{\infty} \delta(t-nT_0) \right] = \sum_{n=-\infty}^{\infty} e^{-j2\pi n \frac{f}{T_0} t} = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta\left(\frac{n}{T_0} - f\right)$$

TCF di segnali periodici

$$y(t) = \sum_{n=-\infty}^{\infty} x(t-nT_0)$$

$$Y(j) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi jt} dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(t-nT_0) \cdot e^{-j2\pi jt} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t') \cdot e^{-j2\pi j t'} dt' \cdot e^{-j2\pi j nT_0} =$$

$$\sum_{n=-\infty}^{\infty} X(j) \cdot e^{-j2\pi j nT_0} = X(j) \sum_{n=-\infty}^{\infty} e^{-j2\pi j nT_0} = X(j) \cdot \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \delta(j - \frac{n}{T_0}) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(j) \delta(j - \frac{n}{T_0}) =$$

$$\frac{1}{T_0} \sum_{n=-\infty}^{\infty} X(\frac{n}{T_0}) \cdot \delta(j - \frac{n}{T_0}) = \boxed{\sum_{n=-\infty}^{\infty} Y_n \cdot \delta(j - \frac{n}{T_0}) = Y(j)}$$

La TCF di un segnale periodico è sempre un traino di δ

Sistemi multidimensionali

Definizione



Proprietà

Linearità

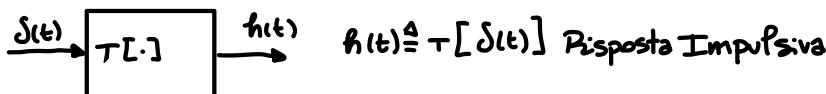
Un sistema è lineare se, dato $x(t) = ax_1(t) + bx_2(t)$ allora
 $y(t) = T[x(t)] = aT[x_1(t)] + bT[x_2(t)]$

Stazionarietà

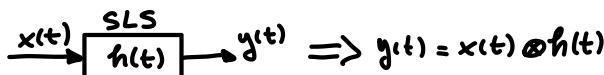
Se $y(t) = T[x(t)]$ allora $y(t-t_0) = T[x(t-t_0)]$

Sistemi Lineari Stazionari (SLS)

Definiamo la risposta impulsiva:



La risposta impulsiva esiste per ogni tipo di sistema, ma nei SLS caratterizza completamente il sistema.
In particolare abbiamo



Dimostrazione

$$y(t) = T[x(t)] = T[x(t) \otimes \delta(t)] = \underset{(t)}{\int} \left[\int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau) d\tau \right] = \int_{-\infty}^{\infty} x(\tau) T[\delta(t-\tau)] d\tau =$$

$$= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = x(t) \otimes h(t)$$

Causalità

$y(t) = T[x(\alpha); \alpha \leq t]$: L'uscita all'istante "t" dipende da istanti precedenti o al più uguali a "t"

Sistemi lineari stazionari e causal

Poniamo $h(t) = \phi \quad \forall t < \phi \quad e \neq \phi \quad \forall t \geq \phi$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^t x(\tau) h(t-\tau) d\tau \Rightarrow y(t) \text{ dipende solo dai valori } \leq t$$

Stabilità BIBO

$$|x(t)| \leq M < \infty \Rightarrow |y(t)| \leq N < \infty \quad \forall t$$

SLS stabili BIBO

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty ; \text{ assoluta integrabilità della } h(t)$$

CS

$$|y(t)| = |T[x(t)]| = |x(t) * h(t)| = \left| \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right| \leq \int_{-\infty}^{\infty} |x(\tau)| \cdot |h(t-\tau)| d\tau \leq M \underbrace{\int_{-\infty}^{\infty} |h(t-\tau)| d\tau}_{\text{è K} < \infty} = MK = N < \infty$$

Memoria

Un sistema è senza memoria se:

$y(t)$ dipende solo ed esclusivamente da $x(t)$

$$y(t) = T[x(\alpha); \alpha = t]$$

Un sistema ha memoria quando l'uscita all'istante "t" dipende anche da valori ad istanti diversi da "t"

Invertibilità

$$\text{Se } y(t) = T[x(t)] \Rightarrow \exists T^{-1}[y(t)] = x(t)$$

Risposta in frequenza

SLS

$$x(t) \rightarrow h(t) \rightarrow y(t)$$

$$x(t) = e^{j2\pi f t} = \cos(2\pi f t) + j \sin(2\pi f t) \quad \text{con } f \text{ generica}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{j2\pi f \tau} \cdot h(t-\tau) d\tau = \int_{-\infty}^{\infty} e^{j2\pi f (t-\tau')} h(\tau') d\tau' = e^{j2\pi f t} \int_{-\infty}^{\infty} h(\tau') \cdot e^{-j2\pi f \tau'} d\tau' = e^{j2\pi f t} \cdot H(f)$$

$t - \tau = \tau'$

$$y(t) = x(t) H(f) \Big|_{x(t) = e^{j2\pi f t}}$$

Definizioni di risposta in frequenza

$$\boxed{1} \quad H(j) \triangleq \left. \frac{y(t)}{x(t)} \right|_{x(t) = e^{j\omega_n j t}}$$

$$\boxed{2} \quad H(j) = TCF[h(t)]$$

$$\boxed{3} \quad H(j) = \frac{Y(j)}{X(j)}$$

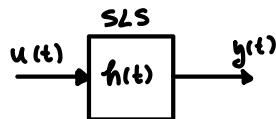
Dimostrazione

$$y(t) = x(t) \otimes h(t) \Rightarrow Y(j) = X(j)H(j) \Rightarrow H(j) = \frac{Y(j)}{X(j)}$$

$$H(j) = \underbrace{|H(j)|}_{\text{Risposta}} \cdot e^{j \underbrace{\angle H(j)}_{\text{in ampiezza}}} \rightarrow \text{Risposta in fase}$$

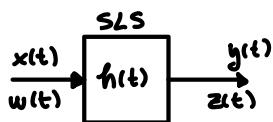
$$Y(j) = X(j) \cdot H(j) = \begin{cases} |Y(j)| = |X(j)| |H(j)| \\ \angle Y(j) = \angle X(j) + \angle H(j) \end{cases}$$

Risposta al gradino



$$\begin{cases} g(t) = u(t) \otimes h(t) = \int_{-\infty}^t h(\alpha) d\alpha \\ h(t) = \frac{d}{dt} g(t) \end{cases}$$

Proprietà dell'integrazione per SLS



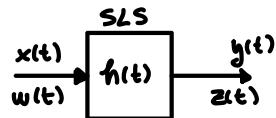
$$y(t) = x(t) \otimes h(t)$$

$$w(t) = \int_{-\infty}^t x(\alpha) d\alpha \quad \text{primitiva di } x(t)$$

$$z(t) = w(t) \otimes h(t) = \overbrace{x(t) \otimes u(t)}^{w(t)} \otimes h(t) = y(t) \otimes u(t)$$

$$z(t) = \int_{-\infty}^t y(\alpha) d\alpha \quad \text{primitiva di } y(t)$$

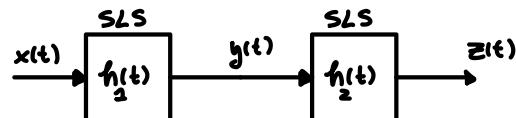
Proprietà della derivazione



Proprietà Integrazione

$$w(t) = \frac{d}{dt} x(t) \Rightarrow x(t) = \int_{-\infty}^t w(\alpha) d\alpha \Rightarrow y(t) = \int_{-\infty}^t z(\alpha) d\alpha \Rightarrow z(t) = \frac{d}{dt} y(t)$$

Sistemi incascata

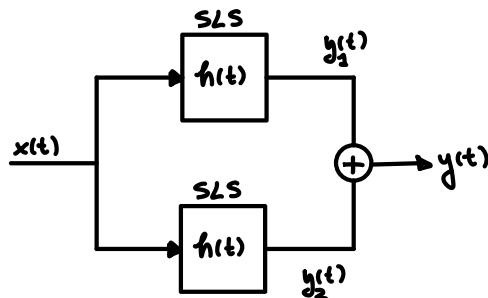


$$y(t) = x(t) \otimes h_2(t)$$

$$z(t) = y(t) \otimes h_2(t) = x(t) \otimes \underbrace{[h_1(t) \otimes h_2(t)]}_{h(t)} = x(t) \otimes h(t) \text{ con } h(t) = h_1(t) \otimes h_2(t) \text{ Risposta impulsiva del sistema a cascata}$$

Sistemi in parallelo

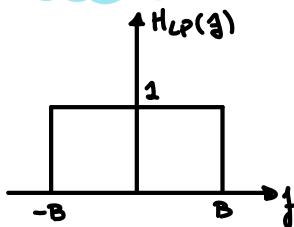
$$H(j) = H_1(j) \cdot H_2(j) \text{ Risposta in frequenza del sistema a cascata}$$



$$y(t) = y_1(t) + y_2(t) = x(t) \otimes h_1(t) + x(t) \otimes h_2(t) = x(t) \otimes \underbrace{[h_1(t) + h_2(t)]}_{h(t)} = x(t) \otimes h(t)$$

Filtri IDEALI

Passa basso di banda B



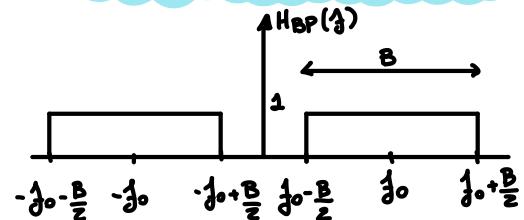
Banda: porzione di spettro calcolata sul semiasse positivo delle frequenze dove lo spettro è diverso da 0

$$B = \left\{ j : H(j) \neq 0, j > 0 \right\}$$

$$\boxed{H_{LP}(j) = \text{rect}\left(\frac{j}{2B}\right)}$$

$$\boxed{h_{LP}(t) = 2B \sin(2Bt)}$$

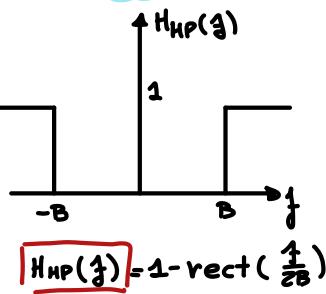
Passa-banda ideale di banda B



$$H_{BP}(f) = \text{rect}\left(\frac{f-f_0}{B}\right) + \text{rect}\left(\frac{f+f_0}{B}\right)$$

$$\begin{aligned} h_{BP}(t) &= B \text{sinc}(Bt) \cdot e^{j\pi f_0 t} + B \text{sinc}(Bt) \cdot e^{-j\pi f_0 t} = 2B \text{sinc}(Bt) \left(\frac{e^{j\pi f_0 t} + e^{-j\pi f_0 t}}{2} \right) = \\ &= 2B \text{sinc}(Bt) \cdot \cos(2\pi f_0 t) \end{aligned}$$

Passa-Alto

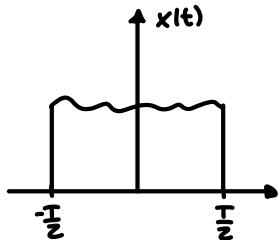


$$H_{HP}(f) = 1 - \text{rect}\left(\frac{f}{2B}\right)$$

$$h_{HP}(t) = \delta(t) - 2B \text{sinc}(2Bt)$$

Relazione tra durata e banda

$x(t)$ a durata uigurosamente limitata, ovvero $x(t) = 0$ per $|t| > \frac{T}{2}$



$$X(f) = \text{TCF}[x(t)] = \text{TCF}[x(t) \cdot \text{rect}\left(\frac{t}{T}\right)] = X(f) \otimes T \text{sinc}(Tf)$$

$X(f)$ ha banda infinita

Per assurdo poniamo che $X(f)$ abbia banda finita:

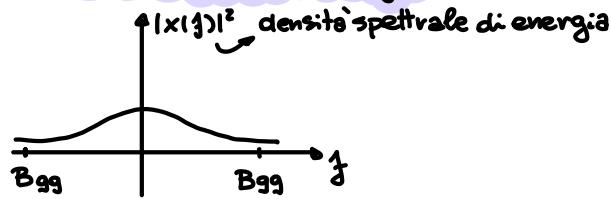
$X(f) \otimes T \text{sinc}(Tf)$ ha banda infinita perché $T \text{sinc}(Tf)$ non si annulla mai

\Rightarrow Per assurdo $X(f)$ ha banda infinita

$x(t)$ a durata finita $\Leftrightarrow X(f)$ a banda infinita

Definizioni "pratiche" di banda

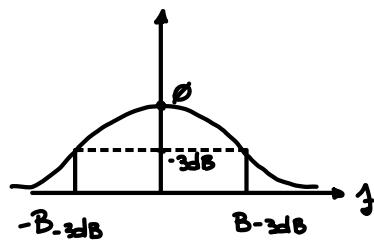
Banda al 99% dell'energia



$$\int_{-\infty}^{\infty} |x(f)|^2 df = E_x$$

$$B_{gg} : \frac{\int_{-B_{gg}}^{B_{gg}} |x(f)|^2 df}{E_x} = 0,99$$

Banda a -3dB



$$B_{-3dB} : X(\phi) - X(B_{-3dB}) = 3dB$$

Distorsioni Lineari

Replica Fedele

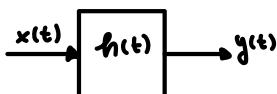
$y(t)$ è una replica fedele di $x(t)$ se $y(t) = K x(t-t_0)$; $K \in \mathbb{C}$, $t_0 \in \mathbb{R}$

Non si hanno distorsioni lineari quando la trasformazione produce alla sua uscita una replica fedele.

In frequenza:

$$Y(f) = K X(f) \cdot e^{-j2\pi f t_0}$$

Filtro Fedele



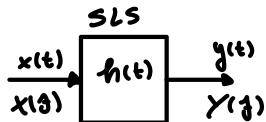
$$h(t) = K \delta(t-t_0)$$

$$y(t) = x(t) \otimes h(t) = K x(t-t_0)$$

Ho un filtro fedele per ogni ingresso

$$H(f) = K \cdot e^{-j2\pi f t_0}$$

Filtraggio di un segnale periodico

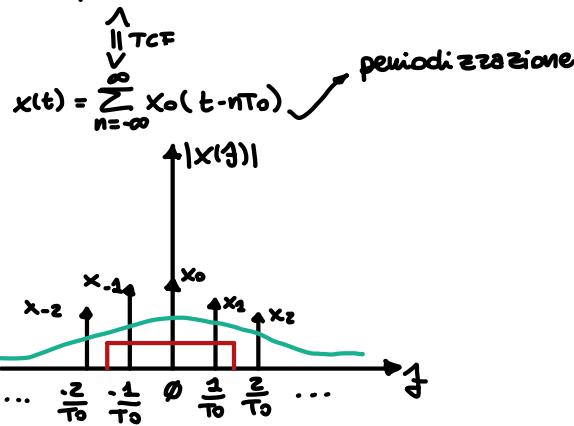


$$\left. \begin{array}{l} y(t) = x(t) * h(t) \\ Y(f) = X(f) \cdot H(f) \end{array} \right\} \text{Vale anche per segnali d'ingresso periodici}$$

$x(t)$ è periodico

$$X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f - \frac{n}{T_0}) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} x_0(\frac{n}{T_0}) \cdot \delta(f - \frac{n}{T_0})$$

$$X_n = \frac{1}{T_0} \cdot x_0(\frac{n}{T_0})$$



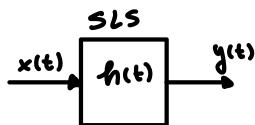
$$Y(f) = \sum_{n=-\infty}^{\infty} X_n H(\frac{n}{T_0}) \cdot \delta(f - \frac{n}{T_0}) = \sum_{n=-\infty}^{\infty} Y_n \cdot \delta(f - \frac{n}{T_0}) \text{ con } Y_n = X_n \cdot H(\frac{n}{T_0})$$

$$y(t) = \sum_{n=-\infty}^{\infty} Y_n \cdot e^{j2\pi n f_0 t} \text{ periodico}$$

In uscita ho sempre un segnale periodico dello stesso periodo

$$x(t) \Rightarrow E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$|X(f)|^2 = S_X(f) = TCF[C_X(\tau)] \text{ con } C_X(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau) dt$$



$$C_y(\tau) = \int_{-\infty}^{\infty} y(t) \cdot y^*(t-\tau) dt$$

$$C_y(\tau) \Big|_{\tau=0} = E_y$$

$$\begin{aligned} S_y(f) &= TCF[C_y(\tau)] = |Y(f)|^2 = Y(f) \cdot Y^*(f) = x(f) H(f) \cdot x^*(f) H^*(f) = x(f) x^*(f) H(f) H^*(f) = \\ &= |x(f)|^2 |H(f)|^2 \end{aligned}$$

$$S_y(j) = S_x(j) |H(j)|^2$$

$$E_y = \int_{-\infty}^{\infty} S_x(j) |H(j)|^2 dj$$

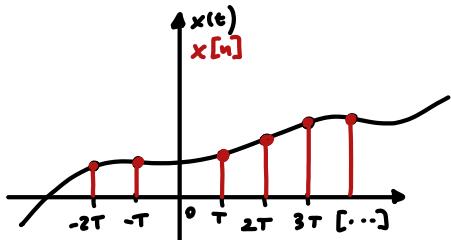
$$\text{ATCF}[S_y(j)] = C_y(\tau)$$

$$\text{ATCF}[S_x(j) \cdot |H(j)|^2] = \text{ATCF}[S_x(j) H(j) H^*(j)] = [C_x(\tau) \otimes h(\tau) \otimes h(-\tau)] = C_y(\tau)$$

Campionamento e Sequenze

$\xrightarrow{T_c}$ $x[n] = x(nT); n \in \mathbb{Z}$

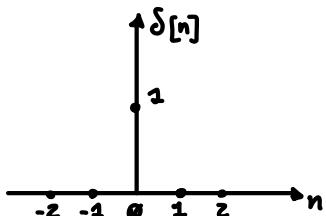
T_c = intervallo di campionamento



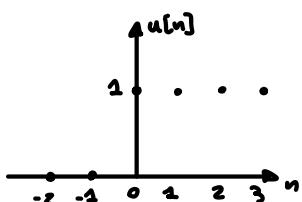
$$F_c \triangleq \frac{1}{T_c}$$

Sequenze Note

① $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$ Delta di Kronecker



② $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$



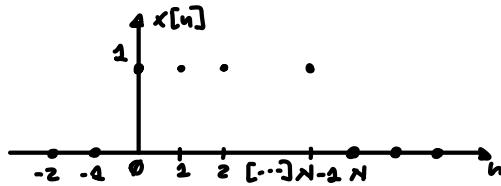
3 Esponenziale

$$x[n] \triangleq a^n u[n], 0 < a < 1$$



4 Rettangolo

$$x[n] = u[n] - u[n-N]$$



5 Oscillazione discinta (complessa)

$$x[n] = \exp[j\pi f_0 n] = \cos(\pi f_0 n) + j \sin(\pi f_0 n)$$

Se $f_0 = \frac{p}{q}; p, q \in \mathbb{Z} \Rightarrow f_0 \in \mathbb{Q} \Rightarrow x[n]$ è periodica, altrimenti no

$$x[n] = x[n-N] \text{ se } f_0 = \frac{p}{q} \text{ con } p, q \in \mathbb{Z}$$

Proprietà

- $\delta[n] = u[n] - u[n-1]$

- $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$

Trasformata di Fourier di una sequenza

$$x[n] \xleftrightarrow{\text{TFS}} \bar{x}(j)$$

$$\bar{x}(j) \triangleq \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi j n T}$$

$$\bar{x}(j) \in \mathbb{C}$$

Proprietà

$\bar{x}(j)$ è periodica di periodo $\frac{1}{T}$

Dimostrazione

$$\bar{x}\left(j - \frac{n}{T}\right) = \sum_{k=-\infty}^{\infty} x[k] e^{-j2\pi\left(j - \frac{n}{T}\right) k T} = \underbrace{\sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j2\pi j k T}}_{\bar{x}(j)} \cdot \underbrace{e^{-j2\pi j nk}}_{1 \forall n, k} = \bar{x}(j)$$

$$x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{x}(\frac{j}{T}) e^{j2\pi j n T} dj \quad |ATFS|$$

$$x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{k=-\infty}^{\infty} x[k] \cdot e^{-j2\pi j k T} \cdot e^{j2\pi j n T} dj = T \sum_{k=-\infty}^{\infty} x[k] \underbrace{\int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{j2\pi j(n-k)T} dj}_{\begin{cases} \frac{1}{\pi} & \text{per } n=k \\ 0 & \text{per } n \neq k \end{cases}} + j \underbrace{\int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sin(2\pi j(n-k)T) dj}_{\phi \psi_{n,k}}$$

$$x[n] = T \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{1}{\pi} \delta[n-k] = x[n]$$

Biunivocità TFS

CS esistenza TFS

$\sum_{n=-\infty}^{\infty} |x[n]| = K < \infty \Rightarrow x[n] \xrightarrow{TFS} \bar{x}(\frac{j}{T}) ; \bar{x}(\frac{j}{T}) \text{ esiste se la serie converge}$

$$|\bar{x}(\frac{j}{T})| = \left| \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j2\pi j n T} \right| \leq \sum_n |x[n]| \cdot \underbrace{|e^{-j2\pi j n T}|}_1 = K < \infty \Rightarrow |\bar{x}(\frac{j}{T})| < \infty \text{ converge}$$

Puendiamo 2 oscillazioni discrete:

$$x_1[n] = e^{j2\pi n F_0 T} ; \quad x_2[n] = e^{j2\pi n (F_0 + \frac{k}{T}) T}$$

$$x_1(t) = e^{-j2\pi F_0 t} \neq x_2(t) = e^{j2\pi F_2 t} \text{ con } F_2 = F_0 + \frac{k}{T}, k \in \mathbb{Z}$$

$$\underline{x_2(t)} \quad \overbrace{\quad}^{\frac{1}{T}} \quad x_2[n] = x_2(nT)$$

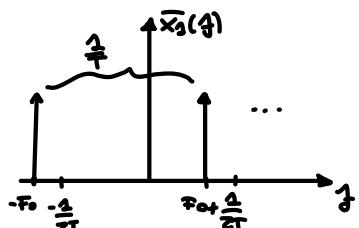
$$x_2[n] = e^{j2\pi n F_0 T} \cdot \underbrace{e^{j2\pi \frac{nK}{T} T}}_1 = x_2[n]$$

$$\bar{x}_1(\frac{j}{T}) = TFS[x_1[n]]$$

Formula di Poisson applicata alla $\delta(\frac{j}{T})$: $\frac{1}{T} \sum_n \delta(\frac{j-n}{T}) = \sum_n e^{-j2\pi j n T}$

$$\bar{x}_2(\frac{j}{T}) = TFS[x_2[n]]$$

$$\bar{x}_2(\frac{j}{T}) = \sum_{n=-\infty}^{\infty} x_2[n] \cdot e^{-j2\pi j n T} = \sum_n e^{j2\pi n F_0 T} \cdot e^{-j2\pi j n T} = \sum_n e^{-j2\pi (j - F_0) n T} = \frac{1}{T} \sum_n \delta(j - F_0 - \frac{n}{T})$$



$$\bar{x}_2(j) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{j\pi(f_0 + \frac{k}{T})nT} = \sum_n e^{j\pi(f_0 + \frac{k}{T})nT} \cdot e^{-j\pi jnT} = \sum_n e^{-j\pi(\frac{1}{T}f_0 - \frac{k}{T})nT} = \frac{1}{T} \sum_n e^{-j\pi(\frac{1}{T}f_0 - \frac{k}{T})nT} \cdot \underbrace{e^{j\pi \frac{k}{T}nT}}_2 =$$

$$= \bar{x}_3(j)$$

Il campionamento può essere ambiguo!

Mediante le condizioni di Nyquist possiamo evitare le ambiguità

Relazione TCF-TFS

$$\underline{x(t)} \xrightarrow{\downarrow T} \underline{x[n]} = x(nT)$$

$$x(t) \xrightarrow{\text{TCF}} X(j) \quad \& \quad x[n] \xrightarrow{\text{TFS}} \bar{x}(j) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\pi f nT}$$

$$\bar{x}(j) = \sum_n x(nT) e^{-j\pi jnT} = \sum_n \int_{-\infty}^{\infty} x(\alpha) \cdot e^{j\pi \alpha nT} d\alpha \cdot e^{-j\pi jnT}$$

$$\downarrow$$

$$x(nT) = \int_{-\infty}^{\infty} x(\alpha) \cdot e^{j\pi \alpha nT} d\alpha$$

$$= \int_{-\infty}^{\infty} x(\alpha) \underbrace{\sum_n e^{-j\pi(j-\alpha)nT}}_{\text{II^2 Poisson}} d\alpha = \frac{1}{T} \sum_n \int_{-\infty}^{\infty} x(\alpha) \delta(j - \alpha - \frac{n}{T}) d\alpha = \frac{1}{T} \sum_n \underbrace{\int_{-\infty}^{\infty} x(\alpha) \delta\left[\overbrace{j - \frac{n}{T}}^j - \alpha\right] d\alpha}_{\text{Convoluzione}} =$$

$$\downarrow$$

$$\frac{1}{T} \sum_n \delta(j - \alpha - \frac{n}{T})$$

$$= \frac{1}{T} \sum_n x(\alpha) \otimes \delta(j - \frac{n}{T}) \Rightarrow \boxed{\bar{x}(j) = \frac{1}{T} \sum_n x(j - \frac{n}{T})}$$

Convoluzione tra sequenze

$$R_{xy}[m] \triangleq \sum_{n=-\infty}^{\infty} x[n] y^*[n-m]$$

$$R_x[m] \triangleq \sum_{n=-\infty}^{\infty} x[n] x^*[n-m] ; \text{ Autocorrelazione}$$

$$\bar{S}_x(j) = \text{TF} [R_x[m]] = \sum_{m=-\infty}^{\infty} R_x[m] e^{-j\pi jmT} = \sum_m \sum_n x[n] x^*[n-m] \cdot e^{-j\pi jmT} =$$

$$\downarrow$$

Densità spettrale di energia di una sequenza

$$= \sum_n x[n] \sum_m x^*[n-m] e^{-j\pi jmT} = \sum_n x[n] \sum_{m'} x^*[m'] e^{-j\pi j(n-m')T} =$$

$$\downarrow n-m=m'$$

$$= \underbrace{\sum_n x[n] \cdot e^{-j\pi jmT}}_{\bar{x}(j)} \underbrace{\left[\sum_{m'} x[m'] e^{-j\pi jm'T} \right]^*}_{\bar{x}^*(j)} \Rightarrow \boxed{\bar{S}_x(j) = |\bar{x}(j)|^2}$$

$$R_x[m] \Big|_{m=0} = \sum_n x[n] x^*[n] = \sum_n |x[n]|^2 = E_{x_n}$$

$$\widehat{S_x}(\frac{j}{T}) = \sum_m R_x[m] \cdot e^{-j\frac{2\pi}{T}m\frac{j}{T}}$$

$$E_x = R_x[m] \Big|_{m=0} = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \widehat{S_x}(\frac{j}{T}) e^{j\frac{2\pi}{T}\frac{j}{T}m\frac{j}{T}} dj \Big|_{m=0} \Rightarrow E_x = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \widehat{S_x}(\frac{j}{T}) dj$$

$$E_x = \sum_n |x[n]|^2$$

Condizione di Nyquist

Segnali a banda rigorosamente limitata

$$X(j) = 0, |j| > B$$

$$\underline{x(t)} \rightarrow^T \underline{x[n]}$$

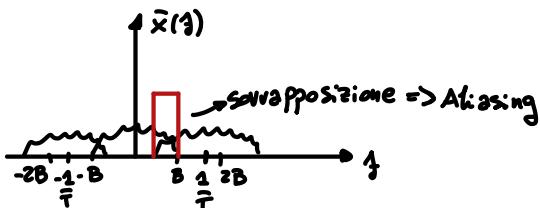
2 casi:

$$\boxed{1} T > \frac{1}{2B}$$

$$\boxed{2} T \leq \frac{1}{2B}$$

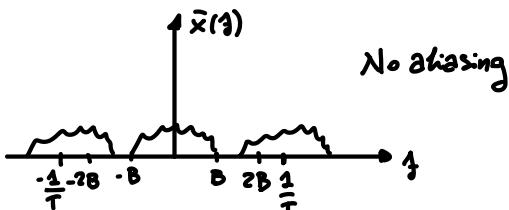
$$\boxed{1} T > \frac{1}{2B}$$

$$\bar{x}(j) = TFS[x[n]] = \frac{1}{T} \sum n x(j - \frac{n}{T}) \text{ con } X(j) = TCF(x(t))$$



$$T > \frac{1}{2B} \Rightarrow \text{Aliasing}$$

$$\boxed{2} T \leq \frac{1}{2B}$$



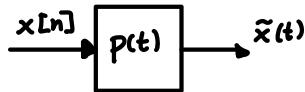
Se $T = \frac{1}{2B}$ ho Aliasing in un solo punto

La condizione di Nyquist per garantire assenza di aliasing è quindi

$$T \leq \frac{1}{2B}$$

Interpolatore

E' il duale di un campionatore



$p(t) \triangleq$ funzione interpolatrice

$$\tilde{x}(t) = \sum_n x[n] p(t-nT)$$

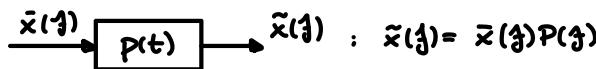
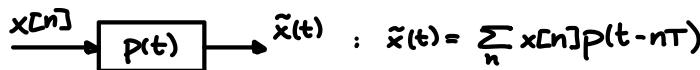
Interpolatore di ordine 1

$$p(t) = (1 - \frac{|t|}{T}) \operatorname{rect}(\frac{t}{\pi T})$$

$$\begin{aligned} \tilde{x}(j) &= \int_{-\infty}^{\infty} \tilde{x}(t) \cdot e^{-j2\pi jt} dt = \int_{-\infty}^{\infty} \sum_n x[n] p(t-nT) \cdot e^{-j2\pi jt} dt = \sum_n x[n] \int_{-\infty}^{\infty} p(t-nT) \cdot e^{-j2\pi jt} dt = \\ &= \sum_n x[n] \int_{-\infty}^{\infty} p(t') e^{-j2\pi j(t'+nT)} dt' = \underbrace{\sum_n x[n] e^{-j2\pi jnT}}_{\bar{x}(j)} \underbrace{\int_{-\infty}^{\infty} p(t') e^{-j2\pi jt'} dt'}_{P(j)} \Rightarrow \boxed{\tilde{x}(j) = \bar{x}(j) P(j)} \end{aligned}$$

$t-nT=t'$

Interpretazione dell'interpolatore



Teorema del campionamento

Ipotesi:

[1] $x(t)$ a banda rigorosamente limitata B

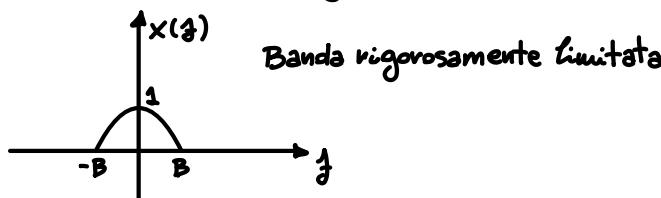
[2] $T: T \leq \frac{1}{2B}$

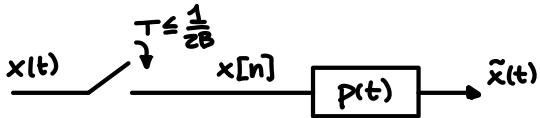
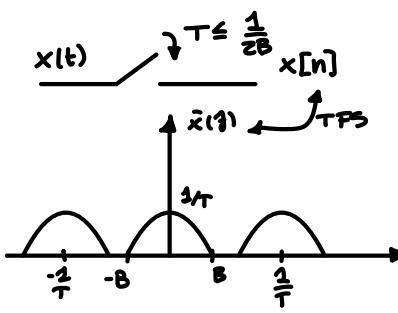
[3] $p(t)$ è un interpolatore cardinale, ovvero: $p(t) = \operatorname{sinc}(2Bt) \Leftrightarrow P(j) = \frac{1}{\pi} \operatorname{rect}(\frac{j}{2B})$

Tesi:

Si può ricostruire esattamente il segnale $x(t)$ a partire dai suoi campioni

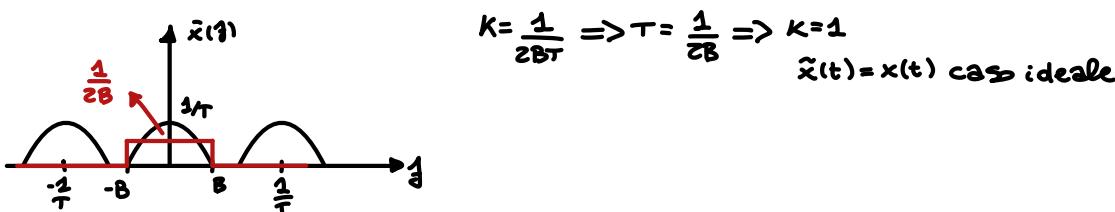
Dimostrazione per via grafica





$$\tilde{x}(j) = \bar{x}(j)P(j) = \frac{1}{2BT} x(j) = k x(j) \Rightarrow \tilde{x}(t) = k x(t) \quad \text{Riplica fedele}$$

$$k = \frac{1}{2BT} \Rightarrow T = \frac{1}{2B} \Rightarrow k=1$$



Teoria della probabilità

Eperimento casuale

1 Spazio campione: insieme dei possibili risultati dell'esperimento

$$\Omega = \{w_1, w_2, \dots, w_n\}$$

2 Evento: sottoinsieme dello spazio campione descritto secondo una certa regola

Proprietà che definiscono un evento:

1 Se A è un evento, anche \bar{A} è un evento

\bar{A} è il complemento di A su Ω

2 Se A e B sono eventi $\Rightarrow A \cup B$ è un evento

Proprietà derivate:

1 $A \cap B$ è un evento

2 $A \cup \bar{A} = \Omega$ evento certo

3 $A \cap \bar{A} = \emptyset$ evento impossibile

Un evento casuale è caratterizzato completamente da:

1 Ω

2 Proprietà degli eventi

3 Legge di probabilità

Probabilità

Definizione Axiomatica (Kolmogorov)

[1] $P(A) \geq 0$

[2] $P(\Omega) = 1$

[3] Se A e B sono 2 eventi mutuamente esclusivi: $\Rightarrow P(A \cup B) = P(A) + P(B)$ e $P(A \cap B) = 0$

Proprietà

[1] $P(\bar{A}) = 1 - P(A)$

[2] $P(\emptyset) = 0$

[3] $0 \leq P(A) \leq 1$

[4] $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Simbologia

$$A \cup B \Rightarrow A + B$$

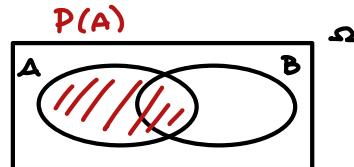
$$A \cap B \Rightarrow AB$$

$P(AB)$: probabilità congiunta di A e B

$P(A), P(B)$

Probabilità condizionata

$$P(A|B) \triangleq \frac{P(AB)}{P(B)}$$



Dopo che si verifica B lo spazio campione si riduce a $B \Rightarrow$ la probabilità che si verifichi A è uguale all'intersezione tra A e B

Definizione classica di probabilità (Pascal)

$$P(A) \triangleq \frac{N_f(A)}{N} \text{ con: } N_f(A): \text{numero casi favorevoli ad } A \\ N: \text{numero casi totali}$$

Definizione Frequentista

$$P(A) \triangleq \lim_{N \rightarrow \infty} \frac{N_A}{N} \text{ con: } N_A: \text{numero risultati favorevoli} \\ N: \text{numero esperimenti fatti}$$

Indipendenza tra due eventi

A è indipendente da B se $P(A|B) = P(A)$

$$\begin{aligned} &\Downarrow \\ P(A|B) &= \frac{P(AB)}{P(B)} = P(A) \\ &\Downarrow \\ P(AB) &= P(A)P(B) \end{aligned}$$

Teorema di Bayes

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} ; P(A), P(B) \neq 0$$

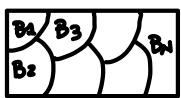
Dimostrazione

$$P(AB) = P(BA)$$

$$P(A|B)P(B) = P(B|A)P(A) \Rightarrow P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Teorema della probabilità totale

Partizione di uno spazio campione



$\{B_i\}$ è una partizione di Ω se

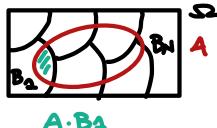
- [1] $B_1 \cup B_2 \cup \dots \cup B_N = \Omega$
- [2] $B_i \cap B_j = \emptyset$ con $i \neq j$ $\forall i, j = 1, \dots, N$

Tesi:

$$P(A) = \sum_{i=1}^N P(A|B_i)P(B_i)$$
 con B_i partizione di Ω

Dimostrazione

$$P(A) = P(A|\Omega) = P(A \sum_{i=1}^N B_i) =$$



$$P(\sum_{i=1}^N AB_i) = P(\underbrace{AB_1 \cup AB_2 \cup \dots \cup AB_N}_{\text{mutuamente esclusivi}}) = P(AB_1) + P(AB_2) + \dots + P(AB_N) = \sum_{i=1}^N P(AB_i) =$$

$$= \sum_{i=1}^N P(A|B_i)P(B_i)$$

Esempio spazio composto

$$\left. \begin{array}{l} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_N \end{array} \right\} \Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N \quad \text{prodotto cartesiano}$$

Esempio:

$$\begin{aligned}\Omega_1 &= \{1, 2, 3, 4, 5, 6\} \\ \Omega_2 &= \{\text{testa, croce}\}\end{aligned}$$

L'esperimento è il lancio simultaneo di un dado e di una moneta

Se i singoli esperimenti sono indipendenti fra loro $\Rightarrow P(A_1 \times A_2 \times \cdots \times A_N) = P(A_1)P(A_2) \cdots P(A_N)$

Prove ripetute

Definisco un esperimento e lo ripeto N volte.

Prove ripetute binarie ed indipendenti,

$$\Omega = \{w_1, w_2\}$$

ogni esperimento è indipendente dagli altri

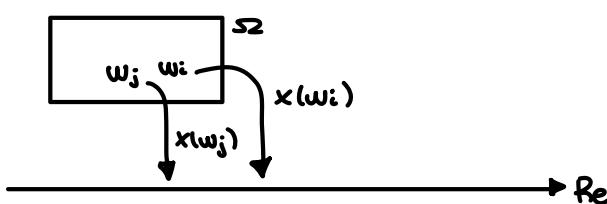
$$w_1 \Rightarrow P(w_1) = p_1$$

$$w_2 \Rightarrow P(w_2) = p_2 = 1 - p_1$$

$$A = \{w_1 \text{ si presenta } K \text{ volte su } N\}$$

$$P(A) = \binom{N}{K} \cdot p_1^K \cdot p_2^{N-K} = \binom{N}{K} p_1^K (1-p_1)^{N-K} \text{ con } \binom{N}{K} = \frac{N!}{K!(N-K)!}$$

Variabili Aleatorie (V.A.)



$x(w_i)$: V.A., trasforma/rappresenta il risultato di un esperimento sull'asse dei numeri reali.

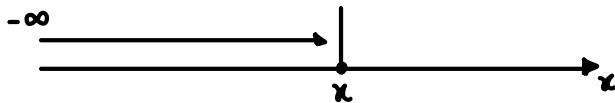
$x(w_i) \rightarrow X$; Si usano lettere maiuscole per rappresentare una V.A.

Descrizione statistica di una V.A.

Funzione di distribuzione di probabilità

$$F_x(x) \stackrel{\text{parametro}}{\equiv} P(X \leq x)$$

↑
V.A.



$X \leq x$ è un evento

Proprietà

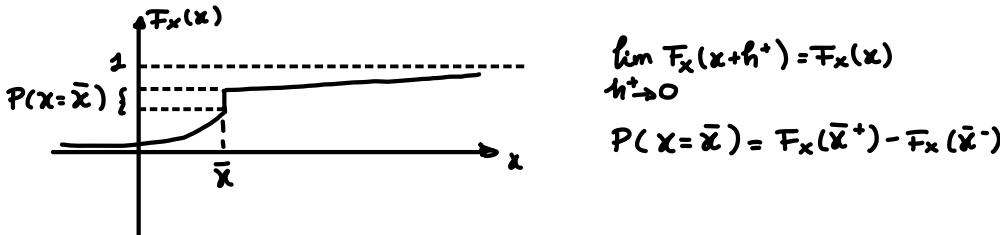
◻ $0 \leq F_x(x) \leq 1$

◻ $\lim_{x \rightarrow \infty} F_x(x) = 1$

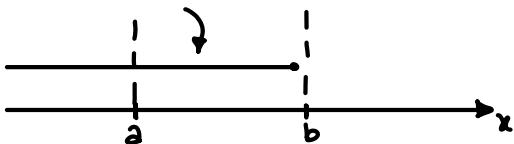
◻ $\lim_{x \rightarrow -\infty} F_x(x) = 0$

◻ $F_x(x)$ è monotona non decrescente: $F_x(x_2) \geq F_x(x_1)$ con $x_2 > x_1$

◻ $F_x(x)$ può avere discontinuità di prima specie



◻ $P(a \leq X \leq b) = F_x(b) - F_x(a) = P(X \leq b) - P(X \leq a)$

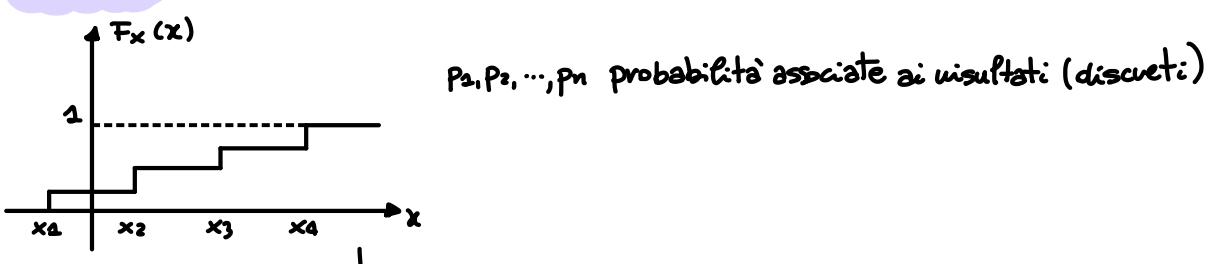


Una V.A. può essere: continua, discreta o mista

Continua

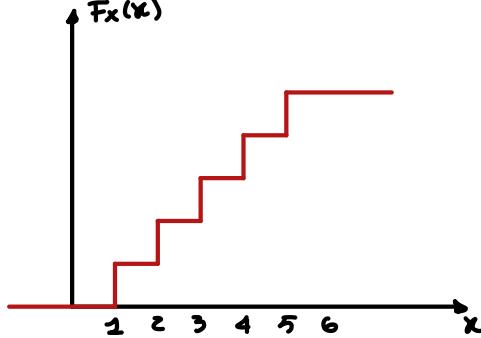
$F_x(x)$ continua

Discreta



Esempio del lancio del dado non truccato:

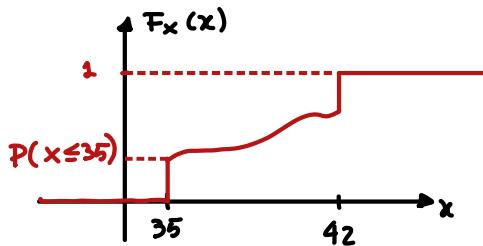
$$F_x(x) = \sum_{i=1}^N p(x=x_i) u(x-x_i) = \sum_{i=1}^6 \frac{1}{6} \cdot u(x-x_i) \quad \text{con } x_i = \{1, 2, 3, \dots, 6\}$$



Mista

$F_x(x)$ ha una componente continua ed una componente discetta

Esempio: temperatura misurata da un termometro clinico.



Densità di probabilità

$$f_x(x) \triangleq \frac{d}{dx} F_x(x)$$

$$F_x(x) = \int_{-\infty}^x f_x(\alpha) d\alpha$$

Proprietà

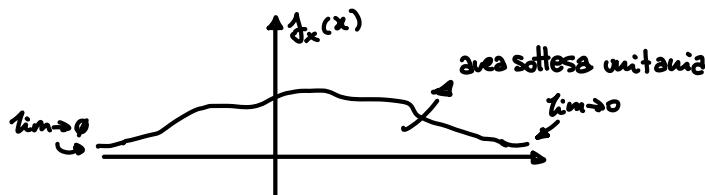
$$\boxed{1} f_x(x) \geq 0$$

$$\boxed{2} P(a \leq x \leq b) = F_x(b) - F_x(a) = \int_{-\infty}^b f_x(\alpha) d\alpha - \int_{-\infty}^a f_x(\alpha) d\alpha = \int_a^b f_x(\alpha) d\alpha$$

$$\boxed{3} \int_{-\infty}^{\infty} f_x(x) dx = 1$$

$f_x(x)$ con x continuo

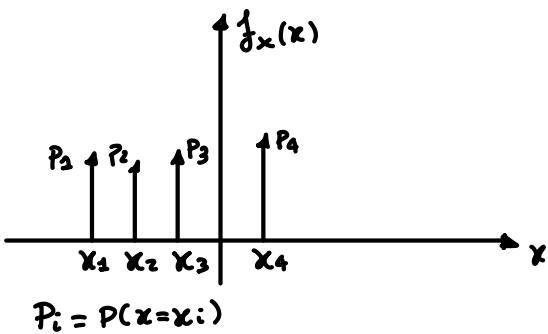
x continuo $\Rightarrow F_x(x)$ continua



$$\lim_{x \rightarrow -\infty} f_x(x) = 0$$

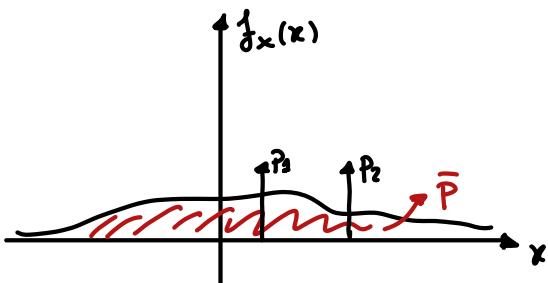
$$\lim_{x \rightarrow \infty} f_x(x) = 0$$

$f_x(x)$ con X discinta



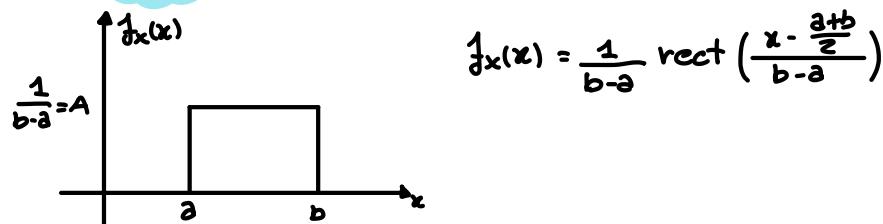
$$\frac{d}{dx} F_x(x) = \frac{d}{dx} \left[\sum_i p_i \cdot u(x-x_i) \right] = \sum_i p_i \frac{d}{dx} u(x-x_i) = \sum_i p_i \delta(x-x_i)$$

$f_x(x)$ con X mista

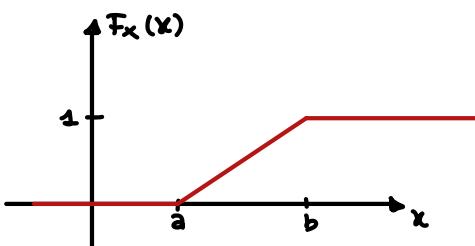


$$p_1 + p_2 + \bar{P} = 1$$

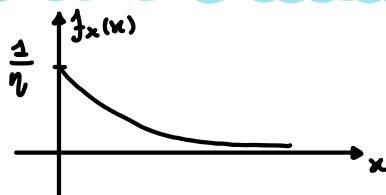
V.A. Uniiformi



$$f_x(x) = \frac{1}{b-a} \text{rect}\left(\frac{x - \frac{a+b}{2}}{b-a}\right)$$



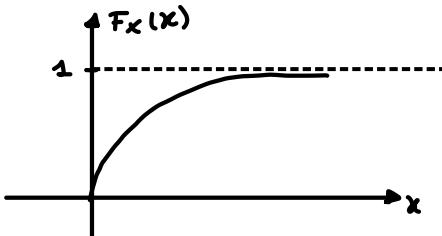
V.A. Esponenziali (unilaterale)



$$f_x(x) = \frac{1}{\eta} e^{-\frac{x}{\eta}} u(x)$$

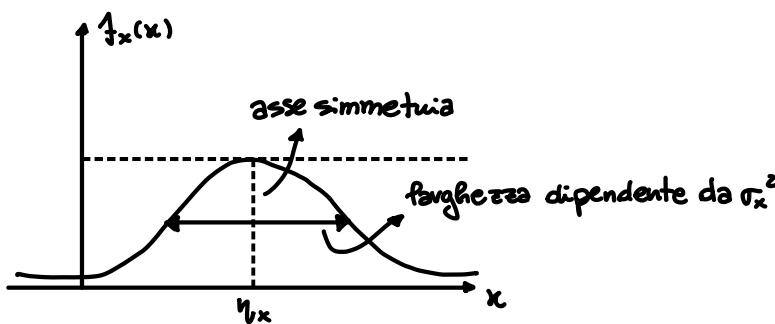
$$\int \frac{1}{\eta} e^{-\frac{x}{\eta}} u(x) dx = \frac{1}{\eta} \int_0^\infty e^{-\frac{x}{\eta}} dx = \frac{1}{\eta} (-\eta) e^{-\frac{x}{\eta}} \Big|_0^\infty = 1$$

$$F_x(x) = \int_0^x \frac{1}{\eta} e^{-\frac{\alpha}{\eta}} d\alpha = \frac{1}{\eta} (-\eta) e^{-\frac{\alpha}{\eta}} \Big|_0^x = (-\alpha) (e^{-\frac{x}{\eta}} - 1) = \begin{cases} 1 - e^{-\frac{x}{\eta}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



V. A. Gaussiana

$$f_x(x) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \cdot e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$



$F_x(x)$ non ha una forma chiusa

$P(x \leq \bar{x}) = F_x(\bar{x})$ non si calcola in forma chiusa

$x \in \mathcal{N}(\mu_x, \sigma_x^2)$
identificano una particolare gaussiana

$x \rightarrow N: x \in \mathcal{N}(0,1)$

Gaussiana Standard

$$\Phi(x) \triangleq \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} d\alpha$$

$$Q(x) \triangleq 1 - \Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} d\alpha : \text{Funzione "coda". Si usa per calcolare la probabilità di errore sul bit nei sistemi di comunicazione numerici}$$

Trasformazione di V.A.

$x \rightarrow y$

$y = T[x]$ definita tramite g del tipo $y = g(x)$

Teorema fondamentale per la trasformazione di V.A.

$$\frac{f_y(y)}{f_x(x)} = \sum_i \frac{f_x(x_i)}{|g'(x_i)|}$$

Se x_i : sono le soluzioni di $x = g(y)$ trasformazione inversa

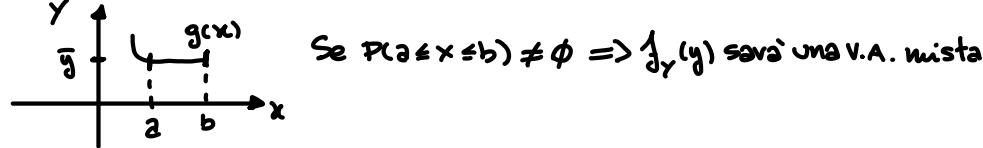
Casi particolari

$$\exists \{x_i\} = \emptyset \Rightarrow f_y(y) = \emptyset$$

$$\exists g(x): g'(x) \Big|_{x=\bar{x}} = \emptyset$$

• minimo o massimo relativo su un unico punto $\Rightarrow f_y(y) \rightarrow \infty$

• $g'(x) = \emptyset$ su un intervallo finito di misura non nulla $\Rightarrow f_y(y)$ in \bar{y} ha una $\delta(y-\bar{y})$ con area $P(a < x < b)$



Indici caratteristici della distribuzione di probabilità

Valor medio

$$\eta_x = E[x] \triangleq \int_{-\infty}^{\infty} x f(x) dx$$

↳ expectation

per V.A. discrete

$$\eta_x = \int_{-\infty}^{\infty} x f_x(x) dx = \int_{-\infty}^{\infty} x \sum_{n=1}^N p_n \delta(x-x_n) dx = \sum_{n=1}^N p_n \int_{-\infty}^{\infty} x \delta(x-x_n) dx = \sum_{n=1}^N p_n x_n$$

Teorema del valore medio

Se $y = g(x)$

$$\begin{cases} \eta_y = E[y] = \int_{-\infty}^{\infty} y f_y(y) dy = \int_{-\infty}^{\infty} g(x) f_x(x) dx = E[g(x)] \\ E[y] = E[g(x)] \end{cases}$$

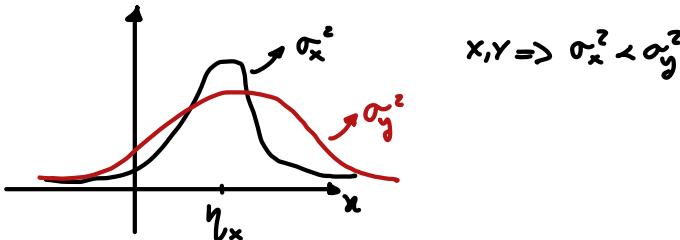
Proprietà di linearità del valore medio

$$Y = \alpha g(x) + \beta h(x); \alpha, \beta \in \mathbb{R} \Rightarrow \eta_Y = \alpha E[g(x)] + \beta E[h(x)]$$

Varianza

$$\sigma_x^2 \triangleq E[(x - \eta_x)^2] = \int_{-\infty}^{\infty} (x - \eta_x)^2 f_x(x) dx$$

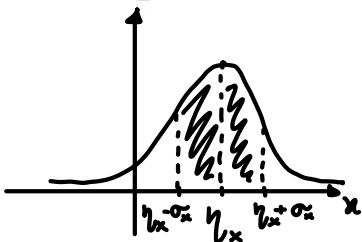
Esempio



Deviazione standard

$$\sigma_x \triangleq \sqrt{\sigma_x^2} \text{ Si usa poiché lineare}$$

Esempio



Varianza Nulla

$$\sigma_x^2 = 0 \Rightarrow f_x(x) = \delta(x - \eta_x) \Rightarrow \text{ottengo un risultato deterministico}$$

Varianza Infinita

$$\sigma_x^2 \rightarrow \infty \Rightarrow \text{Massimo livello di casualità}$$

Valore quadratico medio

$$m_x^2 \triangleq E[x^2] = \int_{-\infty}^{\infty} x^2 f_x(x) dx$$

Relazione tra η_x , σ_x^2 , m_x^2

$$\begin{aligned} \sigma_x^2 &= E[(x - \eta_x)^2] = E[x^2 + \eta_x^2 - 2x\eta_x] = E[x^2] + E[\eta_x^2] - E[2\eta_x x] = m_x^2 + \eta_x^2 - 2\eta_x E[x] = \\ &= m_x^2 + \eta_x^2 - 2\eta_x^2 = \boxed{m_x^2 - \eta_x^2 = \sigma_x^2} \end{aligned}$$

V.A. Gaussiane

$$x \in \mathcal{N}(\eta_x, \sigma_x^2)$$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}$$

$$E[x] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} dx = \int_{-\infty}^{\infty} y + \eta_x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{y^2}{2\sigma_x^2}} dy =$$

$$= \int_{-\infty}^{\infty} y \cdot \left(\frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{y^2}{2\sigma_x^2}} \right) dy + \eta_x \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{y^2}{2\sigma_x^2}} dy}_{1 \text{ poiché Gaussiana}} \Rightarrow E[x] = \eta_x$$

Varianza

$$E[(x-\eta_x)^2] = \int_{-\infty}^{\infty} (x-\eta_x)^2 \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} dx = \int_{-\infty}^{\infty} y^2 \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{y^2}{2\sigma_x^2}} dy =$$

$$= -\sigma_x^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} y \cdot \frac{-y}{\sigma_x^2} \cdot e^{-\frac{y^2}{2\sigma_x^2}} dy$$

$$\frac{d}{dy} \left[e^{-\frac{y^2}{2\sigma_x^2}} \right] = -\frac{1}{2\sigma_x^2} \cdot e^{-\frac{y^2}{2\sigma_x^2}} \cdot (-2y)$$

$$= -\sigma_x^2 \left\{ \underbrace{\left[\frac{1}{\sqrt{2\pi\sigma_x^2}} y \cdot e^{-\frac{y^2}{2\sigma_x^2}} \right]_{-\infty}^{\infty}}_{\phi} - \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy}_{\sigma_x^2} \right\} = \boxed{\sigma_x^2 = E[(x-\eta_x)^2]}$$

$$x \in \mathcal{N}(\eta_x, \sigma_x^2)$$

Le V.A. Gaussiane sono completamente descritte da valore medio e varianza

V.A. Gaussiane generali e standard

$$N \in \mathcal{N}(0,1)$$

$$X = \sigma_x N + \eta_x \text{ trasformazione di una V.A.}$$

$$f_x(x) = \frac{f_N(n)}{|g'(n)|} \Big|_{n=\frac{x-\eta_x}{\sigma_x}} \quad \text{con } n = \frac{x-\eta_x}{\sigma_x} \quad \rightarrow g^{-1}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} \Rightarrow \mathcal{N}(\eta_x, \sigma_x^2)$$

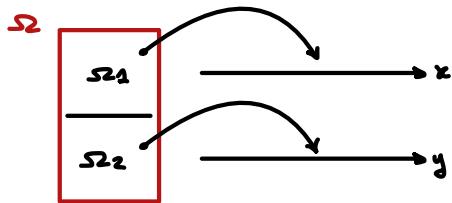
Possiamo ottenere ogni Gaussiana con trasformazione della standard (N)

Espansimento Composto

$$\Omega = \Omega_1 \times \Omega_2$$

$$A = A_1 \times A_2$$

Sistemi di 2 V.A.



$$F_{xy}(x,y) \triangleq P(X \leq x, Y \leq y)$$

↳ funzione di distribuzione congiunta

$$f_{xy} \triangleq \frac{\partial}{\partial x \partial y} F_{xy}(x,y)$$

↳ densità di probabilità congiunta

Densità di probabilità condizionata

$$f_{y|x}(y|x) \triangleq \frac{f_{xy}(x,y)}{f_x(x)}$$

Indipendenza di 2 V.A.

$$f_{y|x}(y|x) = \frac{f_{xy}(y,x)}{f_x(x)} = f_y(y) \Rightarrow f_{xy}(x,y) = f_x(x) \cdot f_y(y)$$

Indici statistici del 2° Ordine

Correlazione tra 2 V.A.

X, Y sono V.A.

$$r_{xy} \triangleq E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{xy}(x,y) dx dy$$

Covarianza

$$C_{xy} \triangleq E[(X - \eta_x)(Y - \eta_y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \eta_x)(y - \eta_y) f_{xy}(x,y) dx dy$$

Relazione tra r_{xy} , C_{xy} e η_x , η_y

$$C_{xy} = E[(x - \eta_x)(y - \eta_y)] = E[xy - \eta_x y - \eta_y x + \eta_x \eta_y] = E[xy] - E[\eta_x y] - E[\eta_y x] + \eta_x \eta_y = r_{xy} - \eta_x \eta_y - \eta_y \eta_x + \eta_x \eta_y \Rightarrow C_{xy} = r_{xy} - \eta_x \eta_y$$

V.A. Inconcrete

V.A. inconcrete $x, y \Leftrightarrow C_{xy} = 0$

Coefficiente di correlazione

$$\rho_{xy} \triangleq \frac{C_{xy}}{\sigma_x \sigma_y}$$

Proprietà:

$$|\rho_{xy}| \leq 1$$

Indipendenza e Inconvenzione

[1] Se x, y sono indipendenti $\Rightarrow x, y$ sono inconcrete

$$f_{xy}(x, y) = f_x(x) f_y(y)$$

$$r_{xy} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{xy}(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_x(x) f_y(y) dx dy = \int_{-\infty}^{\infty} x f_x(x) dx \cdot \int_{-\infty}^{\infty} y f_y(y) dy = \eta_x \eta_y$$

$$\Rightarrow C_{xy} = r_{xy} - \eta_x \eta_y = 0$$

[2] inconvenzione \Rightarrow indipendenza

Controesempio:

$$x, y; \theta \in \mathcal{U}[\theta, 2\pi]$$

$$f_\theta(\theta) = \frac{1}{2\pi} \text{rect}\left(\frac{\theta - \pi}{2\pi}\right)$$

$$x = \cos(\theta)$$

$$y = \sin(\theta)$$

$$r_{xy} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{xy}(x, y) dx dy$$

$$= E[g(\theta) h(\theta)] = \text{Tezema valore medio}$$

$$= \int_{-\infty}^{\infty} \cos(\theta) \sin(\theta) f_\theta(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta) \sin(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sin(2\theta) d\theta = 0$$

$$\Rightarrow \boxed{r_x r_y = \phi}$$

$$C_{xy} = r_x r_y - \eta_x \eta_y$$

$$\eta_x = E[x] = \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta) d\theta = \phi$$

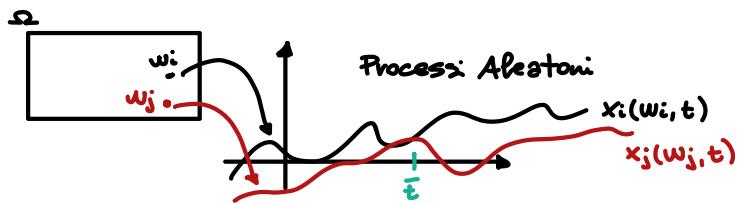
$$\eta_y = E[y] = \frac{1}{2\pi} \int_0^{2\pi} \sin(\theta) d\theta = \phi$$

$\Rightarrow C_{xy} = \phi$ Sempre rebbe esserci incoviazione.

$$x^2 + y^2 = \cos^2(\theta) + \sin^2(\theta) = 1 \Rightarrow x^2 = 1 - y^2 \text{ Ho dipendenza quadratica!}$$

Questo accade perché l'incoviazione è legata solo alla dipendenza lineare tra V.A. Sul resto non dà informazioni.

Segnali Aleatori



$w_i \rightarrow x_i(w_i, t) \rightarrow x_i(t)$ Realizzazione del processo aleatorio

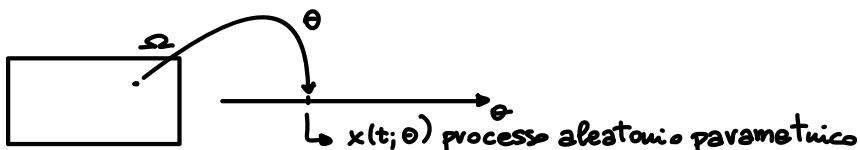
$x(w, t) \rightarrow x(t)$ processo aleatorio

Se fissiamo $t = \bar{t} \Rightarrow x(\bar{t})$ è una V.A.

Processi Aleatori Parametrici

$$x(t) = A \cos(2\pi f_0 t + \theta) \quad \text{Posso avere molte V.A.}$$

\downarrow
V.A. \hookrightarrow V.A.!



Caratterizzazione statistica di processi aleatori

Distribuzione di probabilità

1° Ordine

$$F_x(x_1; t_1) \triangleq P(x(t_1) \leq x_1)$$

2° Ordine

$$F_x(x_1, x_2; t_1, t_2) \triangleq P(x(t_1) \leq x_1, x(t_2) \leq x_2)$$

Si può generalizzare per induzione all'ordine N

Densità di probabilità

1° Ordine

$$f_x(x; t) \triangleq \frac{d}{dx} F_x(x; t)$$

2° Ordine

$$f_x(x_1, x_2; t_1, t_2) \triangleq \frac{d^2}{dx_1 dx_2} F_x(x_1, x_2; t_1, t_2)$$

Indici statistici

1° Ordine

Value Medio

$$\eta_x(t) \triangleq E[x(t)] = \int_{-\infty}^{\infty} x f_x(x; t) dx$$

Potenza Media Statistica

$$P_x(t) \triangleq E[x^2(t)] = \int_{-\infty}^{\infty} x^2 f_x(x; t) dx$$

Varianza del Processo

$$\sigma_x^2(t) = E[(x(t) - \eta_x(t))^2] = \int_{-\infty}^{\infty} (x - \eta_x(t))^2 f_x(x; t) dx$$

$$\sigma_x^2(t) = P_x(t) - \eta_x^2(t)$$

Indici del 2° Ordine

Autocorrelazione

$$R_x(t_1, t_2) \triangleq E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_x(x_1, x_2; t_1, t_2) dx_1 dx_2$$

Autocovarianza

$$\begin{aligned} C_x(t_1, t_2) &\triangleq E[(x(t_1) - \mu_x(t_1))(x(t_2) - \mu_x(t_2))] = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_1 - \mu_x(t_1))(x_2 - \mu_x(t_2)) f_x(x_1, x_2; t_1, t_2) dx_1 dx_2 \end{aligned}$$

$$C_x(t_1, t_2) = R_x(t_1, t_2) - \mu_x(t_1)\mu_x(t_2)$$

Stazionarietà

In senso stretto:

$$f_x(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = f_x(x_1, x_2, \dots, x_n; t_1 + \Delta t, t_2 + \Delta t, \dots, t_n + \Delta t)$$

\Rightarrow Invarianza alle traslazioni temporali, Δt qualsiasi.

1° Ordine

Si estrae una sola V.A.

$$f_x(x_1; t_1) = f_x(x_1; t_1 + \Delta t) = f_x(x_1) \text{ non dipende dal tempo}$$

2° Ordine

Si estraggono 2 V.A.

$$f_x(x_1, x_2; t_1, t_2) = f_x(x_1, x_2; t_1 + \Delta t, t_2 + \Delta t) = f_x(x_1, x_2; t_1 - t_2)$$

Stazionarietà indici del 1° Ordine

$$\mu_x(t) = E[x(t)] = \int_{-\infty}^{\infty} x \frac{1}{\int_{-\infty}^{\infty} f_x(x; t) dx} dx = \int_{-\infty}^{\infty} x f_x(x) dx = \mu_x \text{ costante}$$

$$P_x(t) = E[x^2(t)] = \int_{-\infty}^{\infty} x^2 \frac{1}{\int_{-\infty}^{\infty} f_x(x; t) dx} dx = \int_{-\infty}^{\infty} x^2 f_x(x) dx = P_x \text{ costante}$$

$$\Rightarrow \sigma_x^2(t) = \sigma_x^2$$

Tutti gli indici statistici del 1° Ordine sono costanti:

Stazionarietà indici del 2° Ordine

$$R_x(t_1, t_2) = E[x(t_1)x(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_x(x_1, x_2; t_1, t_2) dx_1 dx_2 = \\ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_x(x_1, x_2; t_1 - t_2) dx_1 dx_2 = R_x(t_1 - t_2) = R_x(\tau) \text{ con } \tau = t_1 - t_2$$

$$C_x(t_1, t_2) = C_x(\tau) = R_x(\tau) - \eta_x^2 \quad \text{Per processi stazionari}$$

Processi stazionari in senso forte (SSL)

Un processo è SSL se:

1 $\eta_x(t) = \eta_x$ costante

2 $R_x(t_1, t_2) = R_x(t_1 - t_2) = R_x(\tau)$

$$\downarrow \\ C_x(t_1, t_2) = C_x(\tau)$$

Proprietà processi SSL

1 $R_x(\tau) = R_x(-\tau)$ pari

$$R_x(-\tau) = E[x(t)x(t - (-\tau))] = E[x(t)x(t + \tau)] \xrightarrow{t + \tau = t'} = E[x(t')x(t' - \tau)] = \underline{R_x(\tau)}$$

2 $R_x(0) = E[x(t)x(t - \tau)] \Big|_{\tau=0} = E[x^2(t)] = P_x$

3 $R_x(0) \geq |R_x(\tau)|$

$$E[(x(t) \pm x(t - \tau))^2] \geq 0$$

$$E[x^2(t) + x^2(t - \tau) \pm 2x(t)x(t - \tau)] \geq 0$$

$$E[x^2(t)] + E[x^2(t - \tau)] \pm \underbrace{2E[x(t)x(t - \tau)]}_{R_x(\tau)} \geq 0 \Rightarrow |R_x(\tau)| \leq P_x = R_x(0)$$

4 Se P_x $R_x(\tau)$ non ha componenti periodiche

$$\lim_{\tau \rightarrow \infty} R_x(\tau) = \eta_x^2$$

$$R_x(\tau) = C_x(\tau) + \eta_x^2$$

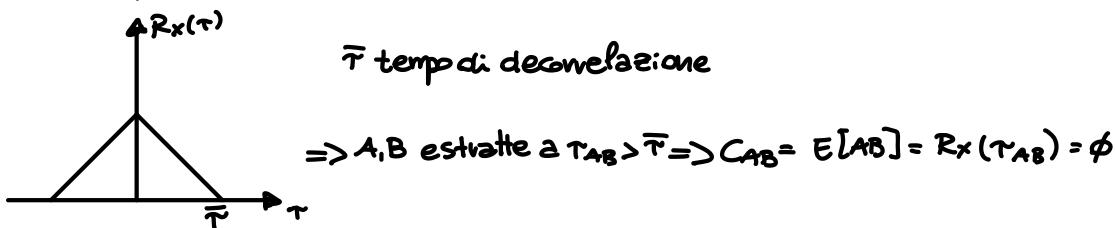
$\lim_{T \rightarrow \infty} C_x(T) = \emptyset$; Si giustifica poiché 2 v.a. estratte a distanza (T) infinite sono incompatibili

||

$$\lim_{T \rightarrow \infty} C_x(T) + \eta_x^2 = \eta_x^2$$

In un processo SSI a media nulla $R_x(\tau)$ determina il tempo di decorrelazione di un processo

Ad esempio



Filtraggio di un processo aleatorio con SLS



$$y(t) = x(t) \otimes h(t)$$

$$y_i(t) = x_i(t) \otimes h(t)$$

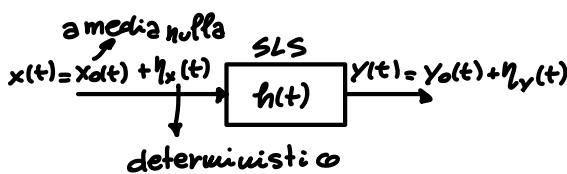
↳ i -esima realizzazione del processo di uscita. $x_i(t)$ è aleatorio, lo sarà anche $y_i(t)$

Indici statistici del processo in uscita

Valor medio $\eta_y(t)$



$$\begin{aligned} \eta_y(t) &= E[y(t)] = E[x(t) \otimes h(t)] = E\left[\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau\right] = \int_{-\infty}^{\infty} E[x(\tau)] h(t-\tau) d\tau = \int_{-\infty}^{\infty} \eta_x(\tau) h(t-\tau) d\tau = \\ &= \eta_x(t) \otimes h(t) \end{aligned}$$



$$E[x(t)] = E[x_d(t)] + E[\eta_x(t)] = \phi + \eta_x(t)$$

$$E[y_o(t)] = E[x_d(t) \otimes h(t)] = E\left[\int_{-\infty}^{\infty} x_d(\tau) h(t-\tau) d\tau\right] = \phi$$

Autocorrelazione



$$\begin{aligned}
 R_y(t_1, t_2) &= E[Y(t_1)Y(t_2)] = E\left[\int_{-\infty}^{\infty} x(\tau_1)h(t_1 - \tau_1)d\tau_1 \int_{-\infty}^{\infty} x(\tau_2)h(t_2 - \tau_2)d\tau_2\right] = \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[x(\tau_1)x(\tau_2)] h(t_1 - \tau_1)d\tau_1 h(t_2 - \tau_2)d\tau_2 = \int_{-\infty}^{\infty} R_x(\tau_1, \tau_2)h(t_1 - \tau_1)d\tau_1 h(t_2 - \tau_2)d\tau_2 = \\
 &= \underbrace{\int_{-\infty}^{\infty} R_x(\tau_1, \tau_2)h(t_1 - \tau_1)d\tau_1}_{R_x'(t_1, t_2)} h(t_2 - \tau_2)d\tau_2 = R_x'(t_1, t_2) \otimes h(t_2) = R_x(t_1, t_2) \otimes h(t_1) \otimes h(t_2)
 \end{aligned}$$

Filtraggio di un SSL con un SLS



$$\boxed{1} \quad \eta_y(t) = \eta_x(t) \otimes h(t) = \eta_x \otimes h(t) = \int_{-\infty}^{\infty} \eta_x(\tau)h(t - \tau)d\tau = \eta_x \int_{-\infty}^{\infty} h(t - \tau)d\tau = \eta_x \int_{-\infty}^{\infty} h(\tau')d\tau' = \eta_x H(\omega) = \boxed{\eta_x H(\omega)}$$

2 Si dimostra che

$$R_y(\tau) = R_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi\omega t} dt \Big|_{\omega=\phi}$$

Processi SSL

Densità spettrale di potenza

$$\downarrow \text{DSP} \\
 S_x(f) \triangleq \lim_{T \rightarrow \infty} E\left[\frac{|X_f(f)|^2}{T}\right] \text{ con } X_f(f) \triangleq TCF[X(t)] = TCF[X(w_i; t)]$$

Per segnali deterministici:

$$S_x(f) \triangleq \lim_{T \rightarrow \infty} \frac{|x(f)|^2}{T}$$

Teorema di Wiener - Khintchine

$$S_x(f) = TCF[R_x(\tau)] = \int_{-\infty}^{\infty} R_x(\tau) \cdot e^{-j2\pi f \tau} d\tau$$

Proprietà

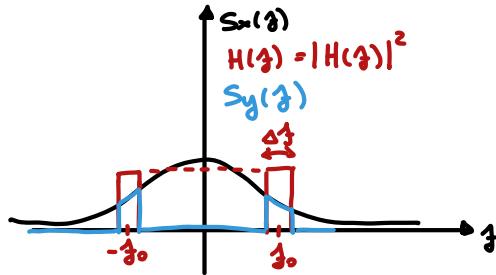
1 $S_x(f)$ è Reale e pari per processi SSL Reali: (Poiché $R_x(\tau)$ è reale e pari)

$$\boxed{2} P_x = E[x^2(t)] = R_x(\phi) = \int_{-\infty}^{\infty} S_x(f) df$$

$$\boxed{3} S_x(f) \geq 0 \forall f$$

Dimostrazione

$$S_y(f) = TCF [R_y(\tau)] = TCF [R_x(\tau) \otimes h(\tau) \otimes h(-\tau)] = S_x(f) \cdot H(f) \cdot H^*(f) = S_x(f) |H(f)|^2$$



$\Delta f \ll 1$ (Banda molto stretta)

$$P_y = \int_{-\infty}^{\infty} S_y(f) df = \int_{-\infty}^{\infty} [S_x(f) \cdot |H(f)|^2] df = \int_{-\infty}^{\infty} S_x(f) df = S_x(f) \cdot 2\Delta f \geq 0 \Rightarrow S_x(f_0) \geq 0 \quad \forall f_0$$

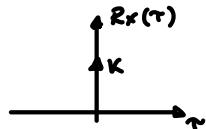
Se $S_x(f)$ fosse negativa avrei una $P_y < 0$ che è un assurdo

Processo di rumore bianco

Un rumore bianco è un processo SSL con:

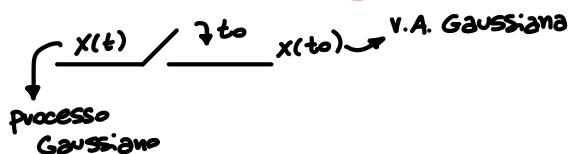
$$\boxed{1} \eta_x = 0$$

$$\boxed{2} R_x(\tau) = K \delta(\tau), \quad K \in \mathbb{R}$$



$$\Rightarrow S_x(f) = K + f$$

Processi Aleatori Gaussiani



$x(t_0) \stackrel{\text{def}}{=} x_0 \in \mathcal{N}(\mu_{x_0}(t_0), \sigma_{x_0}^2(t_0))$ in generale dipende dal tempo

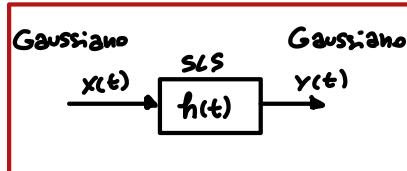
Processo SSL Gaussiano $\Rightarrow f_x(x)$ non dipende dallo tempo

$$\begin{aligned} \mu_{x_0}(t_0) &= \mu_x \\ R_x(t_1, t_2) &= R_x(\tau) \\ P_x &= R_x(\phi) \\ \sigma_x^2 &= P_x - \mu_x^2 \end{aligned}$$

Processi Gaussiani e Bianchi

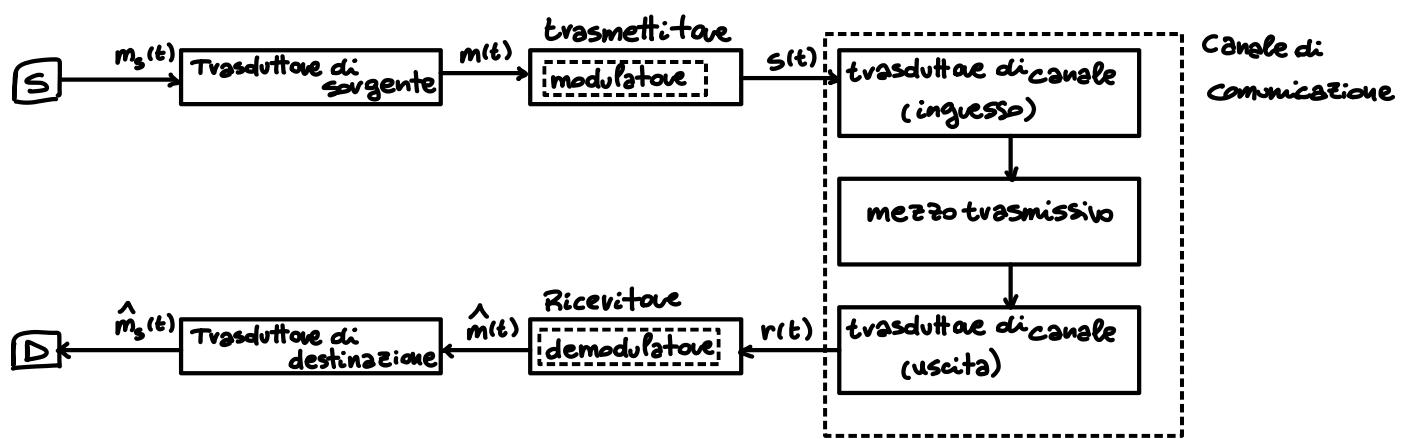
$$\mathbb{E}_x = 0; R_x(\tau) = K \delta(\tau) \text{ (Bianco)}$$

$x(t)$ Gaussiano $\Rightarrow x_0 = x(t_0) \forall t_0 \Rightarrow x_0 \in \mathcal{N}(\phi, \sigma^2)$



Un processo in uscita da un SLS con in ingresso un Gaussiano è un Gaussiano.

Schema generale di un sistema di comunicazione



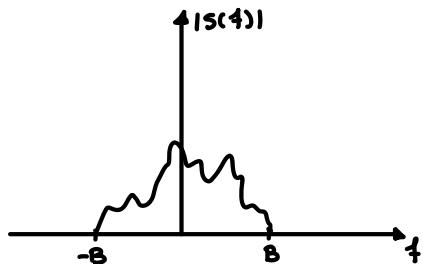
$m_s(t), \hat{m}_s(t)$: segnali fisici

$m(t), \hat{m}(t)$: segnali elettrici

$s(t), r(t)$: segnali elettrici

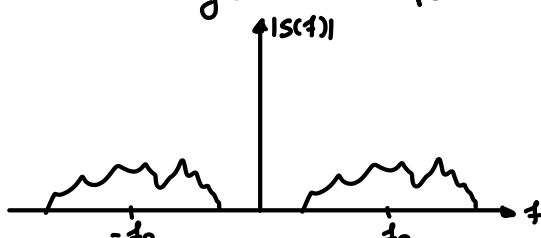
Banda base

Se $s(t)$ è un segnale in banda base:



Banda Passante

se $s(t)$ è un segnale in banda passante:



Banda stretta e larga

Un segnale si dice a banda larga se: $f_0 \leq 2B$

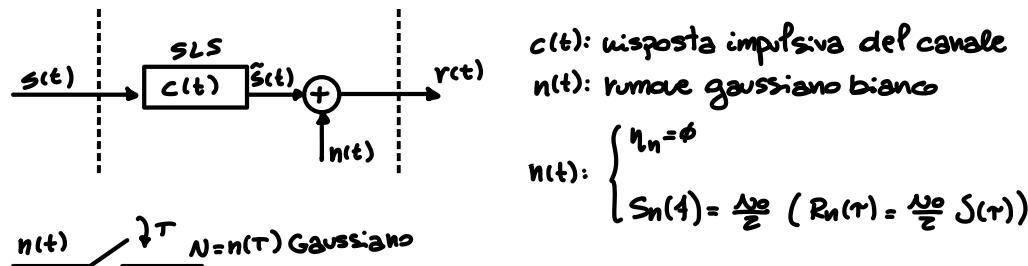
Un segnale si dice a banda stretta se: $f_0 \gg B$

Canale di comunicazione

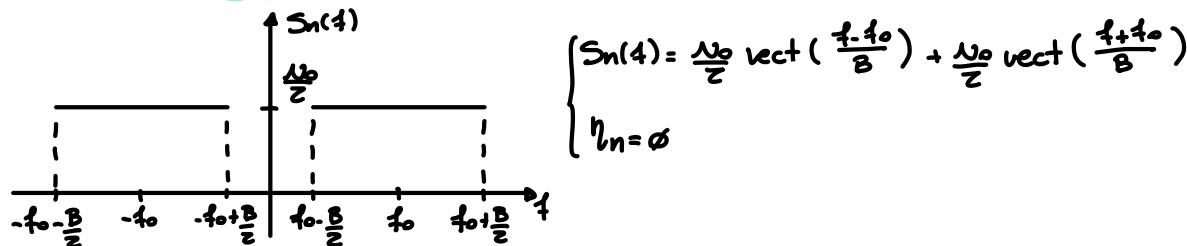
$r(t) \neq s(t)$ per effetto del canale di comunicazione:

- Distorsione
- Introduzione di disturbo (Rumore)

Modello di canale di comunicazione



Rumore bianco in banda



Canale Ideale

$$r(t) = s(t) \Rightarrow \begin{cases} n(t) = \emptyset \\ c(t) = S(t) \end{cases}$$

$$r(t) = s(t) \otimes c(t) + n(t)$$

Sistema di comunicazione Numerico

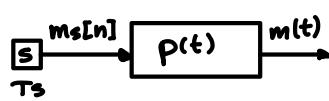


T_s : periodo di segnalazione della sorgente

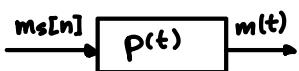
$$m_s[n] \in A_s = \{\alpha_1, \alpha_2, \dots, \alpha_M\}, M \geq 2$$

alfabeto

$m_s[n]$ è come il campionamento di un processo aleatorio



$m(t) \xrightarrow{T_s} m_s[n]$ è una sequenza incognita



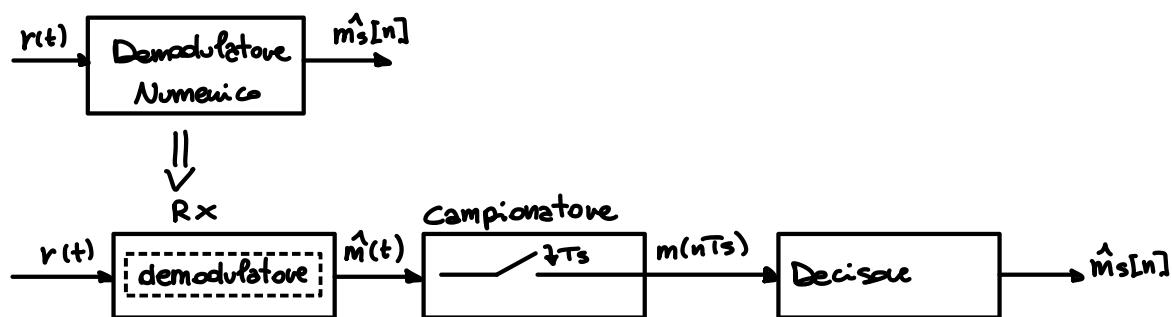
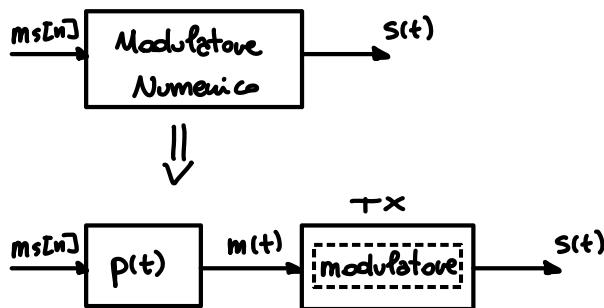
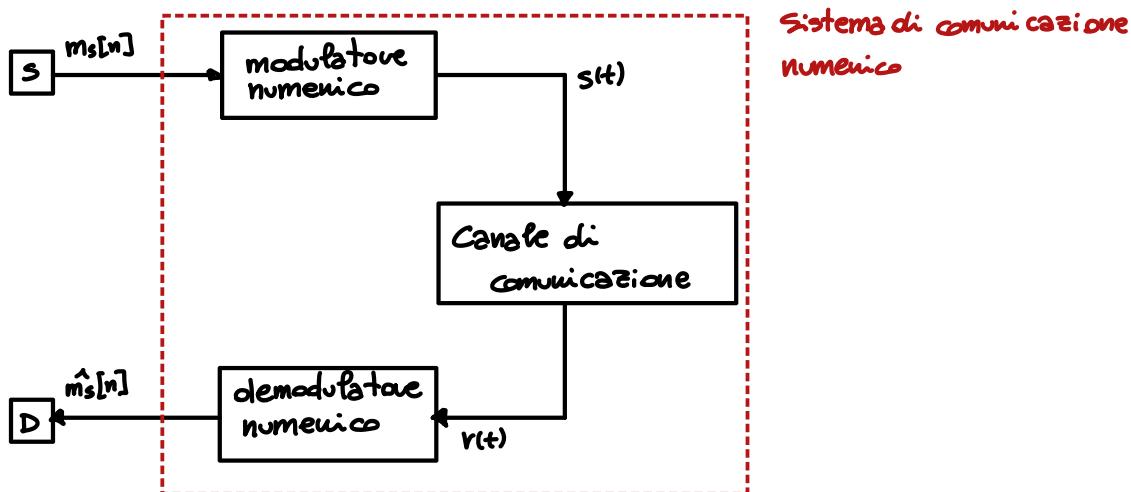
$$m(t) = \sum_n m[n] p(t - nT_s)$$

$p(t)$ è responsabile delle caratteristiche di $m(t)$

$R_s \triangleq \frac{1}{T_s}$: Rate di erogazione dei simboli. Si misura in baudrate [$\frac{\text{simboli}}{\text{s}}$]

Se i simboli sono binari corrisponde al bitrate: $R_b = R_s \cdot M \left[\frac{\text{bit}}{\text{s}} \right]$

Se $A_s = \{\alpha_1, \dots, \alpha_M\}$ con M potenza di 2 $\Rightarrow R_b = \frac{\log_2 M}{T_s}$



Performance

Sistema di comunicazione numerico ideale:

$$\hat{m}_s[n] = m_s[n] + n$$

Nella realtà:

$$\exists \bar{n} : \hat{m}_s[\bar{n}] \neq m_s[\bar{n}]$$

↓ ↓
 simbolo simbolo
 deciso trasmesso

Probabilità di transizione



$$P\{i|j\} \triangleq P\{\hat{m}_s = \alpha_i | m_s = \alpha_j\} \text{ con } \alpha_i, \alpha_j \in A_s$$

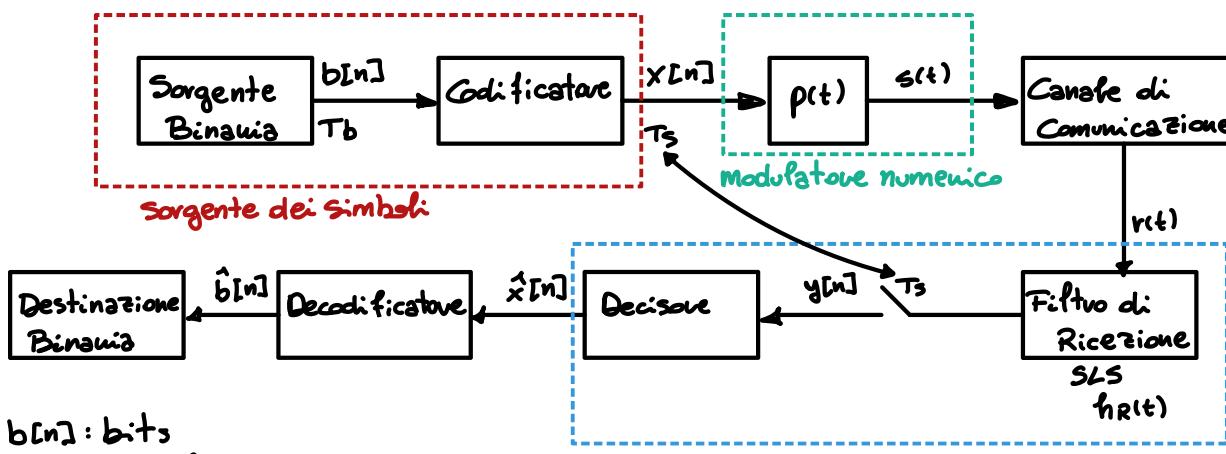
Un sistema di comunicazione numerico è caratterizzato completamente dal punto di vista statistico quando sono note le $P\{i|j\}$ per i, j

Probabilità d'errore sul simbolo

$$P_E(M) \triangleq P\{\hat{m}_s[n] \neq m_s[n]\}$$

↳ Numero simboli in A_s

Modulazioni numeriche in banda base



$b[n]$: bits

$x[n]$: simboli

$$y[n] = y(nT_s)$$

Trasmettitore



$A_s(M)$, $M = 2^K$ con K intero

$$T_s = T_b \cdot \log_2(M)$$

$$x[n] \Rightarrow P\{x[n] = \alpha_i\} \quad \forall i = 0, \dots, M$$

Simboli equi probabili:

$$P\{x[n] = \alpha_i\} = P\{x[n] = \alpha_j\} = 1/M$$

$$s(t) = \sum_n x[n] p(t - nT_s)$$

$s(t)$ è di fatto un segnale aleatorio, poiché sequenza di simboli aleatori. Verrà trattato come un processo aleatorio e ne calcoleremo:

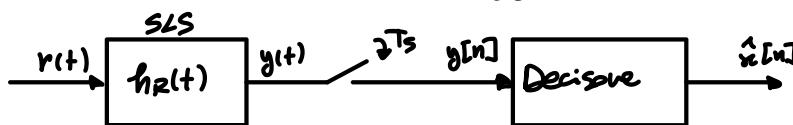
- Valore Medio
- Densità Spettrale di Potenza

$$S_s(f) = \text{DSP}[s(t)]$$

$$P_s(f) = \int_{-\infty}^{\infty} S_s(f) dt$$

Ricevitore

Demodulatore numerico in banda base:



$$P_e(b) = P(\hat{b}[n] \neq b[n])$$

$$P_e(\text{simbolo}) = P(\hat{x}[n] \neq x[n])$$

$$\text{Bit Error Rate} = P_e(b) \cdot \text{bit-rate} \left[\frac{b}{s} \right]_{(R_b)}$$

Calcolo della probabilità di errore sul simbolo

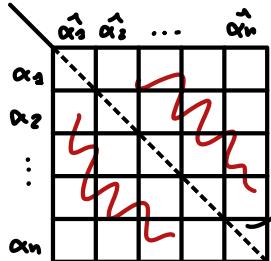
Note:

- Probabilità a priori di trasmissione dei simboli

- Probabilità di transizione

Dal teorema della probabilità totale

$$P_E(M) = P(\hat{x}[n] \neq x[n]) = \sum_{i=1}^M \sum_{\substack{j=1 \\ j \neq i}}^M P(\hat{x}[n] = \alpha_j | x[n] = \alpha_i) P(x[n] = \alpha_i)$$



$P(i|j) \rightarrow$ tutti gli elementi fuori dalla diagonale (simbolo deciso \neq simbolo inviato)

$P(i|i)$

Formato di modulazione equi energia

Energia simbolo trasmesso

$$\alpha_i \Rightarrow E_{S_i} = Es \left|_{x[n]=\alpha_i} \cdot \int_{-\infty}^{\infty} |S_i(t)|^2 dt \right.$$

Siamo in condizioni di equi energia se $E_{S_i} = Es \forall i$

Performance dei sistemi di comunicazione numerici

1) Probabilità d'errore su bit e/o simbolo

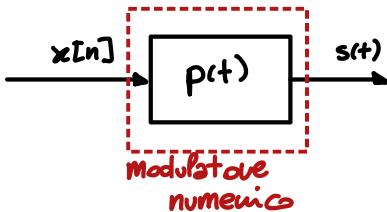
2) Efficienza energetica $\eta_p \triangleq \frac{1}{SNR}$; $SNR = \frac{P_s}{P_n} \rightarrow$ potenza segnale utile / potenza di rumore
Si calcola fissando $P_E(b)$

potenza segnale utile
 $SNR = \frac{P_s}{P_n} \rightarrow$ potenza di rumore
Signal Noise Ratio

3) Efficienza Spettrale: $\eta_B \triangleq \frac{Rb}{B_T}$

Si calcola fissando $P_E(b)$

Pulse Amplitude Modulation (PAM)



Proprietà

1) $s(t) = \sum_n x[n] p(t - nT_s)$

2) M simboli (≥ 2) equiprobabili: $\alpha_i \in As = \{\alpha_1, \dots, \alpha_N\} \Rightarrow \alpha_i = 2i - 1 - M$

Proprietà di $s(t)$

1 $E[s(t)] = E\left[\sum_n x[n] p(t-nT_s)\right] = \sum_n E[x[n]] p(t-nT_s) = \emptyset$, poiché

$$E[x[n]] = \sum_{i=1}^M P[x[n]=\alpha_i] \cdot \alpha_i = \frac{1}{M} \sum_{i=1}^M \alpha_i = \emptyset$$

\hookrightarrow valori medi per
V.A. discrete \hookrightarrow α_i simmetrici

$\Rightarrow s(t)$ è un processo aleatorio a valore medio nullo

2 $S_s(4) = \frac{1}{T_s} \overline{|S_x(4)|^2}$

$$\overline{S_x(4)} = T_s F S [R_x[m]]$$

$$R_x[m] = E[x[n] x[n-m]] = C_x[m] + \eta_x^2 [m]$$

Nel caso della PAM:

$$\eta_x[m] = \emptyset \Rightarrow R_x[m] = C_x[m]$$

\Rightarrow simboli erogati indipendenti \Rightarrow incoppati

Se 2 V.A. indipendenti sono incoppiate:

$$C_x[m] = \sigma_x^2 \cdot S[m]$$

$$C_x[m] = E[(x[n] - \eta_x)(x[n-m] - \eta_x)] , \text{ se } m = \emptyset \Rightarrow E[(x[n] - \eta_x)^2] = \sigma_x^2$$

$$\Rightarrow C_x[m] = \begin{cases} \sigma_x^2 & m = \emptyset \\ \emptyset & m \neq \emptyset \end{cases}$$

$$R_x[m] = C_x[m] = \sigma_x^2 S[m]$$

$$\overline{S_x(4)} = T_s F S [R_x[m]] = \sigma_x^2$$

Quindi, per una PAM:

$$S_s(4) = \frac{\sigma_x^2}{T_s} |P(4)|^2$$

PAM Binaria (2-PAM)

$$M=2$$

$$s(t) = \sum_n x[n] p(t-nT_s)$$

$$x[n] \in A_S = \{-1, 1\}$$

$\checkmark \quad \alpha_i = 2i - 1 - M$

$$T_s = T_b$$

$$Z \cdot PAN \Rightarrow E_{S_1} = \int_{-\infty}^{\infty} (-2)^3 p^2(t-nT_s) dt = \int_{-nT_s}^{t'=t-nT_s} p^2(t') dt' = \bar{E}_p$$

$$E_{S_2} = \int_{-\infty}^{\infty} 2^2 p^2(t-nT_s) dt = \int_{-nT_s}^{t=t-nT_s} p^2(t') dt' = \bar{E}_p$$

$$E[s(t)] = \emptyset$$

$$S_S(f) = \frac{\sigma_n^2}{T_s} |P(f)|^2 = \frac{1}{T_b} \cdot |P(f)|^2$$

$$\sigma_x^2 = E[x^2[n]] = \boxed{1}$$

$$P_S(f) = \int_{-\infty}^{\infty} S_S(f) df = \frac{1}{T_b} \int_{-\infty}^{\infty} |P(f)|^2 df = \frac{E_p}{T_b}$$

$$B_f = B_p$$

$$\eta_B = \frac{R_b}{B_f} = \frac{1}{T_b \cdot B_p}$$

Energia media per simbolo trasmesso

Caso generale PAN

$$\alpha_i \Rightarrow E_{S_i} = \int_{-\infty}^{\infty} \alpha_i^2 p^2(t-nT_s) dt = \alpha_i^2 E_p$$

$$E_S = E \left[\int_{-\infty}^{\infty} x^2[n] p^2(t-nT_s) dt \right]$$

↳ media statistica calcolata su tutti i possibili simboli dell'alfabeto

$$= \int_{-\infty}^{\infty} E[x^2[n]] p^2(t-nT_s) dt$$

$$= E[x^2[n]] \int_{-\infty}^{\infty} p^2(t-nT_s) dt = \boxed{E[x^2[n]] \cdot E_p}$$

Elementi che influiscono le prestazioni dei sistemi di comunicazione numerici

Presenza di interferenza inter-simbolica (ISI)

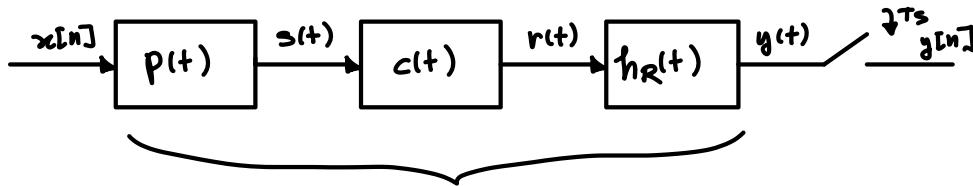


Assenza di ISI: $y[n] = f(x[n])$

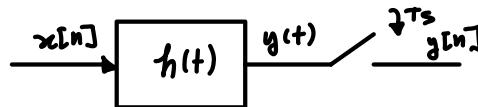
Presenza di ISI: $y[n] = f(\dots, x[n-1], x[n], x[n+1], \dots)$

Alla presenza di ISI contribuiscono:

- $p(t)$
- $c(t)$
- $h_R(t)$



Interpolatore Equivalente



$$Con h(t) = p(t) \otimes c(t) \otimes h_R(t)$$

$$y(t) = \underbrace{\bar{x}(t)}_{S(f)} \underbrace{P(f) C(f) H_R(f)}_{R(f)} \Rightarrow y(t) = \bar{x}(t) H(t) \Rightarrow y(t) = \sum_n x[n] h(t - nT_s)$$

$$\Rightarrow y[k] = y(kT_s) = \sum_n x[n] h((k-n)T_s)$$

$$y[k] = \underbrace{x[k] h(0)}_{\text{componente che dipende da } k\text{-esimo simbolo}} + \sum_{n \neq k} x[n] h((k-n)T_s)$$

↳ ISI

Criterio di Nyquist

① Forma nel Tempo

$$Assenza di ISI \Leftrightarrow h(kT_s) = c \delta[k]$$

$$y[x] = x[k] h(0) + \sum_{n \neq k} x[n] \underbrace{h((k-n)T_s)}_{=0} \Rightarrow \text{No ISI}$$

② Forma in frequenza

$$h(kT_s) = c \cdot \delta[k] \Rightarrow H(f) = c = \frac{1}{T_s} \sum_n H(f - nT_s) \Rightarrow \boxed{\sum_n H(f - nT_s) = c' \text{ costante}}$$

Condizione limite assenza di ISI

T_s : periodo di segnalazione dei simboli

B_c : banda del canale

B_c : banda massima utilizzabile per ottenere il massimo delle prestazioni del sistema

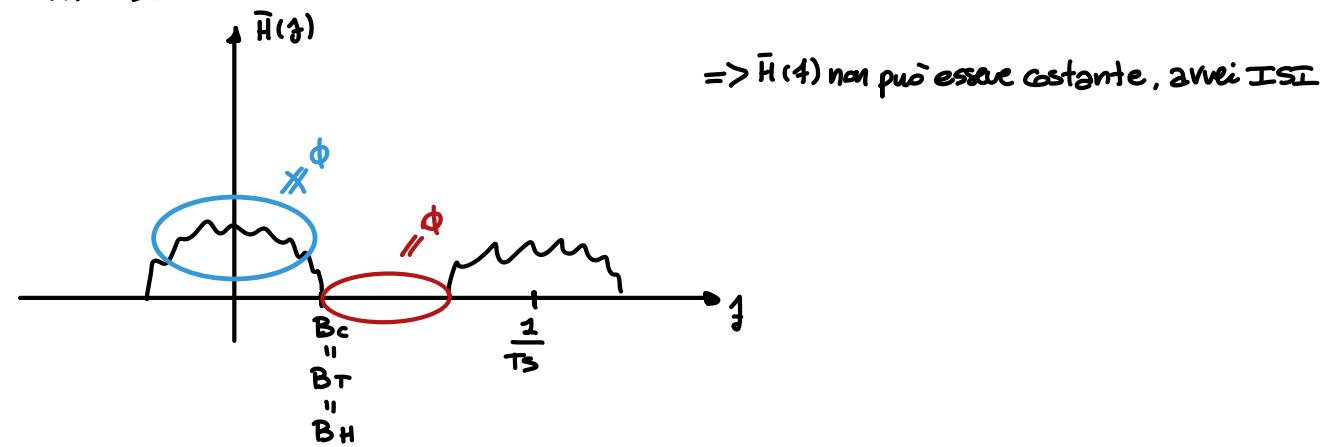
Poniamo $B_T = B_c$

Non è possibile eliminare l'ISI se:

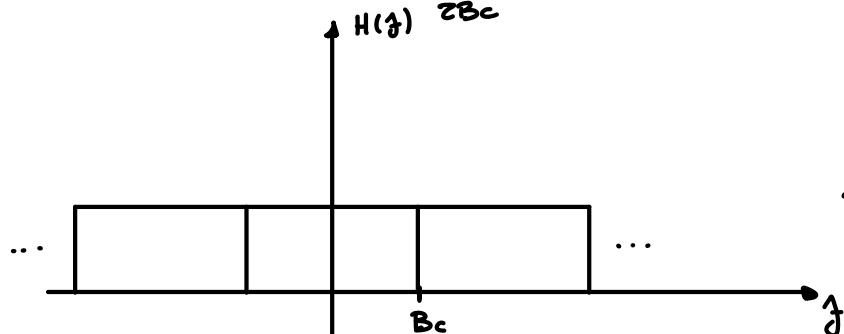
$$T_s < \frac{1}{2B_c}$$

$\bar{H}(f) = C$ per avere modulazione in assenza di ISI

$H(f)$ ha banda B_c



Condizione limite: $T_s = \frac{1}{2B_c}$



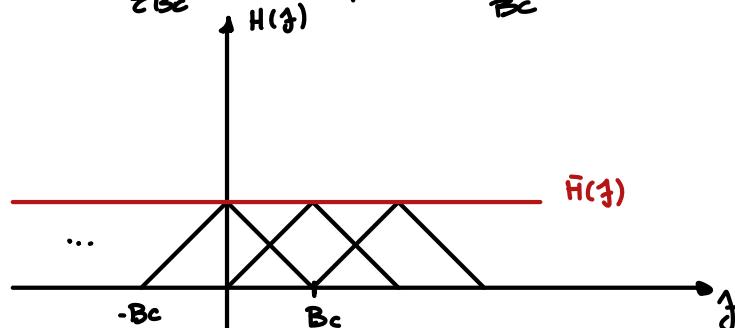
$$T_s = \frac{1}{2B_c} \Rightarrow 1 \text{ sol: } H(f) = C \text{rect}\left(\frac{f}{2B_c}\right)$$

$$p(t), c(t), h_R(t) \Rightarrow h(t) = 2C B_c \text{sinc}(2B_c t)$$

$$h(nT_s) = 2C B_c \text{sinc}\left(2B_c \cdot \frac{n}{2B_c}\right) = 2C B_c S[n]$$

Questa soluzione non è ottimale, poiché per effetto di un campionamento non ideale o per piccoli effetti di distorsioni si introduce un ISI significativo, essendo la sinc una funzione che va a 0 lentamente.

Poniamo $T_s > \frac{1}{2B_c}$, ad esempio $T_s = \frac{1}{B_c}$



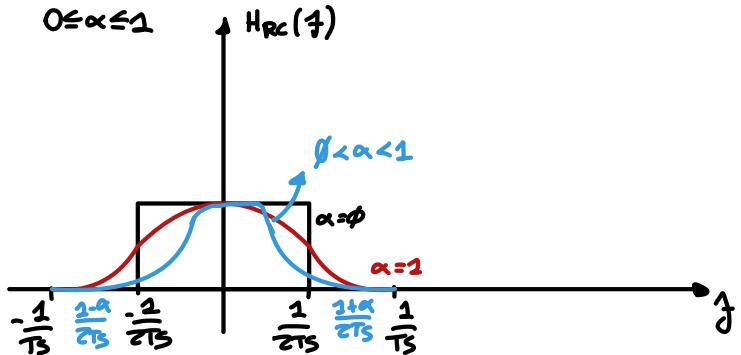
$$H(f) = C \left(1 - \frac{|f|}{B_c}\right) \text{rect}\left(\frac{f}{2B_c}\right)$$

$$h(t) = C B_c \text{sinc}^2(B_c t)$$

Coseno Rialzato

$$H_{RC} \begin{cases} T_S \\ \frac{T_S}{2} \left[1 - \sin \left(\frac{\pi T_S}{\alpha} (1/f) \cdot \frac{1}{e^{fT_S}} \right) \right] \\ \phi \end{cases}$$

$0 \leq |f| \leq \frac{1-\alpha}{2T_S}$
 $\frac{1-\alpha}{2T_S} \leq |f| \leq \frac{1+\alpha}{2T_S}$
 $|f| \geq \frac{1+\alpha}{2T_S}$

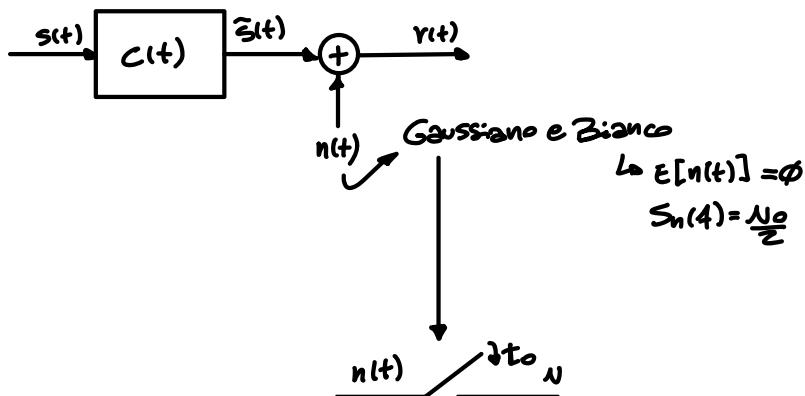


$$\bar{H}_{RC}(f) = \frac{1}{T_S} \sum_n H_{RC}\left(f - \frac{n}{T_S}\right) = C \quad \forall \alpha$$

$$h_{RC}(t) = \text{sinc}\left(\frac{t}{T_S}\right) \frac{\cos\left(\frac{\alpha\pi t}{T_S}\right)}{\left(1 - \frac{2\alpha t}{T_S}\right)^2}$$

$$h_{RC}(t) \rightarrow \phi \sim \frac{1}{t^3}$$

Presenza di Rumore



$$n = n(t_0) \in \mathcal{N}(0, \sigma_n^2)$$

$$r(t) = s(t) \otimes c(t) + n(t) = \sum_n x[n] \tilde{p}(t - nT_S) + n(t) \text{ con } \tilde{p}(t) = p(t) \otimes c(t)$$

$$\begin{aligned} \tilde{s}(t) &= s(t) \otimes c(t) = \sum_n x[n] p(t - nT_S) \otimes c(t) = \int_{-\infty}^{\infty} \sum_n x[n] p(t - nT_S) c(t - \tau) d\tau = \\ &= \sum_n x[n] \int_{-\infty}^{\infty} p(\tau') c[(t - nT_S) - \tau'] d\tau' = \sum_n x[n] [\underbrace{p(t) \otimes c(t)}_{\tilde{p}(t)}] \Big|_{t=t-nT_S} = \sum_n x[n] \tilde{p}(t - nT_S) \end{aligned}$$

$t' = \tau - nT_S$

$$r(t) = \underbrace{\sum_n x[n] \tilde{p}(t - nT_S)}_{\text{componente utile}} + n(t)$$

\hookrightarrow rumore

Filtro Adattato

Filtro ottimo che minimizza l'effetto del rumore.

Massimizza l'SNR ad un determinato istante temporale quando in ingresso:

① È noto il segnale utile

② È presente un rumore Bianco



Segnale Noto

Basta sì a noto a meno della sua ampiezza (ignoro il simbolo)

Ovvero: $\tilde{p}(t)$ Noto

$$h_{FA}(t) = K \cdot \tilde{p}(T_s \cdot t) \Rightarrow H_{FA}(f) = K \tilde{P}^*(f) \cdot e^{-j2\pi f T_s}$$

\downarrow
arbitraria

$$|H_{FA}(f)| = K |\tilde{P}(f)|$$

$$MA \times SNR \stackrel{?}{=} \minima P_e$$

In assenza di ISI il decisore si basa solo su $y[n]$:

$$\hat{x}[n] = f(y[n]): \text{Decisore Single-shot}$$

Strategia di decisione che minimizza la prob. d'errore:

$$P_e \triangleq P(\hat{x}[n] \neq x[n])$$

criterio a min $P_e \Leftrightarrow$ criterio Maximum a Posteriori Probability (MAP)

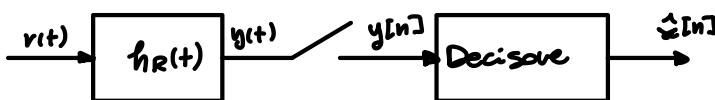
$$\hat{x} = \max_{i=1, \dots, N} P(x=\alpha_i | y)$$

\hookrightarrow campione osservato

Se i simboli sono equiprobabili ho il criterio della massima verosimiglianza:

$$\hat{x} = \max_{i=1, \dots, N} \sum_y P(y | x=\alpha_i)$$

PAM in banda base



Rumore Gaussiano Bianco ed assenza di ISI:

$$y[n] = h(o)x[n] + n_u[n] \text{ con } h(o) = p(t) \otimes c(t) \otimes h_R(t) \Big|_{t=o}$$

$$x[n] \in A_S = \{\alpha_1, \dots, \alpha_N\}$$

$$n_u[n] \in \mathcal{N}(0, \sigma_{n_u}^2); \quad \sigma_{n_u}^2 = P_n = \frac{N_0}{2} E h_R$$

$$\Pr(y| x=\alpha_i) \in \mathcal{N}\left(\underbrace{h(o)\alpha_i}_{h_{nu}}, \underbrace{\frac{N_0}{2} E h_R}_{\sigma_{nu}^2}\right)$$

$$\Pr(y| x=\alpha_i) = \frac{1}{\sqrt{2\pi\sigma_{nu}^2}} \cdot e^{-\frac{(y-h_{nu})^2}{2\sigma_{nu}^2}}$$

→ funzioni di verosimiglianza

$$\min P_e(u) \Rightarrow \max_i \Pr(y| x=\alpha_i)$$

↓
α_i: equiprobabili

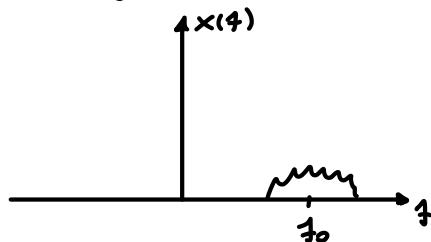
$$\hat{x} = \underset{\alpha_i}{\operatorname{argmax}} \frac{1}{\sqrt{2\pi\sigma_{nu}^2}} \cdot e^{-\frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2}} = \underset{\alpha_i}{\operatorname{argmax}} e^{-\frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2}} = \underset{\alpha_i}{\operatorname{argmax}} \left[-\frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2} \right] =$$

$$= \underset{\alpha_i}{\operatorname{argmin}} \frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2} = \underset{\alpha_i}{\operatorname{argmin}} (y-h(o)\alpha_i)^2 = \boxed{\underset{\alpha_i}{\operatorname{argmin}} |y-h(o)\alpha_i|}$$

Sistemi di comunicazione numerici in banda passante

Segnali passa-banda

È un segnale il cui spettro è contenuto all'interno di un intervallo (Banda) nell'intorno di una frequenza f_0



$$s(t) = a(t) \cos(2\pi f_0 t + \theta(t))$$

↳ fase di $s(t)$

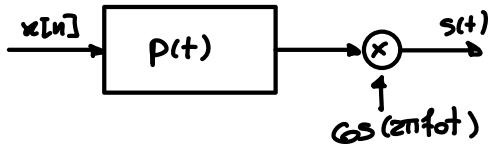
↳ inutopponibile di $s(t)$

$$s(t) = \operatorname{Re} \left\{ \tilde{s}(t) \cdot e^{j(2\pi f_0 t + \theta(t))} \right\} = \operatorname{Re} \left\{ \tilde{s}(t) e^{j2\pi f_0 t} \right\} \text{ con } \tilde{s}(t) = s(t) \cdot e^{j\theta(t)}$$

\hookrightarrow insiluppo complesso di $s(t)$

PAM in Banda Passante

$$s(t) = \sum_n x[n] p(t-nT_s) \cos(2\pi f_0 t)$$



$p(t)$ definito da un segnale noto come nella banda base

$$x[n] \in A_s = \{\alpha_1, \dots, \alpha_M\} \text{ reali; } \alpha_i = z_{i-1-N} \text{ come nella PAM Standard}$$

$$s_i(t) = \alpha_i p(t-nT_s) \cos(2\pi f_0 t)$$

$$\tilde{s}_i(t) = \alpha_i p(t-nT_s)$$

Stesso spazio dei simboli della PAM in banda base

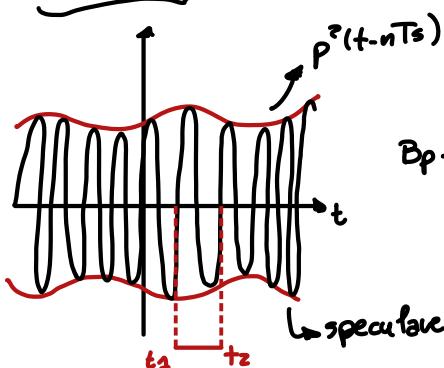
Energia media per simbolo trasmesso

$$S_n(t) = x[n] p(t-nT_s) \cos(2\pi f_0 t)$$

$$E_S = E \left[\int_{-\infty}^{\infty} S_n^2(t) dt \right] = \int_{-\infty}^{\infty} E[x^2[n]] p^2(t-nT_s) \cos^2(2\pi f_0 t) dt =$$

$$= E[x^2[n]] \int_{-\infty}^{\infty} p^2(t-nT_s) \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi f_0 t) \right] dt =$$

$$= \frac{1}{2} E[x^2[n]] E_p + \frac{1}{2} E[x^2[n]] \underbrace{\int_{-\infty}^{\infty} p^2(t-nT_s) \cos(4\pi f_0 t) dt}_{B_p \ll T_0 \Rightarrow \int_{-\infty}^{\infty} p^2(t-nT_s) \cos(4\pi f_0 t) dt \approx 0}$$



$$B_p \ll T_0 \Rightarrow \int_{-\infty}^{\infty} p^2(t-nT_s) \cos(4\pi f_0 t) dt \approx 0$$

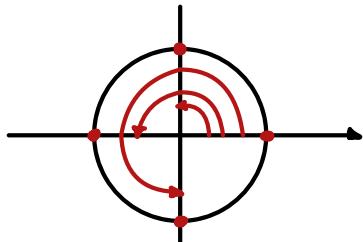
$$\Delta t = K \cdot \frac{T_0}{f_0}$$

$\int_{t_1}^{t_2} (\cdot) \approx 0$

$$\Rightarrow E_S = \frac{1}{2} E[x^2[n]] E_p$$

Phase Shifting Key (PSK)

$$s(t) = \sum_n p(t-nT_s) \cos(2\pi f_0 t + \theta[n]) \text{ con } \theta[n] \in A_S = \{\theta_1, \dots, \theta_M\}$$



$$\theta_i = \frac{\pi}{M} \cdot (i-1)$$

$$\tilde{s}_i(t) = p(t-nT_s) e^{j\theta_i}$$

Ri prova:

$$\operatorname{Re}\{\tilde{s}_i(t) \cdot e^{j2\pi f_0 t}\} = \operatorname{Re}\{p(t-nT_s) \cdot e^{j(2\pi f_0 t + \theta_i)}\} = p(t-nT_s) \cos(2\pi f_0 t + \theta_i)$$

Quadrature Amplitude Modulation (QAM)

$$s(t) = \sum_n x_c[n] p(t-nT_s) \cos(2\pi f_0 t) - x_s[n] p(t-nT_s) \sin(2\pi f_0 t)$$

$$s_n(t) = x_c[n] p(t-nT_s) \cos(2\pi f_0 t) - x_s[n] p(t-nT_s) \sin(2\pi f_0 t)$$

$$\tilde{s}_n(t) = \underbrace{(x_c[n] + j x_s[n])}_{\text{simbolo}} p(t-nT_s)$$

$$\operatorname{Re}\{\tilde{s}_n(t) \cdot e^{j2\pi f_0 t}\} = \operatorname{Re}\{(x_c[n] + j x_s[n]) p(t-nT_s) e^{j2\pi f_0 t}\} =$$

$$\operatorname{Re}\{x_c[n] p(t-nT_s) \cos(2\pi f_0 t) + j x_c[n] p(t-nT_s) \sin(2\pi f_0 t) + \\ + j x_s[n] p(t-nT_s) \cos(2\pi f_0 t) - x_s[n] p(t-nT_s) \sin(2\pi f_0 t)\}$$

$$x_c[n] \in A_S^{(c)} = \{\alpha_1^{(c)}, \dots, \alpha_{M_c}^{(c)}\}$$

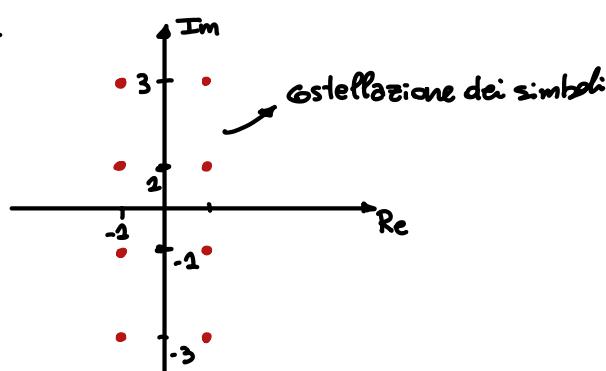
$$x_s[n] \in A_S^{(s)} = \{\alpha_1^{(s)}, \dots, \alpha_{M_s}^{(s)}\}$$

In generale $M_c \neq M_s$

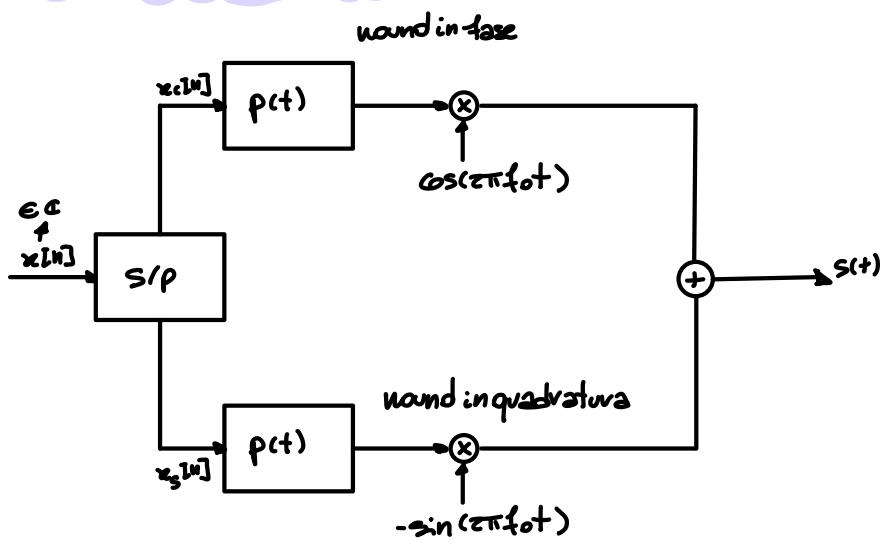
QAM Standard

$$\alpha_i^{(c)} = 2i-1-M_c$$

$$\alpha_i^{(s)} = 2i-1-M_s$$



Schema a blocchi di un modulatore QAM



Energia media per simbolo trasmesso

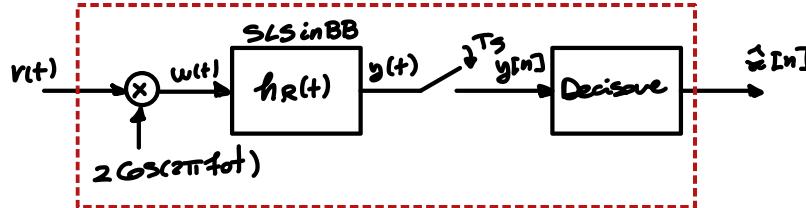
$$\begin{aligned}
 E_s &= E \left[\int_{-\infty}^{\infty} (x_c[n] p(t-nT_s) \cos(2\pi f_0 t) - x_s[n] p(t-nT_s) \sin(2\pi f_0 t))^2 dt \right] = \\
 &= \int_{-\infty}^{\infty} E[x_c^2[n]] p^2(t-nT_s) \cos^2(2\pi f_0 t) dt + \int_{-\infty}^{\infty} E[x_s^2[n]] p^2(t-nT_s) \sin^2(2\pi f_0 t) dt + \\
 &\quad - \int_{-\infty}^{\infty} E[x_c[n] x_s[n]] p^2(t-nT_s) \sin(2\pi f_0 t) \cos(2\pi f_0 t) dt = \\
 &= \frac{1}{2} E[x_c^2[n]] E_p + \frac{1}{2} E[x_s^2[n]] E_p - \underline{\phi} \\
 &\quad \xrightarrow{\text{①}} E[x_c[n] x_s[n]] = \phi \text{ poiché } x_c \text{ e } x_s \text{ sono indipendenti:} \\
 &\quad \xrightarrow{\text{②}} \int_{-\infty}^{\infty} p^2(t-nT_s) \frac{2 \sin(2\pi f_0 t) \cos(2\pi f_0 t)}{\sin(4\pi f_0 t)} dt = \phi
 \end{aligned}$$

$$E_s = \frac{1}{2} [E[x_c^2[n]] + E[x_s^2[n]]] E_p$$

DSP per PAM B.P. e QAM

$$S_s(f) = \frac{1}{T} \sum_{n=1}^N S_{x_n}(f) \cdot \left[p^2(f-f_0) + p^2(f+f_0) \right]$$

Dicenatore di una PAM in B.P.



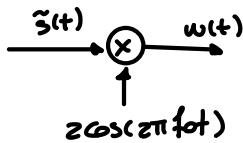
Demodulatore numerico in BB

$$r(t) = \tilde{s}(t) + n(t) = s(t) \otimes c(t) + n(t)$$

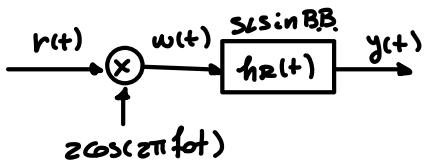
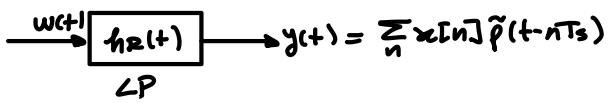
Caso in assenza di rumore

$$\begin{aligned}
 s(t) &= \sum_n x[n] p(t-nT_s) \cos(2\pi f_0 t) \\
 \tilde{s}(t) &= \int_{-\infty}^{\infty} \sum_n x[n] p(t-nT_s) \cos(2\pi f_0 \tau) c(t-\tau) d\tau = \\
 &\quad c(t) \triangleq 2 \tilde{c}(t) \cos(2\pi f_0 t) \\
 &= 2 \sum_n x[n] \int_{-\infty}^{\infty} p(t-nT_s) \tilde{c}(t-\tau) \cos(2\pi f_0 \tau) \cos(2\pi f_0 (t-\tau)) d\tau = \\
 &= \sum_n x[n] \int_{-\infty}^{\infty} p(t-nT_s) \tilde{c}(t-\tau) [\cos(2\pi f_0 t) + \cos(4\pi f_0 \tau - 2\pi f_0 t)] d\tau = \\
 &= \sum_n x[n] \cos(2\pi f_0 t) \underbrace{\int_{-\infty}^{\infty} p(t-nT_s) \tilde{c}(t-\tau) d\tau}_{\approx \phi} + \sum_n x[n] \underbrace{\int_{-\infty}^{\infty} p(t-nT_s) \tilde{c}(t-\tau) \cos(4\pi f_0 \tau + \kappa) d\tau}_{\approx 0} \\
 \Rightarrow \tilde{s}(t) &= \sum_n x[n] \int_{-\infty}^{\infty} p(t-nT_s) \tilde{c}(t-\tau) d\tau \cos(2\pi f_0 t) = \\
 &\quad p(t-nT_s) \otimes c(t) \triangleq \tilde{p}(t-nT_s) \\
 &= \sum_n x[n] \tilde{p}(t-nT_s) \cos(2\pi f_0 t)
 \end{aligned}$$

$\tilde{s}(t)$ ha la struttura di un segnale PAM in B.P.



$$\begin{aligned}
 w(t) &= z \tilde{s}(t) \cos(2\pi f_0 t) = \sum_n x[n] \tilde{p}(t-nT_s) z \cos^2(2\pi f_0 t) = \\
 &\quad \underbrace{\sum_n x[n] \tilde{p}(t-nT_s)}_{B.B.} + \underbrace{\sum_n x[n] \tilde{p}(t-nT_s) \cos(4\pi f_0 t)}_{\text{componente a } 2f_0}
 \end{aligned}$$



$y(t) = \sum_n x[n] \tilde{p}(t - nT_s) \text{ con } \tilde{p}(t) = p(t) \otimes \tilde{c}(t) \otimes h_R(t)$

$r(t) = \sum_n x[n] \tilde{p}(t - nT_s) + n(t)$

\hookrightarrow per ora lo trascuriamo

$\tilde{p}(t) = p(t) \otimes \tilde{c}(t)$

$c(t) = z \tilde{c}(t) \cos(2\pi f_0 t)$

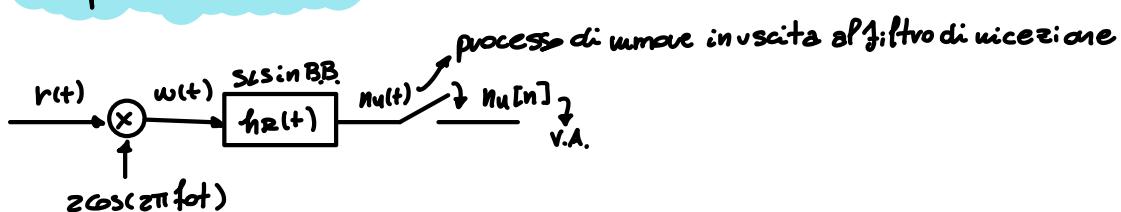
$y(t)$ è identico a quello della BB, eccetto che $\tilde{c}(t)$ sostituisce $c(t)$

$y[n] = \sum_k x[k] \tilde{p}(nT_s - kT_s) + \text{rumore}$

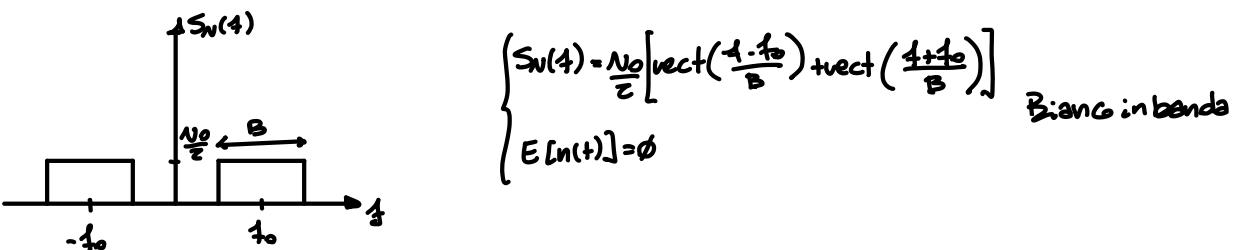
$y[n] = h(0)x[n] + \underbrace{\sum_{k \neq n} x[k] \tilde{p}((n-k)T_s)}_{\text{ISI}} + \text{rumore}$

L'assenza di ISI si verifica ancora con il criterio di Nyquist

Componente di rumore



$n(t)$ Gaussiano e Bianco in banda



$R_n(\tau) \xrightarrow{\text{TF}} S_n(f)$

$R_n(\tau) = \frac{N_0}{2} B \text{sinc}(B\tau) \left[e^{j2\pi f_0 \tau} + e^{-j2\pi f_0 \tau} \right] = N_0 B \text{sinc}(B\tau) \cos(2\pi f_0 \tau), \quad \tau = t_2 - t_1$

$$w(t) = z n(t) \cos(2\pi f_0 t)$$

$$\begin{aligned}
 R_w(t_1, t_2) &\stackrel{d}{=} E[w(t_1) w(t_2)] = 4E[n(t_1) n(t_2)] \underbrace{\cos(2\pi f_0 t_1) \cos(2\pi f_0 t_2)}_{=} = \\
 &= 4N_0 B \text{sinc}(B(t_1 - t_2)) \cos(2\pi f_0(t_1 - t_2)) \left[\frac{1}{2} \cos(2\pi f_0(t_1 + t_2)) + \frac{1}{2} \cos(2\pi f_0(t_1 - t_2)) \right] = \\
 &= 2N_0 B \text{sinc}(B(t_1 - t_2)) \left[\frac{1}{2} \cos(4\pi f_0 t_1) + \frac{1}{2} \cos(4\pi f_0 t_2) + \frac{1}{2} + \frac{1}{2} \cos(4\pi f_0(t_1 + t_2)) \right] = \\
 &= N_0 B \text{sinc}(B(t_1 - t_2)) \underbrace{\left[1 + \cos(4\pi f_0 t_1) + \cos(4\pi f_0 t_2) + \cos(4\pi f_0(t_1 + t_2)) \right]}_{\text{Componenti a } 2f_0}
 \end{aligned}$$

$R_{nu}(t_1, t_2) = R_w(t_1, t_2) \otimes h_R(t_1) \otimes h_R(t_2)$; h_R è un LP che taglia le componenti a $2f_0$, quindi:

$$R_{nu}(t_1, t_2) = N_0 B \text{sinc}(B\tau) \otimes h(\tau) \otimes h(-\tau) = R_{nu}(\tau) \Rightarrow n_u(t) \text{ è SS}$$

Per la FAN Standard

$$S_n(t) = \frac{1}{T} \frac{\sigma_n^2}{T} (P^2(t-f_0) + P^2(t+f_0))$$

Nel passaggio da B.B. a B.P. la banda raddoppia

Consideriamo il segnale in B.B. di banda $B \Rightarrow h_R(t)$ ha banda B

$$\Rightarrow S_{nu}(t) = N_0 \text{rect}\left(\frac{t}{2B}\right) |H_R(t)|^2 = N_0 |H_R(t)|^2$$

Il risultato è lo stesso che si ottiene con un mixer in BB con DSP del rumore in ingresso doppia

$$P_{nu} = \int_{-\infty}^{\infty} S_{nu}(t) dt = \int_{-\infty}^{\infty} N_0 |H_R(t)|^2 dt = N_0 E[h_R] \quad \text{Doppia rispetto a quella in B.B.}$$

$$E[w(t)] = E[n(t) z \cos(2\pi f_0 t)] = \phi \quad \hookrightarrow E[n(t)] = \phi$$

$$\eta_{n_u}(t) = \eta_w(t) \otimes h_R(t) = \phi$$

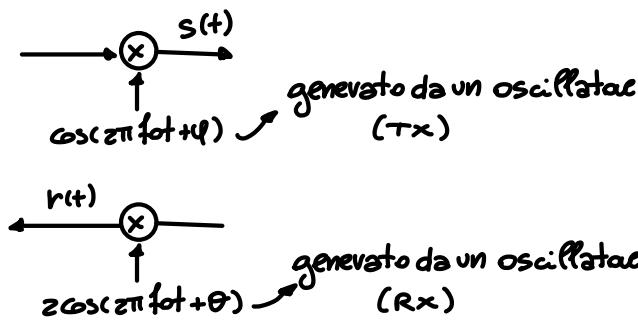
$n_u(t)$ è Gaussiano

$$n_u(t) \Big|_{t=nTs} = n_u[n] \in \mathcal{N}(0, P_{nu})$$

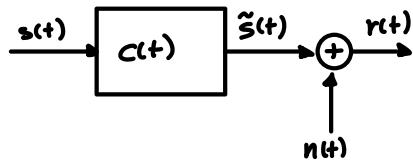
Sincronismo di fase

Problema che si verifica in B.P.

In generale $\varphi \neq \theta$



$$s(t) = \sum_n x[n] p(t-nT_s) \cos(2\pi f_0 t + \varphi)$$



$$\begin{aligned} \tilde{s}(t) &= \int_{-\infty}^{\infty} \sum_n x[n] p(\tau - nT_s) \cos(2\pi f_0 \tau + \varphi) \cdot z \tilde{c}(t - \tau) \cos(2\pi f_0 (t - \tau)) d\tau \\ &= \sum_n x[n] \int_{-\infty}^{\infty} p(\tau - nT_s) \tilde{c}(t - \tau) [\cos(2\pi f_0 t + \varphi) + \cos(4\pi f_0 \tau - 2\pi f_0 t + \varphi)] d\tau \\ &= \sum_n x[n] \cos(2\pi f_0 t + \varphi) \cdot \tilde{p}(t - nT_s) + \underbrace{\sum_n x[n] \int_{-\infty}^{\infty} p(\tau - nT_s) \tilde{c}(t - \tau) \cos(4\pi f_0 \tau + \kappa) d\tau}_{\rightarrow \phi} \\ \Rightarrow \tilde{s}(t) &= \sum_n x[n] \tilde{p}(t - nT_s) \cos(2\pi f_0 t + \varphi) \end{aligned}$$

$$\begin{aligned} w(t) &= \tilde{s}(t) \cdot 2 \cos(2\pi f_0 t + \theta) = \sum_n x[n] \tilde{p}(t - nT_s) [\cos(\varphi - \theta) + \cos(4\pi f_0 t + \varphi + \theta)] \\ &= \sum_n x[n] \tilde{p}(t - nT_s) \cos(\varphi - \theta) + \underline{\text{termine a } 2f_0}, \\ &\quad \hookrightarrow \text{filtrato da } h_R(t) \end{aligned}$$

$$\begin{aligned} y(t) &= \sum_n x[n] \tilde{p}(t - nT_s) \underbrace{\cos(\varphi - \theta)}_{\leq 1} + n_u(t) \\ &\quad \hookrightarrow \text{potrei ottenere } \varphi - \theta = \pi/2 \text{ e perdere l'informazione del simbolo} \end{aligned}$$

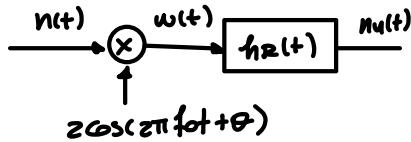
$$y[n] = h(n) x[n] \cos(\varphi - \theta) + \underbrace{\sum_{k \neq n} y[k] h((k-n)T_s) \cos(\varphi - \theta)}_{\text{SSI}}$$

Possiamo usare il criterio di Nyquist poiché la condizione sull'SSI non dipende da φ e θ

Supponiamo assenza di ISI:

$$\begin{aligned} y[n] &= x[n] h(\omega) \cos(\varphi - \theta) + n_u[n] \\ h'(\omega) &\triangleq h(\omega) \cos(\varphi - \theta) \\ &= x[n] h'(\omega) + n_u[n] \end{aligned}$$

Rumore in presenza di una fase nell'oscillazione del demodulatore



$$w(t) = h(t) \cdot 2 \cos(2\pi f_0 t + \theta)$$

$$\begin{aligned} P_{ww}(t_1, t_2) &= E[w(t_1)w(t_2)] = 4E[n(t_1)n(t_2)] \cos(2\pi f_0 t_1 + \theta) \cos(2\pi f_0 t_2 + \theta) = \\ &= 4E[n(t_1)n(t_2)] \left[\frac{1}{2} \cos(2\pi f_0(t_1+t_2) + 2\theta) + \frac{1}{2} \cos(2\pi f_0(t_2-t_1)) \right] \\ E[n(t_1)n(t_2)] &= R_n(t_1-t_2) = zB \operatorname{sinc}(zB(t_2-t_1)) \cos(2\pi f_0(t_2-t_1)) \end{aligned}$$

$$\begin{cases} \cos(4\pi f_0 t_1 + 2\theta) \\ \cos(4\pi f_0 t_2 - 2\theta) \\ \cos(4\pi f_0(t_2-t_1)) \end{cases} \quad \text{filtre}$$

[1]

$$\Rightarrow S_{n_u}(f) = N_0 \operatorname{rect}\left(\frac{f}{z_B}\right) |H_R(f)|^2 \text{ indip da } \theta$$

$$\Rightarrow P_{n_u} = N_0 E_{H_R}$$

$$\begin{aligned} y[n] &= x[n] h'(\omega) + n_u[n] \\ &\quad \hookrightarrow \text{si riduce se } \varphi \neq \theta \\ \text{IF SNR diminuisce se } &\varphi \neq \theta \end{aligned}$$

QAM

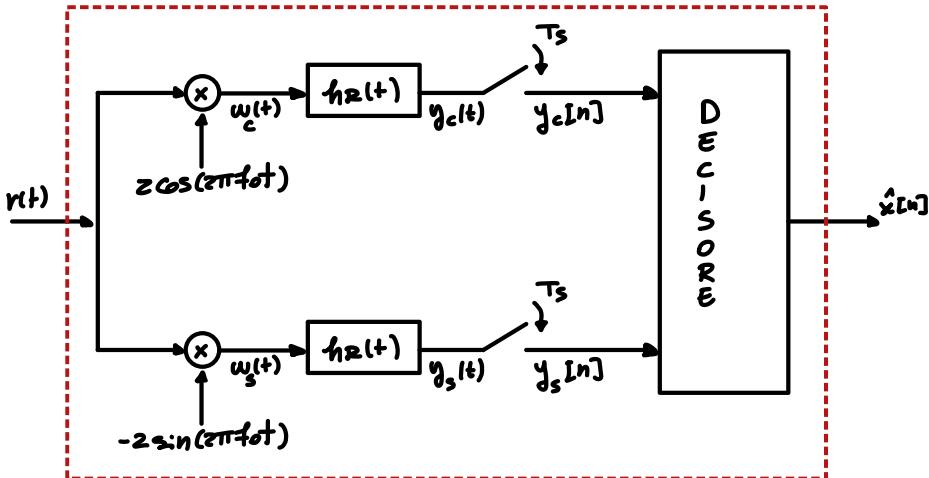
$$s(t) = \sum_n x_c[n] p(t-nT_s) \cos(2\pi f_0 t) - x_s[n] p(t-nT_s) \sin(2\pi f_0 t)$$

$$r(t) = \underbrace{s(t)}_{\tilde{s}(t)} \otimes \tilde{c}(t) + n(t)$$

$$\hat{p}(t) = p(t) \otimes \tilde{c}(t)$$

$$\tilde{s}(t) = \sum_n x_c[n] \hat{p}(t-nT_s) \cos(2\pi f_0 t) - x_s[n] \hat{p}(t-nT_s) \sin(2\pi f_0 t)$$

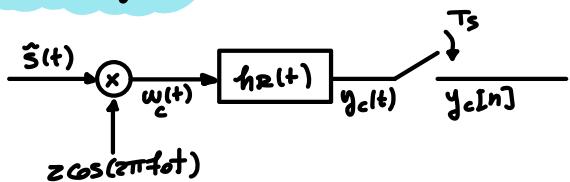
$$r(t) = \sum_n x_c[n] \hat{p}(t-nT_s) \cos(2\pi f_0 t) - x_s[n] \tilde{p}(t-nT_s) \sin(2\pi f_0 t) + n(t)$$



Demodulatore Numerico QAM

$$\tilde{x}[n] = x_c[n] + j x_s[n]$$

Ramo in fase



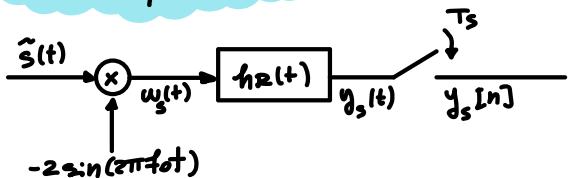
$$\tilde{s}(t) = \sum_n x_c[n] \tilde{p}(t-nT_s) \cos(2\pi f_0 t) - x_s[n] \tilde{p}(t-nT_s) \sin(2\pi f_0 t)$$

$$\begin{aligned} w_c(t) &= \sum_n [x_c[n] \tilde{p}(t-nT_s) \cos(2\pi f_0 t) 2 \cos(2\pi f_0 t) \\ &\quad - x_s[n] \tilde{p}(t-nT_s) \sin(2\pi f_0 t) 2 \cos(2\pi f_0 t)] \\ &= \sum_n x_c[n] \tilde{p}(t-nT_s) [1 + \cos(4\pi f_0 t)] - x_s[n] \tilde{p}(t-nT_s) \sin(4\pi f_0 t) \end{aligned}$$

$$y_c(t) = w_c(t) \otimes h_R(t) = \sum_n x_c[n] h(t-nT_s) \text{ con } h(t) = \tilde{p}(t) \otimes h_R(t)$$

$y_c(t)$ dipende solo da $x_c[n]$

Ramo in quadratura



$$\begin{aligned} w_c(t) &= \sum_n [x_c[n] \tilde{p}(t-nT_s) \cos(2\pi f_0 t) (-2 \sin(2\pi f_0 t))] \\ &\quad - x_s[n] \tilde{p}(t-nT_s) \sin(2\pi f_0 t) (-2 \sin(2\pi f_0 t)) \\ &= \sum_n -x_c[n] \tilde{p}(t-nT_s) \sin(4\pi f_0 t) + x_s[n] \tilde{p}(t-nT_s) [1 - \cos(4\pi f_0 t)] \end{aligned}$$

$$y_s(t) = \sum_n x_s[n] \tilde{p}(t-nT_s) \otimes h_R(t) = \sum_n x_s[n] h(t-nT_s)$$

A parità di banda usata, la QAM ha efficienza spettrale doppia rispetto alla PAN in B.P.

FASE

$$\underline{y_c(t)} \xrightarrow{T_s} \underline{y_c[k]} = \sum_n x_c[n] h((k-n)T_s)$$

QUADRATURA

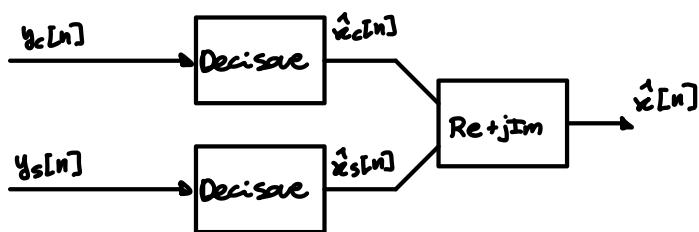
$$\underline{y_s(t)} \xrightarrow{T_s} \underline{y_s[k]} = \sum_n x_s[n] h((k-n)T_s)$$

$$y_c[n] = h(\sigma) x_c[k] + \underbrace{\sum_n x_c[n] h((k-n)T_s)}_{\text{ISI}}$$

$$y_s[n] = h(\sigma) x_s[k] + \underbrace{\sum_n x_s[n] h((k-n)T_s)}_{\text{ISI}}$$

$$\hat{x}[n] = \begin{pmatrix} \hat{x}_c[n] \\ \hat{x}_s[n] \end{pmatrix} = \hat{x}_c[n] + j \hat{x}_s[n]$$

Decisore



P_E nella 4-QAM

$$x_c[n] \in A_S^{(c)} = \left\{ \alpha_1^{(c)}, \alpha_2^{(c)} \right\}$$

$$x_s[n] \in A_S^{(s)} = \left\{ \alpha_1^{(s)}, \alpha_2^{(s)} \right\}$$

1 Verifica assenza ISI su entrambi i canali

$h(t)$ è identico sui 2 canali, la condizione di Nyquist vale su entrambi.

In assenza di ISI:

$$y_c[n] = h(\sigma) x_c[n] + n_c[n]$$

$$y_s[n] = h(\sigma) x_s[n] + n_s[n]$$

$$P_E^{(c)} = P(\hat{x}_c \neq x_c) = P(\hat{x}_c = \alpha_2^{(c)} | x_c = \alpha_2^{(c)}) P(x_c = \alpha_2^{(c)}) + P(\hat{x}_c = \alpha_1^{(c)} | x_c = \alpha_1^{(c)}) P(x_c = \alpha_1^{(c)})$$

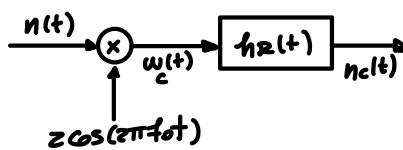
$$P_E^{(s)} = P(\hat{x}_s \neq x_s) = P(\hat{x}_s = \alpha_2^{(s)} | x_s = \alpha_2^{(s)}) P(x_s = \alpha_2^{(s)}) + P(\hat{x}_s = \alpha_1^{(s)} | x_s = \alpha_1^{(s)}) P(x_s = \alpha_1^{(s)})$$

$$P_E = 1 - (1 - P_E^{(c)}) (1 - P_E^{(s)})$$

Probabilità d'errore sul simbolo

Rimane

Ramo in fase



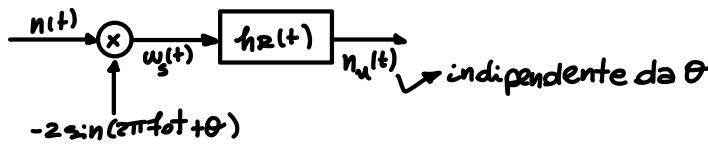
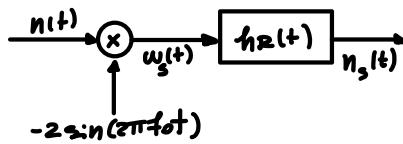
$$S_{nc}(t) = \text{No vect} \left(\frac{1}{2B} \right)$$

$$E[n(t)] = \emptyset$$

$$P_{nc} = \text{No ESR}$$

Noto dallo studio della PAN in B.P.

Ramo in quadratura



$$-z \sin(2\pi f_0 t + \theta) = z \cos(2\pi f_0 t); \theta = \pi/2$$

$$\Rightarrow S_{ns} = \text{No vect} \left(\frac{1}{2B} \right)$$

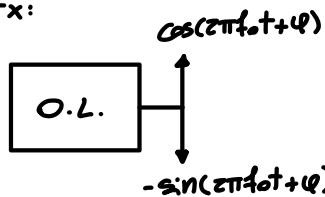
$$E[n_s(t)] = \emptyset$$

$$P_{ns} = \text{No ESR}$$

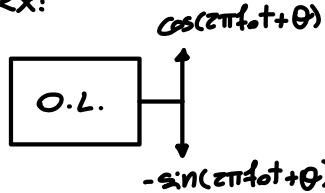
$$\Rightarrow P_{ns} = P_{nc} \text{ ma } n_c[n] \neq n_s[n]!$$

Sincronismo di fase per una QAM

TX:

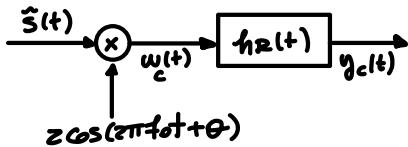


RX:



$$\tilde{s}(t) = \sum_n x_c[n] \tilde{p}(t-nT_s) \cos(2\pi f_0 t + \varphi) - x_s[n] \tilde{p}(t-nT_s) \sin(2\pi f_0 t + \varphi)$$

In fase



$$W_c(t) = \sum_n x_c[n] \tilde{p}(t-nT_s) \cos(2\pi f_0 t + \varphi) Z \cos(2\pi f_0 t + \theta) +$$

$$- x_s[n] \tilde{p}(t-nT_s) \sin(2\pi f_0 t + \varphi) Z \cos(2\pi f_0 t + \theta)$$

$$= \sum_n x_c[n] \tilde{p}(t-nT_s) [\cos(\varphi - \theta) + \cos(4\pi f_0 t + \varphi + \theta)] +$$

$$- x_s[n] \tilde{p}(t-nT_s) [\sin(\varphi - \theta) + \sin(4\pi f_0 t + \varphi + \theta)]$$

$$y_c(t) = \underbrace{\sum_n x_c[n] \tilde{p}(t-nT_s) \cos(\varphi - \theta)}_{\text{componente utile attenuata}} - \underbrace{\sum_n x_s[n] \tilde{p}(t-nT_s) \sin(\varphi - \theta)}_{\text{componente di cross-talk}}$$

Nel ramo in quadratura avviene lo stesso, mutatis mutandis.