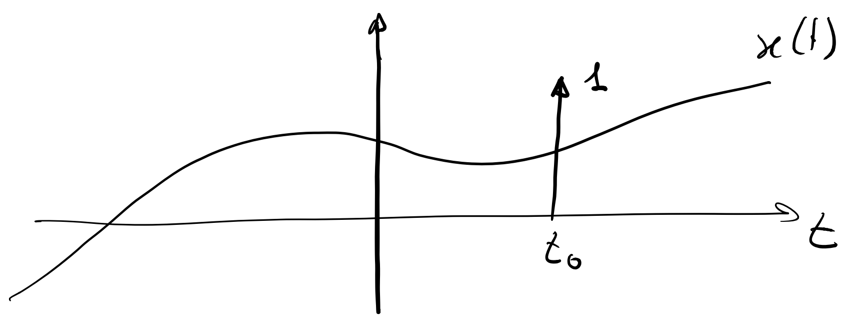
PROPRIETÀ INTEGRALI DELLA DELTA DI DIRAC

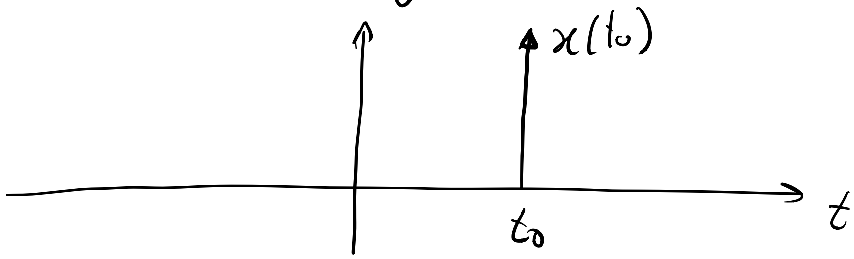
-) TRASLAZIONE

$$\hookrightarrow \int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

$$\hookrightarrow \underline{x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)}$$




$\Downarrow$




5) CONVOLUZIONE

$$x(t) \otimes \delta(t) = x(t)$$

per la parità

Dim

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} x(\tau)\,\delta(\tau-t)\,d\tau$$

$$= x(t)$$

per la proprietà campionatrice

$\Rightarrow$ La Delta di Dirac è l'elemento neutro per l'operatore convoluzione

$$\Rightarrow \boxed{x(t) \otimes \delta(t-t_0) = x(t-t_0)}$$

Dim

$$\int_{-\infty}^{+\infty} x(\tau)\, \delta\left[(t-t_0)-\tau\right] d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, \delta\left[\tau - \underbrace{(t-t_0)}_{t'}\right] d\tau = x(t') = x(t-t_0)$$

6) Campionamento con cambio di scala

$$\int_{-\infty}^{+\infty} x(t)\, \delta(\alpha t)\, dt = \frac{x(0)}{|\alpha|} \quad (\alpha \neq 0)$$

Dim

$\alpha > 0$

$$\int_{-\infty}^{+\infty} x(t)\, \delta(\alpha t)\, dt \qquad \alpha t = t'$$

$$= \int_{-\infty}^{+\infty} x\left(\frac{t'}{\alpha}\right) \delta(t') \frac{dt'}{\alpha} = \underbrace{\frac{1}{\alpha} x(0)} \qquad (\alpha > 0)$$

$\alpha < 0$

$$\int_{-\infty}^{+\infty} x(t)\, \delta(\alpha t)\, dt \qquad -|\alpha| t = t'$$

$$= \int_{+\infty}^{-\infty} x\left(\frac{t'}{\alpha}\right) \delta(t') \left(-\frac{dt'}{|\alpha|}\right) = \frac{1}{|\alpha|} \int_{-\infty}^{+\infty} x\left(\frac{t'}{\alpha}\right) \delta(t')\, dt'$$

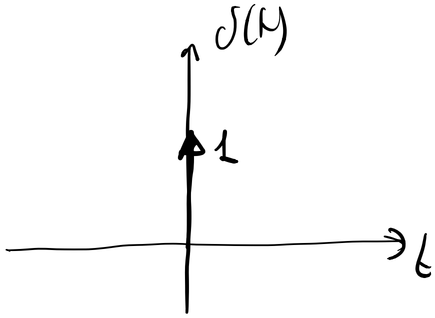
$$= \underbrace{\frac{1}{|\alpha|} x(0)} \qquad (\alpha < 0)$$

$$= \frac{1}{|\alpha|} x(0)$$

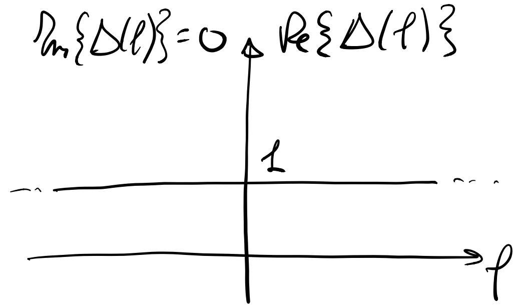
∘) TRASFORMATA CONTINUA DI FOURIER DELLA DELTA DI DIRAC

$$\boxed{\delta(t) \overset{TCF}{\Longleftrightarrow} 1 \quad \forall f}$$

$$\int_{-\infty}^{+\infty} \delta(t)\, e^{-j 2\pi f t}\, dt = \int_{-\infty} \underbrace{e^{-j 2\pi f t}}_{x(t)} \delta(t)\, dt = 1$$

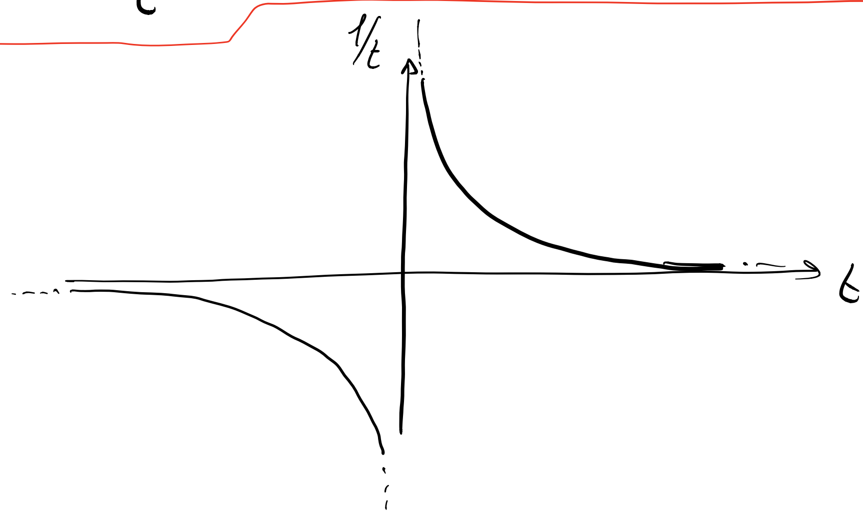


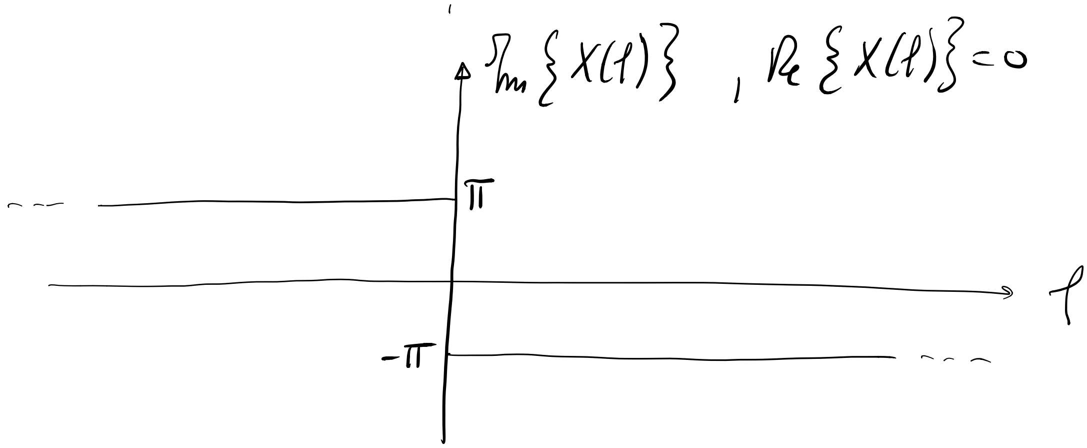

TCF $\iff$




·) TRASFORMATA DEL SEGNALE $\frac{1}{t}$

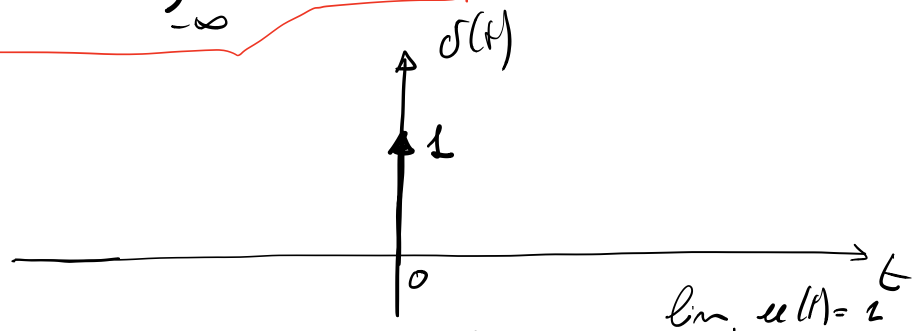
$$\boxed{x(t) = \frac{1}{t} \iff X(f) = -j\pi \,\mathrm{sgn}(f)}$$



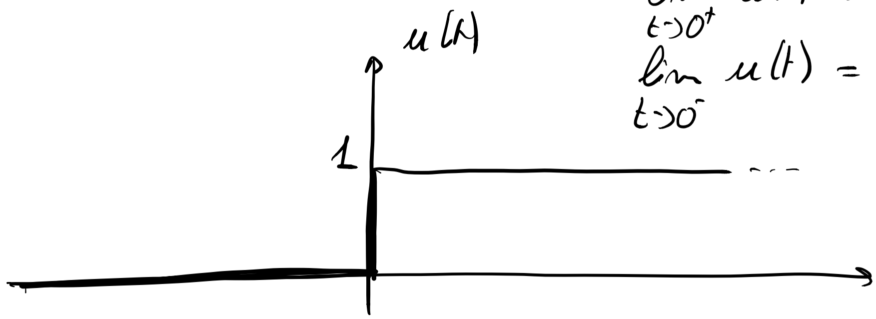
-) RELAZIONE TRA IL GRADINO $u(t)$ E LA DIRAC $\delta(t)$

$$\boxed{u(t) = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha}$$

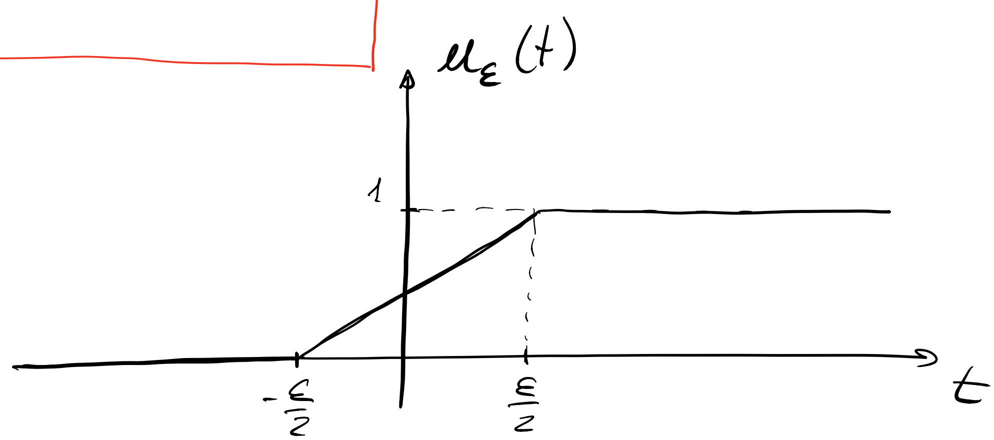



$\lim_{t \to 0^+} u(t) = 1$

$\lim_{t \to 0^-} u(t) = 0$

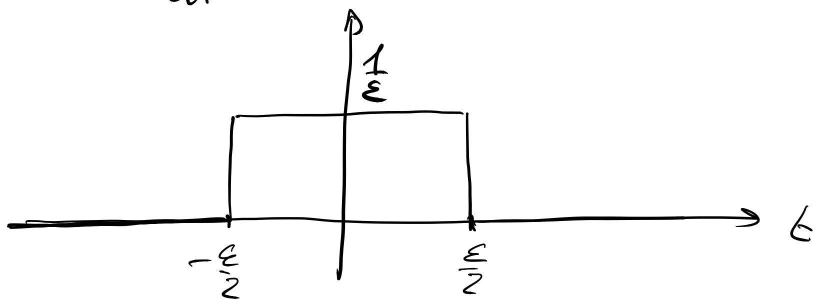



$$\boxed{\frac{d}{dt} u(t) = \delta(t)}$$




$$u(t) = \lim_{\varepsilon \to 0} u_\varepsilon(t)$$

$$\delta_\varepsilon(t) = \frac{d}{dt} u_\varepsilon(t)$$




$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{d}{dt} u_\varepsilon(t)$$

$$= \frac{d}{dt} \lim_{\varepsilon \to 0} u_\varepsilon(t) = \frac{d}{dt} u(t)$$

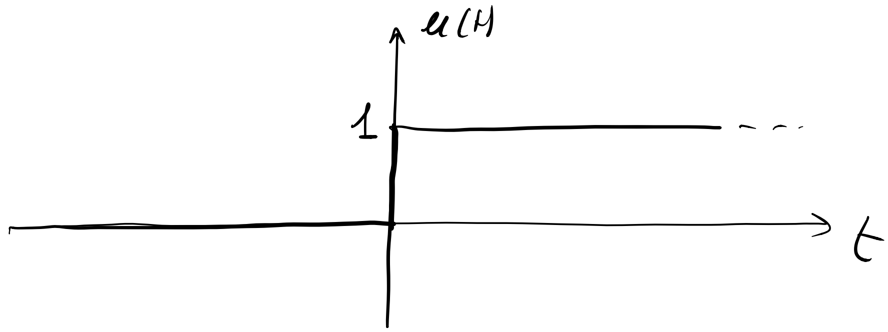
·) TRASFORMATA DEL GRADINO

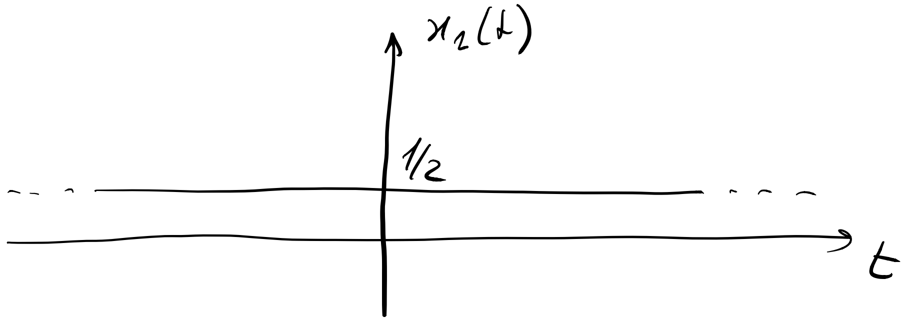
$$U(f) = \int_{-\infty}^{+\infty} u(t)\, e^{-j2\pi f t}\, dt = \int_0^{+\infty} e^{-j2\pi f t}\, dt$$

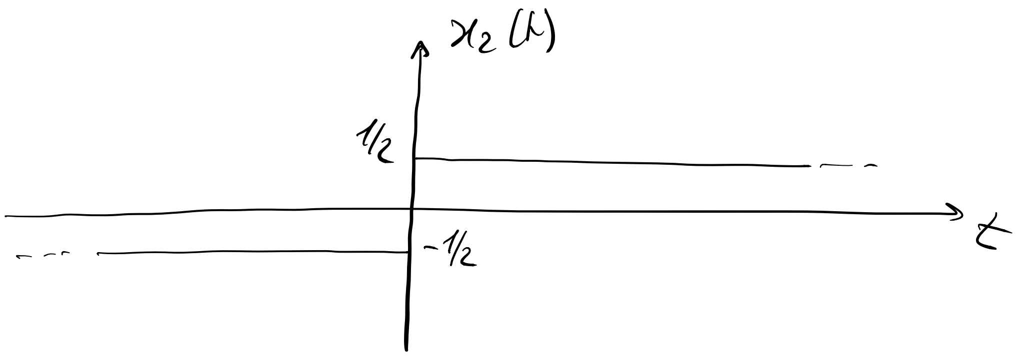
$$= -\frac{1}{j2\pi f}\, e^{-j2\pi f t} \Big|_0^{+\infty} = \text{ind.}$$

altra strada

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\text{sgn}(t)$$




$$u(t) = x_1(t) + x_2(t)$$

$$U(f) = X_1(f) + X_2(f)$$

$$X_1(f) = ?$$

$$x_1(t) = \frac{1}{2} \Rightarrow X_1(f) = ?$$

$$\delta(t) \Longleftrightarrow 1$$

DUALITÀ

$$1 \ \forall t \Longleftrightarrow \delta(-f) = \delta(f)$$

$$X_1(f) = \frac{1}{2}\delta(f)$$


$$x_2(t) = \frac{1}{2}\,\mathrm{sgn}(t) \Rightarrow X_2(f) = ?$$

$$\frac{1}{t} \overset{TCF}{\Longleftrightarrow} -j\pi\,\mathrm{sgn}(f)$$

$$-j\pi\,\mathrm{sgn}(t) \overset{TCF}{\Longleftrightarrow} \frac{1}{-f}$$

$$\Downarrow$$

$$\mathrm{sgn}(t) \overset{TCF}{\Longleftrightarrow} \frac{1}{(-j\pi)(-f)}$$

$$\frac{1}{2}\,\mathrm{sgn}(t) \overset{TCF}{\Longleftrightarrow} \frac{1}{j2\pi f}$$

$$\boxed{U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}}$$

-) TEOREMA DELL'INTEGRAZIONE COMPLETO

$$\text{Ip:}\quad \begin{cases} x(t) \overset{TCF}{\Longleftrightarrow} X(f) \\ y(t) = \displaystyle\int_{-\infty}^{t} x(\alpha)\,d\alpha \end{cases}$$

$$\text{Th:}\quad \boxed{Y(f) = \underbrace{\frac{X(0)}{2}\delta(f)}_{} + \frac{X(f)}{j2\pi f}}$$

compare quando

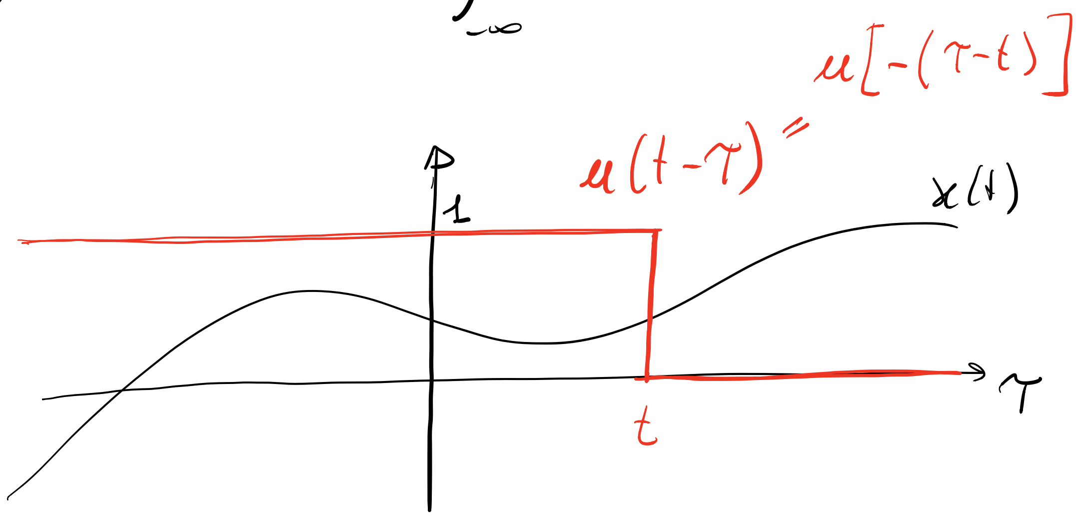
$$\int_{-\infty}^{+\infty} x(t)\,dt \neq 0 \quad \Rightarrow \quad \int_{-\infty}^{+\infty} x(t)\,dt = X(0)$$

$$X(f)\Big|_{f=0} = X(0) = \int_{-\infty}^{+\infty} x(t)\,e^{-j2\pi f t}\,dt\Big|_{f=0} = \int_{-\infty}^{+\infty} x(t)\,dt$$

Dim.

$$y(t) = \int_{-\infty}^{t} x(\alpha)\,d\alpha = \int_{-\infty}^{t} x(\tau)\,d\tau$$

$$x(t) \otimes u(t) = \int_{-\infty}^{+\infty} x(\tau)\,u(t-\tau)\,d\tau$$




$$= \int_{-\infty}^{t} x(\tau)\,d\tau = y(t) = x(t) \otimes u(t)$$

$$Y(f) = X(f)\,U(f) = \frac{1}{2} X(f)\,\delta(f) + \frac{X(f)}{j2\pi f}$$

$$U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

$$= \frac{1}{2} X(0)\,\delta(f) + \frac{X(f)}{j2\pi f}$$

$$\Rightarrow \quad \delta(t) \overset{TCF}{\Longleftrightarrow} 1 \quad \forall f$$

$$\Rightarrow \quad \boxed{\delta(t-t_0) \overset{TCF}{\Longleftrightarrow} e^{-j2\pi f t_0}}$$

$$\int_{-\infty}^{+\infty} \delta(t-t_0)\, e^{-j2\pi f t}\, dt = e^{-j2\pi f t_0}$$

$$e^{-j2\pi f_0 t} \overset{TCF}{\Longleftrightarrow} \delta(-f-f_0)$$

$$e^{j2\pi(-f_0)t} \overset{TCF}{\Longleftrightarrow} \delta(f+f_0)$$

$$\boxed{e^{j2\pi f_0 t} \overset{TCF}{\Longleftrightarrow} \delta(f-f_0)}$$

$$\boxed{e^{-j2\pi f_0 t} \overset{TCF}{\Longleftrightarrow} \delta(f+f_0)}$$

Esempio: <u>Calcolo della TCF di sinusoidi</u>

TCF DI UN COSENO

$$x(t) = A\cos(2\pi f_0 t)$$

energia inf. periodica

$$X(f) = \int_{-\infty}^{+\infty} A\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt$$

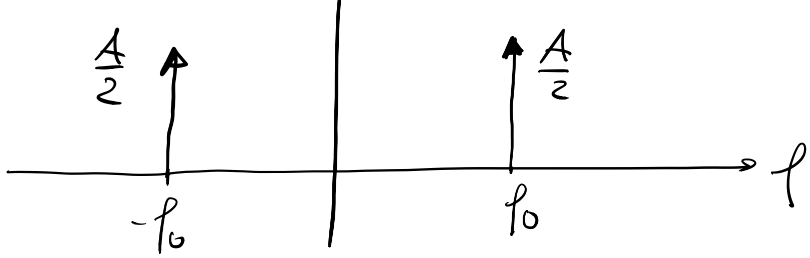
$$= \frac{A}{2}\int_{-\infty}^{+\infty} \left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) e^{-j2\pi f t}\, dt$$

$$= \frac{A}{2}\int_{-\infty}^{+\infty} e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt + \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \frac{A}{2}\, TCF\left[e^{j2\pi f_0 t}\right] + \frac{A}{2}\, TCF\left[e^{-j2\pi f_0 t}\right]$$

$$= \frac{A}{2}\,\delta(f-f_0) + \frac{A}{2}\,\delta(f+f_0)$$ reale e pari

$\mathrm{Re}\{X(f)\}$ , $\mathrm{Im}\{X(f)\} = 0$



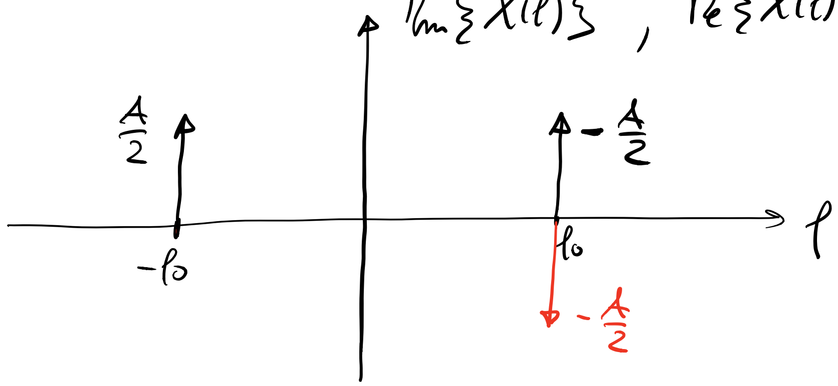

TCF DI UN SENO

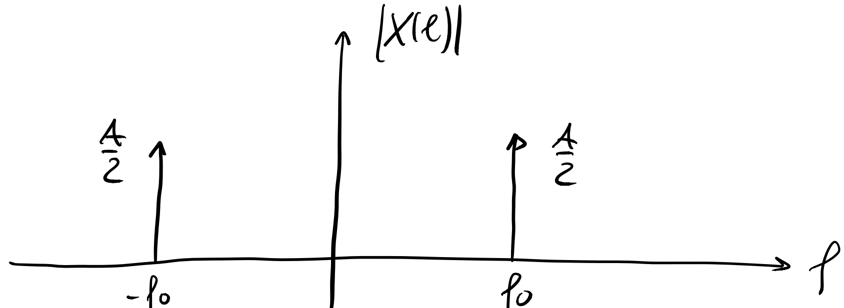
$$\int_{-\infty}^{+\infty} A\sin(2\pi f_0 t)\, e^{-j2\pi f t}\, dt$$

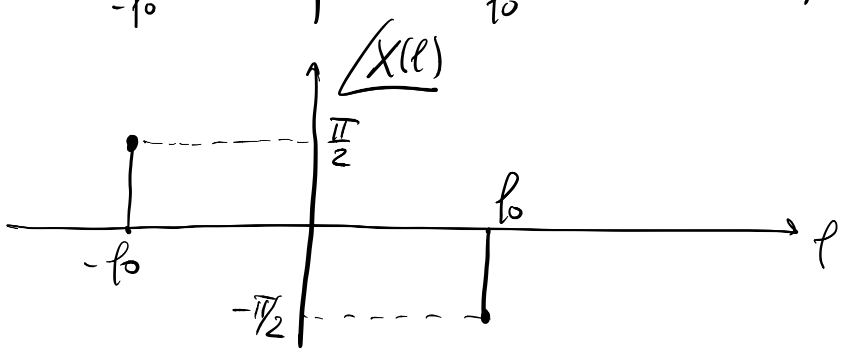
$$= \frac{A}{2j}\int_{-\infty}^{+\infty} e^{j2\pi f_0 t} e^{-j2\pi f t}\, dt - \frac{A}{2j}\int_{-\infty}^{+\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t}\, dt$$

$$= \frac{A}{2j}\,\delta(f-f_0) - \frac{A}{2j}\,\delta(f+f_0)$$

imm e dispari




TCF DI UNA SINUSOIDE

$$\int_{-\infty}^{+\infty} A\cos(2\pi f_0 t + \varphi)\, e^{-j2\pi f t}\, dt$$

$$= \frac{Ae^{j\varphi}}{2}\,\delta(f-f_0) + \frac{Ae^{-j\varphi}}{2}\,\delta(f+f_0)$$

$$\begin{tikzcd} x(t) \arrow[rr, bend left, "TCF"] \arrow[rr, phantom, "\overset{TCF}{\Longleftrightarrow}"] & & X(f) \arrow[ll, bend left, "ATCF"] \end{tikzcd}$$

Dimostrazione della biunivocità della TCF

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

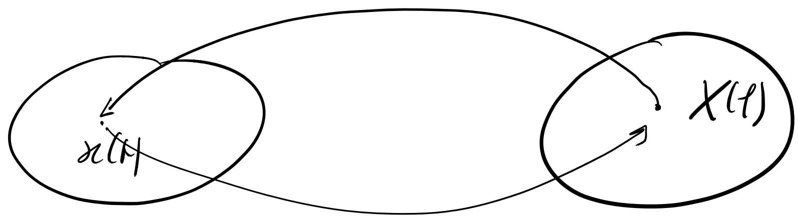
$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

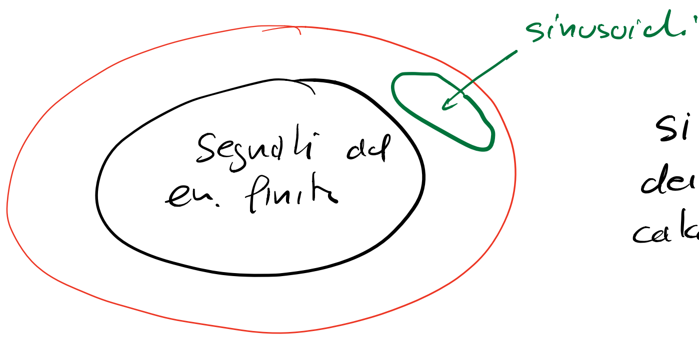
$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{\underset{(t)}{-\infty}}^{+\infty} \underbrace{\int_{\underset{(\alpha)}{-\infty}}^{+\infty} X(\alpha)\, e^{j2\pi\alpha t}\, d\alpha}_{x(t)}\; e^{-j2\pi f t}\, dt$$

$$= \int_{\underset{(\alpha)}{-\infty}}^{+\infty} X(\alpha) \underbrace{\int_{\underset{(t)}{-\infty}}^{+\infty} 1\cdot e^{-j2\pi\underbrace{(-\alpha+f)}_{f'}t}\, dt}_{TCF[1]|_{f'=\alpha-f}}\, d\alpha \Rightarrow \delta(f') = \underset{\delta(f-\alpha)}{\overset{\shortparallel}{\delta(-\alpha+f)}}$$

$$= \int_{-\infty}^{+\infty} X(\alpha)\, \delta(f-\alpha)\, d\alpha = X(f) \otimes \delta(f) = X(f)$$




si allarga la classe dei segnali di cui possiamo calcolare la TCF

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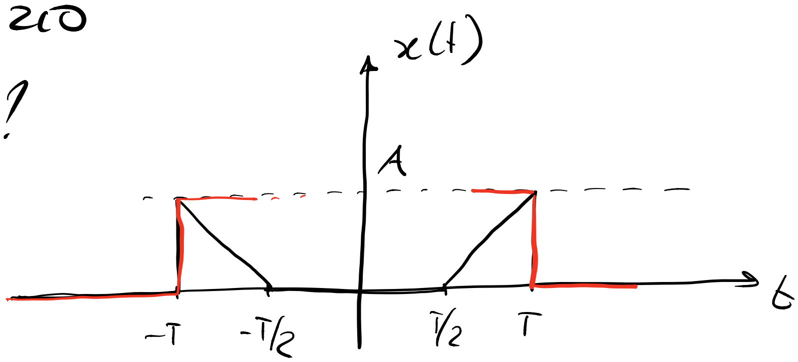
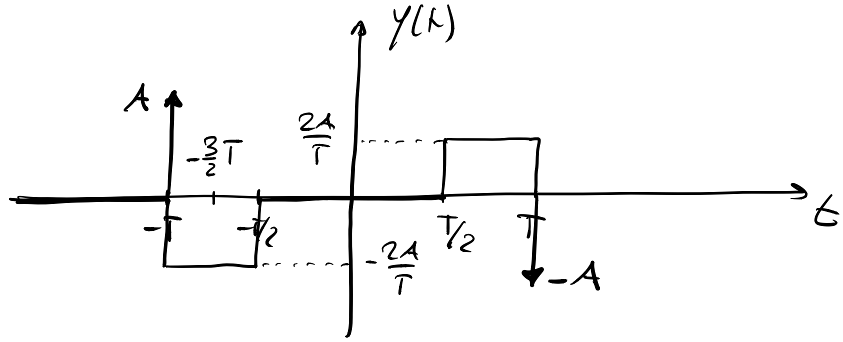
## .) ESERCIZIO

$X(f) = ?$




Teorema dell'integrazione completo

$$y(t) = \frac{d}{dt} x(t) \quad \Rightarrow \quad x(t) = \int_{-\infty}^{t} y(\alpha)\, d\alpha$$




$$X(f) = \frac{Y(0)}{2}\delta(f) + \frac{Y(f)}{j2\pi f}$$

$$y(t) = A\,\delta(t-(-T)) - \frac{2A}{T}\,\mathrm{rect}\left(\frac{t-(-\frac{3}{2}T)}{T/2}\right)$$

$$+ \frac{2A}{T}\,\mathrm{rect}\left(\frac{t-\frac{3}{2}T}{T/2}\right) - A\,\delta(t-T)$$

$$Y(f) = TCF\left[y(t)\right] = A\,e^{j2\pi fT} - \frac{2A}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}\,\mathrm{sinc}\left(\frac{T}{2}f\right)e^{j2\pi\frac{3}{2}Tf}$$

$$+ \frac{2A}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}\,\mathrm{sinc}\left(\frac{T}{2}f\right)e^{-j2\pi\frac{3}{2}Tf} - A\,e^{-j2\pi fT}$$

$$= A\left[e^{j2\pi fT} - e^{-j2\pi fT}\right] - A\,\mathrm{sinc}\left(\frac{T}{2}f\right)\left[e^{j3\pi fT} - e^{-j3\pi fT}\right]$$

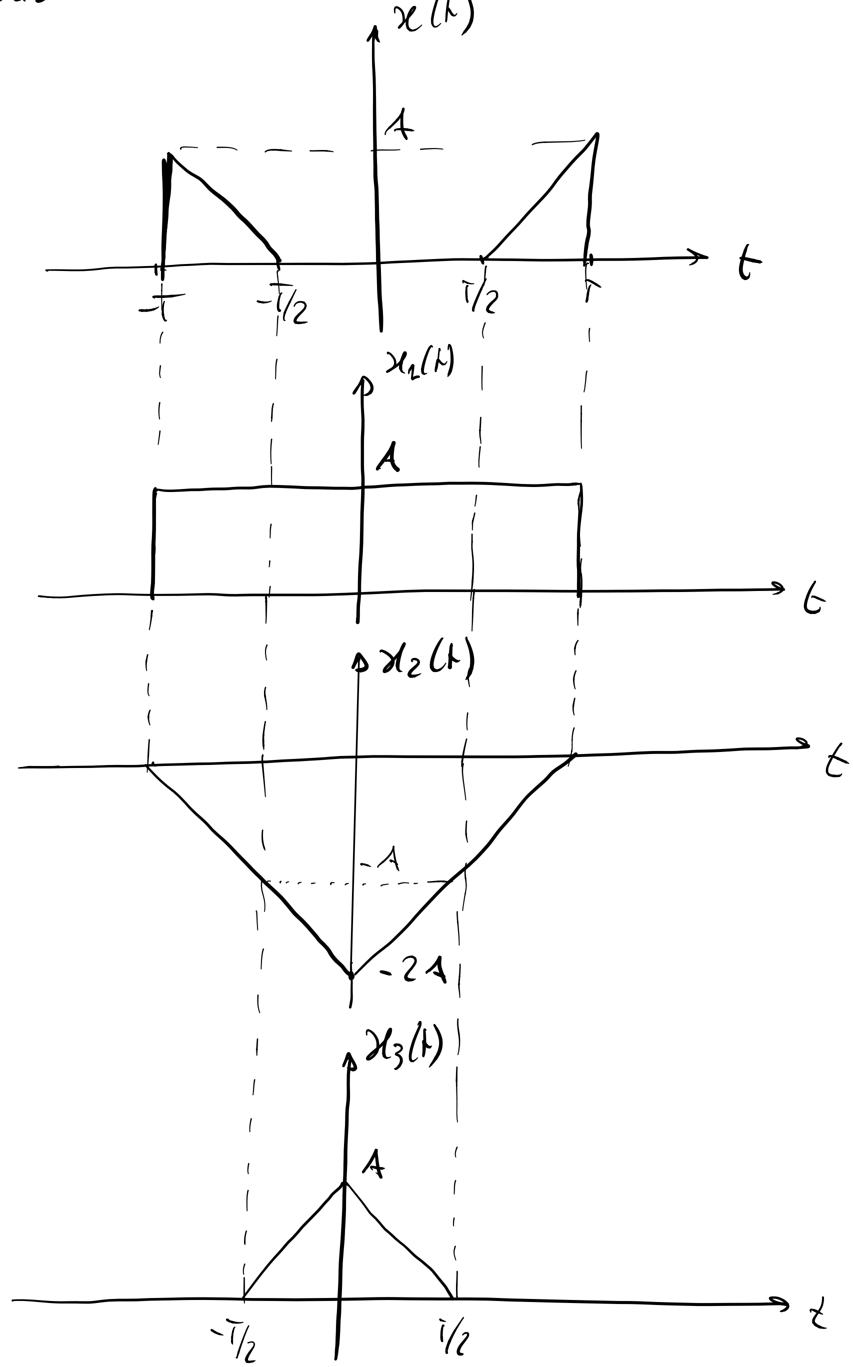
$$= \underbrace{j2}A\sin(2\pi fT) - \underbrace{j2}A\,\mathrm{sinc}\left(\frac{T}{2}f\right)\sin(3\pi fT)$$

$$X(f) = \cancel{\frac{\overset{=0}{Y(0)}}{2}\delta(f)} + \frac{Y(f)}{j2\pi f}$$

$$X(f) = \frac{A}{\pi f}\sin(2\pi fT) - \frac{A}{\pi f}\,\mathrm{sinc}\left(\frac{T}{2}f\right)\sin(3\pi fT)$$

$$= \boxed{2AT\,\mathrm{sinc}(2fT) - 3AT\,\mathrm{sinc}\left(\frac{T}{2}f\right)\mathrm{sinc}(3fT)}$$

altra strada




$$x_1(t) + x_2(t) + x_3(t) = x(t)$$

$$x_1(t) = A \, \mathrm{rect}\left(\frac{t}{2T}\right) \iff 2AT \, \mathrm{sinc}(2Tf)$$

$$x_2(t) = -2A\left(1 - \frac{|t|}{T}\right) \mathrm{rect}\left(\frac{t}{2T}\right) \iff -2AT \, \mathrm{sinc}^2(Tf)$$

$$x_3(t) = A\left(1 - \frac{|t|}{T/2}\right) \mathrm{rect}\left(\frac{t}{T}\right) \iff \frac{AT}{2} \, \mathrm{sinc}^2\left(\frac{T}{2} f\right)$$

$$X(f) = 2AT\left[\mathrm{sinc}(2Tf) - \mathrm{sinc}^2(Tf)\right] + \frac{AT}{2} \, \mathrm{sinc}^2\left(\frac{T}{2} f\right)$$

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-) ESERCIZIO – MODULAZIONE DI AMPIEZZA

$$y(t) = x(t) \cos(2\pi f_0 t)$$

$$x(t) = AB \, \mathrm{sinc}^2(Bt)$$

Svolgimento:

I) teo. della modulaz.

$$Y(f) = \frac{1}{2} X(f-f_0) + \frac{1}{2} X(f+f_0)$$

$$X(f) = TCF[x(t)]$$

II) $y(t) = x(t) \cdot z(t)$ , $z(t) = \cos(2\pi f_0 t)$

$$Y(f) = X(f) \otimes Z(f)$$

$$X(f) = A\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Z(f) = TCF[\cos(2\pi f_0 t)] = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$Y(f) = A\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \otimes \left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right]$$

per le proprietà delle distribuz.

$$= A\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \otimes \frac{1}{2}\delta(f-f_0) +$$

$$+ A\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \otimes \frac{1}{2}\delta(f+f_0)$$

$$= \frac{A}{2}\left(1 - \frac{|f-f_0|}{B}\right) \mathrm{rect}\left(\frac{f-f_0}{2B}\right) +$$

$$+ \frac{A}{2}\left(1 - \frac{|f+f_0|}{B}\right) \mathrm{rect}\left(\frac{f+f_0}{2B}\right)$$

$$= \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$

.) ESERCIZIO — TCF DI SINUSOIDI MULTICOMPONENTE

$$x_1(t) = A\cos(2\pi f_1 t)$$
$$x_2(t) = B\sin(2\pi f_2 t)$$

$$x(t) = x_1(t) + x_2(t)$$

.) $TSF[x(t)] = ?$

.) $TCF[x(t)] = ?$

Svolgimento

TSF

·) mi devo procurare $T_0$ per $x(t)$

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi n f_0 t}\, dt \qquad f_0 = \frac{1}{T_0}$$

$x(t)$ è periodico ??

$f_1$ , $f_2$

·) affinché $x(t)$ sia periodico deve esistere una $f_0$ tale che $f_1$ e $f_2$ siano un multiplo di $f_0$

$$f_1 = K_1 f_0 \qquad K_1, K_2 \text{ int. positivi}$$
$$f_2 = K_2 f_0$$

$$\frac{f_1}{f_2} = \frac{K_1}{K_2} \Rightarrow \text{numero razionale}$$

$\Downarrow$

$x(t)$ è periodico

$\Rightarrow$ La TCF mi permette di calcolare la $X(f)$ indipendentemente dai valori di $f_1$ e $f_2$, quindi indipendentemente dal fatto che $x(t)$ sia periodico o meno

$$TCF[x(t)] = X(f) = X_1(f) + X_2(f)$$

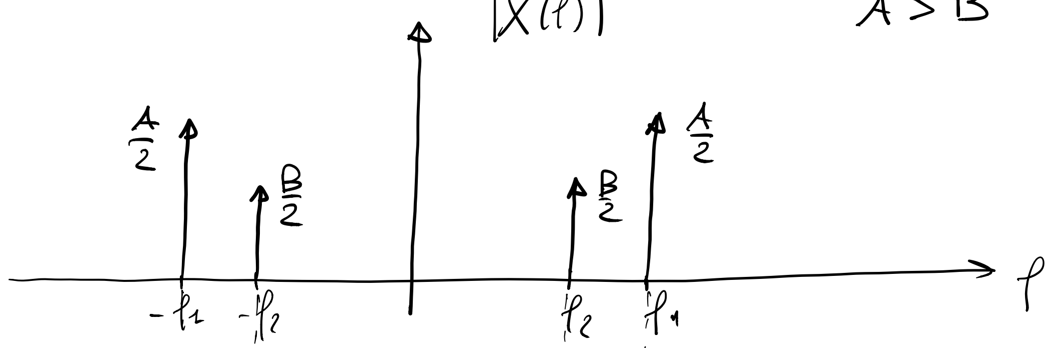
$$X_1(f) = \frac{A}{2}\delta(f-f_1) + \frac{A}{2}\delta(f+f_1)$$

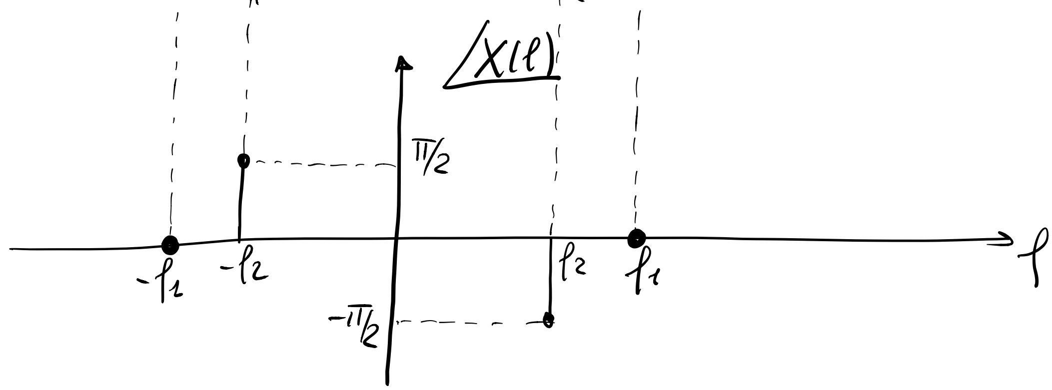
$$X_2(f) = \frac{B}{2j}\delta(f-f_2) - \frac{B}{2j}\delta(f+f_2)$$

Disegnare il grafico

$f_1 > f_2$

$A > B$

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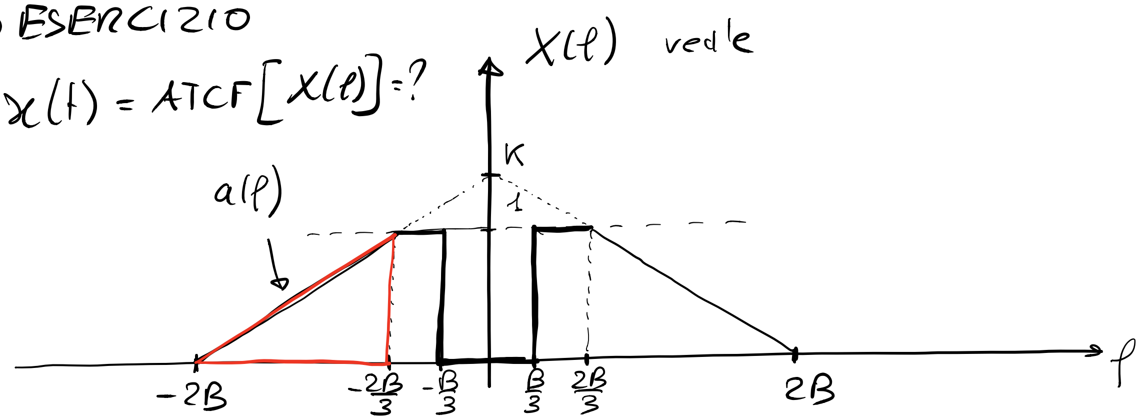
) ESERCIZIO

$x(t) = ATCF[X(f)] = ?$

$X(f)$ reale

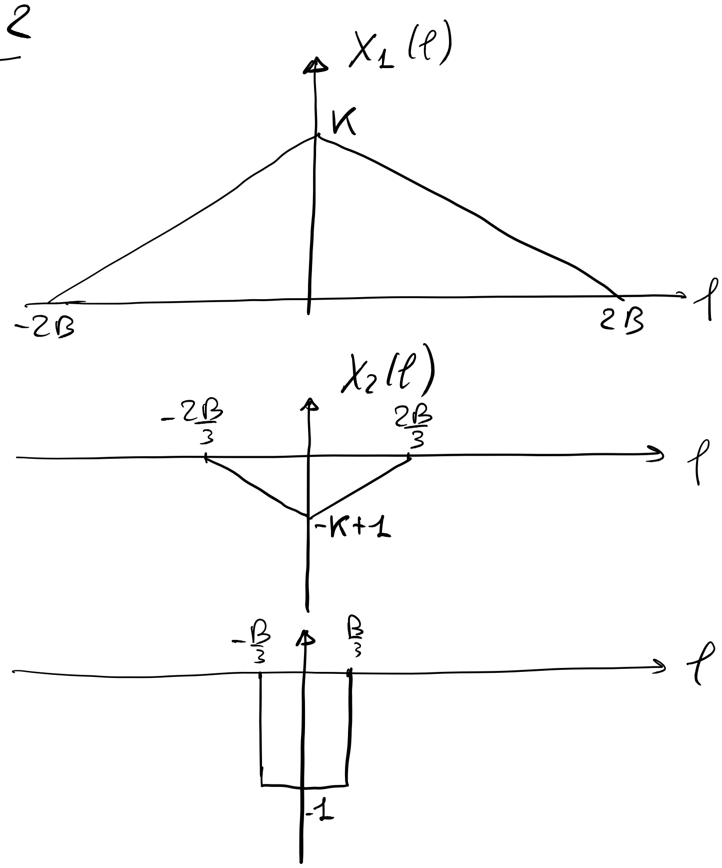


Svolgimento

I) derivazione + teo. int.

II) composizione di segnali con trasf. nota

Strada #2




$$a(f) = c f + K = \frac{3}{4B} f + K$$

$$c = \frac{1}{\left(2B - \frac{2B}{3}\right)} = \frac{3}{4B}$$

$$\Rightarrow \frac{3}{4B} \cdot (-2B) + K = 0 \Rightarrow K = \frac{3}{2}$$

$$X_1(f) = \frac{3}{2}\left(1 - \frac{|f|}{2B}\right)\mathrm{rect}\left(\frac{f}{4B}\right)$$

$$X_2(f) = -\frac{1}{2}\left(1 - \frac{|f|}{\frac{2B}{3}}\right)\mathrm{rect}\left(\frac{f}{\frac{4B}{3}}\right)$$

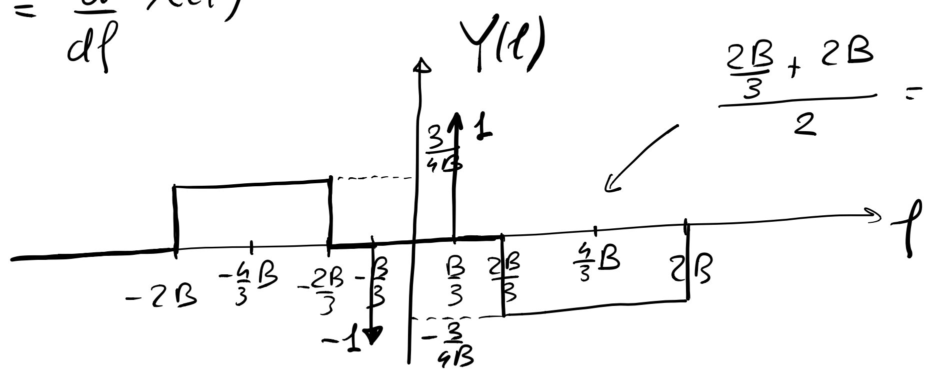
$$X_3(f) = -\,\mathrm{rect}\left(\frac{f}{\frac{2B}{3}}\right)$$

$$x(f) = x_1(t) + x_2(t) + x_3(t)$$

$$= \boxed{3B\,\mathrm{sinc}^2(2Bt) - \frac{B}{3}\,\mathrm{sinc}^2\left(\frac{2B}{3}t\right) - \frac{2B}{3}\,\mathrm{sinc}\left(\frac{2B}{3}t\right)}$$

## Strada #1

$$Y(f) = \frac{d}{df}X(f)$$




$$\frac{\frac{2B}{3} + 2B}{2} = \frac{B}{3} + B = \frac{4}{3}B$$

$$Y(f) = \frac{3}{4B}\,\mathrm{rect}\left(\frac{f + \frac{4}{3}B}{\frac{4}{3}B}\right) - \delta\left(f + \frac{B}{3}\right) + \delta\left(f - \frac{B}{3}\right)$$

$$- \frac{3}{4B}\,\mathrm{rect}\left(\frac{f - \frac{4}{3}B}{\frac{4}{3}B}\right)$$

$$y(t) = \frac{3}{\cancel{4B}} \cdot \cancel{\frac{4}{3}B} \operatorname{sinc}\left(\frac{4}{3}Bt\right)\left[\underbrace{e^{-j2\pi\frac{4}{3}Bt} - e^{j2\pi\frac{4}{3}Bt}}_{-\sin}\right]$$

$$+ e^{j2\pi\frac{B}{3}t} - e^{-j2\pi\frac{B}{3}t}$$

$$= -j2 \operatorname{sinc}\left(\frac{4}{3}Bt\right)\sin\left(\frac{8}{3}\pi Bt\right) + j2\sin\left(\frac{2}{3}\pi Bt\right)$$

$y(0) = 0 \Rightarrow$ applico il teo dell'int. in freq.

$$x(t) = -\frac{y(t)}{j2\pi t} = \frac{8}{3}B \operatorname{sinc}\left(\frac{4}{3}Bt\right)\frac{\sin\left(\frac{8}{3}\pi Bt\right)}{\frac{8}{3}B\pi t} - \frac{2}{3}B\frac{\sin\left(\frac{2}{3}\pi Bt\right)}{\frac{2}{3}B\pi t}$$

$$= \boxed{\frac{8}{3}B\operatorname{sinc}\left(\frac{4}{3}Bt\right)\operatorname{sinc}\left(\frac{8}{3}Bt\right) - \frac{2}{3}B\operatorname{sinc}\left(\frac{2}{3}Bt\right)}$$