$f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$

<u>Casi particolari</u>

1) $m = n = 1$ (da vedere Analisi 1)

2) A - intervallo $\subset \mathbb{R}$, $m \geq 1$ curve parametriche
$n = 1$

$A = I$ — intervallo

$\gamma : I \to \mathbb{R}^m$

$\gamma(t) = (x_1(t), \dots, x_m(t))$

$t_0 \in I$

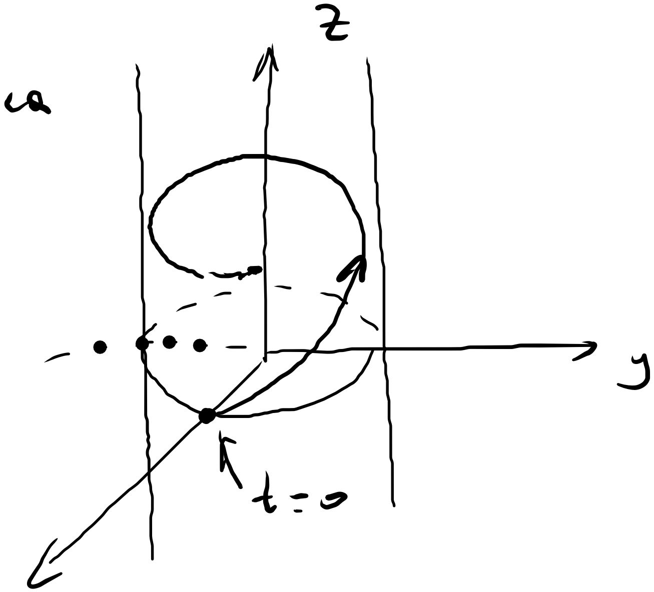
$\dot{\gamma}(t_0) \neq 0 \Rightarrow \dot{\gamma}(t_0)$ è tangente al sostegno della curva nel punto $\gamma(t_0)$ e quindi determina la retta tangente

Es $\gamma(t) = (a\cos t, a\sin t, v t) \subset \mathbb{R}^3$

$a, v \in \mathbb{R}$
$a > 0, v > 0$

$I = [0, 2\pi]$

(un arco di) elica cilindrica



$t_0 = \frac{\pi}{4}$

$\dot{\gamma}(t) = (-a\sin t, a\cos t, v) \neq 0$ curva regolare

$\dot{\gamma}(t_0) = \left(-a\frac{\sqrt{2}}{2}, a\frac{\sqrt{2}}{2}, v\right)$

$\gamma(t_0) = \left(a\frac{\sqrt{2}}{2}, a\frac{\sqrt{2}}{2}, v\frac{\pi}{4}\right)$

Retta tangente all'elica nel punto $\gamma(t_0)$ è

$$r(t) = \gamma(t_0) + \dot{\gamma}(t_0)(t - t_0) \qquad t \in \mathbb{R},$$

ovvero

$$z(t) = \begin{pmatrix} \frac{a\sqrt{2}}{2} \\ \frac{a\sqrt{2}}{2} \\ v\frac{\pi}{4} \end{pmatrix} + \begin{pmatrix} -\frac{a\sqrt{2}}{2} \\ \frac{a\sqrt{2}}{2} \\ v \end{pmatrix} (t - t_0), \quad t \in \mathbb{R}$$

con coordinate

$$\begin{cases} x(t) = \frac{a\sqrt{2}}{2} - \frac{a\sqrt{2}}{2}(t-t_0) \\ y(t) = \frac{a\sqrt{2}}{2} + \frac{a\sqrt{2}}{2}(t-t_0) \\ z(t) = v\frac{\pi}{4} + v(t-t_0) \end{cases}$$

Lunghezza d'arco della elica

$$\ell = \int_0^{2\pi} |\dot{\gamma}(t)|\, dt = \int_0^{2\pi} \sqrt{a^2 \sin^2 t + a^2 \cos^2 t + v^2}\, dt =$$

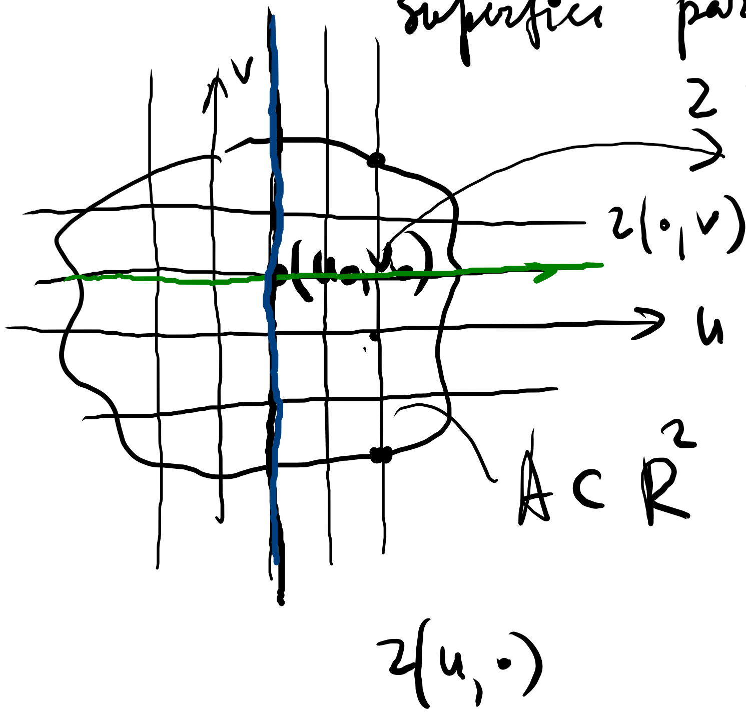
$$= \int_0^{2\pi} \sqrt{a^2 + v^2}\, dt = 2\sqrt{a^2+v^2}\,\pi$$

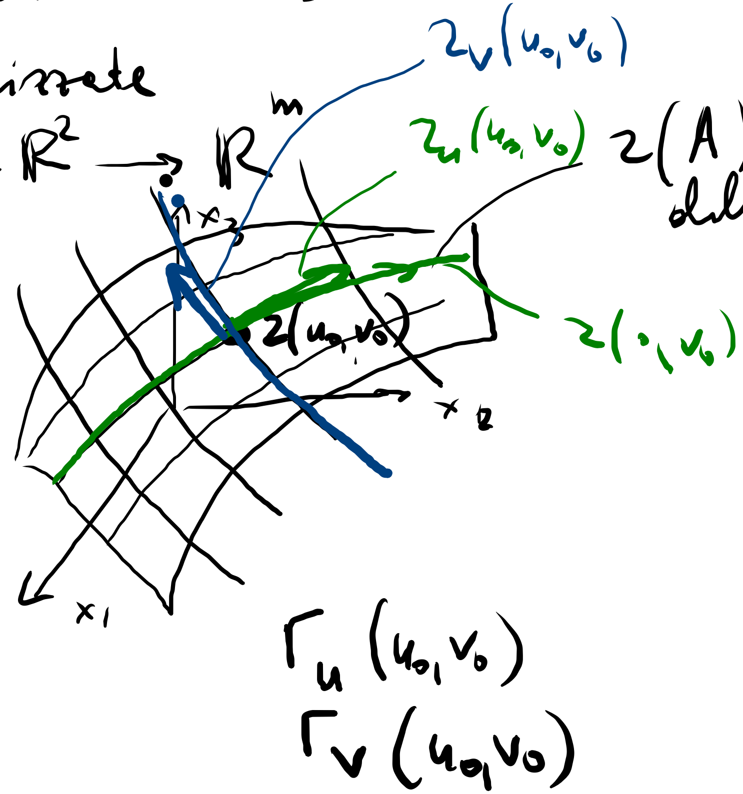
3) $n = 2$ $\quad m > 2$ $\quad$ (spesso $m = 3$)

Superfici parametrizzate

$r : A \subset \mathbb{R}^2 \to \mathbb{R}^m$

$r(A)$ sostegno della superficie





(*) Se $r_u(u_0, v_0)$ e $r_v(u_0, v_0)$ sono linearmente indipendenti,

Allora determiniamo il piano tangente a $r(A)$ nel punto $r(u_0, v_0)$

Eq. parametrica del piano tangente

$$\underset{(x_1,\dots,x_n)^T}{\overset{\shortparallel}{x}} = r(u_0,v_0) + \left(r_u(u_0,v_0) \,\middle|\, r_v(u_0,v_0) \right) \begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix} =$$

$$= r(u_0,v_0) + r_u(u_0,v_0)(u-u_0) + r_v(u_0,v_0)(v-v_0)$$

Come verificare la condizione $(*)$?

- $\left(r_u(u_0,v_0) \,\middle|\, r_v(u_0,v_0) \right)$ - matrice $\underset{\text{righe}}{m} \times \underset{\text{colonne}}{2}$

$(*) \iff \operatorname{rank} \left(r_u(u_0,v_0) \,\middle|\, r_v(u_0,v_0) \right) = 2$

- se $m=3$ $\quad (*) \iff r_u(u_0,v_0) \times r_v(u_0,v_0) \neq 0$

Osservazione: $m=3$

$$r_u(u_0,v_0) \times r_v(u_0,v_0) =: \bar{n}(u_0,v_0)$$

vettore normale a $r(A)$ nel punto $r(u_0,v_0)$

$$\bar{n}(u_0,v_0) \cdot (\overset{\overset{(x_1,x_2,x_3)}{\shortparallel}}{x} - r(u_0,v_0)) = 0$$

— eq. nonparam. del piano tangente

Esempi

1) $A = [0, 2\pi] \times [0, \pi]$

$\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$

longitudine   colatitudine

$$r(\theta, \varphi) := (R \sin\varphi \cos\theta, R \sin\varphi \sin\theta, R\cos\varphi)$$

$R > 0$

sostegno $r(A) = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R^2 \} =$

$= \partial B_R(0) \subset \mathbb{R}^3$

$$r_\theta(\theta,\varphi) = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi \\ 0 \end{pmatrix} \qquad r_\varphi(\theta,\varphi) = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\sin\theta\cos\varphi \\ -R\sin\varphi \end{pmatrix}$$

Calcoliamo

- rank $\underbrace{\begin{pmatrix} -R\sin\theta\sin\varphi & R\cos\theta\cos\varphi \\ R\cos\theta\sin\varphi & R\sin\theta\cos\varphi \\ 0 & -R\sin\varphi \end{pmatrix}}_{3\times 2}$

3 righe, 2 colonne

- $r_\theta \times r_\varphi$

$$r_\theta \times r_\varphi = \det \begin{vmatrix} e_1 & e_2 & e_3 \\ -R\sin\theta\sin\varphi & R\cos\theta\sin\varphi & 0 \\ R\cos\theta\cos\varphi & R\sin\theta\cos\varphi & -R\sin\varphi \end{vmatrix} \begin{matrix} \\ \leftarrow r_\theta \\ \leftarrow r_\varphi \end{matrix}$$

$$= e_1\left(-R^2\cos\theta\sin^2\varphi\right) - e_2\left(R^2\sin\theta\sin^2\varphi\right) +$$

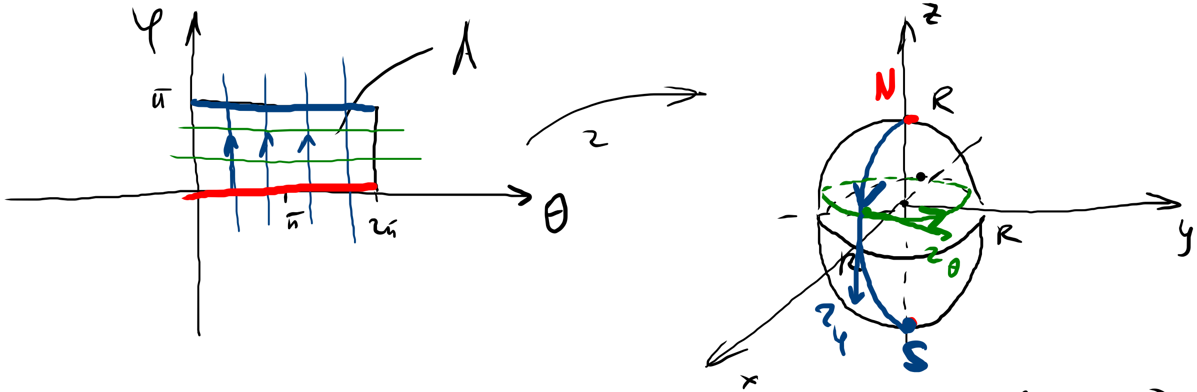
$$+ e_3\left(-R^2\sin^2\theta\sin\varphi\cos\varphi - R^2\cos^2\theta\sin\varphi\cos\varphi\right)$$

$$= R^2\left(-e_1\sin^2\varphi\cos\theta - e_2\sin^2\varphi\sin\theta - e_3\sin\varphi\cos\varphi\right)$$

$$|r_\theta \times r_\varphi| = \sqrt{R^4\left((-\sin^2\varphi\cos\theta)^2 + (-\sin^2\varphi\sin\theta)^2 + (-\sin\varphi\cos\varphi)^2\right)} =$$

$$= R^2\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = R^2\sqrt{\sin^2\varphi} =$$

$$= R^2|\sin\varphi| = 0 \quad \text{sse} \quad \varphi \in \{0, \pi\}$$



L'eq del piano tangente nel punto $r\left(\frac{\pi}{4},\frac{\pi}{3}\right)$

$$r\left(\frac{\pi}{4},\frac{\pi}{6}\right) = \begin{pmatrix} R\frac{\sqrt{2}}{4} \\ R\frac{\sqrt{2}}{4} \\ R\frac{\sqrt{3}}{2} \end{pmatrix} \qquad r_\theta\left(\frac{\pi}{4},\frac{\pi}{6}\right) = \begin{pmatrix} -R\frac{\sqrt{2}}{4} \\ R\frac{\sqrt{2}}{4} \\ 0 \end{pmatrix}$$

$$r_\varphi\left(\frac{\pi}{4},\frac{\pi}{6}\right) = \begin{pmatrix} R\frac{\sqrt{6}}{4} \\ R\frac{\sqrt{6}}{4} \\ -R/2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R\sqrt{2}/4 \\ R\sqrt{2}/4 \\ R\sqrt{3}/2 \end{pmatrix} + \begin{pmatrix} -R\sqrt{2}/4 \\ R\sqrt{2}/4 \\ 0 \end{pmatrix}\left(\theta - \frac{\pi}{4}\right) + \begin{pmatrix} R\sqrt{6}/4 \\ R\sqrt{6}/4 \\ -R/2 \end{pmatrix}\left(\varphi - \frac{\pi}{6}\right)$$

eq. parametrica del piano tangente

$(\theta,\varphi)\in\mathbb{R}^2$

Ovvero

$$\begin{cases} x = R\sqrt{2}/4 - R\sqrt{2}/4\left(\theta - \frac{\pi}{4}\right) + R\sqrt{6}/4\left(\varphi - \frac{\pi}{6}\right) \\ y = R\sqrt{2}/4 + R\sqrt{2}/4\left(\theta - \frac{\pi}{4}\right) + R\sqrt{6}/4\left(\varphi - \frac{\pi}{6}\right) \\ z = R\sqrt{3}/2 - R/2\left(\varphi - \pi/6\right) \end{cases}$$

$$(\theta, \varphi) \in \mathbb{R}^2$$

Alternativamente: $\Gamma_\theta \times \Gamma_\varphi\left(\frac{\pi}{4}, \frac{\pi}{6}\right) = -R^2 \begin{pmatrix} \sqrt{2}/8 \\ \sqrt{2}/8 \\ \sqrt{3}/4 \end{pmatrix}$

$$\Gamma\left(\frac{\pi}{4}, \frac{\pi}{6}\right) = R \begin{pmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{pmatrix}$$

L'eq nonparametrica del piano tangente

$$-R^2 \begin{pmatrix} \sqrt{2}/8 \\ \sqrt{2}/8 \\ \sqrt{3}/4 \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - R \begin{pmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{pmatrix} \right) = 0$$

$$\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ 2\sqrt{3} \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{3}/2 \end{pmatrix} \right) = 0$$

$$\sqrt{2}\left(x - \frac{\sqrt{2}}{4}\right) + \sqrt{2}\left(y - \sqrt{2}/4\right) + 2\sqrt{3}\left(z - \frac{\sqrt{3}}{2}\right) = 0.$$

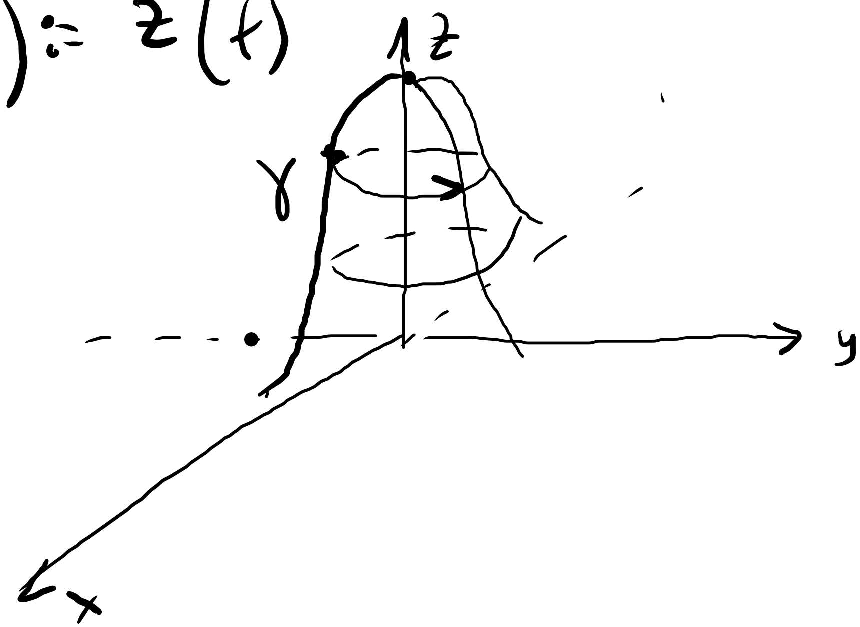
Es 2) Curva γ nel piano $(x\,z)$ di eq parametrica

$$x = \tilde{x}(t), \quad t \in I \subset \mathbb{R}$$
$$z = \tilde{z}(t),$$

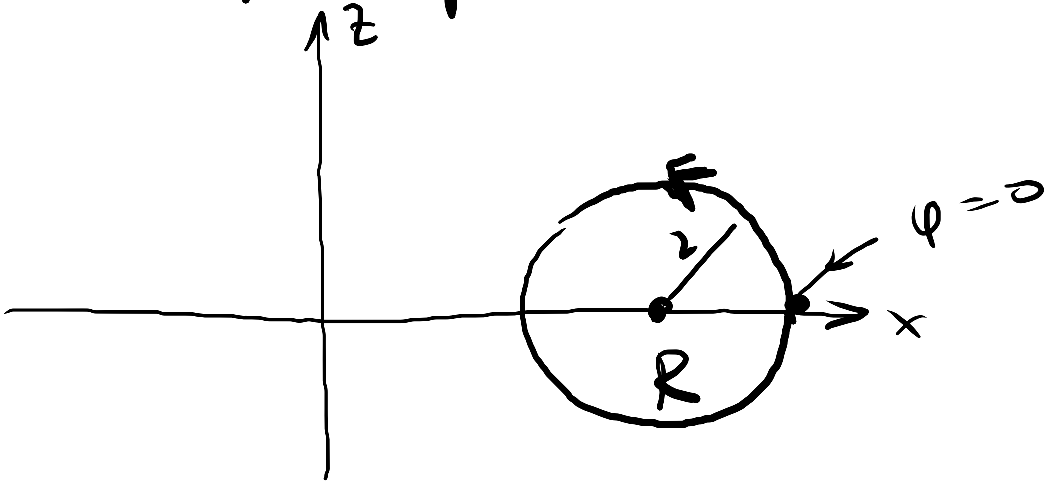
intervallo

Definiamo una superficie (di rotazione)

$$\begin{cases} x(t,\theta) := \tilde{x}(t)\cos\theta \\ y(t,\theta) := \tilde{x}(t)\sin\theta \\ z(t,\theta) := \tilde{z}(t) \end{cases} \qquad \begin{array}{l} t \in I \\ \theta \in [0, 2\pi] \end{array}$$



Esempio particolare



$R > 2 > 0$

numeri

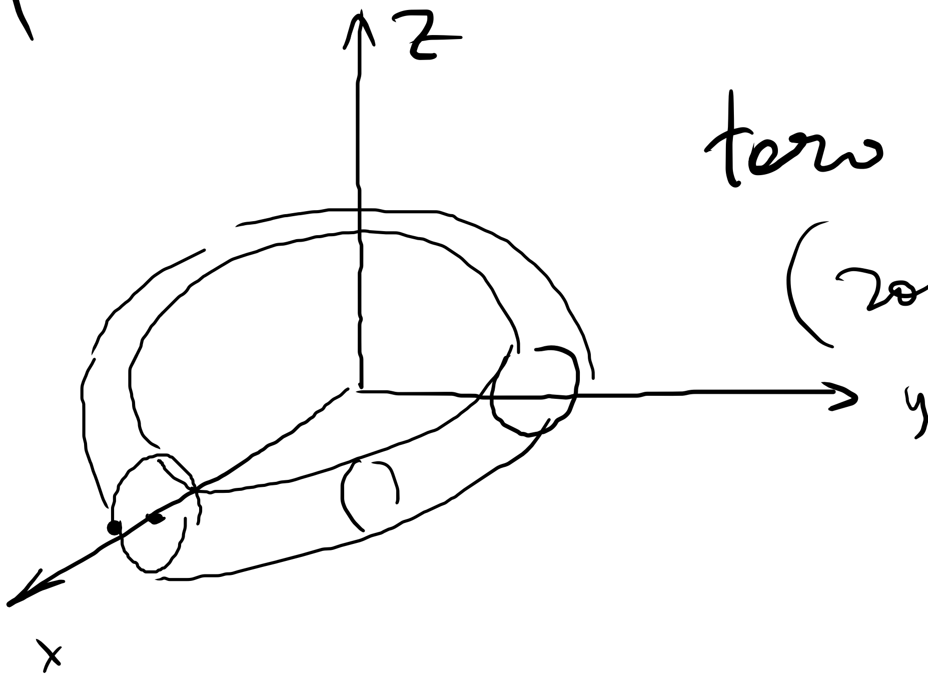
$$\begin{cases} x = R + 2\cos\varphi \\ z = 2\sin\varphi \end{cases}$$

$\varphi \in [0, 2\pi]$

N.B $(x-R)^2 + z^2 = 2^2$

Superficie di rotazione

$$T(\varphi,\varphi): \begin{cases} x(\varphi,\theta) := (R + r\cos\varphi)\cos\theta \\ y(\varphi,\theta) := (R + r\cos\varphi)\sin\theta \\ z(\varphi,\theta) := r\sin\varphi \end{cases}$$



toro di rotazione

(rotation torus)

$$\bar{r}_\varphi(\varphi,\theta) = \big((-2\sin\varphi)\cos\theta,\ (-2\sin\varphi)\sin\theta,\ 2\cos\varphi\big)$$

$$\bar{r}_\theta(\varphi,\theta) = \big(-(R+2\cos\varphi)\sin\theta,\ (R+2\cos\varphi)\cos\theta,\ 0\big)$$

$$n(\theta,\varphi) := \bar{r}_\varphi \times \bar{r}_\theta = \begin{vmatrix} e & e_2 & e_3 \\ -2\sin\varphi\cos\theta & -2\sin\varphi\sin\theta & 2\cos\varphi \\ -(R+2\cos\varphi)\sin\theta & (R+2\cos\varphi)\cos\theta & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \\ -(R+2\cos\varphi)\sin\theta & (R+2\cos\varphi)\cos\theta & 0 \end{vmatrix} =$$

$$= 2\big(-e_1(R+2\cos\varphi)\cos\varphi\cos\theta + e_2(R+2\cos\varphi)\cos\varphi\sin\theta$$
$$+ e_3(R+2\cos\varphi)(-\sin\varphi\cos^2\theta - \sin\varphi\sin^2\theta)\big)$$

$$= 2(R+2\cos\varphi)(-\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi)$$

$$|\bar{r}_\varphi \times \bar{r}_\theta| = 2(R+2\cos\varphi)\sqrt{\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi}$$

$$= 2(R+2\cos\varphi)\sqrt{\underbrace{\cos^2\varphi + \sin^2\varphi}_{=1}} =$$

$$= 2(R+2\cos\varphi) > 0.$$

Esercizio Scrivere l'eq. del piano tangente al toro di rotazione nel punto $\overline{z}\left(0, \frac{\pi}{2}\right)$

(ovvero con $\varphi_0 = 0$, $\theta_0 = \frac{\pi}{2}$)

sia parametrica che non parametrica