591AA 21/22 – COMPITO, LEZIONI 10, 11 E 12

Data di scadenza: Questo compito non sarà raccolto per la valutazione. Invece, circa una settimana dopo che è stato assegnato, le soluzioni saranno pubblicate.

La maggior parte di questi problemi sono trascritti da "Schaum's Outlines, Linear Algebra, 3rd ed", che include anche le soluzioni e molti problemi simili.

Problema 1. [4.19–4.21, pg. 143–144, pdf pg. 151–152 (Lezione 10)]. Determina se i seguenti vettori sono linearmente indipendenti:

(a) $(1,1,2)$, $(2,3,1)$, $(4,5,5)$. [4.19]
(b) $(1,2,5)$, $(2,5,1)$, $(1,5,2)$. [4.20]
(c) $(1,2,5)$, $(1,3,1)$, $(2,5,7)$, $(3,1,4)$. [4.20]
(d) $f(t) = \sin(t)$, $g(t) = \cos(t)$, $h(t) = t$ (elementi dello spazio delle funzioni $\mathbb{R} \to \mathbb{R}$)

Suggerimento per (d): Sia $s(x) = af(x)+bf(x)+ch(x)$. Se $s = 0$ allora $s(0) = s(\pi/2) = s(\pi) = 0$. Interpreta questo come un sistema lineare di equazioni.

Problema 2. [4.13–4.14, pg 142, pdf pg. 150 (Lezione 10)]

(a) Verifica che i seguenti vettori generino $\mathbb{R}^3$: $\{(1,1,1),(1,2,3),(1,5,8)\}$.
(b) Determinare le condizioni su a,b,c in modo che (a,b,c) sia un elemento del sottospazio generato da $\{(1,2,0),(-1,1,2),(3,0,4)\}$.

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Problema 3. [4.24–4.25, pg 145, pdf pg. 153 (Lezione 10)]

(a) Determina se i seguenti vettori sono una base di $\mathbb{R}^3$: $\{(1,1,1),(1,2,5),(5,3,4)$. In caso contrario, determinare la dimensione dello spazio che generano.
(b) Determina se i seguenti vettori sono una base di $\mathbb{R}^4$:

$$\{(1,1,1,1),(1,2,3,2),(2,5,6,4),(2,6,8,5)\}$$

In caso contrario, determinare la dimensione dello spazio che generano.
(c) Determina se i seguenti polinomi sono una base di $P_3[t]$: $\{1, 1+t, 1+t+2t^2, 1+t+2t^2+3t^3\}$

Rouché

Problema 4. (Lezione 11) Spiega usando l'eliminazione gaussiana perch il seguente risultato, chiamato Teorema di Rouch-Capelli, vero: Spiega usando l'eliminazione gaussiana perch il seguente risultato, chiamato Teorema di Rouch-Capelli, è vero:

Teorema. *Un sistema di equazioni lineari con n variabili ha soluzione se e solo se il rango della sua matrice di coefficienti A è uguale al rango della sua matrice aumentata $[A \mid b]$*

Nota: Questo esercizio non è difficile. Se rimani bloccato, prova alcuni esempi per vedere perch il teorema funziona.

Problema 5. (Lezione 11)

(a) Come abbiamo discusso a pagina 2 della lezione 6, la proiezione su un piano in $\mathbb{R}^3$ definisce una mappe lineare $L : \mathbb{R}^3 \to \mathbb{R}^3$. Trova la matrice di L rispetto alla base standard di $\mathbb{R}^3$ nel caso del piano $x + 2y + 2z = 0$. [Ricorda, devi ottenere un vettore unitario per la formula nella lezione 6].

(b) Come abbiamo discusso a pagina 3 della lezione 6, la riflessione su un piano in $\mathbb{R}^3$ definisce una mappa lineare $L : \mathbb{R}^3 \to \mathbb{R}^3$. Trova la matrice di L rispetto alla base standard di $\mathbb{R}^3$ nel caso del piano $x + 2y + 2z = 0$. [Ricorda, devi ottenere un vettore unitario per la formula nella lezione 6].

(c) Trova la matrice della mappa lineare

$$L : P_3[x] \to P_3[x], \qquad L(f) = x\frac{df}{dx} - f$$

rispetto alla base $\{1, x, x^2, x^3\}$.

Problema 6. (Lezione 11) Trova il rango di ciascuna delle mappe lineari del problema 5. Calcola la dimensione del kernel di ogni mappa lineare del problema 5 usando il teorema del rango.

Problema 7. [3.18, pg. 98, pdf pg 107 (Lezione 11)]

(a) Determinare la forma echelon ridotta delle seguenti matrici:

$$A = \begin{pmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & -9 & 6 \\ 0 & 2 & 3 \\ 0 & 9 & 7 \end{pmatrix}$$

(b) La matrice A è equivalente per righe alla matrice

No, le forme canoniche
di riga sono diverse

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Problema 8. (Lezione 12) Siano U e V spazi vettoriali di dimensione finita. Verifica che

$$\dim U \times V = \dim U + \dim V$$

Problema 9. (Lezione 12) Sia U il sottospazio di $\mathbb{R}^5$ generato dai vettori

$$\{(1, -1, -1, -2, 0), (1, -2, -2, 0, -3), (1, -1, -2, -2, 1)\}$$

Sia W il sottospazio di $\mathbb{R}^5$ generato dai vettori

$$\{(1, -2, -3, 0, -2), (1, -1, 2, -3, -4), (1, -1, -2, 2, 5)\}$$

(a) Trova una base di $U + W$. [Questo può essere fatto utilizzando l'algoritmo per trovare la base di uno spazio riga o l'algoritmo per trovare la base di uno spazio colonna discusso nella lezione 9].

(b) Trova una base di $U \cap W$. Questo può essere fatto descrivendo U e W come lo spazio nullo di una coppia di matrici e quindi utilizzando l'algoritmo a pagina 11 della lezione 12.

In alternativa, puoi risolvere le parti (a) e (b) contemporaneamente utilizzando l'algoritmo Zassenhaus a pagina 11 della lezione 12.

where $u_i, w_j \in S$ and $a_i, b_j \in K$. Then

$$u + v = \sum_i a_i u_i + \sum_j b_j w_j \qquad \text{and} \qquad ku = k\left(\sum_i a_i u_i\right) = \sum_i k a_i u_i$$

belong to span(S) since each is a linear combination of vectors in S. Thus span(S) is a subspace of V.

(ii) Suppose $u_1, u_2, \ldots, u_r \in S$. Then all the u_i belong to W. Thus all multiples $a_1u_1, a_2u_2, \ldots, a_ru_r \in W$, and so the sum $a_1u_1 + a_2u_2 + \ldots + a_ru_r \in W$. That is, W contains all linear combinations of elements in S, or, in other words, span(S) $\subseteq W$, as claimed.

LINEAR DEPENDENCE

4.17. Determine whether or not u and v are linearly dependent, where:

(*a*) $u = (1, 2)$, $v = (3, -5)$, (*c*) $u = (1, 2, -3)$, $v = (4, 5, -6)$

(*b*) $u = (1, -3)$, $v = (-2, 6)$, (*d*) $u = (2, 4, -8)$, $v = (3, 6, -12)$

Two vectors u and v are linearly dependent if and only if one is a multiple of the other.

(*a*) No. (*b*) Yes; for $v = -2u$. (*c*) No. (*d*) Yes, for $v = \frac{3}{2}u$.

4.18. Determine whether or not u and v are linearly dependent where:

(*a*) $u = 2t^2 + 4t - 3$, $v = 4t^2 + 8t - 6$, (*b*) $u = 2t^2 - 3t + 4$, $v = 4t^2 - 3t + 2$,

(*c*) $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}, v = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$, (*d*) $u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, v = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Two vectors u and v are linearly dependent if and only if one is a multiple of the other.

(*a*) Yes; for $v = 2u$. (*b*) No. (*c*) Yes, for $v = -4u$. (*d*) No.

4.19. Determine whether or not the vectors $u = (1, 1, 2)$, $v = (2, 3, 1)$, $w = (4, 5, 5)$ in $\mathbf{R}^3$ are linearly dependent.

Sia A la matrice le cui colonne sono i vettori dati. Allora, i vettori sono linearmente dipendenti se e solo se Ax=0 ha una soluzione diversa da zero.

Method 1. Set a linear combination of u, v, w equal to the zero vector using unknowns x, y, z to obtain the equivalent homogeneous system of linear equations and then reduce the system to echelon form. This yields

$$x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z\begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 2y + 4z &= 0 \\ x + 3y + 5z &= 0 \\ 2x + y + 5z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + 4z &= 0 \\ y + z &= 0 \end{aligned}$$

The echelon system has only two nonzero equations in three unknowns; hence it has a free variable and a nonzero solution. Thus u, v, w are linearly dependent.

Method 2. Form the matrix A whose columns are u, v, w and reduce to echelon form:

Applicare l'eliminazione gaussiana ad A e vedere se ci sono variabili indipendenti

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 è una variabile indipendente, e quindi i vettori sono linearmente dipendenti

The third column does not have a pivot; hence the third vector w is a linear combination of the first two vectors u and v. Thus the vectors are linearly dependent. (Observe that the matrix A is also the coefficient matrix in Method 1. In other words, this method is essentially the same as the first method.)

Method 3. Form the matrix B whose rows are u, v, w, and reduce to echelon form:

Un insieme di vettori riga S è linearmente indipendente se e solo se la dimensione dello spazio riga è uguale alla cardinalità di S.

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the echelon matrix has only two nonzero rows, the three vectors are linearly dependent. (The three given vectors span a space of dimension 2.)

4.20. Determine whether or not each of the following lists of vectors in $\mathbf{R}^3$ is linearly dependent:

(*a*) $u_1 = (1, 2, 5)$, $u_2 = (1, 3, 1)$, $u_3 = (2, 5, 7)$, $u_4 = (3, 1, 4)$,

(*b*) $u = (1, 2, 5)$, $v = (2, 5, 1)$, $w = (1, 5, 2)$,

(*c*) $u = (1, 2, 3)$, $v = (0, 0, 0)$, $w = (1, 5, 6)$.

Se V è uno spazio vettoriale a dimensione finita, allora ogni sottoinsieme linearmente indipendente di V ha al massimo dim V elementi.

(*a*) Yes, since any four vectors in $\mathbf{R}^3$ are linearly dependent.

(*b*) Use Method 2 above; that is, form the matrix A whose columns are the given vectors, and reduce the matrix to echelon form:

Utilizzare i metodi 2 o 3 del problema precedente (4.19). Questo è il metodo 2

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -9 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{bmatrix}$$

Every column has a pivot entry; hence no vector is a linear combination of the previous vectors. Thus the vectors are linearly independent.

(*c*) Since $\mathbf{0} = (0, 0, 0)$ is one of the vectors, the vectors are linearly dependent.

$0u + 1v + 0w = 0$

4.21. Show that the functions $f(t) = \sin t$, $g(t) \cos t$, $h(t) = t$ from $\mathbf{R}$ into $\mathbf{R}$ are linearly independent.

Set a linear combination of the functions equal to the zero function $\mathbf{0}$ using unknown scalars x, y, z, that is, set $xf + yg + zh = \mathbf{0}$; and then show $x = 0, y = 0, z = 0$. We emphasize that $xf + yg + zh = \mathbf{0}$ means that, for every value of t, we have $xf(t) + yg(t) + zh(t) = 0$.

Thus, in the equation $x \sin t + y \cos t + zt = 0$:

(i)	Set $t = 0$	to obtain	$x(0) + y(1) + z(0) = 0$	or	$y = 0$.
(ii)	Set $t = \pi/2$	to obtain	$x(1) + y(0) + z\pi/2 = 0$	or	$x + \pi z/2 = 0$.
(iii)	Set $t = \pi$	to obtain	$x(0) + y(-1) + z(\pi) = 0$	or	$-y + \pi z = 0$.

The three equations have only the zero solution, that is, $x = 0$, $y = 0$, $z = 0$. Thus f, g, h are linearly independent.

Se la funzione $a\sin(t) + b\cos(t) + ct = 0$ è zero allora

$$\begin{pmatrix} \sin(t_1) & \cos(t_1) & t_1 \\ \sin(t_2) & \cos(t_2) & t_2 \\ \sin(t_3) & \cos(t_3) & t_3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

per qualsiasi scelta di valori t_1, t_2, t_3. Dobbiamo solo trovare tre valori di t in modo che la matrice abbia rango 3.

4.13. Show that the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (1, 5, 8)$ span $\mathbf{R}^3$.

We need to show that an arbitrary vector $v = (a, b, c)$ in $\mathbf{R}^3$ is a linear combination of u_1, u_2, u_3. Set $v = xu_1 + yu_2 + zu_3$, that is, set

$$(a, b, c) = x(1, 1, 1) + y(1, 2, 3) + z(1, 5, 8) = (x + y + z, \quad x + 2y + 5z, \quad x + 3y + 8z)$$

Form the equivalent system and reduce it to echelon form:

$$\begin{aligned} x + y + z &= a \\ x + 2y + 5z &= b \\ x + 3y + 8z &= c \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + z &= a \\ y + 4z &= b - a \\ 2y + 7c &= c - a \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + y + z &= a \\ y + 4z &= b - a \\ -z &= c - 2b + a \end{aligned}$$

The above system is in echelon form and is consistent; in fact,

$$x = -a + 5b - 3c, \quad y = 3a - 7b + 4c, \quad z = a + 2b - c$$

is a solution. Thus u_1, u_2, u_3 span $\mathbf{R}^3$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & * \\ 1 & 2 & 5 & * \\ 1 & 3 & 8 & * \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & * \\ 0 & 1 & 4 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

$\Longrightarrow$ $Ax = b$ ha una soluzione per ogni vettore b

In alternativa, il rango della matrice è tre, quindi le colonne sono una base di $\mathbb{R}^3$.

4.14: $Ax = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 2 & 1 & 0 & b \\ 0 & 2 & -4 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 3 & -6 & b-2a \\ 0 & 2 & -4 & c \end{array}\right)$$

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$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 1 & -2 & (b-2a)/3 \\ 0 & 1 & -2 & c/2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 3 & a \\ 0 & 1 & -2 & (b-2a)/3 \\ 0 & 0 & 0 & c/2 - (b-2a)/3 \end{array}\right)$$

$$\therefore \quad c/2 - (b-2a)/3 = 0 \Longrightarrow \boxed{3c - 2b + 4a = 0}$$

ho scritto
$$\underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 4 \end{pmatrix}}_{A} \Longrightarrow rkA = 3 \Longrightarrow \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right\}$$

è una base di $\mathbb{R}^3$

4.23. Show that the vectors $u = (1+i,\ 2i)$ and $w = (1,\ 1+i)$ in $\mathbf{C}^2$ are linearly dependent over the complex field $\mathbf{C}$ but linearly independent over the real field $\mathbf{R}$.

Recall that two vectors are linearly dependent (over a field K) if and only if one of them is a multiple of the other (by an element in K). Since

$$(1+i)w = (1+i)(1,\ 1+i) = (1+i,\ 2i) = u$$

u and w are linearly dependent over $\mathbf{C}$. On the other hand, u and w are linearly independent over $\mathbf{R}$, since no real multiple of w can equal u. Specifically, when k is real, the first component of $kw = (k,\ k+ki)$ must be real, and it can never equal the first component $1+i$ of u, which is complex.

BASIS AND DIMENSION

4.24. Determine whether or not each of the following form a basis of $\mathbf{R}^3$:

(a), (b) No

B = base V

$\Rightarrow |B| = \dim V$

$\dim \mathbb{R}^3 = 3$

(*a*) $(1,1,1), (1,0,1)$; (*c*) $(1,1,1), (1,2,3), (2,-1,1)$;

(*b*) $(1,2,3), (1,3,5), (1,0,1), (2,3,0)$; (*d*) $(1,1,2), (1,2,5), (5,3,4)$.

(*a* and *b*) No, since a basis of $\mathbf{R}^3$ must contain exactly 3 elements because $\dim \mathbf{R}^3 = 3$.

(*c*) The three vectors form a basis if and only if they are linearly independent. Thus form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

(c), (d) Forma una base le cui righe sono i vettori dati. Riga ridurre e contare il numero di pivot.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

In alternativa, puoi formare la matrice le cui colonne sono i vettori dati. Applicare l'eliminazione gaussiana e contare il numero di pivot.

The echelon matrix has no zero rows; hence the three vectors are linearly independent, and so they do form a basis of $\mathbf{R}^3$.

(*d*) Form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row; hence the three vectors are linearly dependent, and so they do not form a basis of $\mathbf{R}^3$.

4.25. Determine whether $(1,1,1,1)$, $(1,2,3,2)$, $(2,5,6,4)$, $(2,6,8,5)$ form a basis of $\mathbf{R}^4$. If not, find the dimension of the subspace they span.

Form the matrix whose rows are the given vectors, and row reduce to echelon form:

Lo stesso metodo è 4.24, usando i vettori di riga.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 4 & 2 \\ 0 & 4 & 6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The echelon matrix has a zero row. Hence the four vectors are linearly dependent and do not form a basis of $\mathbf{R}^4$. Since the echelon matrix has three nonzero rows, the four vectors span a subspace of dimension 3.

4.26. Extend $\{u_1 = (1,1,1,1), u_2 = (2,2,3,4)\}$ to a basis of $\mathbf{R}^4$.

Problema 4.

Teorema: Un sistema di equazioni lineari con n variabili ha soluzione se e solo se il rango della sua matrice di coefficienti A è uguale al rango della sua matrice aumentata [A|b].

Dimostrazione:

$\boxed{rk(A) = rk(A|b) \Longrightarrow \text{il sistema ha una soluzione.}}$

$$(A|b) \xrightarrow[\text{Gauss}]{\text{elim.}} (A'|b') = \left(\begin{array}{c|c} A'' & b'' \\ \hline 0 & 0 \end{array}\right)$$

l'ultima riga di A'' contiene un pivot } Altrimenti $rk\, A \neq rk(A|b)$

$\Longrightarrow Ax=b$ ha una soluzione

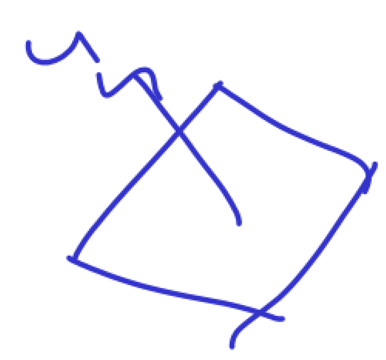
$\boxed{\text{il sistema ha una soluzione} \Longrightarrow rk(A) = rk(A|b)}$

$$(A|b) \xrightarrow[\text{Gauss}]{\text{elim}} \left(\begin{array}{c|c} A'' & b'' \\ \hline 0 & b''' \end{array}\right)$$

l'ultima riga di A'' contiene un pivot

$Ax=b$ ha soluzione $\Longrightarrow b'''=0 \Longrightarrow rk(A) = rk(A|b)$

Problema 5.



$x+2y+2z=0$

$N=(1,2,2)$

$n = N/|N| = \frac{1}{3}(1,2,2)$

(a) $P(v) = v - (v,n)\,n$

$$P(e_1) = \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{9}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)\begin{pmatrix}1\\2\\2\end{pmatrix}$$
$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}8\\-2\\-2\end{pmatrix}$$

$$P(e_2) = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{1}{9}\left(\left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)\right)\begin{pmatrix}1\\2\\2\end{pmatrix}$$
$$= \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{2}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}-2\\5\\-4\end{pmatrix}$$

$$P(e_3) = \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{1}{9}\left(\begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)\begin{pmatrix}1\\2\\2\end{pmatrix}$$
$$= \begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{2}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}-2\\-4\\5\end{pmatrix}$$

$$M = \left(P(e_1) \quad P(e_2) \quad P(e_3)\right)$$
$$= \frac{1}{9}\begin{pmatrix}8 & -2 & -2\\ -2 & 5 & -4\\ -2 & -4 & 5\end{pmatrix}$$

Problema 5

(b) $\quad R(v) = v - 2(v,u)u \qquad u = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}$

$$R(e_1) = \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{2}{9}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\end{pmatrix}\right)\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \frac{2}{9}\begin{pmatrix}1\\2\\2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}7\\-4\\-4\end{pmatrix}$$

$$R(e_2) = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{2}{9}\left(\begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right)\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{4}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}-4\\1\\-8\end{pmatrix}$$

$$R(e_3) = \frac{1}{9}\begin{pmatrix}-4\\-8\\1\end{pmatrix} \quad \text{(stesso m.d.)}$$

$$m = \frac{1}{9}\begin{pmatrix}7 & -4 & -4\\ -4 & 1 & -8\\ -4 & -8 & 1\end{pmatrix}$$

(c) $\quad L(f) = x f' - f$

$$L(1) = -1, \quad L(x) = 0, \quad L(x^2) = x^2, \quad L(x^3) = 2x^3$$

$$m = \begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}$$

Problema 6:

Rango: applica l'eliminazione gaussiana a ciascuna matrice e trova il numero di pivot

dim(ker) = (#numero di colonne) - (rango) (*a*) Use $a_{11} = 1$ as a pivot to obtain 0's below a_{11}, that is, apply the row operations "Replace R_2 by $-2R_1 + R_2$" and "Replace R_3 by $-3R_1 + R_3$"; and then use $a_{23} = 4$ as a pivot to obtain a 0 below a_{23}, that is, apply the row operation "Replace R_3 by $-5R_2 + 4R_3$". These operations yield

$$A \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The matrix is now in echelon form.

(*b*) Hand calculations are usually simpler if the pivot element equals 1. Therefore, first interchange R_1 and R_2. Next apply the operations "Replace R_2 by $4R_1 + R_2$" and "Replace R_3 by $-6R_1 + R_3$"; and then apply the operation "Replace R_3 by $R_2 + R_3$". These operations yield

$$B \sim \begin{bmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in echelon form.

3.16. Describe the *pivoting* row-reduction algorithm. Also describe the advantages, if any, of using this pivoting algorithm.

The row-reduction algorithm becomes a pivoting algorithm if the entry in column j of greatest absolute value is chosen as the pivot a_{1j_1} and if one uses the row operation

$$(-a_{ij_1}/a_{1j_1})R_1 + R_i \to R_i$$

The main advantage of the pivoting algorithm is that the above row operation involves division by the (current) pivot a_{1j_1}, and, on the computer, roundoff errors may be substantially reduced when one divides by a number as large in absolute value as possible.

3.17. Let $A = \begin{bmatrix} 2 & -2 & 2 & 1 \\ -3 & 6 & 0 & -1 \\ 1 & -7 & 10 & 2 \end{bmatrix}$. Reduce A to echelon form using the pivoting algorithm.

First interchange R_1 and R_2 so that -3 can be used as the pivot, and then apply the operations "Replace R_2 by $\frac{2}{3}R_1 + R_2$" and "Replace R_3 by $\frac{1}{3}R_1 + R_3$". These operations yield

$$A \sim \begin{bmatrix} -3 & 6 & 0 & -1 \\ 2 & -2 & 2 & 1 \\ 1 & -7 & 10 & 2 \end{bmatrix} \sim \begin{bmatrix} -3 & 6 & 0 & -1 \\ 0 & 2 & 2 & \frac{1}{3} \\ 0 & -5 & 10 & \frac{5}{3} \end{bmatrix}$$

Now interchange R_2 and R_3 so that -5 can be used as the pivot, and then apply the operation "Replace R_3 by $\frac{2}{5}R_2 + R_3$". We obtain

$$A \sim \begin{bmatrix} -3 & 6 & 0 & -1 \\ 0 & -5 & 10 & \frac{5}{3} \\ 0 & 2 & 2 & \frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} -3 & 6 & 0 & -1 \\ 0 & -5 & 10 & \frac{5}{3} \\ 0 & 0 & 6 & 1 \end{bmatrix}$$

The matrix has been brought to echelon form using partial pivoting.

3.18. Reduce each of the following matrices to row canonical form:

$$(a)\quad A = \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{bmatrix}, \qquad (b)\quad B = \begin{bmatrix} 5 & -9 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 7 \end{bmatrix}$$

(*a*) First reduce A to echelon form by applying the operations "Replace R_2 by $-2R_1 + R_2$" and "Replace R_3 by $-4R_1 + R_3$", and then applying the operation "Replace R_3 by $-R_2 + R_3$". These operations yield

$$A \sim \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 3 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$$

Now use back-substitution on the echelon matrix to obtain the row canonical form of A. Specifically, first multiply R_3 by $\frac{1}{4}$ to obtain the pivot $a_{34} = 1$, and then apply the operations "Replace R_2 by $2R_3 + R_2$" and "Replace R_1 by $-6R_3 + R_1$". These operations yield

$$A \sim \begin{bmatrix} 2 & 2 & -1 & 6 & 4 \\ 0 & 0 & 3 & -2 & 5 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Now multiply R_2 by $\frac{1}{3}$, making the pivot $a_{23} = 1$, and then apply "Replace R_1 by $R_2 + R_1$", yielding

$$A \sim \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Finally, multiply R_1 by $\frac{1}{2}$, so the pivot $a_{11} = 1$. Thus we obtain the following row canonical form of A:

$$A \sim \begin{bmatrix} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

(*b*) Since B is in echelon form, use back-substitution to obtain

$$B \sim \begin{bmatrix} 5 & -9 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & -9 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & -9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The last matrix, which is the identity matrix I, is the row canonical form of B. (This is expected, since B is invertible, and so its row canonical form must be I.)

3.19. Describe the Gauss–Jordan elimination algorithm, which also row reduces an arbitrary matrix A to its row canonical form.

The Gauss–Jordan algorithm is similar in some ways to the Gaussian elimination algorithm, except that here each pivot is used to place 0's both below and above the pivot, not just below the pivot, before working with the next pivot. Also, one variation of the algorithm first *normalizes* each row, that is, obtains a unit pivot, before it is used to produce 0's in the other rows, rather than normalizing the rows at the end of the algorithm.

3.20. Let $A = \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 1 & 1 & 4 & -1 & 3 \\ 2 & 5 & 9 & -2 & 8 \end{bmatrix}$. Use Gauss–Jordan to find the row canonical form of A.

Use $a_{11} = 1$ as a pivot to obtain 0's below a_{11} by applying the operations "Replace R_2 by $-R_1 + R_2$" and "Replace R_3 by $-2R_1 + R_3$". This yields

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 3 & 1 & -2 & 1 \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix}$$

Multiply R_2 by $\frac{1}{3}$ to make the pivot $a_{22} = 1$, and then produce 0's below and above a_{22} by applying the

Problema 8:

Siano $B_U = \{u_1, \dots, u_m\}$ e $B_V = \{v_1, \dots, v_n\}$ basi di U e V

$\Longrightarrow u = a_1 u_1 + \dots + a_m u_m, \quad v = b_1 v_1 + \dots + b_n v_n$

$\Longrightarrow (u, v) = a_1 (u_1, 0) + \dots + a_m (u_m, 0) + b_1 (0, v_1) + \dots + b_n (0, v_n)$

$\Longrightarrow \text{span}\left((u_1, 0), \dots, (u_m, 0), (0, v_1), \dots, (0, v_n) \right) = U \times V$

Supporre che $0 = a_1 (u_1, 0) + \dots + a_m (u_m, 0) + b_1 (0, v_1) + \dots + b_n (0, v_n)$

$= (a_1 u_1 + \dots + a_m u_m, \ b_1 v_1 + \dots + b_n v_n) = (0, 0)$

$\Longrightarrow a_1 u_1 + \dots + a_m u_m = 0, \quad b_1 v_1 + \dots + b_n v_n = 0.$

$\Longrightarrow a_1 = \dots = a_m = 0, \quad b_1 = \dots = b_n = 0$

(B_U, B_V base di U, V)

Problema 9. Applicare l'algoritmo di Zassenhaus

$$\left(\begin{array}{ccccc|ccccc}
1 & -1 & -1 & -2 & 0 & 1 & -1 & -1 & -2 & 0 \\
1 & -2 & -2 & 0 & -3 & 1 & -2 & -2 & 0 & -3 \\
1 & -1 & -2 & -2 & 1 & 1 & -1 & -2 & -2 & 1 \\
\hline
1 & -2 & -3 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 2 & -3 & -4 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -2 & 2 & 5 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Base di $U + W$

Base di $U \cap W$

$$\left(\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 7 & 0 & 0 & -\frac{11}{5} & 0 & \frac{11}{5} \\
0 & 1 & 0 & 0 & 6 & 0 & 0 & -\frac{13}{5} & 0 & \frac{13}{5} \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & \frac{4}{5} & 0 & -\frac{4}{5} \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{27}{5} & -4 & -\frac{12}{5} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{21}{5} & -2 & -\frac{1}{5}
\end{array}\right)$$