INDICI CARATTERISTICI DI V.A. GAUSSIANE

$$X \in \mathcal{N}(\eta_x, \sigma_x^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}$$

⇒ VALOR MEDIO

$$E[X] = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} dx$$

$$x - \eta_x = y \qquad \text{sostituzione}$$

$$= \int_{-\infty}^{+\infty} (y+\eta_x) \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy =$$

$$\in \mathcal{N}(0, \sigma_x^2)$$

$$= \int_{-\infty}^{+\infty} y \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy + \eta_x \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy}_{=1}$$

dispari pari

$$\Downarrow$$

$$\lim_{K \to \infty} \int_{-K}^{K} y \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy = 0$$

$$\Rightarrow \boxed{E[X] = \eta_x}$$

il valor medio è proprio il parametro η_x

⇒ VARIANZA

$$E[(X-\eta_x)^2] = \int_{-\infty}^{+\infty} (x-\eta_x)^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} dx$$

$$x - \eta_x = y$$

$$= \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}} dy$$

$$= -\sigma_X^2 \int_{-\infty}^{+\infty} y \frac{1}{\sqrt{2\pi\sigma_X^2}} \underbrace{\left[\left(-\frac{y}{\sigma_X^2}\right) e^{-\frac{y^2}{2\sigma_X^2}}\right]}_{= \frac{d}{dy}\left[e^{-\frac{y^2}{2\sigma_X^2}}\right]} dy$$

$$= -\sigma_X^2 \left[\underbrace{y \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}} \Bigg|_{-\infty}^{+\infty}}_{=0} - \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{y^2}{2\sigma_X^2}} dy}_{=1} \right]$$

$$\Rightarrow \lim_{y \to \pm\infty} y\, e^{-\frac{y^2}{2\sigma_X^2}} = 0$$

$$\Rightarrow \boxed{E\left[(X-\eta_X)^2\right] = \sigma_X^2}$$

LA VARIANZA È PROPRIO IL PARAMETRO σ_X^2

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}}$$

es

$$f_X(x) = \frac{1}{\sqrt{6\pi}} e^{-\frac{(x-2)^2}{6}} \Rightarrow$$

Calcolare η_X e σ_X^2

$$= \frac{1}{\sqrt{2\pi \cdot 3}} e^{-\frac{(x-2)^2}{2 \cdot 3}} \qquad \eta_X = 2 \quad \sigma_X^2 = 3$$

RELAZIONE TRA VA GAUSSIANE STANDARD E NON-STANDARD

$$N \in \mathcal{N}(0,1)$$

$$X \in \mathcal{N}(\eta_X, \sigma_X^2) \qquad \eta_X \neq 0 \text{ e/o } \sigma_X^2 \neq 1$$

$$f_N(n) = \frac{1}{\sqrt{2\pi}} e^{-\frac{n^2}{2}} \qquad (\sigma_X > 0)$$

$$X = g(N) = \sigma_X N + \eta_X \qquad \text{trasf. lineare}$$

$$N = \frac{X - \eta_X}{\sigma_X} \qquad \text{unica sol. inversa} \quad \{n_i\} = n$$

$$f_X(x) = \frac{f_N(n)}{|g'(n)|} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x-\eta_X}{\sigma_X}\right)^2}{2}}}{\sigma_X}$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sigma_X} e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}} = \underbrace{\frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}}}$$

$$\Phi(x) = F_X(x) \qquad X \in \mathcal{N}(0,1)$$

$$= \int_{-\infty}^{x} f_X(\alpha)\, d\alpha = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}\, dx$$

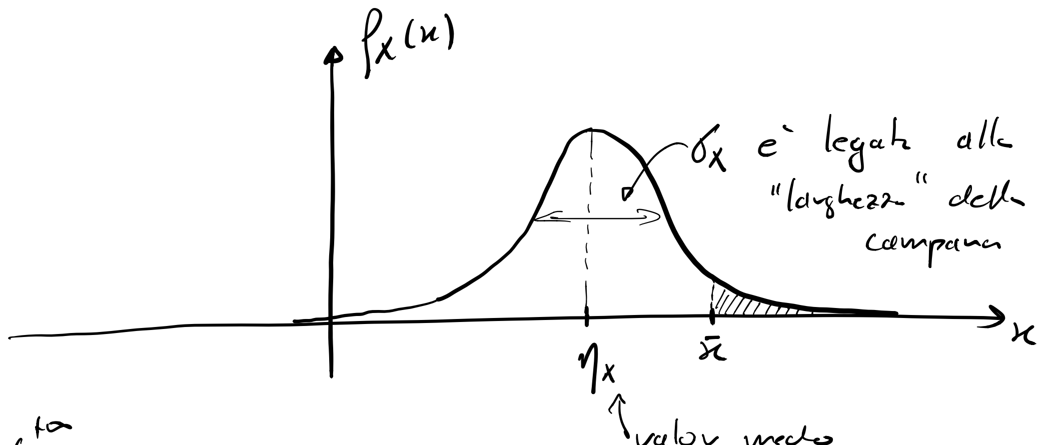
$$Q(x) = 1 - \Phi(x) = \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}\, dx$$

$\downarrow$

sarà la funzione che utilizzeremo per calcolare la prob. di errore in sistemi di com. numerici

$$Q(x) = \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\alpha^2}{2}}\, d\alpha$$

$$X \in \mathcal{N}(\eta_x, \sigma_x^2)$$



$$\int_{\bar{x}}^{+\infty} f_X(x)\, dx = ?$$

$$Q(\bar{n}) = 1 - \Phi(\bar{n})$$

la riteniamo pre-calcolata (tabulata)

$$\int_{\bar{x}}^{+\infty} f_X(x)\,dx = Q\left(\frac{\bar{x} - \eta_X}{\sigma_X}\right)$$

$$n = \frac{x - \eta_X}{\sigma_X}$$

$$\Downarrow$$

$$\int_{\bar{n}}^{+\infty} f_N(n)\,dn = Q(\bar{n})$$

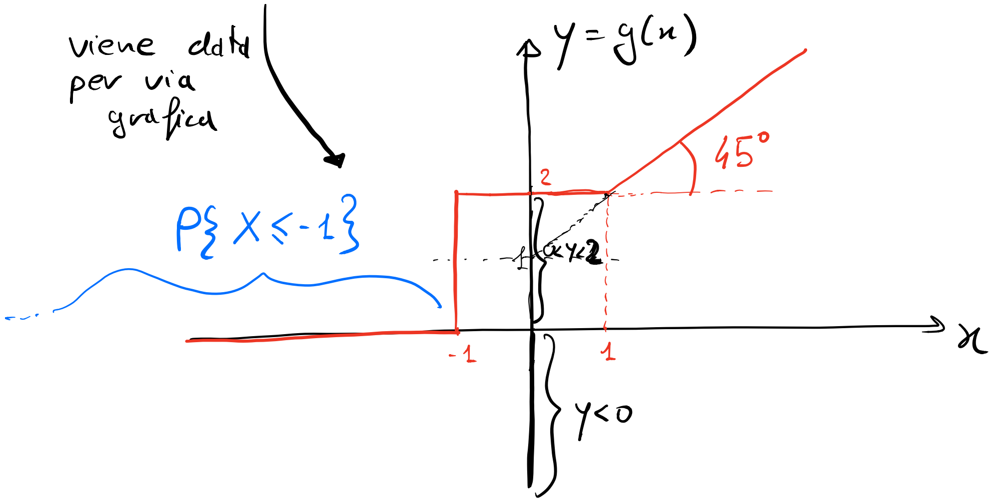
$$\bar{n} = \frac{\bar{x} - \eta_X}{\sigma_X}$$

ESERCIZIO - TRASF. DI V.A.

$$X \in \mathcal{N}(0, 4) \qquad \eta_X = 0, \quad \sigma_X^2 = 4$$

$$Y = g(X) \Rightarrow \boxed{f_Y(y) = ?}$$

viene data per via grafica



Soluzione

$$f_Y(y) = \sum_{i=1}^{N} \frac{f_X(y_i)}{|g'(y_i)|}$$

$\{y_i\}$ sono le soluzioni del problema inverso $x = g^{-1}(y)$

$y<0 \Rightarrow f_Y(y) = 0$

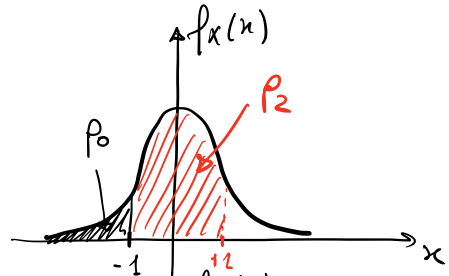
$y=0 \Rightarrow f_Y(y) = p_0\,\delta(y)$

$$p_0 = P\{X \leq -1\}$$

$$p_0 = \int_{-\infty}^{-1} \frac{1}{\sqrt{2\pi 4}} e^{-\frac{x^2}{2\cdot 4}}\,dx$$

$0<y<2 \Rightarrow f_Y(y) = 0$

$y=2 \Rightarrow f_Y(y) = p_2\,\delta(y-2)$

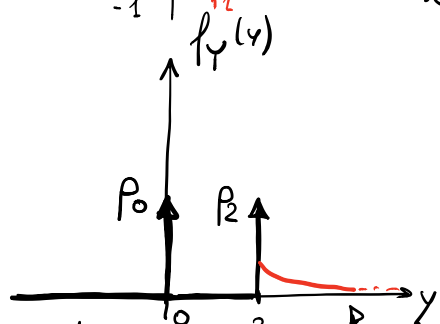


$y>2 \quad y = g(x) = x+1$

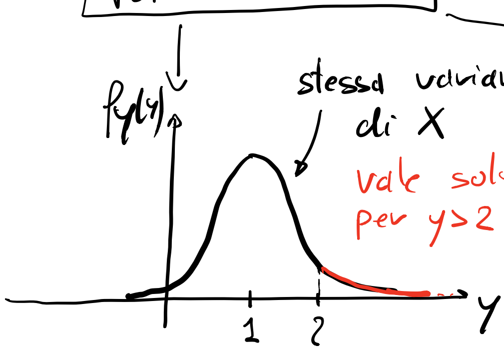
$$x = y-1$$

$$|g'(x)| = |1| = 1$$

$$f_Y(y) = \frac{\frac{1}{\sqrt{8\pi}} e^{-\frac{(y-1)^2}{8}}}{1} = \underbrace{\frac{1}{\sqrt{8\pi}} e^{-\frac{(y-1)^2}{8}}}$$



$$p_2 = \int_{-1}^{1} \frac{1}{\sqrt{8\pi}} e^{-\frac{x^2}{8}}\,dx$$



stessa varianza di X

vale solo per $y>2$

ESERCIZIO – 09 APRILE 2018 es # 1

X ha una ddp esp negativa con parametro λ, quindi con

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x}{\lambda}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

1) Prob. che la durata sia > 1000 ore è pari a $e^{-1} \simeq 0.368$ <= osservato

$\Downarrow$

calcolare λ

$$P\{X > 1000\} = e^{-1}$$

indica la durata di una lampadina in ore

$$P\{X \leq 1000\} = F_X(1000)$$

$$P\{X \leq x\} \triangleq F_X(x)$$

$x = 1000$

$X \leq 1000$ è l'evento complementare di

$X > 1000$

$$\Rightarrow P\{X > 1000\} = 1 - P\{X \leq 1000\}$$

$$P\{X > 1000\} = 1 - F_X(1000) = e^{-1}$$

$$F_X(1000) = 1 - e^{-1}$$

$$1 - e^{-\frac{1000}{\lambda}} = 1 - e^{-1}$$

$$e^{-\frac{1000}{\lambda}} = e^{-1}$$

$$\Rightarrow \boxed{\lambda = 1000}$$

2) $\lambda = 1000$, x_0 : prob. di una durata minore di x_0 sia uguale a 0.09

tempo

$$P\{X \leqslant x_0\} = 0.09$$

$$F_X(x_0) = 0.09$$

$$1 - e^{-\frac{x_0}{1000}} = 0.09$$

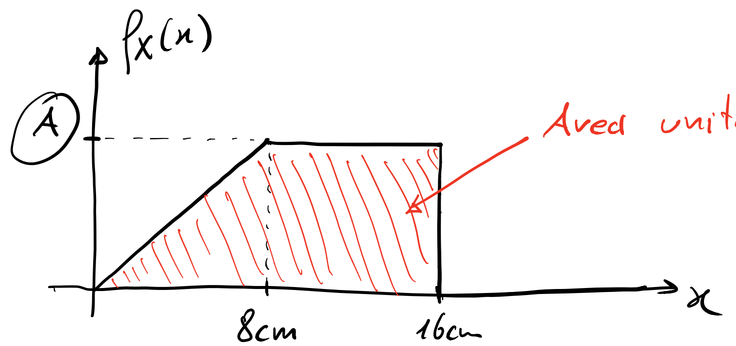
$$e^{-\frac{x_0}{1000}} = 0.91$$

$$-\frac{x_0}{1000} = \ln(0.91) \Rightarrow x_0 = -1000 \ln(0.91)$$

$\simeq 94$ ore

ESERCIZIO – 20/02/2020 Es # 1

X V.A. con ddp   X = distanza dal centro



1) Calcolare il valore di A

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$\frac{A \cdot \overset{4}{\cancel{8}}}{\cancel{2}} + A \cdot 8 = 1 \Rightarrow 12A = 1 \Rightarrow \boxed{A = \frac{1}{12}}$$

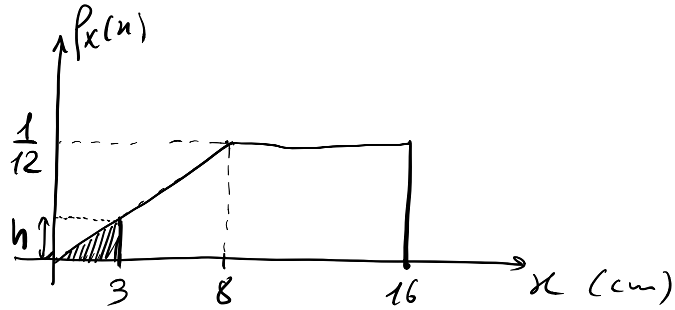
2) Calcolare la prob. che si colpisca il bersaglio entro i 3 cm di distanza dal centro

$$P\{X \leq 3\,cm\}$$

$$F_X(x)\Big|_{x=3cm} \Leftarrow \text{non ce l'abbiamo}$$

$$f_X(x) \Leftarrow \text{ce l'ho} \Rightarrow P\{X \leq 3cm\} = \int_{-\infty}^{3cm} f_X(x)\,dx$$

$$= \int_0^{3cm} f_X(x)\,dx = \frac{1}{32} \cdot 3 \cdot \frac{1}{2} = \boxed{\frac{3}{64}}$$



$$h \Rightarrow \underbrace{\frac{1}{12} \cdot \frac{1}{8} = \frac{h}{3}}_{\text{triangoli simili}} \Rightarrow h = \frac{3}{\cancel{12}_{4} \cdot 8} = \frac{1}{32}$$

3) Calcolare la stessa prob. al punto 2, sapendo però che la freccetta non è andata più in là di 10 cm.

$$P\{X \leq 3cm \mid \underbrace{X \leq 10m}_{\text{osservato}}\}$$

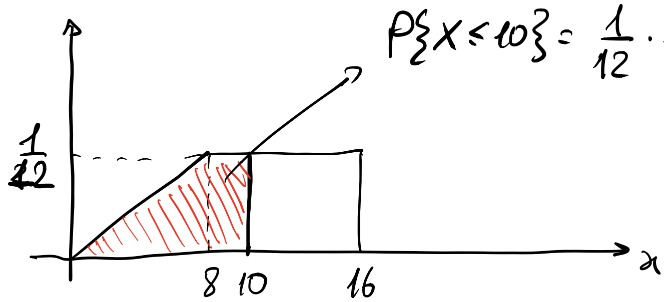
$$P\{A \mid B\} \stackrel{\Delta}{=} \frac{P\{AB\}}{P\{B\}}$$

$$P\{X \leq 3 \mid X \leq 10\} = \frac{P\{X \leq 3, X \leq 10\}}{P\{X \leq 10\}} = \frac{P\{X \leq 3\}}{P\{X \leq 10\}}$$

$$P\{X \leq 10\} = ?$$

$$P\{X \leq 3\} = \frac{3}{64}$$

$$P\{X \leq 10\} = \frac{1}{12} \cdot \frac{8}{2} + \frac{1}{12} \cdot 2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$



$$P\{X \leq 3 \mid X \leq 10\} = \frac{\frac{3}{64}}{\frac{1}{2}} = \boxed{\frac{3}{32}}$$

CORRELAZIONE E COVARIANZA DI DUE VARIABILI ALEATORIE

$$X \qquad Y$$

$$R_{XY} \triangleq E[XY] \qquad \text{correlazione}$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x\, y\, f_{XY}(x, y)\, dx\, dy$$

$f_{XY}(x, y)$ ddp congiunta

$$C_{XY} \triangleq E\left[(X-\eta_X)(Y-\eta_Y) \right]$$ COVARIANZA

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x-\eta_X)(y-\eta_Y)\, f_{XY}(x,y)\, dx\, dy$$

<u>coefficiente di correlazione</u>

$$\rho_{XY} \triangleq \frac{C_{XY}}{\sigma_X \sigma_Y}$$

INDIPENDENZA ED INCORRELAZIONE

$\Rightarrow$ INCORRELAZIONE : due V.A. X ed Y sono <u>INCORRELATE</u> se $C_{XY} = 0$

<u>COVARIANZA</u>

N.B. $C_{XY} = 0 \Rightarrow$ INCORRELAZIONE

~~$R_{XY} = 0 \Rightarrow$ INCORRELAZIONE~~

INDIPENDENZA : nasce a priori e non e' legata ad un parametro statistico

Se due variabili sono indipendenti

$$\Downarrow$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

X e Y sono indipendenti

$$\Downarrow$$

X e Y sono INCORRELATE

NON È VERO IL CONTRARIO !!!