Calcolo differenziale (del I ordine)

$$f : A \subset \mathbb{R}^n \to \mathbb{R}^m$$

$x_0 \in A$ differenziabilità di f in x_0:

$$(1) \qquad f(x) = \underbrace{f(x_0) + \overbrace{Df(x_0)(x-x_0)}^{\text{lineare in } x}}_{\text{affine in } x} + o(\|x-x_0\|), \quad x \to x_0$$

$Df(x_0)$: matrice $m \times n$ $\qquad x, x_0 \in A \subset \mathbb{R}^n$

$$Df(x_0) = \begin{pmatrix} \nabla f_1(x_0) \\ \vdots \\ \nabla f_m(x_0) \end{pmatrix} = \begin{pmatrix} f_{1,1}(x_0) & \cdots & f_{1,n}(x_0) \\ \vdots & & \vdots \\ f_{m,1}(x_0) & \cdots & f_{m,n}(x_0) \end{pmatrix} \text{ matrice jacobiana}$$

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$$

Casi particolari

1°) $m = n = 1$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|)$$

$$x \to 0$$

2°) $n = 1,\ m \geq 1$ curve parametriche

$$z(t) = (x_1(t), \dots, x_n(t))^T = z(t_0) + \dot{z}(t_0)(t - t_0) + $$

$$+ \ o(|t - t_0|),$$

$$t \to t_0$$

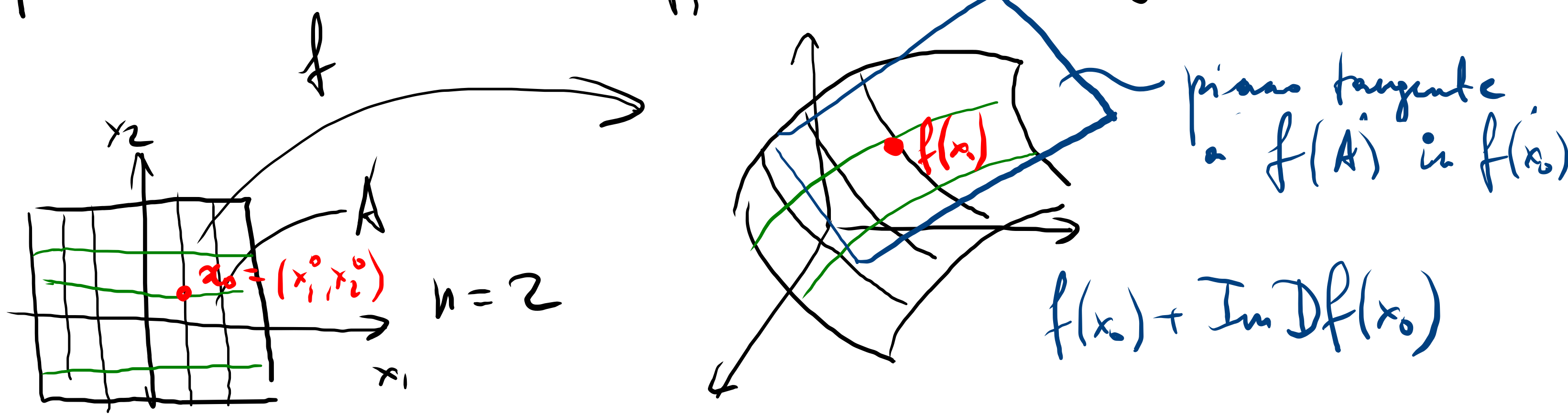
$$\dot{z}(t) = (\dot{x}_1(t), \dots, \dot{x}_m(t))$$

3°) $n \geq 1$, $m = 1$ funzioni scalari

$$f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) + o(|x - x_0|)$$

$$x \to x_0$$

$f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ differenziabile in $x_0 \in A$



Importante da ricordare!

•) $f : A \subset \mathbb{R}^n \to \mathbb{R}$ differenziabile in $x_0 \in A \Rightarrow$

$\forall v \in \mathbb{R}^n, |v| = 1 \quad \exists \frac{\partial f}{\partial v}(x_0) := f'_t(x+tv)\Big|_{t=0}$

th del differenziale totale

•) $\not\Leftarrow$, ma se $\frac{\partial f}{\partial x_i}$ sono continue in un intorno di x_0 per ogni $i = 1, \dots, n$, allora f è differenziabile in x_0

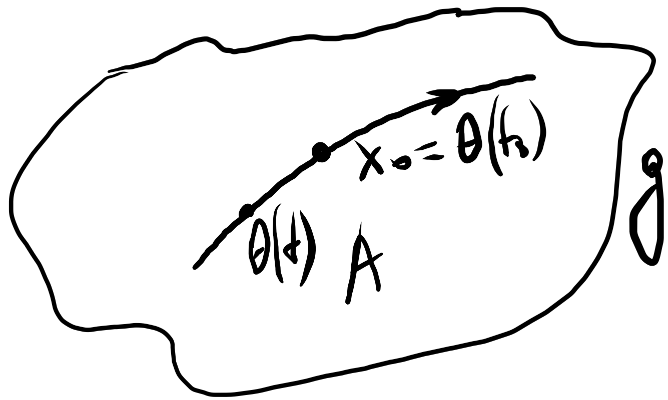
In particolare, se $f \in C^1(A) \Rightarrow f$ è differenziabile ovunque in A

Differenziale / derivate di funzione composta

$f: A \subset \mathbb{R}^n \to \mathbb{R}$

aperto

$x_0 \in A$



$g(t) := f(\theta(t)) = (f \circ \theta)(t) =$

$= f(\theta_1(t), \dots, \theta_n(t))$

$\theta: I \subset \mathbb{R} \to A \subset \mathbb{R}^n$ curva

$\theta(t) = (\theta_1(t), \dots, \theta_n(t))$

I intervallo aperto

$t_0 \in I \qquad \theta(t_0) = x_0$

$g(t) := f(\theta(t))$, $g : \underset{\substack{\cap \\ \mathbb{R}}}{I} \to \mathbb{R}$

Th ("chain rule" / regola della catena)

Sia f differenziabile in x_0 e
θ —"— —"— in t_0, allora

$$g'(t_0) = \frac{d}{dt} f(\theta(t)) \Big|_{t=t_0} = \nabla f(\theta(t_0)) \cdot \dot{\theta}(t_0)$$

Dimostrazione

$t \to t_0 \Rightarrow \theta(t) \to \theta(t_0) = x_0$

$$f(\theta(t)) = f(x_0) + \nabla f(x_0)\cdot(\theta(t) - x_0) + o(|\theta(t) - x_0|) =$$

$$= f(\theta(t_0)) + \nabla f(\theta(t_0))\cdot(\theta(t) - \theta(t_0)) + o(|\theta(t) - \theta(t_0)|)$$

$$= f(\theta(t_0)) + \nabla f(\theta(t_0))\cdot(\dot\theta(t_0)(t-t_0) + o(|t-t_0|)) +$$

$$+ o(|\dot\theta(t_0)(t-t_0) + o(|t-t_0|)|)$$

$$= f(\theta(t_0)) + \nabla f(\theta(t_0))\cdot\dot\theta(t_0)(t-t_0) + o(|t-t_0|)$$

$$\theta(t) = \theta(t_0) + \dot\theta(t_0)(t-t_0) + o(|t-t_0|)$$

$t \to t_0$

$$o(|\dot\theta(t_0)(t-t_0) + o(|t-t_0|)|) = o(|t-t_0|)$$

$$|\dot\theta(t_0)(t-t_0) + o(|t-t_0|)| \le |\dot\theta(t_0)||t-t_0| + o(|t-t_0|)$$

$$o(|\dot\theta(t_0)||t-t_0|) = o(|t-t_0|)$$

$$g(t) = f(\theta(t)) = \underset{\underset{g(t_0)}{\parallel}}{f(\theta(t_0))} + \underbrace{\nabla f(\theta(t_0)) \cdot \dot{\theta}(t_0)}\,(t-t_0) + o(|t-t_0|)$$

per $t \to t_0$

$$\boxed{g'(t_0) = \nabla f(\theta(t_0)) \cdot \dot{\theta}(t_0)}$$

"Regola della catena"

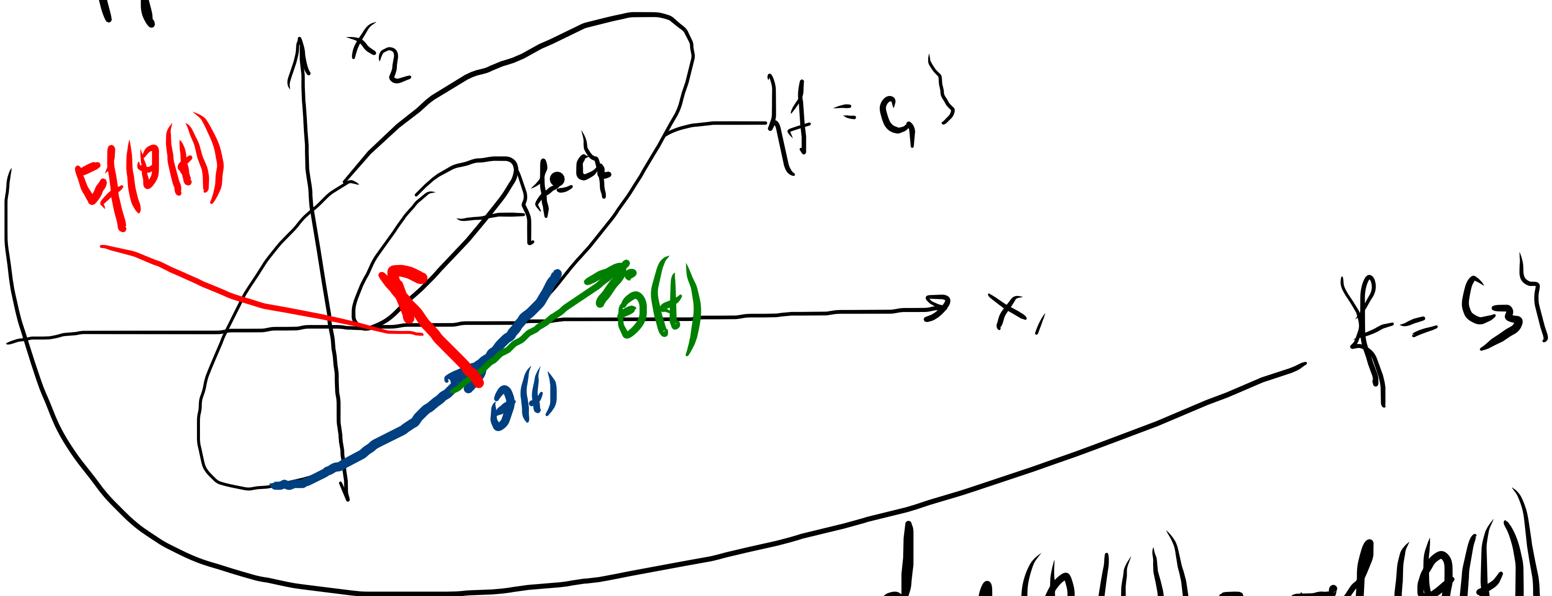
Chain rule

Caso particolare $n=1$: $g'(t_0) = f'(\theta(t_0))\,\dot{\theta}(t_0)$

$$\left((f \circ \theta)(t)\right)'\Big|_{t=t_0} = f'(\theta(t_0))\,\theta'(t_0)$$

Conseguenza banale ma curiosa della regola delle catene:

Supponiamo che $f(\theta(t)) = \text{const}$



$$0 = \frac{d}{dt} f(\theta(t)) = \underline{\nabla f(\theta(t)) \cdot \dot\theta(t)}$$

Ovvero, $\left\{ \begin{array}{l} \nabla f(x_0) \perp \text{piano tangente all'insieme di livello} \\[2em] \text{se } f \text{ è differenziabile e ha senso di parlare del piano tangente all'insieme di livello} \end{array} \right.$

(!)

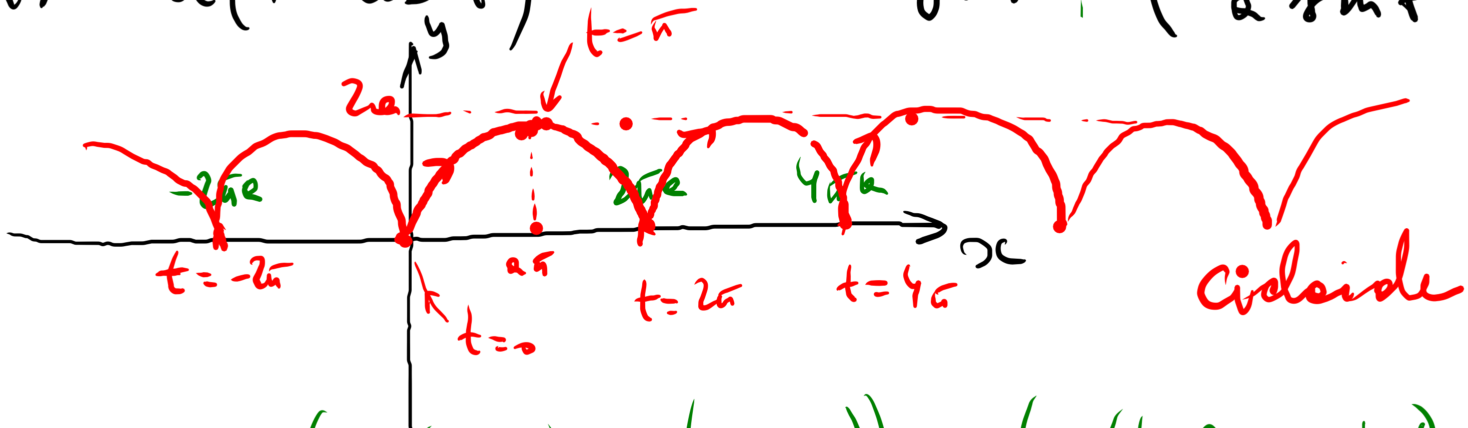
Esempi ed esercizi

1°) $\gamma(t) = (x(t), y(t))$, $t \in \mathbb{R}$

$$\begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases}$$

$a > 0$

$$\dot{\gamma}(t) = \begin{pmatrix} a(1-\cos t) \\ a \sin t \end{pmatrix}$$



$$\gamma(t+2\pi) = (x(t+2\pi), y(t+2\pi)) = (a(t+2\pi - \sin t), a(1-\cos t))$$

$$= \gamma(t) + 2\pi a(1,0) = \gamma(t) + 2\pi a \bar{e}_1$$

1°) lunghezza l di un solo arco di cicloide

$$\gamma(t) = \begin{pmatrix} a(t - \sin t) \\ a(1-\cos t) \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\dot\gamma(t) = \begin{pmatrix} a(1-\cos t) \\ a \sin t \end{pmatrix}$$

$$l = \int_0^{2\pi} |\dot\gamma(t)|\, dt = \int_0^{2\pi} \sqrt{a^2(1-\cos t)^2 + a^2 \sin^2 t}\, dt$$

$$= a \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = a\sqrt{2} \int_0^{2\pi} \sqrt{1-\cos t}\, dt =$$

ved. p. succ.

$$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - (1-\cos^2 t) = 2\cos^2 t - 1$$

$$\cos t = 2\cos^2 t/2 - 1.$$

$$1 - \cos t = 2 - 2\cos^2 t/2 = 2(1-\cos^2 t/2) = 2\sin^2 t/2$$

$$l = a\sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2 t/2}\,dt = 2a\int_0^{2\pi}\sqrt{\sin^2 t/2}\,dt =$$

$$= 2a\int_0^{2\pi}\left|\sin t/2\right|dt = 2a\int_0^{2\pi}\sin \underbrace{t/2}_{\tau}\,dt =$$

$$= 2a\int_0^{\pi}\sin\tau\, 2\,d\tau = 4a\int_0^{\pi}\sin\tau\,d\tau =$$

$\tau = t/2$
$t = 2\tau$
$dt = 2d\tau$

$$= -4a\cos\tau\Big|_0^{\pi} = 2\cdot 4a = \underline{\underline{8a}}$$

2°) Trovare l'eq della retta tangente alla cicloide in $t = \pi/2$

$$\dot{\gamma}\left(\frac{\pi}{2}\right) = \begin{pmatrix} a\left(1-\cos\frac{\pi}{2}\right) \\ a \sin\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix}$$

$$\gamma\left(\frac{\pi}{2}\right) = \begin{pmatrix} a\left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) \\ a\left(1-\cos\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} a\left(\frac{\pi}{2}-1\right) \\ a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\left(\frac{\pi}{2}-1\right) \\ a \end{pmatrix} + \begin{pmatrix} a \\ a \end{pmatrix}\left(t - \quad\right), \quad t \in \mathbb{R}$$

o

$$\begin{cases} x(t) = a\left(\frac{\pi}{2}-1\right) + a\left(t-\frac{\pi}{2}\right), & t \in \mathbb{R} \\ y(t) = a + a\left(t-\frac{\pi}{2}\right), \end{cases}$$

sostegno in forma parametrica

eq. parametrica di una retta

$$t - \pi/2 = \frac{y-a}{a} \Rightarrow x = a\left(\frac{\pi}{2}-1\right) + y - a$$

$$\Updownarrow$$

$$\boxed{x - y = a\frac{\pi}{2}}$$

sostegno in forma non parametrica

(!) 3°| Trovare l'eq delle retta tangente alla cicloide in $t=0$.

$$\dot{\gamma}(0) = \begin{pmatrix} a(1-\cos 0) \\ a \sin 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\gamma(t) = \gamma(0) + \dot{\gamma}(0)(t-0) + o(t-0) =$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} + o(t) =$$

$$= \underline{\begin{pmatrix} 0 \\ 0 \end{pmatrix} + o(t)}$$

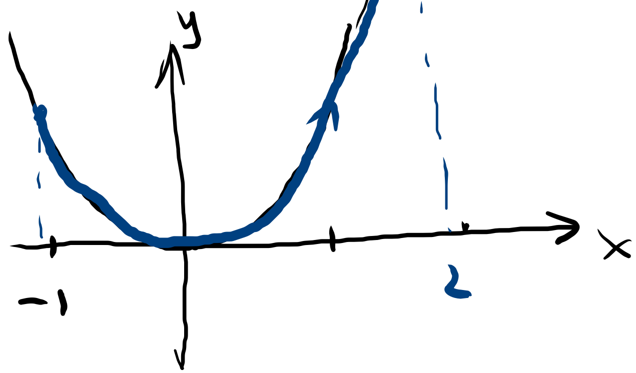
$$\gamma(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + o(t) \qquad t \to 0$$

Non c'è retta tangente

Def: $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ si chiama
(I intervallo)
regolare, se $\gamma \in C^1(I)$,
ed inoltre $\dot{\gamma}(t) \neq 0 \quad \forall t \in I$

N.B. •) cicloide non è regolare
•) circonferenza $\begin{cases} x(t) = a \cos t \\ y(t) = a \sin t \end{cases}$
è regolare

2). $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$ $\quad t \in [-1, 2]$



$y = x^2$

$$l = \int_{-1}^{2} \left\| \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \right\| dt = \int_{-1}^{2} \left\| \begin{pmatrix} 1 \\ 2t \end{pmatrix} \right\| dt =$$

$$= \int_{-1}^{2} \sqrt{1 + 4t^2}\, dt = \dots$$

da fare a casa

In generale $y = f(x)$, $x \in [a, b]$

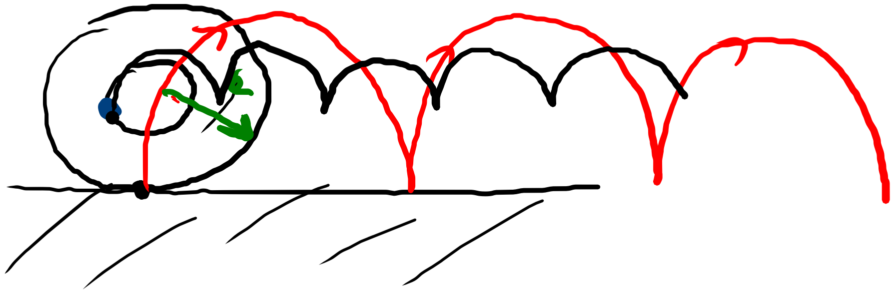
il grafico può essere parametrizzato, ad. es., come

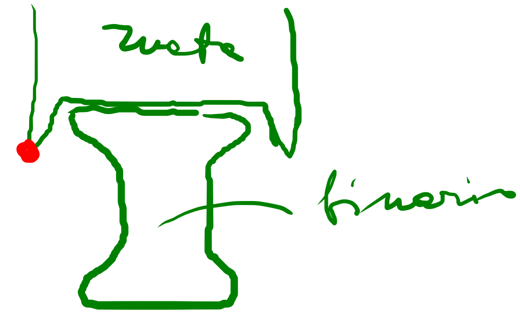
$$\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases} \qquad t \in [a, b]$$

Quindi:

La lunghezza d'arco del grafico di una funzione f è

$$L = \int_a^b \left| \begin{pmatrix} 1 \\ f'(t) \end{pmatrix} \right| dt = \int_a^b \sqrt{1 + f'^2(t)}\, dt$$



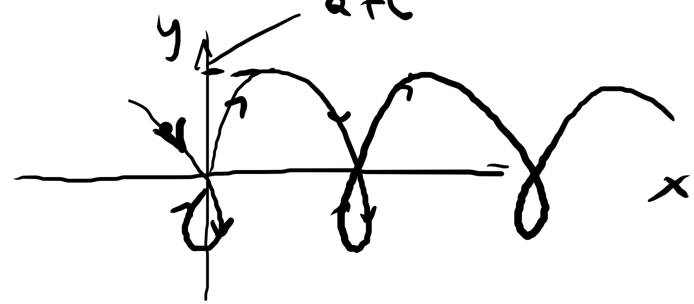


$$\begin{cases} x(t) = at - b \sin t \\ y(t) = a - b \cos t \end{cases}$$

•) $b = a \Rightarrow$ cicloide

•) $b < a$

•) $b > a$



$$\dot{\gamma} = \begin{pmatrix} a - b \cos t \\ b \sin t \end{pmatrix}$$

$b \sin t = 0 \Rightarrow t = k\pi$

$y(t) = a - b \cos\left((2k+1)\pi\right) =$

$= \underline{a + b}$

Un altro caso particolare: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$

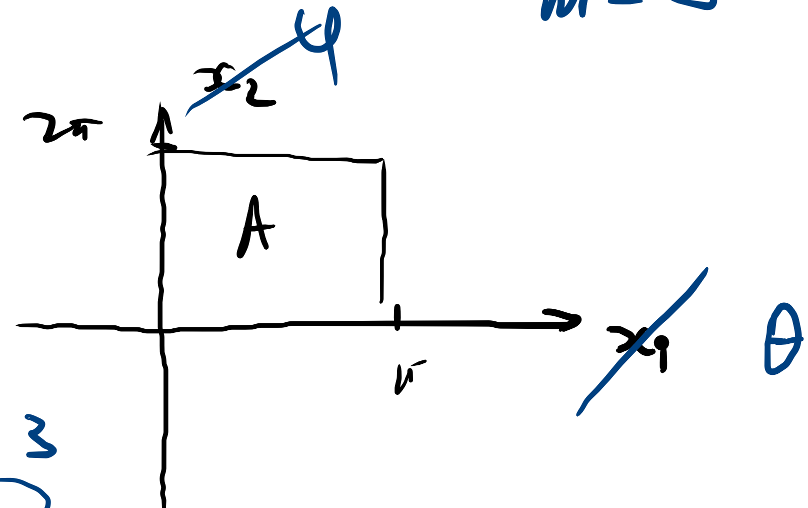
$n = 2$

molto spesso $m = 3$

Superficie parametrizzate

Esempi

$$A = [0, 2\pi] \times [0, \pi]$$



$$f(\theta, \varphi) = \begin{pmatrix} x(\theta, \varphi) \\ y(\theta, \varphi) \\ z(\theta, \varphi) \end{pmatrix} \in \mathbb{R}^3$$

$$\begin{cases} x = R\cos\theta\sin\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\varphi \end{cases}$$

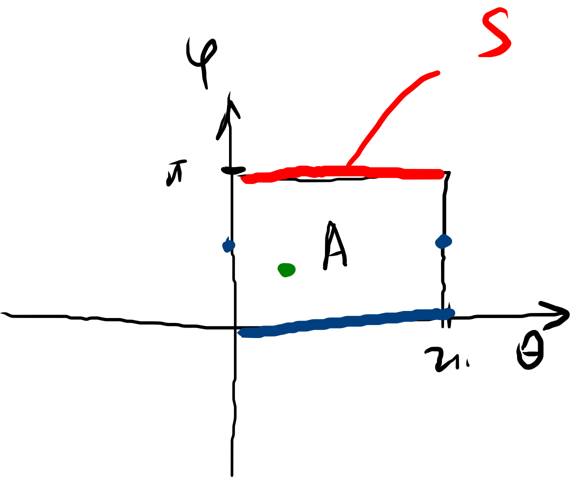
$R > 0$

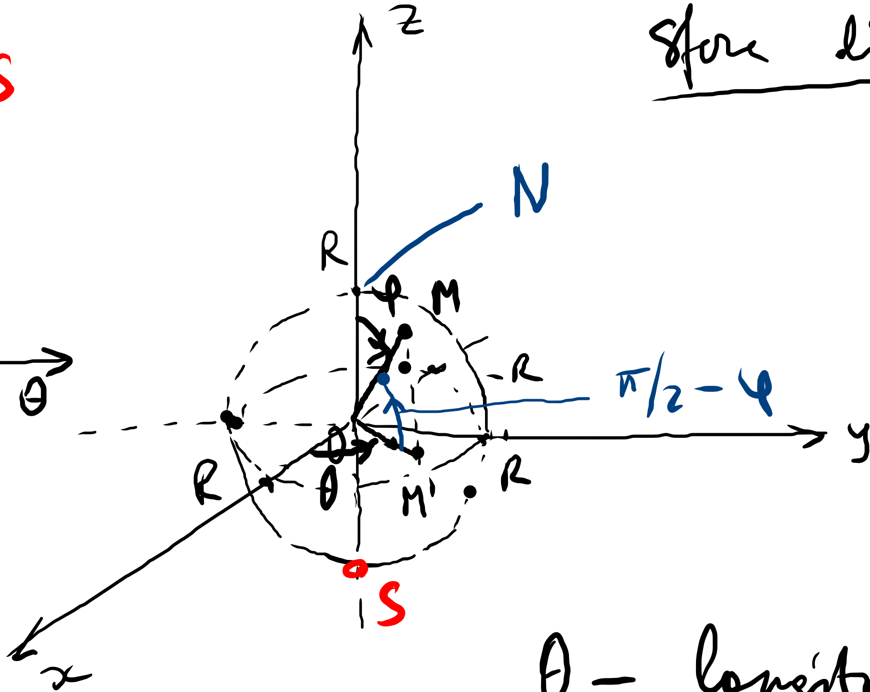
$$x^2+y^2+z^2 = \underline{R^2\cos^2\theta\sin^2\varphi} + \underline{R^2\sin^2\theta\sin^2\varphi} + R^2\cos^2\varphi =$$

$$= R^2\sin^2\varphi + R^2\cos^2\varphi = R^2$$

$$x^2+y^2+z^2 = R^2$$

Sfera di raggio R



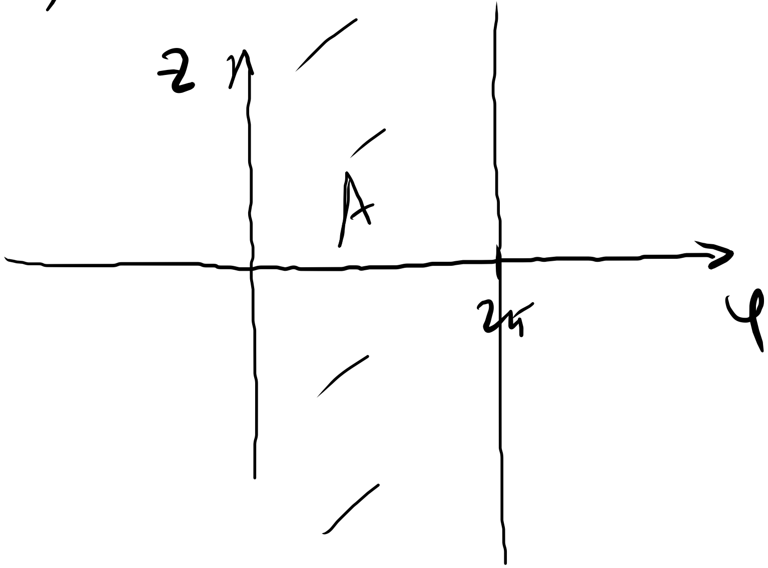


$M(x, y, z)$

$M' = \text{proj}_{(x,y)} M$

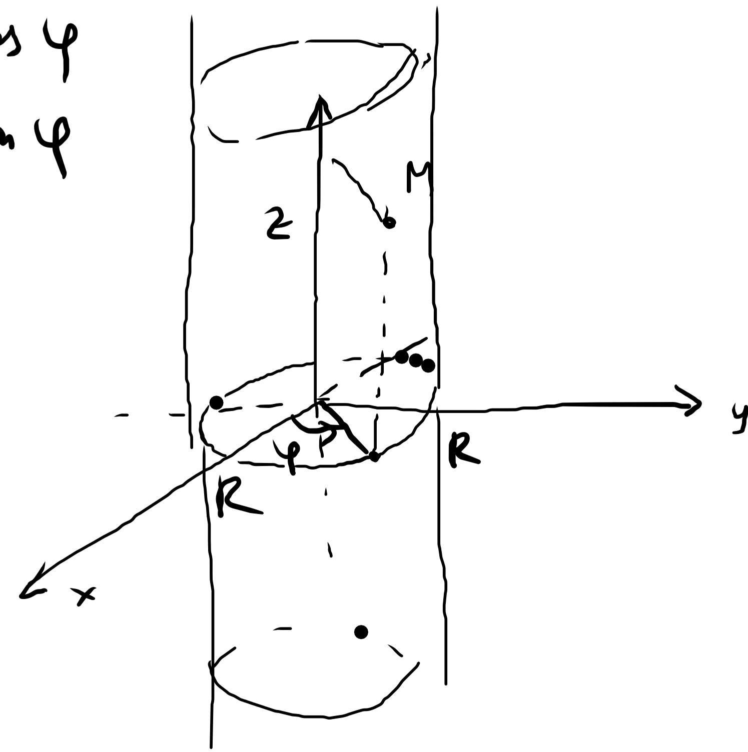
$\left.\begin{array}{l} \theta - \text{longitudine} \\ \pi/2 - \varphi - \text{latitudine} \\ \varphi - \text{colatitudine} \end{array}\right\}$ Coordinate sferiche

2) $A = [0, 2\pi] \times \mathbb{R} \subset \mathbb{R}^2$



$$u(\varphi, z) = \left(x_1(\varphi, z), x_2(\varphi, z), x_3(\varphi, z) \right)$$

$$\begin{cases} x_1 = R\cos\varphi \\ x_2 = R\sin\varphi \\ x_3 = z \end{cases}$$



(z, φ) – coordinate cilindriche