591AA 21/22 – COMPITO, LEZIONI 8 E 9

Data di scadenza: Questo compito non sarà raccolto per la valutazione. Invece, circa una settimana dopo che è stato assegnato, le soluzioni saranno pubblicate.

Questi problemi sono trascritti da "Schaum's Outlines, Linear Algebra, 3rd ed", che include anche le soluzioni e molti problemi simili. Puoi trovare questo libro su Scribed e altrove. Purtroppo non sono riuscito a trovare una traduzione italiana di questo libro.

Problema 1. [3.22, pg. 100, pdf pg. 108] Risolvi ciascuno dei seguenti sistemi lineari utilizzando l'eliminazione gaussiana:

(a)

$$\begin{aligned} x + 2y - z &= 3 \\ x + 3y + z &= 5 \\ 3x + 8y + 4z &= 17 \end{aligned}$$

(b)

$$\begin{aligned} x - 2y + 4z &= 2 \\ 2x - 3y + 5z &= 3 \\ 3x - 4y + 6z &= 7 \end{aligned}$$

(c)

$$\begin{aligned} x + y + 3z &= 1 \\ 2x + 3y - z &= 3 \\ 5x + 7y + z &= 7 \end{aligned}$$

Problema 2. [3.23, pg. 101, pdf pg. 109] Trova la soluzione generale del seguente sistema lineare. Siano x_3 e x_5 le variabili indipendenti

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - 2x_4 + 4x_5 &= 1 \\ 2x_1 + 5x_2 - 8x_3 - x_4 + 6x_5 &= 4 \\ x_1 + 4x_2 - 7x_3 + 5x_4 + 2x_5 &= 8 \end{aligned}$$

Problema 3. [3.26, pg 103, pdf pg. 111] Trova una base per le soluzioni del seguente sistema lineare: [Metodo: Algoritmo per trovare la base di $\ker(A)$, Lezione 9, pg 6].

$$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 - 7x_3 + 4x_4 &= 0 \\ 5x_1 + 10x_2 - 11x_3 + 6x_4 &= 0 \end{aligned}$$

Nota: L'autore dice qualcosa di sbagliato nella soluzione della parte (b) di questo problema nel libro. Lo spazio vettoriale zero $\{0\}$ ha dimensione zero. La base di $\{0\}$ è l'insieme vuoto.

Problema 4. [4.41, pg. 150, pdf pg. 158] Trova una base per lo spazio delle righe di ciascuna delle seguenti matrici.

(a)

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{pmatrix}$$

(b)

$$B = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

[Metodo: Algoritmo per trovare una base di r(A), pg 5, Lezione 9.]

Problema 5. [4.30 & 4.31, pg 147, pdf pg 155] Trova una base per lo span lineare dei seguenti insiemi di vettori

(a)

$$u_1 = \begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix} u_3 = \begin{pmatrix} 3 \\ 8 \\ -3 \\ -5 \end{pmatrix}$$

(b)

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 4 \\ -2 \\ 6 \\ 8 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \\ 2 \\ 6 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 4 \\ 5 \\ 1 \\ 8 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 2 \\ 7 \\ 3 \\ 3 \\ 9 \end{pmatrix}$$

[Metodo: Algoritmo per trovare una base di Im(A), pg. 6, Lezione 9]

Problema 6. [7.11, pg. 258, pdf 266]: Trova una base per il complemento ortogonale dello span lineare dei seguenti vettori in $\mathbb{R}^5$:

$$u = (1, 2, 3, -1, 2), \quad v = (2, 4, 7, 2, -1)$$

Usa il prodotto scalare standard su $\mathbb{R}^5$: $((x_1, \dots, x_5), (y_1, \dots, y_5)) = x_1 y_1 + \cdots + x_5 y_5$.
[Metodo: Algoritmo per trovare il complement ortogonale, pg. 7, Lezione 9]

Problema 7. [4.10 & 4.11, pg. 141, pdf pg 149] Verificare che i seguenti sottoinsiemi di funzioni W siano un sottospazio dello spazio vettoriale dato V:

(a) $V = \mathbb{R}[x]$, $W =$ polinomi di grado maggiore o uguale a 6 e il polinomio zero. [Il grado del polinomio zero è indefinito perché non ha termini].
(b) $V = \{\text{funzioni } f : \mathbb{R} \to \mathbb{R}\}$ $(= \mathbb{R}^{\mathbb{R}})$, $W = \{f \in V \mid f(1) = f(3)\}$.
(c) $V = \{\text{funzioni } f : \mathbb{R} \to \mathbb{R}\}$, $W = \{f \in V \mid f(-x) = f(x)\}$.

operations "Replace R_3 by $-9R_2 + R_3$" and "Replace R_1 by $2R_2 + R_1$". These operations yield

$$A \sim \begin{bmatrix} 1 & -2 & 3 & 1 & 2 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 9 & 3 & -4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Finally, multiply R_3 by $\frac{1}{2}$ to make the pivot $a_{34} = 1$, and then produce 0's above a_{34} by applying the operations "Replace R_2 by $\frac{2}{3}R_3 + R_2$" and "Replace R_1 by $\frac{1}{3}R_3 + R_1$". These operations yield

$$A \sim \begin{bmatrix} 1 & 0 & \frac{11}{3} & -\frac{1}{3} & \frac{8}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{11}{3} & 0 & \frac{17}{6} \\ 0 & 1 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

which is the row canonical form of A.

SYSTEMS OF LINEAR EQUATIONS IN MATRIX FORM

3.21. Find the augmented matrix M and the coefficient matrix A of the following system:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3y - 4z + 7x &= 5 \\ 6z + 8x - 9y &= 1 \end{aligned}$$

First align the unknowns in the system, and then use the aligned system to obtain M and A. We have

$$\begin{aligned} x + 2y - 3z &= 4 \\ 7x + 3y - 4z &= 5; \\ 8x - 9y + 6z &= 1 \end{aligned} \qquad \text{then} \qquad M = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 7 & 3 & -4 & 5 \\ 8 & -9 & 6 & 1 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 1 & 2 & -3 \\ 7 & 3 & -4 \\ 8 & -9 & 6 \end{bmatrix}$$

3.22. Solve each of the following systems using its augmented matrix M:

$$\begin{aligned} x + 2y - z &= 3 \\ x + 3y + z &= 5 \\ 3x + 8y + 4z &= 17 \end{aligned} \qquad \begin{aligned} x - 2y + 4z &= 2 \\ 2x - 3y + 5z &= 3 \\ 3x - 4y + 6z &= 7 \end{aligned} \qquad \begin{aligned} x + y + 3z &= 1 \\ 2x + 3y - z &= 3 \\ 5x + 7y + z &= 7 \end{aligned}$$

(a) (b) (c)

(*a*) Reduce the augmented matrix M to echelon form as follows:

(1) Applicare l'eliminazione gaussiana alla matrice aumentata

$$M = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 1 & 3 & 1 & 5 \\ 3 & 8 & 4 & 17 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 2 & 7 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

Non ci sono variabili indipendenti

Now write down the corresponding triangular system

(2) Risolvi il sistema di matrici scalina risultante

$$\begin{aligned} x + 2y - z &= 3 \\ y + 2z &= 2 \\ 3z &= 4 \end{aligned}$$

and solve by back-substitution to obtain the unique solution

$$x = \tfrac{17}{3}, \quad y = -\tfrac{2}{3}, \quad z = \tfrac{4}{3} \qquad \text{or} \qquad u = \left(\tfrac{17}{3}, -\tfrac{2}{3}, \tfrac{4}{3}\right)$$

Alternately, reduce the echelon form of M to row canonical form, obtaining

$$M \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & \frac{13}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{17}{3} \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{bmatrix}$$

This also corresponds to the above solution.

(*b*) First reduce the augmented matrix M to echelon form as follows:

$$M = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 2 & -3 & 5 & 3 \\ 3 & -4 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 2 & -6 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Questo sistema
non ha soluzione

The third row corresponds to the degenerate equation $0x + 0y + 0z = 3$, which has no solution. Thus "DO NOT CONTINUE". The original system also has no solution. (Note that the echelon form indicates whether or not the system has a solution.)

(*c*) Reduce the augmented matrix M to echelon form and then to row canonical form:

$$M = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 2 & -14 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \end{array}\right]$$

0 0 0 | 0

z è una variabile
indipendente

Questo autore
elimina zero
righe

(The third row of the second matrix is deleted, since it is a multiple of the second row and will result in a zero row.) Write down the system corresponding to the row canonical form of M and then transfer the free variables to the other side to obtain the free-variable form of the solution:

$$\begin{aligned} x + \quad 10z &= 0 \\ y - \quad 7z &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= -10z \\ y &= 1 + 7z \end{aligned}$$

Here z is the only free variable. The parametric solution, using $z = a$, is as follows:

$$x = -10a, \quad y = 1 + 7a, \quad z = a \qquad \text{or} \qquad u = (-10a,\ 1 + 7a,\ a)$$

a=qualsiasi scalare

3.23. Solve the following system using its augmented matrix M:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 - 2x_4 + 4x_5 &= 1 \\ 2x_1 + 5x_2 - 8x_3 - \quad x_4 + 6x_5 &= 4 \\ x_1 + 4x_2 - 7x_3 + 5x_4 + 2x_5 &= 8 \end{aligned}$$

Reduce the augmented matrix M to echelon form and then to row canonical form:

Lo stesso metodo
dell'ultimo problema

$$M = \left[\begin{array}{ccccc|c} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 4 \\ 1 & 4 & -7 & 5 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 2 & -4 & 7 & -2 & 7 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & -2 & 4 & 1 \\ 0 & 1 & -2 & 3 & -2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 0 & 8 & 7 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 24 & 21 \\ 0 & 1 & -2 & 0 & -8 & -7 \\ 0 & 0 & 0 & 1 & 2 & 3 \end{array}\right]$$

Le variabili dipendenti sono
x_1, x_2 e x_4

Write down the system corresponding to the row canonical form of M and then transfer the free variables to the other side to obtain the free-variable form of the solution:

$$\begin{aligned} x_1 + \qquad x_3 + \quad 24x_5 &= 21 \\ x_2 - 2x_3 - \quad 8x_5 &= -7 \\ x_4 + 2x_5 &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= 21 - x_3 - 24x_5 \\ x_2 &= -7 + 2x_3 + 8x_5 \\ x_4 &= 3 - 2x_5 \end{aligned}$$

Risolvi lavorando
a ritroso dall'ultima
equazione

Here x_1, x_2, x_4 are the pivot variables and x_3 and x_5 are the free variables. Recall that the parametric form of the solution can be obtained from the free-variable form of the solution by simply setting the free variables equal to parameters, say $x_3 = a$, $x_5 = b$. This process yields

$$x_1 = 21 - a - 24b, \quad x_2 = -7 + 2a + 8b, \quad x_3 = a, \quad x_4 = 3 - 2b, \quad x_5 = b$$

or

$$u = (21 - a - 24b,\ -7 + 2a + 8b,\ a,\ 3 - 2b,\ b)$$

which is another form of the solution.

3.25. Let $u_1 = (1, 2, 4)$, $u_2 = (2, -3, 1)$, $u_3 = (2, 1, -1)$ in $\mathbf{R}^3$. Show that u_1, u_2, u_3 are orthogonal, and write v as a linear combination of u_1, u_2, u_3, where: (*a*) $v = (7, 16, 6)$, (*b*) $v = (3, 5, 2)$.

Take the dot product of pairs of vectors to get

$$u_1 \cdot u_2 = 2 - 6 + 4 = 0, \quad u_1 \cdot u_3 = 2 + 2 - 4 = 0, \quad u_2 \cdot u_3 = 4 - 3 - 1 = 0$$

Thus the three vectors in $\mathbf{R}^3$ are orthogonal, and hence Fourier coefficients can be used. That is, $v = xu_1 + yu_2 + zu_3$, where

$$x = \frac{v \cdot u_1}{u_1 \cdot u_1}, \qquad y = \frac{v \cdot u_2}{u_2 \cdot u_2}, \qquad z = \frac{v \cdot u_3}{u_3 \cdot u_3}$$

(*a*) We have

$$x = \frac{7 + 32 + 24}{1 + 4 + 16} = \frac{63}{21} = 3, \qquad y = \frac{14 - 48 + 6}{4 + 9 + 1} = \frac{-28}{14} = -2, \qquad z = \frac{14 + 16 - 6}{4 + 1 + 1} = \frac{24}{6} = 4$$

Thus $v = 3u_1 - 2u_2 + 4u_3$.

(*b*) We have

$$x = \frac{3 + 10 + 8}{1 + 4 + 16} = \frac{21}{21} = 1, \qquad y = \frac{6 - 15 + 2}{4 + 9 + 1} = \frac{-7}{14} = -\frac{1}{2}, \qquad z = \frac{6 + 5 - 2}{4 + 1 + 1} = \frac{9}{6} = \frac{3}{2}$$

Thus $v = u_1 - \frac{1}{2}u_2 + \frac{3}{2}u_3$.

3.26. Find the dimension and a basis for the general solution W of each of the following homogeneous systems:

$$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ 3x_1 + 6x_2 - 7x_3 + 4x_4 &= 0 \\ 5x_1 + 10x_2 - 11x_3 + 6x_4 &= 0 \end{aligned} \qquad\qquad \begin{aligned} x - 2y - 3z &= 0 \\ 2x + y + 3z &= 0 \\ 3x - 4y - 2z &= 0 \end{aligned}$$

(*a*) (*b*)

Applicare l'eliminazione gaussiana

$$\begin{pmatrix} 2 & 4 & -5 & 3 \\ 3 & 6 & -7 & 4 \\ 5 & 10 & -11 & 6 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & -5 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(*a*) Reduce the system to echelon form using the operations "Replace L_2 by $-3L_1 + 2L_2$", "Replace L_3 by $-5L_1 + 2L_3$", and then "Replace L_3 by $-2L_2 + L_3$". These operations yield:

$$\begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ x_3 - x_4 &= 0 \\ 3x_3 - 3x_4 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x_1 + 4x_2 - 5x_3 + 3x_4 &= 0 \\ x_3 - x_4 &= 0 \end{aligned}$$

Le variabili indipendenti sono x_2 e x_4

The system in echelon form has two free variables, x_2 and x_4, so $\dim W = 2$. A basis $[u_1, u_2]$ for W may be obtained as follows:

(1) Set $x_2 = 1$, $x_4 = 0$. Back-substitution yields $x_3 = 0$, and then $x_1 = -2$. Thus $u_1 = (-2, 1, 0, 0)$.

(2) Set $x_2 = 0$, $x_4 = 1$. Back-substitution yields $x_3 = 1$, and then $x_1 = 1$. Thus $u_2 = (1, 0, 1, 1)$.

(1) Sia $x_2 = 1$, $x_4 = 0$ $\to u_1$

(2) Sia $x_4 = 1$, $x_2 = 0$ $\to u_2$

(*b*) Reduce the system to echelon form, obtaining

$$\begin{aligned} x - 2y - 3z &= 0 \\ 5y + 9z &= 0 \\ 2y + 7z &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x - 2y - 3z &= 0 \\ 5y + 9z &= 0 \\ 17z &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & -2 & -3 \\ 2 & 1 & 3 \\ 3 & -4 & -2 \end{pmatrix} \to \to \begin{pmatrix} 1 & -2 & -3 \\ 0 & 5 & 9 \\ 0 & 0 & 17 \end{pmatrix}$$

There are no free variables (the system is in triangular form). Hence $\dim W = 0$, and W has no basis. Specifically, W consists only of the zero solution, that is, $W = \{0\}$.

Non ci sono variabili indipendenti, quindi 0 è l'unica soluzione

La base di W è l'insieme vuoto

$$B = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Base delle soluzioni per la parte (a)

3.27. Find the dimension and a basis for the general solution W of the following homogeneous system using matrix notation:

Stiamo risolvendo Ax=0, quindi x deve essere un vettore colonna, non un vettore riga.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 &= 0 \\ 3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 &= 0 \end{aligned}$$

Show how the basis gives the parametric form of the general solution of the system.

By (*a*), W is nonempty, and, by (*b*), the operations of vector addition and scalar multiplication are well defined for W. Axioms $[A_1]$, $[A_4]$, $[M_1]$, $[M_2]$, $[M_3]$, $[M_4]$ hold in W since the vectors in W belong to V. Thus we need only show that $[A_2]$ and $[A_3]$ also hold in W. Now $[A_2]$ holds since the zero vector in V belongs to W by (*a*). Finally, if $v \in W$, then $(-1)v = -v \in W$, and $v + (-v) = 0$. Thus $[A_3]$ holds.

4.9. Let $V = \mathbf{R}^3$. Show that W is not a subspace of V, where:
(*a*) $W = \{(a, b, c) : a \geq 0\}$, (*b*) $W = \{(a, b, c) : a^2 + b^2 + c^2 \leq 1\}$.

In each case, show that Theorem 4.2 does not hold.

(*a*) W consists of those vectors whose first entry is nonnegative. Thus $v = (1, 2, 3)$ belongs to W. Let $k = -3$. Then $kv = (-3, -6, -9)$ does not belong to W, since -3 is negative. Thus W is not a subspace of V.

(*b*) W consists of vectors whose length does not exceed 1. Hence $u = (1, 0, 0)$ and $v = (0, 1, 0)$ belong to W, but $u + v = (1, 1, 0)$ does not belong to W, since $1^2 + 1^2 + 0^2 = 2 > 1$. Thus W is not a subspace of V.

4.10. Let $V = \mathbf{P}(t)$, the vector space of real polynomials. Determine whether or not W is a subspace of V, where:

(*a*) W consists of all polynomials with integral coefficients.

(*b*) W consists of all polynomials with degree ≥ 6 and the zero polynomial.

Sì. Se sommi polinomi di grado ≥ 6, il risultato è 0 o un polinomio di grado ≥ 6. Lo stesso per la moltiplicazione scalare.

(*c*) W consists of all polynomials with only even powers of t.

(*a*) No, since scalar multiples of polynomials in W do not always belong to W. For example,

$$f(t) = 3 + 6t + 7t^2 \in W \qquad \text{but} \qquad \tfrac{1}{2}f(t) = \tfrac{3}{2} + 3t + \tfrac{7}{2}t^2 \notin W$$

(*b*) and (*c*). Yes. Since, in each case, W contains the zero polynomial, and sums and scalar multiples of polynomials in W belong to W.

4.11. Let V be the vector space of functions $f : \mathbf{R} \to \mathbf{R}$. Show that W is a subspace of V, where:

(*a*) $W = \{f(x) : f(1) = 0\}$, all functions whose value at 1 is 0.

(*b*) $W = \{f(x) : f(3) = f(1)\}$, all functions assigning the same value to 3 and 1.

(*c*) $W = \{f(t) : f(-x) = -f(x)\}$, all *odd functions*.

Let $\hat{0}$ denote the zero polynomial, so $\hat{0}(x) = 0$ for every value of x.

(*a*) $\hat{0} \in W$, since $\hat{0}(1) = 0$. Suppose $f, g \in W$. Then $f(1) = 0$ and $g(1) = 0$. Also, for scalars a and b, we have

$$(af + bg)(1) = af(1) + bg(1) = a0 + b0 = 0$$

Thus $af + bg \in W$, and hence W is a subspace.

Sì. (b) è chiuso rispetto a combinazioni lineari finite.

(*b*) $\hat{0} \in W$, since $\hat{0}(3) = 0 = \hat{0}(1)$. Suppose $f, g \in W$. Then $f(3) = f(1)$ and $g(3) = g(1)$. Thus, for any scalars a and b, we have

$$(af + bg)(3) = af(3) + bg(3) = af(1) + bg(1) = (af + bg)(1)$$

Thus $af + bg \in W$, and hence W is a subspace.

Lo stesso modo

(*c*) $\hat{0} \in W$, since $\hat{0}(-x) = 0 = -0 = -\hat{0}(x)$. Suppose $f, g \in W$. Then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Also, for scalars a and b,

$$(af + bg)(-x) = af(-x) + bg(-x) = -af(x) - bg(x) = -(af + bg)(x)$$

Thus $ab + gf \in W$, and hence W is a subspace of V.

4.30. Let W be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$u_1 = (1, -2, 5, -3), \qquad u_2 = (2, 3, 1, -4), \qquad u_3 = (3, 8, -3, -5)$$

L'autore pensa a $\mathbb{R}^4$ come vettori di riga e trova una base per lo spazio delle righe. Otteniamo solo due pivot, quindi aggiungiamo altri due vettori di riga per ottenere 4 pivot

$$\begin{pmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(*a*) Find a basis and dimension of W. (*b*) Extend the basis of W to a basis of $\mathbf{R}^4$.

(*a*) Apply Algorithm 4.1, the row space algorithm. Form the matrix whose rows are the given vectors, and reduce it to echelon form:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Big\} \text{ Base} \implies \dim = 2$$

The nonzero rows $(1, -2, 5, -3)$ and $(0, 7, -9, 2)$ of the echelon matrix form a basis of the row space of A and hence of W. Thus, in particular, $\dim W = 2$.

(*b*) We seek four linearly independent vectors, which include the above two vectors. The four vectors $(1, -2, 5, -3)$, $(0, 7, -9, 2)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are linearly independent (since they form an echelon matrix), and so they form a basis of $\mathbf{R}^4$, which is an extension of the basis of W.

4.31. Let W be the subspace of $\mathbf{R}^5$ spanned by $u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$, $u_4 = (1, 4, 5, 1, 8)$, $u_5 = (2, 7, 3, 3, 9)$. Find a subset of the vectors that form a basis of W.

Qui usiamo vettori di colonna, non vettori di riga. L'algoritmo per trovare una base consiste nell'applicare l'eliminazione gaussiana, quindi selezionare i vettori colonna nella matrice originale che corrispondono ai pivot nella matrice ridotta.

Here we use Algorithm 4.2, the Casting-out algorithm. Form the matrix M whose columns (not rows) are the given vectors, and reduce it to echelon form:

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot positions are in columns C_1, C_3, C_5. Hence the corresponding vectors u_1, u_3, u_5 form a basis of W, and $\dim W = 3$.

4.30 usando i vettori colonna:

$$\left(\begin{array}{ccc|cccc} 1 & 2 & 3 & 1 & 0 & 0 & 0 \\ -2 & 3 & 8 & 0 & 1 & 0 & 0 \\ 5 & 1 & -3 & 0 & 0 & 1 & 0 \\ -3 & -4 & -5 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cccc} 1 & 0 & -1 & 0 & 0 & \frac{4}{17} & \frac{1}{17} \\ 0 & 1 & 2 & 0 & 0 & -\frac{3}{17} & -\frac{5}{17} \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{17} & \frac{9}{17} \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

Collezione originale di vettori

Unire una base di vettori per fare la parte (b)

Scrivo qui la forma canonica di riga per avere una risposta univoca. È sufficiente applicare l'eliminazione gaussiana per ottenere una matrice scalina.

Base $\operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 3 \\ 8 \\ -3 \\ -5 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 \\ -2 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix}\right\}$

Base $\mathbb{R}^4$: $\left\{\begin{pmatrix} 1 \\ -2 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right\}$

By Theorem 4.15, we can delete from $S \cup B$ each vector that is a linear combination of preceding vectors to obtain a basis B' for V. Since S is linearly independent, no u_k is a linear combination of preceding vectors. Thus B' contains every vector in S, and S is part of the basis B' for V.

4.40. Prove Theorem 4.17: Let W be a subspace of an n-dimensional vector space V. Then $\dim W \le n$. In particular, if $\dim W = n$, then $W = V$.

Since V is of dimension n, any $n+1$ or more vectors are linearly dependent. Furthermore, since a basis of W consists of linearly independent vectors, it cannot contain more than n elements. Accordingly, $\dim W \le n$.

In particular, if $\{w_1, \ldots, w_n\}$ is a basis of W, then, since it is an independent set with n elements, it is also a basis of V. Thus $W = V$ when $\dim W = n$.

RANK OF A MATRIX, ROW AND COLUMN SPACES

4.41. Find the rank and basis of the row space of each of the following matrices:

$$(a)\quad A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}, \qquad (b)\quad B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}.$$

(a) Row reduce A to echelon form:

$$A \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = \{(1 \ 2 \ 0 \ -1), (0 \ 2 \ -3 \ -1)\}$

(1) Applicare l'eliminazione gaussiana
(2) Le righe diverse da zero sono una base dello spazio delle righe
(3) Rango = dimensione dello spazio riga = 2

The two nonzero rows $(1, 2, 0, -1)$ and $(0, 2, -3, -1)$ of the echelon form of A form a basis for rowsp(A). In particular, rank(A) = 2.

(b) Row reduce B to echelon form:

Lo stesso metodo. Il rango è 2

$$B \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$B = \{(1 \ 3 \ 1 \ -2 \ -3), (0 \ 1 \ 2 \ 1 \ -1)\}$

The two nonzero rows $(1, 3, 1, -2, -3)$ and $(0, 1, 2, 1, -1)$ of the echelon form of B form a basis for rowsp(B). In particular, rank(B) = 2.

4.42. Show that $U = W$, where U and W are the following subspaces of $\mathbf{R}^3$:

$$U = \text{span}(u_1, u_2, u_3) = \text{span}(1, 1, -1),\ (2, 3, -1),\ (3, 1, -5)\}$$
$$W = \text{span}(w_1, w_2, w_3) = \text{span}(1, -1, -3),\ (3, -2, -8),\ (2, 1, -3)\}$$

Form the matrix A whose rows are the u_i, and row reduce A to row canonical form:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

ORTHOGONALITY, ORTHONORMAL COMPLEMENTS, ORTHOGONAL SETS

7.10. Find k so that $u = (1, 2, k, 3)$ and $v = (3, k, 7, -5)$ in $\mathbf{R}^4$ are orthogonal.

First find

$$\langle u, v\rangle = (1, 2, k, 3) \cdot (3, k, 7, -5) = 3 + 2k + 7k - 15 = 9k - 12$$

Then set $\langle u, v\rangle = 9k - 12 = 0$ to obtain $k = \frac{4}{3}$.

7.11. Let W be the subspace of $\mathbf{R}^5$ spanned by $u = (1, 2, 3, -1, 2)$ and $v = (2, 4, 7, 2, -1)$. Find a basis of the orthogonal complement $W^{\perp}$ of W.

We seek all vectors $w = (x, y, z, s, t)$ such that

$$\begin{aligned} \langle w, u\rangle &= x + 2y + 3z - s + 2t = 0 \\ \langle w, v\rangle &= 2x + 4y + 7z + 2s - t = 0 \end{aligned}$$

Eliminating x from the second equation, we find the equivalent system

$$\begin{aligned} x + 2y + 3z - s + 2t &= 0 \\ z + 4s - 5t &= 0 \end{aligned}$$

The free variables are y, s, and t. Therefore

(1) Set $y = -1$, $s = 0$, $t = 0$ to obtain the solution $w_1 = (2, -1, 0, 0, 0)$.
(2) Set $y = 0$, $s = 1$, $t = 0$ to find the solution $w_2 = (13, 0, -4, 1, 0)$.
(3) Set $y = 0$, $s = 0$, $t = 1$ to obtain the solution $w_3 = (-17, 0, 5, 0, 1)$.

The set $\{w_1, w_2, w_3\}$ is a basis of $W^{\perp}$.

W è l'insieme dei vettori $w = (w_1, w_2, w_3, w_4, w_5)$ tali che
$(w, (1, 2, 3, -1, 2)) = w_1 + 2w_2 + 3w_3 - w_4 + 2w_5 = 0$
$(w, (2, 4, 7, 2, -1)) = 2w_1 + 4w_2 + 7w_3 + 2w_4 - w_5 = 0$
Quindi, dobbiamo trovare il kernel (spazio nullo) della matrice $\begin{pmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{pmatrix}$

7.13. Let S consist of the following vectors in $\mathbf{R}^4$:

$$u_1 = (1, 1, 0, -1), \quad u_2 = (1, 2, 1, 3), \quad u_3 = (1, 1, -9, 2), \quad u_4 = (16, -13, 1, 3)$$

(*a*) Show that S is orthogonal and a basis of $\mathbf{R}^4$.
(*b*) Find the coordinates of an arbitrary vector $v = (a, b, c, d)$ in $\mathbf{R}^4$ relative to the basis S.

(*a*) Compute

$$\begin{aligned} &u_1 \cdot u_2 = 1 + 2 + 0 - 3 = 0, \qquad u_1 \cdot u_3 = 1 + 1 + 0 - 2 = 0, \qquad u_1 \cdot u_4 = 16 - 13 + 0 - 3 = 0 \\ &u_2 \cdot u_3 = 1 + 2 - 9 + 6 = 0, \qquad u_2 \cdot u_4 = 16 - 26 + 1 + 9 = 0, \qquad u_3 \cdot u_4 = 16 - 13 - 9 + 6 = 0 \end{aligned}$$

Thus S is orthogonal, and hence S is linearly independent. Accordingly, S is a basis for $\mathbf{R}^4$ since any four linearly independent vectors form a basis of $\mathbf{R}^4$.