Password per accesso al portale Elearn

commum21

SEGNALI PERIODICI

Def. $x(t)$ è periodico

$$\Updownarrow$$

$$x(t) = x(t - nT_0)$$

$n \in \mathbb{Z}$

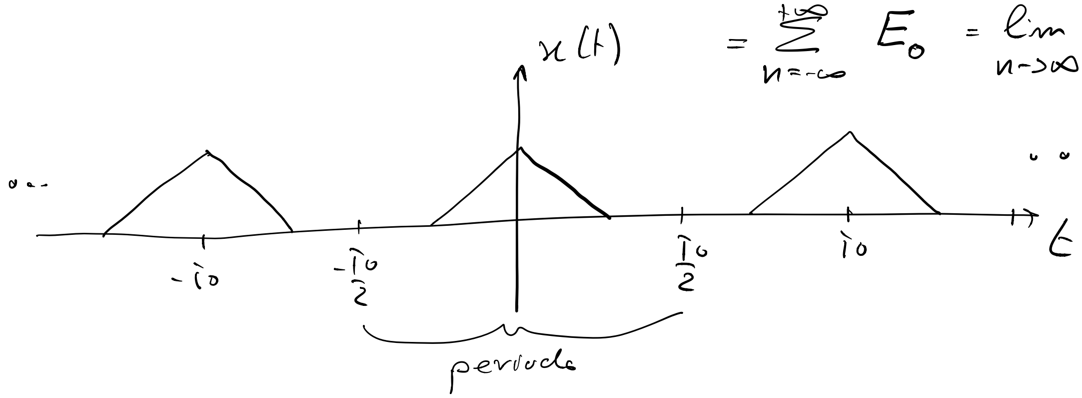
$T_0 \in \mathbb{R}^+$ periodo

$f_0 \triangleq \frac{1}{T_0}$ frequenza

ENERGIA

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt = \infty$$

$$= \sum_{n=-\infty}^{+\infty} E_0 = \lim_{n \to \infty} n E_0$$

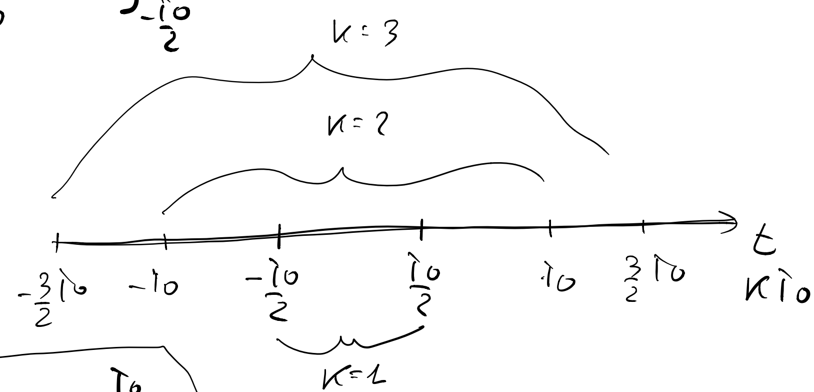


POTENZA MEDIA

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

$$= \lim_{\substack{T \to \infty \\ KT_0 \to \infty \\ K \to \infty}} \frac{1}{KT_0} \int_{-\frac{KT_0}{2}}^{\frac{KT_0}{2}} |x(t)|^2 \, dt$$

$$= \cancel{\lim_{K \to \infty}} \frac{1}{\cancel{K}T_0} \cancel{K} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 \, dt =$$



$$\Rightarrow \boxed{P_x = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 \, dt}$$

$\Rightarrow$ VALOR MEDIO

$$x_m = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \, dt$$

vale la stessa dim. fatta per la potenza media

$\Rightarrow$ Un segnale periodico può essere scomposto in una serie di cosinusoidi

$x(t)$ periodico

$$x(t) = \sum_{n=0}^{+\infty} A_n \cos(2\pi n f_0 t + \varphi_n)$$

generica componente n-esima

Ampiezza A_n

frequenza $n f_0 = \frac{n}{T_0}$

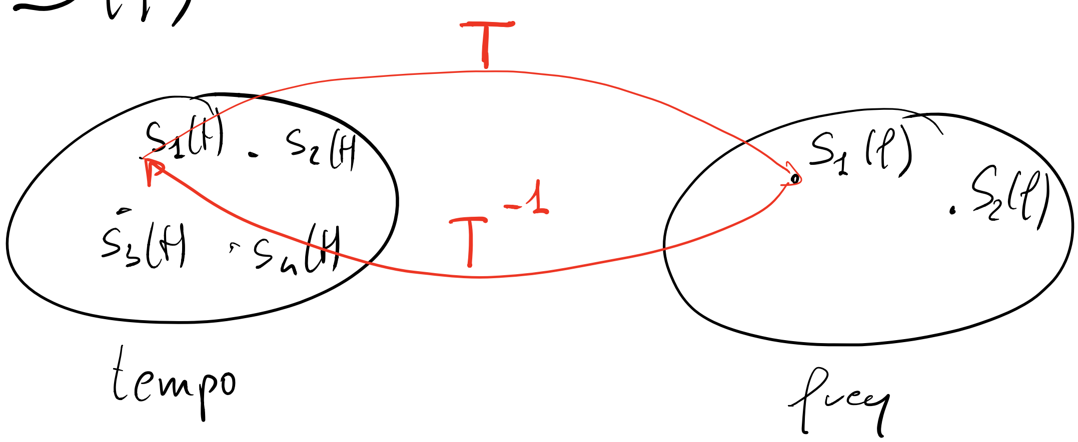
fase φ_n

$f_0 = \frac{1}{T_0}$ freq. base (fondamentale)

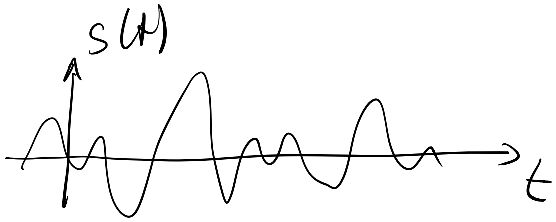
frequenze

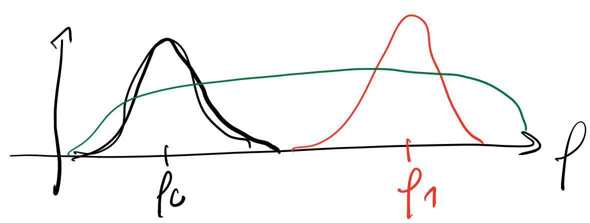
$s(t)$

$S(f)$



TRASFORMAZIONE DI FOURIER





$$s(t) \Longleftrightarrow S(f)$$

Trasformazione biunivoca

Trasformata di Fourier

$$s(t) \Rightarrow S(f)$$

Anti-trasformata di Fourier

$$S(f) \Rightarrow s(t)$$

TRASFORMATA SERIE DI FOURIER

- Si applica a segnali periodici
- Criterio di Dirichelet

Definizione

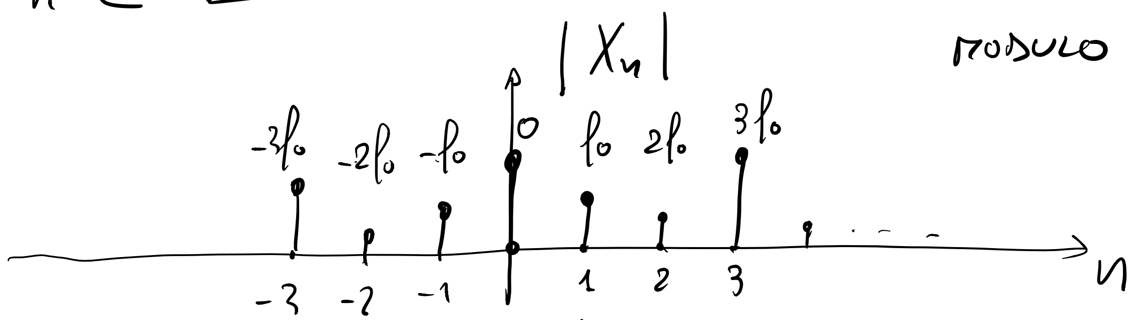
$$\boxed{X_n \triangleq \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}$$

TRASFORMATA DI FOURIER (TSF)

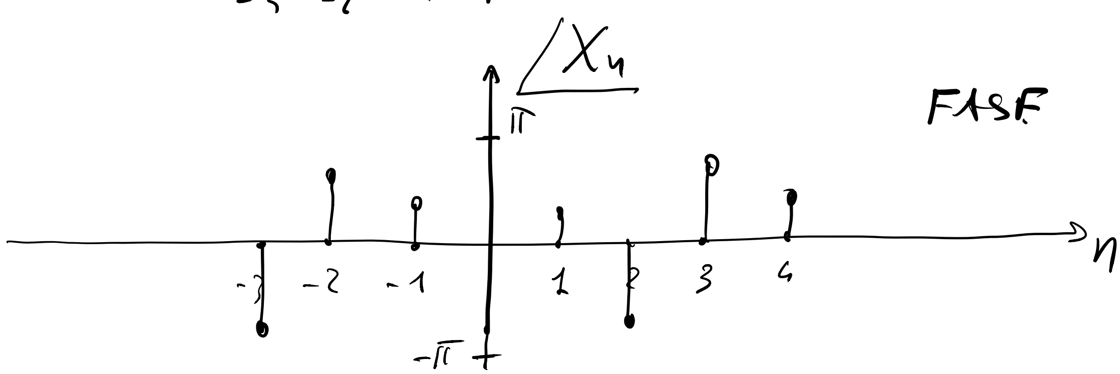
X_n è una sequenza di valori complessi

$n \in \mathbb{Z}$

MODULO



FASE



SPETTRO = rappresentazione in frequenza

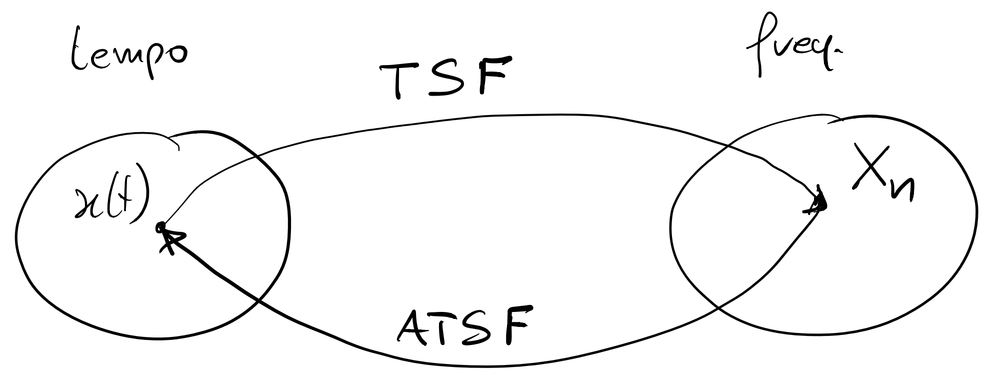
SPETTRO DI UN SEGNALE = rappres. in freq. di un segnale

Trasf. di Fourier $\Rightarrow$ Equazione di analisi

ANTITRASFORMATA SERIE DI FOURIER (ATSF)

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} X_n \, e^{j 2\pi n f_0 t}}$$

Equazione di sintesi



$$x(t) \overset{TSF}{\Longleftrightarrow} X_n$$

Dimostrazione di biunivocità

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt = X_n$$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n\, e^{j2\pi n f_0 t}$$

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underbrace{\sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi k f_0 t}}_{x(t)}\, e^{-j2\pi n f_0 t}\, dt \overset{?}{=} X_n$$

$$= \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_k \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(k-n)f_0 t}\, dt$$

$$e^{j\varphi} = \cos\varphi + j\sin\varphi$$

$$\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos\left(2\pi(k-n)f_0 t\right) dt}_{\substack{k\neq n \Rightarrow 0 \\ k=n \Rightarrow T_0}} + j\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin\left(2\pi(k-n)f_0 t\right) dt}_{\substack{k\neq n \Rightarrow 0 \\ k=0 \Rightarrow 0}}$$

$$= \frac{1}{\cancel{T_0}} X_n \cancel{T_0} = X_n \qquad \text{c.v.d.}$$

SPETTRO DI UN COSENO

$$\boxed{x(t) = A\cos(2\pi f_0 t)} \qquad \underline{\underline{T_0}} = \frac{1}{f_0}$$

$$x(t) \stackrel{?}{=} x(t - nT_0)$$

determinare il periodo del segnale periodico

$$x(t-nT_0) = A\cos\left(2\pi f_0 (t - nT_0)\right)$$

$$= A\cos(2\pi f_0 t - 2\pi \cancel{f_0} n \cancel{T_0})$$

$$= A\cos(2\pi f_0 t - 2\pi n)$$

$$= A\cos(2\pi f_0 t)$$

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt$$

devo det. il periodo!

$$= \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \cdot e^{-j2\pi n f_0 t}\, dt$$

$$\cos\alpha = \frac{e^{j\alpha} + e^{-j\alpha}}{2}$$

$$= \frac{\cos\alpha + \cancel{j\sin\alpha} + \cos\alpha \cancel{- j\sin\alpha}}{2} = \cos\alpha$$

$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(1-n)f_0 t}\,dt \;+$$

$$+ \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi(1+n)f_0 t}\,dt$$

$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos\left(2\pi(1-n)f_0 t\right) + j\sin\left(2\pi(1-n)f_0 t\right) dt$$

$n = 1 \Rightarrow T_0$ $\qquad\qquad$ $\forall n \Rightarrow 0$

$n \neq 1 \Rightarrow 0$

$$+ \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \ldots (1+n) \quad - j\sin \ldots (1+n)$$

$n = -1 \Rightarrow T_0$ $\qquad\qquad$ $\forall n \Rightarrow 0$

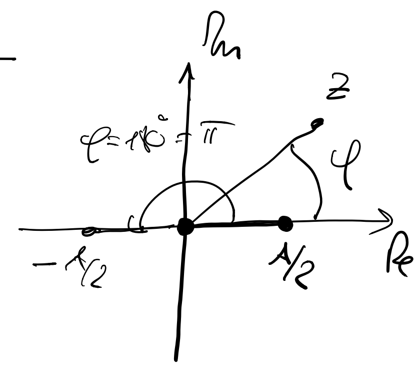
$n \neq -1 \Rightarrow 0$

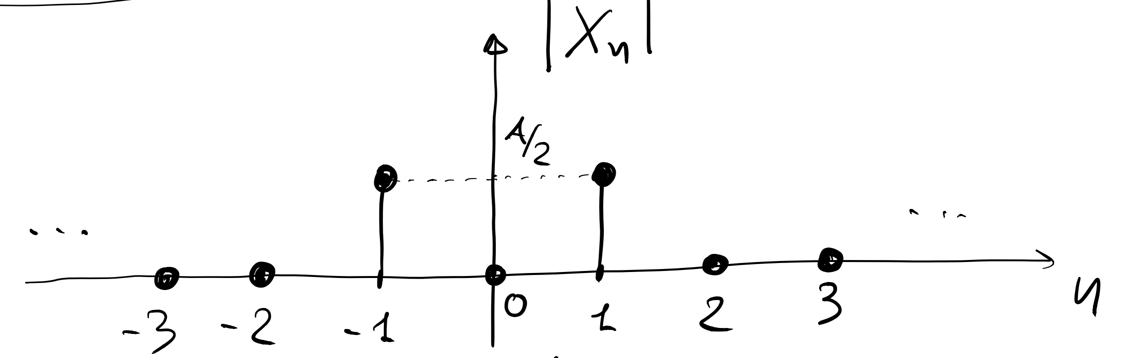
$\Rightarrow \dfrac{A}{2\cancel{T_0}} \cancel{T_0}$ $\quad n = 1 \quad$ I° int.

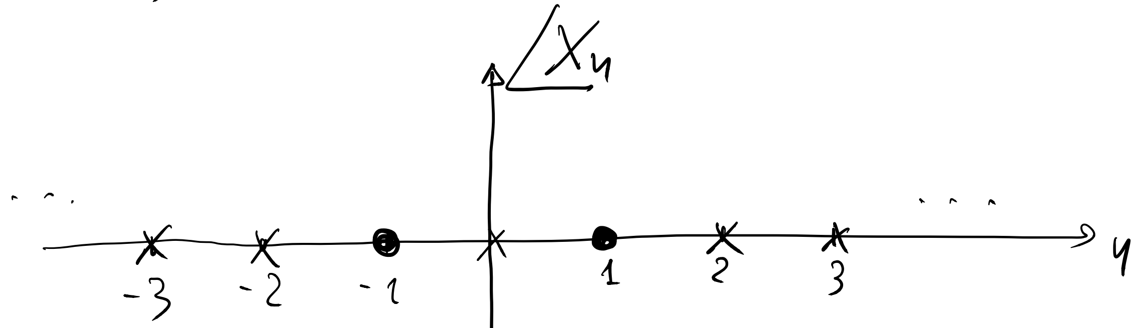
$\Rightarrow \dfrac{A}{2T_0} T_0$ $\quad n = -1 \quad$ II° int

$\Rightarrow 0 \quad n \neq \pm 1$

$$\boxed{X_n = \begin{cases} \dfrac{A}{2} e^{j\varphi}\Big|_{\varphi=0} & n = 1 \\ \dfrac{A}{2} e^{j\varphi}\Big|_{\varphi=0} & n = -1 \\ 0 & n \neq \pm 1 \end{cases}}$$







SPETTRO DI UN SENO

$$x(t) = A \sin(2\pi f_0 t)$$

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A \sin(2\pi f_0 t)\, e^{-j 2\pi n f_0 t}\, dt$$

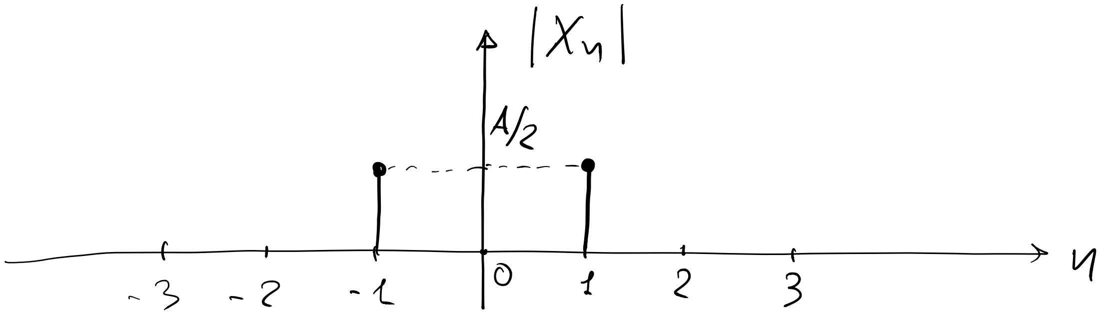
$$\sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$= \frac{A}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j 2\pi f_0 t} - e^{-j 2\pi f_0 t}}{2j}\, e^{-j 2\pi n f_0 t}\, dt$$

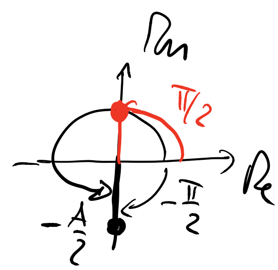
$$= \frac{A}{j 2 T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j 2\pi (1-n) f_0 t} - e^{-j 2\pi (1+n) f_0 t}\, dt$$

$$\Rightarrow X_n = \begin{cases} \frac{A}{j 2 T_0} T_0 & n = 1 \\ \frac{A}{j 2 T_0} (-T_0) & n = -1 \\ 0 & n \neq \pm 1 \end{cases}$$

$$\Rightarrow X_n = \begin{cases} \frac{A}{j2} & n=1 \\ -\frac{A}{j2} & n=-1 \\ 0 & n\neq \pm 1\end{cases}$$

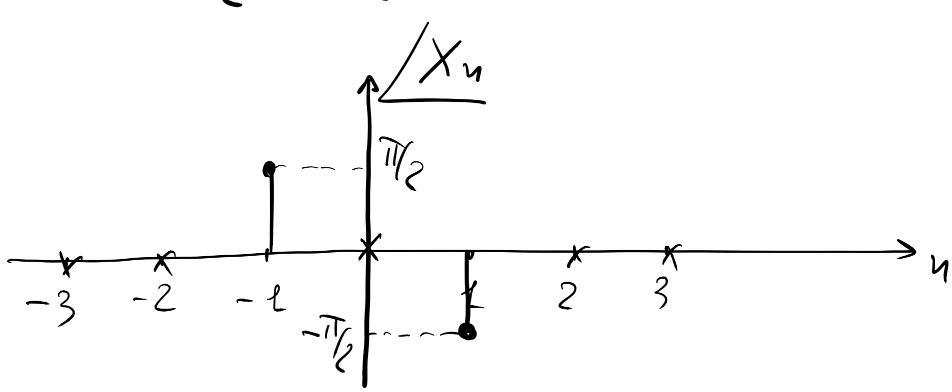


$$\frac{A}{j2} = 0 - j\frac{A}{2} = \frac{A}{2} e^{-j\frac{\pi}{2}}$$



$$\frac{A}{j2}\cdot\frac{j}{j} = \frac{jA}{-1\cdot 2} = -j\frac{A}{2}$$

$$-\frac{A}{j2} = 0 + j\frac{A}{2} = \frac{A}{2}e^{j\frac{\pi}{2}}$$



Le uniche componenti frequenziali non-nulle si trovano a $\pm f_0$

PROPRIETÀ DEL TSF

) LINEARITÀ

$a, b \in \mathbb{C}$

$$\text{Hyp.}\begin{cases} z(t) = a\,x(t) + b\,y(t) \\ x(t) \text{ e } y(t) \text{ sono periodici di } T_0 \\ x(t) \text{ e } y(t) \text{ sono trasformabili} \end{cases}$$

$$\Downarrow$$

$$\text{Th: } \boxed{Z_n = a\,X_n + b\,Y_n}$$

Dim.

$$Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \big(a\,x(t) + b\,y(t)\big)\, e^{-j2\pi n f_0 t}\, dt$$

$$= a \cdot \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_n} + b \cdot \underbrace{\frac{1}{T_0}\int y(t) \ldots dt}_{Y_n}$$

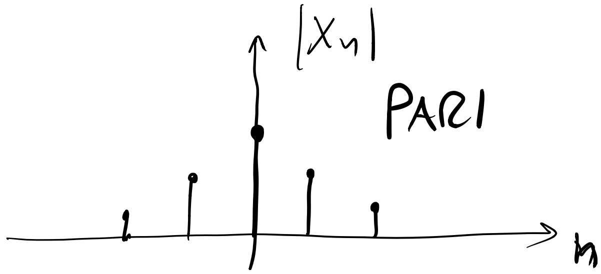
$\Rightarrow$ SIMMETRIA HERMITIANA

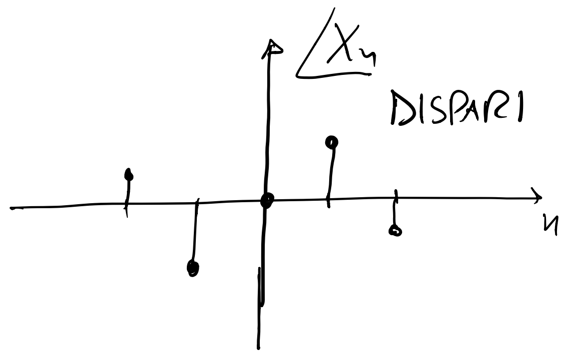
-) $x(t)$ è periodico e trasf. secondo TSF
-) $x(t)$ è reale $\quad x(t) = x^*(t)$

$$\Downarrow$$

$$\boxed{X_{-n} = X_n^*}$$

$$|X_{-n}| = |X_n| \qquad \angle X_{-n} = -\angle X_n$$





Dimostrazione

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt$$

$$(z^*)^* = z$$

$$(a \cdot b)^* = a^* b^*$$

$$= \left[\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underset{\underset{x(t)}{\parallel}}{x^*(t)}\, e^{-j 2\pi n f_0 t} \right]^*$$

$$= \left[\underbrace{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j 2\pi n f_0 t}}_{X_n} \right]^* = X_n^* = X_{-n}$$

$\Rightarrow$ SEGNALI PERIODICI REALI E PARI

- $x(t)$ periodico e trasformabile
- $x(t)$ è reale $\Rightarrow$ $x(t) = x^*(t)$
- $x(t)$ è pari $\Rightarrow$ $x(t) = x(-t)$

$$\Downarrow$$

$X_{-n} = X_n$ è reale e pari

Dim.

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt \qquad t' = -t$$

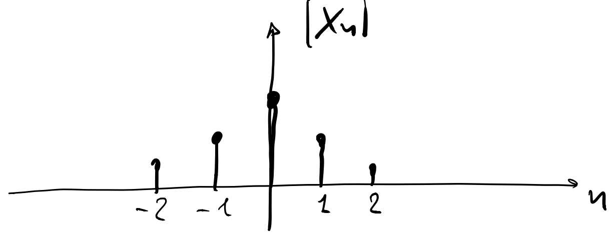
$$= \frac{1}{T_0}\int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} x(-t')\, e^{j2\pi n f_0 (-t')}\, (-dt')$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = X_n$$

$$X_{-n} = X_n \overset{\text{parità}}{\Rightarrow}$$
$$X_{-n} = X_n^* \quad \Rightarrow \quad X_n = X_n^* \quad \text{reale}$$

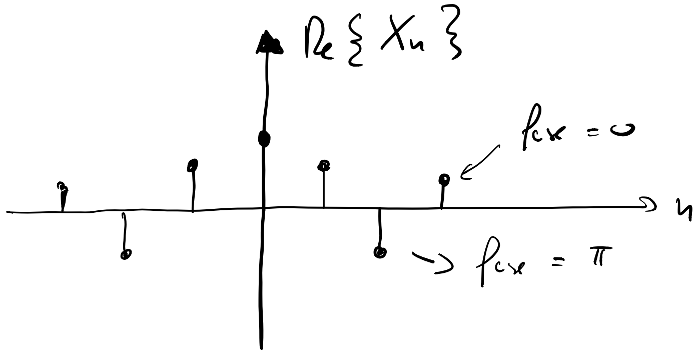
<u>X_n è reale e pari</u>

$x(t)$ reale e pari $\Rightarrow$ X_n reale e pari



La fase di X_n $\Rightarrow$ $\angle X_n$ può assumere solo valori pari a "0" o "π"

Quando X_n è reale conviene rappresentarla in fase e quadratura (Reale + Immaginaria)



$$\mathrm{Im}\{X_n\} = 0$$

SEGNALE REALE E DISPARI

$$x(t) = -x(-t)$$

$$\Downarrow$$

$$X_{-n} = -X_n$$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt \qquad t' = -t$$

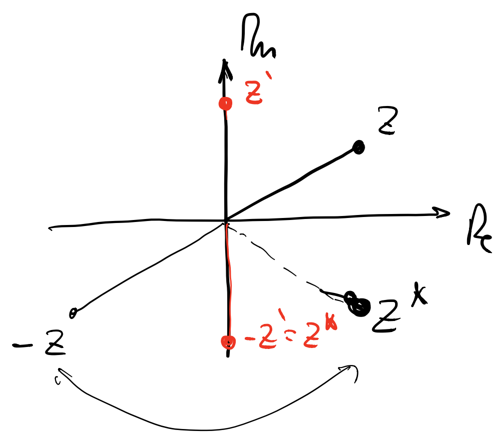
$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underbrace{x(-t')}_{= -x(t')}\, e^{-j2\pi n f_0 t'}\, dt'$$

$$= -\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = -X_n$$

$$X_{-n} = -X_n$$
$$X_{-n} = X_n^* \Rightarrow X_n^* = -X_n$$

$\Downarrow$

X_n è immaginario



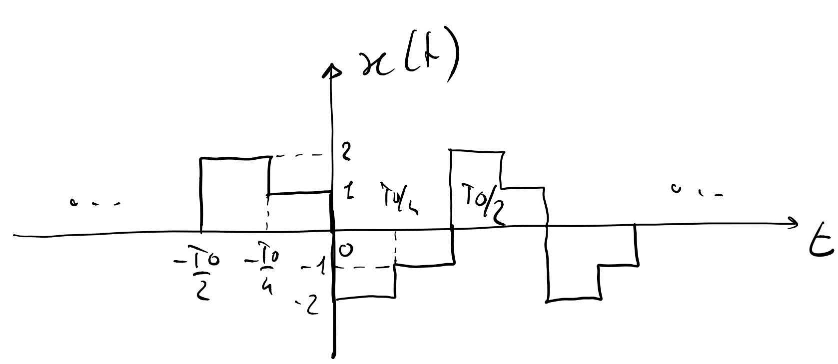
$\Rightarrow x(t)$ è reale e dispari

$\Rightarrow X_n$ è dispari ed immaginaria

SEGNALI PERIODICI ALTERNATIVI

$$x(t) = -x\left(t - \frac{T_0}{2}\right)$$

Esempio



$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

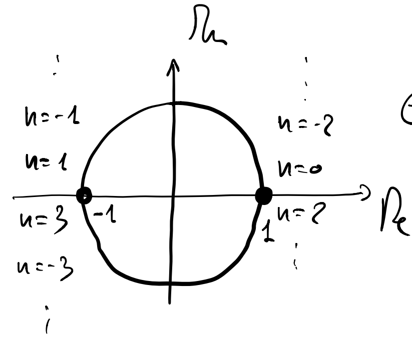
$$= \frac{1}{T_0} \underbrace{\int_{-\frac{T_0}{2}}^{0} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{t' = t + \frac{T_0}{2}} + \int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x\left(t' - \frac{T_0}{2}\right) e^{-j2\pi n f_0 \left(t' - \frac{T_0}{2}\right)}\, dt' + (\cdot)$$

$$= \frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} -x(t')\, e^{-j2\pi n f_0 t'}\, dt'\, e^{j\cancel{2}\pi n \cancel{f_0} \frac{\cancel{T_0}}{\cancel{2}}} + (\cdot)$$

$$= -\frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt'\, e^{j\pi n} + (\cdot)$$

$$= \frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt' \left(1 - \underbrace{e^{j\pi n}}\right)$$



$e^{j\varphi}$

$$\boxed{\begin{array}{lll} n \text{ pari} & \Rightarrow & X_n = 0 \\ n \text{ dispari} & \Rightarrow & X_n = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} x(t)\, e^{-j 2\pi n f_0 t}\, dt \end{array}}$$

ESERCIZI

$$x(t) = A \cos(2\pi \underset{f_0}{f_0} t + \varphi) + B \sin(4\pi \underset{2f_0}{f_0} t)$$

·) Calcolare lo spettro di $x(t)$

·) Disegnare spettro di ampiezza e fase

Svolgimento

1) È periodico?

$$x(t) = x(t - nT_0)$$

periodo $T_0 = \frac{1}{f_0}$ $\Rightarrow$ f_0 = fondamentale

$2f_0$

$$x(t - nT_0) = A \cos(2\pi f_0 (t - nT_0) + \varphi) + B \sin(4\pi f_0 (t - nT_0))$$

$$= A \cos(2\pi f_0 t + \varphi - 2\pi \cancel{f_0} n \cancel{T_0}) + B \sin(4\pi f_0 t - 4\pi \cancel{f_0} n \cancel{T_0})$$

$$= A \cos(2\pi f_0 t + \varphi) + B \sin(4\pi f_0 t)$$ periodico di T_0

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \Big[\underbrace{A\cos(2\pi f_0 t + \varphi)}_{x_1(t)} + \underbrace{B\sin(4\pi f_0 t)}_{x_2(t)}\Big] e^{-j2\pi n f_0 t}\, dt$$

-) Applico la linearità

$$X_n = X_{1n} + X_{2n}$$

$$X_{1n} = TSF[x_1(t)], \quad X_{2n} = TSF[x_2(t)]$$

$$X_{1n} = \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j(2\pi f_0 t + \varphi)} + e^{-j(2\pi f_0 t + \varphi)}}{2}\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{A}{2T_0} e^{j\varphi}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi f_0 (1-n) t}\, dt + \frac{A}{2T_0} e^{-j\varphi}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi(1+n) f_0 t}\, dt$$

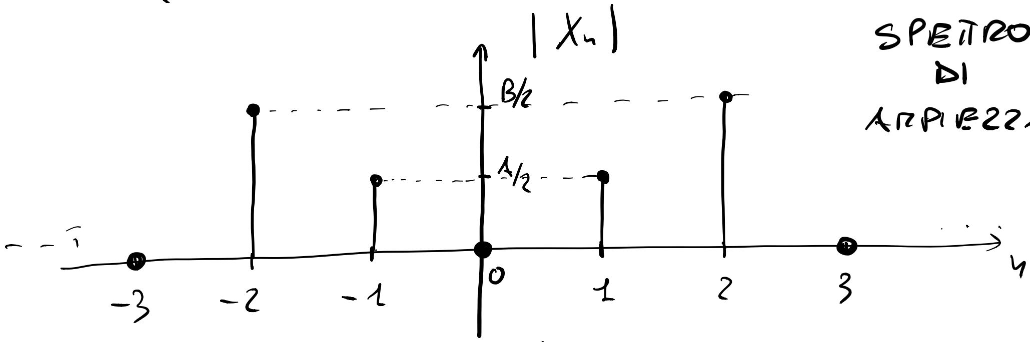
$$X_{1n} = \begin{cases} \frac{A}{2} e^{j\varphi} & n = 1 \\ \frac{A}{2} e^{-j\varphi} & n = -1 \\ 0 & n \neq \pm 1 \end{cases}$$

$$X_{2n} = \frac{B}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j2\pi(2f_0)t} - e^{-j2\pi(2f_0)t}}{2j}\, e^{-j2\pi n f_0 t}\, dt$$

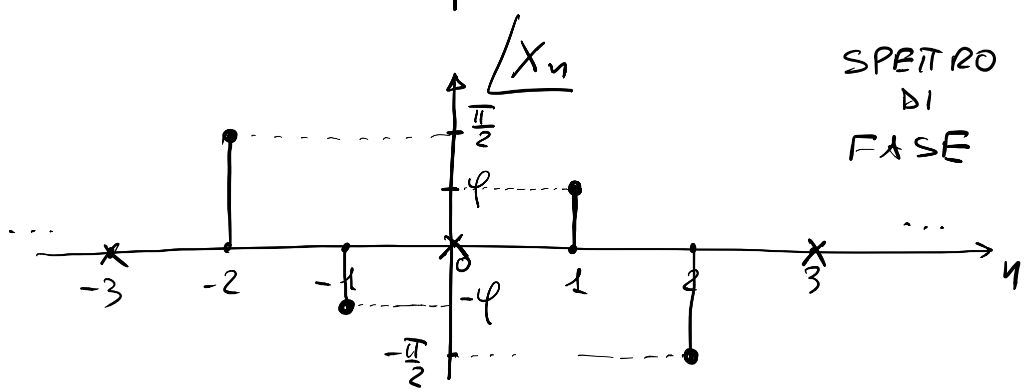
$$= \frac{B}{j2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(2-n)f_0 t}\,dt - \frac{B}{j2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi(2+n)f_0 t}\,dt$$

$$X_{2n} = \begin{cases} \frac{B}{j2\cancel{T_0}}\cdot\cancel{T_0} & n = 2 \\ -\frac{B}{j2\cancel{T_0}}\cdot\cancel{T_0} & n = -2 \\ 0 & n \neq \pm 2 \end{cases}$$

$$X_n = \begin{cases} \frac{A}{2}e^{j\varphi} & n = 1 \\ \frac{A}{2}e^{-j\varphi} & n = -1 \\ \frac{B}{2}e^{-j\frac{\pi}{2}} & n = 2 \\ \frac{B}{2}e^{j\frac{\pi}{2}} & n = -2 \\ 0 & n \neq \pm 1, \pm 2 \end{cases}$$



SPETTRO DI AMPIEZZA



SPETTRO DI FASE