Università di Pisa
Dipartimento di Ingegneria dell'informazione

Ingegneria Informatica

Analisi matematica II

Lezioni online - Maggio/Giugno 2020

A.A 2019-2020

Lavagne scritte da Longo durante la DAD

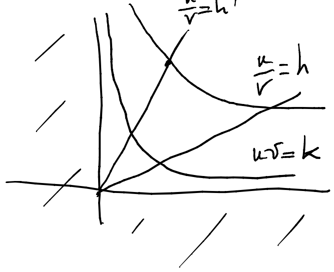
Indice - Lezioni Marzo/Aprile 2020

Note Title 5/5/2020

$$(u,v) \xrightarrow{T} \Big(\underset{x}{uv},\ \underset{y}{\frac{u}{v}}\Big) \qquad v \neq 0$$

$$\det \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -2\frac{u}{v}$$

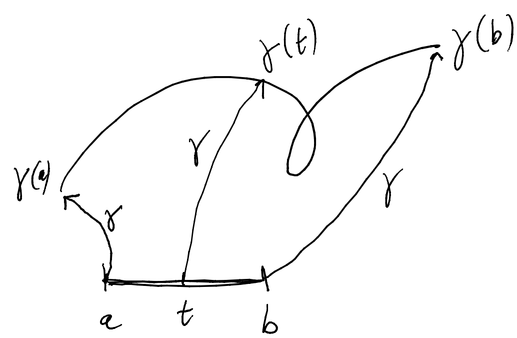
Si può applicare il th. inversa locale se $u \neq 0$

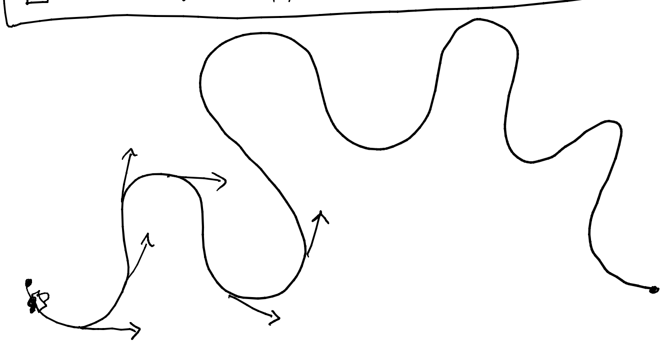


$x = \text{cost}$ $\quad uv = \text{cost}$ $\qquad T(u,v)$

$\boxed{y = \text{cost}}$ $\quad \frac{u}{v} = \text{cost}$

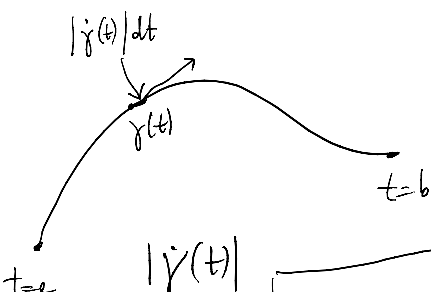
LUNGHEZZA DI UNA CURVA





1 ora $\quad |v| = \text{cost} = 10\,\text{km/h}$

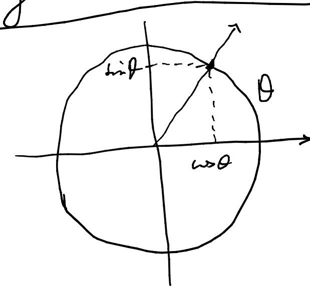
$\Longrightarrow$ lunghezza $= 10\,\text{Km}$

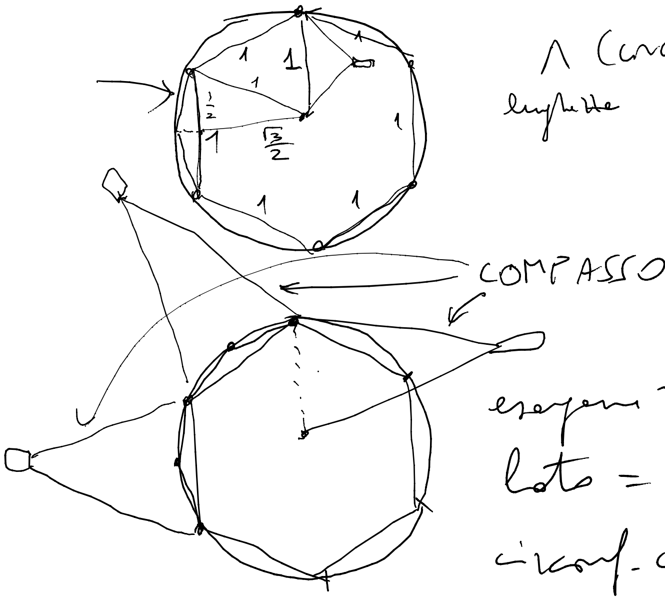


$|\dot\gamma(t)|$

$$\boxed{\Lambda(\gamma) = \int_a^b |\dot\gamma(t)|\,dt}$$

lunghezza





Λ (circonf.) $\geq$ perimetro dell'esagono
lunghezza

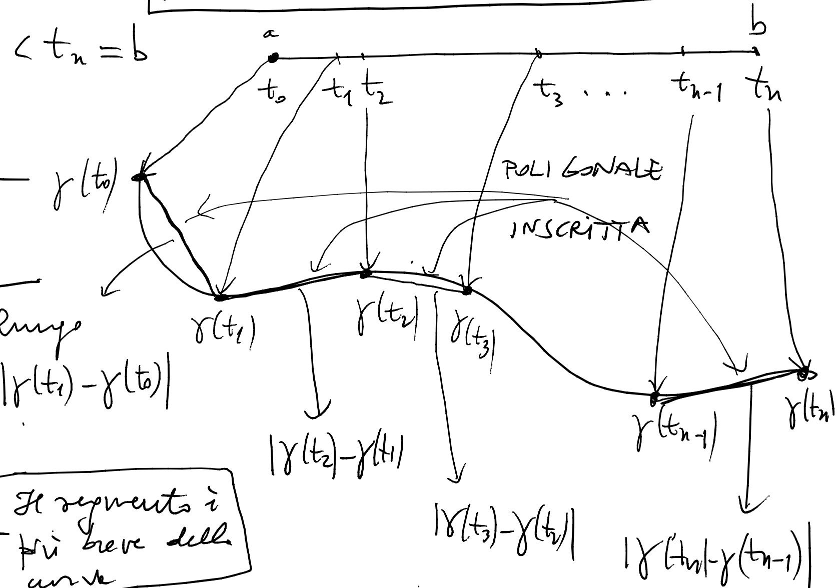
$$2\pi \geq 6 \Rightarrow \pi \geq 3$$

esagono regolare
lato = raggio
circonf. circoscritta

IDEA: (Brisone, Antifonte, Eudosso, Archimede)
APPROSSIMARE LA CURVA CON UNA SPEZZATA.

partizione su $[a,b]$
$a = t_0 < t_1 < t_2 < \ldots < t_n = b$
$\Pi = \{t_0, t_1, \ldots, t_n\}$ con ↑

RETTIFICABILITA' = LUNGHEZZA FINITA



Lunghezza della poligonale inscritta

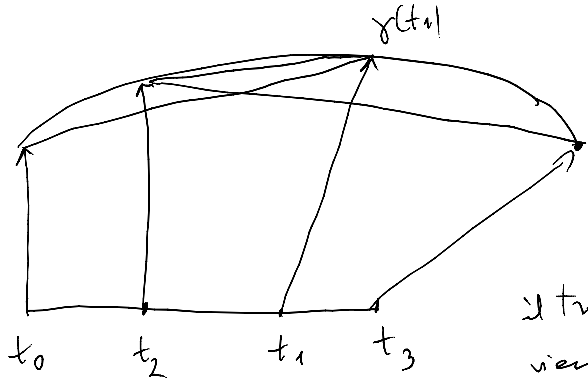
$$\Lambda(\Pi) = \sum_{i=0}^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|$$

$$\boxed{\Lambda(\Pi) \leq \Lambda(\gamma)}$$
?

Il segmento è più breve della curva

$\sup\{\Lambda(\Pi) : \Pi$ è una partiz. di $[a,b]\}$

$$\boxed{\Lambda(\gamma) = \sup_{\Pi} \Lambda(\Pi)}$$ DEFINIZIONE



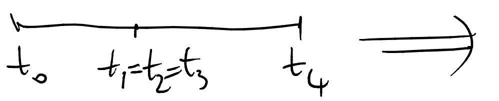
se la sequenza $t_0, t_1, \dots, t_n$ non è crescente si ottiene un valore

SCORRETTO

il tratto di curva fra $\gamma(t_2)$ e $\gamma(t_1)$ viene "preso tre volte", invece di una.

usare $t_0 \le t_1 \dots \le t_n$ non comporta variazioni con il caso $t_0 < t_1 < \dots < t_n$

$$\sum |\gamma(t_{i+1}) - \gamma(t_i)|$$

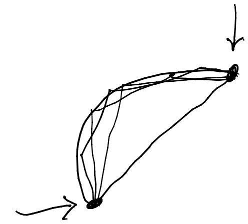


$\Longrightarrow$

$$|\gamma(t_2)-\gamma(t_1)|=0$$
$$|\gamma(t_3)-\gamma(t_2)|=0$$

$\boxed{\gamma \text{ è rettificabile se } \Lambda(\gamma) < \infty}$ DEFINIZIONE

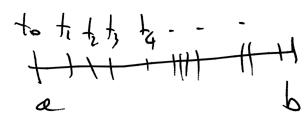
$\Pi = \{a, b\}$, $t_0 < t_1$ $\qquad \Lambda(\gamma) \ge |\gamma(b) - \gamma(a)|$



sup $\Lambda(\Pi)$ al variare di Π partiz. di $[a,b]$

Metro da sarto

Modello di curve rettificabili



Teorema Le curve parametriche di classe C^1 sono rettificabili, e inoltre

$$\Lambda(\gamma) \le \int_a^b |\dot{\gamma}(t)|\,dt$$

$\gamma \in C^1 \iff \begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{pmatrix} = \gamma$ ha componenti $C^1[a,b]$

→ Sia $a=t_0<t_1<\dots<t_n=b$ una partizione arbitraria di $[a,b]$ $\widehat{\Pi}$

$$\to\ \Lambda(\Pi)=\sum_{0}^{n-1}\big|\underbrace{\gamma(t_{i+1})-\gamma(t_i)}\big| \overset{\text{th. fondam. calcolo integrale}}{=}$$

$$=\sum_{0}^{n-1}\left|\int_{t_i}^{t_{i+1}}\dot\gamma(t)\,dt\right| \le$$

$$\le \sum_{0}^{n-1}\int_{t_i}^{t_{i+1}}\underbrace{|\dot\gamma(t)|}_{\in C^0}dt \overset{\substack{\text{additività dell'}\\ \text{integrale}\\ \cup[t_i,t_{i+1}]=[a,b]}}{=}$$

$$=\boxed{\int_a^b|\dot\gamma(t)|\,dt}$$

Maggiorante per $\{\Lambda(\Pi);\ \Pi \text{ partiz}\}$

$$\left|\int_a^b\right| \le \int_a^b|\ |\qquad a<b$$

$$\gamma_j(t_{i+1})-\gamma_j(t_i)=$$

$$=\int_{t_i}^{t_{i+1}}\dot\gamma_j(t)\,dt$$

$$\int_{t_i}^{t_{i+1}}\dot\gamma(t)\approx$$

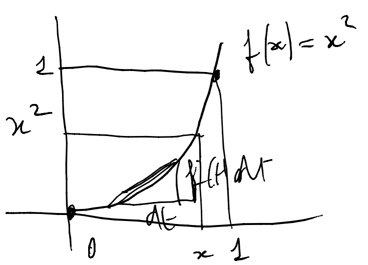
$$\Big|\overbrace{\sum \underbrace{\dot\gamma(\xi)}_{\in\mathbb{R}^N}\underbrace{(t_{i+1}-t_i)}_{\in\mathbb{R}^+}}\Big|\le$$

tri ang.

$$\le\sum|\dot\gamma(\xi)(t_{i+1}-t_i)|=$$

$$=\sum(t_{i+1}-t_i)\,|\dot\gamma(\xi)|$$

$$\approx \int_{t_i}^{t_{i+1}}|\dot\gamma(t)|\,dt$$



Λ grafico di $f(x)=x^2$ relativo ad $x\in[0,1]$?

$$\gamma(t)=\begin{pmatrix}t\\ f(t)\end{pmatrix}=\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\qquad \begin{aligned}x(t)&=t\\ y(t)&=f(t)\end{aligned}$$

$$t\in[0,1]$$

$$\gamma(t)=\begin{pmatrix}t\\ t^2\end{pmatrix}\in C^1[0,1]\implies$$

$$\implies \Lambda(\gamma)<\infty$$

$$\dot\gamma=\begin{pmatrix}1\\ 2t\end{pmatrix}$$

$$\Lambda(\gamma)=\int_0^1|\dot\gamma(t)|\,dt=$$

$$=\int_0^1\sqrt{1+4t^2}\,dt=\dots\dots$$

$$\boxed{\Lambda(\operatorname{graph} f)=\int_a^b\sqrt{1+[f'(t)]^2}\,dt}$$

Note Title 5/6/2020

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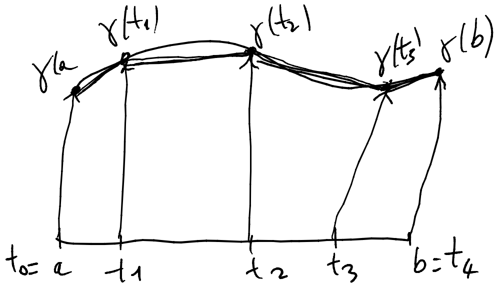
$\boxed{\text{PAGINE.DM.UNIPI.IT/ALAN}}$

$\gamma : [a,b] \longrightarrow \mathbb{R}^N$ curva parametrica. $\gamma[a,b]$ sostegno di γ ($\mathrm{Im}\,\gamma$)

Π partizione di $[a,b]$ $\quad \Pi = \{ \underline{a = t_0 < t_1 < \cdots\cdots < t_{n-1} < t_n = b} \}$

$\{t_0, t_1, \dots, t_{n-1}, t_n\} \; : \; t_i < t_{i+1} \; \forall i = 0..n-1$

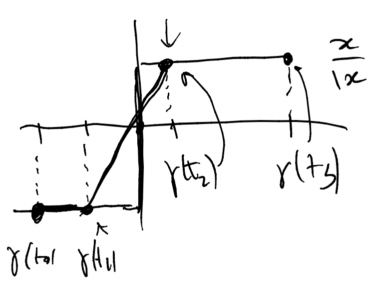


$$\Lambda(\Pi) \equiv \underbrace{\sum_0^{n-1} |\gamma(t_{i+1}) - \gamma(t_i)|}$$

$$\Lambda(\gamma) = \sup_{\substack{\Pi \text{ partiz.} \\ \text{di } [a,b]}} \Lambda(\Pi)$$

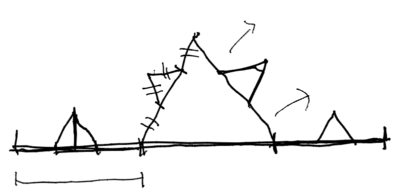
γ si dice rettificabile se $\Lambda(\gamma) = \sup_{\Pi} \Lambda(\Pi) < \infty$ $\quad \gamma(t) \begin{pmatrix} t \\ f(t) \end{pmatrix}$

non rettificabile se $\underline{\Lambda(\gamma) = +\infty}$



γ_0 segmento lung. 1

γ_1 4 segmenti lunghi 1/3



$\gamma_2 =$

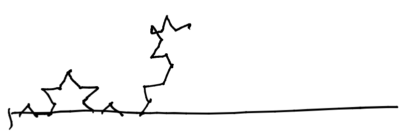
γ_0 ha lunghezza 1

γ_1 ha lunghezza $\frac{4}{3}$

$\gamma_2 \quad \frac{16}{9}$

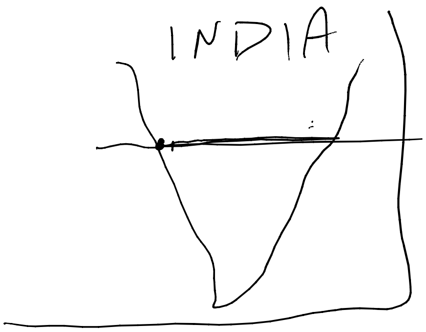
$\vdots$

γ_n ha lunghezza $\left(\frac{4}{3}\right)^n$



KOCH

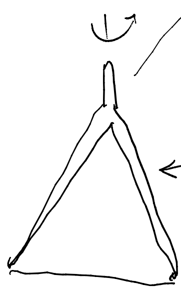
"Fiocco di neve"



MARE

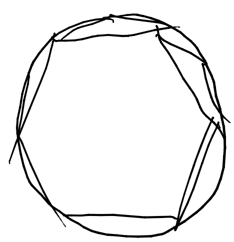
"CALCOLO" della lunghezza delle coste occidentali della

$\boxed{\text{SCOZIA}}$



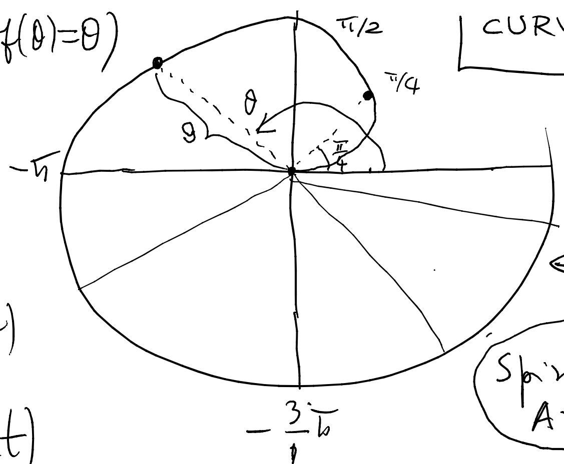
compasso a punte fisse, per riportare distanze

rappresenta una distanza reale in scala

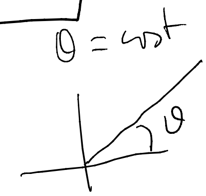


CURVE IN COORDINATE POLARI

$\rho = \theta \quad (f(\theta) = \theta)$



$\theta = \text{cost}$

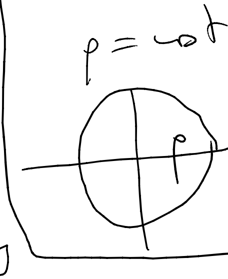


$$\rho = f(\theta)$$

1 "giro" $[0, 2\pi]$

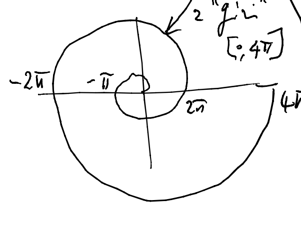
Spirale di Archimede

$\rho = \text{cost}$



$$\gamma(t) = \begin{cases} \rho = f(t) \\ \theta = g(t) \end{cases}$$

2 "giri" $[0, 4\pi]$



$$\Lambda(\gamma) = \int_a^b |\dot{\gamma}(t)|\, dt$$

COORDINATE CARTESIANE

$$\begin{cases} x = \rho \cos\theta \\ y = \rho \sin\theta \end{cases}$$

$\rho = \theta \qquad \begin{aligned} x &= \rho\cos\theta \overset{\rho=\theta}{=} \theta\cos\theta \\ y &= \rho\sin\theta = \theta\sin\theta \end{aligned}$

$$\boxed{\varphi(\theta) = \begin{pmatrix} \theta\cos\theta \\ \theta\sin\theta \end{pmatrix} \qquad \theta \in [0, 2\pi]}$$

$$\dot\varphi = (\cos\theta - \theta\sin\theta,\ \sin\theta + \theta\cos\theta)$$

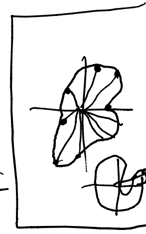
$$|\dot\varphi(\theta)| = \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = \sqrt{1+\theta^2}$$

$$\Lambda = \int_0^{2\pi} \overset{d\ell}{|\dot\varphi(\theta)|\, d\theta} = \int_0^{2\pi} \sqrt{1+\theta^2}\, d\theta$$

$\rho = f(\theta) \qquad x = f(\theta)\cos\theta$

$f: \mathbb{R} \to \mathbb{R}^+ \qquad y = f(\theta)\sin\theta$

$$\varphi(\theta) = (f(\theta)\cos\theta,\ f(\theta)\sin\theta) = f(\theta)(\cos\theta, \sin\theta)$$



$$\dot\varphi(\theta) = \dot f(\theta)\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + f(\theta)\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \dot f(\theta)\cos\theta - f(\theta)\sin\theta \\ \dot f(\theta)\sin\theta + f(\theta)\cos\theta \end{pmatrix}$$

$$|\dot\varphi(\theta)| = \sqrt{[\dot f(\theta)]^2 + f^2(\theta)}$$

$$\gamma(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix} \iff \begin{aligned} \rho &= f(t) \\ \theta &= g(t) \end{aligned} \qquad \left.\begin{matrix}\rho \\ \theta\end{matrix}\right\} \text{funzioni del parametro } t$$

$t \in [a,b]$

$x(t) = \rho\cos\theta$

$y(t) = \rho\sin\theta$

$$\sqrt{\dot x^2 + \dot y^2} = \sqrt{(\dot\rho\cos\theta - \rho\dot\theta\sin\theta)^2 + (\dot\rho\sin\theta + \rho\dot\theta\cos\theta)^2} =$$

$$= \sqrt{\dot\rho^2 + \rho^2\dot\theta^2} \longleftarrow$$

$$\checkmark \quad \Lambda(\gamma) = \int_a^b \sqrt{\dot f^2(t) + f^2(t)\dot g^2(t)}\, dt$$

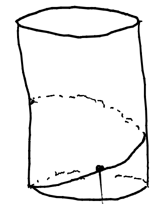
$$\rho = f(\theta) \qquad \begin{cases} \theta = t \\ \rho = f(t) \end{cases} \qquad \boxed{\dot\theta = 1 \quad \dot\rho = \dot f(t) \quad \rho = f(t)}$$

ELICA CILINDRICA

$$\begin{cases} \rho = 1 \\ \theta = t \\ z = t \end{cases} \qquad t \in [0, 2\pi]$$

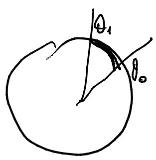
$$x = 1\cos t \qquad y = 1\sin t \qquad z = t \qquad t \in [0, 2\pi]$$

equazioni polari d'un'ELICA cilindrica



$$\varphi(t) = (\cos t, \sin t, t) \qquad \dot\varphi(t) = (-\sin t, \cos t, 1)$$

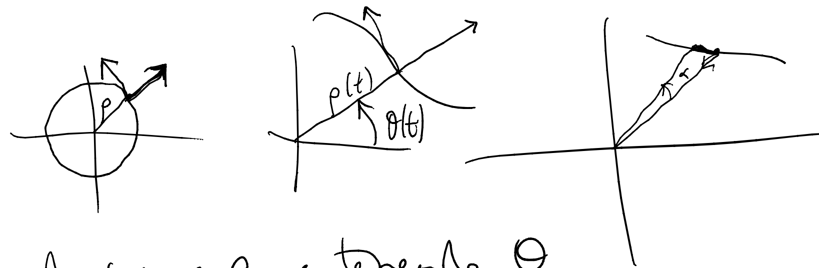
$$\Lambda(\gamma) \equiv \Lambda(\varphi) = \int_0^{2\pi} \underbrace{\sqrt{1^2 + \sin^2 t + \cos^2 t}}_{|\dot\varphi|^2}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \boxed{2\sqrt{2}\,\pi}$$

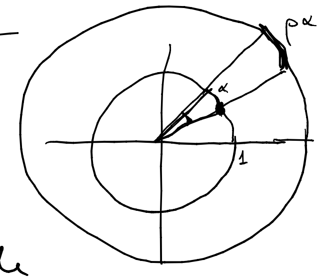


$R(\theta_1 - \theta_0)$

Note Title 5/8/2020

JITSI MEET





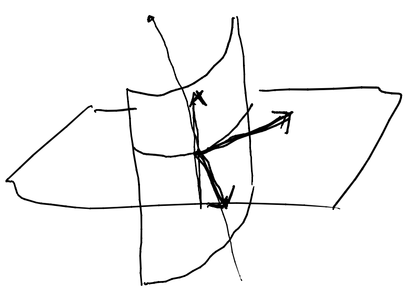
facendo variare ρ e tenendo θ costante

$\dot{\rho}(t)\,dt$ spostamento in direzione radiale

$\dot{\theta}(t)\,dt$ variazione dell'angolo

$\rho\dot{\theta}\,dt$ spostamento in direz. tangenz.

$$ds = \sqrt{\dot{\rho}^2\,dt^2 + \rho^2\dot{\theta}^2\,dt^2} = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2}\,dt .$$



lo spostam. verticale sarà $\dot{z}\,dt$

lo spost. radiale sarà $\dot{\rho}\,dt$

lo spost. tangenziale sarà $\rho\dot{\theta}\,dt$

sono ortogonali a due a due

e dunque lo spostamento totale sarà (Pitagora)

$$ds = \sqrt{\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2} \qquad \Lambda(\gamma) = \int ds = \int \sqrt{\dot{\rho}^2 + \dot{z}^2 + \rho^2\dot{\theta}^2}\,dt$$

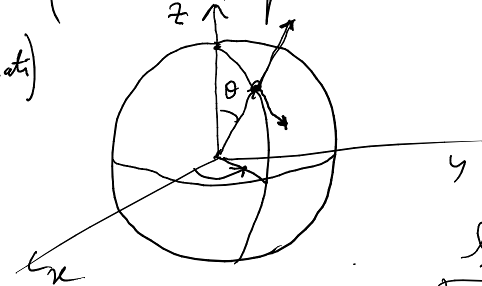
$$\sqrt{\dot{x}^2 + \dot{y}^2} \longrightarrow \sqrt{1 + f'(t)^2}$$

ove $\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$

ρ distanza dall'origine $\Rightarrow$ $\dot\rho\,dt$ è lo spostam. radiale

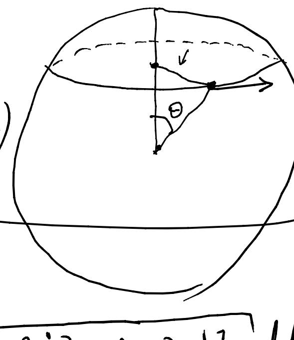
Facendo variare θ (ρ, φ fissati)

$\rho\dot\theta\,dt$ è uno spostamento in direzione del meridiano (geografico) per il punto (spostam. Nord-Sud)



raggio del parallelo per il punto

$\rho\sin\theta\,\dot\varphi\,dt$ è lo spostamento (Est_Ovest)



line element

elemento d' linea

$$ds = \sqrt{\dot\rho^2 + \rho^2\dot\theta^2 + \rho^2\sin^2\theta\,\dot\varphi^2}\,dt$$

lunghezza d' una curva

$$\gamma(t) = (\rho(t), \theta(t), \varphi(t)) \qquad \Lambda(\gamma) = \int_a^b \sqrt{\dot\rho^2 + \rho^2\dot\theta^2 + \rho^2\sin^2\theta\,\dot\varphi^2}\,dt$$

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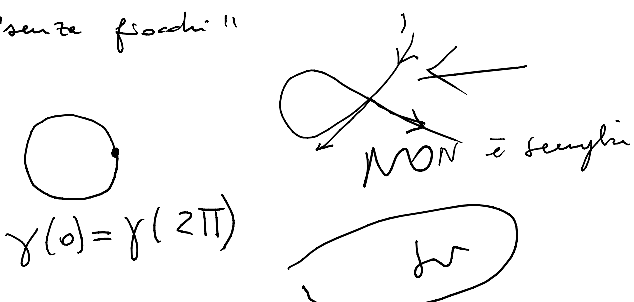
curva semplice (e chiacchiere) "senza fiocchi"

$\gamma : [a,b] \to \mathbb{R}^n$ è semplice se $\boxed{\gamma \text{ è iniettiva in }]a,b[}$

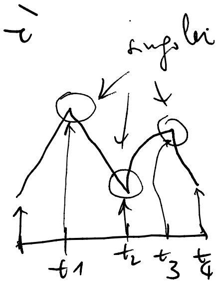


$\gamma(0) = \gamma(2\pi)$

γ si dice CHIUSA se $\gamma(a) = \gamma(b)$

ES. La circonferenza $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $t \in [0, 2\pi]$ è semplice e chiusa

γ si dice REGOLARE se $|\dot\gamma(t)| \neq 0 \ \forall t \in [a,b]$



γ si dice Generalmente regolare / regolare a tratti $\quad$ γ è regolare su $[t_i, t_{i+1}]$ $\forall i$

$\Pi = \{t_0 < t_1 \dots < t_n\}$

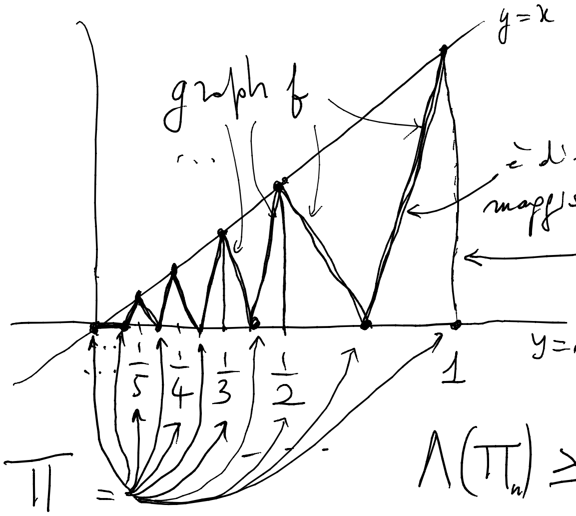
Note Title 5/12/2020

Esempio di curva di classe C^0 non rettificabile

$$\gamma : [0,1] \to \mathbb{R}^2$$

$$\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

$$\gamma(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

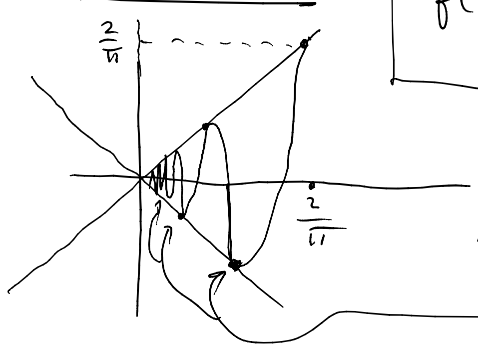


$$\boxed{\sup_n \Lambda(\Pi_n) = +\infty}$$

$$\Lambda(\Pi_n) \geq \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}}_{\text{n-esima somma parziale della serie armonica}}$$

Π_n si ottiene da $[0, \frac{1}{n}, \frac{1}{n-1}, \dots, \frac{1}{2}, 1]$ intercalando un punto arbitrario in ogni intervallo]

Alternative



$$f(t) = \begin{cases} 0 & t = 0 \\ t \sin\frac{1}{t} & t \in]0, \frac{2}{\pi}] \end{cases}$$

$$|f(t)| = |t| \left|\sin\frac{1}{t}\right| \leq |t|$$

$$\sin\frac{1}{t} = 1 \qquad \frac{1}{t} = \frac{\pi}{2} + 2k\pi$$

$$\sin\frac{1}{t} = -1 \quad \frac{1}{-\frac{\pi}{2} + 2k\pi} \qquad t = \frac{1}{\frac{\pi}{2} + 2k\pi} \sim O\left(\frac{1}{k}\right)$$

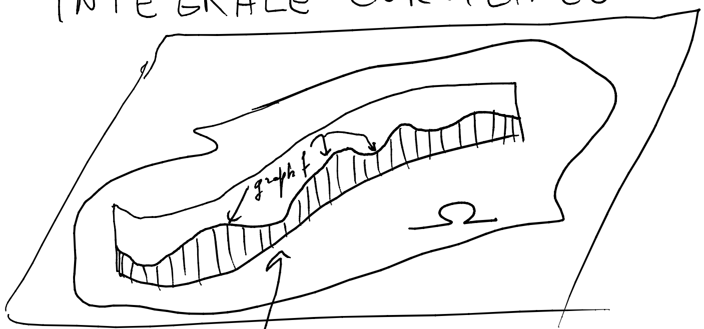
stesso conto di prima

$$\gamma \in C^1 \Rightarrow \Lambda(\gamma) < +\infty \qquad \text{BV}$$

$$\gamma \in C^0 \not\Rightarrow \Lambda(\gamma) < +\infty$$

γ è rettificabile se e solo se le sue componenti sono BV (Bounded Variation)

INTEGRALE CURVILINEO



$\gamma[a,b]$

$$\boxed{f : \Omega \to \mathbb{R}} \quad \Omega \subseteq \mathbb{R}^n$$

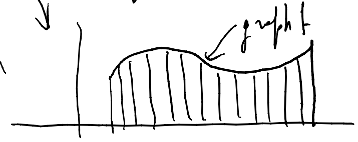
$$\gamma : [a,b] \longrightarrow \Omega \quad \text{gener. regolare}$$

$$\int_\gamma f \, d\gamma \qquad \int_\gamma f \, d\ell$$

deformato

disteso

$$\int_\gamma f \, d\gamma \equiv \int_a^b f(\gamma(t)) \underbrace{\overbrace{|\dot\gamma(t)| \, dt}^{d\gamma}}_{\substack{\text{distanza curvilinea}\\ \text{percorsa dal punto}\\ \text{nell'intervallino } dt}}$$



se $f \equiv 1$

$$\int_a^b 1 \, |\dot\gamma(t)| \, dt = \Lambda(\gamma)$$

$$\underbrace{f(x,y) = x}_{\text{dato}} \qquad \underbrace{\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}}_{\text{dato}} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad f : \mathbb{R}^2 \to \mathbb{R}$$

$$\gamma : \mathbb{R} \to \mathbb{R}^2 \qquad \underbrace{t \in [0,1]}_{\text{dato}}$$

$$\int_\gamma f \, d\gamma = \int_0^1 \underset{\substack{\uparrow \\ f(\gamma(t))}}{t} \sqrt{1+4t^2} \, dt = \qquad f(\gamma(t)) = t$$

$$|\dot\gamma(t)| = \sqrt{1^2 + (2t)^2}$$

$$= \frac{1}{8} \int_0^1 \sqrt{1+4t^2} \, \underbrace{8t \, dt} = \ldots\ldots$$

$1+4t^2 = u$

$du = 8t \, dt$

$$\gamma(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

$$f(\gamma(t)) \equiv f(\underbrace{t, t^2}_{\gamma(t)})$$

ESEMPIO

$$f(x,y,z) = x^2 e^y \sin z \qquad \gamma(t) = \begin{pmatrix}\cos t\\ \sin t\\ t\end{pmatrix} \qquad t\in[0,2\pi]$$

f composto γ vuol dire mettere $\cos t$ al posto di x, $\sin t$ al posto di y e t al posto di z.

$$\boxed{\int_0^{2\pi} \underbrace{\cos^2 t}_{x^2}\, \underbrace{e^{\sin t}}_{e^y}\, \underbrace{\sin t}_{\sin z} \sqrt{(-\sin t)^2+(\cos t)^2+1}\; dt}$$

ESERCIZIO (di analisi 1 e 2)

$\boxed{\text{Calcolare l'integrale nel caso in cui } f(x,y,z) = x e^y \sin z}$

$\gamma:[a,b]\to\mathbb{R}^n$ è detta EQUIVALENTE a $\sigma:[c,d]\to\mathbb{R}^n$

$\boxed{\gamma,\sigma\in C^1}$ se esiste $\rho:[c,d]\to[a,b]$ invertibile (C^1) tale che $\boxed{\dot\rho\neq 0 \ \forall t\in[c,d]}$

$$\sigma(s) = \gamma(\rho(s)) \qquad \dot\sigma(s) = \dot\gamma(\rho(s))\dot\rho(s)$$

da cui

$$\Lambda(\sigma) = \int_c^d |\dot\sigma(s)|\,ds =$$

$$= \int_c^d |\dot\rho(s)\,\dot\gamma(\rho(s))|\,ds =$$

$$\underset{\dot\rho>0}{=} \int_c^d |\dot\gamma(\rho(s))|\,\dot\rho(s)\,ds =$$

$$\underset{\rho(s)=t}{=} \int_{\rho(c)=a}^{\rho(d)=b} |\dot\gamma(t)|\,dt = \Lambda(\gamma)$$

$[c,d] = [0,\pi]$

$[a,b] = [0,2\pi]$

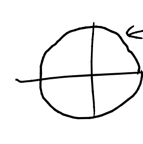
$\rho(t) = 2t$

$\gamma(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix} \quad t\in[0,2\pi]$

$\sigma(s) = \begin{pmatrix}\cos 2s\\ \sin 2s\end{pmatrix} \quad s\in[0,\pi]$

$\dot\rho>0$ stesso verso

$\dot\rho<0$ verso opposto



$\dot\rho<0 \quad \forall s\in[c,d] \qquad \rho(c)=b \qquad \rho(d)=a$

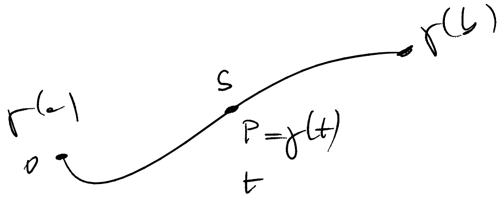
$$\int_c^d |\dot\rho(s)\dot\gamma(\rho(s))|\,ds = \int_c^d |\dot\rho(s)|\,|\dot\gamma(\rho(s))|\,ds \overset{\dot\rho<0}{=} -\int_c^d |\dot\gamma(\rho(s))|\,\dot\rho(s)\,ds =$$

$$\underset{\rho(s)=t}{=} -\int_{\rho(c)=b}^{\rho(d)=a}|\dot\gamma(t)|\,dt = -\int_b^a|\dot\gamma(t)|\,dt = \int_a^b|\dot\gamma(t)|\,dt$$

ASCISSA CURVILINEA

Pietre MILIARI

METRO FLESSIBILE (da sarto)



$\dot\gamma\in C^0$

$\gamma\in C^1$

$$s(t) = \int_a^t |\dot\gamma(s)|\,ds$$

$$\underset{\overset{\shortparallel}{0}}{s(t)-s(a)} = \int \dot s(t)\,dt = \Lambda(\gamma|_{[a,t]}) \;\overset{\shortparallel}{\text{strada percorsa dall'origine a }\gamma(t)}. \qquad \dot s(t) = |\dot\gamma(t)|$$

FORMULA DI TAYLOR

Note Title 5/13/2020

DISPENSE AN 2.8 e AN. 2.9

DERIVATE di ordine superiore $\quad \underline{f:\Omega\to\mathbb{R} \quad \Omega\subseteq\mathbb{R}^n}$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)=\frac{\partial^2 f}{\partial x_i\partial x_j} \qquad i,j=1..n$$

$$\partial_{x_i}(\partial_{x_j}f)=\partial_{x_ix_j}f$$

$i,j,k=1..n$

$$\partial_{x_ix_jx_k}f=\partial_{x_i}(\partial_{x_j}(\partial_{x_k}f))$$

$f_{x_1} \quad f_{x_2} \quad \cdots \quad f_{x_n}$

$\partial_{x_1} \quad \partial_{x_2} \quad \partial_{x_n}$

$\underbrace{(f_{x_1})_{x_1}}_{= f_{x_1x_1}} \quad (f_{x_1})_{x_2} = f_{x_1x_2} \cdots (f_{x_1})_{x_n} = f_{x_1x_n}$

Una funzione genera n derivate parziali, n^2 derivate seconde, n^3 derivate terze ...

SOLO ENUNCIATO Th. (CLAIRAUT-SCHWARZ) $f\in C^2$ allora

pronuncia: Clerò scvarz

$$\boxed{f_{x_ix_j}\equiv f_{x_jx_i} \quad i,j=1..n}$$

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}$$

uguali

solo (3) derivate da calcolare: f_{xx}, f_{yy}, ed una sola fra f_{xy} e f_{yx} (che sono uguali)

$f(x,y)=x\cos y \quad f_x=\cos y \quad f_y=-x\sin y$

$f_{xx}=0 \quad f_{yy}=-x\cos y \quad f_{xy}=-\sin y$

$f_{yx}=(-x\sin y)_x=-\sin y$

$$\begin{matrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz}\end{matrix}$$

6 invece di 9

$f_{xzyx}=(((f_x)_z)_y)_x$

da cui $\to f_{xzyx}=$

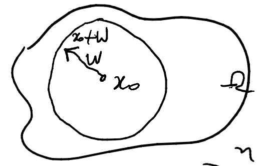
$=f_{xzxy}=f_{xxzy}=$

$=f_{xxyz}$

clairaut appl. f_{xz}

formula di Taylor

$w = (w_1, \dots, w_n)$



$x_0 \in \mathring{\Omega}$ (interno ad Ω)

$\Omega \subseteq \mathbb{R}^n \qquad x_0, w \in \mathbb{R}^n$

$$f(x_0 + w) = \sum_{k=0}^{N} \frac{1}{k!} \sum_{i_1 \dots i_k = 1}^{n} \left[f_{x_{i_1} x_{i_2} \dots x_{i_k}}(x_0)\, w_{i_1} w_{i_2} \dots w_{i_k} \right] + R^N(w)$$

dim. spazio

grado del polinomio di Taylor

PEANO

$$\lim_{w \to 0} \frac{R^N(w)}{|w|^N} = 0$$

LAGRANGE

$$x_0 \to x_0 + \xi w \qquad \xi \in [0,1]$$

$f : \Omega \to \mathbb{R} \qquad \Omega \subseteq \mathbb{R}^3$

x_1, x_2, x_3

$i_1 = 2 \quad i_2 = 2 \quad i_3 = 1$

$$f_{x_{i_1} x_{i_2} x_{i_3}} = f_{x_2 x_2 x_1} \qquad w_{i_1} w_{i_2} w_{i_3} = w_2 w_2 w_1 = w_1 w_2^2$$

$$f(x_0 + w) = \underbrace{f(x_0)}_{\text{term. grado } 0} + \underbrace{\frac{1}{1!} \sum_{i=1}^{n} f_{x_i}(x_0) w_i}_{df(x_0, w)} + \underbrace{\frac{1}{2!} \sum_{i,j=1}^{n} f_{x_i x_j}(x_0) w_i w_j} + \dots$$

$$\dots\ f_{x_i x_j}(x_0)\, w_i w_j + f_{x_j x_i}(x_0)\, w_j w_i + \dots$$

$$= 2 f_{x_i x_j}(x_0)\, w_i w_j$$

nel caso $n = 2$, ponendo

$$x_1 = x \qquad x_2 = y \qquad w = (h, k)$$

si ottiene

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + f_x(x_0, y_0) h + f_y(x_0, y_0) k +$$

$$+ \frac{1}{2} \left[f_{xx}(x_0, y_0) h^2 + f_{yy}(x_0, y_0) k^2 + 2 f_{xy}(x_0, y_0) hk \right] +$$

$$+ R^2(h, k)$$

$\left(\sum_{1}^{n} w_i \partial_{x_i} \right)^2$ ove $\partial_{x_i} \partial_{x_j} = \partial_{x_i x_j}$

$f: \mathbb{R} \to \mathbb{R}$

$$f(x_0+w) = \sum_0^N \frac{1}{k!} f^{(k)}(x_0) w^k + R^N(w)$$

VERSIONE CHE RICHIEDE IL NUMERO DI DERIVATE STRETTAMENTE INDISPENSABILE

$f: \Omega \to \mathbb{R}$ $\Omega \subseteq \mathbb{R}^n$ - n variabili $x_1 \dots x_n$.

$w \in \mathbb{R}^n$ uguali $x_0 \in \Omega$

(1) $k \equiv (k_1, k_2, \dots, k_n)$ $k_i \geq 0$ intero

multiindice

k_i numero di derivate da fare rispetto a x_i

(2) $k! \equiv k_1! \, k_2! \dots k_n!$ $(0! = 1)$

Es. $k_1 = 7$ $k_2 = 0$ $k_3 = 0 \dots k_6 = 0$ 6 variabili indip.

$$\frac{\partial^{(7)} f}{\partial x_1^7} = f_{x_1 x_1 x_1 x_1 x_1 x_1 x_1} \quad k = (7,0,0,0,0,0) \quad k! = 7! \underbrace{0!\,0!\,0!\,0!\,0!}_{1} = 7!$$

(3) $|k| \equiv k_1 + k_2 + \dots + k_n$

(4) $$\partial^k f = \frac{\partial^{|k|} f}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}$$

FORMULA DI TAYLOR in più VARIABILI

$$f(x_0+w) = \sum_{|k| \leq N} \frac{1}{k!} \partial^k f(x_0) \, w^k + R^N(w) =$$

(5) $$w^k = w_1^{k_1} w_2^{k_2} \dots w_n^{k_n}$$

$$= \sum_{\substack{k_1+k_2+\dots+k_n \leq N \\ k_i \text{ interi } \geq 0}} \frac{1}{k_1! \dots k_n!} \frac{\partial^{|k|} f(x_0)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} w_1^{k_1} \dots w_n^{k_n} + R^N(w_1 \dots w_n)$$

? Come elencarli?

Esempio

$N = 4$ $n = 3$

k_1	k_2	k_3	
4	0	0	$\Leftrightarrow f_{x_1 x_1 x_1 x_1}$
3	1	0	
3	0	1	
2	2	0	
2	1	1	
2	0	2	
1	3	0	
1	2	1	
1	1	2	
1	0	3	
0	4	0	
0	3	1	
0	2	2	
0	1	3	
0	0	4	

tutti i multiindici di modulo 4

$\boxed{\text{Ordine "decrescente" somma delle "cifre" ristrette}}$

PER OGNI MULTIINDICE DELL'ELENCO VA CALCOLATO IL TERMINE CORRISPONDENTE della formula

AD ESEMPIO $k_1=3 \quad k_2=1 \quad k_3=0$ DA' $k=(3,1,0)$ e

$|k|=3+1+0=4 \quad k!=3!\,1!\,0!=6$, da cui il termine relativo a tale scelta è

$$\frac{1}{6}\underbrace{f_{x_1x_1x_1x_2}(x_0)}_{\partial^{(k)}f(x_0)}\underbrace{w_1^3w_2}_{w^k}$$

lakeshoreguardian.com

If General Motors Built Cars Like Microsoft...

3 minutes

November 2017 > Smile Awhile

At a computer expo (COMDEX), Bill Gates reportedly compared the computer industry with the auto industry and stated, "If GM had kept up with technology like the computer industry has, we would all be driving twenty-five-dollar cars that got 1,000 miles to the gallon." In response to Bill's comments, General Motors issued a press release stating (supposedly by Mr. Welch himself), "Yes, but would you want your car to crash twice a day?"

Then others added these comments:

1. Every time they repainted the lines on the road you would have to buy a new car.

2. Occasionally, your car would die on the freeway for no reason, and you would just accept this, restart and drive on.

3. Occasionally, executing a maneuver such as a left turn would cause your car to stop and fail, and you would have to re-install the engine. For some strange reason, you would accept this, too.

4. You could only have one person in the car at a time, unless you bought Car95 or CarNT . But then you would have to buy more seats.

5. Macintosh would make a car that was powered by the sun, was reliable, five times as fast, twice as easy to drive, but would only run on five percent of the roads.

6. The Macintosh car owners would get expensive Microsoft upgrades to their cars, which would make their cars run much slower.

7. The oil, gas and alternator warning lights would be replaced by a single "general car fault" warning light.

8. New seats would force everyone to have the same size butt.

9. Before going off, the airbag system would say, "Are you sure?"

10. If you were involved in a crash, you would have no idea what happened.

11. Occasionally, for no reason whatsoever, your car would lock you out and refuse to let you in until you simultaneously lifted the door handle, turned the key and grabbed hold of the radio antenna.

12. GM would require all car buyers to also purchase a deluxe set of *Rand McNally* road maps (now a GM subsidiary), even though they neither needed nor wanted them. Attempting to delete this option would immediately cause the car's performance to diminish by 50 percent or more. Moreover, GM would become the target of

investigation by the Justice Department.

13. Every time GM would introduce a new model, car buyers would have to learn how to drive all over again because none of the controls would operate in the same manner as the old car.

14. You'd press the "start" button to shut off the engine.

MASSIMI E MINIMI

Note Title 5/15/2020

UN PO' D'ANALISI 1

$f'(x_0)=0 \qquad f''(x_0)>0 \quad x_0$ min. locale

$f''(x_0)<0 \quad x_0$ max locale

$f''(x_0)=0 \quad$ MARE DI GUAI!

$$f(x_0+w) = f(x_0) + f'(x_0)w + \frac{1}{2}f''(x_0)w^2 + R_2(w)$$

$f'(x_0)=0$

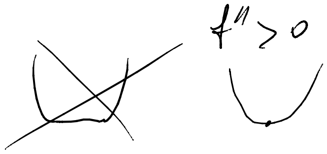
$$\underbrace{f(x_0+w)-f(x_0)}_{\substack{\geq 0 \text{ min. locale} \\ \leq 0 \text{ max locale}}} = \underset{\geq 0}{w^2}\underbrace{\left[\frac{1}{2}f''(x_0) + \frac{R_2(w)}{w^2}\right]}$$

min. / max locale in un intorno

ha il segno di $f''(x_0)$

$\frac{R_2(w)}{w^2} \to 0$, $w \to 0$

$f''>0$



Perm. segno

[ ] ha lo stesso segno del suo limite $\frac{1}{2}f''(x_0)$

$f \in C^2(\Omega) \qquad \Omega \subseteq \mathbb{R}^n \qquad x_0 \in \mathring{\Omega} \qquad \nabla f(x_0)=0 \quad x_0$ critico!

$$f(x_0+w) = f(x_0) + \underbrace{\cancel{\sum_{i=1}^{n} f_{x_i}(x_0)w_i}}_{df(x_0,w)} + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_ix_j}(x_0)w_iw_j + R_2(w)$$

$\nabla f(x_0)=0$

$$f(\underbrace{x_0+w}_{x}) - f(x_0) = \underset{\geq 0}{|w|^2}\left[\frac{1}{2}\frac{\sum_{i,j=1}^{n} f_{x_ix_j}(x_0)w_iw_j}{|w|^2} + \frac{R_2(w)}{|w|^2}\right]$$

$\frac{R_2(w)}{|w|^2} \to 0$, $w \to 0$ visto di Peano

Clairaut-Schwarz. $\quad f_{x_ix_j} = f_{x_jx_i} \qquad \underline{\sum \underbrace{f_{x_ix_j}(x_0)}_{\in\mathbb{R}} w_iw_j}$

forma quadratica simmetrica reale, che verifica

$$\boxed{\lambda|w|^2 \leq \sum_{i,j} a_{ij}w_iw_j \leq \Lambda|w|^2}$$

con λ min autovalore $\quad \Lambda$ max autovalore

$$\lambda \le \frac{\sum_{i,j=1}^{n} f_{x_i x_j}(x_0) w_i w_j}{|w|^2} \le \Lambda$$

ove λ, Λ sono il min e il massimo autovalore

della matrice $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ Matrice Hessiana (HESSE)

$\lambda > 0 \iff$ hessiana è definita positiva.

$$\left[\frac{1}{2}\underbrace{\frac{\sum f_{x_i x_j} \cdots}{|w|^2}}_{\ge \frac{1}{2}\lambda > 0} + \underbrace{\frac{R_2(w)}{|w|^2}}_{w \to 0 \ \to 0}\right]$$

$\left|\frac{R_2(w)}{|w|^2}\right| < \frac{1}{2}\lambda \ (= \varepsilon)$

per $|w|$ abbast. piccolo

$\Lambda < 0$ hessiana definita negativa

$$\underbrace{\left[\underbrace{\frac{1}{2}\frac{\sum_{i,j} f_{x_i x_j} \cdots}{|w|^2}}_{\le \frac{\Lambda}{2}} + \underbrace{\frac{R_2(w)}{|w|^2}}_{\to 0}\right]}_{\text{ha il segno di } \Lambda}$$

$\varepsilon = \left|\frac{\Lambda}{2}\right| \ \exists \delta :$

$|w| < \delta$

$\left|\frac{R_2(w)}{|w|^2}\right| < \varepsilon$

Th.

Se $\nabla f(x_0) = 0$ $f \in C^2(\Omega)$ Ω aperto $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ è

definita positiva $\Rightarrow$ x_0 è un minimo locale

definita negativa $\Rightarrow$ x_0 è un massimo locale

$$\boxed{\text{Forme indef} \quad \lambda < 0 \quad \Lambda > 0}$$

$$\lambda |w|^2 \leq \sum a_{ij} w_i w_j \leq \Lambda |w|^2$$

vale = sugli autovettori di λ

uguaglianza sugli autovettori di Λ

$$\left[\frac{1}{2} \underbrace{\frac{\sum_{ij} f_{x_i x_j} w_i w_j}{|w|^2}}_{= \lambda/2} + \frac{R_2(w)}{|w|^2} \right]$$

$\varepsilon = \left| \frac{\lambda}{2} \right|$

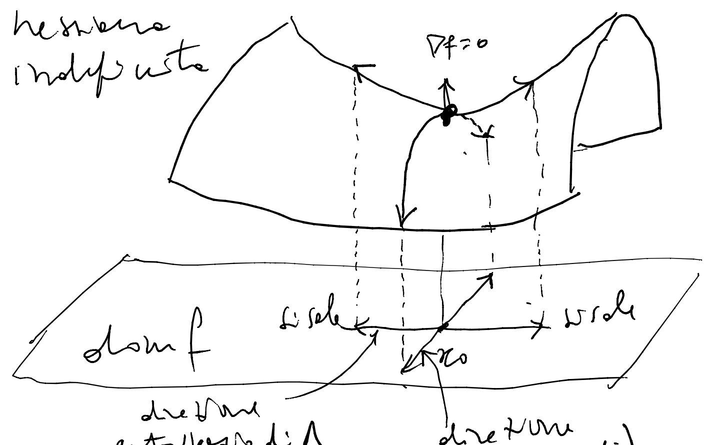
$[\ \ ] < 0$ per $|w|$ piccolo

Se si sceglie come w un autovettore di λ il primo rapporto vale $\lambda < 0$ $\quad [\ \ ] < 0$

Se si sceglie come w un autovettore di Λ il primo rapporto vale $\Lambda > 0$ e $[\ \ ] > 0$.

PUNTO DI SELLA (NON DEGENERE)

hessiana indefinita

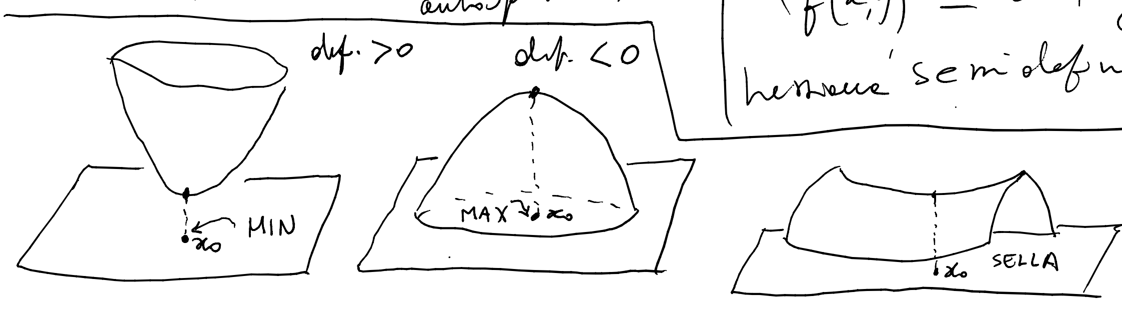


hessiana indefinita

$$f(x,y) = xy$$
$$= x^2 - y^2$$

$$\begin{cases} f(x,y) = x^2 - y^4 \\ f(x,y) = x^2 + y^4 \end{cases}$$

hessiana semidefinita ≥ 0



PUNTI CRITICI: CLASSIFICAZIONE

Note Title 5/19/2020

$\nabla f(x_0)=0$ CN per un estremo

$f\in C^2(B_\rho(x_0))$

$$f(x_0+w)-f(x_0)=\frac{1}{2}\sum_{i,j=1}^{n}\underbrace{f_{x_ix_j}(x_0)}_{\in\mathbb{R}}\,w_i w_j + R_2(w)$$

$$Hf(x_0)=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}(x_0)\right)$$

Se H NON HA AUTOVALORE NULLO (H NON DEGENERE)

H def. >0 x_0 min (locale) H def <0 x_0 max (locale)
H indefinita x_0 è una sella (locale) } non degeneri

---

x^2+y^4

$\nabla f=\begin{pmatrix}2x\\4y^3\end{pmatrix}$

che si annulla solo in $(0,0)$ (unico punto critico)

in $(0,0)$ x^2+y^4 ha minimo

$Hf(0,0)=\begin{pmatrix}2&0\\0&12y^2\end{pmatrix}\Big|_{\substack{x=0\\y=0}}=\begin{pmatrix}2&0\\0&0\end{pmatrix}$

diagonale con autovalori 2 e $\underline{\underline{0}}$
DEGENERE

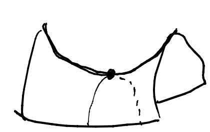
x^2-y^4

$\nabla f=\begin{pmatrix}2x\\-4y^3\end{pmatrix}$

$Hf=\begin{pmatrix}2&0\\0&-12y^3\end{pmatrix}_{\substack{x=0\\y=0}}=\begin{pmatrix}2&0\\0&0\end{pmatrix}$

sull'asse $y=0$
$(0,0)$ è di minimo

sull'asse $x=0$
$(0,0)$ è di massimo



x^2+y^3

$\frac{1}{2}\overset{f_{xx}(0)}{2}\,w_1^2=$
$=w_1^2$

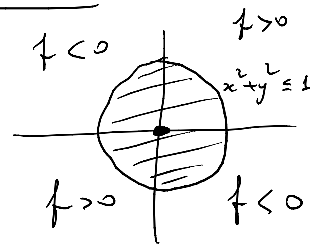
Spostandosi sull'autospazio dell'autovalore 0 (cioè l'asse y) il complesso dei termini di II grado è nullo

Equivale al caso $f'(x_0)=0$ dell'Analisi I

Strategie per la determin. degli estremi globali

$f(x)=x$ $\quad [0,1]$ $\quad$ 0 è min 1 è max $\quad \underline{f'(0)=f'(1)=1}$

$f(x,y)=\underline{xy}$ $\qquad \max_{\min} f(x,y)$, $\quad x^2+y^2\le 1$



$f_x=y$

$f_y=x$

$$\boxed{\nabla f=0 \Leftrightarrow (x,y)=(0,0)}$$

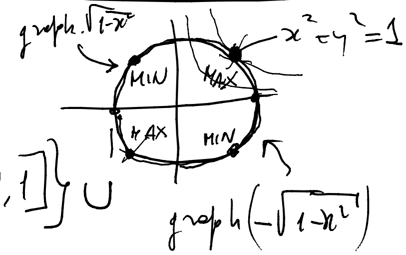
$(0,0)$ sella non degenere

$$Hf(x,y)=\begin{pmatrix}0&1\\1&0\end{pmatrix}=H(0,0) \qquad \begin{vmatrix}0&1\\1&0\end{vmatrix}=-1 \qquad H \text{ indefinito}$$

prodotto degli autovalori

COME STUDIARE GLI ESTREMI SULLA FRONTIERA

$f(x,y)$ sulla circonf. unitaria $x^2+y^2=1$



1) "caso cartesiano" $\{(x,\overbrace{\sqrt{1-x^2}}^{\varphi(x)}),\ x\in[-1,1]\}\cup$

$\cup\{(x,-\sqrt{1-x^2}),\ x\in[-1,1]\}$

$h_1(x)=f(x,\overbrace{\sqrt{1-x^2}}^{\varphi(x)})$ su $[-1,1]$

1 variabile su un intervallo

$$\boxed{\begin{array}{c}\text{DINI}\\ f(x,\varphi(x))\equiv 0\end{array}}$$

$h_2(x)=f(x,-\sqrt{1-x^2})$ su $[-1,1]$

Ogni punto della circonferenza $x^2+y^2=1$ sarà del tipo $(x,\sqrt{1-x^2})$ oppure $(x,-\sqrt{1-x^2})$ con $x\in[-1,1]$.

$h_1(x) = x\sqrt{1-x^2}$ su $[-1,1]$

$$h_1'(x) = \sqrt{1-x^2} + x\cdot\frac{-2x}{2\sqrt{1-x^2}} = \frac{1-x^2-x^2}{\sqrt{1-x^2}}$$ che ha lo stesso segno

di $1-2x^2$ su $]-1,1[$

$$\begin{array}{ccccc} - & & + & & - \\ \hline & -\frac{1}{\sqrt{2}} & & \frac{1}{\sqrt{2}} & \end{array}$$ segno della h_1'

minimo locale in $-\frac{1}{\sqrt{2}}$

max locale in $1/\sqrt{2}$

$h_1\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = -\frac{1}{2}$

$h_1\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = \frac{1}{2}$

$h_1(1) = h(-1) = 0$ e dunque

punto di max globale in $\frac{1}{\sqrt{2}}$
valore max $\frac{1}{2}$

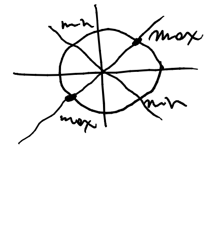
punto di min. globale in $-\frac{1}{\sqrt{2}}$
valore min $-\frac{1}{2}$

Lo stesso studio per $h_2(x) = f\left(x, -\sqrt{1-x^2}\right)$ dà

punto di massimo in $\left(-\frac{1}{\sqrt{2}}\right)$ di valore $\frac{1}{2}$

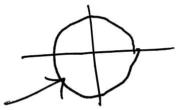
" " min in $\left(\frac{1}{\sqrt{2}}\right)$ " " $-\frac{1}{2}$

<u>Conclusioni</u> $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ e $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ sono punti di massimo globale sulla frontiera e $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ e $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ sono punti di minimo globale sulla frontiera.



$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = \frac{1}{2}$

$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \quad t \in [-\pi, \pi]$



sostegno (immagine) di γ

$\boxed{\text{Caso parametrico}}$

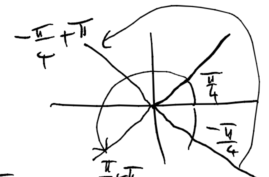
$$\boxed{h(t) = f(\gamma(t)) \quad t \in [-\pi, \pi]}$$

$$\boxed{h(t) = \cos t \sin t} = \frac{1}{2}\sin 2t$$

$$\max_{\min} \underset{x^2+y^2=1}{f} = \max_{\min} \underset{[-\pi,\pi]}{h}$$

$h(t)$ ha max se $2t = \frac{\pi}{2}$

" min se $2t = -\frac{\pi}{2}$



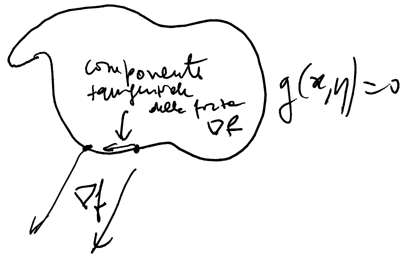
$\sin 2t = 1$

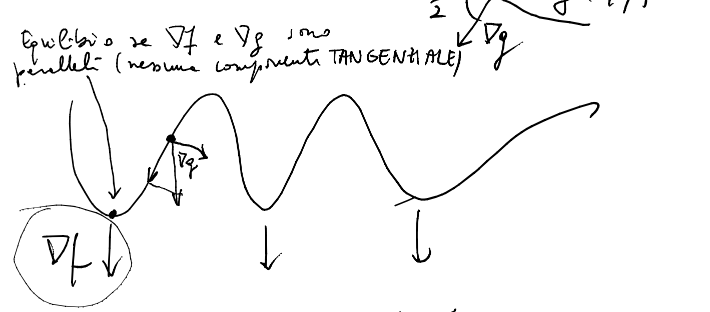
$2t = \frac{\pi}{2} + 2\pi \quad t = \frac{\pi}{4} + \pi$

$\boxed{\text{MOLTIPLICATORI DI LAGRANGE}}$

$\{g(x,y) = 0\}$ = frontiera di $\Omega = \operatorname{dom} f$

$x^2 + y^2 - 1 = 0 \qquad \boxed{g(x,y) = x^2 + y^2 - 1}$





$\exists \lambda : \nabla f = \lambda \nabla g \quad \lambda$ moltiplicatore di Lagrange

I punti di max o min di $f(x,y)$ <u>vincolati</u> all'insieme $\{g(x,y)=0\}$ sono le soluzioni di

$$\begin{cases}\nabla f = \lambda \nabla g \leftarrow \\ g = 0 \leftarrow\end{cases}$$

$$f(x,y) = xy \qquad \nabla f = \begin{pmatrix} y \\ x \end{pmatrix}$$

$$g(x,y) = x^2+y^2-1 \qquad \nabla g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\begin{cases}\begin{pmatrix} y \\ x \end{pmatrix} - 2\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ x^2+y^2-1=0\end{cases}$$

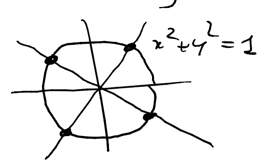
$$\begin{cases} y = 2\underbrace{\lambda x} \\ x = 2\underbrace{\lambda y} \\ x^2+y^2 = 1 \leftarrow \end{cases}$$ incognite x, y, λ

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ x^2+y^2=1 \end{cases}$$

$\lambda \neq 0$, $x \neq 0$, $y \neq 0$

dividendo la prima eq. per la seconda si ottiene $\dfrac{y}{x} = \dfrac{x}{y} \Leftrightarrow \dfrac{y^2-x^2}{xy}$

ho zeri se $y = \pm x$



prime due eq.

$\lambda = 0 \Rightarrow x=0 \quad y=0$ $\qquad (0,0)$ unica soluzione prime due equazioni ma NON della $\underline{III}$

$x=0$ I eq $\Rightarrow y=0$

$y=0$ II eq $\Rightarrow x=0$

$$\max_{\Pi}\,/\,\min_{\Pi} f \qquad \Pi = \left\{ x : \begin{cases} g_1(x)=0 \\ \vdots \\ g_k(x) = 0 \end{cases} \right\}$$

Gli estremi di f su $\Gamma = \{ g_1(x)=0, \ldots, g_k(x)=0 \}$

sono fra le soluzioni di

moltiplicatori di Lagrange

$$\begin{cases} \nabla f + \sum \lambda_i \nabla g_i = 0 & n \text{ equazioni} \\ \boxed{\begin{matrix} g_1(x)=0 \\ \vdots \\ g_k(x)=0 \end{matrix}} & k \text{ equazioni} \end{cases}$$

Estremi vincolati AN 2.7

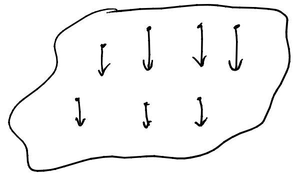
CAMPI VETTORIALI E FORME DIFFERENZIALI LINEARI

Note Title 5/20/2020

<u>MATEMATICA</u> <u>Problema della primitiva</u> $y' = f$

y incognita, f nota

<u>FISICA</u>



$$\Omega \subseteq \mathbb{R}^N$$

$$f : \Omega \to \mathbb{R} \qquad \nabla f : \underset{\subseteq \mathbb{R}^N}{\Omega} \to \mathbb{R}^N.$$

$A : \Omega \to \mathbb{R}^N$, $\Omega \subseteq \mathbb{R}^N$ sarà detto <u>CAMPO (VETTORIALE)</u>

$df : \underset{x_0 \in}{\Omega} \times \underset{w \in}{\mathbb{R}^N} \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^N$ $\quad w \to df(x_0, w)$ è LINEARE $\forall x_0 \in \Omega$.

$\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^N$ $\quad w \to \alpha(x_0, w)$ è LINEARE $\forall x_0 \in \Omega$

FORMA (DIFFERENZIALE LINEARE)

PROBLEMA DELLA PRIMITIVA

1) Se A è un campo su Ω, esiste $f : \Omega \to \mathbb{R}$ tale che $\nabla f \equiv A$ su Ω?

2) Se α è una forma su Ω, esiste $f : \Omega \to \mathbb{R}$ tale che $df \equiv \alpha$ su $\Omega \times \mathbb{R}^N$

Se $\exists f : \Omega \to \mathbb{R}$: $\nabla f \equiv A$ su Ω, A si dirà <u>integrabile</u> ed f si dirà <u>primitiva</u> di A.

Se $\exists f : \Omega \to \mathbb{R} : df \equiv \alpha$ su $\Omega \times \mathbb{R}^N$ α si dirà <u>integrabile</u> ed f si dirà <u>primitiva</u> di α

A integrabile si dirà anche 1) CONSERVATIVO, 2) POTENZIALE

α integrabile si dirà anche 1) ESATTA

f si dirà POTENZIALE

CAMPI E FORME ASSOCIATI

$$\boxed{df(x_0, w) = \nabla f(x_0) w}$$

$\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$ fissato x_0

$w \to \alpha(x_0, w)$ lineare da $\mathbb{R}^N \to \mathbb{R}$

$\exists u \in \mathbb{R}^N : \alpha(x_0, w) = u\, w$ prodotto scalare

Il campo associato ad α è definito ponendo $A(x_0) = u$

$$\boxed{\alpha(x_0, w) = A(x_0)\, w}$$

α ed A sono detti ASSOCIATI se

$$\sum_{i=1}^{N} A_i(x_0)\, w_i = \sum_{i=1}^{N} A_i(x_0)\, dx_i$$

<u>Esempio di campo</u> $A = \begin{pmatrix} \sin xy \\ 1 + \arctg x^2 y \end{pmatrix}$ campo su $\mathbb{R}^2$

Esempio di forma $\alpha(x, dx) = A_1(\underbrace{x_1 \ldots x_n}_{x})\, dx_1 + A_2(\underbrace{x_1 \ldots x_n}_{x})\, dx_2 + \ldots + A_n(\underbrace{x_1 \ldots x_n}_{x})\, dx_n$

$\alpha(x, y, dx, dy) =$

$= \sin xy\, dx + (1 + \arctg x^2 y)\, dy = \begin{pmatrix} \sin xy \\ 1 + \arctg x^2 y \end{pmatrix} \underbrace{\begin{pmatrix} dx \\ dy \end{pmatrix}}_{w}$

A ed α sono ASSOCIATI

→ $\boxed{f:\ \nabla f = A}$ $f \in C^1$ $\boxed{\text{Accade se } \exists f \in C^1 : \ f_{x_i}(x) = A_i(x) \ \forall x \in \Omega}$

A è integrabile

→ $df(x_0, w) = \nabla f(x_0)\, w = A(x_0)\, w = \alpha(x_0, w)$

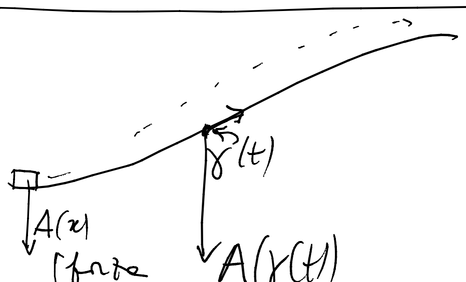
$\boxed{\alpha = df \quad A = \nabla f}$ associato ad A

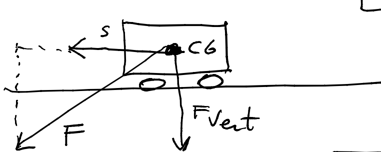
∇f è il campo associato al df

Campo associato alla forma $x\,dx + y\,dy = \alpha(\underbrace{x,y}_{\text{punto}}, \underbrace{dx,dy}_{w})$

è $A(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$

$f(x) = \int_a^x g(t)\,dt$ $f'(x) = g(x)$ $g \in C^0[a,b]$





→ $\boxed{L = F \cdot s}$ prodotto scalare

$\boxed{|F||s|\cos\widehat{Fs}}$

Integrale del campo A sulla curva

REGOLARE $\gamma : [a,b] \to \mathbb{R}^N$

$$\int_\gamma A = \int_a^b A(\gamma(t)) \underbrace{\dot\gamma(t)\,dt}_{ds}$$

prodotto scalare fra i vettori $A(\gamma(t))$ e $\dot\gamma(t)$

Integrale di α sulla curva γ $\equiv$ l'integrale del suo campo associato.

$$\alpha(x,w) = A(x)w \qquad \int_\gamma \alpha \equiv \int_\gamma A = \int_a^b A(\gamma(t))\dot\gamma(t)\,dt$$

$$A = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

associati

$$\alpha = -\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$$

$$\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$$

$$dx_1 = \dot\gamma_1(t)\,dt \qquad dx_2 = \dot\gamma_2(t)\,dt \;\ldots\; dx_n = \dot\gamma_n(t)\,dt$$

$$\gamma = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix} \quad t \in [0, 2\pi]$$

$$\begin{pmatrix}x\\ y\end{pmatrix} = \gamma(t) \qquad \dot\gamma = \begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}$$

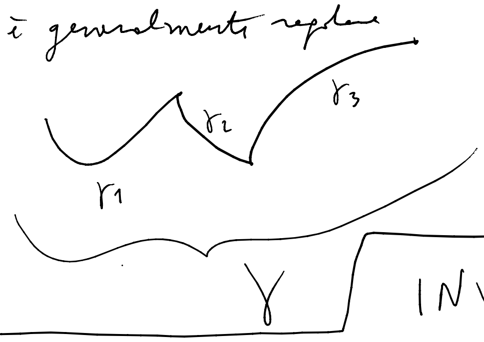
$$A(\gamma(t))\dot\gamma(t) =$$

$$= \underbrace{-\frac{\sin t}{1}}_{A_1(\gamma(t))} \cdot \underbrace{(-\sin t)}_{\dot\gamma_1(t)} + \underbrace{\frac{\cos t}{1}}_{A_2(\gamma(t))} \underbrace{\cos t}_{\dot\gamma_2(t)} =$$

$$= 1$$

$$\int_0^{2\pi} A(\gamma(t))\dot\gamma(t)\,dt = \int_0^{2\pi} 1 = 2\pi$$

se γ è generalmente regolare



$$\int_\gamma A = \int_{\gamma_1} A + \int_{\gamma_2} A + \int_{\gamma_3} A$$

INVARIANZA DELL'INTEGRALE

$$\sigma(t) = \gamma(\rho(t)) \qquad \boxed{\dot\rho > 0} \qquad \sigma, \gamma \text{ regolari}$$

$$\rho : [c,d] \to [a,b] \qquad \rho(c) = a \qquad \rho(d) = b$$

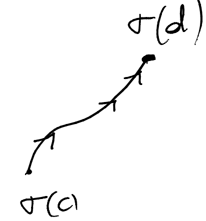
$$\int_\sigma A = \int_c^d A(\sigma(t))\,\dot\sigma(t)\,dt =$$

$$= \int_c^d A(\gamma(\rho(t)))\,\dot\gamma(\rho(t))\,\overbrace{\underbrace{\dot\rho(t)}_{\in\mathbb{R}}\,dt}^{ds} \underset{s=\rho(t)}{=} \int_{\rho(c)=a}^{\rho(d)=b} A(\gamma(s))\,\dot\gamma(s)\,ds = \int_\gamma A$$

prodotto scalare

$\dot{\rho} < 0 \qquad \rho : [c,d] \to [\underset{\rho(c)}{\overset{\shortparallel}{b}}, \underset{\rho(d)}{\overset{\shortparallel}{a}}]$ $\qquad c < d \quad a < b$

$$\int_\sigma A = \int_b^a A(\gamma(s))\,\dot{\gamma}(s)\,ds = -\int_\gamma A$$



C.N. (e S.) perché $A \in C^0(\Omega)$ sia integrabile è che $\int_\gamma A$ dipende **SOLO** dagli estremi di γ e **NON** dal cammino

---

C.N. se A è integrabile e $A \in C^0 \Rightarrow \int_\gamma A$ dipende solo dagli estremi e non dal cammino

---

$A : \Omega \to \mathbb{R}^n \quad \boxed{A \in C^k(\Omega) \iff A_i \in C^k(\Omega)}$

A è integrabile $\Rightarrow \exists\, f : \Omega \to \mathbb{R} \;:\; \nabla f \equiv A$ su Ω.

Poiché $A \in C^0 \Rightarrow \nabla f \in C^0 \Rightarrow f_{x_i} \in C^0 \Rightarrow \underline{f \in C^1(\Omega)}$

$\underline{\gamma \in C^1[a,b] \to \Omega}$

$$\int_\gamma A = \int_a^b A(\gamma(t))\,\dot{\gamma}(t)\,dt \overset{A = \nabla f}{=} \int_a^b \underbrace{\nabla f(\gamma(t))\,\dot{\gamma}(t)}_{\frac{d}{dt}[f(\gamma(t))]}\,dt = $$

Th. Torricelli scalare (in $\mathbb{R}$)

$$= \underset{\text{estremi di } \gamma}{f(\gamma(b)) - f(\gamma(a))} \underset{\text{Fisica}}{=} \boxed{\text{differenza di potenziale}}$$

INTEGRABILITA' DI CAMPI E FORME

Note Title 5/22/2020

C.N. perché $\underline{A \in C^0}$ sia integrabile è che

$\forall \gamma_1, \gamma_2 \in [0,1]$ $\boxed{\int_{\gamma_1} A = \int_{\gamma_2} A}$ se $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$

$\gamma_1 : [a,b] \to \text{dom} A$

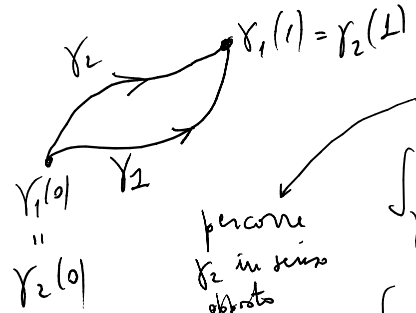
$\gamma_2 : [c,d] \to \text{dom} A$

$\gamma_1(a) = \gamma_2(c)$

$\gamma_1(b) = \gamma_2(d)$

$\| $

$f(\gamma_1(1)) - f(\gamma_1(0)) = \text{diff. di potenziale}$



$\sigma(t) = \gamma_2(1-t)$

percorre γ_2 in senso opposto

$\int_{\gamma_1 \oplus \gamma_2} = \int_{\gamma_1} + \int_{\gamma_2}$

$\int_{\gamma_1 \oplus \sigma} = \int_{\gamma_1} + \int_{\sigma} = -\int_{\gamma_2}$

$\int_{\gamma_1} A = \int_{\gamma_2} A$

$\int_{\sigma} = -\int_{\gamma_2}$ $\dot{\rho} < 0$

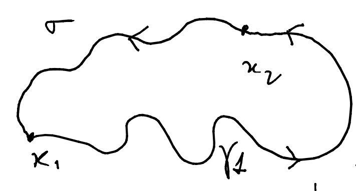
$\Downarrow$

$\boxed{\int_{\gamma_1 \oplus \sigma} = \int_{\gamma_1} - \int_{\gamma_2} = 0}$

C.N. (e S) $A \in C^0$ sia integrabile è che

$\forall \gamma$ chiusa $\gamma : [0,1] \to \text{dom} A$ si ha

$\boxed{\int_{\gamma} A = 0}$



$\int_{\gamma_1} A = -\int_{\sigma} A$

$\|$

$\int_{\gamma_1} A = \int_{\gamma_2} A$

γ_2 come σ orientata all'inverso da x_2 verso x_1

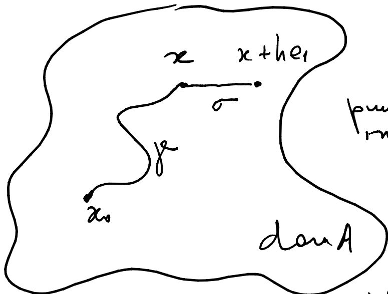
TEOREMA DI TORRICELLI

C. S. perché $\boxed{A \in C^0}$ sia integrabile è che

$\int_{\gamma} A$ non dipende dal sostegno di γ (cammino), ma solo dagli estremi.

DIM. $F(x) = \int A = \underbrace{\int_a^b A(\gamma(t))\dot\gamma(t)\,dt}_{\text{esiste perché l'integrando è continuo}}$ $\quad\Big|\quad \int_a^x f(t)\,dt \overset{f\in C^0}{=} F(x) \Downarrow F'(x) = f(x)$

$\gamma_{x_0 x}$ ← punto iniziale (x_0), punto finale (x)



DICO CHE: $F(x)$ è un potenziale (PRIMITIVA) di A, cioè $\nabla F \equiv A$ sul dom A

$\Updownarrow$

$\boxed{F_{x_i} \equiv A_i \text{ su dom} A}$

$\gamma \oplus \sigma$ congiunge x_0 e $x+he_1$

$F_{x_1} \equiv A_1$ su dom A

$$\lim_{h\to 0} \frac{F(x+he_1)-F(x)}{h} \overset{?}{=} \lim_{h\to 0}\frac{1}{h}\Big[\int_{\gamma_{x_0x}\oplus\sigma} A - \int_{\gamma_{x_0x}} A\Big] =$$

(curve fra x_0 e $x+he_1$)

$$= \lim_{h\to 0}\frac{1}{h}\Big[\cancel{\int_{\gamma_{x_0x}} A} + \int_\sigma A - \cancel{\int_{\gamma_{x_0x}} A}\Big] = \lim_{h\to 0}\frac{1}{h}\Big[\int_\sigma A\Big] =$$

$$= \lim_{h\to 0}\frac{1}{h}\int_0^h A(\underbrace{x+te_1}_{\sigma})\cdot \underbrace{e_1}_{\dot\sigma}\,dt =$$

$x+te_1$, $t\in[0,h]$ è il segmento di estremo iniziale x ed estremo finale $x+he_1$

$$\begin{pmatrix}A_1\\A_2\\\vdots\\A_n\end{pmatrix}\cdot\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = A_1$$

$$= \lim_{h\to 0}\underbrace{\frac{1}{h}\int_0^h A_1(x+te_1)\,dt}_{\text{media integrale di funzione continua}} =$$

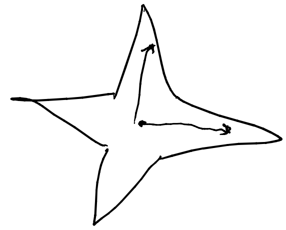
$$= \lim_{h\to 0}\boxed{A_1(x+\xi e_1)} \qquad \boxed{\xi\in[0,h]} \quad \xi?$$

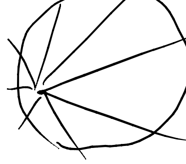
$\| \Leftarrow h\to 0 \Rightarrow \xi\to 0 \Rightarrow x+\underbrace{\xi e_1}_{\to 0}\to x$

$A_1(x)$

$\lim_{x\to a} f(x) \to f(a)$ se $f\in C^0$

Th. media integrale per integrandi <u>CONTINUE</u>





CAMPI IRROTAZIONALI (E FORME CHIUSE)

Note Title 5/26/2020

$\sigma(t) = \gamma(p(t))$ $\boxed{\dot p > 0}$ $\int_\gamma A = \int_\sigma A$ A qualsiasi

A integrabile (C^0) $\int_\gamma A = \int_\sigma A$ purché $\gamma(a) = \sigma(c)$, $\gamma(b) = \sigma(d)$

$\gamma : [a,b] \to \operatorname{dom} A$

$\sigma : [c,d] \to \operatorname{dom} A$

$A \in C^0$ è integrabile $\iff \int_\gamma A = 0 \quad \forall \gamma$ chiusa, ~~regolare~~ a tratti

INVARIANZA DELL'INTEGRALE

$A \in C^1$ A integrabile ($\exists f : \operatorname{dom} A \to \mathbb{R} : \nabla f \equiv A$ sul $\operatorname{dom} A$)

$\boxed{f_{x_i} \equiv A_i}$ $\quad (f_{x_i})_{x_j} = (A_i)_{x_j}$

$\|$ CLAIRAUT-SCHWARZ

$(f_{x_j})_{x_i} = (A_j)_{x_i}$

$\Longrightarrow \boxed{(A_i)_{x_j} = (A_j)_{x_i} \quad \forall i,j}$

CONDIZIONE DEL ROTORE

ESEMPIO

$A(x,y) = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ $\quad A_1 = y$, $A_2 = x$ $\quad (A_1)_{x_2} = (A_2)_{x_1}$

$\frac{\partial y}{\partial y} = 1 \qquad \frac{\partial x}{\partial x} = 1$

$xy = f(x,y)$ $\quad x_1 = x$, $x_2 = y$

CN perché $A \in C^1$ sia integrabile è che

cond. del rotore $\quad (A_i)_{x_j} = (A_j)_{x_i} \quad \forall i,j = 1..n$

$\boxed{(A_i)_{x_j} - (A_j)_{x_i} = 0}$

DEFINIZIONE

Se $A \in C^1$ verifica la condizione $(A_i)_{x_j} = (A_j)_{x_i}$ $\forall i,j$ si dirà IRROTAZIONALE, e la sua forma associata si dice CHIUSA

ESEMPIO: ESISTONO CAMPI IRROTAZIONALI NON INTEGRABILI

$$A(x,y) = \left(\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} \right)$$

l'integrale su $\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $t \in [0, 2\pi]$ non è nullo $\Rightarrow$ A non è integrabile, ma verifica le condizioni del rotore

---

$$y\,dx - x\,dy = \alpha(x,y;dx,dy)$$

$$\frac{\partial y}{\partial y} = 1 \neq \frac{\partial(-x)}{\partial x} = -1$$

la forma non è chiusa

ROTORE

$$\operatorname{rot} A = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \overset{A}{\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}} = \begin{pmatrix} \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \\ -\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \\ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{pmatrix}$$

∇ prodotto vettore

rotore di A

$\boxed{\nabla \times A} = 0 \qquad (A_3)_y - (A_2)_z = 0$

$A_i \qquad A_j$

$$\boxed{(A_j)_{x_i} = (A_i)_{x_j}} \quad \forall\, i,j$$

$$(x^2y\ ,\ xz\ ,\ x^2+y^2+z^2) = A(x,y,z)$$

$\quad A_1 \qquad\quad A_2 \qquad\quad A_3$

$$\frac{\partial A_1}{\partial y} \stackrel{?}{=} \frac{\partial A_2}{\partial x} \qquad\qquad \frac{\partial A_1}{\partial y} = x^2 \quad \frac{\partial A_2}{\partial x} = z$$

$\quad\ x_2 \qquad\quad x_1$

SONO DIVERSE

il campo è ROTAZIONALE

(cioè NON IRROTAZIONALE)

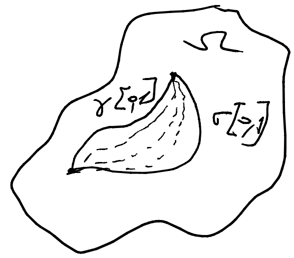
$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \stackrel{?}{=} \frac{\partial}{\partial x}\left(-\frac{x}{x^2+y^2}\right)$$

SÌ

VERIFICARE!

DEFORMAZIONE DI CURVE (OMOTOPIA)

$\gamma : [0,1] \to \Omega \qquad \sigma : [0,1] \to \Omega$



$\gamma(0) = \sigma(0) \qquad$ γ, σ continue

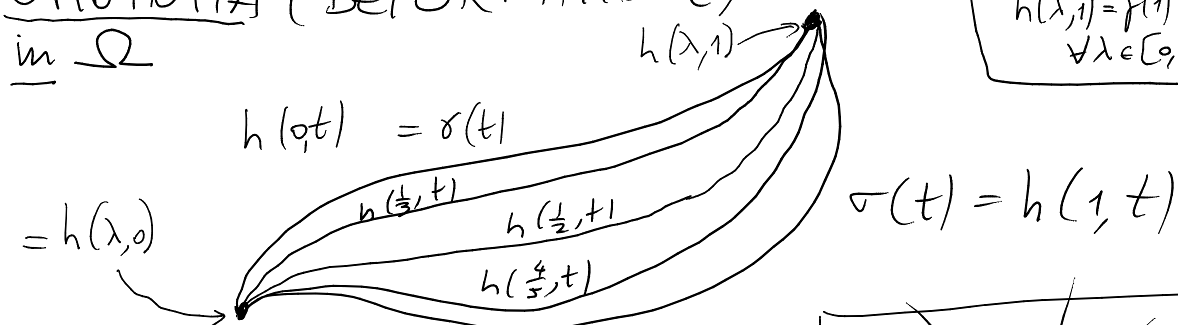
$\gamma(1) = \sigma(1)$

$h : [0,1]\times[0,1] \to \Omega$ con $\boxed{1)\ h \text{ continua}}$

$$\boxed{2)\ h(0,t)=\gamma(t)\ \forall t\in[0,1] \qquad h(1,t)=\sigma(t)\ \forall t\in[0,1]}$$

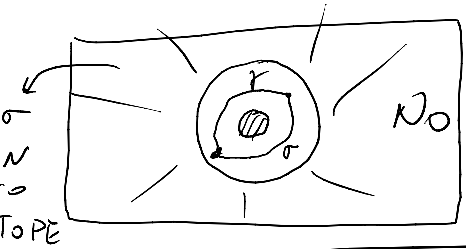
<u>OMOTOPIA</u> (DEFORMAZIONE) <u>di</u> γ <u>e</u> σ <u>in Ω</u>

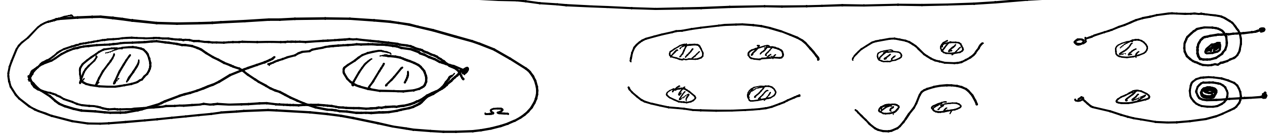
$$\boxed{\begin{array}{l} 3)\ h(\lambda,0)=\gamma(0)=\sigma(0) \\ h(\lambda,1)=\gamma(1)=\sigma(1) \\ \forall \lambda\in[0,1] \end{array}}$$



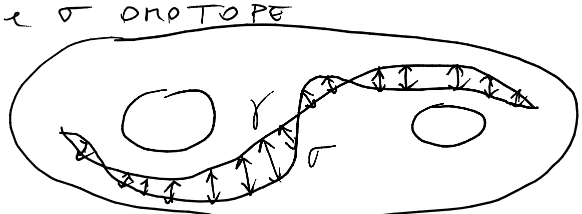
$h(\lambda, t)$ $\quad \lambda\in[0,1]$, $t\in[0,1]$

γ e σ NON SONO OMOTOPE





γ e σ OMOTOPE



LE FRECCE DEFORMANO $\gamma(\bar{t})$ in $\sigma(\bar{t})$ per vari $\bar{t}\in[0,1]$

$\lambda \to h(\lambda, t)$ t fissato

$t \to h(\lambda,t)$ curva che interpola γ e σ per il valore λ

Se A è irrotazionale, $A: \Omega \to \mathbb{R}^N$, $\Omega \subseteq \mathbb{R}^N$

e se $\gamma: [0,1]\to\Omega$, $\sigma: [0,1]\to\Omega$, continue,

$\gamma(0)=\sigma(0)$ e $\gamma(1)=\sigma(1)$, omotope in Ω, <u>allora</u>

$$\boxed{\int_\gamma A = \int_\sigma A}$$

TEOREMA DI INVARIANZA OMOTOPICA

SOLO ENUNCIATO

DIM PRODI ANALISI II

L'INVARIANZA OMOTOPICA DELL'INTEGRALE

vale anche per le curve chiuse $\gamma, \sigma : [0,1] \to \Omega$

$\gamma(0) = \gamma(1) \qquad \sigma(0) = \sigma(1)$ continue

<u>OMOTOPIA</u> di γ e σ in Ω

$h : [0,1] \times [0,1] \to \Omega$

$h(0,t) = \gamma(t)$

$h(1,t) = \sigma(t)$

$t \to h(\lambda, t)$ è chiusa $\forall \lambda \in [0,1]$

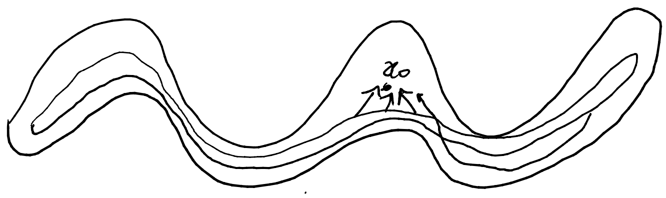
$h(\lambda, 0) = h(\lambda, 1) \quad \forall \lambda \in [0,1]$

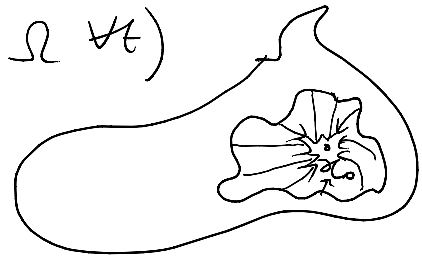
$\Rightarrow$ SE A è IRROTAZIONALE $\Rightarrow$ $\boxed{\int_\gamma A = \int_\sigma A}$

<u>DEF.</u> Ω si dice SEMPLICEMENTE CONNESSO SE OGNI CURVA CHIUSA A VALORI IN Ω E' OMOTOPA AD UNA CURVA COSTANTE $\sigma(t) = x_0 \in \Omega$

INSIEMI SEMPLICEMENTE CONNESSI

Ω si dirà SEMPLICEMENTE CONNESSO se ogni curva chiusa $\gamma : [0,1] \to \Omega$ è omotopa, in Ω, ad una curva costante ($\sigma(t) = x_0 \in \Omega \ \forall t$)

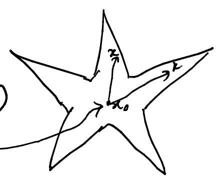


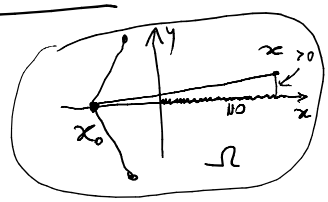


Ω si dirà STELLA se $\exists\, x_0 \in \Omega : \overline{x_0 x} \subseteq \Omega \ \forall x \in \Omega$

segmento

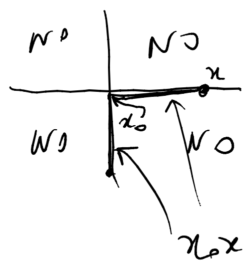
x_0 POLO

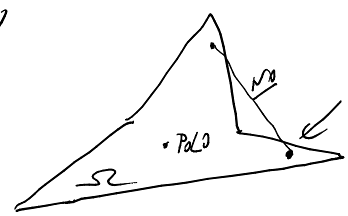




$\Omega = \mathbb{R}^2 \setminus \{(x,0) : x > 0\}$
STELLA

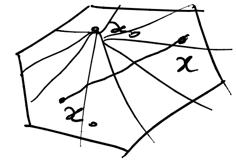
ASSE X $\cup$ ASSE Y





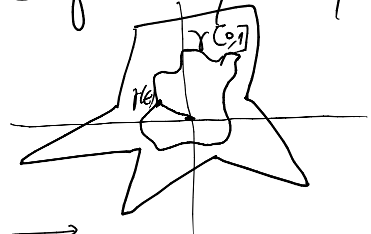
NON è UN POLO

Ω è CONVESSO $\Rightarrow$ Ω è stella



Th. Ω STELLA $\Rightarrow$ Ω sempl. connesso.

DIM. Supponiamo $\boxed{x_0 = 0}$ e sia γ una qualunque curva chiusa a valori in Ω



$$h(\lambda, t) = \underbrace{(1-\lambda)\gamma(t)}_{\text{punto del segmento } \overrightarrow{0\gamma(t)}} \quad \forall \lambda \in [0,1] \quad \forall t \in [0,1]$$

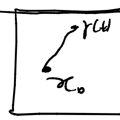
$h : [0,1] \times [0,1] \to \Omega$

h è continua (prodotto di funzioni continue $(1-\lambda)$ e $\gamma(t)$)

$h(0,t) = \gamma(t)$ $(\lambda=0)$ $\quad h(1,t) \equiv 0$

h è a valori in Ω? $h(\lambda,t)$ è un punto del segmento di estremi 0 e $\gamma(t)$ perché $(1-\lambda) \in [0,1]$. Poiché Ω è stella $(1-\lambda)\gamma(t) \in \Omega$ se $\gamma(t) \in \Omega$

Se x_0 non fosse 0?
Si usano i segmenti fra x_0 e $\gamma(t)$



$$h(\lambda,t) = x_0 + (1-\lambda)\left[\gamma(t) - x_0\right]$$

OMOTOPIA DI $\gamma(t)$ sulla curva costante $\sigma(t) \equiv x_0$

<u>OSSERV.</u> Se $\sigma(t) \equiv x_0 \ \forall t$

allora $\int_\sigma A = 0$ $\forall$ campo $A \in C^0$ $\quad \sigma : [0,1] \to \mathrm{dom} A$

$$\int_0^1 A(\sigma(t))\, \dot\sigma(t)\, dt = 0 \quad \text{perché } \dot\sigma \equiv 0$$

Sia $A \in C^1(\Omega)$, Ω semplicemente connesso. Allora A è integrabile se e solo se A è irrotazionale

$\Updownarrow$

CNS ($A \in C^1(\Omega)$, Ω sempl. connesso) sia integrabile è che

$$(A_i)_{x_j} \equiv (A_j)_{x_i} \quad \forall i,j = 1..N$$

<u>ESEMPIO</u>

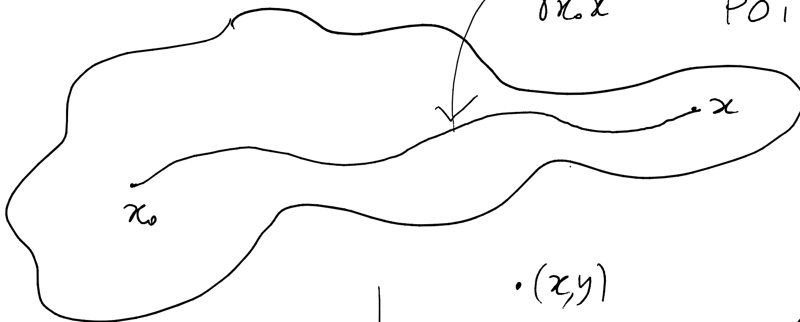
$\alpha = x\,dx + y\,dy$ $\qquad \boxed{A = \begin{pmatrix} x \\ y \end{pmatrix}}$ è integrabile sul suo dominio.

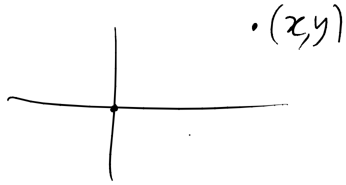
$\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ $\qquad A : \mathbb{R}^2 \to \mathbb{R}^2$

Il dominio del campo è CONVESSO $\Rightarrow$ è semplicemente connesso.

$$\boxed{\frac{\partial}{\partial y} x = 0 \qquad \frac{\partial}{\partial x} y = 0 \qquad \text{chiuse}}$$

A integrabile $\qquad f(x) = \int_{\gamma_{x_0 x}} A \qquad$ COSTRUZIONE DEL POTENZIALE





$$x_0 = (0,0)$$

$$\boxed{A = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A : \mathbb{R}^2 \to \mathbb{R}^2, \quad \mathbb{R}^2 \text{ convesso}}$$

$$f(x,y) = \int_{\overrightarrow{(0,0)(x,y)}} A \qquad \gamma(t) = t\begin{pmatrix} x \\ y \end{pmatrix} \quad t \in [0,1]$$

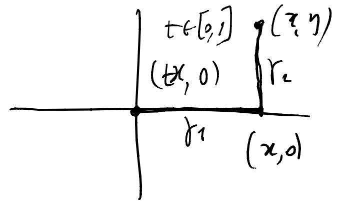
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \gamma(t) = \begin{pmatrix} tx \\ ty \end{pmatrix} \quad t \in [0,1] \qquad f(x,y) = \int_0^1 (tx)x + (ty)y \, dt =$$

$$\dot{\gamma}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A(\gamma(t)) = \begin{pmatrix} tx \\ ty \end{pmatrix}$$

$$= \int_0^1 (x^2+y^2)\, t \, dt = (x^2+y^2) \frac{1}{2} t^2 \Big|_0^1 =$$

$$= \frac{1}{2}(x^2+y^2)$$



$$\int_{\gamma_1} A + \int_{\gamma_2} A = \int_0^1 \overbrace{\begin{pmatrix} tx \\ 0 \end{pmatrix}}^{A(\gamma_1(t))} \cdot \overbrace{\begin{pmatrix} x \\ 0 \end{pmatrix}}^{\dot{\gamma}_1} + \int_0^1 \overbrace{\begin{pmatrix} 0 \\ ty \end{pmatrix}}^{A(\gamma_2(t))} \overbrace{\begin{pmatrix} 0 \\ y \end{pmatrix}}^{\dot{\gamma}_2}$$

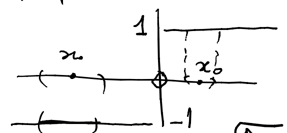
DETERMINARE TUTTE LE PRIMITIVE

$f, g : \Omega \to \mathbb{R}$ $\boxed{\nabla f \equiv \nabla g \equiv A}$

primitive d'$A : \Omega \to \mathbb{R}^N$

$\boxed{\nabla(f-g) \equiv 0}$ $h = f-g$ è una funzione con gradiente ident. nullo

$\varphi'(x) \equiv 0 \Rightarrow \varphi$ costante $\varphi(x) = \dfrac{x}{|x|}$ def. in $\mathbb{R} \setminus \{0\}$



localmente costante MA NON COSTANTE

$\varphi' \equiv 0$

$\varphi(x) - \varphi(y) = (x-y)\underset{0}{\underset{\shortparallel}{\varphi'(\xi)}} \Rightarrow \varphi(x) = \varphi(y)$

Ω aperto connesso $f : \Omega \to \mathbb{R}$ $\nabla f \equiv 0$ su Ω

Allora f è costante.

DIM. $\forall x, y \ \exists \gamma : [0,1] \to \Omega$: γ continua $\gamma(0) = x$ $\gamma(1) = y$

Se Ω è aperto si può scegliere γ regolare a tratti

$\to h(t) = f(\gamma(t))$ $\nabla f \equiv 0 \Rightarrow \nabla f$ è continua

$f \in C^1$ ($\Rightarrow$ differ.)

$\to h'(t) = \underbrace{\nabla f(\gamma(t))}_{\equiv 0} \dot\gamma(t) \equiv 0$

$h' \equiv 0$ su $[0,1]$ intervallo

$\Rightarrow h$ è costante in $[0,1]$

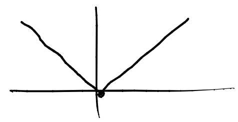
$h : [0,1] \to \mathbb{R}$ $\gamma : [0,1] \to \Omega$ $f : \Omega \to \mathbb{R}$

$h(0) = h(1)$

$f(x) = f(\gamma(0))$ $f(\gamma(1)) = f(y)$

non C^1

$$\gamma(t) = \begin{pmatrix} t \\ |t| \end{pmatrix}$$



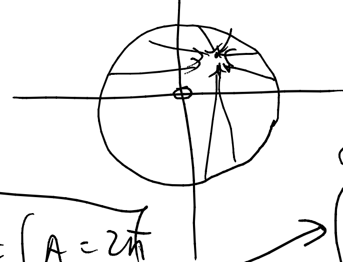
$$\sigma(t) = \begin{pmatrix} t^3 \\ |t|^3 \end{pmatrix} \qquad \dot\sigma(0) = \lim_{t\to 0} \frac{\sigma(t)-\sigma(0)}{t} = \lim_{t \to 0}$$

C^1

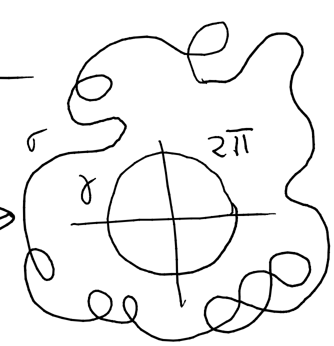
$$\frac{|t|^3 - 0}{t} \longrightarrow 0 \qquad \boxed{\dot\sigma(0) = 0}$$

$$A = \left(\frac{y}{x^2+y^2} , \frac{-x}{x^2+y^2} \right)$$

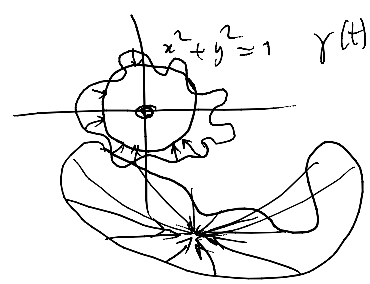
$$\operatorname{dom} A = \mathbb{R}^2 \setminus \{(0,0)\}$$



$$\boxed{\int_\gamma A = \int_\sigma A = 2\pi}$$



$$\nabla \frac{1}{x^2+y^2} = \begin{pmatrix} a(x,y) \\ b(x,y) \end{pmatrix} = A$$



$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ $\quad t \in (0, 2\pi)$

$$f(x,y) = \frac{1}{x^2+y^2}$$

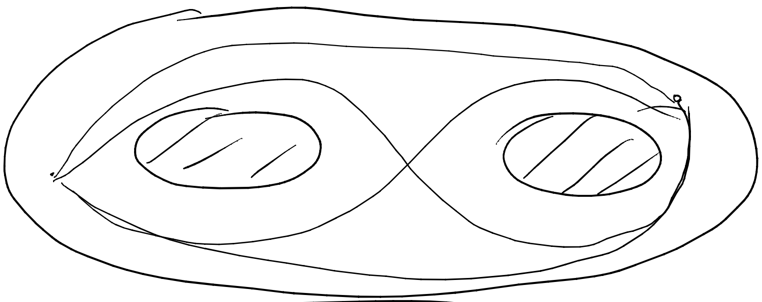
$\nabla f(x,y)$ è integrabile e f è una primitiva

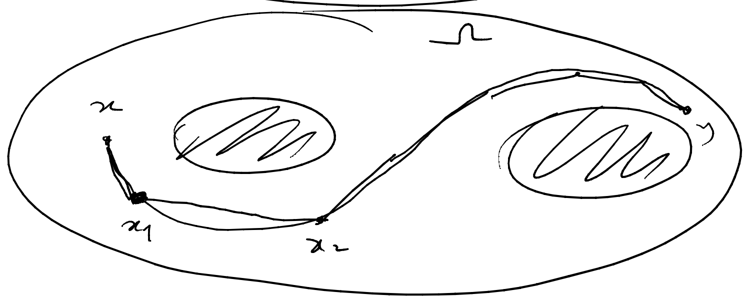
$$\int_{\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}} A = \int_0^{2\pi} \frac{-2\cos t}{1}(-\sin t) + \frac{-2\sin t}{1}(\cos t)\, dt =$$

$$= \int_0^{2\pi} 2 \sin t \cos t - 2 \sin t \cos t \, dt = 0$$

$$a(x,y) = \boxed{\frac{-2x}{(x^2+y^2)^2}}$$

$$b(x,y) = \boxed{\frac{-2y}{(x^2+y^2)^2}}$$

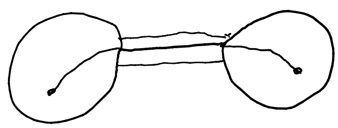




$\nabla f = 0$ in Ω

$f(x_1) = f(x)$

$f(x_2) = f(x_1) = f(x)$

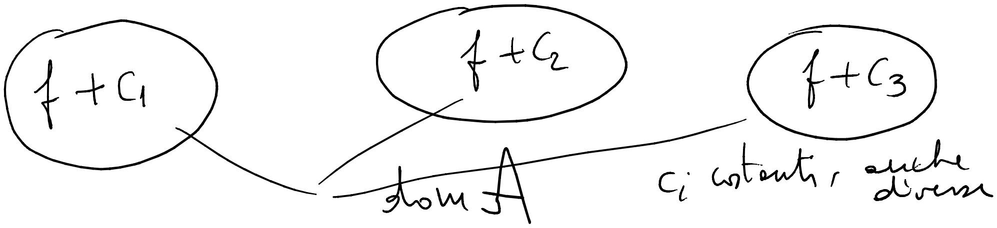


Se f è una primitiva di A

Tutte le altre sono del tipo $f + \varphi$

dove $\boxed{\nabla \varphi = 0}$

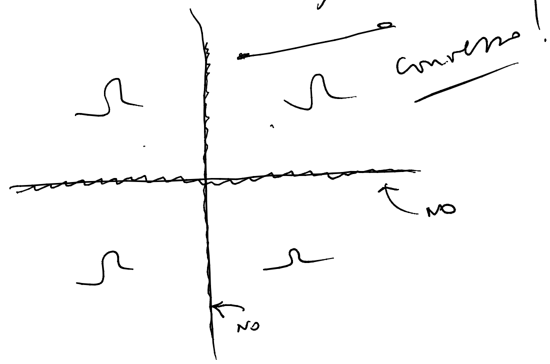
φ è costante su OGNI PARTE CONNESSA di dom A



CAMPI E FORME IN PRATICA

Note Title 5/29/2020

$$\left(\frac{1}{x^2y}\,,\ \frac{1}{xy^2}\right) = A(x,y) \qquad A : \underbrace{\mathbb{R}^2\setminus\{xy=0\}}_{\Omega} \to \mathbb{R}^2$$



dom $A = \mathrm{I}_{q.} \cup \mathrm{II}_q \cup \mathrm{III}_q \cup \mathrm{IV}_q$

dom A è sconnesso, però è unione di 4 insiemi CONVESSI ($\Rightarrow$ STELLA $\Rightarrow$ semplic. connessi)

$$\frac{\partial}{\partial y}\left(\frac{1}{x^2y}\right) = -\frac{1}{x^2y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{xy^2}\right) = -\frac{1}{x^2y^2}$$

sono uguali

A irrotazionale $\Longrightarrow$ A è integrabile

$\Longrightarrow$

$\exists\ f_1$ I quadrante

$\exists\ f_2$ II quadrante

$\exists\ f_3$ III quadrante

$\exists\ f_4$ IV quadrante

: $\boxed{\nabla f_i \equiv A \text{ su il quadrante } i}$

$$f(x) = \begin{cases} f_1 & \mathrm{I}\,q. \\ f_2 & \mathrm{II}\,q. \\ f_3 & \mathrm{III}\,q \\ f_4 & \mathrm{IV}\,q. \end{cases} \qquad \nabla f \equiv A \text{ sul dom} A$$

$\exists f$:

$$\begin{cases} \boxed{f_x = \dfrac{1}{x^2y}} \\ f_y = \dfrac{1}{xy^2} \end{cases}$$

$$\boxed{f(x,y) = -\frac{1}{xy} + c(y)} \quad \longleftarrow \text{ sostituendo}$$

derivando rispetto a y

$$f_y(x,y) = \frac{1}{xy^2} + c'(y) \longrightarrow c'(y) = 0, \quad c = \text{costante}$$

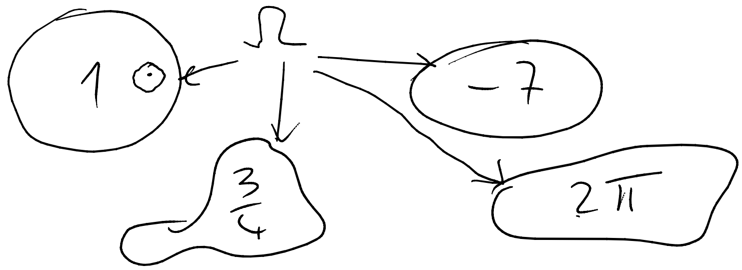
$f(x,y) = -\dfrac{1}{xy} + \text{cost}$ sono primitive di A. SONO TUTTE? NO!

La differenza di due qualunque primitive di A è una funzione con gradiente nullo sull'unione dei quadranti.

$\varphi = c_2$	$\varphi = c_1$
$\varphi = c_3$	$\varphi = c_4$

Tutte le primitive sono del tipo

$$f(x,y) = \begin{cases} -\frac{1}{xy} + c_1 & \text{sul I quadrante} \\ -\frac{1}{xy} + c_2 & \text{" II "} \\ -\frac{1}{xy} + c_3 & \text{su III "} \\ -\frac{1}{xy} + c_4 & \text{" IV "} \end{cases}$$



Tutte le primitive di $\frac{1}{x}$

$$\frac{x}{|x|} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

$\alpha(x,y,dx,dy) = \cos y\, dx - (x \sin y + 1)\, dy$

$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

(x,y) (dx,dy)

associato

$$A(x,y) = \begin{pmatrix} \cos y \\ -x \sin y - 1 \end{pmatrix}$$

$\mathbb{R}^2$ è sempl. connesso

rotore $\frac{\partial}{\partial y}(\cos y) = -\sin y$

$\qquad\qquad \|$

$\frac{\partial}{\partial x}(-x\sin y - 1) = -\sin y$

campo irrotazionale

$\to$ La forma è esatta (integrabile)

$$\begin{cases} f_x = \cos y \\ f_y = -x \sin y - 1 \end{cases}$$

$f_x = \cos y \Rightarrow \boxed{f = x \cdot \cos y + c(y)}$

$\boxed{f_y = -x \sin y + c'(y)}$

$f_y = -x \sin y - 1$

sottraendo membro a membro

$$\boxed{c'(y) = -1} \Rightarrow c(y) = -y$$

$$f(x,y) = x\cos y - y$$

1 primitiva

dom. è aperto e connesso

$x\cos y - y + c$ $c \in \mathbb{R}$ sono TUTTE LE PRIMITIVE

$\alpha = xy\,dx + a(x,y)\,dy$ $\exists a$? ~~tale~~ che α sia chiusa? oppure esatta?

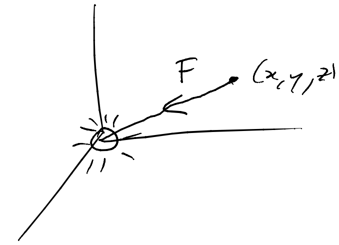
$$\frac{\partial}{\partial y}(xy) = \frac{\partial}{\partial x}a(x,y) \Rightarrow a_x = x \Rightarrow \boxed{a = \frac{1}{2}x^2 + c(y)}$$

DOMINIO $\mathbb{R}^2$ sempl. connesso

POTENZIALE NEWTONIANO

$$F = \underset{\uparrow}{-}\ \underbrace{\frac{1}{x^2+y^2+z^2}}_{\text{modulo della forza di gravità}} \cdot \underbrace{\frac{1}{(x^2+y^2+z^2)^{1/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{versore della posizione}}$$

verso della forza di gravità



$\boxed{GMm = 1}$

$$F = -\frac{1}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$$

Il campo è irrotazionale? Sì!

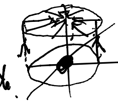
Il dominio è sempl. connesso?

rotore

$$\frac{\partial F_1}{\partial y} \ \| \ ??? \quad \text{sì}$$
$$\frac{\partial F_2}{\partial x}$$

$$\operatorname{dom} F = \mathbb{R}^3 \setminus \{(0,0,0)\}$$

Le curve chiuse può essere allontanate dall'origine prima di deformarle in una costante.



Sì $\Rightarrow$ Il camp. pr. è integrabile

$\mathbb{R}^2 \setminus \{(0,0)\}$ NO!

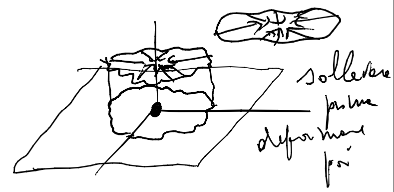
$$A(x,y,z) = -\frac{1}{(x^2+y^2+z^2)^{3/2}}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

<u>A irrotazionale</u>

$$\frac{\partial}{\partial y}\left(-\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) \overset{?}{=} \frac{\partial}{\partial x}\left(-\frac{y}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$-x\,\frac{\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\,2y}{(x^2+y^2+z^2)^3} = -y\,\frac{\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\,2x}{(x^2+y^2+z^2)^3}$$

A irrotaz. su un dominio sempl. connesso
l'omotopia prima sposta la curva e poi la deforma in un punto.



$f : \mathbb{R}^3 - \{(0,0,0)\} \to \mathbb{R} \;\Big|\; \underline{\nabla f = A}$ su dom $A = \mathbb{R}^3 \setminus \{(0,0,0)\}$

$$\begin{cases} (1) & f_x = -\dfrac{x}{(x^2+y^2+z^2)^{3/2}} \\ (2) & f_y = -\dfrac{y}{(x^2+y^2+z^2)^{3/2}} \\ (3) & f_z = -\dfrac{z}{(x^2+y^2+z^2)^{3/2}} \end{cases}$$

$\underset{\text{in } dx}{\overset{\text{integrando}}{\Longrightarrow}}$ $f(x,y,z) = -\frac{1}{2}\int \frac{2x}{(x^2+y^2+z^2)^{3/2}}\,dx + c(y,z) =$

è la derivata di

$$\underset{(x^2+y^2+z^2)=t}{=} c(y,z) - \frac{1}{2}\cdot\int t^{-\frac{3}{2}}\,dt =$$

$$= c(y,z) - \frac{1}{2}\left[\frac{1}{-\frac{1}{2}}\,t^{-\frac{1}{2}}\right] =$$

$$= c(y,z) + \frac{1}{(x^2+y^2+z^2)^{1/2}}$$

$$\boxed{f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} + c(y,z)} \Rightarrow \boxed{f_y = c_y(y,z) - \frac{y}{(x^2+y^2+z^2)^{3/2}}}$$

confrontando con 2) si ha $c_y(y,z) = 0$

C è costante in y (non necess. costante in z)

Dunque $c(y,z) = d(z)$

$$\boxed{f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} + d(z)}$$

derivando rispetto a z e confrontando con 3) si ottiene

$d' \equiv 0 \Rightarrow d =$ costante

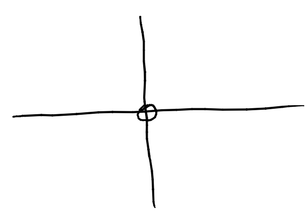
$$\boxed{f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}}$$ 1 primitiva

poiché dom A è connesso, tutte le altre sono

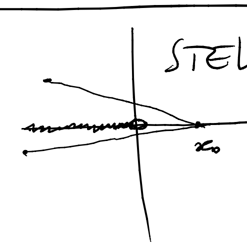
$$\boxed{\frac{1}{\sqrt{x^2+y^2+z^2}} + C}$$

$$A = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

Campo irrotaz. NON INTEGRABILE



STELLA



$$\boxed{\begin{cases} 1)\ f_x = \dfrac{-y}{x^2+y^2} \\ 2)\ f_y = \dfrac{x}{x^2+y^2} \end{cases}}$$

INTEG. DI A nel suo dominio

$\nabla f \equiv A$ in dom A

$$1) \Rightarrow \quad f(x,y) = -y \int \frac{1}{x^2+y^2}\,dx + c(y)$$

$$= \boxed{-\ \operatorname{arctg} \frac{x}{y}} + c(y)$$

dom $f \subset$ dom A strett.

è una primitiva definita per $y \neq 0$

$y>0$ convesso

dom f $\quad \frac{\pi}{2} \quad \frac{\pi}{2} \quad x>0,\ y \to 0^+$

$-\frac{\pi}{2}$

dom f $\quad x>0,\ y \to 0^-$

$y<0$ convesso

$-\operatorname{arctg}\frac{y}{x}$ è anch'essa una primitiva

$\operatorname{arctg} t + \operatorname{arctg}\frac{1}{t} = \frac{\pi}{2} \quad t>0$

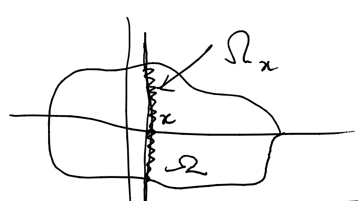
$-\operatorname{arctg}\frac{x}{y} \quad x>0,\ y>0 \qquad -\operatorname{arctg}\frac{x}{y} + \pi \quad x>0,\ y<0$

CAMPI E FORME VII

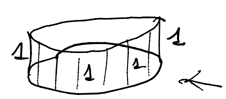
$$\frac{1}{y^2}\int \frac{-y}{\left(\frac{x}{y}\right)^2+1}\,dx$$

$\frac{x}{y} = t \qquad dx = y\,dt$

$$\frac{1}{y^2}\int \frac{-y}{t^2+1}\,y\,dt = \boxed{-\operatorname{arctg}\frac{x}{y}}$$



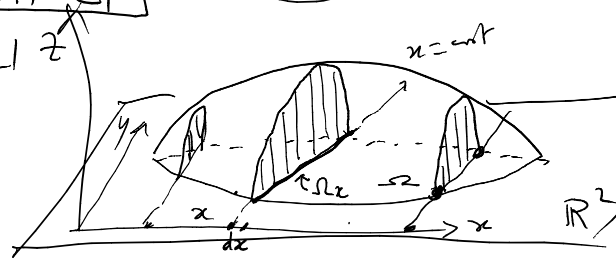
$\Omega \subseteq \mathbb{R}^2 \qquad \int_\Omega 1\,dx\,dy$



vol cilindro = area base

INTEGRALI MULTIPLI

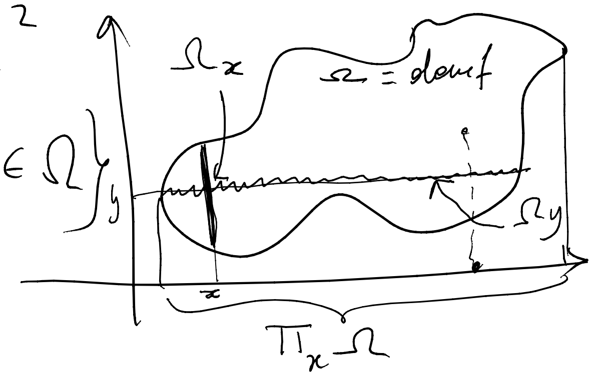
FUBINI - TONELLI



Area sezione di Ω con il piano $x=t$ $= \left[\int_{\Omega_x} f(x,y)\,dy\right] dx$

$$\int_\Omega f\,dx\,dy = \int \left[\int_{\Omega_x} f(x,y)\,dy\right] dx$$

$f:\Omega\to\mathbb{R}$ $\quad \Omega\subseteq\mathbb{R}^2$



$$\Pi_x\Omega = \{x\in\mathbb{R} : \exists y\in\mathbb{R}\ (x,y)\in\Omega\}$$

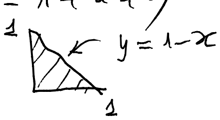
$$\Omega_x = \{y\in\mathbb{R} : (x,y)\in\Omega\}$$

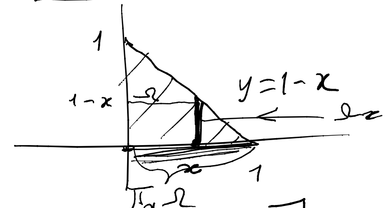
$$\iint_\Omega f(x,y)\,dx\,dy = \int_{\Pi_x\Omega}\Big[\underbrace{\int_{\Omega_x} f(x,y)\,dy}_{\text{funzione di } x}\Big]dx =$$

$$= \int_{\Pi_y\Omega}\Big[\int_{\Omega_y} f(x,y)\,dx\Big]dy$$

Th. integrali ITERATI $\quad$ FUBINI f integrabile
TONELLI $f\geq 0$

$\int_\Omega 1+x+y\,dx\,dy$ $\quad f(x,y)=1+x+y$ $\quad \boxed{\text{ESEMPIO}}$

$\Omega =$ 



|| Fubini-Tonelli

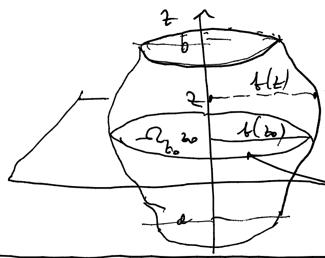
$$\int_0^1 dx \underbrace{\int_0^{1-x} dy\,(1+x+y)}_{\text{funzione della sola } x} =$$

$\Pi_x\Omega = [0,1]$

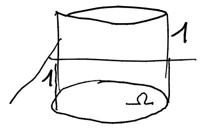
$\Omega_x = [0,1-x]$

$$= \int_0^1 dx\Big[(1+x)y+\frac12 y^2\Big]_0^{1-x} = \int_0^1 dx\Big(\underbrace{(1+x)\underbrace{(1-x)}_{y}+\frac12(1-x)^2}_{\text{primitiva in } y=(1-x)} - \underbrace{0}_{\text{primitiva in } y=0}\Big) =$$

$$= \int_0^1 dx\Big[1-x^2+\frac12(1-x)^2\Big] = \Big[x-\frac13x^3+\frac12\Big(-\frac13(1-x)^3\Big)\Big]_0^1 = 1-\frac13 - \Big(-\frac16\Big)$$



$f \geq 0$ **SOLIDI DI ROTAZIONE**



area Ω = volume del cilindro di base Ω e altezza 1

cerchio di centro $(0,0,z_0)$ e raggio $f(z_0)$

per generare il solido si fa ruotare il grafico di f attorno all'asse z

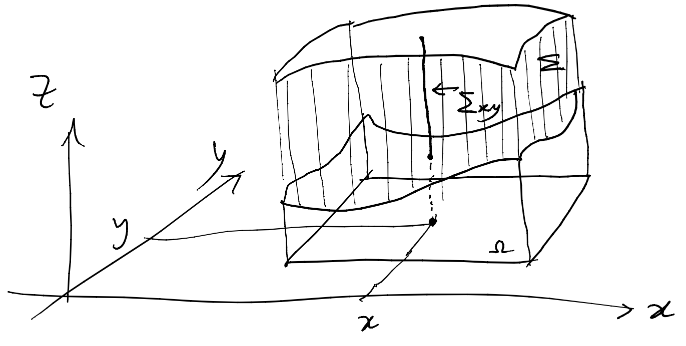
$\Omega_{z_0} =$ ↑ area $\Omega_{z_0} = \pi f^2(z_0)$

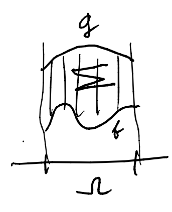
elemento di volume è $\pi f^2(z)\,dz \quad \forall z \in [a,b]$

Il volume totale $= \boxed{\pi \int_a^b f^2(z)\,dz}$

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : z \in [a,b] \quad \sqrt{x^2+y^2} < f(z)\}$$

$$\mathrm{vol}\,\Omega = \int_\Omega 1 = \int_a^b dz \int_{\Omega_z} 1\,dx\,dy = \int_{\Pi_z\Omega} dz\,\pi \underbrace{f^2(z)}_{\text{quadrato del raggio}}$$





$f(x,y) \quad f : \Omega$

$g(x,y) \quad g : \Omega$

$f \leq g$

$$\Sigma = \{(x,y,z) \in \mathbb{R} : (x,y) \in \Omega \quad f(x,y) \leq z \leq g(x,y) \quad \forall x,y \in \Omega\}$$

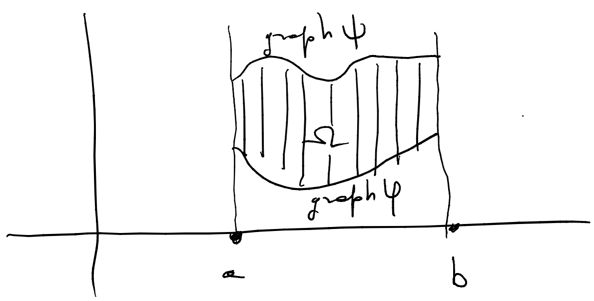
$\boxed{\mathrm{vol}\,\Sigma}$ $\quad \Pi_{x,y}\Sigma = \Omega \quad f(x,y) \leq z \leq g(x,y) \ : \ z \in [f(x,y), g(x,y)]$

$$\| \quad \int_\Sigma 1\,dx\,dy\,dz = \int_{\Pi_{xy}\Sigma = \Omega} dx\,dy \underbrace{\int_{\Sigma_{xy}} \underbrace{f(x,y,z)}_{1}\,dz}_{g(x,y)-f(x,y)} = \boxed{\int_\Omega dx\,dy\,[g(x,y)-f(x,y)]}$$

ampiezza dell'intervallo intersezione di Ω con la retta verticale in (x,y)

Ω è un dominio NORMALE rispetto all'asse x

se $\boxed{\exists\, a<b \quad e \quad \varphi<\psi : [a,b]\to\mathbb{R} \text{ tali che } \quad \Omega=\{(x,y) : x\in[a,b] \quad y\in[\varphi(x),\psi(x)]\}}$



$$\int_\Omega f = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$$

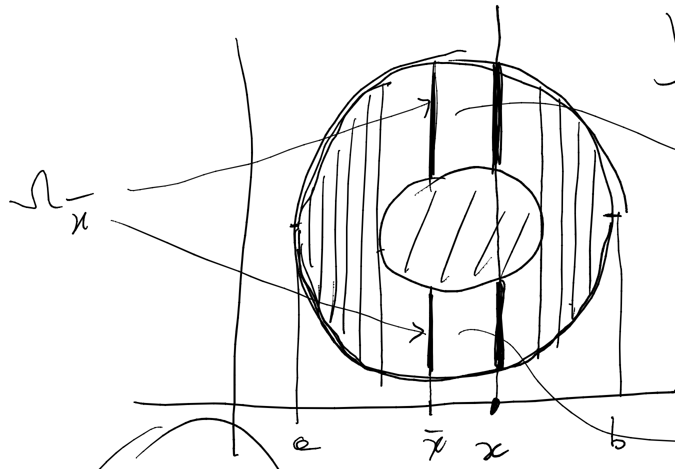
$$[a,b]=\Pi_x \Omega$$

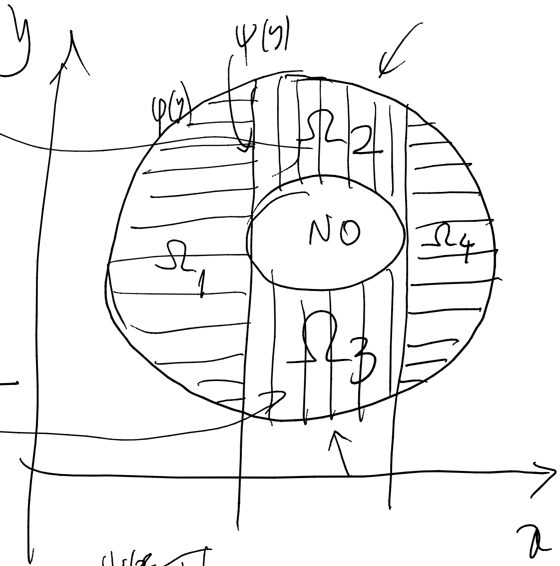
Ω è normale rispetto a y se

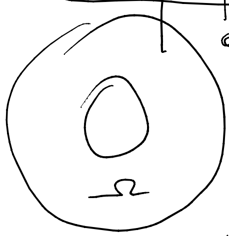
$\boxed{\exists\, a<b \quad \exists\, \varphi,\psi : [a,b]\to\mathbb{R},\ \varphi<\psi : \quad \Omega=\{(x,y) : y\in[a,b] \quad x\in[\varphi(y),\psi(y)]\}}$

$$\int_\Omega f\,dx\,dy = \int_a^b dy \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx$$

$$\int_{\Omega_1\cup\Omega_2\cdots\cup\Omega_n} f = \sum_1^n \int_{\Omega_i}$$ se Ω_i non hanno punti interni a comune.





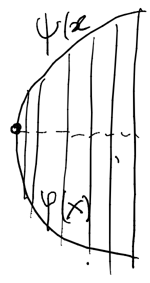


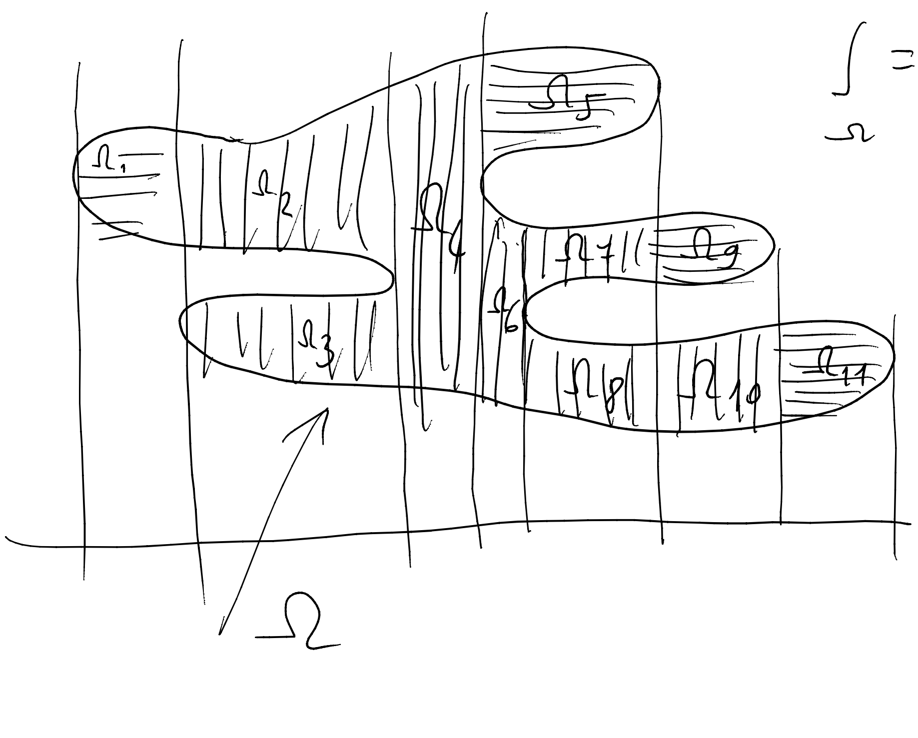
è unione

di $\Omega_1, \Omega_2, \Omega_3, \Omega_4$

privi di punti interni

comuni





$$\int_\Omega = \sum_1^{11} \int_{\Omega_i}$$

CAMBIO DI VARIABILI

$\Omega \subseteq \mathbb{R}^n$ $\quad \Omega$ aperto

$x \in \mathbb{R}^n$

$dx \equiv dx_1 dx_2 \dots dx_n$

$x = g(y) \qquad y \in \mathbb{R}^n$

$g \in C^1(\Sigma)$

$g : \Sigma \to \mathbb{R}^n$

g sia invertibile con inversa C^1

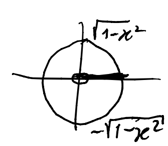
$\Sigma = g^{-1}(\Omega)$

$$\int_\Omega f(x)\,dx = \qquad x = g(y) \quad dx = |\det g'(y)|\,dy$$

$$= \int_{\underbrace{g^{-1}(\Omega)}_{\Sigma}} f(g(y))\, \Big|\underbrace{\det g'(y)}_{\det \frac{\partial(x_1 \dots x_n)}{\partial(y_1 \dots y_n)}}\Big|\, dy$$

$$g(g^{-1}(\Omega)) = \Omega$$

$$g'(y) = \frac{\partial(x_1, \dots, x_n)}{\partial(y_1 \dots y_n)}$$



$$\boxed{\begin{matrix} 0 < \rho < 1 \\ 0 < \theta < 2\pi \end{matrix}} \quad g^{-1}(\Omega)$$

$$\int x^2 + y^2 \, dx\, dy$$

$\Omega = \{x^2 + y^2 < 1\}$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \rho = \sqrt{x^2+y^2}$$

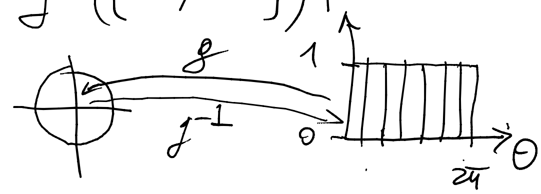
$$\det \frac{\partial(x,y)}{\partial(\rho,\theta)} = \rho$$

$$\int \underbrace{\rho^2}_{f(x,y)} \cdot \underbrace{\rho}_{|\det g'|}\, d\rho =$$

$$= \int_0^{2\pi} d\theta \boxed{\int_0^1 \rho^3\, d\rho} =$$

$$= \int_0^{2\pi} d\theta \;\; \frac{1}{4}\rho^4\Big|_0^1 = \int_0^{2\pi} d\theta\; \frac{1}{4} = \frac{1}{4} 2\pi = \frac{\pi}{2}$$

$g^{-1}(\{x^2+y^2 \le 1\})$



$$\int_{\{y>0;\ x^2+y^2\leq 1\}} \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy \overset{?}{=}$$

(GAUSS - GREEN - OSTROGRADSKIJ)

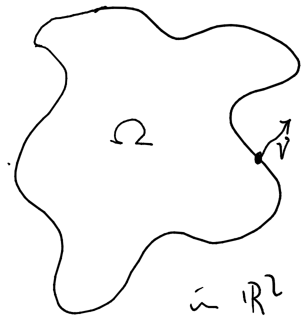
$$\underbrace{\int_{\Omega} f_x(x,y)\,dx\,dy}_{\text{integrale doppio}} = \underbrace{\int_{\partial\Omega^+} \overbrace{f(x,y)\,dy}^{\text{forma diff.}}}_{\text{integrale di una forma su una curva}} \leftarrow$$

$\partial\Omega^+$ (sostegno di) una curva perimetrica PERCORSA IN SENSO ANTIORARIO

$$\boxed{\int_{\partial\Omega} f\,\nu_1\, dl}$$

$dl \to |\dot\gamma(t)|\,dt$

ν versore normale esterno a $\partial\Omega$



$|\nu| = 1$

$\nu = (\nu_1, \nu_2, \dots, \nu_n)$ in $\mathbb{R}^n$

in $\mathbb{R}^2$ $\nu = (\nu_1, \nu_2)$

$$\boxed{\int_{\partial\Omega} f(\gamma(t)) \cdot \frac{\dot\gamma_2(t)}{\cancel{\sqrt{\dot\gamma_1^2(t) + \dot\gamma_2^2(t)}}} \cdot \cancel{|\dot\gamma(t)|}\,dt}$$

$$\parallel$$

$$\int_{\partial\Omega^+} f\,dy$$

$\gamma(t)$

vettore tangente $\tau(t) = \begin{pmatrix} \dot\gamma_1(t) \\ \dot\gamma_2(t) \end{pmatrix}$

vettore normale esterno $\boxed{n = \begin{pmatrix} \dot\gamma_2(t) \\ -\dot\gamma_1(t) \end{pmatrix}}$ $\nu = \dfrac{n}{|n|}$ VERSORE normale esterno

$$\int_{\Omega} f_y\,dx\,dy = -\underbrace{\int_{\partial\Omega^+} f(x,y)\,dx}_{\text{forma diff.}} = \underbrace{\int_{\partial\Omega} f\,\nu_2\,dl}_{\text{integrale curvilineo della funzione } f\nu_2}$$

$\operatorname{div} A =$ A campo

$= (A_1)_{x_1} + (A_2)_{x_2} + \dots + (A_n)_{x_n}$

$\operatorname{div} A = \nabla \cdot A$

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \vdots \\ \partial_{x_n} \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix}$$

$$\int_\Omega \operatorname{div} A \, dx_1 \dots dx_n = \int_{\partial\Omega} A \cdot \nu \, d\sigma$$

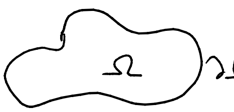
$\boxed{\text{Th. GAUSS}}$

Th. della

$\boxed{\text{FEYNMAN}}$

LEZIONI DI FISICA

Area racchiusa da una curva piana



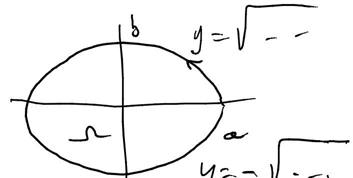
$$\text{Area di } \Omega = \int_\Omega 1 \, dxdy = \int_{\partial\Omega^+} x \, dy$$

$$= -\int_{\partial\Omega^+} y \, dx$$

NON È CHIUSA

$$\text{area di } \Omega = \int_{\partial\Omega^+} \frac{1}{2} (x dy - y dx)$$

NON E' CHIUSA



$$\gamma(t) = \begin{cases} a \cos t = x \\ b \sin t = y \end{cases} \quad t \in [0, 2\pi] \qquad \frac{x^2}{a^2} = \cos^2 t \qquad \frac{y^2}{b^2} = \sin^2 t$$

$$\iint_\Omega 1 \, dxdy = \int_{\partial\Omega^+} x \, dy = \qquad \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

$y = b \sin t$

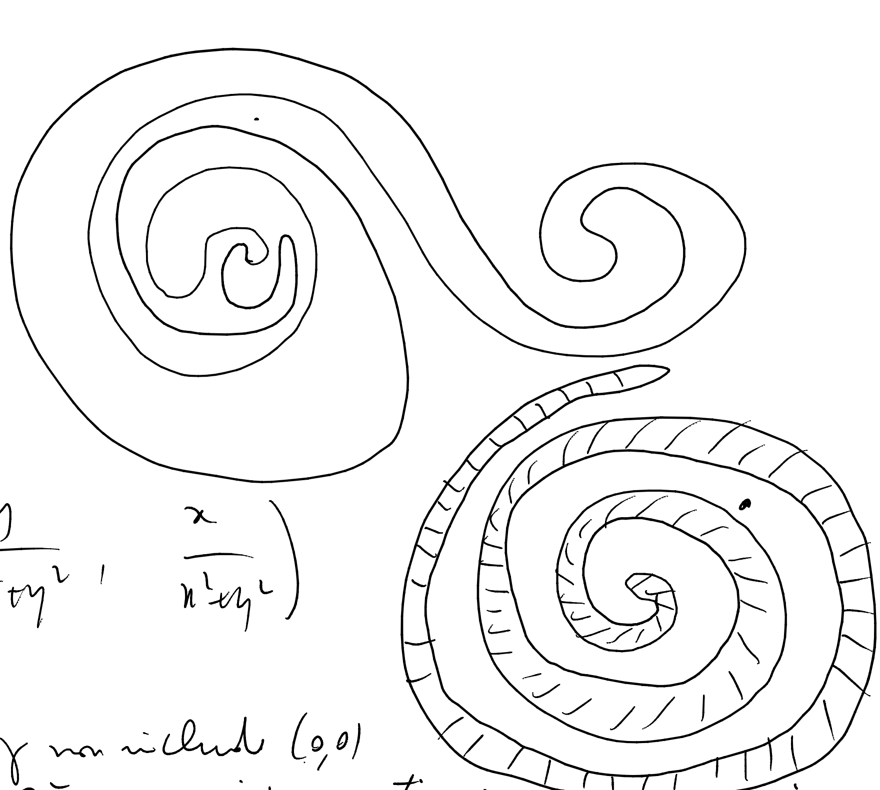
$dy = b \cos t \, dt$

$$= \int_0^{2\pi} \underbrace{a \cos t}_{x} \, b \cos t \, dt =$$

$\cos 2t = 2\cos^2 t - 1$

$$= ab \int_0^{2\pi} \cos^2 t \, dt = \qquad \cos^2 t = \frac{1}{2} + \frac{1}{2} \cos 2t$$

$$= ab \int_0^{2\pi} \frac{1}{2} + ab \int_0^{2\pi} \frac{1}{2} \cos 2t = \pi ab + \frac{ab}{2} \underbrace{\left[\frac{1}{2} \sin 2t\right]_0^{2\pi}}_{= 0} = \pi ab$$



$$\int_\gamma \left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$$

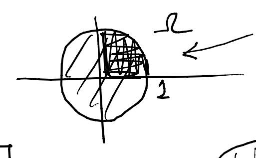
è nullo se γ non include $(0,0)$

è multiplo di 2π se gira in senso antiorario attorno all'origine

---

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy \qquad \Omega = \{x^2+y^2 \le 1\}$$

$[0,\pi/2]$



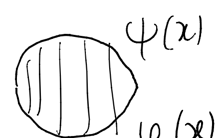
coord. polari

(piane)

$$\frac{1}{\sqrt{x^2+y^2}} = k \qquad \boxed{k>0}$$

$$\sqrt{x^2+y^2} = \frac{1}{k}$$

$$x^2+y^2 = \frac{1}{k^2}$$



$x \in [-1,1] \quad y \in \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right]$

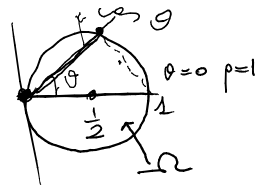
$$\boxed{\int_0^{2\pi} d\theta} \int_0^1 d\rho \left(\underbrace{\frac{1}{\cancel{\rho}}}_{f \text{ in coord. polari}} \ \underbrace{\cancel{\rho}}_{\text{jacobiano delle coord. polari}}\right) = 2\pi \cdot 1$$

$$x^2+y^2 \le 1$$

$$\rho^2\cos^2\theta + \rho^2\sin^2\theta \le 1$$

$$\rho^2 \le 1 \Rightarrow \boxed{\rho \le 1}$$

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy =$$



cerchio di centro $\left(\frac{1}{2}, 0\right)$ e raggio $\frac{1}{2}$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta} d\rho \, \frac{1}{\rho}\cdot\rho =$$

$$\Omega = \left\{ \left(x-\frac{1}{2}\right)^2 + y^2 \leq \frac{1}{4} \right\}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos\theta = \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) = 2$$

$$x^2 - x + \cancel{\tfrac{1}{4}} + y^2 \leq \cancel{\tfrac{1}{4}}$$

$$\Omega = \left\{ \underbrace{x^2+y^2} - x \leq 0 \right\}$$

in coordinate polari

$$\underbrace{\rho^2\cos^2\theta + \rho^2\sin^2\theta} - \rho\cos\theta \leq 0 \qquad \boxed{\rho > 0}$$

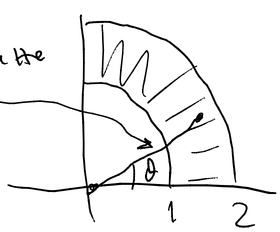
$$\rho^2 \leq \rho\cos\theta \rightarrow 0 < \rho \leq \cos\theta$$

1) $\cos\theta > 0$ $\quad \boxed{\theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}$

2) $\boxed{\rho \in \left]0, \cos\theta\right[}$

---

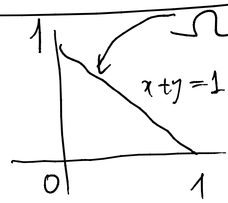
la lunghezza è ρ



$$\theta \in \left[0, \frac{\pi}{2}\right]$$

$$\rho \in [1, 2]$$

$$\int_\Omega \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy =$$



$$\begin{cases} x > 0 \\ y > 0 \\ x + y < 1 \end{cases}$$

1) $\rho\cos\theta > 0$
2) $\rho\sin\theta > 0$
3) $\rho(\cos\theta + \sin\theta) < 1$

$\boxed{\rho > 0}$ sempre

da 1) $\cos\theta > 0$
2) $\sin\theta > 0$
$\longrightarrow \theta \in \left[0, \frac{\pi}{2}\right]$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{1}{\cos\theta+\sin\theta}} d\rho \, \frac{1}{\rho}\,\rho =$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\cos\theta + \sin\theta}\, d\theta =$$

formule parametriche

$t = \operatorname{tg}\frac{\theta}{2} \qquad \theta = 2\operatorname{arctg} t$

$d\theta = \frac{2}{1+t^2}\,dt$

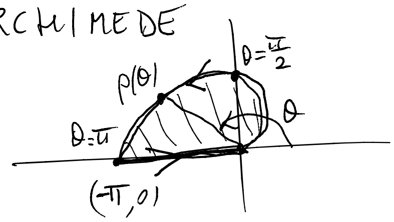
$\sin\theta = \frac{2t}{1+t^2} \qquad \cos t = \frac{1-t^2}{1+t^2}$

$\longrightarrow$ 3) $\rho < \frac{1}{\cos\theta + \sin\theta}$

$$= \int_0^{\tg\frac{\pi}{4}} \frac{1}{\frac{1-t^2+2t}{\cancel{1+t^2}}} \cdot \frac{2}{\cancel{1+t^2}}\, dt = \boxed{2\int_0^1 \frac{dt}{1-t^2+2t}} \longrightarrow$$

funzione razionale: integrazione standard

SPIRALE D'ARCHIMEDE



$\boxed{\rho = \theta}$ $\quad \rho \in [0,\theta]$

$\theta \in [0,\pi]$

$\boxed{\int_\Omega 1\, dx} = \text{volume di } \Omega = \int_{\Phi^{-1}(\Omega)} \rho\, d\rho d\theta$

$$\int_0^\pi d\theta \int_0^\theta \rho\, d\rho = \int_0^\pi d\theta \left[\frac{1}{2}\rho^2\right]_0^\theta = \int_0^\pi \frac{1}{2}\theta^2\, d\theta = \frac{1}{6}\theta^3\Big|_0^\pi = \frac{1}{6}\pi^3$$

$$\int_{\gamma_{\Omega^+}} x\,dy = \qquad \begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \text{ in generale}$$

SPIRALE D'ARCHIMEDE

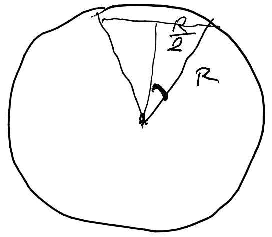
$$\begin{cases} x(\theta) = \theta\cos\theta \\ y(\theta) = \theta\sin\theta \end{cases} \quad \theta \in [0,\pi]$$

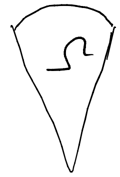
$$= \int_0^\pi \theta\cos\theta\,(\sin\theta + \theta\cos\theta)\, d\theta \; +$$

$\dot{y}(\theta) = \sin\theta + \theta\cos\theta$

$$\cancel{+ \int_{\text{segmento } \overrightarrow{(-\pi,0)(0,0)}} x\,dy}$$

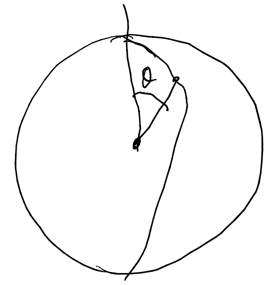
$$\boxed{\begin{array}{l} \sigma(t) = \begin{cases} -\pi + t \\ 0 \end{cases} \quad t \in [0,\pi] \\ \Downarrow \\ \dot{\sigma}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad dy = 0 \cdot dt \\ \text{parametrizzazione del segmento sull'asse } x \end{array}}$$





coord. sferiche

jacobiano $\rho^2 \sin\theta$



$$\text{vol}\ \Omega = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{6}} d\theta \int_0^1 d\rho\ \underbrace{\rho^2 \sin\theta}_{\text{jacobiano}} =$$

$$= \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \underbrace{\int_0^{\frac{\pi}{6}} \sin\theta\, d\theta}_{-\cos\theta\Big|_0^{\frac{\pi}{6}}} \underbrace{\int_0^1 \rho^2 d\rho}_{\frac{1}{3}} = \frac{\cancel{2}}{3}\pi\left(\frac{2-\sqrt{3}}{\cancel{2}}\right) = \frac{\pi}{3}\left(2-\sqrt{3}\right)$$

$$2\pi \quad \left(-\frac{\sqrt{3}}{2}+1\right)$$

$$\Phi : \Delta \to \mathbb{R}^3 \qquad \Phi(u,v) = \begin{pmatrix} \Phi_1(u,v) \\ \Phi_2(u,v) \\ \Phi_3(u,v) \end{pmatrix}$$

$$\nu = \Phi_u \times \Phi_v$$

$$\Phi_u = \begin{pmatrix} (\Phi_1)_u \\ (\Phi_2)_u \\ (\Phi_3)_u \end{pmatrix}$$

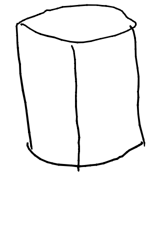
$$\nu \neq 0 \quad \forall (u,v) \in \Delta$$

UNA SUPERFICIE PARAMETRICA

Φ si dice REGOLARE se 1) $\Phi \in C^1(\Delta)$

2) $\nu \neq 0$ dappertutto su Δ

3) Φ sia iniettiva su $\mathring{\Delta}$



$$\text{Area di } \phi \equiv \int_\Delta |\nu|\, du\, dv \equiv \int_\Delta |\phi_u \times \phi_v|\, du\, dv$$

"simile" a $\int |\dot\gamma(t)|\, dt$

$f : \Omega \to \mathbb{R}$ $\qquad$ $\phi : \Delta \to \Omega$ regolare $\qquad$ Integr. curvil. $\int_a^b f(\gamma)\,|\dot\gamma(t)|\, dt$

$$\int_\phi f\, ds \equiv \int_\Delta \underbrace{f(\phi(u,v))}\ \underbrace{|\phi_u(u,v) \times \phi_v(u,v)|\, du\, dv}_{ds}$$

$|\nu(u,v)|\, du\, dv$

$$\phi(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$ rappr. param. del grafico di f

$(u,v) \in \Omega = \operatorname{dom} \phi$

$$\gamma(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

$$\phi_u = \begin{pmatrix} 1 \\ 0 \\ f_u(u,v) \end{pmatrix} \qquad \phi_v = \begin{pmatrix} 0 \\ 1 \\ f_v(u,v) \end{pmatrix}$$

$$\nu(u,v) = \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix} = \begin{pmatrix} -f_u \\ -f_v \\ 1 \end{pmatrix}$$

$$\int_a^b \sqrt{1 + f'^2(x)}\, dx$$

$$\text{area } \phi = \int_\Omega \sqrt{1 + f_u^2 + f_v^2}\, du\, dv$$

Note Title 6/5/2020

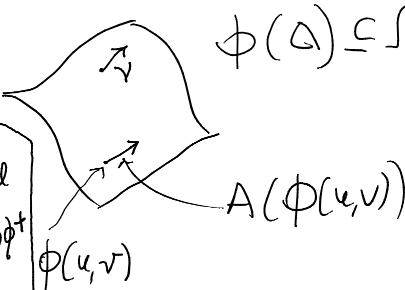
Teorema di Stokes

A campo def. in $\Omega \subseteq \mathbb{R}^3$ $\left(A : \Omega \to \mathbb{R}^3 \right)$

$A \in C^1(\Omega)$

$\phi : \Delta \to \Omega$

$\phi(\Delta) \subseteq \Omega$



ν versore normale

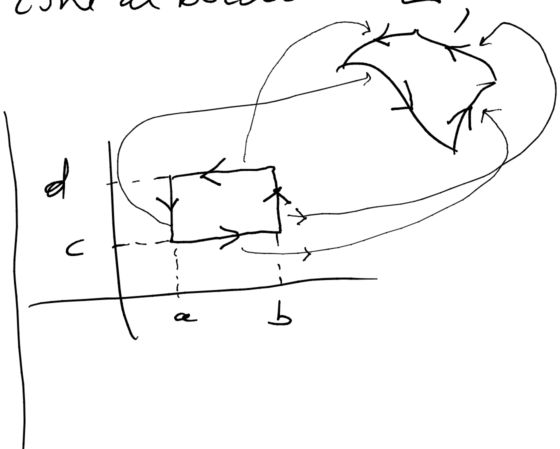
$$\underbrace{\int_{\phi} (\nabla \times A)\, \nu \, dS}_{\text{flusso del rotore}} = \underbrace{\int_{\partial\phi^+} A}_{\text{integrale del campo } A \text{ sulla curva } \partial\phi^+}$$

$$\nu(u,v) = \frac{\phi_u(u,v) \times \phi_v(u,v)}{|\phi_u \times \phi_v|}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \partial_y A_3 - \partial_z A_2 \\ -(\partial_x A_3 - \partial_z A_1) \\ \partial_x A_2 - \partial_y A_1 \end{pmatrix}$$

$\partial\phi^+$ è la curva parametrica ottenuta da ϕ considerandone la restrizione al bordo di Δ PERCORSO IN SENSO ANTIORARIO

Es. $\Delta = [a,b] \times [c,d]$

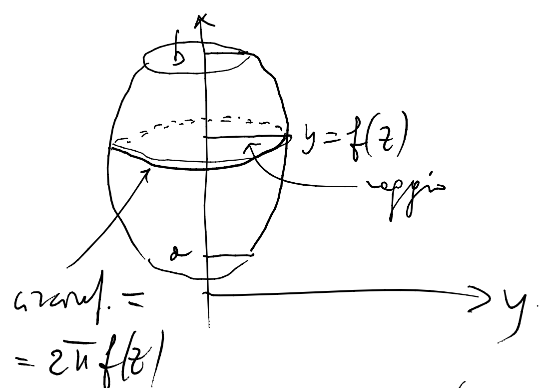


Se ν è orientato come $\phi_u \times \phi_v$ allora il bordo di Δ va orientato in senso ANTIORARIO

Superficie di solidi di rotazione

Volume $\pi f^2(z)\,dz$

Vol. totale $\displaystyle\int_a^b \pi f^2(z)\,dz$



$\rho = f(z)$

Rappres. parametrica del solido di rotazione

$$\rho = f(z) \qquad \begin{pmatrix} \theta \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} f(z)\cos\theta \\ f(z)\sin\theta \\ z \end{pmatrix} \longrightarrow \phi : \Delta \to \mathbb{R}^3$$

$$\theta \in [0, 2\pi] \qquad z \in [a, b] \qquad \Delta = [0, 2\pi] \times [a, b]$$

$$v(\theta, z) = \begin{pmatrix} -f(z)\sin\theta \\ f(z)\cos\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} f'(z)\cos\theta \\ f'(z)\sin\theta \\ 1 \end{pmatrix} = \begin{pmatrix} f(z)\cos\theta \\ -(-f(z)\sin\theta) \\ -f(z)f'(z) \end{pmatrix}$$

$\boxed{\text{vettore normale standard}}$

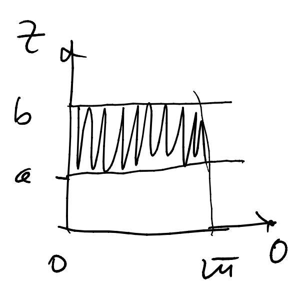
$$|v| = \sqrt{f^2(z)\cos^2\theta + f^2(z)\sin^2\theta + f^2(z)f'^2(z)} = f(z)\sqrt{1 + f'^2(z)}$$

$\boxed{|v|\,d\theta\,dz = dS}$

$f > 0$

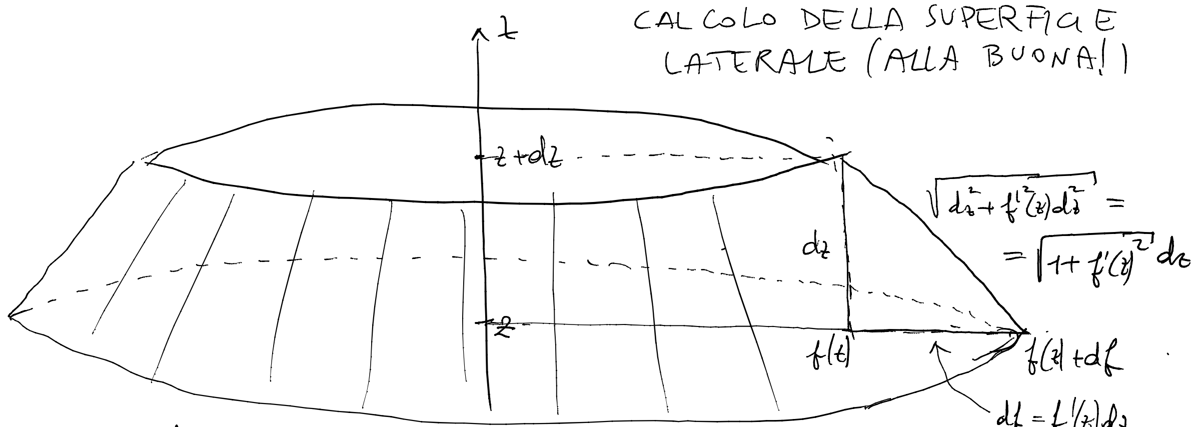
$$\text{Area} = \int_0^{2\pi} d\theta \int_a^b dz\; f(z)\sqrt{1 + f'^2(z)} =$$

$$= \int_\Delta |v|\,d\theta\,dz$$



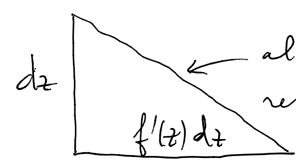
$2\pi f(z)$

CALCOLO DELLA SUPERFICIE LATERALE (ALLA BUONA!)



circonferenza di base $= 2\pi f(z)$

La porzione di superficie evidenziata ha "base" $2\pi \underbrace{f(z)}_{\text{raggio}}$ e "altezza" quella ricavata dal teorema di Pitagora applicato a

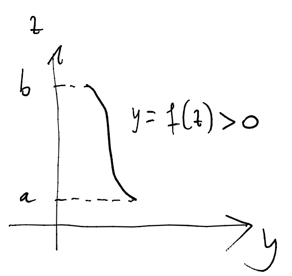


altezza effettiva della striscia relativa all'incremento verticale dz, $\sqrt{1+f'^2(z)}\,dz$

$$= 2\pi \int_a^b f(z)\sqrt{1+f'^2(z)}\,dz$$

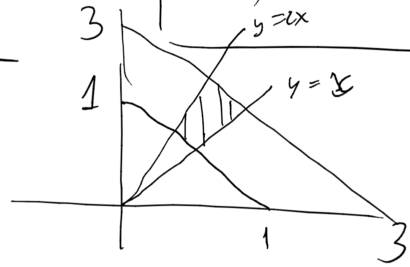
PER I SOLIDI DI ROTAZIONE

$$\boxed{\begin{aligned} &\text{VOLUME} = \pi \int_a^b f^2(z)\,dz \\ &\text{SUPERFICIE LATERALE} = 2\pi \int_a^b f(t)\sqrt{1+f'^2(z)}\,dz \end{aligned}}$$



$$\boxed{\int_\Omega \frac{dx\,dy}{xy} \qquad \Omega = \left\{(x,y)\in\mathbb{R}^2 \ \ \frac{0<x<y<2x}{1<x+y<3}\right\}}$$

$\boxed{\text{CAMBI DI VARIABILE}}$



$x+y=$

$$\boxed{1<\frac{y}{x}<2 \qquad 1<x+y<3}$$

$u \qquad v$

$$u=\frac{y}{x} \qquad v=x+y$$

$$y=xu \qquad v=x+ux=x(1+u) \Rightarrow \boxed{x=\frac{v}{1+u}}$$

$$\boxed{y=\frac{v}{1+u}u}$$

$$T:(u,v)\to(x,y)$$

$$T'=\begin{pmatrix} -\frac{v}{(1+u)^2} & \frac{1}{1+u} \\ \frac{v(1+u)-uv}{(1+u)^2} & \frac{u}{1+u} \end{pmatrix}$$

$\det T'=$

$$=-\frac{uv}{(1+u)^3}-\frac{v}{(1+u)^3}=-\frac{v}{(1+u)^2}\neq 0$$

$$\int_T \frac{dx\,dy}{xy}=\int_1^2 du\int_1^3 dv\,\cancel{\frac{1+u}{v}}\,\frac{\cancel{1+u}}{uv}\,\cancel{\frac{v}{(1+u)^2}}=\int_1^2 du\int_1^3 dv\,\frac{1}{uv}=$$

$$=\int_1^2 du\,\frac{1}{u}\int_1^3\frac{1}{v}dv=\left|\lg|v|\Big|_1^3\cdot\lg|u|\Big|_1^2\right|=\lg 2\lg 3$$

$$\boxed{\begin{array}{l}\text{Calcolare l'area di}\\ \Omega\ \left\{(x,y)\in\mathbb{R}^2 : y>0,\ x+y>0,\ x^2+y^2<3\sqrt{x^2+y^2}-3x\right\}\end{array}}$$

Ω, in coordinate polari, diventa

buone in coord. polari

$$\begin{cases}\overbrace{\rho\sin\theta}^{y}>0\\ \underbrace{\rho(\cos\theta+\sin\theta)}_{x+y}>0\end{cases}$$

$$\underline{\rho^2<3\rho-3\underbrace{\rho\cos\theta}_{x}}\qquad\boxed{\rho>0}$$

$\Downarrow$

$$\rho<3-3\cos\theta=3(1-\cos\theta)$$

$\Downarrow$

$\rho>0\quad\sin\theta>0\quad\theta\in[0,\pi]$

ρ appartiene all'intervallo (variabile con θ)

$$\boxed{0<\rho<3(1-\cos\theta)}$$

$$\cos\theta+\sin\theta>0$$

$$\boxed{\cos\theta>-\sin\theta}$$

$\theta\in\left[\frac{\pi}{2},\frac{3}{4}\pi\right]$ $\quad|\cos|<|\sin|$

I e II membro sono negativi

se $\theta\in\left[\frac{3}{4}\pi,\pi\right]$ $\quad|\cos|>|\sin|$ $\Rightarrow$ la dis. è falsa

$\cos\theta>-\sin\theta$ è verificata (in $[0,\pi]$) solo

per $\theta\in\left[0,\frac{3}{4}\pi\right]$

$$\int_0^{\frac{3}{4}\pi}d\theta\int_0^{3(1-\cos\theta)}1\cdot\rho\,d\rho=\int_0^{\frac{3}{4}\pi}\frac{1}{2}\left[9(1-\cos\theta)^2\right]d\theta$$

$$9-18\cos\theta+9\cos^2\theta$$

funzione integranda per l'area

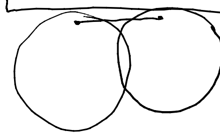
jacobiano coordinate polari

$\int\cos^2\theta$ formule di duplicazione

$$\Omega = \{\underbrace{x^2 + y^2 + 4x \le 0}\}^{\Omega_1} \cup \{x > 0 \,;\, y < |x|\}^{\Omega_2}$$

sommando e sottraendo 4

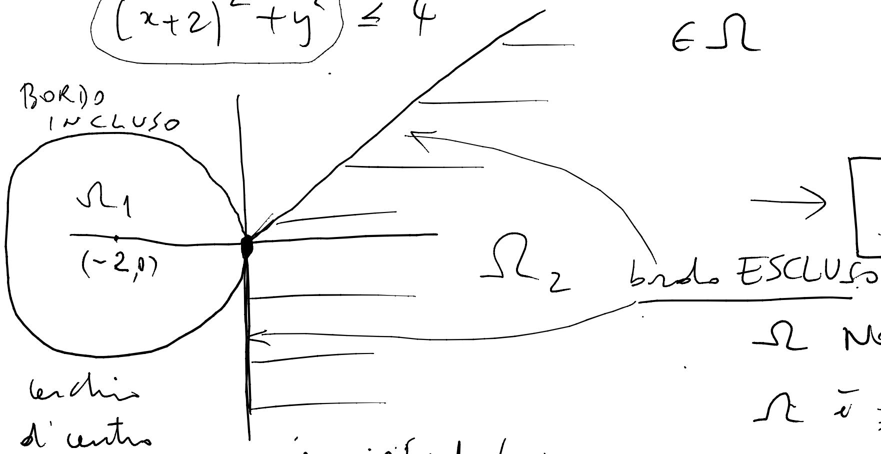
Ω è
- CONVESSO
- SEMPLICEMENTE CONNESSO
- CONNESSO



$$\underbrace{x^2 + 4x + 4}_{} + y^2 \le 4$$

$$(x+2)^2 + y^2 \le 4$$

$$\boxed{(0,0) \in \Omega_1}$$
$$\Downarrow$$
$$\in \Omega$$



Cerchio di centro $(-2,0)$ e raggio 2, incluso il bordo $(\le)$

$$y < |x|$$

$$\rightarrow \boxed{y < x}$$

Ω NON E' CONVESSO

Ω è stellato e quindi sempl. connesso

rispetto a qualsiasi polo $(x_0, y_0) = (0,0)$

$$\lim_{(0,0)} \frac{\sin(x^2-y^2)}{x^2+y^2} \qquad \frac{\sin(x^2-y^2)}{x^2-y^2} \cdot \boxed{\frac{x^2-y^2}{x^2+y^2}}$$

$x^2 - y^2 \ne 0$

$= 0$ se $x^2 - y^2 = 0$

1

0-omogenea NON COSTANTE

$= 1$ se $y = 0$

$= -1$ se $x = 0$

$$\boxed{\text{LIMITE}} \quad \text{NON ESISTE}$$

$$\begin{cases} x = u^2 - v \\ y = u - v^2 \end{cases} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} 2u & -1 \\ 1 & -2v \end{pmatrix}$$

INVERTIBILITA' LOCALE

$$\det J = -4uv + 1$$

$$\left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{4}, 1\right) \qquad \left(\frac{1}{2}, \frac{1}{2}\right)$$

$uv = \frac{1}{6}$ $\boxed{\text{sì}}$ $\qquad uv = \frac{1}{4}$ NO $\qquad uv = \frac{1}{4}$ NO

$$\boxed{uv = \frac{1}{4}}$$

NON SI PUO' APPLICARE IL TH. INV. LOCALE

N.A.

$f(x,y) = |1-\cos xy|^{2/3}$ in $(0,0)$

CONTINUITA'
DERIVABILITA'
DIFFERENZIABILITA'

1) $f(x,y)$ è continua in $(0,0)$ (composta di f. continue)

2) $f_x(0,0) = 0$
$f_y(0,0) = 0$ perché f è costante (e fa 0) sugli assi.

$\exists\, \nabla f(0,0) = \begin{pmatrix}0\\0\end{pmatrix}$ $\qquad w=(h,k)$

3) $$\lim_{h,k\to 0} \frac{f(h,k)-f(0,0)-\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix}}{\underbrace{\sqrt{h^2+k^2}}_{|w|}} = \lim_{h,k\to 0} \frac{|1-\cos hk|^{2/3}}{\sqrt{h^2+k^2}} =$$

ha a che fare con $1-\cos\boxed{} \to$ cosa che va a zero. Limiti notevoli

$hk=t$

$$\left(\frac{1}{2}\right)^{\frac{2}{3}} \leftarrow \frac{|1-\cos hk|^{2/3}}{(hk)^{2\cdot\frac{2}{3}}} \cdot \frac{(hk)^{\frac{4}{3}}}{\sqrt{h^2+k^2}}$$

$(hk)^{\frac43} \to 2\cdot\frac{4}{3}$ - omog. $\frac{8}{3}$-omog.

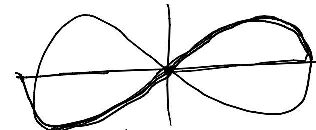
$\sqrt{h^2+k^2}$ — 1-omogenea sempre definita su $\partial B(0,1)$

$$\lim_{t\to 0}\frac{1-\cos t}{t^2} = \frac{1}{2}$$

$\frac{8}{3} - 1 > 0$ la funz. è mai nulla su $h^2+k^2=1$

$\Rightarrow$ Th. funzioni omog. grado >0 $\Rightarrow$ $\lim = 0$

$\theta \in [0,\pi]$, $\rho = 1+\sin\theta$



$$\Lambda = \int_a^b \sqrt{\dot\rho^2 + \rho^2\dot\theta^2} \longrightarrow \begin{cases}\rho = 1+\sin t\\ \theta = t\end{cases}$$

LUNGHEZZA IN COORDINATE POLARI

$$\Lambda = \int_0^\pi \sqrt{\cos^2 t + (1+\sin t)^2\cdot 1}\,dt = \int_0^\pi \sqrt{\cos^2 t + \sin^2 t + 2\sin t + 1}\,dt$$

$$= \int_0^\pi \sqrt{2}\sqrt{1+\sin t}\,dt \underset{u=\frac{\pi}{2}-t}{=}$$

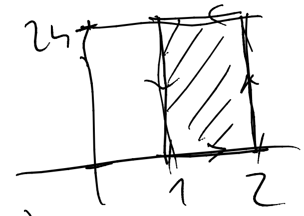
$\cos t =$
$= \cos 2\left(\frac{t}{2}\right) = 2\cos^2\frac{t}{2} - 1$

$$f(x,y,z)=\sqrt{x^2+y^2} \quad \text{estesa a} \quad \text{INTEGRALE SUPERFICIALE}$$

$$\phi(u,v)=\begin{pmatrix} u^2\cos v \\ u^2 \sin v \\ v \end{pmatrix} \qquad \underbrace{u\in[1,2] \quad v\in[0,2\pi]}$$

$$\Delta=[1,2]\times[0,2\pi]$$

$$\int_\phi f=\int_\Delta f(\phi(u,v))\,|\phi_u\times\phi_v|\,du\,dv$$



$$\int_1^2 du\int_0^{2\pi} dv\,\underbrace{\sqrt{u^4\cos^2 v+u^4\sin^2 v}}_{f(\phi(u,v))}\;\underbrace{2u\sqrt{1+u^4}}_{|\nu|}$$

$$\phi_u\rightarrow\begin{pmatrix} 2u\cos v \\ 2u\sin v \\ 0 \end{pmatrix}\times\begin{pmatrix} -u^2\sin v \\ u^2\cos v \\ 1 \end{pmatrix}\leftarrow\phi_v \;=\nu=\begin{pmatrix} 2u\sin v \\ -2u\cos v \\ 2u^3\cos^2 v+2u^3\sin^2 v \end{pmatrix}=2\begin{pmatrix} u\sin v \\ -u\cos v \\ u^3 \end{pmatrix}$$

$$|\nu|=2\sqrt{u^2\sin^2 v+u^2\cos^2 v+u^6}\underset{\substack{u\in[1,2]\\ u>0}}{=}2u\sqrt{1+u^4}$$

$$(*)=\int_1^2 du\int_0^{2\pi} dv\,\underbrace{\cancel{\sqrt{u^4}}}_{u^2}\,2u\sqrt{1+u^4}=\frac{1}{2}\int_0^{2\pi} dv\int_1^2 du\,\sqrt{1+u^4}\,\overbrace{\boxed{2u^3}\cdot 2}^{=\frac{1}{2}\frac{d(1+u^4)}{du}}$$

$(1+u^4)'=4u^3 \qquad 1+u^4=t$