·) TEOREMA DELLA DERIVAZIONE IN FREQUENZA

Ip:
$$\begin{cases} Y(f) = \dfrac{d}{df} X(f) \\ x(t) \overset{TCF}{\Longleftrightarrow} X(f) \end{cases}$$

Th: $y(t) = -j2\pi t \, x(t)$

Dim.

$$Y(f) = \frac{d}{df} X(f) = \frac{d}{df}\left[\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt\right]$$

$$= \int_{-\infty}^{+\infty} x(t) \frac{d}{df}\left[e^{-j2\pi f t}\right] dt$$

$$= \int_{-\infty}^{+\infty} x(t) \left(-j2\pi t\right) e^{-j2\pi f t}\, dt$$

$$= \underbrace{\int_{-\infty}^{+\infty} -j2\pi t \, x(t)\, e^{-j2\pi f t}}_{TCF\left[\underbrace{-j2\pi t \, x(t)}_{y(t)}\right] = Y(f)} = Y(f)$$

-) TEOREMA DELL' INTEGRAZIONE IN FREQ

$$\text{Ip:}\begin{cases} Y(f) = \displaystyle\int_{-\infty}^{f} X(\alpha)\, d\alpha \\ x(t) \overset{TCF}{\Longleftrightarrow} X(f) \\ \displaystyle\int_{-\infty}^{+\infty} X(\alpha)\, d\alpha = 0 \end{cases}$$

$$\text{Th:} \quad y(t) = -\frac{x(t)}{j2\pi t}$$

Dim.

$$Y(f) = \int_{-\infty}^{f} X(\alpha)\, d\alpha \Rightarrow X(f) = \frac{d}{df}\left[Y(f)\right]$$

$$\Downarrow$$

$$y(t) = -\frac{x(t)}{j2\pi t} \Leftarrow x(t) = -j2\pi t\, y(t)$$

$$\lim_{t \to 0} y(t) = K < \infty \Rightarrow x(t)\Big|_{t=0} = 0$$

$$\int_{-\infty}^{+\infty} X(\alpha)\, d\alpha = x(0) = 0$$

⇒ CONVOLUZIONE (o prodotto di convoluzione)

$$\boxed{\begin{aligned} z(t) &= x(t) \otimes y(t) \\ &\triangleq \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau \end{aligned}}$$

⇒ TEOREMA DELLA CONVOLUZIONE

$$Ip: \begin{cases} x(t) \overset{TCF}{\Longleftrightarrow} X(f) \quad , \quad y(t) \overset{TCF}{\Longleftrightarrow} Y(f) \\ z(t) = x(t) \otimes y(t) \end{cases}$$

$$Th: \quad Z(f) = X(f)\, Y(f)$$

Dim.

$$Z(f) = \int_{-\infty}^{+\infty} z(t)\, e^{-j2\pi f t}\, dt = \underset{(t)}{\int_{-\infty}^{+\infty}} \underset{(\tau)}{\int_{-\infty}^{+\infty}} x(\tau)\, y(t-\tau)\, d\tau\, e^{-j2\pi f t}\, dt$$

$$= \underset{(\tau)}{\int_{-\infty}^{+\infty}} x(\tau) \underset{(t)}{\int_{-\infty}^{+\infty}} y(t-\underset{t_0}{\tau})\, e^{-j2\pi f t}\, dt\, d\tau$$

$$= \underset{(\tau)}{\int_{-\infty}^{+\infty}} x(\tau)\; Y(f)\, e^{-j2\pi f \tau}\, d\tau = Y(f) \underbrace{\int_{-\infty}^{+\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau}_{X(f)} = X(f)\, Y(f)$$

- Proprietà della convoluzione

- Commutativa

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

Dim.

$$\int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau \qquad t-\tau = \tau'$$

$$= \int_{-\infty}^{+\infty} x(t-\tau')\, y(\tau')\, d\tau' = \int_{-\infty}^{+\infty} y(\tau')\, x(t-\tau')\, d\tau'$$

$$= y(t) \otimes x(t)$$

- Distributiva

$$x(t) \otimes [y(t) + z(t)] =$$

$$= x(t) \otimes y(t) + x(t) \otimes z(t)$$

Dim

$$\int_{-\infty}^{+\infty} x(\tau) [y(t-\tau) + z(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau + \int_{-\infty}^{+\infty} x(\tau)\, z(t-\tau)\, d\tau$$

·) Associativa

$$x(t) \otimes [y(t) \otimes z(t)] = [x(t) \otimes y(t)] \otimes z(t)$$

$$\parallel$$

$$x(t) \otimes y(t) \otimes z(t)$$

Dim

$$\int_{-\infty}^{+\infty} x(\tau) \int_{-\infty}^{+\infty} y(\alpha)\, z[(t-\tau)-\alpha]\, d\alpha\, d\tau$$

$$\vdots$$

Altra dim

$$x(t) \otimes [y(t) \otimes z(t)] \overset{TCF}{\Longleftrightarrow} X(f) \cdot [Y(f) \cdot Z(f)]$$

$$\parallel$$

$$[x(t) \otimes y(t)] \otimes z(t) \overset{TCF}{\Longleftrightarrow} [X(f) \cdot Y(f)] \cdot Z(f)$$

·) TEOREMA DEL PRODOTTO

$$\text{Ip:} \begin{cases} z(t) = x(t) \cdot y(t) \\ x(t) \overset{TCF}{\Longleftrightarrow} X(f) \quad , \quad y(t) \overset{TCF}{\Longleftrightarrow} Y(f) \end{cases}$$

$$\text{Th:} \quad Z(f) = X(f) \otimes Y(f)$$

Dim

$$Z(f) = \int_{-\infty}^{+\infty} x(t)\, y(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f) \Rightarrow x(t) = \int_{-\infty}^{+\infty} X(\alpha)\, e^{j2\pi\alpha t}\, d\alpha$$

$$= \int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} X(\alpha)\, e^{j2\pi\alpha t}\, d\alpha}_{x(t)}\; y(t)\, e^{-j2\pi f t}\, dt$$

(t) (α)

$$= \int_{-\infty}^{+\infty} X(\alpha) \underbrace{\int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi(f-\alpha)t}\, dt}_{Y(f-\alpha)}\, d\alpha$$

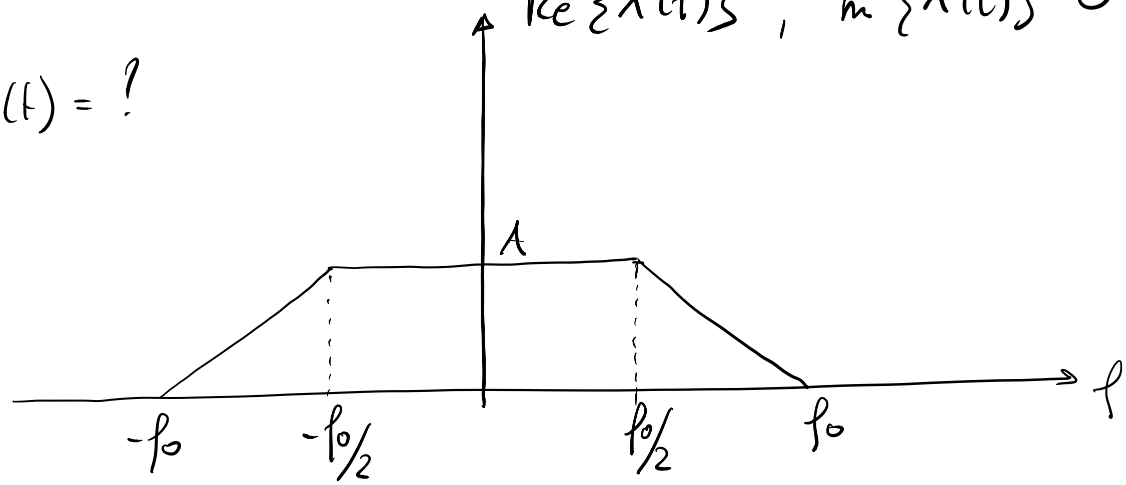
(α) (t)

$$= \int_{-\infty}^{+\infty} X(\alpha)\, Y(f-\alpha)\, d\alpha = X(f) \otimes Y(f)$$

·) ESERCIZIO

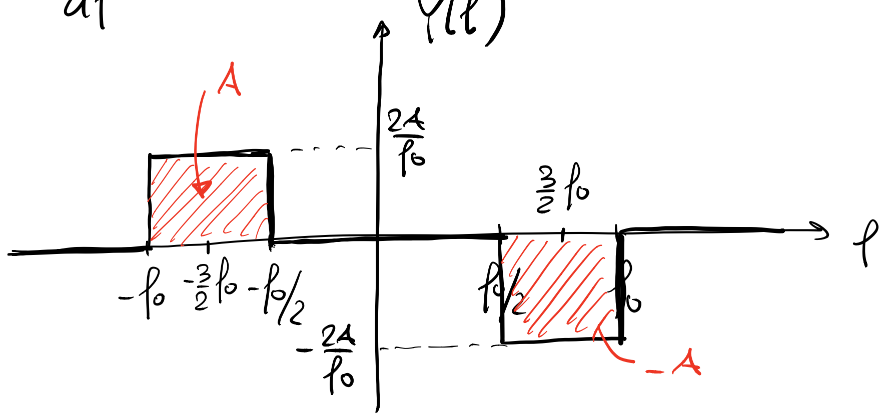
$Re\{X(f)\}$, $Im\{X(f)\}=0$

$x(t) = ?$



$x(t) = ATCF\left[X(f)\right]$

$$Y(f) = \frac{d}{df} X(f)$$



$$X(f) = \int_{-\infty}^{f} Y(\alpha)\, d\alpha \Rightarrow x(t) = -\frac{y(t)}{-j2\pi t}$$

$$\int_{-\infty}^{+\infty} Y(\alpha)\, d\alpha = 0$$

$$y(t) = \text{ATCF}\left[Y(f) \right]$$

centro della rect

$$Y(f) = \frac{2A}{f_0} \text{rect}\left(\frac{f - \left(-\frac{3}{2} f_0\right)}{f_0/2} \right) - \frac{2A}{f_0} \text{rect}\left(\frac{f - \frac{3}{2} f_0}{f_0/2} \right)$$

"durata" della rect

$$y(t) = \frac{\cancel{2A}}{\cancel{f_0}} \frac{f_0}{\cancel{2}} \text{sinc}\left(\frac{f_0}{2} t\right) e^{-j2\pi \frac{3}{2} f_0 t} + $$

$$- \frac{\cancel{2A}}{\cancel{f_0}} \frac{f_0}{\cancel{2}} \text{sinc}\left(\frac{f_0}{2} t\right) e^{j2\pi \frac{3}{2} f_0 t}$$

$$\text{rect}\left(\frac{t}{T}\right) \Leftrightarrow T \, \text{sinc}(Tf)$$

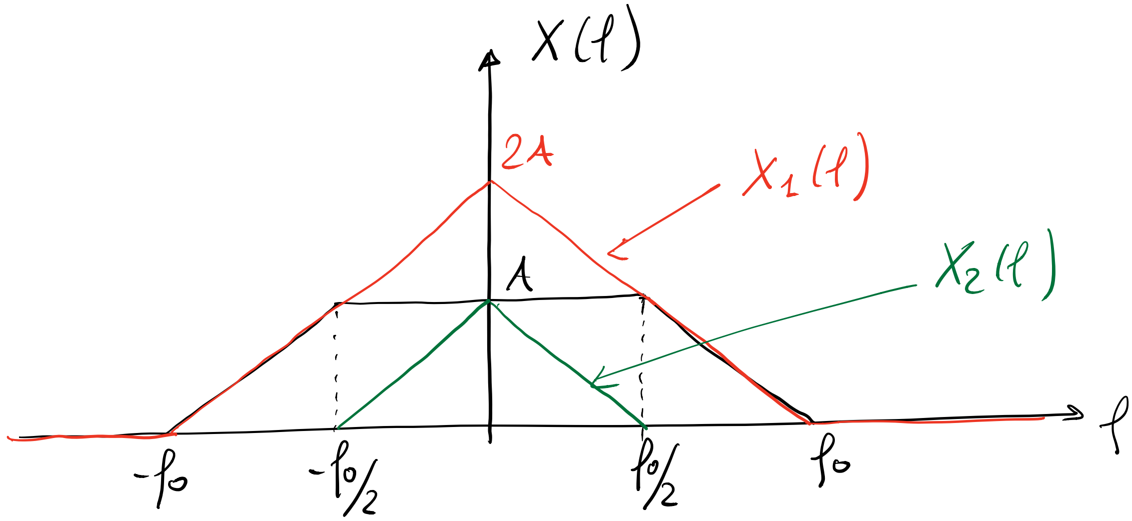
$$\left[\begin{array}{l} x(t - t_0) \Leftrightarrow X(f) e^{-j2\pi f t_0} \\ X(f - f_0) \Leftrightarrow x(t) e^{j2\pi f_0 t} \end{array} \right]$$

$$y(t) = A \, \text{sinc}\left(\frac{f_0}{2} t\right) \left[e^{-j3\pi f_0 t} - e^{j3\pi f_0 t} \right]$$

$$x(t) = -\frac{y(t)}{j2\pi t} = \frac{A}{\pi t} \text{sinc}\left(\frac{f_0}{2} t\right) \left[\frac{e^{j3\pi f_0 t} - e^{-j3\pi f_0 t}}{2j} \right]$$

$$= 3Af_0 \text{sinc}\left(\frac{f_0}{2} t\right) \frac{\sin(3\pi f_0 t)}{3\pi f_0 t} = \boxed{3A f_0 \, \text{sinc}\left(\frac{f_0}{2} t\right) \text{sinc}(3 f_0 t)}$$

·) Altra soluzione



$$X(f) = X_1(f) - X_2(f)$$

$$X_1(f) = 2A\left(1 - \frac{|f|}{f_0}\right)\mathrm{rect}\left(\frac{f}{2f_0}\right)$$

$$X_2(f) = A\left(1 - \frac{|f|}{f_0/2}\right)\mathrm{rect}\left(\frac{f}{f_0}\right)$$

$$x_1(t) = 2A f_0 \,\mathrm{sinc}^2(f_0 t)$$

$$2A\left(1 - \frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right) \overset{TCF}{\Longleftrightarrow} 2T\,\mathrm{sinc}^2(Tf)$$

$$2A f_0 \,\mathrm{sinc}^2(f_0 t) \Longleftrightarrow 2A\left(1 - \frac{|f|}{f_0}\right)\mathrm{rect}\left(\frac{-f}{2f_0}\right)$$

$$x_2(t) = A\frac{f_0}{2}\,\mathrm{sinc}^2\left(\frac{f_0}{2}t\right)$$

$$x(t) = x_1(t) - x_2(t) =$$

$$= \boxed{2Af_0\,\mathrm{sinc}^2(f_0 t) - \frac{Af_0}{2}\,\mathrm{sinc}^2\left(\frac{f_0}{2}t\right)}$$

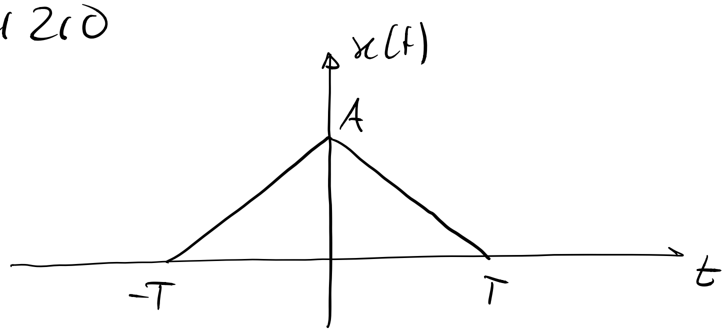
$$= \boxed{3Af_0\,\mathrm{sinc}\left(\frac{f_0}{2}t\right)\mathrm{sinc}(3f_0 t)}$$

$$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

$$\mathrm{sinc}(x_1)\,\mathrm{sinc}(x_2) = \frac{\sin(\pi x_1)}{\pi x_2}\,\frac{\sin(\pi x_2)}{\pi x_2}$$

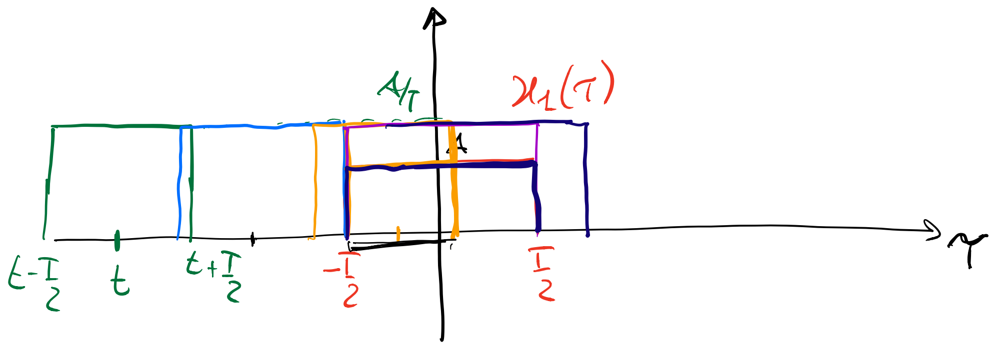
. . . . .

-) ESERCIZIO



$$x(t) = A\left(1 - \frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right) \Longleftrightarrow AT\,\mathrm{sinc}^2(Tf)$$

$$x(t) = x_1(t) \otimes x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$



$$x_1(t) = A\, \mathrm{rect}\left(\frac{t}{T}\right)$$

$$x_2(t) = \frac{K}{T}\, \mathrm{rect}\left(\frac{t}{T}\right)$$

$K = 1$

Ip $t < -T$

$x_1(\tau)\, x_2(t-\tau) = 0$

↳ $x(t) = 0$

$-T \leq t \leq 0$

$A\left(1 + \frac{t}{T}\right)$

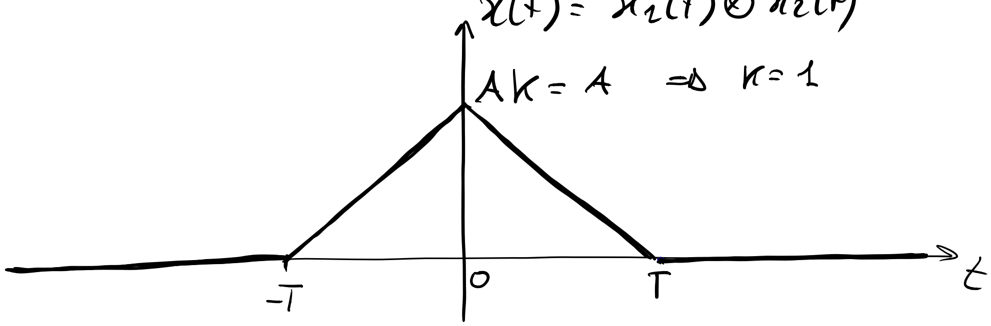
$0 \leq t \leq T$

$A\left(1 - \frac{t}{T}\right)$

$t > T \Rightarrow 0$

$x(t) = x_1(t) \otimes x_2(t)$

$AK = A \Rightarrow K = 1$



$$A\left(1-\frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right) = x(t)$$

$$x(t) = x_1(t) \otimes x_2(t)$$

$$x_1(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$x_2(t) = \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$X(f) = X_1(f)\,X_2(f) = AT\,\mathrm{sinc}(fT)\cdot\frac{1}{\cancel{T}}\cancel{T}\,\mathrm{sinc}(fT)$$

$$= AT\,\mathrm{sinc}^2(fT)$$

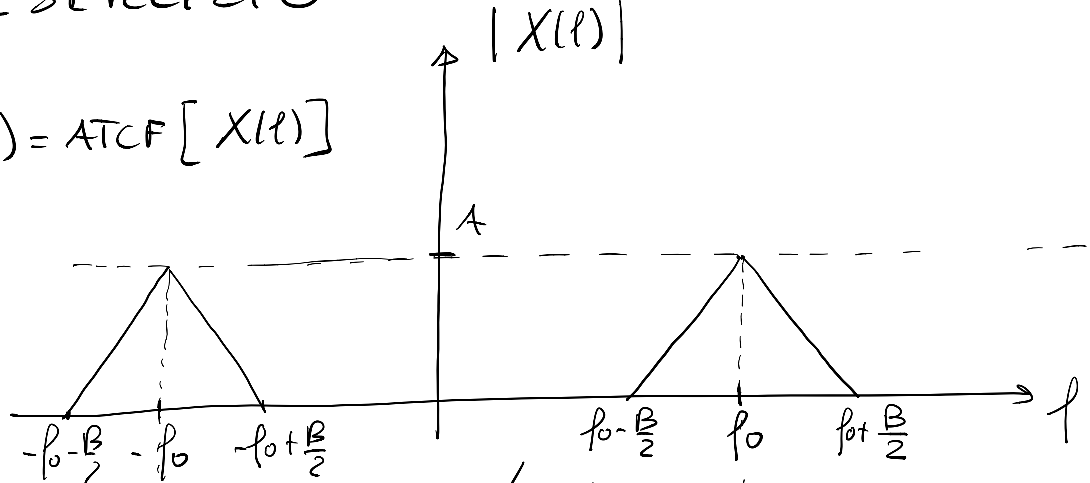
Da ricordare

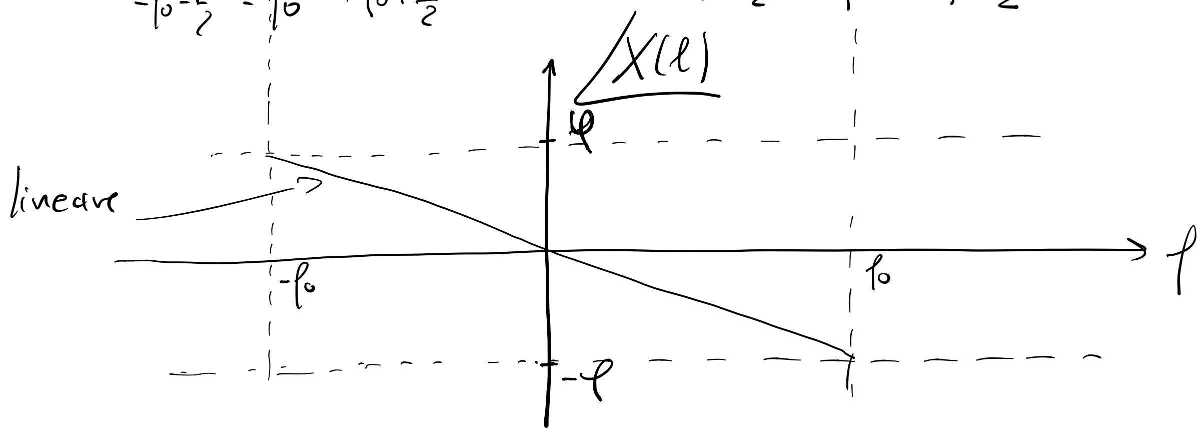
$$\boxed{\begin{array}{l} A\left(1-\frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right) \overset{TCF}{\Longleftrightarrow} AT\,\mathrm{sinc}^2(fT) \\ A f_0\,\mathrm{sinc}^2(f_0 t) \overset{TCF}{\Longleftrightarrow} A\left(1-\frac{|f|}{f_0}\right)\mathrm{rect}\left(\frac{f}{2f_0}\right) \end{array}}$$

$$\boxed{\begin{array}{l} A\,\mathrm{rect}\left(\frac{t}{T}\right) \overset{TCF}{\Longleftrightarrow} AT\,\mathrm{sinc}(fT) \\ A f_0\,\mathrm{sinc}(f_0 t) \overset{TCF}{\Longleftrightarrow} A\,\mathrm{rect}\left(\frac{f}{f_0}\right) \end{array}}$$

-) ESERCIZIO

$x(t) = ATCF\left[X(f)\right]$



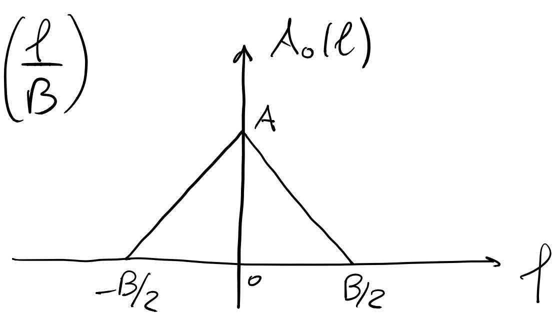


$$X(f) = |X(f)|\, e^{j\angle X(f)}$$

replica a dx replica a sx

$$A(f) = |X(f)| = A_0(f-f_0) + A_0(f+f_0)$$

$$A_0(f) = A\left(1 - \frac{|f|}{B/2}\right) \text{rect}\left(\frac{f}{B}\right)$$



$$F(f) = -\frac{\varphi}{f_0} f$$

$$X(f) = \left[A_0(f-f_0) + A(f+f_0) \right] e^{-j \frac{\varphi}{f_0} f}$$

$$= \left[A\left(1 - \frac{|f-f_0|}{B/2}\right) \mathrm{rect}\left(\frac{f-f_0}{B}\right) + A\left(1 - \frac{|f+f_0|}{B/2}\right) \mathrm{rect}\left(\frac{f+f_0}{B}\right) \right]$$

$$\cdot\, e^{-j 2\pi f t_0} \quad , \quad \boxed{t_0 = \frac{\varphi}{2\pi f_0}}$$

$$X(f) = A(f)\, e^{-j 2\pi f t_0}$$

dal teo. del ritardo

$$x(t) = a(t - t_0)$$

$$\Rightarrow a(t) = \mathrm{ATCF}\left[A(f) \right]$$

$$A(f) = A\left(1 - \frac{|f-f_0|}{B/2}\right) \mathrm{rect}\left(\frac{f-f_0}{B}\right) + A\left(1 - \frac{|f+f_0|}{B/2}\right) \mathrm{rect}\left(\frac{f+f_0}{B}\right)$$

$$= A_0(f - f_0) + A_0(f + f_0)$$

$$a(t) = a_0(t)\, e^{+j 2\pi f_0 t} + a_0(t)\, e^{-j 2\pi f_0 t}$$

$$= 2 a_0(t) \cos(2\pi f_0 t)$$

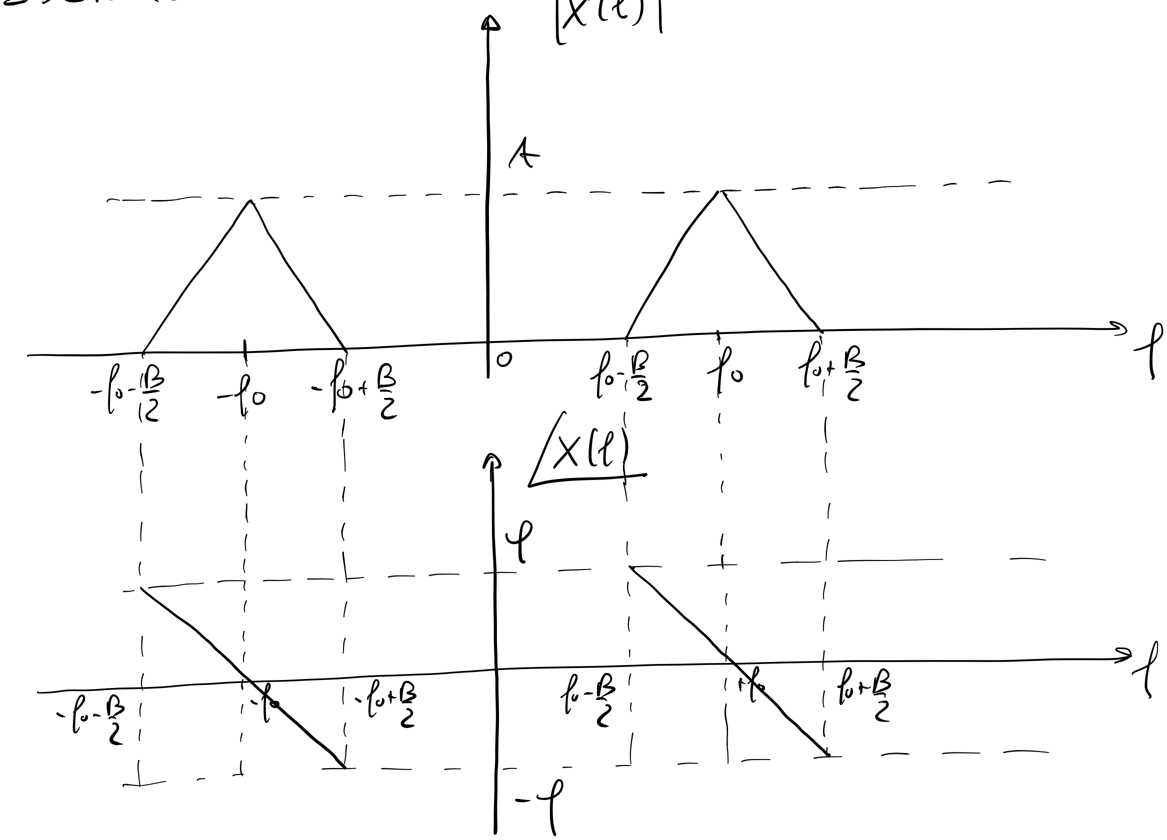
$$x(t) = 2\, a_0(t - t_0) \cos\left[2\pi f_0 (t - t_0)\right]$$

$$a_0(t) = \mathrm{ATCF}\left[A_0(f)\right] = \mathrm{ATCF}\left[A\left(1 - \frac{|f|}{B/2}\right) \mathrm{rect}\left(\frac{f}{B}\right) \right]$$

$$= \frac{AB}{2} \mathrm{sinc}^2\left(\frac{B}{2} t\right)$$

$$\boxed{x(t) = AB\, \mathrm{sinc}^2\left[\frac{B}{2}(t - t_0)\right] \cos\left[2\pi f_0 (t - t_0)\right], \quad t_0 = \frac{\varphi}{2\pi f_0}}$$

-) ESERCIZIO

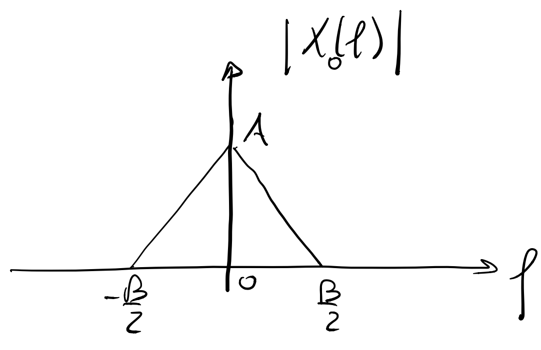


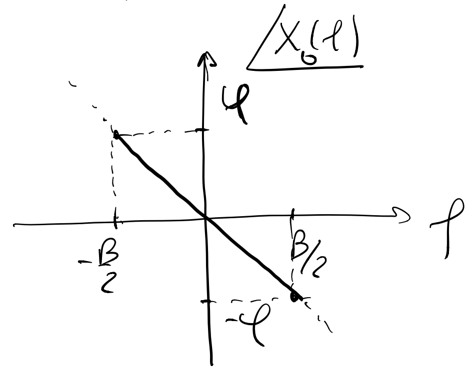
$$x(t) = \text{ATCF}\left[X(f)\right]$$

Soluzione

$$X_0(f) = |X_0(f)|\, e^{j \angle X_0(f)}$$

$$|X_0(f)| = A(f) = A\left(1 - \frac{|f|}{B/2}\right)\text{rect}\left(\frac{f}{B}\right)$$





$$\angle X_0(f) = -\frac{\varphi}{B/2} f = -2\pi f t_0 \quad , \quad t_0 = \frac{\varphi}{\pi B}$$

$$X(f) = X_0(f-f_0) + X_0(f+f_0)$$

$$x(t) = x_0(t)\, e^{j2\pi f_0 t} + x_0(t)\, e^{-j2\pi f_0 t}$$

$$= 2\, x_0(t) \cos(2\pi f_0 t)$$

$$x_0(t) = \text{ATCF}\left[X_0(f) \right]$$

$$= \text{ATCF}\left[A(f)\, e^{-j2\pi f t_0} \right]$$

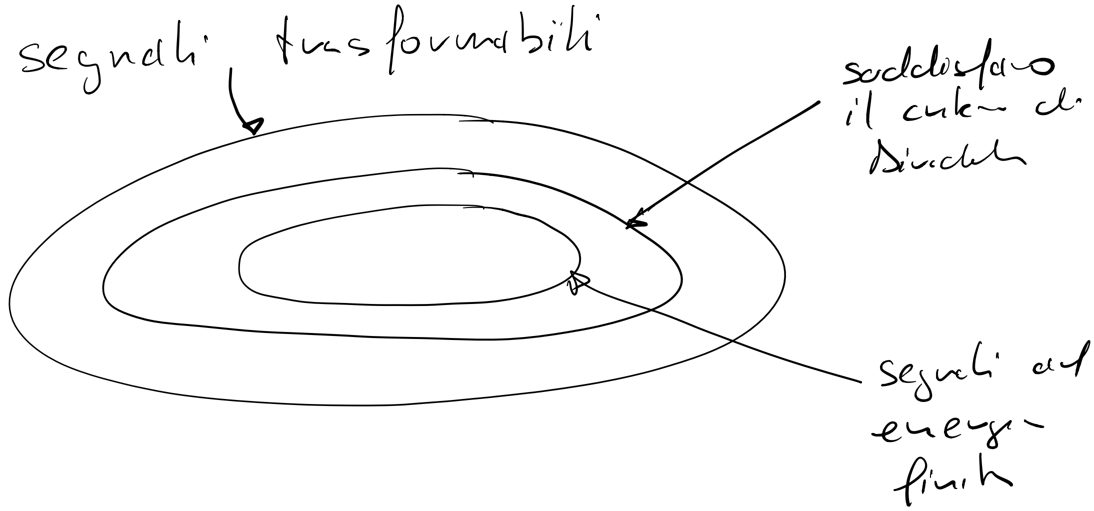
$$= a(t-t_0)$$

$$a(t) = \text{ATCF}\left[A(f) \right] = \frac{AB}{2} \operatorname{sinc}^2\left(\frac{B}{2} t\right)$$

$$x_0(t) = \frac{AB}{2} \operatorname{sinc}^2\left[\frac{B}{2}(t-t_0) \right]$$

$$\boxed{x(t) = \underbrace{AB \operatorname{sinc}^2\left[\frac{B}{2}(t-t_0) \right]}_{\text{inviluppo segnale modulante}} \underbrace{\cos(2\pi f_0 t)}_{\text{oscillazione}}}$$

TCF



$\Rightarrow$ Allarghiamo la classe dei segnali anche a quelli ad energia infinita

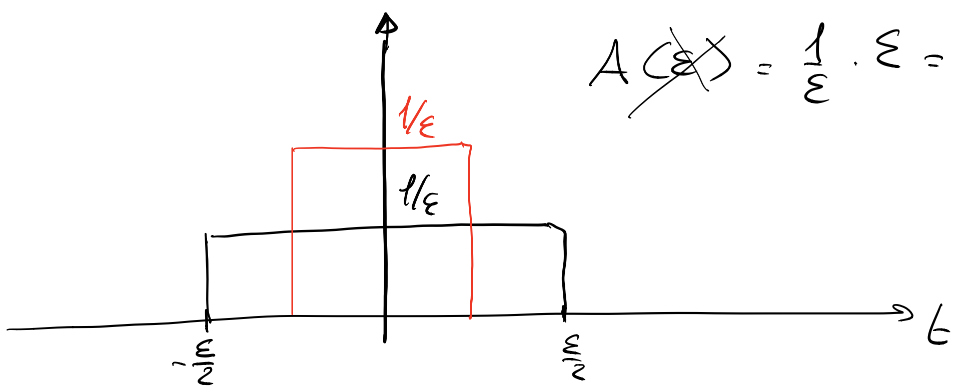
$\Downarrow$

GENERALIZZAZIONE DELLA TCF definendo la "funzione" delta di Dirac e calcolando la sua TCF

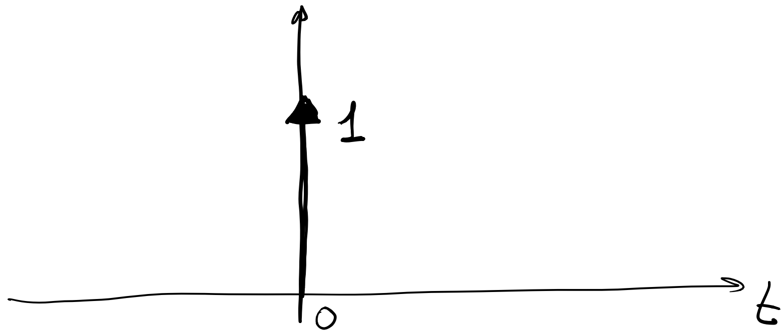
$\Rightarrow$ DEFINIZIONE DI DELTA DI DIRAC

$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}\,\mathrm{rect}\left(\frac{t}{\varepsilon}\right)$$

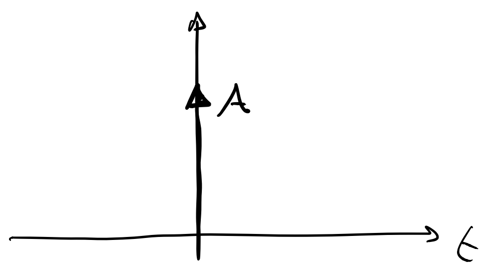
$$A(\cancel{\varepsilon}) = \frac{1}{\varepsilon}\cdot\varepsilon = 1$$



$$\delta(t) \triangleq \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$



$A\,\delta(t)$



$$A\,\delta(t) = \lim_{\varepsilon \to 0} A\,\delta_\varepsilon(t) = A \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

$$\int_{-\infty}^{+\infty} A\,\delta(t)\,dt = A \underbrace{\int_{-\infty}^{+\infty} \delta(t)\,dt}_{1} = A$$

PROPRIETÀ

1) $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$

Dim

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = \int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt$$

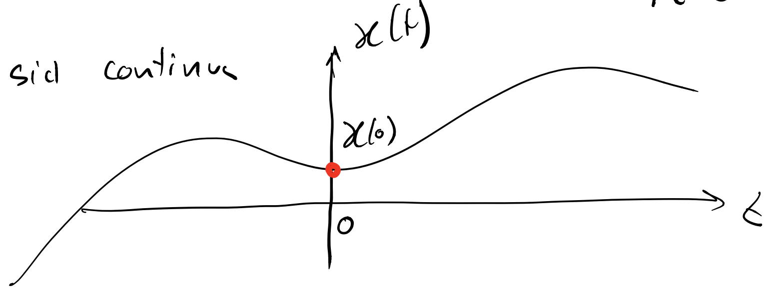
$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \varepsilon = 1$$

2) PROPRIETÀ CAMPIONATRICE

$$\int_{-\infty}^{+\infty} x(t)\, \delta(t)\, dt = x(t)\Big|_{t=0} = x(0)$$

Ip. $x(t)$ sia continua in $t=0$



Dim.

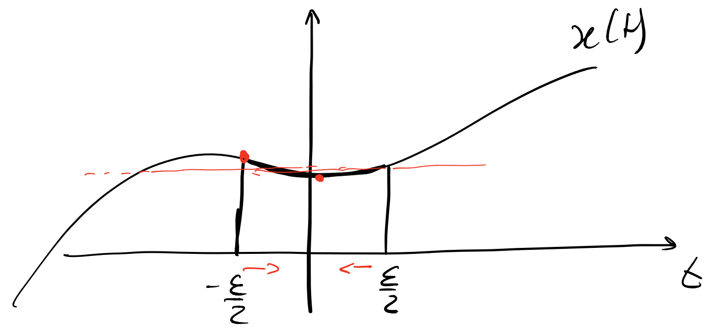
$$\int_{-\infty}^{+\infty} x(t) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} x(t) \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x(t)\, dt$$

teo. della media

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \cancel{\varepsilon} \cdot x(\bar{t}) = \lim_{\varepsilon \to 0} x(\bar{t}) \qquad -\frac{\varepsilon}{2} \le \bar{t} \le \frac{\varepsilon}{2}$$

$= x(0)$



$\Rightarrow$ PROPRIETÀ INTEGRALI DELLA DELTA DI DIRAC

I) $\delta(t) = \delta(-t)$ PARITÀ

$$\int_{-\infty}^{+\infty} \delta(-t)\, x(t)\, dt = \int_{-\infty}^{+\infty} \delta(t)\, x(t)\, dt$$

significato di parità in senso integrale

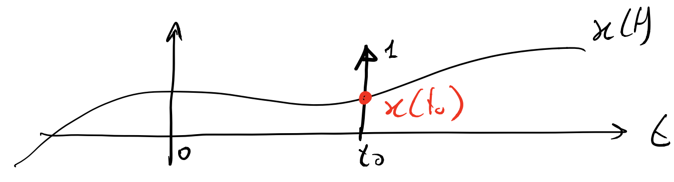
Dim

$$\int_{-\infty}^{+\infty} \delta(-t)\, x(t)\, dt \qquad t' = -t$$

$$= \int_{-\infty}^{+\infty} \delta(t')\, x(-t')\, dt' = x(-t')\Big|_{t'=0} = x(0)$$

II) Traslazione

$$\int_{-\infty}^{+\infty} x(t)\, \delta(t-t_0)\, dt = x(t_0)$$



Dim

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt \qquad t - t_0 = t'$$

$$= \int_{-\infty}^{+\infty} x(t'+t_0)\,\delta(t')\,dt' = x(t'+t_0)\Big|_{t'=0} = x(t_0)$$