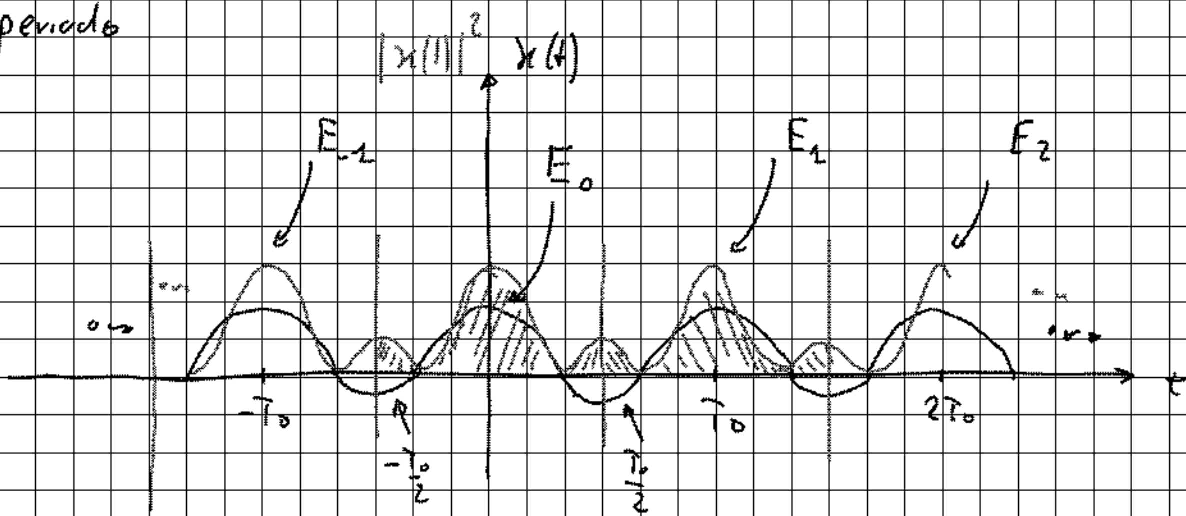
SEGNALI PERIODICI

$$x(t) = x(t - kT_0) \quad , \quad k \in \mathbb{Z} \quad , \quad T_0 \in \mathbb{R}^+$$

T_0 = periodo



Energia

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} E_k = \lim_{k \to \infty} kE = +\infty$$

Potenza media

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt = \lim_{k \to \infty} \frac{1}{kT_0} \int_{-\frac{kT_0}{2}}^{\frac{kT_0}{2}} |x(t)|^2 dt =$$

$$= \cancel{\lim_{k \to \infty}} \frac{1}{\cancel{k}T_0} \cancel{k} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt = \boxed{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt = P_x}$$

$$\boxed{v_m = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, dt}$$

ANALISI DI FOURIER

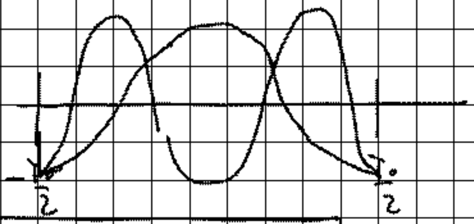
$$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} A_k \cos(2\pi k f_0 t + \varphi_k)} \qquad f_0 \triangleq \frac{1}{T_0}$$

$k = 0 \Rightarrow A_0$

$$v_m = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{+\infty} A_k \cos(2\pi k f_0 t + \varphi_k)\, dt =$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A_0 \cos(\varphi_0) + \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} A_k \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi k f_0 t + \varphi_k)\, dt}_{=0} =$$

$$= \frac{1}{\cancel{T_0}} A_0 \cos(\varphi_0)\cancel{T_0} = A_0\cos(\varphi_0) = A_0'$$



$$\boxed{X_n \triangleq \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}$$

TRASFORMATA SERIE DI FOURIER (TSF)

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi n f_0 t}}$$

ANTITRASFORMATA SERIE DI FOURIER $\left(\begin{matrix} ATSF \\ TSF^{-1}\end{matrix}\right)$

$$x(t) \overset{TSF}{\Longleftrightarrow} X_n$$

$$TSF\left[ATSF\left[A\right]\right] = A$$

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{+\infty} X_k e^{j2\pi k f_0 t} e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_k \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi \underset{m}{(k-n)} f_0 t}\, dt =$$

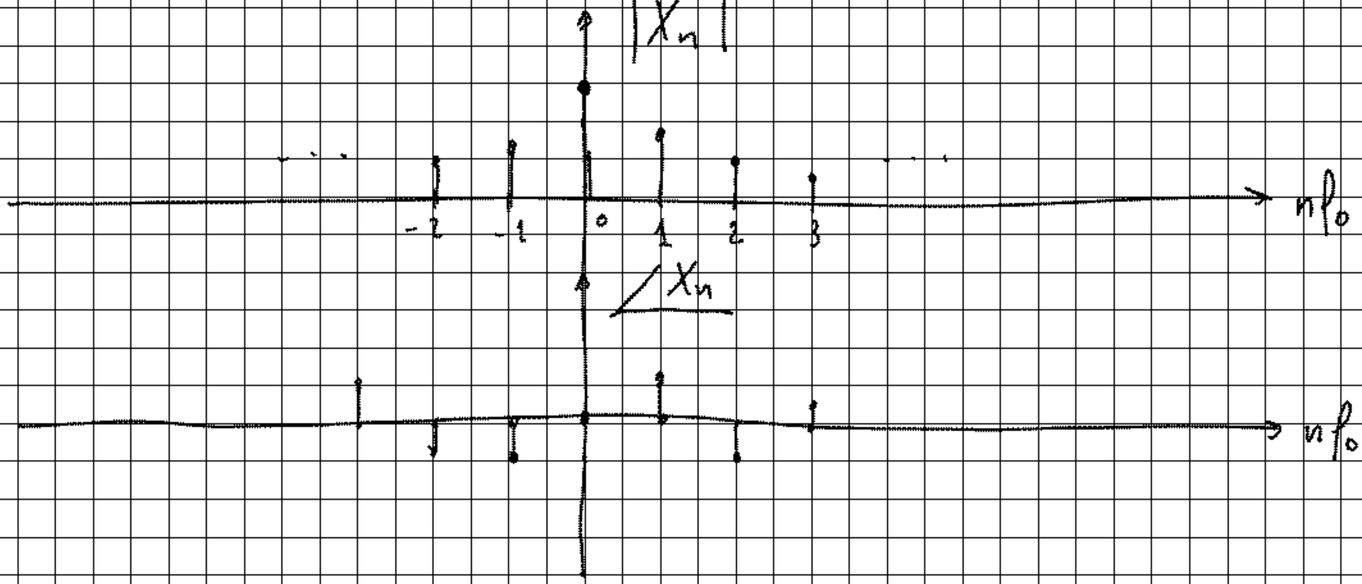
$$= \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_k \left[\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi m f_0 t)\, dt}_{=0} + j\underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin(2\pi m f_0 t)\, dt}_{=0}\right] \quad (m \neq 0$$

$$\Rightarrow m = 0 \quad \frac{1}{\cancel{T_0}}\sum_{k=-\infty}^{+\infty} X_k \cancel{T_0} \qquad m = 0 \Rightarrow k = n$$

$$= X_n$$

SPETTRO DI UN SEGNALE PERIODICO

$X_n \triangleq$ spettro del segnale



SPETTRO DI UN COSENO

$$x(t) = A\cos(2\pi f_0 t)$$

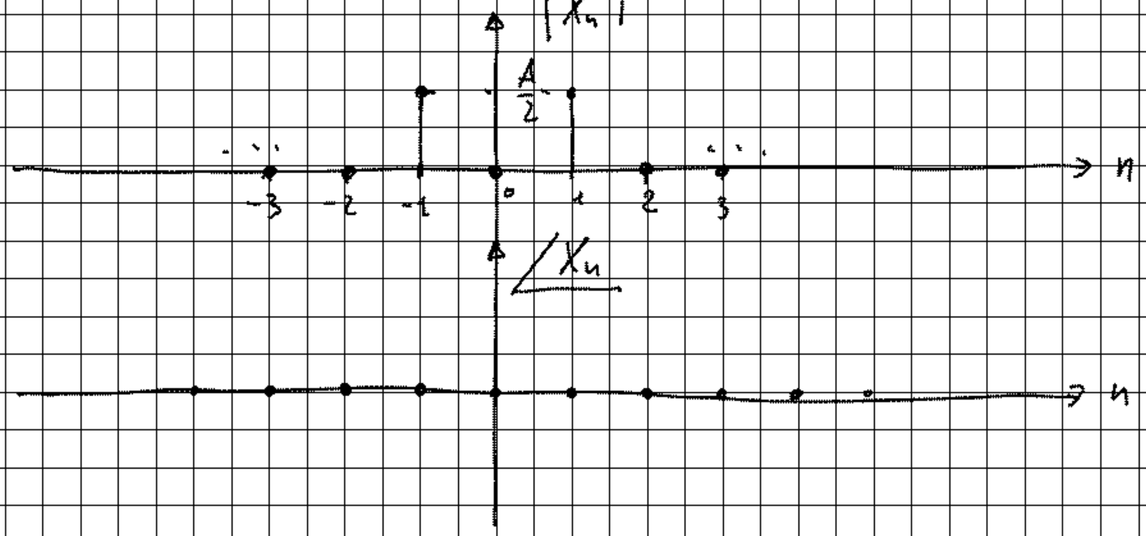
$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$\cos(\alpha) = \frac{e^{j\alpha}+e^{-j\alpha}}{2} = \frac{\cos\alpha + \cancel{j\sin\alpha} + \cos\alpha - \cancel{j\sin\alpha}}{2} = \cos(\alpha)$$

$$= \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[\frac{e^{j2\pi f_0 t}+e^{-j2\pi f_0 t}}{2}\right] e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{A}{2T_0}\left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi\underbrace{(1-n)}_{m} f_0 t}\, dt + \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi\underbrace{(-1-n)}_{m'} f_0 t}\, dt\right]$$

$$\begin{cases} n = 1 \Rightarrow X_1 = \frac{A}{2T_0}T_0 = \frac{A}{2} \\ n = -1 \longrightarrow X_{-1} = \frac{A}{2\cancel{T_0}}\cancel{T_0} = \frac{A}{2} \\ n \neq \pm 1 \Rightarrow X_n = 0 \end{cases}$$



SPETTRO DI UN SENO

$$x(t) = A \sin 2\pi f_0 t$$

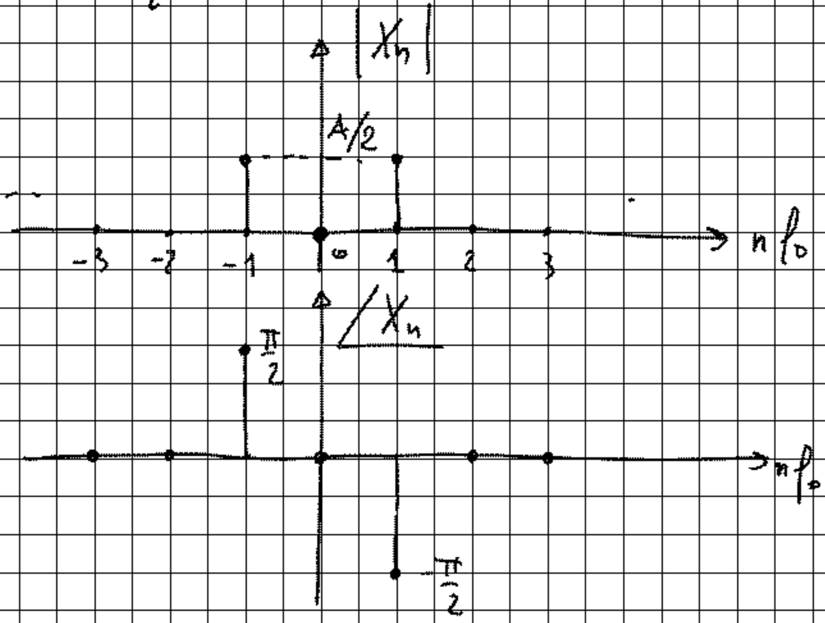
$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A \sin(2\pi f_0 t)\, e^{-j 2\pi n f_0 t}\, dt =$$

$$\sin \alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$= \frac{A}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[\frac{e^{j 2\pi f_0 t} - e^{-j 2\pi f_0 t}}{2j} \right] e^{-j 2\pi n f_0 t}\, dt$$

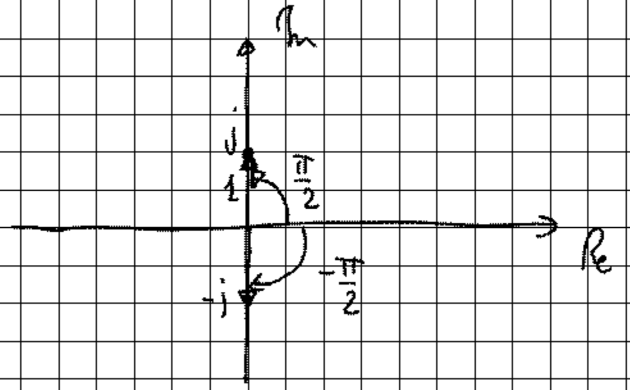
$$= \frac{A}{2j T_0} \left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j 2\pi (1-n) f_0 t}\, dt - \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j 2\pi (-1-n) f_0 t}\, dt \right]$$

$$\begin{cases} X_1 = \dfrac{A}{2j \cancel{T_0}} \cancel{T_0} = \dfrac{A}{2j} \\ X_{-1} = -\dfrac{A}{2j \cancel{T_0}} \cancel{T_0} = -\dfrac{A}{2j} \\ X_n = 0 \quad n \neq \pm 1 \end{cases}$$



$$z = a + jb \quad a = 0,\ b = 1$$

$$z = a + jb \quad a = 0,\ b = -1$$



$$j \Rightarrow e^{j\frac{\pi}{2}}$$

$$-j \Rightarrow e^{-j\frac{\pi}{2}}$$

$$X_1 = \frac{A}{2j}\cdot\frac{j}{j} = \frac{Aj}{-2} = \frac{A}{2}\cdot(-j) = \frac{A}{2}\, e^{-j\frac{\pi}{2}}$$

modulo: $\frac{A}{2}$ fase: $-\frac{\pi}{2}$

$$X_{-1} = -\frac{A}{2j} = \frac{A}{2}\, e^{j\frac{\pi}{2}}$$

PROPRIETA' DELLA TSF

·) LINEARITA'

$$x(t) \Longleftrightarrow X_n$$

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

$$\Updownarrow \qquad\qquad \Updownarrow \qquad\qquad \Updownarrow$$

$$X_n = a_1 X_{1_n} + a_2 X_{2_n}$$

Dim.

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-j2\pi n f_0 t}\, dt$$

$$= a_1 \cdot \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_1(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_{1_n}} + a_2 \cdot \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_2(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_{2_n}}$$

$$= a_1 X_{1_n} + a_2 X_{2_n}$$

TSF di SEGNALI REALI e PERIODICI

$x(t) = x^*(t)$ (segnale reale)

$X_n = X_{-n}^*$ simmetria hermitiana

$X_{-n} = X_n^*$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt =$$

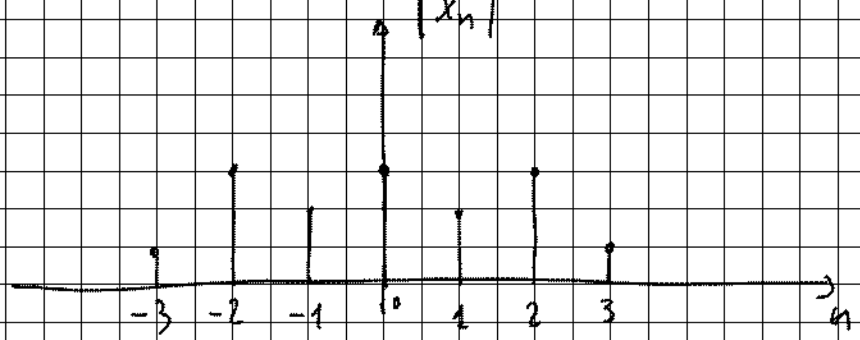
$$= \left[\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^*(t)\, e^{-j2\pi n f_0 t}\, dt\right]^* =$$

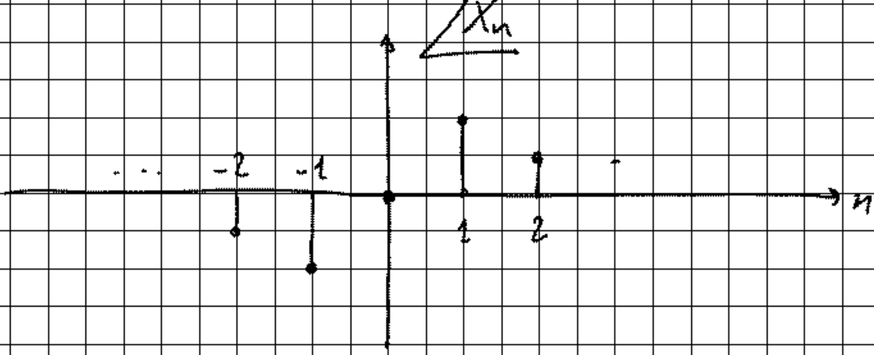
$$= \left[\underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_n}\right]^* = X_n^*$$

$X_n = X_{-n}^*$

$|X_n| = |X_{-n}^*| = |X_{-n}|$

$\angle X_n = \angle X_{-n}^* = -\angle X_{-n}$





TSF di SEGNALI REALI E PARI

$$x(t) = x(-t)$$

$$X_n = X_{-n}$$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt = \qquad (t' = -t)$$

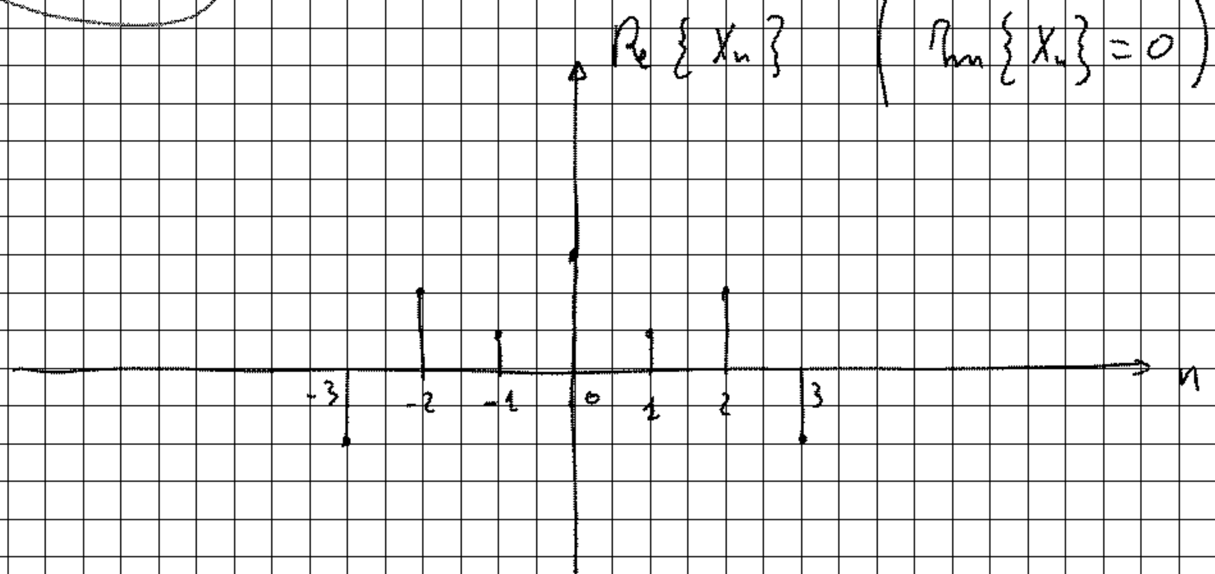
$$= \frac{1}{T_0}\int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} x(-t')\, e^{-j2\pi(-n)f_0(-t')}\,(-dt') =$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = X_n$$

SPETTRO PARI

SPETTRO REALE

$$X_{-n} = X_n = X_n^*$$



TSF di SEGNALI REALI E DISPARI

$$x(t) = -x(-t)$$

X_n immaginario puro e con simmetria dispari

$$Re\{X_n\} = 0 \quad , \quad X_{-n} = -X_n$$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt = \qquad (t' = -t)$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(-t')\, e^{-j2\pi n f_0 t'}\, dt' =$$

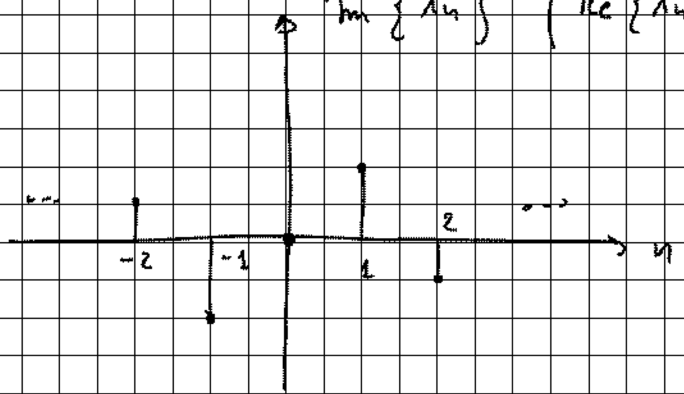
$$= -\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = -X_n$$

SIMM DISP.

$$X_{-n} = -X_n = X_n^*$$

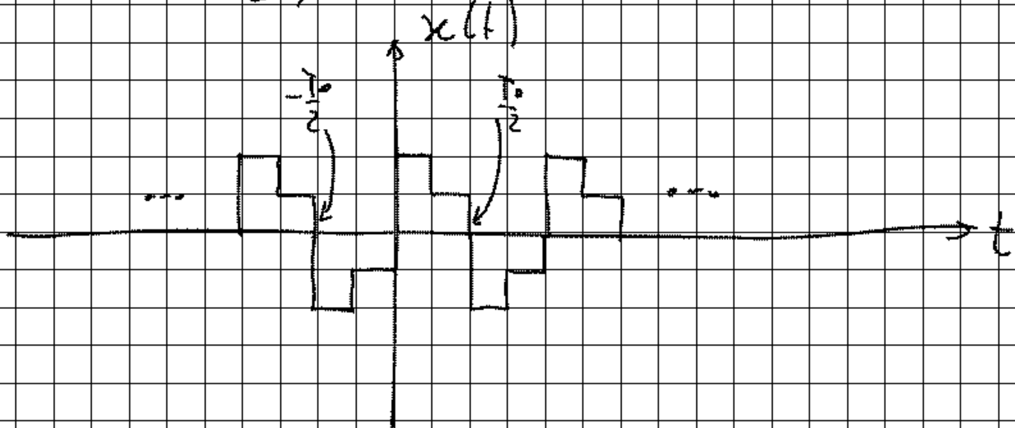
IMM. PURO

$\mathrm{Im}\{X_n\}$ $(\mathrm{Re}\{X_n\}=0)$



TSF di SEGNALI ALTERNATIVI

$$x(t) = -x\left(t - \frac{T_0}{2}\right)$$



$X_n = 0$, per n pari

$X_n \neq 0$, per n dispari

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{0} x(t)\, e^{-j2\pi n f_0 t}\, dt + \frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

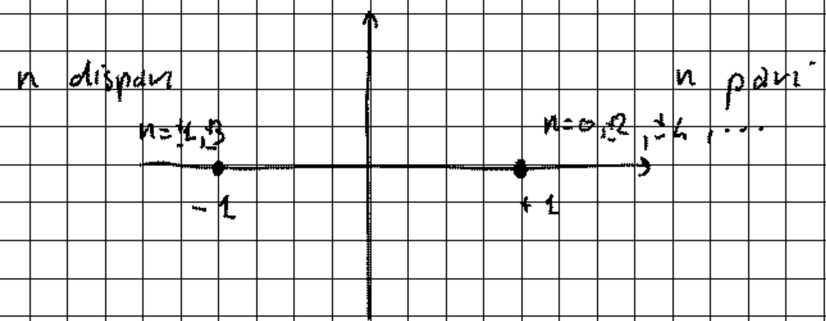
$$= -\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{0} x\left(t - \frac{T_0}{2}\right) e^{-j2\pi n f_0 t}\, dt + \left(\quad \cdot \quad \right) =$$

$$\Rightarrow \quad t' = t - \frac{T_0}{2} \quad \Rightarrow \quad t = t' + \frac{T_0}{2}$$

$$= -\frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 \left(t' + \frac{T_0}{2}\right)}\, dt' + \left(\quad \cdot \quad \right) =$$

$$= -\frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, e^{-j\pi n}\, dt + \left(\quad \cdot \quad \right) =$$

$$= \frac{1}{T_0} \int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt \left(\underbrace{1 - 1 e^{-j\pi n}} \right)$$



$$\begin{cases} X_n = 0 & n \text{ pari} \\ X_n = \dfrac{2}{T_0} \displaystyle\int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt \end{cases}$$

PERIODICIZZAZIONE DI UN SEGNALE APERIODICO

$x_0(t)$ aperiodico $\qquad x_0(t) \neq x_0(t - nT_0), \ \forall T_0$

$$x(t) = \sum_{n=-\infty}^{+\infty} x_0(t + nT_0) \quad , \quad T_0 \text{ periodo di } x(t)$$

$$x(t - kT_0) = \sum_{n=-\infty}^{+\infty} x_0(t - kT_0 + nT_0) =$$

$$= \sum_{n=-\infty}^{+\infty} x_0\left(t-(k-n)T_0\right) \qquad (n' = n-k)$$

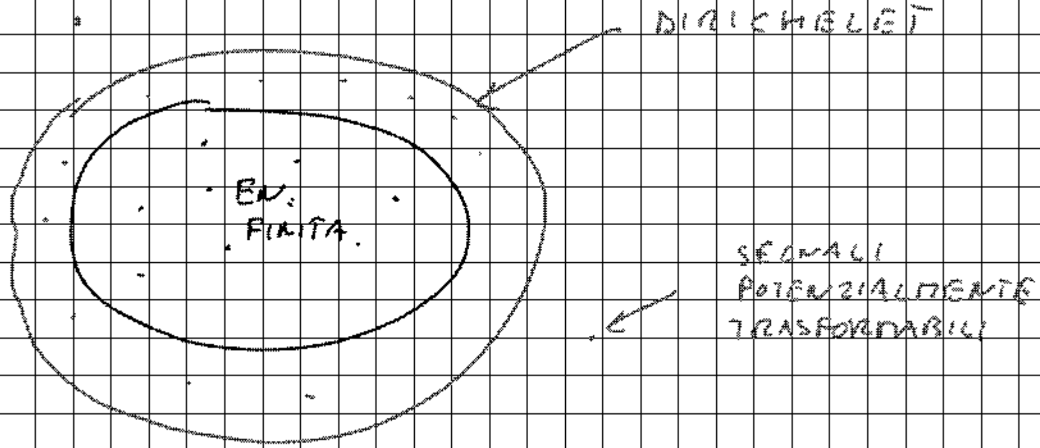
$$= \sum_{n'=-\infty}^{+\infty} x_0\left(t+n'T_0\right) = x(t) \Rightarrow x(t) \text{ è periodico di periodo } T_0$$

TRASFORMATA CONTINUA DI FOURIER

CRITERIO DEI SEGNALI AD ENERGIA FINITA

$x(t)$ ammette trasformata continua di Fourier (TCF) se (condizione sufficiente)

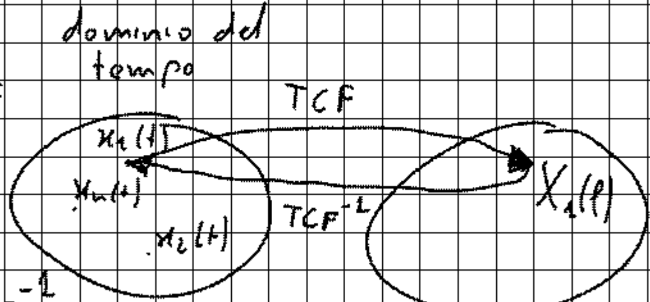
$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty$$



$X(f)$: $f \in \mathbb{R}$, $X \in \mathbb{C}$

$$\boxed{X(f) \triangleq \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt} \quad \text{TCF}$$

$$\boxed{x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df} \quad \text{TCF}^{-1} \ (\text{ATCF})$$



$X(f)$ e' lo spettro di $x(t)$

$|X(f)|$ spettro di ampiezza

$\angle X(f)$ spettro di fase

$$|X(f)| = \sqrt{\mathrm{Re}^2\{X(f)\} + \mathrm{Im}^2\{X(f)\}} \quad , \quad \angle X(f) = \mathrm{arctg} \frac{\mathrm{Im}\{X(f)\}}{\mathrm{Re}\{X(f)\}}$$

SIMMETRIE DELLO SPETTRO

·) HERMITIANA

se $x(t)$ è reale

$$\Rightarrow \quad X(-f) = X^*(f) \quad \Rightarrow \quad X(f) = X^*(-f)$$

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(-f)t}\, dt =$$

$$= \Bigg[\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt}_{X(f)} \Bigg]^* = X^*(f)$$

·) REALE E PARI

se $x(t)$ è reale e pari

$$x(t) \in \mathbb{R} \quad , \quad x(t) = x(-t)$$

$$X(f) \in \mathbb{R} \quad , \quad X(f) = X(-f)$$

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(-f)t}\, dt = \qquad (t' = -t)$$

$$= \int_{-\infty}^{+\infty} x(-t')\, e^{-j2\pi f t'}\, dt' =$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt' = X(f)$$

·) REALE E DISPARI

$$x(t) \in \mathbb{R} \quad , \quad x(-t) = -x(t)$$

$$X(f) \in \mathbb{I} \quad , \quad X(-f) = -X(f)$$

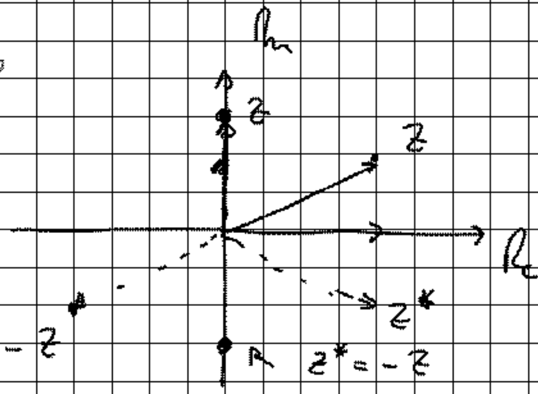
$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(-f)t}\, dt = \int_{-\infty}^{+\infty} x(-t')\, e^{-j2\pi f t'}\, dt' =$$

$$= -\int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt' = -X(f)$$

SIM DISP

IMM PURO

$$X(-f) = -X(f) = X^*(f)$$



TEOREMI SULLA TCF

) LINEARITÀ

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

$$x_1(t) \overset{TCF}{\Longleftrightarrow} X_1(f)$$

$$x_2(t) \overset{TCF}{\Longleftrightarrow} X_2(f)$$

$$\Rightarrow \quad X(f) = a_1 X_1(f) + a_2 X_2(f)$$

Dim.

$$X(f) = \int_{-\infty}^{+\infty} \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-j2\pi f t}\, dt =$$

$$= a_1 \underbrace{\int_{-\infty}^{+\infty} x_1(t)\, e^{-j2\pi f t}\, dt}_{X_1(f)} + a_2 \underbrace{\int_{-\infty}^{+\infty} x_2(t)\, e^{-j2\pi f t}\, dt}_{X_2(f)}$$

$$= a_1 X_1(f) + a_2 X_2(f)$$

·) DUALITÀ

se $x(t) \overset{TCF}{\Longleftrightarrow} X(f)$ $\qquad$ $f(t) \overset{TCF}{\Longrightarrow} g(f)$

$\Rightarrow X(t) \Longleftrightarrow x(-f)$ $\qquad$ $g(t) \Longrightarrow f(-f)$

Dim

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$f \to t \quad , \quad t \to f$

$$X(t) = \int_{-\infty}^{+\infty} x(f)\, e^{-j2\pi t f}\, df = \qquad (f' = -f)$$

$$= \int_{-\infty}^{+\infty} x(-f')\, e^{-j2\pi t(-f')}\, df' = \qquad (f' = f)$$

$$= \int_{-\infty}^{+\infty} x(-f)\, e^{j2\pi f t}\, df$$

$$TCF^{-1}\left[x(-f)\right] = X(t)$$

$$TCF\left[X(t)\right] = x(-f)$$

·) RITARDO

$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$

$y(t) = x(t - t_0)$

$\Rightarrow TCF\left[y(t)\right] = X(f)\, e^{-j2\pi f t_0}$

Dim.

$$Y(f) = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt = \qquad \begin{pmatrix} t - t_0 = t' \\ dt = dt' \end{pmatrix}$$

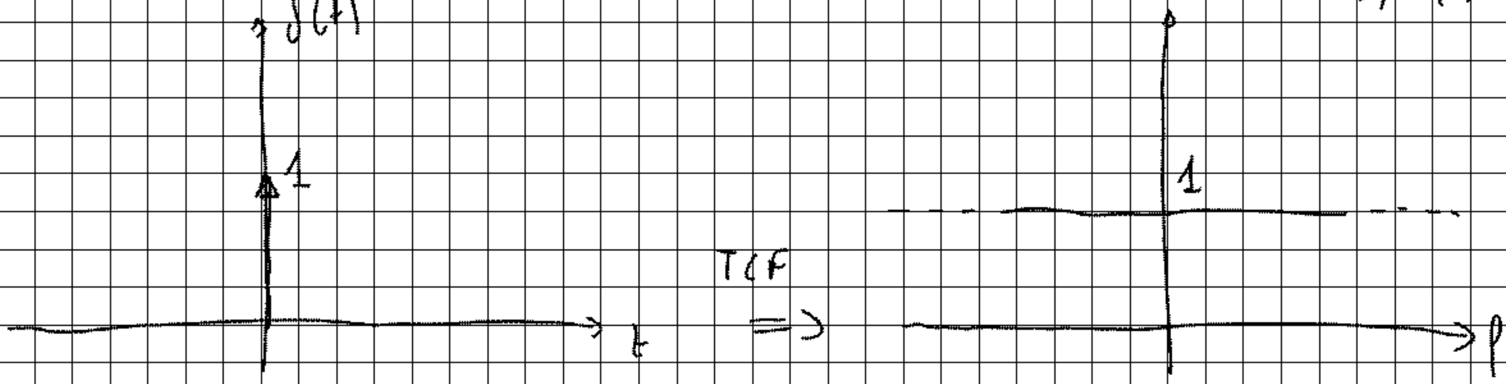
$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f (t'+t_0)}\, dt' =$$

$$= \underbrace{\int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt'}_{X(f)} \cdot e^{-j2\pi f t_0} = X(f)\, e^{-j2\pi f t_0}$$

TRASF. CONT. DI FOURIER DELLA $\delta(t)$

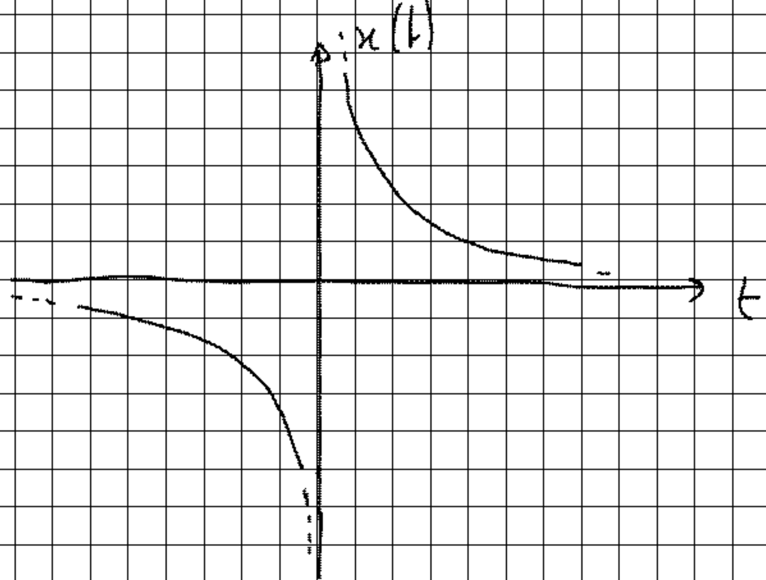
$$\Delta(f) = TCF[\delta(t)]$$

$$\Delta(f) = \int_{-\infty}^{+\infty} \delta(t) e^{-j2\pi f t} dt = 1$$



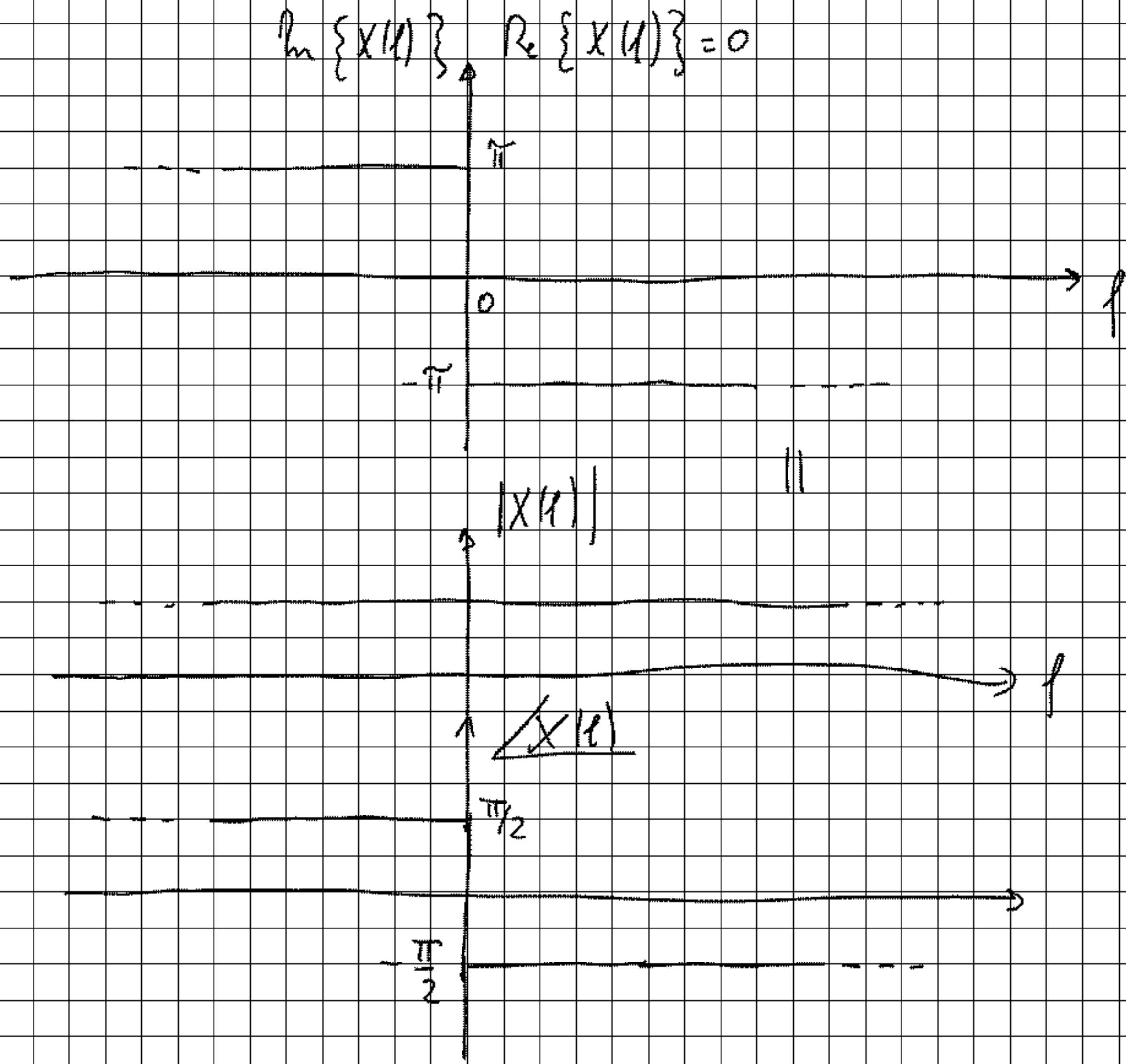
TRASFORMATA DELLA FUNZIONE $\frac{1}{t}$

$$x(t) = \frac{1}{t}$$



$$X(f) = TCF[x(t)] = -j\pi \, sgn(f)$$

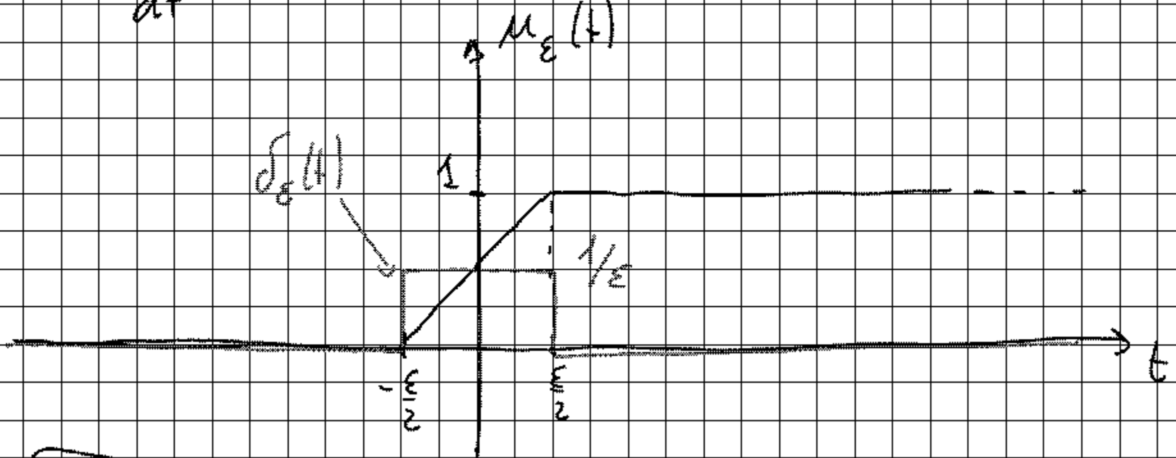
$$sgn(f) = \begin{cases} +1 & f \geq 0 \\ -1 & f < 0 \end{cases} , \quad \begin{cases} +1 & f > 0 \\ 0 & f = 0 \\ -1 & f < 0 \end{cases}$$



RELAZIONI TRA $\delta(t)$ e $u(t)$

$$\delta(t) = \frac{d}{dt}\, u(t)$$

$$\delta_\varepsilon(t) = \frac{d}{dt}\, u_\varepsilon(t)$$



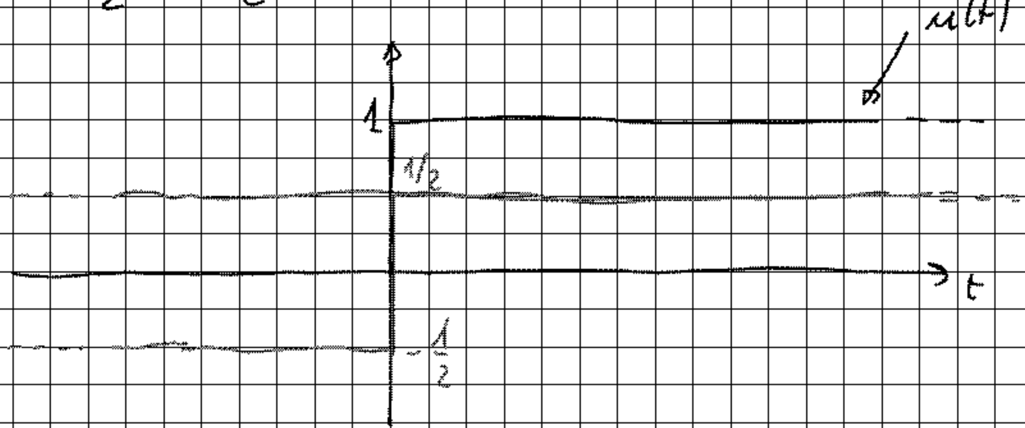
$$\lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \delta(t) = \lim_{\varepsilon \to 0} \frac{d}{dt}\, u_\varepsilon(t) = \frac{d}{dt} \lim_{\varepsilon \to 0} u_\varepsilon(t) = \frac{d}{dt}\, u(t)$$

TCF di $u(t)$

$$U(f) = TCF\left[u(t) \right]$$

$$U(f) = \int_{-\infty}^{+\infty} u(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{+\infty} e^{-j2\pi f t}\, dt$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t)$$



$$U(f) = TCF\left[\frac{1}{2}\right] + TCF\left[\frac{1}{2}\,\mathrm{sgn}(t)\right] = \boxed{\frac{1}{2}\,\delta(f) + \frac{1}{j2\pi f}}$$

$$\Rightarrow TCF\left[1\right]$$

$$TCF\left[\delta(t)\right] = 1$$

$$TCF\left[1\right] = \delta(-f) = \delta(f)$$

$$\Rightarrow TCF\left[\frac{1}{2}\,\mathrm{sgn}(t)\right]$$

$$\Rightarrow TCF\left[\frac{1}{t}\right] = -j\pi\,\mathrm{sgn}(f)$$

$$\Rightarrow -\frac{1}{j2\pi}\, TCF\left[\frac{1}{t}\right] = \frac{1}{2}\,\mathrm{sgn}(f)$$

x dualità $\Rightarrow TCF\left[\frac{1}{2}\,\mathrm{sgn}(t)\right] = -\frac{1}{j2\pi}\cdot\left(\frac{1}{-f}\right) = \frac{1}{j2\pi f}$

TEOREMA DI INTEGRAZIONE COMPLETO

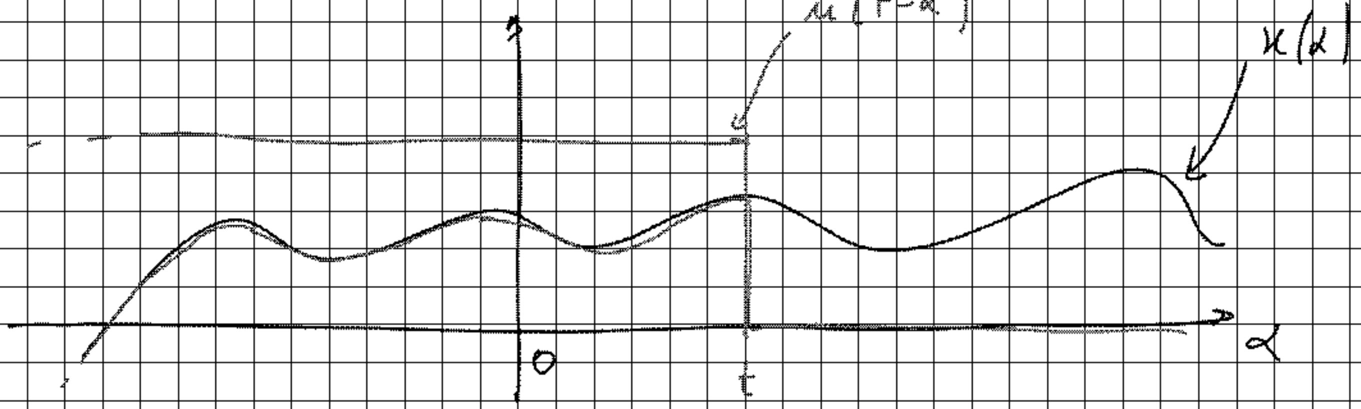
$$x(t) \overset{TCF}{\longleftrightarrow} X(f)$$

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha \quad , \quad \cancel{X(0) = 0 \quad \text{eq} \quad \int_{-\infty}^{+\infty} x(t)\, dt = 0}$$

$$\boxed{Y(f) = \frac{1}{2} X(0)\, \delta(f) + \frac{X(f)}{j2\pi f}}$$

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha = x(t) \otimes u(t) =$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, u(t-\alpha)\, d\alpha$$



$$Y(f) = X(f)\, U(f) = X(f) \cdot \left[\frac{1}{2} \delta(f) + \frac{1}{j2\pi f} \right] =$$

$$= \frac{X(0)}{2} \delta(f) + \frac{X(f)}{j2\pi f}$$

.) TCF di $\delta(t - t_0)$

$$x(t) = \delta(t - t_0)$$

$$X(f) = \int_{-\infty}^{+\infty} \delta(t - t_0)\, e^{-j2\pi f t}\, dt = \boxed{e^{-j2\pi f t_0}}$$

$$TCF^{-1}\left[\delta(f-f_0)\right] = e^{-j2\pi(-t)f_0} = e^{j2\pi f_0 t}$$

$$\int_{-\infty}^{+\infty} \delta(f-f_0)\, e^{j2\pi f t}\, df = e^{j2\pi f_0 t}$$

·) TCF DI UN COSENO

$$x(t) = A\cos(2\pi f_0 t)$$

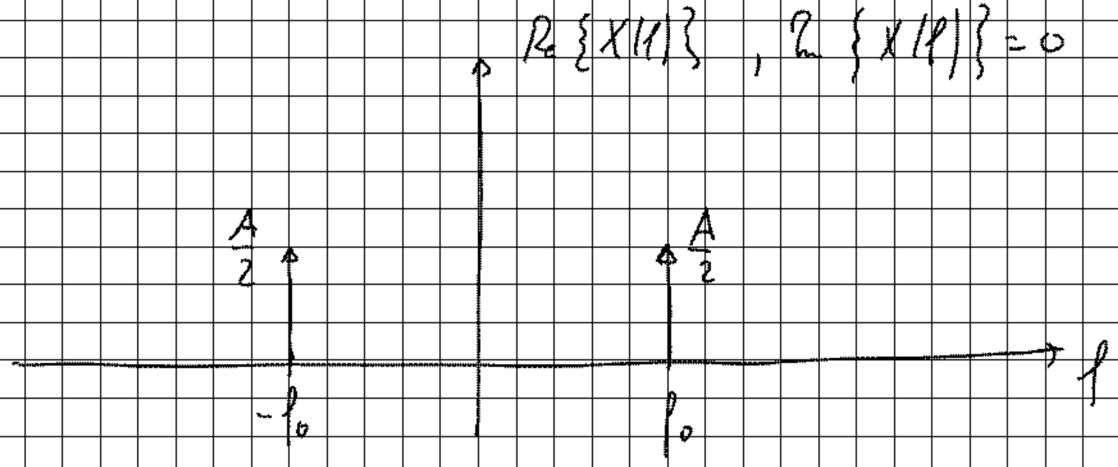
$$X(f) = \int_{-\infty}^{+\infty} A\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= A\int_{-\infty}^{+\infty} \left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right] e^{-j2\pi f t}\, dt =$$

$$= \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t}\, dt + \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi(f+f_0)t}\, dt$$

$$= \frac{A}{2}\int_{-\infty}^{+\infty} 1\cdot e^{-j2\pi f' t}\, dt + \frac{A}{2}\int_{-\infty}^{+\infty} 1\cdot e^{-j2\pi f'' t}\, dt$$

$$= \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0) = X(f)$$



TCF DI UN SENO

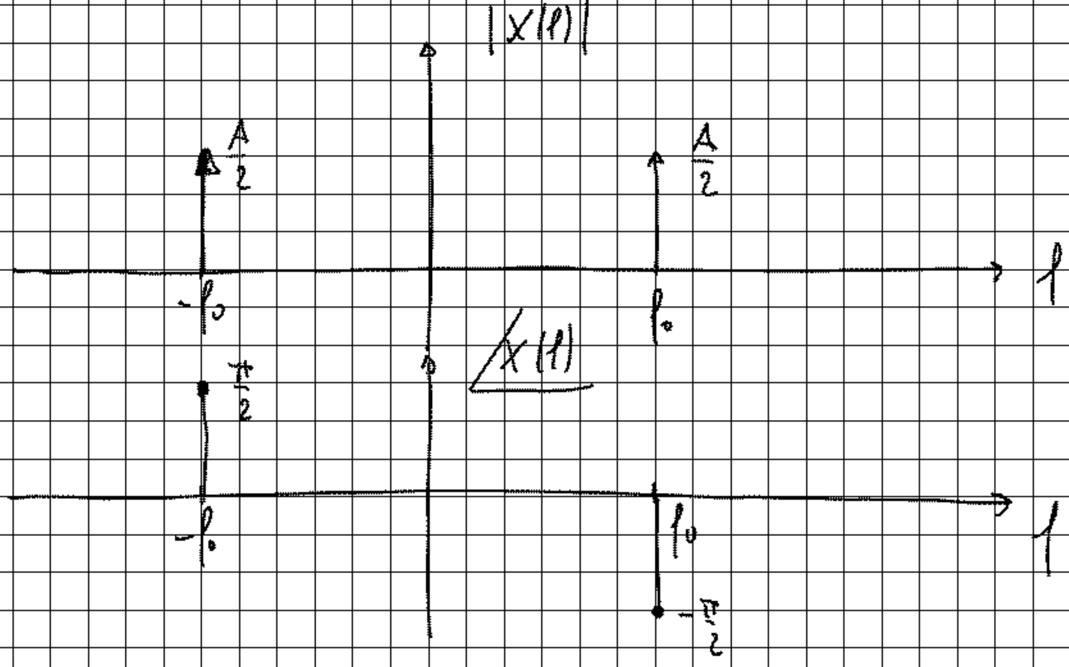
$$x(t) = A \sin(2\pi f_0 t)$$

$$X(f) = \int_{-\infty}^{+\infty} A \sin(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= A \int_{-\infty}^{+\infty} \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\, e^{-j2\pi f t}\, dt =$$

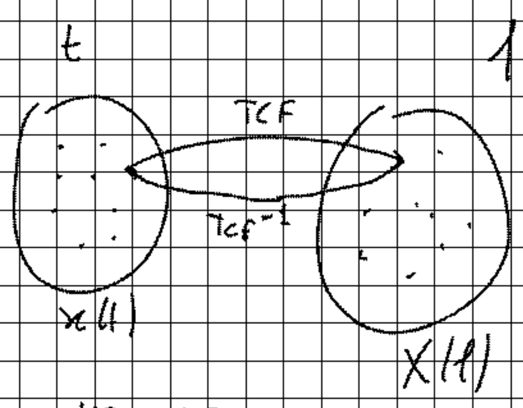
$$= \frac{A}{2j} \int_{-\infty}^{+\infty} e^{-j2\pi (f-f_0)t}\, dt - \frac{A}{2j} \int_{-\infty}^{+\infty} e^{-j2\pi (f+f_0)t}\, dt$$

$$= \frac{A}{2j}\, \delta(f-f_0) - \frac{A}{2j}\, \delta(f+f_0) = \frac{A}{2} e^{-j\frac{\pi}{2}}\, \delta(f-f_0) + \frac{A}{2} e^{j\frac{\pi}{2}}\, \delta(f+f_0)$$



$$x(t) \xrightarrow{TCF} X(f)$$

TCF^{-1} $(ATCF)$



Dim

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df = \underset{(f)}{\int_{-\infty}^{+\infty}} \underset{(\alpha)}{\int_{-\infty}^{+\infty}} x(\alpha)\, e^{-j2\pi f \alpha}\, d\alpha\, e^{j2\pi f t}\, df$$

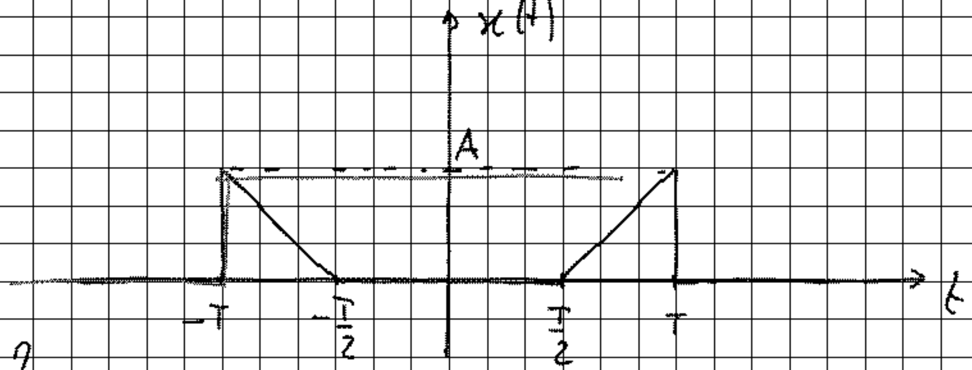
$$x(t) = TCF^{-1}\left[TCF\left[x(t) \right] \right]$$

$$= \int_{-\infty \, (\alpha)}^{+\infty} x(\alpha) \int_{-\infty \, (f)}^{+\infty} e^{-j 2\pi f (-t+\alpha)} \, df \, d\alpha$$

$$= \int_{-\infty \, (\alpha)}^{+\infty} x(\alpha)\, \delta(\alpha - t)\, d\alpha = x(t)$$

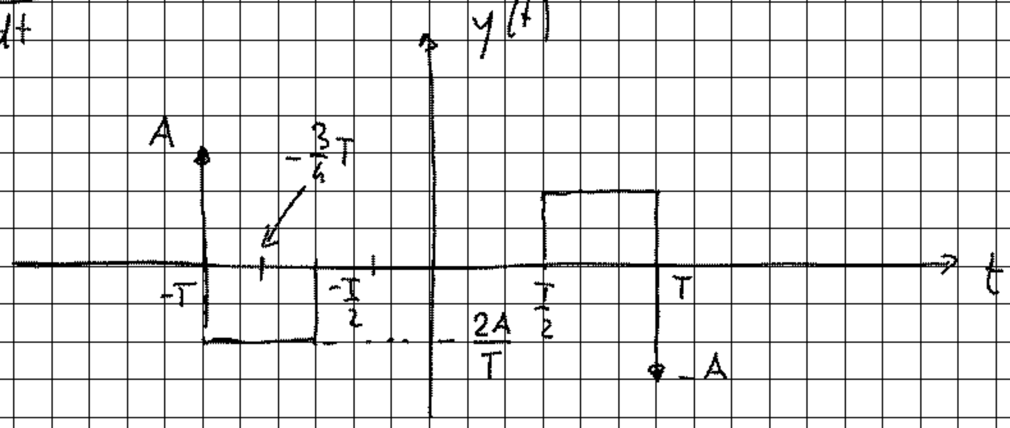
ESEMPI DI UTILIZZO DELLA TCF G

FUNZIONI LINEARI A TRATTI



$X(f) = ?$

$$y(t) = \frac{d}{dt} x(t)$$



$$Y(f) = TCF\left[y(t) \right]$$

$$x(t) = \int_{-\infty}^{t} y(\alpha)\, d\alpha \implies X(f) = \frac{Y(0)}{2}\delta(f) + \frac{Y(f)}{j2\pi f}$$

$$y(t) = A\delta(t+T) - \frac{2A}{T} \mathrm{rect}\left(\frac{t+\frac{3}{4}T}{T/2}\right) + \frac{2A}{T}\mathrm{rect}\left(\frac{t-\frac{3}{4}T}{T/2}\right) - A\delta(t-T)$$

$$Y(f) = A\, e^{j2\pi f T} - \frac{\cancel{2}A}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}\,\mathrm{sinc}\left(\frac{T}{2}f\right) e^{-j2\pi f\left(-\frac{3}{4}T\right)} +$$

$$+ \frac{\cancel{2}A}{\cancel{T}}\frac{\cancel{T}}{\cancel{2}}\,\mathrm{sinc}\left(\frac{T}{2}f\right) e^{-j2\pi f\frac{3}{4}T} - A\, e^{-j2\pi f T} =$$

$$= A\left(e^{j2\pi f T} - e^{-j2\pi f T}\right) - A\,\mathrm{sinc}\left(\frac{T}{2}f\right)\left(e^{j\frac{3}{2}\pi f T} - e^{-j\frac{3}{2}\pi f T}\right)$$

$$Y(0) = 0$$

$$X(f) = \frac{Y(f)}{j2\pi f} = \frac{A}{j2\pi f}\left(e^{j2\pi f T} - e^{-j2\pi f T}\right) +$$

$$- \frac{A}{j2\pi f}\,\mathrm{sinc}\left(\frac{T}{2}f\right)\left(e^{j\frac{3}{2}\pi f T} - e^{-j\frac{3}{2}\pi f T}\right) =$$

$$= \frac{\cancel{j}2AT}{\cancel{j}2\pi f T}\sin(2\pi f T) - \frac{\cancel{j}\cancel{2}A}{\cancel{j}\cancel{2}\pi f}\,\mathrm{sinc}\left(\frac{T}{2}f\right)\sin\left(\frac{3}{2}\pi f T\right) =$$

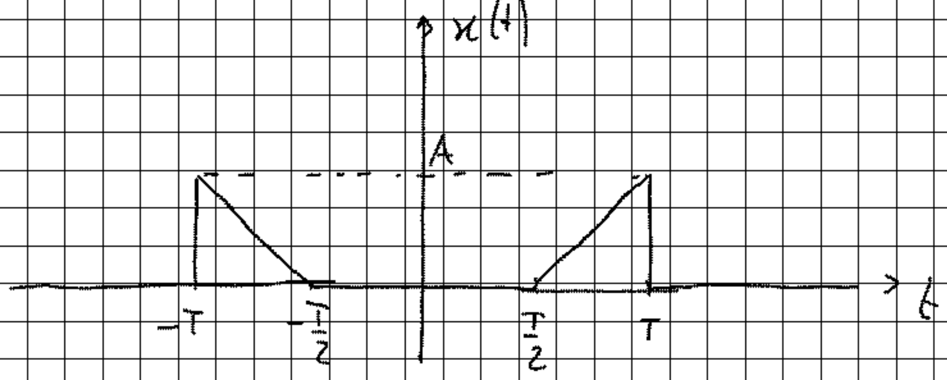
$$= \boxed{2AT\,\mathrm{sinc}(2fT) - \frac{3}{2}AT\,\mathrm{sinc}\left(\frac{3}{2}fT\right)\mathrm{sinc}\left(\frac{T}{2}f\right)}$$

$$\mathrm{rect}\left(\frac{t}{T'}\right) \Leftrightarrow T'\,\mathrm{sinc}(T'f)$$

$$T' = \frac{T}{2} \quad \Rightarrow \quad \mathrm{rect}\left(\frac{t}{T/2}\right) \Leftrightarrow \frac{T}{2}\,\mathrm{sinc}\left(\frac{T}{2}f\right)$$

$$x(t-t_0) \Leftrightarrow X(f)\, e^{-j2\pi f t_0} \quad \Rightarrow \quad t_0 = -\frac{3}{4}T$$

SOLUZIONE ALTERNATIVA



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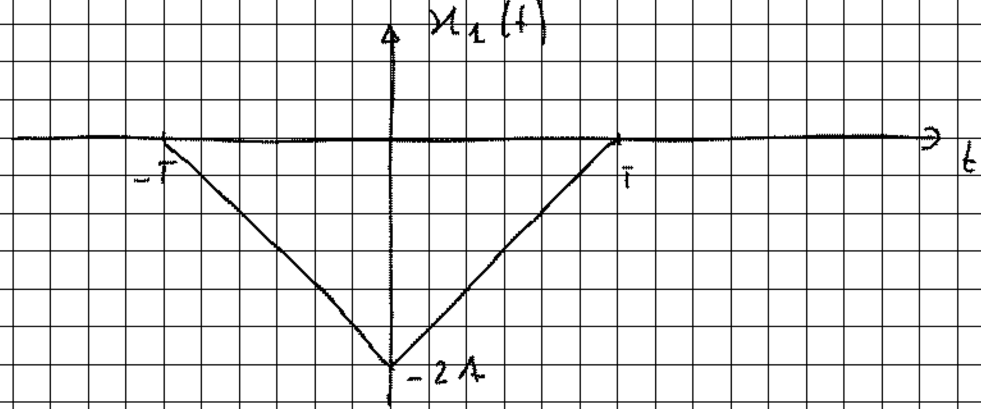

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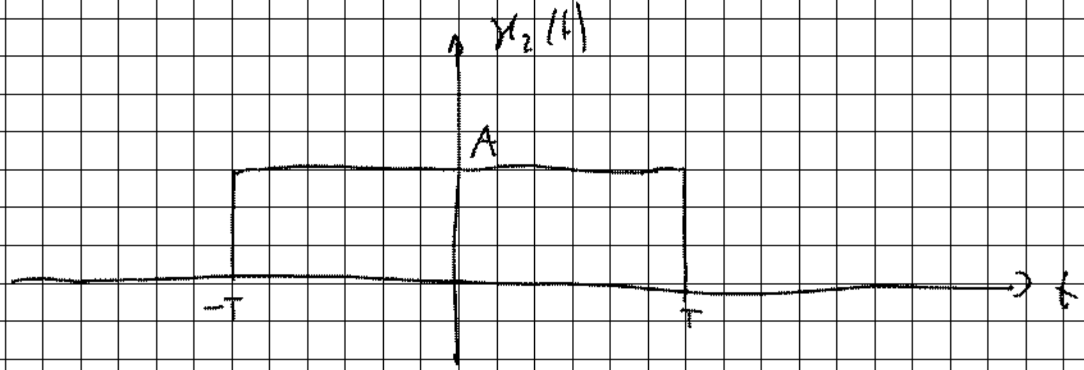

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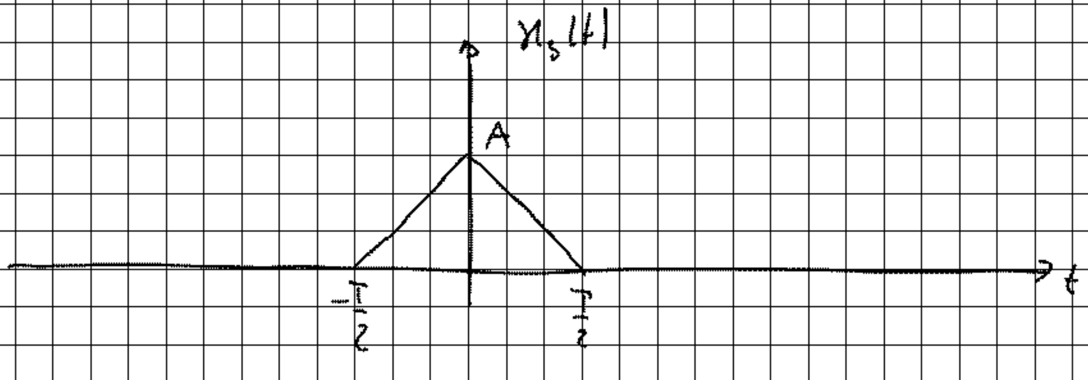
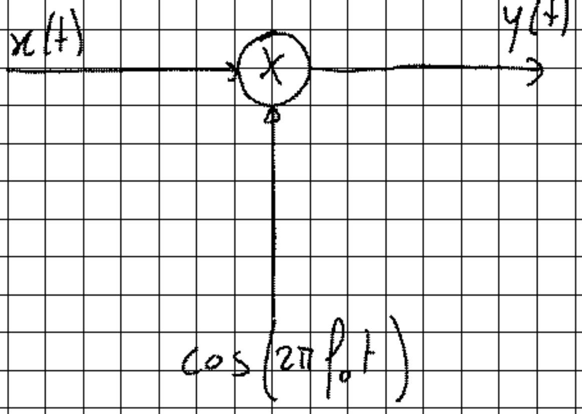


$$X(f) = X_1(f) + X_2(f) + X_3(f)$$

MODULAZIONE DI AMPIEZZA (AM)

Amplitude Modulation



$$x(t) = AB\,\mathrm{sinc}^2(Bt)$$

$$y(t) = x(t)\cos(2\pi f_0 t)$$

$$Y(f) = X(f) \otimes TCF\left[\cos(2\pi f_0 t)\right]$$

$$X(f) = A\left(1 - \frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$TCF\left[\cos(2\pi f_0 t)\right] = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$$

$$Y(f) = X(f) \otimes \left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right] =$$

$$= \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$

$$x(t) = \left(1 - \frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right)$$

$$X(f) = T\,\mathrm{sinc}^2(Tf)$$

$$y(t) = T'\,\mathrm{sinc}^2(T't) \overset{TCF}{\Longrightarrow} Y(f) = \left(1 - \frac{|-f|}{T'}\right)\mathrm{rect}\left(\frac{-f}{2T'}\right)$$

$$T' = B$$

MODULAZIONE CON FASE

$$x(t) = AB \operatorname{sinc}^2(Bt)$$

$$y(t) = x(t) \cos(2\pi f_0 t + \varphi)$$

$$Y(f) = X(f) \otimes \left[\frac{e^{j\varphi}}{2} \delta(f - f_0) + \frac{e^{-j\varphi}}{2} \delta(f + f_0) \right]$$

$$= \frac{e^{j\varphi}}{2} X(f - f_0) + \frac{e^{-j\varphi}}{2} X(f + f_0)$$

$$\varphi = -\frac{\pi}{2} \Rightarrow \cos \Rightarrow \sin$$

$$\frac{e^{-j\frac{\pi}{2}}}{2} \Rightarrow \frac{1}{2j} \quad , \quad \frac{e^{j\frac{\pi}{2}}}{2} = \frac{j}{2} = -\frac{1}{j2}$$

SOMMA DI SINUSOIDI

$$TSF\left[x_1(t) + x_2(t) \right] = TSF\left[x_1(t) \right] + TSF\left[x_2(t) \right]$$

$$x(t) = A_1 \cos(2\pi f_1 t) + A_2 \cos(2\pi f_2 t) \quad , \quad f_1, f_2 \in \mathbb{R}$$

$$X(f) = \frac{A_1}{2} \left[\delta(f - f_1) + \delta(f + f_1) \right] + \frac{A_2}{2} \left[\delta(f - f_2) + \delta(f + f_2) \right]$$

RELAZIONI TRA TSF e TCF

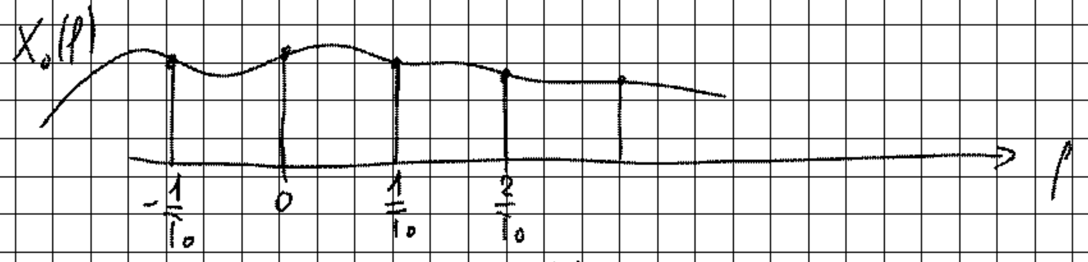
Periodocizzazione di un segnale

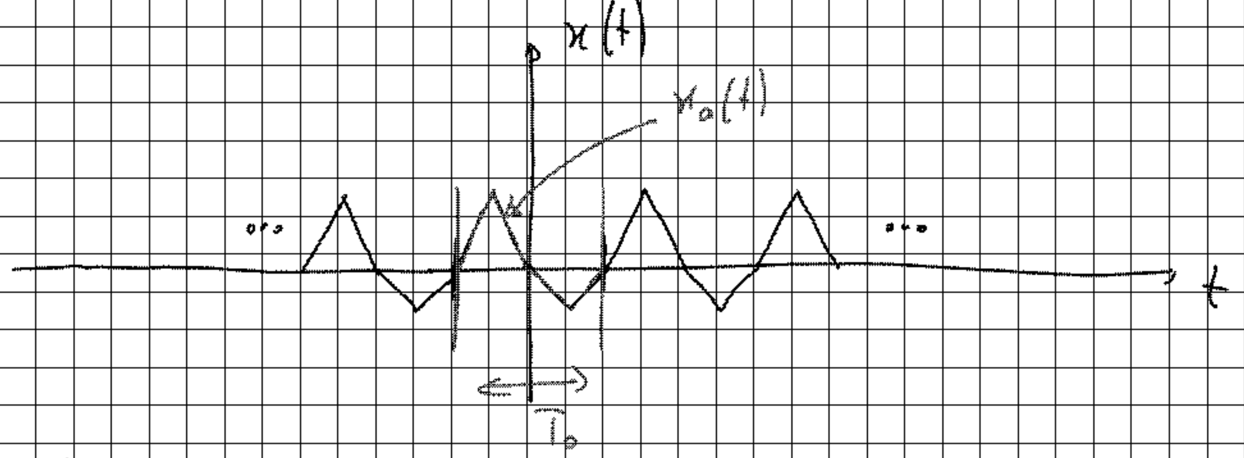
$x_0(t)$ aperiodico

$$x(t) = \sum_{n=-\infty}^{+\infty} x_0(t - nT_0)$$

$$\boxed{TSF\left[x(t)\right] = X_n = \frac{1}{T_0} X_0\left(\frac{n}{T_0}\right)}$$

RELAZIONE TRA TSF e TCF





1) $X_0(f)$

2) $\frac{1}{T_0} X_0\left(\frac{n}{T_0}\right) = X_n$

$f_0 \triangleq \frac{1}{T_0}$

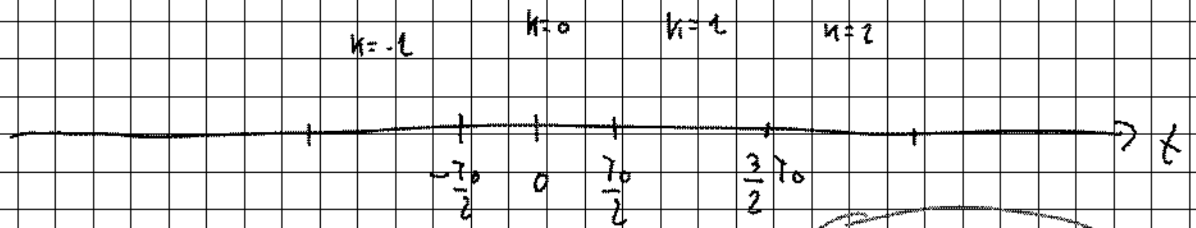
Dim

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0}\sum_{k=-\infty}^{+\infty}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_0(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt = \quad \left(t - kT_0 = t'\right)$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2}+kT_0}^{\frac{T_0}{2}+kT_0} x_0(t')\, e^{-j2\pi n f_0 (t'+kT_0)}\, dt'$$



$$= \frac{1}{T_0} \underbrace{\int_{-\infty}^{+\infty} x_0(t')\, e^{-j2\pi n f_0 t'}\, dt'}_{X_0\left(\frac{n}{T_0}\right)} \; e^{-j2\pi n k \cancel{f_0 T_0}} = 1$$

$$=$$

$$= \frac{1}{T_0} X_0\left(\frac{n}{T_0}\right)$$

FORMULE DI POISSON

I^a FORMULA

$$\boxed{\sum_{n=-\infty}^{+\infty} x(t-nT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X\left(\frac{n}{T_0}\right) e^{j2\pi n \frac{t}{T_0}}}$$

Dim.

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t-nT_0)$$

$$Y_n = TSF\left[y(t)\right] \Rightarrow y(t) = \sum_{n=-\infty}^{+\infty} Y_n\, e^{j2\pi n f_0 t}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_0} X\left(\frac{n}{T_0}\right) e^{j2\pi n \frac{t}{T_0}} = \sum_{n=-\infty}^{+\infty} x(t-nT_0)$$

II^a FORMULA

$$\boxed{\sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f n T_0} = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X\left(f-\frac{n}{T_0}\right)}$$

Dim a valle della definizione della TFS

APPL. DELLA I^a F. d. P. ALLA $\delta(t)$

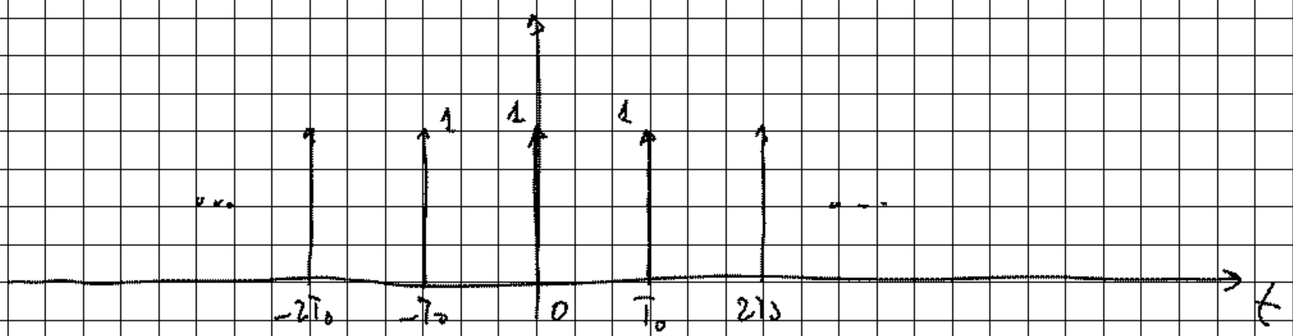
$$\sum_{n=-\infty}^{+\infty} \delta(t-nT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} e^{j2\pi n \frac{t}{T_0}}$$

APPL. DELLA II^a F. d. P. ALLA $\delta(f)$

$$\frac{1}{T_0}\sum_{n=-\infty}^{+\infty} \delta\left(f-\frac{n}{T_0}\right) = \sum_{n=-\infty}^{+\infty} e^{-j2\pi f n T_0}$$

TCF DI UN TRENO DI DELTA DI DIRAC

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT_0)$$

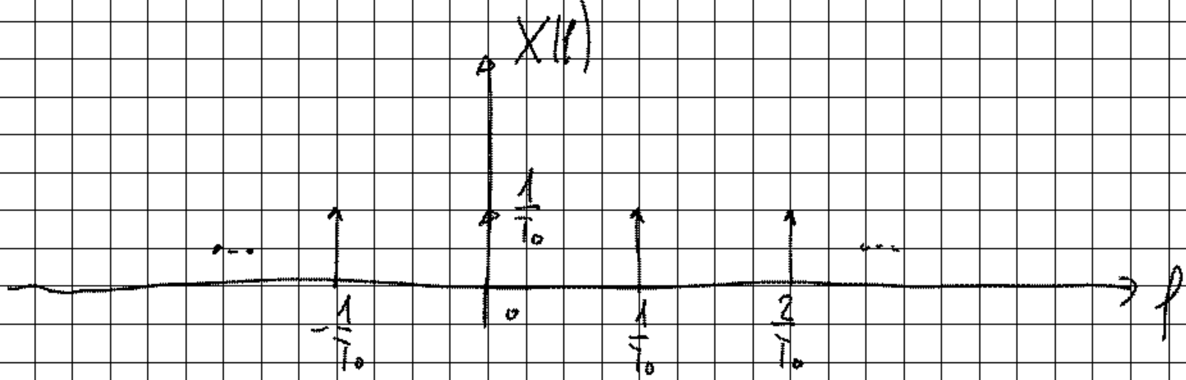


$$X(f) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(t-nT_0)\, e^{-j2\pi f t}\, dt =$$

$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(t-nT_0)\, e^{-j2\pi f t}\, dt =$$

$$= \sum_{n=-\infty}^{+\infty} e^{-j2\pi f n T_0} = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_0}\right)$$

$$\sum_{n=-\infty}^{+\infty} \delta(t - nT_0) \overset{TCF}{\Longleftrightarrow} \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_0}\right)$$



TCF DI SEGNALI PERIODOCIZZATI

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

$$Y(f) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(t - nT_0)\, e^{-j2\pi f t}\, dt =$$

$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(t - nT_0)\, e^{-j2\pi f t}\, dt = \qquad (t - nT_0 = t')$$

$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f (t' + nT_0)}\, dt' =$$

$$= \sum_{n=-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt'}_{X(f)}\; e^{-j2\pi f n T_0} =$$

$$= X(f) \sum_{n=-\infty}^{+\infty} e^{-j2\pi f n T_0} = X(f) \sum_{n=-\infty}^{+\infty} \frac{1}{T_0} \delta\left(f - \frac{n}{T_0}\right)$$

$$= \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X\left(\frac{n}{T_0}\right) \delta\left(f - \frac{n}{T_0}\right) = Y(f)$$

$$= \sum_{n=-\infty}^{+\infty} Y_n\, \delta\left(f - \frac{n}{T_0}\right) = Y(f)$$

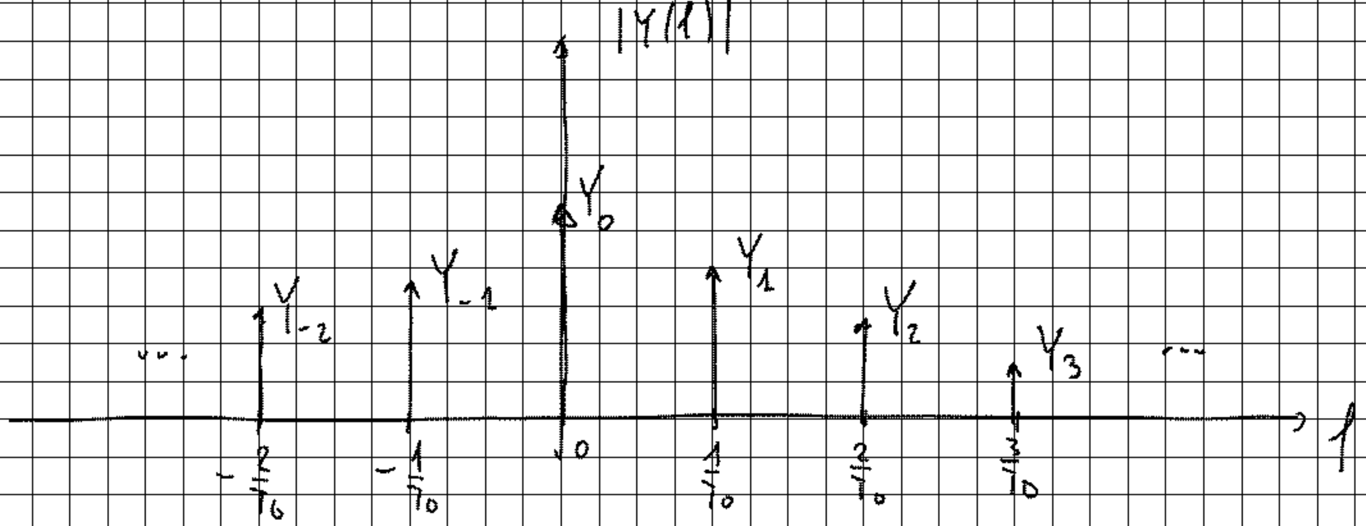
$$y(t) \Leftrightarrow \boxed{Y(f) = \sum_{n=-\infty}^{+\infty} Y_n \, \delta\left(f - \frac{n}{T_0}\right)}$$

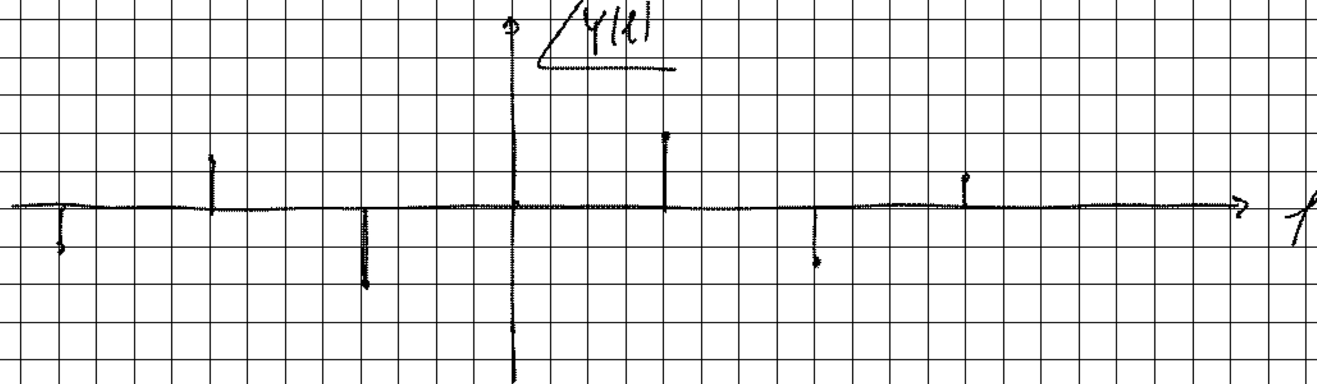
$$\boxed{Y_n = \frac{1}{T_0} Y_0\left(\frac{n}{T_0}\right)}$$

PROCEDIMENTO

$$\boxed{y_0(t) : \sum_{n=-\infty}^{+\infty} y_0(t - nT_0) = y(t)}$$

$$\boxed{Y_0(f) = TCF\left[y_0(t) \right]}$$





TEOREMA DI PARSEVAL PER SEGNALI PERIODICI

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \sum_{n=-\infty}^{+\infty} X_n Y_n^*$$

Dim

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi n f_0 t} \quad y^*(t)\, dt$$

$$= \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X_n \left[\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi n f_0 t}\, dt \right]^* \Rightarrow Y_n^*$$

$$= \sum_{n=-\infty}^{+\infty} X_n Y_n^*$$

POTENZA DI UN SEGNALE PERIODICO

$$P_x = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\, dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, x^*(t)\, dt$$

$$= \sum_{n=-\infty}^{+\infty} X_n X_n^* = \sum_{n=-\infty}^{+\infty} |X_n|^2$$

TEOREMA DI PARSEVAL PER SEGNALI PERIODICI TRAMITE TCF

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \sum_{n=-\infty}^{+\infty} X_n Y_n^* =$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_0} X_0\left(\frac{n}{T_0}\right) \cdot \frac{1}{T_0} Y_0^*\left(\frac{n}{T_0}\right) = \frac{1}{T_0^2} \sum_{n=-\infty}^{+\infty} X_0\left(\frac{n}{T_0}\right) Y_0^*\left(\frac{n}{T_0}\right)$$

$$x(t) = \sum_{n=-\infty}^{+\infty} x_0(t - nT_0) \quad , \quad x_0(t) \overset{TCF}{\Leftrightarrow} X_0(f)$$

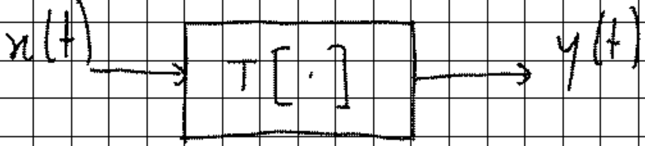
$$y(t) = \sum_{n=-\infty}^{+\infty} y_0(t - nT_0) \quad , \quad y_0(t) \overset{TCF}{\Leftrightarrow} Y_0(f)$$

$$y(t) \overset{TCF}{\Rightarrow} Y(f) \quad , \quad y(t) \text{ è periodico}$$

$$Y(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} Y_0\left(\frac{n}{T_0}\right) \delta\left(f - \frac{n}{T_0}\right)$$

POTENZA $\Rightarrow$ $P_y = \sum_{n=-\infty}^{+\infty} \frac{1}{T_0^2} \left| Y_0\left(\frac{n}{T_0}\right) \right|^2$

SISTEMI



·) LINEARITÀ

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

$$y(t) = T[x(t)] = a_1 y_1(t) + a_2 y_2(t)$$

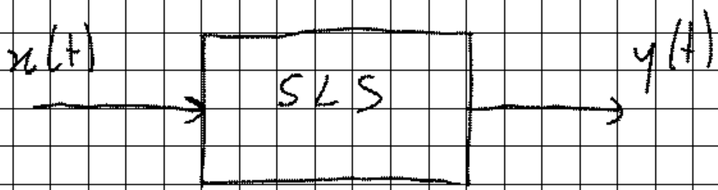
$$y_1(t) = T[x_1(t)] \quad , \quad y_2(t) = T[x_2(t)]$$

·) STAZIONARIETÀ

$$y(t) = T[x(t)]$$

$$y(t-t_0) = T[x(t-t_0)]$$

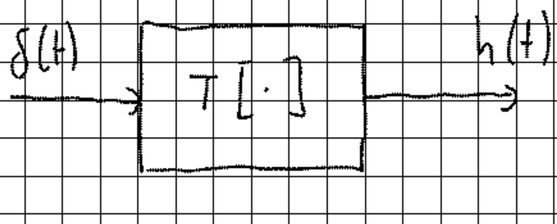
SISTEMI LINEARI e STAZIONARI (SLS)



$$y(t) = x(t) \otimes h(t)$$

$h(t) \triangleq$ risposta impulsiva del sistema

$$h(t) \triangleq T[\delta(t)]$$



$$y(t) = x(t) \otimes h(t)$$

Dim

$$y(t) = T[x(t)] = T[x(t) \otimes \delta(t)] =$$

$$= T\left[\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] =$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau =$$

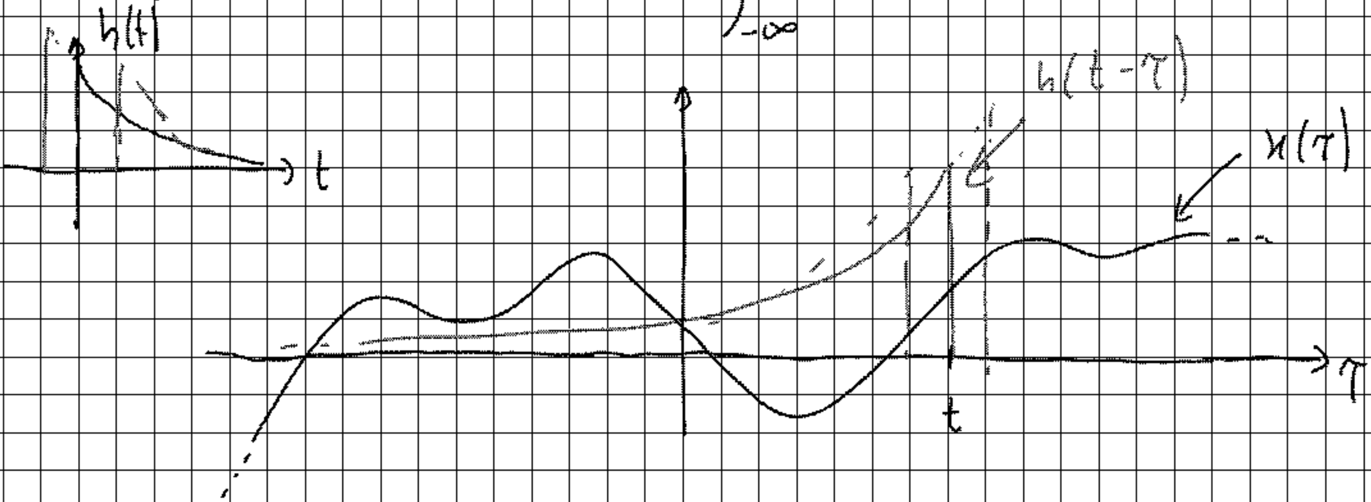
$$= \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) \otimes h(t)$$

) CAUSALITÀ

$$y(t) = T[x(\alpha);\ \alpha \le t]$$

Per i SLS $\Rightarrow$ $h(t) = 0$, $t < 0$

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$



$$y(t) = \int_{-\infty}^{t} x(\tau)\,h(t-\tau)\,d\tau$$

·) STABILITÀ BIBO (Bounded Input Bounded Output)

se $\max\{|x(t)|\} = K < \infty$

allora $\max\{|y(t)|\} = M < \infty$

Per i sistemi SLS

$$\text{se} \quad \int_{-\infty}^{+\infty} |h(t)|\, dt = M < \infty \Rightarrow \max\{|y(t)|\} = N < \infty$$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \right| \leq \int_{-\infty}^{+\infty} |x(\tau)|\,|h(t-\tau)|\, d\tau \leq$$

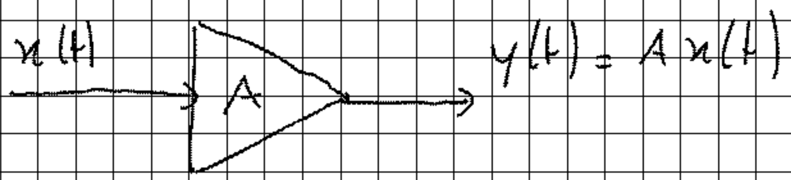
$$\leq \int_{-\infty}^{+\infty} K\, |h(t-\tau)|\, d\tau = K \int_{-\infty}^{+\infty} |h(\tau')|\, d\tau' =$$

$$= K M \quad \Rightarrow \quad |y(t)| \leq N$$

·) MEMORIA

SISTEMA SENZA MEMORIA (ISTANTANEO)

$$y(t) = T[x(\alpha);\ \alpha = t]$$



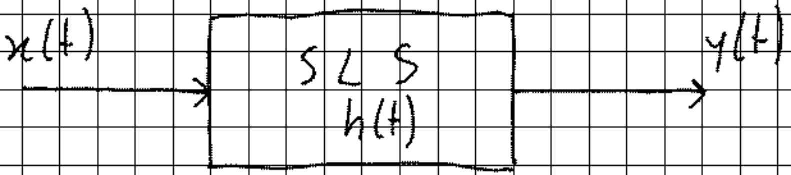
N.B. UN SIST. ISTANT. È ANCHE CAUSALE

·) INVERTIBILITÀ

se $y(t) = T[x(t)]$

e se $x(t) = T^{-1}[y(t)]$ $\Rightarrow$ SIST È INVERT.

RISPOSTA IN FREQUENZA



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$x(t) = e^{j2\pi f t}$$

$$y(t) = \int_{-\infty}^{+\infty} e^{j2\pi f \tau}\, h(t-\tau)\, d\tau = \quad (t-\tau = \tau')$$

$$= \int_{-\infty}^{+\infty} e^{j2\pi f (t-\tau')}\, h(\tau')\, d\tau' =$$

$$= e^{j2\pi f t} \underbrace{\int_{-\infty}^{+\infty} h(\tau')\, e^{-j2\pi f \tau'}\, d\tau'}_{H(f)} =$$

$$= e^{j2\pi f t}\, H(f) = x(t)\, H(f) \Big|_{x(t) = e^{j2\pi f t}} = y(t)$$

1) $H(f)$: $x(t) = e^{j2\pi f t} \Rightarrow y(t) = x(t)\, H(f)$

$$\Rightarrow \boxed{H(f) = \frac{y(t)}{x(t)} \Big|_{x(t) = e^{j2\pi f t}}}$$

2) $\boxed{H(f) = TCF\left[h(t)\right]}$

3) $\boxed{H(f) = \frac{Y(f)}{X(f)}}$, $X(f) = TCF\left[x(t)\right]$, $Y(f) = TCF\left[y(t)\right]$

Dim

$$y(t) = x(t) \otimes h(t)$$

dopo TCF ...

$$\boxed{Y(f) = X(f)\, H(f)}$$

$$H(f) = \frac{Y(f)}{X(f)}$$

RISPOSTA IN AMPIEZZA DI UN SLS

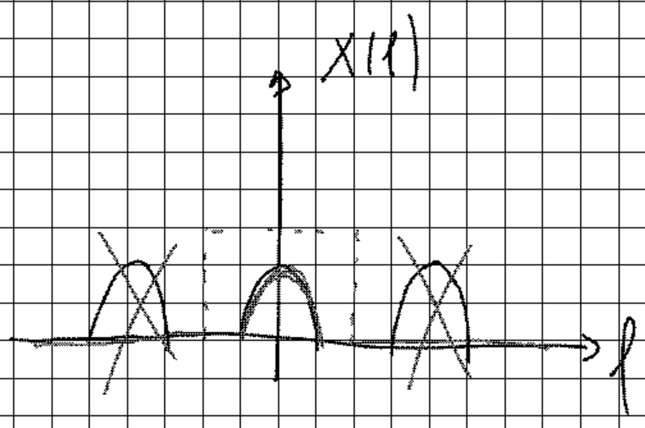
$$A(f) = |H(f)|$$

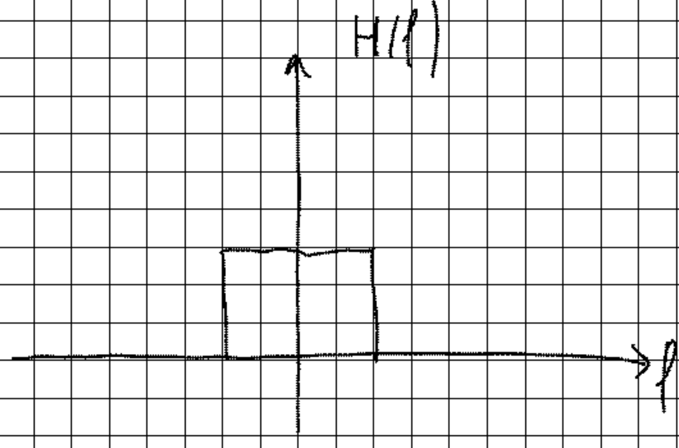
RISPOSTA IN FASE DI UN SLS

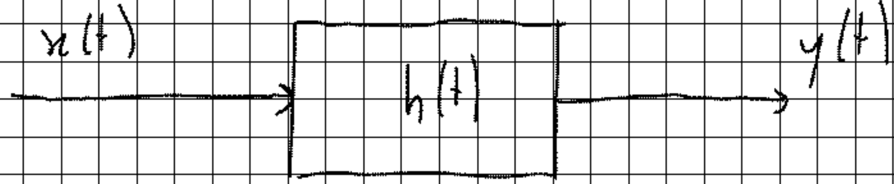
$$\varphi(f) = \angle H(f)$$

$$|Y(f)| = |X(f)| \cdot |H(f)|$$

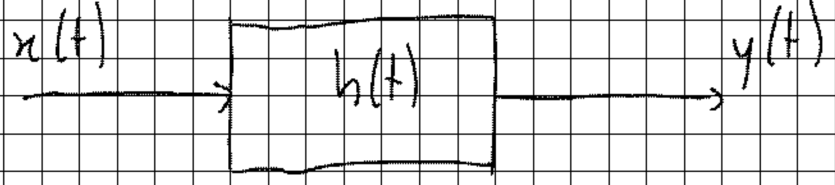
$$\angle Y(f) = \angle X(f) + \angle H(f)$$







RISPOSTA AL GRADINO



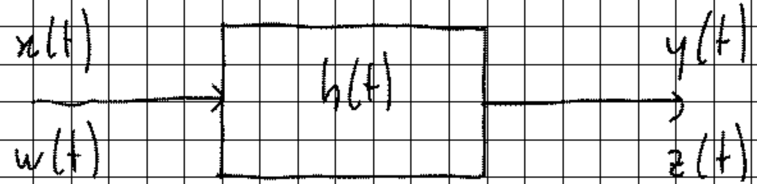
$$x(t) = u(t)$$

$$y(t) = u(t) \otimes h(t) = \int_{-\infty}^{t} h(\alpha)\, d\alpha$$

$$u(t) = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha \quad \Rightarrow \quad g(t) = \int_{-\infty}^{t} h(\alpha)\, d\alpha$$

$$\delta(t) = \frac{d}{dt}\left[u(t)\right] \qquad\qquad h(t) = \frac{d}{dt}\left[g(t)\right]$$

PROPR. DI INTEGRAZIONE PER I SLS



$$y(t) = x(t) \otimes h(t)$$

$$w(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha$$

$$z(t) = w(t) \otimes h(t) = [\underbrace{x(t) \otimes u(t)}_{w(t)}] \otimes h(t) =$$

$$= x(t) \otimes [u(t) \otimes h(t)] =$$

$$= x(t) \otimes [h(t) \otimes u(t)] =$$

$$= [x(t) \otimes h(t)] \otimes u(t) = y(t) \otimes u(t) = z(t)$$

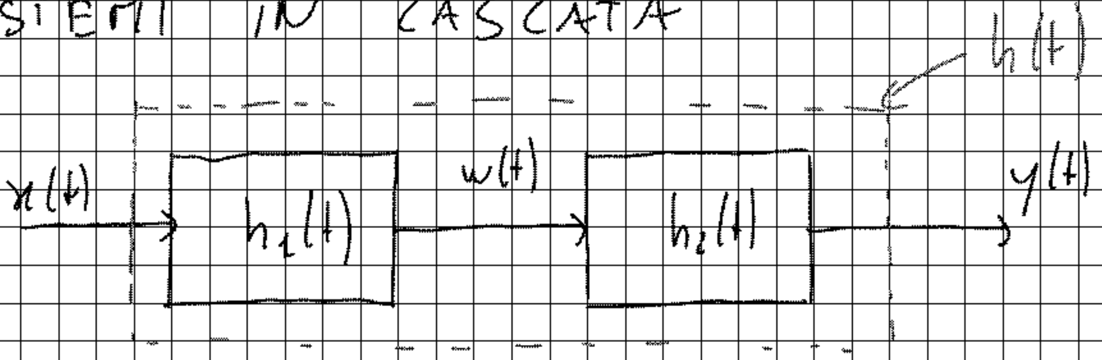
primitiva di $y(t)$ $\downarrow$ $z(t)$

PROPR. DI DERIVAZIONE PER I SLS

$$x(t) = \frac{d}{dt} w(t) \quad \Rightarrow \quad y(t) = \frac{d}{dt} z(t)$$

segue dalla dim. precedente

SISTEMI IN CASCATA



$$y(t) = x(t) \otimes \Big[\overbrace{h_1(t) \otimes h_2(t)}^{h(t)} \Big] =$$

$$= x(t) \otimes h(t) \quad , \quad \boxed{h(t) = h_1(t) \otimes h_2(t)}$$

$$w(t) = x(t) \otimes h_1(t)$$

$$y(t) = w(t) \otimes h_2(t) = \big[x(t) \otimes h_1(t) \big] \otimes h_2(t)$$

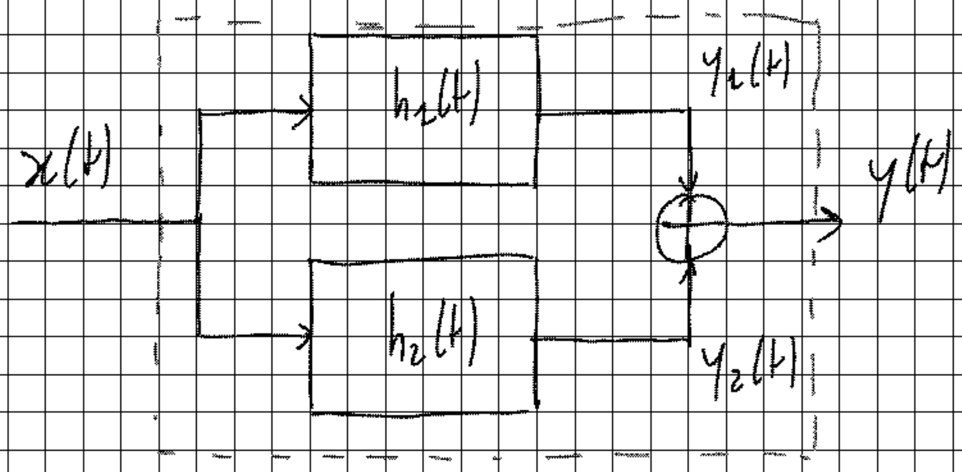
$$= x(t) \otimes \Big[\underbrace{h_1(t) \otimes h_2(t)}_{h(t)} \Big]$$

IN FREQUENZA

$$Y(f) = X(f)\, H_1(f)\, H_2(f) = X(f)\, H(f)$$

$$H(f) \triangleq H_1(f)\, H_2(f) = TCF \big[h_1(t) \otimes h_2(t) \big]$$

SISTEMI IN PARALLELO



$$y(t) = y_1(t) + y_2(t) = x(t) \otimes h_1(t) + x(t) \otimes h_2(t) =$$

$$= x(t) \otimes \underbrace{\left[h_1(t) + h_2(t) \right]}_{h(t)}$$

$$h(t) = h_1(t) + h_2(t)$$

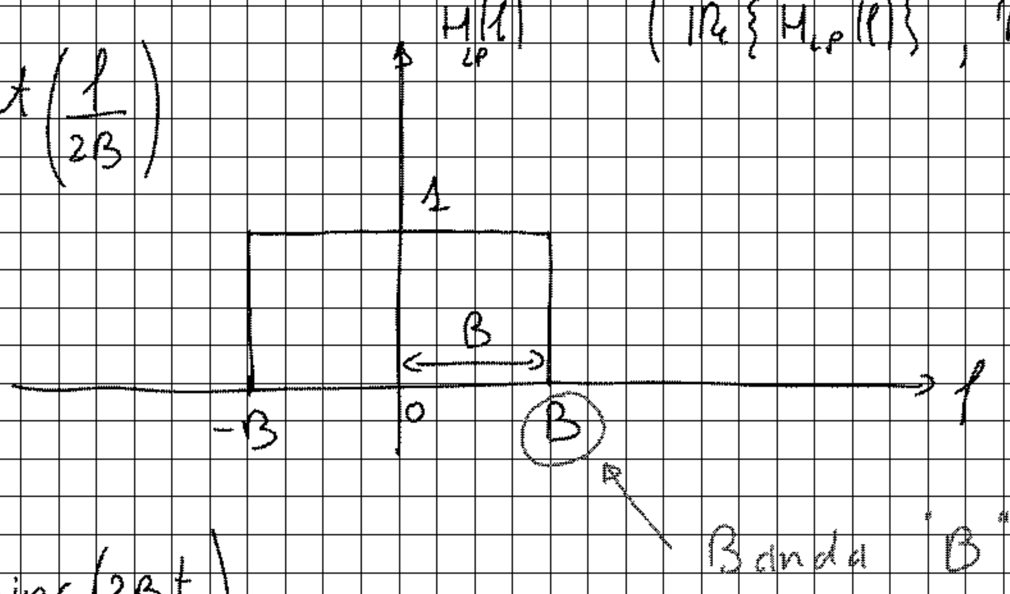
$$H(f) = H_1(f) + H_2(f)$$

FILTRI IDEALI

-) FILTRO PASSA BASSO

$$\underset{\text{(Low Pass)}}{H_{LP}(f)} = \mathrm{rect}\left(\frac{f}{2B}\right)$$

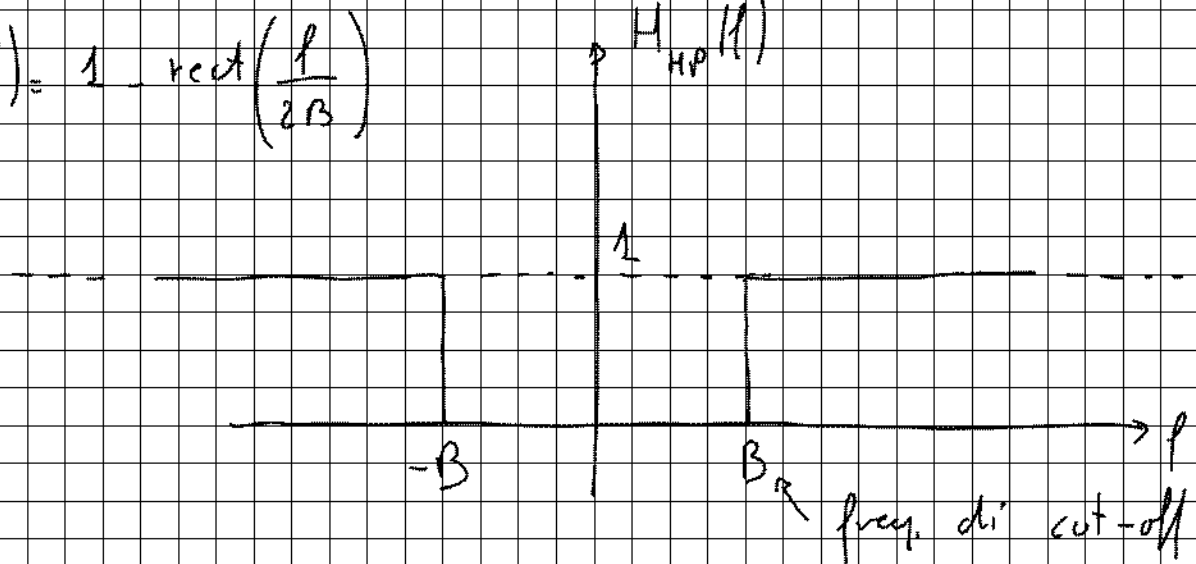
$\left(\mathrm{Re}\{ H_{LP}(f) \},\ \mathrm{Im}\{\cdot\} = 0 \right)$



$$h(t) = 2B\ \mathrm{sinc}(2Bt)$$

·) FILTRO PASSA ALTO

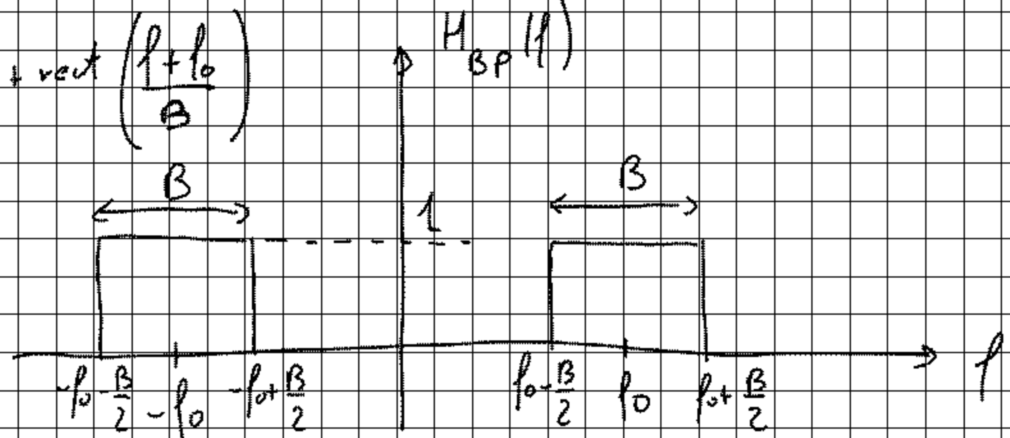
$$H_{HP}(f) = 1 - rect\left(\frac{f}{2B}\right)$$



$$h_{HP}(t) = \delta(f) - 2B\ sinc(2Bt)$$

·) FILTRO PASSA BANDA

$$H_{BP}(f) = rect\left(\frac{f-f_0}{B}\right) + rect\left(\frac{f+f_0}{B}\right)$$

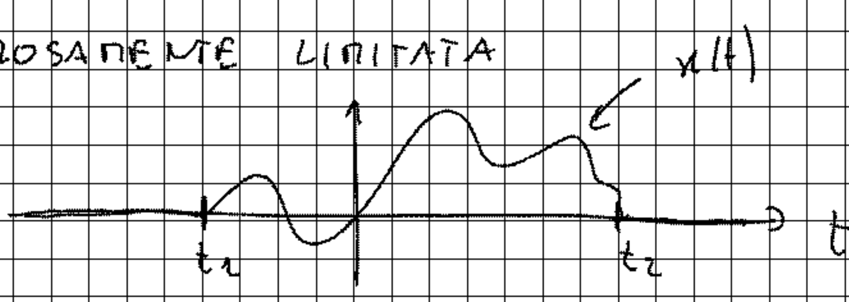


$$h_{BP}(t) = B\ sinc(Bt)\,e^{j2\pi f_0 t} + B\ sinc(Bt)\,e^{-j2\pi f_0 t} =$$

$$= 2B\ sinc(Bt)\cos(2\pi f_0 t)$$

DURATA E BANDA DI UN SEGNALE

·) DURATA RIGOROSAMENTE LIMITATA



$$\begin{cases} x(t) \neq 0 & t_1 \leq t \leq t_2 \\ x(t) = 0 & \text{altrove} \end{cases}$$

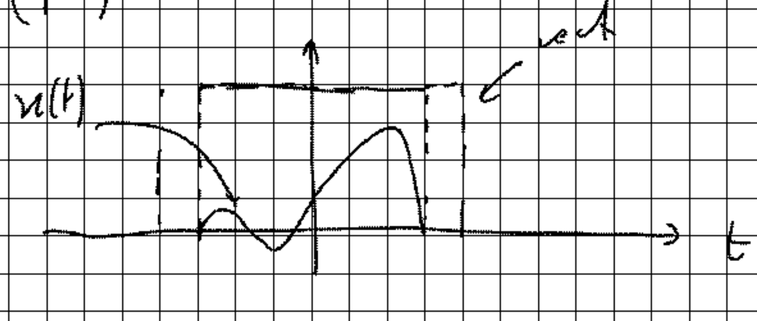
·) BANDA RIGOROSAMENTE LIMITATA

$$x(t): \begin{cases} X(f) \neq 0 & f_1 \leq f \leq f_2 \\ X(f) = 0 & \text{altrove} \end{cases}$$

RELAZIONE TRA DURATA e BANDA

"Se un segnale ha durata rigorosamente limitata la sua banda è infinita e viceversa"

$$x(t) \cdot \text{rect}\left(\frac{t}{T}\right) = x(t)$$



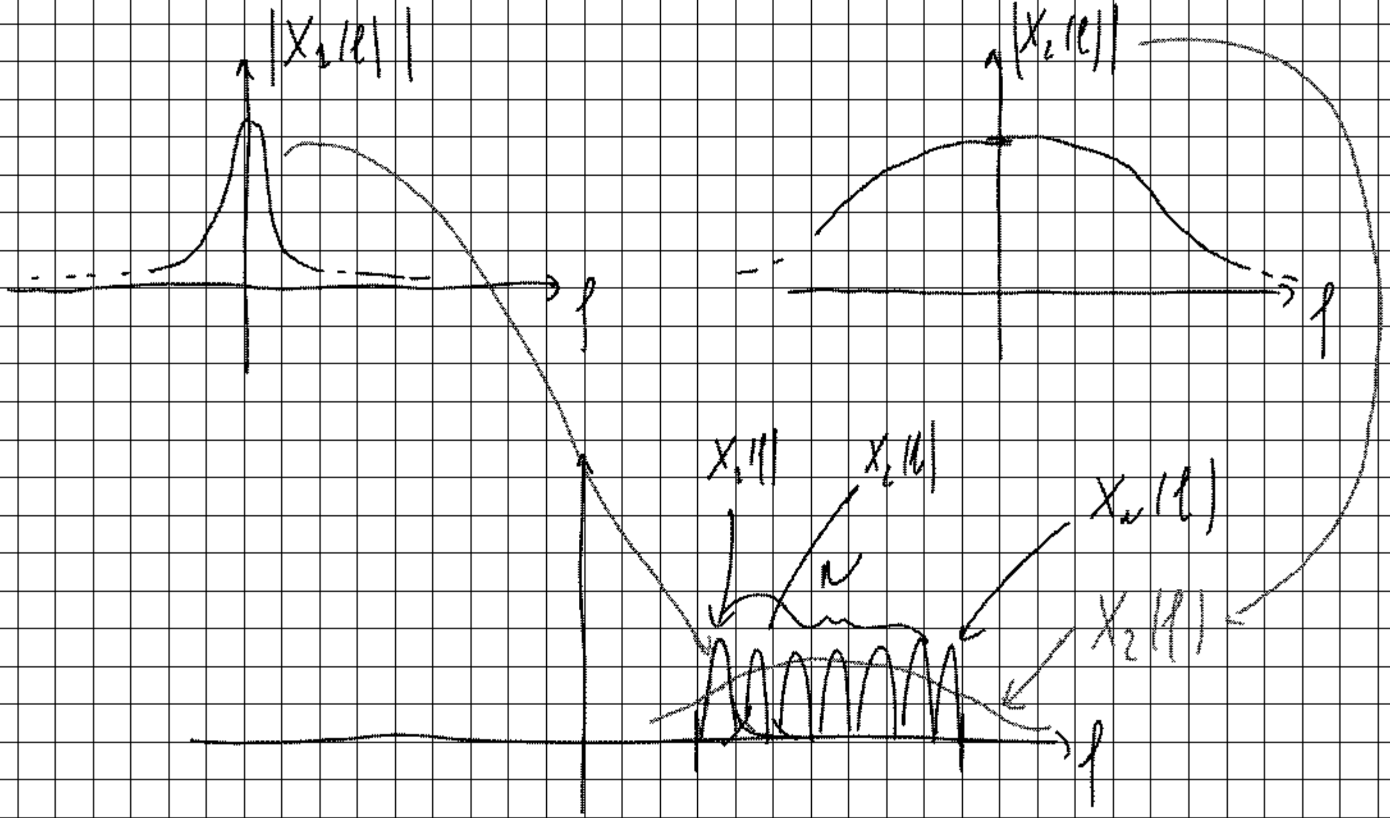
$$X(f) = TCF[x(t)] = TCF[x(t)] \otimes TCF\left[\text{rect}\left(\frac{t}{T}\right)\right] =$$
$$= X(f) \otimes T\,\text{sinc}(Tf)$$

Dim x assurdo:

1) Suppongo $X(f)$ a banda rig. lim.

2) La convoluzione con uno spettro a banda infinita produce uno spettro a banda infinita (nega l'ipotesi)

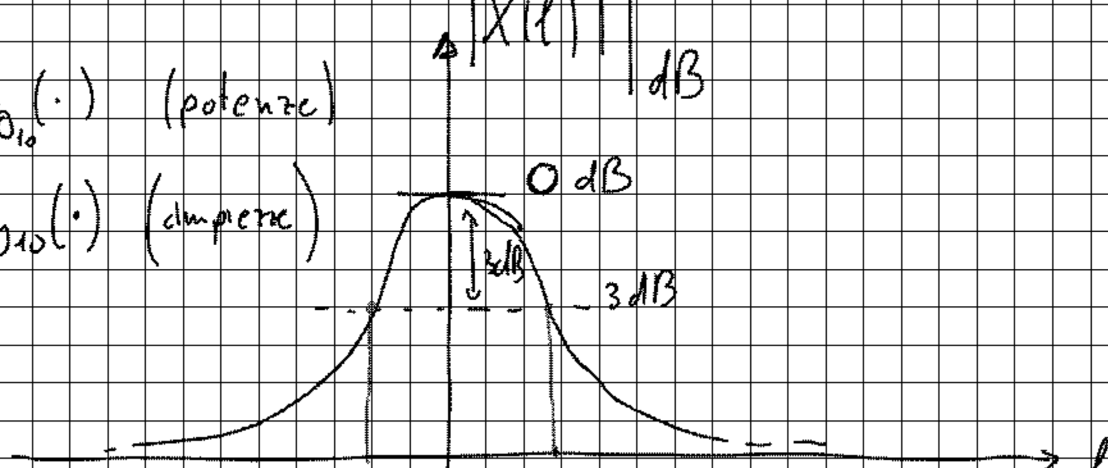
PROBLEMA DELLA DEFINIZIONE DI BANDA



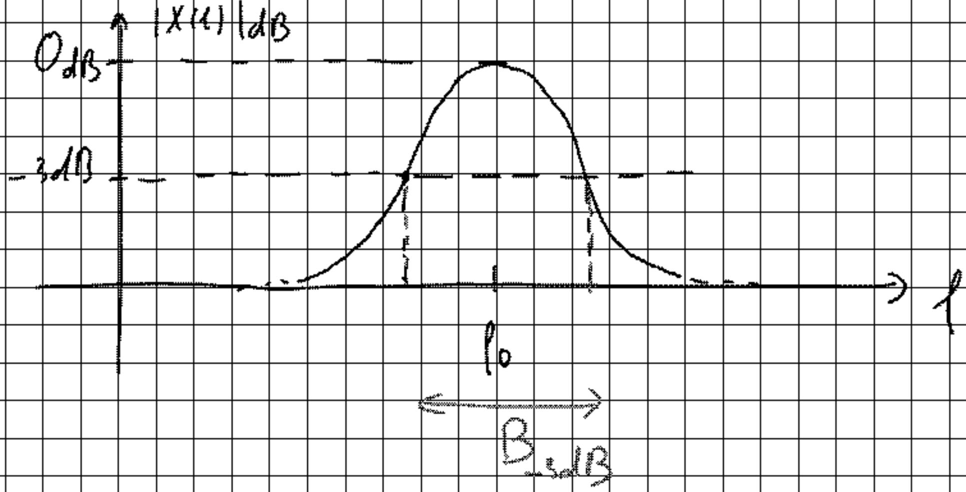
·) BANDA a $-3\,dB$

$dB \triangleq 10 \log_{10}(\cdot)$ (potenze)

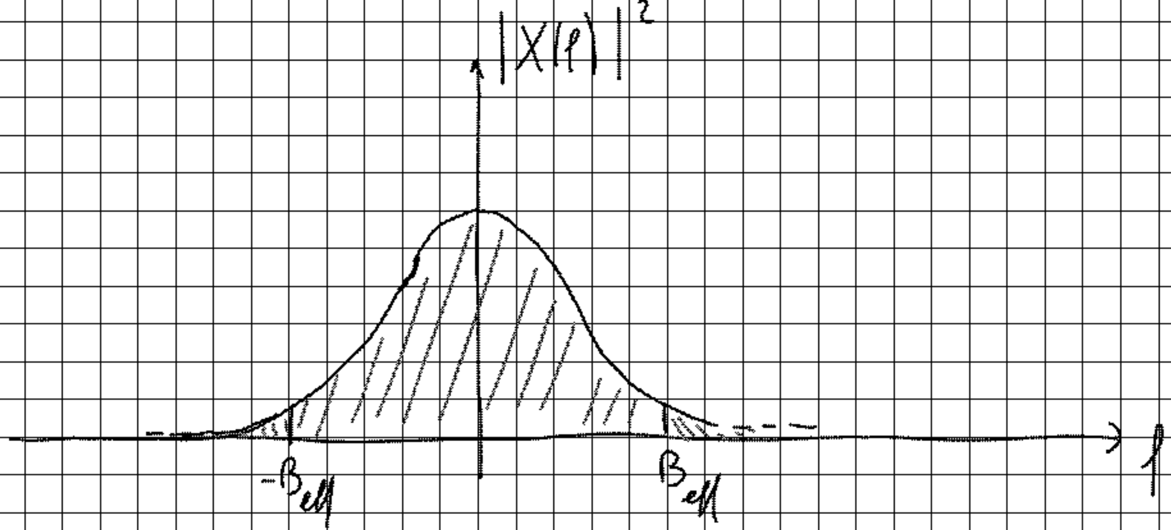
$dB \triangleq 20 \log_{10}(\cdot)$ (ampiezze)



$B_{-3dB} \triangleq f_{MAX} : |X(f)| \geq -3dB \;\; f \leq f_{MAX}$ $\quad (\max |X(f)| = 0\,dB)$



·) BANDA al 99% dell' ENERGIA (BANDA EFFICACE)



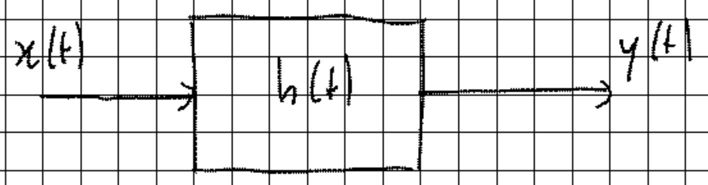
TEOREMA DI PARSEVAL PER SEGNALI APERIODICI

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$

$$B_{eff} : \int_{-B_{eff}}^{B_{eff}} |X(f)|^2 \, df = 0{,}99 \, E_x$$

LE STESSE DEFINIZIONI POSSO ESSERE DATE PER LA DURATA

DISTORSIONI LINEARI



REPLICA "FEDELE" DI UN SEGNALE

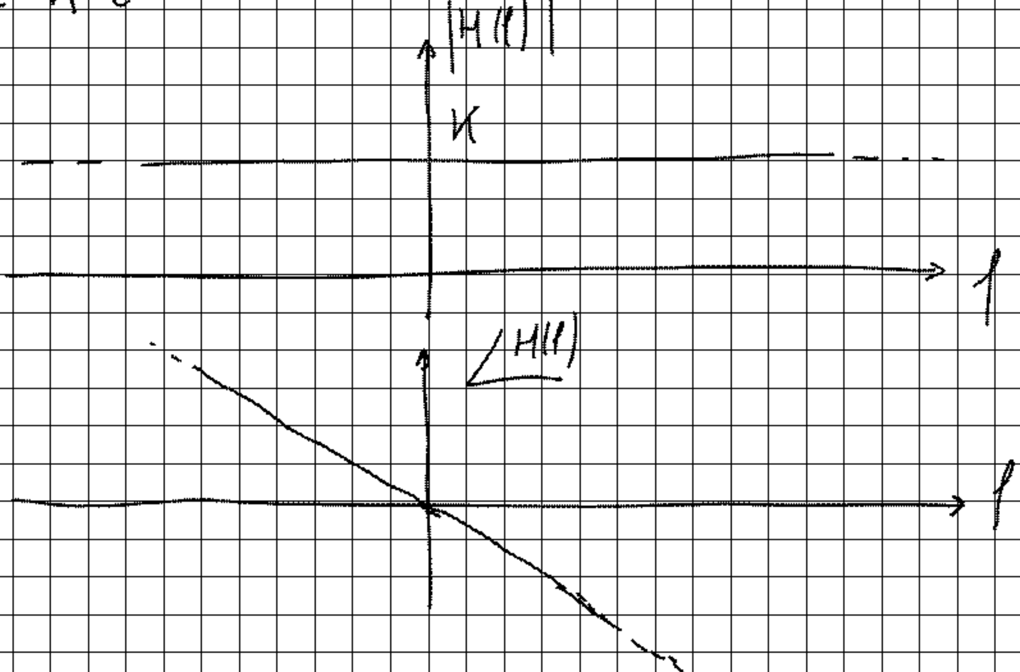
$$y(t) = K\,x(t-t_0)$$

$$y(t) = K\,x(t-t_0)$$

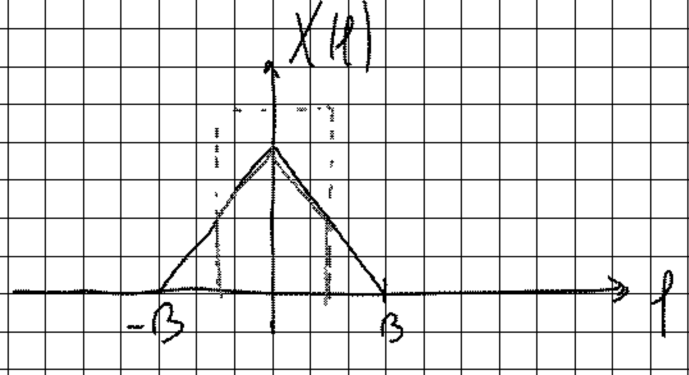
$$y(t) = x(t) \otimes h(t) = x(t) \otimes K\,\delta(t-t_0) = K\,x(t-t_0)$$

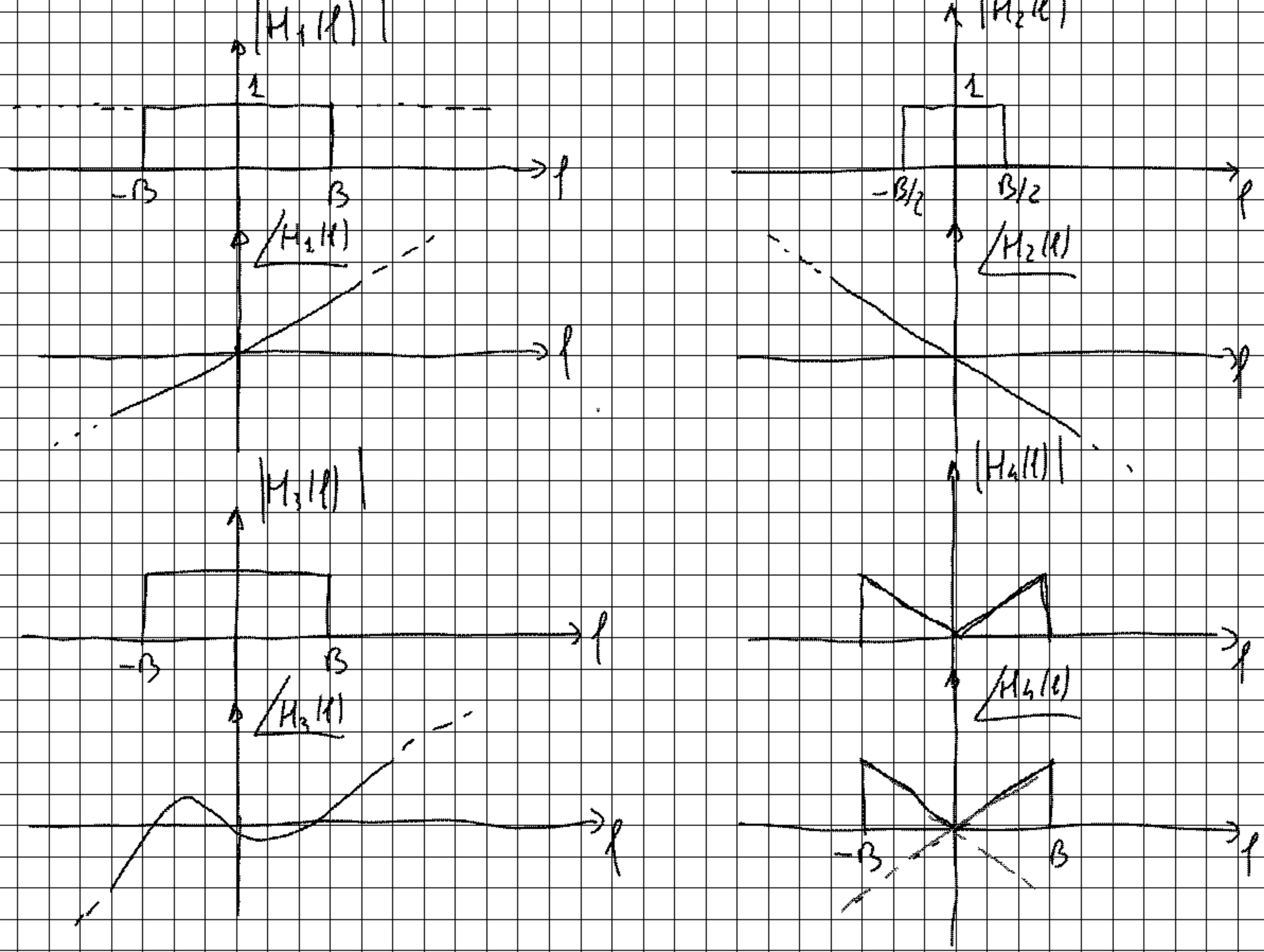
$$h(t) = K\,\delta(t-t_0)$$

$$H(f) = K\,e^{-j2\pi f t_0}$$



ESEMPIO





1) $|X(f)|\,|H_1(f)| = |X(f)|$ NON HO DIST. DI AMP

$\angle X(f) + \angle H_1(f) = \angle Y(f) = 0 + 2\pi f t_0 \Rightarrow$

$y(t) = x(t + t_0)$

$Y(f) = X(f)\, e^{j2\pi f t_0} = X(f)\, e^{-j2\pi f(-t_0)}$

2) $|Y(f)| \neq |X(f)| \Rightarrow Y(f) \neq K|X(f)|$

$\angle Y(f) = \angle X(f) + 2\pi f t_0$

FILTRAGGIO DI SEGNALI PERIODICI

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$x(t)$ e' periodico di T_0

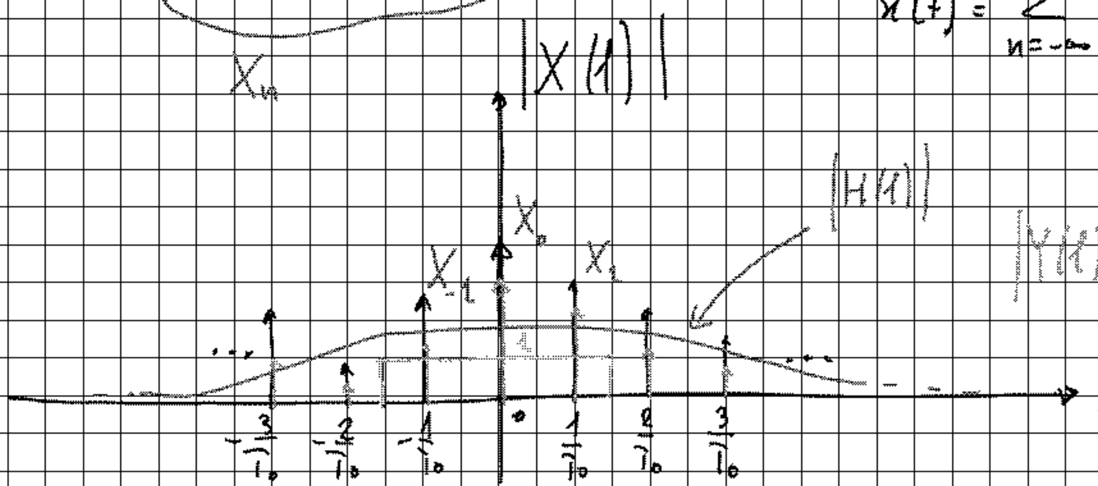
$$y(t) = x(t) \otimes h(t)$$

$$Y(f) = X(f)\,H(f)$$

$$X(f) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X_0\left(\frac{n}{T_0}\right)\delta\left(f-\frac{n}{T_0}\right) \quad , \quad X_0(f) \overset{TCF}{\Longleftrightarrow} x_0(t)$$

X_n

$$x(t) = \sum_{n=-\infty}^{+\infty} x_0(t-nT_0)$$



$|Y(f)| = |X(f)|\cdot|H(f)|$

$$Y(f) = \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X_0\left(\frac{n}{T_0}\right)\delta\left(f-\frac{n}{T_0}\right)\cdot H(f) =$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{+\infty} X_0\left(\frac{n}{T_0}\right) H\left(\frac{n}{T_0}\right)\delta\left(f-\frac{n}{T_0}\right) =$$

$$= \sum_{n=-\infty}^{+\infty} \underbrace{X_n\, H\left(\frac{n}{T_0}\right)}_{Y_n}\delta\left(f-\frac{n}{T_0}\right)$$

$$= \sum_{n=-\infty}^{+\infty} Y_n\,\delta\left(f-\frac{n}{T_0}\right) \overset{TCF}{\Longleftrightarrow} y(t)$$

ANALISI ENERGETICA DI SEGNALI APERIODICI

Correlazione tra due segnali

$$C_{xy}(\tau) \triangleq \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt$$

Autocorrelazione

$$C_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt$$

Proprieta' della autocorrelazione

1) $C_x(0) = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = E_x$

2) $C_x^*(\tau) = C_x(-\tau)$

Dim

$$C_x^*(\tau) = \left[\int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt\right]^* =$$

$$= \int_{-\infty}^{+\infty} x^*(t)\, x(t-\tau)\, dt = \qquad (t-\tau = t')$$

$$= \int_{-\infty}^{+\infty} x^*(t'+\tau)\, x(t')\, dt' = \int_{-\infty}^{+\infty} x(t')\, x^*[t'-(-\tau)]\, dt'$$

$$= C_x(-\tau)$$

TCF DELLA AUTOCORRELAZIONE

$$S_x(f) = \int_{-\infty}^{+\infty} C_x(\tau)\, e^{-j2\pi f\tau}\, d\tau =$$

$$= \int_{-\infty}^{+\infty}_{(\tau)} \int_{-\infty}^{+\infty}_{(t)} x(t)\, x^*(t-\tau)\, dt\; e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty}_{(t)} x(t) \int_{-\infty}^{+\infty}_{(\tau)} x^*(t-\tau)\, e^{-j2\pi f\tau}\, d\tau\, dt = \quad (t-\tau=\tau')$$

$$= \int_{-\infty}^{+\infty}_{(t)} x(t) \int_{-\infty}^{+\infty}_{(\tau)} x^*(\tau')\, e^{-j2\pi f(t-\tau')}\, d\tau'\, dt$$

$$= \underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt}_{X(f)} \left[\underbrace{\int_{-\infty}^{+\infty} x(\tau')\, e^{-j2\pi f\tau'}\, d\tau'}_{X(f)} \right]^*$$

$$= \boxed{X(f)\, X^*(f) = |X(f)|^2 = S_x(f)}$$ DENSITÀ SPETTRALE DI ENERGIA

$$\int_{-\infty}^{+\infty} S_x(f)\, df = \left.\int_{-\infty}^{+\infty} S_x(f)\, e^{j2\pi f\tau}\, df\right|_{\tau=0} =$$

$$= C_x(0) = E_x$$

CALCOLO DELL'ENERGIA DI UN SEGNALE APERIODICO

I) $C_x(0) = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$

II) $\int_{-\infty}^{+\infty} S_x(f) \, df = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$

TEOREMA DI PARSEVAL PER SEGNALI APERIODICI

$$\int_{-\infty}^{+\infty} x(t) \, y^*(t) \, dt = \int_{-\infty}^{+\infty} X(f) \, Y^*(f) \, df$$

Dim

$$\int_{-\infty}^{+\infty} x(t) \, y^*(t) \, dt = \underset{(t)}{\int_{-\infty}^{+\infty}} \underbrace{\underset{(f)}{\int_{-\infty}^{+\infty}} X(f) \, e^{j2\pi f t} \, df}_{x(t)} \, y^*(t) \, dt$$

$$= \underset{(f)}{\int_{-\infty}^{+\infty}} X(f) \underset{(t)}{\int_{-\infty}^{+\infty}} y^*(t) \, e^{j2\pi f t} \, dt \, df =$$

$$= \int_{-\infty}^{+\infty} X(f) \left[\underbrace{\int_{-\infty}^{+\infty} y(t) \, e^{-j2\pi f t} \, dt}_{Y(f)} \right]^* df = \int_{-\infty}^{+\infty} X(f) \, Y^*(f) \, df$$

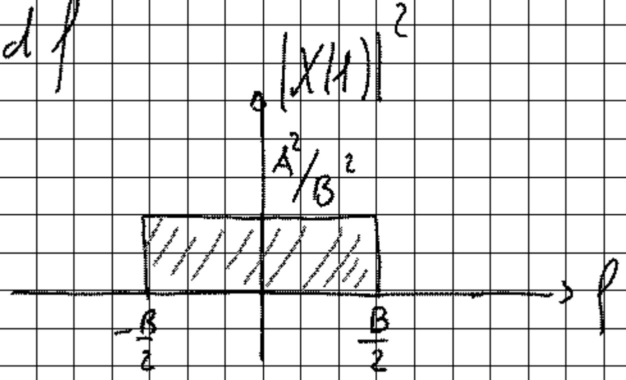
ESEMPIO

Calcolare l'energia del segnale

$$x(t) = A\ \mathrm{sinc}(Bt)$$

I) $$E_x = \int_{-\infty}^{+\infty} A^2\ \mathrm{sinc}^2(Bt)\ dt$$

II) $$E_x = \int_{-\infty}^{+\infty} \left\{ TCF\left[A\ \mathrm{sinc}(Bt)\right] \right\}^2 df$$



$$X(f) = A \cdot \frac{1}{B}\ \mathrm{rect}\left(\frac{f}{B}\right)$$

$$E_x = \int_{-\infty}^{+\infty} \left| \frac{A}{B}\ \mathrm{rect}\left(\frac{f}{B}\right) \right|^2 df = \frac{A^2}{B^2} \cdot B = \frac{A^2}{B}$$

RELAZIONE TRA CORRELAZIONE E CONVOLUZIONE

$$\boxed{C_{xy}(\tau) = x(\tau) \otimes y^*(-\tau)}$$

$$\int_{-\infty}^{+\infty} x(\alpha)\ y^*\left[-(\tau-\alpha)\right] d\alpha =$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\ y^*(\alpha-\tau)\ d\alpha = C_{xy}(\tau)$$

FILTRAGGIO E ANALISI ENERGETICA

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) \otimes h(t)$$

$$C_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt$$

$$C_y(\tau) = \int_{-\infty}^{+\infty} y(t)\, y^*(t-\tau)\, dt$$

$$\Rightarrow \boxed{\begin{aligned} C_y(\tau) &= C_x(\tau) \otimes h(\tau) \otimes h(-\tau) \\ S_y(f) &= S_x(f)\, H(f)\, H^*(f) = S_x(f) \left| H(f) \right|^2 \end{aligned}}$$

Dim

$$S_y(f) = \left| Y(f) \right|^2 = \left| X(f)\, H(f) \right|^2 = \left| X(f) \right|^2 \left| H(f) \right|^2 =$$

$$= S_x(f) \left| H(f) \right|^2 = S_x(f)\, H(f)\, H^*(f)$$

$$TCF^{-1}\left[S_y(f) \right] = C_y(\tau) = C_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

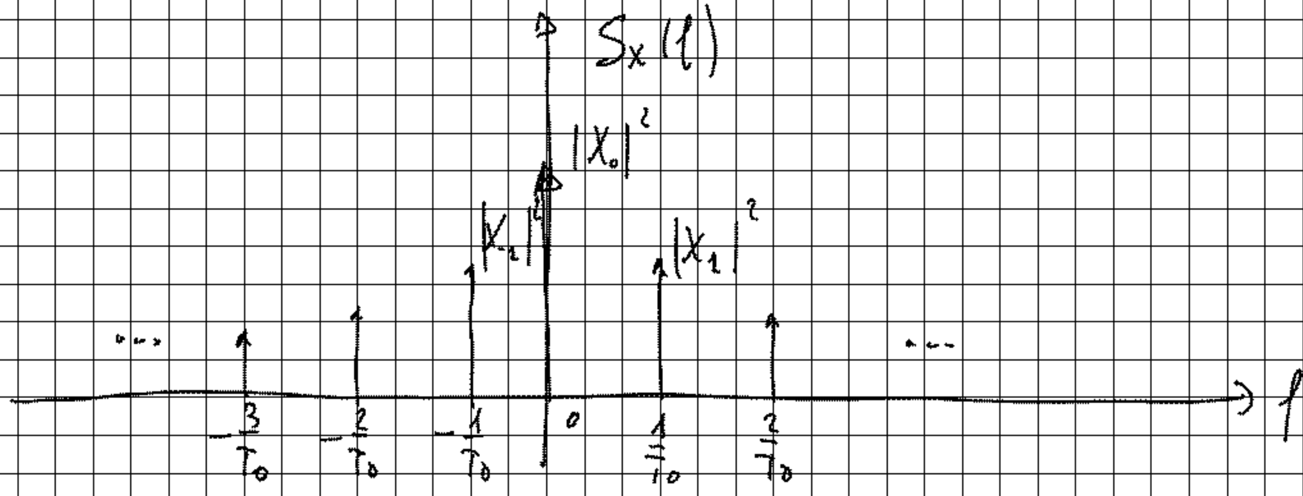
SEGNALI PERIODICI

$$S_x(f) = \left| X(f) \right|^2 \Rightarrow S_x(f) \triangleq \sum_{n=-\infty}^{+\infty} \left| X_n \right|^2 \delta\left(f - \frac{n}{T_0}\right)$$

aperiodici — periodici di T_0

$$\Rightarrow P_x = \int_{-\infty}^{+\infty} S_x(f)\, df = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left| X_n \right|^2 \delta\left(f - \frac{n}{T_0}\right) df$$

$$= \sum_{n=-\infty}^{+\infty} |X_n|^2 \underbrace{\int_{-\infty}^{+\infty} \delta\left(f - \frac{n}{T_0}\right) df}_{1 \ \forall n} = \sum_{n=-\infty}^{+\infty} |X_n|^2 = P_x$$



SEGNALI APERIODICI A POTENZA FINITA

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, dt =$$

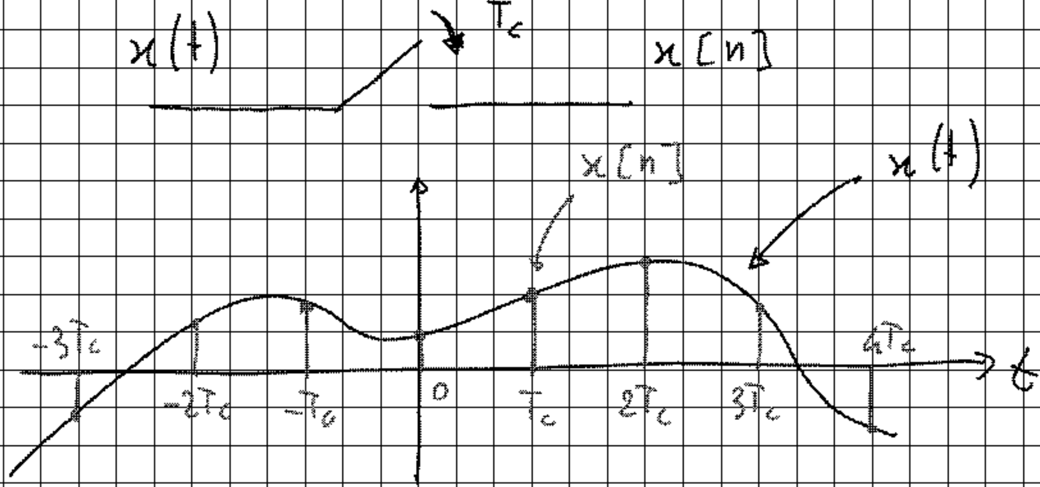
$$= \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{|x_T(t)|^2}{T} \, dt = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{|x_T(t)|^2}{T} \, dt$$

$$= \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} \, df = \int_{-\infty}^{+\infty} S_x(f) \, df$$

$$S_x(f) \triangleq \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} \quad , \quad X_T(f) \overset{TCF}{\Leftrightarrow} x_T(t)$$

$$S_y(f) = S_x(f) \, |H(f)|^2$$

SEGNALI CAMPIONATI



$F_c \triangleq \frac{1}{T_c}$ freq. di camp.

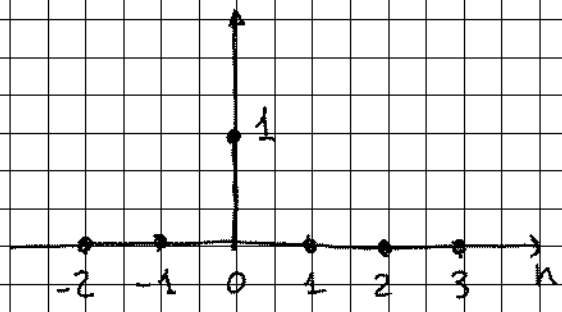
T_c = intervallo di camp.

SEQUENZE

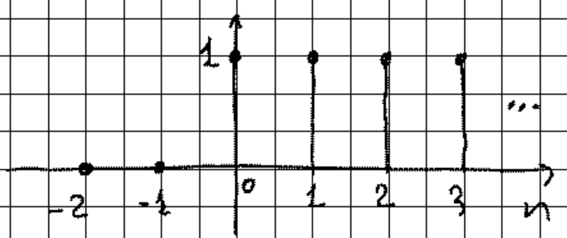
SEQUENZE NOTEVOLI

•) $\delta[n]$ <u>delta di Kronecker</u>

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

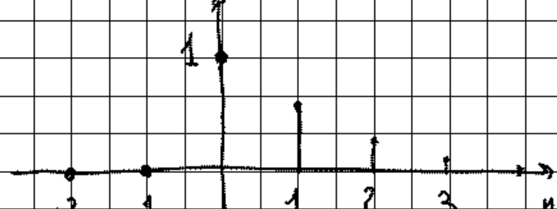


•) $u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ <u>gradino</u>

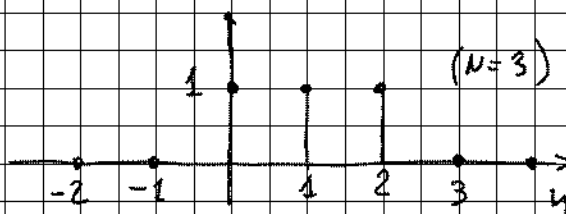


•) $x[n] = a^n \, u[n]$ <u>esponenziale</u>

$0 < a < 1$



•) $x[n] = u[n] - u[n-N]$ <u>rettangolo</u>



•) $x[n] = \exp\left(j 2\pi F_0 n\right)$ oscillazione complessa discreta

$F_0 \in \mathbb{Q} \Rightarrow F_0 = \frac{p}{q}$, $p, q \in \mathbb{N}$ (affinché sia periodico)

Proprietà: relazione tra $\delta[n]$ e $u[n]$

$$\delta[n] = u[n] - u[n-1]$$

$$u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$$

TRASFORMATA DI FOURIER DI UNA SEQUENZA

$$x[n] \overset{TFS}{\Longleftrightarrow} \bar{X}(f)$$

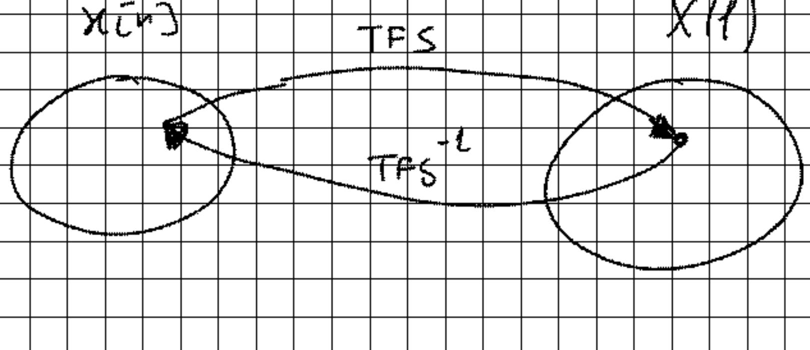
$$\boxed{\bar{X}(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f n T}} \quad TFS$$

$$\boxed{x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, e^{j2\pi f n T}\, df} \quad TFS^{-1}$$

$$\bar{X}(f) = \bar{X}\left(f - \frac{n}{T}\right) \qquad n \in \mathbb{Z}$$

$$\bar{X}\left(f - \frac{n}{T}\right) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi\left(f - \frac{n}{T}\right)kT} =$$

$$= \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi f k T}}_{\bar{X}(f)} \cdot \underbrace{e^{j2\pi \frac{n}{\cancel{T}} k \cancel{T}}}_{=1} = \bar{X}(f)$$



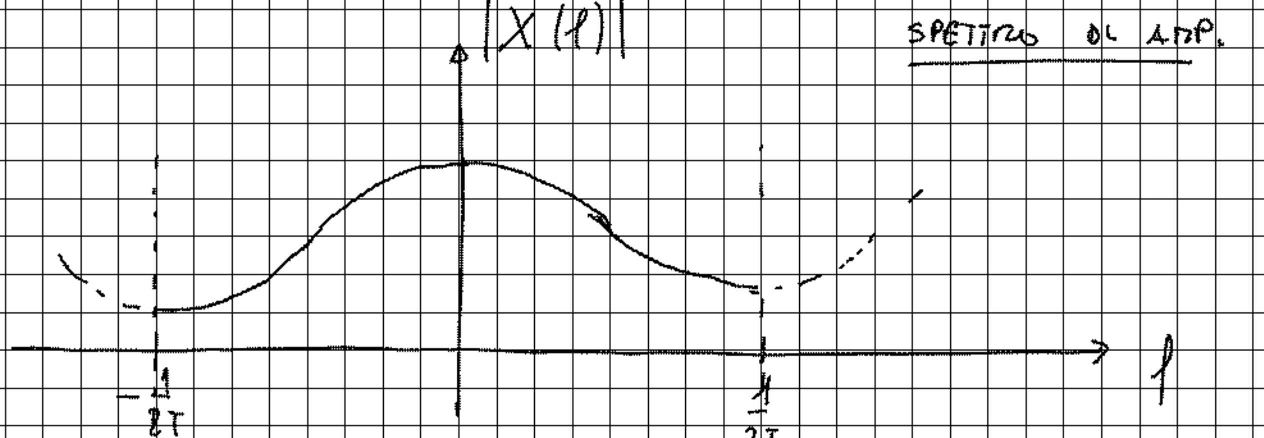
$$x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \overline{X}(f)\, e^{j2\pi f n T}\, df =$$

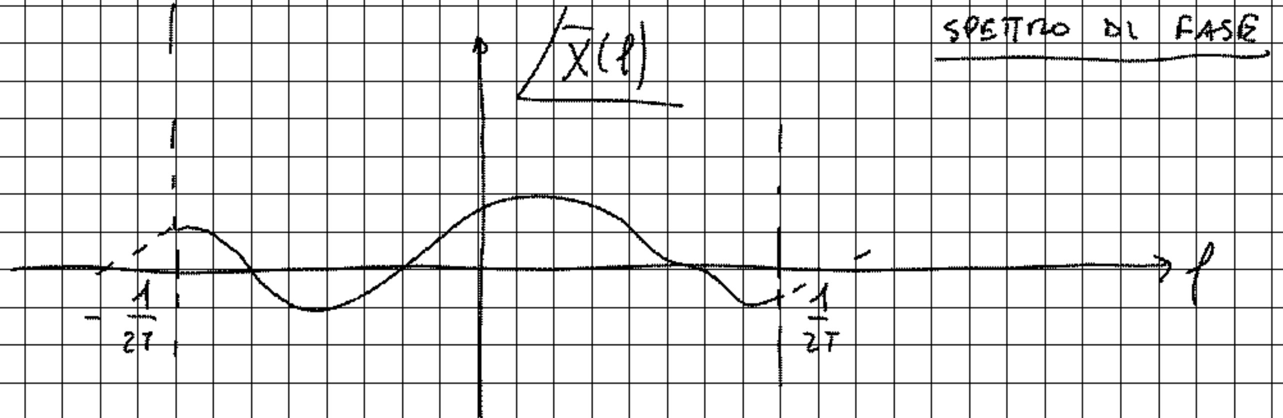
$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi f k T} \cdot e^{j2\pi f n T}\, df =$$

$$= T \sum_{k=-\infty}^{+\infty} x[k] \underbrace{\int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{j2\pi f (n-k) T}\, df}_{= \begin{cases} 0 & \Rightarrow n \neq k \\ \frac{1}{T} & \Rightarrow n = k \end{cases}} =$$

$$= \cancel{T} \cdot \frac{1}{\cancel{T}}\, x[n] = x[n]$$

RAPPRESENTAZIONE GRAFICA DELLA TFS





$$x_1[n] = e^{j 2\pi n F_0 T} \qquad x_2[n] = e^{j 2\pi n \left(F_0 + \frac{m}{T}\right) T}$$

$$F_0 \qquad F_0 + \frac{m}{T}, \quad m \in \mathbb{Z}$$

$$x_2[n] = e^{j 2\pi n F_0 T} \cdot \underbrace{e^{j 2\pi n m}}_{= 1} = e^{j 2\pi n F_0 T} = x_1[n]$$

$$\bar{X}_1(f) = \sum_{n=-\infty}^{+\infty} x_1[n] e^{-j 2\pi f n T} = \sum_{n=-\infty}^{+\infty} e^{-j 2\pi (f - F_0) n T} =$$

$$= \boxed{\sum_{n=-\infty}^{+\infty} \delta\left(f - F_0 - \frac{n}{T}\right)}$$

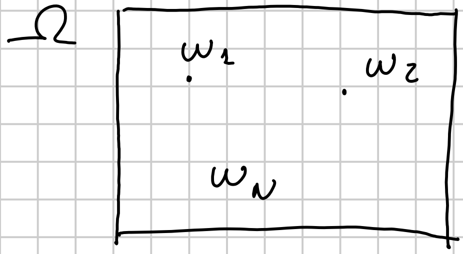
$$\bar{X}_2(f) = \sum_{n=-\infty}^{+\infty} x_2[n] e^{-j 2\pi f n T} = \sum_{n=-\infty}^{+\infty} e^{-j 2\pi \Big(\underbrace{f - F_0 - \frac{m}{T}}_{f'}\Big) n T} =$$

$$= \sum_{n=-\infty}^{+\infty} \delta\Big(\underbrace{f - F_0 - \frac{m}{T}}_{f'} - \frac{n}{T}\Big) = \left(n + m = n' \right)$$

$$= \boxed{\sum_{n'=-\infty}^{+\infty} \delta\left(f - F_0 - \frac{n'}{T}\right)}$$

TEORIA DELLA PROBABILITÀ

ESPERIMENTO CASUALE : il risultato non è predicibile



Ω = SPAZIO CAMPIONE

ω_i = possibile risultato

EVENTO : sottoinsieme dello spazio campione che soddisfa le seguenti proprietà

- Se A è un evento $\Rightarrow \bar{A}$ è un evento dove $\bar{A}$ è il complemento di A su Ω
- Se A e B sono due eventi $\Rightarrow A \cup B$ è un evento

Proprietà discendenti dalle precedenti

- se A e B sono eventi $\Rightarrow A \cap B$ è un evento
- $A \cup \bar{A} = \Omega$ EVENTO CERTO
- $A \cap \bar{A} = \emptyset$ EVENTO IMPOSSIBILE

CARATTERIZZAZIONE DI UN ESP CASUALE

necessità di :

1) Definizione di uno spazio campione
2) Definizione di eventi (descrizione)
3) Legge di probabilità

PROBABILITÀ

Definizione assiomatica (Kolmogorov)

1) $P\{A\} \geq 0$ (non-negativa)

2) $P\{\Omega\} = 1$ (normalizzazione)

3) Se due eventi A e B sono disgiunti $(A \cap B = \emptyset)$
$\Rightarrow P\{A \cup B\} = P\{A\} + P\{B\}$

PROPRIETÀ

- $P\{\bar{A}\} = 1 - P\{A\}$
- $P\{\emptyset\} = 0$
- $0 \leq P\{A\} \leq 1$
- $P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$

SIMBOLOGIA

$A \cup B \iff A + B$

$A \cap B \iff AB$

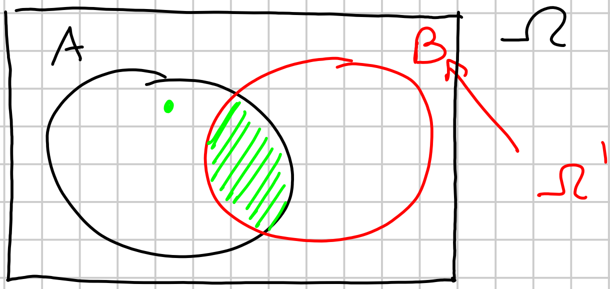
NOMENCLATURA

$P\{AB\}$ Probabilità congiunta

$P\{A\}, P\{B\}$ Probabilità marginali

PROBABILITÀ CONDIZIONATA

$P\{A | B\}$ Prob. dell'evento A condizionata dall'evento B



$$\boxed{P\{A|B\} \triangleq \frac{P\{AB\}}{P\{B\}}}$$

$$P\{A\} \overset{?}{=} P\{A|\Omega\} = \frac{P\{A\Omega\}}{P\{\Omega\}} = \frac{P\{A\}}{1}$$

DEFINIZIONE CLASSICA DI PROBABILITÀ (Pascal)

$$P\{A\} \triangleq \frac{N_F}{N}$$

DEFINIZIONE DI VON MISES (FREQUENTISTA)

$$P\{A\} \triangleq \lim_{N\to\infty} \frac{N_F}{N}$$

1) $P\{A\} \geq 0$ verificata poiché $N_F \geq 0$, $N > 0$

2) $P\{\Omega\} = \lim_{N\to\infty} \frac{N}{N} = 1$

3) $P\{A \cup B\} = \lim_{N\to\infty} \frac{N_A + N_B}{N} = \lim_{N\to\infty} \frac{N_A}{N} + \lim_{N\to\infty} \frac{N_B}{N}$

$$= P\{A\} + P\{B\}$$

INDIPENDENZA TRA EVENTI

$$P\{A|B\} = P\{A\} \qquad P\{A\},\ P\{B\} \neq 0$$

$$P\{A\} = P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

$$\Rightarrow \quad P\{AB\} = P\{A\}P\{B\}$$

TEOREMA DI BAYES

$$\boxed{P\{A|B\} = \frac{P\{B|A\}P\{A\}}{P\{B\}}}$$

$P\{A\}, P\{B\} \neq 0$

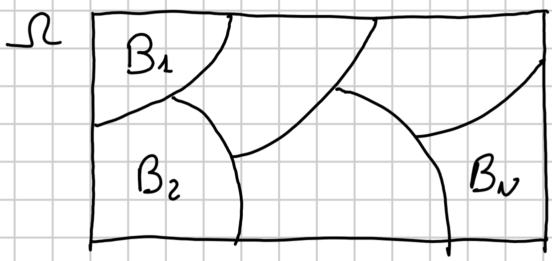
Dim.

$$P\{AB\} = P\{BA\}$$

$$P\{A|B\}P\{B\} = P\{B|A\}P\{A\}$$

$$P\{A|B\} = \frac{P\{B|A\}P\{A\}}{P\{B\}}$$

PARTIZIONE DI UNO SPAZIO CAMPIONE



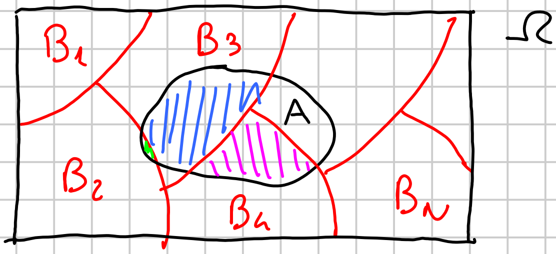
·) $\Omega = \sum_{i=1}^{N} B_i = B_1 + B_2 + \cdots + B_N$

·) $B_i \cap B_k = 0 \qquad i \neq k$

$\Rightarrow \quad \{B_i\}_{i=1,\dots,N}$ PARTIZIONE DI Ω

TEOREMA DELLA PROBABILITÀ TOTALE

$P\{A\}$



$$\boxed{P\{A\} = \sum_{i=1}^{N} P\{A|B_i\}\, P\{B_i\}}$$

se $\{B_i\}$ è una partizione di Ω !!

Dim

$$P\{A\} = P\{A\Omega\}$$

$$P\{A\Omega\} = P\{A \sum_{i=1}^{N} B_i\} = P\{\sum_{i=1}^{N} AB_i\} =$$

$$= \sum_{i=1}^{N} P(AB_i) = \sum_{i=1}^{N} P\{A|B_i\}\, P\{B_i\}$$

$AB_i \cap AB_k = \phi \qquad i \neq k$

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

ESPERIMENTO ALEATORIO COMPOSTO

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_N$$

$$A = A_1 \times A_2 \times \cdots \times A_N$$

$P\{A\}$

$P\{A_1\}, P\{A_2\}, \ldots, P\{A_N\}$

ESPERIMENTI INDIPENDENTI

$$\Rightarrow P\{A\} = P\{A_1\} \cdot P\{A_2\} \cdots P\{A_N\}$$

In generale la $P\{A\}$ non e' direttamente ricavabile dalle $P\{A_i\}$

PROVE RIPETUTE BINARIE E INDIPENDENTI

$$P\{w_0\} = p_0$$

$$P\{w_1\} = p_1 = 1 - p_0$$

Es. lancio di una moneta

Qual'e' la probabilita' che esca "testa" k volte su n

Bernoulli $\Rightarrow$ $\boxed{P = \binom{n}{k} p_0^{k} (1-p_0)^{n-k}}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

ESEMPIO (1)

.) moneta perfetta $\Rightarrow$ $p_0 = P\{\text{"testa"}\} = 0.5$

$p_1 = P\{\text{"croce"}\} = 1 - p_0 = 0.5$

.) " truccata $\Rightarrow$ $p_0 = 0.8$, $p_1 = 0.2$

Viene presa "a caso" una delle due monete e viene lanciata 10 volte. Si osserva che esce per 5 volte "testa".

$\Rightarrow$ Qual'è la probabilità che è stata presa la moneta "perfetta"

Soluzione

$$P\{A|B\} = ?$$

A = scelta della moneta "perfetta"

B = 5 volte "testa" su 10 lanci

Applichiamo il Teorema di Bayes

$$P\{A|B\} = \frac{P\{B|A\}\,P\{A\}}{P\{B\}}$$

$P\{A\} = 0.5$

$\Rightarrow$ $P\{B|A\}$

$\Rightarrow$ $P\{B\}$

Calcolo della $P\{B|A\}$

Uso Bernoulli

$p_0 = 0.5$, $n = 10$, $k = 5$

$$P\{B|A\} = \binom{n}{k} p_0^{\,k} (1-p_0)^{n-k} = \frac{10!}{5!\,5!} \cdot 0.5^5 \cdot 0.5^5$$

$$= 252 \cdot 0.5^{10} \cong 0.246$$

$$P\{B\} = P\{B|A\}P\{A\} + P\{B|C\}P\{C\}$$

$C =$ scelta della moneta truccata $= \bar{A}$

$P\{C\} = 0.5 \qquad (P\{A\} + P\{C\} = 1)$

$P\{B|C\} = ?$

Uso Bernoulli $(p_0 = 0.8, \ n = 10, \ k = 5)$

$$P\{B|C\} = \binom{n}{k} p_0^k (1-p_0)^{n-k} =$$

$$= 252 \cdot 0.8^5 \cdot 0.2^5 \cong 0.0264$$

$$P\{B\} \cong 0.246 \cdot 0.5 + 0.0264 \cdot 0.5 = 0.136$$

$$P\{A|B\} = \frac{0.246 \cdot 0.5}{0.136} \cong 0.903$$

ESEMPIO 2

In una classe ci sono 30 persone. Quanto scommetteresti che almeno 2 persone siano nate nello stesso giorno dell'anno

$A =$ almeno 2 persone su 30 sono nate lo stesso giorno dell'anno

$$P\{A\} = 1 - P\{\bar{A}\}$$

$\bar{A} =$ tutti sono nati in giorni diversi

$P\{\bar{A}\}$

Ragioniamo per induzione

$P_n = \dfrac{1}{365}$ probabilità di essere nato in un particolare giorno

$$P_2\{\bar{A}\} = \frac{364}{365} \qquad (1 - p_n)$$

$$P_3\{\bar{A}\} = \frac{364}{365} \cdot \frac{363}{365}$$

$$\vdots$$

$$P_K\{\bar{A}\} = \frac{365-1}{365} \cdot \frac{365-2}{365} \cdot \ldots \cdot \frac{365-K+1}{365}$$

$$= \frac{(N-1)!}{N^{K-1}\,(N-K)!}$$

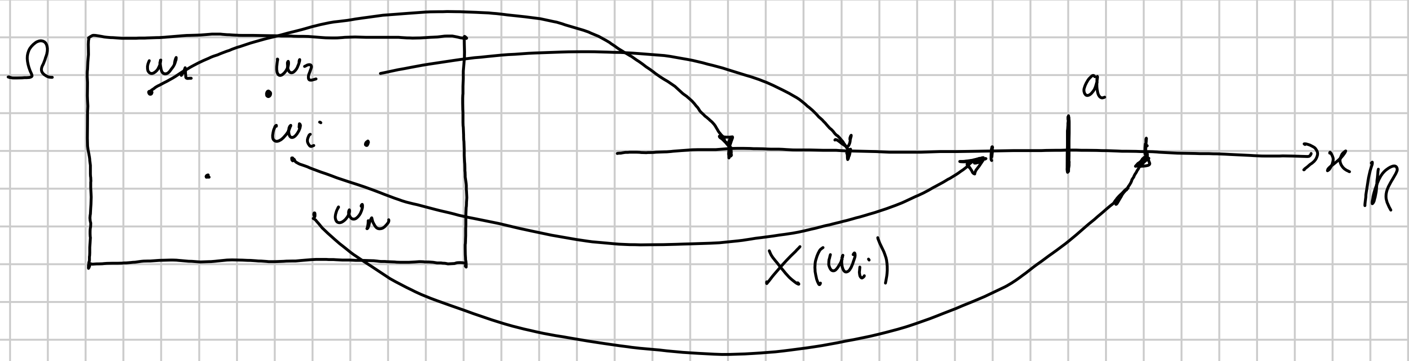
$N =$ nr. giorni dell'anno

$P_K\{\bar{A}\} =$ probabilità che K persone in una classe siano nate tutte in giorni diverse

$$P\{\bar{A}\} = P_K\{\bar{A}\}\Big|_{K=30} \simeq 0.27$$

$$P\{A\} = 1 - P_K\{\bar{A}\}\Big|_{K=30} \simeq 0.73$$

VARIABILI ALEATORIE



$X(w_i)$ è la v.a. (X)

se $X(w_i) \leq a$, con a arbitrario, è un EVENTO

$$\boxed{P\{X \leq a\} = F_X(x)}$$

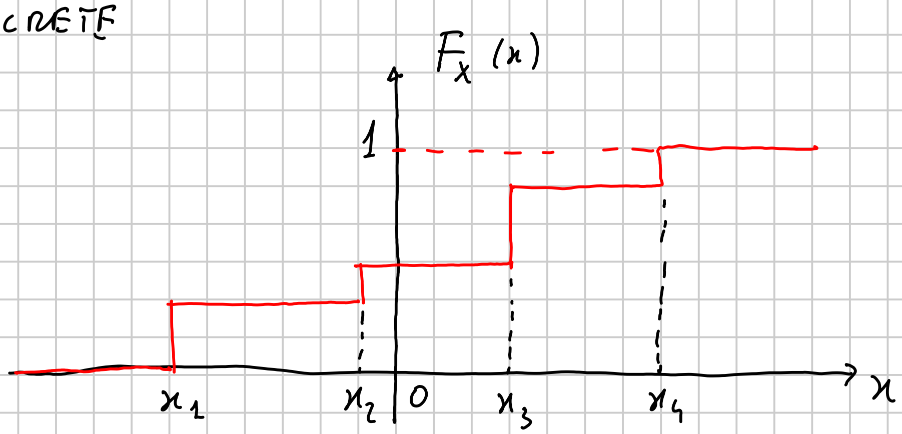
FUNZIONE DISTRIBUZIONE DI PROBABILITÀ DI UNA V.A.

PROPRIETÀ

- $0 \leq F_X(x) \leq 1$
- $\lim_{x \to +\infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$
- se $x_2 > x_1 \Rightarrow F_X(x_2) \geq F_X(x_1)$ monotona non decrescente
- $\lim_{h \to 0^+} F_X(x+h) = F_X(x)$ (continuità da destra)
- Se si presenta una disc. di I^a specie in $x = \bar{x}$
  $\Rightarrow F_X(\bar{x}^+) - F_X(\bar{x}^-) = P\{x = \bar{x}\}$
- $P\{a < X \leq b\} = F_X(b) - F_X(a)$

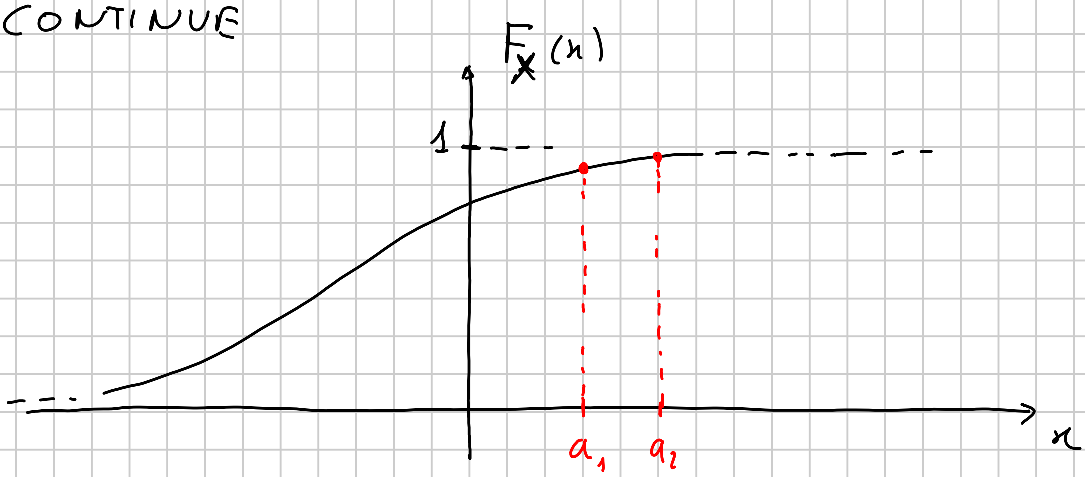
V.A. DISCRETE, CONTINUE O MISTE

V.A. DISCRETE

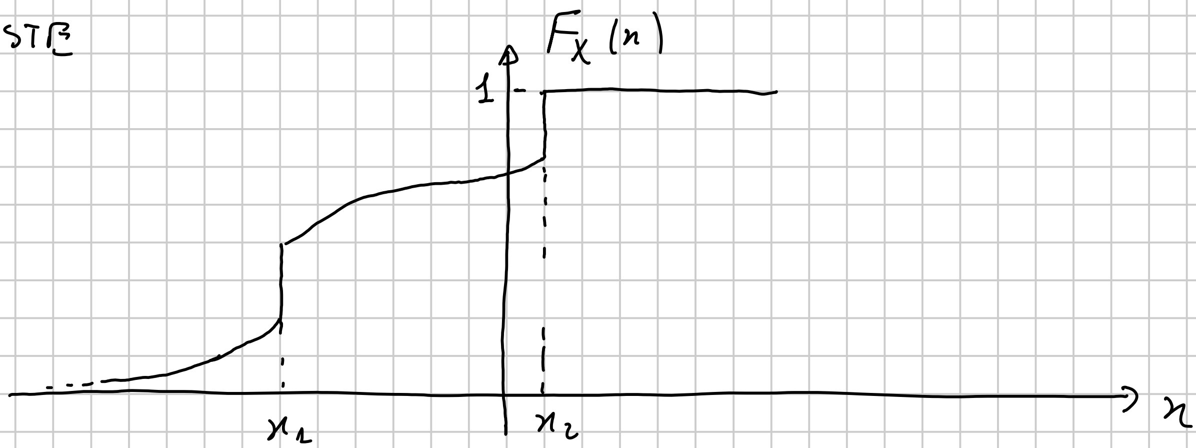


$P_k = P\{x = x_k\}$

V.A. CONTINUE



V.A MISTE



DENSITÀ DI PROBABILITÀ DI UNA V.A.

$$\boxed{f_X(x) \triangleq \frac{d}{dx} F_X(x)} \qquad ddp$$

$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

PROPRIETÀ

·) $f_X(x) \geq 0 \quad \forall x$

·) $P\{a < X \leq b\} = F_X(b) - F_X(a) =$

$$= \int_{-\infty}^{b} f_X(x)\, dx - \int_{-\infty}^{a} f_X(x)\, dx = \int_{a}^{b} f_X(x)\, dx$$

$$\cdot) \int_{-\infty}^{+\infty} f_X(x)\,dx = 1 = \lim_{x\to+\infty} F_X(x) = \lim_{x\to+\infty} \int_{-\infty}^{x} f_X(\alpha)\,d\alpha$$

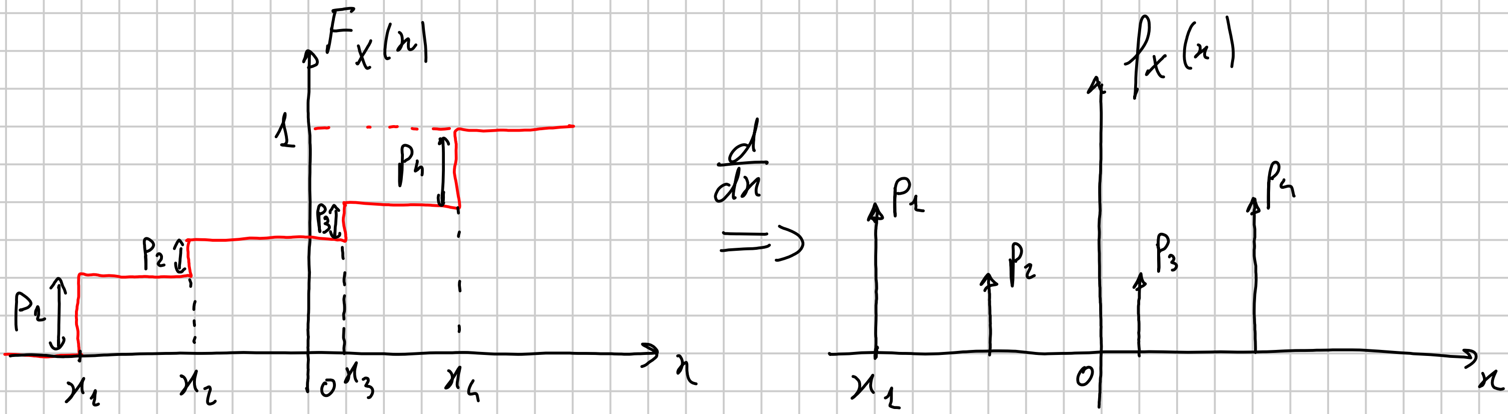
CONCETTO DI DDP

$$P\{\bar{x} < X < \bar{x}+\Delta x\} = \int_{\bar{x}}^{\bar{x}+\Delta x} f_X(\alpha)\,d\alpha \simeq f_X(\bar{x})\,\Delta x$$

Δx molto piccolo

$$f_X(\bar{x}) \simeq \frac{P\{\bar{x} < X \leq \bar{x}+\Delta x\}}{\Delta x}$$

DDP di V.A. discrete



$$p_k = P\{X = x_k\}$$

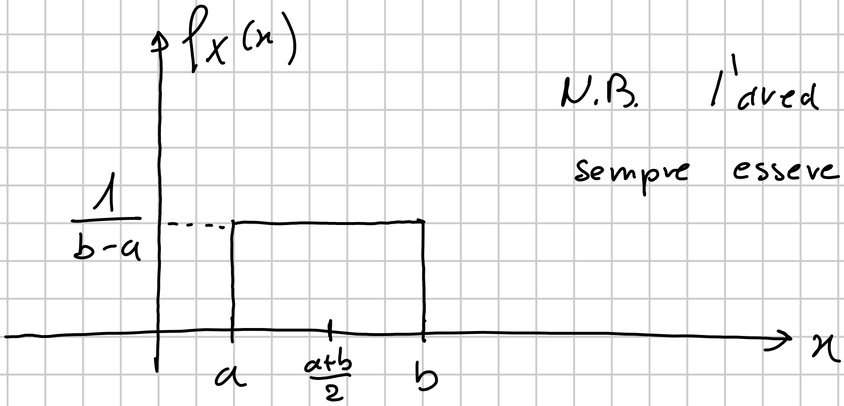
$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \sum_{k=1}^{N} p_k\, u(x - x_k) =$$

$$= \sum_{k=1}^{N} p_k\, \delta(x - x_k)$$

V.A. UNIFORMI

uniforme in (a,b) se

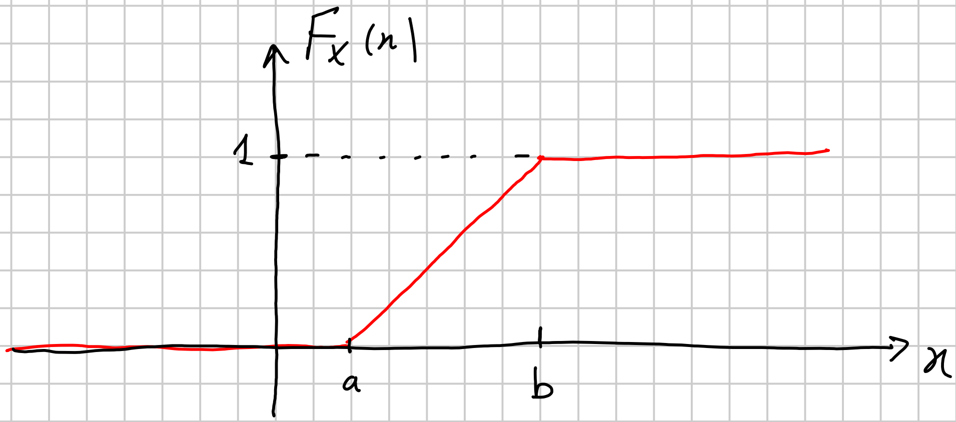
$$f_X(x) = \frac{1}{b-a} \operatorname{rect}\left(\frac{x - \frac{a+b}{2}}{b-a}\right)$$



N.B. l'area deve sempre essere unitaria

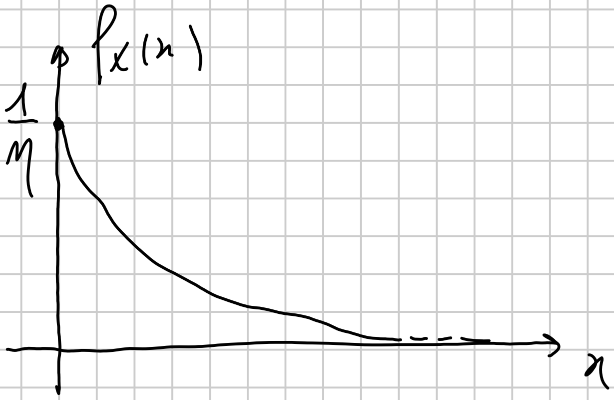
$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha = \frac{1}{b-a}\int_{-\infty}^{x} \mathrm{rect}\left(\frac{\alpha - (a+b)/2}{b-a}\right) d\alpha$$

$$= \frac{1}{b-a}\int_{a}^{x} d\alpha \quad (x \leq b) = \frac{x-a}{b-a}$$



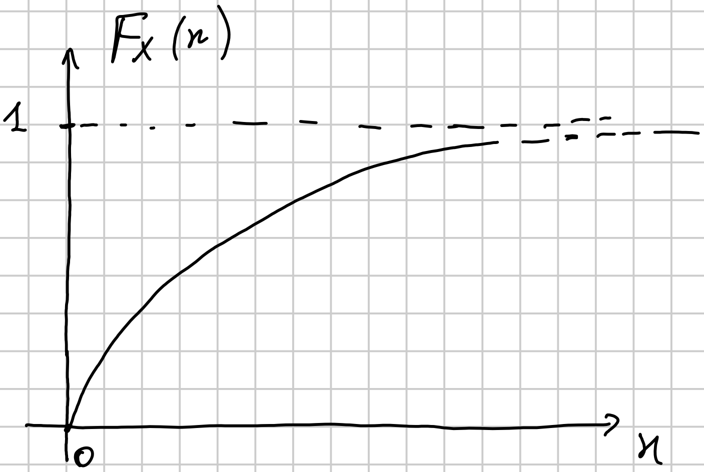
V.A. ESPONENZIALE (UNILATERA)

$$f_X(x) = \frac{1}{\eta} e^{-\frac{x}{\eta}} u(x)$$



$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\eta} e^{-\frac{\alpha}{\eta}} u(x)\, d\alpha =$$

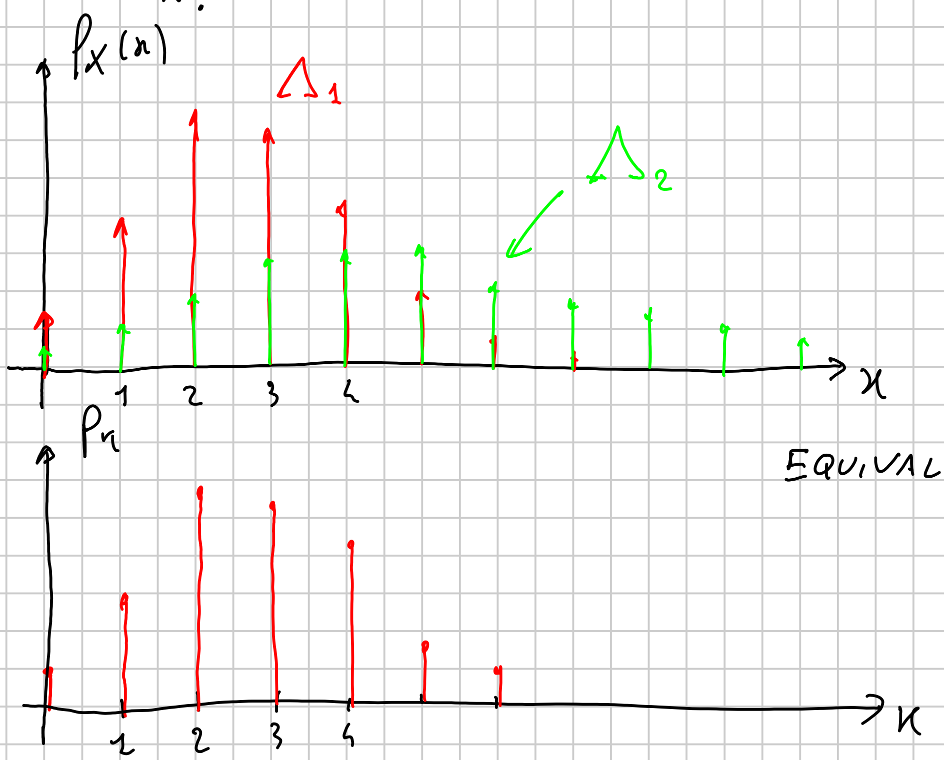
$$= \frac{1}{\eta}\int_{0}^{x} e^{-\frac{\alpha}{\eta}} d\alpha = \frac{1}{\eta}(-\eta)\, e^{-\frac{\alpha}{\eta}}\Big|_{0}^{x} = 1 - e^{-\frac{x}{\eta}}$$



V.A. DI POISSON (DISCRETA)

$$f_X(x) = \sum_{k=0}^{N} p_k\, \delta(x-k)$$

$$p_k = \frac{\Lambda^k}{k!} e^{-\Lambda}$$



EQUIVALENTI

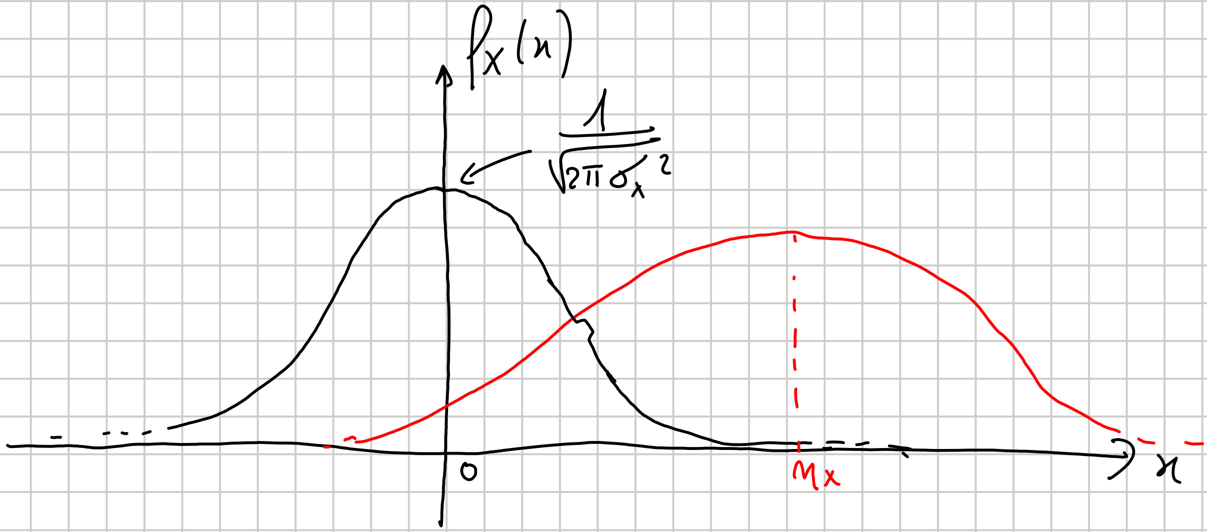
V.A. GAUSSIANE

$$p_X(x) \triangleq \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}$$

η_x

V.A. GAUSSIANA

$$X \in \mathcal{N}(\eta_x, \sigma_x^2)$$



V.A. GAUSSIANA STANDARD (NORMALE)

$$p_N(n) = \frac{1}{\sqrt{2\pi}} e^{-\frac{n^2}{2}} \Rightarrow X \in \mathcal{N}(0,1)$$

DISTRIBUZIONE DI PROBABILITÀ DI UNA V.A. GAUSSIANA

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(\alpha-\eta_x)^2}{2\sigma_x^2}} d\alpha$$

NON ESISTE UNA FORMA CHIUSA

$$F_N(n) = \int_{-\infty}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} d\alpha \triangleq \Phi(n) \quad \text{(STANDARD)}$$

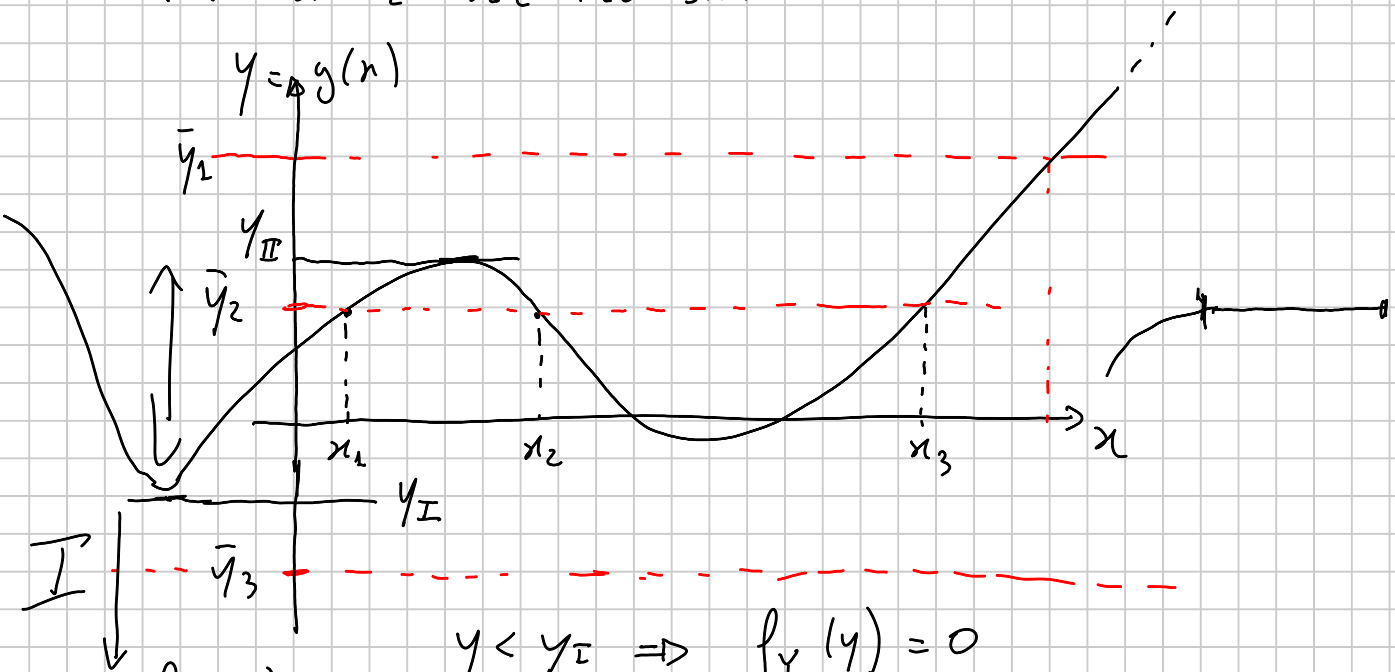
TEOREMA FONDAMENTALE PER LA TRASF. DI V.A.

$\Rightarrow Y = g(X) \qquad (y = g(x))$

$\Rightarrow p_Y(y) = ?$

$$f_Y(y) = \sum_{i=1}^{N} \frac{f_X(x_i)}{|g'(x_i)|} \quad , \{x_i\} = \text{soluzioni del problema inverso } x_i = g^{-1}(y)$$

APPLICAZIONE DEL TEOREMA

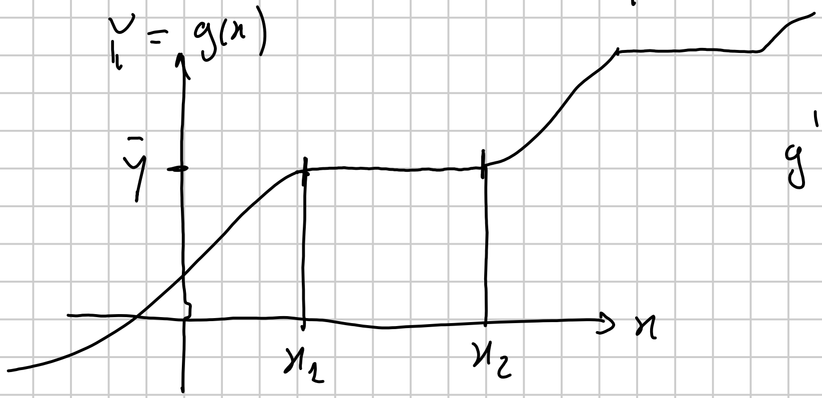


$$f_Y(y) \begin{cases} y < y_I \Rightarrow f_Y(y) = 0 \\ y_I \leq y < y_{II} \Rightarrow \sum_{i=1}^{3} \frac{f_X(x_i)}{|g'(x_i)|} \\ y > y_{II} \Rightarrow f_Y(y) = \frac{f_X(x_i)}{|g'(x_i)|} \end{cases}$$

CASI PARTICOLARI

$\Rightarrow g'(x) = 0$
- $\rightarrow f_X(x) = 0$
- $\rightarrow f_X(x) \neq 0 \Rightarrow f_Y(y) \Rightarrow +\infty$

$\Rightarrow g'(x) = 0$ su un intervallo finito



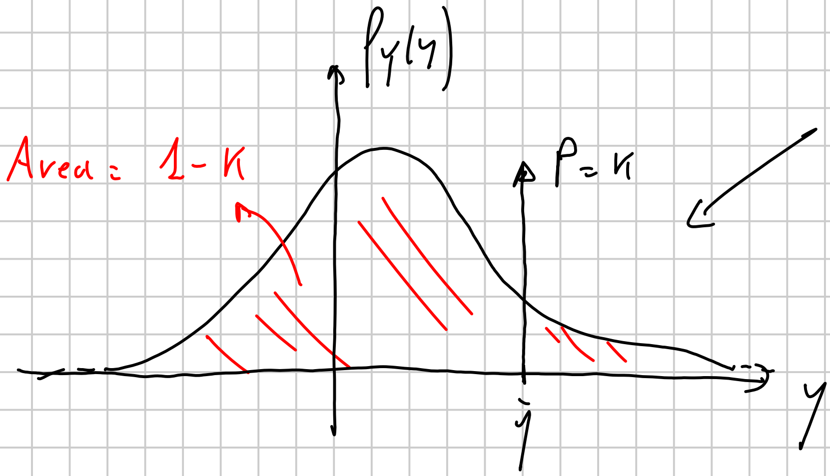
$g'(x) = 0, \; x_1 \leq x \leq x_2$

$$P\{x_1 < X \leq x_2\} = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\, dx = K$$

$$0 \leq K \leq 1$$

$$P\{Y = \bar{y}\} = P\{x_1 < X \leq x_2\} = K$$

$\Rightarrow$ $f_Y(y)$ è mista



dd.p di una v.a. mista

INDICI CARATTERISTICI DI V.A.

VALOR MEDIO

$$E[X] = \eta_X$$ "Expectation", "Valore atteso"

$$\eta_X \triangleq \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

VALOR MEDIO PER V.A. DISCRETE

$$\eta_X = \int_{-\infty}^{+\infty} x \underbrace{\sum_{n=1}^{N} p_n\, \delta(x - x_n)}_{\text{ddp di v.a discrete}} dx =$$

$$= \sum_{n=1}^{N} p_n \int_{-\infty}^{+\infty} x\, \delta(x - x_n)\, dx =$$

$$= \sum_{n=1}^{N} p_n x_n$$

TEOREMA DEL VALOR MEDIO

$$\eta_X = E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

$$\eta_Y = E[Y] \quad , \quad Y = g(X)$$

$$\eta_Y = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy = \boxed{\int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx = E[g(X)]}$$

PROPRIETÀ DI LINEARITÀ

$$Y = \alpha\, g(X) + \beta\, h(X)$$

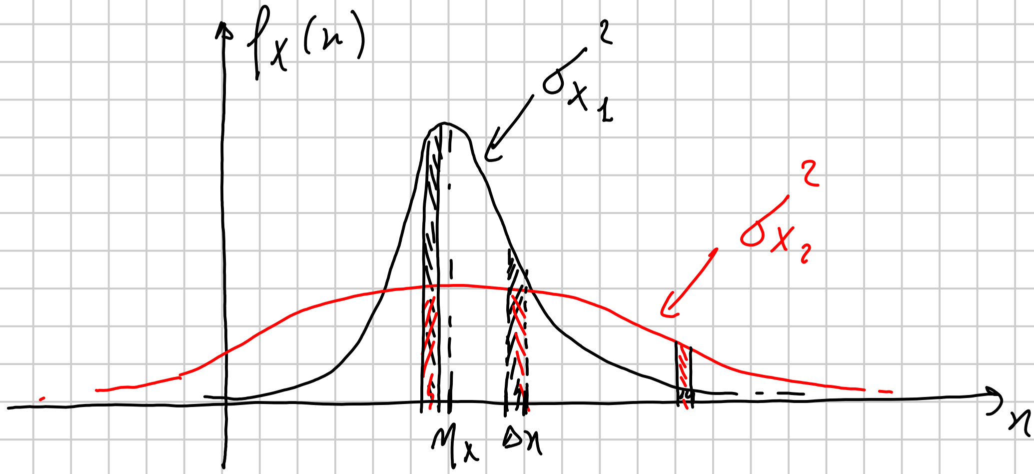
$$E[Y] = \int_{-\infty}^{+\infty} [\alpha g(x) + \beta h(x)]\, f_X(x)\, dx =$$

$$= \alpha \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx + \beta \int_{-\infty}^{+\infty} h(x)\, f_X(x)\, dx$$

$$E[Y] = \alpha\, E[g(X)] + \beta\, E[h(X)]$$

VARIANZA DI UNA V.A.

$$\sigma_X^2 \triangleq E[(X - \eta_X)^2] = \int_{-\infty}^{+\infty} (x - \eta_X)^2 f_X(x)\, dx$$



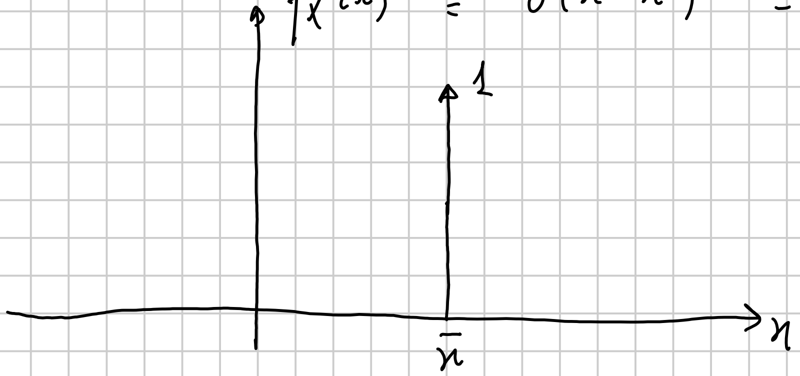
DEVIAZIONE STANDARD

$$\sigma_X \triangleq \sqrt{\sigma_X^2} = \sqrt{E\left[(X-\eta_X)^2\right]}$$

CASO LIMITE => VARIANZA NULLA (AL LIMITE CON IL DETERMINISTICO)

$$f_X(x) = \delta(x-\bar{x}) \Rightarrow \sigma_X^2 = 0$$

$$\eta_X = \bar{x}$$



VALOR QUADRATICO MEDIO

$$m_X \triangleq E\left[X^2\right] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx$$

RELAZIONE TRA σ_X e m_X

$$\sigma_X^2 = E\left[(X-\eta_X)^2\right] = E\left[X^2 + \eta_X^2 - 2\eta_X X\right]$$

$$= E\left[X^2\right] + \eta_X^2 \underbrace{E[1]}_{1} - 2\eta_X \underbrace{E[X]}_{\eta_X} =$$

$$= m_X + \eta_X^2 - 2\eta_X^2 = \boxed{m_X - \eta_X^2 = \sigma_X^2}$$

CALCOLO DEL VALOR MEDIO DI UNA GAUSSIANA

$$E[X] = ? \qquad X \in \mathcal{N}(\eta_X, \sigma_X^2)$$

$$E[x] = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} dx = \begin{pmatrix} x-\eta_x = y \\ dx = dy \end{pmatrix}$$

$$= \int_{-\infty}^{+\infty} (y+\eta_x) \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy =$$

$$= \underbrace{\int_{-\infty}^{+\infty} (y)\left(\frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}}\right) dy}_{=0} + \eta_x \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy}_{=1}$$

$$\Rightarrow E[X] = \eta_x$$

VARIANZA DI UNA V.A. GAUSSIANA

$$E[(X-\eta_x)^2] = \int_{-\infty}^{+\infty} (x-\eta_x)^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} dx$$

$$(x-\eta_x = y)$$

$$= \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy =$$

$$= \int_{-\infty}^{+\infty} y \cdot y \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} dy =$$

$$\left(\Rightarrow \frac{d}{dy}\left[e^{-\frac{y^2}{2\sigma_x^2}}\right] = -\frac{2y}{2\sigma_x^2} e^{-\frac{y^2}{2\sigma_x^2}} = -\frac{y}{\sigma_x^2} e^{-\frac{y^2}{2\sigma_x^2}}\right)$$

$$= -\sigma_x^2 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}}\, y \cdot \left(-\frac{y}{\sigma_x^2}\, e^{-\frac{y^2}{2\sigma_x^2}}\right) dy$$

$$= \underbrace{-\sigma_x^2 \frac{1}{\sqrt{2\pi\sigma_x^2}}\, y\, e^{-\frac{y^2}{2\sigma_x^2}}}_{=0} \Bigg|_{-\infty}^{+\infty} - \left(-\sigma_x^2 \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{y^2}{2\sigma_x^2}}\, dy}_{1}\right)$$

$$= \sigma_x^2 = E\left[(X-\eta_x)^2\right]$$

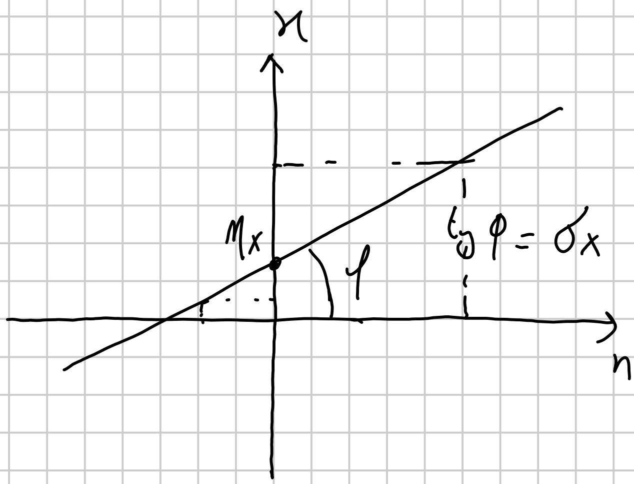
$$X \in \mathcal{N}(\eta_x, \sigma_x^2)$$

RELAZIONE TRA V.A. GAUSSIANE GENERICHE e V.A. GAUSSIANE STANDARD

$$X \in \mathcal{N}(\eta_x, \sigma_x^2)$$

$$N \in \mathcal{N}(0, 1)$$

$$\boxed{X = \sigma_x N + \eta_x}$$



$$f_X(x) = \sum_{i=1}^{N} \frac{f_N(n_i)}{|g'(n_i)|}$$

$\cdot)\ \{n_i\} \Rightarrow n = \dfrac{x-\eta_x}{\sigma_x}$

$$f_X(x) = \frac{\frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\frac{x-\eta_x}{\sigma_x}\right)^2}{2}}}{|\sigma_x|} = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}$$

$$\boxed{\begin{matrix}\text{SORG.}\\ \text{DI V.A G. S.}\end{matrix}} \xrightarrow{N} \boxed{X=\sigma_x N+m_x} \xrightarrow{X}$$

GENERAZIONE DI V.A. GAUSSIANE GENERICHE

CALCOLO DELLE PROBABILITÀ DI V.A. GAUSSIANE

$$P\{X \leq x\} \triangleq F_X(x)$$

PER V.A. GAUSSIANE STANDARD

$$\Phi(x) \triangleq \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2}{2}} \, d\alpha$$

$$X = \sigma_x N + m_x$$

$$N = \frac{X - m_x}{\sigma_x} \quad \Rightarrow \quad n = \frac{x - m_x}{\sigma_x}$$

$$P\{X \leq x\} = P\left\{N \leq \frac{x - m_x}{\sigma_x}\right) = \Phi\left(\frac{x - m_x}{\sigma_x}\right)$$

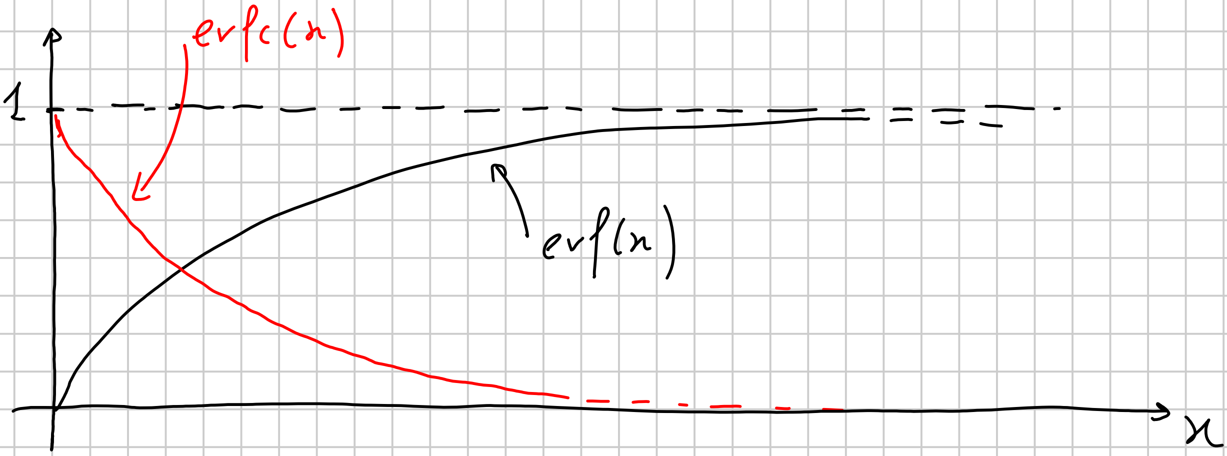
$$P\{a < X \leq b\} = \Phi\left(\frac{b - m_x}{\sigma_x}\right) - \Phi\left(\frac{a - m_x}{\sigma_x}\right)$$

FUNZIONE "ERF" - ERROR FUNCTION

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\alpha^2} \, d\alpha$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-\alpha^2} \, d\alpha$$

COMPLEMENTARI ERROR FUNCTION



$$\Phi(x) = F_N(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

$$P\{a < X \leq b\} = \Phi\left(\frac{b-\eta_x}{\sigma_x}\right) - \Phi\left(\frac{a-\eta_x}{\sigma_x}\right) =$$

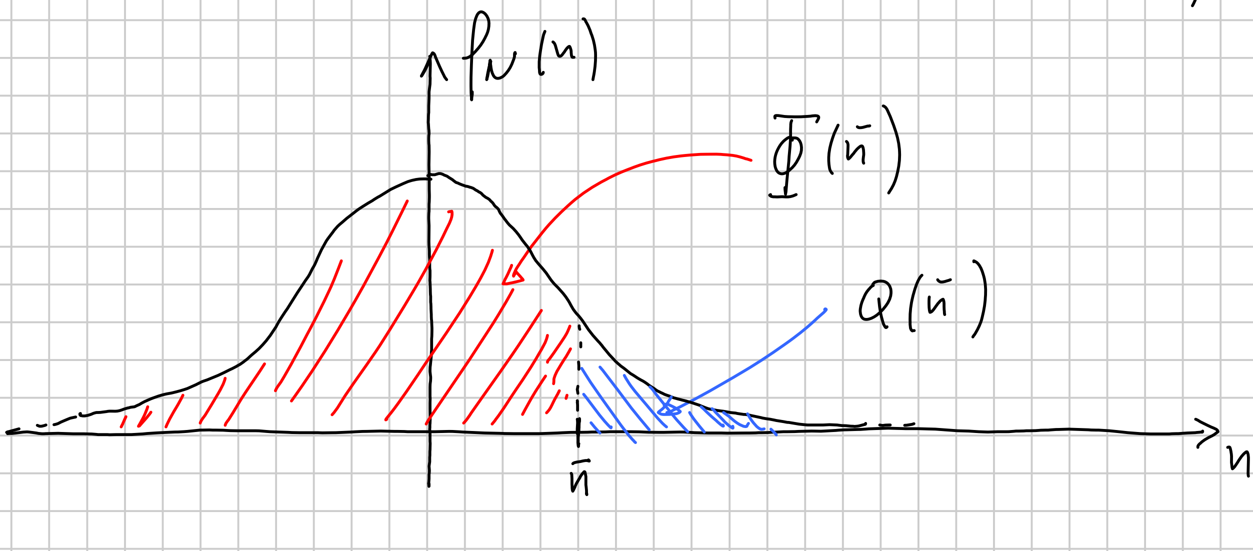
$$= \cancel{\frac{1}{2}} + \frac{1}{2}\,\mathrm{erf}\left(\frac{b-\eta_x}{\sqrt{2\sigma_x^2}}\right) - \cancel{\frac{1}{2}} - \frac{1}{2}\,\mathrm{erf}\left(\frac{a-\eta_x}{\sqrt{2\sigma_x^2}}\right) =$$

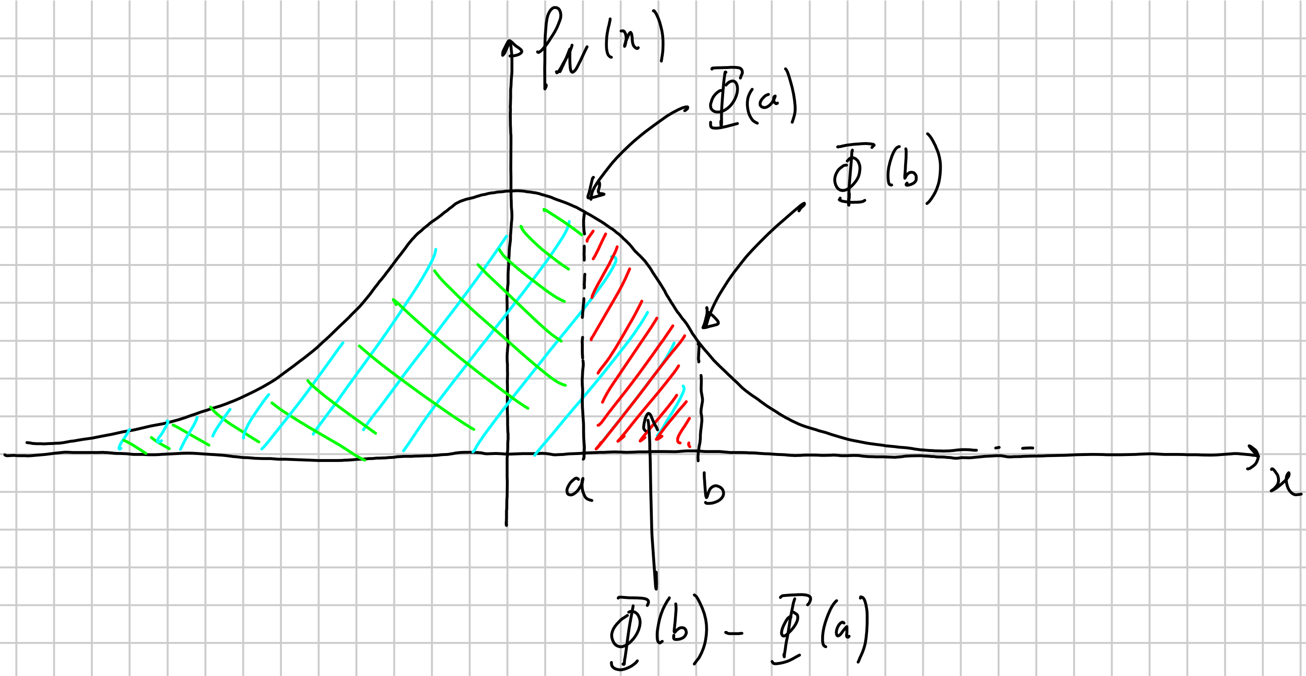
$$= \frac{1}{2}\,\mathrm{erf}\left(\frac{b-\eta_x}{\sqrt{2\sigma_x^2}}\right) - \frac{1}{2}\,\mathrm{erf}\left(\frac{a-\eta_x}{\sqrt{2\sigma_x^2}}\right)$$

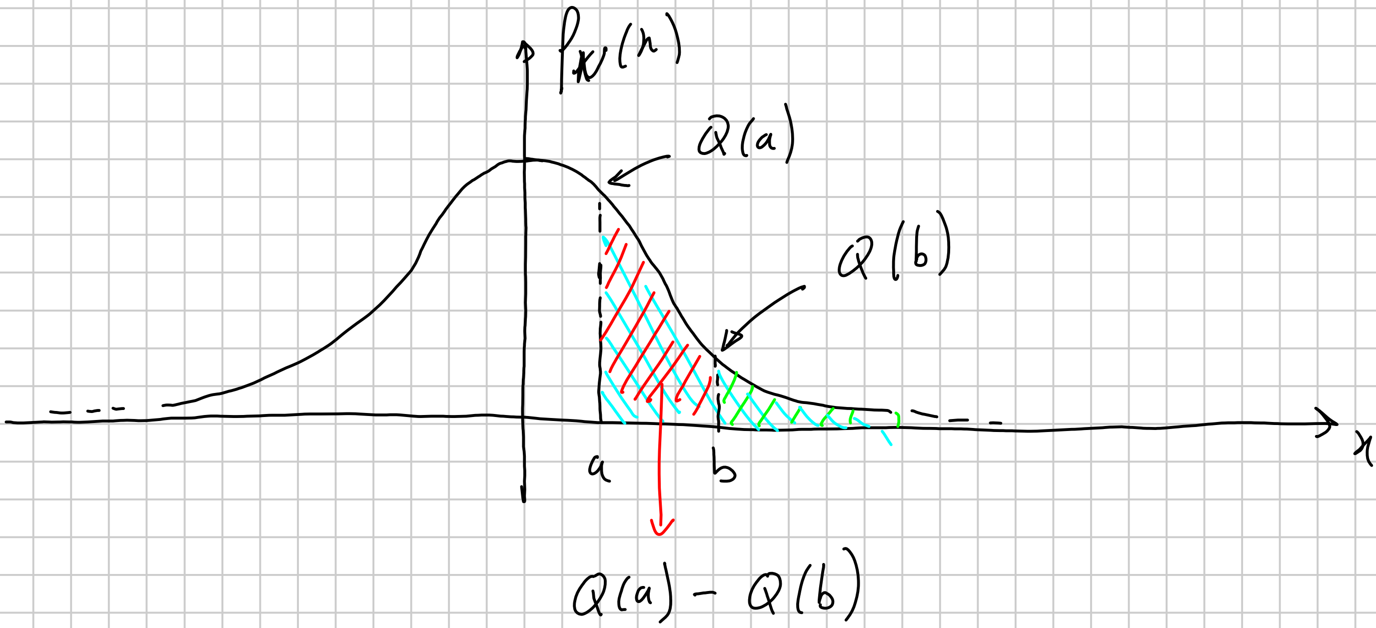
FUNZIONE "Q" — QUEUE "CODA"

$$Q(x) \triangleq 1 - \Phi(x) = 1 - \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) =$$

$$= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

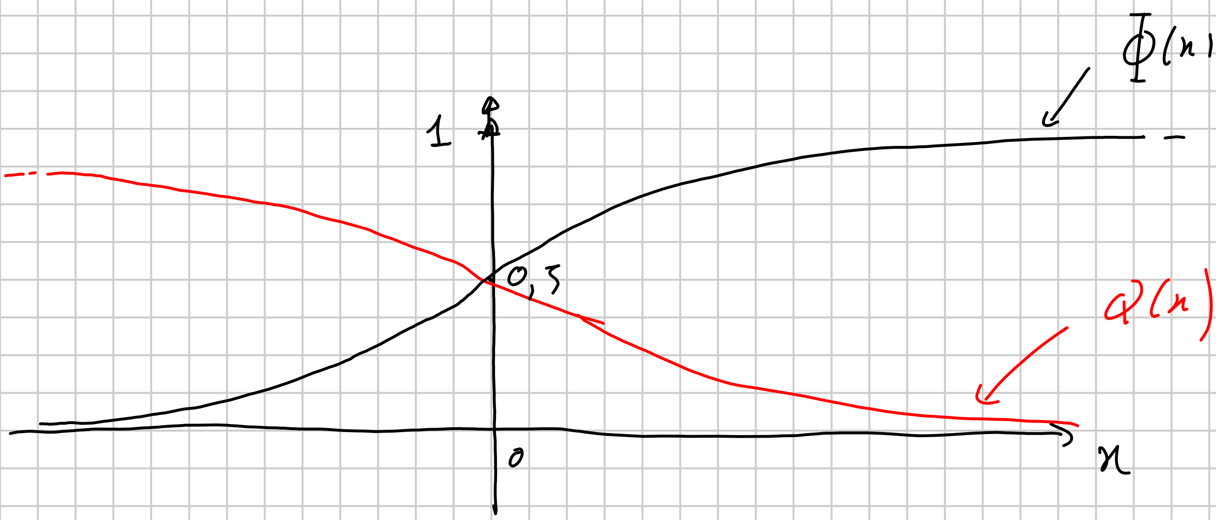






$$\Phi(b) - \Phi(a) = 1 - Q(b) - \left[1 - Q(a) \right] =$$

$$= \cancel{1} - Q(b) - \cancel{1} + Q(a) = Q(a) - Q(b)$$



V.A. CONDIZIONATE

$$P\{X \leq x\} = F_X(x)$$

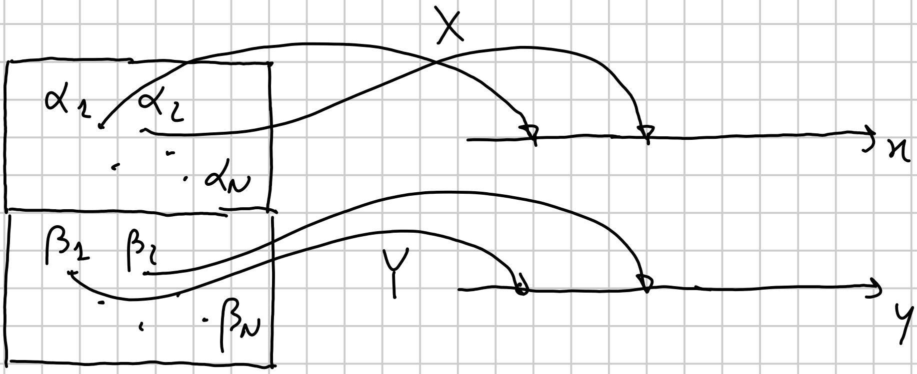
$$P\{X \leq x \mid B\} \triangleq F_{X|B}(x|B) = \frac{P\{X \leq x, B\}}{P\{B\}}$$

D.D.P. DI V.A. CONDIZIONATE

$$f_{X|B}(x|B) \triangleq \frac{d}{dx} F_{X|B}(x|B)$$

VALGONO TUTTE LE PROPRIETÀ VISTE PER LE V.A. NON CONDIZIONATE

SISTEMI DI DUE VARIABILI ALEATORIE



DESCRIZIONE CONGIUNTA DI DUE ESPERIMENTI

$$\boxed{F_{XY}(x,y) \triangleq P\{X \leq x, Y \leq y\}}$$

DISTRIBUZIONE DI PROBABILITÀ CONGIUNTA

$$F_X(x), F_Y(y) \not\Rightarrow F_{XY}(x,y)$$

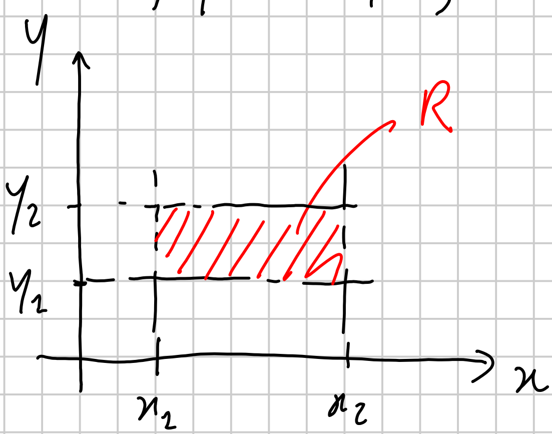
$$F_{XY}(x,y) \Rightarrow F_X(x), F_Y(y)$$

PROPRIETÀ

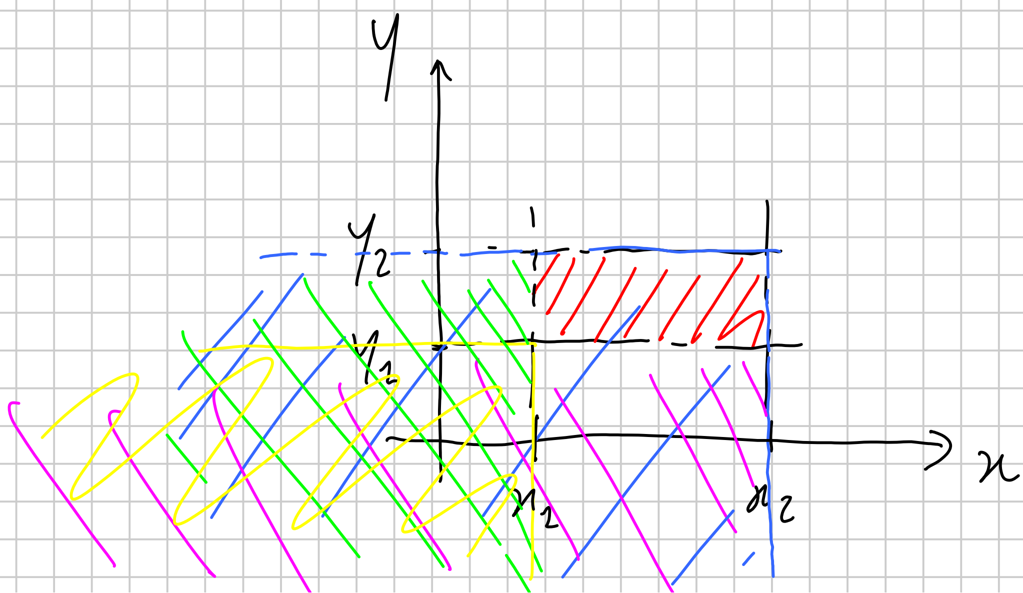
-) $0 \leq F_{XY}(x,y) \leq 1$
-) $F_{XY}(x, y_0)$ è monotona non decrescente e continua dx

- $F_{XY}(x_0, y)$ è monotona non decrescente e continua dx
- $F_{XY}(-\infty, y) = P\{X \leq -\infty, Y \leq y\} = 0$
- $F_{XY}(x, -\infty) = 0$
- $F_{XY}(-\infty, -\infty) = 0$
- $F_X(x) = F_{XY}(x, +\infty) = P\{X \leq x, Y \leq +\infty\} = P\{X \leq x\}$
- $F_Y(y) = F_{XY}(+\infty, y)$
- $F_{XY}(+\infty, +\infty) = 1 = P\{X \leq +\infty, Y \leq +\infty\}$
- La probabilità dell'evento "rettangolare"

$$R = \{x_1 < X \leq x_2, \; y_1 < Y \leq y_2\}$$



$$P\{R\} = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$



$$P\{\bar{x} < X < \bar{x}+\Delta x \ , \ \bar{y} < Y \leq \bar{y}+\Delta y\} =$$

$$= F_{XY}(\bar{x}+\Delta x, \bar{y}+\Delta y) - F_{XY}(\bar{x}, \bar{y}+\Delta y) - F_{XY}(\bar{x}+\Delta x, \bar{y})$$
$$+ F_{XY}(\bar{x}, \bar{y}) =$$

$$= F_{XY}(\bar{x}+\Delta x, \bar{y}+\Delta y) - F_{XY}(\bar{x}, \bar{y}+\Delta y) +$$
$$- \left[F_{XY}(\bar{x}+\Delta x, \bar{y}) - F_{XY}(\bar{x}, \bar{y}) \right] =$$

$$\simeq \frac{\partial}{\partial x} F_{XY}(x, \bar{y}+\Delta y)\,\Delta x - \frac{\partial}{\partial x} F_{XY}(x, \bar{y})\,\Delta x =$$

$$= \left[\frac{\partial}{\partial x} F_{XY}(x, \bar{y}+\Delta y) - \frac{\partial}{\partial x} F_{XY}(x, \bar{y}) \right] \Delta x =$$

$$= \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)\,\Delta x\,\Delta y =$$

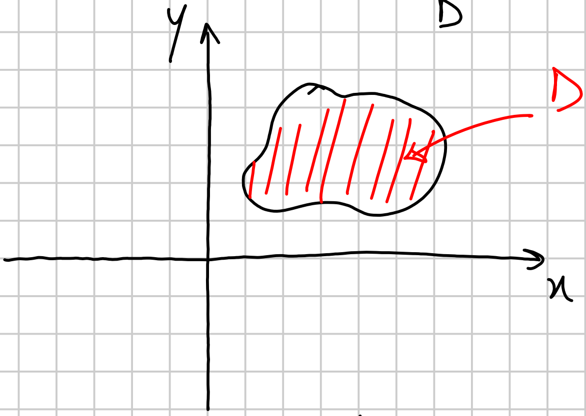
$$\Rightarrow f_{XY}(x, y) \triangleq \frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)$$

$$P\{x < X \leq x+\Delta x \ , \ y < Y \leq y+\Delta y\} \cong f_{XY}(x, y)\,\Delta x\,\Delta y$$

ddp congiunta di X e Y

$$f_{XY}(x,y) \triangleq \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

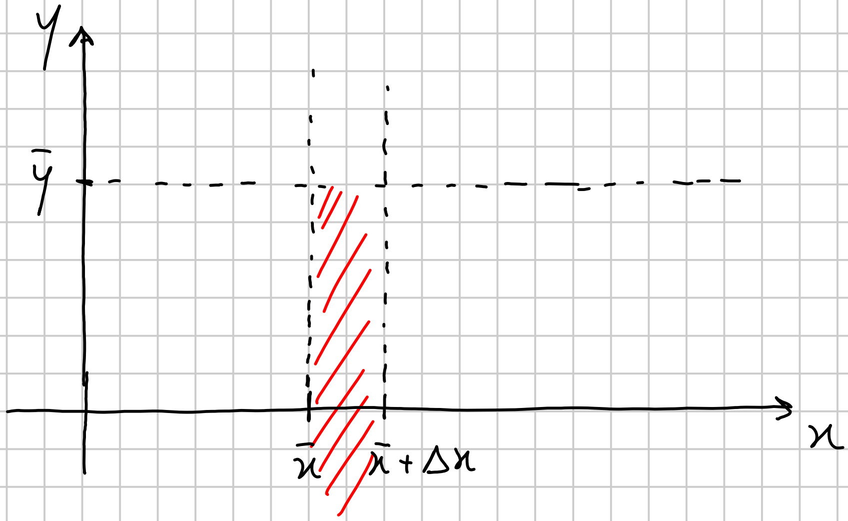
- $f_{XY}(x,y) \geq 0 \qquad \forall x,y$
- $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$
- $\displaystyle f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$
- $\displaystyle f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$
- $\displaystyle P\{(X,Y) \in D\} = \iint_D f_{XY}(x,y)\,dx\,dy$



- $\displaystyle F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(\alpha,\beta)\,d\alpha\,d\beta$

DISTRIBUZIONE DELLA V.A. Y CONDIZIONATA AD X

$$F_{Y|X}(y|x) \triangleq \frac{\int_{-\infty}^{y} f_{XY}(x,\beta)\,d\beta}{f_X(x)}$$



$$F_{Y|X}(y|x) = P\{Y \le y \mid X = x\} = \frac{P\{Y \le y,\, X = x\}}{P\{X = x\}}$$

$$P\{Y \le \bar{y} \mid \bar{x} < X \le \bar{x}+\Delta x\} = \frac{P\{Y \le \bar{y},\, \bar{x} < X \le \bar{x}+\Delta x\}}{P\{\bar{x} < X \le \bar{x}+\Delta x\}}$$

$$= \frac{\int_{-\infty}^{y}\int_{x}^{x+\Delta x} f_{XY}(\alpha,\beta)\, d\alpha\, d\beta}{\int_{x}^{x+\Delta x} f_X(x)\, dx} =$$

$$\simeq \frac{\int_{-\infty}^{y} f_{XY}(x,\beta)\, d\beta \cdot \cancel{\Delta x}}{f_X(x)\, \cancel{\Delta x}} = \frac{\int_{-\infty}^{y} f_{XY}(x,\beta)\, d\beta}{f_X(x)}$$

D.D.P. CONDIZIONATA

$$f_{Y|X}(y|x) \triangleq \frac{d}{dy} F_{Y|X}(y|x) =$$

$$= \frac{d}{dy} \frac{\int_{-\infty}^{y} f_{XY}(x,\beta)\, d\beta}{f_X(x)} = \frac{f_{XY}(x,y)}{f_X(x)}$$

INDIPENDENZA TRA X e Y

$$f_{Y|X}(y|x) = f_Y(y)$$

$$\Rightarrow \quad f_Y(y) = \frac{f_{XY}(x,y)}{f_X(x)}$$

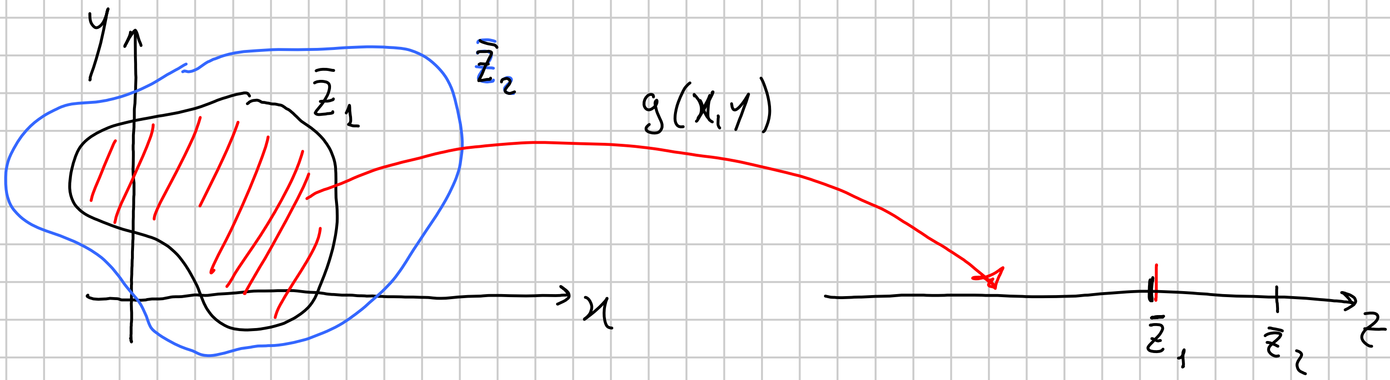
$$\Rightarrow \quad f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

SOLO NEL CASO DI INDIPENDENZA!

TRASFORMAZIONE DI UNA COPPIA DI V.A.

$$Z = g(X,Y)$$

$$F_Z(z) = P\{Z \leq z\} = P\{g(X,Y) \leq z\}$$



$$\boxed{F_Z(z) = \iint\limits_{R(z)} f_{XY}(x,y)\, dx\, dy}$$

$R(z)$ è la regione del piano (x,y) per cui $g(x,y) \leq z$

D.D.P. di Z

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

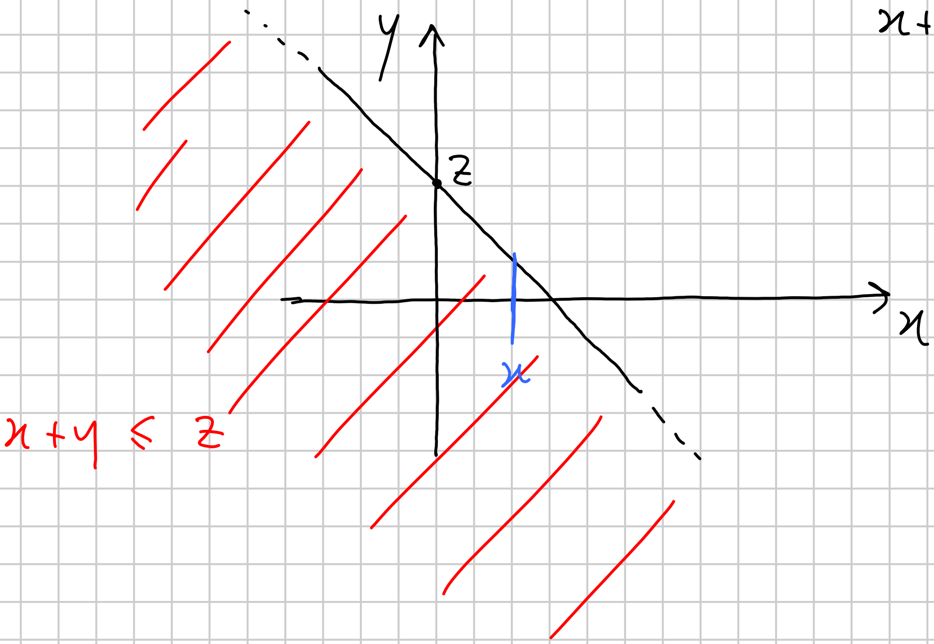
ESEMPIO

$Z = X + Y$

$f_{XY}(x,y)$

$f_Z(z) = ?$

$$F_Z(z) = \iint_{R(z)} f_{XY}(x,y)\, dx\, dy$$



$x + y = z \Rightarrow y = -x + z$

$$F_Z(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dx\, dy = \quad \begin{pmatrix} y' = y + x \end{pmatrix}$$

$y = y' - x$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{z} f_{XY}(x, y'-x)\, dx\, dy'$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_{-\infty}^{+\infty} \int_{-\infty}^{z} f_{XY}(x, y'-x)\, dx\, dy'$$

$$= \int_{-\infty}^{+\infty} \frac{d}{dz} \int_{-\infty}^{z} f_{XY}(x, y'-x)\, dx\, dy' =$$

$$= \int_{-\infty}^{+\infty} f_{XY}(x, z-x)\, dx$$

Se X e Y sono INDIPENDENTI

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z-x)\, dx = f_X(z) \otimes f_Y(z)$$

INDICI CARATTERIZZANTI COPPIE DI V.A.

X, Y

$$\boxed{r_{XY} \triangleq E[XY] = \iint xy\, f_{XY}(x,y)\, dx\, dy}$$ CORRELAZIONE

$$\boxed{\begin{aligned} c_{XY} &\triangleq E[(X-\eta_X)(Y-\eta_Y)] = \\ &= \iint (x-\eta_X)(y-\eta_Y)\, f_{XY}(x,y)\, dx\, dy \end{aligned}}$$ COVARIANZA

·) $c_{XY} = r_{XY} - \eta_X \eta_Y$

$$c_{XY} = \iint [xy - \eta_X y - \eta_Y x - \eta_X \eta_Y]\, f_{XY}(x,y)\, dx\, dy =$$

$$= \iint xy\, f_{XY}(x,y)\, dx\, dx + \quad \Rightarrow \quad r_{XY}$$

$$-\eta_X \iint y\, f_{XY}(x,y)\, dx\, dy + \quad \Rightarrow \quad -\eta_X \eta_Y$$

$$-\eta_Y \iint x\, f_{XY}(x,y)\, dx\, dy + \quad \Rightarrow \quad \cancel{-\eta_X \eta_Y}$$

$$+\eta_X \eta_Y \underbrace{\iint f_{XY}(x,y)\, dx\, dy}_{1} \quad \Rightarrow \quad \cancel{+\eta_X \eta_Y}$$

$$-\eta_X \int_{(y)} y \underbrace{\left[\int_{(x)} f_{XY}(x,y)\, dx \right]}_{f_Y(y)} dy = -\eta_X \underbrace{\int_{-\infty}^{+\infty} y\, f_Y(y)\, dy}_{\eta_Y} = -\eta_X \eta_Y$$

SIGNIFICATO DI COVARIANZA

INCORRELAZIONE TRA 2 V.A.

$$\boxed{X, Y \text{ sono incorrelate se } C_{XY} = 0}$$

COEFFICIENTE DI CORRELAZIONE

$$\rho_{XY} \triangleq \frac{C_{XY}}{\sigma_X \sigma_Y}$$

·) $-1 \leq \rho_{XY} \leq 1$, $|\rho_{XY}| \leq 1$

$X_1, Y_1 \Rightarrow C_{X_1 Y_1} = 10 \quad \sigma_X = 5 \quad \sigma_Y = 5$

$X_2, Y_2 \Rightarrow C_{X_2 Y_2} = 5 \quad \sigma_X = 3 \quad \sigma_Y = 2$

$$\rho_{X_1 Y_1} = \frac{10}{25} \quad , \quad \rho_{X_2 Y_2} = \frac{5}{6}$$

$$\rho_{X_2 Y_2} > \rho_{X_1 Y_1}$$

·) $|\rho_{XY}| = 1 \Rightarrow Y = aX + b$

·) $|\rho_{XY}| = 0 \Rightarrow C_{XY} = 0$

RELAZIONE TRA INCORRELAZIONE E INDIPENDENZA

Se X e Y SONO INDIPENDENTI ALLORA SONO ANCHE INCORRELATE

INDIPENDENZA $\Rightarrow$ INCORRELAZIONE

$$C_{XY} \stackrel{?}{=} 0$$

$$C_{XY} = r_{XY} - \eta_X \eta_Y$$

$$r_{XY} = \iint xy\, f_{XY}(x,y)\, dx\, dy = \iint xy\, f_X(x)\, f_Y(y)\, dx\, dy =$$

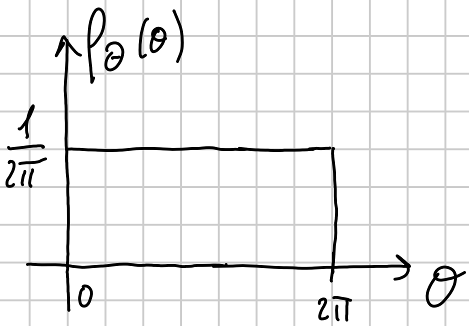
$$= \underbrace{\int x\, f_X(x)\, dx}_{\eta_X} \underbrace{\int y\, f_Y(y)\, dy}_{\eta_Y} = \eta_X \eta_Y$$

$$C_{XY} = \eta_X \eta_Y - \eta_X \eta_Y = 0$$

INCORRELAZIONE $\not\Longrightarrow$ INDIPENDENZA

Esempio

$$\Theta \in U[0, 2\pi) \Rightarrow f_\Theta(\theta) = \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta - \pi}{2\pi}\right)$$



Definiamo la trasformazione

$$\begin{cases} X = \cos \Theta \\ Y = \sin \Theta \end{cases} \qquad g(\Theta)$$

$$C_{XY} \stackrel{?}{=} 0$$

$$C_{XY} = r_{XY} - \eta_X \eta_Y$$

$$r_{XY} = E[XY] = \int_{-\infty}^{+\infty} \cos\theta \cdot \sin\theta \, f_\Theta(\theta) \, d\theta =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos\theta \sin\theta \, d\theta = \frac{1}{4\pi} \int_0^{2\pi} \sin(2\theta) \, d\theta = 0$$

$$\sin 2\theta = \sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

$$r_{XY} = 0$$

$$\eta_X = \int_{-\infty}^{+\infty} \cos\theta \, f_\Theta(\theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos\theta \, d\theta = 0$$

$$\eta_Y = \int_{-\infty}^{+\infty} \sin\theta \, f_\Theta(\theta) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \sin\theta \, d\theta = 0$$

$C_{XY} = 0 - 0 \cdot 0 = 0 \Rightarrow$ X e Y sono incorrelate

$$X^2 + Y^2 = 1 \Rightarrow X = \pm\sqrt{1-Y^2}$$

SISTEMI DI N V.A. (VETTORE ALEATORIO)

$X_1, X_2, \dots, X_N$

Distribuzione di probabilità congiunta

$$F_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N) \triangleq P\{X_1 \leq x_1, X_2 \leq x_2, \dots, X_N \leq x_N\}$$

D.D.P

$$f_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N) \triangleq \frac{\partial^N}{\partial x_1 \partial x_2 \cdots \partial x_N} F_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N)$$

D.D.P marginali

$$f_{X_1}(x_1) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N) \, dx_2 \, dx_3 \cdots dx_N$$

$$f_{X_j}(x_j) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N)\, dx_1 \cdots dx_{j-1}\, dx_{j+1} \cdots dx_N$$

D.D.P. CONDIZIONATA

$$f_{\{X_i\} | \{X_j\}}(\{x_i\} | \{x_j\}) \triangleq \frac{f_{X_1, \ldots, X_N}(x_1, \ldots, x_N)}{f_{\{X_j\}}(\{x_j\})}$$

$$\{X_i\} \cap \{X_j\} = \phi$$

ESEMPIO CON 5 V.A.

$$f_{X_1, X_2, X_4 | X_3, X_5}(x_1, x_2, x_4 | x_3, x_5) = \frac{f_{X_1 \ldots X_5}(x_1, \ldots, x_5)}{f_{X_3 X_5}(x_3, x_5)}$$

INDIPENDENZA DI N V.A.

$$f_{\{X_i\} | \{X_j\}}(\{x_i\} | \{x_j\}) = f_{\{X_i\}}(\{x_i\})$$

comunque si scelgano $\{x_i\}$ e $\{x_j\}$

Se N V.A. sono indipendenti

$$f_{X_1, \ldots, X_N}(x_1, \ldots, x_N) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_N}(x_N)$$

VETTORE ALEATORIO

$$\underline{X} \triangleq [X_1, X_2, \ldots, X_N]^T = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

- $F_{\underline{X}}(\underline{x}) \triangleq F_{X_1, \ldots, X_N}(x_1, x_2, \ldots, x_N)$
- $f_{\underline{X}}(\underline{x}) \triangleq f_{X_1, \ldots, X_N}(x_1, \ldots, x_N)$

INDICI CARATTERISTICI DI UN VETTORE ALEATORIO

$$\underline{m}_{\underline{X}} \triangleq E[\underline{X}] = \left[E[X_1], E[X_2], \dots, E[X_N] \right]^T$$

VALOR MEDIO DI UN VETT. AL.

MATRICE DI CORRELAZIONE

$$\underline{\underline{R}}_{\underline{X}} \triangleq \begin{bmatrix} r_{X_1X_1} & r_{X_1X_2} & \cdots & r_{X_1X_N} \\ \vdots & & & \\ r_{X_NX_1} & r_{X_NX_2} & \cdots & r_{X_NX_N} \end{bmatrix} \triangleq E\left[\underline{X}\,\underline{X}^T\right]$$

$$m_{X_i} = E\left[X_i^2\right]$$

VALORI QUADRATICI MEDI

$$= E\left[\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \left[X_1, X_2, \dots, X_N \right] \right]$$

MATRICE DI COVARIANZA

$$\underline{\underline{C}}_{\underline{X}} = E\left[\left(\underline{X} - \underline{m}_{\underline{X}}\right)\left(\underline{X} - \underline{m}_{\underline{X}}\right)^T \right] =$$

$$= \begin{bmatrix} c_{X_1X_1} & c_{X_1X_2} & \cdots & c_{X_1X_N} \\ \vdots & & & \vdots \\ c_{X_NX_1} & c_{X_NX_2} & \cdots & c_{X_NX_N} \end{bmatrix}$$

$$E\left[\left(X_i - m_{X_i}\right)^2\right] = \sigma_{X_i}^2$$

TRASFORMAZIONE DI UN VETTORE ALEATORIO

$$\underline{Y} = \underline{g}(\underline{X}) \quad , \quad \underline{X} = [X_1, \dots, X_N]$$

$$\underline{Y} = [Y_1, \dots, Y_M]$$

$$\underline{g} = [g_1(\underline{X}), g_2(\underline{X}), \dots, g_M(\underline{X})]$$

$$\underline{Y} = [Y_1 = g_1(\underline{X}), Y_2 = g_2(\underline{X}), \dots, Y_M = g_M(\underline{X})]$$

TEOREMA FONDAMENTALE GENERALIZZATO

$$f_{\underline{Y}}(\underline{y}) = \sum_{i=1}^{N} \frac{f_{\underline{X}}(\underline{x}_i)}{|\det \underline{\underline{J}}(x_i)|} \qquad N = M$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial g_1(\underline{x})}{\partial x_1} & \cdots & \frac{\partial g_1(\underline{x})}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial g_N(\underline{x})}{\partial x_1} & \cdots & \frac{\partial g_N(\underline{x})}{\partial x_N} \end{bmatrix}$$

PER 2 V.A.

$$f_{Y_1 Y_2}(y_1, y_2) = \sum_{i=1}^{N} \frac{f_{X_1, X_2}(x_{1i}, x_{2i})}{|\det \underline{\underline{J}}(x_{1i}, x_{2i})|}$$

VETTORI ALEATORI GAUSSIANI

N V.A. CONGIUNTAMENTE GAUSSIANE

$$\boxed{f_{\underline{X}}(\underline{x}) \triangleq \frac{1}{\sqrt{(2\pi)^N \det \underline{\underline{C}}_{\underline{X}}}} e^{-\frac{1}{2}(\underline{X}-\underline{\eta}_{\underline{X}})^T \underline{\underline{C}}_{\underline{X}}^{-1}(\underline{X}-\underline{\eta}_{\underline{X}})}}.$$

$\underline{\eta}_{\underline{X}}$, $\underline{\underline{C}}_{\underline{X}}$

·) N V.A. CONG. GAUSSIANE INCORRELATE

$\Downarrow$

INDIPENDENZA

Dim x 2 V.A. CONG GAUSSIANE INCORRELATE

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{\sqrt{(2\pi)^2 \sigma_{X_1}^2 \sigma_{X_2}^2}} e^{-\frac{1}{2}\left[\frac{(x_1-\eta_{X_1})^2}{\sigma_{X_1}^2} + \frac{(x_2-\eta_{X_2})^2}{\sigma_{X_2}^2}\right]}$$

$$\underline{\underline{C}}_{\underline{X}} = \begin{bmatrix} \sigma_{X_1}^2 & 0 \\ 0 & \sigma_{X_2}^2 \end{bmatrix} \qquad \det\left\{\underline{\underline{C}}_X\right\} = \sigma_{X_1}^2 \sigma_{X_2}^2$$

$$[x_1-\eta_{X_1} , x_2-\eta_{X_2}] \begin{bmatrix} 1/\sigma_{X_1}^2 & 0 \\ 0 & 1/\sigma_{X_2} \end{bmatrix} \begin{bmatrix} x_1-\eta_{X_1} \\ x_2-\eta_{X_2} \end{bmatrix} =$$

$$= \left[\frac{x_1-\eta_{X_1}}{\sigma_{X_1}^2} , \frac{x_2-\eta_{X_2}}{\sigma_{X_2}^2}\right] \begin{bmatrix} x_1-\eta_{X_1} \\ x_2-\eta_{X_2} \end{bmatrix}$$

$$= \frac{(x_1 - \eta_{x_1})^2}{\sigma_{x_1}^2} + \frac{(x_2 - \eta_{x_2})^2}{\sigma_{x_2}^2}$$

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi \sigma_{x_1}^2}} e^{-\frac{(x_1 - \eta_{x_1})^2}{2\sigma_{x_1}^2}} \cdot$$

$$\cdot \frac{1}{\sqrt{2\pi \sigma_{x_2}^2}} e^{-\frac{(x_2 - \eta_{x_2})^2}{2\sigma_{x_2}^2}}$$

$$= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \quad \Rightarrow \text{INDIPENDENZA}$$

•) $\underline{X}$ cong. gaussiana

$$\underline{X}' = [X_1, \ldots, X_M] \qquad M < N$$

$\underline{X}'$ è ancora un vettore aleatorio Gaussiano

(Non sono necessariamente le prime "M")

$$\Rightarrow \underline{X}' = \{X_i\} \qquad \{X_i\} \in \underline{X}$$

N.B. $\underline{X}' = X_i \Rightarrow$ D.D.P. MARGINALI GAUSSIANE

•) $\underline{Y} = a\underline{X} + b$ e se $\underline{X}$ è Gaussiano allora anche $\underline{Y}$ è Gaussiano

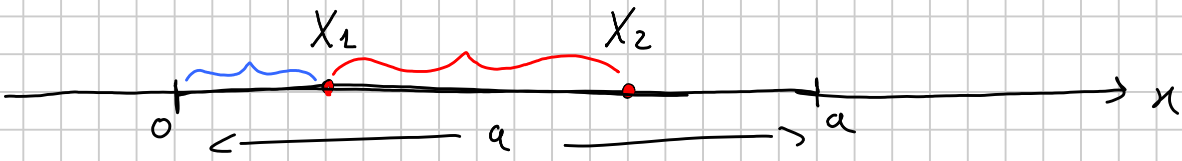
$$\underline{\eta}_{\underline{Y}} = \underline{\underline{A}}\, \underline{\eta}_{\underline{X}} + \underline{B} \qquad \underline{\underline{A}} = \text{matrice di trasformaz.}$$

$$\underline{B} = \text{vettore}$$

$$\underline{\underline{C}}_{\underline{Y}} = A\, \underline{\underline{C}}_{\underline{X}}\, \underline{\underline{A}}^T$$

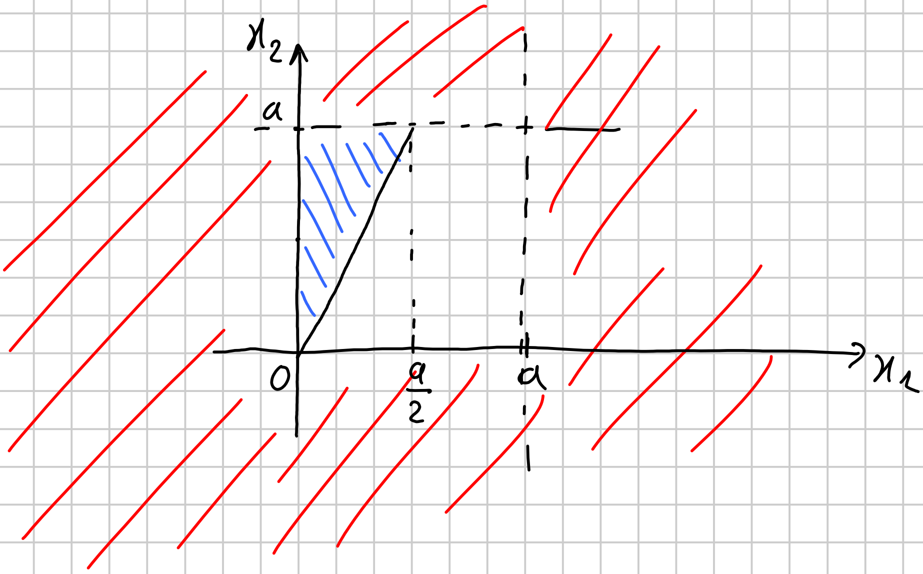
•) $\{X_i\} \mid \{X_j\}$ è ancora cong. Gaussiano comunque si scelgano gli insiemi

ESERCIZIO 1



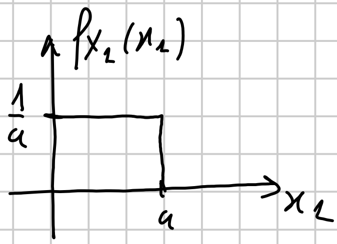
$$P\{X_2 - X_1 > X_1\} = ?$$

$$P\{X_2 > 2X_1\}$$



$$P\{D\} = \iint_D f_{XY}(x_1, x_2)\, dx_1 dx_2$$

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$$



$$f_{X_1}(x_1) = \frac{1}{a}\, \mathrm{rect}\left(\frac{x_1 - a/2}{a}\right)$$

$$f_{X_2}(x_2) = \frac{1}{a}\, \mathrm{rect}\left(\frac{x_1 - a/2}{a}\right)$$

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{a^2}\, \mathrm{rect}\left(\frac{x_1 - a/2}{a}\right) \mathrm{rect}\left(\frac{x_2 - a/2}{a}\right)$$

$$P\{X_2 > 2X_1\} = \frac{1}{a^2}\, a \cdot \frac{a}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

ESERCIZIO 2

X_1, X_2 indipendenti

$X_1 \in \mathcal{N}(0, \sigma^2)$

$X_2 \in \mathcal{N}(0, \sigma^2)$

$$\begin{cases} g_1: A = \sqrt{X_1^2 + X_2^2} & A \geq 0 \\ g_2: \Phi = tg^{-1} \dfrac{X_2}{X_1} & -\pi \leq \Phi \leq \pi \end{cases}$$

SOLUZIONE

$$f_{A\Phi}(a, \varphi) = \sum_{i=1}^{N} \frac{f_{X_1X_2}(x_{1i}, x_{2i})}{\left|\det \underline{\underline{J}}(x_{1i}, x_{2i})\right|}$$

SOLUZIONI INVERSE

$$\begin{cases} X_1 = A \cos \Phi \\ X_2 = A \sin \Phi \end{cases}$$

$$\underline{\underline{J}} = \begin{bmatrix} \dfrac{\partial g_1(\underline{x})}{\partial x_1} & \dfrac{\partial g_1(\underline{x})}{\partial x_2} \\ \\ \dfrac{\partial g_2(\underline{x})}{\partial x_1} & \dfrac{\partial g_2(\underline{x})}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial g_1(\underline{x})}{\partial x_1} = \frac{\partial}{\partial x_1}\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} = \frac{1}{2}\left(x_1^2 + x_2^2\right)^{-\frac{1}{2}} \cdot 2x_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_1(\underline{x})}{\partial x_2} = \frac{\partial}{\partial x_2}\left(x_1^2 + x_2^2\right)^{\frac{1}{2}} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_2(\underline{x})}{\partial x_1} = \frac{1}{1+\frac{x_2^2}{x_1^2}} \cdot x_2 \left(-x_1^{-2}\right) = \frac{-x_2}{x_1^2+x_2^2}$$

$$\frac{\partial g_2(\underline{x})}{\partial x_2} = \frac{1}{1+\frac{x_2^2}{x_1^2}} \cdot \frac{1}{x_1} = \frac{x_1}{x_1^2+x_2^2}$$

$$\frac{\partial g_1(\underline{x}_i)}{\partial x_1} = \frac{a\cos\varphi}{a} = \cos\varphi$$

$$\frac{\partial g_1(\underline{x}_i)}{\partial x_2} = \frac{a\sin\varphi}{a} = \sin\varphi$$

$$\frac{\partial g_2(\underline{x}_i)}{\partial x_1} = \frac{-a\sin\varphi}{a^2} = -\frac{\sin\varphi}{a}$$

$$\frac{\partial g_2(\underline{x}_i)}{\partial x_2} = \frac{a\cos\varphi}{a^2} = \frac{\cos\varphi}{a}$$

$$\underline{\underline{J}}(x_i) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\frac{\sin\varphi}{a} & \frac{\cos\varphi}{a} \end{bmatrix}$$

$$\left|\det\left\{\underline{\underline{J}}(x_i)\right\}\right| = \left|\frac{\cos^2\varphi}{a} + \frac{\sin^2\varphi}{a}\right| = \left|\frac{1}{a}\right| = \frac{1}{a}$$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{\sqrt{(2\pi)^2\sigma^2\sigma^2}} e^{-\frac{x_1^2+x_2^2}{2\sigma^2}}$$

$$f_{A\Phi}(a,\varphi) = \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} \qquad 0 \le a \le +\infty$$

$$= \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)$$

$$f_A(a) = \int_{-\infty}^{+\infty} f_{A\Phi}(a,\varphi)\, d\varphi =$$

$$= \int_{-\pi}^{\pi} \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)\, d\varphi = \frac{a}{\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)$$

RAYLEIGH

$$f_\Phi(\varphi) = \int_0^{+\infty} \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}}\, da = \qquad (a^2 = b)$$

$$2a\, da = db$$

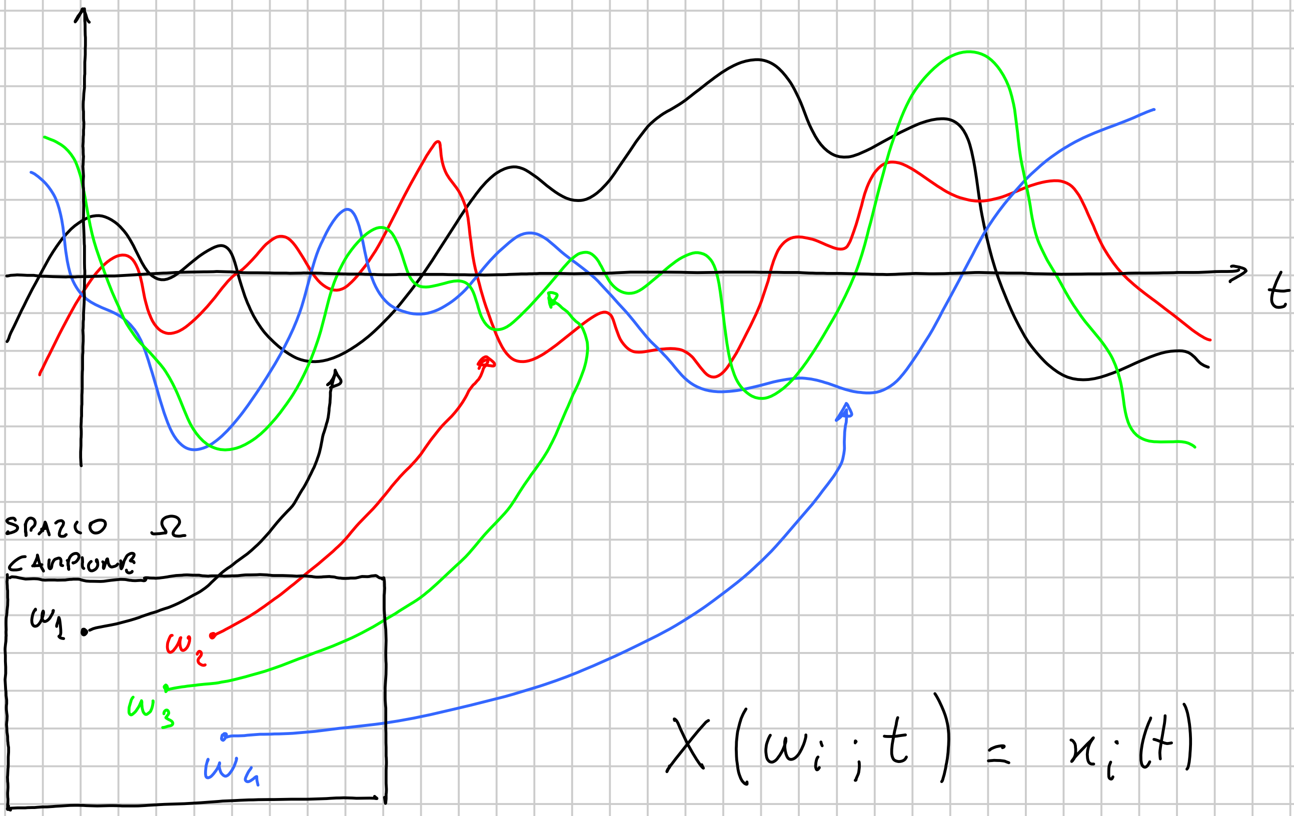
$$= \int_0^{+\infty} \frac{1}{4\pi\sigma^2} e^{-\frac{b}{2\sigma^2}}\, db =$$

$$= \frac{1}{4\pi\sigma^2} \left[-2\sigma^2 e^{-\frac{b}{2\sigma^2}} \right]_0^{+\infty} =$$

$$= -\frac{1}{2\pi} e^{-\frac{b}{2\sigma^2}} \Big|_0^{+\infty} = 0 + \frac{1}{2\pi} = \frac{1}{2\pi} \qquad (-\pi \leq \varphi \leq \pi)$$

$$\Phi \in U[-\pi, \pi] \Rightarrow f_\Phi(\varphi) = \frac{1}{2\pi} \mathrm{rect}\left(\frac{\varphi}{2\pi}\right)$$

SEGNALI ALEATORI



$$X(\omega_i\,;t) = x_i(t)$$

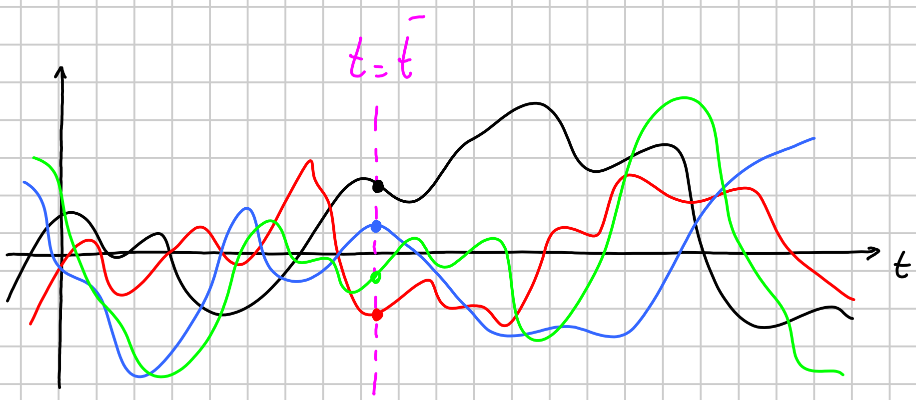
PROCESSO ALEATORIO

$$X(\omega\,;t) \Rightarrow X(t)$$

la corrispondenza tra il risultato di un esperimento casuale ed una realizzazione di un segnale aleatorio

Possiamo vedere una v.a. come un processo ad un determinato istante temporale $\bar{t}$

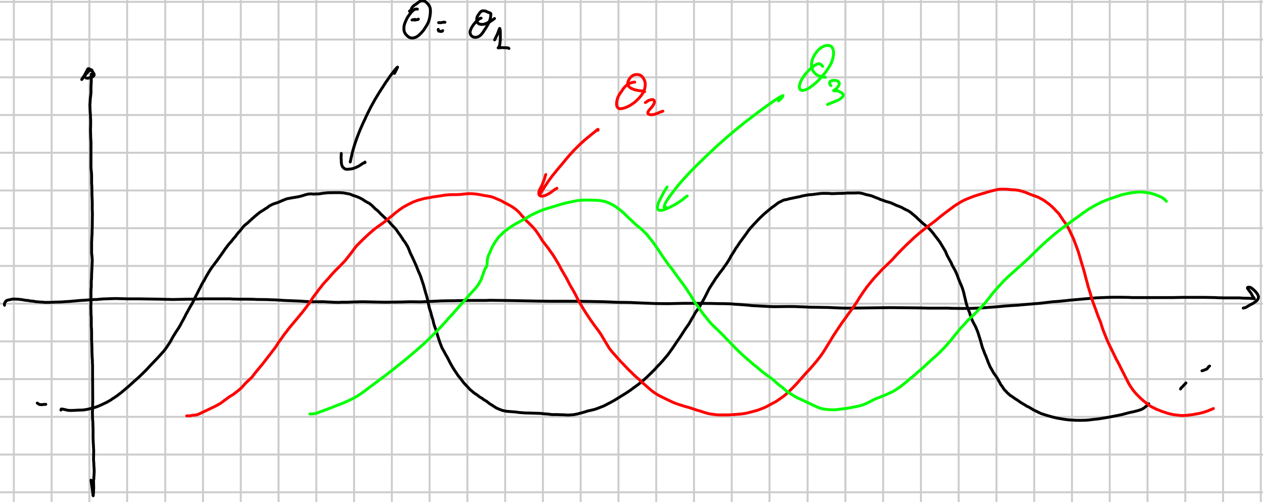
$$\text{v.a.} \quad X(\omega\,;\bar{t}) \Rightarrow X(\bar{t})$$



PROCESSI PARAMETRICI

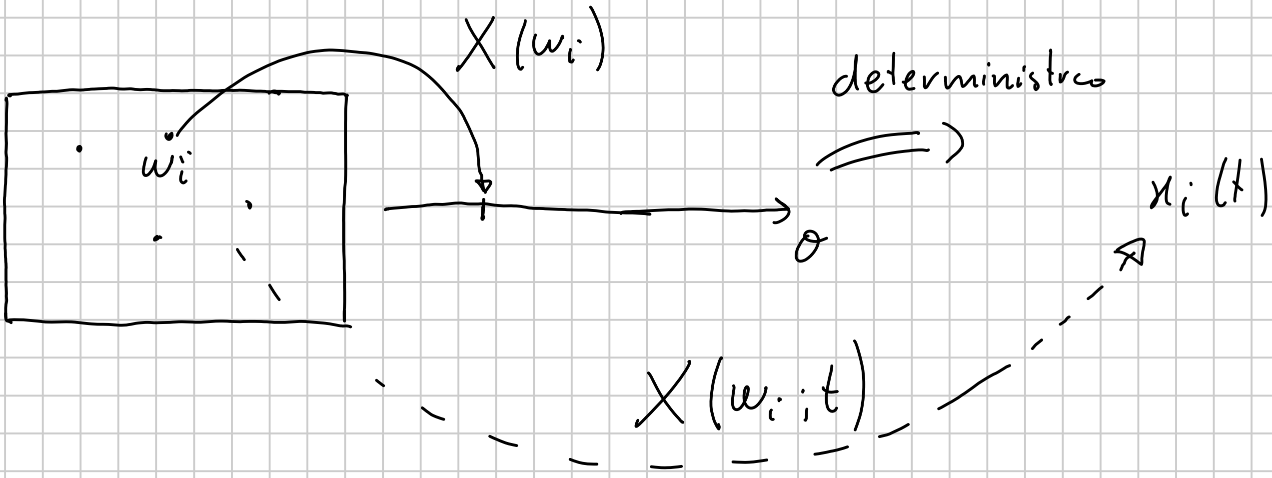
$$X(t) = a \cos(2\pi f_0 t + \Theta)$$

V.A.



$$X(t) = A \cos(2\pi f_0 t + \Theta)$$

V.A. V.A.



CARATTERIZZAZIONE STATISTICA DI PROCESSI ALEATORI

DISTRIBUZIONE DI PROBABILITÀ DEL I° ORDINE

$$X(t)\Big|_{t=\bar{t}} = X(\bar{t})$$

$$F_X(x_1; t_1) \triangleq P\{X(t_1) \leq x_1\}$$

DISTRIBUZIONE DI PROBABILITÀ DEL II° ORDINE

$$F_X(x_1, x_2; t_1, t_2) \triangleq P\{X(t_1) \leq x_1 \,,\; X(t_2) \leq x_2\}$$

$$\vdots$$

DISTRIBUZIONE DI PROBABILITÀ DI ORDINE N

$$F_X(x_1, \dots, x_N; t_1, \dots, t_N) \triangleq P\{X(t_1) \leq x_1, \dots, X(t_N) \leq x_N\}$$

UN PROCESSO ALEATORIO È COMPLETAMENTE CARATTERIZZATO DAL PUNTO DI VISTA STATISTICO SE È NOTA LA FUNZIONE DISTRIBUZIONE DI PROBABILITÀ DI ORDINE N, CON N ARBITRARIO

DENSITÀ DI PROBABILITÀ DI ORDINE N

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) \triangleq \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_X(x_1, \dots, x_N; t_1, \dots, t_N)$$

$\Rightarrow$ I° ORDINE

$$f_X(x_1; t_1) = \frac{\partial}{\partial x_1} F_X(x_1, t_1) = \frac{d}{dx_1} F_X(x_1, t_1)$$

$\Rightarrow$ II° ORDINE

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

INDICI STATISTICI DI PROCESSI ALEATORI DEL I° ORDINE

VALOR MEDIO

$X(\bar{t})$

$$\eta_X(\bar{t}) \triangleq E\left[X(\bar{t})\right] \triangleq \int_{-\infty}^{+\infty} x\, f_X(x;\bar{t})\, dx$$

$$\Downarrow$$

$$\eta_X(t) \triangleq E\left[X(t)\right] \triangleq \int_{-\infty}^{+\infty} x\, f_X(x;t)\, dx$$

POTENZA MEDIA STATISTICA

$$P_X(t) \triangleq E\left[X^2(t)\right] \triangleq \int_{-\infty}^{+\infty} x^2 f_X(x;t)\, dx$$

$P_x(t) = |x(t)|^2$, se $x(t)$ e' reale $P_x(t) = x^2(t)$
(per segnali deterministici)

VARIANZA

$$\sigma_X^2(t) \triangleq E\left[\left(X(t)-\eta_X(t)\right)^2\right] \triangleq \int_{-\infty}^{+\infty} \left(x-\eta_X(t)\right)^2 f_X(x;t)\, dx$$

RELAZIONE TRA $\eta_X(t)$, $P_X(t)$ e $\sigma_X^2(t)$

$$\sigma_X^2(t) = P_X(t) - \eta_X^2(t)$$

Dim:

$$\sigma_X^2(t) = E\left[\left(X(t)-\eta_X(t)\right)^2\right] = E\left[X^2(t) + \eta_X^2(t) - 2X(t)\eta_X(t)\right]$$

$$= E\left[X^2(t)\right] + \eta_X^2(t) - 2\eta_X(t)E\left[X(t)\right] =$$

$$= P_X(t) + \eta_X^2(t) - 2\eta_X(t)\,\eta_X(t)$$
$$= P_X(t) - \eta_X^2(t)$$

AUTOCORRELAZIONE (INDICE DEL II° ORDINE)

$$X(t) \Longrightarrow X(t_1),\ X(t_2)$$

$$R_X(t_1,t_2) \triangleq E\left[X(t_1)\,X(t_2)\right] =$$
$$= \int_{-\infty}^{+\infty} x_1 x_2\, f_X(x_1,x_2;t_1,t_2)\,dx_1\,dx_2$$

AUTOCOVARIANZA

$$C_X(t_1,t_2) \triangleq E\left[\left(X(t_1)-\eta_X(t_1)\right)\left(X(t_2)-\eta_X(t_2)\right)\right]$$
$$= \int_{-\infty}^{+\infty} \left(x_1-\eta_X(t_1)\right)\left(x_2-\eta_X(t_2)\right) f_X(x_1,x_2;t_1,t_2)\,dx_1\,dx_2$$

RELAZIONE TRA $R_X(t_1,t_2)$, $C_X(t_1,t_2)$, $\eta_X(t_1)$, $\eta_X(t_2)$

$$C_X(t_1,t_2) = R_X(t_1,t_2) - \eta_X(t_1)\,\eta_X(t_2)$$

STAZIONARIETÀ DI PROCESSI ALEATORI

- Stazionarietà in senso stretto SSS
- " " " lato SSL

·) SSS

$$X(t_1, t_2, \dots, t_N)$$

$$X(t_1+\Delta t, t_2+\Delta t, \dots, t_N+\Delta t) \qquad \forall \Delta t$$

$$\Rightarrow f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1+\Delta t, \dots, t_N+\Delta t)$$

ddp I° ordine

$$f_X(x;t) = f_X(x; t+\Delta t) \qquad \forall \Delta t$$

$$\overset{!}{=} f_X(x) \qquad \text{NON DIPENDE DAL TEMPO !}$$

ddp II° ordine

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1+\Delta t, t_2+\Delta t)$$

$$\overset{!}{=} f_X(x_1, x_2; t_2 - t_1)$$

$$\Delta t = -t_1$$

$$f_X(x_1, x_2; 0, t_2 - t_1)$$

INDICI I° ORD. DI PROCESSI SSS

$$\eta_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \eta_X$$

$$P_X(t) = E[X^2(t)] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = P_X$$

$$\sigma_X^2(t) = \sigma_X^2 = P_X - \eta_X^2$$

INDICI DEL II° ORDINE DI UN PROCESSO SSS

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{+\infty} x_1 x_2 f_X(x_1, x_2; t_1 - t_2)\, dx_1 dx_2$$

$$= R_X(t_1 - t_2)$$

$$C_X(t_1, t_2) = C_X(t_1 - t_2)$$

STATISTICHE DI ORDINE N : dipendono dalle differenze tra gli istanti temporali

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1 - t_2, t_2 - t_3, \dots, t_{N-1} - t_N)$$

N.B. la stazionarietà di ordine N implica la stazionarietà di tutti gli ordini inferiori ad N. Dimostrabile applicando la proprietà delle ddp marginali

esempio:

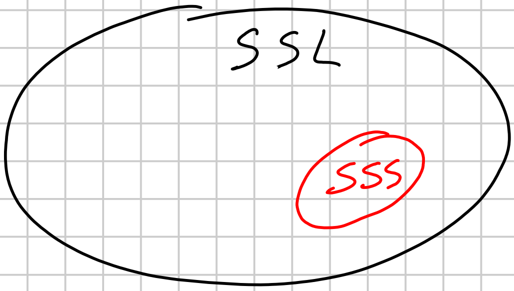
$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 - t_2) = f_X(x_1, x_2; \tau)$$

$\Downarrow$ SOLO IN UN VERSO

$$f_X(x_1; t_1) = \int_{-\infty}^{+\infty} f_X(x_1, x_2; t_1, t_2)\, dx_2 = f_X(x)$$

·) SSL

$$\left.\begin{aligned} &\cdot)\ \eta_X(t) = \eta_X \\ &\cdot)\ R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau) \end{aligned}\right\}\ \text{SSL}$$



AUTOCOVARIANZA DI UN PROCESSO SSL

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \eta_X(t_1)\,\eta_X(t_2)$$
$$= R_X(\tau) - \eta_X^2 = C_X(\tau)$$

PROPRIETÀ DELLA AUTOCORRELAZIONE DI PROCESSI SSL

1) $R_X(\tau) = R_X(-\tau)$

Dim $R_X(\tau) = E\left[X(t)\,X(t-\tau)\right] =$ $\quad (t - \tau = t')$

$$= E\left[X(t'+\tau)\,X(t')\right] = E\left[X(t')\,X(t'-(-\tau))\right]$$
$$= R_X(-\tau)$$

2) $R_X(0) = E\left[X(t)\,X(t)\right] = E\left[X^2(t)\right] = P_X \geq 0$

3) $R_X(0) \geq |R_X(\tau)| \quad \forall\ \tau$

Dim: $E\left[\left(X(t) \pm X(t-\tau)\right)^2\right] \geq 0$

$$E\left[X^2(t)\right] + E\left[X^2(t-\tau)\right] \pm 2E\left[X(t)\,X(t-\tau)\right] \geq 0$$

$$P_X + P_X \pm 2R_X(\tau) \geq 0$$

$$\cancel{2}P_X \pm \cancel{2}R_X(\tau) \geq 0 \implies P_X \geq \pm R_X(\tau)$$

$$\implies P_X \geq |R_X(\tau)|$$
$$= R_X(0) \geq |R_X(\tau)|$$

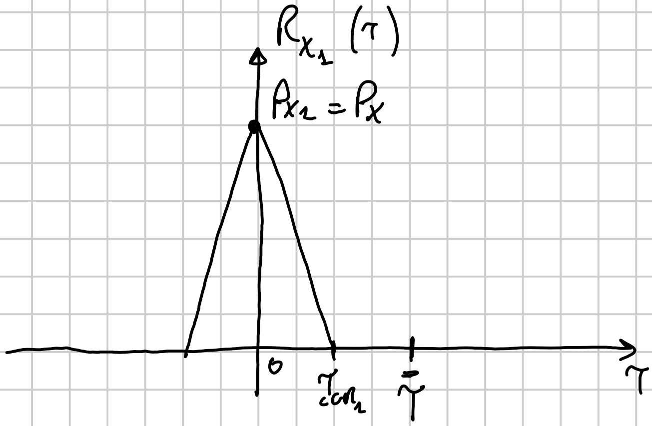
4) Se la $R_X(\tau)$ non contiene componenti periodiche

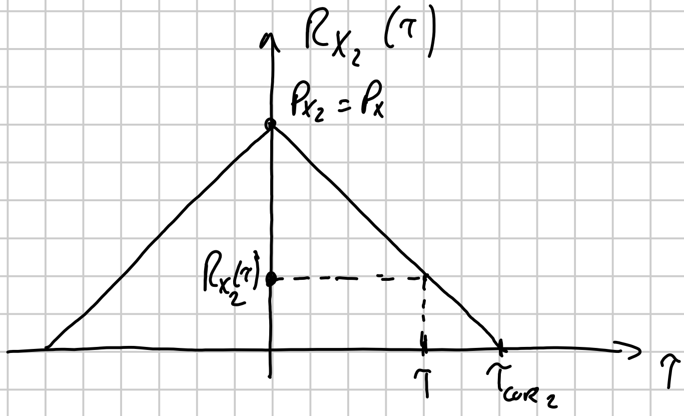
$$\lim_{\tau \to \infty} R_X(\tau) = \eta_X^2$$

Giustificazione: $R_X(\tau) = C_X(\tau) + \eta_X^2$

$$\lim_{\tau \to \infty} \underbrace{C_X(\tau)}_{0} + \eta_X^2$$

SIGNIFICATO DELLA AUTOCORRELAZIONE

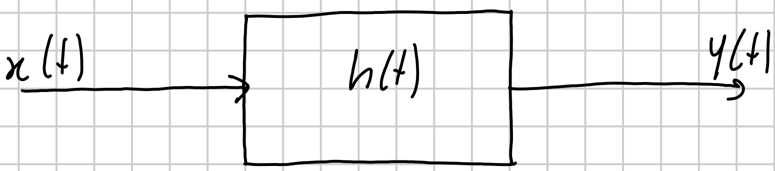




τ_{COR} : $\tau > \tau_{cor}$ $R_X(\tau) = 0$

↳ TEMPO DI DECORRELAZIONE

FILTRAGGIO DI UN PROCESSO ALEATORIO



$$y(t) = x(t) \otimes h(t)$$

$x(t) + D(t)$

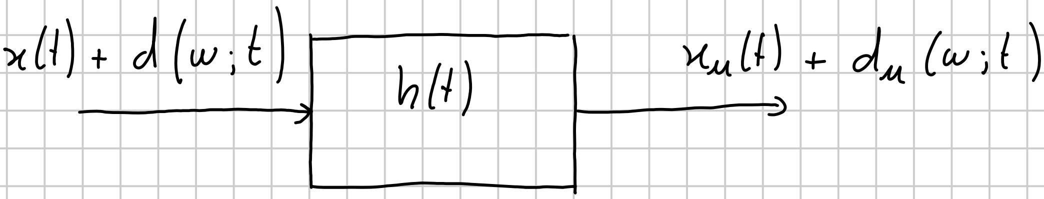
$x(t)$: segnale utile

$D(t)$: disturbo (aleatorio)

$$y(t) = x_u(t) + D_u(t)$$

$$x_u(t) = x(t) \otimes h(t)$$

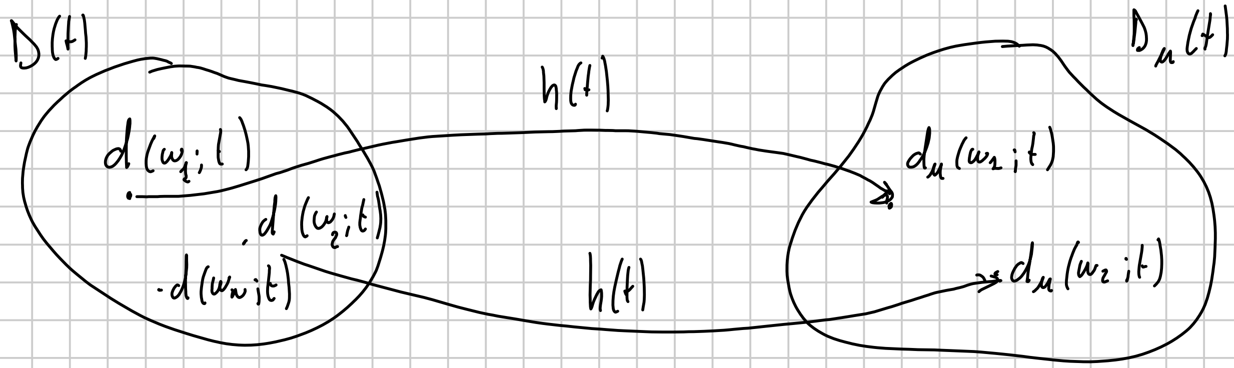
$$D_u(t) = ?$$



$d(w;t)$ è una realizzazione del processo aleatorio in ingresso al sistema

$$d_u(w;t) = d(w;t) \otimes h(t)$$

$$Y(t) = X(t) \otimes h(t)$$



DESCRIZIONE STATISTICA COMPLETA DEL PROCESSO IN INGRESSO + CONOSCENZA DELLA RISPOSTA IMPULSIVA

$$\not\Downarrow$$

DESCRIZ. STAT. COMPLETA DEL PROCESSO IN USCITA

VALOR MEDIO DEL PROCESSO DI USCITA

$$\eta_Y(t) = E[Y(t)] = E[X(t) \otimes h(t)] =$$

$$= E\left[\int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha\right] = \int_{-\infty}^{+\infty} E[X(\alpha)]\, h(t-\alpha)\, d\alpha$$

$$= \int_{-\infty}^{+\infty} \eta_X(\alpha)\, h(t-\alpha)\, d\alpha = \eta_X(t) \otimes h(t)$$

Se $\eta_X(t) = 0 \Rightarrow \eta_Y(t) = 0$

$$X(t) = X_0(t) + \eta_X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = Y_0(t) + \eta_Y(t)$$

$$\eta_Y(t) = \eta_X(t) \otimes h(t)$$
$$Y_0(t) = X_0(t) \otimes h(t)$$

AUTOCORRELAZIONE DI UN PROCESSO IN USCITA AD UN SLS

$$R_Y(t_1, t_2) = E[Y(t_1)\, Y(t_2)] =$$

$$= E\left[\int_{-\infty}^{+\infty} X(\alpha)\, h(t_1-\alpha)\, d\alpha \int_{-\infty}^{+\infty} X(\beta)\, h(t_2-\beta)\, d\beta\right] =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[X(\alpha)\, X(\beta)]\, h(t_1-\alpha)\, h(t_2-\beta)\, d\alpha\, d\beta =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\alpha, \beta)\, h(t_1-\alpha)\, d\alpha\, h(t_2-\beta)\, d\beta$$

$$= \int_{-\infty}^{+\infty} \left[R_X(t_1, \beta) \otimes h(t_1) \right] h(t_2 - \beta)\, d\beta$$
(β)

$$= R_X(t_1, t_2) \otimes h(t_1) \otimes h(t_2)$$

FILTRAGGIO DI PROCESSI SSL

VALOR MEDIO

$$\eta_Y(t) = \eta_X(t) \otimes h(t) = \int_{-\infty}^{+\infty} \eta_X(\alpha)\, h(t-\alpha)\, d\alpha =$$

$$= \eta_X \int_{-\infty}^{+\infty} h(t-\alpha)\, d\alpha = \eta_X \int_{-\infty}^{+\infty} h(\alpha)\, d\alpha = \eta_X\, H(0)$$

$$H(0) = H(f)\Big|_{f=0} = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt \Big|_{f=0} = \int_{-\infty}^{+\infty} h(t)\, dt$$

AUTOCORRELAZIONE

$$R_Y(t_1, t_2) = R_Y(t, t-\tau) = E\left[Y(t)\, Y(t-\tau) \right]$$

$$= E\left[\int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha \int_{-\infty}^{+\infty} X(\beta)\, h(t-\tau-\beta)\, d\beta \right]$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E\left[X(\alpha)\, X(\beta) \right] h(t-\alpha)\, h(t-\tau-\beta)\, d\beta\, d\alpha$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_X(\alpha-\beta)\, h(t-\alpha)\, h(t-\tau-\beta)\, d\beta\, d\alpha$$

$$\Rightarrow \quad \alpha - \beta = \xi$$

$$= \underset{(\beta)}{\int_{-\infty}^{+\infty}} \underset{(\xi)}{\int_{-\infty}^{+\infty}} R_x(\xi)\, h(t-\beta-\xi)\, d\xi\, h(t-\tau-\beta)\, d\beta$$

$$= \int_{-\infty}^{+\infty} \left[R_x(t-\beta) \otimes h(t-\beta) \right] h(t-\beta-\tau)\, d\beta$$

$$\Rightarrow \quad t - \beta = \beta'$$

$$= \int_{-\infty}^{+\infty} \left[R_x(\beta') \otimes h(\beta') \right] h\left[-(\tau-\beta')\right] d\beta'$$

$$= R_x(\tau) \otimes h(\tau) \otimes h(-\tau) = R_x(\tau) \otimes r_h(\tau)$$

$$r_h(\tau) = h(\tau) \otimes h(-\tau) = \int_{-\infty}^{+\infty} h(t)\, h(t-\tau)\, dt$$

DENSITÀ SPETTRALE DI POTENZA MEDIA (STATISTICA) PER PROCESSI ALMENO STAZIONARI IN SENSO LATO

Per segnali deterministici a potenza finita

$$S_x(f) = \lim_{T \to \infty} \frac{|X_T(f)|^2}{T}$$

Per segnali aleatori (processi)

$$S_x(w;f) = \lim_{T \to \infty} \frac{|X_T(w;f)|^2}{T}$$

$$S_X(f) = E\left[S_X(\omega; f) \right] \triangleq \lim_{T \to \infty} \frac{E\left[|X_T(\omega; f)|^2 \right]}{T}$$

TEOREMA DI WIENER - KHINTCHINE

$$\boxed{S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau)\, e^{-j2\pi f \tau}\, d\tau}$$

PROPRIETÀ DELLA DSP

1) $S_X(f)$ è reale e pari

Dim: ·) $R_X(\tau)$ è reale e pari

·) TCF di un segnale reale e pari è anche essa reale e pari

2) $P_X = \int_{-\infty}^{+\infty} S_X(f)\, df$

Dim. $P_X = E\left[X^2(t)\right] = R_X(0) = \left. \int_{-\infty}^{+\infty} S_X(f)\, e^{j2\pi f \tau}\, df \right|_{\tau=0}$

$$= \int_{-\infty}^{+\infty} S_X(f)\, df$$

3) $S_X(f) \geq 0 \quad \forall f$

FILTRAGGIO DI UN PROCESSO ALEATORIO E DENSITÀ SPETTRALE DI POTENZA

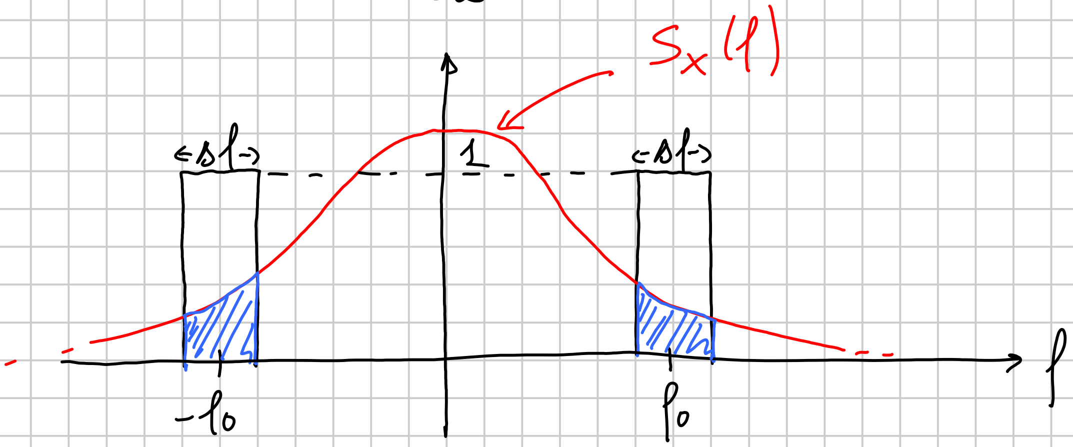
$$\xrightarrow[S_X(f)]{X(t)} \boxed{\ h(t)\ } \xrightarrow[S_Y(f)]{Y(t)}$$

$$S_Y(f) = TCF\left[R_Y(\tau)\right] = TCF\left[R_X(\tau) \otimes h(\tau) \otimes h(-\tau)\right]$$

Supponendo $h(t)$ reale

$$= S_X(f)\, H(f)\, H^*(f) = S_X(f)\, |H(f)|^2$$

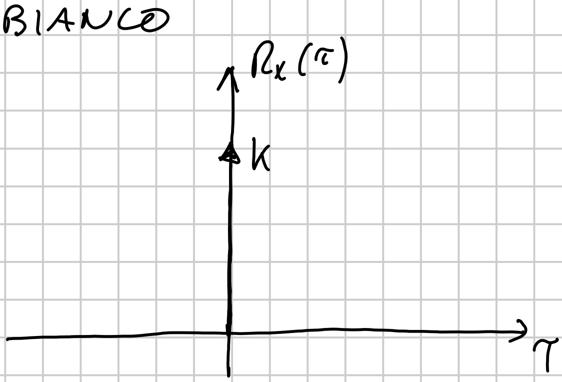
$$P_Y = \int_{-\infty}^{+\infty} S_Y(f)\, df = \int_{-\infty}^{+\infty} S_X(f)\, |H(f)|^2\, df$$



$P_Y \geq 0$

$P_Y \simeq 2 S_X(f_0) \cdot \Delta f \geq 0 \qquad \forall f_0$

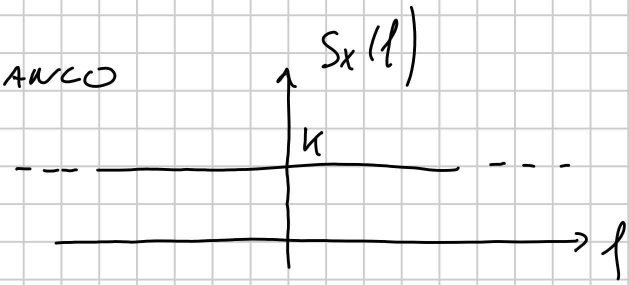
PROCESSO DI RUMORE BIANCO



·) $\eta_X = 0$

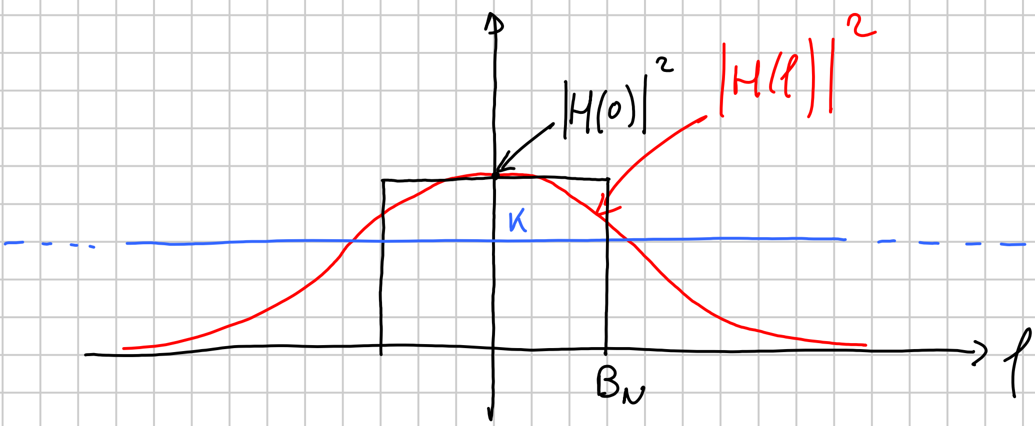
·) $R_X(\tau) = K\, \delta(\tau)$

DSP DI UN RUMORE BIANCO



$S_X(f) = K$

BANDA EQUIVALENTE DI RUMORE DI UN FILTRO



$$P_Y = \int_{-\infty}^{+\infty} K\,|H(f)|^2\,df = K\int_{-\infty}^{+\infty} |H(f)|^2\,df$$

$$B_N : 2B_N|H(0)|^2 \cdot K = P_Y$$

$$2B_N|H(0)|^2 \cdot \cancel{K} = \cancel{K}\int_{-\infty}^{+\infty} |H(f)|^2\,df$$

$$\Rightarrow \boxed{B_N \triangleq \frac{\int_{-\infty}^{+\infty} |H(f)|^2\,df}{2|H(0)|^2}}$$

PROCESSI ALEATORI GAUSSIANI

Se si estrae un N-upla di v.a. dal processo

$$\underline{X} = \{ X(t_1), X(t_2), \dots, X(t_N) \}$$

ddp di $\underline{X}$ è congiuntamente Gaussiana comunque si scelgano gli istanti $t_1, \dots, t_N$ e $\forall N$.

$$\Downarrow$$

PROCESSO è GAUSSIANO

Per caratterizzare dal punto di vista statistico un processo aleatorio Gaussiano ed in maniera completa basta conoscere $\eta_X(t)$ e $R_X(t_1, t_2)$.

$$f_{\underline{X}}(x_1, \dots, x_N ; t_1, \dots, t_N) =$$

$$= \frac{1}{\sqrt{(2\pi)^N \det(\underline{\underline{C}}_{\underline{X}})}} e^{-\frac{1}{2}(\underline{x} - \underline{\eta})^T \underline{\underline{C}}_{\underline{X}}^{-1} (\underline{x} - \underline{\eta}_{\underline{X}})}$$

$$\underline{\eta}_{\underline{X}} \triangleq \begin{bmatrix} \eta_X(t_1) \\ \eta_X(t_2) \\ \vdots \\ \eta_X(t_N) \end{bmatrix}, \quad \underline{\underline{C}}_{\underline{X}} \triangleq \begin{bmatrix} C(t_1,t_1) & C(t_1,t_2) & \cdots & C(t_1,t_N) \\ C(t_2,t_1) & C(t_2,t_2) & \cdots & C(t_2,t_N) \\ \vdots & \vdots & & \\ C(t_N,t_1) & C(t_N,t_2) & \cdots & C(t_N,t_N) \end{bmatrix}$$

$$C_X(t_i, t_j) \qquad i,j = 1, \dots, N$$

$$\parallel$$

$$R_X(t_i, t_j) - \eta_X(t_i)\,\eta_X(t_j)$$

STAZIONARIETÀ DI PROCESSI GAUSSIANI

$$\boxed{\begin{matrix} SSL & \Rightarrow & SSS \\ SSL & \Leftarrow\!\Rightarrow & SSS \end{matrix}}$$

Dim.

$$\underline{X} = \{ X(t_1), \dots, X(t_N) \}$$

$$\underline{X}' = \{ X(t_1+\Delta t), \dots, X(t_N+\Delta t) \}$$

$$\boxed{f_{\underline{X}}(\underline{x}; \underline{t}) = f_{\underline{X}'}(\underline{x}'; \underline{t})}$$

$$SSL \Rightarrow \quad \eta_X(t) = \eta_X$$

$$\underline{\eta}_{\underline{X}} = \begin{bmatrix} \eta_X \\ \eta_X \\ \vdots \\ \eta_X \end{bmatrix} \qquad \underline{\eta}_{\underline{X}'} = \begin{bmatrix} \eta_X \\ \eta_X \\ \vdots \\ \eta_X \end{bmatrix}$$

$$R(t_1, t_2) = R(t_1 - t_2) = R(\tau)$$

$$C(t_1, t_2) = C(t_1 - t_2) = C(\tau)$$

$$\underline{\underline{C}}_{\underline{X}} \overset{?}{=} \underline{\underline{C}}_{\underline{X}'}$$

$$\underline{\underline{C}}_{\underline{X}} = \begin{bmatrix} C(0) & C(t_1 - t_2) & \cdots & C(t_1 - t_N) \\ \vdots & \ddots & & \\ & & C(0) & \\ & & & \ddots \\ C(t_N - t_1) & \cdots & \cdots & C(0) \end{bmatrix}$$

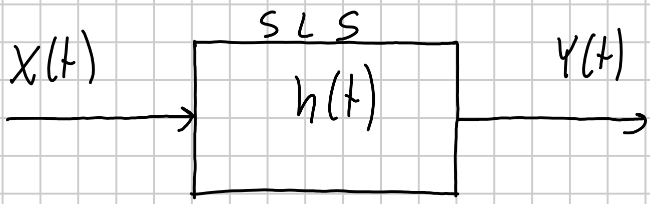
$$\underline{\underline{C}}_{\underline{x}'} = \begin{bmatrix} C(t_1+\cancel{\Delta t}-(t_1+\cancel{\Delta t})) & C(t_1+\cancel{\Delta t}-(t_2+\cancel{\Delta t})) & \cdots & C(t_1+\cancel{\Delta t}-(t_N+\cancel{\Delta t})) \\ \vdots & & & \\ C(t_N+\cancel{\Delta t}-(t_1+\cancel{\Delta t})) & - - - & - - - & C(t_N+\cancel{\Delta t}-(t_N+\cancel{\Delta t})) \end{bmatrix}$$

$$\underline{\underline{C}}_{x'} = \underline{\underline{C}}_{x}$$

Se $\underline{\eta}_{\underline{x}} = \underline{\eta}_{\underline{x}'}$ e $\underline{\underline{C}}_{\underline{x}} = \underline{\underline{C}}_{\underline{x}'}$

allora $f_{\underline{x}}(\underline{x};\underline{t}) = f_{\underline{x}'}(\underline{x}';\underline{t})$

FILTRAGGIO DI PROCESSI GAUSSIANI



$$Y(t) = X(t) \otimes h(t) = \int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha$$

Se il processo $X(t)$ e' Gaussiano
allora anche il processo $Y(t)$ e' Gaussiano

Dim. $\qquad d\alpha \to \Delta\alpha$

$$Y(t) = \int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha \simeq \sum_{n=-\infty}^{+\infty} X(n\Delta\alpha)\, h(t-n\Delta\alpha)\, \Delta\alpha$$

$$\begin{cases} Y(t_1) = \displaystyle\sum_{n=-\infty}^{+\infty} X(n\Delta\alpha)\, h(t_1-n\Delta\alpha)\,\Delta\alpha \\ \vdots \qquad \vdots \\ Y(t_N) = \displaystyle\sum_{n=-\infty}^{+\infty} X(n\Delta\alpha)\, h(t_N-n\Delta\alpha)\,\Delta\alpha \end{cases}$$

$$\begin{cases} Y(t_1) = \sum_{n=-\infty}^{+\infty} \beta_1(n)\, X(n\Delta\alpha) & , \quad \beta_1(n) = h(t_1 - n\Delta\alpha)\,\Delta\alpha \\ \vdots & \qquad \vdots \\ Y(t_N) = \sum_{n=-\infty}^{+\infty} \beta_N(n)\, X(n\Delta\alpha) & , \quad \beta_N(n) = h(t_N - n\Delta\alpha)\,\Delta\alpha \end{cases}$$

$$\underline{Y} = \underline{\underline{A}}\, \underline{X}$$

$\underline{Y}$ è un vettore aleatorio Gaussiano

$\Rightarrow$ $Y(t)$ è un processo aleatorio Gaussiano

Conoscenza statistica completa di un processo $X(t)$ in ingresso ad un SLS di cui conosco la $h(t)$

$$\Downarrow$$

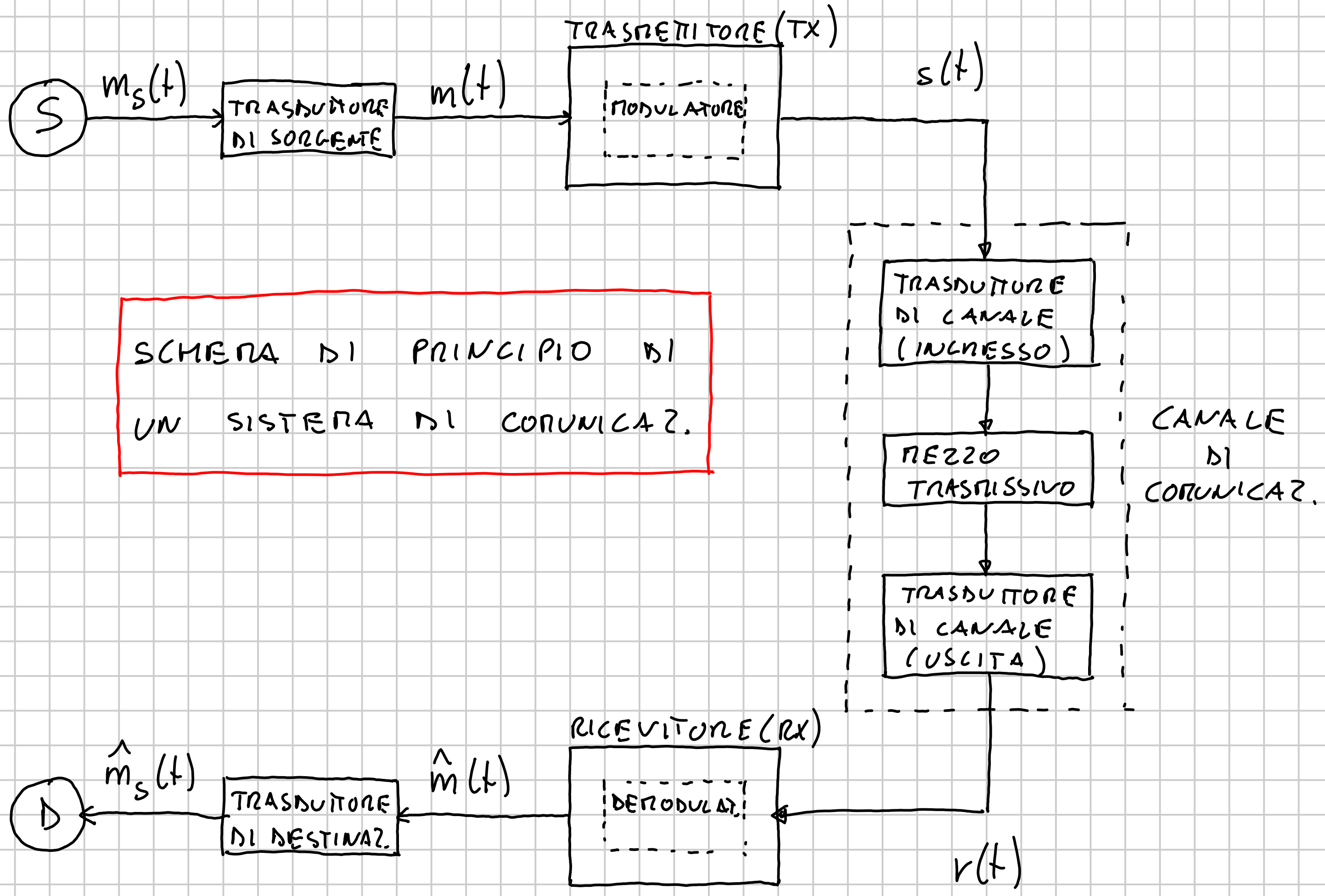
conoscenza statistica completa del processo $Y(t)$ in uscita al filtro previo il calcolo di $\eta_Y(t)$ e $R_Y(t_1, t_2)$

$$\eta_Y(t) = \eta_X(t) \otimes h(t)$$
$$R_Y(t_1, t_2) = R_X(t_1, t_2) \otimes h(t_1) \otimes h(t_2)$$

INTRODUZIONE AI SISTEMI DI COMUNICAZIONE

Sistemi di comunicazione punto - punto: 1 nodo sorgente ed 1 nodo destinazione.

SCHEMA DI PRINCIPIO DI UN SISTEMA DI COMUNICAZ.



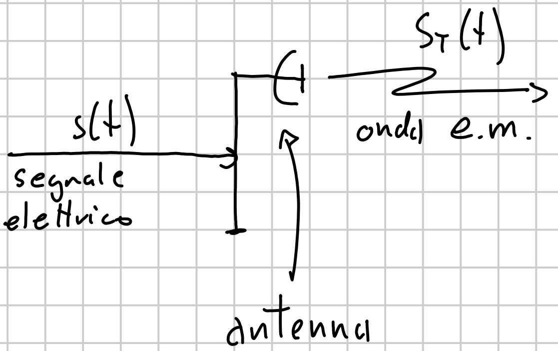
TRASMETTITORE

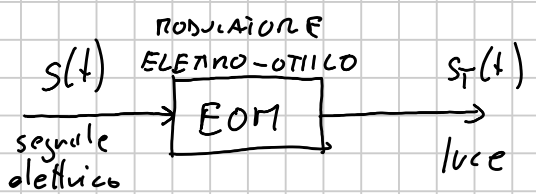
1) TRASL. IN FREQ. (MODULAZIONE)

2) SAGOMATURA

$s(t)$ = segnale trasmesso (analogico)

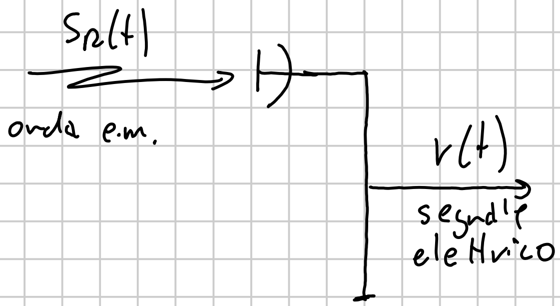
TRASDUTTORE DI CANALE (INGRESSO)

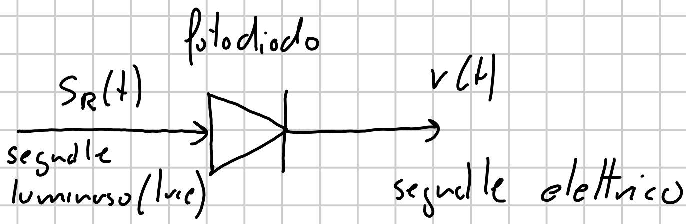




TRASDUTTORE DI CANALE (USCITA)

Riporta il segnale su un supporto fisico di tipo elettrico





RICEVITORE

1) Operazione di sagomatura inversa
2) Traslazione in freq. inversa

$$\boxed{\begin{array}{l}\text{Se } r(t) = s(t) \\ \text{allora } \hat{m}(t) = m(t)\end{array}}$$

3) Eliminare o ridurre il più possibile i disturbi introdotti dal canale di comunicazione

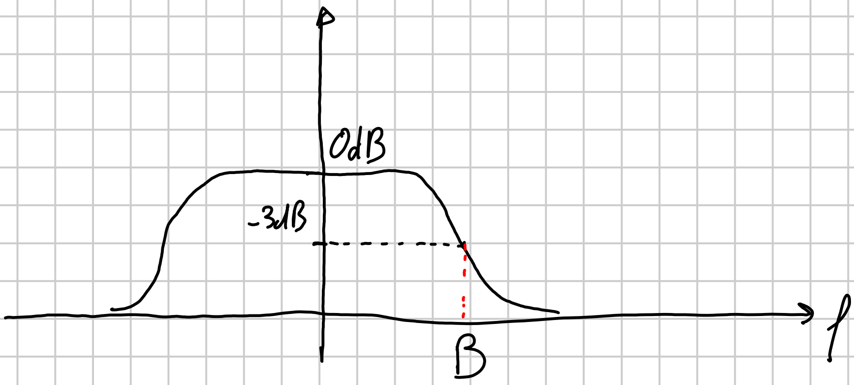
BANDA PASSANTE E LARGHEZZA DI BANDA DI UN CANALE DI COMUNICAZIONE

BANDA PASSANTE : $\{ f : f_{min} \leq f \leq f_{max} \}$

LARGHEZZA DI BANDA : $B = f_{max} - f_{min}$

FREQ CENTRALE : $f_0 = \dfrac{f_{max} + f_{min}}{2}$

In banda base



BANDA BASE : $\{ f : 0 \leq f \leq B \}$

LARGHEZZA DI BANDA : B

BANDA LARGA E BANDA STRETTA

BANDA LARGA : $f_0 \leq 2B$

BANDA STRETTA: $f_0 > 2B$

Esempi

·) Doppino telefonico:

$$\left.\begin{array}{ll} \text{BANDA PASSANTE: (zona lineare)} & 300\ \text{Hz} \div 4\ \text{KHz} \\ B: & 3.7\ \text{KHz} \\ f_0: & 2.15\ \text{KHz} \end{array}\right\} \text{BANDA LARGA}$$

·) Etere (DVB-T)

$$\left.\begin{array}{ll} B: & 8\ \text{MHz} \\ f_0: & 400\ \text{MHz} \end{array}\right\} \text{BANDA STRETTA}$$

CANALE RADIO

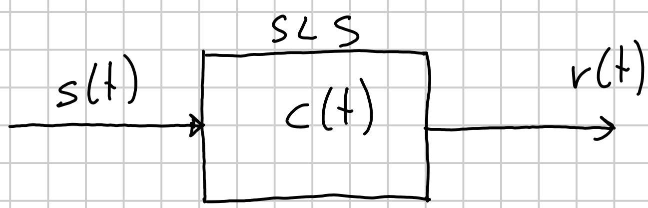
1) Passa-banda (in banda passante)
2) Tipicamente a banda stretta
3) I trasduttori sono antenne

$$\frac{\lambda}{10} \quad , \quad \lambda = \frac{c}{f_0}$$

LF (Low Freq.)	30 - 300 KHz (1 - 10 Km)	Radiolocalizzazione
MF (Medium Freq.)	300 - 3000 KHz (100 - 1000 m)	Radionavigazione, radiodiffusion
HF (High Freq)	3 - 30 MHz (10 - 100 m)	Comunicazione a grande distanza (riflessione ionosf.)
VHF (Very High Freq.)	30 - 300 MHz (1 - 10 m)	radio FM
UHF (Ultra High Freq)	300 - 3000 MHz (0.1 - 1 m)	Segnali TV, Telefonia mobile
SHF (Super High Freq.)	3 - 30 GHz (1 - 10 cm)	Comunic. satellitari, comunic. punto-punto

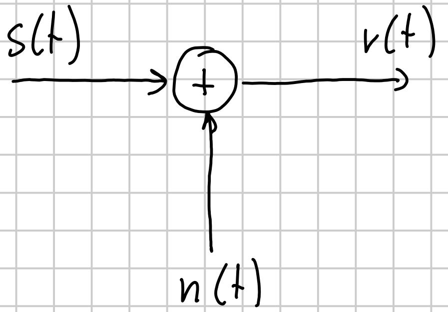
DISTURBI INTRODOTTI DAL CANALE DI COMUNICAZIONE

.) Distorsioni : lineare e stazionario



$$r(t) = s(t) \otimes c(t)$$

•) Rumore: additivo

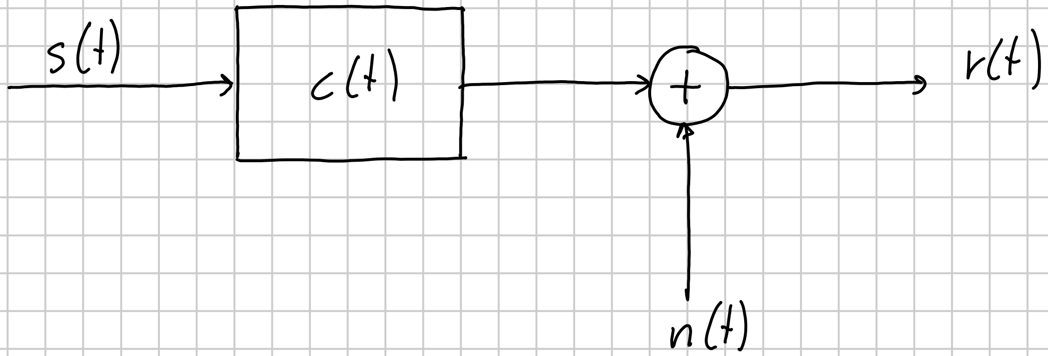


$$r(t) = s(t) + n(t)$$

segnale utile

rumore (aleatorio)

Modello complessivo semplificato del canale di comunic.



$$r(t) = \underbrace{s(t) \otimes c(t)}_{\text{deterministica}} + n(t)$$

aleatoria

CANALE IDEALE

$$\left.\begin{array}{l} \cdot)\ c(t) = \delta(t) \\ \cdot)\ n(t) = 0 \end{array}\right\} \quad r(t) = s(t)$$

SISTEMI DI COMUNICAZIONE ANALOGICI

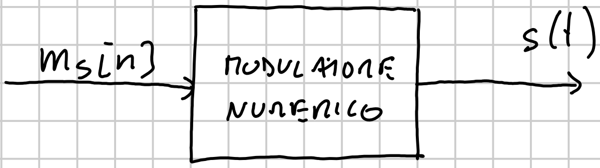
•) $m(t)$, $\hat{m}(t)$ sono segnali analogici

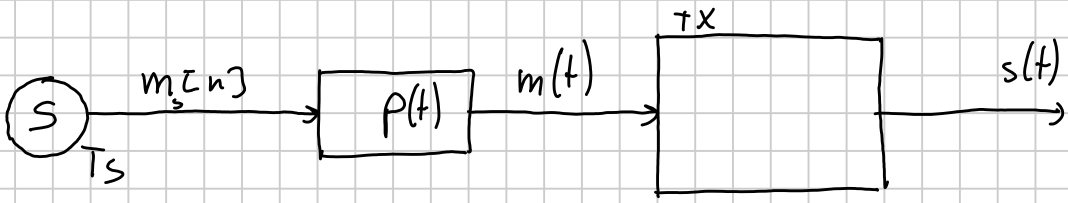
•) ovviamente anche $m_s(t)$ e $\hat{m}_s(t)$ sono analogici

•) $s(t)$ e $r(t)$ sono anche essi analogici

SISTEMI DI COMUNICAZIONE NUMERICI

·) $m_s(t)$, $\hat{m}_s(t)$ sequenze numeriche $\Rightarrow$ $m_s[n]$, $\hat{m}_s[n]$





$$m(t) = \sum_{n=-\infty}^{+\infty} m_s[n]\, p(t - nT_s)$$

T_s = intervallo di segnalazione della sorgente

$$p(t) = \text{rect}\left(\frac{t}{T_s}\right)$$

$m(t)$ è un segnale analogico

$\hookrightarrow$ segnale numerico

sorgente emette una sequenza di simboli $\in A_s$

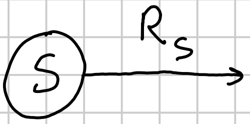
$A_s \triangleq \{\alpha_1, \alpha_2, \dots, \alpha_M\}$, $M \geq 2$

$M_s(t) \Rightarrow M_s[n] = M_s(nT_s)$ sequenza aleatoria di sorgente

$m_s[n]$ è una realizzazione di $M_s[n]$

$\hat{M}_s[n]$ sequenza aleatoria di destinazione

$M_s[n]$, $\hat{M}_s[n] \in A_s$

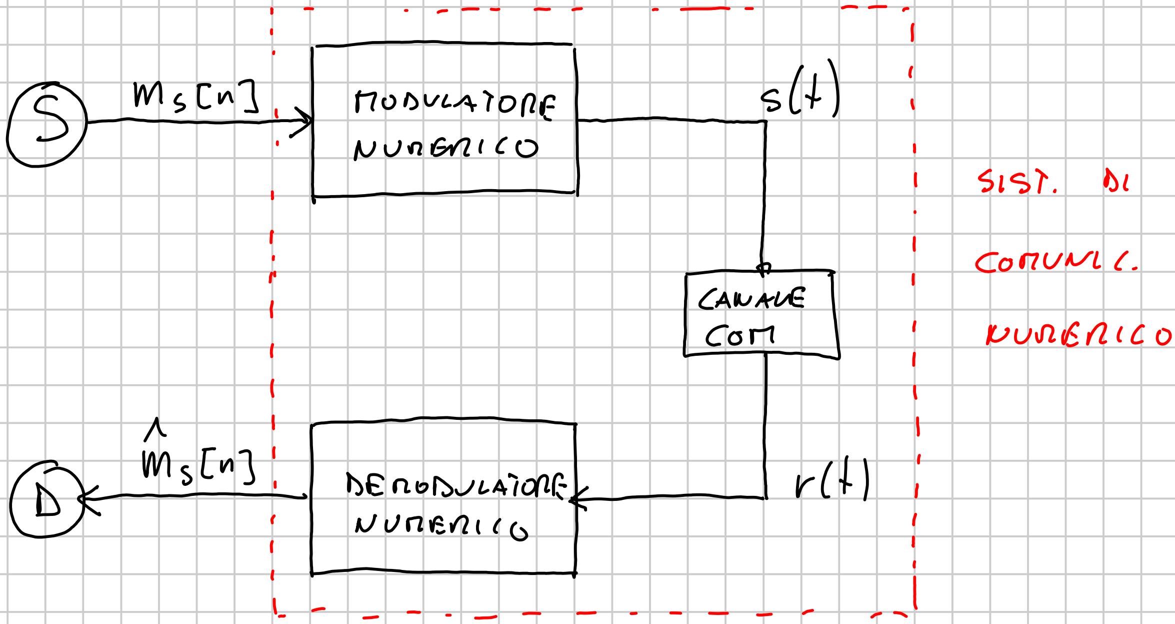


$R_s \triangleq \dfrac{1}{T_s}$ Tasso di evogazione della sorgente

SORGENTE + CODIFICA BINARIA

$R_b = \dfrac{\log_2 M}{T_s}$, M è la cardinalità di A_s

Tasso di evogazione binario



- MODULATORE NUMERICO : INTERP + TRASL. FREQ + SAGOMATURA
- DEMODULATORE NUMERICO : SAGOMATURA INVERSA + TRASL. FREQ. INV. + CONV. ANAL./DIG.

SCOPO DEL SIST. DI COM. NUM

$\hat{m}_s[n]$ "il più vicino possibile a" $m_s[n]$

IDEALE

$$\hat{m}_s[n] = m_s[n]$$

$$m_s[n] = \text{"}i\text{"} \quad , \quad \hat{m}_s[n] = \text{"}j\text{"}$$

Prob. di transizione $= P_{ij}\{ \hat{m}_s[n] = \text{"}j\text{"} \mid m[n] = \text{"}i\text{"} \} = P\{j|i\}$

i, j sono l'iesimo simbolo e il j-esimo simbolo

Se i simboli di A_s sono M allora dovrò conoscere M^2 Prob. di transiz. per caratterizzare completamente il sist. di com. numerico dal punto di vista statistico.

SIST. DI COMUNICAZIONE IDEALE

$$\begin{cases} P\{j|i\} = 1 & i = j \\ P\{j|i\} = 0 & i \neq j \end{cases}$$

$$P\{j|i\}\Big|_n = P\{\hat{m}_s[n] = "j" \mid m_s[n] = i\}$$

$$P\{j|i\}\Big|_n = P\{j|i\} \qquad \forall n \qquad \text{STAZIONARIETÀ}$$

PROBABILITÀ DI ERRORE SUL SIMBOLO M-ARIO

$$P_E(M) \triangleq P\{\hat{m}_s[n] \neq m_s[n]\}$$

QoS

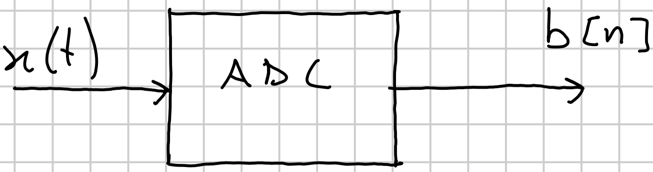
$$P_E(M) \leq P_E^{(MAX)}$$

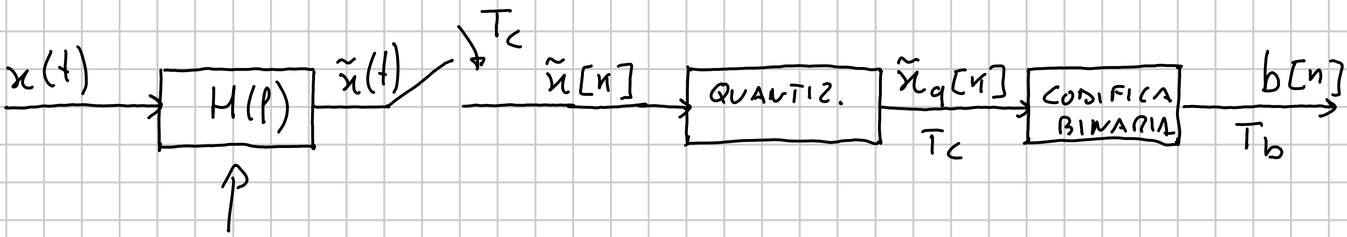
Telefonia $\Rightarrow$ $P_E^{(MAX)} \simeq 10^{-3}$

Dati $\Rightarrow$ $P_E^{(MAX)} \simeq 10^{-7}$

- DUALISMO TRA POTENZA E BANDA
- VANTAGGI E SVANTAGGI

CONVERSIONE ANALOGICO/DIGITALE (ADC)

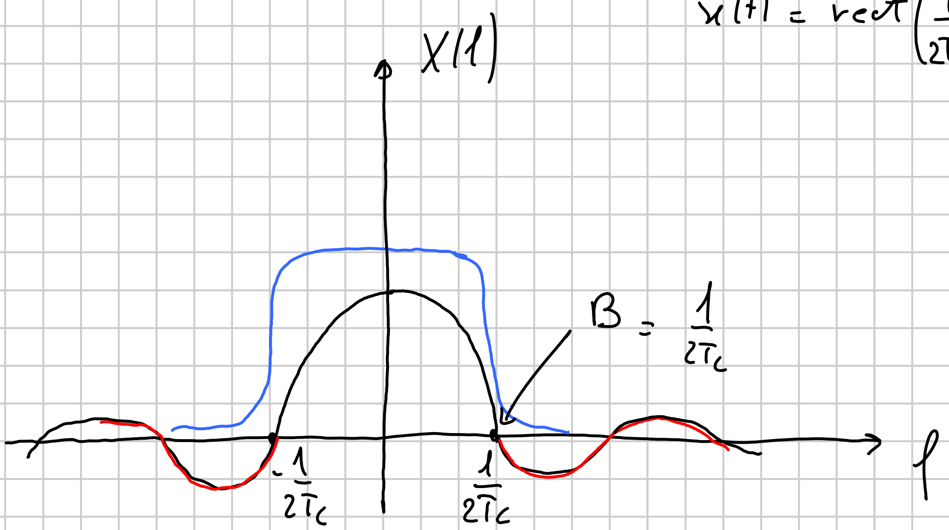




FILTRO ANTI-ALIASING

Esempio

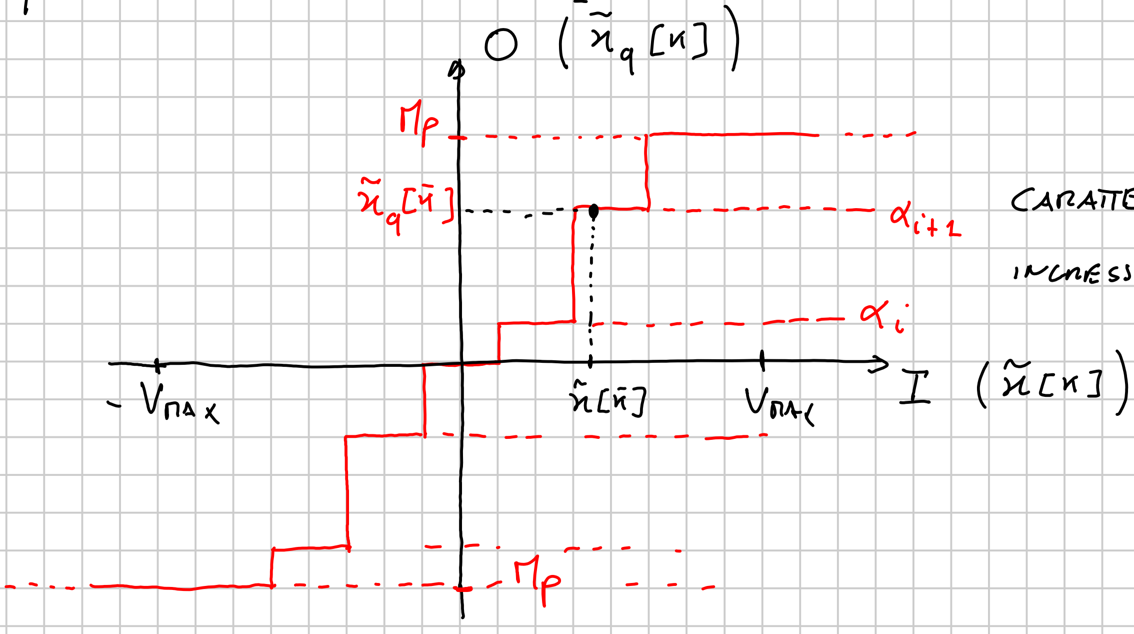
$$x(t) = \mathrm{rect}\left(\frac{t}{2T_c}\right)$$



QUANTIZZATORE

Sistema non-lineare istantaneo

$$y(t) = T\left[\, x(\alpha)\,;\ \alpha = t \,\right]$$



CARATTERISTICA INGRESSO/USCITA

DINAMICA IN INGRESSO : $2 V_{max}$

DINAMICA IN USCITA : $2 M_p$

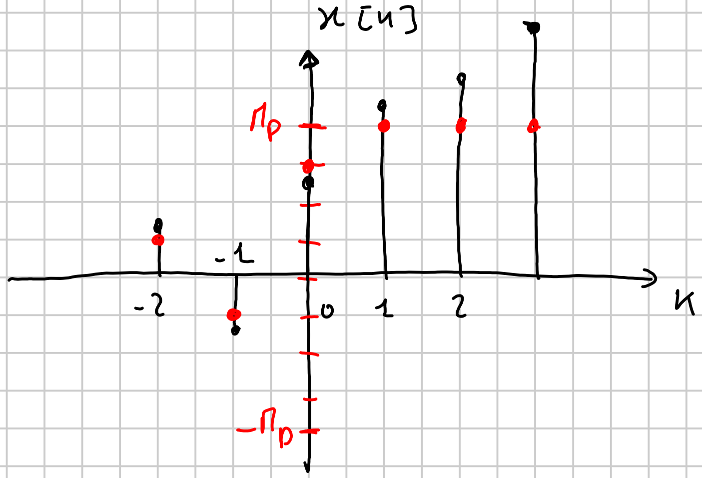
$M_p > V_{max}$, $M_p < V_{max}$, $M_p = V_{max}$

QUANTIZZATORI SCALARI

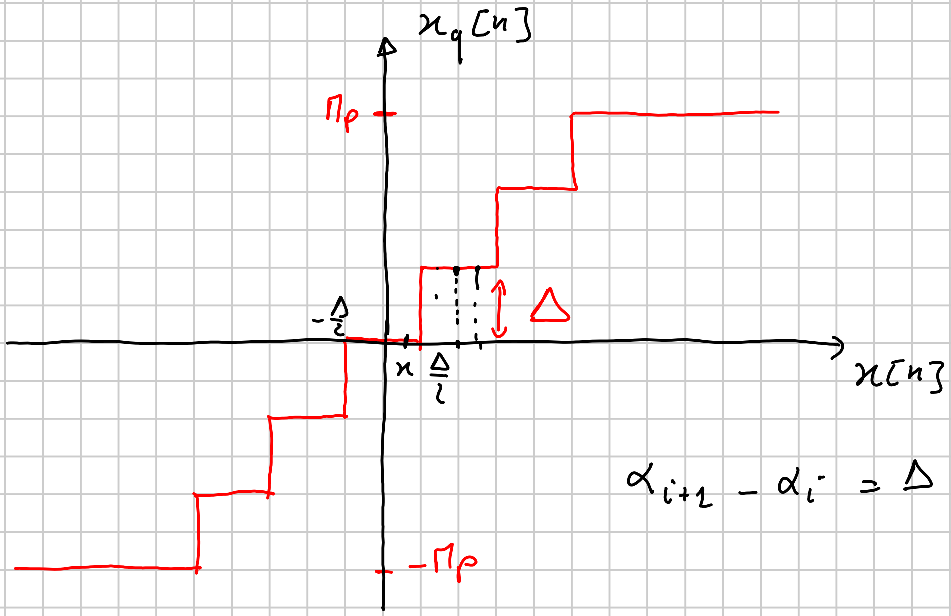
$$x_q[n] = Q\left(x[n] \right)$$

QUANTIZZATORI VETTORIALI

$$x_q[n] = Q\left(\dots, x[n-1], x[n], x[n+1], \dots \right)$$



QUANTIZZATORE SCALARE UNIFORME



$$\alpha_{i+1} - \alpha_i = \Delta \quad \forall i$$

$\Delta = \dfrac{2 M_p}{M-1}$, M = nr. livelli del quantizzatore

nr. di simboli nell'alfabeto A_s

PRESTAZIONI DI QUANTIZZATORI SCALARI UNIFORMI

$$x[n] \Rightarrow x_q[n]$$

$$e[n] = x[n] - x_q[n]$$

$$SQER \triangleq \frac{E[x^2[n]]}{E[e^2[n]]}$$

$x[n]$ processo stazionario

$$SQER = \frac{E[x^2]}{E[e^2]} \quad , \quad x[n] = x_q[n] + e[n]$$

$$SQER = \frac{\int_{-\infty}^{+\infty} x^2 f_X(x)\, dx}{\int_{-\infty}^{+\infty} (x - Q(x))^2 f_X(x)\, dx}$$

Esempio

$$f_X(x) = \frac{1}{2V_{max}} \operatorname{rect}\left(\frac{x}{2V_{max}}\right) \qquad x \in \mathcal{U}[-V_{max}, V_{max}]$$

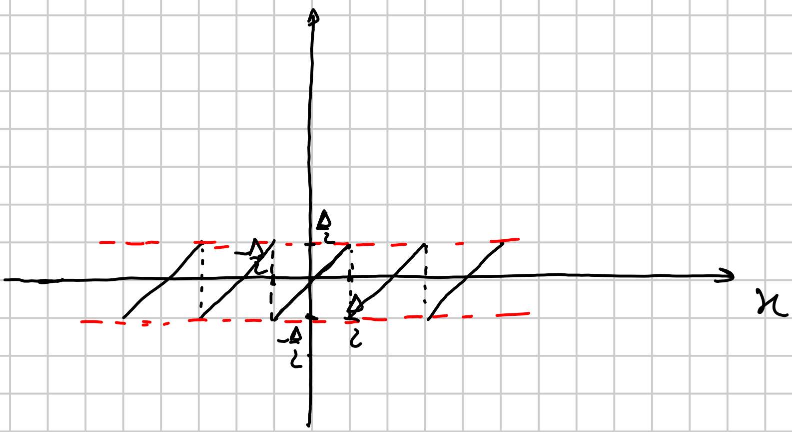
$$E[x^2] = \int_{-\infty}^{+\infty} x^2 \frac{1}{2V_{max}} \operatorname{rect}\left(\frac{x}{2V_{max}}\right) dx =$$

$$= \frac{1}{2V_{max}} \int_{-V_{max}}^{V_{max}} x^2\, dx = \frac{1}{2V_{max}} \left.\frac{x^3}{3}\right|_{-V_{max}}^{V_{max}} =$$

$$= \frac{1}{2V_{max}} \left(\frac{V_{max}^3}{3} + \frac{V_{max}^3}{3}\right) = \frac{V_{max}^2}{3}$$

$$E[e^2] = \int_{-\infty}^{+\infty} e^2 \, f_E(e) \, de$$

$$f_E(e) = ?$$



$$f_E(e) = \frac{1}{\Delta} \, \mathrm{rect}\left(\frac{e}{\Delta}\right)$$

$$E[e^2] = \int_{-\infty}^{+\infty} e^2 \, \frac{1}{\Delta} \, \mathrm{rect}\left(\frac{e}{\Delta}\right) de = \frac{1}{\Delta} \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 \, de$$

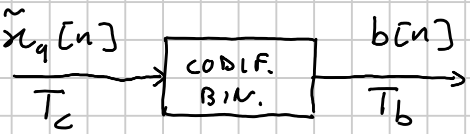
$$= \frac{1}{\Delta} \left. \frac{e^3}{3} \right|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{1}{3\Delta}\left(\frac{\Delta^3}{8} + \frac{\Delta^3}{8}\right) = \frac{\Delta^2}{12}$$

$$\Rightarrow \quad SQER\left(V_{max} = M_p\right) = \frac{\frac{\cancel{M_p}^{2}}{3}}{\frac{4\cancel{M_p^2}}{\cancel{12}(M-1)^2}} = (M-1)^2$$

- $n = \log_2 M$ se M è una potenza di 2

- $n = \log_2 M'$ se M non è una potenza di 2, M' è la minor potenza di 2, $M' \geq M$

CODIFICATORE BINARIO

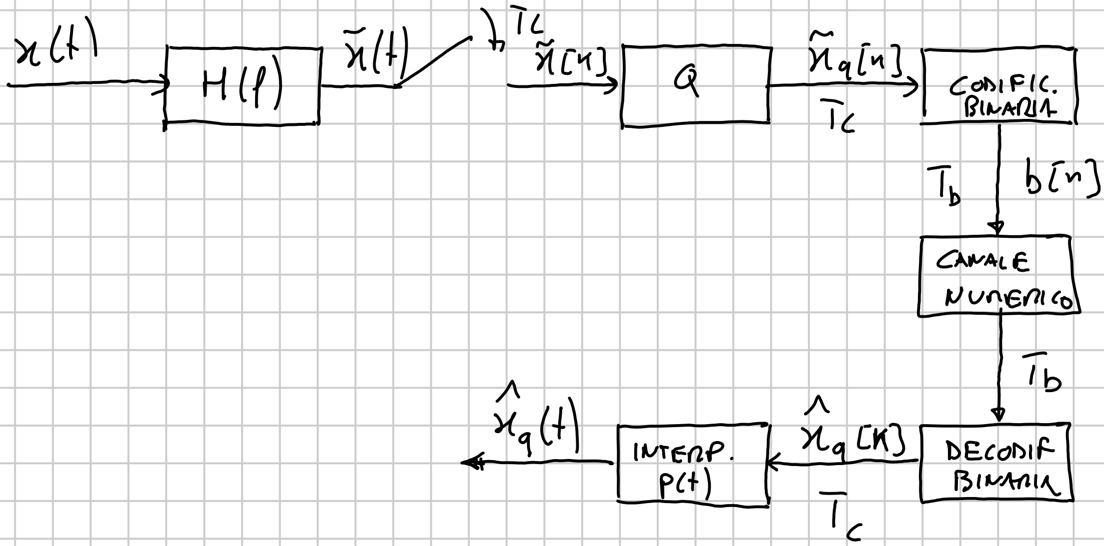


$$\tilde{x}_q[n] \xrightarrow[T_c]{} \boxed{\text{CODIF. BIN.}} \xrightarrow[T_b]{} b[n]$$

$$T_b = \frac{T_c}{\log_2 M}$$

$$R_b = \frac{1}{T_b} = \frac{\log_2 M}{T_c}$$

PCM (Pulse Code Modulation)



$\hat{x}_q(t)$ soffre dei seguenti peggioramenti

1) Effetti distorsivi di $H(f)$ (filtro anti-aliasing) + aliasing residuo
2) Quantizzazione
3) Effetti distorsivi e presenza di rumore introdotti dal canale di comunicazione

STANDARD PCM EUROPEO PER LA TELEFONIA

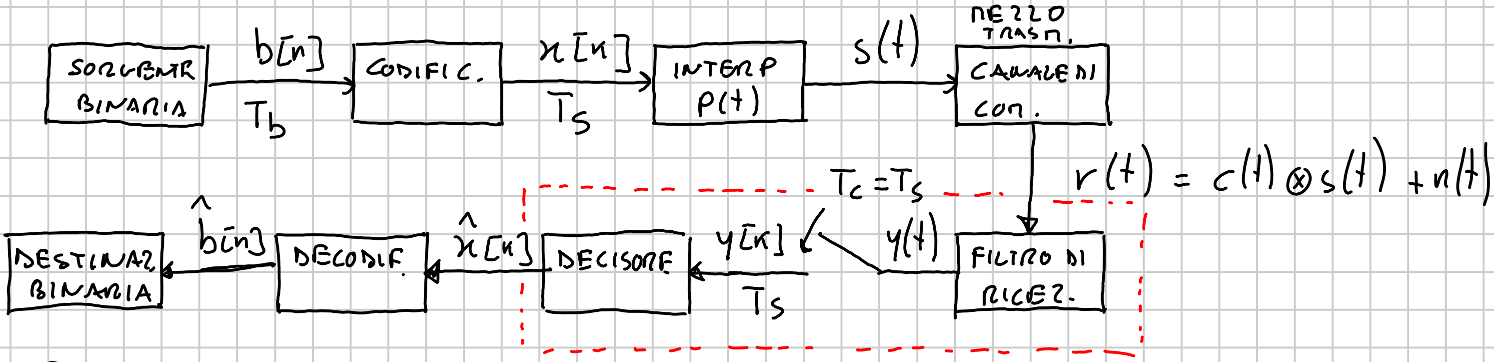
$H(f) \Rightarrow B = 4\ \text{KHz}$

$T_c = \frac{1}{2B} \Rightarrow f_c = 2B = 8\ \text{KHz}$

$M = 256 \Rightarrow n = 8$

$R_b = 8 \cdot 10^3 \cdot 8 = 64\ \text{Kbit/s}$

MODULAZIONI NUMERICHE IN BANDA BASE



DEFINIZIONI

·) $T_s = \log_2 M \, T_b$

·) Simboli $x[n]$ equiprobabili

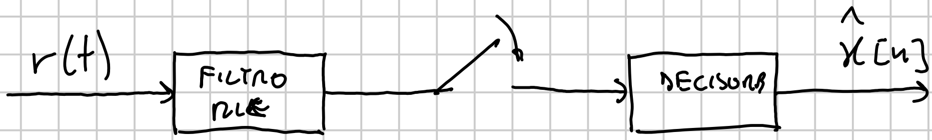
$$P\{\alpha_i\} = \frac{1}{M} \quad , \quad \forall i$$

·) INTERPOLATORE

$$s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s)$$

N.B. l'interpolatore fa da modulatore numerico

·) DEMODULATORE NUMERICO



·) Energia dell'impulso

$$E_p = \int_{-\infty}^{+\infty} p^2(t)\, dt = \int_{-\infty}^{+\infty} |P(f)|^2 \, df$$

·) Banda dell'impulso $\Rightarrow B_p$ (dipende da $P(f)$)

·) $s(t)$ è una possibile realizzazione di un processo aleatorio $S(t)$ stazionario

$$R_S(\tau) \quad , \quad S_S(f) \quad \Rightarrow \quad B_T = \text{banda del segnale trasmesso}$$

·) Energia per simbolo e per bit

$$E_S = T_S P_S = T_S \int_{-\infty}^{+\infty} S_S(f)\, df = T_S R_S(0)$$

$$E_b = \frac{E_S}{\log_2 M} = \frac{T_S P_S}{\log_2 M}$$

PRESTAZIONI DI UN SISTEMA DI COMUN. NUMERICO

$$P_{E_b} = P\{\hat{b}[n] \neq b[n]\}$$ BEP (Bit error probability)
BER (" " rate)

$$P_{E_S} = P\{\hat{x}[n] \neq x[n]\}$$

$$P_{E_S}(M) = \sum_{i=1}^{M} \sum_{\substack{j=1 \\ j \neq i}}^{M} P\{\hat{x}[k] = \alpha_i \,,\, x[k] = \alpha_j\} =$$

$$= 1 - \sum_{i=1}^{M} P\{\hat{x}[n] = \alpha_i \,,\, x[k] = \alpha_i\}$$

$$= \sum_{i=1}^{M} \sum_{\substack{j=1 \\ j \neq i}}^{M} P\{\hat{x}[k] = \alpha_i \mid x[k] = \alpha_j\}\, P\{\alpha_j\}$$

Per simboli equiprobabili $\Rightarrow P_{E_S}(M) = \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{M} P(i|j)$

·) Simboli equienergia

$$E_S(i) = \int_{-\infty}^{+\infty} s_i^2(t)\, dt \quad , \quad s_i(t) \text{ è il segnale trasmesso relativamente al simbolo i-esimo}$$

$$E_S(i) = E_S$$

·) simboli ortogonali

$$\int_{-\infty}^{+\infty} s_i(t)\, s_j(t)\, dt = 0 \quad , \forall\, i \neq j$$

EFFICIENZA ENERGETICA

Fissata una BEP si definisce efficienza energetica la quantità

$$\eta_P \triangleq \frac{1}{SNR} \quad , \quad SNR \triangleq \frac{P_S}{P_N}$$

che garantisce tale BEP.

EFFICIENZA SPETTRALE

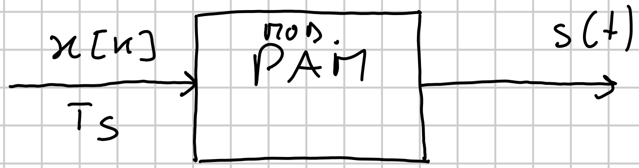
Fissato un flusso di bit (tasso di erogazione) R_b, si definisce efficienza spettrale la quantità

$$\eta_B \triangleq \frac{R_b}{B_T}$$

dove B_T è la banda del segnale trasmesso

PAM (PULSE AMPLITUDE MODULATION)

M-PAM (o PAM M-aria)



CONDIZIONI CHE DEFINISCONO UNA M-PAM

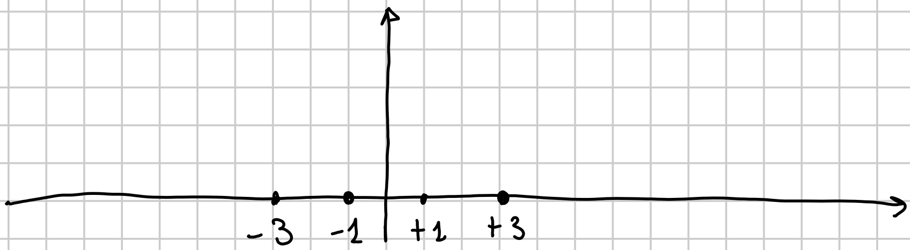
.) $s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s)$

.) $x[n] \in A_s \{\alpha_1, \dots, \alpha_M\}$ $\qquad M \geq 2$

$$\alpha_i = 2i - M - 1 \quad , \quad i = 1, \dots, M$$

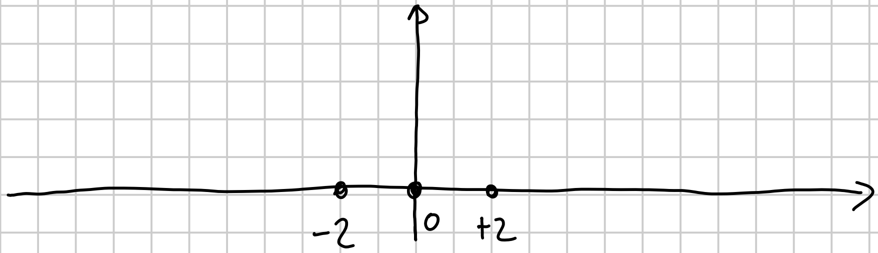
$M = 4 \Rightarrow$ 4-PAM

$\alpha_1 = -3 \; ; \; \alpha_2 = -1 \; ; \; \alpha_3 = +1 \; ; \; \alpha_4 = +3$



$M = 3 \Rightarrow$ 3-PAM

$\alpha_1 = -2 \; ; \; \alpha_2 = 0 \; ; \; \alpha_3 = +2$



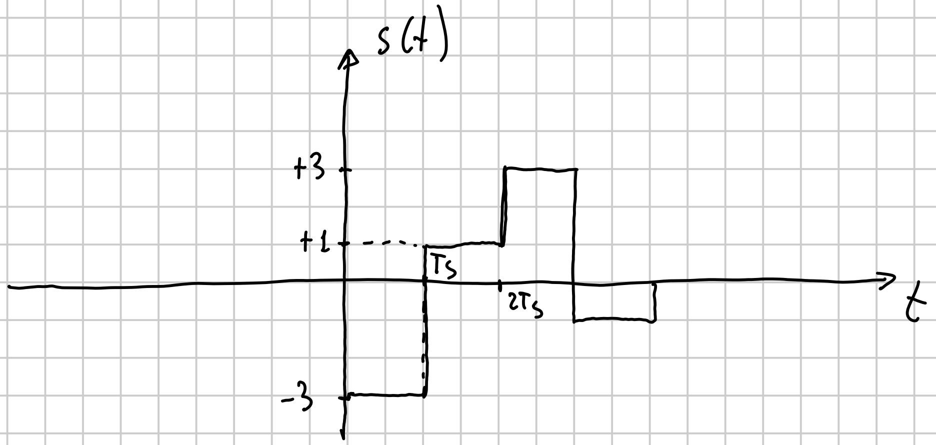
M PARI $\Rightarrow A_s = \{\pm 1, \pm 3, \dots\}$

M DISPARI $\Rightarrow A_s = \{0, \pm 2, \pm 4, \dots\}$

$$E_{s_i} = \int_{-\infty}^{+\infty} s_i^2(t)\, dt = \int_{-\infty}^{+\infty} \alpha_i^2\, p^2(t)\, dt =$$

$$= \int_{-\infty}^{+\infty} (2i - M - 1)^2\, p^2(t)\, dt = (2i - M - 1)^2\, E_p$$

$$\underline{x} = [x[1], x[2], \dots, x[4]] = [-3, +1, +3, -1] \qquad p(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right)$$



PROPRIETÀ DELLA M-PAM

1) Se i simboli sono equiprobabili

$$E[s(t)] = E\left[\sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s)\right] =$$

$$= \sum_{k=-\infty}^{+\infty} \underbrace{E[x[k]]}_{=0}\, p(t - kT_s) = 0$$

2) $$S_s(f) = \frac{\sigma_x^2}{T_s} |P(f)|^2$$

$$\sigma_x^2 = E[x^2[k]] = \frac{(M^2 - 1)}{3}$$

$$R_s(\tau) = E[s(t)\, s^*(t - \tau)]$$

$$= E\left[\sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s) \sum_{n=-\infty}^{+\infty} x^*[n]\, p^*(t - \tau - nT_s)\right]$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} E\left[x[k]\, x^*[n] \right] p(t-kT_s)\, p^*(t-\tau-nT_s)$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R_x[k-n]\, p(t-kT_s)\, p^*(t-\tau-nT_s)$$

(dip. anche da t)

$$\bar{R}_s(\tau) \triangleq \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} R_s(\tau, t)\, dt =$$

$$= \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} R_s(\tau, t)\, dt$$

$$= \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R_x[k-n]\, p(t-kT_s)\, p^*(t-\tau-nT_s)\, dt$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} R_x[k-n] \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} p(t-kT_s)\, p^*(t-\tau-nT_s)\, dt$$

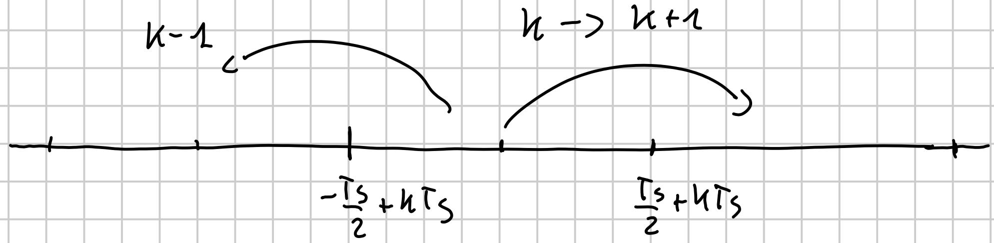
$k-n = n' \Rightarrow n = k-n'$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} R_x[n'] \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} p(t-kT_s)\, p^*(t-\tau-kT_s+n'T_s)\, dt$$

$t - kT_s = t'$

$$= \frac{1}{T_s} \sum_{n'=-\infty}^{+\infty} R_x[n'] \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_s}{2}+kT_s}^{\frac{T_s}{2}+kT_s} p(t')\, p^*(t'-\tau+n'T_s)\, dt'$$

$$= \frac{1}{T_s} \sum_{n'=-\infty}^{+\infty} R_x[n'] \int_{-\infty}^{+\infty} p(t')\, p^*(t'-\tau+n'T_s)\, dt'$$



$$= \frac{1}{T_s} \sum_{n'=-\infty}^{+\infty} R_x[n'] \int_{-\infty}^{+\infty} P(f)\, P^*(f)\, e^{-j2\pi f \tau}\, e^{j2\pi f n' T_s}\, df$$

$$= \frac{1}{T_s} \int_{-\infty}^{+\infty} \sum_{n'=-\infty}^{+\infty} R_x[n'] |P(f)|^2 e^{-j2\pi f \tau}\, e^{j2\pi f n' T_s}\, df$$

$-n' = n \qquad f' = -f$

$$= \frac{1}{T_s} \int_{-\infty}^{+\infty} \underbrace{\sum_{n=-\infty}^{+\infty} R_x[n]\, e^{-j2\pi f n' T_s}}_{S_x(f)} |P(f')|^2 e^{j2\pi f' \tau}\, df'$$

$$= \frac{1}{T_s} \int_{-\infty}^{+\infty} \underbrace{S_x(f)\, |P(f')|^2}\, e^{j2\pi f' \tau}\, df'$$

$$\bar{R}_s(\tau) = \frac{1}{T_s}\, TCF^{-1}\left[S_x(f)\, |P(f)|^2 \right]$$

$$TCF\left[\bar{R}_s(\tau) \right] = S_s(f) = \frac{1}{T_s} S_x(f)\, |P(f)|^2$$

M-PAM $\Rightarrow$ simboli indipendenti $\Rightarrow$ incorrelati

$$R_x[n] = R_x[0]\, \delta[n] = (\sigma_x^2 + \underset{0}{\eta_x^2})\, \delta[n] = \sigma_x^2\, \delta[n]$$

$\Downarrow$

$$S_x(f) = \sigma_x^2$$

$\Downarrow$

$$S_s(f) = \frac{\sigma_x^2}{T_s} \underbrace{|P(f)|^2}_{\text{d.s.energia di } p(t)}$$

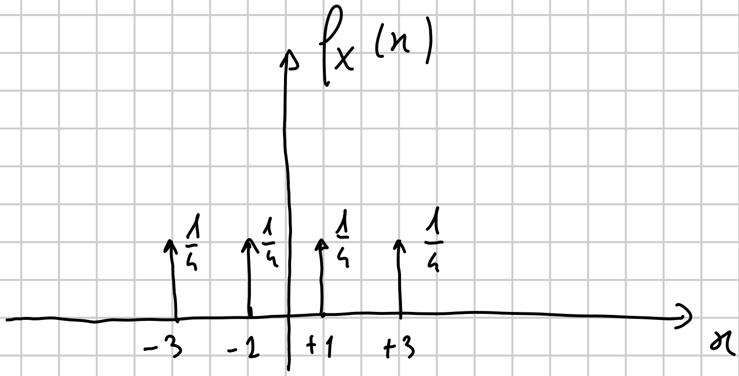
$\downarrow$

$s(t)$ ha la stessa occupazione di banda di $p(t)$

$\cdot)\ P_s = \int_{-\infty}^{+\infty} S_s(f)\, df = \frac{\sigma_x^2}{T_s} E_p$

Calcolo di σ_x^2

$$\sigma_x^2 = \int_{-\infty}^{+\infty} (x - \eta_x)^2 f_x(x)\, dx$$



$$\sigma_x^2 = \int_{-\infty}^{+\infty} x^2 f_x(x)\, dx = \frac{1}{M} \sum_{i=1}^{M} \alpha_i^2 = \frac{1}{M} \sum_{i=1}^{M} (2i - M - 1)^2$$

$$\Rightarrow \sum_{i=1}^{M} i = \frac{M(M+1)}{2} \quad , \quad \sum_{i=1}^{M} i^2 = \frac{M(M+1)(2M+1)}{6}$$

$$= \frac{(M-1)^2}{3}$$

$$S_s(f) = \frac{(M-1)^2}{3T_s} |P(f)|^2$$

$$P_s = \frac{(M-1)^2}{3T_s} E_p$$

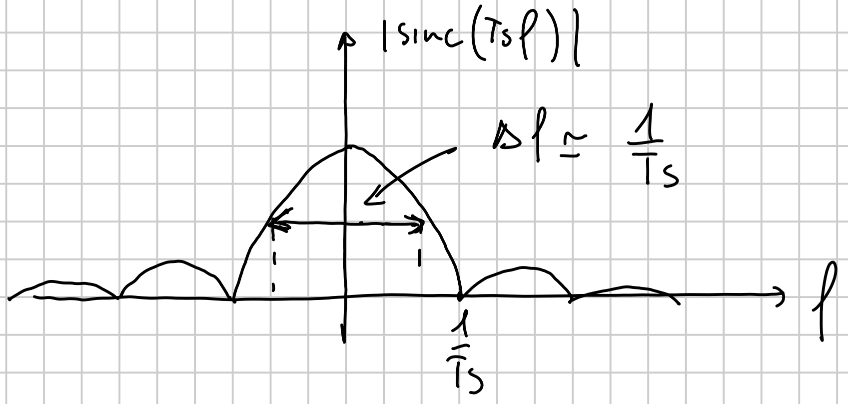
EFFICIENZA SPETTRALE

$$\eta_B = \frac{R_b}{B_T} = \frac{R_b}{B_p} = \frac{\log_2 M}{T_s B_p}$$

B_p = banda di $p(t)$

Es.

$$p(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right) \Rightarrow P(f) = T_s\, \mathrm{sinc}(T_s f)\, e^{-j\pi f T_s}$$



$$B_p = \frac{1}{2T_s}$$

PAM BINARIA (BPSK = Binary Phase Shift Keying)

·) $M = 2$

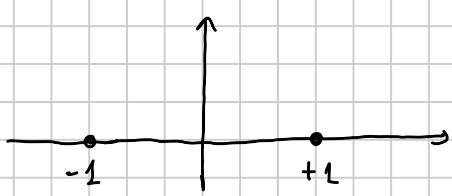
$$s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s) \quad , \; x[k] \in A_s = \{\pm 1\}$$

·) $E_{s_i} = \int_{-\infty}^{+\infty} s_i^2(t)\, dt = \int_{-\infty}^{+\infty} (\pm 1)^2\, p^2(t)\, dt = E_p \quad \forall i$

simboli equienergia

·) $T_s = T_b$

·) Simboli si dicono "antipodali"



·) $E[s(t)] = 0$ (per simboli equiprobabili)

·) $S_s(f) = \frac{1}{T_b} |P(f)|^2$

·) $P_s = \frac{E_p}{T_b}$ ·) $\eta_B = \frac{1}{T_b B_p}$

SEGNALAZIONE ON-OFF (OOK - On Off Keying)

$$A_s \equiv \{0, 1\}$$

$$s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t-kT_s) \quad , \quad x[k] \in A_s$$

$$T_s = T_b$$

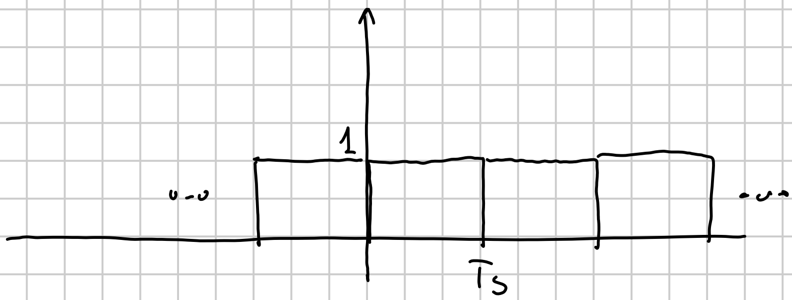
$$p(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right)$$

$$\begin{cases} s_1(t) = 0 \\ s_2(t) = p(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right) = \mathrm{rect}\left(\frac{t - T_b/2}{T_b}\right) \end{cases}$$

$$\begin{cases} E_{s_1} = 0 \\ E_{s_2} = T_s = T_b \end{cases}$$

$$E[s(t)] = E\left[\sum_{k=-\infty}^{+\infty} x[k]\, p(t-kT_b)\right] =$$

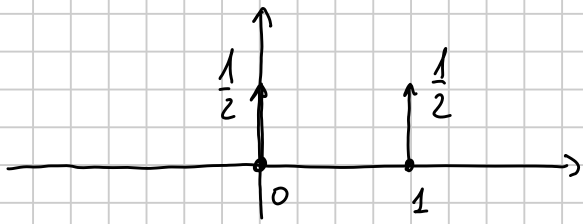
$$= \sum_{k=-\infty}^{+\infty} E[x[n]]\, p(t-kT_b) = \frac{1}{2}\sum_{k=-\infty}^{+\infty} p(t-kT_b) = \frac{1}{2}$$



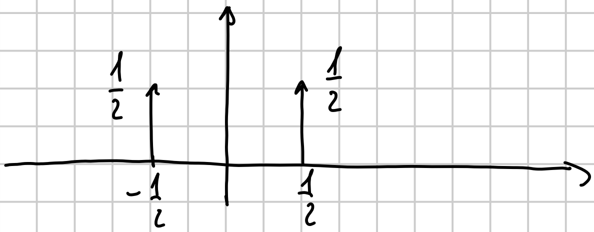
- $E_s = \frac{1}{2}E_{s_1} + \frac{1}{2}E_{s_2} = \frac{1}{2}E_{s_2} = \frac{T_b}{2}$
- $P_s = \frac{E_s}{T_s} = \frac{T_b}{2}\cdot\frac{1}{T_b} = \frac{1}{2}$
- $S_s(f) = \frac{S_x(f)}{T_s}|P(f)|^2$
- $R_x[n] = \sigma_x^2\,\delta[n] + \eta_x^2$

$$\eta_x = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\sigma_x^2 = \int_{-\infty}^{+\infty} (x-\eta_x)^2 f_x(x)\, dx = \frac{1}{4}$$



$$\Downarrow$$



$$\frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{2}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

·) $S_x(f) = \frac{1}{4} + \frac{1}{4T_b}\delta(f)$

$$T_b \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} S_x(f)\, e^{j2\pi f n T_b}\, df = T_b \int_{-\frac{1}{2T_b}}^{\frac{1}{2T_b}} \left[\frac{1}{4} + \frac{1}{4T_b}\delta(f)\right] e^{j2\pi f n T_b}\, df$$

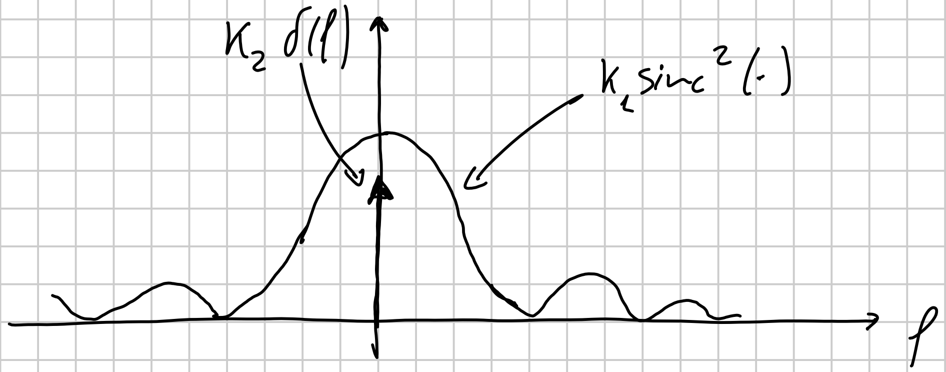
$$= T_b \cdot \frac{1}{4}\cdot\frac{1}{T_b} + T_b \cdot \frac{1}{4T_b}\cdot 1\, \delta[n] =$$

$$= \frac{1}{4} + \frac{1}{4}\delta[n]$$

·) $S_s(f) = \frac{\left(\frac{1}{4} + \frac{1}{4T_b}\delta(f)\right)}{T_b} \cdot |P(f)|^2$

$$= \frac{1}{T_b}\left(\frac{1}{4} + \frac{1}{4T_b}\delta(f)\right) \cdot T_b^2\, \mathrm{sinc}^2(T_b f) =$$

$$= T_b\left(\frac{1}{4} + \frac{1}{4T_b}\delta(f)\right) \mathrm{sinc}^2(T_b f)$$

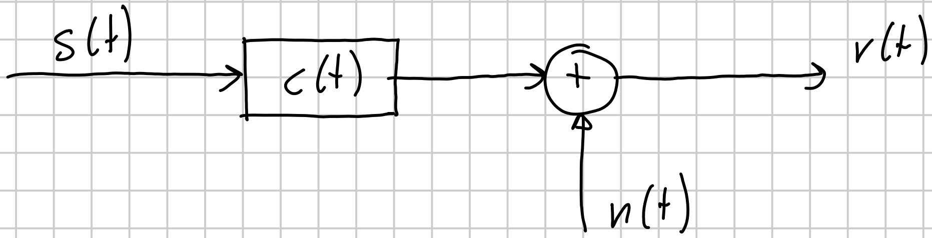


$$\eta_B = \frac{\log_2 M}{T_s B_T} = \frac{1}{T_b B_p}$$

(identica alla PAM bipolare
2-PAM
BPSK)

— o —

PRESTAZIONI DI UN SISTEMA DI COMUNICAZIONE NUMERICO IN BANDA BASE



$$r(t) = \underbrace{s(t) \otimes c(t)}_{\text{fenomeno distorsivo}} + \underbrace{n(t)}_{\text{presenza di rumore}}$$

1) INTERFERENZA INTER-SIMBOLICA (ISI) (Inter-symbol interf.)
causata dalle distorsioni di canale

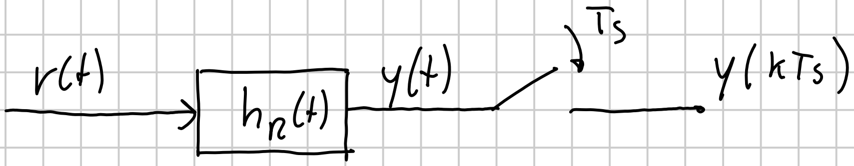
2) PRESENZA DI RUMORE ADDITIVO

ISI

Supponiamo che non sussista rumore

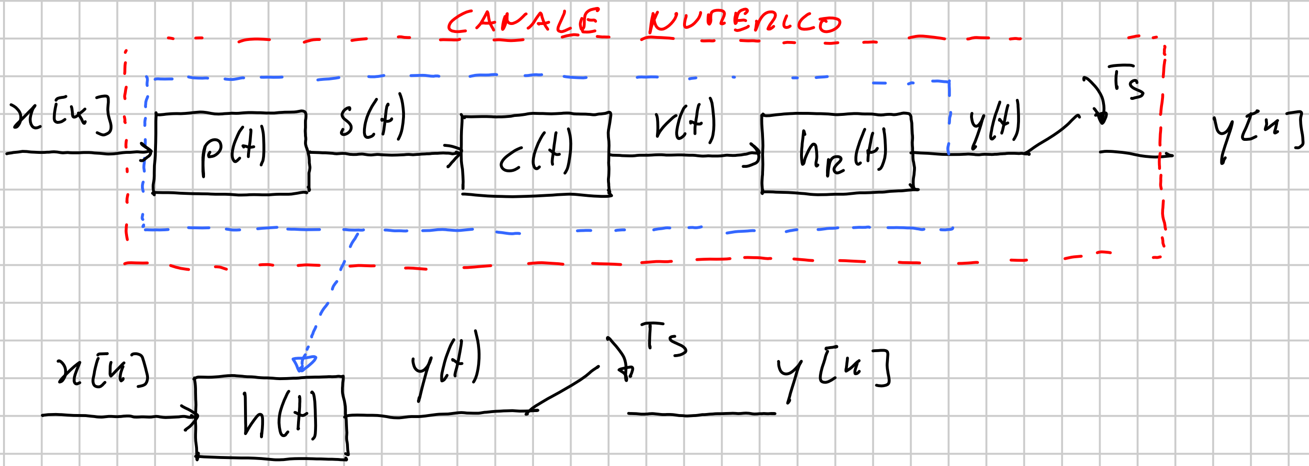
$$r(t) = s(t) \otimes c(t)$$

ISI



ASSENZA DI ISI $\Rightarrow$ $y(kT_s) = f(x[k])$

PRESENZA DI ISI $\Rightarrow$ $y(kT_s) = f(\dots, x[k-1], x[k], x[k+1], \dots)$



$$h(t) = p(t) \otimes c(t) \otimes h_R(t)$$

Dim.

$$H(f) = P(f)\, C(f)\, H_R(f)$$

$$y(t) = r(t) \otimes h_R(t) \Rightarrow Y(f) = R(f)\, H_R(f)$$

$$r(t) = s(t) \otimes c(t) \Rightarrow R(f) = S(f)\, C(f)$$

$$S(f) = \bar{X}(f)\, P(f)$$

$$Y(f) = \bar{X}(f) \underbrace{P(f)\, C(f)\, H_R(f)}_{H(f)}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} x[k]\, h(t-kT_s)$$

$$\Rightarrow y(nT_s) = \sum_{k=-\infty}^{+\infty} x[k]\, h((n-k)T_s)$$

$n = k$

$$\Rightarrow y(nT_s) = \underbrace{x[n]\, h(0)}_{\text{componente utile}} + \underbrace{\sum_{\substack{k=-\infty \\ k \neq n}}^{+\infty} x[k]\, h((n-k)T_s)}_{ISI}$$

CANALE CON <u>ISI</u> E POSSIBILITÀ DI ELIMINARLO

1) T_S minimo per cui è possibile eliminare completamente l'ISI

2) Determinare le condizioni per una modulazione M-PAM in modo da eliminare l'ISI

CASO ASSURDO

$p(t)$ di durata finita $\Rightarrow$ elimino l'ISI

↳ Banda illimitata

Banda del mezzo trasmissivo (CANALE) limitata

$\Rightarrow$ Gli impulsi $p(t)$ <u>devono</u> avere durata illimitata

I° CRITERIO DI NYQUIST

Supposta B_C definita

ASSENZA DI ISI $\Longleftrightarrow$ $h[kT_s] = \begin{cases} 1, & k=0 \\ 0, & k\neq 0 \end{cases}$ DOMINIO DEL TEMPO

$$\Updownarrow$$

$$\sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = T_s$$ DOMINIO DELLA FREQUENZA

Dim.

$$y(kT_s) = x[k]\,h[0] + \sum_{n=-\infty}^{+\infty} x[n]\,\underbrace{h\big((k-n)T_s\big)}$$

$$y(kT_s) = x[k]\cdot 1 + \sum_{n=-\infty}^{+\infty} x[n]\cdot 0 = x[k]$$

In freq.

$$\underset{\delta[k]}{\overset{}{h[k]}} \overset{TFS}{\Longrightarrow} \bar{H}(f) = 1$$

$$\bar{H}(f) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right) = 1$$

$$\Downarrow$$

$$\sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right) = T_s$$

CANALE AFFETTO DA ISI

- B_c definito
- $C(f)$ è a banda rigorosamente limitata

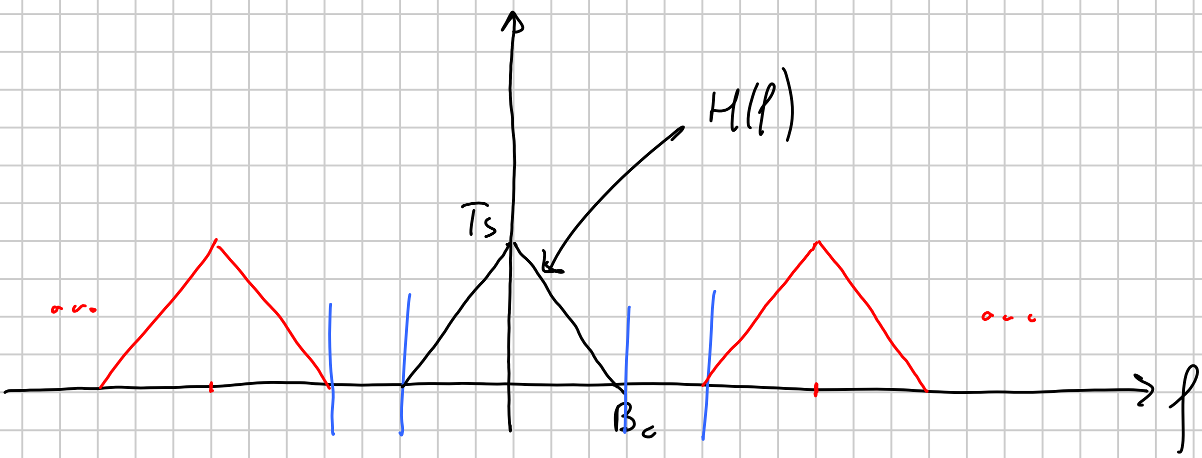
$$C(f) = 0 \qquad |f| > B_c$$

- $B_T = B_c$

$T_s < \dfrac{1}{2B_c} \Rightarrow$ NON POSSO ELIMINARE L'ISI

(NON POSSO AVERE TRASMISSIONE PRIVA DI ISI)

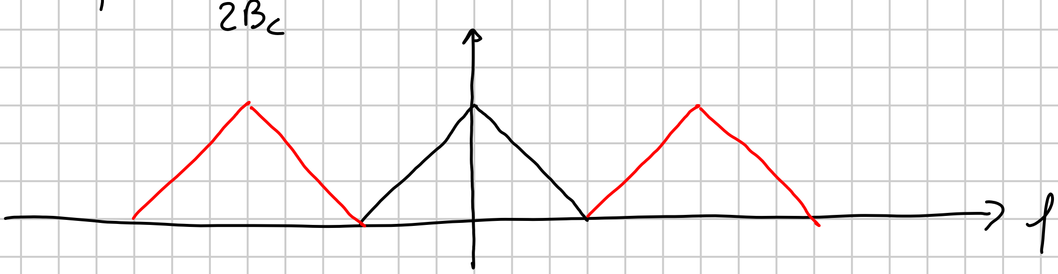
$H(f) = P(f)\, C(f)\, H_R(f) \Rightarrow$ banda rig. limitata



$$\sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right) = T_s$$ NON È MAI OTTENIBILE

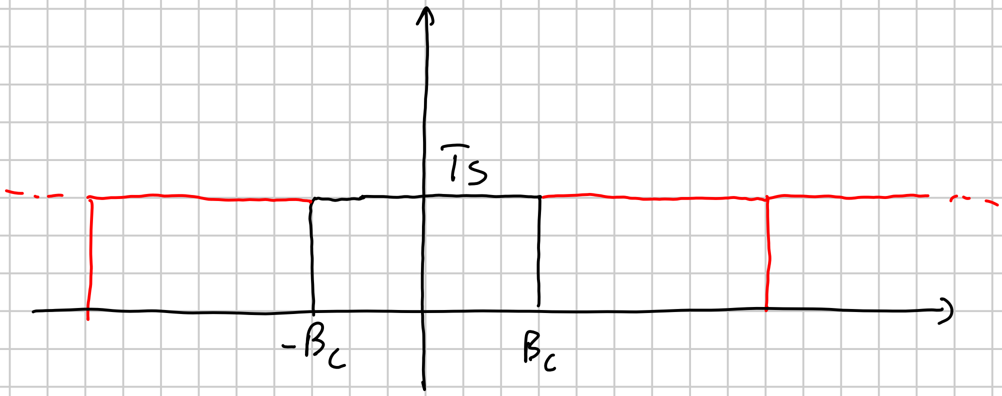
$$T_s < \frac{1}{2B_c}$$

$$T_{s(\text{minimo})} = \frac{1}{2B_c}$$



$$T_s = \frac{1}{2B_c} \Rightarrow H(f) = T_s \, \mathrm{rect}(f T_s) = \frac{1}{2B_c} \, \mathrm{rect}\left(\frac{f}{2B_c}\right)$$

unica soluzione



$$\sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = T_s \Rightarrow \text{ASSENZA DI ISI}$$

PRESTAZIONI DEI SISTEMI DI COMUNICAZIONE NUMERICI IN BANDA BASE

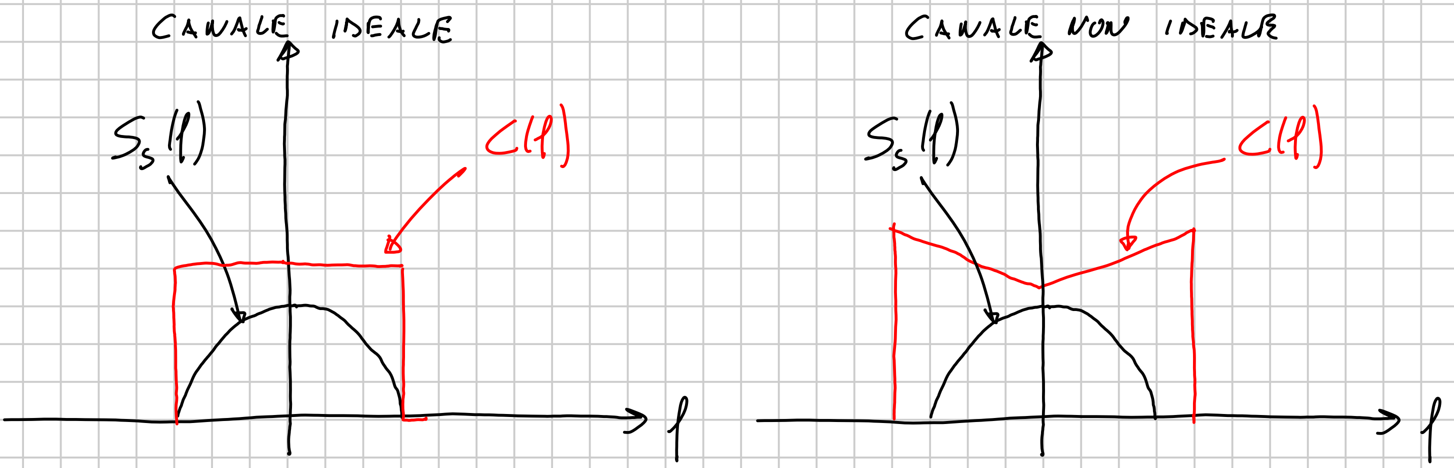
Nel valutare le prestazioni dei sistemi di comunicazioni numerici in banda base considereremo due fenomeni peggiorativi

1) INTERFERENZA INTER-SIMBOLO
2) PRESENZA DI RUMORE

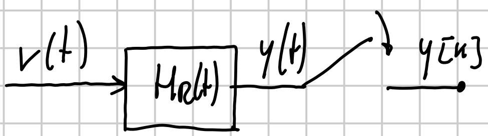
Per il momento ignoriamo il rumore e ci concentriamo sul primo problema.

INTERFERENZA INTER-SIMBOLICA (ISI)

Il primo fenomeno è causato dalla non perfetta risposta in frequenza del canale di trasmissione, e quindi dalle distorsioni lineari introdotte da questo.



Il risultato è che il campione estratto al ricevitore dal segnale ricevuto al k-esimo istante non dipende solo dal k-esimo simbolo.



ASSENZA DI ISI

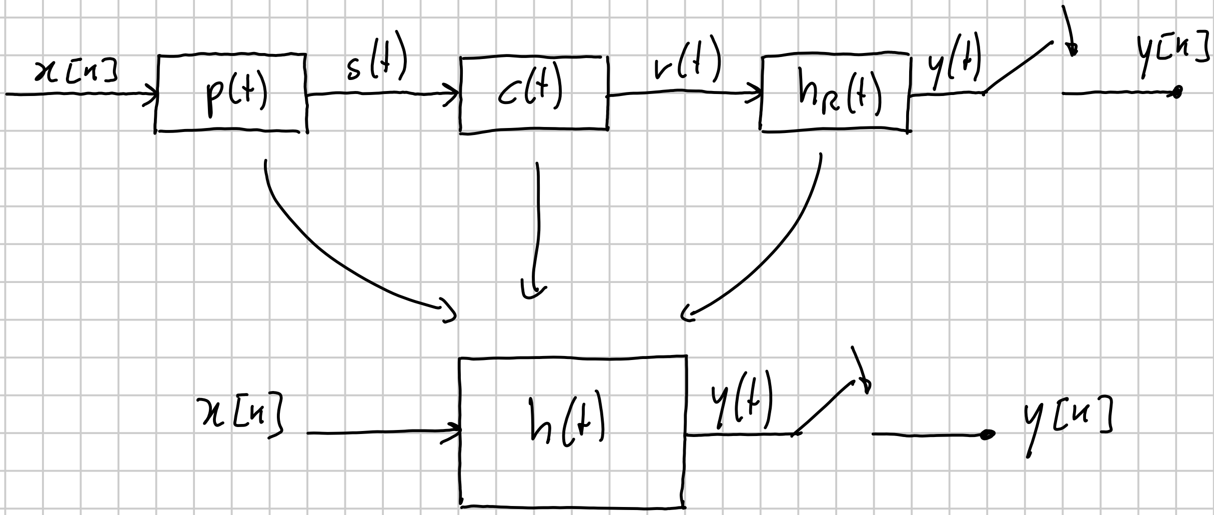
$$y[n] = f\left(x[n]\right)$$

PRESENZA DI ISI

$$y[n] = f\left(\dots, x[n-1], x[n], x[n+1], \dots\right)$$

Per valutare gli effetti dell'ISI si devono considerare

- il sagomatore in trasmissione $p(t)$
- la risposta impulsiva del canale $c(t)$
- il filtro in ricezione



$$h(t) = p(t) \otimes c(t) \otimes h_R(t)$$

Dimostrazione:

$$Y(f) = R(f)\, H_R(f) = S(f)\, C(f)\, H_R(f) = \bar{X}(f)\, P(f)\, C(f)\, H_R(f)$$

$$= \bar{X}(f)\, H(f) \quad , \quad H(f) = P(f)\, C(f)\, H_R(f)$$

$$y(t) = \sum_{n=-\infty}^{+\infty} x[n]\, h(t - nT_s)$$

$$y[n] = y(kT_s) = \sum_{n=-\infty}^{+\infty} x[n]\, h\big((k-n)T_s\big) =$$

$$= \underbrace{x[k]\, h(0)}_{\substack{\text{componente} \\ \text{utile}}} + \underbrace{\sum_{\substack{n=-\infty \\ n \neq k}}^{+\infty} x[n]\, h\big((k-n)T_s\big)}_{\text{ISI}}$$

CANALE CON ISI

Un canale con Banda B_c in generale introduce ISI.
Ci sono due aspetti di cui ci occuperemo:

1) Determinazione del T_s minimo che può essere adottato al fine di ottenere una sequenza campionata priva di ISI

2) Determinare le condizioni sotto le quali è possibile trasmettere un segnale M-PAM attraverso un canale non ideale in modo che non vi sia ISI nella sequenza campionata.

Nel risolvere i due problemi riterremo la $c(t)$ fissata e $p(t)$ e $h_R(t)$ variabili, in quanto determinabili dal progettista.

UN APPROCCIO NON PERSEGUIBILE : IMPULSI DI DURATA FINITA

Trasmettere impulsi di durata finita crea un segnale trasmesso con banda illimitata. Questo è in contrasto con la limitatezza messa a disposizione dal canale di trasmissione ($B_c < \infty$)

$\Rightarrow$ Gli impulsi $p(t)$ devono avere <u>durata infinita</u>

I° CRITERIO DI NYQUIST PER LA TRASMISSIONE PRIVA DI ISI

$$h(kT_s) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases} \qquad \text{DOMINIO DEL TEMPO}$$

$$\sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = T_s \quad \forall f \qquad \text{DOMINIO DELLA FREQ.}$$

Dimostrazione:

Il I° criterio di Nyquist nel dominio del tempo garantisce l'assenza di ISI in quanto

$$y[k] = x[k]\,h(0) + \underbrace{\sum_{\substack{n=-\infty \\ n\neq k}}^{+\infty} x[n]\, h\big((n-k)T_s\big)}_{=0} = y[k] = h(0)\, x[k]$$

DIPENDENZA SOLO DAL SIMBOLO $x[k]$

La relazione in frequenza si ottiene come trasformazione

$$h[k] = \delta[k] \quad \Longleftrightarrow \quad \bar{H}(f) = 1 \qquad \forall f$$

$$\bar{H}(f) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = 1 \qquad \forall f$$

$$\Downarrow$$

$$\sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = T_s \qquad \forall f$$

TRASMISSIONE PRIVA DI ISI

Supponiamo sia assegnato un canale a banda rigorosamente limitata, con banda B_C.

$$C(f) = 0 \qquad |f| > B_C$$

e supponiamo che $B_T = B_C$, ovvero che il segnale trasmesso occupa tutta la banda messa a disposizione dal canale.

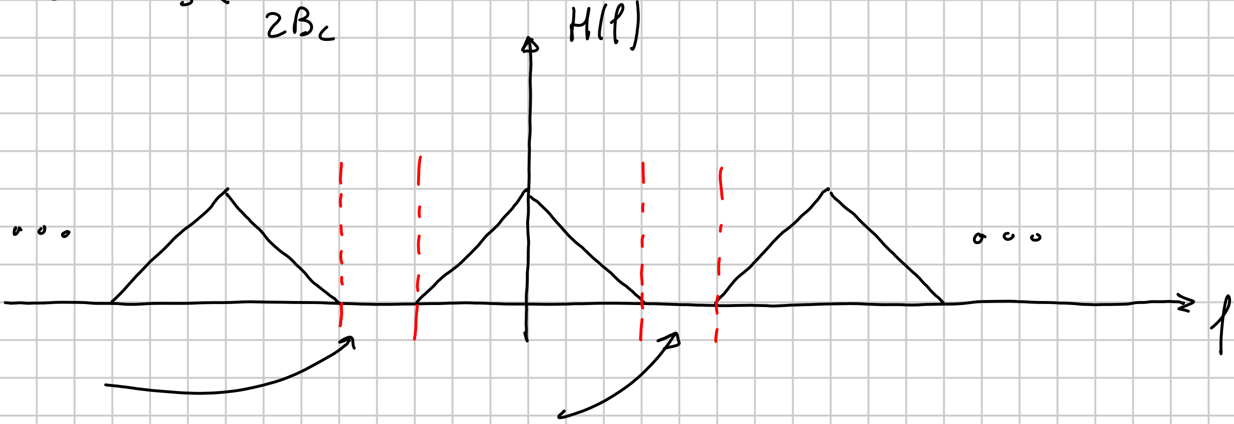
Allora si verificano le seguenti:

1) NON È POSSIBILE in alcun modo eliminare l'ISI quando

$$T_s < \frac{1}{2 B_C}$$

Dim:

Quando $T_s < \frac{1}{2B_c}$



Esistono degli intervalli frequenziali dove $\tilde{H}(f) = 0$, per cui non può mai accadere che $\bar{H}(f) = 1 \quad \forall f$

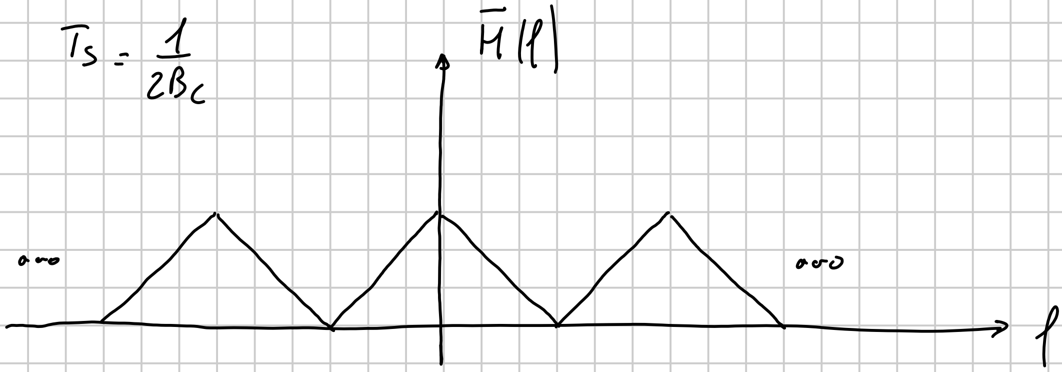
2) IL PIÙ PICCOLO VALORE DI T_s che permette di eliminare l'ISI è

$$T_s^{(min)} = \frac{1}{2B_c}$$

$$f_s^{(max)} = \frac{1}{T_s^{(min)}} = 2B_c = f_N \qquad \text{(frequenza di Nyquist)}$$

Dim:

Quando $T_s = \frac{1}{2B_c}$

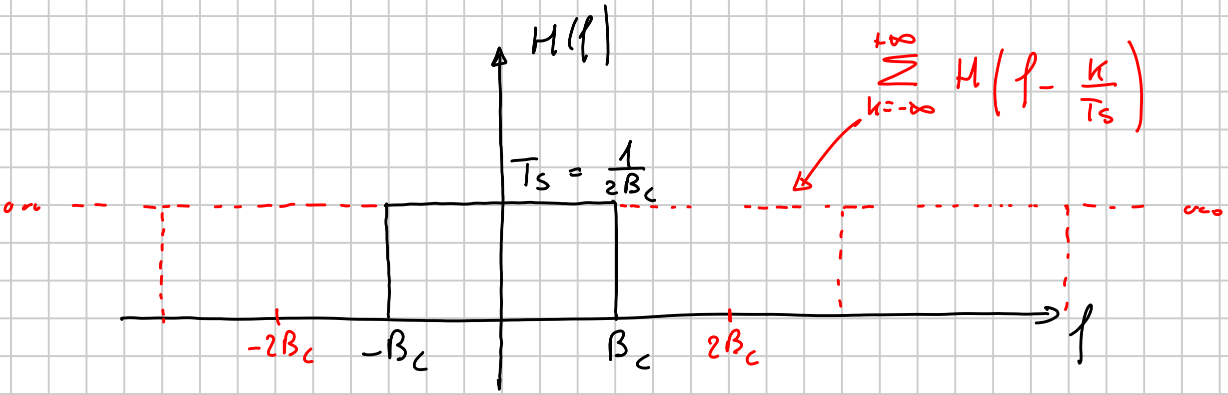


Non esistono intervalli di frequenze dove $\bar{H}(f) = 0$

3) Nel caso valga la condizione $T_s = \frac{1}{2B_c}$, allora l'unica funzione di trasferimento che permette di eliminare completamente l'ISI è

$$H(f) = \frac{1}{2B_c} \mathrm{rect}\left(\frac{f}{2B_c}\right) \iff h(t) = \mathrm{sinc}(2B_c t)$$

Dim:



Si nota anche che la funzione $\text{sinc}(2B_c t)$ si annulla quando $t = \frac{K}{2B_c} = KT_s$ con $K \neq 0$

per cui

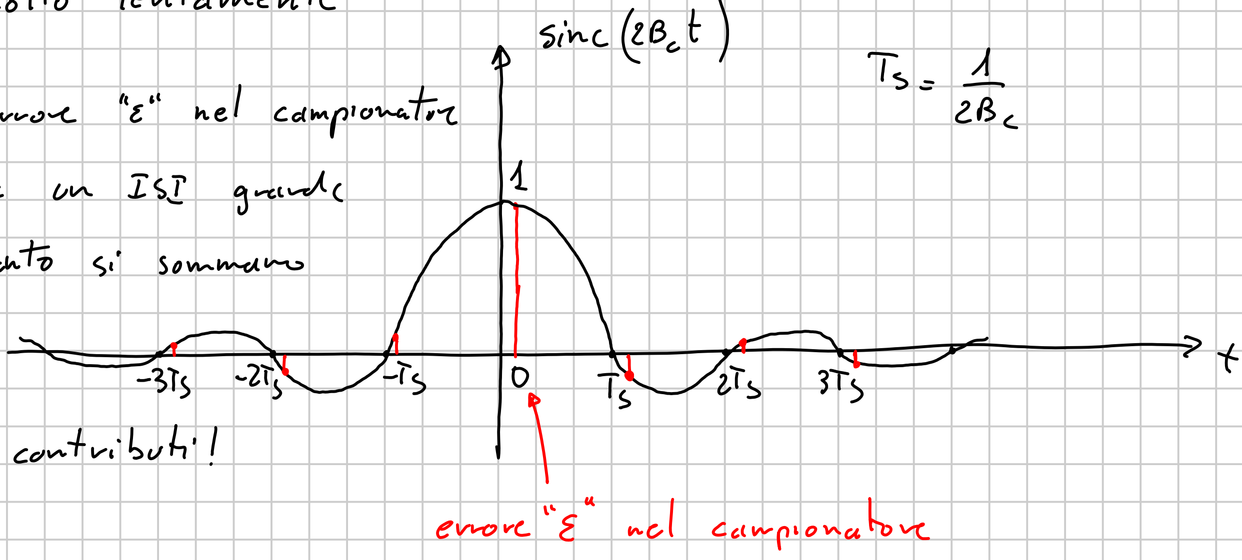
$$h[KT_s] = \text{sinc}\left(2B_c \frac{K}{2B_c}\right) = \text{sinc}(K) = \begin{cases} 1 & K=0 \\ 0 & K\neq 0 \end{cases}$$

LIMITI DI APPLICABILITÀ DELLA FUNZIONE DI TRASFERIMENTO RETTANGOLARE

1) Realizzabilità di una funzione di trasferimento rettangolare
Risposte in frequenza ideali come quella rettangolare non sono fisicamente realizzabili (Criterio di Paley-Wiener).

2) Piccoli errori di campionamento provocano un ISI molto grande poiché la funzione $\text{sinc}(2B_c t)$ decresce molto lentamente

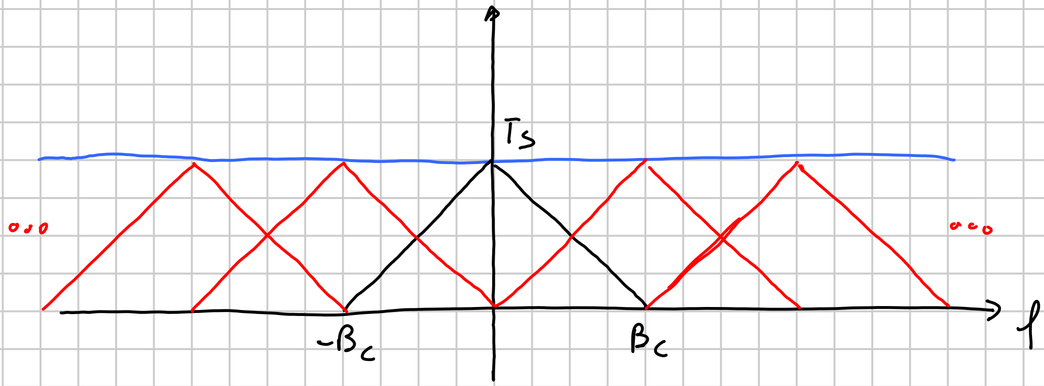
Un errore "ε" nel campionatore induce un ISI grande in quanto si sommano molti contributi!



Rilassando la condizione $T_S = \frac{1}{2B_C}$, ovvero ammettendo

$$T_S > \frac{1}{2B_C}$$

si ottiene il seguente effetto:



La sovrapposizione permette di definire una classe di infinite funzioni di trasferimento che soddisfano il I° criterio di Nyquist.

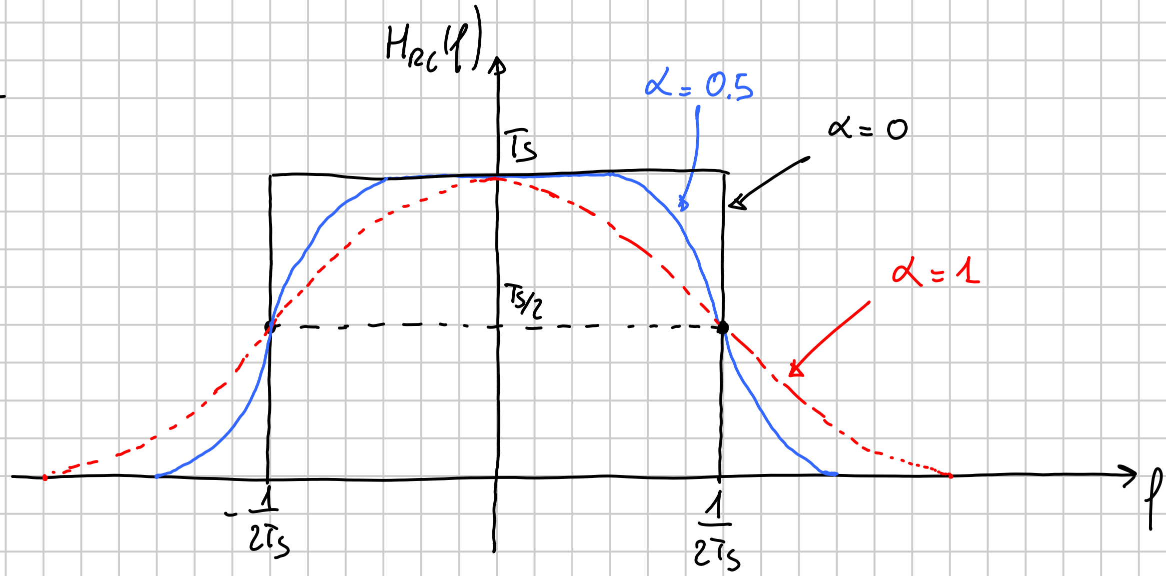
In questo caso però $B_C > \frac{1}{2T_S}$, per cui, a parità di T_S, c'è bisogno di una banda disponibile nel canale che è maggiore di quella che occorre con la funzione di trasferimento rettangolare.

FILTRO A COSENO RIALZATO (Raised Cosine - RC)

$$H_{RC}(f) = \begin{cases} T_S & 0 \leq |f| \leq \frac{1-\alpha}{2T_S} \\ \frac{T_S}{2}\left[1 - \sin\left(\frac{\pi T_S}{\alpha}\left(|f| - \frac{1}{2T_S}\right)\right)\right] & \frac{1-\alpha}{2T_S} \leq |f| \leq \frac{1+\alpha}{2T_S} \\ 0 & |f| \geq \frac{1+\alpha}{2T_S} \end{cases}$$

$0 \leq \alpha \leq 1$

$0 \leq \alpha \leq 1$



α = coefficiente di roll-off

Proprietà

- Quando $\alpha = 0$ il coseno rialzato coincide con la funzione di trasferimento rettangolare

- La banda B_H è direttamente ottenibile

$$B_H = \frac{1+\alpha}{2T_S}$$

La $h_{RC}(t)$ è calcolabile in forma chiusa:

$$h_{RC}(t) = \mathrm{sinc}\left(\frac{t}{T_S}\right) \frac{\cos\left(\frac{\alpha \pi t}{T_S}\right)}{\left(1 - \frac{2\alpha t}{T_S}\right)^2}$$

$$\Downarrow$$

$$h_{RC}(kT_S) = \delta[k]$$

- soddisfa il I° criterio di Nyquist nel tempo, per cui garantisce assenza di ISI

- decresce per $|t| \to \infty$ come $1/|t|^3$ per $\alpha > 0$, quindi molto più velocemente del caso $\alpha = 0$ (rettangolare)

ECCESSO DI BANDA E EFFICIENZA SPETTRALE DEI SISTEMI M-PAM CON COSENO RIALZATO

$$p(t) \otimes c(t) \otimes h_R(t) = h_{RC}(t)$$

sagomatore in tx

risposta impuls. del canale

filtro in ricezione

risposta impulsiva del tipo a coseno rialzato

Efficienza spettrale del canale di comunicazione numerico

$$\eta_B = \frac{\log_2 M}{B_T T_S} = \frac{\log_2 M}{B_H T_S} = \boxed{\frac{2\log_2 M}{1+\alpha}}$$

Considerazioni:

·) l'efficienza spettrale, a parità di M, decresce al crescere del coefficiente di roll-off (α)

·) La robustezza del sistema di comunicazione numerico all'ISI aumenta al crescere di α

$\Rightarrow$ TRADE-OFF tra robustezza all'ISI e eff. spettrale

VALORI OTTIMALI in corrispondenza di $\alpha \simeq 0.4$

Eccesso di banda richiesto dall'adozione del coseno rialzato

$$\Delta B_H = B_H - \frac{1}{2T_S} = \frac{\alpha}{2T_S}$$

PRESTAZIONI DI UN SISTEMA DI COMUNICAZIONE NUMERICO IN BANDA BASE IN PRESENZA DI RUMORE

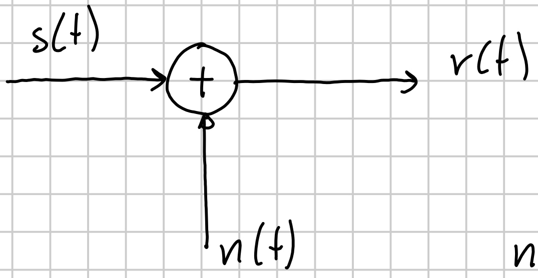
Capacità di canale

La capacità C di un canale di comunicazione è definita come il massimo valore che può assumere il tasso binario di segnalazione $R_b = \frac{1}{T_b}$ al variare di tutte le possibili coppie MODULATORE / DEMODULATORE sotto il vincolo che la Probabilità di errore sia esattamente nulla

$$\begin{cases} C \triangleq \max \{ R_b \} \\ P_E(b) = P\{ \hat{b}[n] \neq b[n] \} = 0 \end{cases}$$

La Capacità di canale C si misura in bit/s ed è un numero non-negativo.
Ovviamente, più è grande la capacità del canale e migliori sono le sue prestazioni.

CAPACITÀ DI CANALE CON RUMORE GAUSSIANO BIANCO ADDITIVO (AWGN)



$n(t)$ Gaussiano Bianco

In questo caso la capacità di canale può essere espressa in forma chiusa ed in dipendenza dei parametri caratteristici del segnale trasmesso e del rumore

$$C = B_T \log_2 \left(1 + \frac{P_S}{N_0 B_T} \right) \qquad \text{(Shannon)}$$

dove

B_T = banda del segnale $s(t)$

P_S = potenza media di $s(t)$

$\frac{N_0}{2}$ = DSP del rumore $n(t)$ (costante essendo bianco)

Considerazioni

·) Fissata B_T

$$\lim_{P_S/N_0 \to 0} C = 0$$

$$\lim_{P_S/N_0 \to \infty} C = +\infty$$

·) Fissato P_S/N_0

$$\lim_{B_T \to 0} C = 0$$

$$\lim_{B_T \to \infty} C = \log_2 e \cdot \frac{P_S}{N_0}$$

·) Riscrivendo la formula di Shannon utilizzando $P_S = E_b R_b$

$$\frac{C}{B_T} = \log_2\left(1 + \frac{E_b}{N_0}\frac{R_b}{B_T}\right)$$

E_b = energia per bit

$R_b \leq C$ (data la definizione di C come max valore di R_b)

SISTEMA DI COMUNICAZIONE NUMERICO IDEALE

Un sistema di comunicazione numerico è detto ideale se soddisfa le seguenti condizioni

1) $R_b = C$

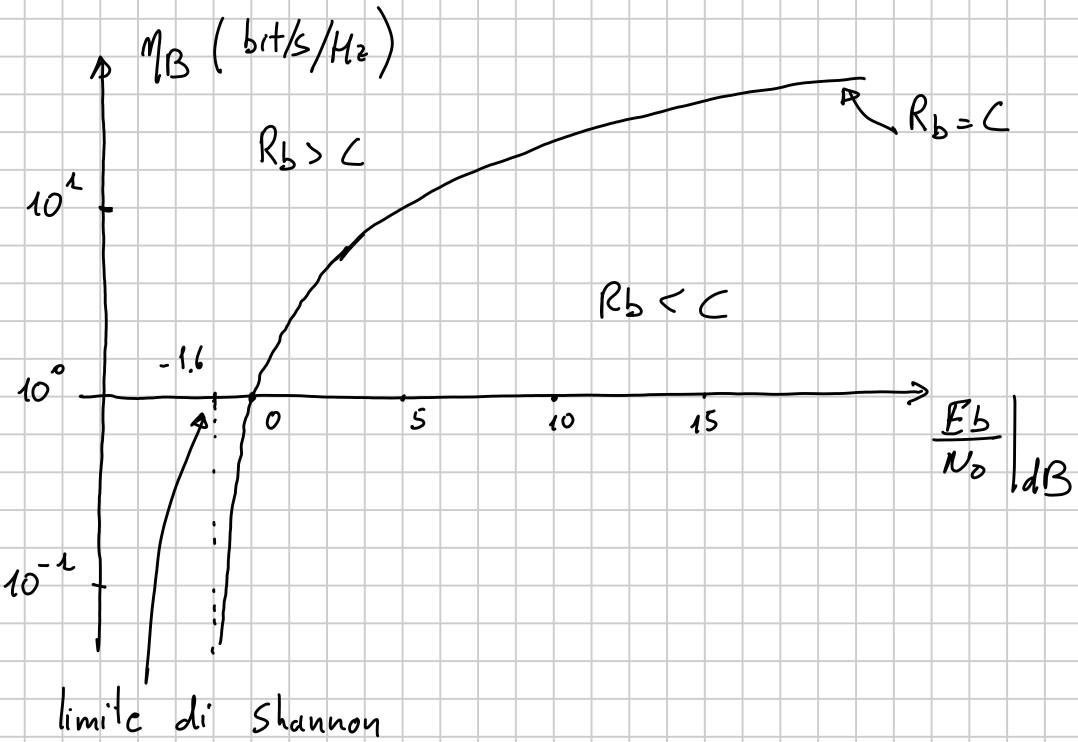
2) $P_E(b) = 0$

In queste condizioni è possibile mettere in relazione la efficienza spettrale con il rapporto E_b/N_0 (legato alla efficienza in potenza)

$$\Rightarrow \quad \eta_B = \log_2\left(1 + \frac{E_b}{N_0}\eta_B\right) \qquad , \quad \eta_B = \frac{R_b}{B_T} = \frac{C}{B_T}$$

$$\Rightarrow \quad 2^{\eta_B} = 1 + \frac{E_b}{N_0}\eta_B$$

$$\Rightarrow \quad \boxed{\frac{E_b}{N_0} = \frac{2^{\eta_B} - 1}{\eta_B}} \qquad , \quad \eta_B \geq 0$$



$\Rightarrow$ La zona sotto la curva $R_c = C$ è la zona ove è possibile avere trasmissioni con probabilità di errore nulla

Considerazioni

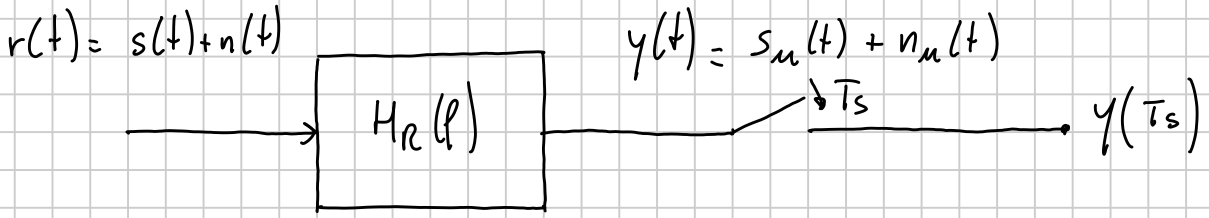
•) $\lim_{\eta_B \to +\infty} \frac{2^{\eta_B} - 1}{\eta_B} = +\infty$

•) $\lim_{\eta_B \to 0} \frac{2^{\eta_B} - 1}{\eta_B} = \log_e 2 \quad (-1.6\ dB)$

Una modulazione è efficiente in potenza se, operando al limite di Shannon $\Rightarrow \frac{E_b}{N_0} = \log_e 2$ (con $\eta_B = 0$), si ottiene una $P_E(b) = 0$

RICEZIONE OTTIMA IN PRESENZA DI RUMORE BIANCO

Per il momento consideriamo solo gli effetti relativi al rumore



Dove $s(t)$ è un segnale di forma nota e $n(t)$ è un rumore additivo bianco

$$s_u(t) = s(t) \otimes h_R(t) \quad , \quad n_u(t) = n(t) \otimes h_R(t)$$

$$y(T_s) = s_u(T_s) + n_u(T_s)$$

Si definisce il rapporto segnale-rumore in uscita al filtro $h_R(t)$ all'istante $t = T_s$ come

$$SNR \triangleq \frac{s_u^2(T_s)}{E[n_u^2(T_s)]}$$

Si definisce ricevitore ottimo il filtro $h_R(t)$ che massimizza il SNR in uscita al filtro.

Nel caso di rumore bianco in ingresso il filtro ottimo prende il nome di FILTRO ADATTATO.

Problema:

1) Derivare il filtro $h_R(t)$ che massimizza l'SNR all'uscita

2) Determinare il valore massimo del SNR all'uscita.

Derivazione del filtro adattato

$$SNR = \frac{s_u^2(T_s)}{E[n_u^2(T_s)]}$$

$$s_u^2(T_s) = \left[\int_{-\infty}^{+\infty} s(\tau)\, h_R(T_s - \tau)\, d\tau \right]^2 = \left[\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df \right]^2$$

$$E[n_u^2(T_s)] = R_{n_u}(0)$$

$$R_{n_u}(\tau) = R_n(\tau) \otimes h_R(\tau) \otimes h_R(-\tau)$$

$$S_{n_u}(f) = S_n(f)\, |H_R(f)|^2 = \frac{N_0}{2} |H_R(f)|^2$$

$$R_{n_u}(0) = \int_{-\infty}^{+\infty} S_{n_u}(f)\, df$$

$$SNR = \frac{\left[\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df \right]^2}{\frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2\, df} = \frac{2}{N_0 E_{h_R}} \left[\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df \right]^2$$

Utilizzando la disuguaglianza di Schwarz si può dimostrare che

$$\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df \quad \text{raggiunge il massimo valore}$$

quando

$$H_R(f)\, e^{j2\pi f T_s} = S^*(f)$$

Quindi

$$\boxed{H_R(f) = S^*(f)\, e^{-j2\pi f T_s}} \iff \boxed{h_R(t) = s(T_s - t)}$$

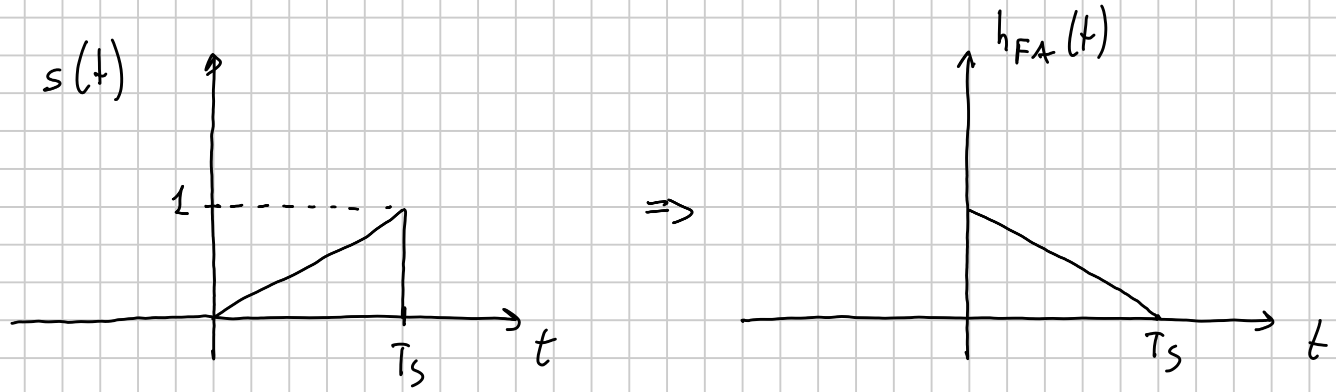
Dalla espressione della risposta impulsiva $h_R(t)$ si deduce il nome di FILTRO ADATTATO, in quanto la sua risp. imp. è adattata al segnale in ingresso al filtro stesso.

Studiando il modulo della risposta in frequenza $H_R(f)$ si deduce che il filtro tende ad amplificare le componenti frequenziali dove è presente il segnale e ad attenuare (o anche eliminare) le componenti frequenziali dove il contributo di segnale è scarso (o addirittura assente).

$$|H_R(f)| = |S(f)|$$

La simbologia per indicare un filtro adattato è $h_{FA}(t)$ o $H_{FA}(f)$

Esempio:



Calcolo del SNR_{MAX}

Il valore del SNR_{MAX} si ottiene per definizione quando si utilizza il F.A.

$$SNR = \frac{2}{N_0 E_s} \cdot E_s^2 = \boxed{\frac{2E_s}{N_0}}$$

N.B. Il SNR non dipende dalla forma del segnale ma solo dalla sua energia. Questo da spazio alla progettazione della forma del segnale indipendentemente dai risultati in termini di SNR.

N.B. C'è inoltre da notare che il massimo del SNR lo si ottiene per qualunque $h_{FA}(t) = K\, s(T_s - t)$, $K \in \mathbb{R}$

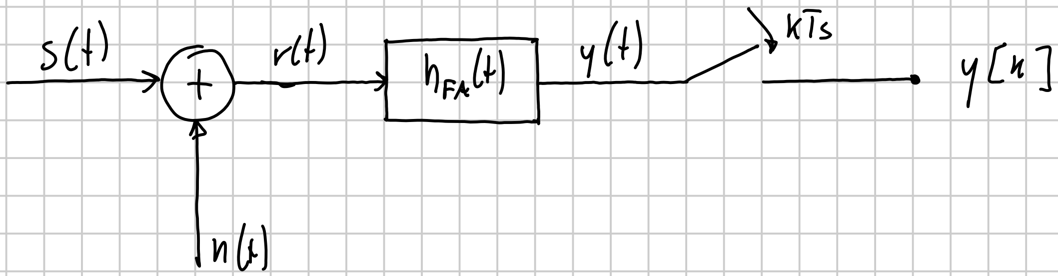
Infatti basta calcolare il SNR

$$SNR = \frac{2}{N_0 \cancel{K^2} E_S} \cancel{K^2} E_S^2 = \frac{2 E_S}{N_0}$$

Quindi, fattori di amplificazione e/o attenuazione non cambiano il risultato. Questo è abbastanza intuitivo, in quanto un fattore costante di amplificazione opera allo stesso modo sul segnale utile e sul rumore, per cui, nel rapporto, i contributi si elidono.

SCHEMA DEL RICEVITORE CON FILTRO ADATTATO

Considerariamo la trasmissione di un simbolo



$$s(t) = \alpha\, p(t - nT_s)$$

$$h_{FA}(t) = K\, p(T_s - t)$$

$$y(t) = s_u(t) + n_u(t)$$

Caratteristiche di $s_u(t)$ e $n_u(t)$

•) $s_u(t) = s(t) \otimes h_{FA}(t) = k\alpha\, p(t) \otimes p(T_s - t)$

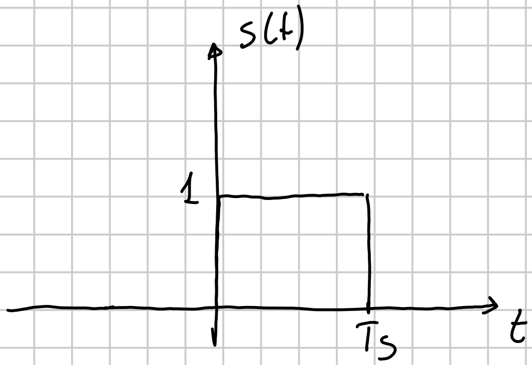
$$= k\alpha \int_{-\infty}^{+\infty} p(\tau)\, p(\tau - (t - T_s))\, d\tau = k\alpha\, C_p(t - T_s)$$

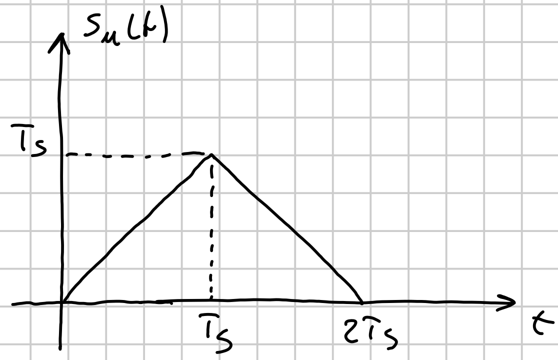
$$C_p(t) = \int_{-\infty}^{+\infty} p(\tau)\, p(\tau - t)\, d\tau$$ autocorrelazione dell'impulso sagomatore

Esempio:

$$s(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right)$$

$$s_u(t) = C_s(t - T_s) = T_s\left(1 - \frac{|t - T_s|}{T_s}\right)\mathrm{rect}\left(\frac{t - T_s}{2T_s}\right)$$





·) $n_u(t) = n(t) \otimes h_{FA}(t)$

$n(t)$ = rumore bianco, Gaussiano, additivo (AWGN)

$E[n(t)] = 0$

N.B. un rumore bianco è per definizione SSL ed essendo anche Gaussiano è anche SSS.

$$R_n(\tau) = \sigma_n^2\,\delta(\tau) = \frac{N_0}{2}\delta(\tau)$$

$n(\bar{t})$ = v.a. con ddp $\displaystyle f_N(n) = \frac{1}{\sqrt{2\sigma_n^2}}\, e^{-\frac{n^2}{2\sigma_n^2}} = \frac{1}{\sqrt{N_0}}\, e^{-\frac{n^2}{N_0}}$

Essendo il filtro in ricezione un filtro lineare e stazionario $n_u(t)$ è un rumore Gaussiano, additivo e stazionario

$E[n_u(t)] = 0$

$$R_{n_u}(\tau) = R_n(\tau) \otimes h_{FA}(\tau) \otimes h_{FA}(-\tau) = \frac{N_0}{2} C_{h_{FA}}(\tau)$$

$C_{h_{FA}}(\tau)$ = autocorrelazione di $h_{FA}(t)$

$$S_{n_u}(f) = \frac{N_0}{2}|H_{FA}(f)|^2 = K^2\frac{N_0}{2}|P(f)|^2$$

$$P_{n_u} = \frac{N_0}{2}E_{H_{FA}} = \frac{N_0}{2}E_p K^2$$

È importante capire se i campioni di rumore sono tra loro correlati o meno.

$$E\left[n_u[k]\, n_u[n] \right] = 0 \qquad \forall\, k \neq n \qquad \text{(INCORRELAZIONE)}$$

N.B. la si può scrivere così poiché $E[n_u[k]] = 0$!!

Questo vuol dire che $R_{n_u}[kT_s] = 0 \quad \forall\, k \neq 0$

$$R_{n_u}[kT_s] = K^2 \frac{N_0}{2}\, C_p[kT_s] = 0$$

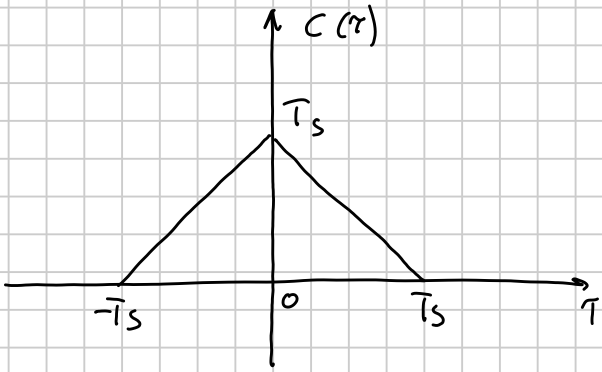
$$\Rightarrow C_p[kT_s] = 0$$

Dobbiamo ricordare che il segnale utile in ingresso al filtro adattato è ottenuto tramite il modulatore in trasmissione per cui è la funzione $p(t)$ che determina la sagoma (forma) del segnale $s(t)$

·) IMPULSO RETTANGOLARE

$$p(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right)$$

$$c_p(\tau) = T_s\left(1 - \frac{|t|}{T_s}\right)\mathrm{rect}\left(\frac{t}{2T_s}\right)$$



$$R_{n_u}(\tau) = K^2 \frac{N_0}{2}\, C_p(\tau)$$

$$R_{n_u}(nT_s) = \begin{cases} K^2 \frac{N_0}{2} T_s & k = 0 \\ 0 & k \neq 0 \end{cases} \Rightarrow \text{campioni di rumore incorrelati}$$

$$\Downarrow$$

indipendenti (Gaussiani)

·) IMPULSO A RADICE DI COSENO RIALZATO

$$P(f) = \text{radice di coseno rialzato} \quad \left(\sqrt{H_{RC}(f)}\right)$$

$$S_{n_u}(f) = K^2 \frac{N_0}{2} |P(f)|^2 = K^2 \frac{N_0}{2} H_{RC}(f)$$

$$R_{n_u}(\tau) = K^2 \frac{N_0}{2} h_{RC}(\tau)$$

$$R_{n_u}(nT_s) = \begin{cases} K^2 \frac{N_0}{2} & K = 0 \\ 0 & K \neq 0 \end{cases}$$

N.B. la $h_{RC}(\tau)$ ha la $\mathrm{sinc}(\cdot)$ che si annulla in multipli di T_s

SNR per bit all'ingresso del ricevitore

Il SNR per bit è un parametro utile per determinare le prestazioni di un ricevitore in quanto tiene in considerazione quantità energetiche sia del segnale utile che del rumore.

$$SNR_b = \frac{E_b}{N_0}$$

$$E_b \triangleq P_S T_b = E\left[x^2[n]\right] T_b \qquad \text{Energia per bit}$$

$$\frac{N_0}{2} = S_n(f)$$

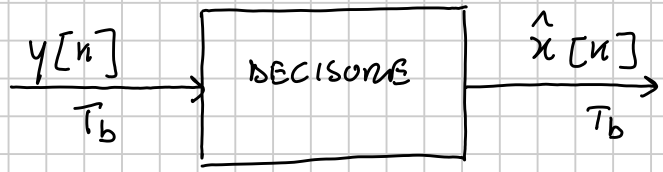
$$\Rightarrow \boxed{SNR_b = \frac{E\left[x^2[n]\right]}{N_0 R_b}} \quad , \quad R_b = \frac{1}{T_b}$$

DECISORE OTTIMO E CRITERIO DELLA MASSIMA VEROSIMIGLIANZA

Il decisore deve mappare i campioni $y[n]$ in simboli dell'alfabeto. I campioni $y[n]$ sono statisticamente indipendenti l'uno dall'altro. Questo è dimostrato dal fatto che

$$y[n] = s_u[n] + n_u[n]$$

dove sia $s_u[n]$ che $n_u[n]$ sono indipendenti.



$$y[n] = s_u[n] + n_u[n] \qquad \hat{x}[n] \in A_s$$

Quindi si può concludere che $\hat{x}[n]$ può essere deciso in base alla sola conoscenza di $y[n]$. Questa decisione si dice di tipo "ad un sol colpo" (one-shot detector).

DECISIONE A MINIMA PROBABILITÀ DI ERRORE

·) Probabilità di errore sul simbolo

$$P_E(M) \triangleq P\{\hat{x}[n] \neq x[n]\}$$

·) Criterio di ottimalità: minimizzazione della $P_E(M)$

Derivazione del decisore ottimo

$$x \triangleq x[n], \quad y \triangleq y[n], \quad n_u \triangleq n_u[n], \quad \hat{x} = \hat{x}[n]$$

CRITERIO A MASSIMA PROBABILITÀ A POSTERIORI E MINIMA PROBABILITÀ DI ERRORE

MAP = Maximum A-posteriori Probability

$$\hat{x} = \max_{i=1,\dots,M} \{P(x = \alpha_i \mid y\}$$

Viene associato ad un osservato y il simbolo dell'alfabeto $\hat{x}$ tale che sia massima la probabilità a posteriori (condizionata) che quel simbolo sia stato trasmesso.

$$\boxed{\text{Se il decisore adotta il criterio MAP allora la probabilità di errore sul simbolo è minima}}$$

Dimostrazione

Definiamo $R(i) \triangleq \{ y \in \mathbb{R} : \hat{x} = \alpha_i \}$, $i = 1, \dots, M$
come la "zona di decisione" del simbolo α_i, ovvero l'insieme dei valori di y per cui si decide per il simbolo α_i.

$$P\{x = \alpha_i | y\} = \frac{f_Y(y | x = \alpha_i) \, P\{x = \alpha_i\}}{f_Y(y)} \qquad \text{(BAYES)}$$

$$P_E(M) = P\{\hat{x} \neq x\} = 1 - P\{\hat{x} = x\} = 1 - \sum_{i=1}^{M} P\{\hat{x} = \alpha_i, x = \alpha_i\}$$

$$= 1 - \sum_{i=1}^{M} P\{\hat{x} = \alpha_i | x = \alpha_i\} P\{x = \alpha_i\} =$$

$$= 1 - \sum_{i=1}^{M} P\{x = \alpha_i\} P\{y \in R(i) | x = \alpha_i\}$$

$$= 1 - \sum_{i=1}^{M} P\{x = \alpha_i\} \int_{y \in R(i)} f_Y(y | x = \alpha_i) \, dy =$$

$$= 1 - \sum_{i=1}^{M} \int_{y \in R(i)} P\{x = \alpha_i\} f_Y(y | x = \alpha_i) \, dy =$$

$$= 1 - \sum_{i=1}^{M} \int_{y \in R(i)} f_Y(y) \, P\{x = \alpha_i | y\} \, dy$$

Per minimizzare la $P_E(M)$ devo scegliere le $R(i)$ in modo tale che, osservato y, sia massima la probabilità a posteriori relativa al simbolo i-esimo.

Si osserva che, se le probabilità a priori sono identiche

$$P\{x = \alpha_i\} = \frac{1}{M} \qquad i = 1, \dots, M$$

allora, dato che $f_Y(y)$ non dipende da "i":

$$\hat{x} = \max_{i=1,\dots,M} \left[\frac{1}{M} \cdot \frac{1}{f_Y(y)} \cdot f_Y(y | x = \alpha_i) \right] = \max_{i=1,\dots,M} \left[f_Y(y | x = \alpha_i) \right]$$

La funzione $f_Y(y|x=\alpha_i)$ viene detta anche

"FUNZIONE DI VEROSIMIGLIANZA"

Infatti il criterio a minima probabilità di errore (o massima probabilità a posteriori) coincide con il criterio a MASSIMA VEROSIMIGLIANZA quando le probabilità a priori $P\{x=\alpha_i\}$ sono identiche.

Nel caso di AGWN

$$y = s_u + n_u = \alpha_i + n_u \quad , \quad n_u \in \mathcal{N}(0, \sigma_{n_u}^2)$$

$$f_Y(y|x=\alpha_i) = f_{n_u}(y-\alpha_i)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{(y-\alpha_i)^2}{2\sigma_{n_u}^2}} \qquad i = 1, \dots, M$$

$$\hat{x} = \max_{i=1,\dots,M} \left[\frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{(y-\alpha_i)^2}{2\sigma_{n_u}^2}} \right] = \min_{i=1,\dots,M} \left\{ (y-\alpha_i)^2 \right\}$$

$$\boxed{\hat{x} = \min_{i=1,\dots,M} \left\{ |y-\alpha_i| \right\} \qquad \text{minimo della distanza Euclidea}}$$

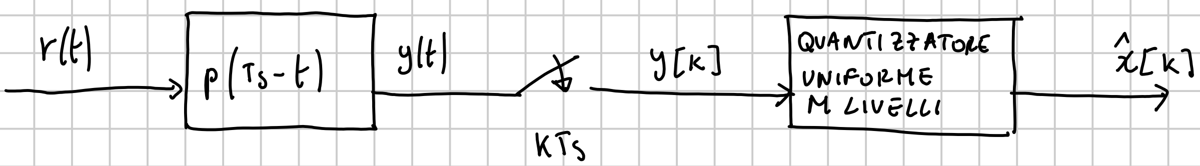
Il decisore ottimo coincide con la scelta del simbolo a distanza Euclidea minima dall'osservato

Le zone di decisione sono quindi stabilite dalla regola di quantizzazione uniforme.

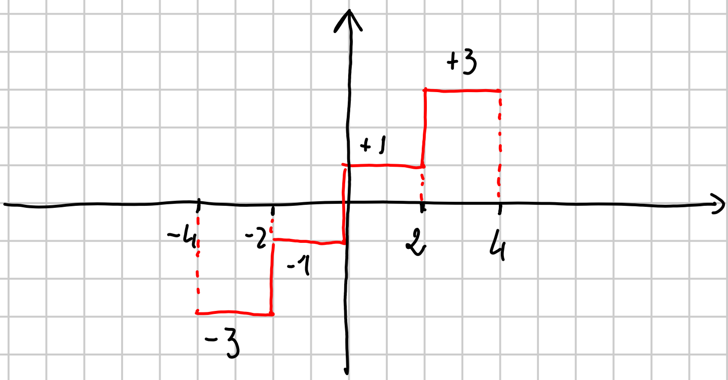
Questo significa che il decisore può essere realizzato con un quantizzatore uniforme

RICEVITORE OTTIMO PER UN SISTEMA DI COMUNICAZIONE PAM

Per un sistema di comunicazione PAM con simboli equiprobabili, il ricevitore ottimo secondo il criterio a minima probabilità di errore è il seguente



Esempio : 4-PAM



QUANTIZZATORE UNIFORME 4 LIVELLI

PROBABILITÀ DI ERRORE DI BIT E DI SIMBOLO

$$P_E(b) = P_r\{\hat{b}[k] \neq b[k]\}$$ bit

$$P_E(M) = P_r\{\hat{x}[k] \neq x[k]\}$$ simbolo

$P_E(M) = P_E(b)$ solo quando l'alfabeto A_s è composto da soli due simboli

Vale però sempre che:

$$\frac{P_E(M)}{\log_2 M} \leq P_E(b) \leq \frac{M/2}{M-1} P_E(M)$$

CODIFICA DI GRAY

$$A_S \triangleq \{ \alpha_1, \dots, \alpha_M \}$$

$$\alpha_i = 2i - M - 1 \qquad i = 1, \dots, M$$

La codifica di Gray associa stringhe di bit a simboli dell'alfabeto in modo che le stringhe di bit relative a due simboli adiacenti α_i e α_{i+1} differiscono al più per 1 bit.

Nel caso di SNR sufficientemente elevato (> 10 dB), l'evento errore consiste generalmente nel decidere per uno dei simboli dell'alfabeto adiacenti a quello trasmesso.

Utilizzando quindi la codifica di Gray e in condizioni di SNR elevato, un errore su un simbolo M-ario ogni N simboli M-ari, si traduce in un errore su una sola cifra binaria ogni N simboli M-ari, cioè ogni $N \log_2 M$ cifre binarie, quindi:

$$P_E(b) \simeq \frac{P_E(M)}{\log_2 M}$$

Esempi:

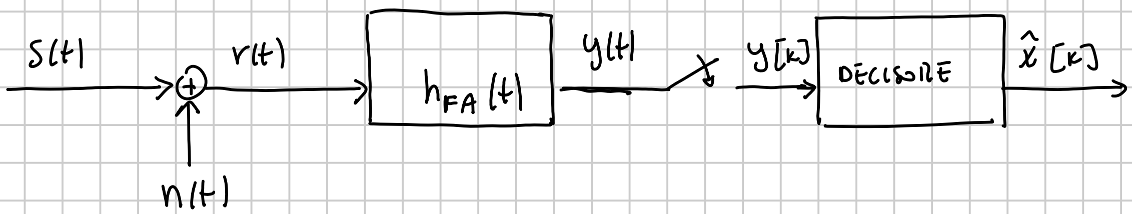
1) 4- PAM

-3	-1	1	3
00	01	11	10

2) 8 - PAM

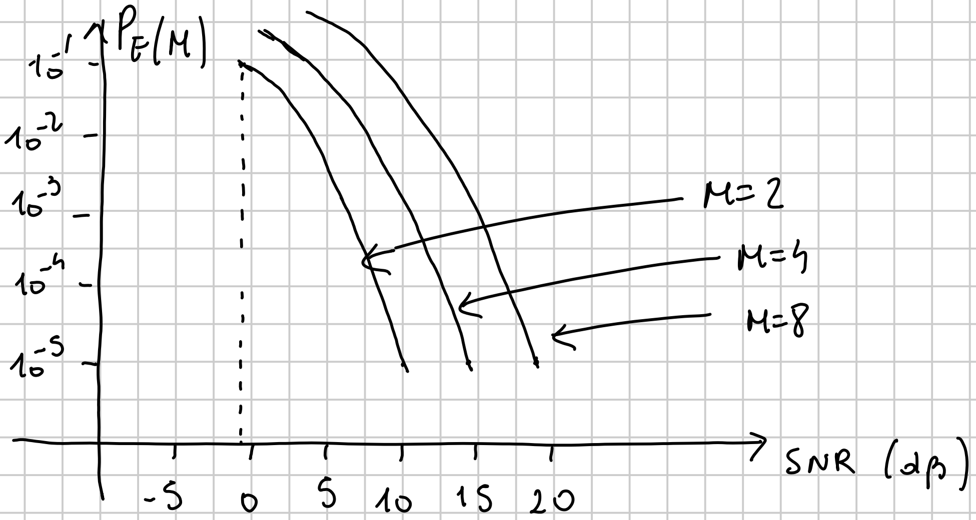
-7	-5	-3	-1	1	3	5	7
000	001	011	010	110	111	101	100

PRESTAZIONI DI UNA M-PAM IDEALE, IN PRESENZA DI SOLO RUMORE.



$$s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s)$$

$$\boxed{P_E(M) = \left(\frac{M-1}{M}\right) \operatorname{erfc}\left(\sqrt{\frac{3 \log_2 M \; SNR}{(M^2-1)}}\right)}$$



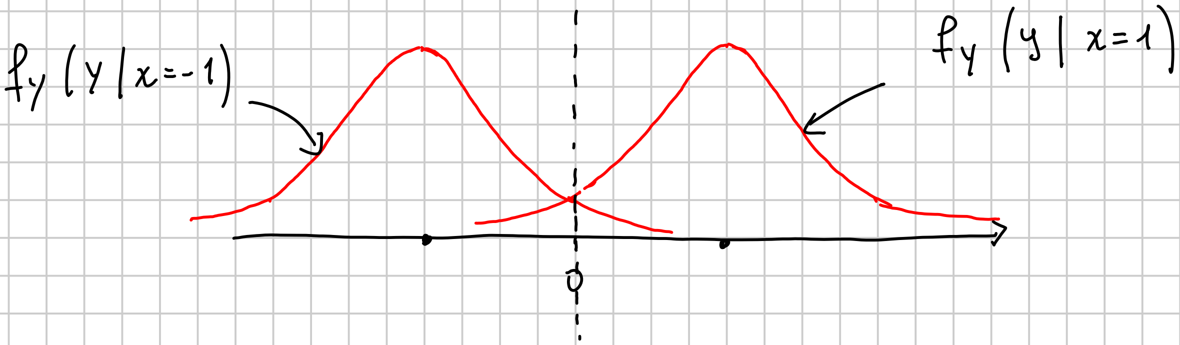
Per SNR > 10 dB ed utilizzando la codifica di Gray

$$\boxed{P_E(b) \simeq \frac{P_E(M)}{\log_2 M}}$$

Per una BPSK (2-PAM)

$$P_E(M) = P_E(b) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

Dimostrazione



$$P_E(b) = Pr\{x=+1\}\cdot Pr\{\hat{x}=-1 \mid x=1\} + Pr\{x=-1\}\cdot Pr\{\hat{x}=1 \mid x=-1\}$$

$$Pr\{\hat{x}=-1 \mid x=1\} = \int_{-\infty}^{0} f_y(y \mid x=1)\, dy$$

$$f_y(y \mid x=1) = \left. \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{(y-h(0)x)^2}{2\sigma_{n_u}^2}} \right|_{x=1}$$

Dopo il campionatore

$$y[k] = x[k]\,h(0) + n_u[k]$$

$$\sigma_{n_u}^2 = \frac{N_0}{2} E_{h_R} = \frac{N_0}{2} h(0)$$

$S_w(f) = \frac{N_0}{2} \Rightarrow$ DSP del processo di rumore in ingresso al filtro in ricezione $h_R(t)$

$$SNR = \frac{h^2(0)}{\frac{N_0}{2} h(0)} = \frac{2h(0)}{N_0}$$

$$P_r\{\hat{x}=-1 \mid x=1\} = 1 - Q\left(\frac{0-h(0)}{\sqrt{\frac{N_0}{2}h(0)}}\right) =$$

$$= Q\left(\sqrt{\frac{2h(0)}{N_0}}\right) = Q\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

Si può dimostrare per simmetria che

$$P_r\{\hat{x}=1 \mid x=-1\} = P_r\{\hat{x}=-1 \mid x=1\} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

Quindi

$$P_E(b) = \frac{1}{2}\cdot\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) + \frac{1}{2}\cdot\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

PRESENZA DI ISI E RUMORE

$$y(t) = \sum_{k=-\infty}^{+\infty} x[k]\, h(t-kT_s) + n_u(t)$$

$$h(t) = p(t) \otimes c(t) \otimes h_R(t)$$

$$n_u(t) = n(t) \otimes h_R(t)$$

$$y[k] = x[k]\, h(0) + I[k] + n_u[k]$$

$$I[k] = \sum_{\substack{n=-\infty \\ n\neq k}}^{+\infty} x[n]\, h\left((k-n)T_s\right)$$

L'approccio da seguire è il seguente:

Il filtro $h_R(t)$ deve essere allo stesso tempo quello che elimina l'ISI e che massimizza il SNR

Questo problema può essere risolto progettando opportunamente $p(t)$ e $h_R(t)$

EQUALIZZATORE ZERO-FORCING

Massimizza il SNR vincolando $I(k) = 0 \quad \forall\, k$

N.B. = È un problema di massimizzazione vincolato per cui la soluzione NON porta alla realizzazione del filtro adattato.

Soluzione:

·) $I[k] = 0$ quando $h(t) = h_{RC}(t)$, Allora

$$P(f)\,C(f)\,H_R(f) = H(f) = H_{RC}(f)\,\underbrace{e^{-j2\pi f T_s}}_{\text{CAUSALITÀ}}$$

Si pone quindi il problema di massimizzare le SNR con il vincolo

$$P(f)\,C(f)\,H_R(f) = H_{RC}(f)\,e^{-j2\pi f T_s}$$

$$\boxed{\begin{gathered}|P(f)| = |H_R(f)| = \sqrt{\frac{H_{RC}(f)}{|C(f)|}} \\ \angle P(f) = \angle H_R(f) = -\pi f T_s - \frac{\angle C(f)}{2}\end{gathered}}$$

N.B. = Se $C(f) = 1$ (CANALE IDEALE) Allora

$$P(f) = H_R(f) = \sqrt{H_{RC}(f)}\; e^{-j2\pi f T_s}$$

Il valore di SNR in tal caso è:

$$\boxed{SNR = \frac{E\left[x^2[n]\right] h^2(0)}{\frac{N_0}{2} \int_{-B_c}^{B_c} \frac{H_{Rc}(f)}{C(f)} \, df}}$$

B_c = banda del canale $C(f)$

$$h(0) = \int_{-\infty}^{+\infty} H(f) \, df$$

$$E_s = P_s T_s = T_s \frac{(M^2 - 1)}{3}$$

Se $C(f) = 1 \Rightarrow SNR = \frac{2 E_s}{N_0}$ FILTRO ADATTATO

CANALE IDEALE

ZERO - FORCING $\Rightarrow$ FILTRO ADATTATO

$$r(t) = \sum_{k=-\infty}^{+\infty} x[k] \, p(t - kT_s) + n(t)$$

$\hookrightarrow$ AWGN

$$h_R(t) = p(T_s - t)$$

$$H_R(f) = P(f) \, e^{-j2\pi f T_s}$$

$$H(f) = P^2(f) \, e^{-j2\pi f T_s} = H_{Rc}(f) \, e^{-j2\pi f T_s}$$

Riassumendo, il problema di eliminare l'ISI e massimizzare il SNR lo si risolve utilizzando il filtro sagomatore $p(t)$ e quello di ricezione, $h_R(t)$, realizzati con la RADICE DI COSENO RIALZATO

Nel caso di canale ideale, la soluzione coincide con il filtro adattato.

MODULAZIONI IN BANDA PASSANTE

SEGNALI PASSA-BANDA

$s(t)$ è passa-banda se

$S(f) \neq 0 \qquad f_0 - \frac{B}{2} < f < f_0 + \frac{B}{2} \qquad e \qquad -f_0 - \frac{B}{2} < f < -f_0 + \frac{B}{2}$

$(B << f_0) \qquad (B < f_0)$

Banda stretta

$$s(t) = a(t) \cos\left[2\pi f_0 t + \theta(t)\right]$$

inviluppo reale di $s(t)$

fase di $s(t)$

$$s(t) = \mathrm{Re}\left\{ \tilde{s}(t) e^{j 2\pi f_0 t} \right\}$$

$\tilde{s}(t) = a(t) e^{j\theta(t)}$ inviluppo complesso di $s(t)$

$\tilde{s}(t)$ deve essere un segnale passa-basso

PROCESSI ALEATORI PASSA-BANDA (STAZIONARI)

$n(t)$ processo di rumore è passa banda se

$S_n(f) \neq 0 \qquad f_0 - \frac{B}{2} \leq f \leq f_0 + \frac{B}{2} \qquad e \qquad -f_0 - \frac{B}{2} \leq f \leq -f_0 + \frac{B}{2}$

$$n(t) = \mathrm{Re}\left\{ \tilde{n}(t) e^{j 2\pi f_0 t} \right\}$$

inviluppo complesso di $n(t)$

$$\tilde{n}(t) = \underbrace{n_c(t)}_{\text{componente in fase}} + j\,\underbrace{n_s(t)}_{\text{componente in quadratura}}$$

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

FUNZIONE DI AUTOCORRELAZIONE

$$R_{nn}(\tau) \triangleq E[n(t)\,n(t-\tau)]$$

$$R_{n_c n_c}(\tau) \triangleq E[n_c(t)\,n_c(t-\tau)]$$

$$R_{n_s n_s}(\tau) \triangleq E[n_s(t)\,n_s(t-\tau)]$$

$$R_{n_c n_s}(\tau) \triangleq -R_{n_s n_c}(\tau) \triangleq E[n_s(t)\,n_c(t-\tau)]$$

$$R_{nn}(\tau) = R_{n_c n_c}(\tau)\cos(2\pi f_0 \tau) - R_{n_c n_s}(\tau)\sin(2\pi f_0 \tau)$$

$$= R_{n_s n_s}(\tau)\cos(2\pi f_0 \tau) + R_{n_s n_c}(\tau)\sin(2\pi f \tau)$$

$$R_{nn}(\tau) = \mathrm{Re}\left\{ R_{\tilde{n}\tilde{n}}(\tau)\, e^{j2\pi f_0 \tau} \right\}$$

$$R_{\tilde{n}\tilde{n}}(\tau) = E[\tilde{n}(t)\,\tilde{n}^*(t-\tau)] = R_{n_c n_c}(\tau) + j\,R_{n_c n_s}(\tau)$$

$$S_n(f) = TCF[R_{nn}(\tau)] = TCF\left[\mathrm{Re}\left\{R_{\tilde{n}\tilde{n}}(\tau)\, e^{j2\pi f_0 \tau}\right\}\right]$$

$$= TCF\left[\frac{1}{2}R_{\tilde{n}\tilde{n}}(\tau)\, e^{j2\pi f_0 \tau} + \frac{1}{2}R^*_{\tilde{n}\tilde{n}}\, e^{-j2\pi f_0 \tau}\right] =$$

$$\boxed{S_n(f) = \frac{1}{2}S_{\tilde{n}}(f-f_0) + \frac{1}{2}S_{\tilde{n}}(f+f_0)}$$ reale e pari

MODULAZIONI LINEARI E PRIVE DI MEMORIA

$$x_1[n] \longrightarrow s_1(t)$$

$$x_2[n] \longrightarrow s_2(t)$$

$$a\,x_1[n] + b\,x_2[n] = x[n]$$

$$\Downarrow$$

$$a\,s_1(t) + b\,s_2(t) = s(t)$$

LINEARITÀ

$$s(t) = f\left[x[n]\right]$$

il segnale modulato nell'intervallo di trasmissione K-esimo dipende solo dal simbolo K-esimo.

1) PAM in banda passante
2) PSK (Phase Shift Keying)
3) QAM (Quadrature Amplitude Modulation)

PAM in banda passante

$$s_m(t) = A_m\, p(t) \cos(2\pi f_0 t)$$

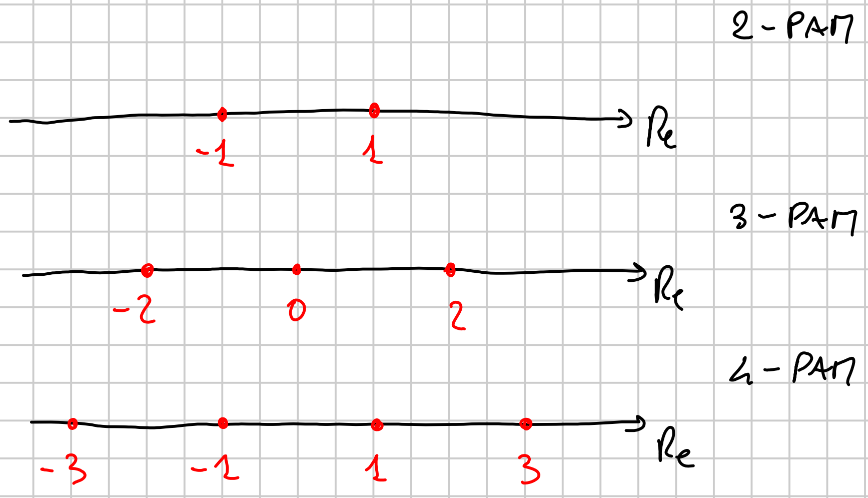
$$A_m \in A_s = \{\alpha_1, \dots, \alpha_M\}$$

$$\alpha_i = 2i - M - 1$$

$p(t)$ = impulso in banda base ("sagomatore")

$$\tilde{s}_m(t) = A_m\, p(t) \Rightarrow s_m(t) = \mathrm{Re}\left\{\tilde{s}_m(t)\, e^{j2\pi f_0 t}\right\}$$

COSTELLAZIONE DEI SIMBOLI

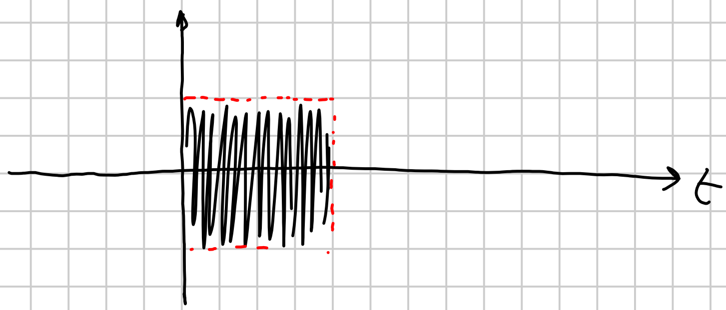


$$E_m = \int s_m^2(t)\, dt = \int A_m^2\, p^2(t) \cos^2(2\pi f_0 t)\, dt =$$

$$\simeq A_m^2 \frac{E_P}{2} \qquad \left(\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha\right)$$

(Simboli non equienerg)

$$\frac{A_m^2}{2} \int p^2(t) \cos(4\pi f_0 t)\, dt \simeq 0$$



PSK (MODULAZIONE DI FASE)

$$s_m(t) = p(t) \cos(2\pi f_0 t + \theta_m)$$

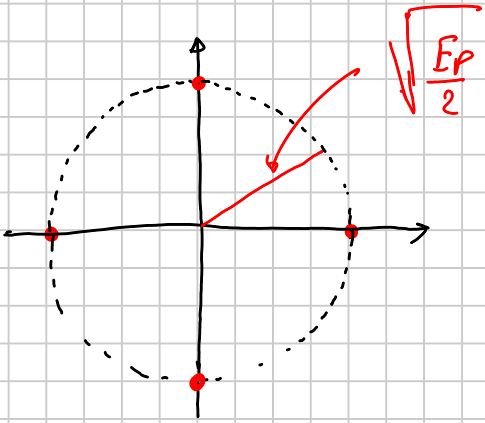
$$\theta_m = \frac{2\pi}{M}(m-1)$$

$$4\text{-PSK} \Rightarrow \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi\right\}$$

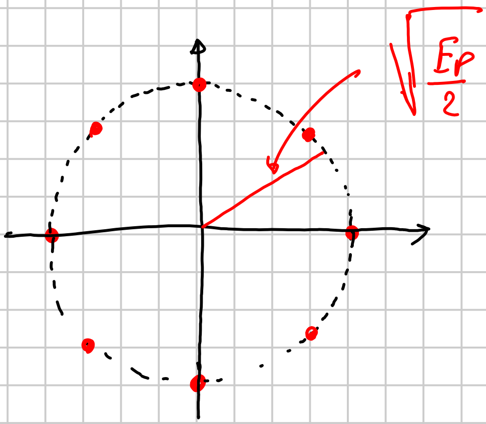
8 - PSK $\Rightarrow$ $\theta \equiv \left\{ 0, \frac{\pi}{4}, \frac{\pi}{\pi}, \frac{3}{4}\pi, \pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{7}{4}\pi \right\}$

$$\tilde{s}_m(t) = p(t)\, e^{j\theta_m} \Rightarrow s_m(t) = \mathrm{Re}\left\{ \tilde{s}(t)\, e^{j2\pi f_0 t} \right\}$$

COSTELLAZIONE DI UNA PSK

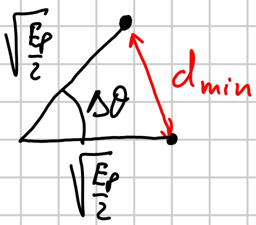


4 - PSK



8 - PSK

$d_{min} = ?$



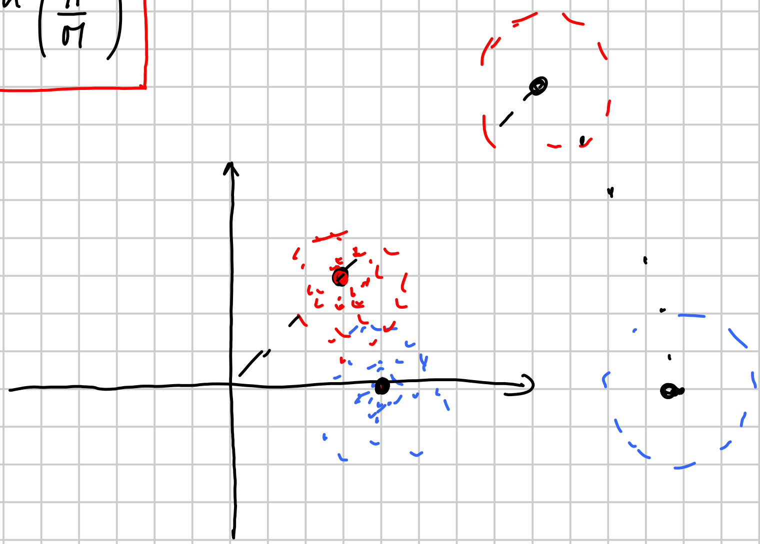
$$\Delta\theta = \frac{2\pi}{M}$$

$$d_{min}^2 = \frac{Ep}{2} + \frac{Ep}{2} - 2\,\frac{Ep}{2}\cos\left(\frac{2\pi}{M}\right) = Ep\left(1 - \cos\left(\frac{2\pi}{M}\right)\right)$$

$$2\sin^2(\alpha) = 1 - \cos(2\alpha) \quad , \quad \alpha = \frac{\pi}{M}$$

$$\boxed{d_{min}^2 = 2 E_p \sin^2\left(\frac{\pi}{M}\right)}$$

$$R_b = \frac{\log_2 M}{T_s}$$



EFFICIENZA ENERGETICA e SPETTRALE SI CONTRASTANO

$$E_m = \int s_m^2(t)\, dt = \int p^2(t) \cos^2(2\pi f_0 t + \theta_m)\, dt$$

$$\simeq \frac{E_P}{2} \qquad \text{(simboli equi-energia)}$$

QAM

$$s_m(t) = \underbrace{A_m^c\, p(t) \cos(2\pi f_0 t)}_{\text{componente in fase}} - \underbrace{A_m^s\, p(t) \sin(2\pi f_0 t)}_{\text{componente in quadratura}}$$

$A_m^c \; , \; A_m^s$

$A_m^c \in A_s^c \equiv \{ \alpha_1^c, \dots, \alpha_{M_c}^c \}$ $\qquad M_c$

$A_m^s \in A_s^s \equiv \{ \alpha_1^s, \dots, \alpha_{M_s}^s \}$ $\qquad M_s$

$$\tilde{s}_m(t) = \left(A_m^c + j A_m^s \right) p(t) \Rightarrow s_m(t) = \mathrm{Re}\left\{ \tilde{s}_m(t)\, e^{j 2\pi f_0 t} \right\}$$

$$E_m = \int s_m^2(t)\, dt = \int \left[A_m^c\, p(t) \cos(2\pi f_0 t) - A_m^s\, p(t) \sin(2\pi f_0 t) \right]^2 dt$$

$$= A_m^{c\,2} \int p^2(t) \cos^2(2\pi f_0 t)\, dt + A_m^{s\,2} \int p^2(t) \sin^2(2\pi f_0 t)\, dt +$$

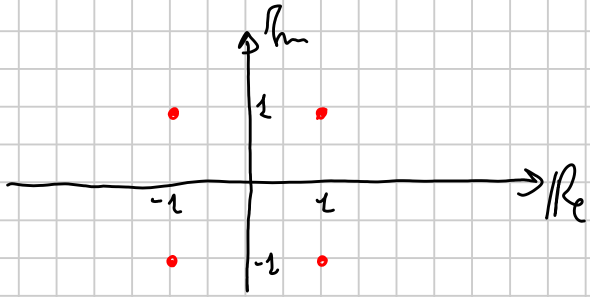
$$- \underbrace{2 A_m^c A_m^s \int \cos(2\pi f_0 t) \sin(2\pi f_0 t)\, dt}_{= 0}$$

$$= A_m^{c\,2} \frac{E_p}{2} + A_m^{s\,2} \frac{E_p}{2} = \left(A_m^{c\,2} + A_m^{s\,2} \right) \frac{E_p}{2}$$

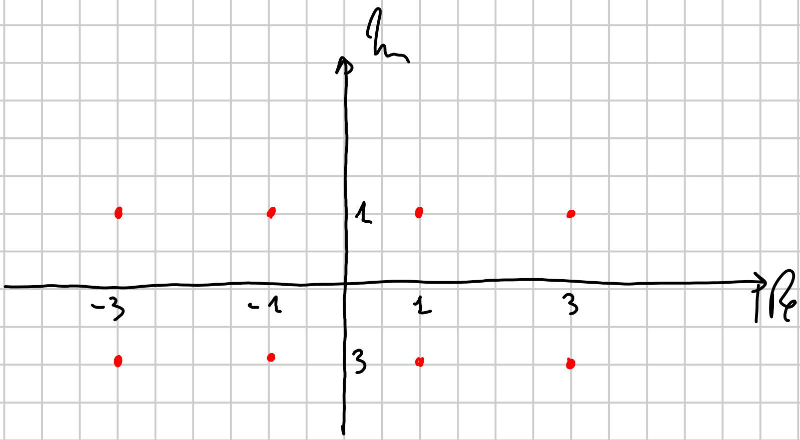
(simboli non equi-energia)

COSTELLAZIONE DEI SIMBOLI DI UN QAM

4 - QAM



8 - QAM



$$\alpha_i^c = 2i - M_c - 1$$

$$\alpha_i^s = 2i - M_s - 1$$

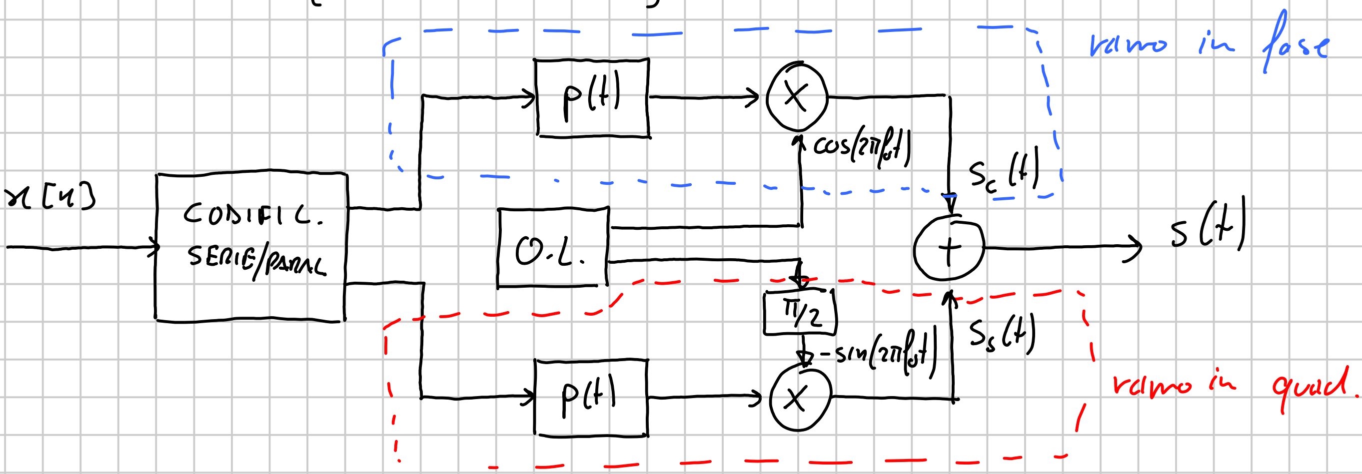
$$M = M_c M_s$$

MODELLO EQUIVALENTE PER PAM, PSK e QAM

$$\tilde{s}(t) = \sum_{k=-\infty}^{+\infty} \tilde{x}[k]\, p(t - kT_s)$$

$$\tilde{x}[k] = \begin{cases} A_m & \text{PAM} \\ e^{j\theta_m} & \text{PSK} \\ A_m^c + j A_m^s & \text{QAM} \end{cases}$$

$$s(t) = \mathrm{Re}\left\{ \tilde{s}(t)\, e^{j2\pi f_0 t} \right\}$$



PAM $\Rightarrow$ viene escluso il ramo in quadratura

QAM $\Rightarrow$ A_m^c , A_m^s

PSK $\Rightarrow$ $A_m^c = \cos\theta_m$, $A_m^s = \sin\theta_m$

PAM $\Rightarrow$ $\tilde{s}(t) = A_m\, p(t)$

QAM $\Rightarrow$ $\tilde{s}(t) = A_m^c\, p(t) + j A_m^s\, p(t)$

PSK $\Rightarrow$ $\tilde{s}(t) = \cos\theta_m\, p(t) + j \sin\theta_m\, p(t) = e^{j\theta_m}\, p(t)$

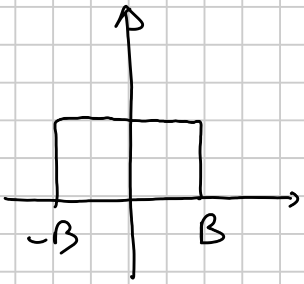
OCCUPAZIONE DI BANDA

$$S_s(f) = \frac{1}{2}\frac{\sigma_x^2}{T_s}\left(P^2(f-f_0) + P^2(f+f_0) \right)$$

N.B. l'occupazione di banda dipende dalle caratteristiche spettrali di $p(t)$

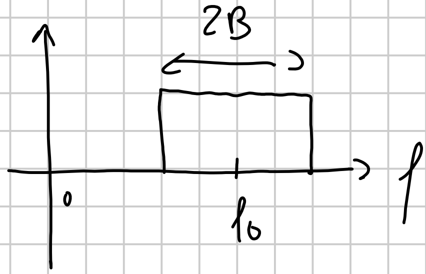
B_p = banda di $p(t)$

$B_T = 2B_p$



$f \to f_0$

$\Rightarrow$

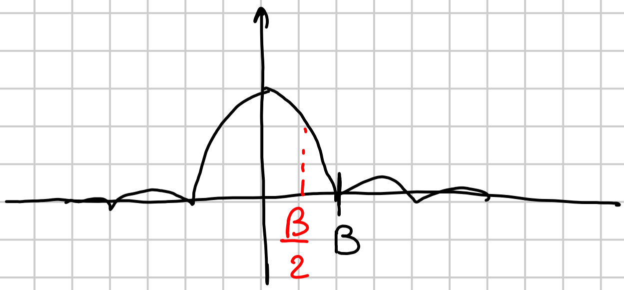


$$\eta_B = \frac{\log_2 M}{T_s B_T} = \frac{\log_2 M}{2 T_s B_p}$$

IMPULSO RETTANGOLARE

$$p(t) = \mathrm{rect}\left(\frac{t - T_s/2}{T_s}\right) \Rightarrow |P(f)| = T_s \,\mathrm{sinc}(T_s f)$$

$$B_p = \frac{1}{T_s}$$



$$B_T = 2 B_p = \frac{2}{T_s}$$

$$\eta_B = \frac{\log_2 M}{\frac{2}{T_s} \cdot T_s} = \frac{\log_2 M}{2}$$

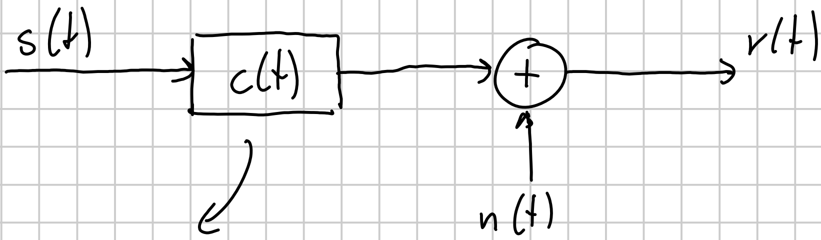
IMPULSO A RADICE DI COSENO RIALZATO

$$P(f) = \sqrt{H_{RC}(f)}$$

$$B_p = \frac{1+\alpha}{T_s} \Rightarrow B_T = \frac{2(1+\alpha)}{T_s}$$

$$\eta_B = \frac{\log_2 M}{T_s} \cdot \frac{T_s}{2(1+\alpha)} = \frac{\log_2 M}{2(1+\alpha)}$$

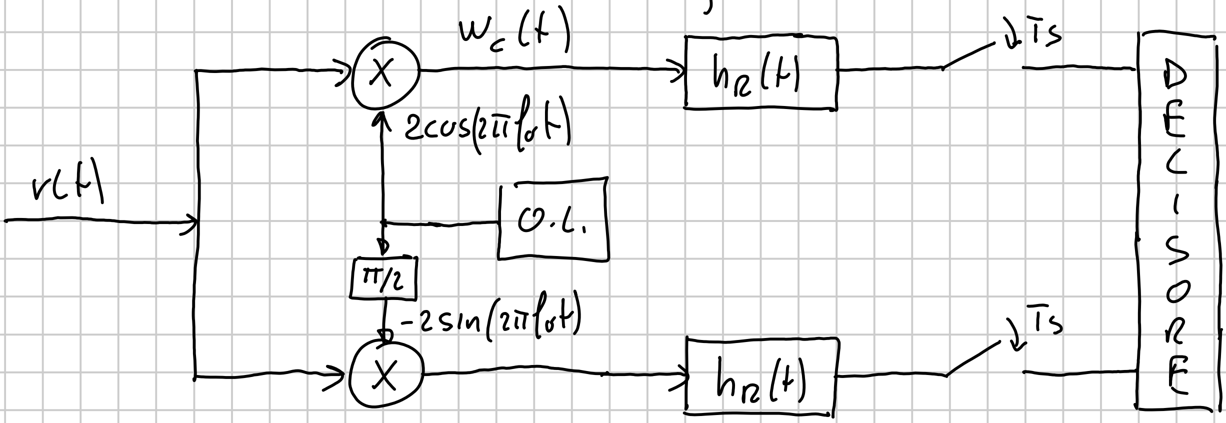
CANALE PASSA-BANDA



$C(f) \neq 0 \quad f_0 - \frac{B}{2} < f < f_0 + \frac{B}{2}$

bianco in banda e Gaussiano

RICEVITORE PER M-PAM, M-PSK e M-QAM



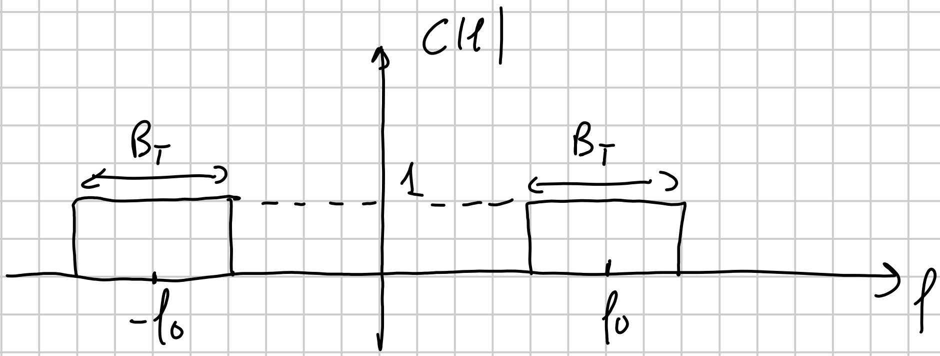
M-PAM

$$s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t - kT_s) \cos(2\pi f_0 t)$$

$$x[k] \in A_s = \{\alpha_1, \dots, \alpha_M\} \quad , \quad \alpha_i = 2i - 1 - M$$

$$r(t) = s(t) \otimes c(t) + n(t)$$

assumo che $c(t)$ sia ideale



$$r(t) = s(t) + n(t) =$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT_s) \cos(2\pi f_0 t) + n(t)$$

$$w_c(t) = 2 s(t) \cos(2\pi f_0 t) + 2 n(t) \cos(2\pi f_0 t)$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT_s) \left(1 + \cos(4\pi f_0 t)\right) +$$

$$+ 2\left[n_c(t) \cos(2\pi f_0 t) - n_s(t) \sin(2\pi f_0 t) \right] \cos(2\pi f_0 t) =$$

$$= \sum_{k=-\infty}^{+\infty} x[n]\, p(t - nT_s) \left(1 + \cos(4\pi f_0 t)\right) +$$

$$+ n_c(t) \left(1 + \cos(4\pi f_0 t)\right) - n_s(t) \sin(2\pi f_0 t) \cos(2\pi f_0 t)$$

$h_R(t)$ → ELIMINARE L'ISI

$h_R(t)$ → MAX SNR

$$y(t) : \quad y[nT_s] = \begin{cases} = 0 & k \neq 0 \\ \neq 0 & k = 0 \end{cases}$$

$$h_R(t) \otimes p(t) = h(t) \Rightarrow h(kT_s) = \begin{cases} = 0 & k \neq 0 \\ \neq 0 & k = 0 \end{cases}$$

$$\left.\begin{aligned} P(f) &= \sqrt{H_{RC}(f)} \\ H_R(f) &= \sqrt{H_{RC}(f)} \end{aligned}\right\} \quad H(f) = H_{RC}(f)$$

N.B. $H_R(f)$ è di tipo passa-basso

$$y(t) = \sum_{k=-\infty}^{+\infty} x[k] \underbrace{h(t-kT_s)} + n_c(t)$$

simboli

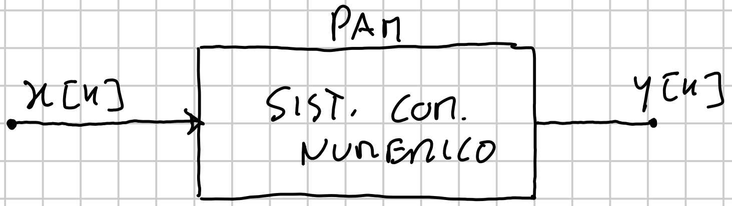
coseno rialzato

rispetto Nyquist

max SNR

rumore bianco banda-base

$$y[kT_s] = x[k] + n_c(kT_s)$$

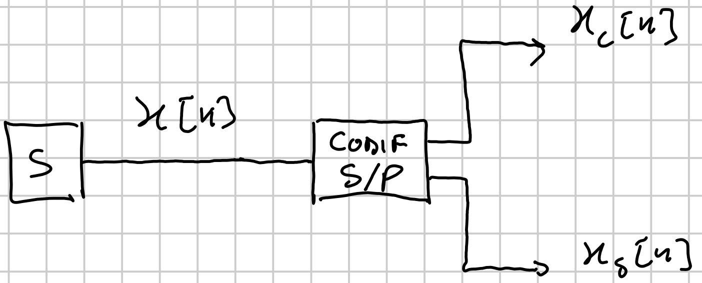


EQUIVALENZA
PAM bb / PAM bp.

QAM

$$s(t) = \sum_{k=-\infty}^{+\infty} x_c[n]\, p(t-kT_s) \cos(2\pi f_0 t) + \qquad I$$

$$- x_s[n]\, p(t-kT_s) \sin(2\pi f_0 t) \qquad Q$$

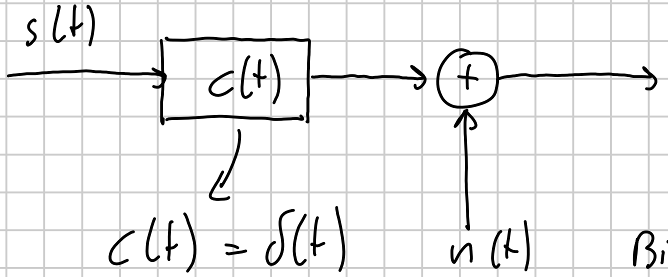


$$x[n] \in A_s = \{\alpha_1, \dots, \alpha_M\}$$

$$x_c[n] \in A_s^c = \{\alpha_1^c, \dots, \alpha_{M_c}^c\}$$

$$x_s[n] \in A_s^s = \{\alpha_1^s, \dots, \alpha_{M_s}^s\}$$

Ipotesi di canale rumoroso non distorcente



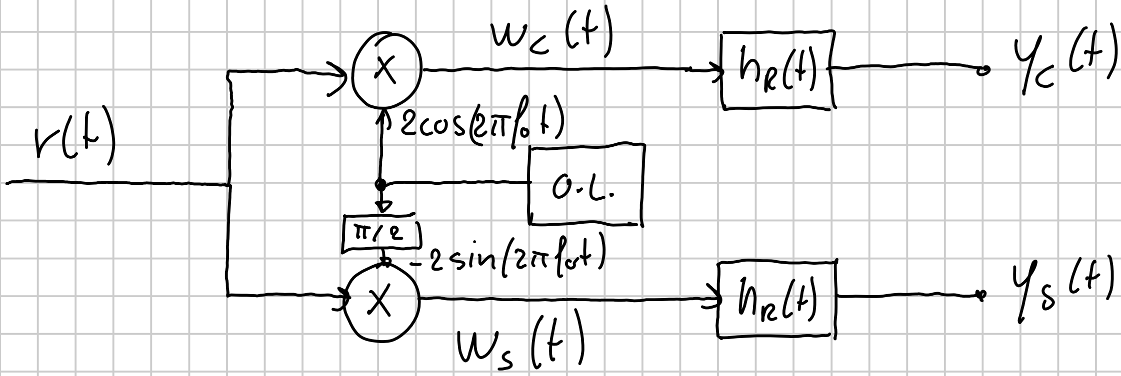
$$r(t) = s(t) + n(t)$$

Bianco in banda
e Gaussiano

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

$$r(t) = \sum_{k=-\infty}^{+\infty} x_c[k]\, p(t-kT_s)\cos(2\pi f_0 t) + n_c(t)\cos(2\pi f_0 t) +$$

$$- \left[\sum_{k=-\infty}^{+\infty} x_s[k]\, p(t-kT_s)\sin(2\pi f_0 t) + n_s(t)\sin(2\pi f_0 t) \right]$$



$$w_c(t) = \sum_{k=-\infty}^{+\infty} \left[x_c[k]\, p(t-kT_s) + n_c(t) \right] \left(1 + \underline{\cos 4\pi f_0 t} \right)$$

$$- \sum_{k=-\infty}^{+\infty} \left[x_s[k]\, p(t-kT_s) + n_s(t) \right] \underline{\sin(4\pi f_0 t)}$$

componenti ad alta freq $(2 f_0)$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\Rightarrow \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$

$$\Rightarrow \sin(2\alpha) + 0 = 2\sin\alpha\cos\alpha$$

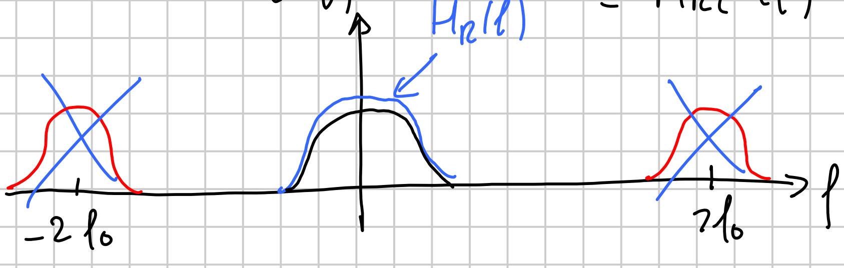
$$\Rightarrow 2\sin(2\pi f_0 t)\cos(2\pi f_0 t) = \sin(4\pi f_0 t)$$

$$p(t) \Rightarrow P(f) = \sqrt{H_{RC}(f)}$$

$$h_R(t) \Rightarrow h_{FA}(t) \Rightarrow H_{FA}(f) = \sqrt{H_{RC}(f)}$$

$$p(t) \otimes h_R(t) = h(t) \Rightarrow H(f) = \sqrt{H_{RC}(f)} \cdot \sqrt{H_{RC}(f)}$$

$$S_{w_c}(f) = H_{RC}(f)$$



$$y_c(t) = w_c(t) \otimes h_R(t) =$$

$$= \sum_{k=-\infty}^{+\infty} x_c[k]\, h(t-kT_s) + n_c(t) \otimes h_R(t)$$

$$h(t-kT_s) = [p(t) \otimes h_R(t)] \otimes \delta(t-kT_s)$$

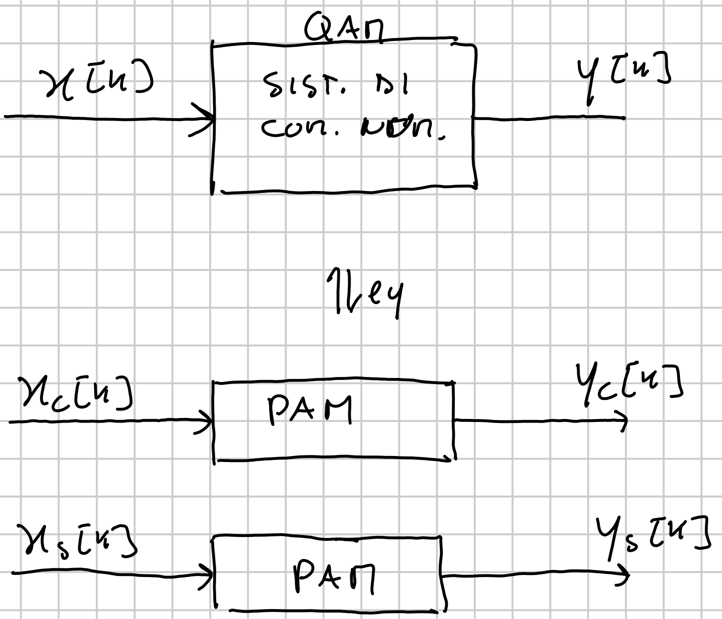
$$p(t-kT_s) \otimes h_R(t)$$

Si può dimostrare in maniera equivalente che:

$$y_s(t) = \sum_{k=-\infty}^{+\infty} x_s[k]\, h(t-kT_s) + n_s(t) \otimes h_R(t)$$

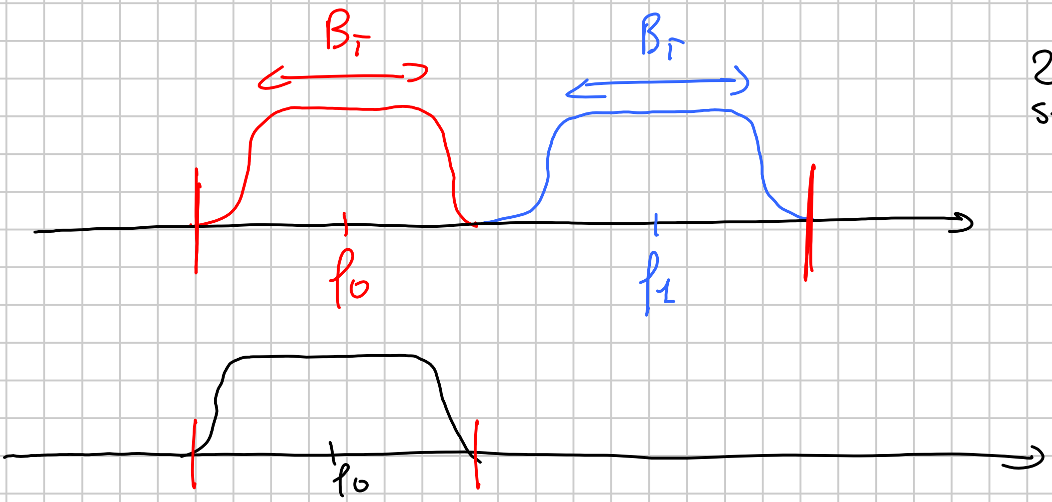
Osservazioni

1)



equivalenza tra QAM e due PAM

2)



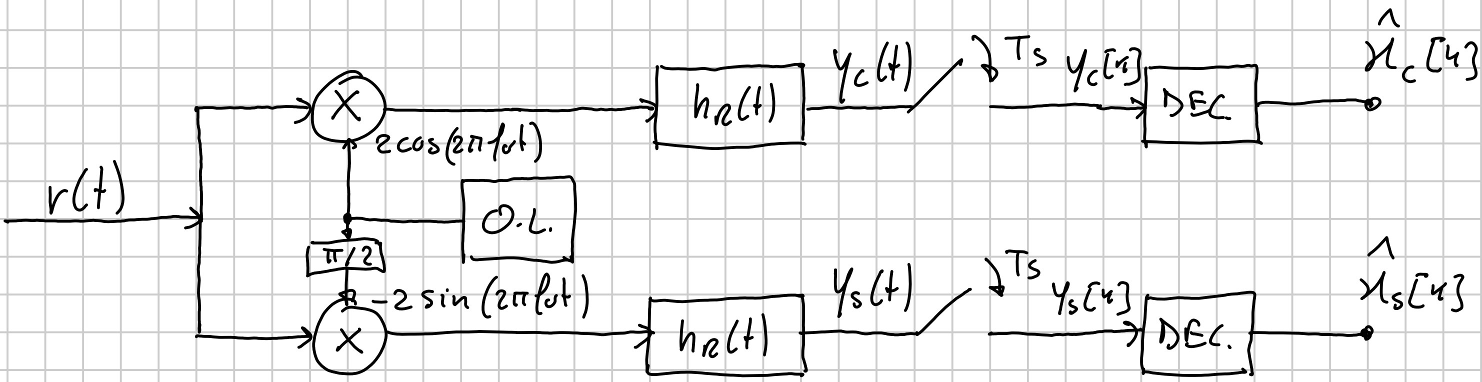
2 PAM separate

$$\eta_B \triangleq \frac{R_b}{B_T}$$

l'efficienza spettrale per una QAM è doppia rispetto ad una PAM

3) Strategia di decisione ottima è la stessa del PAM. La applico due volte.

SCHEMA EQUIVALENTE DEL RICEVITORE OTTIMO



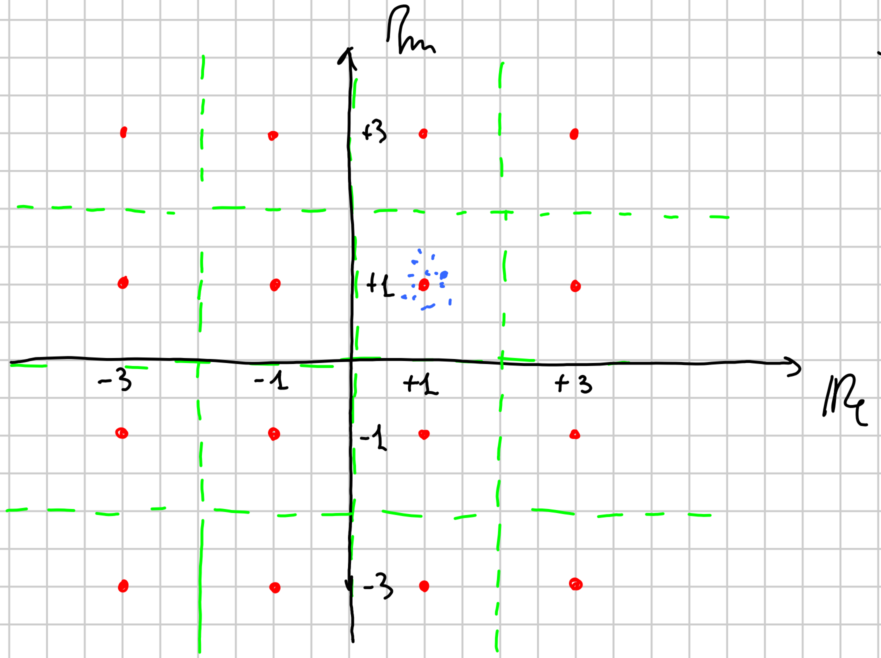
Osservazioni

1) Se i simboli sono equiprobabili il decisore è un quantizzatore uniforme

2) Caso 4-QAM: equivalente a due PAM binarie.

ZONE DI DECISIONE

Esempio: 16-QAM



$$y_c[n] = x_c[n]\,\overset{1}{\overset{\parallel}{h[0]}} + n_{uc}[n]$$

$$n_{uc}(t) = n_c(t) \otimes h_R(t)$$

$$y_s[n] = x_s[n]\,\overset{1}{\overset{\parallel}{h[0]}} + n_{us}[n]$$

$$x[n] = x_c[n] + j\,x_s[n]$$

$$y[n] = x_c[n] + j\,x_s[n] + n_{uc}[n] + j\,n_{us}[n]$$

CALCOLO DELLA PROBABILITÀ DI ERRORE SUL SIMBOLO

$$P_E^{QAM}(M) = ?$$

$$P_E^{PAM}(M_c) = P_c$$

$$P_E^{PAM}(M_s) = P_s$$

$$P_E^{QAM}(M) = P_c(1-P_s) + P_s(1-P_c) + P_c P_s$$

$P_c\,,\ P_s \ll 1$

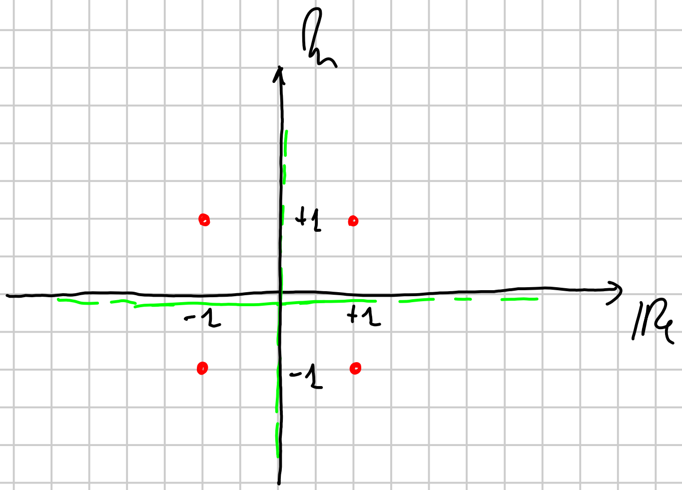
$$\boxed{P_E^{QAM}(M) \simeq P_c + P_s}$$

Nel caso in cui $P_c = P_s = P \Rightarrow P_E^{QAM}(M) \simeq 2P$

Se si adotta la codifica di Gray su entrambe le componenti (I e Q) e se il SNR e' sufficientemente alto :

$$P_E^{QAM}(b) \simeq \frac{P_E^{QAM}(M)}{\log_2 M}$$

4 - QAM



La 4-QAM e' equivalente ad una doppia 2-PAM (BPSK)

$$P_E^{PAM}(M_c) = P_E^{PAM}(2) = P_E^{PAM}(b)$$

$$P_E^{PAM}(M_s) = P_E^{PAM}(b)$$

$$P_E^{QAM}(4) = 2\,P_E^{PAM}(b) = 2 \cdot \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) =$$

$$= \operatorname{erf}\left(\sqrt{SNR}\right)$$

$$\boxed{P_E^{QAM}(b) \cong \frac{P_E^{QAM}(s)}{2} = \frac{1}{2}\operatorname{erfc}\left(\sqrt{SNR}\right)}$$

PROBLEMA DELLA FASE

$$s(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t-kT_s)\cos(2\pi f_0 t + \theta_0) \qquad (\text{PAM})$$

V.A. $\mathcal{U}[0; 2\pi)$

$$s(t) = \sum_{k=-\infty}^{+\infty} x_c[k]\, p(t-kT_s)\cos(2\pi f_0 t + \theta_0) +$$

$$- x_s[k]\, p(t-kT_s)\sin(2\pi f_0 t + \theta_0)$$

Ip: Canale rumoroso non distorcente

$$r(t) = \sum_{k=-\infty}^{+\infty} x_c[k]\, p(t-kT_s)\cos(2\pi f_0 t + \theta_0) + n_c(t)\cos(2\pi f_0 t)$$

$$- \left[x_s[k]\, p(t-kT_s)\sin(2\pi f_0 t + \theta_0) + n_s(t)\sin(2\pi f_0 t)\right]$$

$$w_c(t) = r(t)\cdot 2\cos(2\pi f_0 t) =$$

$$= \sum_{k=-\infty}^{+\infty} x_c[k]\, p(t-kT_s)\left[\underline{\cos(4\pi f_0 t + \theta_0)} + \cos\theta_0\right] +$$

$$+ n_c(t)\left[1 + \underline{\cos(4\pi f_0 t)}\right] +$$

$$- \left[\sum_{k=-\infty}^{+\infty} x_s[k]\, p(t-kT_s) \left[\sin(4\pi f_0 t + \theta_0) + \sin\theta_0 \right] + n_s(t) \sin(4\pi f_0 t) \right]$$

$$y_c(t) = w_c(t) \otimes h_R(t) =$$

$$= \underbrace{\sum_{k=-\infty}^{+\infty} x_c[k]\, h(t-kT_s)}_{\text{segnale utile}} \cos\theta_0 + n_c(t) \otimes h_R(t)$$

$$- \underbrace{\sum_{k=-\infty}^{+\infty} x_s[k]\, h(t-kT_s) \sin\theta_0}_{\text{cross-talk}}$$

Si può dimostrare che si arriva ad un risultato analogo per il calcolo di $y_s(t)$

$$y_c(t) = \sum_{k=-\infty}^{+\infty} \left[\underbrace{x_c[k] \cos\theta_0 - x_s[k] \sin\theta_0}_{} \right] h(t-kT_s)$$

$$+ n_c(t) \otimes h_R(t)$$

COMPENSAZIONE DEL TERMINE DI FASE

·) Si trasmette una sequenza nota (di "training")

Esempio

·) Trasmetto il simbolo "1" sul canale in quadratura e non trasmetto niente sul canale in fase.

·) Osservo il campione $y_c[k]$

$$y_c[k] = x_c[k]\, h[0] \cos\theta_0 - x_s[k]\, h[0] \sin\theta_0 + n_{kc}[k]$$

$$= -x_s[k]\, \underset{\underset{1}{\Downarrow}}{h[0]} \sin\theta_0 + n_{uc}[k]$$

$$y_c[k] = -\underset{\underset{1}{\parallel}}{x_s[k]} \sin\theta_0 + n_{uc}[k] = -\sin\theta_0 + n_u[k]$$

Se $\quad n_{uc}[k] \ll 1 \qquad\qquad \text{SNR} \gg 1$

$$y_c[k] \simeq -\sin\theta_0 \quad \Rightarrow \quad \hat{\theta}_0 = -\arcsin\left[y_c[k]\right]$$

SNR è basso

.) Trasmetto N volte il simbolo "1"

$$\underline{y}_c = \begin{bmatrix} \sin\theta_0 + n_{uc}[1] \\ \sin\theta_0 + n_{uc}[2] \\ \vdots \\ \sin\theta_0 + n_{uc}[N] \end{bmatrix}$$

$$\widehat{\sin\theta_0} = \frac{1}{N} \sum_{k=1}^{N} y_c[k]$$

$$\hat{\theta}_0 = \arcsin\left[\frac{1}{N} \sum_{k=1}^{N} y_c[k]\right]$$

ESERCITAZIONE 29-02-2012

TRASFORMATA SERIE DI FOURIER

Esercizio #1

$$x(t) = A\cos(2\pi f_0 t + \varphi) + B\sin(4\pi f_0 t)$$

-) calcolare lo spettro di $x(t)$

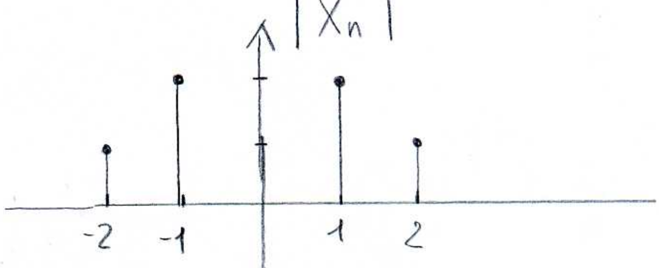
-) Disegnare spettro di ampiezza e fase.

$x(t)$ reale e periodico $\rightleftarrows$ X_n con simmetria Hermitiana.

EQUAZIONE DI SINTESI

$$x(t) = \frac{A}{2}\left[e^{j2\pi f_0 t} e^{j\varphi} + e^{-j2\pi f_0 t} e^{-j\varphi}\right] + \frac{B}{2j}\left[e^{j4\pi f_0 t} - e^{-j4\pi f_0 t}\right] =$$

$$= \frac{A}{2} e^{j\varphi} e^{j2\pi f_0 t} + \frac{A}{2} e^{-j\varphi} e^{-j2\pi f_0 t} + \frac{B}{2} e^{-j\pi/2} e^{j4\pi f_0 t} - \frac{B}{2} e^{-j\pi/2} e^{-j4\pi f_0 t} =$$

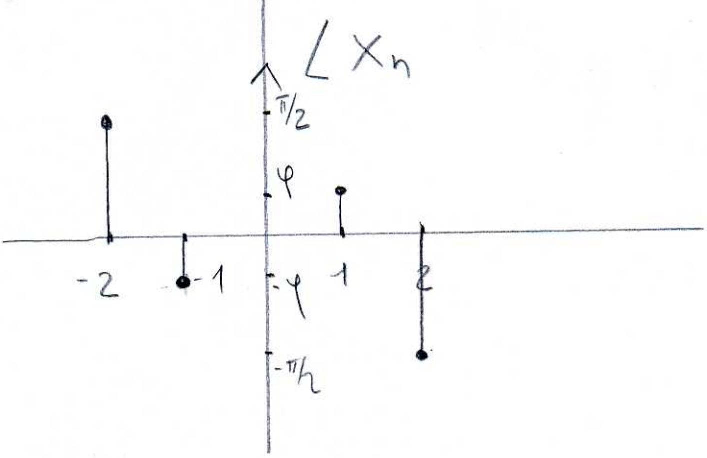


$$X_1 = \frac{A}{2} e^{j\varphi}$$

$$X_{-1} = \frac{A}{2} e^{-j\varphi}$$

$$X_2 = \frac{B}{2} e^{-j\pi/2}$$

$$X_{-2} = \frac{B}{2} e^{j\pi/2}$$



Esercizio #2

$$z(t) = A\cos(2\pi f_0 t + \varphi)\cdot\cos(4\pi f_0 t)$$

$z(t)$ reale e periodico $\rightleftarrows$ Z_n con simmetria Hermitiana

EQUAZIONE DI SINTESI

$$z(t) = \frac{A}{4}\left[e^{j2\pi f_0 t + j\varphi} + e^{-j2\pi f_0 t - j\varphi}\right]\cdot\left[e^{j4\pi f_0 t} + e^{-j4\pi f_0 t}\right] =$$

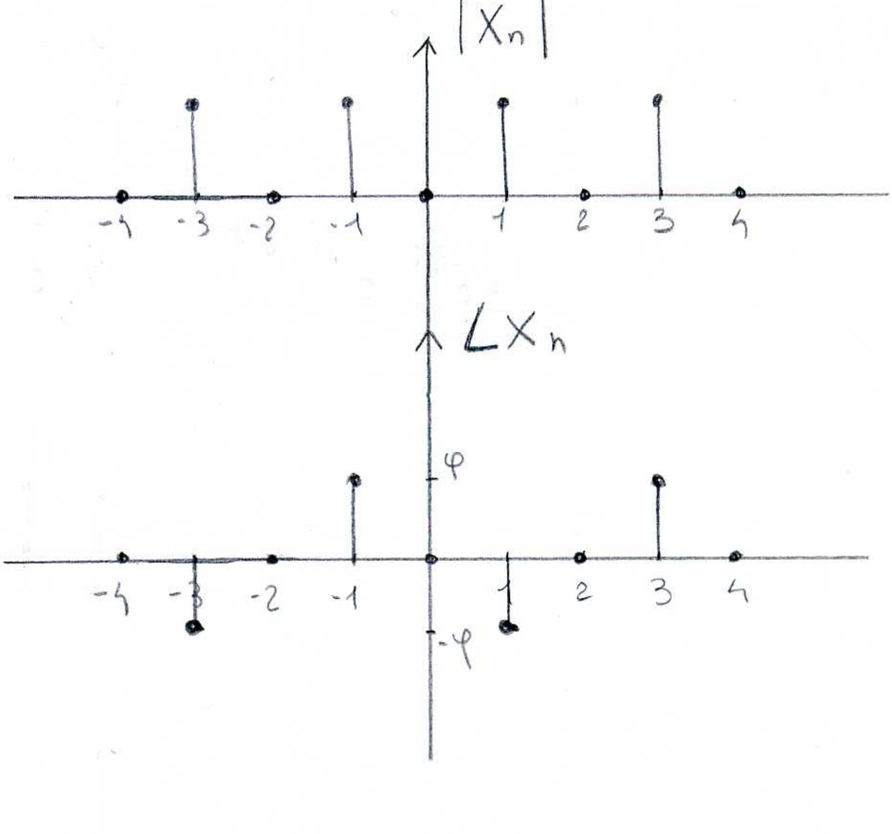
$$= \frac{A}{4}\left[e^{j6\pi f_0 t + j\varphi} + e^{-j2\pi f_0 t + j\varphi} + e^{j2\pi f_0 t - j\varphi} + e^{-j6\pi f_0 t - j\varphi}\right]$$

$$X_1 = \frac{A}{4}e^{-j\varphi}$$

$$X_{-1} = \frac{A}{4}e^{j\varphi}$$

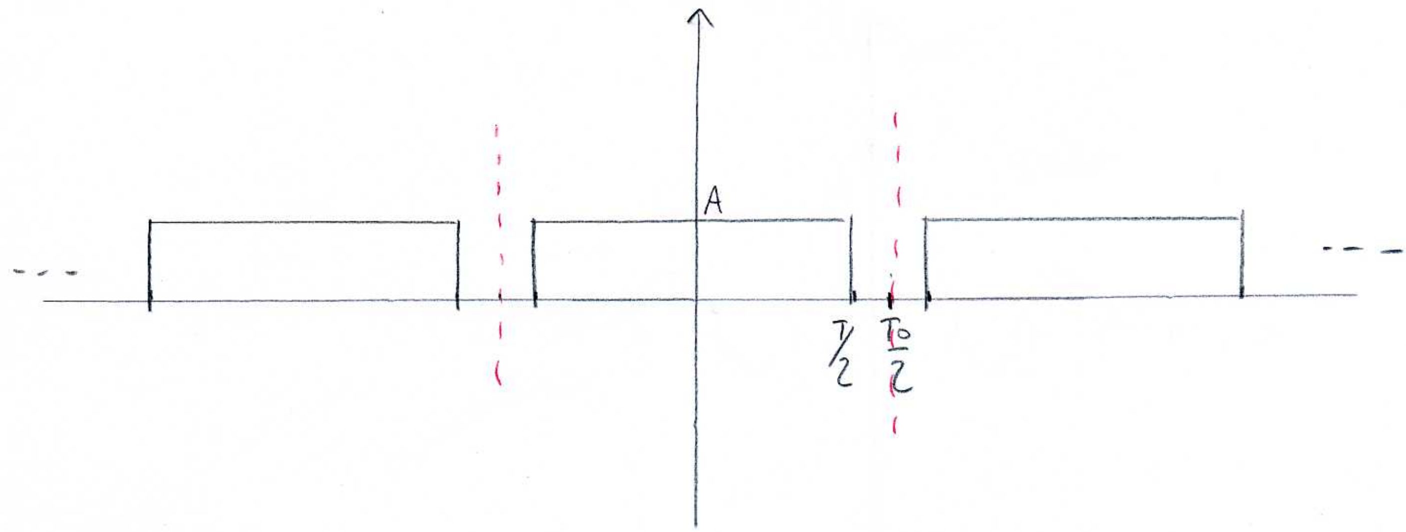
$$X_3 = \frac{A}{4}e^{j\varphi}$$

$$X_{-3} = \frac{A}{4}e^{-j\varphi}$$



Esercizio #3

TRENO DI IMPULSI RETTANGOLARI



$$x_0(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) \qquad x(t) = A\sum_{k=-\infty}^{+\infty} \mathrm{rect}\left(\frac{t-kT_0}{T}\right)$$

~~Sea~~ $x(t)$ reale periodico e pari $\rightleftharpoons$ $\mathcal{Im}\{X_n\}=0$; $X_n = X_{-n}$

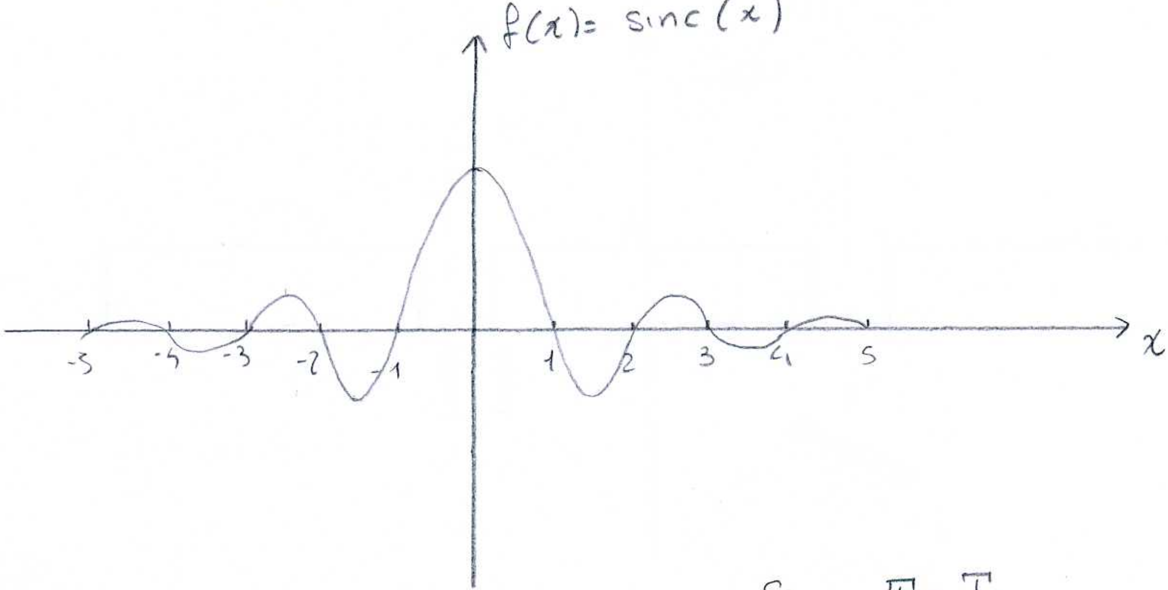
EQUAZIONE DI ANALISI

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi n f_0 t}\,dt = \frac{1}{T_0}\int_{-T/2}^{T/2} A\, e^{-j2\pi n f_0 t}\,dt =$$

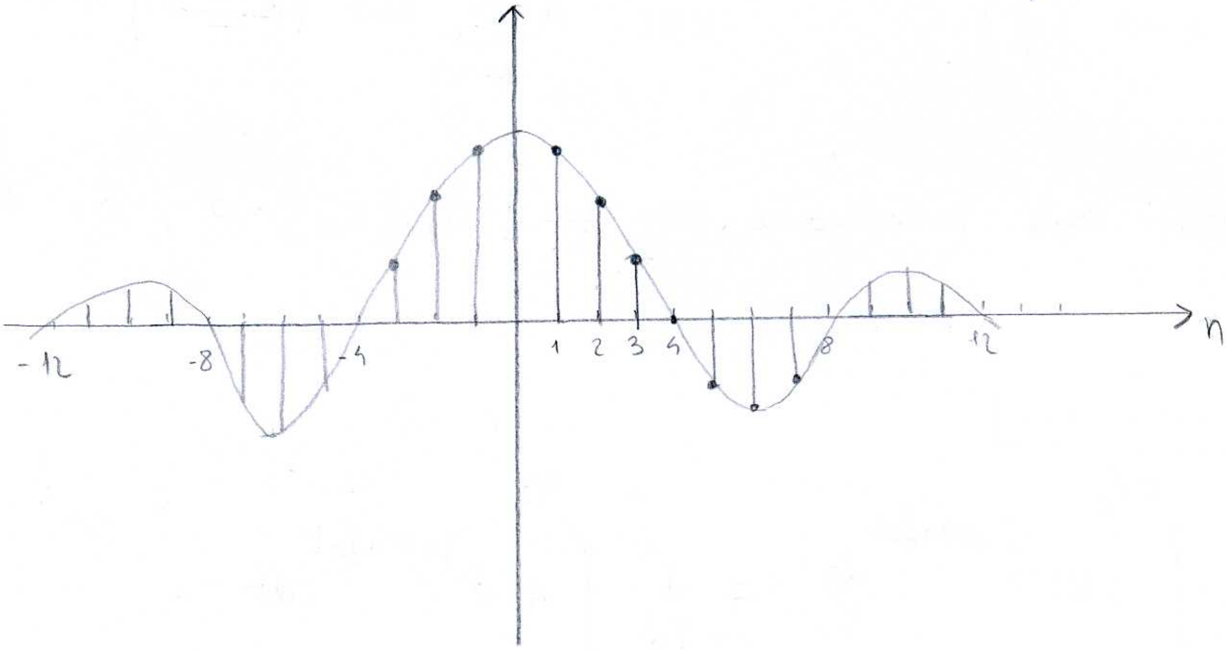
$$= \frac{A}{T_0}\left(-\frac{e^{-j2\pi f_0 n t}}{j2\pi n f_0}\right)_{-T/2}^{T/2} = -\frac{A}{j2\pi n}\left(e^{-j2\pi f_0 n \frac{T}{2}} - e^{j2\pi n f_0 \frac{T}{2}}\right) =$$

$$= \frac{A}{\pi n}\left(\frac{e^{j\pi n f_0 T} - e^{-j\pi n f_0 T}}{2j}\right) = \frac{A}{\pi n}\sin\left(\pi n f_0 T\right) =$$

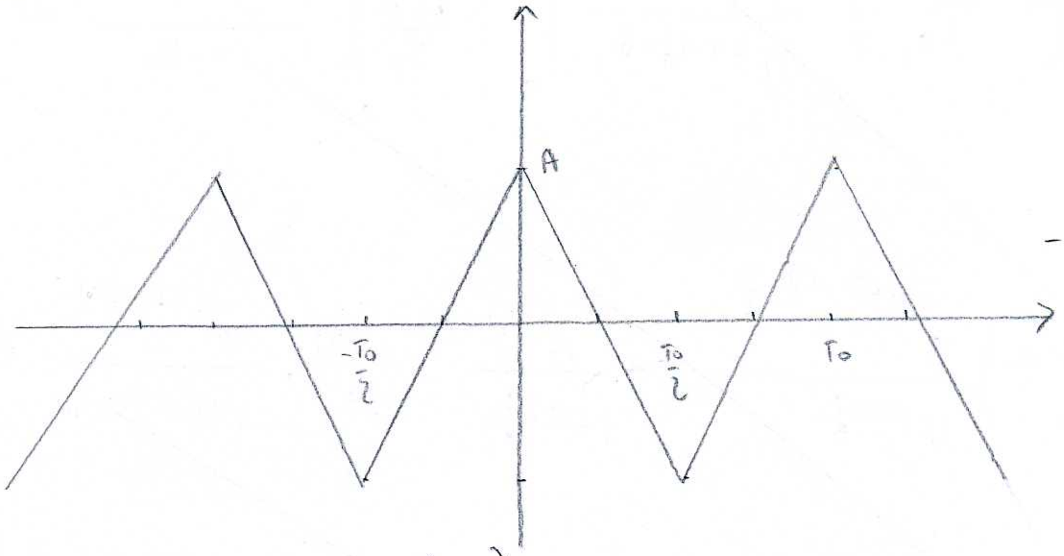
$$X_n = A f_0 T \cdot \frac{\sin(n \pi f_0 T)}{n \pi f_0 T} = A f_0 T \cdot \mathrm{sinc}(n f_0 T)$$



Se $T = \frac{T_0}{4}$



Esercizio #4



$$x_0(t) = \left(A - \frac{4A}{T_0}|t| \right) \text{rect}\left(\frac{t}{T_0}\right) \qquad x(t) = -x(t - T_0/2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)$$

$x(t)$ periodico, reale, pari e alternativo $\rightleftharpoons$ $X_n = X_{-n}$; $\mathrm{Im}\{X_n\} = 0$

$X_n = 0$ per n pari

La TSF per segnali alternativi è

$$X_n = \begin{cases} 0 & \text{per } n \text{ pari} \\ \frac{2}{T_0} \int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t}\, dt & \text{per } n \text{ dispari} \end{cases}$$

$$X_n = \frac{2}{T_0} \int_0^{T_0/2} \left(A - \frac{4A}{T_0} t \right) e^{-j2\pi n f_0 t}\, dt =$$

$$X_n = \frac{2}{T_0}\left[\left(A - \frac{4At}{T_0}\right)\frac{e^{-j2\pi n f t}}{-j2\pi n f_0}\Bigg|_0^{T_0/2} + \int_0^{T_0/2} \frac{4A}{T_0}\,\frac{e^{-j2\pi n f t}}{-j2\pi n f_0}\,dt\right] =$$

$$= -\frac{1}{j\pi n}\left[(A - 2A)\,e^{-j\pi n} - A + \frac{4A}{T_0}\,\frac{e^{-j2\pi n f t}}{-j2\pi n f_0}\Bigg|_0^{T_0/2}\right] =$$

$$= -\frac{A}{j\pi n}\left[e^{-j\pi n} - 2e^{-j\pi n} - 1 - \frac{2}{j\pi n}\left(e^{-j\pi n} - 1\right)\right] =$$

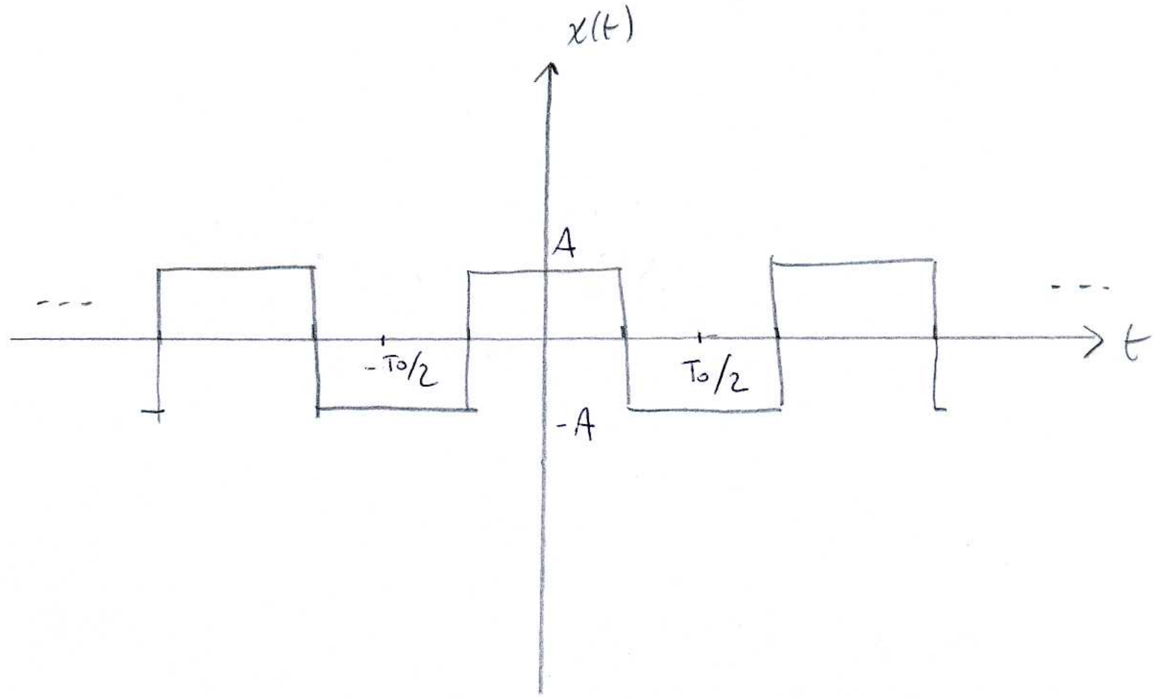
$$= -\frac{A}{j\pi n}\left[-e^{-j\pi n} - 1 - \frac{2}{j\pi n}\left(e^{-j\pi n} - 1\right)\right] =$$

$$= \frac{A}{j\pi n}\left(e^{-j\pi n} + 1\right) - \frac{2A}{\pi^2 n^2}\left(e^{-j\pi n} - 1\right)$$

per n dispari $e^{-j\pi n} = -1$

$$X_n = \frac{4A}{\pi^2 n^2}$$

Esercizio 5



$x(t)$ pari, reale, periodico e determinato $\rightleftarrows$ $X_n = X_{-n}$; $\Im m\{x_n\} = 0$

$X_n = 0$ per n pari

$$X_k = \frac{2}{T_0} \int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{2}{T_0} \int_0^{T_0/4} A\, e^{-j2\pi n f_0 t}\, dt - \frac{2}{T_0} \int_{T_0/4}^{T_0/2} A\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{2A}{T_0} \left[\left. \frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0} \right|_0^{T_0/4} - \left. \frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0} \right|_{T_0/4}^{T_0/2} \right] =$$

$$= \frac{\cancel{2}A}{-j\cancel{2}\pi n \cancel{f_0}\cancel{T_0}} \left[e^{-j\cancel{2}\pi n \cancel{f_0} \frac{\cancel{T_0}}{\cancel{4}2}} - 1 - \left(e^{-j\cancel{2}\pi n \cancel{f_0} \frac{\cancel{T_0}}{\cancel{2}}} - e^{-j\cancel{2}\pi n \cancel{f_0} \frac{\cancel{T_0}}{\cancel{4}2}} \right) \right] =$$

$$= \frac{jA}{\pi n} \left[e^{-j\frac{\pi n}{2}} - 1 - e^{-j\pi n} + e^{-j\pi n/2} \right] =$$

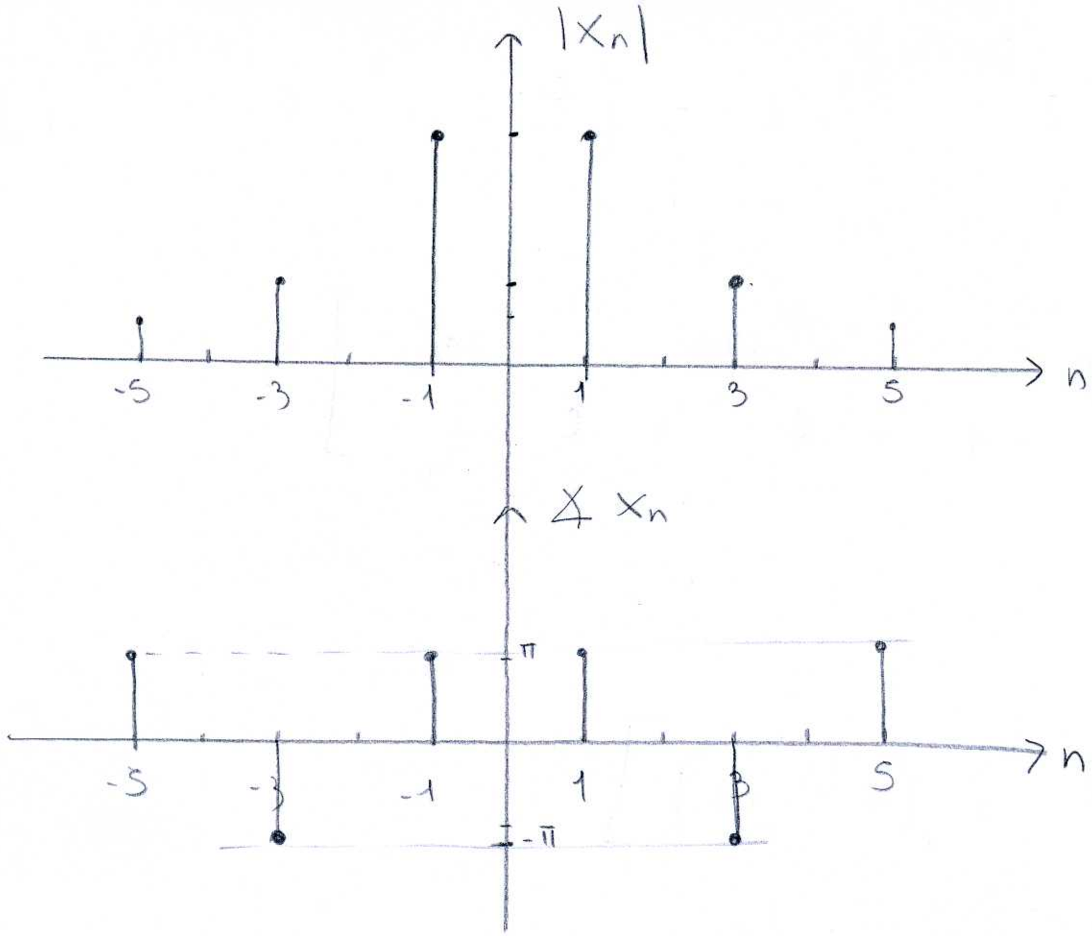
$$= -\frac{jA}{\pi n} \left[-2e^{-j\frac{\pi n}{2}} + \left(e^{-j\pi n} + 1 \right) \right]$$

per n dispari $e^{-j\pi n} = -1$

$$X_n = +\frac{2A}{\pi n} j(-j)^n = \frac{2A}{\pi n} (-1)^{\frac{n-1}{2}}$$

$X_1 = \frac{2A}{\pi}$ $\quad X_3 = -\frac{2A}{3\pi}$ $\quad X_5 = \frac{2A}{5\pi}$

$X_{-1} = \frac{2A}{\pi}$ $\quad X_{-3} = -\frac{2A}{3\pi}$ $\quad X_{-5} = \frac{2A}{5\pi}$



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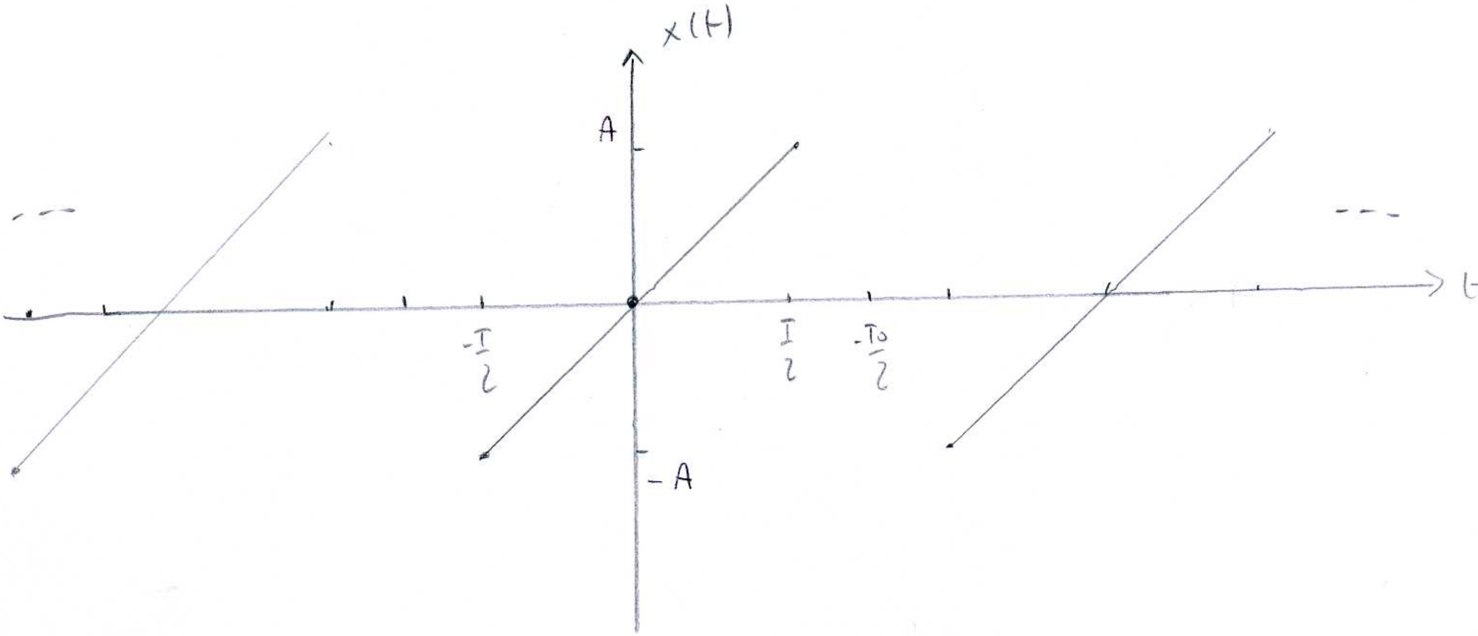
Esercizio #6



a) Determinare l'espressione di $x(t)$ in un periodo

b) calcolarne lo spettro.

a) $x_0(t) = \frac{2A}{T} t \, \text{rect}\left(\frac{t}{T}\right)$

b) $x(t)$ reale, periodico e dispari $\rightleftarrows$ $\Re\{X_n\} = 0$; $X_n = -X_{-n}$

$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi n f_0 t} dt = \frac{1}{T_0} \int_{-\frac{T}{2}}^{T/2} \frac{2A}{T} t e^{-j2\pi n f_0 t} dt =$$

$$= \frac{2A}{T T_0} \left[\frac{t \, e^{-j2\pi n f_0 t}}{-j2\pi n f_0} \Bigg|_{-\frac{T_0}{2}}^{T_0/2} - \int_{-\frac{T_0}{2}}^{T_0/2} \frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0} dt \right] =$$

$$= \frac{8A}{T T_0} \cdot \frac{1}{-j8\pi n f_0} \left[\frac{T}{2} \left(e^{-j2\pi n f_0 \frac{T}{2}} + e^{j2\pi n f_0 \frac{T}{2}} \right) + \left. \frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0} \right|_{-\frac{T}{2}}^{T/2} \right] =$$

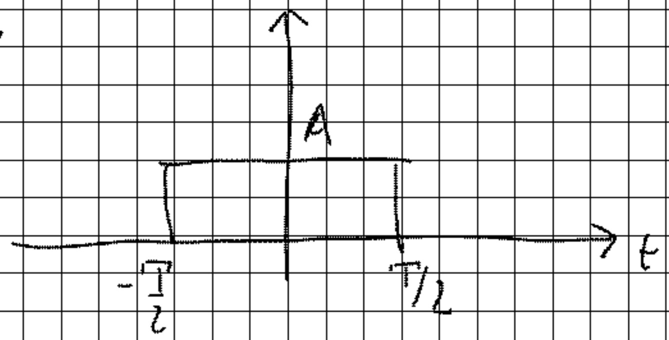
$$= + \frac{jA\cancel{T}}{\pi n \cancel{T}} \cos(\pi n f_0 T) - \frac{jA}{\pi^2 n^2 f_0 T} \sin(\pi n f_0 T) =$$

$$= j\frac{A}{\pi n} \left[\cos(n\pi f_0 T) - \frac{\sin(\pi n f_0 T)}{\pi n f_0 T} \right] =$$

$$= j\frac{A}{\pi n} \left[\cos(\pi n f_0 T) - \operatorname{sinc}(n f_0 T) \right]$$

07 / 03 / 2012

Esercizio #1



$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

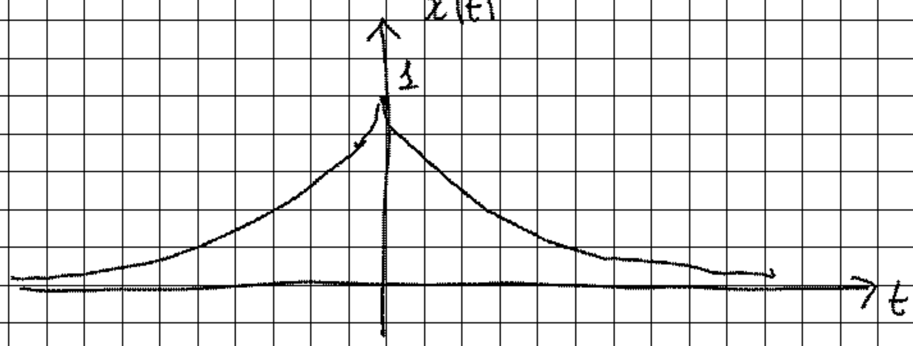
$$X(f) = \int_{-\frac{T}{2}}^{T/2} A\, e^{-j2\pi f t}\, dt = A \left. \frac{e^{-j2\pi f t}}{(-j2\pi f)} \right|_{-\frac{T}{2}}^{T/2}$$

$$= \frac{A}{(-j2\pi f)} \left(e^{-j\pi f T} - e^{+j\pi f T} \right) = \frac{AT}{\pi f T} \sin(\pi f T) =$$

$$= AT \operatorname{sinc}(fT)$$

$$x(t) = \operatorname{rect}\left(\frac{t}{T}\right) \rightleftarrows X(f) = T \operatorname{sinc}(fT)$$

Esercizio #2



$$x(t) = e^{-|t|}$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{0} e^{t}\, e^{-j2\pi f t}\, dt + \int_{0}^{+\infty} e^{-t}\, e^{-j2\pi f t}\, dt =$$

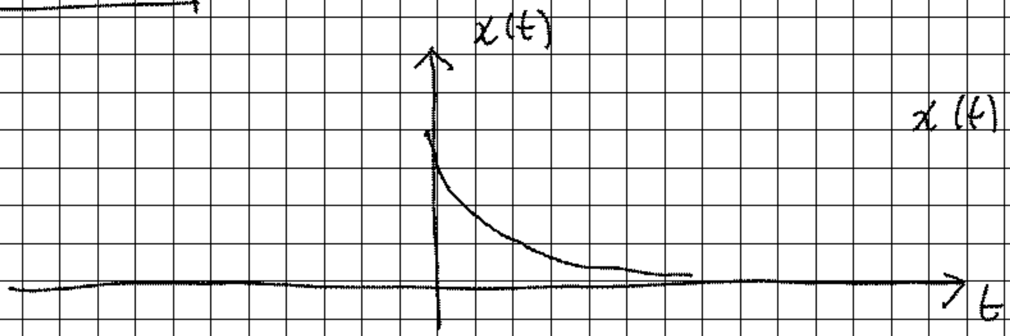
$$= \int_{-\infty}^{0} e^{t(1-j2\pi f)}\, dt + \int_{0}^{+\infty} e^{-t(1+j2\pi f)}\, dt =$$

$$= \left.\frac{e^{t(1-j2\pi f)}}{(1-j2\pi f)}\right|_{-\infty}^{0} + \left.\frac{e^{-t(1+j2\pi f)}}{-(1+j2\pi f)}\right|_{0}^{+\infty}$$

$$= \left(\frac{1}{1-j2\pi f} - 0\right) + \left(0 + \frac{1}{1+j2\pi f}\right) =$$

$$X(f) = \frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} = \frac{1+\cancel{j2\pi f}+1-\cancel{j2\pi f}}{1+4\pi^2 f^2} = \frac{2}{1+4\pi^2 f^2}$$

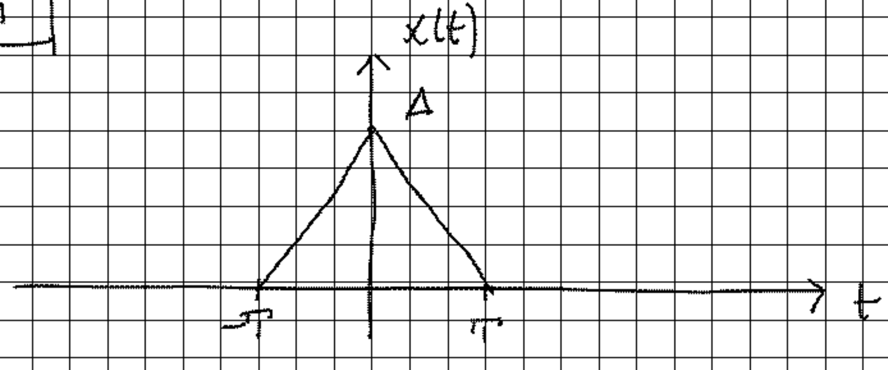
Esercizio #3



$$x(t) = e^{-t}\, u(t)$$

$$X(f) = \int_0^{+\infty} e^{-t}\, e^{-j2\pi f t}\, dt = \frac{1}{1+j2\pi f}$$

Esercizio #4



$$x(t) = A\left(1-\frac{|t|}{T}\right) \mathrm{rect}\left(\frac{t}{2T}\right)$$

I) $$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-T}^{0} A\left(1+\frac{t}{T}\right) e^{-j2\pi f t}\, dt + \int_{0}^{T} A\left(1-\frac{t}{T}\right) e^{-j2\pi f t}\, dt =$$

$$\int_{-T}^{0} A\left(1+\frac{t}{T}\right) e^{-j2\pi f t}\,dt = A\left(1+\frac{t}{T}\right)\frac{e^{-j2\pi f t}}{(-j2\pi f)}\Bigg|_{-T}^{0} - \int_{-T}^{0}\frac{A}{T}\frac{e^{-j2\pi f t}}{(-j2\pi f)}\,dt =$$

$$= \frac{A}{(-j2\pi f)} - 0 - \frac{A}{T}\frac{e^{-j2\pi f t}}{(-j2\pi f)^2}\Bigg|_{-T}^{0} =$$

$$= \boxed{-\frac{A}{j2\pi f} + \frac{A\left(1-e^{2j\pi f T}\right)}{T\,4\pi^2 f^2}}$$

$$\int_{0}^{T} A\left(1-\frac{t}{T}\right) e^{-j2\pi f t}\,dt = A\left(1-\frac{t}{T}\right)\frac{e^{-j2\pi f t}}{(-j2\pi f)}\Bigg|_{0}^{T} + \frac{A}{T}\int_{0}^{T}\frac{e^{-j2\pi f t}}{(-j2\pi f)}\,dt =$$

$$= 0 + \frac{A}{j2\pi f} + \frac{A}{T}\frac{e^{-j2\pi f t}}{(-j2\pi f)^2}\Bigg|_{0}^{T} =$$

$$= \boxed{\frac{A}{j2\pi f} - \frac{A}{T\,4\pi^2 f^2}\left(e^{-j2\pi f T} - 1\right)}$$

$$X(f) = -\cancel{\frac{A}{i2\pi f}} + \frac{A}{T4\pi^2 f^2}\left(1 - e^{i2\pi fT}\right) + \cancel{\frac{A}{i2\pi f}} - \frac{A}{T4\pi^2 f^2}\left(e^{-i2\pi fT} - 1\right) =$$

$$= \frac{A}{4\pi^2 f^2 T}\left[1 - e^{i2\pi fT} + 1 - e^{-i2\pi fT}\right] =$$

$$= \frac{A}{4\pi^2 f^2 T}\left(1 - e^{i2\pi fT}\right)\left(1 - e^{-i2\pi fT}\right) =$$

$$= \frac{A}{4\pi^2 f^2 T}\,\cancel{e^{i\pi fT}}\left(e^{-i\pi fT} - e^{+i\pi fT}\right)\cancel{e^{-i\pi fT}}\left(e^{+i\pi fT} - e^{-i\pi fT}\right) =$$

$$= \frac{A}{\pi^2 f^2 T\,(2i)(-2i)}\boxed{\left(e^{-i\pi fT} - e^{+i\pi fT}\right)\left(e^{+i\pi fT} - e^{-i\pi fT}\right)} =$$

$$= \frac{A}{\pi^2 f^2 T}\left(\frac{e^{-i\pi fT} - e^{+i\pi fT}}{-2i}\right)\left(\frac{e^{+i\pi fT} - e^{-i\pi fT}}{2i}\right) =$$

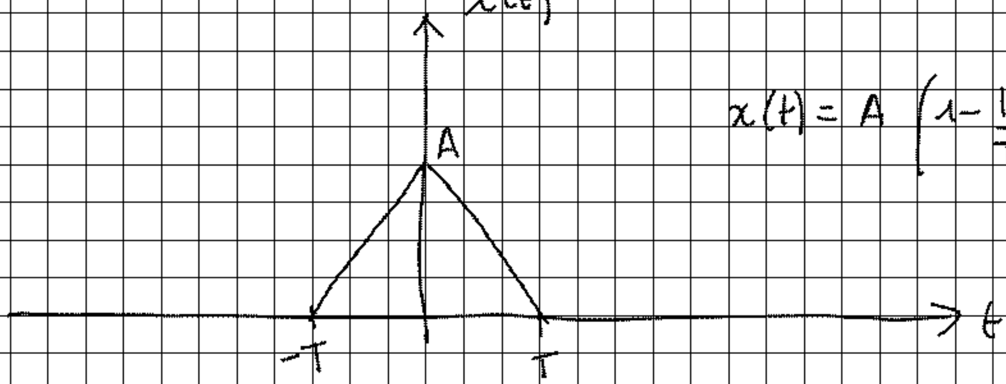
$$= \frac{A}{\pi^2 f^2 T}\sin(\pi fT)\cdot\sin(\pi fT)$$

$$= AT \frac{\sin(\pi f T)}{\pi f T} \cdot \frac{\sin(\pi f T)}{\pi f T} = AT \operatorname{sinc}^2(fT)$$

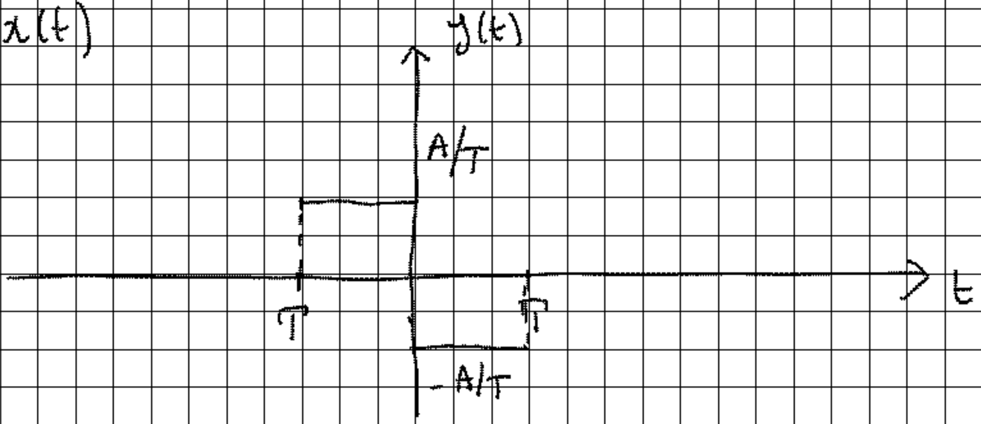
$$x(t) = \underbrace{\left(1 - \frac{|t|}{T}\right) \operatorname{rect}\left(\frac{t}{2T}\right)}_{x(t)} \rightleftharpoons X(f) = T \operatorname{sinc}^2(fT)$$

II)

$$x(t) = A\left(1 - \frac{|t|}{T}\right) \operatorname{rect}\left(\frac{t}{2T}\right)$$



$$y(t) = \frac{\partial}{\partial t} x(t)$$



·) Teorema di derivazione

$$y(t) = \frac{\partial}{\partial t} x(t) \rightleftharpoons Y(f) = j2\pi f X(f)$$

$$X(f) = \frac{Y(f)}{j2\pi f}$$

$$y(t) = \frac{A}{T} \, \mathrm{rect}\left(\frac{t+T/2}{T}\right) - \frac{A}{T} \, \mathrm{rect}\left(\frac{t-T/2}{T}\right)$$

$$y_0(t) = \mathrm{rect}\left(\frac{t}{T}\right) \rightleftharpoons Y_0(f) = T \, \mathrm{sinc}(fT)$$

$$y(t) = \frac{A}{T}\left[y_0(t+T/2) - y_0(t-T/2) \right]$$

) Teorema del ritardo

$$Y(f) = \frac{A}{T} \, Y_0(f) \, e^{+j2\pi f \frac{T}{2}} - \frac{A}{T} \, Y_0(f) e^{-j2\pi f \frac{T}{2}}$$

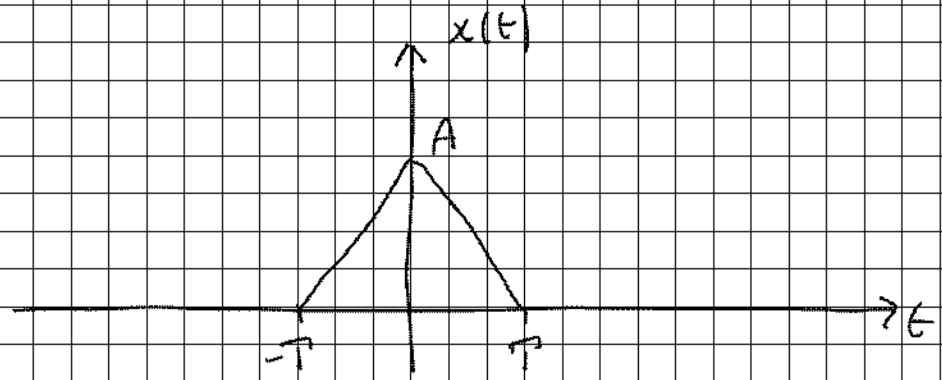
$$Y(f) = \frac{A}{T} T \, \mathrm{sinc}(fT) \left(e^{+j\pi f T} - e^{-j\pi f T} \right) \cdot \frac{2j}{2j} =$$

$$= A \, 2j \, \mathrm{sinc}(fT) \cdot \sin(\pi f T)$$

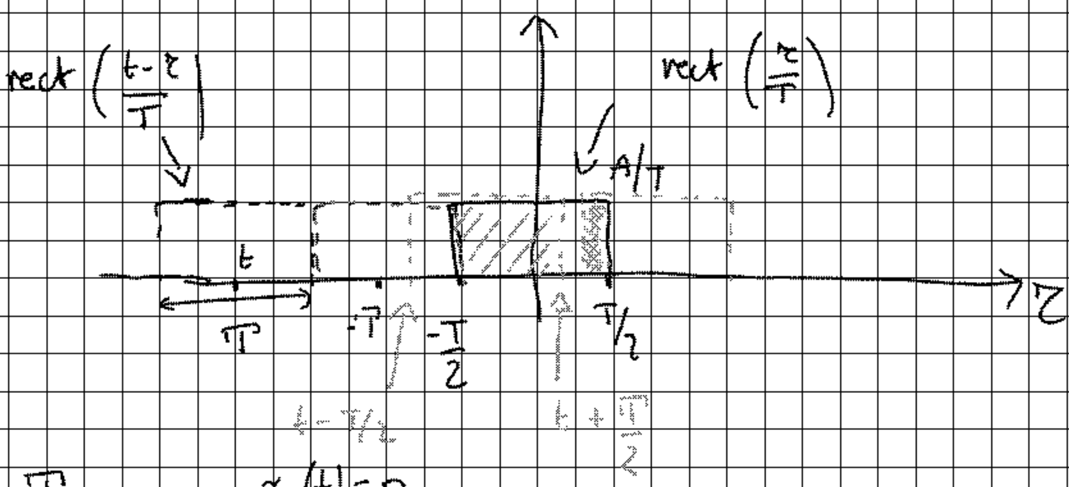
$$X(f) = \frac{Y(f)}{j2\pi f} = \frac{A}{\pi f} \, \mathrm{sinc}(fT) \, \sin(\pi f T) =$$

$$= AT \, \mathrm{sinc}^2(fT)$$

III)



$$x(t) = \frac{A}{T}\left(\operatorname{rect}\left(\frac{t}{T}\right) \otimes \operatorname{rect}\left(\frac{t}{T}\right)\right)$$



$t < -T \qquad x(t) = 0$

$-T \leq t < 0 \qquad \left(t + \frac{T}{2} - \left(-\frac{T}{2}\right)\right) \cdot \frac{A}{T} = (t+T)\frac{A}{T} = A\left(1 + \frac{t}{T}\right)$

$0 \leq t < T \qquad \left(\frac{T}{2} - \left(t - \frac{T}{2}\right)\right)\frac{A}{T} = (T - t)\frac{A}{T} = A\left(1 - \frac{t}{T}\right)$

$t \geq T \qquad x(t) = 0$

$$x(t) = A\left(1 - \frac{|t|}{T}\right) \operatorname{rect}\left(\frac{t}{2T}\right)$$

$$x(t) = \frac{A}{T} \operatorname{rect}\left(\frac{t}{T}\right) \otimes \operatorname{rect}\left(\frac{t}{T}\right)$$

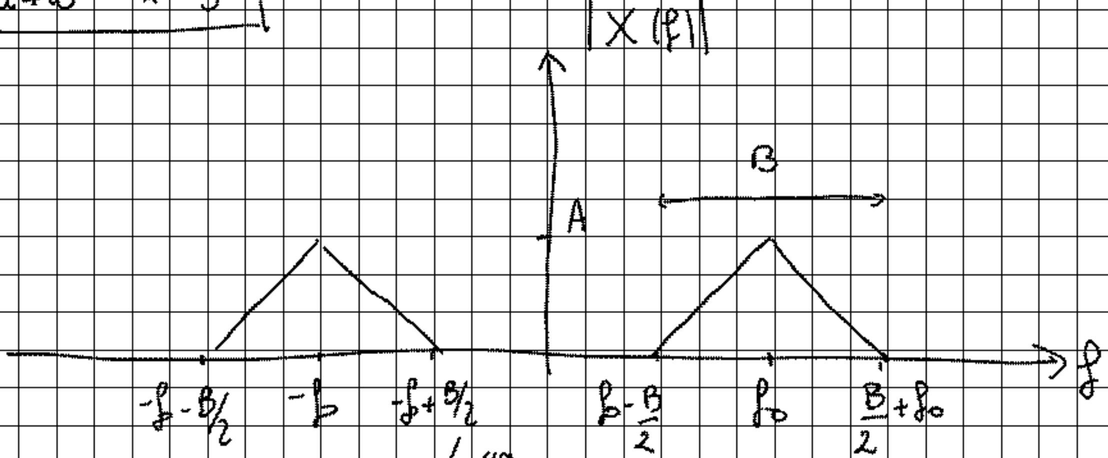
) teorema della convoluzione

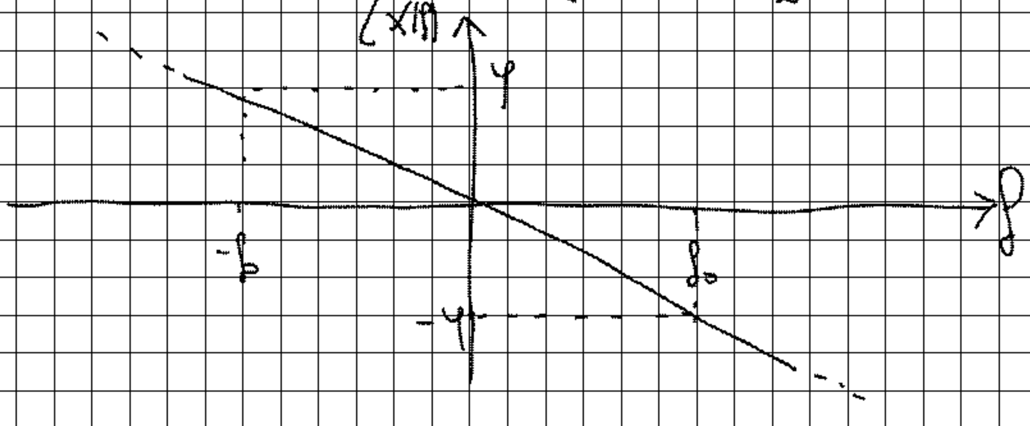
$$z(t) = x(t) \otimes y(t) \rightleftharpoons Z(f) = X(f) \cdot Y(f)$$

$$X(f) = \frac{A}{T} \, T \operatorname{sinc}(fT) \cdot T \operatorname{sinc}(fT) =$$

$$= AT \operatorname{sinc}^2(fT)$$

Esercizio # 5)





$$X(f) = |X(f)| \, e^{j\angle X(f)}$$

$$|X(f)| = X_1(f - f_0) + X_1(f + f_0)$$

$$X_1 = A\left(1 - \frac{|f|}{B/2}\right) \operatorname{rect}\left(\frac{f}{B}\right)$$

$$\angle X(f) = a f + b \qquad b = 0 \qquad a = -\frac{\varphi}{f_0}$$

$$\angle X(f) = -\frac{\varphi}{f_0} f$$

$$X(f) = \underbrace{\left[X_1(f - f_0) + X_1(f + f_0) \right]}_{A(f)} e^{-j\frac{\varphi}{f_0} f}$$

$$X(f) = A(f)\, e^{-j2\pi f t_0} \qquad t_0 = \frac{\varphi}{2 f_0 \pi}$$

·) teorema del ritardo

$$x(t) = a(t - t_0)$$

·) $A(f) = X_1(f - f_0) + X_1(f + f_0)$

$$X_1(f) = A\left(1 - \frac{|f|}{B/2}\right) \operatorname{rect}\left(\frac{f}{B}\right)$$

·) teorema della modulazione

$$y(t) = x(t) \cdot \cos(2\pi f_0 t)$$

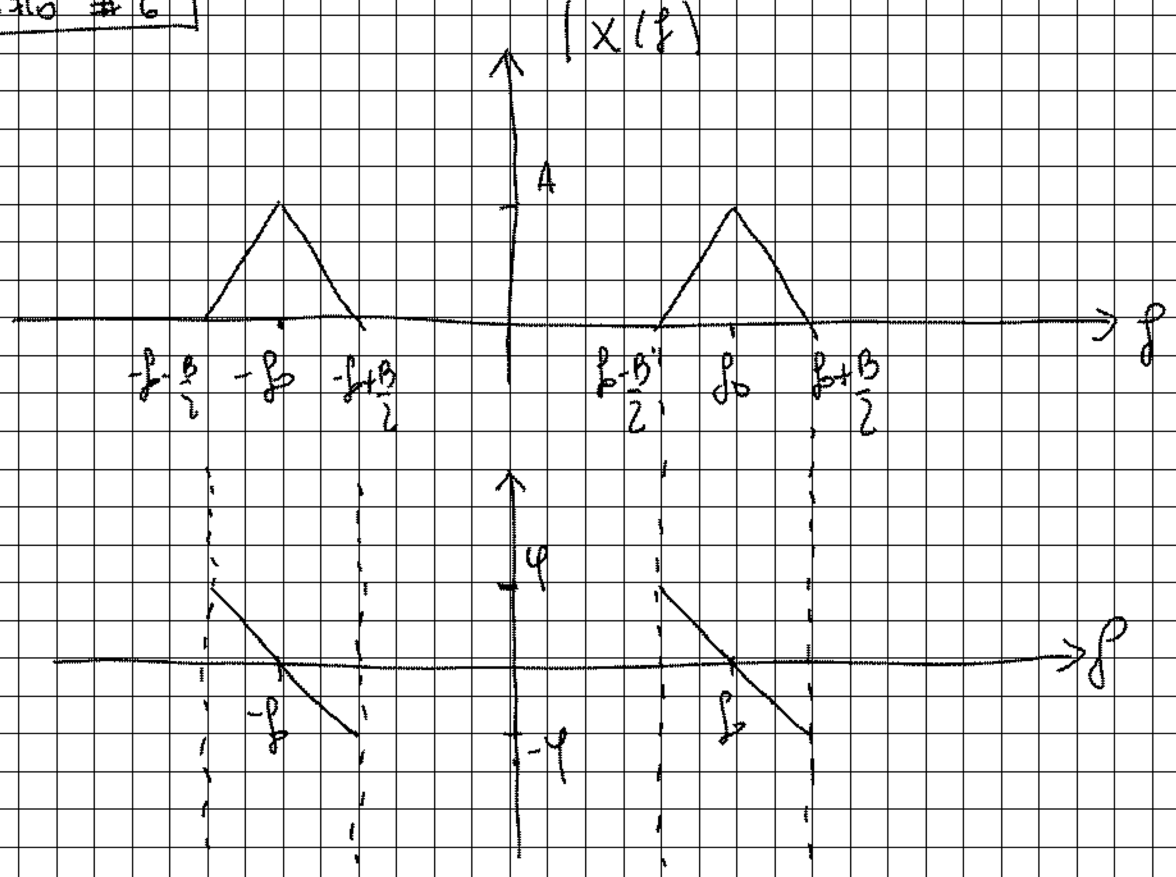
$$Y(f) = \frac{X(f - f_0) + X(f + f_0)}{2}$$

$$a(t) = 2\cos(2\pi f_0 t)\, x_1(t)$$

$$x_1(t) = A \cdot \frac{B}{2} \operatorname{sinc}^2\left(t \frac{B}{2}\right)$$

$$x(t) = AB\cos\left(2\pi f_0 (t-t_0)\right)\,\mathrm{sinc}^2\left(\frac{B}{2}(t-t_0)\right)$$

Esercizio #6



$$X(f) = X_0(f-f_0)\,e^{j\varphi_1(f)} + X_0(f+f_0)\,e^{j\varphi_2(f)}$$

$$X_0(f) = A\left(1-\frac{|f|}{B/2}\right)\mathrm{rect}\left(\frac{f}{B}\right)$$

$$\varphi_1(f) = \varphi_0(f-f_0) = -\frac{2\varphi}{B}(f-f_0)$$

$$\varphi_2(f) = \varphi_0(f+f_0) = -\frac{2\varphi}{B}(f+f_0)$$

$$X(f) = X_0(f-f_0)\,e^{j\varphi_0(f-f_0)} + X_0(f+f_0)\,e^{j\varphi_0(f+f_0)}$$

$$Y(f) = X_0(f)\, e^{i\varphi_0(f)}$$

$$X(f) = Y(f - f_0) + Y(f + f_0)$$

·) teorema della modulazione

$$\boxed{x(t) = 2\cos(2\pi f_0 t)\, y(t)}$$

$$Y(f) = X_0(f) \cdot e^{-j\frac{2\varphi}{B}f} = X_0(f)\, \boxed{e^{-j2\pi f t_0}} \qquad t_0 = \frac{\varphi}{B\pi}$$

·) teorema del ritardo

$$z(t - t_0) \rightleftharpoons Z(f)\, e^{-j2\pi f t_0}$$

$$y(t) = x_0(t - t_0)$$

$$x_0(t) = \frac{AB}{2} \operatorname{sinc}^2\left(t\frac{B}{2}\right)$$

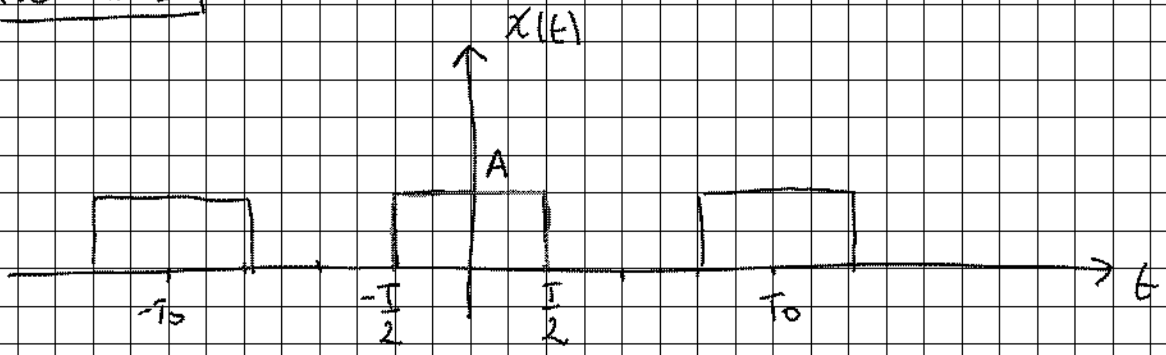
$$x(t) = AB \cos(2\pi f_0 t) \operatorname{sinc}^2\left((t - t_0)\frac{B}{2}\right)$$

elisa.giusti@iet.unipi.it

050 - 22 17 673

Esercitazione SEGNALI PERIODICIZZATI

Esercizio # 1



- $X(f)$
- X_n

$$X(f) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right) \delta\left(f - \frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right)$$

$$x_0(t) = A \operatorname{rect}\left(\frac{t}{T}\right)$$

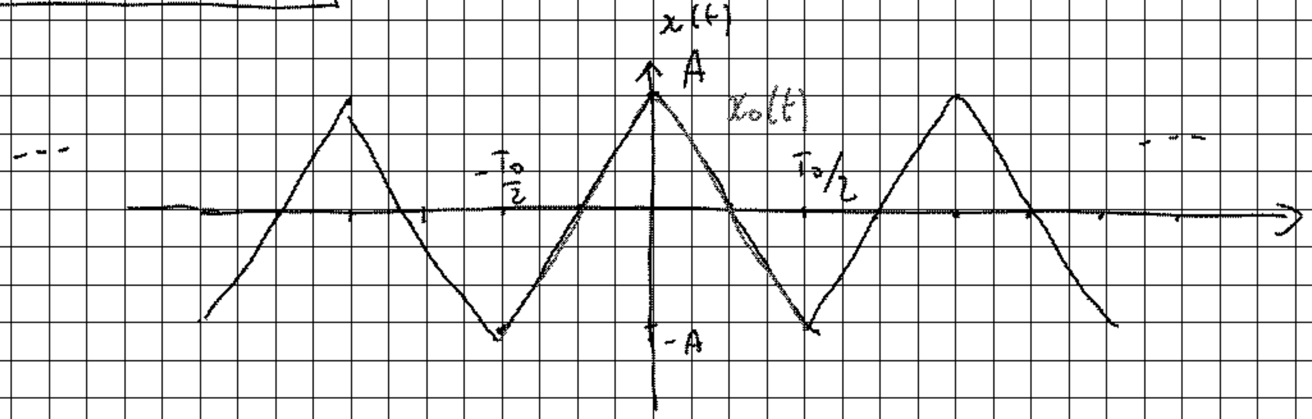
$$X_0(f) = AT \operatorname{sinc}(fT)$$

$$X(f) = \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} AT \operatorname{sinc}\left(\frac{k}{T_0} T\right) \delta\left(f - \frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} AT \operatorname{sinc}\left(\frac{k}{T_0} T\right)$$

Esercizio # 2

ONDA TRIANGOLARE



Determinare $X(f)$ e X_k

$$x_o(t) = 2A\left(1-\frac{|t|}{T_0/2}\right)\text{rect}\left(\frac{t}{T_0}\right) - A\,\text{rect}\left(\frac{t}{T_0}\right)$$

$$X_o(f) = 2A\frac{T_0}{2}\,\text{sinc}^2\left(f\frac{T_0}{2}\right) - AT_0\,\text{sinc}(fT_0)$$

$$\boxed{X_k = \frac{1}{T_0}X_o\left(\frac{k}{T_0}\right)}$$

$$X_k = A\,\text{sinc}^2\left(\frac{\cancel{T_0}}{2}\cdot\frac{k}{\cancel{T_0}}\right) - A\,\text{sinc}\left(\cancel{T_0}\frac{k}{\cancel{T_0}}\right) =$$

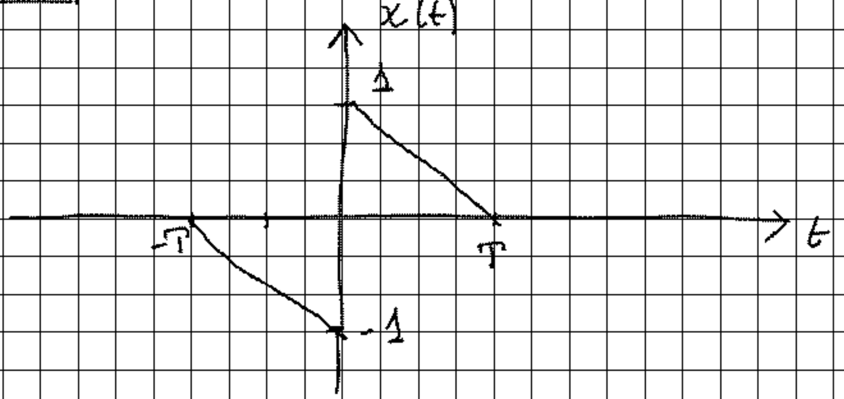
$$= A\,\text{sinc}^2\left(\frac{k}{2}\right) - A\,\text{sinc}(k) = \begin{cases} 0 & \text{per } k \text{ pari} \\ \frac{4}{k^2\pi^2} & \text{per } k \text{ dispari}\end{cases}$$

$$\text{sinc}(k) = 0 \quad \forall\, k = \pm 1, \pm 2, \pm 3, \dots$$

$$\text{sinc}^2\left(\frac{k}{2}\right) = \frac{\sin^2\left(\frac{k\pi}{2}\right)}{\left(\frac{k\pi}{2}\right)^2} = \begin{cases} 0 & \text{per } k \text{ pari} \\ \frac{4}{k^2\pi^2} & \text{per } k \text{ dispari}\end{cases}$$

$$X(f) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_o\left(\frac{k}{T_0}\right)\delta\left(f-\frac{k}{T_0}\right)$$

Esercizio #3

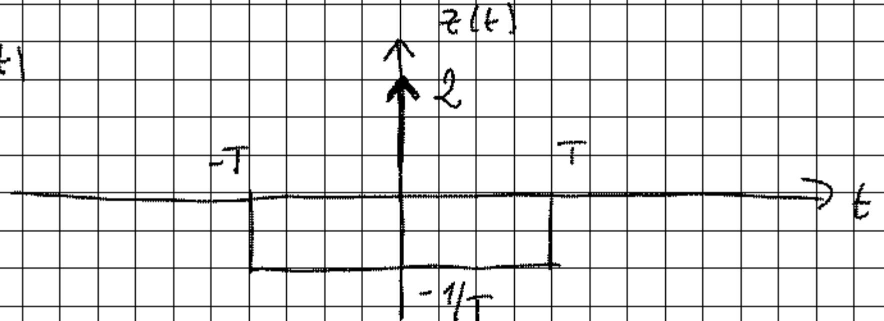


$$y(t) = \sum_{n=-\infty}^{+\infty} x(t - 4Tn) \qquad T_0 = 4T$$

Determinare $Y(f)$ e Y_k

- Teorema di derivazione

$$z(t) = \frac{\partial}{\partial t} x(t)$$



$$x(t) = \int_{-\infty}^{t} z(t)\, dt$$ la funzione integrale di $z(t)$ deve essere $x(t)$

$$z(t) = -\frac{1}{T} \operatorname{rect}\left(\frac{t}{2T}\right) + 2\delta(t)$$

$$Z(f) = -\frac{1}{\cancel{T}} 2\cancel{T} \operatorname{sinc}(f2T) + 2 = -2\operatorname{sinc}(f2T) + 2$$

$\boxed{Z(0) = 0} \Rightarrow$ è valida l'ipotesi di applicabilità del teorema di derivazione incompleto.

$$Z(f) = j2\pi f X(f)$$

$$X(f) = \frac{Z(f)}{j2\pi f} = \frac{2\left[1 - \mathrm{sinc}(f 2T)\right]}{j2\pi f} \qquad T_0 = 4T$$

$$Y_k = \frac{1}{T_0} X\left(\frac{k}{T_0}\right) = \frac{1}{4T} X\left(\frac{k}{4T}\right) =$$

$$= \frac{\left[1 - \mathrm{sinc}\left(\frac{k}{4T} 2T\right)\right]}{j\pi \frac{k}{\cancel{4T}}} \cdot \frac{1}{\cancel{4T}} =$$

$$= \frac{1 - \mathrm{sinc}\left(\frac{k}{2}\right)}{j\pi k}$$

$$Y(f) = \sum_{k=-\infty}^{+\infty} \frac{1 - \mathrm{sinc}(k/2)}{j\pi k} \cdot \delta\left(f - \frac{k}{4T}\right)$$

<u>Esercizio 4</u>

$$s(t) = \mathrm{sinc}\left(2Bt - 1/2\right) + \mathrm{sinc}\left(2Bt + 1/2\right)$$

Si calcolino Energia e Potenza dei segnali

$$x(t) = s(t) \sin(2\pi B t)$$

$$y(t) = \sum_{k=-\infty}^{+\infty} s(t - 2k/B)$$

$$s(t) = \text{sinc}\left(2B\left(t - \overset{t_0}{\frac{1}{4B}}\right)\right) + \text{sinc}\left(2B\left(t + \overset{t_0}{\frac{1}{4B}}\right)\right)$$

$$S(f) = \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) e^{-i2\pi f \frac{1}{4B}} + \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) e^{i2\pi f \frac{1}{4B}} =$$

$$= \frac{1}{B}\,\text{rect}\left(\frac{f}{2B}\right) \cdot \cos\left(2\pi f \frac{1}{4B}\right) = \frac{1}{B}\,\text{rect}\left(\frac{f}{2B}\right) \cos\left(\frac{\pi f}{2B}\right)$$

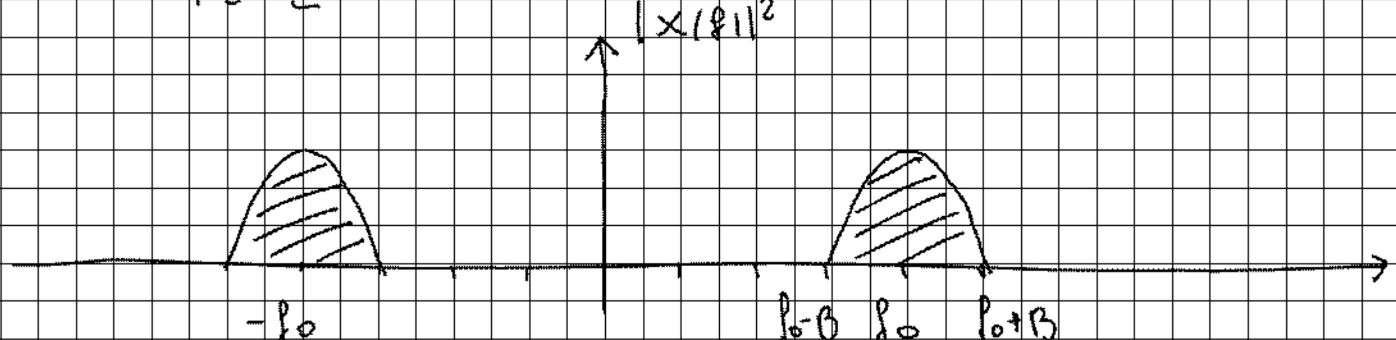
$$x(t) = s(t) \sin(2\pi B t)$$

$$X(f) = \frac{1}{2i}\left[S(f-f) - S(f+f_0)\right] =$$

$$= \frac{1}{2iB}\left[\text{rect}\left(\frac{f-f_0}{2B}\right) \cos\left(\frac{\pi(f-f_0)}{2B}\right) - \text{rect}\left(\frac{f+f_0}{2B}\right)\cos\left(\frac{\pi(f+f_0)}{2B}\right)\right]$$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$|X(f)|^2 = \frac{1}{4B^2}\left[\text{rect}\left(\frac{f-f_0}{2B}\right)\cos^2\left(\frac{\pi(f-f_0)}{2B}\right) + \text{rect}\left(\frac{f+f_0}{2B}\right)\cos^2\left(\frac{\pi(f+f_0)}{2B}\right)\right]$$

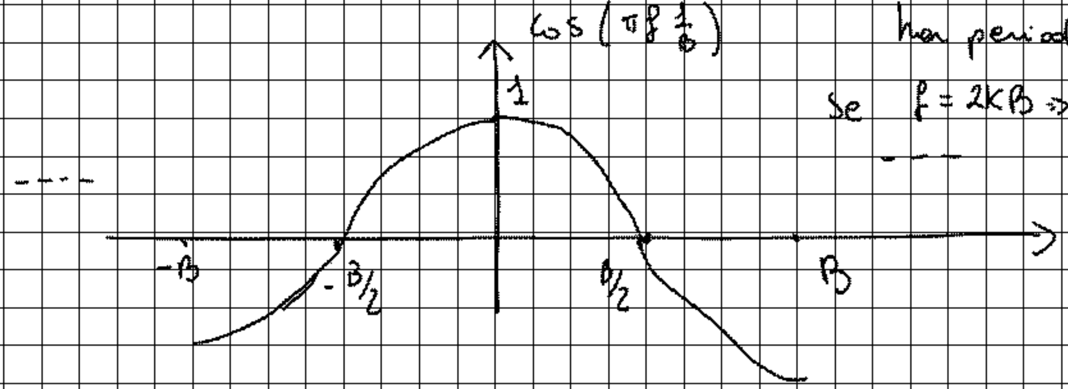


$$E_x = \frac{2}{4B^2} \int_{f_0-B}^{f_0+B} \cos^2\left(\frac{\pi (f-f_0)}{2B}\right) df \qquad f - f_0 = f'$$

$$E_x = \frac{2}{4B^2} \int_{-B}^{B} \cos^2\left(\frac{\pi f'}{2B}\right) df' \qquad \cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$$

$$E_x = \frac{1}{2B^2} \int_{-B}^{B} \left[\frac{1}{2} + \frac{1}{2}\cos\left(\pi f' \frac{1}{B}\right)\right] df'$$

$$E_x = \frac{1}{2B^2} \int_{-B}^{B} \frac{1}{2}\, df' = \frac{1}{2B^2} \left.\frac{f}{2}\right|_{-B}^{B} = \frac{1}{4B^2}\, 2B = \frac{1}{2B}$$



ha periodo $2B$

se $f = 2kB \Rightarrow \cos(2k\pi)$

$$E_x = \frac{1}{2B} \qquad P_x = 0$$

$$y(t) = \sum_k x\left(t - \frac{2k}{B}\right)$$

$$E_y = \infty$$

$$P_y = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} |y(t)|^2 = \sum_{k=-\infty}^{+\infty} |Y_k|^2$$

$$Y_k = \frac{1}{T_0} S\left(\frac{k}{T_0}\right) = \frac{B}{2} S\left(\frac{kB}{2}\right) =$$

$$= \frac{B}{2} \frac{1}{B} \operatorname{rect}\left(\frac{k\, B/2}{2B}\right) \cos\left(\frac{\pi k\, B/2}{2B}\right) = \frac{1}{2} \operatorname{rect}\left(\frac{k}{4}\right) \cos\left(\frac{\pi}{4} k\right)$$

$$Y_k = \begin{cases} 1/2 & k = 0 \\ \frac{\sqrt{2}}{4} & k = \pm 1 \\ 0 & \text{altrove} \end{cases}$$

$$P_y = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Esercizi su SISTEMI

Esercizio #1

Sia data la trasformazione ingresso-uscita di un sistema

$$y(t) = \int_{+T}^{t} x(\alpha)\, d\alpha$$

Dire se il sistema gode delle seguenti proprietà:

1) Linearità
2) stazionarietà
3) stabilità (BIBO)
4) istantaneità

Svolgimento

1) $x(t) = a\, x_1(t) + b\, x_2(t)$

$$y(t) = T\left[x(t)\right] = a\, T\left[x_1(t)\right] + b\, T\left[x_2(t)\right]$$

$$y(t) = \int_{T}^{t} \left[a\, x_1(t) + b\, x_2(t)\right] dt =$$

per la proprietà di linearità dell'integrale

$$= \int_{T}^{t} a\, x_1(t)\, dt + \int_{T}^{t} b\, x_2(t)\, dt =$$

$$= a\, T\left[x_1(t)\right] + b\, T\left[x_2(t)\right]$$

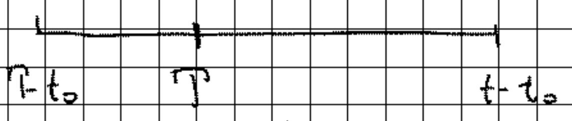
LINEARE

2) STAZIONARIETÀ

$$y(t-t_0) = T\left[x(t-t_0)\right]$$

$$\int_{T}^{t} x(\alpha - t_0)\, d\alpha = \qquad \boxed{\alpha - t_0 = \beta}$$

$$= \int_{T-t_0}^{t-t_0} x(\beta)\, d\beta = \int_{T-t_0}^{T} x(\beta)\, d\beta + \int_{T}^{t-t_0} x(\beta)\, d\beta =$$



$$= \int_{T-t_0}^{T} x(\beta)\, d\beta + y(t-t_0) \qquad \boxed{\text{NON STAZIONARIO}}$$

3) STABILITÀ

$$|x(t)| \le M \qquad |y(t)| \le K$$

Dimostriamo l'INSTABILITÀ del sistema

$$x(\alpha) = M$$

$$\int_{+T}^{t} x(\alpha)\,d\alpha = \int_{T}^{t} M\,d\alpha = M\,(t-T) = y(t)$$

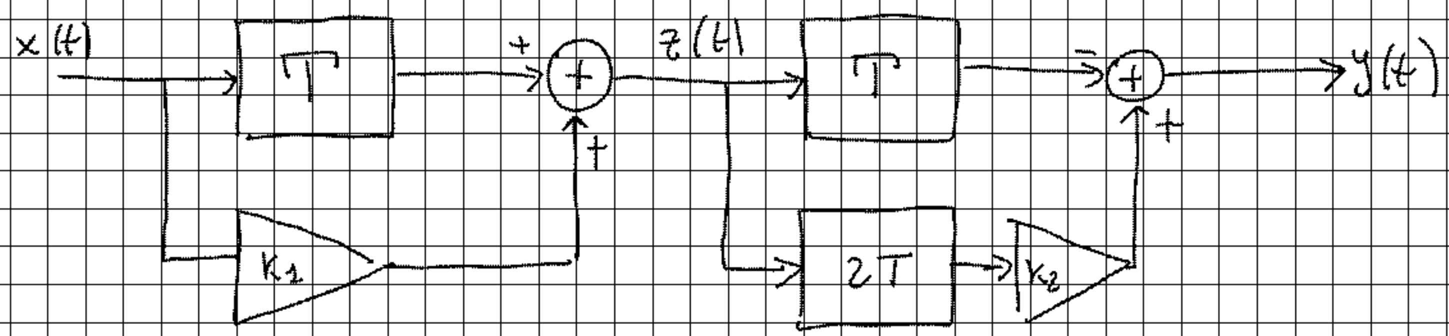
$\boxed{\text{NON È STABILE}}$

4) ISTANTANEITÀ

$$y(t) = \int_{T}^{t} x(\alpha)\,d\alpha$$

$\boxed{\text{CON MEMORIA}}$

Esercizio #2 STUDIO DI SISTEMA



Determinare l'espressione di $y(t)$ e dire se il sistema è:

- stazionario
- lineare
- senza memoria
- causale
- stabile

$$z(t) = x(t-T) + k_1\,x(t)$$

$$y(t) = -z(t-T) + k_2\,z(t-2T)$$

$$y(t) = -\Big(x(t-2T) + k_1\,x(t-T)\Big) + k_2\Big(x(t-3T) + k_1\,x(t-2T)\Big)$$

$$y(t) = -k_1 x(t-T) + x(t-2T)(-1 + k_1 k_2) + k_2 x(t-3T)$$

1) STAZIONARIETÀ

$$x(t-t_0)$$

$$-k_1 x(t-t_0-T) + x(t-t_0-2T)(k_1k_2-1) + k_2 x(t-t_0-3T) =$$

$$= y(t-t_0)$$

STAZIONARIO

2) LINEARITÀ

$$x(t) = a x_1(t) + b x_2(t)$$

$$y(t) = -k_1 x(t-T) + x(t-2T)(k_1k_2-1) + k_2 x(t-3T)$$

$$-k_1 \Big(a x_1(t-T) + b x_2(t-T) \Big) + \Big(a x_1(t-2T) + b x_2(t-2T) \Big)(k_1k_2-1)$$

$$+ k_2 \Big(a x_1(t-3T) + b x_2(t-3T) \Big) =$$

$$= \underbrace{\Big(-k_1 a x_1(t-T) + a x_1(t-2T)(k_1k_2-1) + a k_2 x_1(t-3T) \Big)}_{a y_1(t)} + \underbrace{\Big(-k_1 b x_2(t-T) + a x_2(t-2T)(k_1k_2-1) + a k_2 x_2(t-3T) \Big)}_{b y_2(t)} =$$

$$= a y_1(t) + b y_2(t) = a T\left[x_1(t)\right] + b T\left[x_2(t)\right]$$

LINEARE

3) MEMORIA

Il sistema è con memoria perché l'uscita all'istante t $(y(t))$ dipende da istanti precedenti a t $(x(t-T), x(t-2T), x(t-3T))$

4) CAUSALITÀ

Il sistema è causale perché $\underbrace{\text{l'uscita}}_{}$ all'istante t non dipende da istanti temporali successivi a t.

5) STABILITÀ

$$|x(t)| \leq M \quad \forall t$$

$$|y(t)| = \left| -k_1 x(t-T) + (k_2 k_1 - 1) x(t-2T) + k_2 x(t-3T) \right| \leq$$

$$\left| -k_1 x(t-T) \right| + \left| (k_2 k_1 - 1) x(t-2T) \right| + \left| k_2 x(t-3T) \right| \leq$$

$$|k_1| M + |k_2 k_1 - 1| M + |k_2| M < +\infty$$

SISTEMA STABILE

- Calcolare la risposta impulsiva e in frequenza del sistema

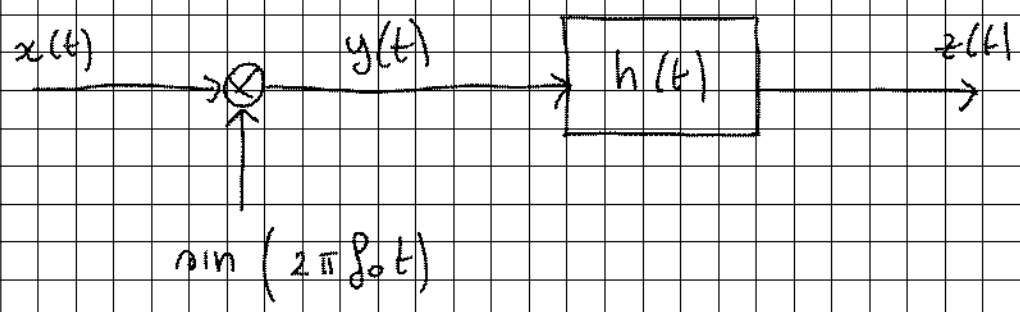
Il sistema è SLS quindi ha senso calcolarne la risposta impulsiva

$x(t) = \delta(t)$

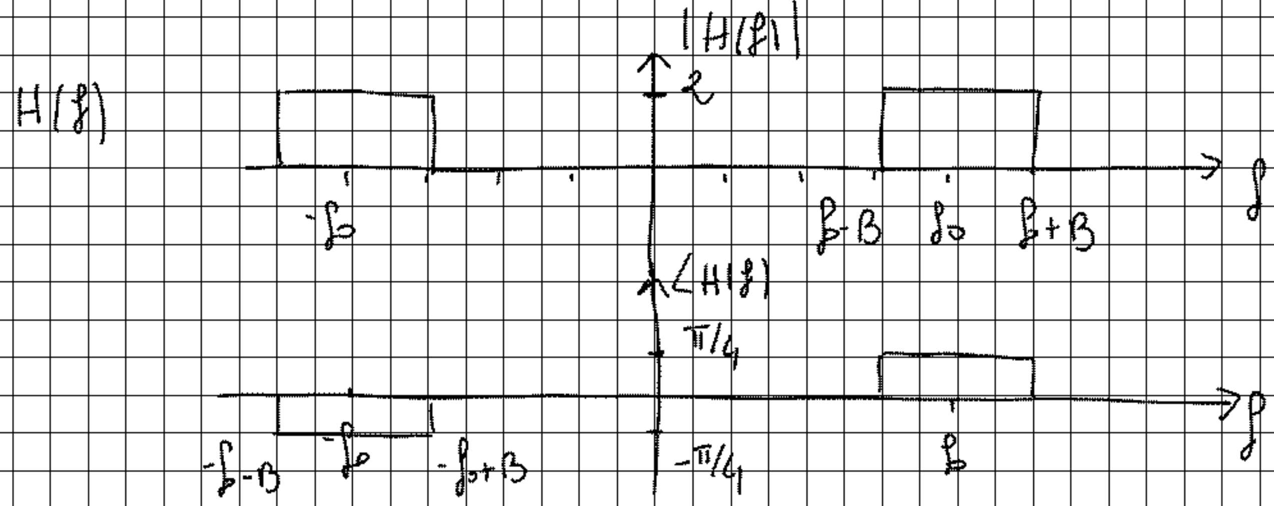
$$y(t) = -k_1\,\delta(t-T) + (k_2 k_1 - 1)\,\delta(t-2T) + k_2\,\delta(t-3T)$$

$$Y(f) = -k_1 \cdot 1 \cdot e^{-j2\pi f T} + (k_2 k_1 - 1)\, e^{-j2\pi f 2T} + k_2\, e^{-j2\pi f 3T}$$

Esercizio #3



$$X(f) = A\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$



- Calcolare espressione di z(t)
- Calcolare l'energia di z(t)

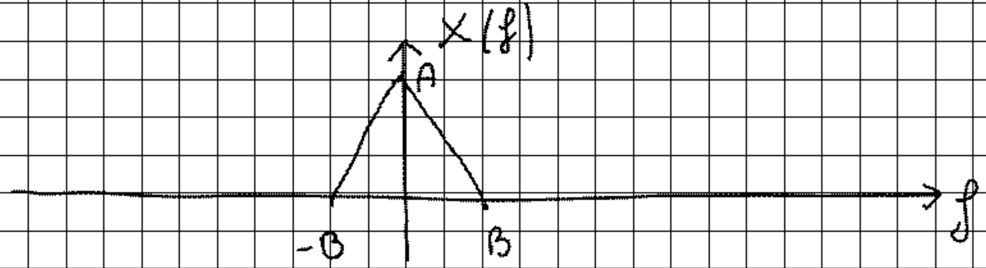
Svolgimento

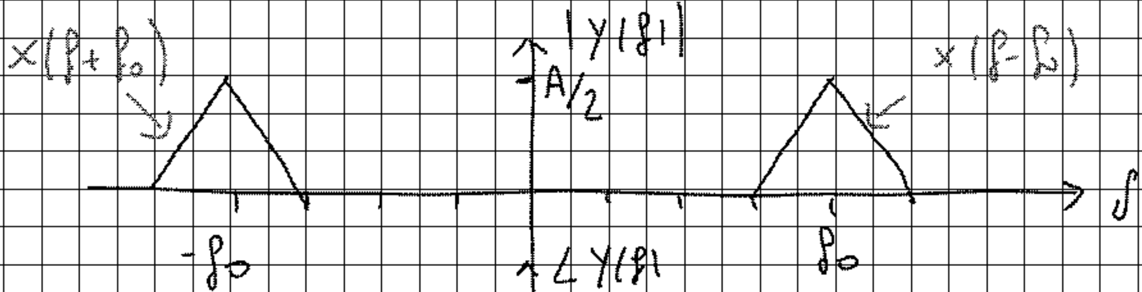
$$y(t) = x(t) \cdot \sin(2\pi f_0 t)$$

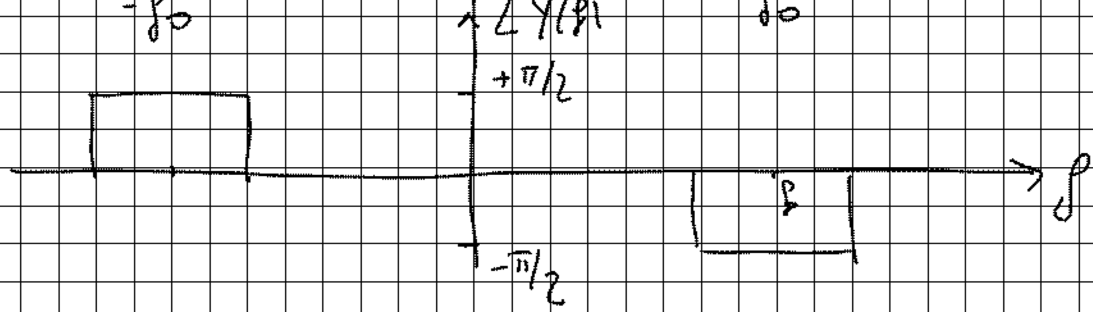
$$Y(f) = X(f) \otimes \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right) =$$

$$= \frac{1}{2j}\left(X(f-f_0) - X(f+f_0)\right) =$$

$$= \frac{e^{-j\pi/2}}{2} X(f-f_0) + \frac{e^{j\pi/2}}{2} X(f+f_0)$$



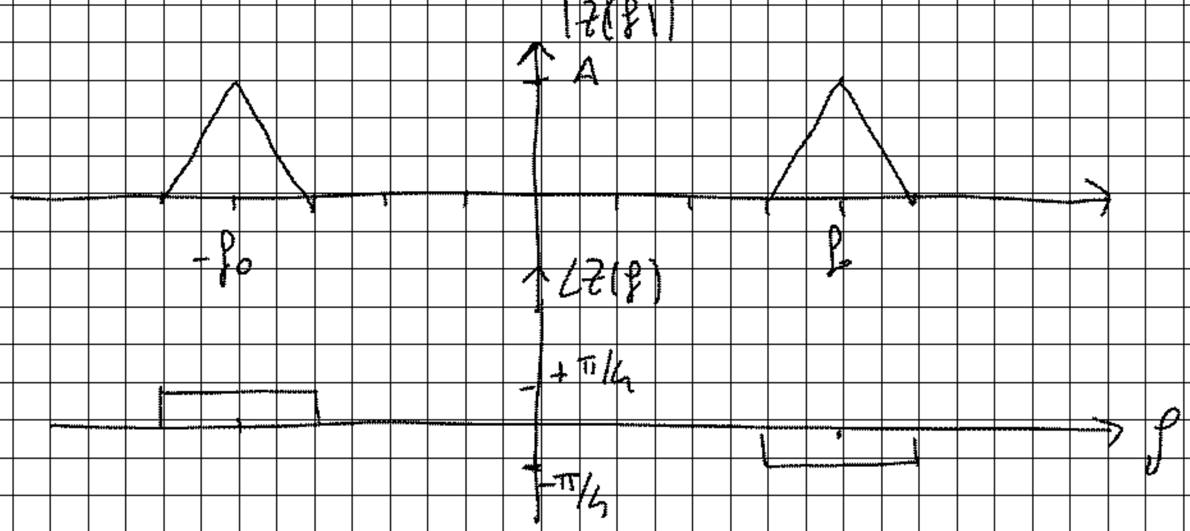




$$Z(f) = Y(f)\,H(f)$$

$$|Z(f)| = |Y(f)|\,|H(f)|$$

$$\angle Z(f) = \angle Y(f) + \angle H(f)$$



$$Z(f) = A\left(1-\frac{|f-f_0|}{B}\right)\mathrm{rect}\left(\frac{f-f_0}{2B}\right)e^{-i\pi/4} + A\left(1-\frac{|f+f_0|}{B}\right)\mathrm{rect}\left(\frac{f+f_0}{2B}\right)e^{i\pi/4}$$

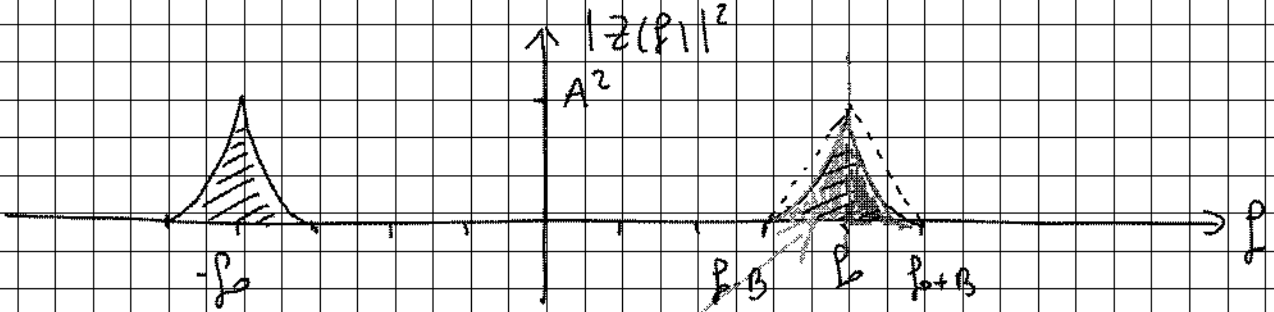
$$Z(f) = A\left(1-\frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right) \otimes \left[\delta(f-f_0)\,e^{-i\pi/4} + \delta(f+f_0)\,e^{i\pi/4}\right]$$

$$z(t) = AB\,\mathrm{sinc}^2(fB)\cdot\left[e^{+i2\pi f_0 t}\,e^{-i\pi/4} + e^{-i2\pi f_0 t}\,e^{i\pi/4}\right] =$$

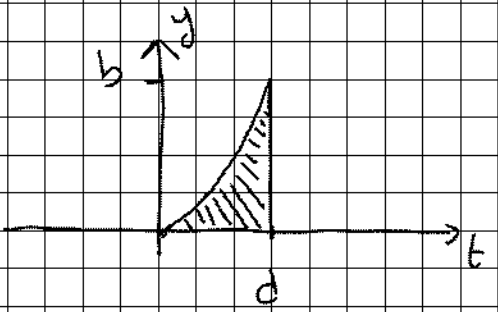
$$= 2AB\,\mathrm{sinc}^2(fB)\cdot\cos(2\pi f_0 t - \pi/4)$$

$$- E_z = \int_{-\infty}^{+\infty} |z(t)|^2\,dt = \int_{-\infty}^{+\infty} |Z(f)|^2\,df$$

$$Z(f) = A\left(1-\frac{|f+f_0|}{B}\right)\mathrm{rect}\left(\frac{f+f_0}{2B}\right)e^{i\pi/4} + A\left(1-\frac{|f-f_0|}{B}\right)\mathrm{rect}\left(\frac{f-f_0}{2B}\right)e^{-i\pi/4}$$



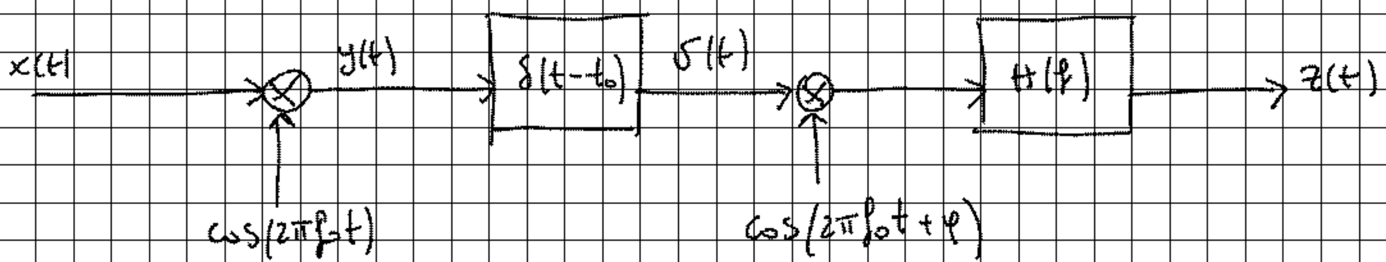
$$E_z = 4 \int_{f_0-B}^{f_0} |Z(f)|^2 \, df$$



$$y = \frac{b}{a^2} t^2 \qquad \int_0^a \frac{b}{a^2} t^2 \, dt = \frac{ab}{3}$$

$$E_z = 4 \frac{A^2 B}{3}$$

Esercizio #4 | SISTEMA WIRELESS



Il sistema rappresenta un sistema di comunicazione di tipo wireless, nel quale il segnale trasmesso $y(t)$ si propaga in spazio libero. Il blocco $\delta(t-t_0)$ schematizza il canale di propagazione. L'onda e.m. ricevuta, $r(t)$, è una copia ritardata ed attenuata di $y(t)$. t_0 è il tempo che l'onda e.m. impiega a percorrere la distanza tra il trasmettitore e il ricevitore.

$$X(f) = \left[1 + \cos\left(\frac{\pi f}{B} \right) \right] \text{rect}\left(\frac{f}{2B} \right)$$

$H(f)$ è un filtro passa basso ideale di banda B

Calcolare:

1) $Y(f)$

2) $W(f)$

3) $Z(f)$

4) Determinare il valore di φ affinché $z(t)$ abbia energia massima

Svolgimento

1) $y(t) = x(t) \cos(2\pi f_0 t)$ teorema della modulazione

$$Y(f) = \frac{X(f-f_0) + X(f+f_0)}{2}$$

$$Y(f) = \frac{1}{2} \left[1 + \cos\left(\frac{\pi (f-f_0)}{B} \right) \right] \text{rect}\left(\frac{f-f_0}{2B} \right) +$$

$$\frac{1}{2} \left[1 + \cos\left(\frac{(f+f_0)\pi}{B} \right) \right] \text{rect}\left(\frac{f+f_0}{2B} \right)$$

2) $v(t) = y(t) \otimes \delta(t-t_0)$

teorema della modulazione

teorema del ritardo

$$V(f) = Y(f) \cdot e^{-j2\pi f t_0}$$

$$w(t) = v(t) \cdot \cos(2\pi f_0 t + \varphi)$$

$$W(f) = V(f) \otimes \left[\frac{1}{2}\delta(f-f_0)e^{j\varphi} + \frac{1}{2}\delta(f+f_0)e^{-j\varphi}\right] =$$

$$= \frac{1}{2}V(f-f_0)e^{j\varphi} + \frac{1}{2}V(f+f_0)e^{-j\varphi}$$

$$W(f) = \frac{1}{2}Y(f-f_0)e^{-j2\pi(f-f_0)t_0}e^{j\varphi} + \frac{1}{2}Y(f+f_0)e^{-j2\pi(f+f_0)t_0}e^{-j\varphi} =$$

$$= \frac{1}{2}e^{-j2\pi f t_0}\left[Y(f-f_0)e^{+j2\pi f_0 t_0 + j\varphi} + Y(f+f_0)e^{-j2\pi f_0 t_0 - j\varphi}\right] =$$

$$= \frac{1}{2}e^{-j2\pi f t_0}\left\{\left[\left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi(f-2f_0)}{B}\right)\right)\mathrm{rect}\left(\frac{f-2f_0}{2B}\right) + \right.\right.$$

$$\left. + \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi f}{B}\right)\right)\mathrm{rect}\left(\frac{f}{2B}\right)\right]e^{j2\pi f_0 t_0 + j\varphi} +$$

$$+\left[\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi f}{B}\right)\right)\mathrm{rect}\left(\frac{f}{2B}\right)+\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi(f+2f_0)}{2B}\right)\right)\mathrm{rect}\left(\frac{f+2f_0}{2B}\right)\right]\cdot e^{-j2\pi f_0 t_0-j\varphi}\Bigg\}$$

Definisco

$$W_{2f_0}(f)=\frac{1}{4}\,e^{-j2\pi f t_0}\,e^{j2\pi f_0 t_0+j\varphi}\left(1+\cos\left(\frac{\pi(f-2f_0)}{B}\right)\right)\mathrm{rect}\left(\frac{f-2f_0}{2B}\right)$$

$$W_{-2f_0}(f)=\frac{1}{4}\,e^{-j2\pi f t_0}\,e^{-j2\pi f_0 t_0-j\varphi}\left(1+\cos\left(\frac{\pi(f+2f_0)}{B}\right)\right)\mathrm{rect}\left(\frac{f+2f_0}{2B}\right)$$

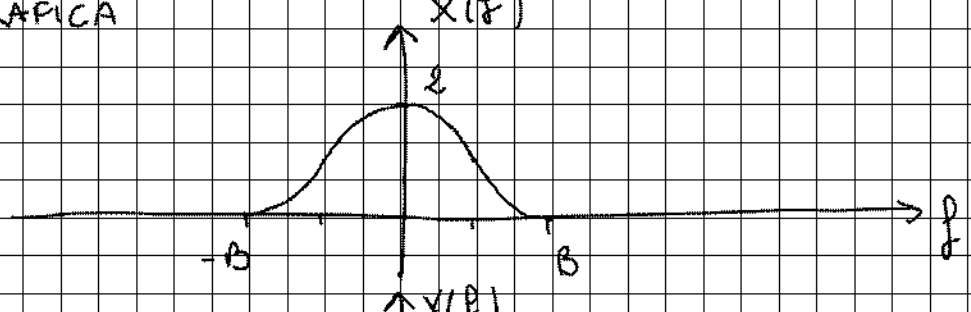
L'espressione di $W(f)$ si può riscrivere quindi come segue:

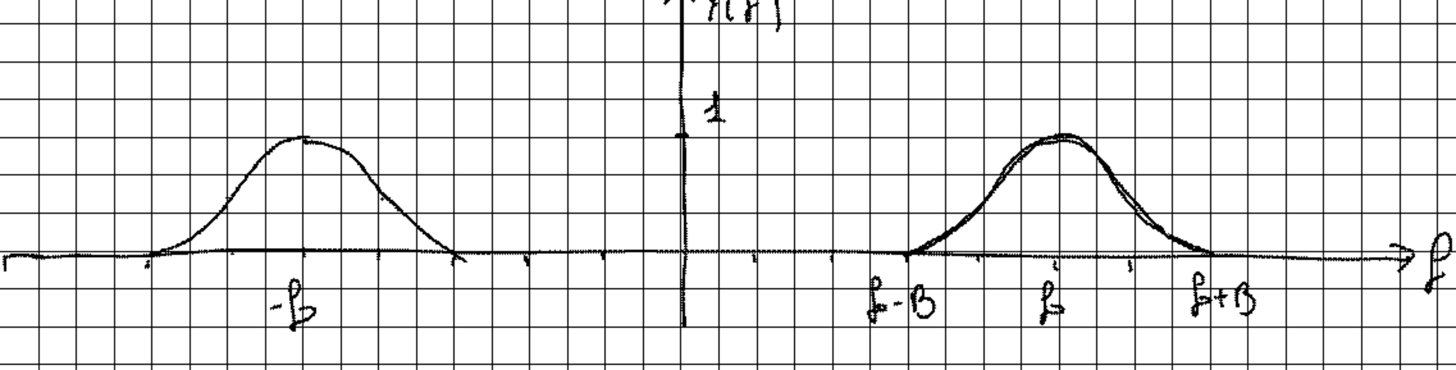
$W(f)=$

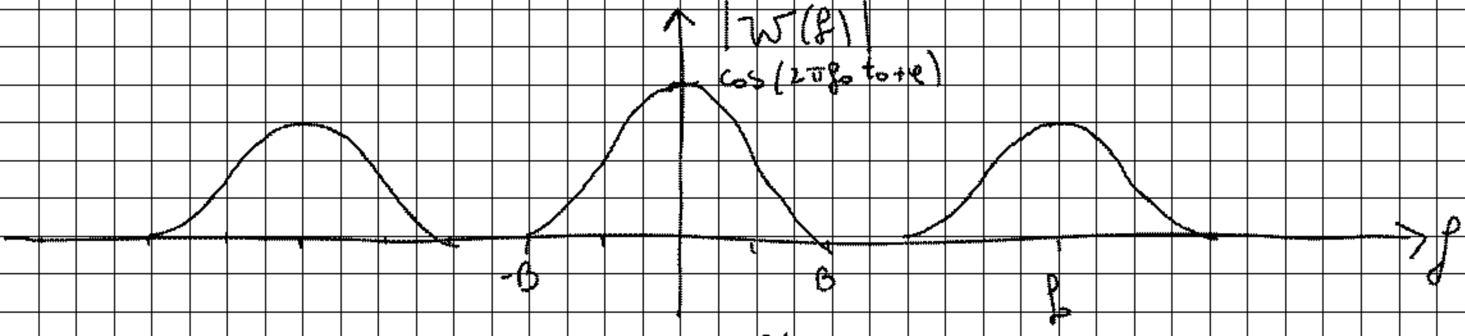
$$=W_{2f_0}(f)+W_{-2f_0}(f)+\frac{1}{2}\,e^{-j2\pi f t_0}\left(1+\cos\left(\frac{\pi f}{B}\right)\right)\cos\left(2\pi f_0 t_0+\varphi\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

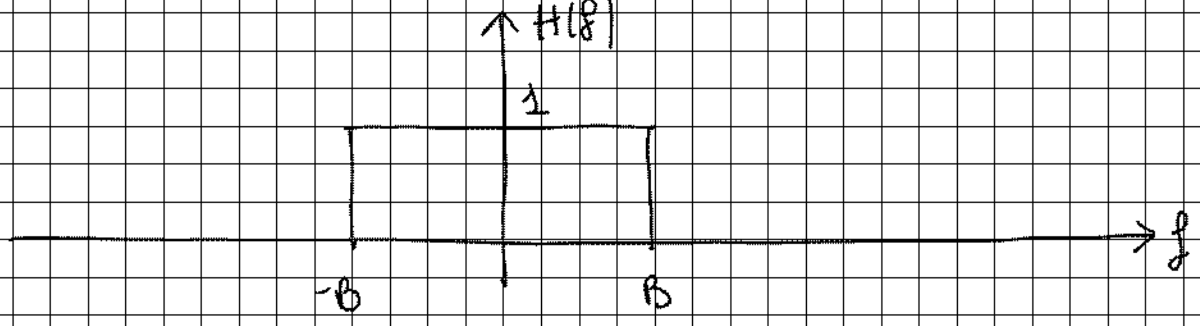
3)

ANALISI GRAFICA









h(t) è un filtro passa basso ideale quindi:

$$Z(f) = \frac{1}{2} e^{-j2\pi f t_0} \cos\left(2\pi f_0 t_0 + \varphi\right)\left(1 + \cos\left(\frac{\pi f}{B}\right)\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Z(f) = \frac{1}{2} \cos\left(2\pi f_0 t_0 + \varphi\right) \cdot X(f)\, e^{-j2\pi f t_0}$$

teorema del ritardo

$$z(t) = \frac{1}{2} \cos\left(2\pi f_0 t + \varphi\right) \cdot x\left(t - t_0\right)$$

z(t) è una replica ritardata e attenuata del segnale trasmesso

$$X(f) = \mathrm{rect}\left(\frac{f}{2B}\right) + \mathrm{rect}\left(\frac{f}{2B}\right) \cos\left(\frac{\pi f}{B}\right) \qquad t_0 = \frac{1}{2B}$$

$$\Updownarrow$$

$$x(t) = 2B\, \mathrm{sinc}\left(t\, 2B\right) + B\, \mathrm{sinc}\left(t\, 2B\right) \otimes \left[\delta\left(t - t_0\right) + \delta\left(t + t_0\right)\right] =$$

$= 2B \operatorname{sinc}(t2B) + B \operatorname{sinc}(2B(t-t_0)) + B \operatorname{sinc}(2B(t+t_0))$

4) $E_z = \int_{-\infty}^{+\infty} |z(t)|^2 dt = \int_{-\infty}^{+\infty} |Z(f)|^2 df$

$Z(f) = \frac{1}{2} \cos(2\pi f_0 t_0 + \varphi)\, x(t-t_0)$

$E_z = \frac{1}{4} \cos^2(2\pi f_0 t_0 + \varphi) \int_{-\infty}^{+\infty} |x(t-t_0)|^2 dt = \frac{1}{4} \cos^2(2\pi f_0 t_0 + \varphi) E_x$

dove E_x è l'energia di $x(t)$. Allora E_z è massima quando $\cos^2(2\pi f_0 t_0 + \varphi) = 1$

$E_{z,max} = \frac{1}{4} E_x$

$\cos^2(2\pi f_0 t_0 + \varphi) = 1 \Rightarrow 2\pi f_0 t_0 + \varphi = k\pi$

$\boxed{\varphi = -2\pi f_0 t_0 + k\pi}$

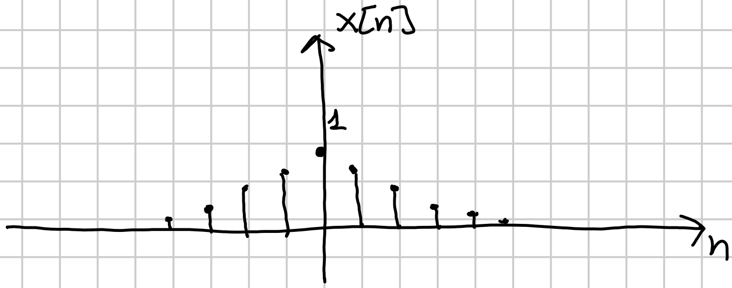
Esercitazione 21 - 03 - 2012 RICEVIMENTO VENERDI

9:30 11:30

Esercizio #1

Calcolare la TFS

$x[n] = a^{|n|}$ $\quad 0 \leq a < 1$



$$x[n] = a^{n} u[n] + a^{-n} u[-n] - x[0]\,\delta[n]$$

$$x[n] = y[n] + y[-n] - x[0]\,\delta[n]$$

TFS di $y[n]$

$$y[n] = \alpha^n u[n]$$

$$\bar{Y}(f) = \sum_{n=0}^{+\infty} \alpha^n e^{-j2\pi f nT} = \sum_{n=0}^{+\infty} \left(\alpha e^{-j2\pi f T}\right)^n$$

$$\boxed{\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q} \qquad |q| < 1}$$

$$\bar{Y}(f) = \frac{1}{1-\alpha e^{-j2\pi f T}}$$

$$x[n] = y[n] + y[-n] - x[0]\,\delta[n]$$

$$\bar{X}(f) = \bar{Y}(f) + \bar{Y}(-f) - x[0]$$

$y[n]$ è una sequenza reale $\Rightarrow \bar{Y}(f)$ è a simmetria Hermitiana

$$\bar{Y}(-f) = \bar{Y}^*(f)$$

$$\bar{X}(f) = \bar{Y}(f) + \bar{Y}^*(f) - x[0] = 2\,\mathrm{Re}\{\bar{Y}(f)\} - x[0]$$

$$\bar{X}(f) = \frac{1}{1-\alpha e^{-j2\pi fT}} + \frac{1}{1-\alpha e^{+j2\pi fT}} - 1 =$$

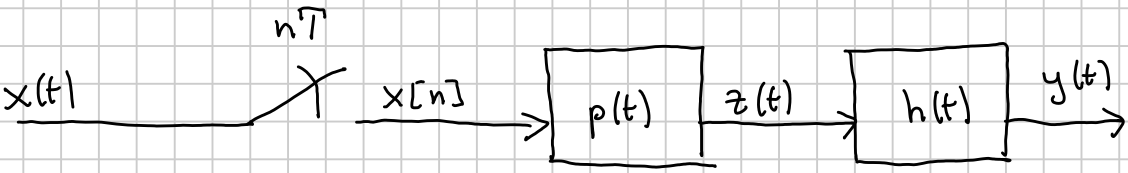
$$= \frac{1-\alpha e^{+j2\pi fT} + 1-\alpha e^{-j2\pi fT} - \left(1-\alpha e^{-j2\pi fT}\right)\left(1-\alpha e^{+j2\pi fT}\right)}{\left(1-\alpha e^{-j2\pi fT}\right)\left(1-\alpha e^{+j2\pi fT}\right)}$$

$$= \frac{1-\cancel{\alpha e^{+j2\pi fT}} + 1-\cancel{\alpha e^{-j2\pi fT}} - 1-\alpha^2+\cancel{2\alpha\cos(2\pi fT)}}{1+\alpha^2-2\alpha\cos(2\pi fT)} =$$

$$= \frac{1-\alpha^2}{1+\alpha^2-2\alpha\cos(2\pi fT)}$$

Esercizio #2

Ricostruzione di un segnale analogico a partire da una sequenza



Ricavare $y(t)$

$$x(t) = \mathrm{sinc}\left(\frac{t}{T}\right)$$

$$p(t) = \mathrm{rect}\left(\frac{t-T/2}{T}\right)$$ INTERPOLATORE A MANTENIMENTO

$$H(f) = \frac{\pi fT}{\sin(\pi fT)}\,\mathrm{rect}(fT)$$

Svolgimento

- $x[n] = \operatorname{sinc}\left(\frac{nT}{T}\right) = \operatorname{sinc}(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

$x[n] = \delta[n]$

- $z(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT) = p(t)$

- $y(t) = z(t) \otimes h(t)$

$Y(f) = Z(f)\, H(f) = P(f)\, H(f)$

- $p(t) = \operatorname{rect}\left(\frac{t - T/2}{T}\right)$

$P(f) = T \operatorname{sinc}(fT)\, e^{-j 2\pi f \frac{T}{2}} = T \operatorname{sinc}(fT)\, e^{-j\pi f T}$

- $Y(f) = T \operatorname{sinc}(fT)\, e^{-j\pi f T} \cdot \frac{\pi f T}{\sin(\pi f T)} \operatorname{rect}(fT) =$

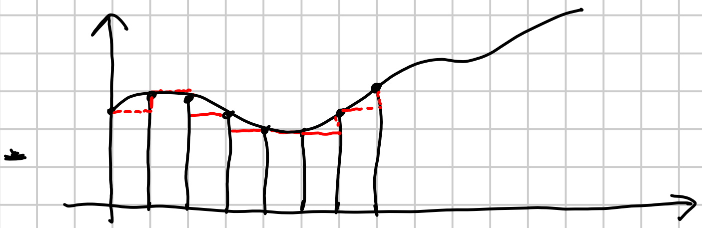
$= T\, \frac{\cancel{\sin(\pi f T)}}{\cancel{\pi f T}}\, e^{-j\pi f T}\, \frac{\cancel{\pi f T}}{\cancel{\sin(\pi f T)}} \operatorname{rect}(fT)$

$Y(f) = T \operatorname{rect}(fT)\, e^{-j\pi f T}$

- $y(t) = \operatorname{sinc}\left(\frac{t - T/2}{T}\right)$ replica non distorta del segnale $x(t)$

Sembra che l'interpolatore a mantenimento non introduce distorsioni!
Se però in ingresso ho un segnale $x(t) = sinc^2(t/T)$ allora l'interpolatore non è in grado di ricostruire il segnale di partenza.

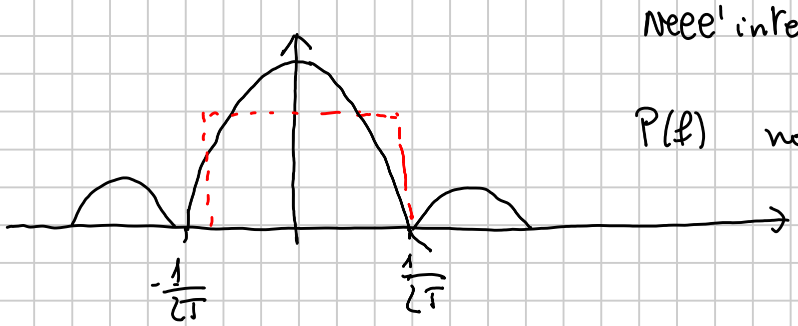
DISTORSIONI INTRODOTTE DALL'INTERPOLATORE A MANTENIMENTO



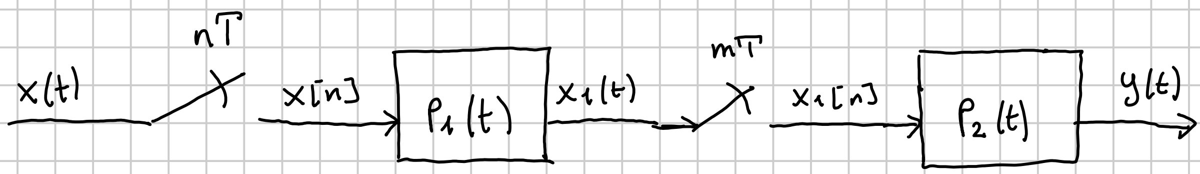
Le discontinuità nel dominio del tempo introducono delle componenti frequentili che non sono presenti nel segnale analogico, denominate IMMAGINI.

$$p(t) = rect\left(\frac{t - T/2}{T}\right) \Rightarrow P(f) = T\, sinc(fT)\, e^{-j 2\pi f T/2}$$

Nell'intervallo base $[-\frac{1}{2T}; \frac{1}{2T}]$ $P(f)$ non è costante.

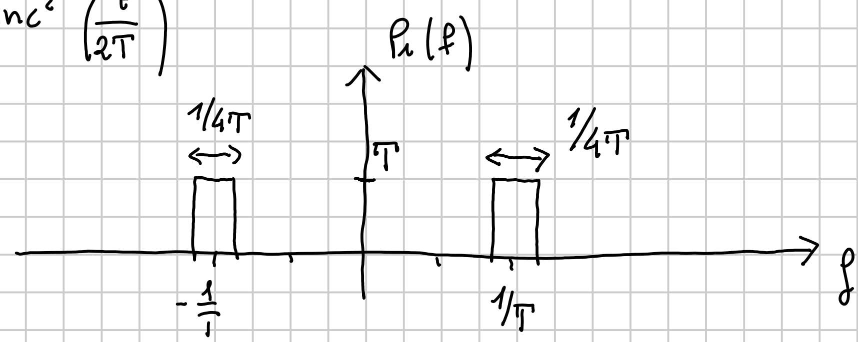


Esercizio #3



$$x(t) = sinc^2\left(\frac{t}{2T}\right)$$

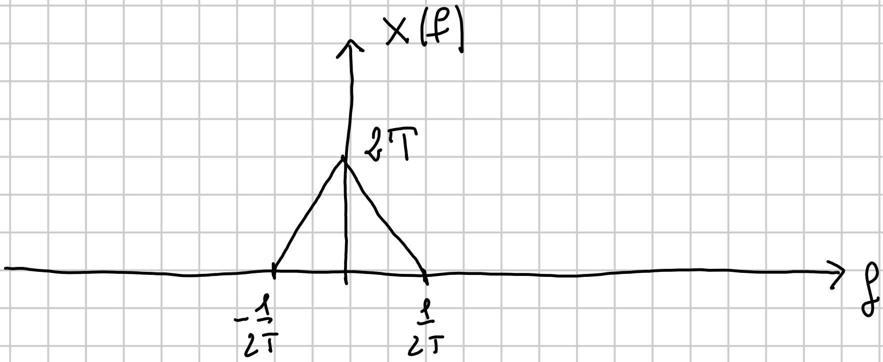
$P_1(f)$



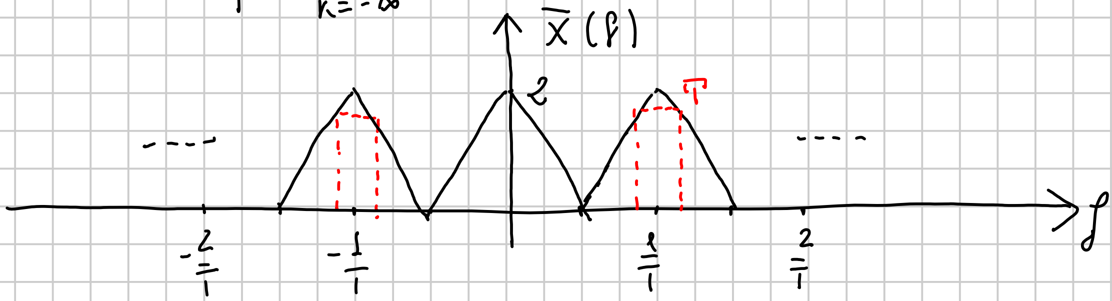
$$P_2(t) = sinc\left(\frac{t}{T}\right)$$

Trovare l'espressione di $y(t)$

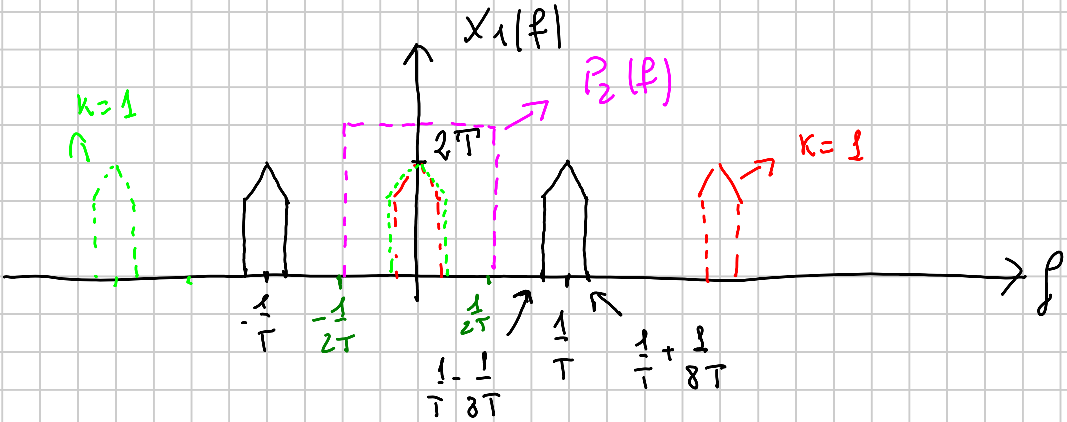
- $X(f) = 2T\left(1 - |f|\,2T\right)\operatorname{rect}(fT)$ $\qquad B = \frac{1}{T}$



- $\overline{X}(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T}\right)$



$X_1(f) = P_1(f)\cdot\overline{X}(f)$



$\overline{X}_1(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_1\left(f - \frac{k}{T}\right)$

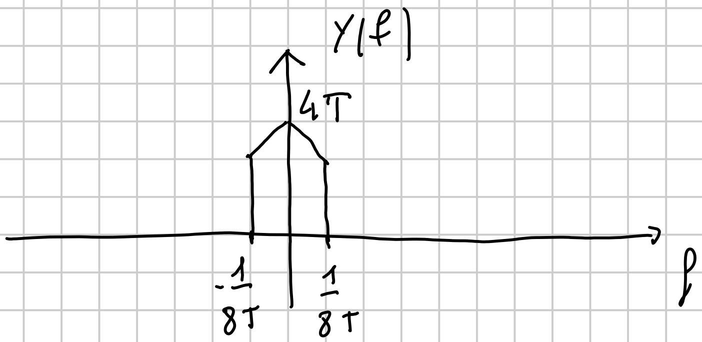
$$\bar{X}_1(f) = 2\,X(f)\,\mathrm{rect}\,(f\,4T) \qquad -\frac{1}{2T} \le f \le \frac{1}{2T}$$

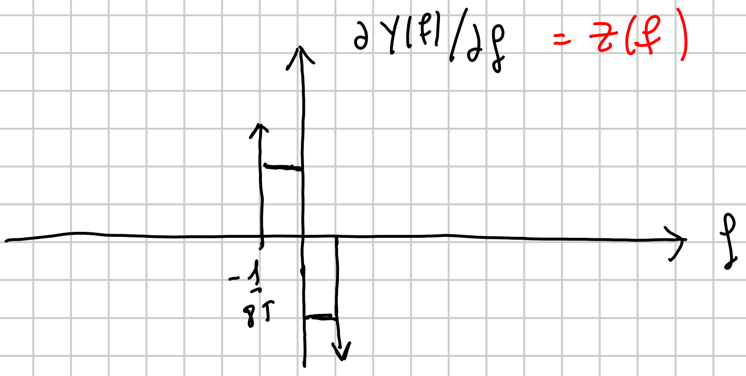
$$p_2(t) = \mathrm{sinc}\left(\frac{t}{T}\right)$$

$$P_2(f) = T\,\mathrm{rect}\,(f\,T)$$

$$Y(f) = P_2(f)\cdot X_1(f) = 2\,X(f)\,\mathrm{rect}\,(f\,4T)\;T\,\mathrm{rect}\,(fT)$$

$$Y(f) = 2T\,X(f)\,\mathrm{rect}\,(f\,4T)$$





$$z(f) = \frac{\partial Y(f)}{\partial f} \Rightarrow z(t) = -j2\pi t\; y(t)$$

Se $z(t)\big|_{t=0} = 0$

$$y(t) = \frac{z(t)}{-j2\pi t}$$

$$z(t)\Big|_{t=0} = 0 \quad \Rightarrow \quad \int_{-\infty}^{+\infty} Z(f) = 0$$

$$Y(f) = 4T \left(1 - \frac{|f|}{1/2T}\right) \text{rect}\left(\frac{f}{1/4T}\right)$$

$$f = \pm \frac{1}{8T} \qquad Y(f)\Big|_{f=\pm\frac{1}{8T}} = 3T$$

$$a = 8T^2$$ pendenza delle rette che compongono $Y(f)$

$$Z(f) = \frac{\partial Y(f)}{\partial f} = 8T^2 \,\text{rect}\left(\frac{f + \frac{1}{16T}}{\frac{1}{8T}}\right) - 8T^2 \,\text{rect}\left(\frac{f - \frac{1}{16T}}{\frac{1}{8T}}\right)$$

$$+ 3T\,\delta\left(f + \frac{1}{8T}\right) - 3T\,\delta\left(f - \frac{1}{8T}\right)$$

$$z(t) = \cancel{8}T^{\cancel{2}} \left(\frac{1}{\cancel{8T}} \,\text{sinc}\left(\frac{t}{8T}\right) e^{j2\pi t\frac{1}{16T}} - \frac{1}{8T}\,\text{sinc}\left(\frac{t}{8T}\right) e^{-j2\pi t\frac{1}{16T}} \right)$$

$$+ 3T\frac{2j}{2j}\left(\delta\left(f + \frac{1}{8T}\right) - \delta\left(f - \frac{1}{8T}\right) \right)$$

$$z(t) = T\frac{2j}{2j}\,\text{sinc}\left(\frac{t}{8T}\right)\left[e^{j2\pi t\frac{1}{16T}} - e^{-j2\pi t\frac{1}{16T}} \right] - 3T \cdot 2j \cdot \sin\left(\pi\frac{t}{8T}\right)$$

$$z(t) = 2j\,T\,\mathrm{sinc}\left(\frac{t}{8T}\right)\cdot \sin\left(\frac{\pi t}{8T}\right) - 6Tj\,\sin\left(\frac{\pi t}{8T}\right)$$

$$y(t) = \frac{z(t)}{-j2\pi t}$$

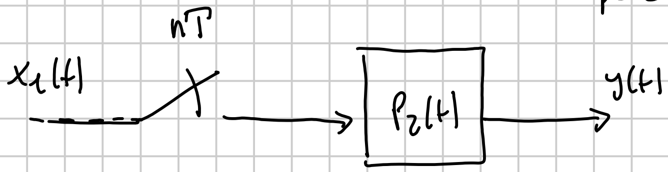
$$y(t) = \frac{2Tj}{-2j\pi t}\,\mathrm{sinc}\left(\frac{t}{8T}\right)\sin\left(\frac{\pi t}{8T}\right) - \frac{6Tj}{-2j\pi t}\,\sin\left(\frac{\pi t}{8T}\right) =$$

$$= -\frac{T}{\pi t}\,\mathrm{sinc}\left(\frac{t}{8T}\right)\sin\left(\frac{\pi t}{8T}\right) + \frac{3T}{\pi t}\,\sin\left(\frac{\pi t}{8T}\right) =$$

$$= -\frac{1}{8}\cdot\frac{1}{\pi t\cdot\frac{1}{8T}}\sin\left(\frac{\pi t}{8T}\right)\mathrm{sinc}\left(\frac{t}{8T}\right) + \frac{3}{8}\,\frac{1}{\pi t\,\frac{1}{8T}}\sin\left(\frac{\pi t}{8T}\right) =$$

$$= -\frac{1}{8}\,\mathrm{sinc}^2\left(\frac{t}{8T}\right) + \frac{3}{8}\,\mathrm{sinc}\left(\frac{t}{8T}\right)$$

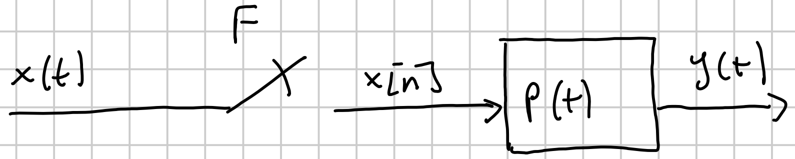
OSSERVAZIONE: $x_1(t)$ e' un segnale modulato. Il sistema quindi a valle di $x_1(t)$ opera come un demodulatore.



$$x_1(t) = x_0(t)\cos(2\pi f_0 t) \qquad f_0 = \frac{k}{T}$$

$$y(t) = a\,x_0(t - t_0)$$

Esercizio #4



- $p(t)$ interpolatore a mantenimento $\quad p(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right)$

$$x(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT) \qquad x(nT) \doteq x[n]$$

$$F = \frac{1}{T} = \frac{10}{\alpha}$$

$$x(t) = e^{-t/\alpha}\, u(t)$$

Calcolare E_x ed E_y

$$E_x = \int_{-\infty}^{+\infty} e^{-2t/\alpha}\, u(t)\, dt = \int_{0}^{+\infty} e^{-2t/\alpha}\, dt = -\frac{\alpha}{2} e^{-2t/\alpha} \Big|_{0}^{+\infty} = \frac{\alpha}{2}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} x[n]\, \mathrm{rect}\left(\frac{t - nT - T/2}{T}\right)$$

$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2\, dt$$

$$E_y = \int_{-\infty}^{+\infty} \sum_n x^2[n] \; \text{rect}\left(\frac{t - nT - T/2}{T}\right) dt$$

$$|y(t)|^2 = \sum_n x[n] \, \text{rect}\left(\frac{t - nT - T/2}{T}\right) \sum_m x^*[m] \, \text{rect}\left(\frac{t - mT - T/2}{T}\right) =$$

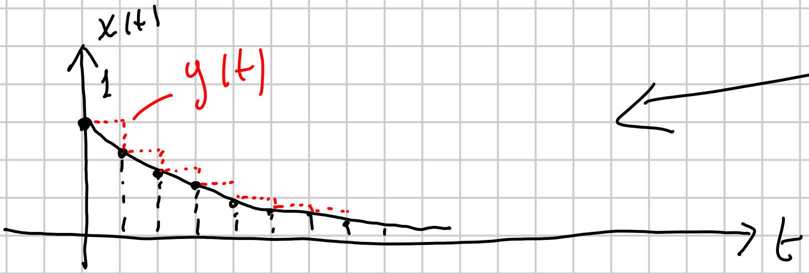
$$= \sum_n x^2[n] \; \text{rect}\left(\frac{t - nT - T/2}{T}\right)$$

$$E_y = \sum_n x^2[n] \int_{-\infty}^{+\infty} \text{rect}\left(\frac{t - nT - T/2}{T}\right) dt = T \sum_{n=0}^{+\infty} \left(e^{-2T/\alpha}\right)^n$$

$$E_y = T \cdot \frac{1}{1 - e^{-2T/\alpha}} = \frac{\frac{\alpha}{10}}{1 - e^{-1/5}} = \frac{\alpha}{2} \, \frac{1}{5\left(1 - e^{-1/5}\right)} =$$

$$= E_x \frac{1}{5\left(1 - e^{-1/5}\right)} \cong 1{,}1 \; E_x$$

È giusto che $E_y > E_x$ perché:



Esercitazione 28-03-2012

Esercizio #1

Una stanza è illuminata da due lampadine in serie. La probabilità che una lampadina sia guasta ad un certo istante è p. Si possono guastare in maniera indipendente l'una dall'altra.

Calcolare la probabilità che la stanza sia buia.

$$\Omega = \{ (F_1, G_2); (G_1, F_2); (G_1, G_2); (F_1, F_2) \}$$

$$A = \{ (F_1, G_2); (G_1, F_2); (G_1, G_2) \}$$

$$Pr\{A\} = ?$$

1) $Pr\{A\} = Pr\{(F_1, G_2)\} + Pr\{(G_1, F_2)\} + Pr\{(G_1, G_2)\}$

2) $Pr\{A\} = 1 - Pr\{(F_1, F_2)\}$

3) $\alpha_1 = \{ (G_1, F_2); (G_1, G_2) \}$ $\qquad Pr\{\alpha_1\} = p$

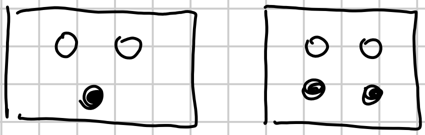
$\alpha_2 = \{ (F_1, G_2); (G_1, G_2) \}$ $\qquad Pr\{\alpha_2\} = p$

$$Pr\{A\} = Pr\{\alpha_1 \cup \alpha_2\} = Pr\{\alpha_1\} + Pr\{\alpha_2\} - Pr\{\alpha_1 \cap \alpha_2\} =$$

$$= p + p - p^2 = 2p - p^2$$

Esercizio #2

Ci sono due urne, una contiene 1 pallina nera e 2 bianche e l'altra 2 bianche e 2 nere



L'esperimento consiste nello scegliere un'urna e poi nello scegliere una pallina.

Qual è la probabilità di estrarre una pallina bianca.

$$\Omega = \{ (U_1, b_{11}) \ (U_1, b_{12}) \ (U_2, b_{21}) \ (U_2, b_{22}) \ (U_1, n_{11}) \ (U_2, n_{12}) \ (U_2, n_{22}) \}$$

$$B = \{ (U_1, b_{11}) \ (U_1, b_{12}) \ (U_2, b_{21}) \ (U_2, b_{22}) \}$$

$$P_r\{B\} = \sum_i P_r\{B | U_i\} \cdot P_r\{U_i\}$$

$$U_1 \cup U_2 = \Omega$$

$$U_1 \cap U_2 = \emptyset$$

$$Pr\{B \mid U_1\} \cdot Pr\{U_1\} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

$$Pr\{B \mid U_2\} \cdot Pr\{U_2\} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$Pr\{B\} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Calcolare la probabilità di aver scelto l'urna 1 sapendo che ho estratto una pallina bianca.

$$Pr\{U_1 \mid B\} = \frac{Pr\{B \mid U_1\} \cdot Pr\{U_1\}}{Pr\{B\}}$$

$$Pr\{B\} = \frac{7}{12}$$

$$Pr\{U_1\} = \frac{1}{2}$$

$$Pr\{B \mid U_1\} = \frac{2}{3}$$

$$Pr\{U_1 \mid B\} = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{12}^{4}}{7} = \frac{4}{7}$$

Esercizio #3

Un trasmettitore trasmette un segnale binario

$Pr\{T_b = 1\} = 0{,}3$

$Pr\{T_b = 0\} = 0{,}7$

$Pr\{E\} = 0{,}01$ probabilità che il bit sia ricevuto erroneamente

Il ricevitore riceve uno 0 logico.
Calcolare la probabilità che sia stato trasmesso uno 0

$$Pr\{T_0 \mid R_0\} = ?$$

Bayes

$$Pr\{T_0 \mid R_0\} = \frac{Pr\{R_0 \mid T_0\}\, Pr\{T_0\}}{Pr\{R_0\}}$$

$$Pr\{R_0 \mid T_0\} = 1 - Pr\{R_0 \mid T_1\} = 1 - 0{,}01 = 0{,}99$$

$$Pr\{T_0\} = 0{,}7$$

$$Pr\{R_0\} = Pr\{R_0 \mid T_0\} \cdot Pr\{T_0\} + Pr\{R_0 \mid T_1\} \cdot Pr\{T_1\} =$$

$$T_1 \cup T_0 = \Omega \qquad T_1 \cap T_0 = \emptyset$$

$$= 0{,}99 \cdot 0{,}7 + 0{,}01 \cdot 0{,}3 = 0{,}696$$

$$\Pr\{T_0 \mid R_0\} = \frac{0{,}99 \cdot 0{,}7}{0{,}696} \cong 0{,}9956$$

Esercizio #4

Lo studente X è sottoposto ad un quiz con M possibili risposte.

Se X ha studiato risponderà correttamente, altrimenti risponderà a caso.

X ha studiato con probabilità p, supponiamo che sottoposto al quiz risponda correttamente.

Qual'è la probabilità che abbia studiato davvero?

$S = \{$ X ha studiato $\}$

$\Pr\{S\} = p$

$C = \{$ X ha risposto correttamente $\}$

$\Pr\{C \mid S\} = 1$

$\bar{S} = \{$ X non ha studiato $\}$

$$\Pr\{C \mid \bar{S}\} = \frac{1}{M}$$

$$S \cap \bar{S} = \emptyset$$

$$S \cup \bar{S} = \Omega$$

$$\Pr\{S \mid C\} = ?$$

Bayes

$$\Pr\{S \mid C\} = \frac{\Pr\{C \mid S\} \cdot \Pr\{S\}}{\Pr\{\varepsilon\}}$$

$$\Pr\{C\} = \Pr\{C \mid S\} \cdot \Pr\{S\} + \Pr\{C \mid \bar{S}\} \cdot \Pr\{\bar{S}\} =$$

$$= 1 \cdot p + \frac{1}{M} \cdot (1-p)$$

$$\Pr\{S \mid C\} = \frac{p}{p + \frac{1}{M}(1-p)}$$

Esercizio #5

Un libro A che ha 200 pagine e un libro B che ha 300 pagine.

Questi due libri sono aperti indipendentemente da due lettori.

Calcolare la probabilità dell'evento

$$\alpha = \{ \text{pag } A > \text{pag } B \} = \{ P_A > P_B \}$$

- I risultati dell'esperimento sono coppie di pagine (P_A, P_B)
- Le coppie favorevoli sono quelle in cui $P_A > P_B$
-

$$\Pr\{P_A > P_B\} = \sum_{n=1}^{300} \Pr\{P_A > P_B \mid P_B = n\} \cdot \Pr\{P_B = n\}$$

$$\Pr\{P_B = n\} = \frac{1}{300}$$

$n \leq 200$ $\qquad \Pr\{P_A > P_B \mid P_B = n\} = \dfrac{200-n}{200}$

$$\Pr\{P_A > P_B\} = \sum_{n=1}^{200} \frac{1}{300} \cdot \frac{200-n}{200} = \frac{1}{300 \cdot 200} \sum_{n=1}^{200} (200-n) =$$

$$= \frac{1}{300 \cdot 200} \sum_{n=0}^{199} n = \frac{1}{300 \cdot 200} \left(\frac{199 \cdot 198}{2} \right)$$

Esercizio # 6

È data una V.A. $X \in \mathcal{N}(0,1)$

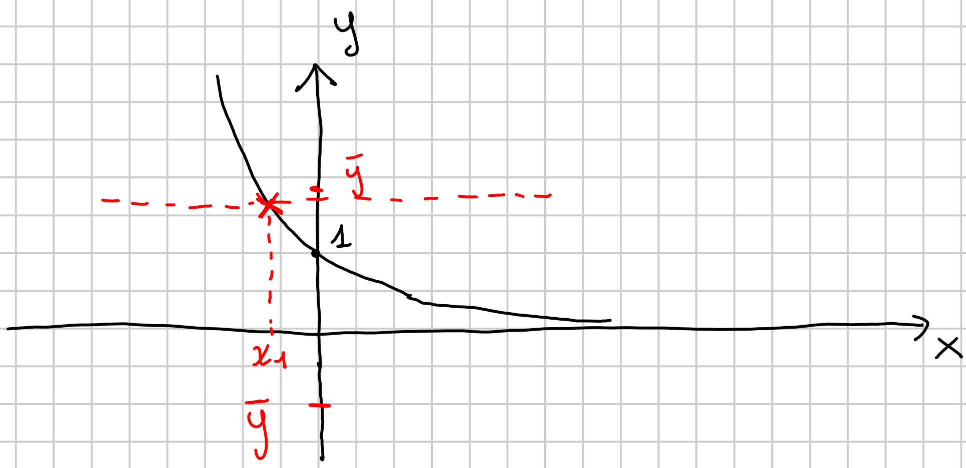
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Determinare $f_Y(y)$ e η_Y della V.A.

$$Y = e^{-X}$$ V.A. LOG NORMALE

$$f_Y(y) = \sum_{i=1}^{m} \frac{f_X(x_i)}{|g'(x_i)|}$$

$$y = g(x_i)$$



Se $\bar{y} < 0$ non ci sono soluzioni $\Rightarrow f_Y(y) = 0 \quad \bar{y} < 0$

Se $\bar{y} > 0$ $\exists$ 1 sola soluzione $x_1 = -\ln(\bar{y})$

$$f_X(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln^2 y}{2}}$$

$$g'(x_1) = -e^{-x_1} = -y$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y^2}} e^{-\frac{\ln^2 y}{2}} u(y)$$

$$\eta_y = E\{Y\} = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\, dy$$

$$\eta_y = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx$$

$$\eta_y = \int_{-\infty}^{+\infty} e^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}+x\right)} dx =$$

$$\frac{x^2}{2} + x = \frac{1}{2}\left[(x+1)^2 - 1\right]$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{+\frac{1}{2}} \cdot e^{-\frac{1}{2}(x+1)^2} dx =$$

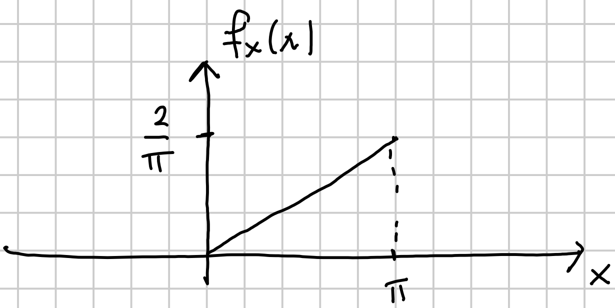
$$= e^{1/2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} dx}_{\mathcal{N}(-1,\,1)} = e^{1/2}$$

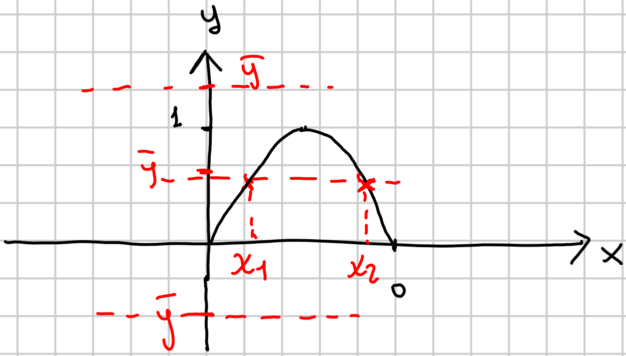
Esercizio #6

Sia X una v.A. tale che

$$f_X(x) = \frac{2x}{\pi^2} \, \mathrm{rect}\left(\frac{x-\pi/2}{\pi}\right)$$

Sia $Y = \sin(X)$ Trovare $f_Y(y)$ e η_Y





·) se $\bar{y} > 1$ e $\bar{y} < 0$ non esistono soluzione dell'equazione

$y = g(x) \Rightarrow f_Y(y) = 0 \quad y > 1$ e $y < 0$

·) se $0 \leq \bar{y} \leq 1$

$x_1 = \arcsin(y)$

$x_2 = \pi - x_1$

·) $g(x) = \sin(x)$

$g'(x) = \cos(x)$

$g'(x_1) = \cos(x_1)$

$g'(x_2) = \cos(\pi - x_1) = -\cos(x_1)$

$$f_X(x_1) = \frac{2x_1}{\pi^2}$$

$$f_X(x_2) = \frac{2(\pi - x_1)}{\pi^2}$$

$$f_Y(y) = \frac{2x_1}{\pi^2} \cdot \frac{1}{\cos(x_1)} + \frac{2(\pi - x_1)}{\pi^2 \cdot \cos(x_1)} = \frac{2\cancel{\pi}}{\pi^2 \cdot \cos x_1} = \frac{2}{\pi \cos x_1} =$$

$$= \frac{2}{\pi \cos(\arcsin(y))}$$

$$\eta_y = \int_{-\infty}^{+\infty} y \cdot f_y(y)\, dy$$

$$\eta_y = \int_{-\infty}^{+\infty} g(x) \cdot f_x(x)\, dx$$

$$\eta_y = \int_0^1 y \cdot \frac{2}{\pi \cos(\operatorname{asin}(y))} \, dy$$

$$z = \operatorname{asin}(y)$$

$$y = \sin z$$

$$dy = \cos z \cdot dz$$

$$y = 0 \Rightarrow z = 0$$

$$y = 1 \Rightarrow z = \pi/2$$

$$= \int_0^{\pi/2} \sin z \, \frac{2}{\pi \cdot \cancel{\cos z}} \cdot \cancel{\cos z}\, dz =$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sin z \, dz = -\frac{2}{\pi} \int_0^{\pi/2} -\sin(z)\, dz = -\frac{2}{\pi} \cos(z) \Big|_0^{\pi/2}$$

$$= -\frac{2}{\pi} \Big(0 - (1) \Big) = \frac{2}{\pi}$$

Esercizio # 7

È data una V. A. $X \in \mathcal{N}(2, 2)$

Si costruisce una variabile

$$Y = \begin{cases} 1 & \text{se } X > 0 \\ -1 & \text{se } X < 0 \end{cases}$$

Calcolare η_y e σ_y^2

$$\eta_y = \sum_{i=1}^{n} y_i \cdot P_i = 1 \cdot P_1 - 1 \cdot P_{-1}$$

$$P_1 = P_r\{Y = 1\}$$

$$P_{-1} = P_r\{Y = -1\}$$

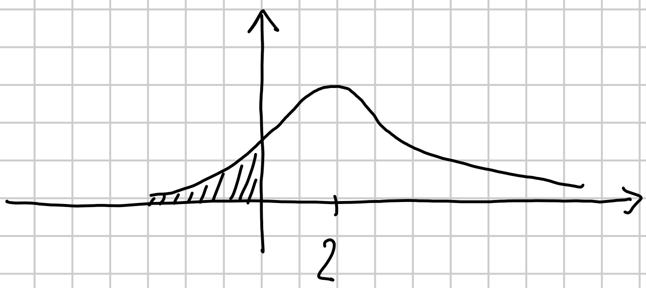
$$P_1 = \int_0^{+\infty} f_x(x)\, dy = P_r\{X > 0\}$$

$$P_{-1} = 1 - P_1$$

$$\eta_y = \int_{-\infty}^{+\infty} g(x) \cdot f_x(x)\, dx =$$

$$y = g(x) = \mathrm{sgn}(x)$$

$$\eta_y = -\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{+\infty} f_x(x)\,dx$$



$$\int_{-\infty}^{0} f_x(x)\,dx = \Phi\left(\frac{0-2}{\sqrt{2}}\right) = \Phi\left(-\frac{2}{\sqrt{2}}\right)$$

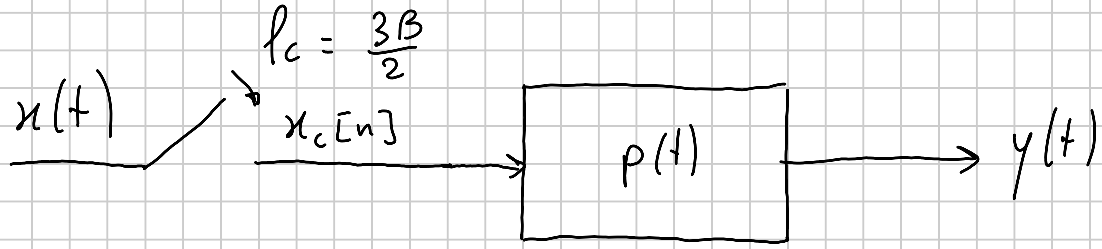
$$\int_{0}^{+\infty} f_x(x)\,dx = 1 - \Phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\eta_y = -\Phi\left(-\frac{2}{\sqrt{2}}\right) + 1 - \Phi\left(-\frac{2}{\sqrt{2}}\right) = 1 - 2\,\Phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\sigma_y^2 = E\left\{(y-\eta_y)^2\right\} = \int_{-\infty}^{+\infty} (y-\eta_y)^2 f_y(y)\,dy =$$

$$= \sum_i (y - \eta_y)^2 \cdot p_i = (1 - \eta_y)^2 \cdot p_1 + (-1 - \eta_y)^2 \cdot p_{-1}$$

ES. 1



$p(t)$ è un interp. cardinale di banda B

$x(t) = B \operatorname{sinc}^2(Bt)$

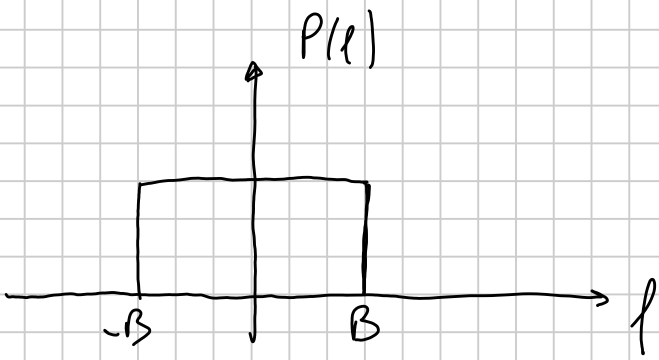
a) $y(t) = ?$

b) $E_y, P_y = ?$

c) $x_c[n]$, $\bar{X}_c(f)$ quando $f_c = B$

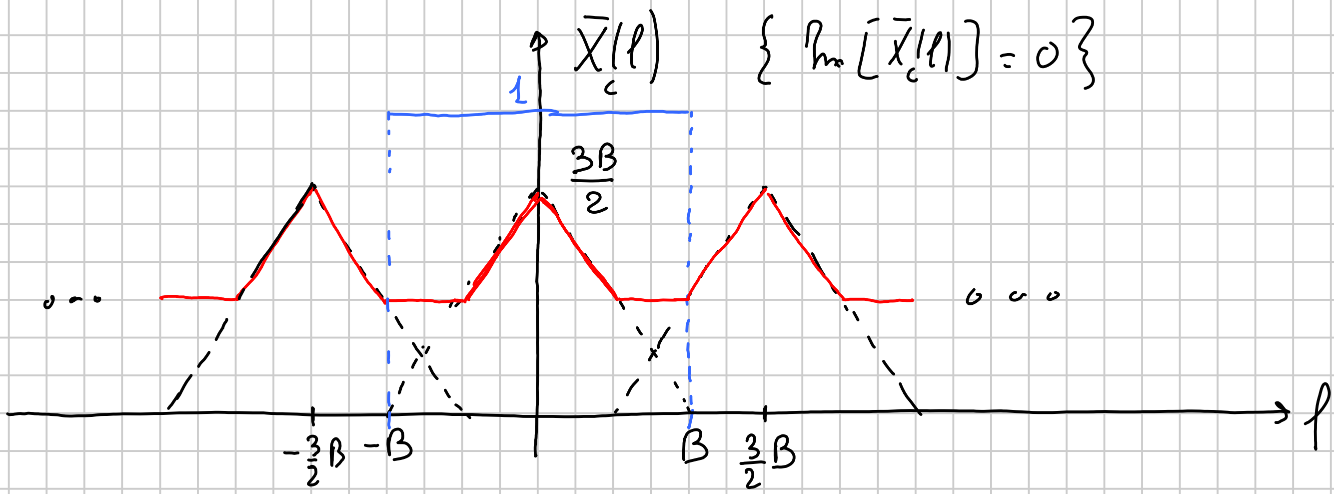
SOLUZIONE

$$P(f) = \operatorname{rect}\left(\frac{f}{2B}\right)$$

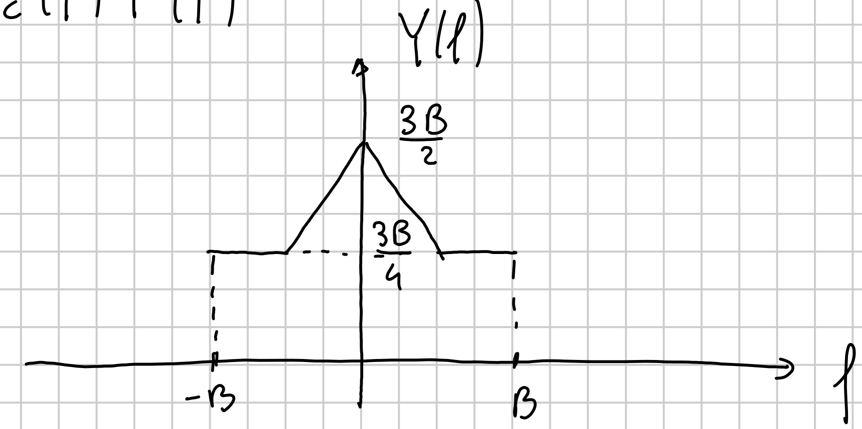


$$X(f) = \left(1 - \frac{|f|}{B}\right) \operatorname{rect}\left(\frac{f}{2B}\right)$$

$$\bar{X}_c(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right) \quad , \quad T = \frac{1}{f_c} = \frac{2}{3B}$$



$$Y(f) = \bar{X}_c(f)\, P(f)$$



$$y(t) = TCF^{-1}\left[Y(f) \right]$$

$$Y(f) = Y_1(f) + Y_2(f)$$

$$Y_1(f) = \frac{3B}{4} \operatorname{rect}\left(\frac{f}{2B}\right) \quad , \quad Y_2(f) = \frac{3B}{4}\left(1 - \frac{|f|}{B/2}\right) \operatorname{rect}\left(\frac{f}{B}\right)$$

$$y_1(t) = \frac{3B}{4} \cdot 2B \operatorname{sinc}(2Bt) = \frac{3B^2}{2} \operatorname{sinc}(2Bt)$$

$$y_2(t) = \frac{3B}{4} \cdot \frac{B}{2} \operatorname{sinc}^2\left(\frac{B}{2}t\right) = \frac{3B^2}{8} \operatorname{sinc}^2\left(\frac{B}{2}t\right)$$

$$y(t) = y_1(t) + y_2(t)$$

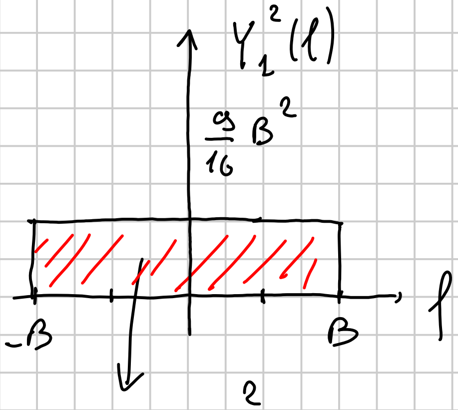
b) $E_y = ?$, $P_y = ?$

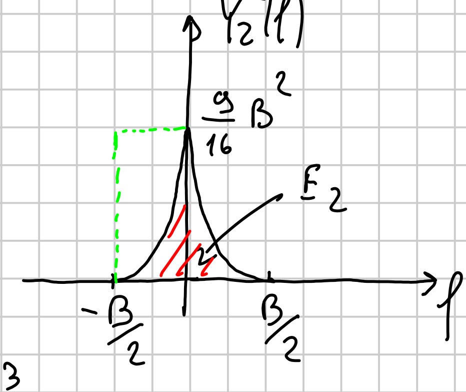
$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2\, dt = \int_{-\infty}^{+\infty} |Y(f)|^2\, df$$

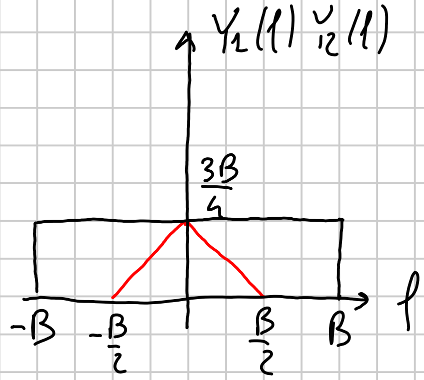
$$Y(f) = Y_1(f) + Y_2(f)$$

$$|Y(f)|^2 = Y^2(f) = Y_1^2(f) + Y_2^2(f) + 2\, Y_1(f)\, Y_2(f)$$

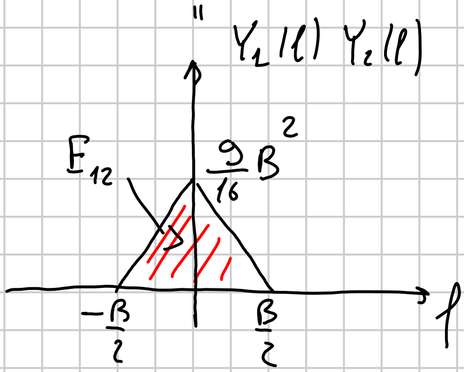
$$E_y = \int_{-\infty}^{+\infty} Y_1^2(f)\, df + \int_{-\infty}^{+\infty} Y_2^2(f)\, df + 2\int_{-\infty}^{+\infty} Y_1(f)\, Y_2(f)\, df$$







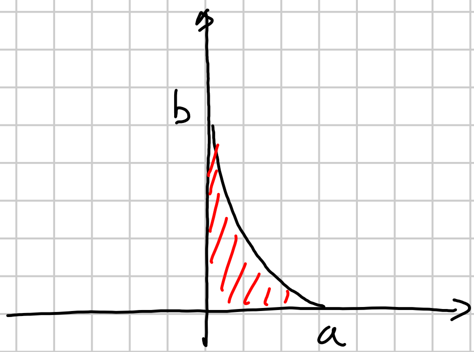
$$E_1 = \frac{9}{16}B^2 \cdot 2B = \frac{9}{8}B^3$$



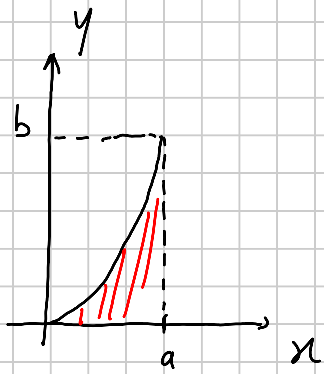
$$E_2 = \frac{2}{3}\,\frac{9}{16}B^2 \cdot \frac{B}{2} = \frac{3}{16}B^3$$

$$E_{12} = \frac{9}{16}B^2 \cdot \frac{B}{2} = \frac{9}{32}B^3$$

$$E = E_1 + E_2 + 2E_{12} = \left(\frac{9}{8} + \frac{3}{16} + \frac{9}{16}\right)B^3 = \frac{15}{8}B^3$$



$\Rightarrow$



$\Downarrow$

$$y = a_1x^2 + bx + c \qquad b = c = 0$$

$$y = a_1x^2$$

$$\int_0^a y(x)\,dx = \int_0^a a_1x^2\,dx = \int_0^a \frac{b}{a^2}x^2\,dx = \frac{b}{a^2}\,\frac{x^3}{3}\Big|_0^a =$$

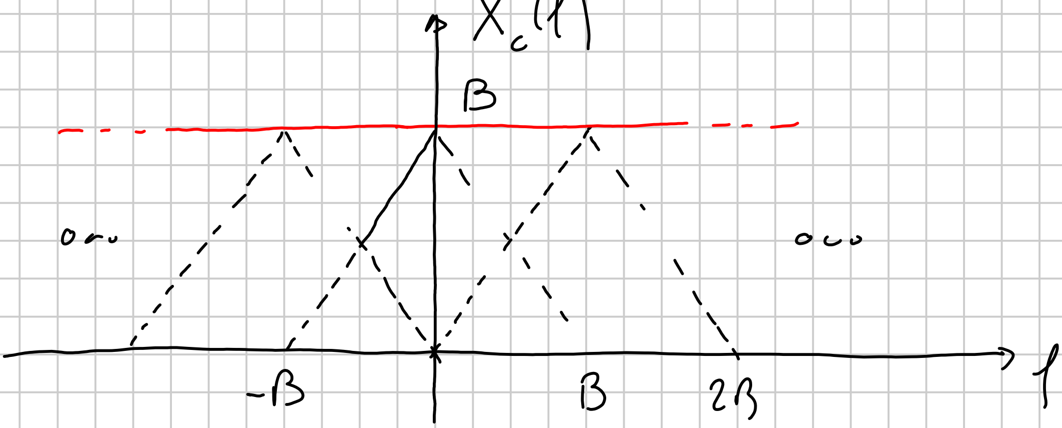
$$= \frac{b}{a^2}\,\frac{a^3}{3} = \boxed{\frac{ab}{3}}$$

$a_1 = ?$ $\qquad x = a \Rightarrow y = b$

$$b = a_1a^2 \Rightarrow a_1 = \frac{b}{a^2}$$

$$E_y < +\infty \Rightarrow P_y = 0$$

c)



$$\bar{X}_c(f) = B \implies x_c[n] = B\,\delta[n]$$

$$\bar{X}_c(f) = \sum_{n=-\infty}^{+\infty} x_c[n]\, e^{-j2\pi f n T} =$$

$$= \sum_{n=-\infty}^{+\infty} B\,\delta[n]\, e^{-j2\pi f n T} = B$$

ESERCIZIO (2)

$X \in \mathcal{U}[0,1]$, $Y \in \mathcal{U}[0,1]$ e indipendenti

$f_{Z|A}(z|A) = ?$

$$\begin{cases} Z = X - Y \\ V = X + Y \end{cases} \quad , \quad A \equiv \{V \leq 1\}$$

SOLUZIONE

$$f_{Z|A}(z|A) \triangleq \frac{d}{dz} F_{Z|A}(z|A)$$

$$F_{Z|A}(z|A) \triangleq \frac{P\{Z \leq z, A\}}{P\{A\}}$$

$$f_{ZV}(z,v) = \sum_{n=1}^{N} \frac{f_{XY}(x_i, y_i)}{|\det \underline{\underline{J}}(x_i;y_i)|} \quad , \quad (x_i, y_i) \text{ sol. inv.}$$

SOL. DEL. PROB. INV.

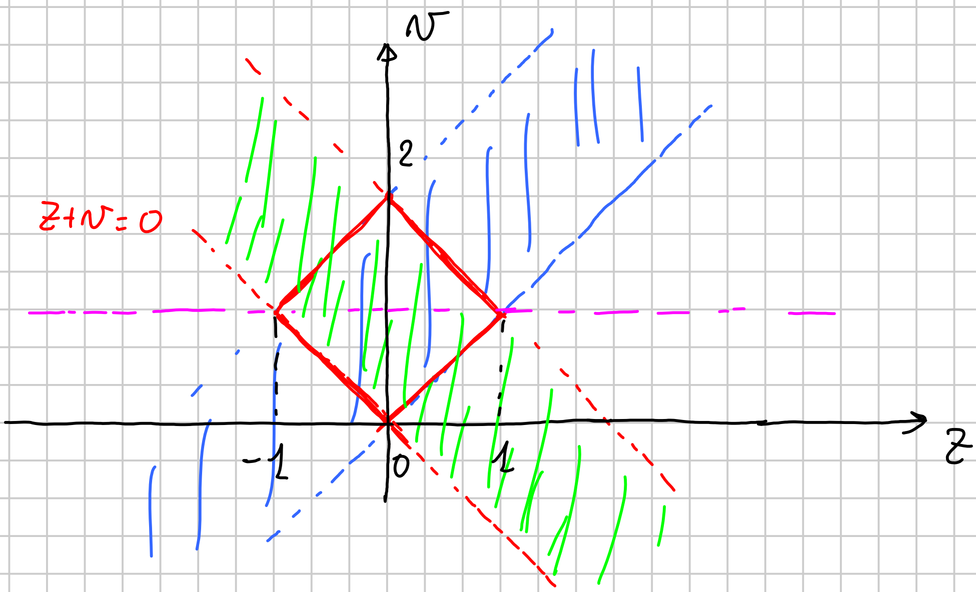
$$\begin{cases} x = \dfrac{z+v}{2} \\ \\ y = \dfrac{v-z}{2} \end{cases}$$

$$\underline{\underline{J}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow |\det\{\underline{\underline{J}}\}| = 2$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \mathrm{rect}\left(\frac{x-1/2}{1}\right) \mathrm{rect}\left(\frac{y-\frac{1}{2}}{1}\right)$$

$$f_{ZV}(z,v) = \frac{1}{2} \cdot \mathrm{rect}\left(\frac{\frac{z+v}{2} - \frac{1}{2}}{1}\right) \mathrm{rect}\left(\frac{\frac{v-z}{2} - \frac{1}{2}}{1}\right)$$

$$= \frac{1}{2}\, \mathrm{rect}\left(\frac{z+v-1}{2}\right) \mathrm{rect}\left(\frac{v-z-1}{2}\right)$$



$$-1 \leq z+v-1 \leq +1 \quad \Rightarrow \quad 0 \leq z+v \leq 2$$

$$-1 \leq v-z-1 \leq 1 \quad \Rightarrow \quad 0 \leq v-z \leq 2$$

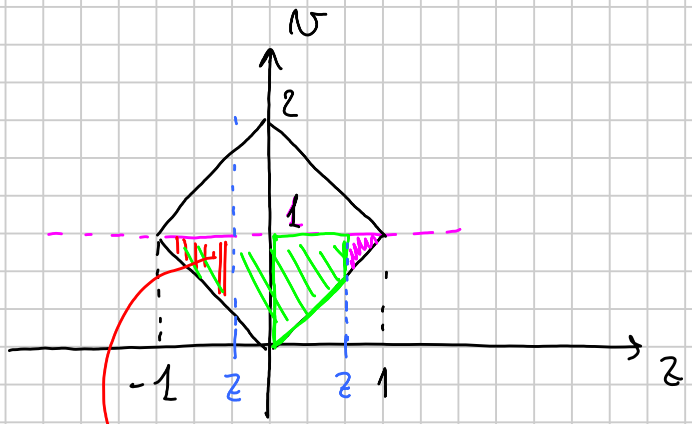
$$P\{A\} = P\{V \leq 1\} = \frac{1}{2} = \int_{-\infty\,(z)}^{+\infty} \int_{-\infty\,(v)}^{1} f_{ZV}(z,v)\, dz\, dv = F_V(1)$$

$$\int_{-\infty}^{+\infty} f_{ZV}(z,v)\, dz = f_V(v)$$

$$P\{V \leq 1\} = \int_{-\infty}^{1} f_V(v)\, dv = F_V(1)$$

$$P\{Z \leq z, A\} = \int_{-\infty}^{z} \int_{-\infty}^{1} f_{ZV}(z,v)\, dz\, dv =$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{1} \frac{1}{2} \operatorname{rect}\left(\frac{z+v-1}{2}\right) \operatorname{rect}\left(\frac{v-z-1}{2}\right) dz\, dv$$



$-1 \leq z \leq 0$

$$P\{Z \leq z, V \leq 1\} = \frac{(1+z)^2}{2} \cdot \frac{1}{2}$$

$0 \leq z \leq 1$

$$P\{Z \leq z, V \leq 1\} = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2} - \frac{(1-z)^2}{2}\right] = \left[1 - \frac{(1-z)^2}{2}\right] \frac{1}{2}$$

$$F_{Z|A}(z|A) = \frac{1}{1/2} \cdot \begin{cases} \frac{(1+z)^2}{4} & -1 \leq z \leq 0 \\ \frac{1}{2}\left(1 - \frac{(1-z)^2}{2}\right) & 0 \leq z \leq 1 \end{cases}$$

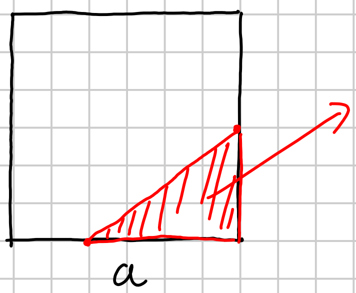
$$F_{Z|A}(z|A) = \begin{cases} \frac{(1+z)^2}{2} & -1 \leq z \leq 0 \\ 1 - \frac{(1-z)^2}{2} & 0 \leq z \leq 1 \end{cases}$$

$$f_{Z|A}(z|A) = \frac{d}{dz} F_{Z|A}(z|A) =$$

$$= \begin{cases} 1+z & -1 \leq z \leq 0 \\ 1-z & 0 \leq z \leq 1 \end{cases}$$

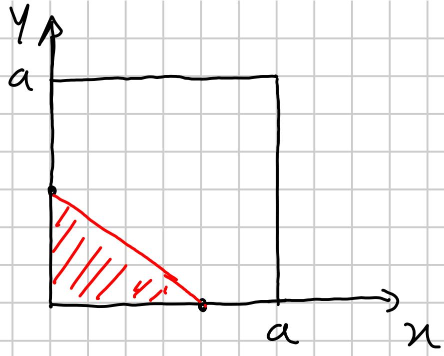
$$f_{Z|A}(z|A) = \begin{cases} 1 - |z| & -1 \leq z \leq 1 \\ 0 & \end{cases}$$

ESERCIZIO 3



$A_T < \frac{1}{8} A_Q$

$A_Q = a^2 \implies P\left\{A_T < \frac{a^2}{8}\right\}$



X, Y ind. $\in \mathcal{U}[0, a]$

$\Rightarrow a = 1$

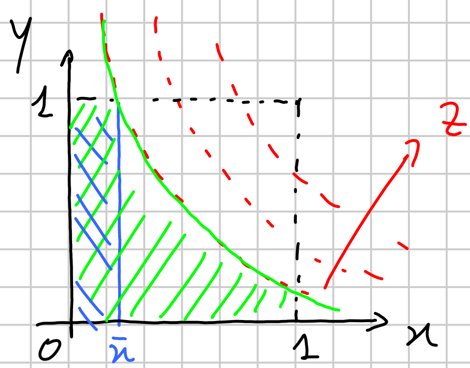
$$X = a \cdot X_1 \quad , \quad X_1 \in \mathcal{U}[0,1]$$
$$Y = a \cdot Y_1 \quad , \quad Y_1 \in \mathcal{U}[0,1]$$

$$A_T = \frac{XY}{2} = \frac{aX_1 \cdot aY_1}{2} = a^2 \frac{X_1 Y_1}{2}$$

$$P\left\{ A_T < \frac{1}{8} A_Q \right\} = P\left\{ a^2 \frac{X_1 Y_1}{2} < \frac{1}{8} a^2 \right\} =$$
$$= P\left\{ \frac{X_1 Y_1}{2} < \frac{1}{8} \right\}$$
$$= P\left\{ X_1 Y_1 < \frac{1}{4} \right\}$$

$Z = X_1 Y_1$

$$P\left\{ Z < \frac{1}{4} \right\} = F_Z\left(\frac{1}{4}\right)$$



$P\{ Z \le z) =$

$= A_{verde} \cdot 1$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \mathrm{rect}\left(\frac{x-1/2}{1}\right) \mathrm{rect}\left(\frac{y-1/2}{1}\right)$$

$$x \cdot y = \frac{1}{4} \Rightarrow y = \frac{1}{4x}$$

$$\bar{x} \Rightarrow 1 = \frac{1}{4\bar{x}} \Rightarrow \bar{x} = \frac{1}{4}$$

$$A_{verde} = \bar{x} \cdot 1 + \int_{\bar{x}}^{1} \frac{1}{4x}\, dx =$$

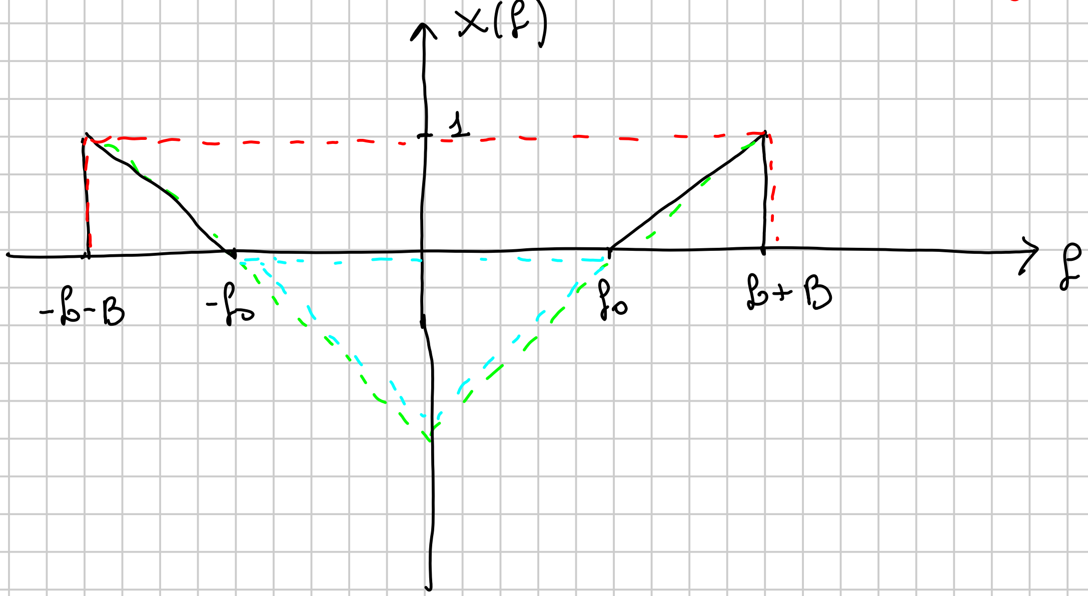
$$= \bar{x} + \frac{1}{4} \ln x \Big|_{\bar{x}}^{1} = \bar{x} + \frac{1}{4}\left[\ln(1) - \ln(\bar{x})\right]$$

$$P\left(Z \leq \frac{1}{4}\right) = P\left\{A_T < \frac{1}{8} A_Q\right\} = \frac{1}{4}\left[1 - \ln(\bar{x})\right]$$

Compitino 2003

Esercizio 1

Si calcoli la TCF^{-1} dello spettro in figura.

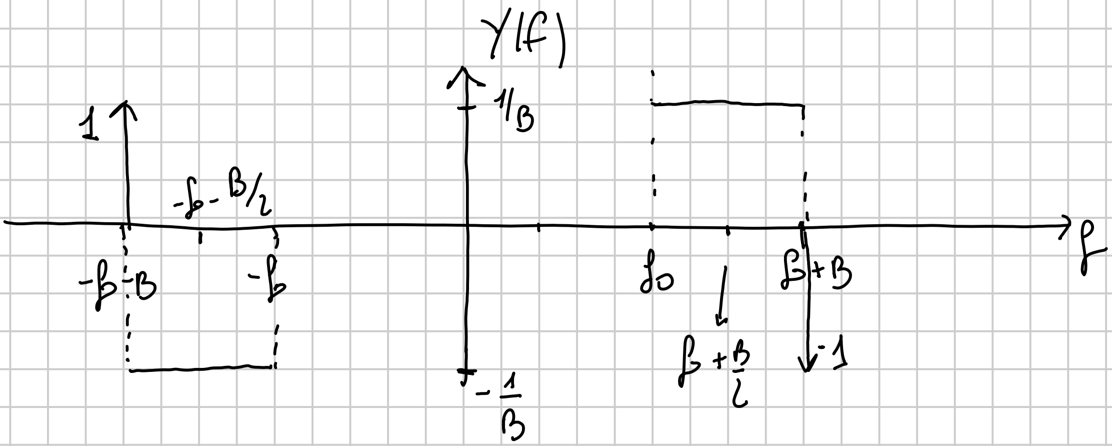


Teorema della derivata

$$Y(f) = \frac{\partial}{\partial f} X(f) \rightleftharpoons y(t) = -j2\pi t \, x(t)$$

Se $y(t)\Big|_{t=0} = 0$ allora $x(t) = -\dfrac{y(t)}{j2\pi t}$

$$y(0) = 0 \qquad \int_{-\infty}^{+\infty} Y(f)\, df = 0$$



$$Y(f) = \delta\left(f + f_0 + B\right) - \delta\left(f - (f_0 + B)\right) -$$

$$- \frac{1}{B} \operatorname{rect}\left(\frac{f + (f_0 + B/2)}{B}\right) + \frac{1}{B} \operatorname{rect}\left(\frac{f - (f_0 + B/2)}{B}\right)$$

$$y(t) = -2j \sin\left(2\pi(f_0 + B)t\right) - \operatorname{sinc}(tB)\, e^{-j2\pi t(f_0 + B/2)} + \operatorname{sinc}(tB)\, e^{+j2\pi t(f_0 + B/2)} =$$

$$= -2j \sin\left(2\pi t\,(f_0 + B)\right) + \operatorname{sinc}(tB)\left(e^{+j2\pi t(f_0 + B/2)} - e^{-j2\pi t(f_0 + B/2)}\right) =$$

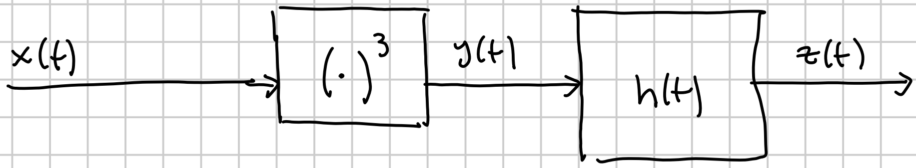
$$= -2j \sin\left(2\pi t\,(f_0 + B)\right) + 2j \operatorname{sinc}(tB) \cdot \sin\left(2\pi t\left(f_0 + \frac{B}{2}\right)\right)$$

$$x(t) = -\frac{y(t)}{2j\pi t} =$$

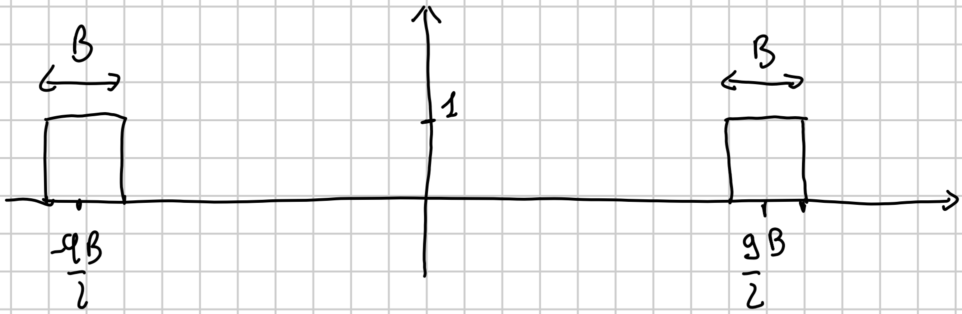
$$= \frac{\sin\left(2\pi t\,(f_0 + B)\right)}{\pi t} - \operatorname{sinc}(tB)\, \frac{\sin\left(2\pi t\,(f_0 + B/2)\right)}{\pi t} =$$

$$= 2(f_0 + B) \operatorname{sinc}\left(2t\,(f_0 + B)\right) - 2\left(f_0 + \frac{B}{2}\right) \operatorname{sinc}(tB) \cdot \operatorname{sinc}\left(2t\,(f_0 + B/2)\right)$$

Esercizio 2



$$x(t) = A\cos(3\pi B t + \theta)$$



- Calcolare $Y(f)$ e $y(t)$ e rappresentare i grafici di modulo e fase

- Determinare l'espressione di $z(t)$

$$y(t) = x^3(t) = A^3\cos^3(3\pi B t + \theta) =$$

$$= A^3\cos(3\pi B t + \theta)\cdot\cos^2(3\pi B t + \theta) =$$

$$= A^3\cos(3\pi B t + \theta)\left(\frac{1}{2} + \frac{1}{2}\cos(6\pi B t + 2\theta)\right) =$$

$$= \frac{A^3}{2}\cos(3\pi B t + \theta) + \frac{A^3}{2}\cos(3\pi B t + \theta)\cos(6\pi B t + 2\theta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$= \frac{A^3}{2}\cos(3\pi Bt+\theta) + \frac{A^3}{2}\left(\frac{1}{2}\cos(9\pi Bt+3\theta) + \frac{1}{2}\cos(3\pi Bt+\theta)\right)$$

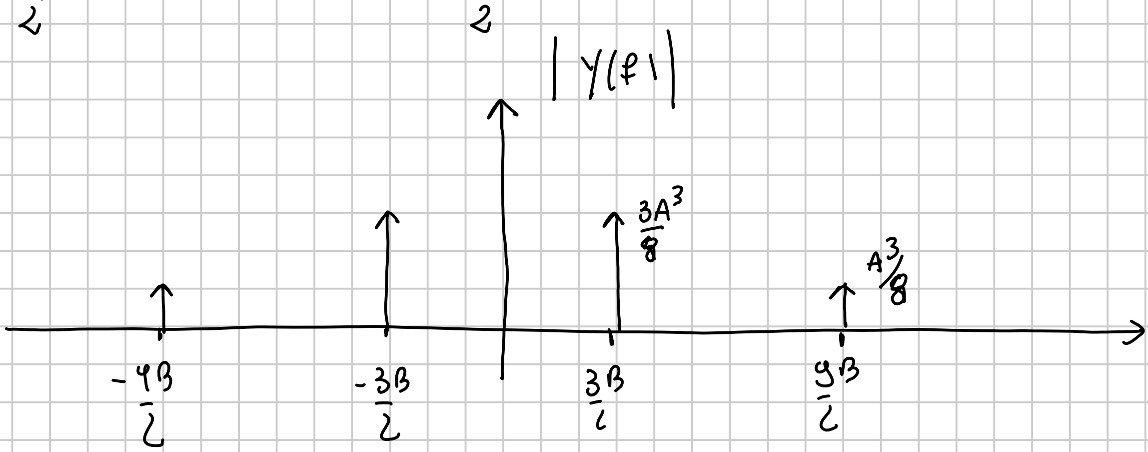
$$= \cos(3\pi Bt+\theta)\left(\frac{A^3}{2}+\frac{A^3}{4}\right) + \frac{A^3}{4}\cos(9\pi Bt+3\theta) =$$

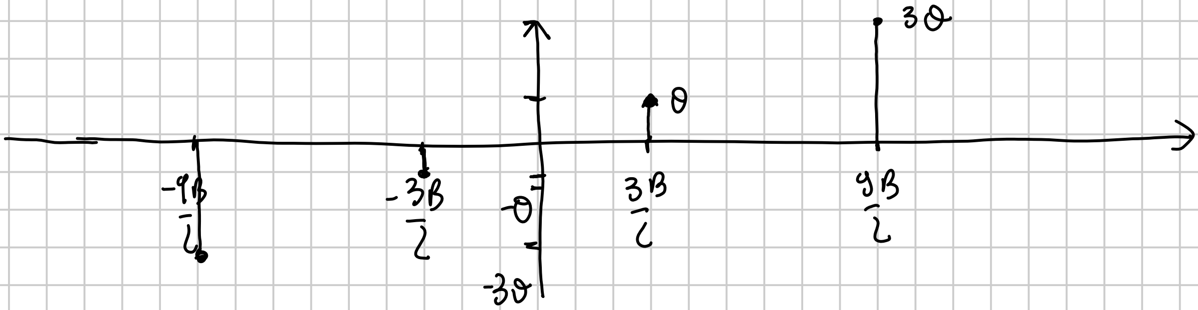
$$= \cos(\underbrace{3\pi Bt+\theta}_{2\pi \frac{3B}{2} t + \theta})\,\frac{3A^3}{4} + \frac{A^3}{4}\cos(\underbrace{9\pi Bt+3\theta}_{2\pi \frac{9B}{2} t + 3\theta})$$

$$Y(f) = \frac{3A^3}{4}\left(\frac{\delta(f-f_{01})e^{j\theta} + \delta(f+f_{01})e^{-j\theta}}{2}\right) +$$

$$\frac{A^3}{4}\left(\frac{\delta(f-f_{02})e^{j3\theta} + \delta(f-f_{02})e^{-j3\theta}}{2}\right)$$

$$f_{01} = \frac{3B}{2} \qquad f_{02} = \frac{9B}{2}$$



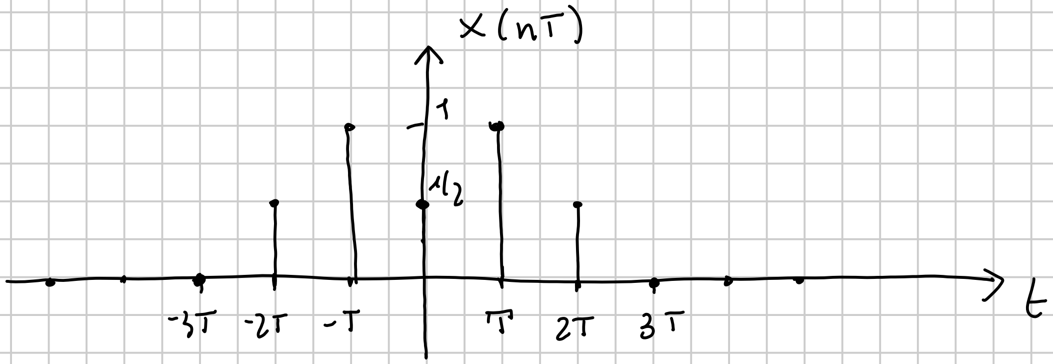


$$Z(f) = \frac{A^3}{8} \left(\delta\left(f - \frac{9B}{2}\right) e^{j3\theta} + \delta\left(f + \frac{9B}{2}\right) e^{-j3\theta} \right)$$

$$z(t) = \frac{A^3}{4} \cos\left(2\pi \frac{9B}{2} t + 3\theta \right)$$

Esercizio 3

Si calcoli la Trasformata Discreta di Fourier (TDF), $\overline{X}(f)$ della sequenza in figura.



$$x[n] = \frac{1}{2}\delta[n] + 1\,\delta[n-1] + 1\,\delta[n+1] + \frac{1}{2}\delta[n-2] + \frac{1}{2}\delta[n+2]$$

Teorema del ritardo

$$\overline{X}(f) = \frac{1}{2} + e^{-j2\pi fT} + e^{+j2\pi fT} + \frac{1}{2} e^{-j2\pi f\cdot 2T} + \frac{1}{2} e^{j2\pi f 2T}$$

$$= \frac{1}{2} + 2\cos(2\pi f T) + \cos(2\pi f 2T)$$

COMPITINO 19- APRILE 2010

Esercizio 1

SIA DATO IL SISTEMA

$$y(t) = |x(t)| + \int_a^t x(\alpha)\, d\alpha$$

DIRE SE IL SISTEMA È:

- LINEARE, STAZIONARIO, CON MEMORIA, STABILE, CAUSALE

- LINEARE

$$x(t) = a x_1(t) + b x_2(t)$$

$$T[x(t)] = |a x_1(t) + b x_2(t)| + \int_a^t (a x_1(t) + b x_2(t))\, dt \neq$$

$$\neq a|x_1(t)| + b|x_2(t)| + \int_a^t a x_1(t)\, dt + \int_a^t b x_2(t)\, dt = a y_1(t) + b y_2(t)$$

NON È LINEARE

- STAZIONARIO

$x(t-t_0) \Rightarrow y(t-t_0)$

$\alpha - t_0 = \alpha'$

$$T\left[x(t-t_0)\right] = |x(t-t_0)| + \int_b^t x(\alpha - t_0)\, d\alpha =$$

$$= |x(t-t_0)| + \int_{b-t_0}^{t-t_0} x(\alpha')\, d\alpha'$$

$$y(t-t_0) = |x(t-t_0)| + \int_b^{t-t_0} x(\alpha)\, d\alpha$$

NON È STAZIONARIO

- MEMORIA

Il sistema ha memoria perché l'uscita dipende da valori precedenti a t.

- CAUSALE

Se $b < t$

Il sistema è causale perché dipende solo da istanti precedenti a t.

- STABILITÀ

$|x(t)| < M$ per ogni t

$$y(t) = |x(t)| + \int_b^t x(\alpha)\,d\alpha$$

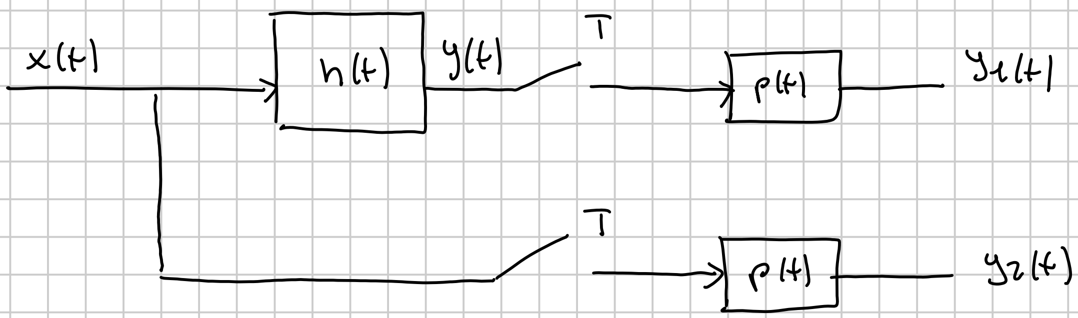
$$|y(t)| = \left| |x(t)| + \int_b^t x(\alpha)\,d\alpha \right| \leq |x(t)| + \left| \int_b^t x(\alpha)\,d\alpha \right| \leq$$

$$|x(t)| + \int_b^t |x(\alpha)|\,d\alpha \leq M + \int_b^t M\,d\alpha \leq N \quad \text{per } t < \infty$$

$$M + (t-b)M = K \quad t < \infty$$

NON STABILE

Esercizio 2



$$x(t) = 2AB\,\mathrm{sinc}^2(2Bt)$$

$$h(t) = 2B\,\mathrm{sinc}(2Bt)$$

$p(t)$ interpolatore cardinale di banda B

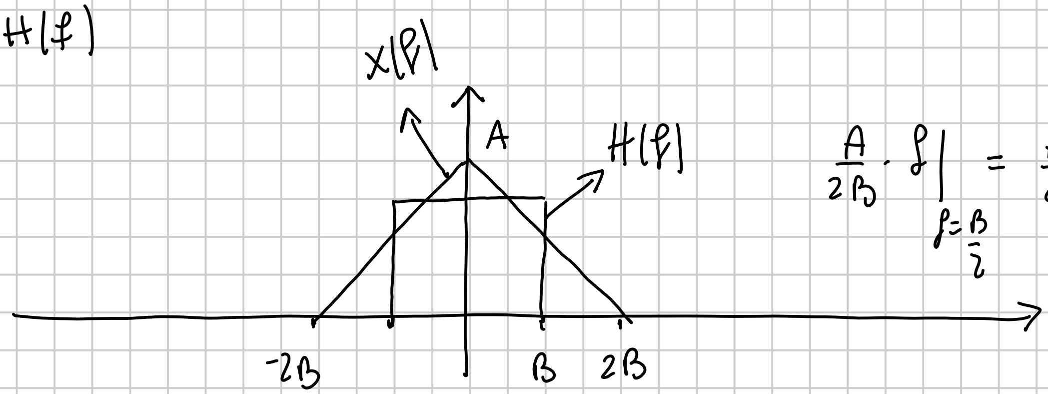
$$T = \frac{1}{2B}$$

- Calcolare l'espressione analitica di $y_1(t)$ e $y_2(t)$
- Calcolare l'energia di $y_1(t)$ e $y_2(t)$
- Calcolare la potenza di $y_1(t)$ e $y_2(t)$

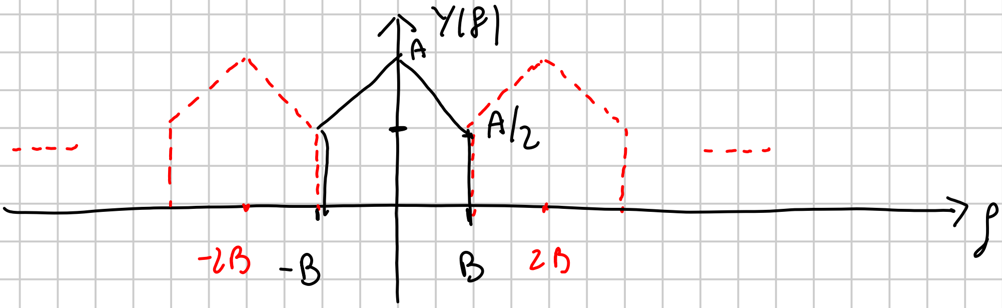
$$X(f) = A\left(1 - \frac{|f|}{2B}\right) \text{rect}\left(\frac{f}{4B}\right)$$

$$H(f) = \text{rect}\left(\frac{f}{2B}\right)$$

$$Y(f) = X(f)\,H(f)$$

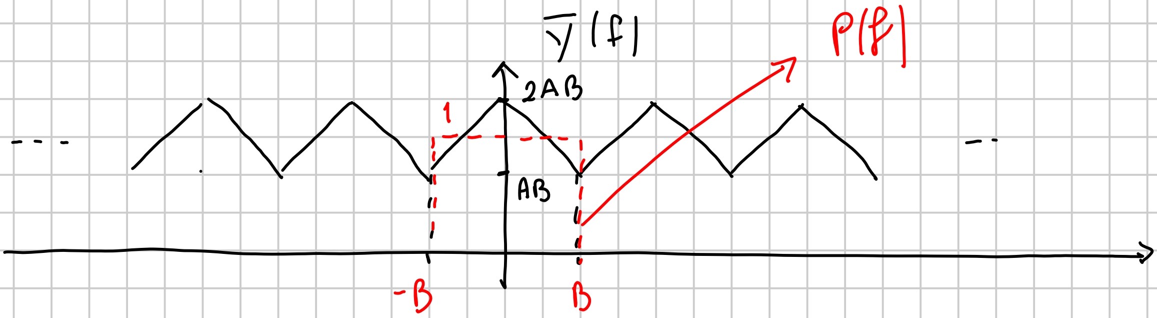


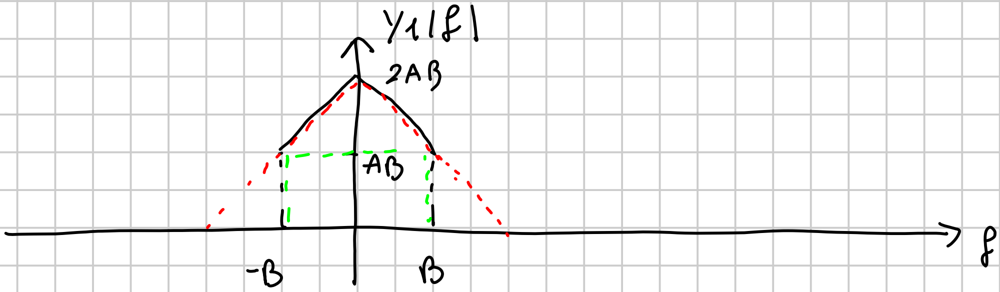
$$\frac{A}{2B} \cdot f\Big|_{f=\frac{B}{2}} = \frac{A}{2}$$



$$T = \frac{1}{2B} \qquad \bar{Y}(f) = \frac{1}{T} \sum_n Y\left(f - \frac{n}{T}\right) =$$

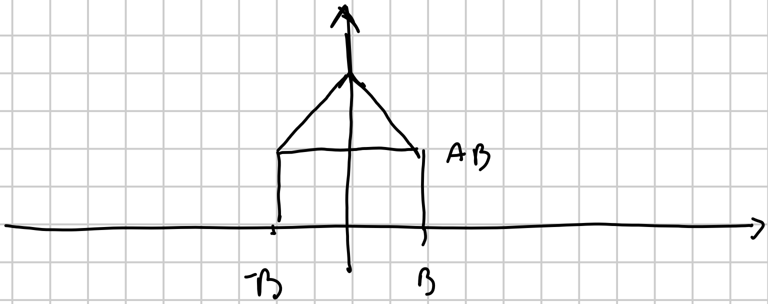
$$= 2B \sum_n Y(f - n2B)$$





$$Y_1(f) = 2AB\left(1-\frac{|f|}{2B}\right)\text{rect}\left(\frac{f}{2B}\right)\cdot\text{rect}\left(\frac{f}{2B}\right) =$$

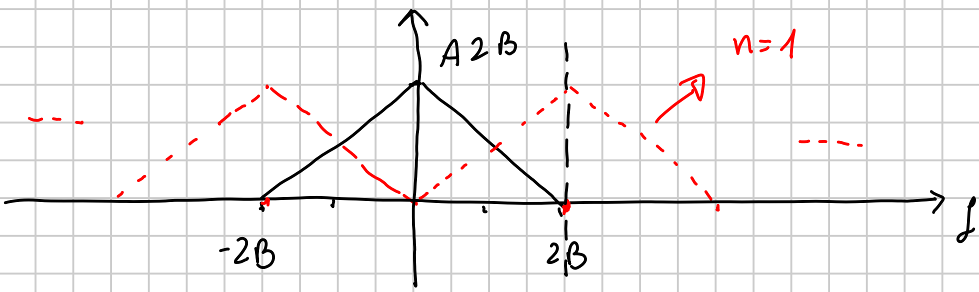
$$= 2AB\left(1-\frac{|f|}{2B}\right)\text{rect}\left(\frac{f}{2B}\right)$$

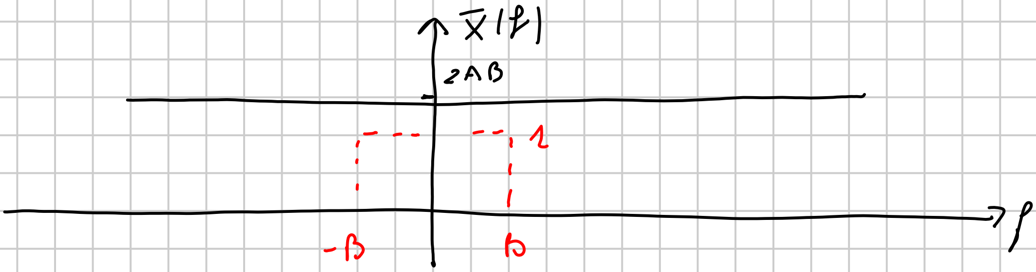


$$Y_1(f) = AB\,\text{rect}\left(\frac{f}{2B}\right) + AB\left(1-\frac{|f|}{B}\right)\text{rect}\left(\frac{f}{2B}\right)$$

$$y_1(t) = A2B^2\,\text{sinc}(t2B) + AB^2\,\text{sinc}^2(tB)$$

$$\bar{X}(f) = 2B\sum_n X(f-2Bn)$$





$$Y_2(f) = \overline{X}(f)\; P(f) \qquad P(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Y_2(f) = 2AB \;\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$y_2(t) = 4AB^2 \;\mathrm{sinc}\left(t\,2B\right)$$

2)

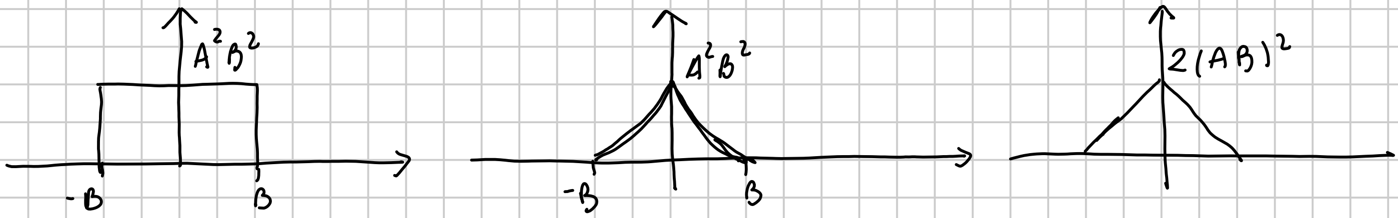
$$Y_1(f) = AB\,\mathrm{rect}\left(\frac{f}{2B}\right) + AB\left(1-\frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Y_2(f) = 2AB\;\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$E_{y_1} = \int_{-\infty}^{+\infty} |Y_1(f)|^2\, df = \int_{-\infty}^{+\infty} (Y_1(f))^2\, df =$$

$$= \int_{-B}^{B} (AB)^2\, df + 2\int_{0}^{B} (AB)^2\left(1-\frac{f}{B}\right)^2 df +$$

$$\int_{-B}^{B} 2\,(AB)^2\left(1-\frac{|f|}{B}\right) df$$



$$\int_{-B}^{B} A^2B^2\, df = 2B \cdot A^2B^2$$

$$2\int_{0}^{B} A^2B^2 \left(1-\frac{f}{B}\right)^2 df = 2A^2B^2 \int_{0}^{B} \left(1+\frac{f^2}{B^2}-\frac{2f}{B}\right) df =$$

$$= 2A^2B^2 \left[f + \frac{f^3}{3B^2} - \frac{\cancel{2}f^2}{\cancel{2}B} \right]_0^B = 2A^2B^2 \left[\cancel{B} + \frac{B}{3} - \cancel{B} \right] =$$

$$= 2A^2B^2 \left[\frac{B}{3} \right] = \frac{2}{3} A^2B^3$$

$$4\int_{0}^{B} A^2B^2 \left(1-\frac{f}{B}\right) df = 4A^2B^2 \int_{0}^{B} \left(1-\frac{f}{B}\right) df =$$

$$= 4A^2B^2 \left[f - \frac{f^2}{2B} \right]_0^B = 4A^2B^2 \left[B - \frac{B}{2} \right] = 2A^2B^3$$

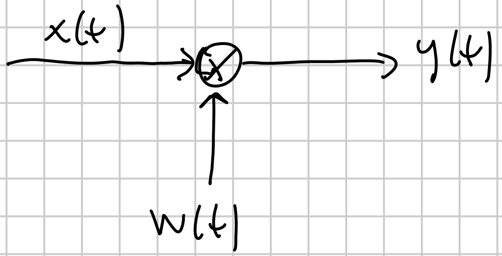
$$E_{Y_1} = 2A^2B^3 + \frac{2}{3}A^2B^3 + 2A^2B^3 = \frac{14}{3}A^2B^3$$

$$E_{y2} = \int_{-\infty}^{+\infty} |Y_2(f)|^2 \, df = \int_{-B}^{B} 4A^2B^2 \, df = 4A^2B^2 \cdot 2B = 8A^2B^3$$

$y_1(t)$ e $y_2(t)$ sono segnali ad energia finita e quindi hanno potenza nulla

$$P_{y1} = P_{y2} = 0$$

Esercizio 3



$$x(t) = \sum_{n=-\infty}^{+\infty} \text{rect}\left(\frac{t-2Tn}{T/2}\right)$$

$$w(t) = \cos\left(\frac{\pi t}{T}\right)$$

Calcolare

X_n, $X(f)$

Y_n, $Y(f)$

Energia e potenza di $y(t)$

$$x_0(t) = \mathrm{rect}\left(\frac{t}{T/2}\right)$$

$$x(t) = \sum_n x_0(t - n2T) \qquad T_0 = 2T$$

$$w(t) = \cos\left(\frac{\pi t}{T}\right) = \cos\left(2\pi t \cdot \frac{1}{2T}\right) \qquad T_0 = 2T$$

$y(t) = x(t) \cdot w(t)$ sarà un segnale periodo $T_0 = 2T$

$$y(t) = \sum_n y_0(t - nT_0)$$

$$Y_n = \frac{1}{T_0} Y_0\left(\frac{n}{T_0}\right)$$

$$X_n = \frac{1}{T_0} X_0\left(\frac{n}{T_0}\right)$$

$$X_0(f) = \frac{T}{2} \mathrm{sinc}\left(f\frac{T}{2}\right)$$

$$X_n = \frac{T}{2} \cdot \frac{1}{2T} \mathrm{sinc}\left(\frac{T}{2} \cdot \frac{n}{2T}\right) = \frac{1}{4} \mathrm{sinc}\left(\frac{n}{4}\right)$$

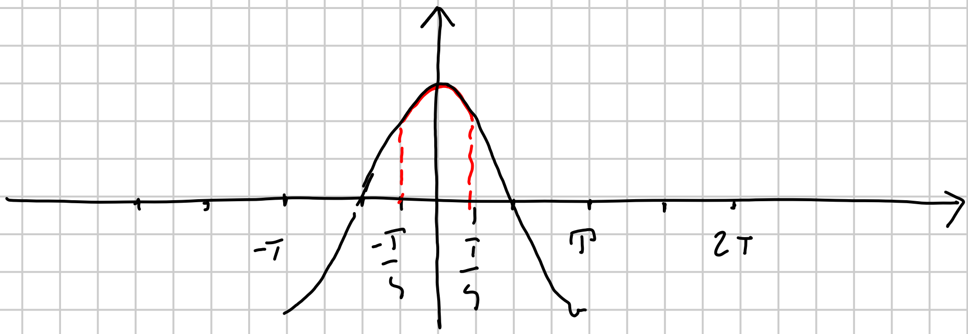
$$X(f) = \sum_{n=-\infty}^{+\infty} X_n \, \delta\left(f - \frac{n}{2T}\right)$$

$$x_0(t) = \mathrm{rect}\left(t/T/2\right)$$

$$w_0(t) = \cos\left(\frac{\pi t}{T}\right) \cdot \mathrm{rect}\left(\frac{t}{T_0}\right) = \cos\left(\frac{\pi t}{T}\right) \mathrm{rect}\left(\frac{t}{2T}\right)$$

$$y_0(t) = x_0(t) \cdot w_0(t) = \cos\left(\frac{\pi t}{T}\right) \mathrm{rect}\left(\frac{t}{2T}\right) \mathrm{rect}\left(\frac{t\,2}{T}\right) =$$

$$= \cos\left(\frac{\pi t}{T}\right) \mathrm{rect}\left(\frac{2t}{T}\right)$$



$$Y_0(f) = \frac{T}{2} \mathrm{sinc}\left(f \frac{T}{2}\right) \otimes \left[\frac{1}{2} \delta\left(f - \frac{1}{2T}\right) + \frac{1}{2} \delta\left(f + \frac{1}{2T}\right)\right] =$$

$$= \frac{T}{4} \mathrm{sinc}\left(\frac{T}{2}\left(f - \frac{1}{2T}\right)\right) + \frac{T}{4} \mathrm{sinc}\left(\frac{T}{2}\left(f + \frac{1}{2T}\right)\right) =$$

$$= \frac{T}{4} \mathrm{sinc}\left(f\frac{T}{2} - \frac{1}{4}\right) + \frac{T}{4} \mathrm{sinc}\left(f\frac{T}{2} + \frac{1}{4}\right)$$

$$Y_n = \frac{1}{T_0} Y_0\left(\frac{n}{T_0}\right) = \frac{1}{2T} Y_0\left(\frac{n}{2T}\right)$$

$$Y_n = \frac{T}{4} \cdot \frac{1}{2T} \mathrm{sinc}\left(\frac{T}{2} \cdot \frac{n}{2T} - \frac{1}{4}\right) + \frac{T}{4} \cdot \frac{1}{2T} \mathrm{sinc}\left(\frac{T}{2} \frac{n}{2T} + \frac{1}{4}\right)$$

$$= \frac{1}{8} \operatorname{sinc}\left(\frac{n-1}{4}\right) + \frac{1}{8} \operatorname{sinc}\left(\frac{n+1}{4}\right)$$

$$Y(f) = \sum_{n=-\infty}^{+\infty} Y_n \, \delta\left(f - \frac{n}{2T}\right)$$

$$E_y = \infty$$

$$P_y = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{T_0/2} |y(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |Y_n|^2$$

$$P_y = \frac{1}{2T} \int_{-T}^{T} |y(t)|^2 dt = \frac{1}{2T} \int_{-\frac{T}{4}}^{T/4} \cdot \cos^2\left(\frac{\pi t}{T}\right) dt =$$

$$= \frac{1}{2T} \int_{-\frac{T}{4}}^{T/4} \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{2\pi t}{T}\right)\right) dt =$$

$$= \frac{1}{2T} \int_{-\frac{T}{4}}^{T/4} \frac{1}{2} dt + \frac{1}{2T} \int_{-\frac{T}{4}}^{T/4} \frac{1}{2} \cos\left(\frac{2\pi t}{T}\right) dt =$$

$$\frac{1}{4T} \cdot \frac{T}{2} + \frac{1}{4T} \sin\left(\frac{2\pi t}{T}\right) \cdot \frac{T}{2\pi} \Bigg|_{-T/4}^{T/4} = \frac{1}{8} + \frac{1}{8\pi} \sin\left(\frac{2\pi t}{T}\right) \Bigg|_{-T/4}^{T/4} =$$

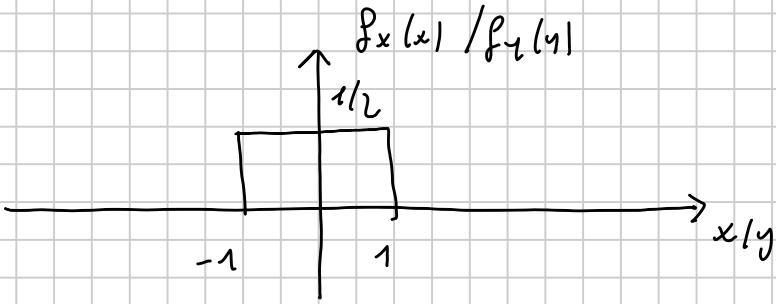
$$= \frac{1}{8} + \frac{1}{4\pi}$$

Esercizio 7.11

Sono assegnate 2 v.A indipendenti

$X \in U(-1,1)$

$Y \in U(-1,1)$



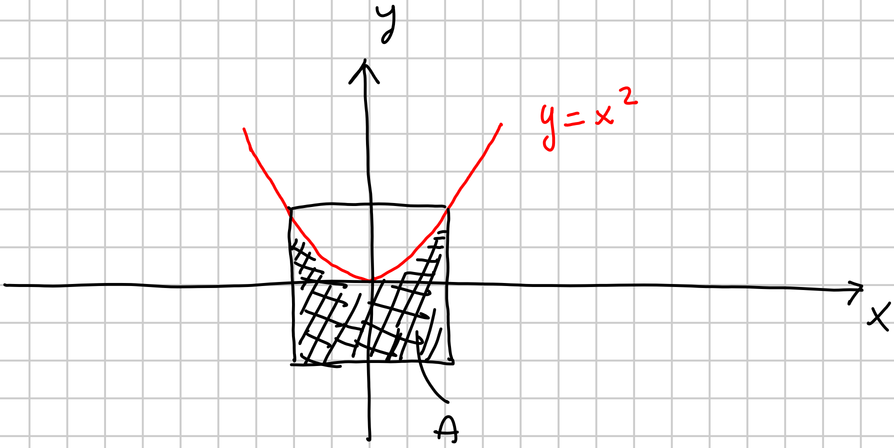
Calcolare la probabilità che l'equazione di secondo grado in α, abbia radici reali

$$\alpha^2 + 2X\alpha + Y = 0$$

$$\Delta = b^2 - 4ac = 4X^2 - 4Y \geq 0$$ affinché abbia radici reali

$$X^2 - Y \geq 0$$

$$Y \leq X^2$$



$$\Pr\{Y \leq X^2\}$$

$$f_{xy} = f_x(x)\, f_y(y)$$

$$\Pr\{ y \leq x^2 \} = \iint_A f_{xy} \, dx\,dy = \frac{1}{2}\cdot\frac{1}{2} \iint_A 1 \, dx\,dy =$$

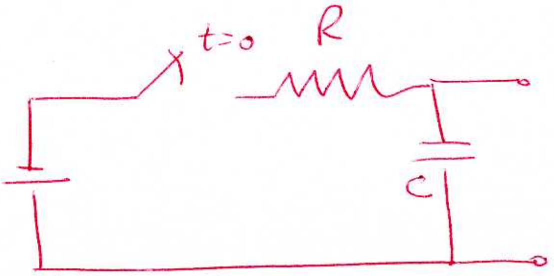
$$= \frac{1}{4} \cdot \left(\text{Area del dominio} \right)$$

$$\int_{-1}^{+1} \int_{-1}^{x^2} \frac{1}{4} \, dy \, dx = \frac{1}{4} \cdot \int_{-1}^{1} y \Big|_{-1}^{x^2} dx$$

$$= \frac{1}{4} \int_{-1}^{1} \left(x^2 + 1 \right) dx = 2 \frac{1}{4} \int_{0}^{1} \left(x^2 + 1 \right) dx =$$

$$= \frac{1}{2} \left[\frac{x^3}{3} + x \right]_0^1 = \frac{1}{2} \left[\frac{1}{3} + 1 \right] = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

Esercizio 7.11



I valori di R e C sono 2 variabili aleatorie $\overbrace{\text{con}}^{\text{indipendenti}}$ con densità di probabilità uniforme tra 0 e $2k\Omega$ per il resistore e tra 0 e $2\mu F$ per il condensatore.

A $t=0$ è applicato al circuito una tensione $V_0 = 1V$.

Calcolare la probabilità che la tensione ai capi del condensatore C, all'istante $t_0 = 1ms$ sia inferiore a $\left(1-\frac{1}{e}\right)V$.

La tensione ai capi del condensatore è

$$V(t) = V_0\left(1-e^{-t/RC}\right)u(t).$$

All'istante $t=t_0$, la tensione ai capi del condensatore è

$$V(t=t_0) = V_0\left(1-e^{-t_0/RC}\right)$$

Questa deve essere minore o uguale a $\left(1-\frac{1}{e}\right)V$

$$V_0\left(1-e^{-t_0/RC}\right) \leq \left(1-\frac{1}{e}\right)$$

$V_0 = 1\,V$, quindi

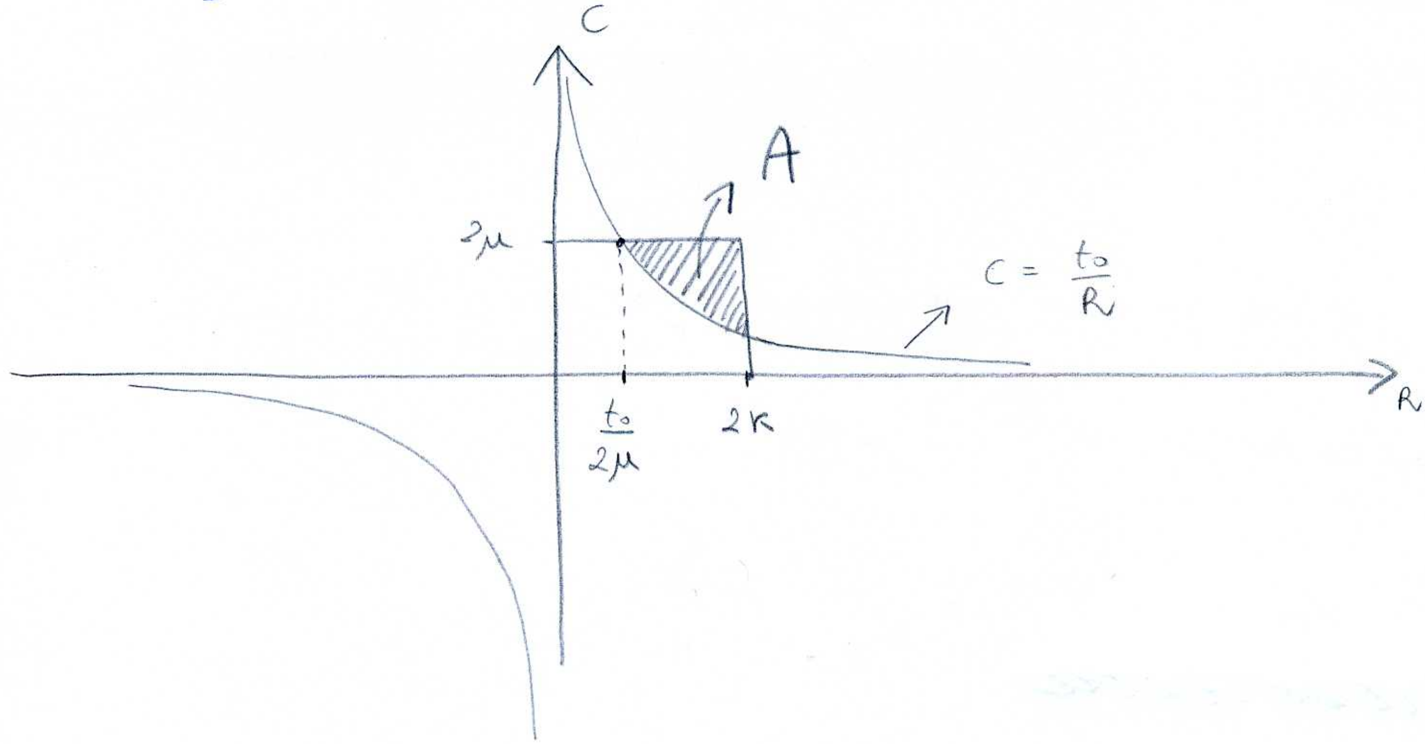
$$1 - e^{-t_0/RC} \leq 1 - e^{-1}$$

Quindi deve essere che

$$\frac{t_0}{RC} = 1 \qquad \text{quindi} \qquad RC \geq t_0$$

Quindi l'evento di cui abbiamo calcolare la probabilita' e'

$$A = \{RC \geq t_0\}$$



$$f_{RC}(r,c) = f_R(r) \cdot f_C(c) \qquad \text{perche' indipendenti}$$

In questo caso $\frac{t_0}{2\mu} < 2K$ infatti

$$\frac{10^{-3}}{2\,10^{-6}} = \frac{1}{2}10^{3} < 2 \cdot 10^{3}.$$

$$Pr\{A\} = \int_{\tau=\frac{t_0}{2\mu}}^{2K} \int_{c=\frac{t_0}{\tau}}^{2\mu} f_{RC}(\tau, c)\, d\tau\, dc =$$

$$= \int_{\frac{t_0}{2\mu}}^{2K} \int_{\frac{t_0}{\tau}}^{2\mu} \frac{1}{2000} \frac{1}{2 \cdot 10^{-6}}\, d\tau\, dc =$$

$$= \frac{1}{4 \cdot 10^{-3}} \int_{\frac{t_0}{2\mu}}^{2K} c \Big|_{\frac{t_0}{\tau}}^{2\mu} d\tau =$$

$$= \frac{1}{4\, 10^{-3}} \int_{\frac{t_0}{2\mu}}^{2K} \left(2\, 10^{-6} - \frac{t_0}{\tau} \right) d\tau =$$

$$= \frac{1}{4\, 10^{-3}} \Big[2\, 10^{-6} \cdot \tau - t_0 \log_e(\tau) \Big]_{\frac{t_0}{2\mu}}^{2K}$$

$$\frac{1}{4\,10^{-3}}\left[2\,10^{-6}\left(2000-\frac{t_0}{2\,10^{-6}}\right)-t_0\left[\log_e(2000)-\log_e\left(\frac{t_0}{2\,10^{-6}}\right)\right]\right]$$

$$=\frac{1}{2}10^{-3}\left(2000-\frac{t_0}{2\,10^{-6}}\right)-\frac{t_0}{4\,10^{-3}}\left(\log_e(2000)-\log_e\left(\frac{t_0}{2\,10^{-6}}\right)\right)=$$

$$=1-\frac{1}{2}10^{-3}\cdot\frac{10^{-3}}{2\cdot 10^{-6}}-\frac{1}{4}\left(\log_e(2000)-\log_e\left(\frac{10^{-3}}{2\,10^{-6}}\right)\right)=$$

$$=1-\frac{1}{4}-\frac{1}{4}\left(\log_e(2000)-\log_e\left(\frac{1}{2\,10^{-3}}\right)\right)=$$

$$=\frac{3}{4}-\frac{1}{4}\left(\log_e(2000)-\log_e\left(\frac{1}{2}\cdot 10^{3}\right)\right)$$

Esercizio # 1

Sia $X(t) = A\cos(2\pi f_0 t + \Theta)$

A variabile aleatoria esponenziale con valor medio η

$\Theta \sim \mathcal{U}(-\pi, \pi)$

A e Θ sono indipendenti.

$$f_A(a) = \frac{1}{\eta} e^{-a/\eta} u(a)$$

$$f_\Theta(\theta) = \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi}\right)$$

Calcolare $\eta_x(t)$, $P_x(t)$ e determinare se è un processo SSL.

1) $\eta_x(t) = E\{X(t)\} = E\{A\cos(2\pi f_0 t + \Theta)\} =$

perché A e Θ sono indipendenti

$= E\{A\} \cdot E\{\cos(2\pi f_0 t + \Theta)\} =$

$= \eta \cdot E\{\cos(2\pi f_0 t + \Theta)\} =$

$$= \eta \cdot \int_{-\infty}^{+\infty} g(\theta) f_\Theta(\theta)\, d\theta = \eta \cdot \int_{-\infty}^{+\infty} \cos(2\pi f_0 t + \theta) \cdot \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi}\right) d\theta =$$

$$= \frac{\eta}{2\pi} \cdot \int_{-\pi}^{\pi} \cos(2\pi f_0 t + \theta)\, d\theta = \frac{\eta}{2\pi} \left[\sin(2\pi f_0 t + \theta)\right]_{-\pi}^{\pi} =$$

$$= \frac{\eta}{2\pi} \left[\sin(2\pi f_0 t + \pi) - \sin(2\pi f_0 t - \pi)\right] =$$

$$= \frac{\eta}{2\pi}\left[-\sin(2\pi f t) - \left(\sin(2\pi f t)\cdot(-1)\right)\right] = 0$$

$$\eta_x(t) = \eta_x = 0$$

2) $$P_x(t) = E\{x^2(t)\} = E\{A^2 \cdot \cos^2(2\pi f t + \Theta)\} = $$

$$\cos^2(\alpha) = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$$

$$= E\left\{A^2\left(\frac{1}{2} + \frac{1}{2}\cos(4\pi f t + 2\Theta)\right)\right\} =$$

$$= E\left\{\frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f t + 2\Theta)\right\} =$$

$$= E\left\{\frac{A^2}{2}\right\} + E\left\{\frac{A^2}{2}\cos(4\pi f t + 2\Theta)\right\} =$$

$$E\{A^2\} = 2\eta^2$$

$$= \eta^2 + E\left\{\frac{A^2}{2}\right\}\cdot E\{\cos(4\pi f t + 2\Theta)\} =$$

$$= \eta^2 + \eta^2 \cdot E\{\cos(4\pi f t + 2\Theta)\} =$$

$$= \eta^2 + \eta^2 \int_{-\pi}^{\pi} \cos(4\pi f t + 2\theta)\, d\theta =$$

$$= \eta^2 + \eta^2 \cdot \frac{1}{2}\left[\sin(4\pi f t + 2\theta)\right]_{-\pi}^{\pi} =$$

$$= \eta^2 + \frac{\eta^2}{2}\left[\sin(4\pi f_0 t + 2\pi) - \sin(4\pi f_0 t - 2\pi)\right] =$$

$\sin(\alpha+\beta) = \sin(\alpha)\cdot\cos(\beta) + \cos(\alpha)\sin(\beta)$

$$= \eta^2 + \frac{\eta^2}{2}\left[\sin(4\pi f_0 t) - \sin(4\pi f_0 t)\right] = \eta^2$$

$$P_x(t) = P_x = \eta^2$$

$X(t)$ è SSL se $\eta_x(t) = \eta_x$ e $R_x(t_1,t_2) = R_x(\tau)$ dove $\tau = t_2 - t_1$.

$$\begin{cases} X(t_1) = A\cos(2\pi f_0 t_1 + \Theta) \\ X(t_2) = A\cos(2\pi f_0 t_2 + \Theta) \end{cases}$$

2 Variabili Aleatorie

$$R_x(t_1,t_2) \stackrel{\Delta}{=} E\{X(t_1)\, X(t_2)\} =$$

$$E\{A\cos(2\pi f_0 t_1 + \Theta)\cdot A\cos(2\pi f_0 t_2 + \Theta)\} =$$

$$= E\{A^2 \cdot \cos(2\pi f_0 t_1 + \Theta)\cos(2\pi f_0 t_2 + \Theta)\} =$$

$$= E\{A^2\}\cdot E\{\cos(2\pi f_0 t_1 + \Theta)\cos(2\pi f_0 t_2 + \Theta)\} =$$

$$\cos(\alpha)\cdot\cos(\beta) = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta)$$

$$= 2\eta^2 \cdot E\left\{\frac{1}{2}\cos\left(2\pi f_0 (t_1-t_2)\right) + \frac{1}{2}\cos\left(2\pi f_0 (t_1+t_2) + 2\Theta\right)\right\} =$$

$$= 2\eta^2\, E\left\{\frac{1}{2}\cos\left(2\pi f_0 (t_1-t_2)\right)\right\} + \underbrace{2\eta^2\, E\left\{\frac{1}{2}\cos\left(2\pi f_0 (t_1+t_2)+2\Theta\right)\right\}}_{= 0} =$$

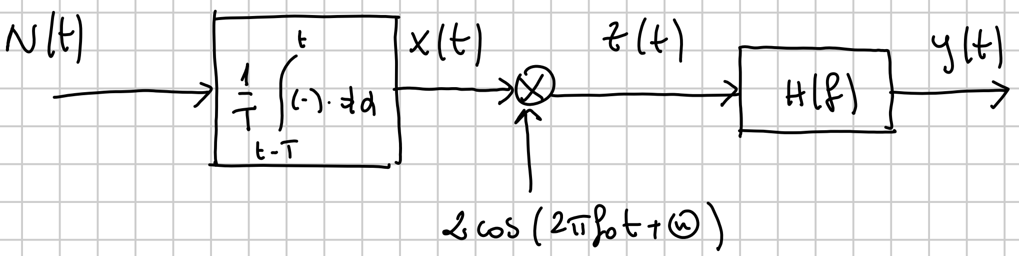
$$= \frac{2\eta^2}{2}\, E\left\{\cos\left(2\pi f_0 (t_1-t_2)\right)\right\} = \eta^2 \cdot \cos\left(2\pi f_0 (t_1-t_2)\right) = \eta^2 \cos(2\pi f_0 \tau)$$

Una proprietà delle funzione di autocorrelazione di processi stazionari è:

$\lim_{\tau\to+\infty} R_x(\tau) = \eta_x^2$ SE $R_x(\tau)$ NON CONTIENE COMPONENTI PERIODICHE.

In questo caso, essendo $R_x(\tau)$ periodica questa proprietà non è infatti verificata.

Esercizio #2



$N(t)$ è un processo SSL con $S_N(f) = \xi$ (processo Gaussiano Bianco)

Θ è una Variabile Aleatoria indipendente da $N(t)$ e $U(-\pi,\pi)$

Calcolare la $S_y(f)$.

- Poiché l'integratore è un sistema Lineare e Stazionario

$X(t)$ sarà un processo Gaussiano e SSL.

$$\eta_X(t) = \eta_N(t) \cdot H_1(0) = 0$$

$$R_X(\tau) = R_N(\tau) \otimes h_1(\tau) \otimes h_1(-\tau)$$

$$S_X(f) = S_N(f) \cdot |H_1(f)|^2$$

$$h_1(t) = \frac{1}{T} \operatorname{rect}\left(\frac{t - T/2}{T}\right)$$

$$H_1(f) = \operatorname{sinc}(fT)\, e^{-j2\pi f \frac{T}{2}} \Rightarrow |H_1(f)|^2 = \operatorname{sinc}^2(fT)$$

$$R_X(\tau) = \xi\, \delta(\tau) \otimes \underbrace{h_1(\tau) \otimes h_1(-\tau)}_{R_h(\tau)} = \xi\, \delta(\tau) \otimes R_h(\tau) =$$

$$= \xi\, R_h(\tau) = \xi \frac{1}{T}\left(1 - \frac{|\tau|}{T}\right) \operatorname{rect}\left(\frac{\tau}{2T}\right)$$

$$S(t) = 2\cos(2\pi f_0 t + \Theta) \qquad \Theta \in U(-\pi, \pi)$$

$$\left.\begin{aligned} \eta_S &= 0 \\ R_S(\tau) &= 2\cos(2\pi f_0 \tau) \end{aligned}\right\} \text{Ricavate nell'esercizio 1}$$

$$z(t) = x(t) \cdot s(t)$$

$$\eta_z(t) = E\{z(t)\} = 2\, E\left\{ x(t) \cdot \cos(2\pi f_0 t + \Theta) \right\} =$$

$$= 2\, E\{x(t)\} \cdot E\{\cos(2\pi f_0 t + \Theta)\} = 0$$

$$R_z(t_1, t_2) = R_z(t, t-\tau) = E\{ z(t)\, z(t-\tau) \} =$$

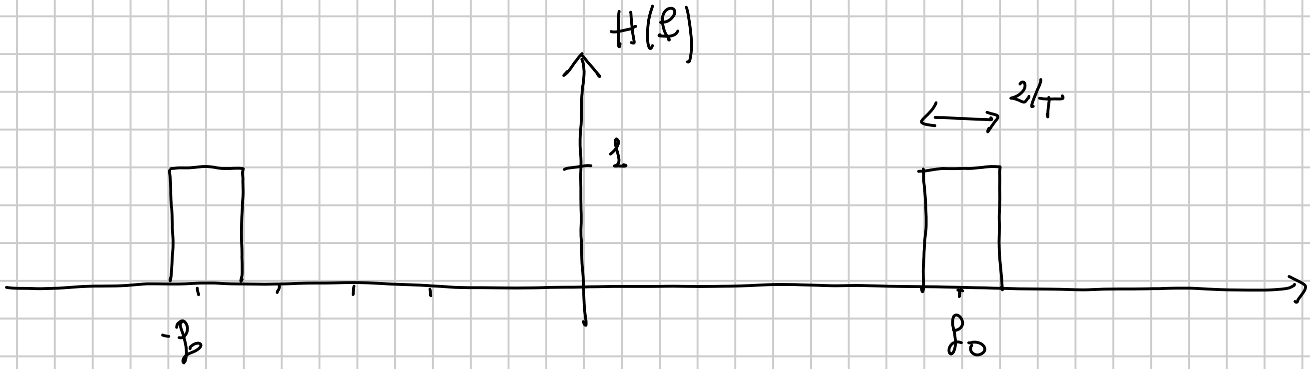
$$= 4\, E\left\{ x(t)\, x(t-\tau) \cos(2\pi f_0 t + \Theta) \cdot \cos(2\pi f_0 (t-\tau) + \Theta) \right\} =$$

Θ e $x(t)$ INDIPENDENTI

$$= 4\, E\{x(t)\, x(t-\tau)\} \cdot E\left\{ \cos(2\pi f_0 t + \Theta) \cdot \cos(2\pi f_0 (t-\tau) + \Theta) \right\} =$$

$$= R_x(\tau) \cdot R_s(\tau) = 2\, R_x(\tau) \cdot \cos(2\pi f_0 \tau)$$

$$R_z(\tau) = 2\, R_x(\tau) \cdot \cos(2\pi f_0 \tau)$$



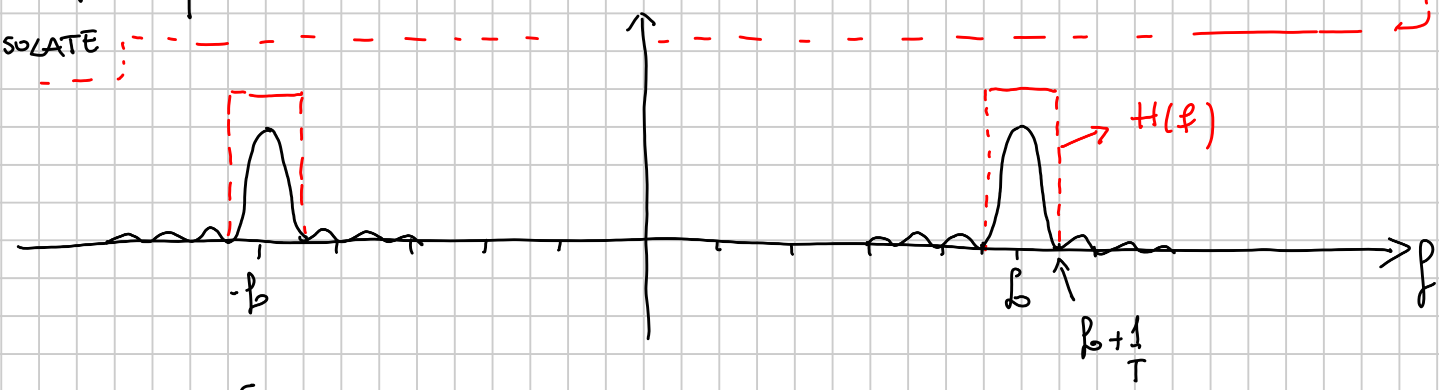
$$S_y(f) = S_z(f) \cdot |H(f)|^2$$

$$R_z(\tau) = 2\, R_x(\tau) \cdot \cos(2\pi f_0 \tau)$$

$$R_x(\tau) = \frac{\xi}{T} \left(1 - \frac{|\tau|}{T}\right) \mathrm{rect}\left(\frac{\tau}{2T}\right)$$

$$S_Z(f) = \xi \left\{ \mathrm{sinc}^2\left((f+f_0)T\right) + \mathrm{sinc}^2\left((f-f_0)T\right) \right\}$$

Se $f_0 \gg \frac{1}{T}$ le due repliche di $S_X(f)$ centrate a $\pm f_0$ possono considerarsi ISOLATE



$$S_Y(f) = \xi \left[\mathrm{sinc}^2\left((f+f_0)T\right) \mathrm{rect}\left(\frac{(f+f_0)T}{2}\right) + \mathrm{sinc}^2\left((f-f_0)T\right) \mathrm{rect}\left(\frac{(f-f_0)T}{2}\right) \right]$$

È possibile ricavare la ddp di 1° ordine di $y(t)$, $f_Y(y;t)$?

OSSERVAZIONE: Condizionatamente ad un valore di Θ, il processo $Y(t)$ è Gaussiano quindi $f_{Y|\Theta}(y|\theta;t)$ è la ddp di una Gaussiana.

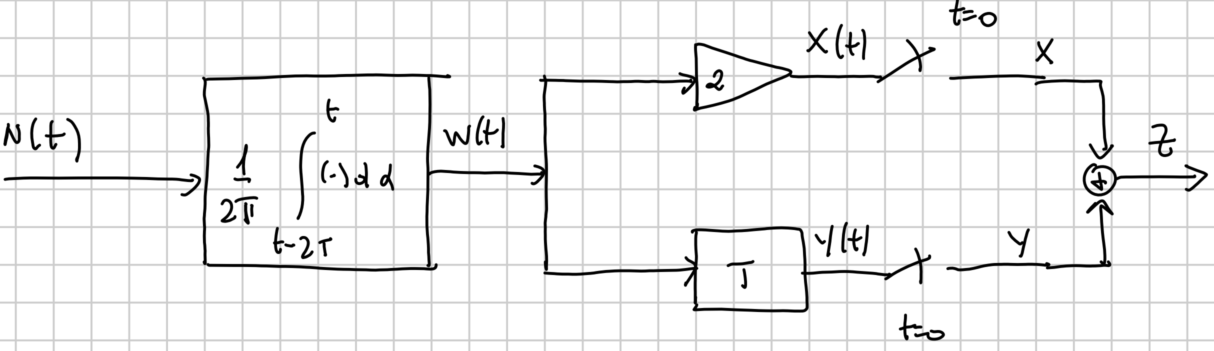
Quindi la ddp congiunta tra le variabili $Y(t)$ e Θ è

$$f_{Y\Theta}(y,\theta;t) = f_{Y|\Theta}(y|\theta;t) \cdot f_\Theta(\theta)$$

È possibile quindi ricavare adesso la ddp marginale dalla congiunta.

$$f_Y(y;t) = \int_{-\infty}^{+\infty} f_{Y\Theta}(y,\theta;t)\, d\theta$$

Esercizio 3



Trovare la ddp di Z

$N(t)$ é Gaussiano Bianco $S_N(f) = 2T$

$$h(t) = \frac{1}{2T} \operatorname{rect}\left(\frac{t-T}{2T}\right)$$

$$X(t) = 2W(t)$$

$$Y(t) = W(t-T)$$

Quando campiono a $t=0$

$$X = 2W(0)$$

$$Y = W(-T)$$

Poiché $W(t)$ é Gaussiano allora X e Y sono Variabili Aleatorie Congiuntamente Gaussiane.

$$Z = X + Y = 2W(0) + W(-T)$$

La combinazione Lineare di Variabili Aleatorie Congiuntamente Gaussiane é una Variabile Aleatoria Gaussiana.

$$\eta_W(t) = \eta_N(t) \cdot H(0) = \eta_N \cdot H(0) = 0$$

$$R_W(\tau) = \left(1 - \frac{|\tau|}{2T}\right) \operatorname{rect}\left(\frac{\tau}{4T}\right)$$

$$E\{z\} = E\{x+y\} = E\{x\} + E\{y\} = 2E\{w(0)\} + E\{w(-T)\} = 0$$

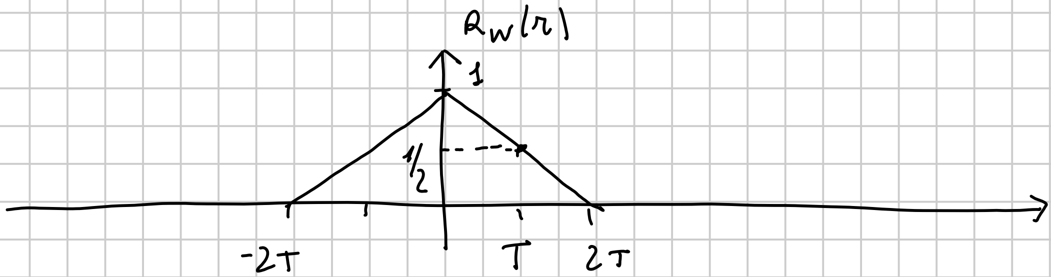
$$\sigma_z^2 = E\{(z-\eta_z)^2\} = E\{z^2\} =$$

$$= E\{[2w(0) + w(-T)]^2\} =$$

$$= E\{4w^2(0) + w^2(-T) + 4w(0)w(-T)\} =$$

$$= E\{4w^2(0)\} + E\{w^2(-T)\} + 4E\{w(0)w(-T)\} =$$

$$= 4R_w(0) + R_w(0) + 4R_w(T)$$



$$R_w(0) = 1$$

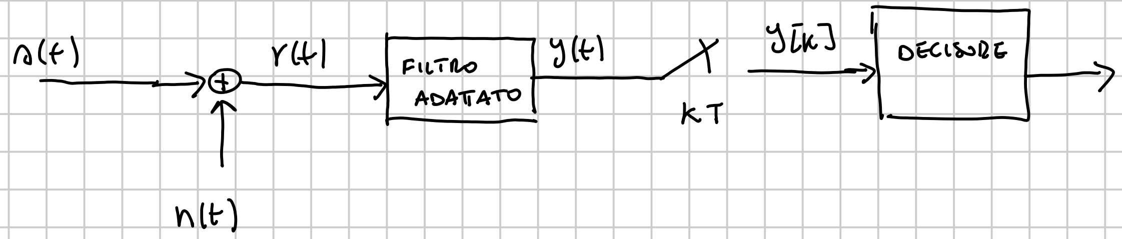
$$R_w(T) = \frac{1}{2}$$

$$\sigma_z^2 = 4 + 1 + 4\cdot\frac{1}{2} = 5 + 2 = 7$$

$$f_z(z;t) = \frac{1}{\sqrt{14\pi}}\, e^{-z^2/14}$$

Esercizio #1

PAM binario



$$s(t) = \sum_{i=-\infty}^{+\infty} x[i]\, p(t - iT)$$

$a_i \in \{\pm 1\}$ equiprobabili e indipendenti

$$Pr\{x[i] = a_1\} = Pr\{x[i] = a_{-1}\} = \frac{1}{2}$$

$n(t)$ è AWGN

$\lambda = 0$

$$\hat{x}[k] = \begin{cases} 1 & \text{se } y[k] \geq \lambda \\ -1 & \text{se } y[k] < \lambda \end{cases}$$ DECISORE

$$h(t) = p(t) \otimes h_{FA}(t)$$

$$r(t) = s(t) + n(t)$$

$$y(t) = s_u(t) + n_u(t)$$

$$y[k] = x[k]\, h(0) + n_u[k]$$

$$\sigma^2_{nu} = \int_{-\infty}^{+\infty} S_{nu}(f)\, df = \int_{-\infty}^{+\infty} S_n(f)\, |H_{FA}(f)|^2\, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_{FA}(f)|^2\, df =$$

$$= (\text{per la proprietà del filtro adattato}) = \frac{N_0}{2} \int_{-\infty}^{+\infty} H(f)\, df = \frac{N_0}{2} h(0)$$

$$SNR = \frac{h^2(0)}{\frac{N_0}{2} h(0)} = \frac{2h(0)}{N_0}$$

$$E\left\{ x^2[n]\, h^2(0) \right\} = h^2(0)\, E\left\{ x^2[n] \right\} = h^2(0) \left\{ \frac{1}{2} \cdot (1)^2 + \frac{1}{2} \cdot (-1)^2 \right\} = h^2(0)$$

$$P_E[b] = Pr\{ x[n] = +1 \}\, Pr\{ \hat{x} = -1 \mid x = +1 \} +$$

$$Pr\{ x = -1 \}\, Pr\{ \hat{x} = 1 \mid x = -1 \} =$$

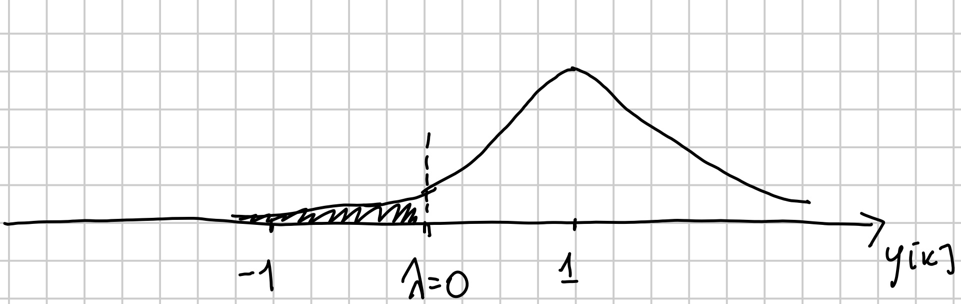
$$= \frac{1}{2} Pr\{ \hat{x} = -1 \mid x = 1 \} + \frac{1}{2} Pr\{ \hat{x} = 1 \mid x = -1 \}$$

$$Pr\{ \hat{x} \mid -1 \mid x = 1 \}$$

$$y[k] = x[k]\, h(0) + n_u(k)$$

$$y[k]\big|_{x=1} = +h(0) + n_u(k) \in \mathcal{N}\left(+h(0), \sigma^2_{nu}\right)$$

$$Pr\{ \hat{x} - 1 \mid x = 1 \} = \int_{-\infty}^{\lambda} f_y(y \mid x = 1)\, dy = \int_{-\infty}^{0} f_y(y \mid x = 1)\, dy$$



$$f_y(y \mid x = 1) = \frac{1}{\sqrt{2\pi \sigma^2_{nu}}}\, e^{-\frac{(y - h(0))^2}{2\sigma^2_n}}$$

$$P\{\hat{x}=-1 \mid x=1\} = 1 - Q\left(\frac{0-h(0)}{\sigma_{nu}}\right) = \qquad \sigma_{nu} = \sqrt{\frac{N_0}{2}\, h(0)}$$

$$= Q\left(\sqrt{\frac{2h(0)}{N_0}}\right) = Q\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

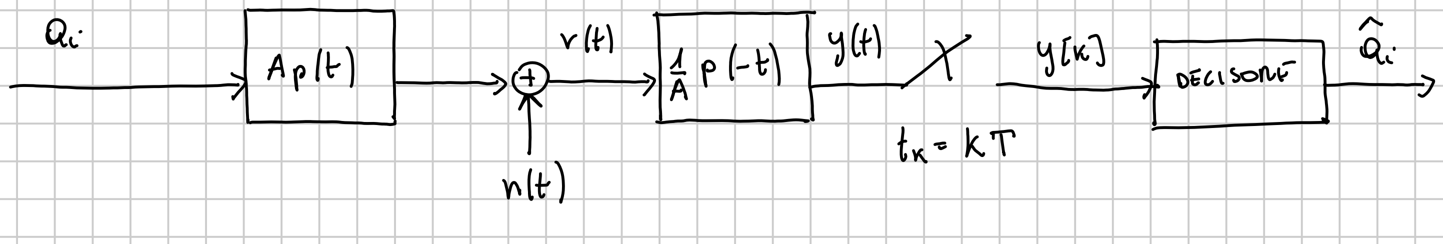
Si dimostra per simmetria che

$$P_r\{\hat{x}=1 \mid x=-1\} = P_r\{\hat{x}=-1 \mid x=1\} = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

$$P_E(b) = \frac{1}{2}\cdot\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) + \frac{1}{2}\,\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)$$

Esercizio #2

sistema di comunicazione PAM a 4 livelli



$$r(t) = \sum_{i=-\infty}^{+\infty} a_i\, A\, p(t - iT) + n(t)$$

$a_i \in \{\pm 3;\ \pm 1\}$ indipendenti ed equiprobabili

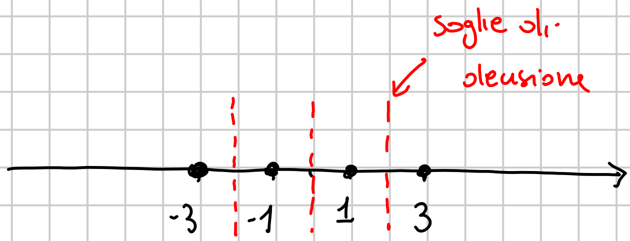
$n(t)$ AWGN $\qquad S_n(f) = \frac{N_0}{2} \quad \forall f$

$p(t)$ ha trasformata di Fourier

$$P(f) = \begin{cases} \sqrt{T}\,\sqrt{1-|fT|} & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

La strategia di decisione è:

$$\hat{a}_k = \begin{cases} 3 & \text{se} \quad y[k] \geq 2 \\ 1 & \text{se} \quad 0 \leq y[k] < 2 \\ -1 & \text{se} \quad -2 \leq y[k] < 0 \\ -3 & \text{se} \quad y[k] < -2 \end{cases}$$



Si determini:

1) Il SNR $= \frac{E_s}{N_0}$ dove E_s è l'energia media per intervallo di segnalazione del segnale trasmesso.

2) La $P_E(M)$ = probabilità di errore in funzione di E_s/N_0

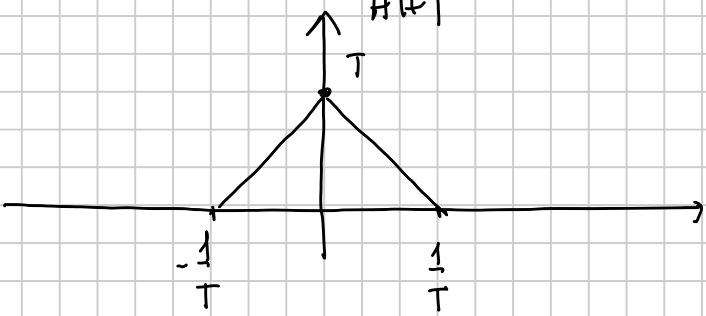
Svolgimento:

Il segnale all'uscita del FA

$$y(t) = \sum_{i=-\infty}^{+\infty} a_i\, h(t-iT) + n(t)$$

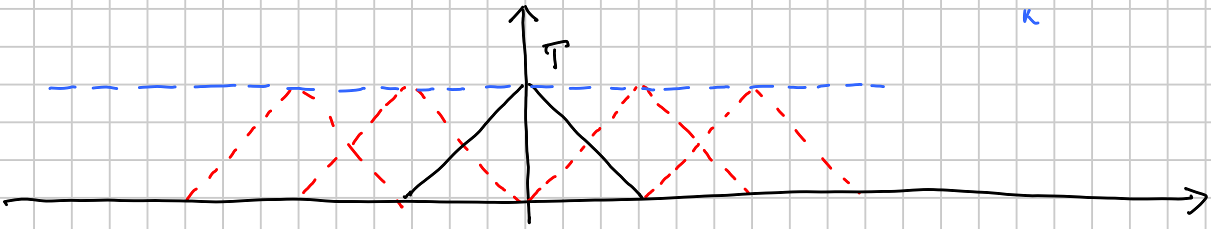
$$h(t) = p(t) \otimes h_{FA}(t) = p(t) \otimes p(-t)$$

$$H(f) = P(f)\, P^*(f) = P(f)\, P(f) = P^2(f) = \begin{cases} T\,(1-|fT|) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$



Soddisfa la condizione di Nyquist!

$$\sum_k H\left(f - k\frac{1}{T}\right)$$



$$\sum_k H\left(f - \frac{k}{T}\right) = T \quad \Rightarrow \quad h(0) = 1$$

$$n_u(t) = n(t) \otimes h_R(t) = \frac{1}{A}\, n(t) \otimes p(-t)$$

$$\sigma^2_{n_u} = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2\, df = \frac{N_0}{2A^2} \underbrace{\int_{-\infty}^{+\infty} |P(f)|^2\, df}_{h(0)} = \frac{N_0}{2A^2}\, h(0) = \frac{N_0}{2A^2}$$

$$E_S = E\left\{ \int_0^T s^2(t)\, dt \right\} = A^2 E\left\{ \int_0^T \sum_i \sum_n a_i a_n\, p(t-iT)\, p(t-nT) \right\} =$$

$$= A^2 \sum_i \sum_n E\{a_i a_n\} \int_0^T p(t-iT)\, p(t-nT)\, dt$$

Osserviamo che

$$E\{a_i\} = \frac{1}{4}\cdot(-3) + \frac{1}{4}(-1) + \frac{1}{4}(1) + \frac{1}{4}(3) = 0$$

$$E\{a_i^2\} = \frac{1}{4}(3)^2 + \frac{1}{4}(-3)^2 + \frac{1}{4}(-1)^2 + \frac{1}{4}(1)^2 = 5$$

Quindi:

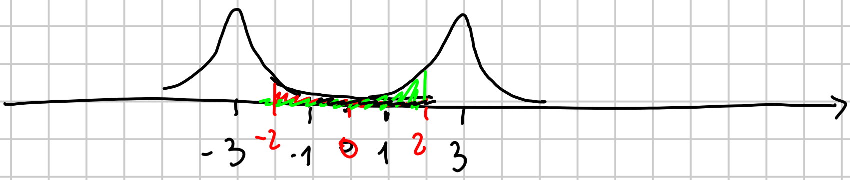
$$E\{a_i a_n\} = \begin{cases} E\{a_i^2\} & \text{se } i=n \\ E\{a_i\}\cdot E\{a_n\} & \text{se } i\neq n \end{cases} = \begin{cases} 5 & \text{se } i=n \\ 0 & \text{se } i\neq n \end{cases}$$

$$E_S = A^2\, 5 \sum_i \int_0^T p^2(t-iT) = 5A^2 \int_{-\infty}^{+\infty} p^2(t)\, dt = 5A^2 \int_{-\infty}^{+\infty} P^2(f)\, df = 5A^2$$

Calcoliamo le probabilità

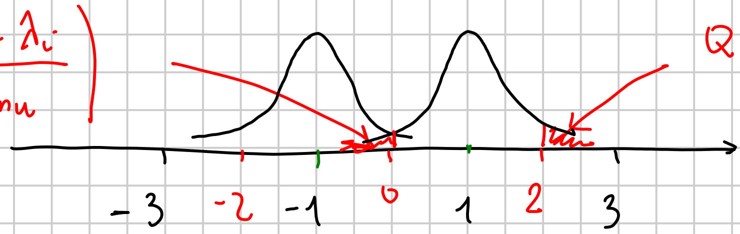
$$P_E = \frac{1}{4}\left\{ Pr\{e \mid a_K=-3\} + Pr\{e \mid a_K=-1\} + Pr\{e \mid a_K=1\} + \right.$$
$$\left. + Pr\{e \mid a_K=3\} \right\}$$

$$y(K) \mid a_K = 3 \qquad \in \mathcal{N}(a_K, \sigma_{nu}^2)$$



$$\Pr\{e \mid a_k = 3\} = \Pr\{e \mid a_k = -3\} = \Pr\{-3 + n_k > -2\} = Q\left(\frac{-2+3}{\sigma_{nu}}\right) = Q\left(\frac{1}{\sigma_{nu}}\right)$$

$$\Pr\{e \mid a_k = 1\} = \Pr\{e \mid a_k = -1\} = \Pr\{-1 + n_k > 0 \,,\, -1 + n_k \leq -2\}$$



$\eta_i =$ valor medio delle Gaussiane

$\lambda_i =$ soglia di decisione

$$\Pr\{e \mid a_k = 1\} = \Pr\{-1 + n_k > 0\} + \Pr\{-1 + n_k \leq -2\} =$$

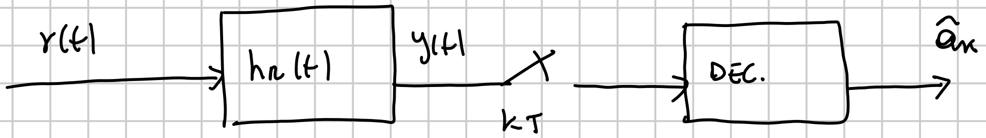
$$= Q\left(\frac{0+1}{\sigma_{nu}}\right) + Q\left(\frac{+2-1}{\sigma_{nu}}\right) = 2Q\left(\frac{1}{\sigma_{nu}}\right)$$

$$P_e = \frac{3}{2} Q\left(\frac{1}{\sigma_{nu}}\right)$$

$$\sigma_{nu}^2 = \frac{N_0}{2A^2} = \frac{5}{5}\frac{N_0}{2A^2} = \frac{5}{2}\frac{N_0}{E_s}$$

$$P_e = \frac{3}{2} Q\left(\sqrt{\frac{2}{5}\frac{E_s}{N_0}}\right)$$

Esercizio #3



$a_i \in \{-e \,;\, 2e\}$ indipendenti ed equiprobabili.

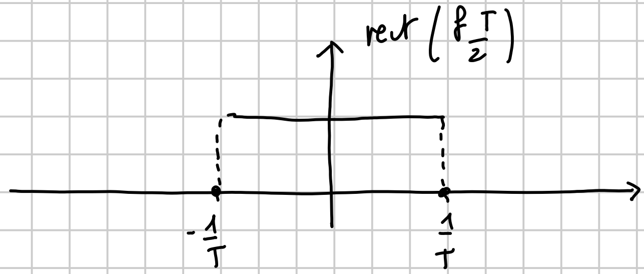
$$r(t) = \underbrace{\sum_i a_i \, p(t - iT)}_{s(t)} + n(t)$$

$S_n(f) = \frac{N_0}{2}$ AWGN

$$P(f) = T \, |f| \, e^{-|f|} \, \text{rect}\left(\frac{fT}{2}\right)$$

Si assume che la rect$(x) = 0$ per $x = \pm \frac{1}{2}$

$$H_R(f) = e^{|f|} \, \text{rect}\left(\frac{fT}{2}\right)$$



$$\hat{a}_k = \begin{cases} -e & x_k \leq \lambda \\ 2e & x_k > \lambda \end{cases}$$

$\lambda = \frac{3e}{2}$

1) E_s del segnale trasmesso

2) $P_r\{e\}$ nel caso in cui $\lambda = \frac{3e}{2}$

3) Il valore ottimo di λ che minimizza la $Pr\{e\}$

① E_s si può calcolare utilizzando la DSP del segnale trasmesso.

2) $E_s = P_T \cdot T$ dove $P_T = \int_{-\infty}^{+\infty} S_s(f)\, df$ e

$$S_s(f) = \frac{1}{T} S_a(f) \, |P(f)|^2 \qquad S_a(f) \triangleq \sum_m R_a(m) \, e^{-j2\pi f m T}$$

$$R_a(m) = E\{a_{i+m} \, a_i\} = \begin{cases} E\{a_i^2\} & m=0 \\ E\{a_{i+m}\} \cdot E\{a_i\} & m \neq 0 \end{cases}$$

funzione di autocorrelazione dei simboli trasmessi a_i

$$E\{a_i\} = \frac{1}{2}(-e) + \frac{1}{2}(2e) = \frac{e}{2}$$

$$E\{a_i^2\} = \frac{1}{2} e^2 + \frac{1}{2} 4e^2 = \frac{5}{2} e^2$$

$$R_a(m) = \begin{cases} \frac{5}{2} e^2 & m=0 \\ \frac{e^2}{4} & m \neq 0 \end{cases} \Rightarrow \boxed{R_a(m) = \frac{9}{4} e^2 \delta(m) + \frac{e^2}{4}}$$

$$S_a(f) = \frac{9}{4} e^2 + \frac{e^2}{4T} \sum_k \delta\left(f - \frac{k}{T}\right)$$

$$S_s(f) = \frac{1}{T} |P(f)|^2 \left[\frac{9}{4} e^2 + \frac{e^2}{4T} \sum_k \delta\left(f - \frac{k}{T}\right)\right]$$

$$E_S = \int_{-\infty}^{+\infty} |P(f)|^2 \cdot \left[\frac{9}{4} e^2 + \frac{e^2}{4T} \sum_k \delta\left(f - \frac{k}{T}\right) \right] =$$

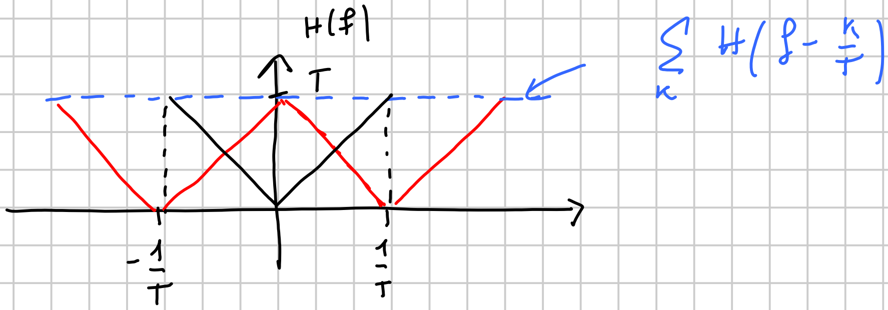
$$= \frac{9}{4} e^2 \int_{-\infty}^{+\infty} |P(f)|^2 df + \frac{e^2}{4T} \sum_k \int_{-\infty}^{+\infty} |P(f)|^2 \, \delta\left(f - \frac{k}{T}\right) df =$$

$$\int_{-\infty}^{+\infty} |P(f)|^2 df = \frac{1}{4} \left[1 - e^{-2/T} \left(1 + \frac{2}{T} + \frac{2}{T^2} \right) \right]$$ (*) SVOLTO ALLA FINE

$$\int_{-\infty}^{+\infty} |P(f)|^2 \, \delta\left(f - \frac{k}{T}\right) df = \left| P\left(\frac{k}{T}\right) \right|^2 \Rightarrow \sum_k \left| P\left(\frac{k}{T}\right) \right|^2 = |P(0)|^2 = 0$$

$$E_S = \frac{9}{4} e^2 \cdot E_P \qquad E_P = \frac{1}{4} \left[1 - e^{-2/T} \left(1 + \frac{2}{T} + \frac{2}{T^2} \right) \right]$$

(2) $H(f) = P(f) \, H_R(f) = T \, |f| \, \mathrm{rect}\left(\frac{fT}{2}\right)$



$$\sum_k H\left(f - \frac{k}{T}\right) = T \Rightarrow \boxed{h(0) = 1}$$

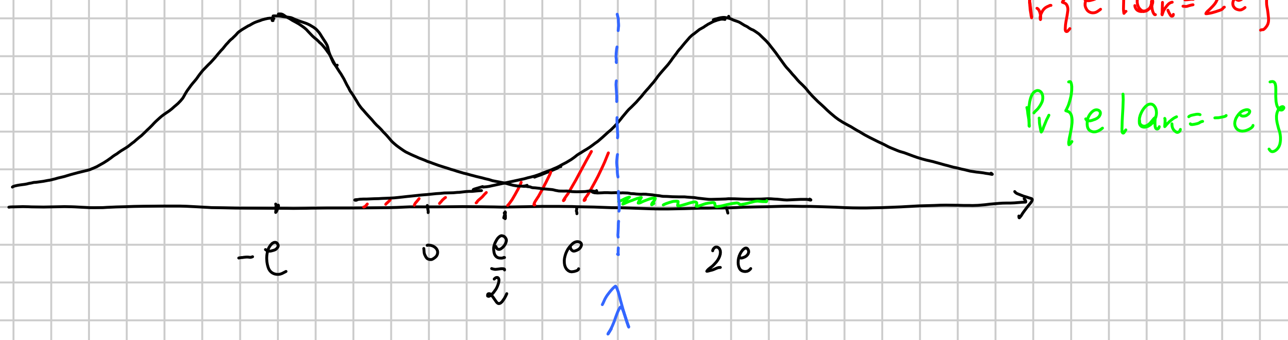
$$y_k = h(0) \, a_k + n_k = a_k + n_k$$

$$n_u(t) = n(t) \otimes h_R(t)$$

$$S_{n_u}(f) = \frac{N_0}{2} \left| H_R(f) \right|^2$$

$$\sigma_{n_u}^2 = \frac{N_0}{2} \int_{-\infty}^{+\infty} \left| H_R(f) \right|^2 df = \frac{N_0}{2} \, 2 \int_0^{+\infty} e^{2f} \, df = \frac{N_0}{2} \left(e^{2/T} - 1 \right)$$

$$P(e) = \frac{1}{2} \Pr\{ e \mid a_K = -e \} + \frac{1}{2} \Pr\{ e \mid a_K = 2e \}$$



(3) $$\Pr\{ e \mid a_K = 2e \} = Q\left(\frac{-\lambda + 2e}{\sigma_{n_u}} \right)$$

$$\Pr\{ e \mid a_K = -e \} = Q\left(\frac{\lambda + e}{\sigma_{n_u}} \right)$$

$$P_E = \frac{1}{2} Q\left(\frac{-\lambda + 2e}{\sigma_{n_u}} \right) + \frac{1}{2} Q\left(\frac{\lambda + e}{\sigma_{n_u}} \right)$$

Si può calcolare la P_E nel caso in cui $\lambda = \frac{3e}{2}$

Per trovare la soglia che minimizza la probabilità di errore si impone che.

$$\frac{\partial P_E}{\partial \lambda} = 0$$

$$Q'(x) \triangleq \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\frac{\partial P_E}{\partial \lambda} = -\cancel{\frac{1}{2}}\left(+\cancel{\frac{1}{\sqrt{2\pi}}}\right) e^{-\frac{(-\lambda+2e)^2}{2\sigma_n^2}} + \cancel{\frac{1}{2}} \left(\cancel{\frac{1}{\sqrt{2\pi}}}\right) e^{-\frac{(\lambda+e)^2}{2\sigma_{nu}^2}}.$$

$$e^{-\frac{(-\lambda+2e)^2}{2\sigma_{nu}^2}} = e^{-\frac{(\lambda+e)^2}{2\sigma_{nu}^2}}$$

$$\frac{(-\lambda+2e)^2}{\cancel{2\sigma_{nu}^2}} = \frac{(\lambda+e)^2}{\cancel{2\sigma_{nu}^2}}$$

$$\cancel{\lambda^2} + 4e^2 - 4\lambda e = \cancel{\lambda^2} + e^2 + 2\lambda e$$

$$\lambda(-4e - 2e) = -4e^2 + e^2$$

$$-6e\lambda = -3e^2$$

$$\lambda = \frac{3e^2}{6e} = \frac{e}{2}$$

Come ci si doveva aspettare λ soglia ottima nel caso di simboli equiprobabili è equidistante dai simboli.

$\textcircled{d}$ l'integrale si risolve per parti

$$\int_0^{1/T} f^2 e^{-2T} \, df = f^2 \left[-\frac{1}{2} e^{-2f} \right]_0^{1/T} - \int_0^{1/T} 2f \left[-\frac{1}{2} e^{-2f} \right] df =$$

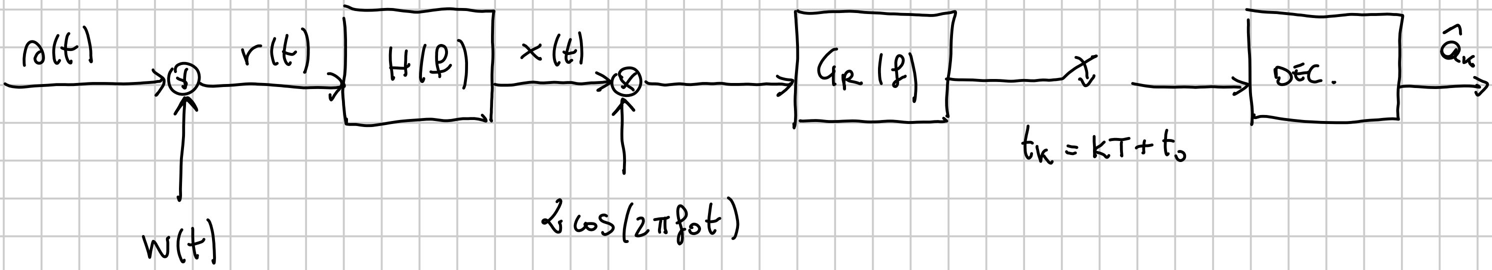
$$= -\frac{f^2}{2} e^{-2f} \Big|_0^{1/T} + \int_0^{1/T} f e^{-2f} \, df =$$

$$= -\frac{f^2}{2} e^{-2f} \Big|_0^{1/T} + \left[-\frac{f}{2} e^{-2f} \right]_0^{1/T} + \int_0^{1/T} \frac{1}{2} e^{-2f} \, df =$$

$$= \left[-\frac{f^2}{2} e^{-2f} \right]_0^{1/T} + \left[-\frac{f}{2} e^{-2f} \right]_0^{1/T} - \frac{1}{4} e^{-2f} \Big|_0^{1/T} =$$

$$= -\frac{1}{2T^2} e^{-2/T} - \frac{1}{2T} e^{-2/T} - \frac{1}{4} e^{-2/T} + \frac{1}{4}$$

Esercizio 3.1



$w(t)$ AWGN

$$s(t) = \sum_i a_i \, p(t - iT) \cos(2\pi f_0 t)$$

$a_i \in \{\pm 1\}$ indipendenti ed equiprobabili

$$P(f) = \begin{cases} \sqrt{T} \cos(\pi f T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$H(f) = \begin{cases} e^{-j[2\pi(f-f_0)\tau + \pi/6]} & |f-f_0| \leq \frac{1}{T} \\ e^{-j[2\pi(f+f_0)\tau - \pi/6]} & |f+f_0| \leq \frac{1}{T} \\ 0 & \text{altrove} \end{cases}$$

$$r(t) = s(t) + w(t)$$

$$x(t) = \sum_i a_i \left[p(t - iT) \cos(2\pi f_0 t) \right] \otimes h(t) + \underbrace{w(t) \otimes h(t)}_{n(t)}$$

$$g(t) = \left(p(t-iT)\cos(2\pi f_0 t)\right) \otimes h(t)$$

$$G(f) = \left[P(f)\, e^{-j2\pi f iT} \otimes \left(\frac{\delta(f-f_0) + \delta(f+f_0)}{2} \right) \right] \cdot H(f) =$$

$$\left[\frac{1}{2} P(f-f_0)\, e^{-j2\pi(f-f_0)\cdot iT} + \frac{1}{2} P(f+f_0)\, e^{-j2\pi(f+f_0)\cdot iT} \right] \cdot H(f) =$$

$$= \frac{1}{2} P(f-f_0)\, e^{-j2\pi(f-f_0) iT} \cdot e^{-j2\pi(f-f_0)\tau}\, e^{-j\pi/6} +$$

$$\frac{1}{2} P(f+f_0)\, e^{-j2\pi(f+f_0) iT}\, e^{-j2\pi(f+f_0)\tau}\, e^{+j\pi/6} =$$

$$= \frac{1}{2} P(f-f_0)\, e^{-j2\pi(f-f_0)(iT+\tau)}\, e^{-j\pi/6} + \frac{1}{2} P(f+f_0)\, e^{-j2\pi(f+f_0)(iT+\tau)}\, e^{+j\pi/6}$$

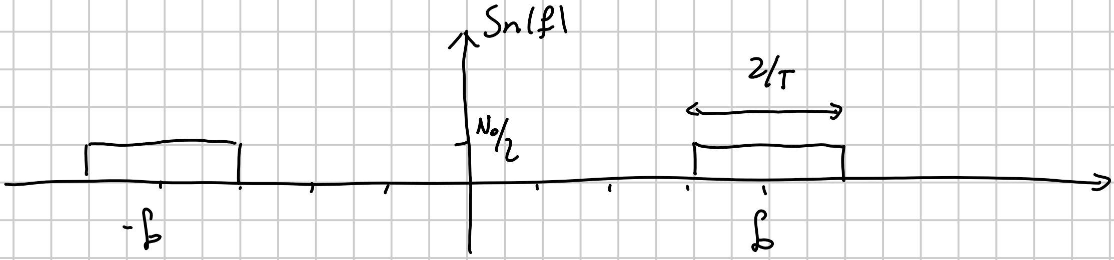
Trasformata Continua di Fourier Inversa

$$g(t) = p(t-iT-\tau)\cos(2\pi f_0 t + \pi/6) =$$

$$= p(t-iT-\tau)\left[\cos(2\pi f_0 t)\cos\frac{\pi}{6} - \sin(2\pi f_0 t)\sin\frac{\pi}{6} \right]$$

$$x(t) = \sum_i a_i\, p(t - iT - \tau)\left[\cos(2\pi f_0 t)\cos \pi/6 - \sin(2\pi f_0 t)\sin(\pi/6)\right] + n(t)$$

$n(t)$ è un processo passa banda, a valor medio nullo, SSL, Gaussiano



$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

$$\Rightarrow z(t) = x(t)\cdot 2\cos(2\pi f_0 t) = z_s(t) + z_n(t)$$

$$z_s(t) = 2\sum_i a_i\, p(t - iT - \tau)\left[\cos^2(2\pi f_0 t)\cos(\pi/6) - \sin(2\pi f_0 t)\cos(2\pi f_0 t)\cdot\sin(\pi/6)\right] =$$

$$= \sum_i a_i\, p(t - iT - \tau)\left[\left(1 + \cos(4\pi f_0 t)\right)\cos(\pi/6) - \sin(4\pi f_0 t)\sin(\pi/6)\right]$$

$$z_n(t) = 2n(t)\cdot\cos(2\pi f_0 t) = \left[n_c(t)\cdot\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)\right]\cdot 2\cos(2\pi f_0 t) =$$

$$= n_c(t)\cdot\left(1 + \cos(4\pi f_0 t)\right) - n_s(t)\sin(4\pi f_0 t)$$

$G_R(f)$ realizza il filtro adattato

$$G_R(f) = P(f) = \begin{cases} \sqrt{T} \cos(\pi f T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$y(t) = z(t) \otimes g_R(t) = y_s(t) + y_n(t)$$

$$y_s(t) = z_s(t) \otimes g_R(t)$$

$$y_n(t) = z_n(t) \otimes g_R(t)$$

$$y_s(t) = \sum_i a_i \cos(\pi/6)\, p(t - iT - \tau) \otimes \underset{\overset{\parallel}{g_R(t)}}{p(t)}$$

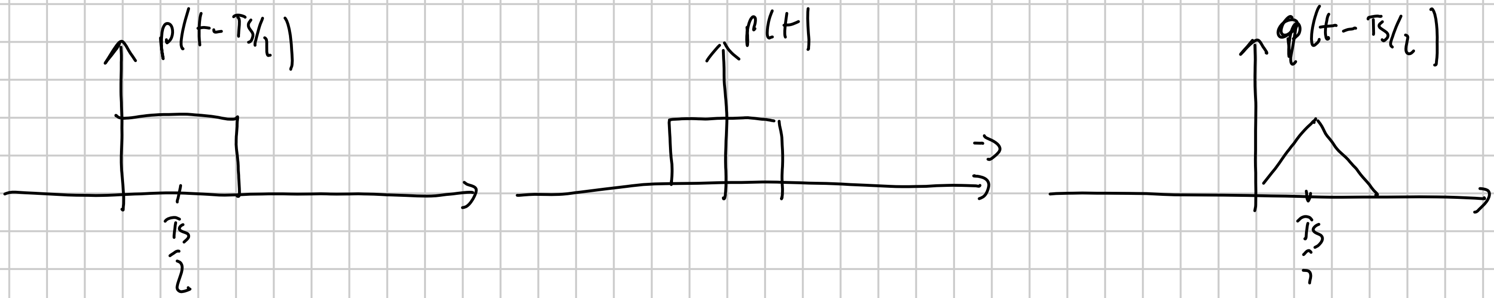
$$y_n(t) = z_n(t) \otimes p(t) = n_c(t) \otimes p(t)$$

$$y(t) = \sum_i a_i \frac{\sqrt{3}}{2}\, p(t - iT - \tau) \otimes p(t) + n_c(t) \otimes p(t)$$

$$q(t) = p(t) \otimes p(t)$$

$$Q(f) = P^2(f) = \begin{cases} T \cos^2(\pi f T/2) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$$



$$Q(f) = \begin{cases} \frac{T}{2}\left(1 + \cos(\pi f T)\right) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

Funzione Raised Cosine

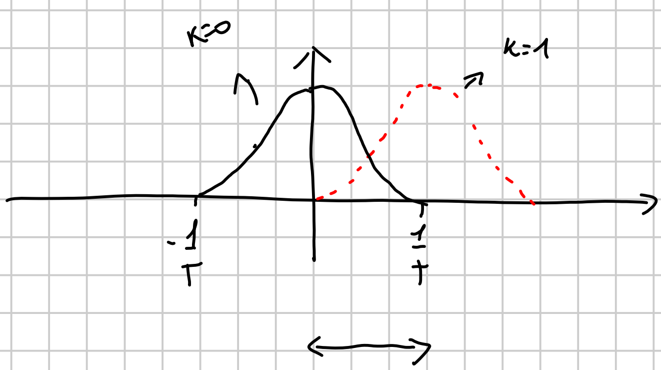
$\alpha = 1$

$$q(kT) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$y(t) = \sum_i a_i \frac{\sqrt{3}}{2} \, q(t - iT - \tau) + n_c(t) \otimes p(t)$$

$t_k = kT + \tau \Rightarrow t_0 = \tau$ istante di campionamento ottimo

$$\sum_k Q\left(f - \frac{k}{T}\right) = T$$



$$\frac{T}{2}\left(1 + \cos(\pi f T)\right) + \frac{T}{2}\left(1 + \cos\left(\pi\left(f - \frac{1}{T}\right)T\right)\right) =$$

$$= \frac{T}{2} + \frac{T}{2}\cos(\pi f T) + \frac{T}{2} + \frac{T}{2}\cos\left(\pi f T - \pi\right) \rightarrow -\frac{T}{2}\cos(\pi f T) = T$$

Se campiono all'istante ottimo

$$y[k] = a_k \cdot \frac{\sqrt{3}}{2} + n_{cp}[k] \qquad n_{cp}(t) = n_c(t) \otimes p(t)$$

DOBBIAMO RICAVARE LA DSP DI $n_{cp}(t)$

Dato un processo $x(t)$ con $\tilde{x}(t)$, il suo inviluppo complesso, SSL e a valor medio nullo, si dimostra che

1) $$x_c(t) \triangleq \Re e\{\tilde{x}(t)\}$$

$$x_s(t) \triangleq \Im m\{\tilde{x}(t)\}$$

sono processi SSL a media nulla

2) $$R_{x_c}(\tau) = R_{x_s}(\tau) \Rightarrow S_{x_c}(f) = S_{x_s}(f) \text{ pari e reale}$$

$$R_{x_s x_c}(\tau) = -R_{x_s x_c}(-\tau) \Rightarrow j\, S_{x_s x_c}(f) = -j\, S_{x_s x_c}(f) \text{ immaginaria e dispari}$$

3) $$x(t) \triangleq \Re e\{\tilde{x}(t)\, e^{j2\pi f t}\}$$ SSL e a media nulla

$$R_{\tilde{x}}(t) = R_{x_c}(\tau) + j\, R_{x_s x_c}(\tau) \qquad (eq.\ 1)$$

$$R_x(t) = R_{x_c}(\tau)\cos(2\pi f \tau) - R_{x_s x_c}(\tau)\sin(2\pi f \tau) \triangleq \Re e\{R_{\tilde{x}}(\tau)\, e^{j2\pi f t}\}$$

la conoscenza di $R_{\tilde{x}}(\tau)$ permette di risalire a $R_{x_c}(\tau)$ e $R_{x_s}(\tau)$

N.B. poiché $R_{x_s x_c}(\tau) = - R_{x_s x_c}(-\tau)$

Allora per $\tau = 0$ deve essere

$$R_{x_s x_c}(0) \triangleq E\{ x_c(t_1)\, x_s(t_1) \} = 0$$

le variabili aleatorie $x_c(t_1)$ e $x_s(t_1)$ sono incorrelate

Antitrasformando (eq. 1)

$$S_{\tilde{x}}(f) = S_{x_c}(f) + j\, S_{x_s x_c}(f)$$

$$\begin{cases} S_{x_c}(f) = S_{x_s}(f) = \dfrac{S_{\tilde{x}}(f) + S_{\tilde{x}}(-f)}{2} \\ \\ j\, S_{x_s x_c}(f) = -j\, S_{x_s x_c}(f) = \dfrac{S_{\tilde{x}}(f) - S_{\tilde{x}}(-f)}{2} \end{cases}$$

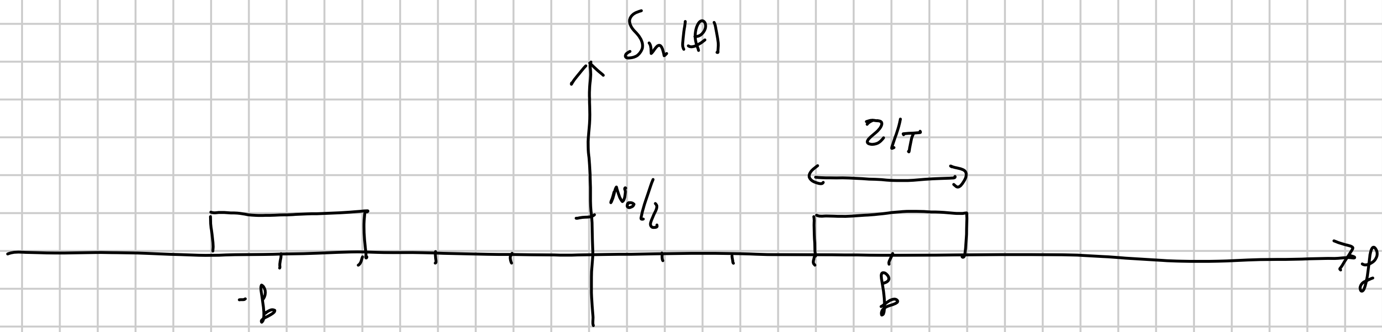
Se il processo è passa banda

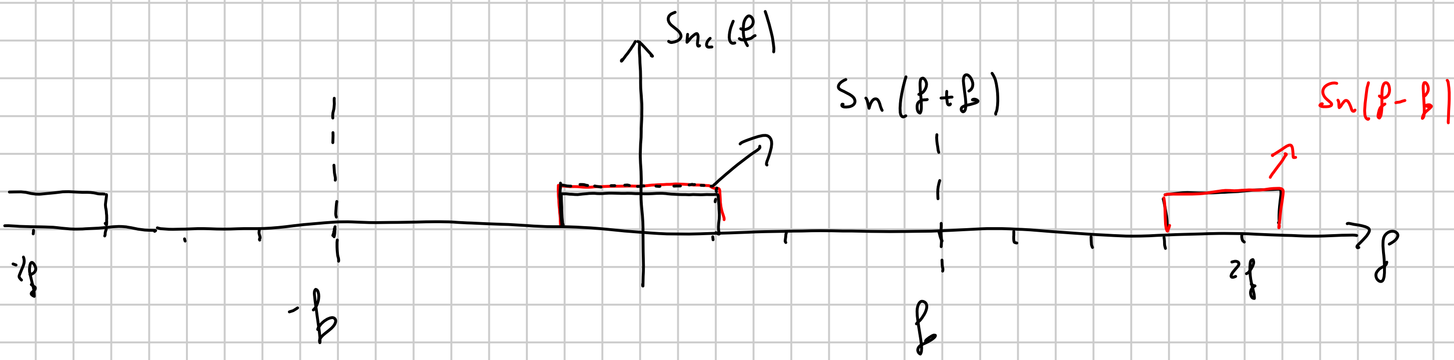
$$S_x(f) = 0 \quad \text{per} \quad |f| \geq 2f_0$$

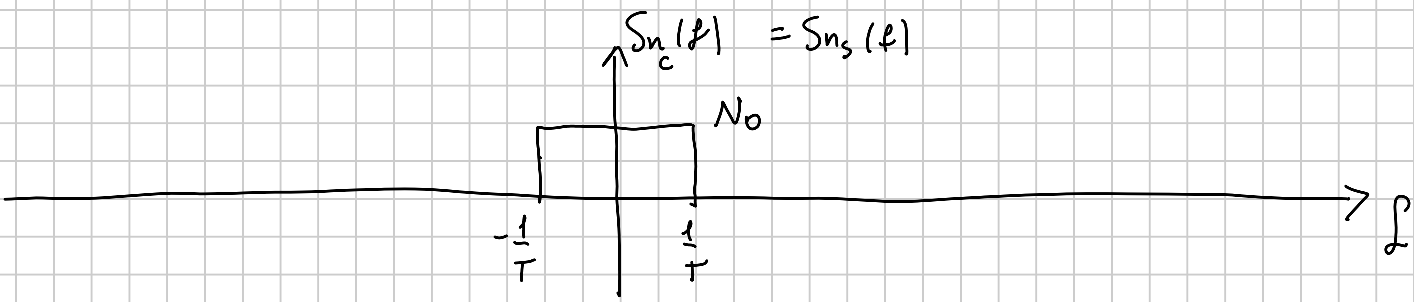
$$S_{\tilde{x}}(f) = \begin{cases} 2\, S_x(f + f_0) & |f| \leq f_0 \\ 0 & \text{altrove} \end{cases}$$

$$S_{x_c}(f) = S_{x_s}(f) = \begin{cases} S_x(f+f_0) + S_x(f-f_0) & |f| \le f_0 \\ 0 & \text{altrove} \end{cases}$$

$$j\, S_{x_s x_c}(f) = \begin{cases} S_x(f+f_0) - S_x(f-f_0) & |f| \le f_0 \\ 0 & \text{altrove} \end{cases}$$







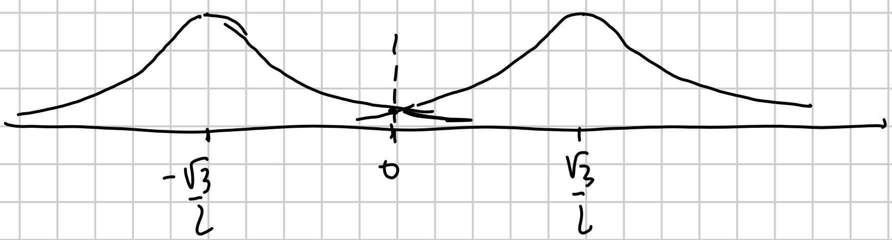
$$S_{x_s x_c}(f) = 0 \qquad \forall f$$

All'uscita del campionatore

$$y[k] = a_k \frac{\sqrt{3}}{2} + n_{cp}[k] \qquad n_{cp}(t) = n_c(t) \otimes p(t)$$

$$S_{n_{cp}}(f) = S_{n_c}(f) \cdot |P(f)|^2$$

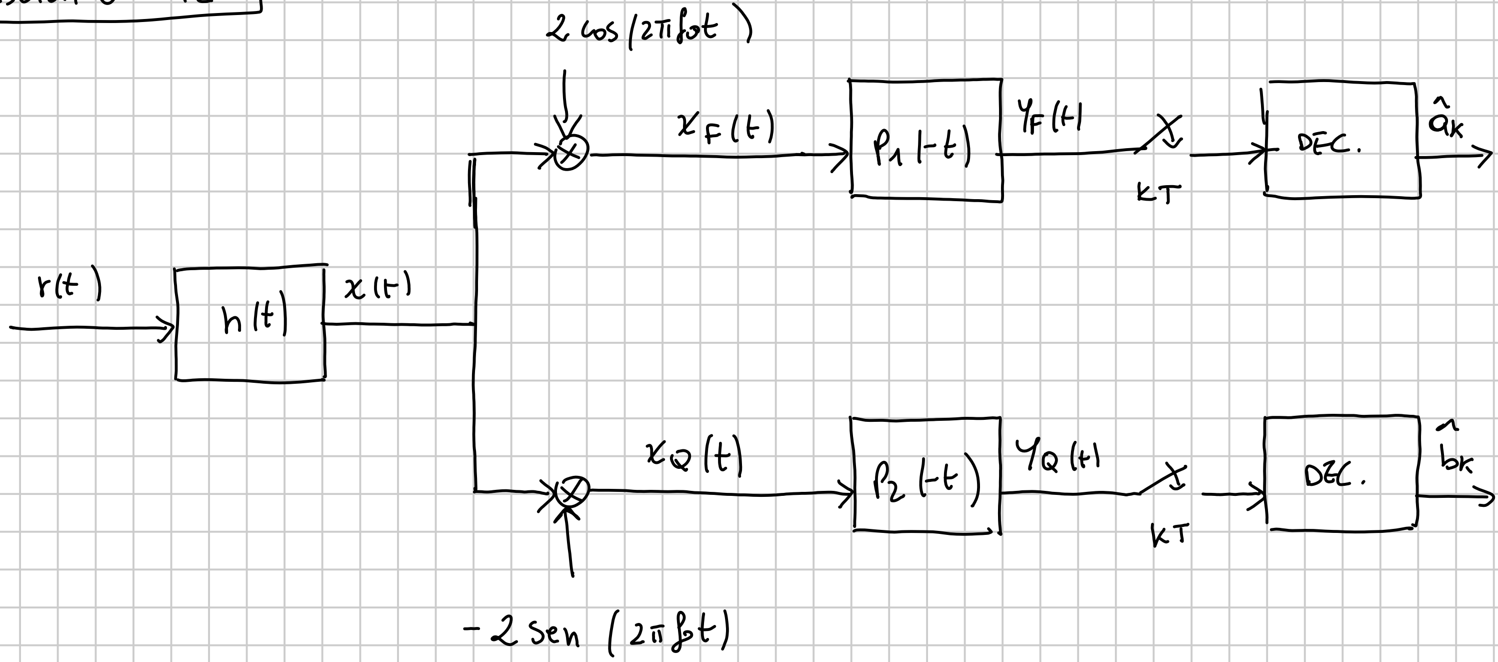
$$\sigma^2_{n_{cp}} = \int_{-\infty}^{+\infty} S_{n_{cp}}(f)\, df = N_0 \int_{-\frac{1}{T}}^{1/T} |P(f)|^2\, df = N_0$$



$$P(e) = \frac{1}{2} P(e \mid a_k = 1) + \frac{1}{2} P(e \mid a_k = -1) = P(e \mid a_k = 1) =$$

$$= Q\left(\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{N_0}}\right) = Q\left(\sqrt{\frac{3}{4N_0}}\right)$$

Esercizio #2



$$r(t) = \sum_i a_i\, p_1(t-iT)\cos(2\pi f_0 t) - \sum_i b_i\, p_2(t-iT)\,\mathrm{sen}(2\pi f_0 t) + w(t)$$

a_i e b_i sono indipendenti $\quad a_i \in \{\pm 2\} \quad b_i \in \{\pm 1\}$

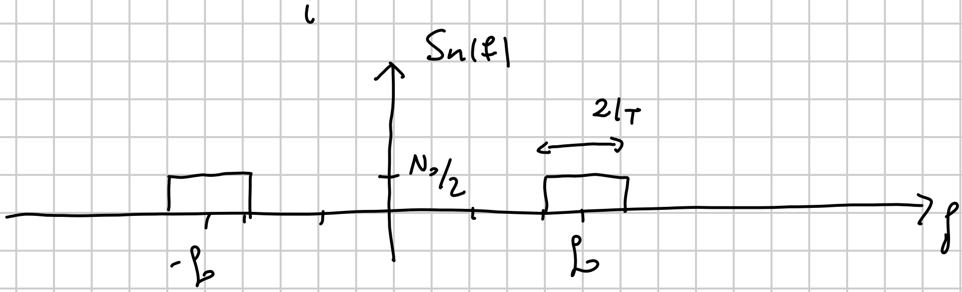
$H(f)$ è passa banda ideale di banda $\frac{2}{T}$ alle frequenze $\pm f_0$

$$P_1(f) = \begin{cases} T\sqrt{|f|} & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$

$$P_2(f) = \begin{cases} \sqrt{T}\,\mathrm{sen}(\pi |f| T/2) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$

$$x(t) = \sum_i a_i\, p_1(t-iT)\cos(2\pi f_0 t) - \sum_i b_i\, p_2(t-iT)\,\mathrm{sen}(2\pi f_0 t) + n(t)$$

$$n(t) = w(t) \otimes h(t)$$



$$x_F(t) = x(t)\cdot 2\cos(2\pi f_0 t) =$$

$$= \sum_i a_i\, p_1(t-iT)\left(1+\cos(4\pi f_0 t)\right) - \sum_i b_i\, p_2(t-iT)\,\mathrm{sen}(4\pi f_0 t) +$$

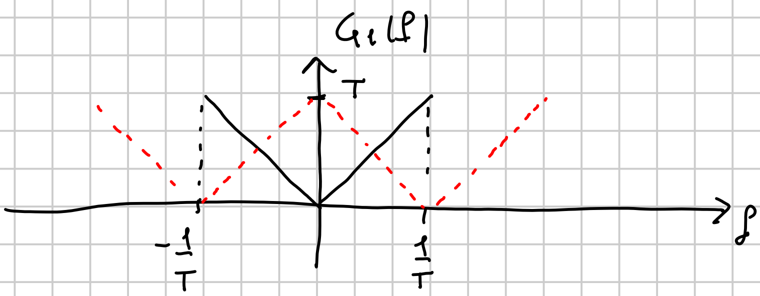
$$+\; n(t)\cdot 2\cos(2\pi f_0 t)$$

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\,\mathrm{sen}(2\pi f_0 t)$$

$$y_F(t) = \sum_i a_i\, p_1(t-iT) \otimes p_1(-t) \quad + \quad n_c(t) \otimes p_1(-t)$$

$$g_1(t) = p_1(t) \otimes p_1(-t)$$

$$G_1(f) = P_1(f)\cdot P_1^*(f) = \begin{cases} T^2\,|f| & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$



$$\sum_k G_1\left(f-\frac{k}{T}\right)=T \qquad \text{ok} \qquad g_1(kT)=\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$$

All' uscita del campionatore

$$y_F[k]=a_k+n_{cp_1}[k] \qquad n_{cp_1}(t)=n_c(t)\otimes p_1(-t)$$

RAMO IN QUADRATURA

$$x_Q(t)=x(t)\left(-2\,\text{sen}\left(2\pi f_0 t\right)\right)=$$

$$\sum_i a_i\,p_1(t-iT)\left(-\,\text{sen}\left(4\pi f_0 t\right)\right)+\sum_i b_i\,p_2(t-iT)\,2\,\text{sen}^2\left(2\pi f_0 t\right)+$$

$$+\,n(t)\left(-2\,\text{sen}\left(2\pi f_0 t\right)\right)$$

$$\text{sen}^2\left(2\pi f_0 t\right)=\frac{1}{2}-\frac{1}{2}\cos\left(4\pi f_0 t\right)$$

$$n(t)=n_c(t)\cos(2\pi f_0 t)-n_s(t)\,\text{sen}(2\pi f_0 t)$$

$$x_Q(t)=-\sum_i a_i\,p_1(t-iT)\,\text{sen}\left(4\pi f_0 t\right)+\sum_i b_i\,p_2(t-iT)\left(1-\cos\left(4\pi f_0 t\right)\right)+$$

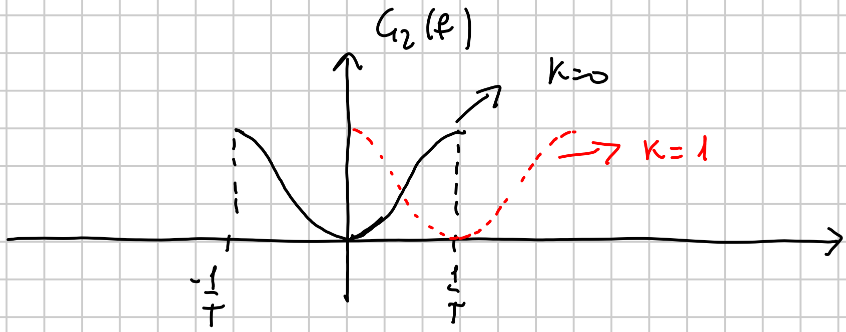
$$n(t)\left(-2\,\text{sen}\left(2\pi f_0 t\right)\right)$$

$$y_Q(t) = \sum_i b_i \, p_2(t - iT) \otimes p_2(-t) + n_s(t)$$

$$g_2(t) = p_2(t) \otimes p_2(-t)$$

$$G_2(f) = P_2(f) \cdot P_2^*(f) = \begin{cases} T \, \text{sen}^2(\pi |f| T/2) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$

$$G_2(f) = \begin{cases} \frac{T}{2}\left(1 - \cos(\pi f T)\right) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$



$$\sum_k G_2\left(f - \frac{k}{T}\right) = T$$

$$\frac{T}{2}\left(1 - \cos(\pi f T)\right) + \frac{T}{2}\left(1 - \cos\left(\pi \left(f - \frac{1}{T}\right) T\right)\right) =$$

$$\frac{T}{2} - \frac{T}{2}\cos(\pi f T) + \frac{T}{2} - \frac{T}{2}\cos(\pi f T - \pi)$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$= \frac{T}{2} - \frac{T}{2}\cancel{\cos(\pi f T)} + \frac{T}{2} + \frac{T}{2}\cancel{\cos(\pi f T)} = T$$

$$g_2(kT) = \begin{cases} 1 & k=0 \\ 0 & k \neq 0 \end{cases}$$

$$y_Q[k] = b_k + n_{sp_2}[k] \qquad n_{sp}(t) = n_s(t) \otimes p_2(-t)$$

All'uscita del campionatore, nei due rami, si ha che

$$y_F[k] = a_k + n_{cp_1}[k] = a_k + \nu_c$$

$$y_Q[k] = b_k + n_{sp_2}[k] = b_k + \nu_s$$

$$S_{\nu_c}(f) = S_{n_c}(f) \cdot |P_1(f)|^2$$

$$S_{\nu_s}(f) = S_{n_s}(f) \cdot |P_2(f)|^2$$

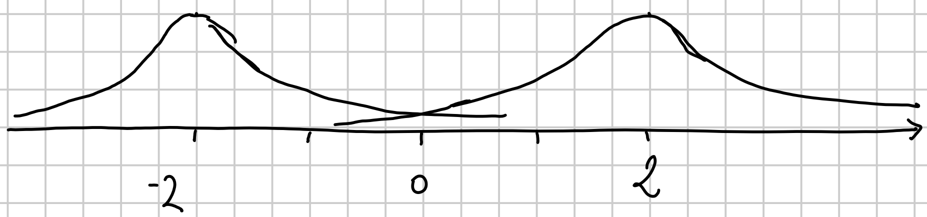
$$S_{n_c}(f) = S_{n_s}(f) = \begin{cases} N_0 & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$\sigma^2_{v_c} = N_0 \int_{-\infty}^{+\infty} |P_1(f)|^2 \, df = N_0$$

$$\int_{-\infty}^{+\infty} |P_1(f)|^2 \, df = \int_{-\infty}^{+\infty} G_1(f) \, df = g_1(0) = 1$$

$$\sigma^2_{v_s} = N_0 \int_{-\infty}^{+\infty} |P_2(f)|^2 \, df = N_0$$

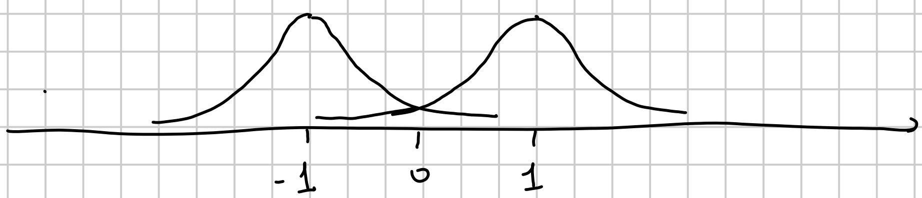
Ramo in fase $\qquad y_F[k]\big|_{a_k} \in \mathcal{N}(a_k, \sigma^2_{v_c})$



$$P_r\{a_k + v_c < 0 \mid a_k = 2\} = P_r\{a_k + v_c > 0 \mid a_k = -2\}$$

$$P_r(e) = P_r\{a_k + v_c < 0 \mid a_k = 2\} = Q\left(\frac{2}{\sqrt{N_0}}\right)$$

RAMO IN QUADRATURA $\qquad y_Q[k]\big|_{b_k} \in \mathcal{N}(b_k, \sigma^2_{v_s})$



$$P_r(e) = Pr\{b_k + V_s < 0 \mid b_k = 1\} = Q\left(\frac{1}{\sqrt{N_0}}\right)$$

Calcoliamo e' energia media per intervallo di segnalazione del segnale trasmesso.

$$s(t) = \sum_i a_i\, p_1(t - iT)\cos(2\pi f_0 t) - \sum_i b_i\, p_2(t - iT)\,\text{sen}(2\pi f_0 t)$$

$$E_s = \int_0^T E\{s^2(t)\}\, dt$$

$$E\{a_i \cdot a_{i+k}\} = \begin{cases} E\{a_i^2\} & k = 0 \\ E\{a_i\} \cdot E\{a_{i+k}\} & k = 0 \end{cases}$$ perché indipendenti

$$E\{a_i\} = (-2) \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 0$$

$$E\{a_i^2\} = \frac{4}{2} + \frac{4}{2} = 4$$

$$R_a[k] = 4\,\delta[k]$$

$$E\{b_i\, b_{i+k}\} = \begin{cases} E\{b_i^2\} = 1 & k = 0 \\ E\{b_i\} \cdot E\{b_{i+k}\} = 0 & k \neq 0 \end{cases}$$

$$R_b[k] = \delta[k]$$

$$E\{a_i\, b_{i+k}\} = E\{a_i\} \cdot E\{b_{i+k}\} = 0 \quad \forall k$$

$$E\left\{ \int_0^T \left(\sum_i a_i\, p_1(t-iT)\cos(2\pi f_0 t) - \sum_i b_i\, p_2(t-iT)\,\text{sen}(2\pi f_0 t) \right) \cdot \right.$$

$$\left(\sum_k a_k\, p_1(t-kT)\cos(2\pi f_0 t) - \sum_k b_k\, p_2(t-kT)\,\text{sen}(2\pi f_0 t) \right) dt$$

$$= \int_0^T \sum_i \sum_k E\{a_i a_k\}\, p_1(t-iT)\, p_1(t-kT)\cos^2(2\pi f_0 t) \; -$$

$$- \int_0^T \sum_i \sum_k \cancel{E\{a_i b_k\}\, p_1(t-iT)}\, p_2(t-kT)\cos(2\pi f_0 t)\cdot \text{sen}(2\pi f_0 t)$$

$$- \int_0^T \sum_i \sum_k \cancel{E\{b_i a_k\}\, p_2(t-iT)}\, p_1(t-kT)\,\text{sen}(2\pi f_0 t)\cdot\cos(2\pi f_0 t)$$

$$+ \int_0^T \sum_i \sum_k E\{b_i b_k\}\, p_2(t-iT)\, p_2(t-kT)\,\text{sen}^2(2\pi f_0 t)$$

$$E_S = \int_0^T \sum_i 4\, p_1^2(t-iT)\cos^2(2\pi f_0 t)\,dt + \int_0^T \sum_i 1\, p_2^2(t-iT)\,\text{sen}^2(2\pi f_0 t)\,dt =$$

$$= 4\int_{-\infty}^{+\infty} p_1^2(t)\cos^2(2\pi f_0 t)\,dt + \int_{-\infty}^{+\infty} p_2^2(t)\,\text{sen}^2(2\pi f_0 t)\,dt$$

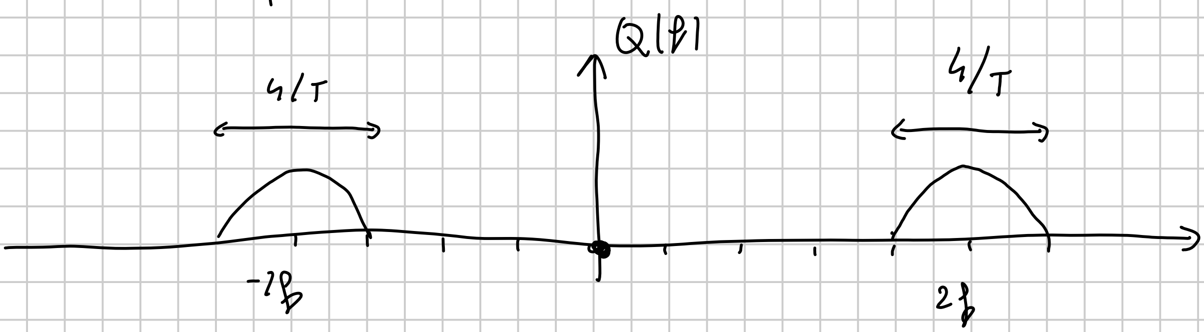
$$= 2\int_{-\infty}^{+\infty} p_1^2(t)\left(1+\cos(4\pi f_0 t)\right)dt + \int_{-\infty}^{+\infty} p_2^2(t)\left(\frac{1}{2}-\frac{1}{2}\,\mathrm{sen}(4\pi f_0 t)\right)dt =$$

$$= 2\int_{-\infty}^{+\infty} p_1^2(t)\,dt + \frac{1}{2}\int_{-\infty}^{+\infty} p_2^2(t)\,dt = 2+\frac{1}{2} = \frac{5}{2}$$

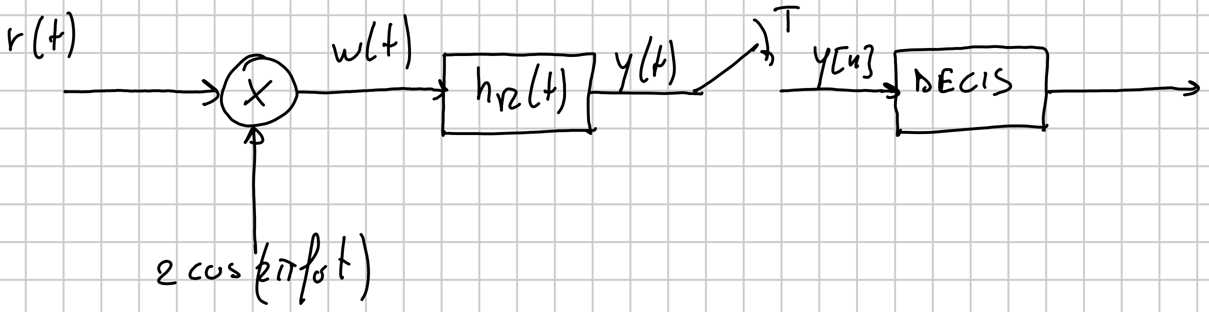
$$\Vert$$

$$2\int_{-\infty}^{+\infty} |P_1(f)|^2\,df + \frac{1}{2}\int_{-\infty}^{+\infty} |P_2(f)|^2\,df = \frac{5}{2}$$

$$\int_{-\infty}^{+\infty} \underbrace{p_1^2(t)\cdot\cos(4\pi f_0 t)}_{q(t)}\,df = Q(0) \qquad \text{se } f_0 \gg \frac{2}{T}$$



Esercizio 2 - 18/09/08



$$r(t) = \sum_{k=-\infty}^{+\infty} x[n]\, p(t-kT)\cos(2\pi f_0 t + \theta) + n(t) \qquad f_0 \gg \frac{1}{T}, \; \theta = \frac{\pi}{3}$$

$x[n] \in A_s = \{\pm 1\}$ equiprob. ed indip.

$$S_n(f) = \frac{N_0}{2}\left[\left(1 - \frac{|f-f_0|}{B}\right)\mathrm{rect}\left(\frac{f-f_0}{2B}\right) + \left(1 - \frac{|f+f_0|}{B}\right)\mathrm{rect}\left(\frac{f+f_0}{2B}\right)\right]$$

B = banda dell' impulso trasmesso

$p(t) \Rightarrow P(f) = \sqrt{T}\,\mathrm{rect}(Tf)$

$h_R(t) = p(t)$, $c(t) = \delta(t)$

1) $E_s = ?$

2) P_{n_u} (Potenza media del rumore in uscita da $h_R(t)$)

3) $P_E(b)$

Soluzione

1) $E_s = ?$

$$s_1(t) = p(t)\cos(2\pi f_0 t + \theta)$$
$$s_2(t) = -p(t)\cos(2\pi f_0 t + \theta)$$

$$E_{s_1} = E_{s_2} = \int_{-\infty}^{+\infty} p^2(t)\cos^2(2\pi f_0 t + \theta)\,dt = \frac{E_p}{2} = \frac{1}{2}$$

$$E_p = \int_{-\infty}^{+\infty} p^2(t)\, dt = \int_{-\infty}^{+\infty} P^2(f)\, df = T \cdot \frac{1}{T} = 1$$

$$E_s = \frac{1}{2} E_{s_1} + \frac{1}{2} E_{s_2} = \frac{1}{2}$$

2) P_{n_u}

$$w(t) = \underbrace{s(t) \cdot 2\cos(2\pi f_0 t)}_{w_s(t)} + \underbrace{n(t) \cdot 2\cos(2\pi f_0 t)}_{w_n(t)}$$

$$w_n(t) = \left[n_c(t) \cos(2\pi f_0 t) - n_s(t) \sin(2\pi f_0 t) \right] \cdot 2\cos(2\pi f_0 t)$$

$$= n_c(t) \left(1 + \cos(4\pi f_0 t)\right) - n_s(t) \sin(4\pi f_0 t)$$

Il filtro $h_R(t)$ è passa-basso per cui rimane solo la componente in b.b.

$$y(t) = s_u(t) + n_u(t)$$

$$n_u(t) = w_n(t) \otimes h_R(t) = n_c(t) \otimes h_R(t)$$

$$S_{n_u}(f) = S_{n_c}(f)\, |H_R(f)|^2$$

$$S_{n_c}(f) = S_n(f+f_0) + S_n(f-f_0) = N_0 \left(1 - \frac{|f|}{B}\right) \text{rect}\left(\frac{f}{2B}\right)$$

$$S_{n_u}(f) = N_0 \cdot \frac{1}{2B} \left(1 - \frac{|f|}{B}\right) \text{rect}\left(\frac{f}{2B}\right)$$

$$B = \frac{1}{2T} \Rightarrow T = \frac{1}{2B}$$

$$P_{n_u} = \int_{-\infty}^{+\infty} S_{n_u}(f)\, df = \frac{N_0}{2B} \cdot B = \frac{N_0}{2}$$

3) $P_E(b) = P_E(2) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) = Q\left(\sqrt{SNR}\right)$

VALIDA SOTTO L'IPOTESI: NO ISI

$$SNR = \frac{E[s_u^2[n]]}{E[n_u^2[n]]} = \frac{E[s_u^2[n]]}{P_{n_u}}$$

Bisogna determinare $s_u[n]$

$$s_u(t) = w_s(t) \otimes h_R(t)$$

$$w_s(t) = 2 \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT) \cos(2\pi f_0 t + \theta) \cos(2\pi f_0 t)$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT) \left[\cos\theta + \underbrace{\cos(4\pi f_0 t + \theta)}_{\text{viene eliminata dal } h_R(t)} \right]$$

$$s_u(t) = \sum_{n=-\infty}^{+\infty} x[n]\, h(t-nT) \cos\theta$$

$$h(t) = p(t) \otimes h_R(t) = p(t) \otimes p(t) = p(t) \otimes p(-t) = c_p(t)$$

$$s_u[n] = x[n]\, c_p(0) \cos\theta + \underbrace{\sum_{\substack{n=-\infty \\ n \neq k}}^{+\infty} x[n]\, c_p((n-k)T) \cos\theta}_{ISI}$$

$$c_p(t) = TCF^{-1}\left[P^2(f) \right] = \mathrm{sinc}\left(\frac{t}{T}\right) \Rightarrow c_p((n-k)T) = 0 \quad n \neq k$$

NO ISI

$$c_p(0) = \int_{-\infty}^{+\infty} P^2(f)\, df = 1$$

$$s_u[n] = x[n] \cos\theta$$

$$E[s_u^2[n]] = \cos^2\theta\, E[x^2[n]] = \cos^2\theta$$

$$SNR = \frac{\cos^2\theta}{N_0/2}$$

$$P_e(b) = Q\left(\sqrt{\frac{\cos^2\theta}{N_0/2}}\right)$$

(Applicabile vista l'assenza di ISI e simboli equip. ed ind.)

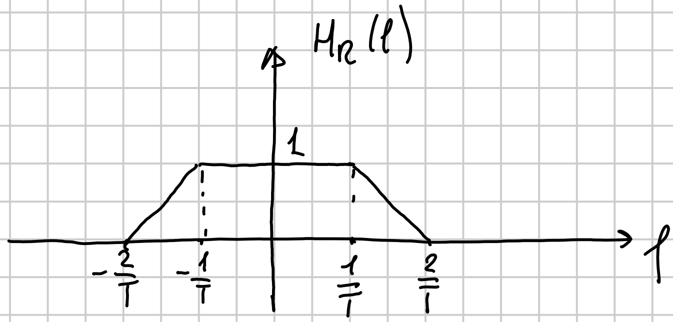
Es. 2 - 12/11/09

$$r(t) = \sum_{k=-\infty}^{+\infty} x[k]\, p(t-kT) + n(t)$$

$x[n] \in A_s = \{0, 4\}$

$n(t)$ è Gaussiano Bianco con $S_n(f) = \frac{N_0}{2}$

$p(t) \Rightarrow P(f) = T \cdot |fT| \operatorname{rect}\left(\frac{fT}{2}\right)$



Strategia di decisione:

$$\hat{x}[n] = \begin{cases} 0 & \text{se } y[n] \le \lambda \\ 4 & \text{se } y[n] > \lambda \end{cases}, \quad \lambda = \frac{3}{2}$$

Calcolare:

1) E_s
2) P_{n_u}
3) $P_E(b)$

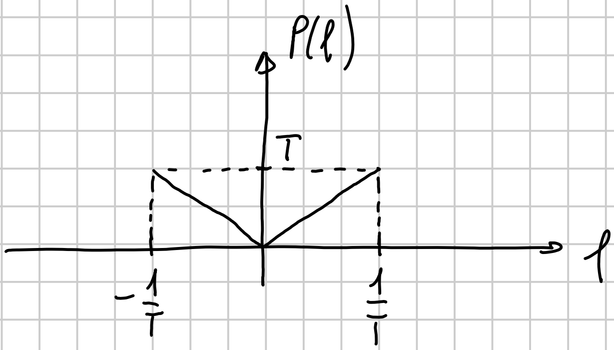
Soluzione

1) $E_s = ?$

$s_1(t) = 0 \quad , \quad s_2(t) = 4p(t)$

$$E_{s_1} = 0$$

$$E_{s_2} = \int_{-\infty}^{+\infty} 16\, p^2(t)\, dt = 16 \int_{-\infty}^{+\infty} P^2(f)\, df = 16 \cdot \frac{2}{3} T^2 \cdot \frac{1}{T} = \frac{32}{3} T$$



$$E_s = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{32}{3} T = \frac{16}{3} T$$

2) $P_{n_u} = ?$

$$S_{n_u}(f) = S_n(f)\, |H_R(f)|^2 = \frac{N_0}{2} |H_R(f)|^2$$

$$P_{n_u} = \int_{-\infty}^{+\infty} S_{n_u}(f)\, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2\, df =$$

$$= \frac{N_0}{2} \cdot \left(\frac{2}{T} + \frac{2}{3} \cdot \frac{1}{T} \right) = \frac{4}{3} \frac{N_0}{T}$$

3) $P_E(b) = ?$

$$\begin{cases} P_E(b) = Q\left(\sqrt{SNR}\right) \\ SNR = \dfrac{E[s_u^2[n]]}{P_{n_u}} \end{cases}$$

(se i simboli sono equip. ind., se c'è assenza di ISI e se la soglia è quella ottima)

$$s_u(t) = \sum_{k=-\infty}^{+\infty} x[n]\, h(t - kT)$$

$$h(t) = p(t) \otimes h_R(t)$$

$$H(f) = P(f)\, H_R(f) = P(f)$$

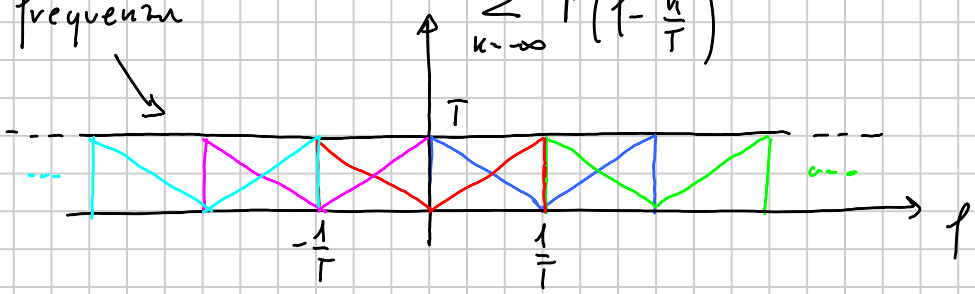
Verifica sull'ISI:

•) $\displaystyle\sum_{k=-\infty}^{+\infty} P\left(f - \frac{k}{T}\right) = \text{cost}$ criterio di Nyquist in freq.

•) $p(kT) = \begin{cases} \neq 0 & \text{se } k = 0 \\ = 0 & \text{altrove} \end{cases}$ criterio di Nyquist nel tempo

Verifica in entrambi i modi

in frequenza



Nel tempo:

$$P(f) = T\,\text{rect}\left(\frac{f}{2/T}\right) - T\left(1 - \frac{|f|}{1/T}\right)\text{rect}\left(\frac{f}{2/T}\right)$$

$$p(t) = 2\,\text{sinc}\left(\frac{t}{T/2}\right) - \text{sinc}^2\left(\frac{t}{T}\right)$$

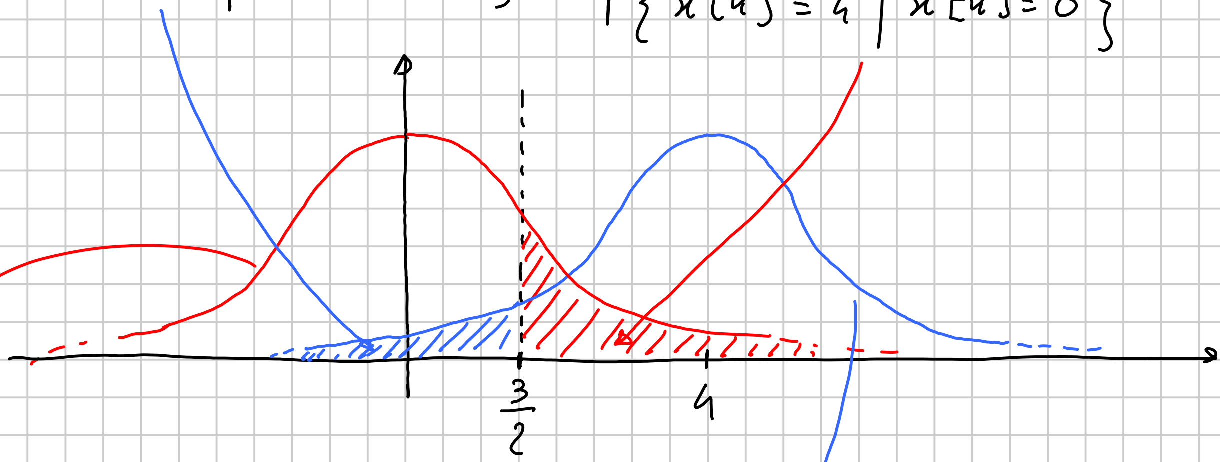
$$p(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$P_E(b) = P_E(2) = P\{\hat{x}[n] = \alpha_1 \mid x[n] = \alpha_2\}\, P\{\alpha_2\} + P\{\hat{x}[n] = \alpha_2 \mid x[n] = \alpha_1\}\, P\{\alpha_1\}$$

$$P\{\alpha_1\} = P\{\alpha_2\} = \frac{1}{2}$$

$$P\{\hat{x}[n]=0 \mid x[n]=4\}$$

$$P\{\hat{x}[n]=4 \mid x[n]=0\}$$



$$f_n\left(\frac{y-0}{\sigma_{n_u}}\right)=\frac{1}{\sqrt{2\pi\sigma_{n_u}^2}}\, e^{-\frac{y^2}{2\sigma_{n_u}^2}} \quad , \quad f_n\left(\frac{y-4}{\sigma_{n_u}}\right)=\frac{1}{\sqrt{2\pi\sigma_{n_u}^2}}\, e^{-\frac{(y-4)^2}{2\sigma_{n_u}^2}}$$

$$f_{Y|\alpha_1}(y|\alpha_1) \in \mathcal{N}(0,\sigma_{n_u}^2)$$

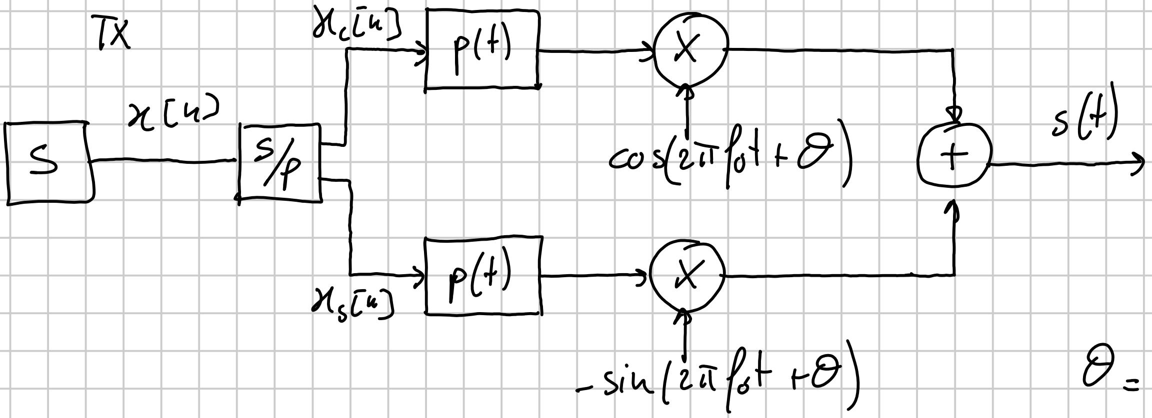
$$f_{Y|\alpha_2}(y|\alpha_2) \in \mathcal{N}(4,\sigma_{n_u}^2)$$

$$\sigma_{n_u}^2 = P_{n_u} = \frac{4}{3}\frac{N_0}{T}$$

$$P_E(b) = \frac{1}{2}\, Q\left(\frac{3/2}{\sigma_{n_u}}\right) + \frac{1}{2}\, Q\left(\frac{4-3/2}{\sigma_{n_u}}\right)$$

Esercizio su QAM

TX



$\theta = \frac{\pi}{4}$

$$p(t) \Rightarrow P(f) = \left[1 + \cos\left(2\pi \frac{f}{B}\right)\right] \text{rect}\left(\frac{f}{2B}\right), \quad B = \frac{2}{T} \ll f_0$$

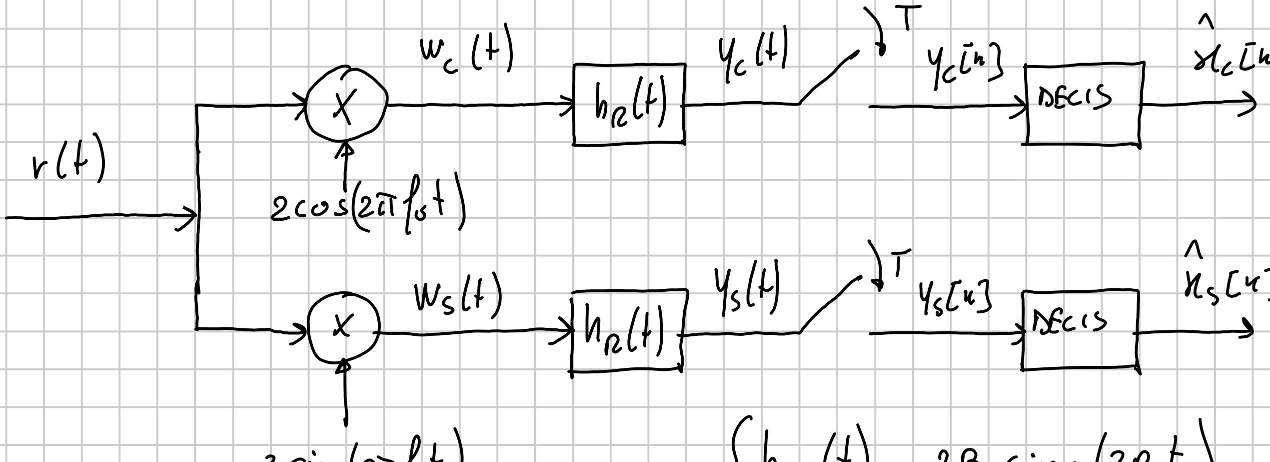
$$x_c[n] \in A_S^c = \{\pm 1\}, \quad x_s[n] \in A_S^s = \{\pm 1\}$$

Canale:

$$c(t) = \delta(t)$$

$$n(t) \Rightarrow S_n(f) = \frac{N_0}{2}\left[\text{rect}\left(\frac{f - f_0}{2B}\right) + \text{rect}\left(\frac{f + f_0}{2B}\right)\right]$$

Ricevitore:



$$\begin{cases} h_R(t) = 2B \,\text{sinc}(2Bt) \\ \lambda_c = \lambda_s = 0 \end{cases}$$

Calcolare

1) $E_S = ?$
2) $\eta_B = ?$
3) $P_{n_{u_c}}, P_{n_{u_s}}$
4) $P_E(4) = ?$

Soluzione

1) $E_s = ?$

$$s_1(t) = p(t)\cos(2\pi f_0 t + \vartheta) - p(t)\sin(2\pi f_0 t + \vartheta) \qquad (1,1)$$
$$s_2(t) = p(t)\cos(2\pi f_0 t + \vartheta) + p(t)\sin(2\pi f_0 t + \vartheta) \qquad (1,-1)$$
$$s_3(t) = -p(t)\cos(2\pi f_0 t + \vartheta) - p(t)\sin(2\pi f_0 t + \vartheta) \qquad (-1,1)$$
$$s_4(t) = -p(t)\cos(2\pi f_0 t + \vartheta) + p(t)\sin(2\pi f_0 t + \vartheta) \qquad (-1,-1)$$

$$E_{s_1} = \int_{-\infty}^{+\infty} s_1^2(t)\,dt = \int_{+\infty}^{+\infty} p^2(t)\left[\cos(2\pi f_0 t) - \sin(2\pi f_0 t)\right]^2 dt$$

$$(\cos\alpha \pm \sin\alpha)^2 = \cos^2\alpha + \sin^2\alpha \pm 2\cos\alpha\sin\alpha =$$
$$= \frac{1}{2} + \frac{1}{2}\cos 2\alpha + \frac{1}{2} - \frac{1}{2}\cos 2\alpha \pm \sin 2\alpha =$$
$$= 1 \pm \sin 2\alpha$$

$$E_{s_1} = \int_{-\infty}^{+\infty} p^2(t)\left(1 - \sin(4\pi f_0 t)\right) dt \simeq E_p$$

$$E_{s_2} = \int_{-\infty}^{+\infty} p^2(t)\left(\cos(2\pi f_0 t) + \sin(2\pi f_0 t)\right)^2 dt =$$
$$= \int_{-\infty}^{+\infty} p^2(t)\left(1 + \sin(4\pi f_0 t)\right) dt \simeq E_p$$

$$s_3(t) = -s_2(t) \Rightarrow E_{s_3} = E_{s_2} \simeq E_p$$

$$s_4(t) = -s_1(t) \Rightarrow E_{s_4} = E_{s_1} \simeq E_p$$

$$E_s = \frac{1}{4}E_{s_1} + \frac{1}{4}E_{s_2} + \frac{1}{4}E_{s_3} + \frac{1}{4}E_{s_4} \simeq E_p$$

$$E_S = E\left[\int_{-\infty}^{+\infty} \left[x_c[n] \cos(2\pi f_0 t) - x_s[n] \sin(2\pi f_0 t) \right]^2 p^2(t)\, dt \right] =$$

$$= \int_{-\infty}^{+\infty} E\left[x_c^2[n] \right] \cos^2(2\pi f_0 t)\, p^2(t)\, dt \; +$$

$$+ \int_{-\infty}^{+\infty} E\left[x_s^2[n] \right] \sin^2(2\pi f_0 t)\, p^2(t)\, dt \; +$$

$$- 2 \int_{-\infty}^{+\infty} \underbrace{E\left[x_c[n]\, x_s[n] \right]}_{=0} \cos(2\pi f_0 t) \sin(2\pi f_0 t)\, p^2(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \left[\frac{1}{2} + \frac{1}{2} \cos(4\pi f_0 t) \right] p^2(t)\, dt \; +$$

$$+ \int_{-\infty}^{+\infty} \left[\frac{1}{2} - \frac{1}{2} \cos(4\pi f_0 t) \right] p^2(t)\, dt \; \approx \; E_p$$

2) $\eta_B = ?$

$$\eta_B = \frac{R_b}{B_T} = \frac{\log_2 M}{B_T\, T}$$

$$R_b = \frac{2}{T} \quad , \quad B_T = \frac{2}{T} \quad \Rightarrow \quad \eta_B = 1$$

3) $P_{n_{u_c}} = ?$, $P_{n_{u_s}} = ?$

$$S_{n_c}(f) = S_{n_s}(f) = N_0 \, \mathrm{rect}\left(\frac{f}{2B} \right)$$

$$S_{n_{u_c}}(f) = S_{n_c}(f) \left| H_R(f) \right|^2 = N_0 \, \mathrm{rect}\left(\frac{f}{2B} \right) \cdot \left(\mathrm{rect} \frac{f}{2B} \right)^2 =$$

$$= N_0 \, \mathrm{rect}\left(\frac{f}{2B} \right) = S_{n_{u_s}}(f)$$

$$P_{n_{u_c}} = P_{n_{u_s}} = 2 B N_0 = \frac{4}{T} N_0$$

4) $P_E(4) = ?$

$$P_E(4) = P_E^c(2)\left(1 - P_E^s(2)\right) + P_E^s(2)\left(1 - P_E^c(2)\right) + P_E^c(2)\, P_E^s(2)$$

$$\left.\begin{aligned} P_E^c(2) &= Q\left(\sqrt{SNR_c}\right) \\ P_E^s(2) &= Q\left(\sqrt{SNR_s}\right) \end{aligned}\right\}$$ vere quando non c'e' ISI, i simboli sono equip. ed ind. e soglia ottima

$$SNR_c = \frac{E\left[s_{u_c}^2[n]\right]}{P_{n_{u_c}}}$$

$$SNR_s = \frac{E\left[s_{u_s}^2[n]\right]}{P_{n_{u_s}}}$$

N.B. al denominatore $E\left[n_{u_c}^2[n]\right] = P_{n_{u_c}}$ essendo il rumore a media nulla (stesso per il ramo in quad.)

→ Verifica sulla assenza di ISI

$$H(f) = H_R(f)\, P(f) = P(f) = H_{RC}(f)$$

$P(f)$ e' a coseno rialzato per cui e' soddisfatto il criterio di Nyquist ⇒ NO ISI

→ Calcolo di $E\left[s_{u_c}^2[n]\right]$

$$s_{u_c}(t) = \sum_{n=-\infty}^{+\infty} x_c[n]\, h(t - nT)$$

$$s_{u_c}[n] = x[n]\, h(0)$$

$$E\left[s_{u_c}^2[n]\right] = E\left[x_c^2[n]\right] h^2(0) = h^2(0) = \left[\int_{-\infty}^{+\infty} H(f)\, df\right]^2 =$$

$$= \left[\int_{-B}^{B} \left[1 + \cos\left(2\pi \frac{f}{B}\right)\right] df\right]^2 = 4B^2$$

$$SNR_c = \frac{4B^2}{2N_0 B} = \frac{2B}{N_0}$$

$SNR_s = \dfrac{2B}{N_0}$ (ripetendo i calcoli ed essendo $E\left[x_s^2[n]\right] = E\left[x_c^2[n]\right] = 1$)

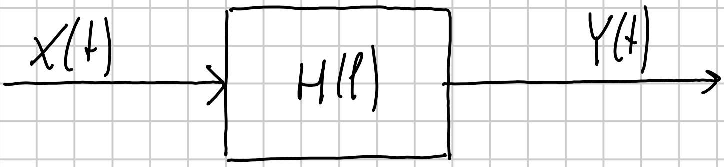
$$P_E^c(2) = P_E^s(2) = Q\left(\sqrt{\frac{2B}{N_0}}\right) = P_E(2)$$

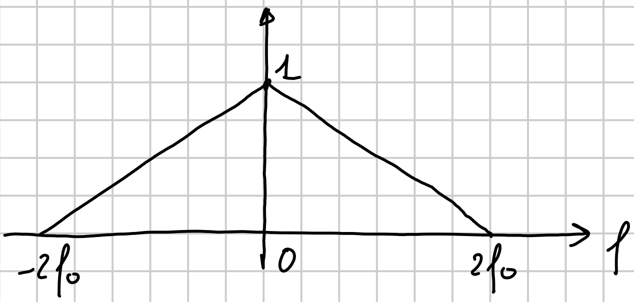
$$P_E(4) = 2\,P_E(2)\left(1 - P_E(2)\right) + P_E^2(2) = 2\,P_E(2) - P_E^2(2)$$

Esercizio su Processi Aleatori Parametrici

$$X(t) = A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)$$

$$A, B \in \mathcal{N}(0, \sigma^2), \quad \sigma^2 = 4 \qquad \text{indipendenti}$$





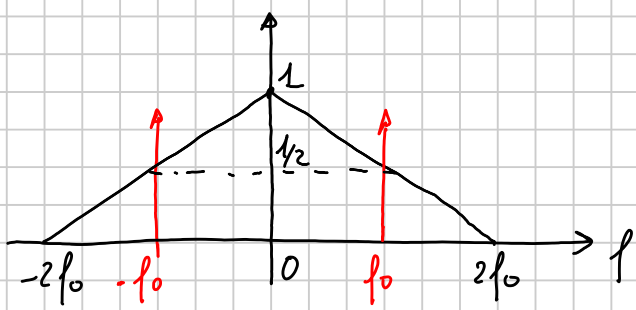
-) Ricavare la $f_Y(y;t)$

Soluzione

Per ogni realizzazione di $X(t)$

$$x_r(t) \Rightarrow X_r(f) = \frac{A}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) + \frac{B}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

$$\Rightarrow Y_r(f) = X_r(f)\,H(f) = \frac{1}{2}X_r(f) \Rightarrow y_r(t) = \frac{1}{2}x_r(t)$$



$$Y(t) = \frac{1}{2} X(t)$$

$$Y = Y(\bar{t}) = A \cos(2\pi f_0 \bar{t}) + B \sin(2\pi f_0 \bar{t}) =$$

$$= K_1 \frac{A}{2} + K_2 \frac{B}{2} \quad , \quad K_1 = \cos(2\pi f_0 \bar{t}) \, , \, K_2 = \sin(2\pi f_0 \bar{t})$$

$$= Y_A + Y_B \quad , \quad Y_A = K_1 \frac{A}{2} \quad , \quad Y_2 = K_2 \frac{B}{2}$$

Y è una V.A. Gaussiana essendo la somma di v.a. Gaussiane. Inoltre essendo A e B indipendenti, lo sono anche Y_A e Y_B. Per cui Y_A e Y_B sono incorrelate.

$$\Rightarrow \eta_{Y_A} = E[K_1 A] = 0$$

$$\Rightarrow \eta_{Y_B} = E[K_2 B] = 0$$

$$\Rightarrow \sigma_{Y_A}^2 = E[K_1^2 A^2] = \frac{K_1^2}{4} \sigma^2$$

$$= \sigma_{Y_B}^2 = E[K_2^2 B^2] = \frac{K_2^2}{4} \sigma^2$$

$$E_Y = E[Y_A + Y_B] = 0$$

$$\sigma_Y^2 = E[(Y_A + Y_B)^2] = E[Y_A^2] + E[Y_B^2] + \underbrace{2 E[Y_A Y_B]}_{\substack{= \\ 0 \\ \text{(incorrelate)}}}$$

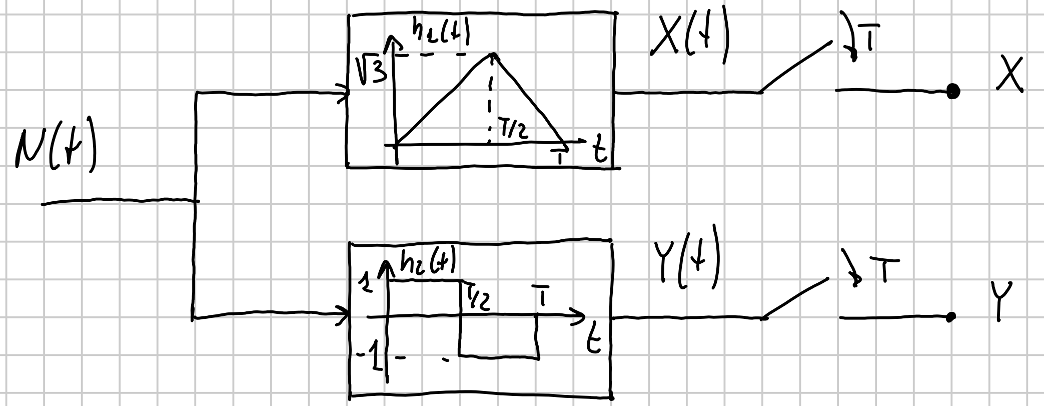
$$= \left(\frac{K_1^2 + K_2^2}{4} \right) \sigma^2 =$$

$$= \left[\cos^2(2\pi f_0 t) + \sin^2(2\pi f_0 t) \right] \frac{4}{4} = 1$$

$$\Rightarrow f_Y(y;t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \in \mathcal{N}(0,1)$$

V.A. GAUSSIANA STANDARD

Esercizio su processi aleatori non-parametrici



$N(t)$ è Gaussiano con $S_n(f) = \frac{N_0}{2}$

- Calcolare la $P\{X > Y\}$
- Calcolare $E[XY]$

Soluzione:

$$\left.\begin{aligned} X &= \int_{-\infty}^{+\infty} N(\tau)\, h_1(T-\tau)\, d\tau \\ Y &= \int_{-\infty}^{+\infty} N(\tau)\, h_2(T-\tau)\, d\tau \end{aligned}\right\} \text{ sono v.a. Gaussiane}$$

Dobbiamo determinare valor medio e varianza

$\eta_n(t) = 0$ essendo $S_n(f) = \frac{N_0}{2}$ (rumore bianco)

Se $\eta_n(t) \neq 0 \Rightarrow S_n(f)$ avrebbe un $\delta(f)$

$$\eta_X(t) = \eta_N(t) \otimes h_1(t) = 0$$

$$\eta_Y(t) = \eta_N(t) \otimes h_2(t) = 0$$

$$S_X(f) = S_N(f)\,|H_1(f)|^2$$

$$P_X = \int_{-\infty}^{+\infty} S_N(f)\,|H_1(f)|^2\, df = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H_1(f)|^2\, df =$$

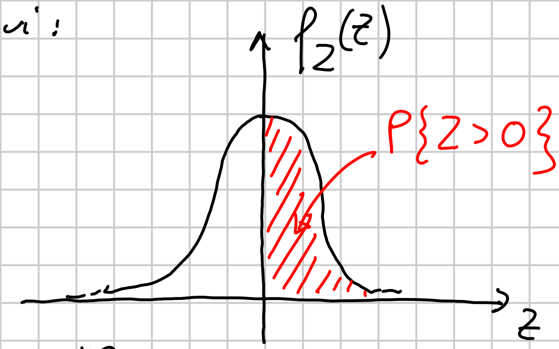
$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} |h_1(t)|^2\, dt = \frac{N_0}{2}\,3\,\frac{2}{3}\,\frac{T}{2} = \frac{N_0 T}{2} = \sigma_X^2$$

$$P_Y = \frac{N_0}{2}\int_{-\infty}^{+\infty} |h_2(t)|^2\, dt = \frac{N_0 T}{2} = \sigma_Y^2$$

$$Z = X - Y \quad \in \mathcal{N}(0, \sigma_Z^2) \quad \Leftarrow E[X-Y] = 0$$

Indipendentemente dal valor di σ_Z^2, $f_Z(z)$ è una Gaussiana a valor medio nullo, per cui:

$$P\{X > Y\} = P\{Z > 0\} = \frac{1}{2}$$



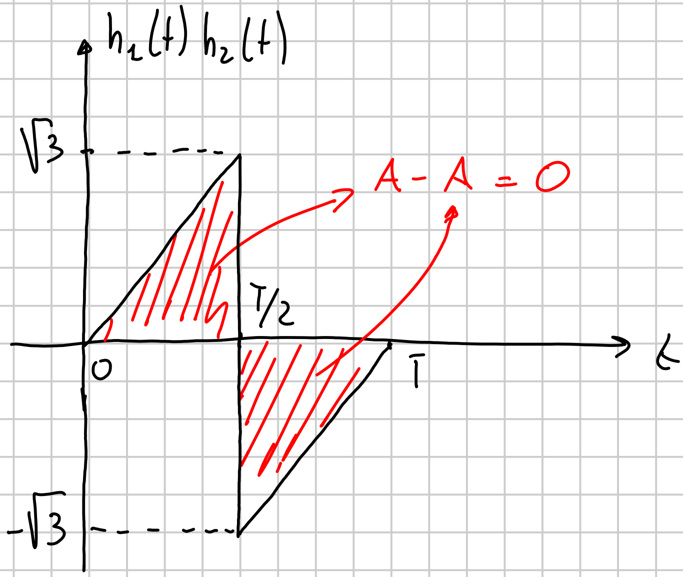
$$E[XY] = E\left[\int_{-\infty}^{+\infty} N(\tau)\, h_1(T-\tau)\, d\tau \int_{-\infty}^{+\infty} N(\alpha)\, h_2(T-\alpha)\, d\alpha\right] =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[N(\tau)N(\alpha)]\, h_1(T-\tau)\, h_2(T-\alpha)\, d\tau\, d\alpha =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{N_0}{2}\,\delta(\tau-\alpha)\, h_1(T-\tau)\, h_2(T-\alpha)\, d\tau\, d\alpha =$$

$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} h_1(T-\alpha)\, h_2(T-\alpha)\, d\alpha =$$

$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} h_1(t)\, h_2(t)\, dt = 0 \qquad \text{(vedi grafico sotto)}$$

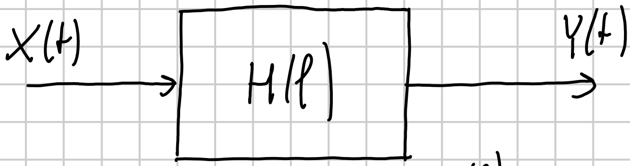


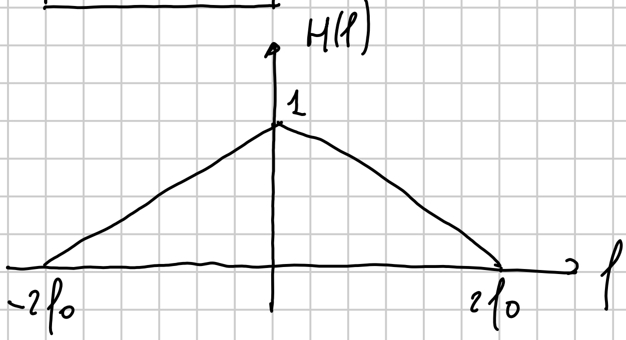
$h_1(t)\, h_2(t)$
$\sqrt{3}$
$-\sqrt{3}$
0
$T/2$
T
t
$A - A = 0$

Es. PROCESSI ALEATORI PARAMETRICI

$$X(t) = A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)$$

$A, B \in \mathcal{N}(0, \sigma^2)$, $\sigma^2 = 4$ indipendenti



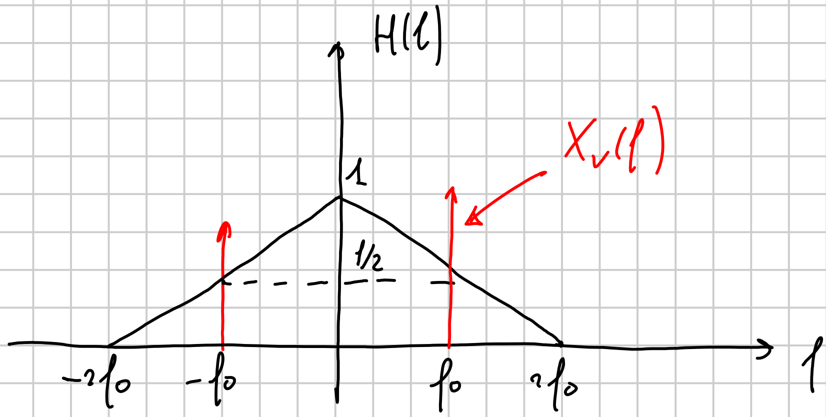


·) Ricavare la $f_Y(y;t)$

Solutione :

$$x_r(t) \Rightarrow X_r(f) = \frac{A}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) + \frac{B}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

$$Y_r(f) = X_r(f)\,H(f) =$$



$$Y_r(f) = \frac{1}{2}X_r(f) \Rightarrow y_r(t) = \frac{1}{2}x_r(t)$$

$$Y(t) = \frac{1}{2}X(t) = \frac{A}{2}\cos(2\pi f_0 t) + \frac{B}{2}\sin(2\pi f_0 t)$$

$$Y = Y(\bar{t}) = \frac{A}{2}\cos(2\pi f_0 \bar{t}) + \frac{B}{2}\sin(2\pi f_0 \bar{t})$$

$$= K_1 A + K_2 B$$

$$K_1 = \frac{1}{2}\cos(2\pi f_0 \bar{t}) \quad , \quad K_2 = \frac{1}{2}\sin(2\pi f_0 \bar{t})$$

$$Y \in \mathcal{N}(\eta_Y, \sigma_Y^2)$$

$$f_Y(y) = f_Y(y;t) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\eta_Y)^2}{2\sigma_Y^2}}$$

$$\eta_Y = E[K_1 A + K_2 B] = K_1 \underset{\overset{\shortparallel}{0}}{E[A]} + K_2 \underset{\overset{\shortparallel}{0}}{E[B]} = 0$$

$$\sigma_Y^2 = E[Y^2] - \eta_Y^2 = E[Y^2] = E[(K_1 A + K_2 B)^2] =$$

$$= K_1^2 E[A^2] + K_2^2 E[B^2] + 2K_1K_2 \underbrace{E[AB]}_{=0} =$$

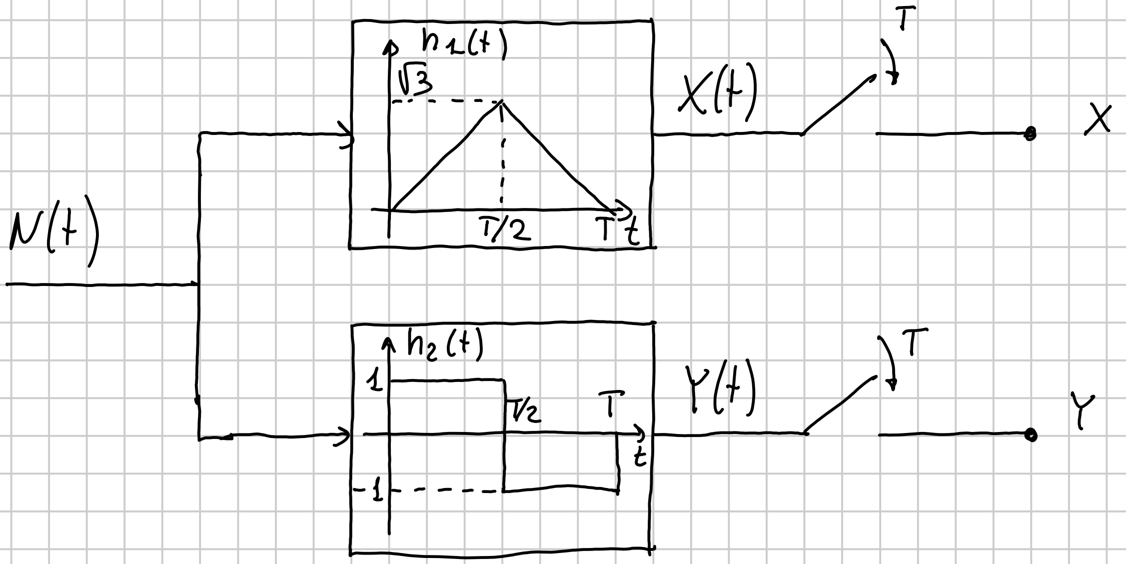
$$= K_1^2\sigma^2 + K_2^2\sigma^2 = (K_1^2 + K_2^2)\sigma^2$$

$$K_1^2 + K_2^2 = \frac{1}{4}\cos^2(2\pi f_0 t) + \frac{1}{4}\sin^2(2\pi f_0 t) = \frac{1}{4}$$

$$\sigma_Y^2 = \frac{1}{4}\cdot 4 = 1$$

$$Y \in \mathcal{N}(0,1) \qquad (\text{STANDARD})$$

ES. PROCESSI ALEATORI NON - PARAMETRICI



$N(t)$ Gaussiano con $S_n(f) = \frac{N_0}{2}$

Calcolare:

- $P\{X > Y\}$
- $E[XY]$

Soluzione:

$$X = X(T) = \left[N(t) \otimes h_1(t)\right]_{t=T} = \int_{-\infty}^{+\infty} N(\tau)\, h_1(T-\tau)\, d\tau$$

$$Y = Y(T) = \int_{-\infty}^{+\infty} N(\tau)\, h_2(T-\tau)\, d\tau$$

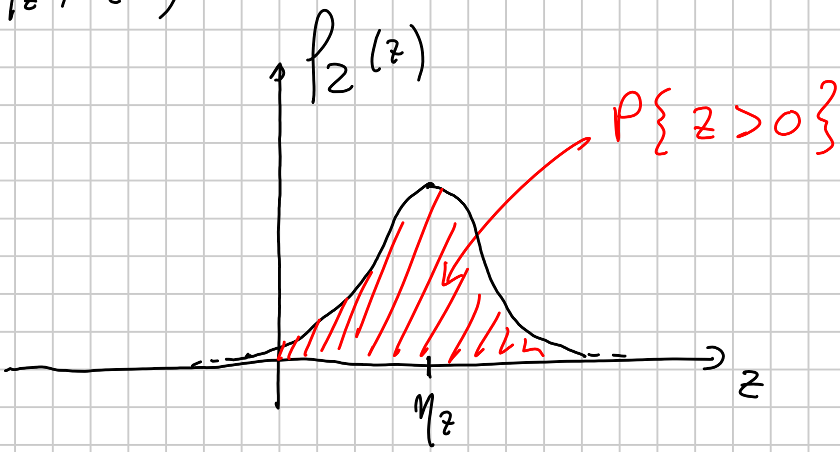
$$X, Y \in \mathcal{N}$$

$$X \in \mathcal{N}(\eta_X, \sigma_X^2) \quad , \quad Y \in \mathcal{N}(\eta_Y, \sigma_Y^2)$$

$$P\{X > Y\}$$

$$Z = X - Y \quad \Rightarrow \quad P\{X > Y\} = P\{Z > 0\}$$

$$z \in \mathcal{N}(\eta_z, \sigma_z^2)$$



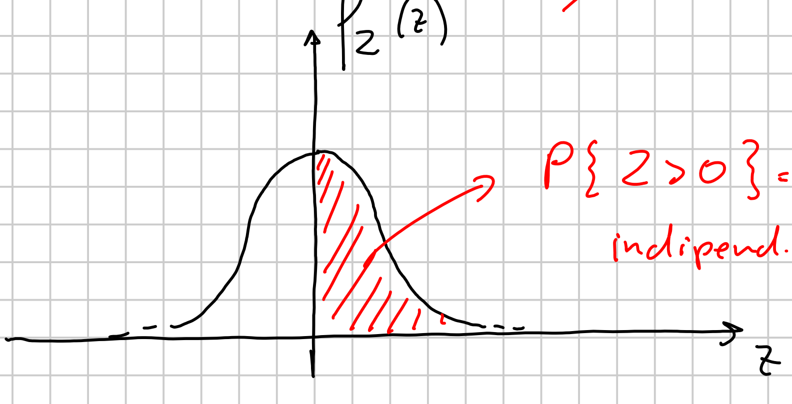
$$\eta_z = E[X - Y] = E\left[\int_{-\infty}^{+\infty} N(\tau)\, h_1(T-\tau)\, d\tau + \right.$$
$$\left. - \int_{-\infty}^{+\infty} N(\tau)\, h_2(T-\tau)\, d\tau \right]$$

$$= E\left[\int_{-\infty}^{+\infty} N(\tau)\left[h_1(T-\tau) - h_2(T-\tau)\right] d\tau\right] =$$

$$= \int_{-\infty}^{+\infty} \underbrace{E[N(\tau)]}_{=0} \left[h_1(T-\tau) - h_2(T-\tau)\right] d\tau = 0$$

$$E[N(\tau)] = 0$$

$$W(t) = N(t) + A \quad \Rightarrow \quad \cancel{S_W(f) = \frac{N_0}{2}}$$



$P\{Z>0\} = P\{X>Y\} = \frac{1}{2}$

indipend. da σ_z^2

2) $E[XY] = E\left[\int_{-\infty}^{+\infty} N(\tau)\, h_1(T-\tau)\, d\tau \int_{-\infty}^{+\infty} N(\alpha)\, h_2(T-\alpha)\, d\alpha \right]$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[N(\tau)N(\alpha)]\, h_1(T-\tau)\, h_2(T-\alpha)\, d\tau\, d\alpha$$

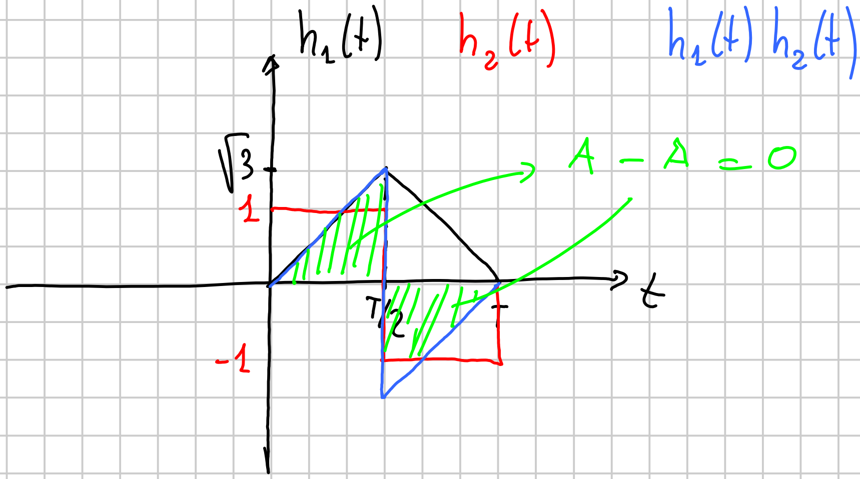
$$E[N(\tau)\, N(\alpha)] = R_N(\beta) = \frac{N_0}{2}\delta(\beta) = \frac{N_0}{2}\delta(\tau-\alpha)$$

$$\beta = \tau - \alpha$$

$$E[XY] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{N_0}{2}\, \delta(\tau-\alpha)\, h_1(T-\tau)\, h_2(T-\alpha)\, d\tau\, d\alpha$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h_1(T-\tau)\, h_2(T-\tau)\, d\tau \qquad (T-\tau = t)$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h_1(t)\, h_2(t)\, dt = 0$$



X e Y incorrelate

$$E[(X-\eta_X)(Y-\eta_Y)] = E[XY] = 0$$

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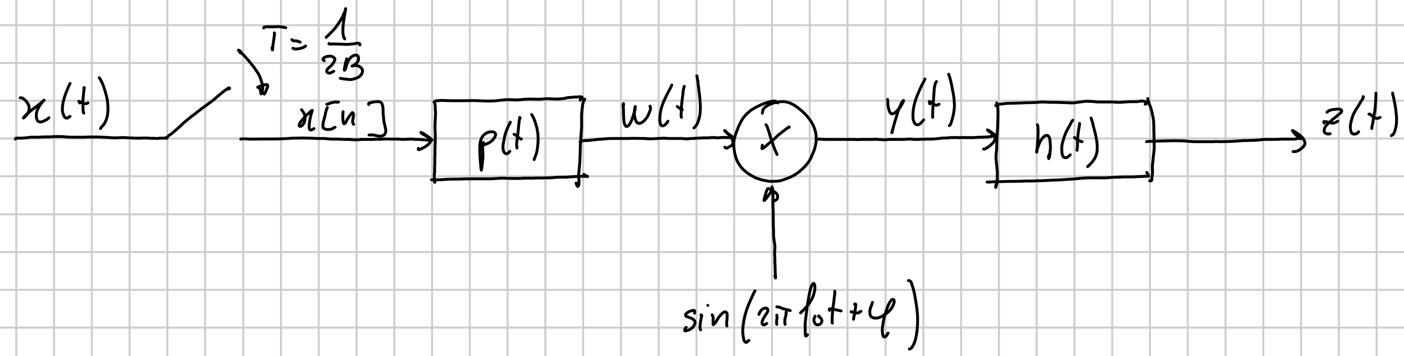


$x(t) = B \operatorname{sinc}^2(Bt)$

$h(t) = 2B \operatorname{sinc}(2Bt)$

$p(t) = 2B \operatorname{sinc}(2Bt) \cos(4\pi Bt)$

Si calcoli

1) $z(t)$

2) E_z, P_z

3) φ tale che E_z sia max

Soluzione:

1) $X(f) = \left(1 - \frac{|f|}{B}\right) \operatorname{rect}\left(\frac{f}{2B}\right)$

$$\bar{X}(f) = \frac{1}{T} \sum_{K=-\infty}^{+\infty} X\left(f - \frac{K}{T}\right) = 2B \sum_{K=-\infty}^{+\infty} X(f - K2B)$$

$$P(f) = \operatorname{rect}\left(\frac{f}{2B}\right) \otimes \left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right] =$$

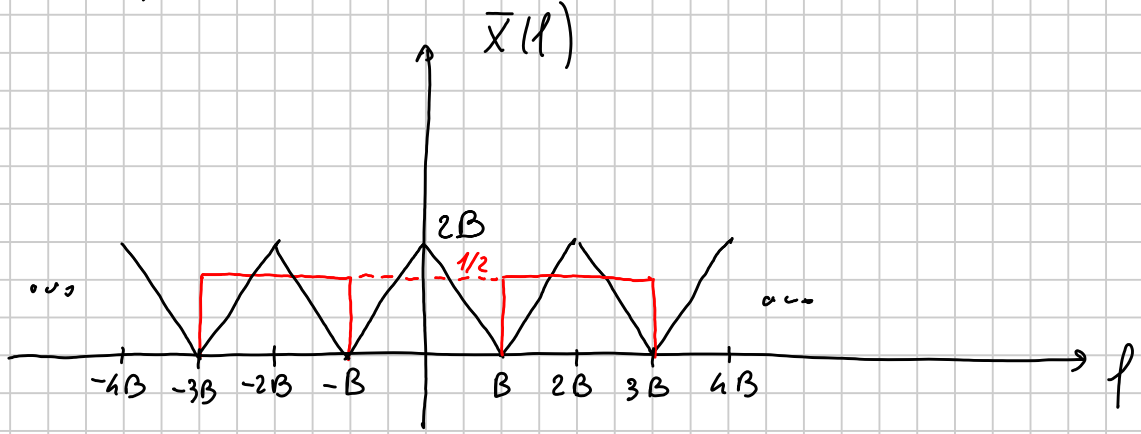
$$= \operatorname{rect}\left(\frac{f-f_0}{2B}\right) + \operatorname{rect}\left(\frac{f+f_0}{2B}\right)$$

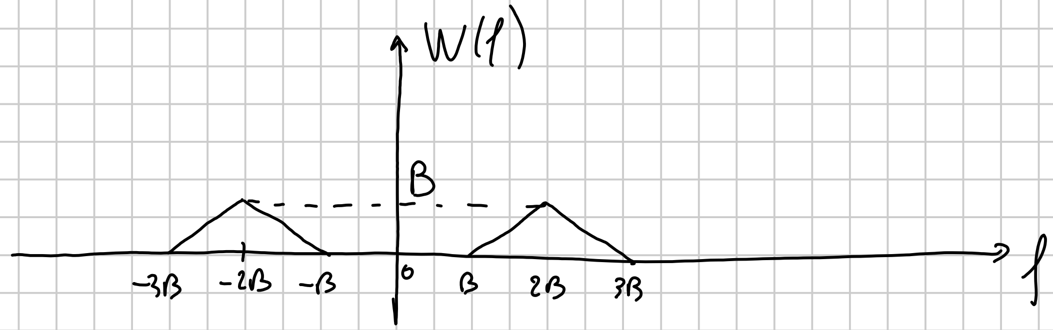
$f_0 = 2B$

$$= \frac{1}{2}\operatorname{rect}\left(\frac{f-2B}{2B}\right) + \frac{1}{2}\operatorname{rect}\left(\frac{f+2B}{2B}\right)$$

$$H(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$W(f) = \bar{X}(f)\, P(f)$$





$$W(f) = B\left(1 - \frac{|f-2B|}{B}\right)\mathrm{rect}\left(\frac{f-2B}{2B}\right) + B\left(1 - \frac{|f+2B|}{B}\right)\mathrm{rect}\left(\frac{f+2B}{2B}\right)$$

$$c(t) = \sin(2\pi f_0 t + \varphi) = \cos(2\pi f_0 t + \varphi') \quad , \quad \varphi' = \varphi - \frac{\pi}{2}$$

$$C(f) = \frac{e^{j\varphi'}}{2}\delta(f-f_0) + \frac{e^{-j\varphi'}}{2}\delta(f+f_0)$$

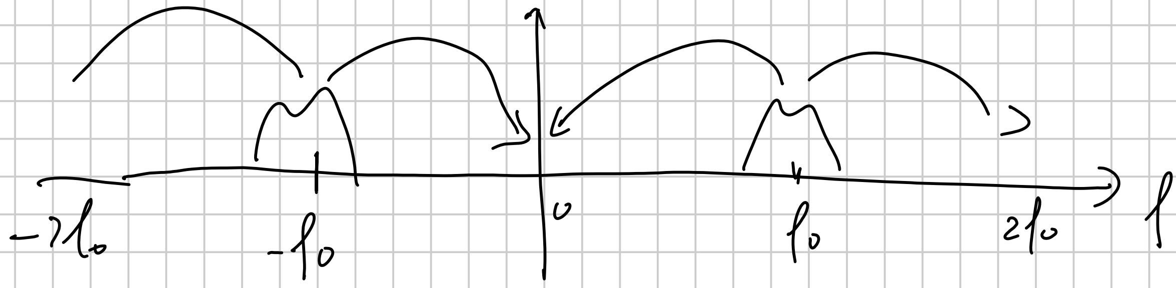
$f_0 = 2B$

$$= \frac{e^{j\varphi'}}{2}\delta(f-2B) + \frac{e^{-j\varphi'}}{2}\delta(f+2B)$$

$$Y(f) = W(f) \otimes C(f) = W(f-2B)\frac{e^{j\varphi'}}{2} + W(f+2B)\frac{e^{-j\varphi'}}{2}$$

$$= B\frac{e^{j\varphi'}}{2}\left(1 - \frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right) + \mathrm{comp}\,(2f_0)$$

$$+ \frac{B e^{-j\varphi'}}{2} \left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) + \mathrm{comp}\,(-2f_0)$$



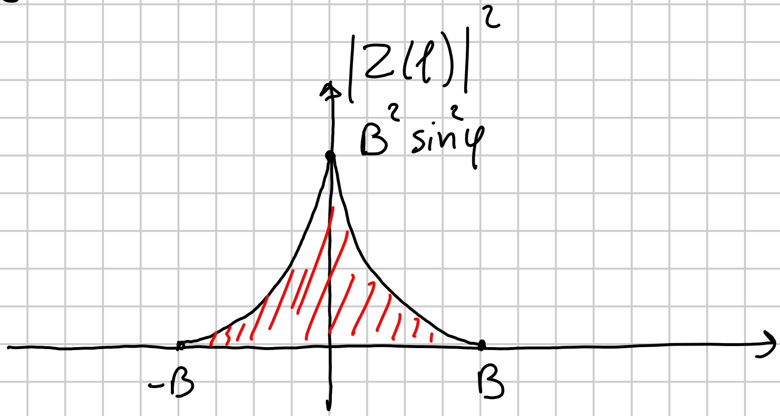
$$Z(f) = \frac{B}{2} \left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \left(e^{j\varphi'} + e^{-j\varphi'}\right)$$

$$= B \left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \cos\varphi'$$

$$= B \left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \sin\varphi$$

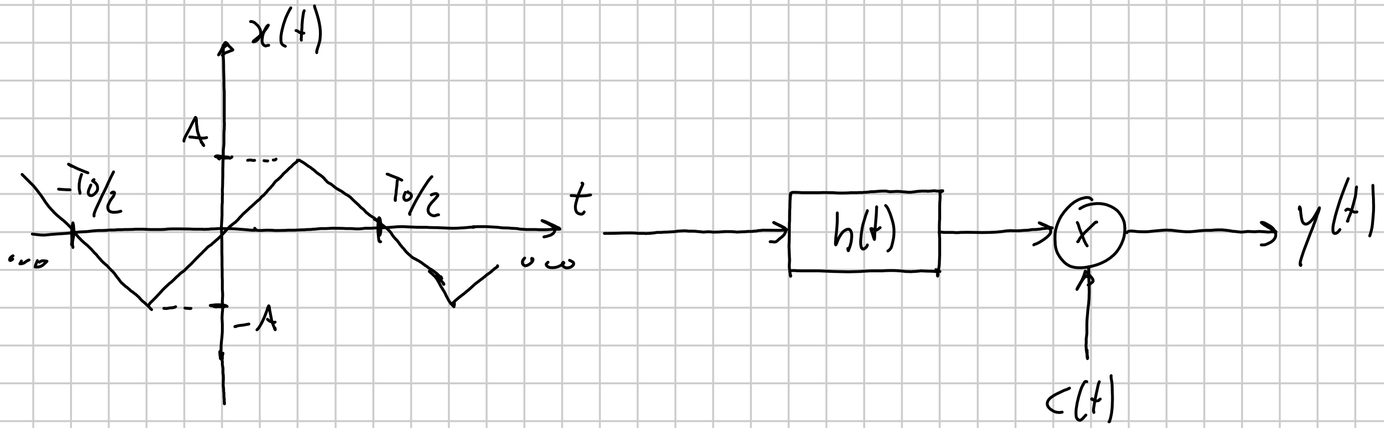
$$z(t) = B^2 \,\mathrm{sinc}^2(Bt) \sin\varphi$$

2) $$E_2 = \int_{-\infty}^{+\infty} |Z(f)|^2 \, df = \frac{2}{3} B^3 \sin^2\varphi \Rightarrow P_2 = 0$$



3) $$E_z = E_z^{(max)} \Rightarrow \varphi = \frac{\pi}{2} + k\pi$$

Es 1 20/02/12



$$h(t) = \frac{3}{T_0}\,\mathrm{sinc}\left(\frac{3t}{T_0}\right)$$

$$c(t) = \sin\left(2\pi\frac{t}{T_0}\right)$$

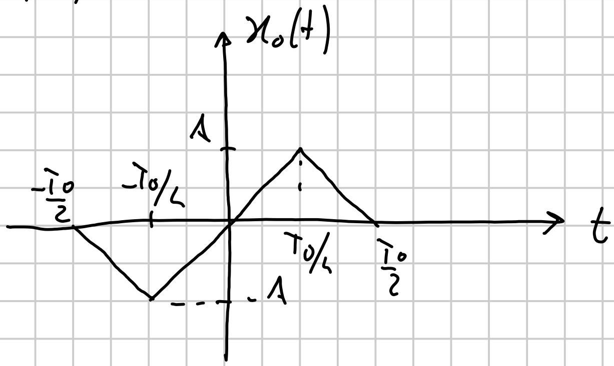
Si calcoli

1) $X(f)$
2) $y(t)$
3) E_y, P_y

Soluzione

1) $$X(f) = \sum_{k=-\infty}^{+\infty} X_k\,\delta\left(f - \frac{k}{T_0}\right)$$

$$X_k = \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right) \quad , \quad X_0(f) = TCF\left[x_0(t)\right]$$



$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)$$

$$x_0(t) = A\left(1 - \frac{|t - T_0/4|}{T_0/4}\right) \mathrm{rect}\left(\frac{t - T_0/4}{T_0/2}\right) +$$

$$- A\left(1 - \frac{|t + T_0/4|}{T_0/4}\right) \mathrm{rect}\left(\frac{t + T_0/4}{T_0/2}\right)$$

$$X_0(f) = A\frac{T_0}{4} \,\mathrm{sinc}^2\left(\frac{T_0}{4} f\right) e^{-j 2\pi f \frac{T_0}{4}} +$$

$$- A\frac{T_0}{4} \,\mathrm{sinc}^2\left(\frac{T_0}{4} f\right) e^{+j 2\pi f \frac{T_0}{4}}$$

$$= A\frac{T_0}{4} \,\mathrm{sinc}^2\left(\frac{T_0}{4} f\right) (-2j) \left(\frac{e^{j 2\pi f \frac{T_0}{4}} - e^{-j 2\pi f \frac{T_0}{4}}}{2j}\right)$$

$$= -j\frac{A T_0}{2} \,\mathrm{sinc}^2\left(\frac{T_0 f}{4}\right) \sin\left(\frac{\pi f T_0}{2}\right)$$

$$X_k = \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right) = \frac{1}{T_0}\left(-j\frac{A T_0}{2}\right) \mathrm{sinc}^2\left(\frac{T_0}{4}\frac{k}{T_0}\right) \sin\left(\frac{\pi T_0}{2}\frac{k}{T_0}\right)$$

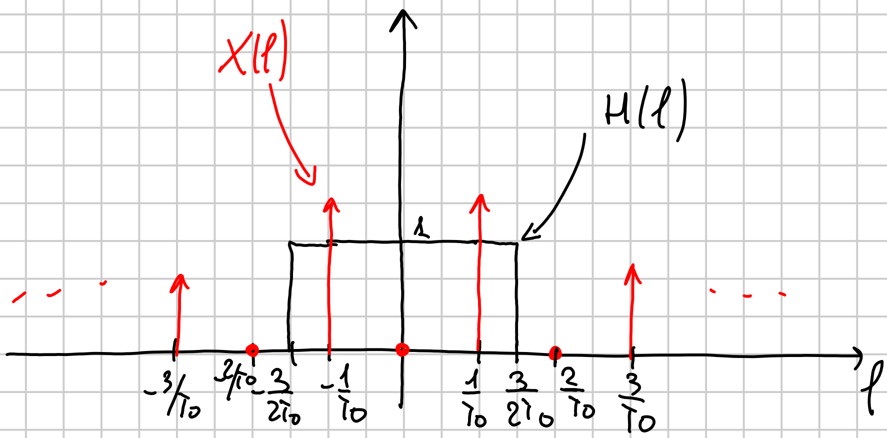
$$= -j\frac{A}{2} \,\mathrm{sinc}^2\left(\frac{k}{4}\right) \sin\left(k\frac{\pi}{2}\right)$$

$$X(f) = \sum_{k=-\infty}^{+\infty} \left(-j\frac{A}{2}\right) \mathrm{sinc}^2\left(\frac{k}{4}\right) \sin\left(k\frac{\pi}{2}\right) \delta\left(f - \frac{k}{T_0}\right)$$

2) $w(t) = x(t) \otimes h(t)$

$$W(f) = X(f)\, H(f)$$

$$H(f) = \mathrm{rect}\left(\frac{f}{3/T_0}\right)$$



$$W(f) = -j\frac{A}{2}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\delta\left(f-\frac{1}{T_0}\right) + j\frac{A}{2}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\delta\left(f+\frac{1}{T_0}\right)$$

$$Y(f) = W(f) \otimes C(f)$$

$$C(f) = \frac{1}{2j}\delta\left(f-\frac{1}{T_0}\right) - \frac{1}{2j}\delta\left(f+\frac{1}{T_0}\right)$$

$$Y(f) = -\frac{A}{4}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\delta\left(f-\frac{2}{T_0}\right) + \frac{A}{4}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\delta(f) +$$

$$+\frac{A}{4}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\delta(f) - \frac{A}{4}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\delta\left(f+\frac{2}{T_0}\right)$$

$$= \frac{A}{4}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\left[2\,\delta(f) - \left(\delta\left(f-\frac{2}{T_0}\right) + \delta\left(f+\frac{2}{T_0}\right)\right)\right]$$

$$y(t) = \frac{A}{4}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\left[2 - 2\cos\left(4\pi\frac{t}{T_0}\right)\right] =$$

$$= \frac{A}{2}\,\mathrm{sinc}^2\left(\frac{1}{4}\right)\left[1 - \cos\left(\frac{4\pi t}{T_0}\right)\right]$$

3) $$P_y = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{A^2}{4}\,\mathrm{sinc}^4\left(\frac{1}{4}\right)\left[1 - \cos\left(\frac{4\pi t}{T_0}\right)\right]^2 dt =$$

$$= \frac{A^2}{4T_0}\,\mathrm{sinc}^4\left(\frac{1}{4}\right)\left[T_0 + \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\left(\frac{1}{2} + \underbrace{\frac{1}{2}\cos\left(\frac{8\pi t}{T_0}\right)}_{0}\right)dt + \right.$$

$$\left. -\underbrace{2\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}}\cos\left(\frac{4\pi t}{T_0}\right)dt}_{=0}\right] \qquad E_y = \infty$$

$$= \frac{A^2}{4T_0}\,\mathrm{sinc}^4\left(\frac{1}{4}\right)\left[T_0 + \frac{T_0}{2}\right] = \frac{A^2}{4\cancel{T_0}}\,\mathrm{sinc}^4\left(\frac{1}{4}\right)\frac{3}{2}\cancel{T_0}$$

Esercizio 1

Siano dati i segnali $x_1(t)=2B\,\text{sinc}(2Bt)$ e $x_2(t)=B\,\text{sinc}^2(Bt)$ in ingresso al sistema in Fig. 1 e siano $h_{BP}(t)=4B\,\text{sinc}(2Bt)\cos(2\pi f_1 t)$ e $h_{LP}(t)=2B\,\text{sinc}(2Bt)$ le risposte impulsive dei filtri relativi. Note le frequenze $f_0=4B$ e $f_1=2B$ a) si calcoli e disegni lo spettro del segnale x(t), b) si calcolino le espressioni analitiche dei segnali $z_1(t)$ e $z_2(t)$, c) si calcolino infine la energia e potenza dei segnali $z_1(t)$ e $z_2(t)$.

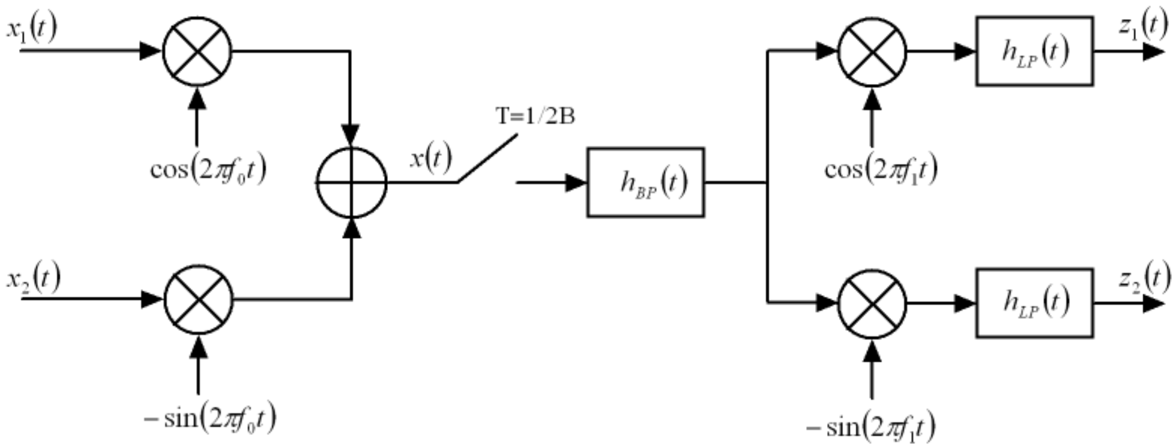


Fig. 1

Esercizio 2

Al ricevitore di Fig. 2 e' presente il segnale PAM in banda base $r(t)=\sum_n x[n]p(t-nT)+n(t)$. Sapendo che i simboli $x[n]$, indipendenti ed equiprobabili, sono appartenenti all'alfabeto $A_s\equiv[-1,2]$, $p(t)=B\text{sinc}^2(Bt)$, $h_R(t)=B\text{sinc}(Bt)$, $n(t)$ e' un processo Gaussiano bianco con DSP $S_n(f)=\frac{N_0}{2}$ e che la soglia del decisore e' fissata a $\lambda=0$, si calcoli:

1) L'energia trasmessa media per simbolo E_S;
2) La potenza media di rumore P_{n_u} all'uscita del filtro in ricezione $h_R(t)$;
3) Si verifichi la condizione di Nyquist;
4) Si determini la probabilità di errore sul bit $P_E(b)$

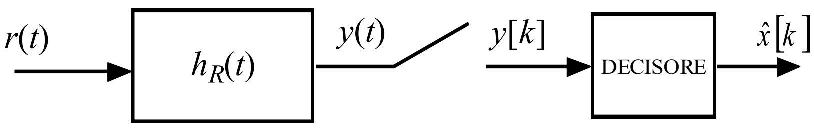


Fig. 2

SOLUZIONE ES. 1 08/06/2012

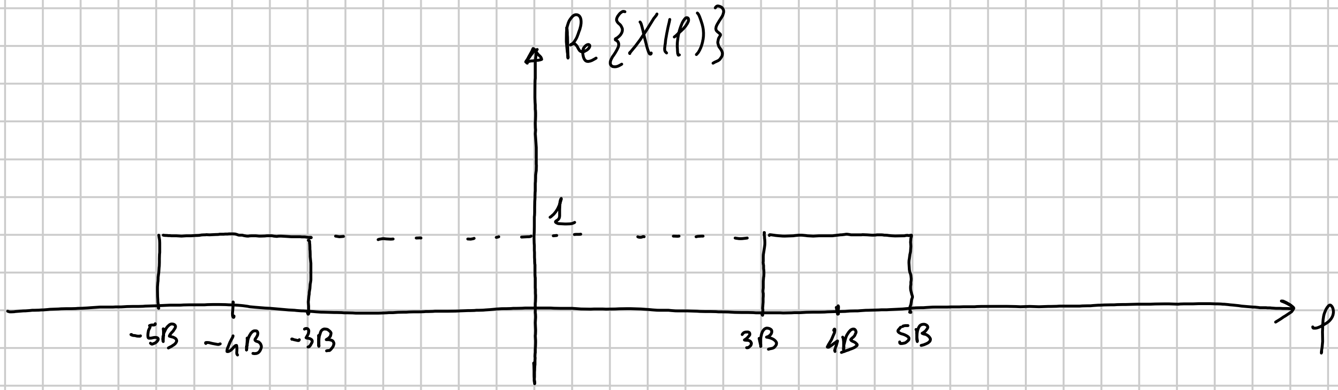
1) $x(t) = x_1(t)\cos(2\pi f_0 t) - x_2(t)\sin(2\pi f_0 t)$

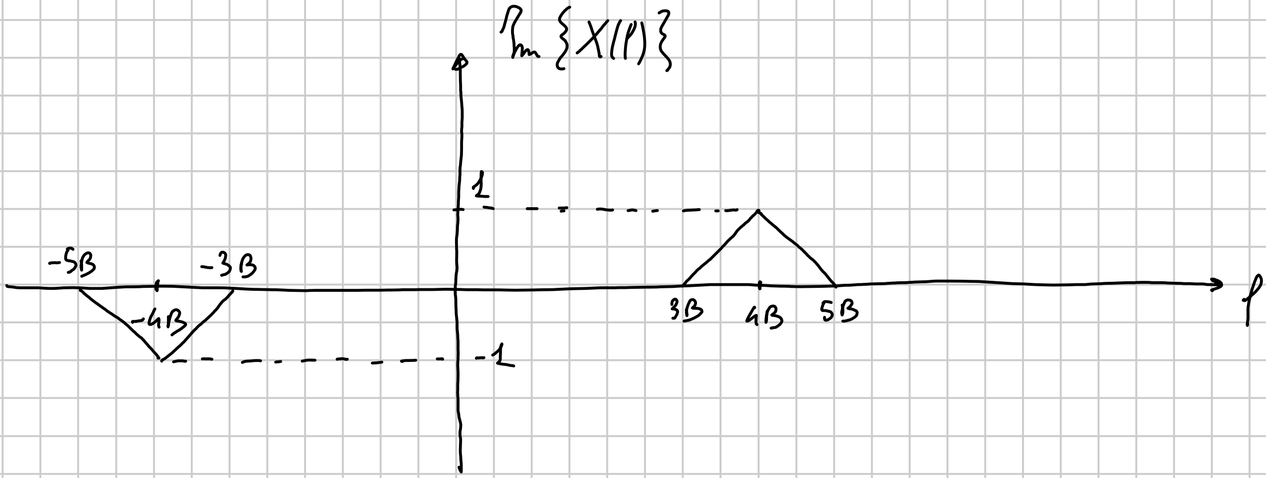
$$X(f) = X_1(f) \otimes \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] +$$

$$- X_2(f) \otimes \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] =$$

$$= \frac{1}{2}\left[X_1(f-f_0) + X_1(f+f_0)\right] + j\frac{1}{2}\left[X_2(f-f_0) - X_2(f+f_0)\right]$$

$$X_1(f) = \mathrm{rect}\left(\frac{f}{2B}\right) \quad , \quad X_2(f) = \left(1 - \frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

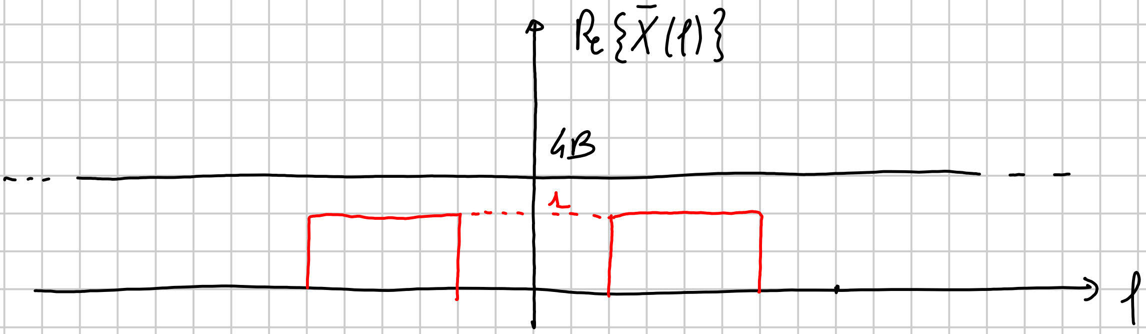


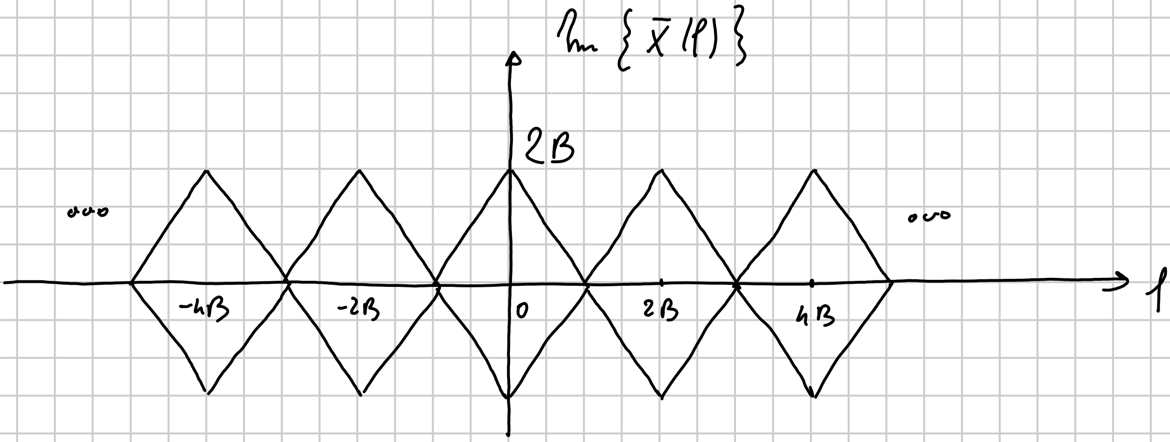


2) $x[n] = x(nT)$

$$\bar{X}(f) = 2B \sum_{n=-\infty}^{+\infty} X(f - n2B) =$$

$$2B \sum_{n=-\infty}^{+\infty} \mathrm{Re}\{X(f-n2B)\} + j\,2B \sum_{n=-\infty}^{+\infty} \mathrm{Im}\{X(f-n2B)\}$$





$$Y(f) = H_{BP}(f)\,\bar{X}(f) = H_{BP}(f)\,\mathrm{Re}\{\bar{X}(f)\} + j\,H_{BP}(f)\,\underbrace{\mathrm{Im}\{\bar{X}(f)\}}_{=0}$$

$$= 4B\left[\mathrm{rect}\left(\frac{f-2B}{2B}\right) + \mathrm{rect}\left(\frac{f+2B}{2B}\right)\right]$$

$$w(t) = y(t)\cos(2\pi f_0 t)$$

$$W_1(f) = \frac{1}{2}Y(f-2B) + \frac{1}{2}Y(f+2B) =$$

$$= 4B\,\mathrm{rect}\left(\frac{f}{2B}\right) + \text{comp. reale a } \pm 2f_1 = \pm 4B$$

$$Z_1(f) = W_1(f)\,H_{LP}(f) = W_1(f)\,\mathrm{rect}\left(\frac{f}{2B}\right) = 4B\,\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$z_1(t) = 8B^2\,\mathrm{sinc}(2Bt)$$

$$W_2(f) = -\frac{1}{2j}\,Y(f-2B) + \frac{1}{2j}\,Y(f+2B) =$$

$$= -\frac{1}{2j}\, 4B\, \mathrm{rect}\left(\frac{f}{2B}\right) + \frac{1}{2j}\, 4B\, \mathrm{rect}\left(\frac{f}{2B}\right) + \text{comp a } \pm 2f_1 = \pm 4B$$

$$Z_2(f) = W_2(f)\, H_{LP}(f) = 0 \quad \Rightarrow \quad z_2(t) = 0$$

3) $E_{z_1} = 16B^2 \cdot 2B = 32B^3 \quad \Rightarrow \quad P_{z_1} = 0$

$E_{z_2} = 0 \quad \Rightarrow P_{z_2} = 0$

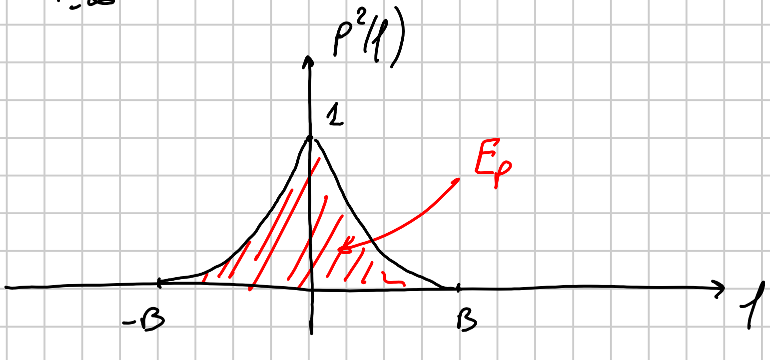
SOLUZIONE ES. 2 08/06/2012

1) $$E_s = E\left[\int_{-\infty}^{+\infty} x^2[n]\, p^2(t)\, dt\right] = \int_{-\infty}^{+\infty} E[x^2[n]]\, p^2(t)\, dt =$$

$$= E[x^2[n]]\, E_p$$

$$E[x^2[n]] = \frac{1}{2}(-1)^2 + \frac{1}{2}(2)^2 = \frac{1}{2} + 2 = \frac{5}{2}$$

$$E_p = \int_{-\infty}^{+\infty} p^2(t)\, dt = \int_{-\infty}^{+\infty} P^2(f)\, df = \frac{2}{3}B$$



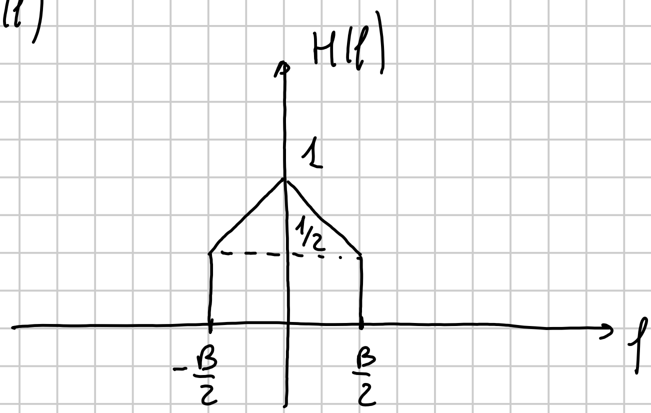
$$E_s = \frac{5}{2} \cdot \frac{2}{3} B = \frac{5}{3} B$$

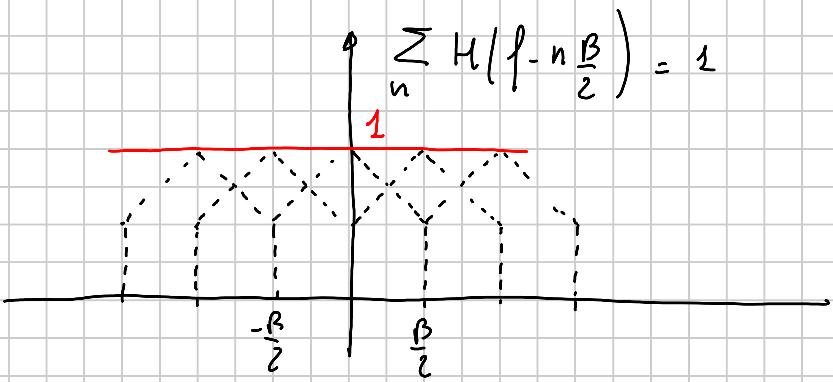
2) $$S_{n_u}(f) = S_n(f)\,|H_R(f)|^2 = \frac{N_0}{2}\, \mathrm{rect}\left(\frac{f}{B}\right)$$

$$P_{n_u} = \int_{-\infty}^{+\infty} S_{n_u}(f)\, df = \frac{N_0 B}{2}$$

3) $\sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T}\right) = K$

$H(f) = P(f)\, H_R(f)$





Cond. di Nyquist verificata

$\Downarrow$

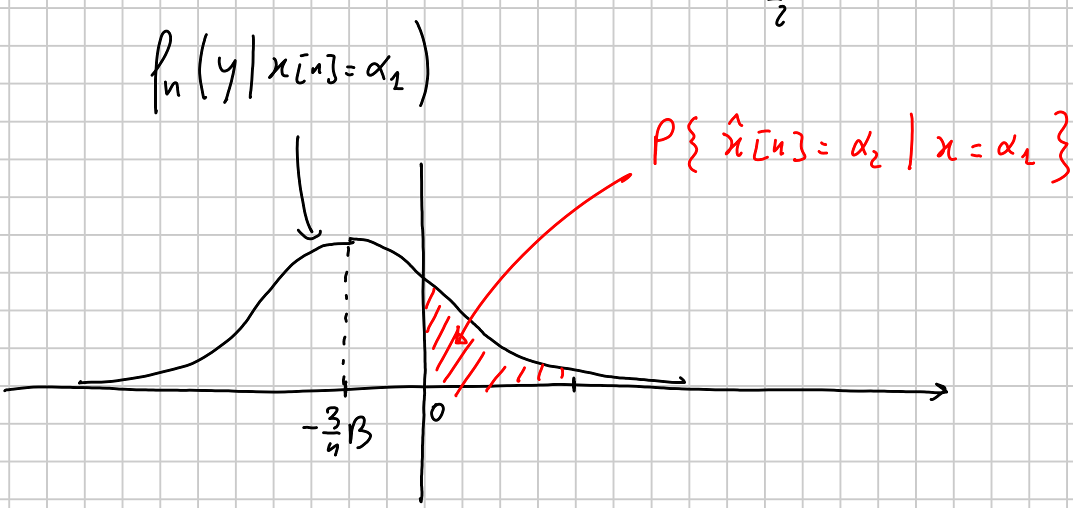
NO ISI

4) $y[n] = x[n]\, h(0) = x[n] \int_{-\infty}^{+\infty} H(f)\, df = \frac{3}{4} B\, x[n]$

$$P_E(2) = P_E(b) = P\{\hat{x}[n] = \alpha_1 \mid x[n] = \alpha_2\}\, \underbrace{P\{\alpha_2\}}_{=\frac{1}{2}} + P\{\hat{x}[n] = \alpha_2 \mid x[n] = \alpha_1\}\, \underbrace{P\{\alpha_1\}}_{=\frac{1}{2}}$$

$\alpha_1 = -1$

$\alpha_2 = 2$



$$P\{\hat{x}[n] = \alpha_2 \mid x[n] = \alpha_1\} = Q\left(\frac{\frac{3}{4}B}{\sqrt{\frac{N_0 B}{2}}}\right) = Q\left(\sqrt{\frac{9}{8}\frac{B}{N_0}}\right)$$

$$P\{\hat{x}[n] = \alpha_1 \mid x[n] = \alpha_1\} = Q\left(\frac{\frac{3}{2}B}{\sqrt{\frac{N_0 B}{2}}}\right) = Q\left(\sqrt{\frac{9}{2}\frac{B}{N_0}}\right)$$

$$P_E(b) = \frac{1}{2}\, Q\left(\sqrt{\frac{9}{8}\frac{B}{N_0}}\right) + \frac{1}{2}\, Q\left(\sqrt{\frac{9}{2}\frac{B}{N_0}}\right)$$

Prova di Comunicazioni Numeriche

3 Luglio 2012

Es. 1 - Sia dato il segnale $x(t) = \sum_{k=-\infty}^{\infty} rect\left(\frac{5}{T_0}(t - kT_0)\right)$ in ingresso al sistema in Fig. 1, dove $h(t) = \frac{4}{T_0} sinc\left(\frac{2}{T_0}t\right) cos\left(5\pi \frac{t}{T_0}\right)$, $T = \frac{T_0}{2}$ e $p(t) = \frac{3}{T_0} sinc\left(\frac{3}{T_0}t\right)$. Calcolare: 1) $X(f)$, 2) $z(t)$ e 3) E_z e P_z.

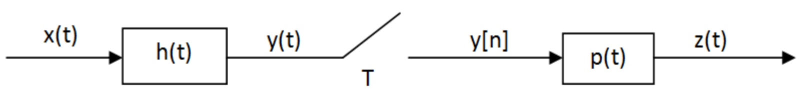


Fig. 1

Es. 2 - In un sistema di comunicazione numerico QAM il segnale all' ingresso del ricevitore è $s(t) = \sum_k x_c[k]\, p(t - kT) \cdot cos(2\pi f_0 t) - \sum_k x_s[k]\, p(t - kT) \cdot sin(2\pi f_0 t)$, dove i simboli $x_c[k] \in A_s^c = \{-1, 2\}$ e $x_s[k] \in A_s^s = \{-1, 1\}$ sono indipendenti ed equiprobabili. L'impulso sagomatore è $p(t) = 2B sinc(2Bt) + B sinc\left(2B\left(t - \frac{1}{2B}\right)\right) + B sinc\left(2B\left(t + \frac{1}{2B}\right)\right)$, $f_0 \gg B$, $T = \frac{1}{B}$. Il canale di propagazione è ideale, quindi $c(t) = \delta(t)$ e la DSP del rumore in ingresso al ricevitore è $S_n(f) = \frac{N_0}{2}\left[rect\left(\frac{f-f_0}{B}\right) + rect\left(\frac{f+f_0}{B}\right)\right]$. Il filtro in ricezione è un filtro passa basso ideale di banda B. Sia per il ramo in fase che per il ramo in quadratura la soglia di decisione è $\lambda = 0$.

Calcolare:

1) L'energia media per intervallo di segnalazione del segnale in ingresso al ricevitore, E_s

2) Calcolare la potenza di rumore in uscita ai filtri in ricezione su entrambi i rami in fase e quadratura, $P_{n_{uc}}$ e $P_{n_{us}}$

3) Calcolare la probabilità di errore sul bit, $P_E(b)$

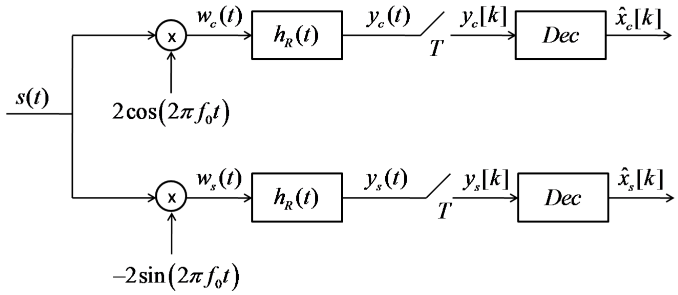


Fig. 2

Prova di Comunicazioni Numeriche

3 Luglio 2012

Es. 1 - Sia dato il segnale $x(t) = \sum_{k=-\infty}^{\infty} rect\left(\frac{5}{T_0}(t - kT_0)\right)$ in ingresso al sistema in Fig. 1, dove $h(t) = \frac{4}{T_0} sinc\left(\frac{2}{T_0}t\right) cos\left(5\pi\frac{t}{T_0}\right)$, $T = \frac{T_0}{2}$ e $p(t) = \frac{3}{T_0} sinc\left(\frac{3}{T_0}t\right)$. Calcolare: 1) $X(f)$, 2) $z(t)$ e 3) E_z e P_z.

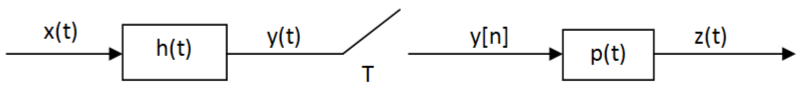


Fig. 1

Es. 2-bis - In un sistema di comunicazione numerico PAM in banda passante il segnale trasmesso è $s(t) = \sum_k x[k]\, p(t - kT) \cdot cos\left(2\pi f_0 t + \frac{\pi}{4}\right)$, dove i simboli $x[k] \in A_s = \{-1, 2\}$ sono indipendenti ed equiprobabili. L'impulso sagomatore è $p(t) = 2Bsinc(2Bt) + Bsinc\left(2B\left(t - \frac{1}{2B}\right)\right) + Bsinc\left(2B\left(t + \frac{1}{2B}\right)\right)$, $f_0 \gg B$, $T = \frac{1}{B}$. Il canale di propagazione è ideale, quindi $c(t) = \delta(t)$ e la DSP del rumore in ingresso al ricevitore è $S_n(f) = \frac{N_0}{2}\left[rect\left(\frac{f-f_0}{B}\right) + rect\left(\frac{f+f_0}{B}\right)\right]$. Il filtro in ricezione è un filtro passa basso ideale di banda B. La soglia di decisione è $\lambda = 0$.

Calcolare:

1) L'energia media per intervallo di segnalazione del segnale trasmesso, E_s

2) Calcolare la potenza di rumore in uscita al filtro in ricezione, P_{n_u}

3) Calcolare la probabilità di errore sul bit, $P_E(b)$

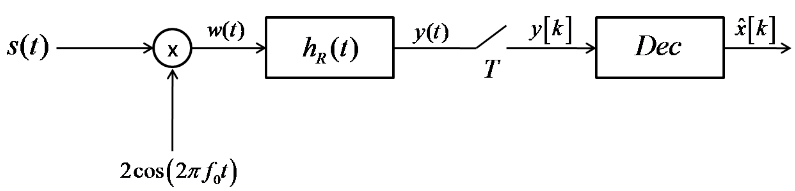


Fig. 2-bis

SOLUZIONE ESERCIZIO 1 - 3/07/2012

1) $$x(t) = \sum_k x_0(t-kT_0)$$

$$x_0(t) = \text{rect}\left(\frac{t}{T_0/5}\right)$$

$$X_0(f) = \frac{T_0}{5}\,\text{sinc}\left(\frac{T_0}{5}f\right)$$

$$X_n = \frac{1}{T_0}\,X_0\left(\frac{n}{T_0}\right) = \frac{1}{5}\,\text{sinc}\left(\frac{n}{5}\right)$$

$$X(f) = \sum_n X_n\,\delta\left(f-\frac{n}{T_0}\right)$$

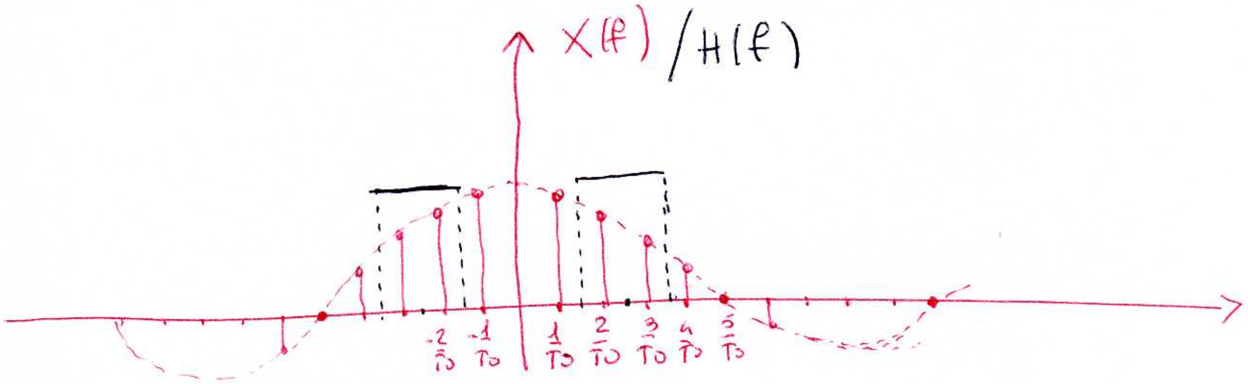
2)

$$h(t) = \frac{4}{T_0}\,\text{sinc}\left(\frac{2}{T_0}t\right)\cos\left(5\pi\frac{t}{T_0}\right) = \frac{4}{T_0}\,\text{sinc}\left(\frac{2}{T_0}t\right)\cos\left(2\pi t\frac{5}{2T_0}\right)$$

$$H(f) = \text{rect}\left(\frac{T_0}{2}f\right) \otimes \left[\delta(f-f_0)+\delta(f+f_0)\right] =$$

$$\boxed{f_0 = \frac{5}{2T_0}}$$

$$= \text{rect}\left(\frac{T_0}{2}\left(f-\frac{5}{2T_0}\right)\right) + \text{rect}\left(\frac{T_0}{2}\left(f+\frac{5}{2T_0}\right)\right)$$



$$Y(f) = \frac{1}{5} \operatorname{sinc}\left(\frac{2}{5}\right) \left[\delta\left(f - \frac{2}{T_0}\right) + \delta\left(f + \frac{2}{T_0}\right) \right] +$$

$$\frac{1}{5} \operatorname{sinc}\left(\frac{3}{5}\right) \left[\delta\left(f - \frac{3}{T_0}\right) + \delta\left(f + \frac{3}{T_0}\right) \right]$$

$$y(t) = \frac{2}{5} \operatorname{sinc}\left(\frac{2}{5}\right) \cos\left(2\pi t \frac{2}{T_0}\right) + \frac{2}{5} \operatorname{sinc}\left(\frac{3}{5}\right) \cos\left(2\pi t \frac{3}{T_0}\right)$$

$$\bar{Y}(f) = \frac{2}{T_0} \sum_{k} Y\left(f - \frac{2}{T_0} k\right)$$

$$P(f) = \operatorname{rect}\left(\frac{T_0}{3} f\right)$$

$$Z(f) = P(f)\,\bar{Y}(f) = \frac{4}{T_0} \operatorname{sinc}\left(\frac{2}{5}\right) \delta(f) + \frac{4}{T_0} \operatorname{sinc}\left(\frac{3}{5}\right) \left[\delta\left(f - \frac{1}{T_0}\right) + \delta\left(f + \frac{1}{T_0}\right.\right.$$

$$z(t) = \frac{4}{T_0} \operatorname{sinc}\left(\frac{2}{5}\right) + \frac{2}{T_0} \operatorname{sinc}\left(\frac{3}{5}\right) \cos\left(2\pi t \cdot \frac{1}{T_0}\right)$$

3) $E_z = +\infty$ perché e' un segnale periodico

$$P_z = \int_{-\infty}^{+\infty} |Z(f)|^2 \, df =$$

$$= \left(\frac{4}{T_0} \operatorname{sinc}\left(\frac{2}{5}\right)\right)^2 + 2 \cdot \left(\frac{4}{T_0} \operatorname{sinc}\left(\frac{3}{5}\right)\right)^2$$

SOLUZIONE ES. 2

1) $E_s = E\left[\int_{-\infty}^{+\infty} s_n^2(t)\, dt\right] = E\left[\int_{-\infty}^{+\infty} x_c^2[n]\, p^2(t) \cos^2(2\pi f_0 t)\, dt\right.$

$+ \int_{-\infty}^{+\infty} x_s^2[k]\, p^2(t) \sin^2(2\pi f_0 t)\, dt - \left.\int_{-\infty}^{+\infty} x_s[n]\, x_c[n]\, p^2(t) \cos(2\pi f_0 t) \cdot \sin(2\pi f_0)\, dt\right]$

$= \int_{-\infty}^{+\infty} E[x_c^2]\, p^2(t) \cos^2(2\pi f_0 t)\, dt +$

$+ \int_{-\infty}^{+\infty} E[x_s^2]\, p^2(t) \sin^2(2\pi f_0 t)\, dt +$

$- \int_{-\infty}^{+\infty} E[x_c]\, \underbrace{E[x_s]}_{=0}\, p^2(t) \cos(2\pi f_0 t) \sin(2\pi f_0 t)\, dt$

$= \frac{1}{2} E[x_c^2] \overbrace{\int_{-\infty}^{+\infty} \cos(4\pi f_0 t)\, p^2(t)\, dt}^{\simeq 0} + \frac{1}{2} E[x_c^2] \int_{-\infty}^{+\infty} p^2(t)\, dt +$

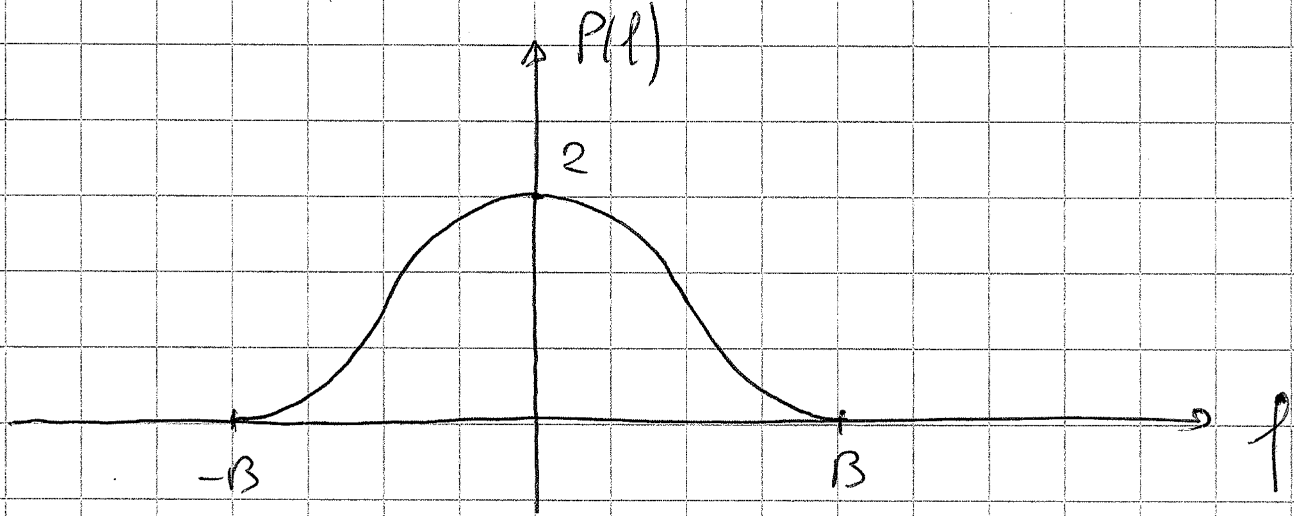
$+ \frac{1}{2} E[x_s^2] \underbrace{\int_{-\infty}^{+\infty} \cos(4\pi f_0 t)\, p^2(t)\, dt}_{\simeq 0} + \frac{1}{2} E[x_s^2] \int_{-\infty}^{+\infty} p^2(t)\, dt =$

$= \frac{1}{2} E[x_c^2] E_p + \frac{1}{2} E[x_s^2] E_p = \frac{1}{2} \cdot \frac{5}{2} \cdot E_p + \frac{1}{2} \cdot 1 \cdot E_p$

$= \frac{7}{4} E_p$

$$E_p = \int_{-\infty}^{+\infty} p^2(t)\,dt = \int_{-\infty}^{+\infty} P^2(f)\,df$$

$$P(f) = \mathrm{rect}\left(\frac{f}{2B}\right) + \mathrm{rect}\left(\frac{f}{2B}\right)\cos\left(\pi f/B\right)$$



$$E_p = \int_{-B}^{B} \left[1 + \cos\left(\pi f/B\right)\right]^2 df =$$

$$= 2B + \underbrace{\int_{-B}^{B} 2\cos\left(\pi f/B\right) df}_{=0} + \frac{1}{2}\cdot 2B + \underbrace{\frac{1}{2}\int_{-B}^{B} \cos\left(2\pi f/B\right) df}_{=0}$$

$$= 3B$$

$$\boxed{E_s = \frac{7}{4}\cdot 3B = \frac{21}{4} B}$$

2) $S_{n_c}(f) = N_0 \operatorname{rect}\left(\frac{f}{B}\right)$

$S_{n_s}(f) = N_0 \operatorname{rect}\left(\frac{f}{B}\right)$

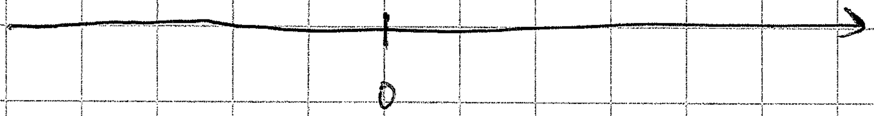
$$S_{n_{uc}}(f) = S_{n_c}(f)\left|H_R(f)\right|^2 = S_{n_c}(f)$$

$$S_{n_{us}}(f) = S_{n_s}(f)\left|H_R(f)\right|^2 = S_{n_s}(f)$$

$$P_{n_{uc}} = \int_{-\infty}^{+\infty} S_{n_c}(f)\, df = \int_{-\infty}^{+\infty} S_{n_s}(f)\, df = P_{n_{us}} = N_0 B$$

~~3)~~

3) $P_E(b)$ sul ramo in fase =



in assenza di ISI

$$y_c[n] = x_c[n]\, h(0) + n_{u_c}[n]$$

$$h(t) = p(t) \otimes c(t) \otimes h_R(t) = p(t) \otimes h_R(t)$$

$$H(f) = P(f)\, H_R(f) = P(f) \quad \text{è di Nyquist} \Rightarrow \text{no ISI}$$

$$h(t) = p(t) \Rightarrow h(0) = p(0) = \int_{-\infty}^{+\infty} P(f)\, df =$$

$$= \int_{-B}^{B} \left[1 + \cos\left(\pi f B\right)\right] df = 2B = h(0)$$

$$y_c[n] = 2B\, x_c[n] + n_{u_c}[n]$$

$$P_E^{(c)}(b) = P\{\hat{x}_c = -2 \mid x_c = 1\}\, P\{x_c = 1\} + P\{\hat{x}_c = 1 \mid x_c = -2\}\, P\{x_c = -2\}$$

3) $P_E(b)$ sul ramo in fase $\left(P_E^c(b)\right)$

$$h(t) = p(t) \otimes c(t) \otimes h_R(t) = p(t) \otimes h_R(t)$$

$$H(f) = P(f)\, H_R(f) = P(f) \quad \text{è di Nyquist} \Rightarrow \text{NO ISI}$$

$$h(t) = p(t) \Rightarrow h(0) = p(0) = \int_{-\infty}^{+\infty} P(f)\, df$$

$$y_c[n] = x_c[n]\, h(0) + n_{x_c}[n]$$

$$h(0) = \int_{-B}^{B} \left[1 + \cos\left(\pi f/B\right)\right] df = 2B$$

$$P_E^c(b) = P\left\{\hat{x}_c = -2 \mid x_c = 1\right\} P\left\{x_c = 1\right\} +$$

$$P\left\{\hat{x}_c = 1 \mid x_c = -2\right\} P\left\{x_c = -2\right\}$$

$$P\left\{x_c = 1\right\} = P\left\{x_c = -2\right\} = \frac{1}{2}$$

$$P\left\{\hat{x}_c = -2 \mid x_c = 1\right\} = Q\left(\frac{2B}{\sqrt{BN_0}}\right) = Q\left(\sqrt{\frac{4B}{N_0}}\right)$$

$$P\left\{\hat{x}_c = 1 \mid x_c = -2\right\} = Q\left(\frac{4B}{\sqrt{BN_0}}\right) = Q\left(\sqrt{\frac{16B}{N_0}}\right)$$

$$P_E^s(b) = P\{\hat{x}_s = -1 \mid x_s = 1\}\, P\{x_s = 1\} +$$

$$P\{\hat{x}_s = 1 \mid x_s = -1\}\, P\{x_s = -1\}$$

$$= P\{\hat{x}_s = 1 \mid x_s = -1\} = Q\left(\sqrt{\frac{4B}{N_0}}\right)$$

Prova di Comunicazioni Numeriche

19 Luglio 2012

Es. 1 - Siano i segnali in ingresso al sistema in Figura 1 definiti come $x_1(t) = 2Bsinc\,(2Bt)$ e $x_2(t) = Bsinc^2\,(Bt)$. Il segnale $y(t)$ è campionato con passo di campionamento $T = 1/f_0$, $p(t)$ è un interpolatore cardinale, quindi $p(t) = sinc\,(BT)$. Si determini:

1) l'espressione di $Y(f)$ e se ne disegni lo spettro di ampiezza e di fase
2) l'espressione di $z(t)$
3) energia, E_z, e potenza P_z, di $z(t)$.

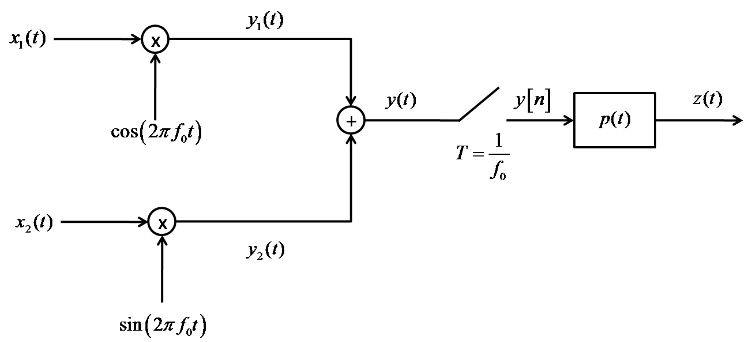


Fig. 1

Es. 2 - In un sistema di comunicazione numerico il segnale trasmesso è $s(t) = \sum_k x\,[k]\,p(t-kT)$, dove i simboli $x\,[k]$ appartengono all'alfabeto $A = \{-2, +3\}$ e $p(t) = 2Bsinc\,(2Bt)$. La risposta impulsiva del canale è $c(t) = Bsinc^2\,(Bt)$. Il canale introduce anche rumore Gaussiano additivo bianco in banda la cui densità spettrale di potenza è $S_N(f) = \frac{N_0}{2}$. Il segnale ricevuto $r(t)$ è in ingresso al ricevitore in Figura 2. La risposta impulsiva del filtro in ricezione è $h_R(t) = 2Bsinc\,(2Bt)$. Il segnale in uscita al filtro in ricezione è campionato con passo di campionamento $T = \frac{1}{B}$ e i campioni costituiscono l'ingresso del decisore che ha soglia di decisione pari a $\lambda = 0$. Determinare:

1) L'energia media per intervallo di segnalazione del segnale trasmesso

2) Calcolare la potenza di rumore in uscita al filtro in ricezione P_{nu}

3) Calcolare la probabilità di errore sul bit, $P_E(b)$

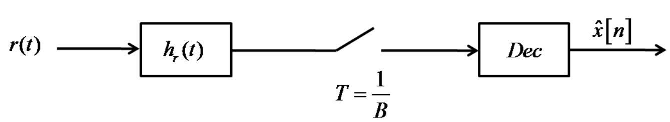
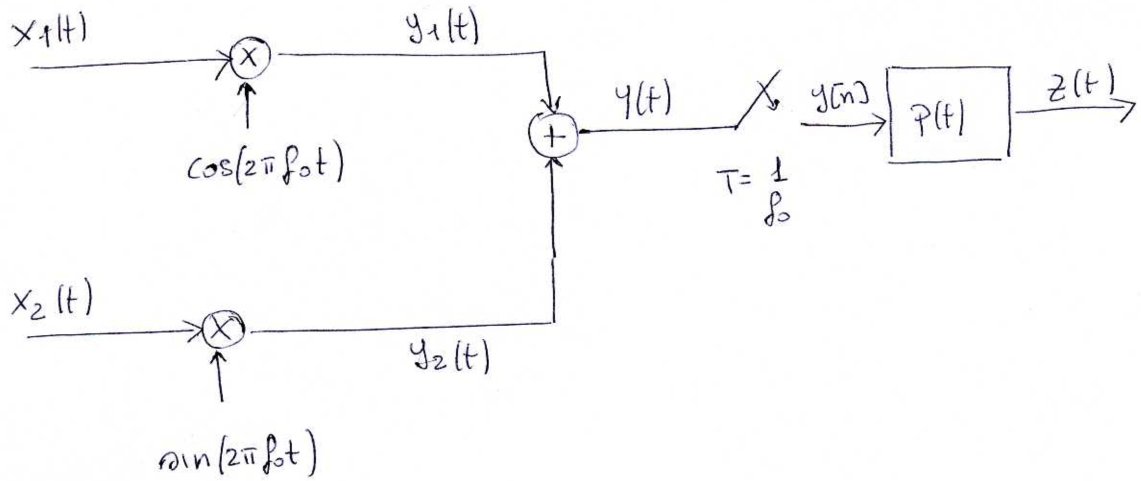


Fig. 2

Esercizio # 1 COMPITO 19/07/2012



$$x_1(t) = 2B \operatorname{sinc}(2Bt) \rightleftharpoons X_1(f) = \operatorname{rect}\left(\frac{f}{2B}\right)$$

$$x_2(t) = B \operatorname{sinc}^2(Bt) \rightleftharpoons X_2(f) = \left(1 - \frac{|f|}{B}\right) \operatorname{rect}\left(\frac{f}{2B}\right)$$

1) $y_1(t) = x_1(t) \cdot \cos(2\pi f_0 t)$

$$Y_1(f) = X_1(f) \otimes \left[\frac{\delta(f-f_0) + \delta(f+f_0)}{2}\right]$$

$$y_2(t) = x_2(t) \cdot \sin(2\pi f_0 t)$$

$$Y_2(f) = X_2(f) \otimes \left[\frac{\delta(f-f_0) - \delta(f+f_0)}{2j}\right]$$

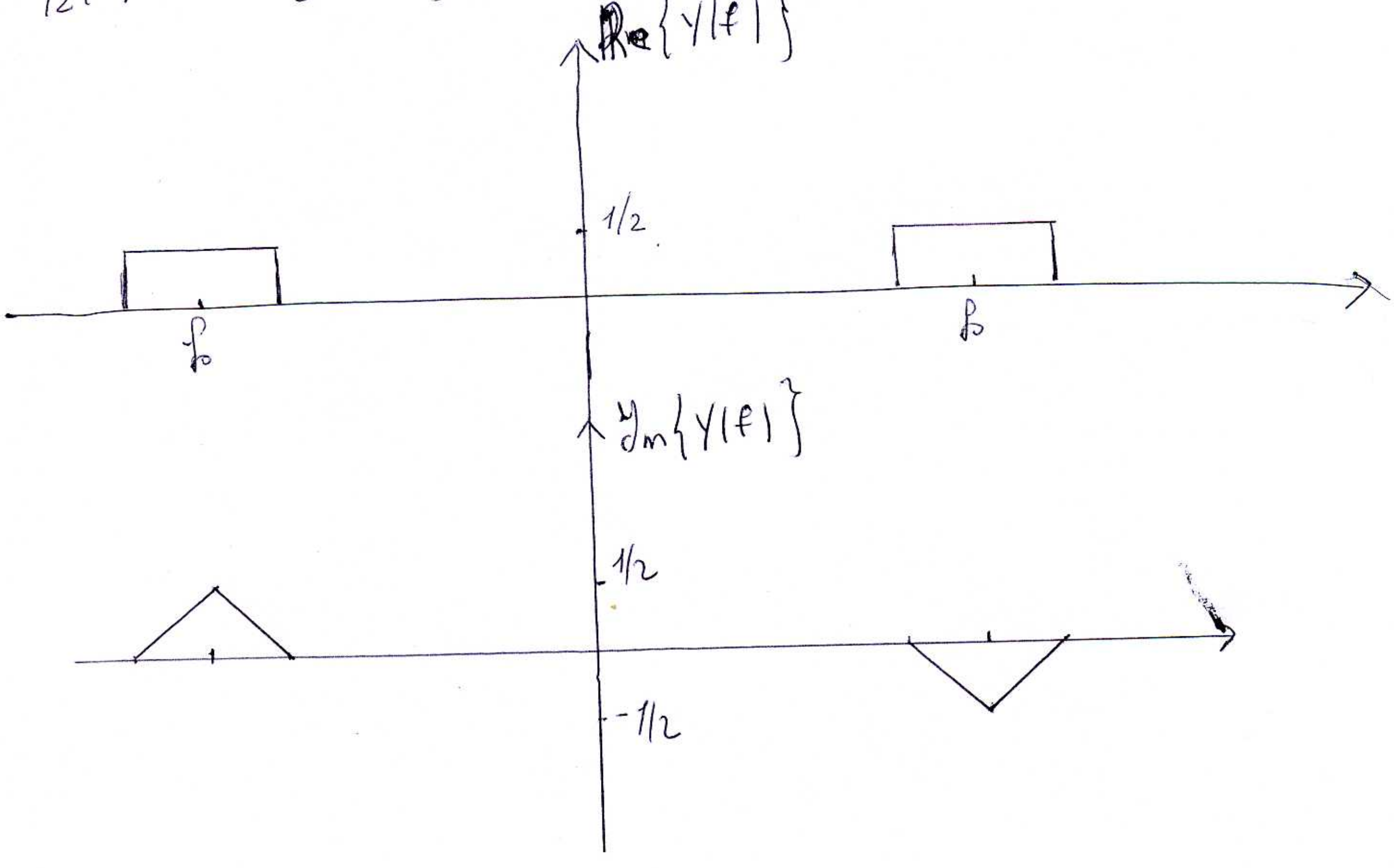
$$Y_1(f) = \frac{1}{2}\left[\mathrm{rect}\left(\frac{f-f_0}{2B}\right) + \mathrm{rect}\left(\frac{f+f_0}{2B}\right)\right]$$

$$Y_2(f) = \frac{j}{2}\left[\left(1-\frac{|f+f_0|}{B}\right)\mathrm{rect}\left(\frac{f+f_0}{2B}\right) - \left(1-\frac{|f-f_0|}{B}\right)\cdot\mathrm{rect}\left(\frac{f-f_0}{2B}\right)\right]$$

$$Y(f) = Y_1(f) + Y_2(f)$$

$$Y_1(f) = \mathrm{Re}\{Y(f)\}$$

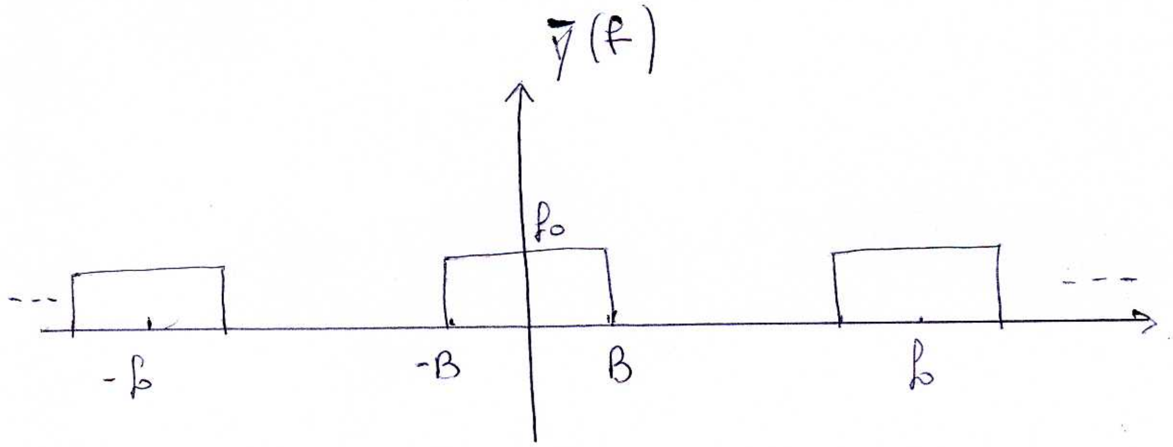
$$Y_2(f) = \mathrm{Im}\{Y(f)\}$$



$$|Y(f)| = \sqrt{Y_1^2(f) + Y_2^2(f)} \qquad \angle Y(f) = \operatorname{atan}\left(\frac{Y_2(f)}{Y_1(f)}\right)$$

2) $$y[n] \rightleftharpoons \bar{Y}(f) = \frac{1}{T} \sum_k Y\left(f - \frac{k}{T}\right) =$$

$$= f_0 \sum_k Y(f - k f_0)$$



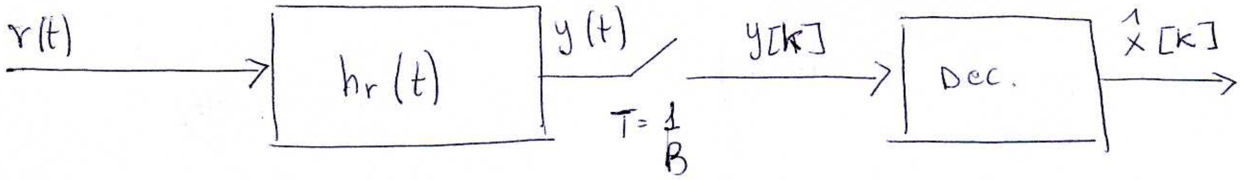
$$\Im m\{\bar{Y}(f)\} = 0$$

$$z(t) = f_0 \operatorname{sinc}(Bt) \qquad E_z = f_0$$

$$\Rightarrow \qquad P_z = 0$$

$$Z(f) = \frac{f_0}{B} \operatorname{rect}\left(\frac{f}{B}\right)$$

Esercizio # 2 compito 19/07/2012



$$s(t) = \sum_k x[k]\, p(t - kT)$$

$x[k] \in \{-2; +3\}$ indipendenti ed equiprobabili.

$$p(t) = 2B \operatorname{sinc}(2Bt)$$

$$c(t) = B \operatorname{sinc}^2(Bt)$$

$$S_N(f) = \frac{N_0}{2}$$

$$h_r(t) = 2B \operatorname{sinc}(2Bt)$$

1) $E_s = \frac{E_{s1}}{2} + \frac{E_{s2}}{2}$

$$E_{s1} = \int_{-\infty}^{+\infty} s_1^2(t)\, dt \qquad s_1(t) = -2 \cdot p(t)$$

$$E_{s2} = \int_{-\infty}^{+\infty} s_2^2(t)\, dt \qquad s_2(t) = 3 \cdot p(t)$$

$$E_p = \int_{-\infty}^{+\infty} p^2(t)\, dt = \int_{-\infty}^{+\infty} P^2(f)\, df = 2B$$

$$P(f) = \text{rect}\left(\frac{f}{2B}\right)$$

$$E_s = \frac{1}{2} \cdot 4 \cdot E_p + \frac{1}{2} \cdot 9 \cdot E_p = 13B$$

2) $y(t) = r(t) \otimes h_r(t) =$

$$= \sum_k x[k]\, h(t-kT) + w(t)$$

$$h(t) = p(t) \otimes c(t) \otimes h_r(t)$$

$$w(t) = n(t) \otimes h_r(t) \qquad S_N(f) = \frac{N_0}{2}$$

$$P_{nu} = \int_{-\infty}^{+\infty} S_w(f)\, df = N_0 B$$

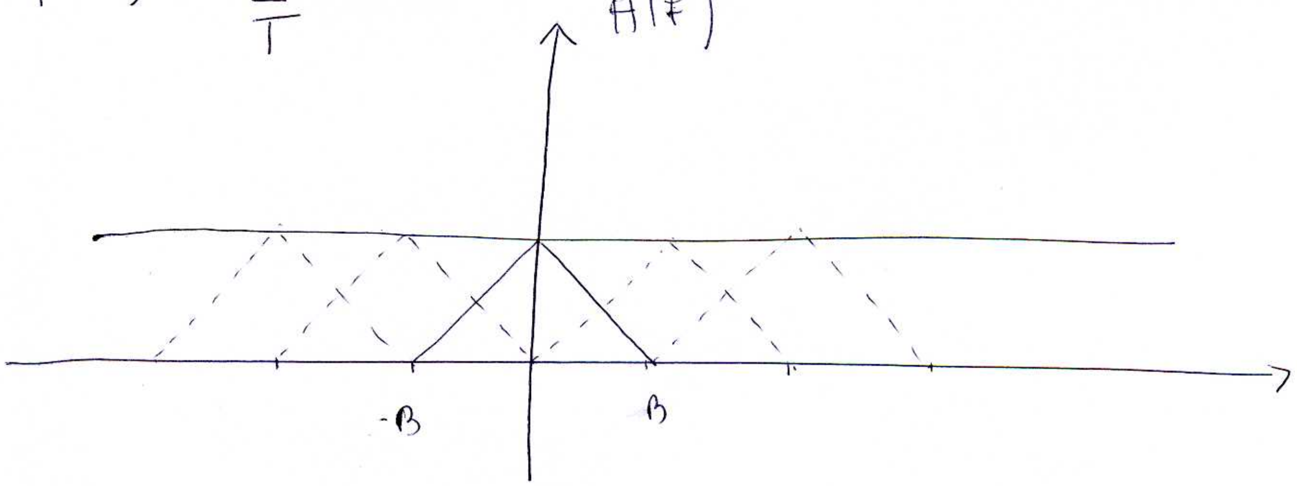
3) Prima di calcolare la $P_E(b)$ e' necessario verificare se e' valida la condizione di Nyquist.

$$h(t) = p(t) \otimes c(t) \otimes h_r(t)$$

$$H(f) = P(f) \cdot C(f) \cdot H_r(f) = \text{rect}\left(\frac{f}{2B}\right) \cdot \left(1 - \frac{|f|}{B}\right) \text{rect}\left(\frac{f}{2B}\right) \cdot \text{rect}\left(\frac{f}{2B}\right)$$

$$H(f) = \left(1 - \frac{|f|}{B}\right) \text{rect}\left(\frac{f}{2B}\right)$$

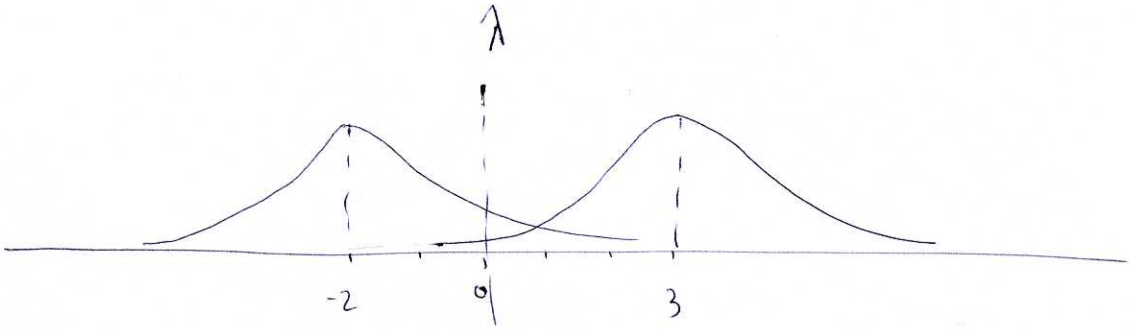
$$\bar{H}(f) = \frac{1}{T}$$



La condizione di Nyquist all'istante di campionamento e' verificata.

$$y[k] = h(0) \cdot x[k] + w[k] = x[k] + w[k]$$

$w[k] \in \mathcal{N}(0, \sigma_{nu})$



$$P_E(b) = \frac{1}{2} \Pr\{\hat{x}[k] = -2 \mid x[k] = 3\} + \frac{1}{2} \Pr\{\hat{x}[k] = 3 \mid x[k] = -2\} =$$

$$= \frac{1}{2} \cdot Q\left(\frac{3}{\sqrt{N_0 T}}\right) + \frac{1}{2} \cdot Q\left(\frac{2}{\sqrt{N_0 T}}\right)$$