$$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$S_X(f) = TCF\left[R_X(\tau)\right]$$

PROPRIETÀ DELLA $S_X(f)$ (DSP)

1) $S_X(f)$ è reale e pari

poichè la $R_X(\tau)$ è reale e pari

2) $\displaystyle\int_{-\infty}^{+\infty} S_X(f)\, df = P_X$

Dim.

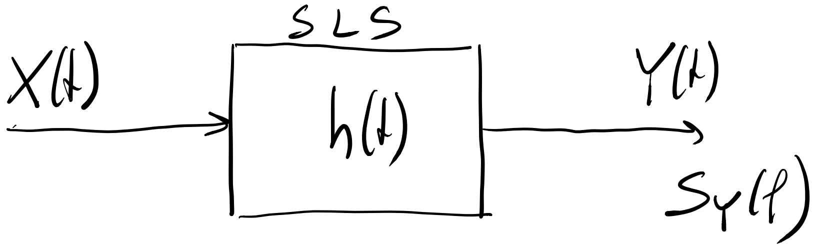
$$P_X \triangleq E\left[X^2(t)\right] = R_X(\tau)\Big|_{\tau=0} = R_X(0)$$

$$\underset{P_X}{\overset{\shortparallel}{R_X(0)}} = R_X(\tau)\Big|_{\tau=0} = \int_{-\infty}^{+\infty} S_X(f)\, \underbrace{e^{j2\pi f\tau}}_{= 1} df \Big|_{\tau=0} = \int_{-\infty}^{+\infty} S_X(f)\, df$$

3) $S_X(f) \geq 0 \quad \forall f$ non-negativa

Dim. in seguito

FILTRAGGIO DI UN PROCESSO SSL E CALCOLO DELLA DSP DEL PROCESSO IN USCITA



$X(t)$ è SSL

$$S_Y(f) = TCF\left[R_Y(\tau) \right]$$

posso già stabilire che $Y(t)$ è SSL

$$= TCF\left[\underbrace{R_X(\tau) \otimes h(\tau) \otimes h(-\tau)}_{R_Y(\tau)} \right]$$

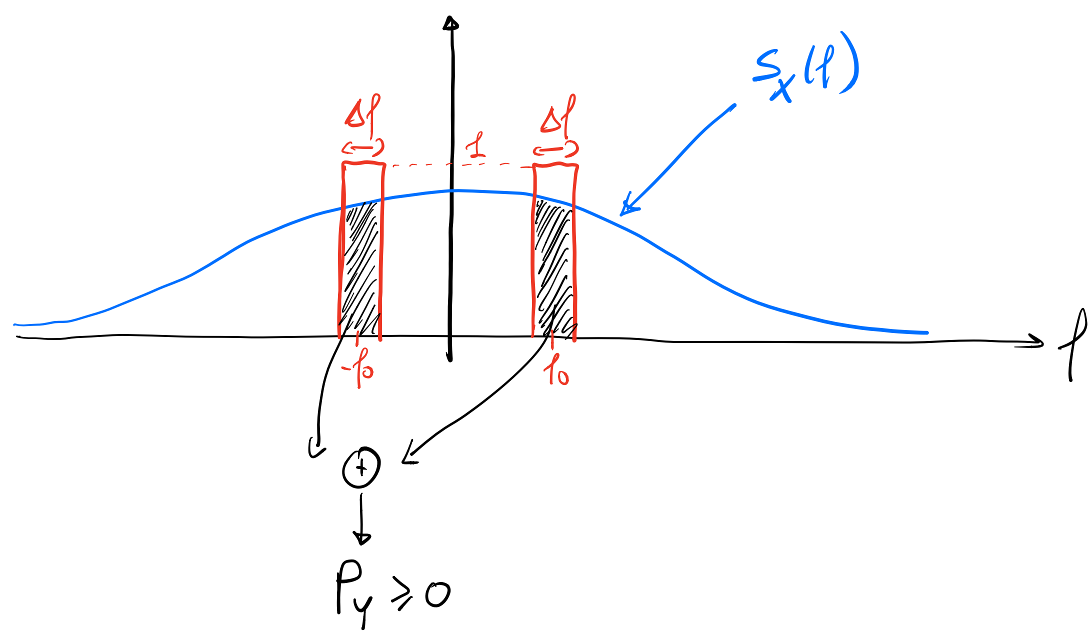
$$= S_X(f)\, H(f)\, H^*(f) = S_X(f)\, |H(f)|^2$$

$$\boxed{S_Y(f) = S_X(f)\, |H(f)|^2}$$

POTENZA MEDIA DEL P.A. IN USCITA

$$P_Y = \int_{-\infty}^{+\infty} S_Y(f)\, df = \int_{-\infty}^{+\infty} S_X(f)\, |H(f)|^2\, df$$

$\Rightarrow P_Y \geq 0$



$\Delta f \to 0 \quad \Rightarrow \quad \underbrace{P_Y}_{\geq 0} = 2 \underbrace{S_X(f_0)}_{\geq 0} \underbrace{\Delta f}_{>0} \qquad \forall f_0$

$\Rightarrow S_X(f_0) \geq 0 \qquad \forall f_0$

$S_X(f) \geq 0 \qquad \forall f$

Dimostrazione della 3ª proprietà delle DSP

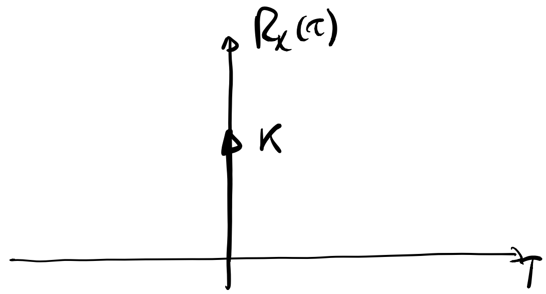
PROCESSO DI RUMORE BIANCO

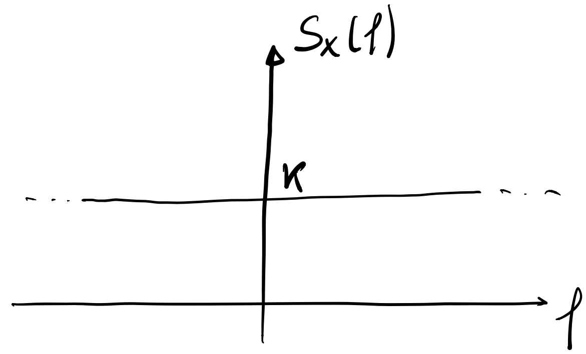
$X(t)$ si dice bianco se

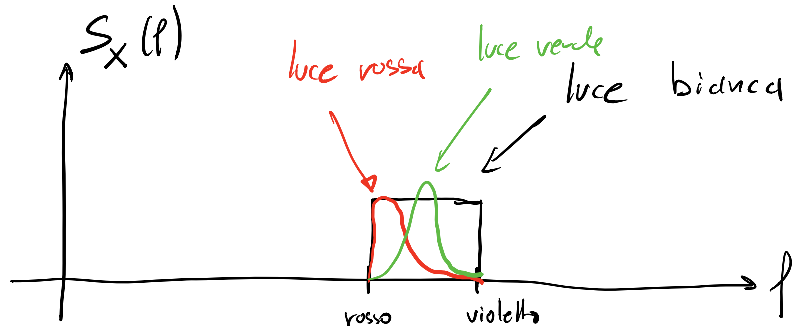
- $\eta_X = 0$
- $R_X(\tau) = K\,\delta(\tau) = C_X(\tau)$

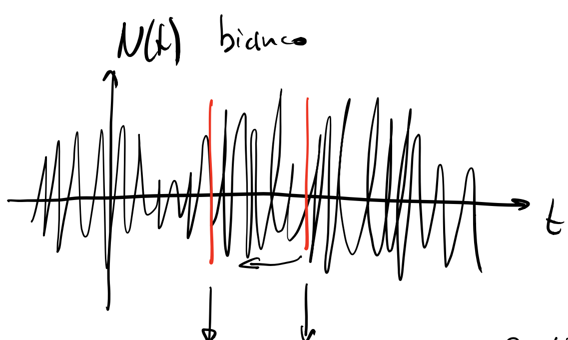
RUMORE BIANCO $\Rightarrow$ SSL

$$S_X(f) = K \quad \forall f$$





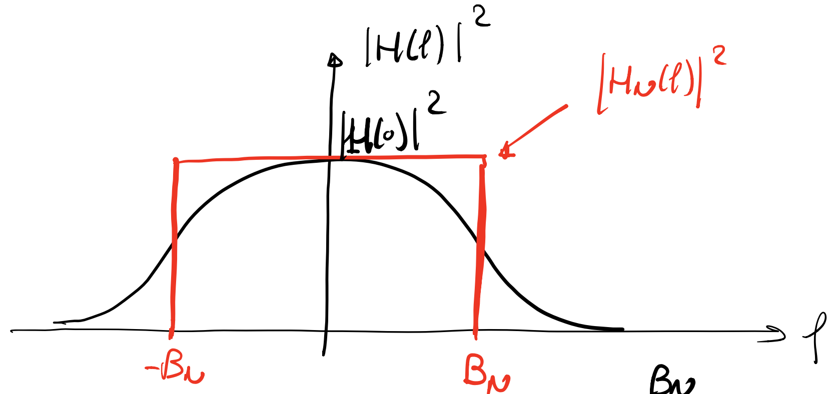




$N(t_2)$ $N(t_1)$ $\Rightarrow$ $R_X(t_2, t_1) = R_X(\tau) = 0$ $\quad \tau \neq 0$

$C_X(t_2, t_1) = C_X(\tau) = 0$

BANDA EQUIVALENTE DI RUMORE DI UN FILTRO



$$B_N: \quad P_Y = \left.\int_{-\infty}^{+\infty} S_X(f)\,|H(f)|^2\,df\right|_{S_X(f)=K} = \left.\int_{-B_N}^{B_N} S_X(f)\,|H(0)|^2\,df\right|_{S_X(f)=K}$$

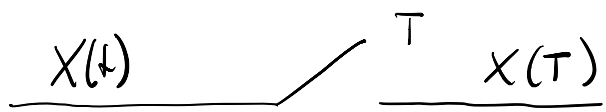
$$\Rightarrow P_Y = K\int_{-\infty}^{+\infty} |H(f)|^2\,df = K\,|H(0)|^2 \cdot 2B_N$$

$$\boxed{B_N \triangleq \frac{\int_{-\infty}^{+\infty} |H(f)|^2\,df}{2\,|H(0)|^2}}$$

⇒ PROCESSI ALEATORI GAUSSIANI

Definizione: un processo aleatorio si dice Gaussiano se, estraendo N V.A., con N arbitrario, ad istanti $t_1, t_2, \dots, t_N$ arbitrari, si ottiene un vettore di dimensioni N di V.A. congiuntamente Gaussiane

⇒ Ai fini pratici :



$X(t)$ Gaussiano

$X(T)$ è una V.A. Gaussiana

⇒ PROPRIETÀ DEI PROCESSI ALEATORI GAUSSIANI

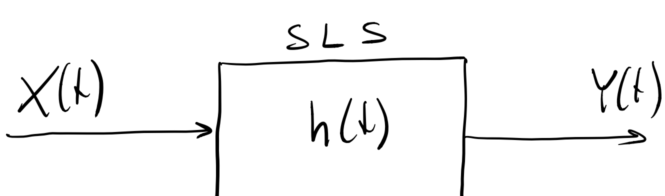
•) La conoscenza completa di un processo aleatorio Gaussiano la si ottiene conoscendo solamente

$$\left.\begin{array}{l} \cdot)\ \eta_X(t) \\ \cdot)\ R_X(t_1, t_2) \end{array}\right\} \text{posso descrivere la ddp di ordine } N$$

•) Un processo aleatorio Gaussiano SSL è anche SSS

$$SSL \iff SSS$$

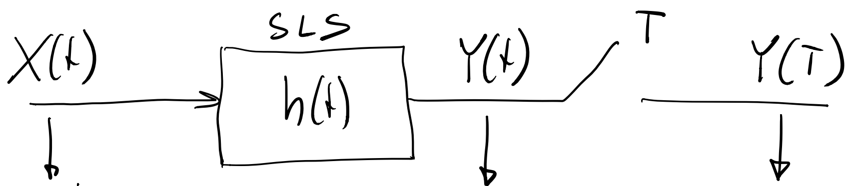
·) FILTRAGGIO DI PROCESSI ALEATORI GAUSSIANI



$X(t)$ è Gaussiano

$Y(t)$ è Gaussiano

Per i sistemi di comunicazione



$\eta_X(t) \longrightarrow \eta_Y(t)$ V.A. Gaussiana

$R_X(t_1, t_2) \longrightarrow R_Y(t_1, t_2)$

Gaussiano $\longrightarrow$ Gaussiano

- RUMORE BIANCO

$$\eta_X(t) = 0$$

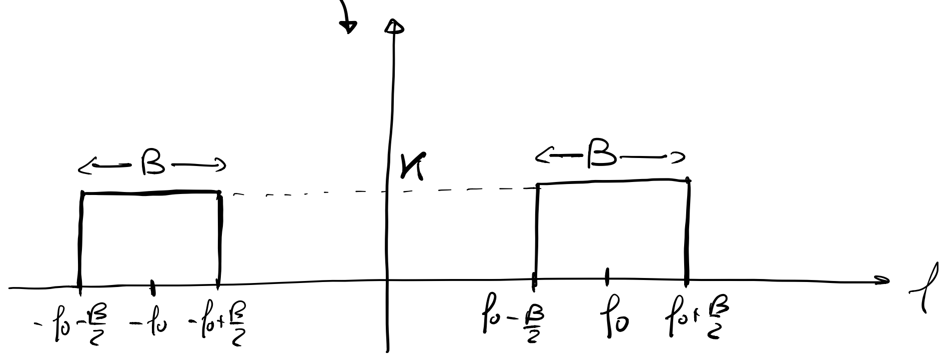
$$R_X(\tau) = K\,\delta(\tau)$$

$$S_X(f) = K \quad \forall f$$

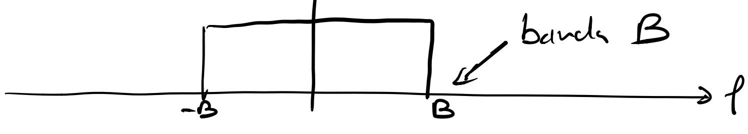
- RUMORE BIANCO IN BANDA

$$\eta_X(t) = 0$$

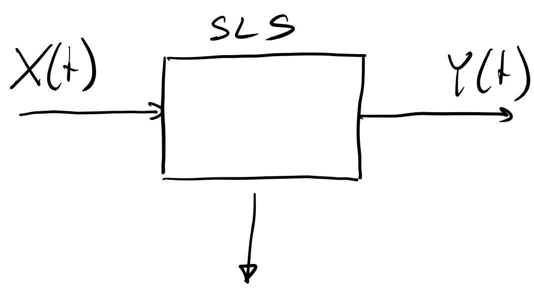
$$S_X(f)$$



per $f_0 = 0$



ESERCIZIO - 05/06/2018



$$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

$X(t)$ è Gaussiano bianco

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

1) $R_Y(\tau)$, $S_Y(f)$ e disegnarne i grafici
2) Calcolare P_X, P_Y
3) Scrivere la ddp di $Y(t_0)$, estratta da $Y(t)$ all'istante t_0
4) Calcolare la probabilità che $Y > \sqrt{\frac{N_0}{2}}$

Soluzione

1) $$\frac{d}{dt}y(t) + 2y(t) = x(t)$$

$$j2\pi f\, Y(f) + 2Y(f) = X(f)$$

$$Y(f)\left[2 + j2\pi f\right] = X(f)$$

$$\frac{Y(f)}{X(f)} = H(f) = \frac{1}{2+j2\pi f}$$

$$S_Y(f) = S_X(f)\left|H(f)\right|^2 = \frac{N_0}{2}\left|\frac{1}{2+j2\pi f}\right|^2 = \frac{N_0}{2}\ \frac{1}{4+4\pi^2 f^2}$$

$$= \frac{N_0}{8}\ \frac{1}{1+\pi^2 f^2}$$

$$R_Y(\tau) = ATCF\left[S_Y(f)\right]$$

$$e^{-|t|} \overset{TCF}{\Longleftrightarrow}$$

$$\int_{-\infty}^{+\infty} e^{-a|t|} e^{-j2\pi f t} dt = \int_{-\infty}^{0} e^{at} e^{-j2\pi f t} dt +$$

$$+ \int_{0}^{+\infty} e^{-at} e^{-j2\pi f t} dt = \int_{-\infty}^{0} e^{(a-j2\pi f)t} dt +$$

$$+ \int_{0}^{+\infty} e^{-(a+j2\pi f)t} dt = \frac{1}{a-j2\pi f} e^{(a-j2\pi f)t}\Big|_{-\infty}^{0} +$$

$$+ \left(-\frac{1}{a+j2\pi f}\right) e^{-(a+j2\pi f)t}\Big|_{0}^{+\infty}$$

$$= \frac{1}{a - j2\pi f} + \frac{1}{a + j2\pi f} = \frac{2a}{a^2 + 4\pi^2 f^2}$$

$$e^{-a|t|} \Longleftrightarrow \frac{2a}{a^2 + 4\pi^2 f^2} = 2a \frac{1}{a^2 + 4\pi^2 f^2} = \frac{2a}{a^2} \frac{1}{1 + \frac{4}{a^2}\pi^2 f^2}$$

$$\boxed{S_Y(f) = \frac{N_0}{8} \frac{1}{1 + \pi^2 f^2}}$$

$$e^{-a|t|} \Longleftrightarrow \frac{2}{a} \frac{1}{1 + \frac{4}{a^2}\pi^2 f^2} \Leftarrow \text{risultato generico}$$

$$a = 2$$

$$R_Y(\tau) = \frac{N_0}{8} e^{-2|\tau|} \Longleftrightarrow \frac{N_0}{8} \frac{1}{1 + \pi^2 f^2} = S_Y(f)$$

$$\boxed{R_Y(\tau) = \frac{N_0}{8} e^{-2|\tau|}}$$

Altra strada

$$H(f) = \frac{1}{2 + j2\pi f} \Longrightarrow h(t) \Rightarrow R_Y(\tau) = R_X(\tau) \otimes h(\tau) \otimes h(-\tau)$$

$$h(t) = \text{ATCF}\left[H(f)\right]$$

$$\int_{-\infty}^{+\infty} e^{-at} u(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{+\infty} e^{-(a+j2\pi f)t}\, dt = -\frac{1}{a+j2\pi f} e^{-(a+j2\pi f)t} \Big|_{0}^{+\infty}$$

$$= \frac{1}{a+j2\pi f}$$

$a = 2$

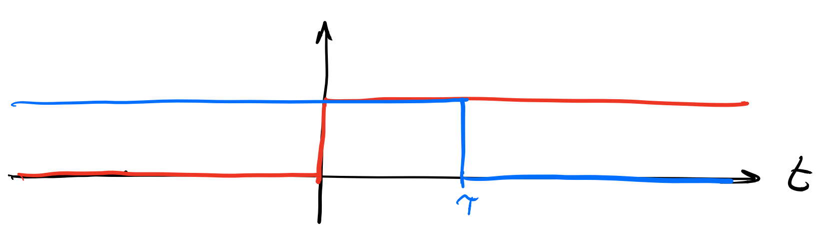
$$h(t) = e^{-2t} u(t) \overset{TCF}{\Longleftrightarrow} \frac{1}{2+j2\pi f} = H(f)$$

$$R_X(\tau) = \frac{N_0}{2} \delta(\tau)$$

$$R_Y(\tau) = \frac{N_0}{2} \underbrace{\delta(\tau) \otimes h(\tau)} \otimes h(-\tau)$$

$$= \frac{N_0}{2} h(\tau) \otimes h(-\tau)$$

$$R_Y(\tau) = \frac{N_0}{2} \int_{-\infty}^{+\infty} e^{-2t} e^{-2(\tau-t)} u(t)\, u(\tau-t)\, dt$$



$\tau < 0 \Rightarrow R_Y(\tau) = 0$

$$R_Y(\tau) = \frac{N_0}{2} \int_0^{\tau} dt \; e^{-2T}$$

da risolvere

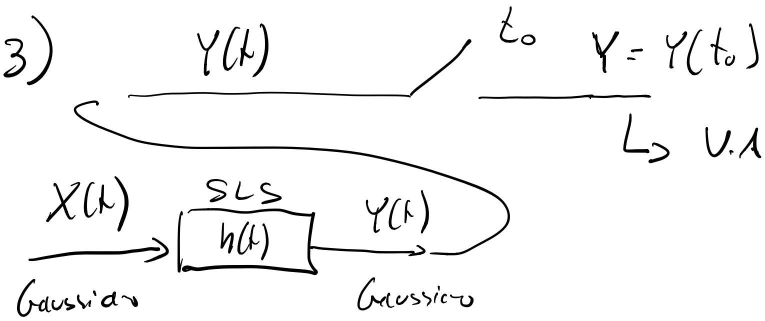
è una strada più complicata della precedente

2) P_X, P_Y

$$P_X = \int_{-\infty}^{+\infty} S_X(f)\, df = \int_{-\infty}^{+\infty} \frac{N_0}{2}\, df = \boxed{\infty}$$

$$P_Y = \begin{cases} R_Y(\tau)\big|_{\tau=0} = R_Y(0) = \boxed{\dfrac{N_0}{8}} \\ \displaystyle\int_{-\infty}^{+\infty} S_Y(f)\, df \end{cases}$$

3)



$\hookrightarrow$ V.A. Gaussiana

ddp $Y = ?$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}}\, e^{-\frac{(y-\eta_Y)^2}{2\sigma_Y^2}}$$

$\eta_Y = ?$

$\sigma_Y^2 = ?$

$$\eta_Y = \eta_Y(t_0)$$

$$\eta_Y(t)\Big|_{t=t_0} = \eta_Y$$

$$\eta_Y(t) = \eta_X(t) \otimes h(t) = 0$$

$\eta_X(t) = 0$ poiche' il processo $X(t)$ e' bianco

$\eta_X(t) = 0$ $\quad R_X(t_2,t_1) = R_X(\tau)$

$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$$\sigma_Y^2 = \sigma_Y^2(t)\Big|_{t=t_0}$$

$$\sigma_Y^2(t) = P_Y(t) - \eta_Y^2(t) = P_Y(t)$$

$$\sigma_Y^2(t_0) = P_Y(t_0) = P_Y = R_Y(0) \quad \text{costante}$$

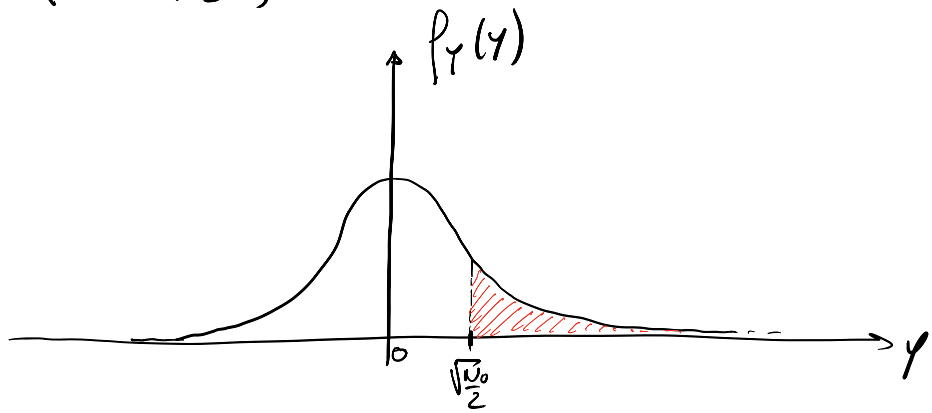
$$\sigma_Y^2 = P_Y = \frac{N_0}{8}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi \frac{N_0}{8}}} e^{-\frac{y^2}{2\frac{N_0}{8}}} = \boxed{\frac{2}{\sqrt{\pi N_0}} e^{-\frac{y^2}{N_0/4}}}$$

$$f_Y(y; t_0) \Rightarrow f_Y(y)$$

non importa dove si campiona il risultato e' indipendente da t_0

4) $P\left(Y > \sqrt{\frac{N_0}{2}}\right)$



$$P\left(Y > \sqrt{\frac{N_0}{2}}\right) = \int_{\sqrt{\frac{N_0}{2}}}^{+\infty} f_Y(y)\, dy$$

$$Q(x) = \int_{x}^{+\infty} f_N(n)\, dn$$

$N \in \mathcal{N}(0, 1)$

$Y \in \mathcal{N}\left(0, \frac{N_0}{8}\right)$

$$N = \frac{Y - \eta_Y}{\sigma_Y}$$

$$x = \frac{\sqrt{\frac{N_0}{2}} - 0}{\sqrt{\frac{N_0}{8}}} = \sqrt{4} = 2$$

$$P\left\{Y > \sqrt{\frac{N_0}{2}}\right\} = P\{N > 2\} = \boxed{Q(2)}$$

$Q(x)$ → tabelle

→ calcolabile con MATLAB o altri tools di calcolo

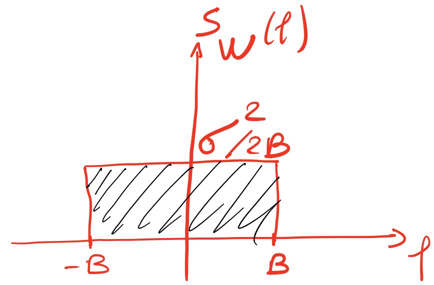
ESERCIZIO - 11/11/2019

$$Y(t) = X(t) + W(t)$$

$X(t) = A$, A v.a. con media nulla e varianza σ_A^2

$W(t)$ = rumore bianco in banda B con potenza σ^2

$W(t)$ è indipend. da $X(t)$



1) Verificare se $Y(t)$ è SSL

2) Calcolare $S_Y(f)$

Soluzione

1) $Y(t)$ SSL ?

$\eta_Y(t) = \eta_Y$ ✓

$R_Y(t_1, t_2) = R_Y(t_1 - t_2) = R_Y(\tau)$

$$\eta_Y(t) = E[Y(t)] = E[X(t) + W(t)] =$$

$$= E[X(t)] + E[W(t)]$$

$$E[X(t)] = E[A] = 0$$

A è una v.a. con media nulla (dato del problema)

$E[W(t)] = 0$ (poiché è un rumore bianco)

$\eta_W = 0$, $R_W(\tau) = K\,\delta(\tau)$

$$\eta_Y(t) = E\left[Y(t)\right] = 0 + 0 = 0 \qquad \text{costante}$$

$$R_Y(t_1, t_2) = E\left[Y(t_1)\,Y(t_2)\right] = E\left[\left[X(t_1)+W(t_1)\right]\left[X(t_2)+W(t_2)\right]\right]$$

$$= E\left[X(t_1)X(t_2)\right] + E\left[X(t_1)\,W(t_2)\right] + E\left[W(t_1)\,X(t_2)\right] + E\left[W(t_1)W(t_2)\right]$$

$$= R_X(t_1, t_2) + \underbrace{E\left[X(t_1)\right]}_{=0}\underbrace{E\left[X(t_2)\right]}_{=0} + \underbrace{E\left[W(t_1)\right]}_{=0}\underbrace{E\left[W(t_2)\right]}_{=0}$$

$$+ R_W(t_1, t_2)$$

$$= R_X(t_1, t_2) + R_W(t_1, t_2)$$

$$\cdot)\; R_X(t_1, t_2) = E\left[X(t_1)X(t_2)\right] = E\left[A^2\right] = \sigma_A^2 + \underbrace{\eta_A^2}_{=0} = \underline{\underline{\sigma_A^2}}$$

$$R_X(\tau) = \sigma_A^2 \qquad \text{costante}$$

$$\boxed{S_W(f) = \frac{\sigma^2}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)}$$

$$\cdot)\; R_W(t_1, t_2) = R_W(\tau) = \sigma^2\,\mathrm{sinc}(2B\tau)$$

$$\begin{cases} R_Y(\tau) = \sigma_A^2 + \sigma^2\,\mathrm{sinc}\left[2B(\tau)\right] & \text{dipende da } \tau = t_1 - t_2 \\ \eta_Y(t) = 0 & \text{costante} \end{cases}$$

$\rightarrow Y(t)$ è SSL

2) $S_Y(f) = ?$

$$S_Y(f) = TCF\left[R_Y(\tau)\right] = \sigma_A^2\,\delta(f) + \frac{\sigma^2}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)$$

ESERCIZIO - 06/06/2017

$U(t)$ Gaussiano SSL

$\eta_U(t) = 0$

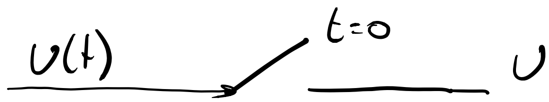
$R_U(\tau) = \sigma_U^2 \operatorname{sinc}(2B\tau)$

1) Si estragga una v.a. $U = U(0)$. Si descriva la ddp di U

2) $Y(t) = U(t) + 3U(t-T)$. Calcolare:

- $S_Y(f)$
- $R_Y(\tau)$

Soluzione



v.a.

$f_U(u) = ?$

$U(t)$ Gaussiano $\Rightarrow U \in \mathcal{N}(\eta_u, \sigma_u^2)$

$U(t)$ a v.m. nullo $\Rightarrow \eta_u(t) = 0 \Rightarrow \eta_u = 0$

$\sigma_u^2(t) = P_u(t) - \eta_u^2(t) = P_u - 0 = P_u = R_U(0)$

$= \sigma_U^2$

$$U \in \mathcal{N}(0, \sigma_U^2)$$

$$\boxed{f_U(u) = \frac{1}{\sqrt{2\pi\sigma_U^2}} e^{-\frac{u^2}{2\sigma_U^2}}}$$

2) $Y(t) = U(t) + 3U(t-T)$

$S_Y(f) = ?$

$$R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$$

$$= E[(U(t_1) + 3U(t_1 - T))(U(t_2) + 3U(t_2 - T))]$$

$$= E[U(t_1)U(t_2)] + 3[U(t_1 - T)U(t_2)] +$$

$$+ 3E[U(t_1)U(t_2 - T)] + 9E[U(t_1 - T)U(t_2 - T)]$$

$$= R_U(t_1 - t_2) + 3R_U(t_1 - T - t_2) +$$

$$+ 3R_U(t_1 - t_2 + T) + 9R_U(t_1 - t_2)$$

$$= 10R_U(t_1 - t_2) + 3R_U(t_1 - t_2 - T) +$$

$$3R_U(t_1 - t_2 + T)$$

$$= 10R_U(\tau) + 3R_U(\tau - T) + 3R_U(\tau + T)$$

$$S_Y(f) = 10S_U(f) + 3S_U(f)e^{-j2\pi fT} + 3S_U(f)e^{j2\pi fT}$$

$$= 10\, S_U(f) + 6\, S_U(f)\, \frac{e^{j2\pi f T} + e^{-j2\pi f T}}{2}$$

$$= S_U(f) \left[10 + 6 \cos(2\pi f T) \right]$$

$$S_U(f) = TCF\left[R_U(\tau) \right] = \frac{\sigma_U^2}{2B} \, \mathrm{rect}\left(\frac{f}{2B} \right)$$

$$\Rightarrow \boxed{S_Y(f) = \left(10 + 6\cos(2\pi f T)\right) \frac{\sigma_U^2}{2B} \, \mathrm{rect}\left(\frac{f}{2B} \right)}$$