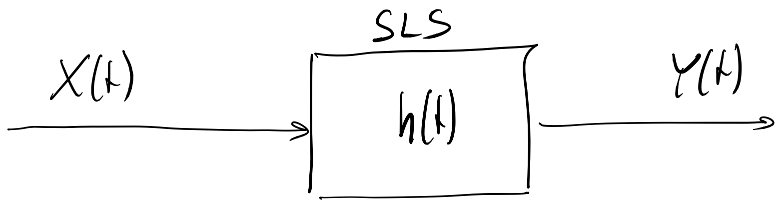
ESERCIZIO - 18/07/2017

$X(t)$

$$S_X(f) = \frac{\sigma_X^2}{2B} \text{rect}\left(\frac{f}{2B}\right)$$



$$h(t) = \frac{1}{2}\delta(t) + \delta(t-T) + \delta(t-2T)$$

1) Calcolare $S_Y(f)$
2) Disegnare il grafico di $S_Y(f)$ per $B = \frac{1}{2T}$
3) Calcolare $R_Y(\tau)$
4) Disegnare il grafico di $R_Y(\tau)$ per $B = \frac{1}{2T}$

Soluzione

1) Due strade

$S_X(f) \Rightarrow R_X(\tau) \Rightarrow R_Y(\tau) = R_X(\tau) \otimes h(\tau) \otimes h(-\tau)$

$\Downarrow$

$S_Y(f) = TCF\left[R_Y(\tau)\right]$

$h(t) \Rightarrow H(f) \Rightarrow S_Y(f) = S_X(f)\left|H(f)\right|^2$

I strada

$$R_X(\tau) = ATCF\left[S_X(f)\right] = ATCF\left[\frac{\sigma_X^2}{2B} \text{rect}\left(\frac{f}{2B}\right)\right]$$

$$= \sigma_X^2 \text{sinc}(2B\tau)$$

$$R_y(\tau) = R_x(\tau) \otimes \underbrace{h(\tau) \otimes h(-\tau)}$$

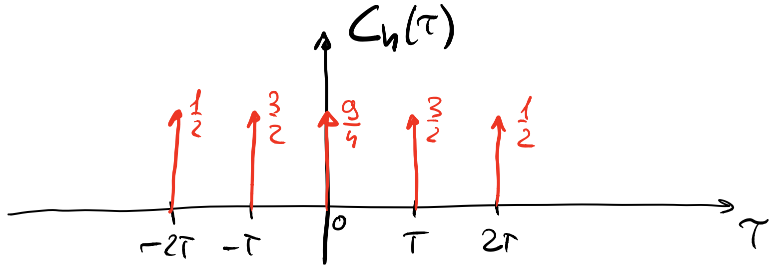
$$C_h(\tau) = \int_{-\infty}^{+\infty} h(t)\, h(t-\tau)\, dt$$

$$h(t) = \frac{1}{2}\delta(t) + \delta(t-T) + \delta(t-2T)$$

$$h(-t) = \frac{1}{2}\delta(t) + \delta(t+T) + \delta(t+2T)$$

$$h(t) \otimes h(-t) = \frac{1}{4}\delta(t) + \frac{1}{2}\delta(t+T) + \frac{1}{2}\delta(t+2T) +$$
$$+ \frac{1}{2}\delta(t-T) + \delta(t) + \delta(t+T) +$$
$$+ \frac{1}{2}\delta(t-2T) + \delta(t-T) + \delta(t)$$

$$= \frac{9}{4}\delta(t) + \frac{3}{2}\delta(t+T) + \frac{1}{2}\delta(t+2T) +$$
$$+ \frac{3}{2}\delta(t-T) + \frac{1}{2}\delta(t-2T)$$



$$R_y(\tau) = R_x(\tau) \otimes C_h(\tau)$$
$$= \frac{1}{2}R_x(\tau-2T) + \frac{3}{2}R_x(\tau-T) + \frac{9}{4}R_x(\tau) + \frac{3}{2}R_x(\tau+T) + \frac{1}{2}R_x(\tau+2T)$$

$$\boxed{R_Y(\tau) = \sigma_x^2\left[\frac{1}{2}\,\mathrm{sinc}\left[2B(\tau+2T)\right] + \frac{3}{2}\,\mathrm{sinc}\left[2B(\tau+T)\right] + \frac{9}{4}\,\mathrm{sinc}\left[2B\tau\right] + \frac{3}{2}\,\mathrm{sinc}\left[2B(\tau-T)\right] + \frac{1}{2}\,\mathrm{sinc}\left[2B(\tau-2T)\right]\right]}$$

$$S_Y(f) = \frac{\sigma_x^2}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)\left[\frac{1}{2}e^{j2\pi f 2T} + \frac{3}{2}e^{j2\pi fT} + \frac{9}{4} + \frac{3}{2}e^{-j2\pi fT} + \frac{1}{2}e^{-j2\pi f2T}\right]$$

$$= \boxed{\frac{\sigma_x^2}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)\left[\frac{9}{4} + \cos(4\pi fT) + 3\cos(2\pi fT)\right]}$$

II^a strada

$$h(t) = \frac{1}{2}\delta(t) + \delta(t-T) + \delta(t-2T)$$

$$H(f) = \frac{1}{2} + e^{-j2\pi fT} + e^{-j4\pi fT}$$

$$|H(f)|^2 = H(f)\,H^*(f) =$$

$$= \left(\frac{1}{2} + e^{-j2\pi fT} + e^{-j4\pi fT}\right)\left(\frac{1}{2} + e^{j2\pi fT} + e^{j4\pi fT}\right)$$

$$= \frac{1}{4} + \frac{1}{2}e^{j2\pi fT} + \frac{1}{2}e^{j4\pi fT} + \frac{1}{2}e^{-j2\pi fT} + 1 + e^{j2\pi fT} + \frac{1}{2}e^{-j4\pi fT} + e^{-j2\pi fT} + 1$$

$$= \frac{9}{4} + \frac{3}{2} e^{j2\pi f T} + \frac{1}{2} e^{j4\pi f T} + \frac{3}{2} e^{-j2\pi f T} + \frac{1}{2} e^{-j4\pi f T}$$

$$= \frac{9}{4} + 3\cos(2\pi f T) + \cos(4\pi f T)$$

$$S_Y(f) = S_X(f)\,|H(f)|^2$$

$$= \boxed{\frac{\sigma_X^2}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)\left[\frac{9}{4} + 3\cos(2\pi f T) + \cos(4\pi f T)\right]}$$

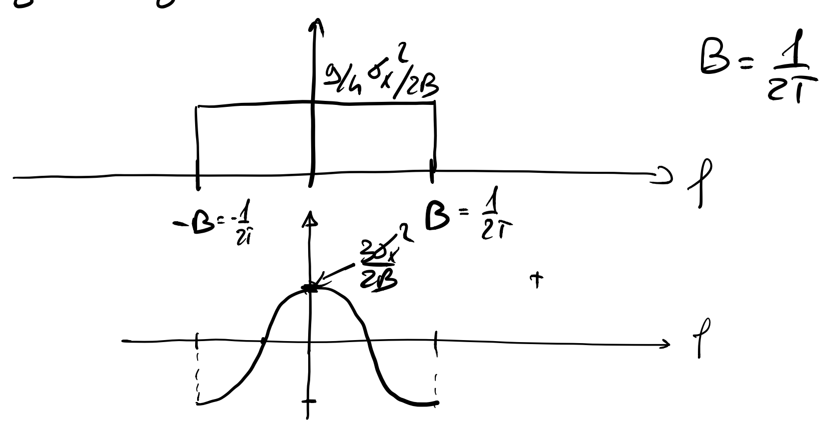
$$R_Y(\tau) = \mathrm{ATCF}\left[S_Y(f)\right]$$

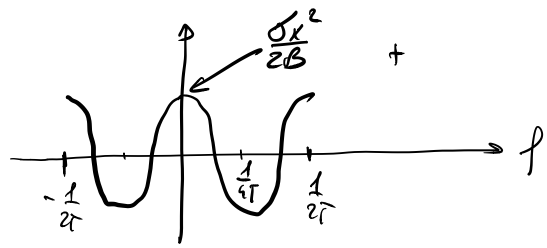
$$= \frac{9}{4}\underbrace{\sigma_X^2\,\mathrm{sinc}(2B\tau)} + \frac{3}{2}\Big[\sigma_X^2\,\mathrm{sinc}\left[2B(\tau - T)\right] +$$

$$+ \sigma_X^2\,\mathrm{sinc}\left[2B(\tau + T)\right]\Big] + \frac{1}{2}\Big[\sigma_X^2\,\mathrm{sinc}\left[2B(\tau - 2T)\right] +$$

$$+ \sigma_X^2\,\mathrm{sinc}\left[2B(\tau + 2T)\right]\Big]$$

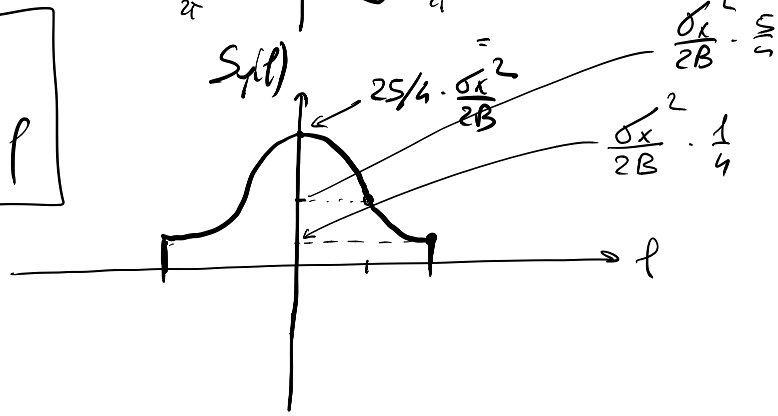
$\Rightarrow$ Si ottengono gli stessi risultati





$S_Y(f) = S_Y(-f)$

$S_Y(f) \geq 0 \quad \forall f$



Il grafico di $R_Y(\tau)$ lo evitiamo (complicato)

ESERCIZIO – 13/01/2020

$C_X(\tau)$ di un processo $X(t)$

$$C_X(\tau) = A\, e^{-\alpha|\tau|} \cos(2\pi f_0 \tau)$$

$$f_X(x;t) \in \mathcal{U}[0, 10]$$

$$\Rightarrow S_X(f) = ?$$

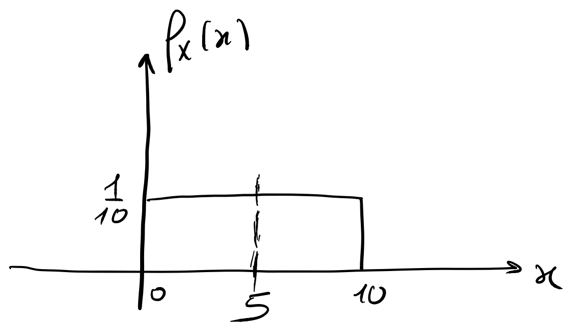
$$S_X(f) = TCF\left[R_X(\tau) \right]$$

$$R_X(\tau) = C_X(\tau) + \eta_X^2(\tau)$$

$$\eta_X(t) = \int_{-\infty}^{+\infty} x\, f_X(x;t)\, dx$$

$f_X(x;t)$ non dipende da "t" poiché è distribuita tra 0 e 10 indipendentemente dal tempo.

$$\eta_X = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx = \int_{-\infty}^{+\infty} x \, \frac{1}{10} \, \mathrm{rect}\left(\frac{x-5}{10}\right) dx$$



$$= \frac{1}{10} \int_0^{10} x \, dx = \frac{1}{10} \frac{x^2}{2} \Big|_0^{10} = \frac{1}{20}(100-0) = \frac{100}{20} = 5$$

$$R_X(\tau) = \underbrace{A \, e^{-\alpha|\tau|} \cos(2\pi f_0 \tau)} + 25$$

$$S_X(f) = \underbrace{TCF\left[A \, e^{-\alpha|\tau|} \cos(2\pi f_0 \tau)\right]} + 25 \, \delta(f)$$

$$S_X'(f) = \frac{A}{2}\left[S_{X_0}(f-f_0) + S_{X_0}(f+f_0)\right]$$

$$S_{X_0}(f) = TCF\left[e^{-\alpha|\tau|}\right]$$

$$S_{X_0}(f) = \int_{-\infty}^{+\infty} e^{-\alpha|\tau|} e^{-j2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{\alpha\tau} e^{-j2\pi f\tau} d\tau + \int_{0}^{+\infty} e^{-\alpha\tau} e^{-j2\pi f\tau} d\tau$$

$$= \int_{-\infty}^{0} e^{(\alpha - j2\pi f)\tau} d\tau + \int_{0}^{+\infty} e^{-(\alpha + j2\pi f)\tau} d\tau$$

$$= \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

$$\Rightarrow S_X'(f) = \frac{A}{2}\left[\frac{2\alpha}{\alpha^2 + 4\pi^2(f-f_0)^2} + \frac{2\alpha}{\alpha^2 + 4\pi^2(f+f_0)^2}\right]$$

$$= \alpha A \left[\frac{1}{\alpha^2 + 4\pi^2(f-f_0)^2} + \frac{1}{\alpha^2 + 4\pi^2(f+f_0)^2}\right]$$

$$\Rightarrow S_X(f) = \alpha A \left[\frac{1}{\alpha^2 + 4\pi^2(f-f_0)^2} + \frac{1}{\alpha^2 + 4\pi^2(f+f_0)^2}\right] + 25\,\delta(f)$$

ESERCIZIO - 17/07/2018

$N(t)$ Gaussiano bianco di banda B con potenza $N_0 B$

$X = X(t)\big|_{t=0} = X(0)$

$h(t) = e^{-2t} u(t)$

1) Calcolare la DSP di $X(t)$ $\Rightarrow$ $S_X(f)$
e la sua potenza P_X

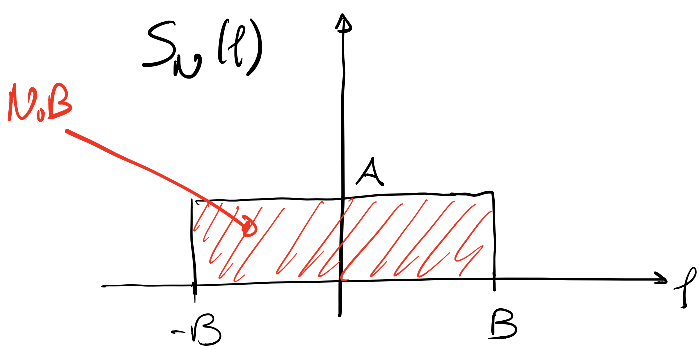
2) Si campioni a $t=0$ e si scriva la ddp di X

3) $X \longrightarrow \boxed{(\cdot)^2} \longrightarrow Y = X^2$

Si calcoli la ddp di Y

Soluzione

1) $S_X(f) = S_N(f)\,|H(f)|^2$



$R_N(\tau) = 2AB\,\mathrm{sinc}(2B\tau)$

$R_X(\tau) = R_N(\tau) \otimes h(\tau) \otimes h(-\tau)$

$A = ?$ $\Rightarrow$ $P_N = N_0 B = \int_{-\infty}^{+\infty} S_N(f)\,df = 2A\cancel{B} = N_0 \cancel{B}$

$$\Downarrow$$

$$A = \frac{N_0}{2}$$

$$S_N(f) = \frac{N_0}{2}\,\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$H(f) = \int_{-\infty}^{+\infty} e^{-2t} u(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_0^{+\infty} e^{-(2+j2\pi f)t}\, dt$$

$$= \frac{1}{2 + j2\pi f}$$

$$|H(f)|^2 = H(f)\, H^*(f) = \frac{1}{2+j2\pi f} \cdot \frac{1}{2-j2\pi f} = \frac{1}{4+4\pi^2 f^2}$$

$$\boxed{S_x(f) = \frac{N_0}{2} \frac{1}{4+4\pi^2 f^2} \operatorname{rect}\left(\frac{f}{2B}\right)}$$

$$P_x = \int_{-\infty}^{+\infty} S_x(f)\, dx = \frac{N_0}{2} \int_{-B}^{B} \frac{1}{4+4\pi^2 f^2}\, df$$

$$\int_{-\infty}^{x} \frac{1}{1+z^2}\, dz = \operatorname{arctg} x$$

$$\Downarrow$$

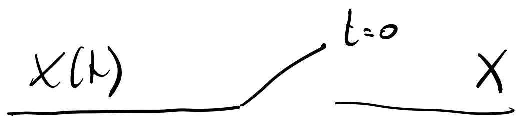
$$\int_{-\infty}^{x} \frac{1}{1+(\alpha z)^2}\, dz = \frac{1}{\alpha} \operatorname{arctg} \frac{x}{\alpha}$$

$$P_x = \frac{N_0}{8} \int_{-B}^{B} \frac{1}{1+\pi^2 f^2}\, df = \frac{N_0}{8} \frac{1}{\pi} \operatorname{arctg} \frac{f}{\pi} \Big|_{-B}^{B}$$

$$= \frac{N_0}{8\pi} \left[\operatorname{arctg}\left(\frac{B}{\pi}\right) - \operatorname{arctg}\left(-\frac{B}{\pi}\right) \right] = \frac{N_0}{4\pi} \operatorname{arctg}\left(\frac{B}{\pi}\right)$$

$$\boxed{P_X = \frac{N_0}{4\pi}\,\mathrm{arctg}\left(\frac{B}{\pi}\right)}$$

2)



$X(t)$ è proc. Gaussiano

X è una V. A. Gaussiana

$$X \in \mathcal{N}(\eta_X, \sigma_X^2)$$

$$\Downarrow$$

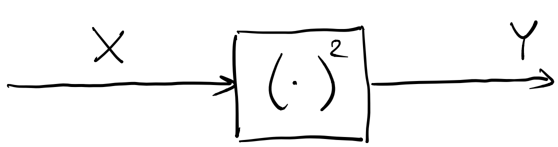
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}}\, e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}}$$

$\eta_X(t) = 0 \Rightarrow \eta_X = 0$ proc è bianco

$$\sigma_X^2 = P_X - \eta_X^2 = P_X$$

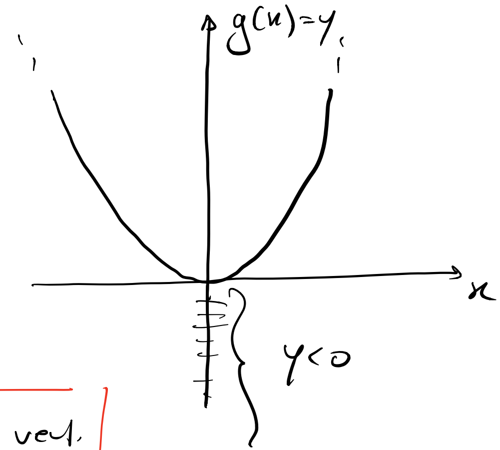
$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi P_X}}\, e^{-\frac{x^2}{2P_X}}}$$

3)



$$g(x) = x^2$$

$$f_Y(y) = ?$$



$$\boxed{y<0 \Rightarrow f_Y(y) = 0}$$

$$\boxed{y=0 \Rightarrow g'(x) = 0 \Rightarrow \text{asintoto vert.}}$$

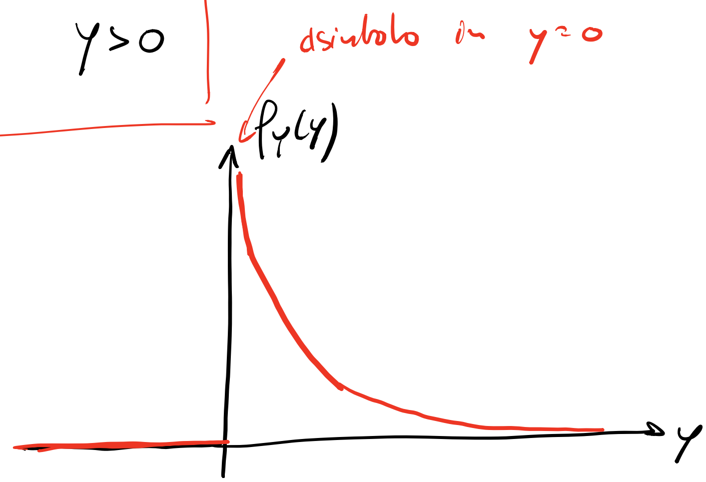
$$y>0 \Rightarrow \text{due sol} \Rightarrow \pm\sqrt{y}$$

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|g'(\sqrt{y})|} + \frac{f_X(-\sqrt{y})}{|g'(-\sqrt{y})|} = \frac{\frac{1}{\sqrt{2\pi P_X}} e^{-\frac{y}{N_0}}}{2\sqrt{y}} + \frac{\frac{1}{\sqrt{2\pi P_X}} e^{-\frac{y}{N_0}}}{|2(-\sqrt{y})|}$$

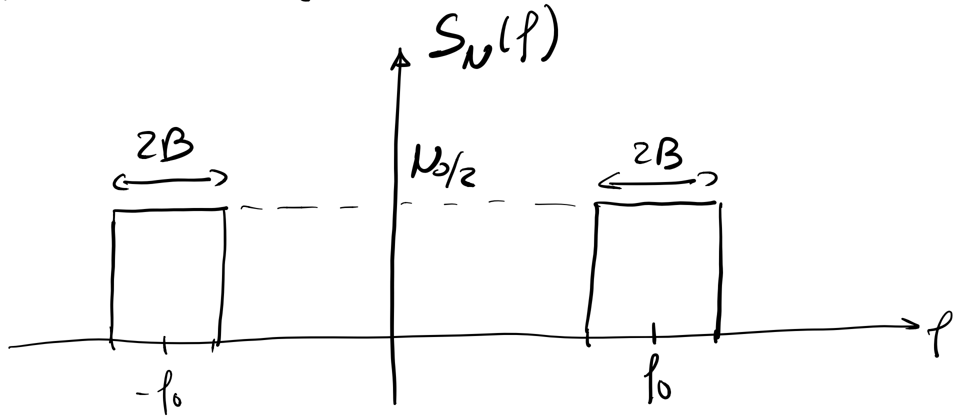
$$g'(x) = 2x$$

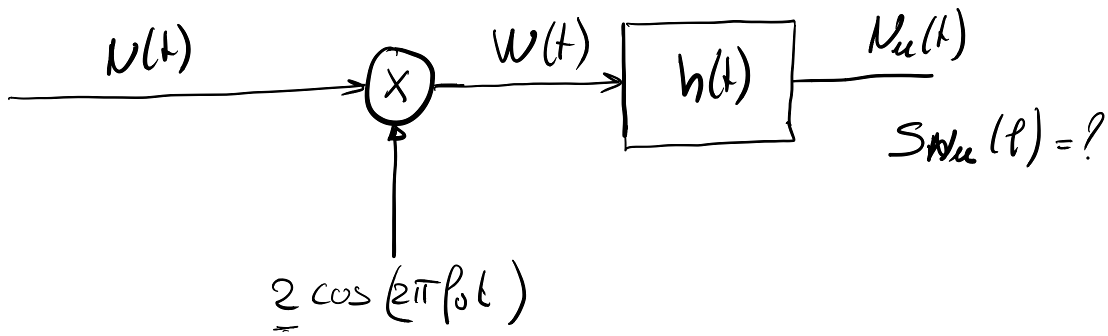
$$= \boxed{\frac{1}{\sqrt{2\pi P_X\, y}}\, e^{-\frac{y}{N_0}} \qquad y>0}$$

$$\lim_{y\to 0^+} f_Y(y) = +\infty$$

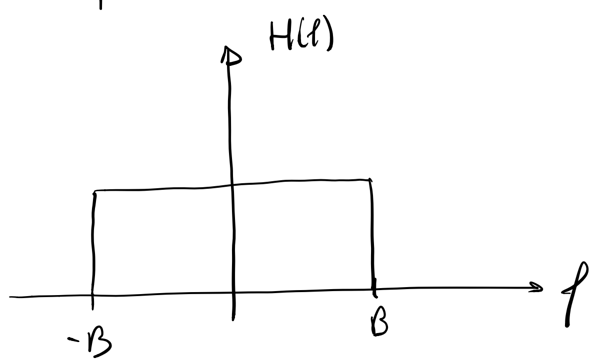


DEMODULAZIONE DI UN RUMORE BIANCO IN BANDA PASSANTE





$h(t)$ è un passa-basso ideale di banda B



$R_N(\tau) = ?$

$$S_N(f) = \frac{N_0}{2}\left[\mathrm{rect}\left(\frac{f-f_0}{2B}\right) + \mathrm{rect}\left(\frac{f+f_0}{2B}\right)\right]$$

$$R_N(\tau) = 2N_0B\,\mathrm{sinc}(2B\tau)\left[\frac{e^{j2\pi f_0\tau}+e^{-j2\pi f_0\tau}}{2}\right]$$

$$= 2N_0B\,\mathrm{sinc}(2B\tau)\cos(2\pi f_0\tau)$$

$$W(t) = 2N(t)\cos(2\pi f_0 t)$$ è ancora Gaussiano

$$\eta_W(t) = 0 = E[W(t)] = E[2N(t)\cos(2\pi f_0 t)] =$$
$$= 2\cos(2\pi f_0 t)\,\underbrace{E[N(t)]}_{0}=0$$

$$R_W(t_1,t_2) = E[W(t_1)W(t_2)]$$

$$= E[2N(t_1)\cos(2\pi f_0 t_1)\,2N(t_2)\cos(2\pi f_0 t_2)]$$

$$= 4\,E[N(t_1)N(t_2)]\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2)$$

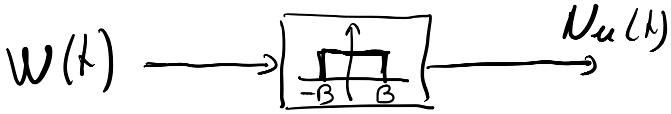
$$= 2\,\underbrace{R_N(t_1-t_2)}_{(\tau)}\left[\cos(2\pi f_0(t_1+t_2))+\cos(2\pi f_0(t_1-t_2))\right]$$

$$R_N(\tau) = 2N_0B\,\mathrm{sinc}(2B\tau)\cos(2\pi f_0\tau)$$

$$= 4N_0B\,\mathrm{sinc}(2B\tau)\cos(2\pi f_0(t_1-t_2))\,\cdot$$
$$\cdot\left[\cos[2\pi f_0(t_1+t_2)] + \cos[2\pi f_0(t_1-t_2)]\right]$$

$$= 2N_0B\,\mathrm{sinc}(2B\tau)\Big[\cos(4\pi f_0 t_1)+\cos(4\pi f_0 t_2) +$$
$$+ 1 + \cos[4\pi f_0(t_1-t_2)]\Big]$$

$$= 2N_0 B \,\mathrm{sinc}(2B\tau) + \text{termini a freq. } 2f_0$$



$$R_{N_u}(t_1,t_2) = \underbrace{R_w(t_1,t_2) \otimes h(t_1)}_{\substack{\text{elimina le comp} \\ \cos(4\pi f_0 t_1) \text{ e} \\ \cos 4\pi f_0 (t_2 - t_1)}} \otimes h(t_2)$$

elimina le comp. $\cos(4\pi f_0 t_2)$ $\cos(4\pi f_0(t_1-t_2))$

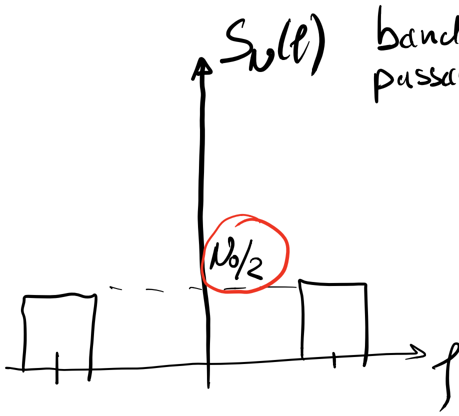
$$f_0 \gg B$$

$$R_{N_u}(t_1,t_2) = 2N_0 B \,\mathrm{sinc}\left[2B(t_1-t_2)\right]$$

$$\Downarrow$$

$$R_{N_u}(\tau) = 2N_0 B \,\mathrm{sinc}(2B\tau)$$

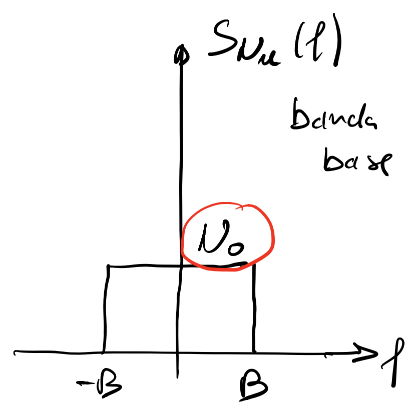
$$S_{N_u}(f) = N_0 \,\mathrm{rect}\left(\frac{f}{2B}\right)$$



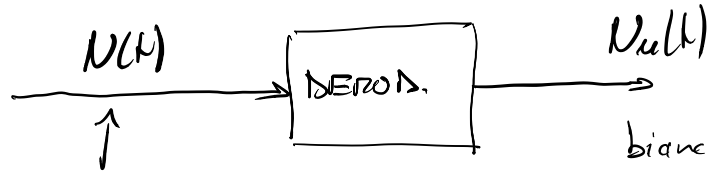
banda passante

DEMOD

$\Longrightarrow$



banda base

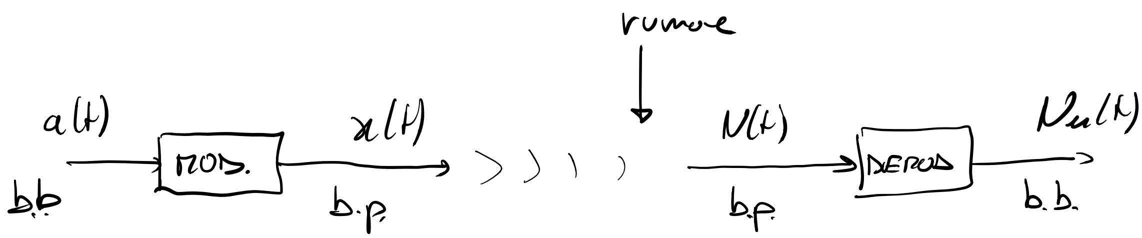


bianco in banda passante

$S_N(f) = \frac{N_0}{2}$ in banda

bianco in banda base

$S_{N_u}(f) = N_0$ in banda



$$x(t) = a(t)\cos(2\pi f_0 t)$$

$$x(t) \Rightarrow y(t) = (x(t)\, 2\cos 2\pi f_0 t) \otimes h(t)$$

passa-basso