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Moodle :

http://www.moodle.iet.unipi.it/moodle_2012-2013/course/category.php?id=9
passwd : comnum-13

TESTI DI RIFERIMENTO :

·) "Teoria dei Segnali", 3ª Edizione, Mc Graw-Hill
M. Luise, G.M. Vitetta

·) "Fondamenti di Comunicazioni", Progetto Leonardo, Esculapio Bologna
E. Baccarelli, N. Cordeschi, M. Biagi

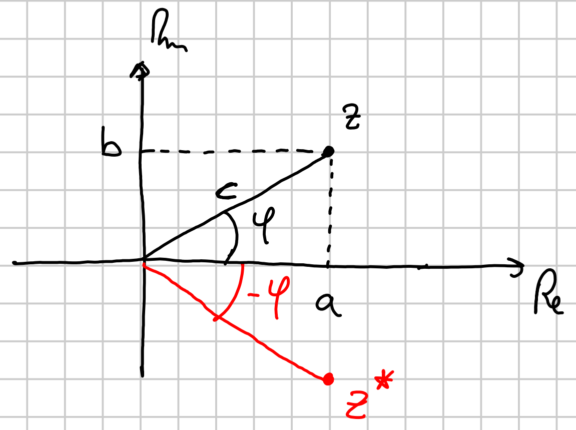
·) "Comunicazioni Elettriche", Edizioni ETS
A.N. D'Andrea

RICHIAMO SUI NUMERI COMPLESSI

$$j = i = \sqrt{-1}$$

·) $z = a + ib$, $a \in \mathbb{R}$, $b \in \mathbb{R}$

$a = \mathrm{Re}\{z\}$
$b = \mathrm{Im}\{z\}$



·) $z = c\, e^{j\varphi}$, $c \in \mathbb{R}$, $\varphi \in \mathbb{R}$

$$\begin{cases} a = c \cos\varphi \\ b = c \sin\varphi \end{cases} \qquad \begin{cases} c = \sqrt{a^2 + b^2} \\ \varphi = \operatorname{arctg}\left(\dfrac{b}{a}\right) \end{cases}$$

$$z = a + ib = c\cos\varphi + i\, c\sin\varphi =$$
$$= c\left(\cos\varphi + i\sin\varphi\right) = c\, e^{j\varphi}$$

$$e^{j\varphi} = \cos\varphi + i\sin\varphi$$

$$z^* = a - ib$$

$$\begin{cases} \mathrm{Re}\{z\} = a = \dfrac{1}{2}\left(z + z^*\right) \\ \mathrm{Im}\{z\} = b = \dfrac{1}{2j}\left(z - z^*\right) \end{cases}$$

OPERAZIONI ALGEBRICHE

$$z = z_1 \pm z_2$$

$$z_1 = a_1 + j b_1 \quad , \quad z_2 = a_2 + j b_2$$

$$z = a_1 + j b_1 \pm a_2 + j b_2 = (a_1 \pm a_2) + j\,(b_1 \pm b_2)$$

$$z = z_1 \cdot z_2 = (a_1 + j b_1) \cdot (a_2 + j b_2) =$$

$$= a_1 a_2 + j a_1 b_2 + j a_2 b_1 - b_1 b_2 =$$

$$= (a_1 a_2 - b_1 b_2) + j\,(a_1 b_2 + a_2 b_1)$$

$$z = z_1 \cdot z_2 = c_1 e^{j\varphi_1} \cdot c_2 e^{j\varphi_2} = c_1 c_2 e^{j(\varphi_1 + \varphi_2)}$$

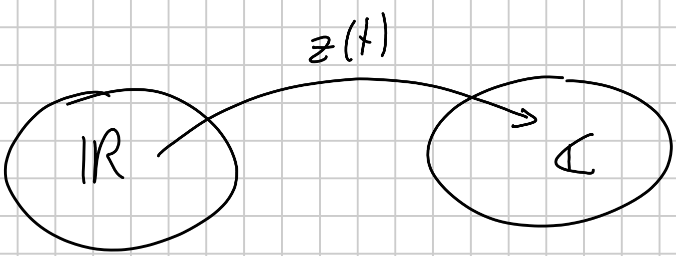
$$z = \frac{z_1}{z_2} = \frac{a_1 + j b_1}{a_2 + j b_2} = \frac{c_1 e^{j\varphi_1}}{c_2 e^{j\varphi_2}} = \frac{c_1}{c_2} e^{j(\varphi_1 - \varphi_2)}$$

$$z \cdot z^* = c e^{j\varphi} \cdot c e^{-j\varphi} = c^2 e^{j(\varphi - \varphi)} = c^2 = |z|^2$$

$$|z| = \sqrt{z \cdot z^*}$$

FUNZIONI COMPLESSE DI VARIABILE REALE

$z(t)$: $t \in \mathbb{R}$, $z(t) \in \mathbb{C}$



$$\int_a^b z(t)\,dt = \int_a^b a(t)\,dt + j \int_a^b b(t)\,dt$$

$$\frac{d}{dt} z(t) = \frac{d}{dt} a(t) + j \frac{d}{dt} b(t)$$

SEGNALI

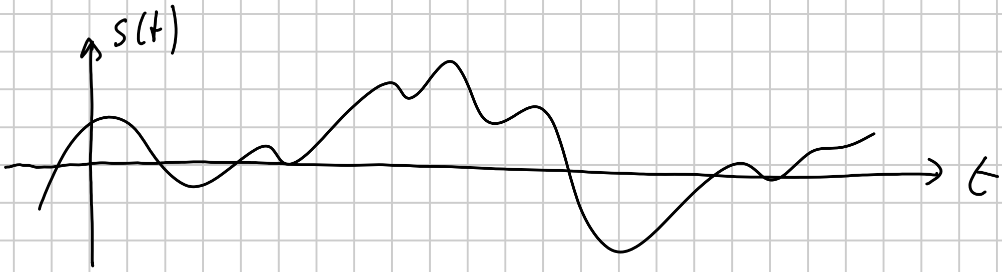
- DETERMINISTICI : segnali perfettamente noti e quindi rappresentabili tramite funzioni analitiche
- ALEATORI : segnali non noti rappresentabili tramite statistiche

DIMENSIONALITÀ

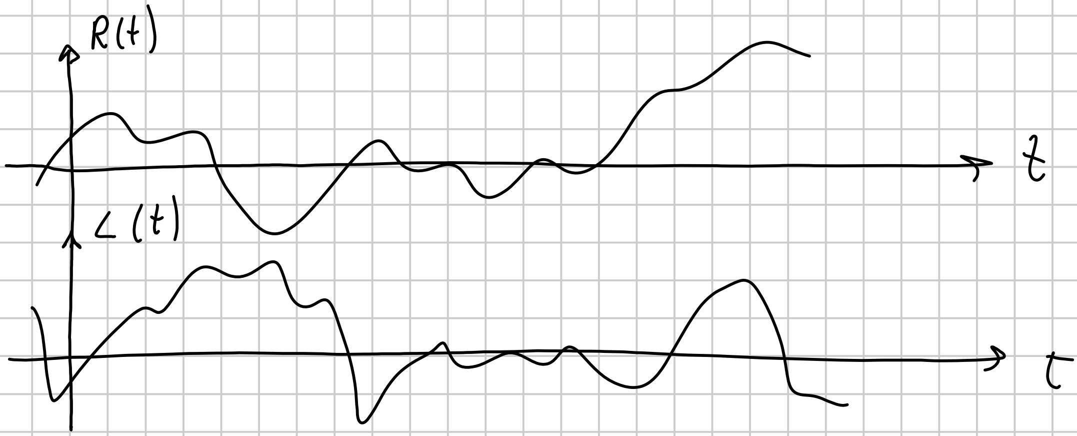
$\underline{s}(\underline{x})$

$\underline{s} \to \mathbb{R}^{m}$ $\quad$ $\underline{x} \to \mathbb{R}^{n}$

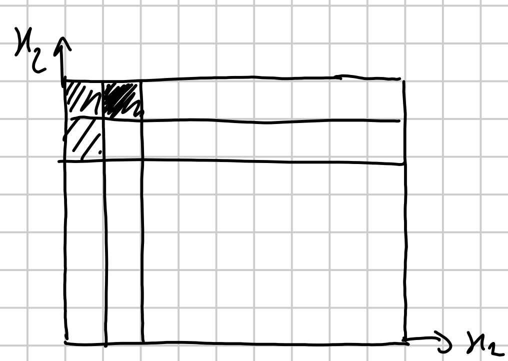
$s(t)$ segnale audio "MONO"



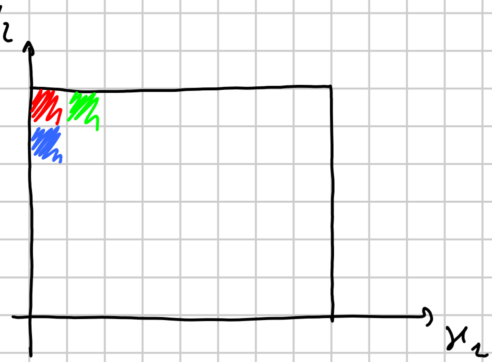
$\underline{s}(t)$ segnale audio "STEREO"



$v(x_1, x_2)$



$\underline{V}(x_1, x_2)$



$\underline{V}(x_1, x_2, t)$

SEGNALI MONODIMENSIONALI V COMPLESSI DI VARIABILE REALE

t (tempo) → continua / discreta

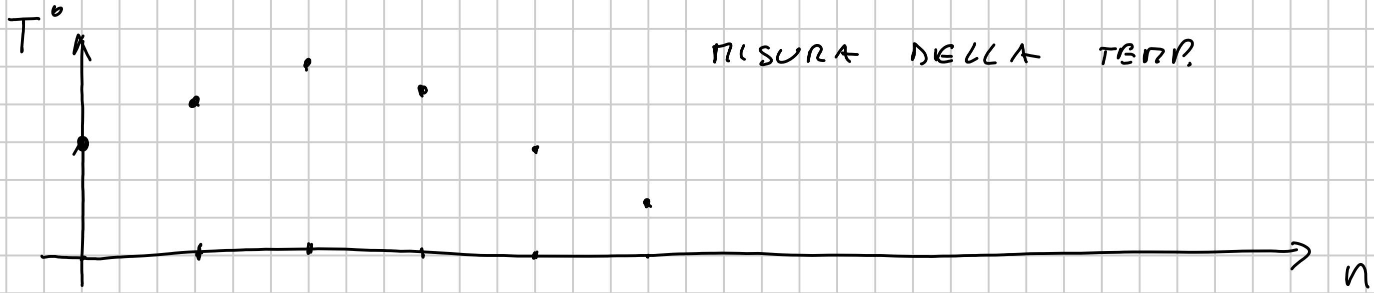
s (ampiezza) → continuo / discreto

	tempo cont.	tempo disc.
ampiezza cont.	ANALOGICI	SEQUENZE
ampiez. discreta	QUANTIZZATI	NUMERICI

$s \in \mathbb{C} \rightarrow a \in \mathbb{R},\ b \in \mathbb{R}$ amp. cont — può assumere tutti i valori

$s \in \mathbb{C} \Rightarrow s \in [s_1, s_2, \dots, s_N]$ — può assumere valori all'interno di un alfabeto finito

- SEGNALI ANALOGICI $\Rightarrow x(t),\ s(t),\ y(t) \quad t \in \mathbb{R}$
- SEQUENZE $\Rightarrow x[n],\ s[n],\ y[n] \quad n \in \mathbb{Z}$

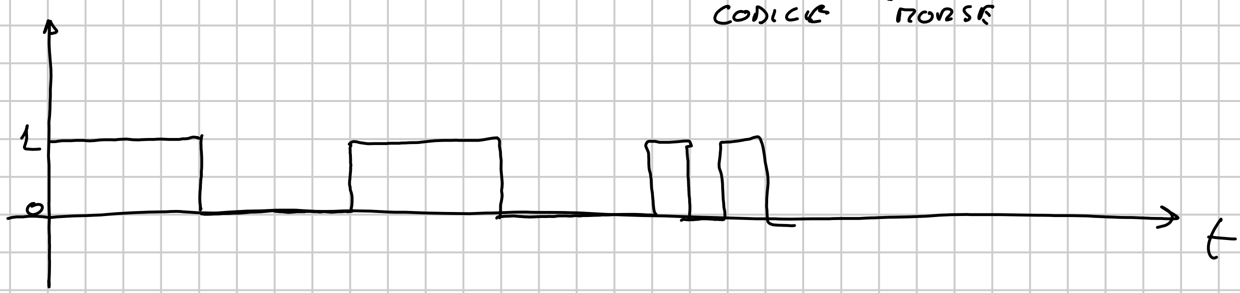


MISURA DELLA TEMP.

CAMPIONAMENTO DI UN SEGNALE ANALOGICO

·) SEGNALI QUANTIZZATI

CODICE "MORSE"



·) SEGNALI NUMERICI

$x(t)$ SEGNALI DET. MONODIMENSIONALI

POTENZA ISTANTANEA

$$P_x(t) \triangleq |x(t)|^2$$

ENERGIA

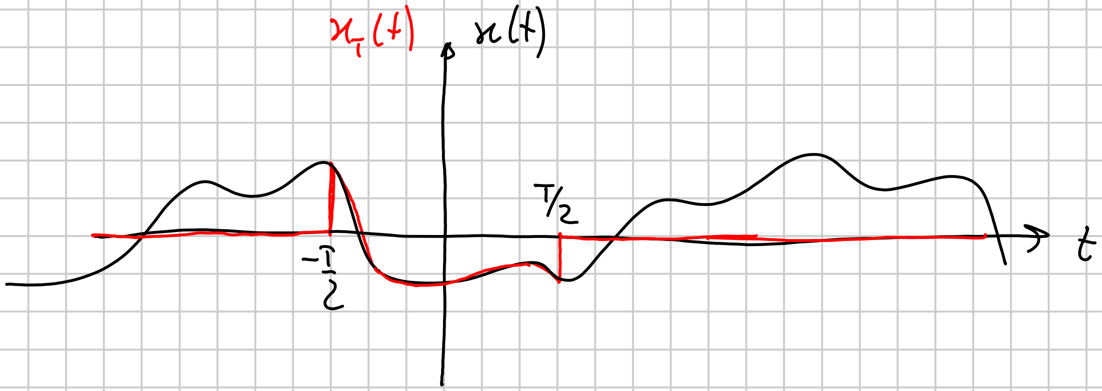
$$\boxed{E_x \triangleq \int_{-\infty}^{+\infty} P_x(t)\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt}$$

SEGNALE FISICO $\Rightarrow E_x < \infty$

$$x(t) = V \qquad -\infty < t < +\infty$$

$$E_x = \int_{-\infty}^{+\infty} V^2\, dt = \infty$$

$$x_T(t) \triangleq \begin{cases} x(t) & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{altrove} \end{cases}$$



$$E_{x_T} < \infty$$

$$\lim_{T \to \infty} E_{x_T} = E_x$$

POTENZA MEDIA

$$P_{x_T} \triangleq \frac{E_{x_T}}{T}$$ P.M. DEL SEGNALE TRONCATO

$$P_x \triangleq \lim_{T\to\infty} P_{x_T} = \lim_{T\to\infty} \frac{E_{x_T}}{T} = \boxed{\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\, dt \triangleq P_x}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} |x_T(t)|^2\, dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |\underbrace{x_T(t)}_{= x(t)}|^2\, dt$$

RELAZIONI TRA ENERGIA E POTENZA MEDIA

$$P_x = \lim_{T\to\infty} \frac{E_{x_T}}{T} = K \Rightarrow \lim_{T\to\infty} E_{x_T} = \infty \Rightarrow E_x = \infty$$

$$E_x = \lim_{T\to\infty} E_{x_T} = K \Rightarrow \lim_{T\to\infty} P_{x_T}\, T = K \Rightarrow \lim_{T\to\infty} P_{x_T} = 0 = P_x$$

VALORE EFFICACE

$$\boxed{x_{eff} \triangleq \sqrt{P_x}}$$

VALOR MEDIO

$$\boxed{x_m \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, dt}$$

SEGNALI TIPICI

·) COSTANTE

$$x(t) = A \quad \forall t \qquad (-\infty < t < +\infty)$$

·) $E_x = \int_{-\infty}^{+\infty} |A|^2 \, dt = \infty$

·) $P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |A|^2 \, dt = \lim_{T \to \infty} \frac{1}{\cancel{T}} |A|^2 \cancel{T} = |A|^2$

·) $x_{eff} = |A|$

·) $x_m = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A \, dt = \lim_{T \to \infty} \frac{1}{\cancel{T}} A \cancel{T} = A$

·) SINUSOIDI

$$x(t) = A \cos\left(2\pi f_0 t + \varphi\right) \qquad A, t, \varphi, f_0 \in \mathbb{R}$$

·) $E_x = \int_{-\infty}^{+\infty} A^2 \cos^2\left(2\pi f_0 t + \varphi\right) dt =$ $\qquad \cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos 2\alpha$

$$= \int_{-\infty}^{+\infty} A^2 \left(\frac{1}{2} + \frac{1}{2} \cos\left(4\pi f_0 t + 2\varphi\right)\right) dt$$

$$= \int_{-\infty}^{+\infty} \frac{A^2}{2} dt + \int_{-\infty}^{+\infty} \frac{1}{2} \cos\left(4\pi f_0 t + 2\varphi\right) dt$$

$$= +\infty + \text{ind. finito} = +\infty$$

$\cdot)\ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A^2 \cos^2 (2\pi f_0 t + \varphi)\, dt =$

$= \lim_{T \to \infty} \frac{1}{\cancel{T}} \frac{A^2}{2} \cancel{T} + \lim_{T \to \infty} \frac{1}{T} \frac{A^2}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos (4\pi f_0 t + 2\varphi)\, dt$

$= \frac{A^2}{2} + \frac{A^2}{2} \lim_{T \to \infty} \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos (4\pi f_0 t + 2\varphi)\, dt}{T} = \frac{A^2}{2}$

$\cdot)\ x_{eff} = \frac{A}{\sqrt{2}}$

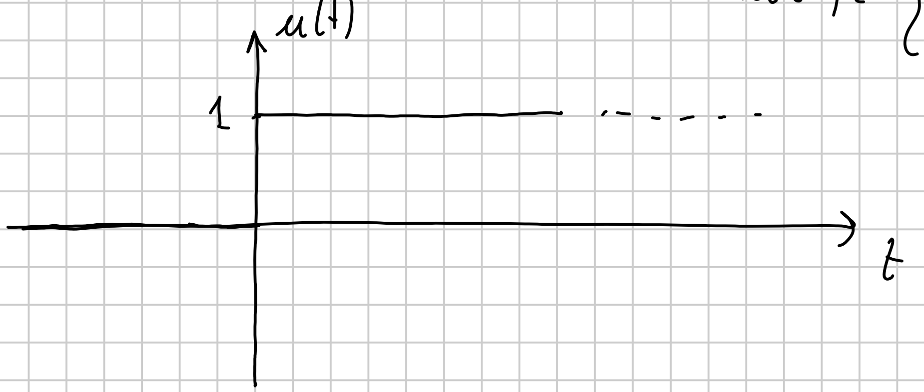
$\cdot)\ x_m = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} A \cos (2\pi f_0 t + \varphi)\, dt =$

$= A \lim_{T \to \infty} \frac{\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos (2\pi f_0 t + \varphi)\, dt}{T} = 0$

-) GRADINO

$x(t) = u(t)$

$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$



$\cdot)\ E_x = \int_{-\infty}^{+\infty} u^2(t)\, dt = \int_{-\infty}^{+\infty} u(t)\, dt = \infty$

$$\cdot)\ P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} 1\, dt =$$

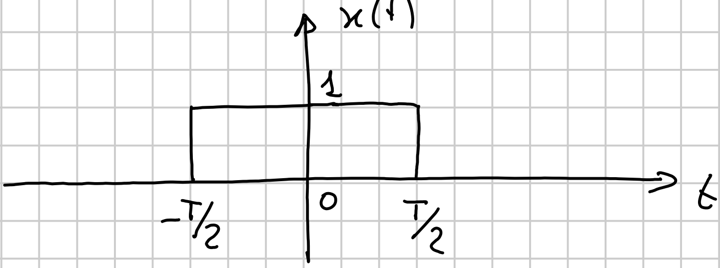
$$= \lim_{T \to \infty} \frac{1}{\cancel{T}} \frac{\cancel{T}}{2} = \frac{1}{2}$$

$$\cdot)\ x_{eff} = \frac{1}{\sqrt{2}}$$

$$\cdot)\ x_m = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\, dt = \frac{1}{2}$$

06/03/2013

·) Rettangolo



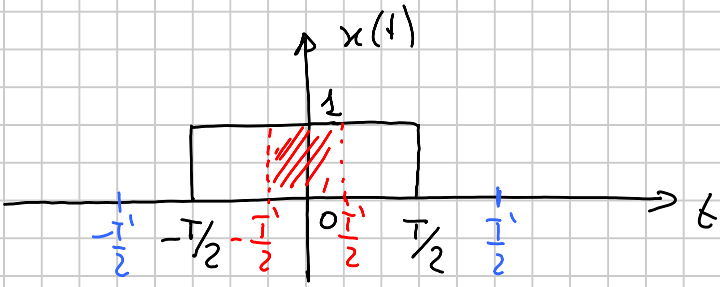
$$x(t) = \begin{cases} 1 & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{altrove} \end{cases} \quad \Longleftrightarrow \quad x(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \, dt = T$$

$$P_x = \lim_{T' \to \infty} \frac{1}{T'} \int_{-\frac{T'}{2}}^{\frac{T'}{2}} |x(t)|^2 \, dt$$

$T' < T \Rightarrow \lim_{T' \to \infty} \frac{1}{T'} T'$

$T' > T \Rightarrow \lim_{T' \to \infty} \frac{1}{T'} T = 0$



$$x_{eff} = 0$$

$$x_m = \lim_{T' \to \infty} \frac{1}{T'} \int_{-\frac{T'}{2}}^{\frac{T'}{2}} x(t) \, dt = 0$$

) ESPONENZIALE UNILATERA

$$x(t) = e^{-t}\,u(t)$$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{altrove} \end{cases}$$

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} e^{-2t}\,dt =$$

$$= \lim_{T\to\infty}\frac{1}{T}\left(-\frac{1}{2}\right)e^{-2t}\Big|_0^{\frac{T}{2}} = \lim_{T\to\infty}\frac{1}{T}\left(-\frac{1}{2}\right)\left[e^{-T}-1\right] =$$

$$= \lim_{T\to\infty}\frac{1}{T}\left(\frac{1}{2}-\frac{1}{2}e^{-T}\right) = \lim_{T\to\infty}\frac{1}{2T} - \frac{1}{2}\lim_{T\to\infty}\frac{e^{-T}}{T} = 0$$

$$E_x = \int_{-\infty}^{+\infty}|x(t)|^2\,dt = \int_0^{+\infty} e^{-2t}\,dt = \left(-\frac{1}{2}\right)e^{-2t}\Big|_0^{+\infty} =$$

$$= \frac{1}{2}$$

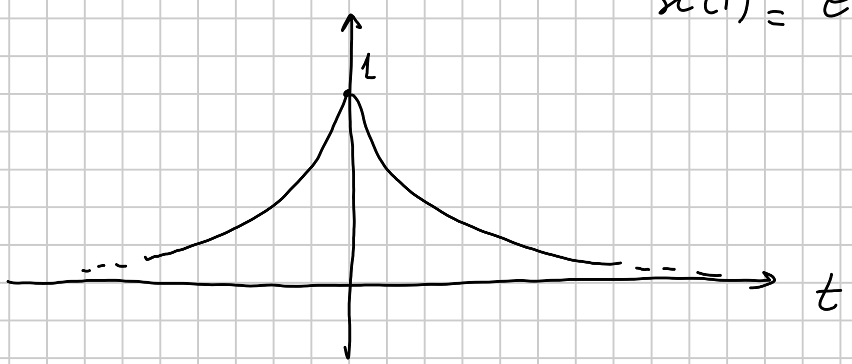
$$x_{eff} = 0$$

$$x_m = \lim_{T\to\infty}\frac{1}{T}\int_0^{\frac{T}{2}} e^{-t}\,dt = \lim_{T\to\infty}\frac{1}{T}(-1)e^{-t}\Big|_0^{\frac{T}{2}}$$

$$= \lim_{T\to\infty}\left(+\frac{1}{T}\right) - \lim_{T\to\infty}\frac{e^{-\frac{T}{2}}}{T} = 0$$

·) Esponenziale bilatera

$$x(t) = e^{-|t|}$$



$$E_x = \int_{-\infty}^{+\infty} e^{-2|t|} \, dt = 2 \int_0^{+\infty} e^{-2t} \, dt =$$

$$= \cancel{2} \left(-\frac{1}{\cancel{2}}\right) e^{-2t} \Big|_0^{+\infty} = 1 - 0 = 1$$

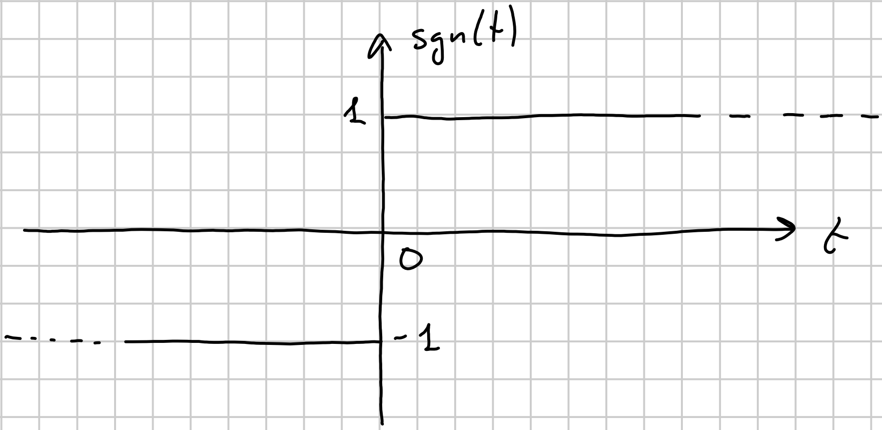
$P_x = 0$ (calcolo praticamente identico a quello fatto per e^{-t})

$x_{eff} = 0$

$x_m = 0$

·) SEGNO

$$x(t) = \text{sgn}(t) = \begin{cases} 1 & t \geq 0 \\ -1 & t < 0 \end{cases}$$



$$E_x = \int_{-\infty}^{+\infty} 1 \, dt = +\infty$$

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \, dt = 1$$

$$x_{eff} = 1$$

$$x_m = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{sgn}(t) \, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^{\frac{T}{2}} 1 \, dt + \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{0} (-1) \, dt$$

$$= \lim_{T \to \infty} \frac{1}{\cancel{T}} \frac{\cancel{T}}{2} - \lim_{T \to \infty} \frac{1}{\cancel{T}} \frac{\cancel{T}}{2} = \frac{1}{2} - \frac{1}{2} = 0$$

Osserviamo che

$$P_x = 0 \Rightarrow x_m = 0$$

$$x(t) = x'(t) + x_m$$

$$x'(t) = x(t) - x_m \Rightarrow x'_m = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left[x(t) - x_m \right] dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \, dt - \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_m \, dt =$$

$$= x_m - x_m = 0$$

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x'(t) + x_m|^2 \, dt = 0$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} (x'(t) + x_m)(x'^*(t) + x_m^*) \, dt$$

$$\stackrel{!}{=} \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x'(t)|^2 + |x_m|^2 + x'(t)\,x_m^* + x'^*(t)\,x_m \, dt$$

$$= P_{x'} + |x_m|^2 + \underbrace{\lim_{T\to\infty} \frac{x_m^*}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x'(t)\,dt}_{=0} + \underbrace{x_m \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x'^*(t)\,dt}_{=0}$$

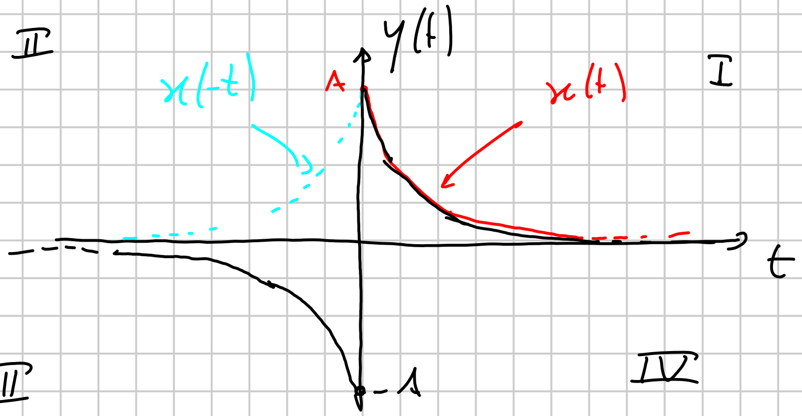
$$P_x = \underbrace{P_{x'}}_{=0} + \underbrace{|x_m|^2}_{x_m = 0} = 0$$

$$\boxed{P_x = 0 \Rightarrow x_m = 0}$$

ESERCIZIO

$$y(t) = x(t) - x(-t)$$

$$x(t) = A e^{-t} u(t) \qquad , \qquad A \in \mathbb{R}$$



$$E_y = \int_{-\infty}^{+\infty} y^2(t)\,dt = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left[x(t) - x(-t)\right]^2 dt$$

$$= \int_{-\infty}^{+\infty} \left[A e^{-t} u(t) - A e^{t} u(-t)\right]^2 dt$$

$$= \int_{-\infty}^{+\infty} A^2 e^{-2t} u(t)\,dt + \int_{-\infty}^{+\infty} A^2 e^{2t} u(-t)\,dt$$

$$= \int_{-\infty}^{+\infty} A^2 \underbrace{u(t)\, u(-t)}_{= 0} \, dt$$

$$= A^2 \int_{-\infty}^{+\infty} e^{-2t} \, dt + \lim_{T \to \infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{0} e^{2t} \, dt$$

$$\Downarrow \quad t' = -t$$

$$= A^2 \int_{-\infty}^{+\infty} e^{-2t} \, dt + A^2 \int_{-\infty}^{+\infty} e^{-2t'} \, dt'$$

$$= 2A^2 \int_{-\infty}^{+\infty} e^{-2t} \, dt = 2A^2 \, \frac{1}{2} = A^2$$

$P_x = 0$ (perché $E_x < \infty$)

$x_{eff} = 0$

$x_m = 0$ (perché $P_x = 0$)

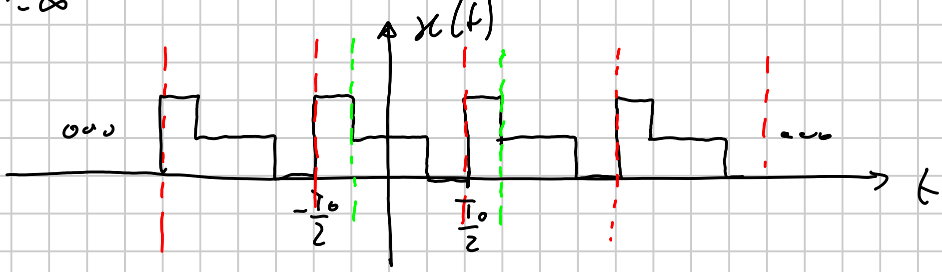
SEGNALI PERIODICI

$$x(t) = x(t - kT_0) \quad , \quad T_0 \in \mathbb{R} \quad , \quad k = \mathbb{Z}$$

T_0 = periodo del segnale

ENERGIA DI UN SEGNALE PERIODICO

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2} + kT_0}^{\frac{T_0}{2} + kT_0} |x(t)|^2 \, dt = \lim_{k \to \infty} k \overbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt}^{A} = \infty$$



$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,dt = \lim_{k\to\infty}\frac{1}{kT_0}\int_{-\frac{kT_0}{2}}^{\frac{kT_0}{2}} |x(t)|^2\,dt$$

$$= \lim_{k\to\infty}\frac{1}{\cancel{k}T_0}\,\cancel{k}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\,dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\,dt$$

$$x_m = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,dt$$

ANALISI DI FOURIER

$x(t)$ periodico che soddisfa le condizioni di Dirichelet allora

$\Downarrow$

$$x(t) = \sum_{n=0}^{+\infty} A_n \cos(2\pi n f_0 t + \varphi_n)$$

SVILUPPO IN SERIE DI FOURIER (FORMA POLARE)

$f_0 \triangleq \frac{1}{T_0}$, T_0 = periodo di $x(t)$

A_n = ampiezza della n-esima componente ($A_n \in \mathbb{R}$)

φ_n = fase " " " ($\varphi_n \in \mathbb{R}$)

comp. #0 $A_0 \cos\varphi$

comp #1 $A_1 \cos(2\pi f_0 t + \varphi_1)$ ARMONICHE

comp #2 $A_2 \cos(4\pi f_0 t + \varphi_2)$

$\vdots$

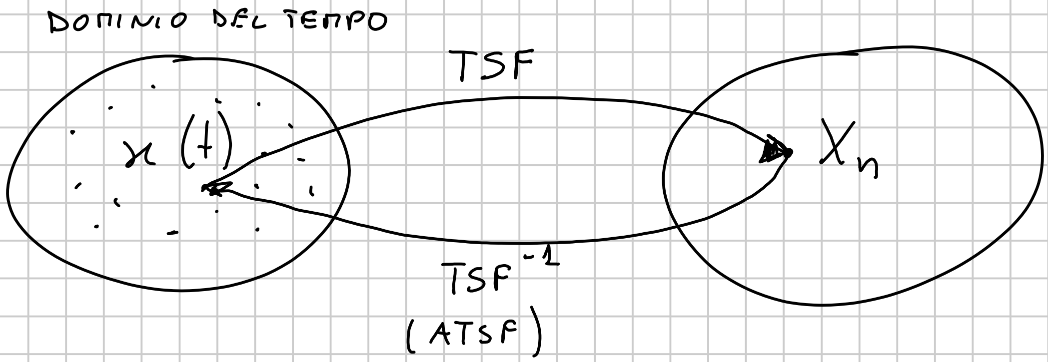
TRASFORMATA SERIE DI FOURIER

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi n f_0 t}}$$

ANTITRASFORMATA SERIE DI FOURIER (EQ. DI SINTESI

$$\boxed{X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\,dt}$$

TRASFORMATA SERIE DI FOURIER EQ. DI ANALISI



$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underbrace{\sum_{k=-\infty}^{+\infty} X_k\, e^{j2\pi k f_0 t}}_{x(t)}\, e^{-j2\pi n f_0 t}\, dt$$

$$= \sum_{k=-\infty}^{+\infty} X_k\, \frac{1}{T_0} \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi (k-n) f_0 t}\, dt}$$

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos\left[2\pi (k-n) f_0 t\right] dt + j \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin\left[2\pi (k-n) f_0 t\right] dt$$

$k = n$

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 1 \cdot dt = T_0 \qquad\qquad \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} 0\, dt = 0$$

$k = n \Rightarrow X_n = X_n$

$k \neq n \Rightarrow k - n = 1 \Rightarrow \displaystyle\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(2\pi f_0 t)\, dt = 0 \;\Big|\; \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin(2\pi f_0 t)\, dt = 0$

$k - n = 2 \Rightarrow \displaystyle\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos(4\pi f_0 t)\, dt = 0 \quad . \; .$

Quindi sopravvive solo la componente $k = n$

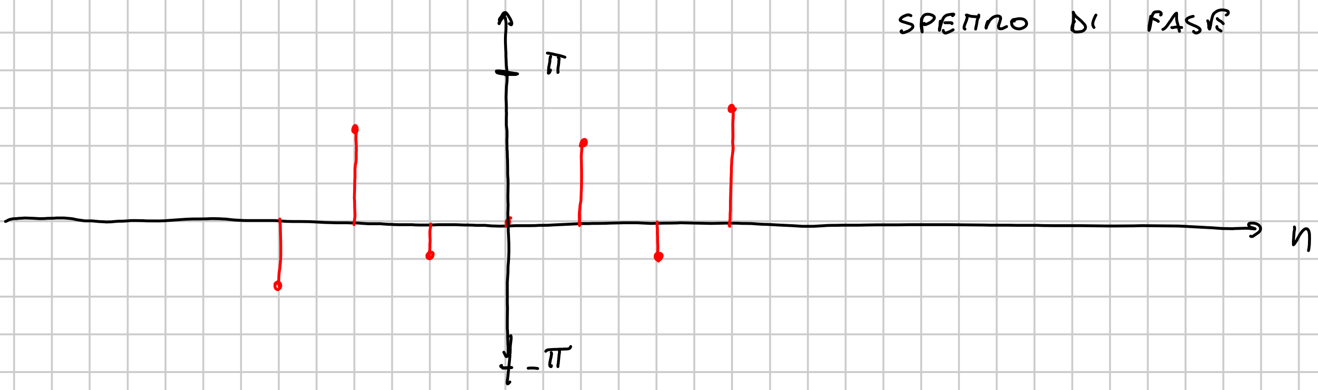
$$\Rightarrow \quad X_n = X_n \qquad \text{c.v.d.}$$

SPETTRO DI UN SEGNALE PERIODICO

$$\{X_n\} = \text{spettro}$$

SPETTRO DI AMPIEZZA $= \{|X_n|\}$

SPETTRO DI FASE $= \{\angle X_n\}$



SPETTRO DI UN COSENO

$$x(t) = A \cos(2\pi f_0 t)$$

$$T_0 = \frac{1}{f_0}$$

$$x(t) \overset{?}{=} x(t - kT_0) = A \cos(2\pi f_0 (t - kT_0)) =$$

$$= A \cos(2\pi f_0 t - 2\pi \cancel{f_0} k \cancel{T_0}) = A \cos(2\pi f_0 t)$$

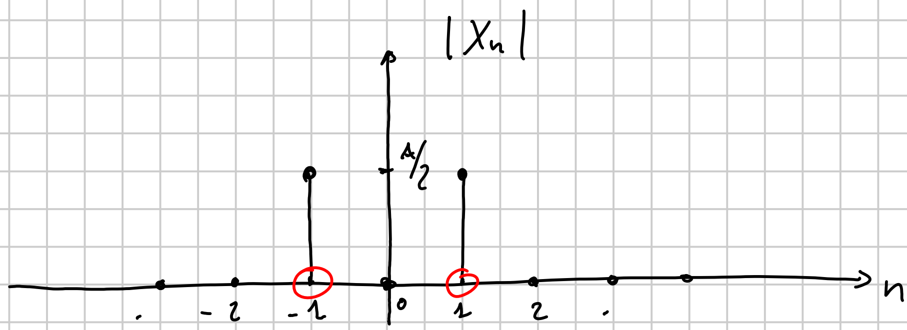
$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt =$$

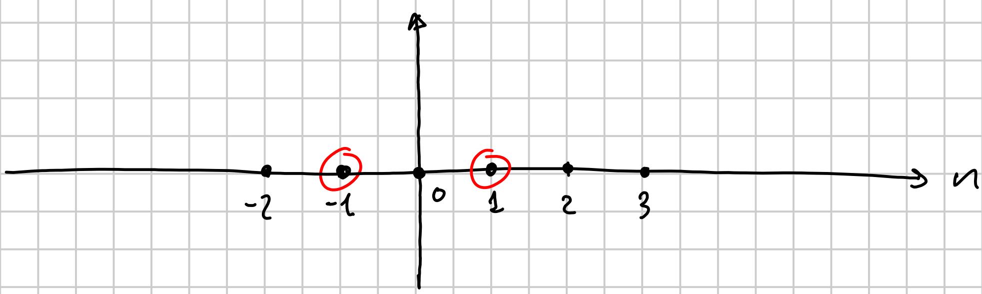
$$= \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\, e^{-j2\pi n f_0 t}\, dt$$

$$\boxed{\begin{aligned} e^{j2\pi f_0 t} &= \cos(2\pi f_0 t) + j\sin(2\pi f_0 t) \\ e^{-j2\pi f_0 t} &= \cos(2\pi f_0 t) - j\sin(2\pi f_0 t) \end{aligned}}$$

$$X_n = \underbrace{\frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(1-n)f_0 t}\, dt}_{\neq 0 \text{ solo per } n=1} + \underbrace{\frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi(1+n)f_0 t}\, dt}_{\neq 0 \text{ solo per } n=-1}$$

$$X_n = \begin{cases} 0 & n \neq \pm 1 \\ A/2 & n = 1 \\ A/2 & n = -1 \end{cases}$$





07/03/2013

SPETTRO DI UN SENO

$$x(t) = A \sin(2\pi f_0 t)$$

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A \sin(2\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt =$$

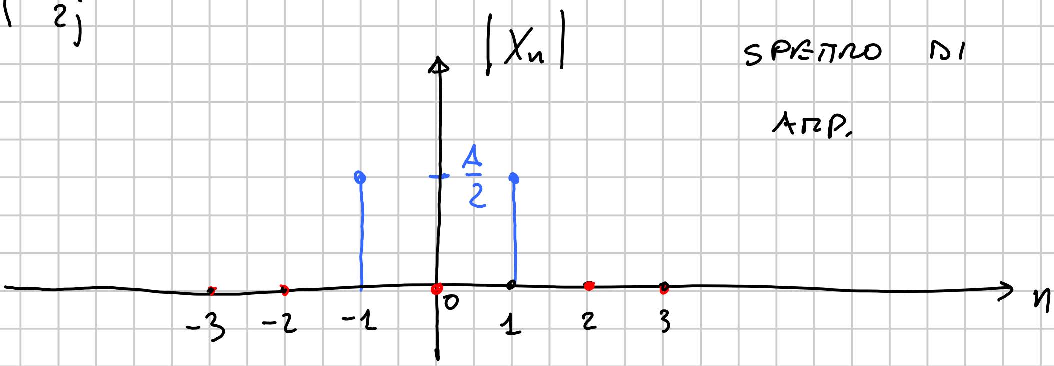
$$= \frac{A}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\, e^{-j2\pi n f_0 t}\, dt$$

$$e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j \sin(2\pi f_0 t)$$

$$e^{-j2\pi f_0 t} = \cos(2\pi f_0 t) - j \sin(2\pi f_0 t)$$

$$= \frac{A}{2jT_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(1-n) f_0 t}\, dt - \frac{A}{2jT_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi(1+n) f_0 t}\, dt$$

$$X_n = \begin{cases} 0 & n \neq \pm 1 \\ \frac{A}{2j} & n = 1 \\ -\frac{A}{2j} & n = -1 \end{cases}$$



SPETTRO DI AMP.

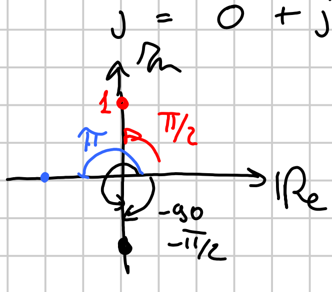
$$X_1 = \frac{A}{2j} = -\frac{A}{2} j = -\frac{A}{2} e^{j\frac{\pi}{2}} = e^{j\pi} \cdot \frac{A}{2} e^{j\frac{\pi}{2}}$$

$$= \frac{A}{2} e^{-j\frac{\pi}{2}}$$

modulo

fase

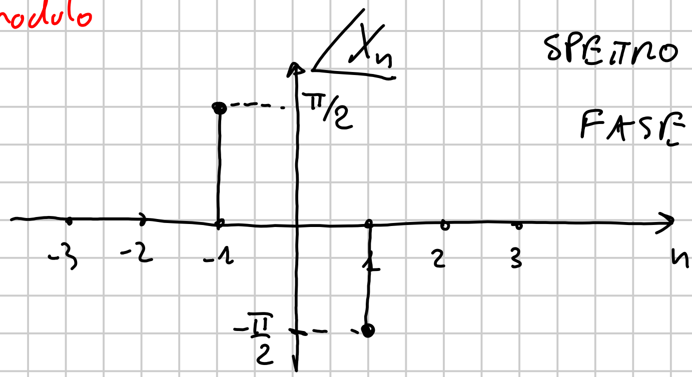
$$j = 0 + j1$$



$$X_{-1} = -\frac{A}{2j} = \frac{A}{2} j = \frac{A}{2} e^{j\frac{\pi}{2}}$$

modulo

fase



SPETTRO DI FASE

PROPRIETÀ DELLA TSF

·) LINEARITÀ

$$z(t) = x(t) + y(t)$$

, $z(t)$, $x(t)$, $y(t)$ sono periodici di T_0 e trasformabili secondo la TSF

$$\Updownarrow \text{TSF}$$

$$Z_n = X_n + Y_n$$

, dove $X_n = TSF[x(t)]$

$Y_n = TSF[y(t)]$

DIM.

$$Z_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} [x(t)+y(t)]\, e^{-j2\pi n f_0 t}\, dt$$

$$= \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_n} + \underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi n f_0 t}\, dt}_{Y_n}$$

c.v.d.

·) SIMMETRIA HERMITIANA

$x(t)$ periodico di T_0 e reale $\Rightarrow$ $x(t) = x^*(t)$

$$\Updownarrow \text{ TSF}$$

$$X_n = X_{-n}^* \quad \Rightarrow \quad |X_n| = |X_{-n}| \;, \quad \angle X_n = -\angle X_{-n}$$

DIM.

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi(-n)f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt$$

$$= \left[\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^*(t)\, e^{-j2\pi n f_0 t}\, dt\right]^* =$$

$$= \left[\underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{X_n}\right]^* = X_n^*$$

$$\Rightarrow X_{-n} = X_n^* \quad \Rightarrow \quad X_n = X_{-n}^* \qquad \text{c.v.d.}$$

·) TSF DI SEGNALI PERIODICI REALI E PARI

PARITÀ: $x(t) = x(-t)$, $x(t) = x^*(t)$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{+j2\pi n f_0 t}\, dt = \ldots (\, t' = -t\,)$$

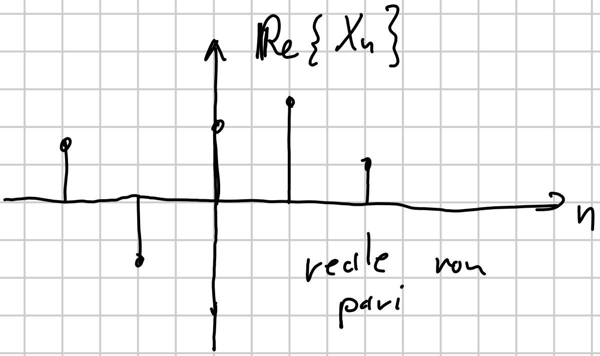
$$= \frac{1}{T_0}\int_{\frac{T_0}{2}}^{-\frac{T_0}{2}} x(-t')\, e^{-j2\pi n f_0 t'}\, (-dt') =$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t')\, e^{-j2\pi n f_0 t'}\, dt' = X_n$$

$$\Rightarrow X_{-n} = X_n = X_{-n}^* \Rightarrow X_{-n} = X_{-n}^* \Rightarrow X_m = X_m^*$$

X_n è reale e pari

Se $x(t)$ è reale e pari $\Rightarrow$ X_n è reale e pari

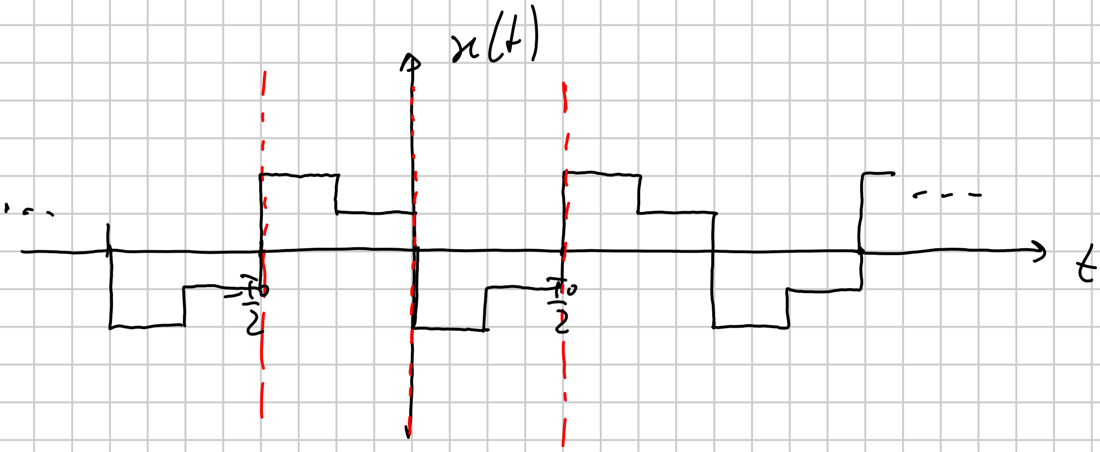


·) $Im\{X_n\} = 0 \quad \forall n$

·) SPETTRO REALE

·) SEGNALI PERIODICI E ALTERNATIVI

$$x(t) = -x\left(t - \frac{T_0}{2}\right)$$



$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{0} x(t)\, e^{-j2\pi n f_0 t}\, dt +$$

$$+ \frac{1}{T_0}\int_{0}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$\Rightarrow t' = t + \frac{T_0}{2} \Rightarrow t = t' - \frac{T_0}{2}$$

$$X_n = \frac{1}{T_0}\int_0^{\frac{T_0}{2}} x\left(t' - \frac{T_0}{2}\right) e^{-j2\pi n f_0\left(t' - \frac{T_0}{2}\right)} dt' + \frac{1}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt$$

$$= \frac{1}{T_0}\int_0^{\frac{T_0}{2}} \left(-x(t')\right) e^{-j2\pi n f_0 t'}\, e^{j2\pi n f_0 \frac{T_0}{2}} dt' + \dots$$

$$= -e^{j n\pi} \cdot \frac{1}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt + \frac{1}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt$$

$$= \left(1 - e^{j n\pi}\right) \frac{1}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt$$

$$e^{j n \pi} \begin{cases} \to 1 & n \text{ pari} \\ \to -1 & n \text{ dispari} \end{cases}$$

$$= \left(1 - (-1)^n\right) \frac{1}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt$$

$$X_n = \begin{cases} 0 & n \text{ pari} \\ \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt & n \text{ dispari} \end{cases}$$

N.B. le componenti pari sono nulle

ESERCIZI

1) SOMMA DI DUE SINUSOIDI

$$x(t) = A\cos\left(2\pi f_0 t + \varphi\right) + B\sin\left(4\pi f_0 t\right)$$

Calcolare la TSF $\Rightarrow X_n$

$$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$X_n = X_{1n} + X_{2n}$$

$$X_{1n} \overset{TSF}{\Longleftrightarrow} x_1(t) = A\cos(2\pi f_0 t + \varphi)$$

$$X_{2n} \overset{TSF}{\Longleftrightarrow} x_2(t) = B\sin(4\pi f_0 t)$$

$$X_{1n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A\cos(2\pi f_0 t + \varphi)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{A}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[e^{j(2\pi f_0 t + \varphi)} + e^{-j(2\pi f_0 t + \varphi)} \right] e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{A}{2T_0} e^{j\varphi} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(1-n) f_0 t}\, dt + \frac{A}{2T_0} e^{-j\varphi} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{-j2\pi(1+n) f_0 t}\, dt$$

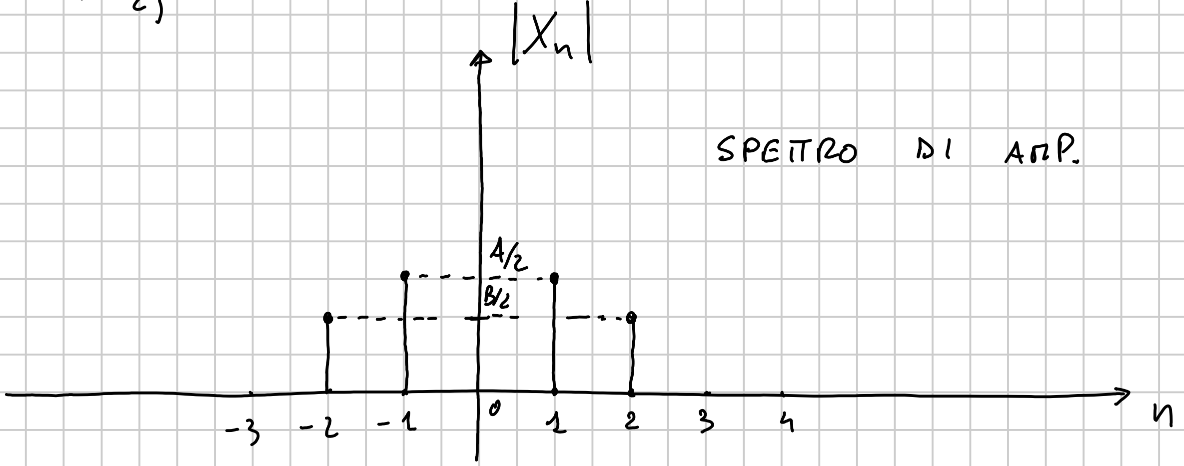
$$\Rightarrow X_{1n} = \begin{cases} 0 & n \neq \pm 1 \\ \frac{A}{2} e^{j\varphi} & n = 1 \\ \frac{A}{2} e^{-j\varphi} & n = -1 \end{cases}$$

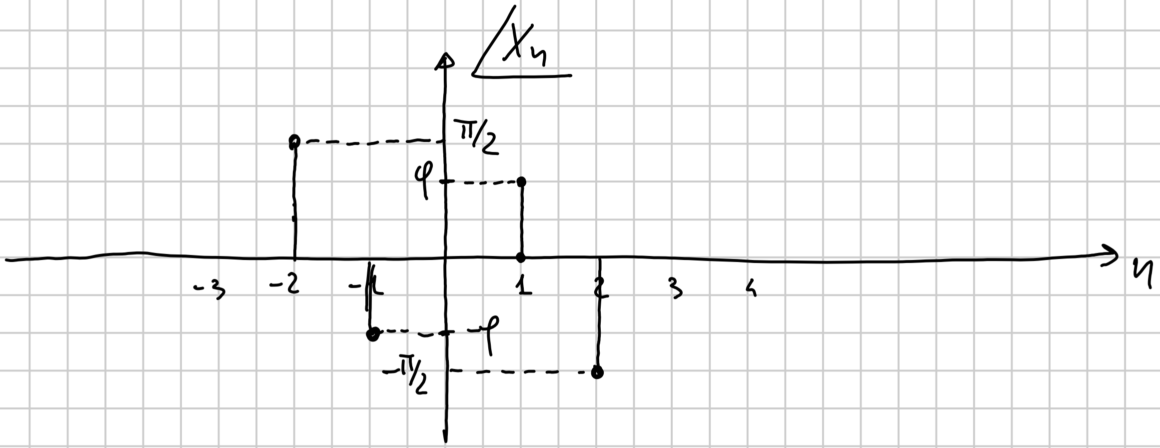
$$X_{2n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} B\sin(4\pi f_0 t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{B}{2jT_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[e^{j4\pi f_0 t} - e^{-j4\pi f_0 t} \right] e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{B}{2jT_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi(2-n) f_0 t} - e^{-j2\pi(2+n) f_0 t}\, dt$$

$$X_{2n} = \begin{cases} 0 & n \neq \pm 2 \\ \dfrac{B}{2j} & n = 2 \\ -\dfrac{B}{2j} & n = -2 \end{cases}$$





2) PRODOTTO DI SINUSOIDI

$$z(t) = A \cos(2\pi f_0 t + \varphi) \cos(4\pi f_0 t)$$

$$z(t) = \frac{A}{2} \cos(6\pi f_0 t + \varphi) + \frac{A}{2} \cos(\pm 2\pi f_0 t \mp \varphi)$$

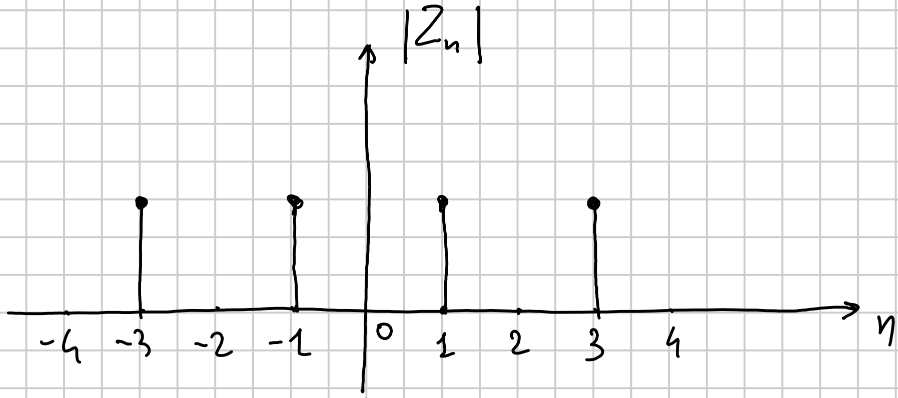
$$\cos\alpha \cdot \cos\beta = \frac{1}{2} \cos(\alpha + \beta) + \frac{1}{2} \cos(\alpha - \beta)$$

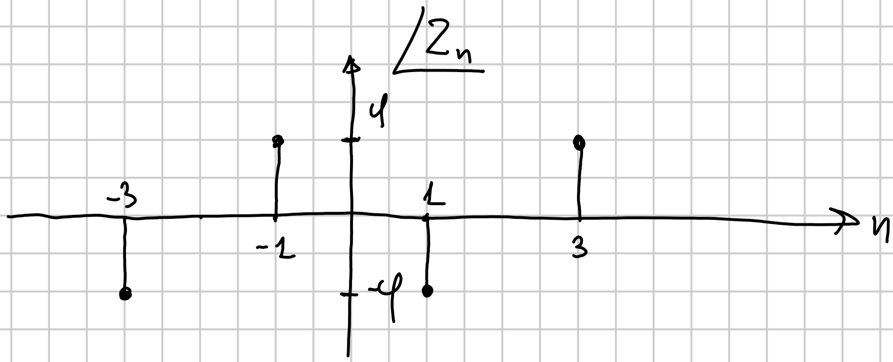
$$Z_n = X_n + Y_n$$

$$X_n = TSF[x(t)] \quad , \quad x(t) = \frac{A}{2} \cos(6\pi f_0 t + \varphi)$$

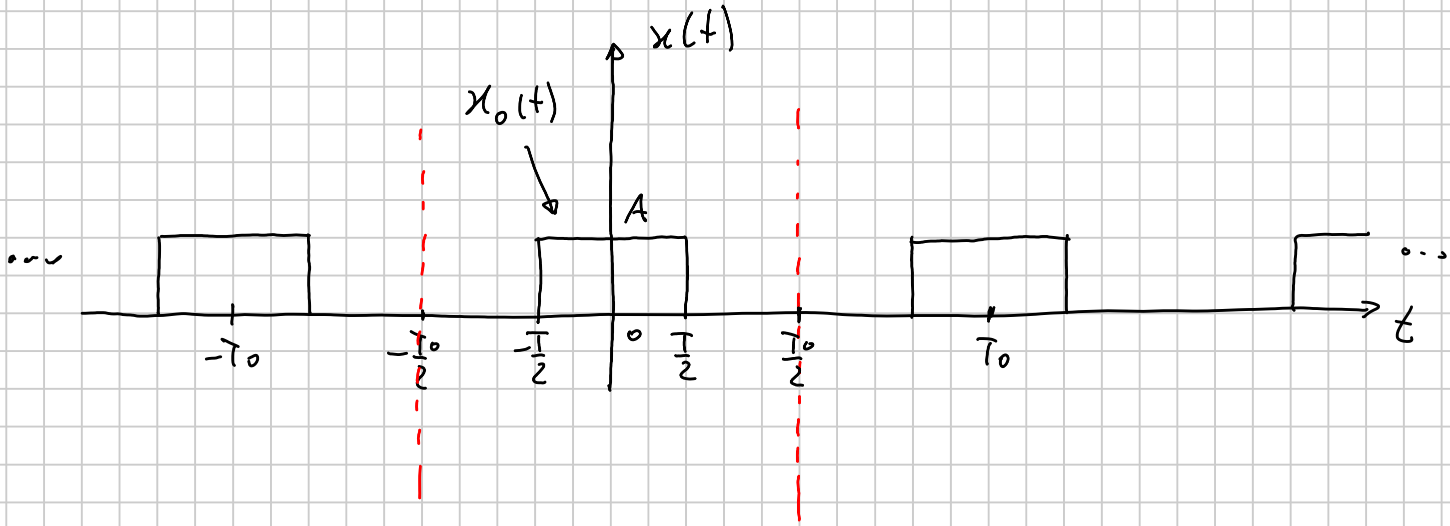
$$Y_n = TSF[y(t)] \quad , \quad y(t) = \frac{A}{2} \cos(2\pi f_0 t - \varphi)$$

$$X_n = \begin{cases} 0 & n \neq \pm 3 \\ A/4\, e^{j\varphi} & n = 3 \\ A/4\, e^{-j\varphi} & n = -3 \end{cases} \quad , \quad Y_n \begin{cases} 0 & n \neq \pm 1 \\ A/4\, e^{-j\varphi} & n = 1 \\ A/4\, e^{j\varphi} & n = -1 \end{cases}$$





3) TRENO DI IMPULSI RETTANGOLARI



$$\begin{cases} x_0(t) = A\, \mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} A & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{altrove} \end{cases} \\ \\ x(t) = \sum_{n=-\infty}^{+\infty} x_0(t - nT_0) \end{cases}$$

$x_0(t)$ aperiodico qualunque

GENERAZIONE DI SEGNALI PERIODICI A PARTIRE DA SEGNALI APERIODICI

$$x(t) = \sum_{n=-\infty}^{+\infty} x_0(t - nT_0)$$

$$x(t) \overset{?}{=} x(t - kT_0) \quad , \quad k \in \mathbb{Z}$$

$$x(t - kT_0) = \sum_{n=-\infty}^{+\infty} x_0(t - kT_0 - nT_0) = \sum_{n=-\infty}^{+\infty} x_0(t - (n+k)T_0)$$

$$n + k = n' \Rightarrow n = n' - k$$

$$x(t - kT_0) = \sum_{n'=-\infty}^{+\infty} x_0(t - n'T_0) = x(t)$$

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A \, \mathrm{rect}\left(\frac{t}{T}\right) e^{-j2\pi n f_0 t} \, dt = \qquad (T < T_0)$$

$$= \frac{A}{T_0} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi n f_0 t} \, dt = \frac{A}{T_0} \left(-\frac{1}{j2\pi n f_0}\right) e^{-j2\pi n f_0 t} \Big|_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$= -\frac{A}{j2\pi n} \left[e^{-j2\pi n f_0 \frac{T}{2}} - e^{j2\pi n f_0 \frac{T}{2}} \right] =$$

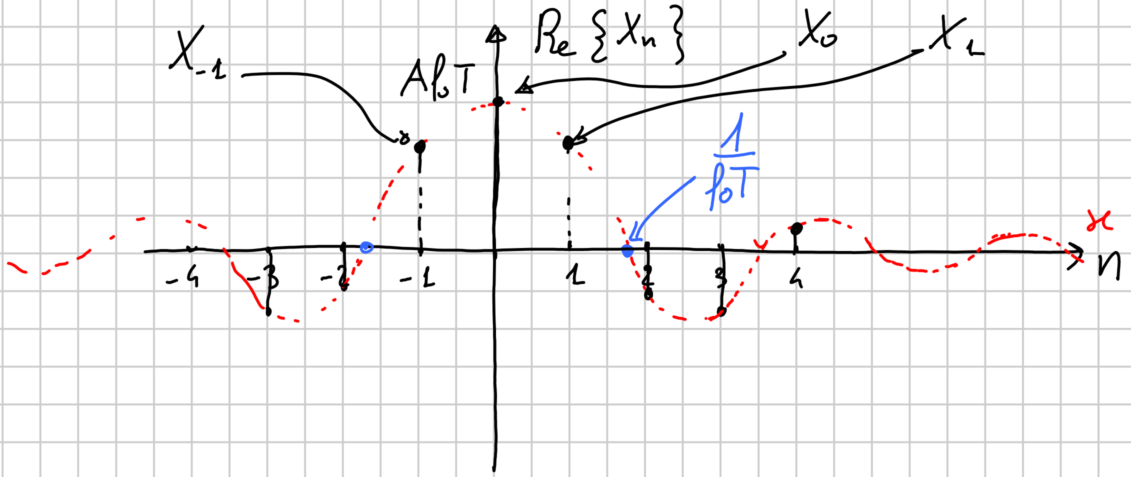
$$= \frac{\cancel{2j}A}{\cancel{j}2\pi n} \left[\frac{e^{j\pi n f_0 T} - e^{-j\pi n f_0 T}}{2j} \right] = \frac{A \cancel{f_0 T}}{\pi n \cancel{f_0 T}} \sin(\pi n f_0 T)$$

$$\Rightarrow \mathrm{sinc}(x) \triangleq \frac{\sin(\pi x)}{\pi x}$$

$$= A f_0 T \, \mathrm{sinc}(n f_0 T) \qquad (x = n f_0 T)$$

$$\mathrm{sinc}(-x) \overset{?}{=} \mathrm{sinc}(x)$$

$$\mathrm{sinc}(-x) = \frac{\sin(-\pi x)}{-\pi x} = \frac{\cancel{-}\sin(\pi x)}{\cancel{-}\pi x} = \frac{\sin(\pi x)}{\pi x} = \mathrm{sinc}(x)$$



$$X_0 = Af_0T\,\mathrm{sinc}(0)$$

$$X_1 = Af_0T\,\mathrm{sinc}(f_0T) \qquad X_{-1} = Af_0T\,\mathrm{sinc}(f_0T)$$

$$X_2 = Af_0T\,\mathrm{sinc}(2f_0T) \qquad \vdots$$

Primo zero della sinc

$$\mathrm{sinc}(x) = \mathrm{sinc}(nf_0T) = \mathrm{sinc}\left(\frac{1}{f_0T}\cdot f_0T\right) = \mathrm{sinc}(1) =$$

$$= \frac{\sin(\pi)}{\pi} = 0$$

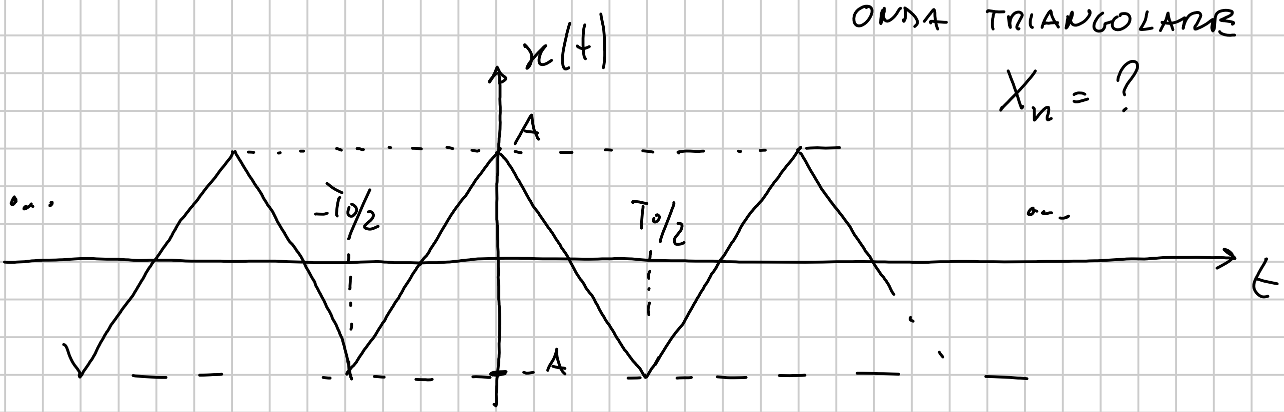
secondo zero della sinc

$$\mathrm{sinc}\left(\frac{2}{f_0T}f_0T\right) = \mathrm{sinc}(2) = \frac{\sin 2\pi}{2\pi} = 0$$

ESERCIZIO x CASA

ONDA TRIANGOLARE

$$X_n = ?$$



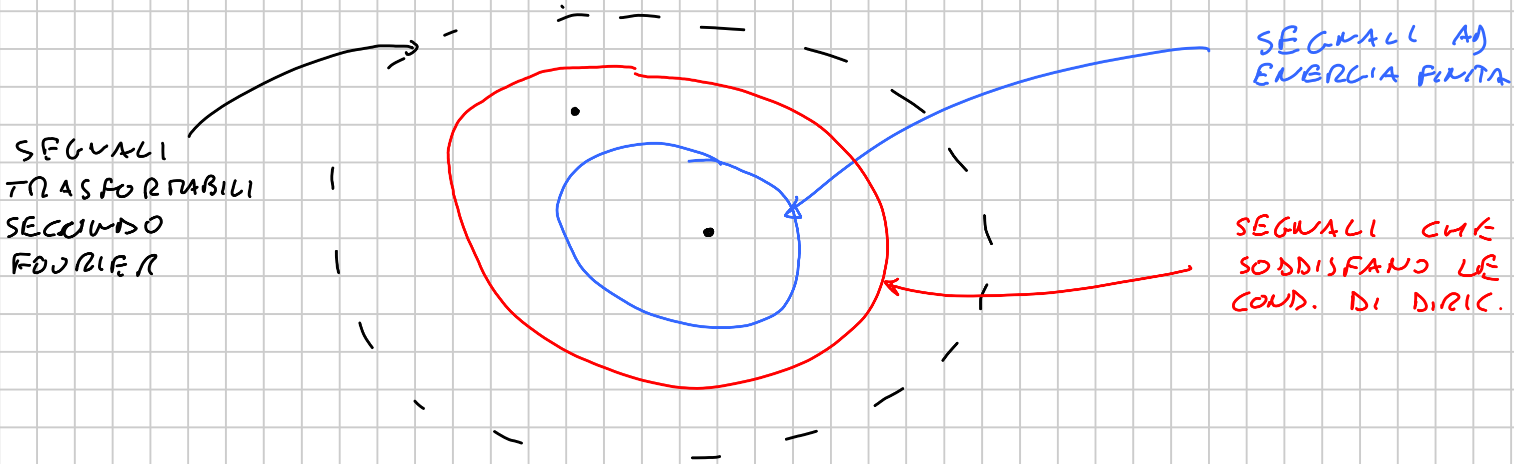
TRASFORMATA CONTINUA DI FOURIER

$x(t)$ segnale aperiodico che soddisfa le seguenti condizioni

o) $\displaystyle\int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty$

oppure

o) CONDIZIONI DI DIRICHELET



TRASFORMATA CONTINUA DI FOURIER (TCF)

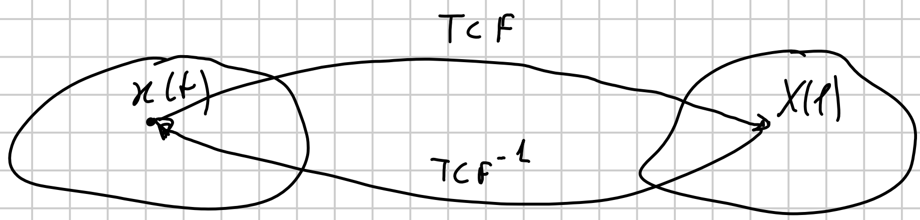
$$\boxed{X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt}$$

(TCF
EQ. DI ANALISI)

$$\boxed{x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df}$$

(TCF^{-1} (ATCF)
EQ. DI SINTESI)

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$



f [Hz] , $\omega = 2\pi f$

$\downarrow$

NOTAZIONE $$X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$X(f) = FT[x(t)]$, $X(f) = TCF[x(t)]$

$x(t) = FT^{-1}[X(f)]$, $x(t) = ATCF[X(f)]$

$x(t) = TCF^{-1}[X(f)]$

SPETTRO DI UN SEGNALE APERIODICO = TCF

SPETTRO DI AMPIEZZA $\Rightarrow$ $|X(f)|$

" " FASE $\Rightarrow$ $\angle X(f)$

SIMMETRIE

·) $x(t)$ è reale $\Rightarrow$ $X(f) = X^*(-f)$

DIM.

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(-f)t}\, dt = \int_{-\infty}^{+\infty} x(t)\, e^{j2\pi f t}\, dt$$

$$= \left[\int_{-\infty}^{+\infty} x^*(t)\, e^{-j2\pi f t}\, dt \right]^* = X^*(f)$$

$X(-f) = X^*(f) \Rightarrow X(f) = X^*(-f)$ SIMMETRIA HERMITIANA

·) $x(t)$ è reale e pari

$\Downarrow$

reale e pari

$X(f) = X(-f) = X^*(-f) \Rightarrow X(f) = X^*(f) = X(-f)$

·) $x(t)$ è reale e dispari

$$x(t) = x^*(t)$$
$$x(-t) = -x(t)$$

$$X(-f) = \int_{-\infty}^{+\infty} x(t)\, e^{+j2\pi f t}\, dt = \dots \qquad t' = -t$$

$$= \int_{-\infty}^{+\infty} x(-t')\, e^{-j2\pi f t'}\, dt' = -\underbrace{\int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt'}_{X(f)}$$

$$= -X(f)$$

$$X(-f) = -X(f) = X^*(f)$$

$X(f)$ immaginaria pura e dispari

TEOREMI SULLA TCF

·) TEOREMA DELLA LINEARITÀ

$$\begin{cases} x(t) = a_1 x_1(t) + a_2 x_2(t) \quad , \quad a_{1,2} \in \mathbb{C} \\ x_1(t) \overset{TCF}{\Longleftrightarrow} X_1(f) \\ x_2(t) \overset{TCF}{\Longleftrightarrow} X_2(f) \end{cases}$$

$$\Downarrow$$

$$X(f) = TCF\left[x(t)\right] = a_1 X_1(f) + a_2 X_2(f)$$

Dim.

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-j2\pi f t}\, dt$$

$$= a_1 \underbrace{\int_{-\infty}^{+\infty} x_1(t)\, e^{-j2\pi f t}\, dt}_{X_1(f)} + a_2 \underbrace{\int_{-\infty}^{+\infty} x_2(t)\, e^{-j2\pi f t}\, dt}_{X_2(f)}$$

c.v.d.

·) T. DELLA DUALITÀ

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$\Downarrow$$

$$X(t) \overset{TCF}{\Longleftrightarrow} x(-f)$$

Dim

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$X(t) = \int_{-\infty}^{+\infty} x(f)\, e^{-j2\pi f t}\, df \qquad (f' = -f)$$

$$X(t) = \int_{-\infty}^{+\infty} x(-f')\, e^{j2\pi f' t}\, df'$$

$$X(t) = TCF^{-1}\left[x(-f') \right]$$

$$TCF\left[X(t) \right] = x(-f) \qquad \text{c.v.d.}$$

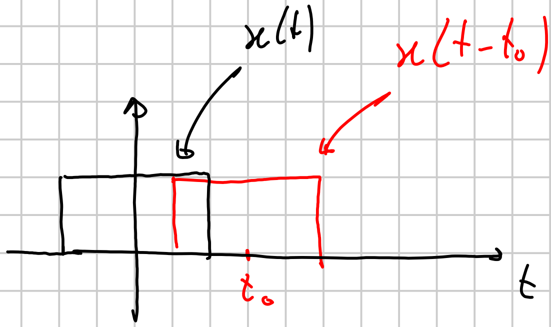
·) T. DEL RITARDO

se $x(t) \overset{TCF}{\Longleftrightarrow} X(f)$

$$\Downarrow$$

allora $x(t-t_0) \overset{TCF}{\Longleftrightarrow} X(f)\, e^{-j2\pi f t_0}$

$t_0 \in \mathbb{R}$



Dim.

$$y(t) = x(t-t_0)$$

$$Y(f) = \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j2\pi f t}\, dt$$

$$t - t_0 = t'$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f (t'+t_0)}\, dt' =$$

$$= \underbrace{\int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt'}_{X(f)}\, e^{-j2\pi f t_0} = X(f)\, e^{-j2\pi f t_0}$$

c.v.d.

·) T. DEL CAMBIAMENTO DI SCALA

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$\Downarrow$$

$$x(at) \overset{TCF}{\Longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$|a|>1$, $|a|<1$

$(a \neq 0)$

Dim.

$a > 0$

$$y(t) = x(at)$$

$$Y(f) = \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(at)\, e^{-j2\pi f t}\, dt \qquad (at = t')$$

$dt' = a\,dt$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f \frac{t'}{a}}\, \frac{dt'}{a} = \frac{1}{a} \underbrace{\int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi \frac{f}{a} t'}\, dt'}_{X\left(\frac{f}{a}\right)}$$

$$= \boxed{\frac{1}{|a|} X\left(\frac{f}{a}\right)}$$

$a < 0$

$$Y(f) = \int_{-\infty}^{+\infty} x(at)\, e^{-j2\pi f t}\, dt = \ldots \quad (t' = at) \Rightarrow dt' = a\,dt$$

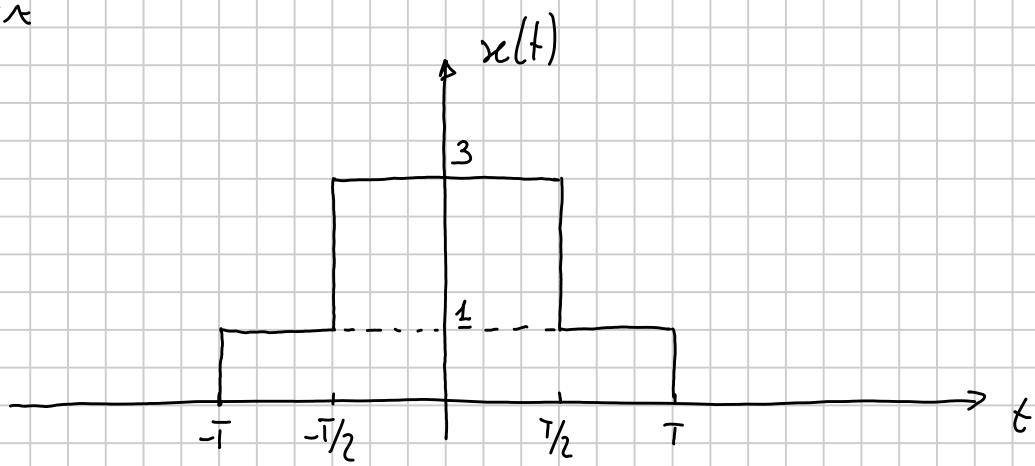
$a = -|a|$

$$= \frac{1}{|a|} \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi \frac{f}{a} t'}\, dt' = \boxed{\frac{1}{|a|} X\left(\frac{f}{a}\right)}$$

c.v.d.

ESEMPI

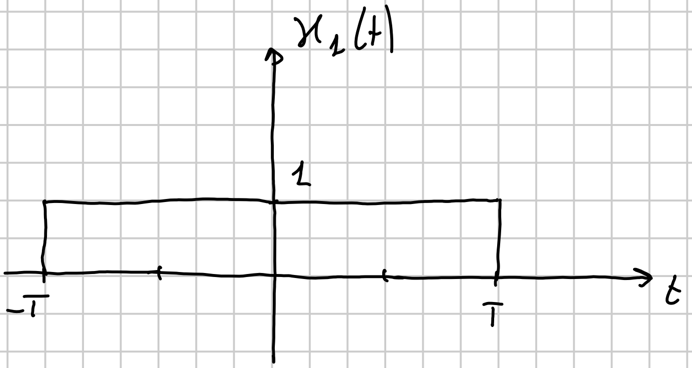
·) LINEARITÀ



$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-T}^{-T/2} e^{-j2\pi f t}\, dt + 3\int_{-T/2}^{T/2} e^{-j2\pi f t}\, dt + \int_{T/2}^{T} e^{-j2\pi f t}\, dt$$

$$x(t) = x_1(t) + x_2(t)$$



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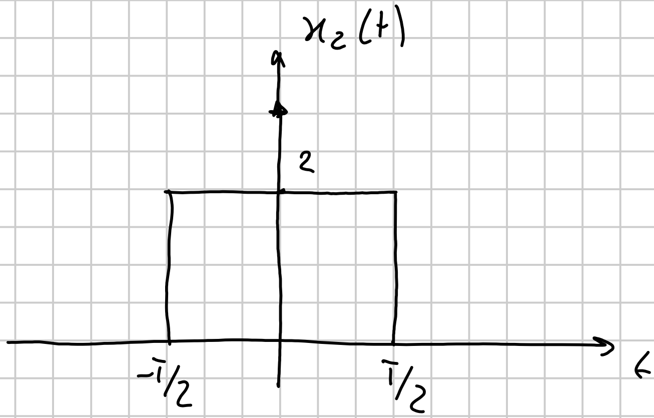


$$X(f) = X_1(f) + X_2(f)$$

$$x_1(t) = \text{rect}\left(\frac{t}{2T}\right) \quad , \quad x_2(t) = 2\,\text{rect}\left(\frac{t}{T}\right)$$

$$x_0(t) = \text{rect}\left(\frac{t}{T}\right)$$

$$x_2(t) = 2\, x_0(t) \Rightarrow X_2(f) = 2\, X_0(f)$$

$$x_1(t) = x_0(t') \quad , \quad t' = \frac{t}{2} \quad , \quad a = \frac{1}{2}$$

$$= \mathrm{rect}\left(\frac{t'}{T}\right) = \mathrm{rect}\left(\frac{t}{2T}\right)$$

$$x_1(t) = x_0(a t) \quad , \quad a = \frac{1}{2}$$

$$X_1(f) = 2\, X_0(2f)$$

$$X(f) = 2\, X_0(2f) + 2\, X_0(f)$$

TCF DI UN RETTANGOLO

$$X(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi f t}\, dt = -\frac{1}{j2\pi f}\, e^{-j2\pi f t}\Big|_{-\frac{T}{2}}^{\frac{T}{2}} =$$

$$= -\frac{1}{j2\pi f}\left[e^{-j\cancel{2}\pi f \frac{T}{\cancel{2}}} - e^{j\cancel{2}\pi f \frac{T}{\cancel{2}}} \right] =$$

$$= \frac{1}{\pi f}\, \frac{e^{j\pi f T} - e^{-j\pi f T}}{2j} = T\, \frac{\sin(\pi f T)}{\pi f T} = T\, \mathrm{sinc}(fT)$$

$$\boxed{\mathrm{rect}\left(\frac{t}{T}\right) \overset{TCF}{\Longleftrightarrow} T\, \mathrm{sinc}(fT)}$$

$$X_0(f) = T\, \mathrm{sinc}(fT)$$

$$X(f) = 2T\, \mathrm{sinc}(2fT) + 2T\, \mathrm{sinc}(fT)$$

- DUALITÀ

$$x(t) = A \operatorname{sinc}(Bt)$$

$$X(f) = \int_{-\infty}^{+\infty} A \operatorname{sinc}(Bt)\, e^{-j2\pi f t}\, dt$$

$$\operatorname{rect}\left(\frac{t}{T}\right) \overset{TCF}{\Longleftrightarrow} T \operatorname{sinc}(fT)$$

$$AT \operatorname{sinc}(tT) \overset{TCF}{\Longleftrightarrow} A \operatorname{rect}\left(\frac{-f}{T}\right) = A \operatorname{rect}\left(\frac{f}{T}\right)$$

$$\Downarrow$$

$$\frac{A \cancel{B}}{\cancel{B}} \operatorname{sinc}(Bt) \overset{TCF}{\Longleftrightarrow} \frac{A}{B} \operatorname{rect}\left(\frac{f}{B}\right)$$

- RITARDO

$$x(t) = \operatorname{rect}\left(\frac{t - t_0}{T}\right)$$

$$x_0(t) = \operatorname{rect}\left(\frac{t}{T}\right) \Rightarrow x(t) = x_0(t - t_0)$$

$$x_0(t) \overset{TCF}{\Longleftrightarrow} X_0(f)$$

$$\Downarrow$$

$$x(t) \overset{TCF}{\Longleftrightarrow} X_0(f)\, e^{-j2\pi f t_0}$$

$$X_0(f) = T \operatorname{sinc}(fT) \Rightarrow X(f) = T \operatorname{sinc}(fT)\, e^{-j2\pi f t_0}$$

-) TEOREMA DELLA MODULAZIONE

segnale modulato

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

segnale modulante

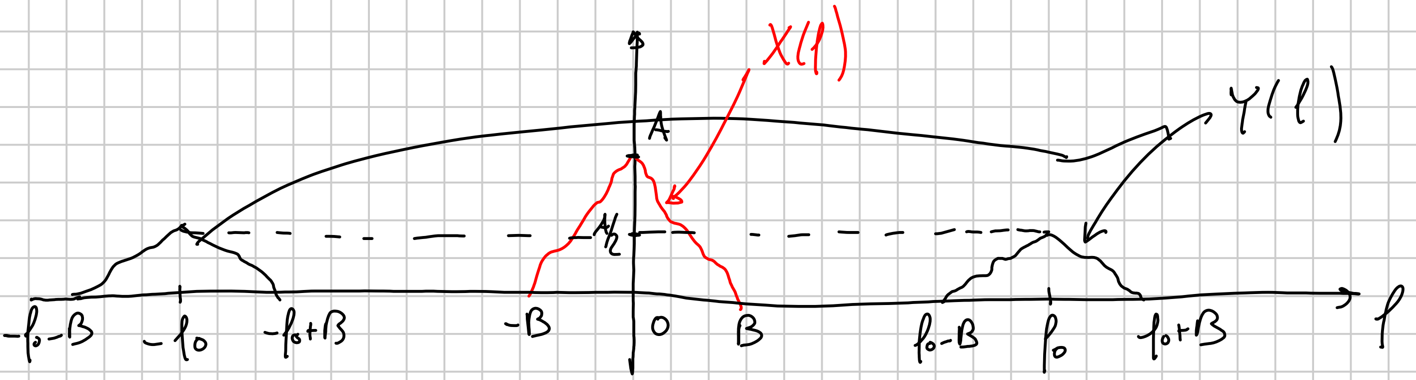
$$y(t) = x(t)\cos(2\pi f_0 t)$$

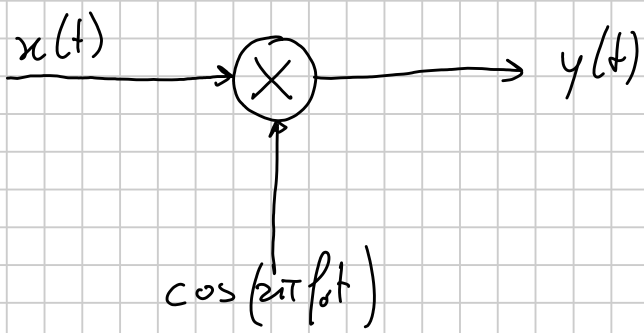
$$Y(f) = \int_{-\infty}^{+\infty} x(t)\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right) e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f-f_0)t}\, dt}_{X(f-f_0)} + \frac{1}{2}\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f+f_0)t}\, dt}_{X(f+f_0)}$$

$$Y(f) = \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$





·) MODULAZIONE CON IL SENO

$$y(t) = x(t)\sin(2\pi f_0 t)$$

$$Y(f) = \frac{1}{2j} X(f-f_0) - \frac{1}{2j} X(f+f_0)$$

·) MODULAZIONE CON COSENO A FASE GENERICA

$$y(t) = x(t)\cos(2\pi f_0 t + \varphi)$$

$$Y(f) = \frac{e^{j\varphi}}{2} X(f-f_0) + \frac{e^{-j\varphi}}{2} X(f+f_0)$$

Dim.

$$Y(f) = \int_{-\infty}^{+\infty} x(t)\cos(2\pi f_0 t + \varphi)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{e^{j(2\pi f_0 t+\varphi)} + e^{-j(2\pi f_0 t+\varphi)}}{2}\right) e^{-j2\pi f t}\, dt$$

$$= \frac{e^{j\varphi}}{2}\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f-f_0)t}\, dt}_{X(f-f_0)} + \frac{e^{-j\varphi}}{2}\underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi(f+f_0)t}\, dt}_{X(f+f_0)}$$

$$= \frac{e^{j\varphi}}{2} X(f-f_0) + \frac{e^{-j\varphi}}{2} X(f+f_0)$$

- MODULAZIONE CON ESP. COMPLESSO

$$y(t) = x(t)\, e^{j2\pi f_0 t}$$

$$Y(f) = X(f - f_0)$$

Dim.

$$Y(f) = \int_{-\infty}^{+\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f-f_0)t}\, dt = X(f-f_0)$$

$$\boxed{\begin{array}{lcl} x(t)\, e^{j2\pi f_0 t} & \Longleftrightarrow & X(f-f_0) \\ x(t-t_0) & \Longleftrightarrow & X(f)\, e^{-j2\pi f t_0} \end{array}}$$

DUALITÀ

- T. DELLA DERIVAZIONE

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$y(t) = \frac{d}{dt} x(t) \overset{TCF}{\Longleftrightarrow} j2\pi f\, X(f)$$

Dim

$$y(t) = \frac{d}{dt} x(t) = \frac{d}{dt}\Big[\underbrace{\int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df}_{x(t)} \Big]$$

$$= \int_{-\infty}^{+\infty} X(f) \Big[\frac{d}{dt} e^{j2\pi f t}\Big] df = \int_{-\infty}^{+\infty} \overbrace{j2\pi f\, X(f)}^{Y(f)}\, e^{j2\pi f t}\, df$$

$$\Rightarrow \quad y(t) = TCF^{-1}\left[Y(f)\right] \Rightarrow Y(f) = TCF\left[y(t)\right]$$

$$\Downarrow$$

$$j2\pi f\, X(f) \Leftrightarrow \frac{d}{dt} x(t) \qquad \text{c.v.d.}$$

·) TEOREMA DELL' INTEGRAZIONE

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha$$

j) $x(t) \overset{TCF}{\Leftrightarrow} X(f)$

jj) $\displaystyle\int_{-\infty}^{+\infty} x(t)\, dt = 0 \Rightarrow X(f)\Big|_{f=0} = 0$

$$X(f)\Big|_{f=0} = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt\,\Big|_{f=0} = \int_{-\infty}^{+\infty} x(t)\, dt$$

$$\Rightarrow \quad Y(f) = \frac{X(f)}{j2\pi f}$$

Dim.

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha \Rightarrow x(t) = \frac{d}{dt} y(t)$$

$$\Downarrow$$

$$X(f) = j2\pi f\, Y(f)$$

$$\Downarrow$$

$$Y(f) = \frac{X(f)}{j2\pi f}$$

·) T. DELLA DERIVAZIONE NEL DOMINIO DELLA FREQ.

$$Y(f) = \frac{d}{df} X(f) \quad , \quad x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$\Downarrow$$

$$y(t) = -j2\pi t \, x(t)$$

Dim

$$Y(f) = \frac{d}{df} X(f) = \frac{d}{df}\left[\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt\right]$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{d}{df} e^{-j2\pi f t}\right] dt = \int_{-\infty}^{+\infty} \underbrace{-j2\pi f\, x(t)}_{y(t)}\, e^{-j2\pi f t}\, dt$$

c.v.d.

·) T. DELL' INTEGRAZIONE NEL DOMINIO DELLA FREQ.

$$Y(f) = \int_{-\infty}^{f} X(\alpha)\, d\alpha$$

j) $x(t) \overset{TCF}{\Longleftrightarrow} X(f)$

jj) $\int_{-\infty}^{+\infty} X(\alpha)\, d\alpha = 0 \Rightarrow x(t)\Big|_{t=0} = 0$

$$x(t)\Big|_{t=0} = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df\Big|_{t=0} = \int_{-\infty}^{+\infty} X(f)\, df$$

$$y(t) = -\frac{x(t)}{j2\pi t}$$

CONVOLUZIONE

13/03/2013

$x(t) \quad , y(t)$

$\Downarrow$

$$z(t) = x(t) \otimes y(t) \triangleq \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau$$

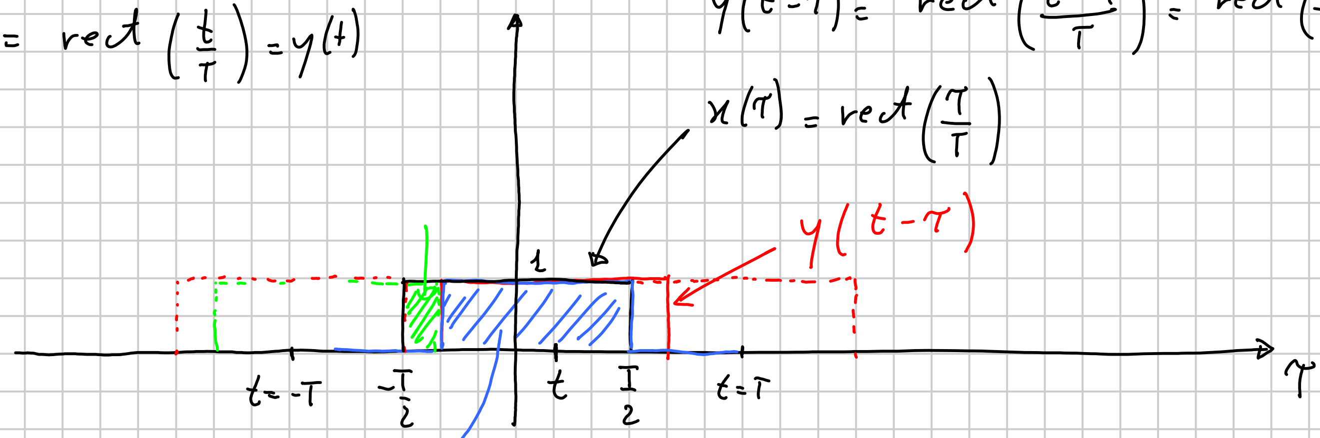
operatore di convoluzione

ESEMPIO

$x(t) = \text{rect}\left(\frac{t}{T}\right) = y(t)$

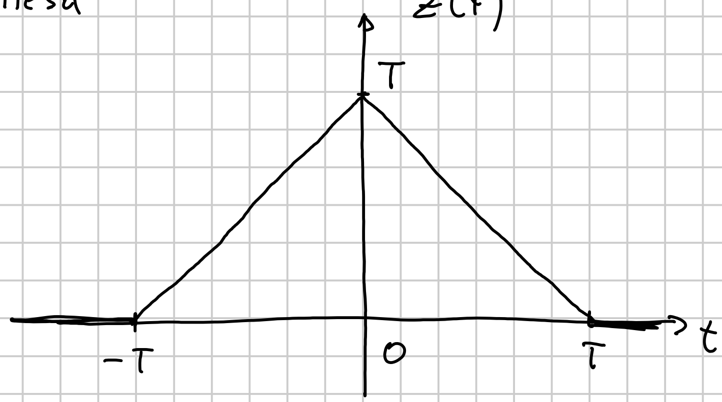
$y(t-\tau) = \text{rect}\left(\frac{t-\tau}{T}\right) = \text{rect}\left(\frac{\tau-t}{T}\right)$

$x(\tau) = \text{rect}\left(\frac{\tau}{T}\right)$



$z(t)$ = area sottesa

$$z(t) = \begin{cases} 0 & t \leq -T \\ 0 & t \geq T \\ T+t & -T \leq t \leq 0 \\ T-t & 0 \leq t \leq T \end{cases}$$



TEOREMA DELLA CONVOLUZIONE

$$z(t) = x(t) \otimes y(t)$$

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f) \quad , \quad y(t) \overset{TCF}{\Longleftrightarrow} Y(f)$$

$$\Downarrow$$

$$Z(f) = X(f)\, Y(f)$$

$$z(t) = x(t) \otimes y(t)$$

$$\begin{array}{ccccc} z(t) & = & x(t) & \otimes & y(t) \\ \Uparrow \scriptstyle TCF^{-1} & & \Downarrow \scriptstyle TCF & & \Downarrow \scriptstyle TCF \\ Z(f) & = & X(f) & \cdot & Y(f) \end{array}$$

Dim.

$$Z(f) = \int_{-\infty}^{+\infty} z(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty\,(t)}^{+\infty} \underbrace{\int_{-\infty\,(\tau)}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau}_{z(t)}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty\,(\tau)}^{+\infty} x(\tau) \underbrace{\int_{-\infty\,(t)}^{+\infty} y(t-\tau)\, e^{-j2\pi f t}\, dt}\, d\tau$$

$$= \int_{-\infty\,(\tau)}^{+\infty} x(\tau)\, Y(f)\, e^{-j2\pi f \tau}\, d\tau$$

$$= \underbrace{\int_{-\infty}^{+\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau}_{X(f)}\, Y(f) = X(f)\, Y(f) \qquad \text{c.v.d.}$$

ALTRE PROPRIETÀ DELLA CONVOLUZIONE

·) COMMUTATIVA

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

Dim.

$$\int_{-\infty}^{+\infty} y(\tau)\, x(t-\tau)\, d\tau = \ldots \quad (t-\tau) = \alpha \;,\; d\alpha = -d\tau$$

$$= \int_{-\infty}^{+\infty} y(t-\alpha)\, x(\alpha)\, d\alpha = \int_{-\infty}^{+\infty} x(\alpha)\, y(t-\alpha)\, d\alpha = x(t) \otimes y(t)$$

c.v.d.

·) DISTRIBUTIVA

$$x(t) \otimes \left[y(t) + z(t) \right] = x(t) \otimes y(t) + x(t) \otimes z(t)$$

Dim.

$$\int_{-\infty}^{+\infty} x(\tau) \left[y(t-\tau) + z(t-\tau) \right] d\tau =$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau + \int_{-\infty}^{+\infty} x(\tau)\, z(t-\tau)\, d\tau = x(t) \otimes y(t) + x(t) \otimes z(t)$$

c.v.d.

·) ASSOCIATIVA

$$x(t) \otimes \left[y(t) \otimes z(t) \right] = \left[x(t) \otimes y(t) \right] \otimes z(t)$$

$$\Downarrow$$

$$x(t) \otimes y(t) \otimes z(t)$$

Dim.

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f) \;,\; y(t) \overset{TCF}{\Longleftrightarrow} Y(f) \;,\; z(t) \overset{TCF}{\Longleftrightarrow} Z(f)$$

$$X(f) \cdot [Y(f) \cdot Z(f)] = [X(f) \cdot Y(f)] \cdot Z(f)$$

$$\Downarrow$$

$$x(t) \otimes [y(t) \otimes z(t)] = [x(t) \otimes y(t)] \otimes z(t)$$

c.v.d.

·) TEOREMA DEL PRODOTTO

$$z(t) = x(t)\, y(t)$$

$$x(t) \overset{TCF}{\Longleftrightarrow} X(f) \quad , \quad y(t) \overset{TCF}{\Longleftrightarrow} Y(f)$$

$$\Downarrow$$

$$Z(f) = X(f) \otimes Y(f)$$

Dim.

$$Z(f) = \int_{-\infty}^{+\infty} z(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(t)\, y(t)\, e^{-j2\pi f t}\, dt$$

$$\Rightarrow y(t) = \int_{-\infty}^{+\infty} Y(\alpha)\, e^{j2\pi \alpha t}\, d\alpha$$

$$Z(f) = \int_{\underset{(t)}{-\infty}}^{+\infty} x(t) \int_{\underset{(\alpha)}{-\infty}}^{+\infty} Y(\alpha)\, e^{j2\pi\alpha t}\, d\alpha\, e^{-j2\pi f t}\, dt$$

$$= \int_{\underset{(\alpha)}{-\infty}}^{+\infty} Y(\alpha) \underbrace{\int_{\underset{(t)}{-\infty}}^{+\infty} x(t)\, e^{-j2\pi \overbrace{(f-\alpha)}^{f'} t}\, dt}_{X(f') = X(f-\alpha)}\, d\alpha$$

$$Z(f) = \int_{-\infty}^{+\infty} Y(\alpha)\, X(f-\alpha)\, d\alpha = Y(f) \otimes X(f) = X(f) \otimes Y(f)$$

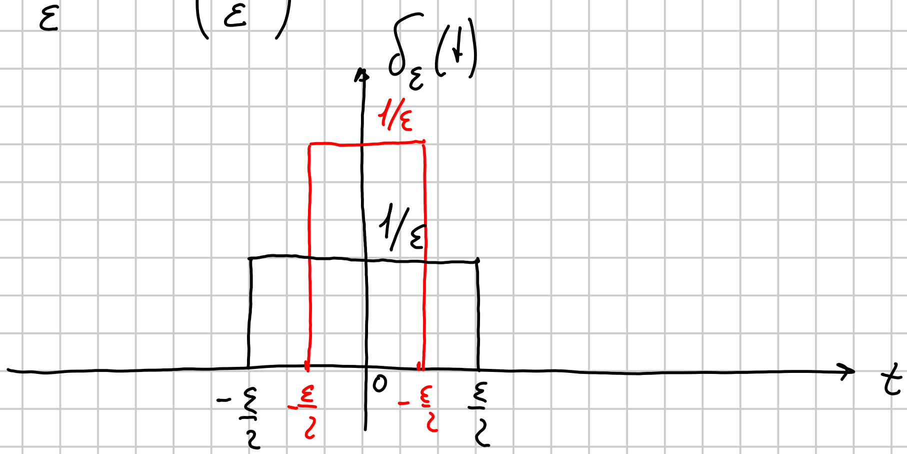
c.v.d.

14/03/2013

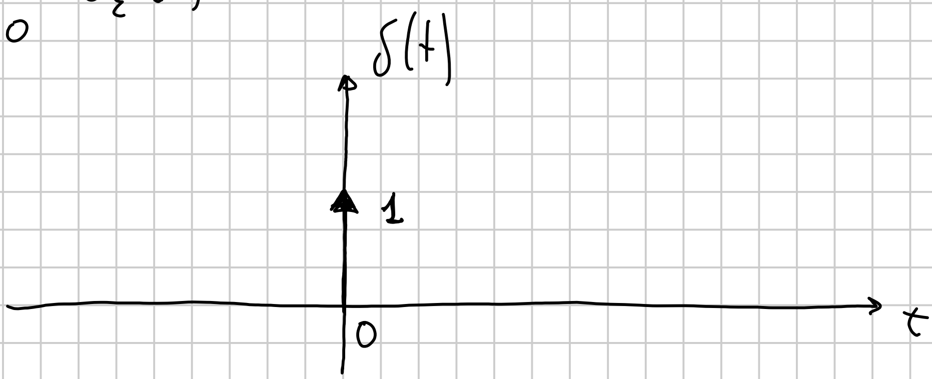
IMPULSO (DELTA) DI DIRAC

$$\delta(t)$$

$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}\,\mathrm{rect}\left(\frac{t}{\varepsilon}\right)$$



$$\delta(t) \triangleq \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$



·) PROPRIETÀ

1) $$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$ AREA UNITARIA

Dim.

$$\int_{-\infty}^{+\infty} \lim_{\varepsilon\to 0} \delta_\varepsilon(t)\,dt = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\cdot\varepsilon = 1$$

2) $$\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = x(0)$$ PROP. CAMPIONATRICE

Dim

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = \int_{-\infty}^{+\infty} x(t) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt =$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x(t)\,dt \underset{\text{TEOREMA DELLA MEDIA}}{=} \lim_{\varepsilon \to 0} x(\tilde{t}) = x(0) \qquad -\frac{\varepsilon}{2} \le \tilde{t} \le \frac{\varepsilon}{2}$$

c.v.d.

3) PARITÀ

$$\int_{-\infty}^{+\infty} \delta(-t)\,dt = \int_{-\infty}^{+\infty} \delta(t)\,dt$$

$$\delta(-t) = \delta(t)$$

Dim

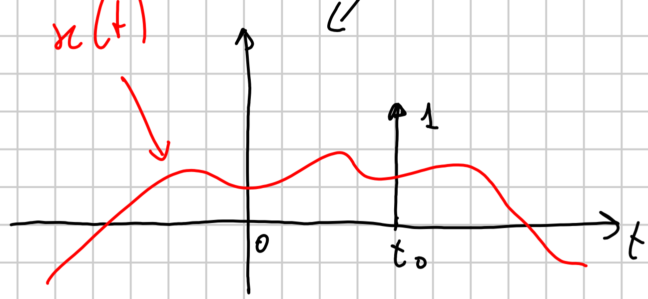
$$\int_{-\infty}^{+\infty} x(t)\,\delta(-t)\,dt = (-t = t')$$

$$= \int_{-\infty}^{+\infty} x(-t')\,\delta(t')\,dt' = x(0)$$

4) TRASLAZIONE

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = (t - t_0 = t')$$

$$= \int_{-\infty}^{+\infty} x(t'+t_0)\,\delta(t')\,dt' = x(t_0)$$



4 bis) $$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{+\infty} x(t_0)\,\delta(t-t_0)\,dt$$

5) CONVOLUZIONE

$$x(t) \otimes \delta(t) = x(t)$$

Dim

$$\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} x(\tau)\,\delta(\tau-t)\,d\tau = x(t)$$

c.v.d.

6) $$\int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt = \frac{x(0)}{|a|} \qquad a \neq 0$$

Dim

$$\int_{-\infty}^{+\infty} x(t)\,\delta(at)\,dt = \qquad (at = t')$$

$a>0$

$$= \int_{-\infty}^{+\infty} x\left(\frac{t'}{a}\right)\delta(t')\,\frac{dt'}{a} = \frac{x(0)}{|a|}$$

$a<0$

$$= \int_{+\infty}^{-\infty} x\left(\frac{t'}{a}\right)\delta(t')\,\frac{-dt'}{|a|} = \int_{-\infty}^{+\infty} x\left(\frac{t'}{a}\right)\delta(t')\,\frac{dt'}{|a|} = \frac{x(0)}{|a|}$$

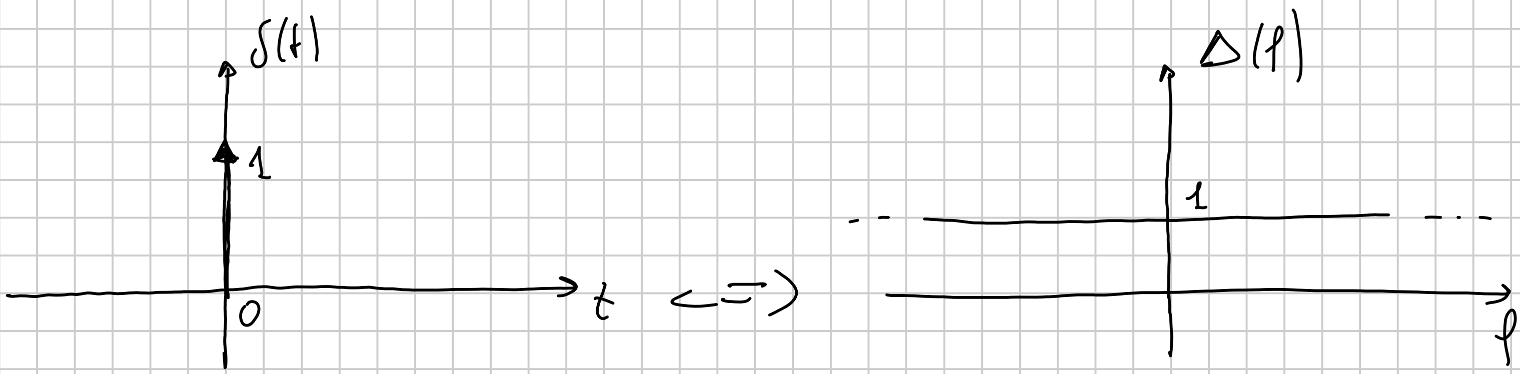
c.v.d.

TRASFORMATA CONTINUA DI FOURIER GENERALIZZATA

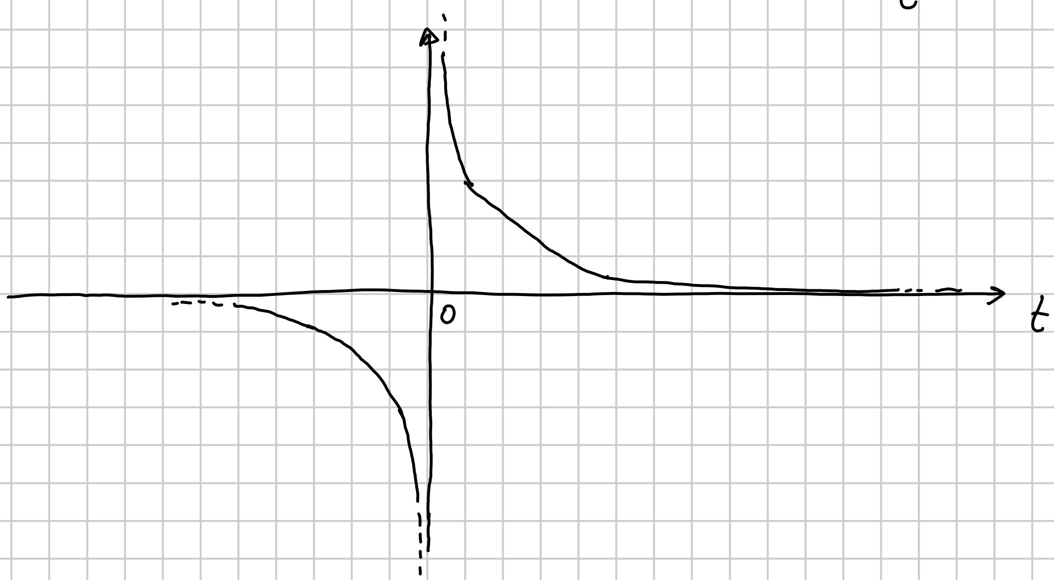
TRASFORMATA DELLA $\delta(t)$

$$TCF[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t)\, \underbrace{e^{-j2\pi f t}}_{x(t)}\, dt = 1$$

$$\delta(t) \overset{TCF}{\Longleftrightarrow} 1$$

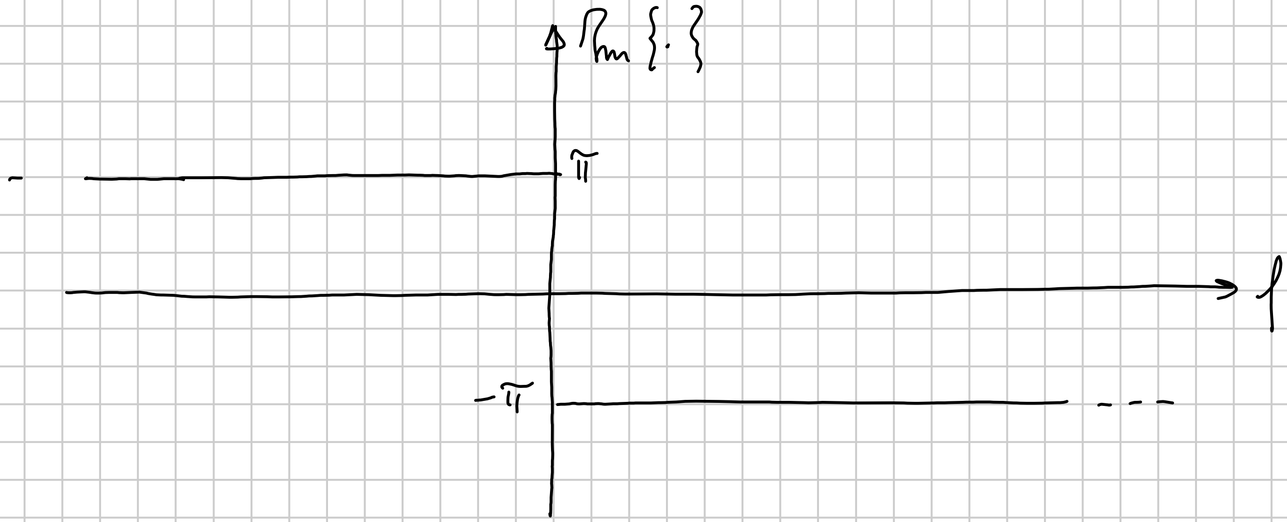


-) TRASFORMATA DELLA FUNZIONE $\frac{1}{t}$



$$TCF\left[\frac{1}{t}\right] = ?$$

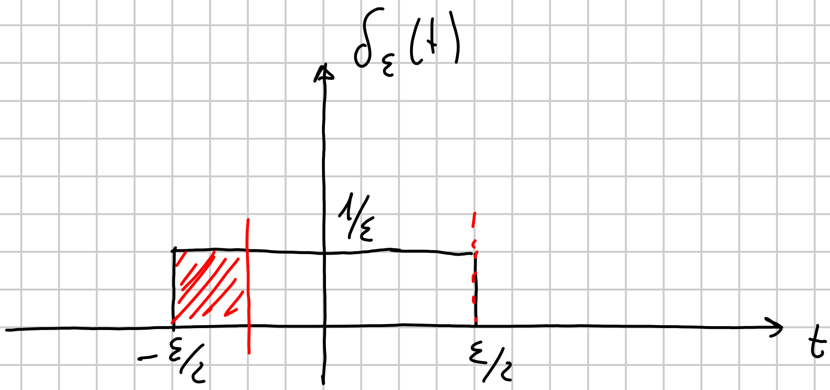
$$\boxed{\frac{1}{t} \overset{TCF}{\Longleftrightarrow} -j\pi\, \mathrm{sgn}(f)}$$



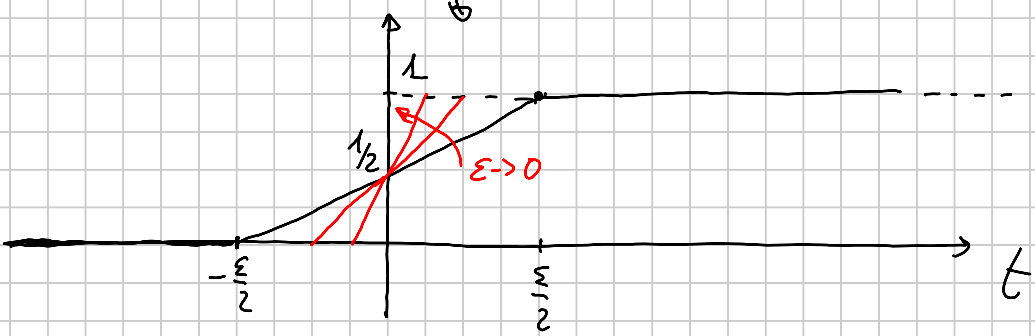
·) RELAZIONE TRA $\delta(t)$ e $u(t)$

$$\boxed{u(t) = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha}$$

Dim



$$u_\varepsilon(t) = \int_{-\infty}^{t} \delta_\varepsilon(\alpha)\, d\alpha$$



$$u(t) \overset{?}{=} \lim_{\varepsilon \to 0} u_\varepsilon(t) = \lim_{\varepsilon \to 0} \int_{-\infty}^{t} \delta_\varepsilon(\alpha)\, d\alpha =$$

$$= \int_{-\infty}^{t} \lim_{\varepsilon \to 0} \delta_\varepsilon(\alpha)\, d\alpha = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha$$

c.v.d.

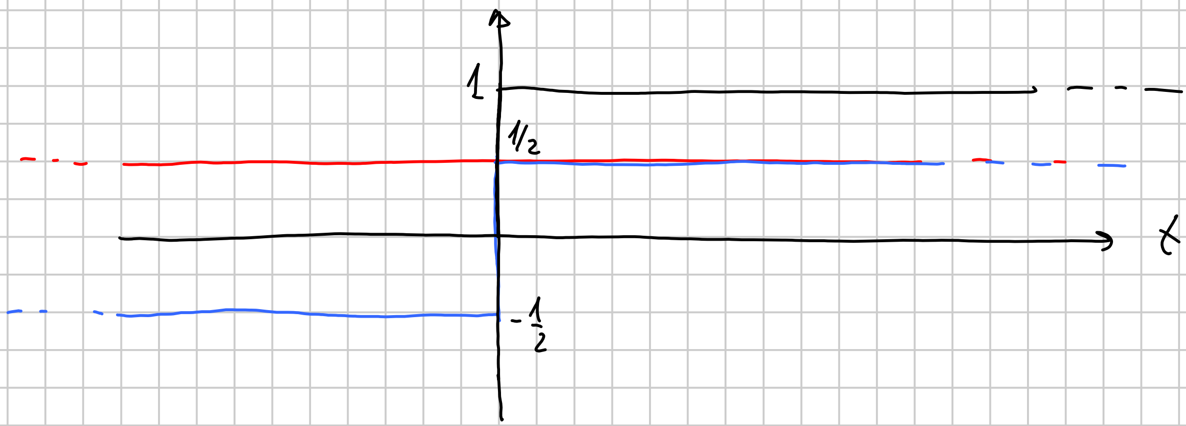
$$\boxed{\delta(t) = \frac{d}{dt} u(t)}$$

·) TRASFORMATA DI $u(t)$

$$U(f) = \int_{-\infty}^{+\infty} u(t)\, e^{-j2\pi f t}\, dt = \int_{0}^{+\infty} e^{-j2\pi f t}\, dt$$

uso una strada alternativa

$$u(t) = \frac{1}{2} + \frac{1}{2}\, \text{sgn}(t)$$



$$TCF\left[\frac{1}{2} + \frac{1}{2}\,\text{sgn}(t)\right] = TCF\left[\frac{1}{2}\right] + TCF\left[\frac{1}{2}\,\text{sgn}(t)\right]$$

j) $$TCF\left[\frac{1}{2}\right] = \int_{-\infty}^{+\infty} \frac{1}{2}\, e^{-j2\pi f t}\, dt = \frac{1}{2}\int_{-\infty}^{+\infty} 1 \cdot e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2}\,\delta(f)$$

$\delta(t) \overset{TCF}{\Longleftrightarrow} 1$

per la dualità

$1 \overset{TCF}{\Longleftrightarrow} \delta(-f) = \delta(f)$

jj) $\frac{1}{t} \overset{TCF}{\Longleftrightarrow} -j\pi\, \text{sgn}(f)$ per la dualità $\Longrightarrow$ $-j\pi\, \text{sgn}(t) \overset{TCF}{\Longleftrightarrow} -\frac{1}{f}$

$$TCF\left[\frac{1}{2}\,sgn(t)\right] = \frac{1}{2}\,TCF\left[sgn(t)\right] = \frac{1}{j2\pi f}$$

$$sgn(t) \overset{TCF}{\Longleftrightarrow} \frac{1}{j\pi f}$$

$$\boxed{U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}}$$

·) TEOREMA DELL'INTEGRAZIONE COMPLETO

TEO. DELL'INTEGRAZIONE DICEVA

$$y(t) = \int_{-\infty}^{t} x(\alpha)\,d\alpha$$

j) $x(t) \overset{TCF}{\Longleftrightarrow} X(f)$

jj) $\cancel{\int_{-\infty}^{+\infty} x(t)\,dt = 0}$ ⟸ RIMUOVIAMO QUESTA COND.

TEOR. DELL' INT. COMPLETO

$$y(t) = \int_{-\infty}^{t} x(\alpha)\,d\alpha \;, \qquad x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$\Downarrow$$

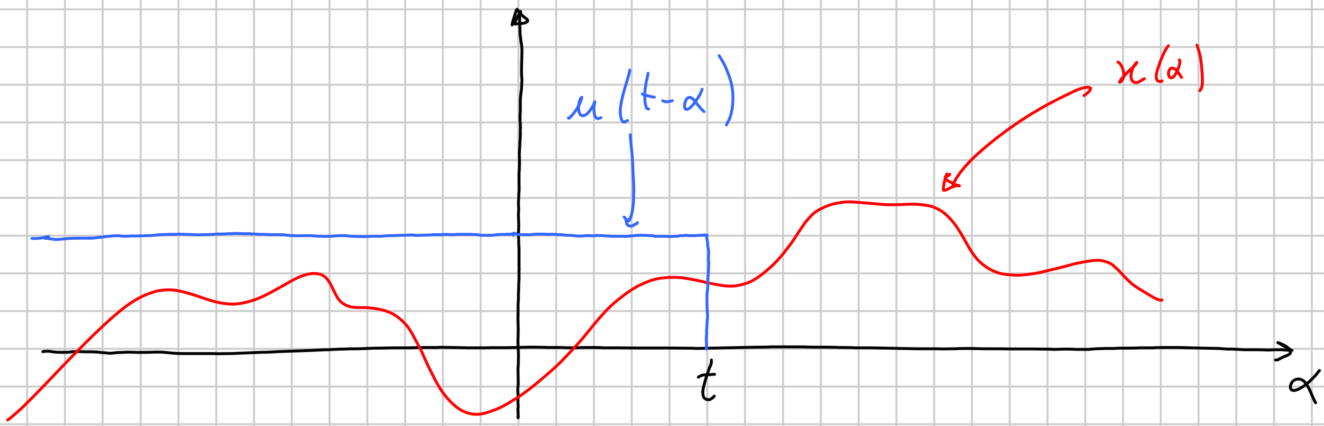
$$Y(f) = \frac{X(0)}{2}\delta(f) + \frac{X(f)}{j2\pi f}$$

TERMINE AGGIUNTIVO

→ TERMINE CHE ERA PRESENTE ANCHE NEL PREC. TEOREMA

Dim.

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha = x(t) \otimes u(t) = \int_{-\infty}^{+\infty \to t} x(\alpha)\, u(t-\alpha)\, d\alpha$$



$$Y(f) = TCF\left[x(t) \otimes u(t)\right] = X(f)\, U(f)$$

$$= X(f) \cdot \left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right] = \frac{X(f)\,\delta(f)}{2} + \frac{X(f)}{j2\pi f}$$

$$= \frac{X(0)}{2}\delta(f) + \frac{X(f)}{j2\pi f} \qquad \text{c.v.d.}$$

TCF DI UN COSENO

$$x(t) = A\cos(2\pi f_0 t)$$

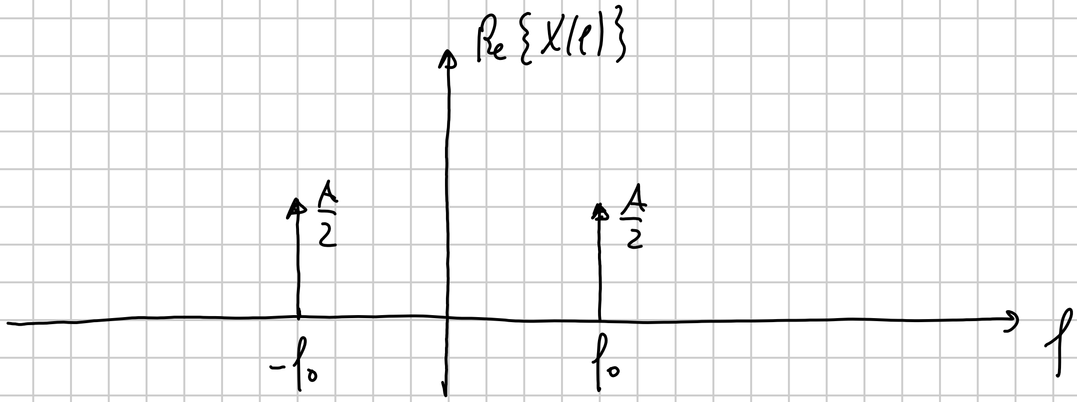
$$X(f) = \int_{-\infty}^{+\infty} A\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt =$$

$$= A\int_{-\infty}^{+\infty} \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\, e^{-j2\pi f t}\, dt =$$

$$= \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t}\, dt + \frac{A}{2}\int_{-\infty}^{+\infty} e^{-j2\pi(f+f_0)t}\, dt$$

$$\int_{-\infty}^{+\infty} 1 \cdot e^{-j2\pi \underbrace{(f-f_0)}_{f'} t} dt = \delta(f') = \delta(f-f_0)$$

$$X(f) = \frac{A}{2}\delta(f-f_0) + \frac{A}{2}\delta(f+f_0)$$



TCF DEL SENO

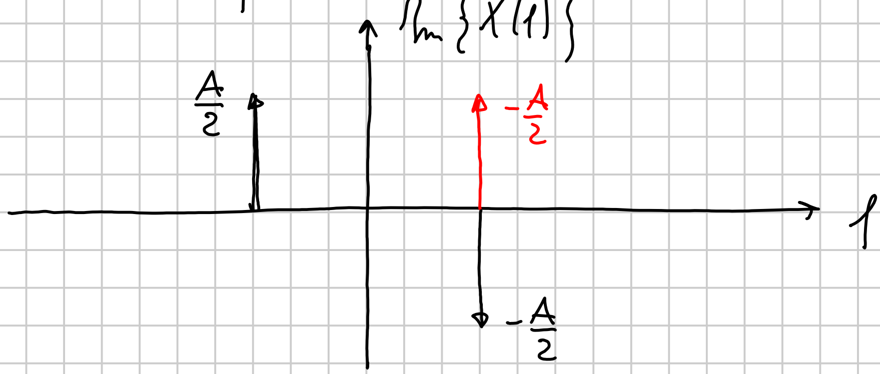
$$x(t) = A \sin(2\pi f_0 t)$$

$$X(f) = \int_{-\infty}^{+\infty} A \sin(2\pi f_0 t)\, e^{-j2\pi f t} dt =$$

$$= A \int_{-\infty}^{+\infty} \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j} e^{-j2\pi f t} dt$$

$$= \frac{A}{2j}\int_{-\infty}^{+\infty} e^{-j2\pi(f-f_0)t} dt - \frac{A}{2j}\int_{-\infty}^{+\infty} e^{-j2\pi(f+f_0)t} dt$$

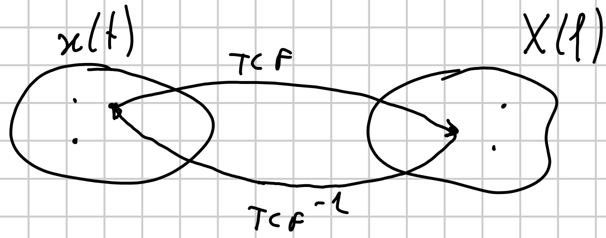
$$X(f) = \frac{A}{2j}\delta(f-f_0) - \frac{A}{2j}\delta(f+f_0)$$



BIUNIVOCITÀ DELLA TCF

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$



$$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df =$$

$$= \int_{-\infty\ (f)}^{+\infty} \int_{-\infty\ (\alpha)}^{+\infty} x(\alpha)\, e^{-j2\pi f \alpha}\, d\alpha\; e^{j2\pi f t}\, df$$

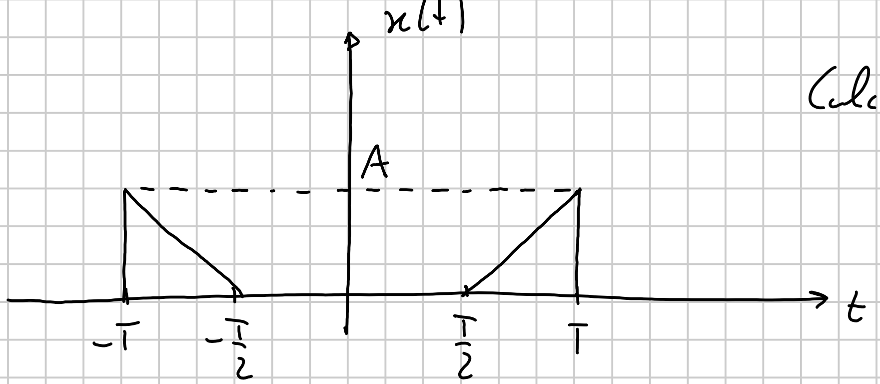
$$= \int_{-\infty\ (\alpha)}^{+\infty} x(\alpha) \int_{-\infty\ (f)}^{+\infty} e^{j2\pi f (t-\alpha)}\, df\, d\alpha$$

$$= \int_{(\alpha)} x(\alpha)\, \delta(t-\alpha)\, d\alpha = x(t) \otimes \delta(t) = x(t)$$

c.v.d.

ESEMPI

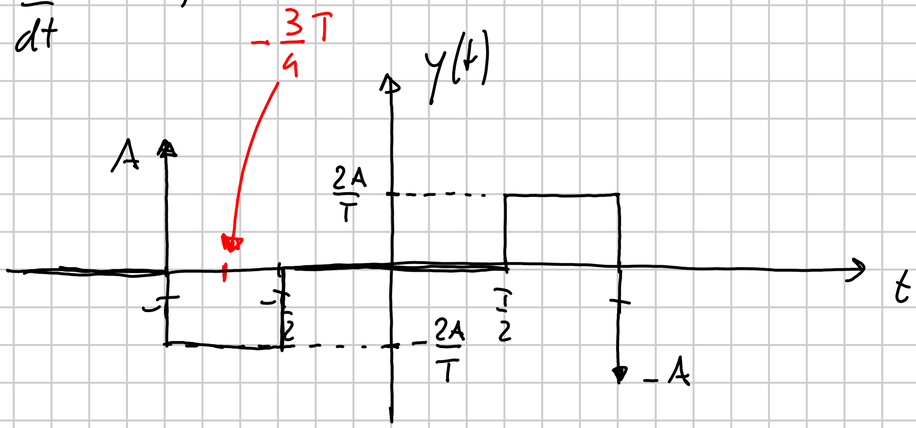
1)



Calcolare la $X(f)$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$y(t) = \frac{d}{dt} x(t)$$



$$x(t) = \int_{-\infty}^{t} y(\alpha)\, d\alpha$$

STEP I $\Rightarrow$ Calcolo $Y(f) = TCF\left[y(t)\right]$

STEP II $\Rightarrow$ $X(f) = \frac{Y(0)}{2}\delta(f) + \frac{Y(f)}{j2\pi f}$

$$Y(f) = Y_1(f) + Y_2(f) + Y_3(f) + Y_4(f)$$

$$Y_n(f) = TCF\left[y_n(t)\right] \qquad n = 1, \ldots, 4$$

$$y_1(t) = A\,\delta(t+T) \qquad y_3(t) = -\frac{2A}{T}\operatorname{rect}\left(\frac{t+\frac{3}{4}T}{T/2}\right)$$

$$y_2(t) = -A\,\delta(t-T) \qquad y_4(t) = \frac{2A}{T}\operatorname{rect}\left(\frac{t-\frac{3}{4}T}{T/2}\right)$$

$$Y_1(f) = A\, e^{-j2\pi f(-T)} = A\, e^{j2\pi f T}$$

$$Y_2(f) = -A\, e^{-j2\pi f T}$$

$$Y_3(f) = -\frac{2A}{T}\, T' \operatorname{sinc}(T' f)\, e^{j\frac{3}{2}\pi f T} \qquad (T' = T/2)$$

$$= -\frac{\cancel{2}A}{\cancel{T}}\,\frac{\cancel{T}}{\cancel{2}} \operatorname{sinc}\left(\frac{T}{2} f\right) e^{j\frac{3}{2}\pi f T}$$

$$Y_4(f) = A \,\mathrm{sinc}\left(\frac{T}{2} f\right) e^{-j\frac{3}{2}\pi f T}$$

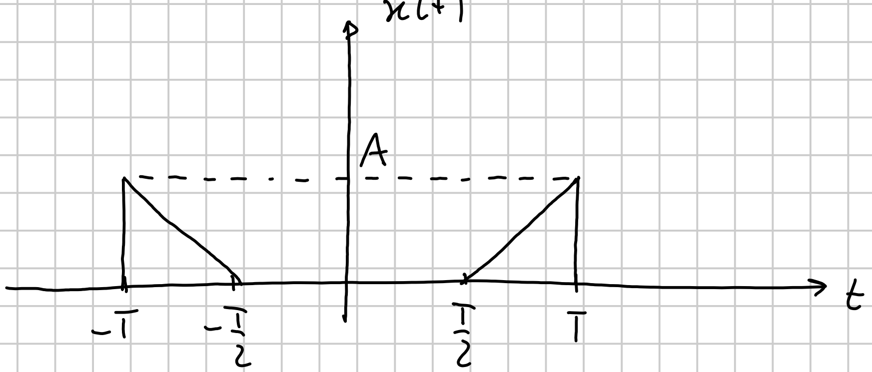
$$Y(f) = 2jA \,\frac{e^{j2\pi f T} - e^{-j2\pi f T}}{2j} - 2jA \,\mathrm{sinc}\left(\frac{T}{2} f\right) \frac{e^{j\frac{3}{2}\pi f T} - e^{-j\frac{3}{2}\pi f T}}{2j}$$

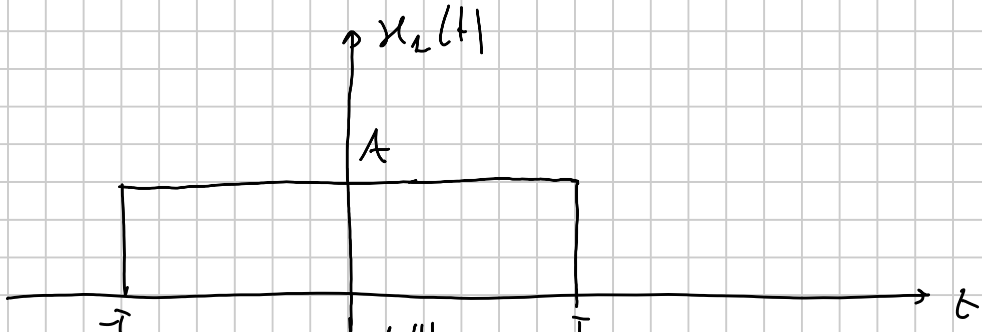
$$= 2jA \sin(2\pi f T) - 2jA \,\mathrm{sinc}\left(\frac{T}{2} f\right) \sin\left(\frac{3}{2}\pi f T\right)$$

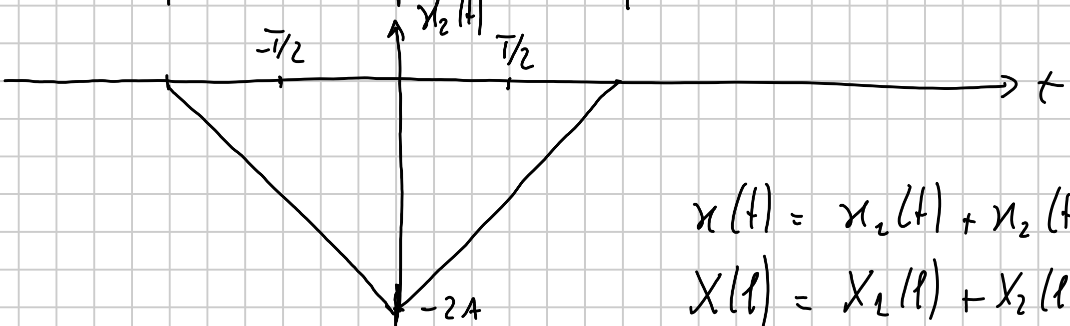
$$Y(0) = 0$$

$$X(f) = \frac{Y(f)}{j2\pi f} = \frac{2AT}{2\pi f T} \sin(2\pi f T) - \frac{\frac{3}{2}AT}{\frac{3}{2}\pi f T} \,\mathrm{sinc}\left(\frac{T}{2} f\right) \sin\left(\frac{3}{2}\pi f T\right)$$

$$= 2AT \,\mathrm{sinc}(2fT) - \frac{3}{2}AT \,\mathrm{sinc}\left(\frac{T}{2} f\right) \mathrm{sinc}\left(\frac{3}{2} f T\right)$$

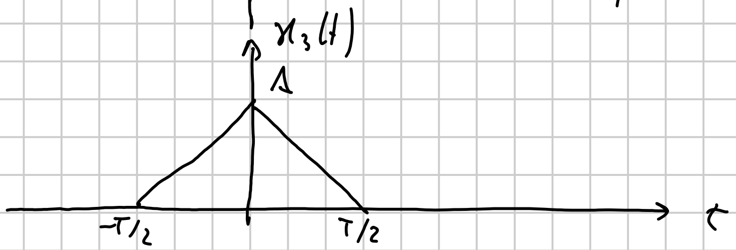




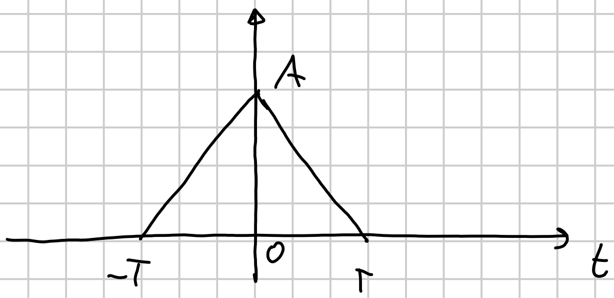


$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

$$X(f) = X_1(f) + X_2(f) + X_3(f)$$



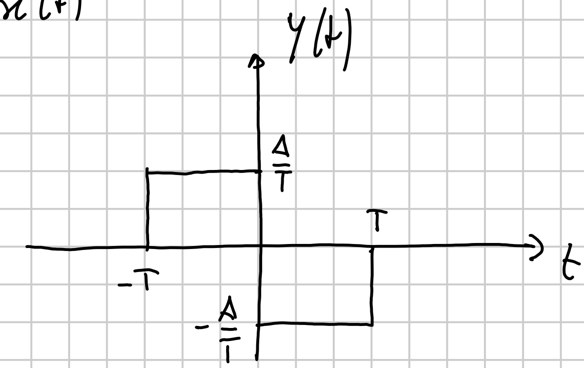
TRASFORMATA DEL TRIANGOLO



$$x(t) = A\left(1 - \frac{|t|}{T}\right)\text{rect}\left(\frac{t}{2T}\right)$$

$$X(f) = ?$$

$$y(t) = \frac{d}{dt}x(t)$$



$$Y(f) = \frac{A}{\cancel{T}}\cancel{T}\,\text{sinc}(Tf)\,e^{j\pi fT} - \frac{A}{\cancel{T}}\cancel{T}\,\text{sinc}(Tf)\,e^{-j\pi fT}$$

$$y(t) = \frac{A}{T}\text{rect}\left(\frac{t+T/2}{T}\right) - \frac{A}{T}\text{rect}\left(\frac{t-T/2}{T}\right)$$

$$Y(f) = 2jA\,\text{sinc}(Tf)\,\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}$$

$$= 2jA\,\text{sinc}(Tf)\sin(\pi fT) \Rightarrow Y(0) = 0$$

$$X(f) = \frac{Y(f)}{j2\pi f} = \frac{AT}{\pi fT}\text{sinc}(Tf)\sin(\pi fT) = AT\,\text{sinc}^2(fT)$$

$$\boxed{\left(1 - \frac{|t|}{T}\right)\text{rect}\left(\frac{t}{2T}\right) \overset{TCF}{\Longleftrightarrow} T\,\text{sinc}^2(fT)}$$

$$x_1(t) = A \,\mathrm{rect}\left(\frac{t}{2T}\right)$$

$$x_2(t) = -2A\left(1 - \frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right)$$

$$x_3(t) = A\left(1 - \frac{|t|}{T/2}\right)\mathrm{rect}\left(\frac{t}{T}\right)$$

$$X_1(f) = A2T \,\mathrm{sinc}(2fT)$$

$$X_2(f) = -2AT\,\mathrm{sinc}^2(fT)$$

$$X_3(f) = A\frac{T}{2}\,\mathrm{sinc}^2\left(\frac{fT}{2}\right)$$

$$x_3(t) = A\left(1 - \frac{|t|}{T'}\right)\mathrm{rect}\left(\frac{t}{2T'}\right) \quad , \; T' = \frac{T}{2}$$

$$\Downarrow$$

$$X_3(f) = AT'\,\mathrm{sinc}^2(fT')$$

2) MODULAZIONE DI AMPIEZZA

$$y(t) = x(t)\cos(2\pi f_0 t) \qquad , \; x(t) = AB\,\mathrm{sinc}^2(Bt)$$

$$Y(f) = \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$

$$y(t) = x(t)\cdot\cos(2\pi f_0 t)$$

$$\Downarrow$$

$$Y(f) = X(f) \otimes \mathrm{TCF}\left[\cos(2\pi f_0 t)\right]$$

$$= X(f) \otimes \left[\frac{1}{2}\delta(\underbrace{f-f_0}_{f'}) + \frac{1}{2}\delta(f+f_0)\right]$$

$$\Rightarrow \int_{-\infty}^{+\infty} X(\alpha)\, \delta\left[\underbrace{(f-f_0)}_{f'} - \alpha\right] d\alpha =$$

$$= \int_{-\infty}^{+\infty} X(\alpha)\, \delta\left[\alpha - \underbrace{(f-f_0)}_{f'}\right] d\alpha = X(f-f_0)$$

$$Y(f) = \frac{1}{2} X(f-f_0) + \frac{1}{2} X(f+f_0)$$

$$X(f) = TCF\left[AB \operatorname{sinc}^2(Bt)\right]$$

$$\left(1 - \frac{|t|}{T}\right) \operatorname{rect}\left(\frac{t}{2T}\right) \overset{TCF}{\Longleftrightarrow} T \operatorname{sinc}^2(fT)$$

DUALITÀ $\Downarrow$

$$AB \operatorname{sinc}^2(Bt) \overset{TCF}{\Longleftrightarrow} A\left(1 - \frac{|f|}{B}\right) \operatorname{rect}\left(\frac{f}{2B}\right)$$

$$Y(f) = \frac{A}{2}\left(1 - \frac{|f-f_0|}{B}\right) \operatorname{rect}\left(\frac{f-f_0}{2B}\right) + \frac{A}{2}\left(1 - \frac{|f+f_0|}{B}\right) \operatorname{rect}\left(\frac{f+f_0}{2B}\right)$$

SEGNALI PERIODICIZZATI

$x_0(t)$ aperiodico

$\Downarrow$

$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0) \quad , k \in \mathbb{Z} \quad , T_0 \in \mathbb{R}^+$$

$x(t)$ segnale periodico di periodo T_0

$$x_0(t) \overset{TCF}{\Longleftrightarrow} X_0(f)$$

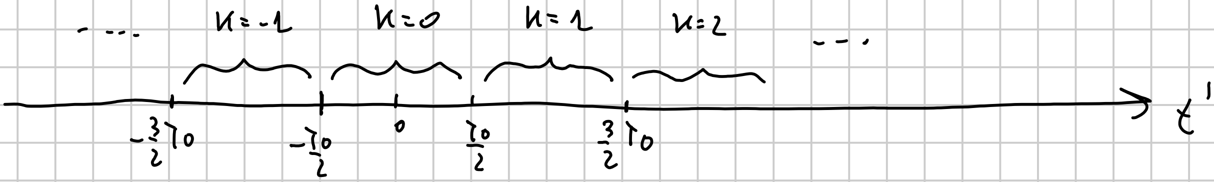
$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt =$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_0(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt = \ldots \left(t - kT_0 = t' \right)$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2} - kT_0}^{\frac{T_0}{2} - kT_0} x_0(t')\, e^{-j2\pi n f_0 (t' + kT_0)}\, dt'$$

$$= \frac{1}{T_0} \sum_{k=-\infty}^{+\infty} \int_{-\frac{T_0}{2} - kT_0}^{\frac{T_0}{2} - kT_0} x_0(t')\, e^{-j2\pi n f_0 t'}\, dt' \; \underbrace{e^{-j2\pi n \cancel{f_0} k \cancel{T_0}}}_{1}$$



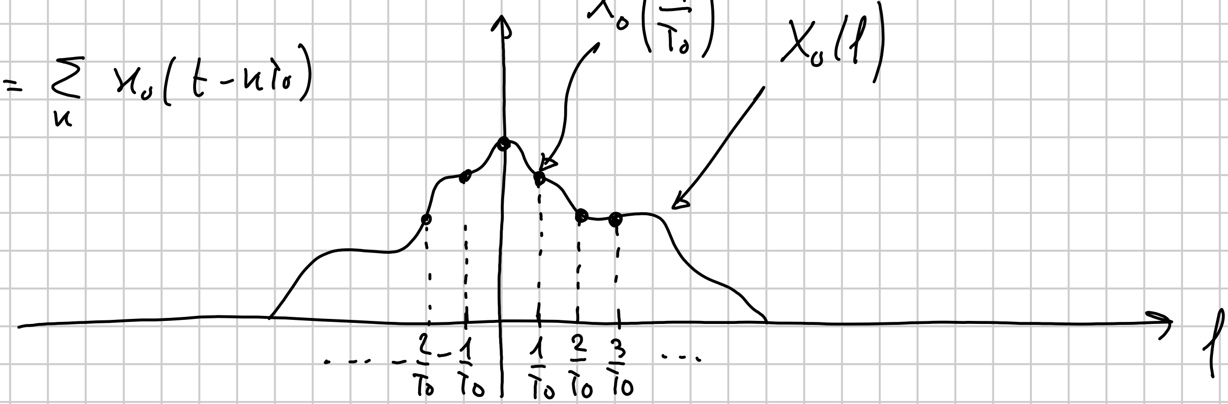
$$= \frac{1}{T_0} \int_{-\infty}^{+\infty} x_0(t')\, e^{-j2\pi n f_0 t'}\, dt' = \frac{1}{T_0} X_0(nf_0) = \frac{1}{T_0} X_0\left(\frac{n}{T_0}\right)$$

$$\boxed{X_n = \frac{1}{T_0} X_0\left(\frac{n}{T_0}\right)}$$

REL. TRA TSF DI UN SEGN. PERIODICIZZATO E LA TCF DEL SEGNALE APERIODICO

$$x_0(t) \overset{TCF}{\Longleftrightarrow} X_0(f)$$

$$x(t) = \sum_k x_0(t - kT_0)$$



FORMULE DI POISSON

I^a
$$\boxed{\sum_{k=-\infty}^{+\infty} x_0(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right) e^{j2\pi n\frac{t}{T_0}}}$$

Dim.

$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)$$

$$x(t) \overset{TSF}{\Longleftrightarrow} X_n \quad \Rightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} X_n e^{j2\pi k f_0 t}$$

$$\sum_{k=-\infty}^{+\infty} x_0(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right) e^{j2\pi k\frac{t}{T_0}}$$

c.v.d.

II^a FORMULA DI POISSON

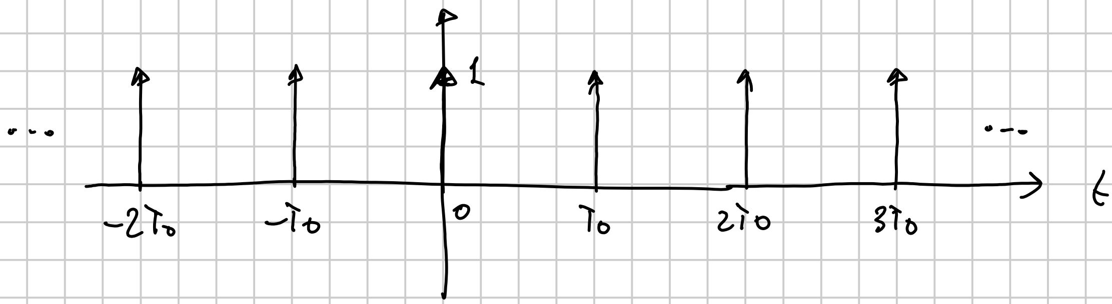
I^a
$$\sum_{k=-\infty}^{+\infty} x_0(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right) e^{j2\pi n\frac{t}{T_0}}$$

x dualità $\quad \frac{1}{T_0} \Rightarrow T_0 \quad , \quad X_0\left(\frac{k}{T_0}\right) \Rightarrow x(kT_0)$

$$\text{II}^a \quad \boxed{T_0 \sum_{k=-\infty}^{+\infty} x(kT_0)\, e^{-j2\pi n f T_0} = \sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T_0}\right)}$$

APPLICAZIONE DELLA I^a FORMULA DI POISSON ALLA $\delta(t)$

$$TCF\left[\sum_{k=-\infty}^{+\infty} \delta(t - kT_0)\right] = ?$$

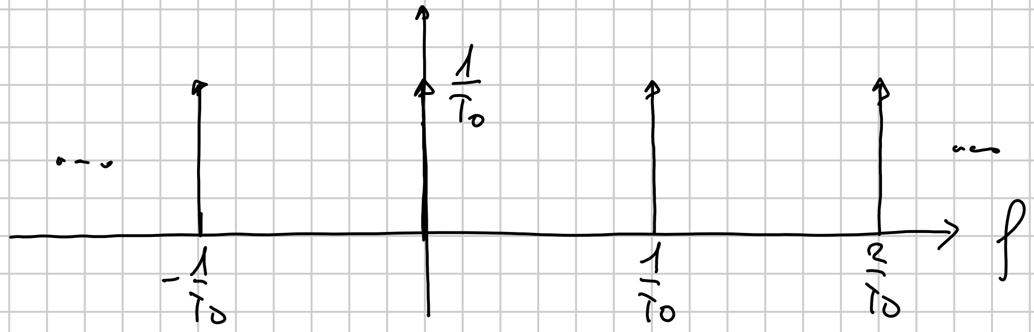


$$\sum_{k=-\infty}^{+\infty} \delta(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} e^{j2\pi n \frac{t}{T_0}}$$

$$TCF\left[\sum_{k=-\infty}^{+\infty} \delta(t - kT_0)\right] = \sum_{k=-\infty}^{+\infty} e^{-j2\pi f k T_0} =$$

$$= \sum_{k=-\infty}^{+\infty} e^{-j2\pi k f T_0} = \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_0}\right)$$

$$\boxed{\sum_{k=-\infty}^{+\infty} \delta(t - kT_0) \overset{TCF}{\Longleftrightarrow} \frac{1}{T_0}\sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{k}{T_0}\right)}$$



TCF DI UN SEGNALE PERIODICO

$x(t)$ periodico di T $\Rightarrow$ $\exists$ infiniti $x_0(t)$:
$\sum_k x_0(t - kT_0) = x(t)$

$x(t) = \sum_k x_0(t - kT_0)$, $x_0(t)$ e' aperiodico

$$x_0(t) \overset{TCF}{\Longleftrightarrow} X_0(f)$$

$$X(f) = ? = TCF[x(t)] = TCF\left[\sum_k x_0(t - kT_0)\right]$$

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \sum_k x_0(t - kT_0)\, e^{-j2\pi f t}\, dt =$$

$$= \sum_k \int_{-\infty}^{+\infty} x_0(t - kT_0)\, e^{-j2\pi f t}\, dt = \qquad (t - kT_0) = t'$$

$$= \sum_k \int_{-\infty}^{+\infty} x_0(t')\, e^{-j2\pi f (t' + kT_0)}\, dt' =$$

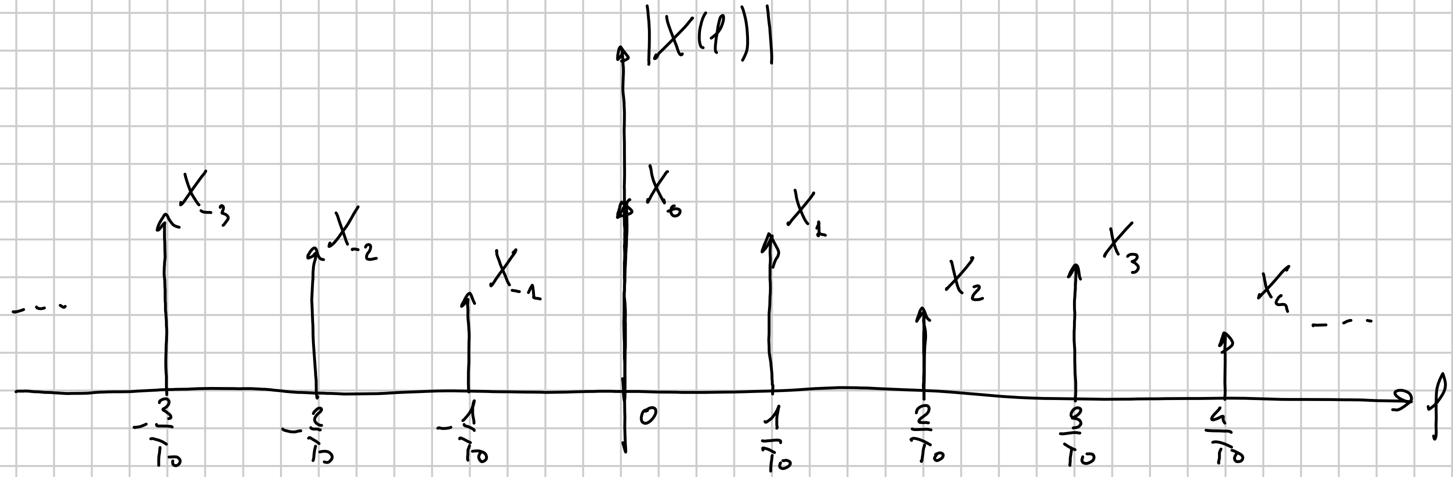
$$= \sum_k \int_{-\infty}^{+\infty} x_0(t')\, e^{-j2\pi f t'}\, dt'\, e^{-j2\pi f k T_0}$$

$$= \underbrace{\int_{-\infty}^{+\infty} x_0(t')\, e^{-j2\pi f t'}\, dt'}_{X_0(f)} \cdot \underbrace{\sum_k e^{-j2\pi f k T_0}}_{\frac{1}{T_0}\sum_k \delta\left(f - \frac{k}{T_0}\right)}$$

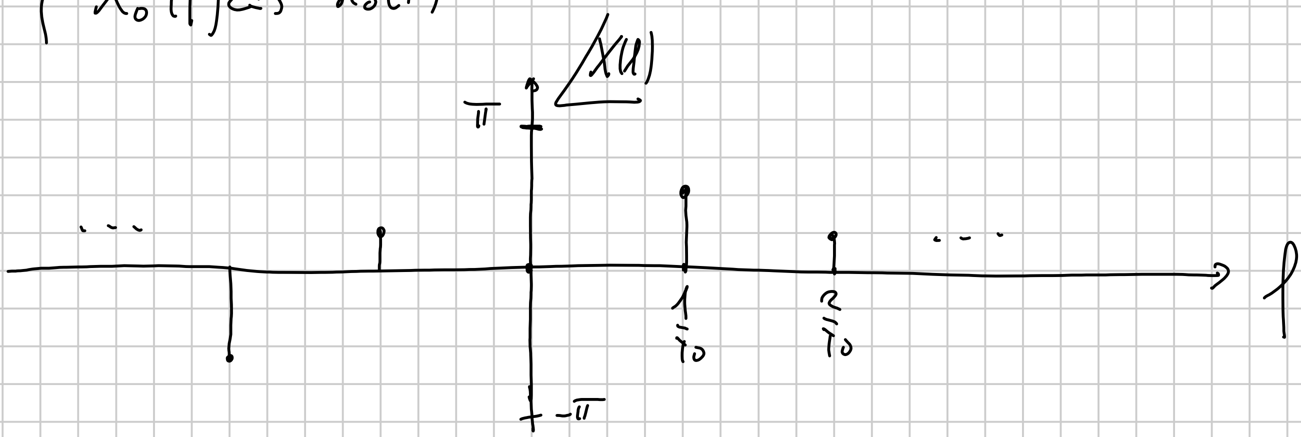
$$X(f) = \frac{1}{T_0} \sum_k X_0(f)\, \delta\left(f - \frac{k}{T_0}\right) = \frac{1}{T_0} \sum_k X_0\left(\frac{k}{T_0}\right) \delta\left(f - \frac{k}{T_0}\right)$$

$$\boxed{\begin{aligned} X(f) &= \sum_{k=-\infty}^{+\infty} X_k\, \delta\left(f - \frac{k}{T_0}\right) \\ X_k &= \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right) \end{aligned}}$$

RELAZIONE TRA TCF e TSF



$$\begin{cases} X_n = \dfrac{1}{T_0} X_0\left(\dfrac{n}{T_0}\right) \\ X_0(f) \overset{TCF}{\Leftrightarrow} x_0(t) \end{cases}$$



TEOREMA DI PARSEVAL PER SEGNALI PERIODICI

$$x(t) \overset{TSF}{\Longleftrightarrow} X_n \quad , \quad y(t) \overset{TSF}{\Longleftrightarrow} Y_n \quad , \quad T_0$$

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \sum_{n=-\infty}^{+\infty} X_n Y_n^*$$

Dim.

$$\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{n=-\infty}^{+\infty} X_n e^{j2\pi n f_0 t}\, y^*(t)\, dt$$

$$= \sum_{n=-\infty}^{+\infty} X_n \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y^*(t)\, e^{j2\pi n f_0 t}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} X_n \Bigg[\underbrace{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi n f_0 t}\, dt}_{Y_n} \Bigg]^* = \sum_{n=-\infty}^{+\infty} X_n Y_n^*$$

$$\Rightarrow \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, y^*(t)\, dt = \frac{1}{T_0^2} \sum_{n=-\infty}^{+\infty} X_0\left(\frac{n}{T_0}\right) Y_0^*\left(\frac{n}{T_0}\right)$$

CALCOLO DELLA POTENZA DI UN SEGNALE PERIODICO

$$P_x = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\, dt = \sum_{n=-\infty}^{+\infty} X_n X_n^* = \sum_{n=-\infty}^{+\infty} |X_n|^2$$

$$|x(t)|^2 = x(t) \cdot x^*(t)$$

DENSITÀ SPETTRALE DI POTENZA PER SEGNALI PERIODICI

$$S_x(f) = \sum_{k=-\infty}^{+\infty} |X_n|^2 \, \delta\left(f - \frac{n}{T_0}\right)$$

$x(t)$ è periodico di T_0

$$P_x = \int_{-\infty}^{+\infty} S_x(f)\, df =$$

$$= \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} |X_n|^2 \, \delta\left(f - \frac{k}{T_0}\right) df =$$

$$= \sum_{k=-\infty}^{+\infty} |X_n|^2 \underbrace{\int_{-\infty}^{+\infty} \delta\left(f - \frac{k}{T_0}\right) df}_{= 1} = \sum_{k=-\infty}^{+\infty} |X_n|^2 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 dt$$

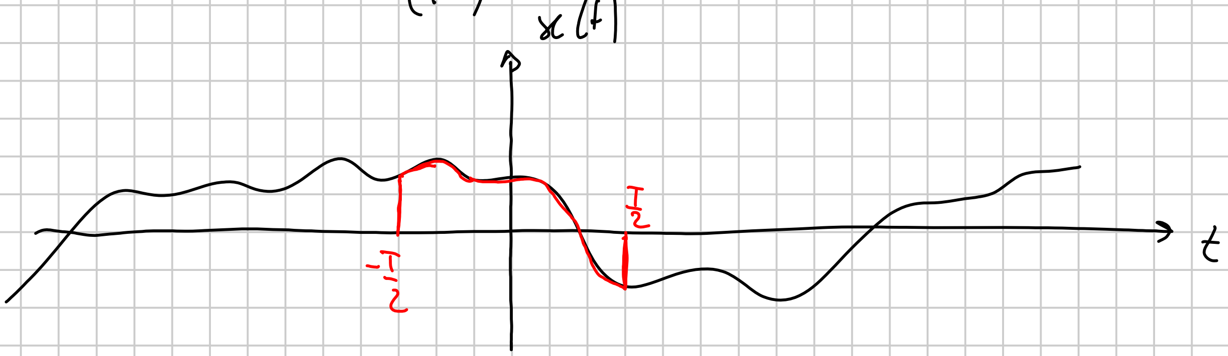
c.v.d.

DENSITÀ SPETTRALE DI POTENZA DI SEGNALI APERIODICI

$$S_x(f) \triangleq \lim_{T \to \infty} \frac{|X_T(f)|^2}{T}$$

$$X_T(f) = TCF\left[x_T(t)\right]$$

$$x_T(t) = x(t) \, \mathrm{rect}\left(\frac{t}{T}\right)$$



ANALISI ENERGETICA DI SEGNALI APERIODICI

CORRELAZIONE TRA SEGNALI

$$C_{xy}(\tau) \triangleq \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt$$

CORRELAZIONE TRA DUE SEGNALI

CROSS – CORRELAZIONE

τ: ritardo temporale (time - lag)

AUTOCORRELAZIONE

$$C_x(\tau) \triangleq \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt$$

PROPRIETÀ DELLA AUTOCORRELAZIONE

j) $C_x(0) = E_x$

$$C_x(\tau)\Big|_{\tau=0} = \int_{-\infty}^{+\infty} x(t)\, x^*(t)\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt = E_X$$

jj) $C_x(\tau) = C_x^*(-\tau)$ SIMMETRIA HERMITIANA

$$C_x(-\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-(-\tau))\, dt =$$

$$= \int_{-\infty}^{+\infty} x(t)\, x^*(t+\tau)\, dt = \ldots \quad (t+\tau = t')$$

$$= \int_{-\infty}^{+\infty} x(t'-\tau)\, x^*(t')\, dt'$$

$$= \int_{-\infty}^{+\infty} x^*(t') \, x(t'-\tau) \, dt'$$

$$= \Bigg[\underbrace{\int_{-\infty}^{+\infty} x(t') \, x^*(t'-\tau) \, dt'}_{C_x(\tau)} \Bigg]^* = C_x^*(\tau)$$

$$C_x(-\tau) = C_x^*(\tau)$$

$$C_x(\tau) = C_x^*(-\tau) \qquad \text{c.v.d.}$$

se $C_x(\tau)$ è reale $\Rightarrow$ $C_x(\tau) = C_x(-\tau)$

simmetria pari

·) TCF DELLA $C_x(\tau)$

$$TCF\left[C_x(\tau) \right] = \int_{-\infty}^{+\infty} C_x(\tau) \, e^{-j2\pi f\tau} \, d\tau$$

$$= \int_{-\infty \, (\tau)}^{+\infty} \underbrace{\int_{-\infty \, (t)}^{+\infty} x(t) \, x^*(t-\tau) \, dt}_{C_x(\tau)} \; e^{-j2\pi f\tau} \, d\tau$$

$$= \int_{-\infty \, (t)}^{+\infty} x(t) \underbrace{\int_{-\infty \, (\tau)}^{+\infty} x^*(t-\tau) \, e^{-j2\pi f\tau} \, d\tau}\, dt = .. \; (t-\tau = \tau')$$

$$= \int_{-\infty}^{+\infty} x(t) \int_{-\infty}^{+\infty} x^*(\tau') \, e^{-j2\pi f(t-\tau')} \, d\tau' \, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \int_{-\infty}^{+\infty} x^*(\tau') e^{+j2\pi f\tau'} d\tau' \, e^{-j2\pi f t} dt$$

$$= \underbrace{\int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt}_{X(f)} \underbrace{\left[\int_{-\infty}^{+\infty} x(\tau') e^{-j2\pi f\tau'} d\tau'\right]^*}_{X^*(f)}$$

$$\Rightarrow \boxed{TCF[C_x(\tau)] = |X(f)|^2 = S_x(f)}$$

DENSITÀ SPETTRALE DI ENERGIA

$$\int_{-\infty}^{+\infty} S_x(f)\, df = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi f\tau} df \Big|_{\tau=0} = C_x(\tau)\Big|_{\tau=0} = E_x$$

COME SI CALCOLA L'ENERGIA DI UN SEGNALE APER.

$x(t)$ aperiodico

$$\hookrightarrow \quad C_x(0) = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

TEOREMA DI PARSEVAL PER SEGNALI APERIODICI

$$\int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{+\infty} X(f)\, Y^*(f)\, df$$

se

$x(t) \overset{TCF}{\Longleftrightarrow} X(f)$

$y(t) \overset{TCF}{\Longleftrightarrow} Y(f)$

Dim

$$\int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt = \underset{(t)}{\int_{-\infty}^{+\infty}} \underset{(f)}{\int_{-\infty}^{+\infty}} X(f) e^{j2\pi f t} df \; y^*(t)\, dt$$

$$= \int_{-\infty}^{+\infty} X(f) \int_{-\infty}^{+\infty} y^*(t)\, e^{j2\pi f t}\, dt\, df = \int_{-\infty}^{+\infty} X(f) \Big[\underbrace{\int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt}_{Y(f)} \Big]^* df$$

(f) (t) (f) (t)

$$= \int_{-\infty}^{+\infty} X(f)\, Y^*(f)\, df \qquad \text{c.v.d.}$$

ESEMPIO

$$x(t) = A \operatorname{sinc}(Bt)$$

Calcolare E_x

$$E_x = \int_{-\infty}^{+\infty} \underbrace{A^2 \operatorname{sinc}^2(Bt)}_{|x(t)|^2}\, dt = \int_{-\infty}^{+\infty} |X(f)|^2\, df$$

$$X(f) = \frac{A}{B} \operatorname{rect}\left(\frac{f}{B}\right)$$

$$E_x = \frac{A^2}{B^2} \int_{-\frac{B}{2}}^{\frac{B}{2}} 1\, df = \frac{A^2}{B^2} \cdot B = \frac{A^2}{B}$$

RELAZIONE TRA CORRELAZIONE E CONVOLUZIONE

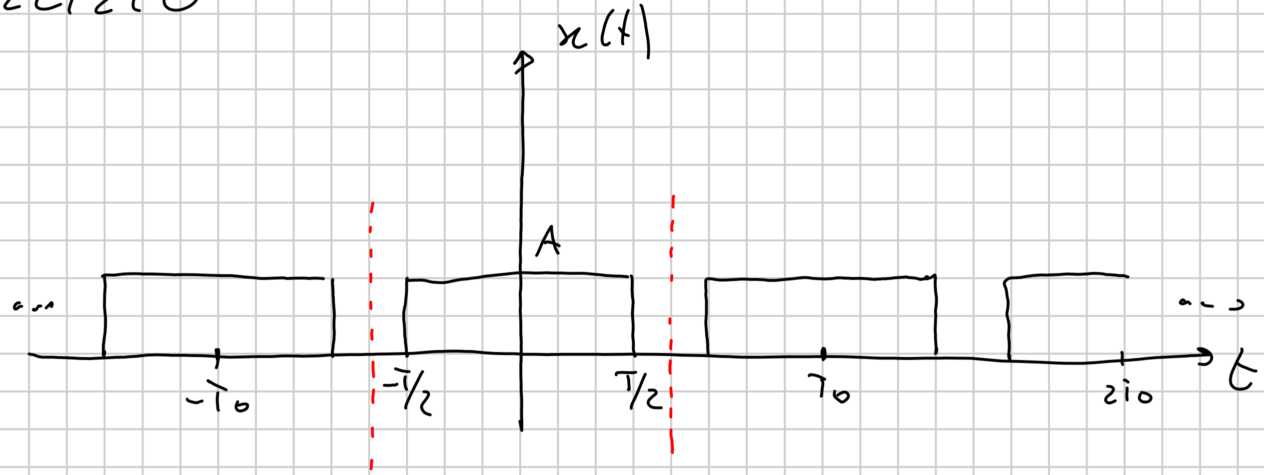
$$C_{xy}(\tau) = x(\tau) \otimes y^*(-\tau)$$

Dim

$$x(\tau) \otimes y^*(-\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*[-(\tau - t)]\, dt =$$

$$= \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt = C_x(\tau)$$

ESERCIZIO



Calcolare la $X(f) = TCF[x(t)]$

$$x(t) = \sum_k x_0(t - kT_0)$$

$$x_0(t) = A\, \mathrm{rect}\left(\frac{t}{T}\right)$$

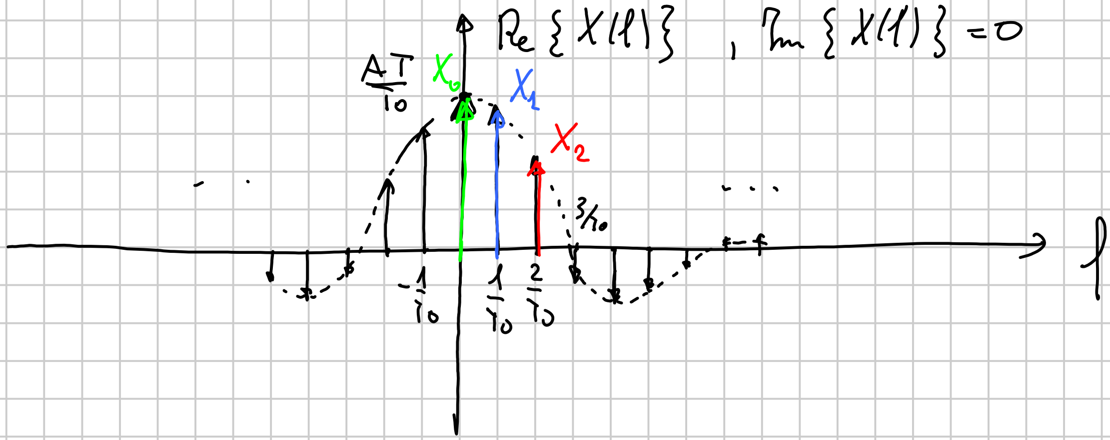
$$X(f) = \underbrace{\frac{1}{T_0} \sum_{k=-\infty}^{+\infty} X_0\left(\frac{k}{T_0}\right)}_{X_k} \delta\left(f - \frac{k}{T_0}\right) = \sum_{k=-\infty}^{+\infty} X_k\, \delta\left(f - \frac{k}{T_0}\right)$$

$$X_0(f) \Rightarrow \frac{1}{T_0} X_0(f)\Big|_{f=\frac{k}{T_0}} = X_k$$

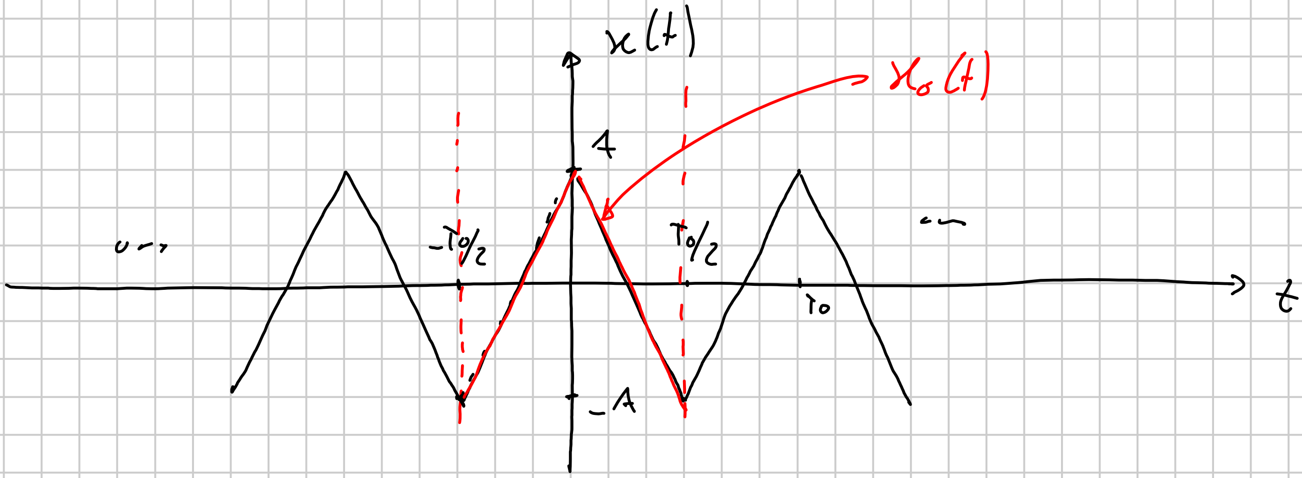
$$X_0(f) = TCF[x_0(t)] = AT\, \mathrm{sinc}(Tf)$$

$$X_k = \frac{1}{T_0} AT\, \mathrm{sinc}\left(\frac{Tk}{T_0}\right) = \boxed{\frac{AT}{T_0}\, \mathrm{sinc}\left(\frac{T}{T_0}k\right)}$$ TSF

$$\boxed{X(f) = \frac{AT}{T_0} \sum_{k=-\infty}^{+\infty} \mathrm{sinc}\left(\frac{T}{T_0}k\right) \delta\left(f - \frac{k}{T_0}\right)}$$ TCF



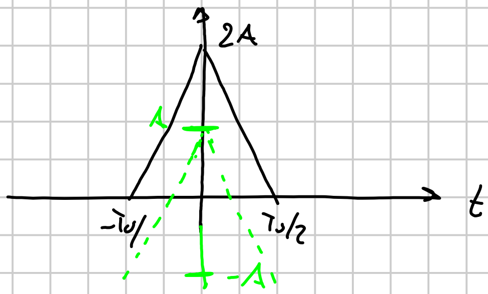
·) ONDA TRIANGOLARE



Calcolare TSF, TCF

$$x(t) = \sum_{k=-\infty}^{+\infty} x_0(t - kT_0)$$

$$x_0(t) = 2A\left(1 - \frac{|t|}{T_0/2}\right)\mathrm{rect}\left(\frac{t}{T_0}\right) - A\,\mathrm{rect}\left(\frac{t}{T_0}\right)$$



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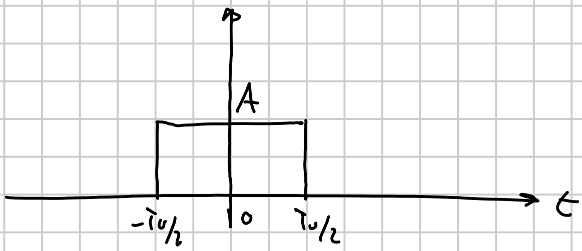


$$TSF[x(t)] = \frac{1}{T_0} X_0\left(\frac{k}{T_0}\right) = X_k$$

$$X_0(f) \Rightarrow X_k = \frac{1}{T_0} X_0(f)\Big|_{f=\frac{k}{T_0}}$$

$$X_0(f) = 2A \cdot \frac{T_0}{2} \operatorname{sinc}^2\left(\frac{T_0}{2} f\right) - A T_0 \operatorname{sinc}(T_0 f)$$

$$X_k = \frac{1}{\cancel{T_0}} \cancel{2}A \frac{\cancel{T_0}}{\cancel{2}} \operatorname{sinc}^2\left(\frac{\cancel{T_0}}{2} \frac{k}{\cancel{T_0}}\right) - \frac{1}{\cancel{T_0}} A \cancel{T_0} \operatorname{sinc}\left(\cancel{T_0} \frac{k}{\cancel{T_0}}\right)$$

$$= \boxed{A \operatorname{sinc}^2\left(\frac{k}{2}\right) - A \operatorname{sinc}(k)} \qquad TSF$$

$$k=0 \Rightarrow X_k\Big|_{k=0} = 0$$

$$k=1 \Rightarrow X_1 = A \operatorname{sinc}^2\left(\frac{1}{2}\right) = A \frac{\sin^2 \pi/2}{(\pi/2)^2} = \frac{4A}{\pi^2}$$

$$k=-1 \Rightarrow X_{-1} = A \operatorname{sinc}^2\left(\frac{1}{2}\right)$$

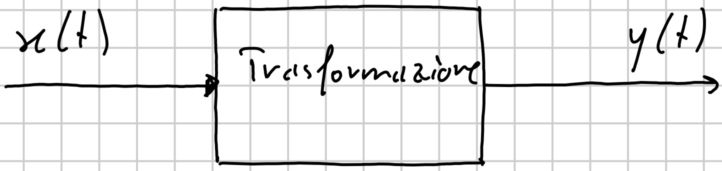
$$k=2 \Rightarrow X_2 = A \operatorname{sinc}^2(1) = 0$$

$X_k = 0$ k è pari

$X_k \neq 0$ k dispari

20/03/2013

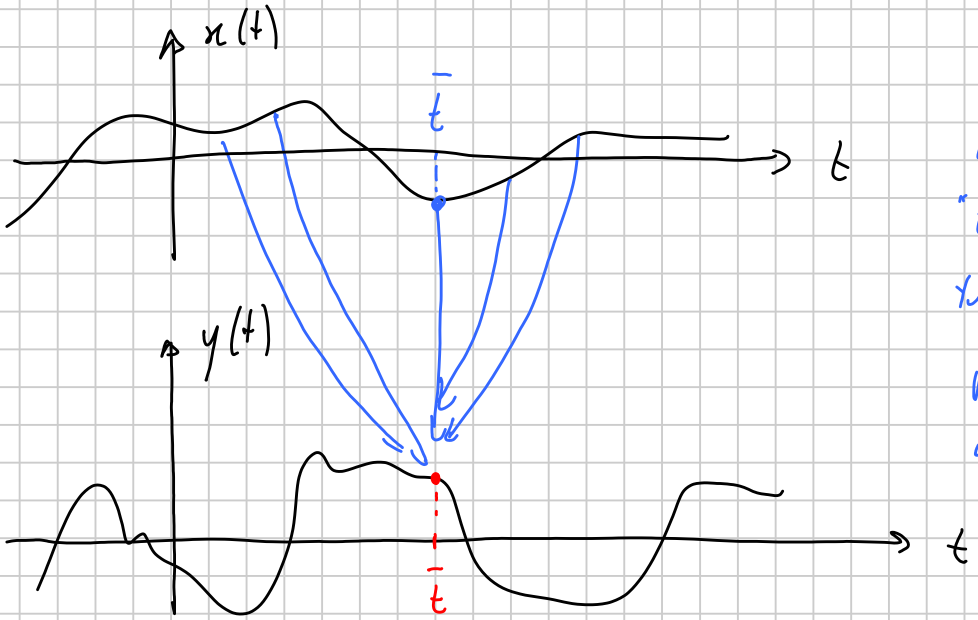
SISTEMI MONODIMENSIONALI LINEARI STAZIONARI A TEMPO CONTINUO



$$y(t) = T\left[x(t)\right]$$

es. $T\left[x(t)\right] = \int_{-\infty}^{t} x(\alpha)\,d\alpha$

l'uscita all'istante "t" dipende da tutti i valori precedenti o uguali a "t"



·) LINEARITÀ

Se $\quad x(t) = a_1 x_1(t) + a_2 x_2(t)$

$$\Downarrow$$

$$y(t) = T\left[x(t)\right] = a_1 T\left[x_1(t)\right] + a_2 T\left[x_2(t)\right]$$

·) STAZIONARIETÀ

Se $\quad y(t) = T\left[x(t)\right]$

$$\Downarrow$$

$$T\left[x(t-t_0)\right] = y(t-t_0)$$

CARATTERIZZAZIONE DEI SISTEMI LINEARI E STAZ.
(SLS)

$h(t) \triangleq T[\delta(t)]$ risposta impulsiva del sistema

$$\Rightarrow \quad y(t) = T[x(t)] \underset{\text{vero per SLS}}{=} x(t) \otimes h(t)$$

$h(t)$ descrive completamente un SLS

Dim.

$$y(t) = T[x(t)] = T[x(t) \otimes \delta(t)] =$$

$$= T\left[\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] =$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) \otimes h(t) \qquad \text{c.v.d.}$$

ALTRE PROPRIETÀ

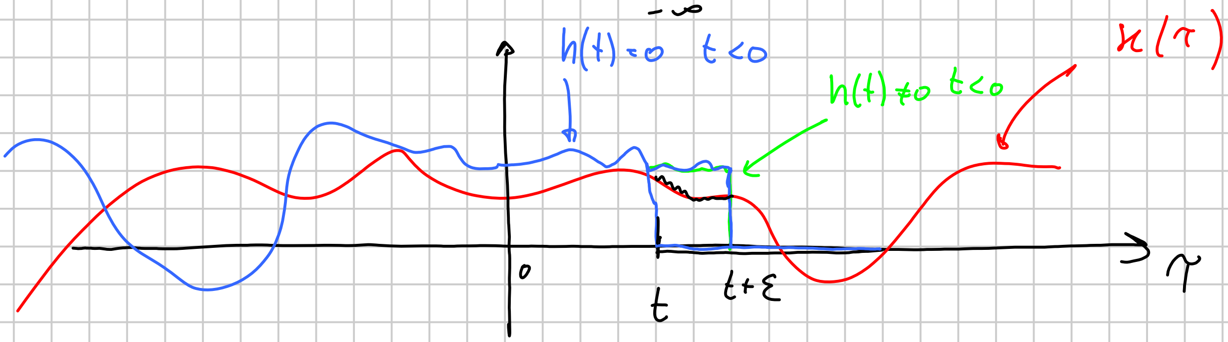
·) CAUSALITÀ

$$y(t) = T[x(\alpha);\ \alpha \leq t]$$

CAUSALITÀ PER SLS

$$\boxed{h(t) = 0 \quad (t < 0)}$$

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$



$$h(t) = 0 \quad t < 0 \quad \Rightarrow \quad h(t-\tau) = 0 \quad \text{per } \tau > t$$

·) STABILITÀ BIBO

Bounded Input Bounded Output

Se $|x(t)| \leq K < \infty \qquad \forall t$

$$\Downarrow$$

$$|y(t)| \leq N < \infty$$

STABILITÀ BIBO PER I SLS

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty \qquad \forall t$$

Dim della cond. di sufficienza

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty \quad \Rightarrow \quad \text{SLS è STABILE BIBO}$$

Dim.

$$\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$$

$$|y(t)| = \left| \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau \right| \leq \int_{-\infty}^{+\infty} |x(\tau)|\, |h(t-\tau)|\, d\tau$$

$$\leq \int_{-\infty}^{+\infty} K\, |h(t-\tau)|\, d\tau = K \underbrace{\int_{-\infty}^{+\infty} |h(\tau')|\, d\tau'}_{M<\infty} \leq KM < \infty$$

c.v.d.

·) MEMORIA

·) Sistema senza memoria (istantaneo)

$$y(t) = T\left[x(\alpha),\ \alpha = t \right]$$

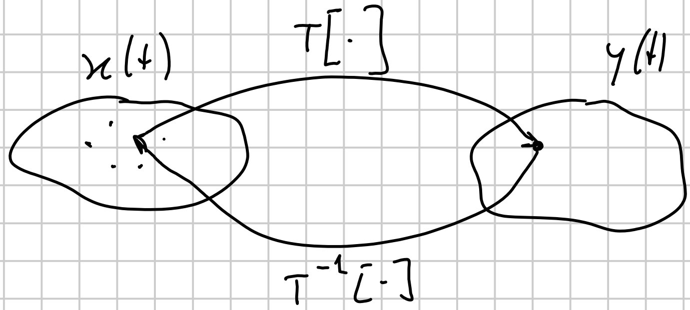
·) Quando la cond. sopra non e' verificata il sistema ha memoria

·) INVERTIBILITA'

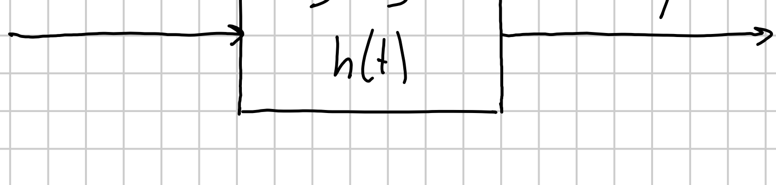
$$y(t) = T\left[x(t) \right]$$

$$\Downarrow$$

$$x(t) = T^{-1}\left[y(t) \right]$$



RISPOSTA IN FREQUENZA



$$x(t) = e^{j2\pi f t}$$

$$y(t) = \int_{-\infty}^{+\infty} e^{j2\pi f \tau} h(t-\tau)\, d\tau = \ldots \quad t-\tau = \tau'$$

$$= \int_{-\infty}^{+\infty} e^{j2\pi f (t-\tau')} h(\tau')\, d\tau' =$$

$$= \int_{-\infty}^{+\infty} e^{-j2\pi f \tau'} h(\tau')\, d\tau' \; e^{j2\pi f t}$$

$$y(t) = \underbrace{e^{j2\pi f t}}_{x(t)} \underbrace{\int_{-\infty}^{+\infty} h(\tau') e^{-j2\pi f \tau'} d\tau'}_{H(f)}$$

Definiamo la risposta in frequenza:

I) $\quad H(f) \triangleq \left. \dfrac{y(t)}{x(t)} \right|_{x(t) = e^{j2\pi f t}}$

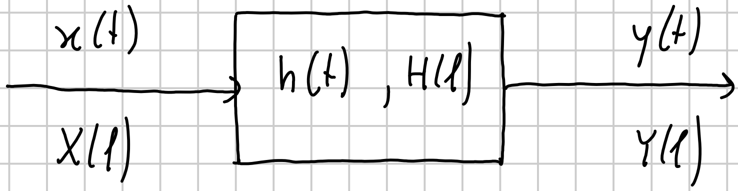
II) $\quad H(f) \triangleq TCF\left[h(t) \right]$

III) $\quad H(f) \triangleq \dfrac{Y(f)}{X(f)}$

Dim. della III

$$y(t) = x(t) \otimes h(t) \xRightarrow{TCF} Y(f) = X(f)\, H(f) \Rightarrow H(f) = \frac{Y(f)}{X(f)}$$

c.v.d.



$$y(t) = x(t) \otimes h(t)$$

$$Y(f) = X(f)\, H(f)$$

$y(t) = ?$ $\Rightarrow$

j) calcolo la $X(f)$

jj) calcolo la $H(f)$

jjj) calcolo $Y(f) = X(f)\, H(f)$

jv) calcolo $TCF^{-1}\left[Y(f) \right]$

RISPOSTA IN AMPIEZZA

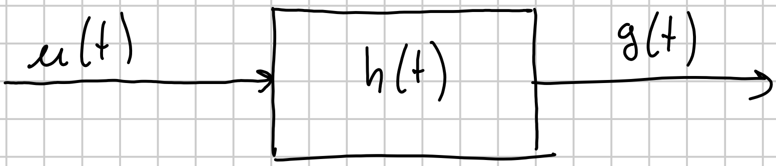
$$A(f) \triangleq |H(f)|$$

RISPOSTA IN FASE

$$\varphi(f) \triangleq tg^{-1} \frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}} = \angle H(f)$$

$$\begin{cases} |Y(f)| = |X(f)| \cdot |H(f)| \\ \angle Y(f) = \angle X(f) + \angle H(f) \end{cases}$$

RISPOSTA DI UN SLS AD UN GRADINO

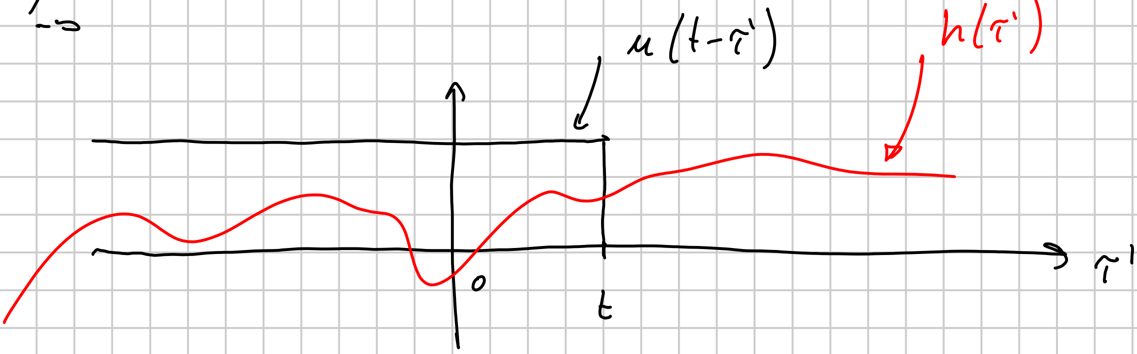


$$g(t) = u(t) \otimes h(t)$$

$$u(t) = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha$$

$$g(t) = u(t) \otimes h(t) = \int_{-\infty}^{+\infty} u(\tau)\, h(t-\tau)\, d\tau = \ldots \quad (t-\tau = \tau')$$

$$= \int_{-\infty}^{+\infty} h(\tau')\, u(t-\tau')\, d\tau' =$$



$$g(t) = \int_{-\infty}^{t} h(\tau')\, d\tau' \quad \Rightarrow \quad h(t) = \frac{d}{dt}\, g(t)$$

PROPRIETÀ DEI SLS

INTEGRAZIONE

$$w(t) \triangleq \int_{-\infty}^{t} x(\alpha)\, d\alpha = x(t) \otimes u(t)$$

$$z(t) = w(t) \otimes h(t) = x(t) \otimes u(t) \otimes h(t) =$$

$$= u(t) \otimes \left[x(t) \otimes h(t)\right] = y(t) \otimes u(t) = \int_{-\infty}^{t} y(\alpha)\, d\alpha$$

·) DERIVAZIONE

$$w(t) = \frac{d}{dt} x(t) \quad \Rightarrow \quad x(t) = \int_{-\infty}^{t} w(\alpha)\, d\alpha$$

$z(t) = ?$

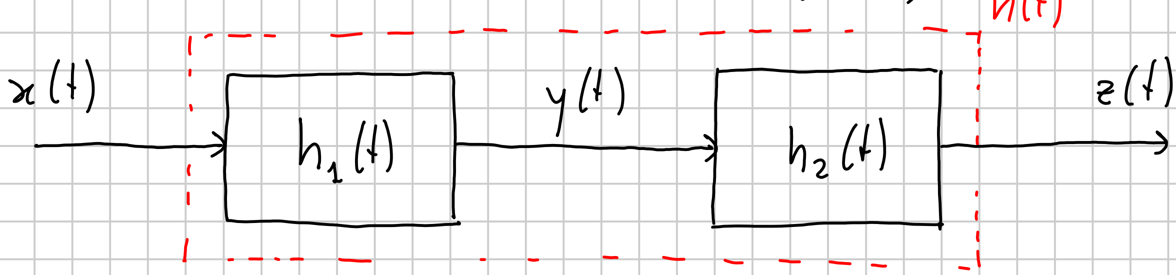
$\Downarrow$ per la prop di int.

$$y(t) = \int_{-\infty}^{t} z(\alpha)\, d\alpha$$

$$\Downarrow$$

$$\boxed{z(t) = \frac{d}{dt} y(t)}$$

SISTEMI IN CASCATA (SERIE)

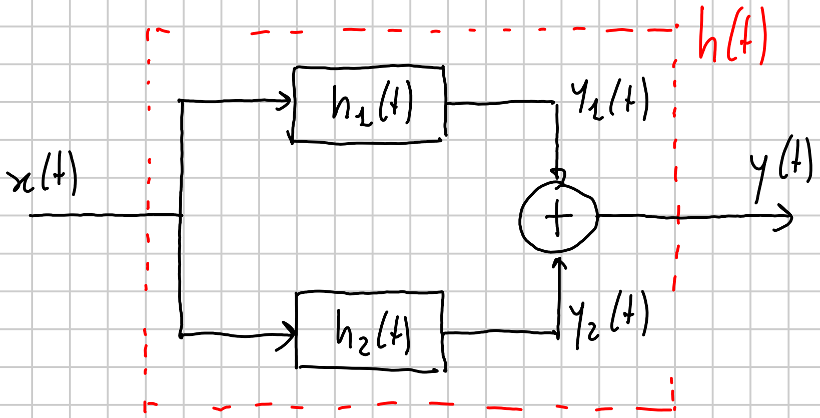


$$z(t) = y(t) \otimes h_2(t) = \underbrace{\left[x(t) \otimes h_1(t)\right]}_{y(t)} \otimes h_2(t)$$

$$= x(t) \otimes \underbrace{\left[h_1(t) \otimes h_2(t)\right]}_{h(t)} = x(t) \otimes h(t)$$

$$h(t) \triangleq h_1(t) \otimes h_2(t) \quad \Rightarrow \quad H(f) = H_1(f)\, H_2(f)$$

SISTEMI IN PARALLELO



$$y(t) = y_1(t) + y_2(t) = x(t) \otimes h_1(t) + x(t) \otimes h_2(t)$$

$$= x(t) \otimes \left[h_1(t) + h_2(t) \right] = x(t) \otimes h(t)$$

$$h(t) \triangleq h_1(t) + h_2(t)$$

$$H(f) = H_1(f) + H_2(f)$$

FILTRI IDEALI

DECIBEL

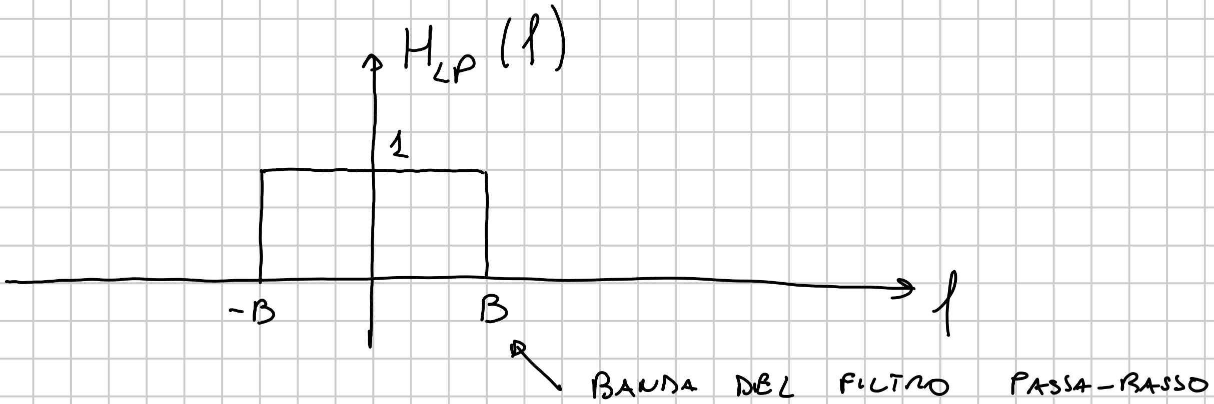
$$dB \triangleq 10 \log_{10} P \qquad \text{(per potenze)}$$

$$dB \triangleq 20 \log_{10} A \qquad \text{(per ampiezze)}$$

$$P = 1000 \Rightarrow dB = 10 \log_{10} 10^3 = 30$$

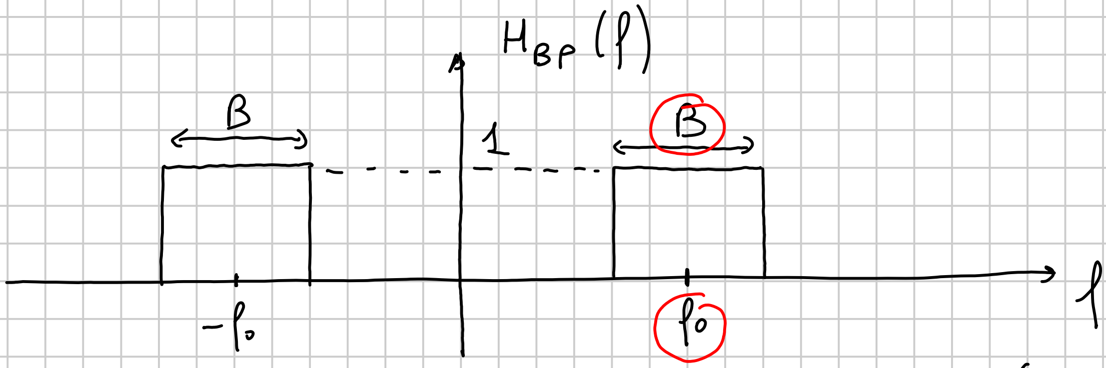
$$A = \sqrt{1000} \Rightarrow dB = 20 \log_{10} 10^{3/2} = 30$$

·) FILTRO PASSA-BASSO (LOW-PASS)



$$H_{LP}(f) \triangleq \mathrm{rect}\left(\frac{f}{2B}\right) \Rightarrow h_{LP}(t) = 2B\,\mathrm{sinc}(2Bt)$$

·) FILTRO PASSA-BANDA (BAND-PASS)



$$H_{BP}(f) = \mathrm{rect}\left(\frac{f-f_0}{B}\right) + \mathrm{rect}\left(\frac{f+f_0}{B}\right) \qquad \left(f_0 > \frac{B}{2}\right)$$

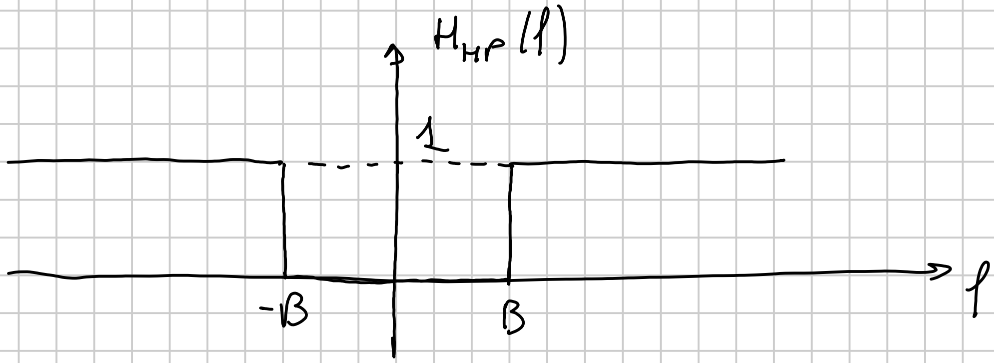
$$h_{BP}(t) = B\,\mathrm{sinc}(Bt)\,e^{+j2\pi f_0 t} + B\,\mathrm{sinc}\,Bt\;e^{-j2\pi f_0 t}$$

$$= 2B\,\mathrm{sinc}(Bt)\cos(2\pi f_0 t)$$

solitamente $f_0 >> B$

fattore $Q \triangleq \dfrac{f_0}{B}$

·) FILTRO PASSA - ALTO (HIGH - PASS)

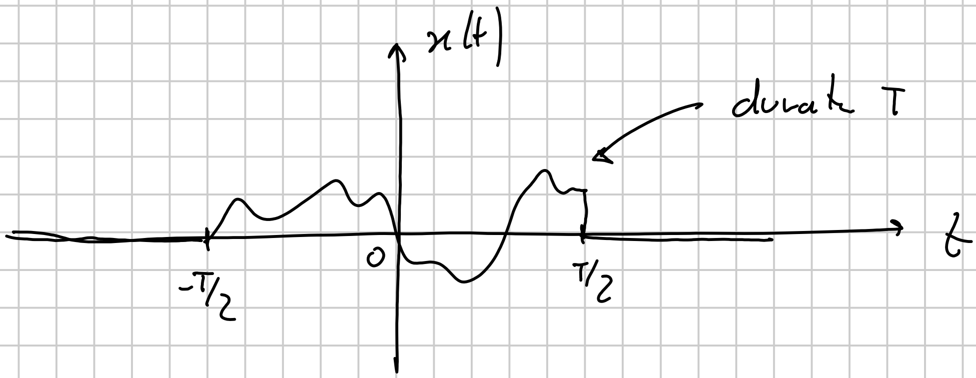


$$H_{HP}(f) \triangleq 1 - \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$h_{HP}(t) = \delta(f) - 2B\,\mathrm{sinc}(2Bt)$$

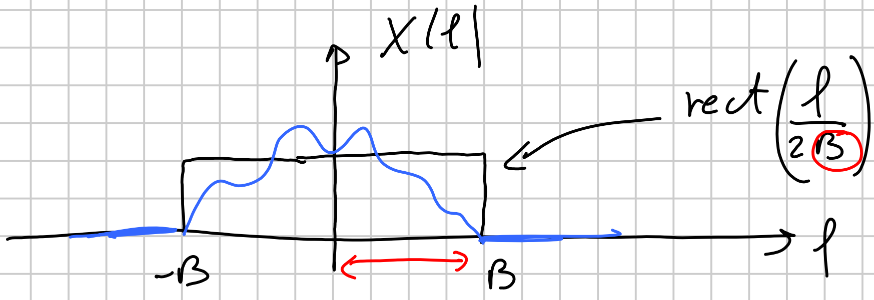
DURATA E BANDA DI UN SEGNALE

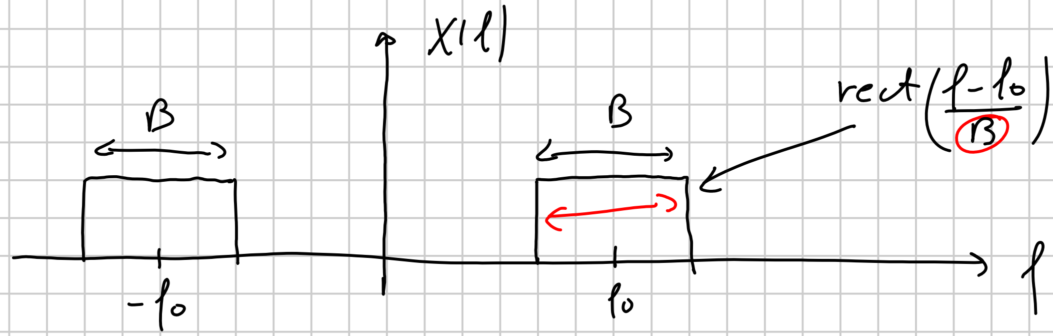
DURATA: l'intervallo temporale all'interno del quale un segnale $x(t)$ è diverso da zero



·) BANDA RIGOROSAMENTE LIMITATA

B = intervallo di frequenze (misurato sul semiasse positivo delle frequenze) per cui $X(f) \neq 0$





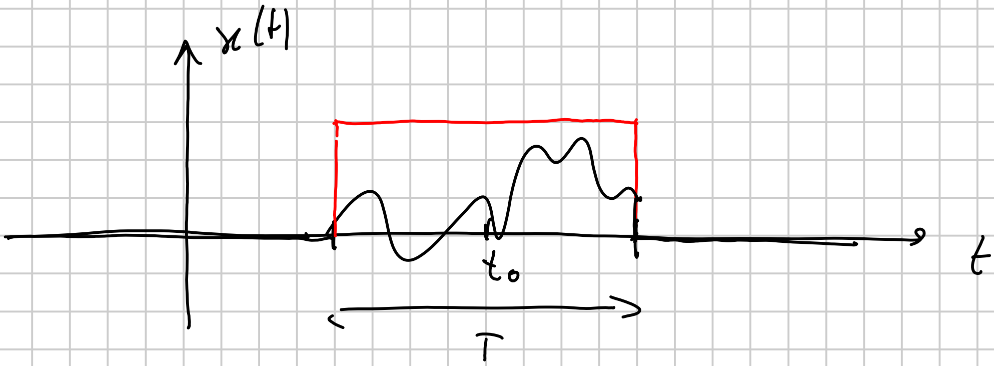
PROPRIETÀ

i) $\begin{cases} x(t) \text{ ha durata limitata} \\ \Downarrow \\ X(f) \text{ ha banda infinita} \end{cases}$

ii) $\begin{cases} X(f) \text{ ha banda rig. lim.} \\ \Downarrow \\ x(t) \text{ ha durata infinita} \end{cases}$

Dim.

$$x(t) = x(t) \cdot \text{rect}\left(\frac{t-t_0}{T}\right)$$

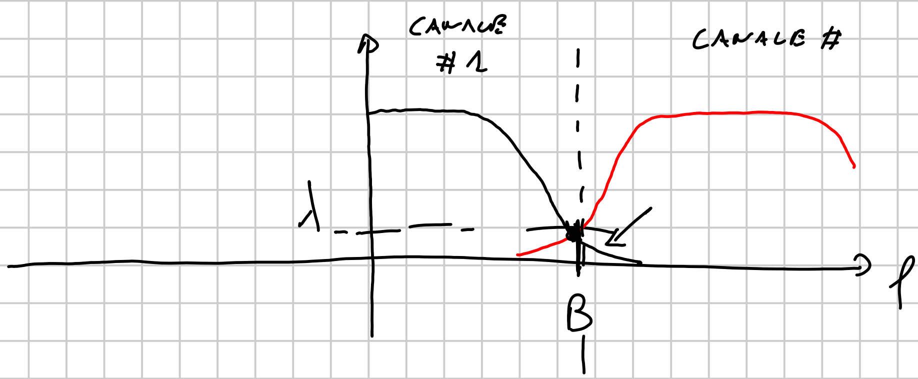


$$X(f) = TCF\left[x(t)\right] = X(f) \otimes T\,\text{sinc}(fT)\, e^{-j2\pi f t_0}$$

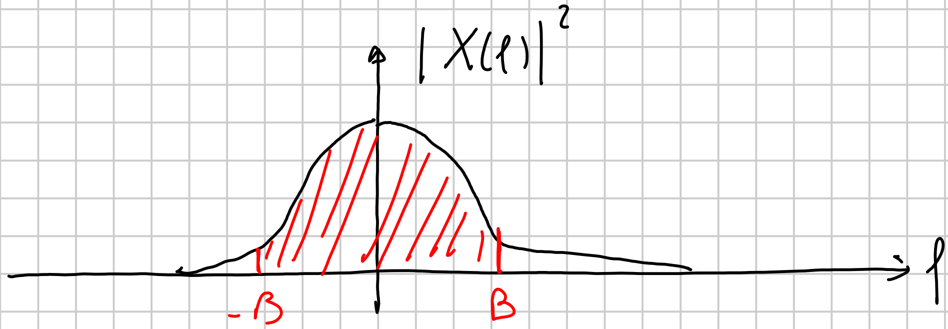
$$|X(f)| = \left|X(f) \otimes T\,\text{sinc}(fT)\, e^{-j2\pi f t_0}\right|$$

il risultato della convoluzione tra uno spettro a banda infinita (sinc) e lo spettro di un segnale arbitrario è necessariamente illimitato (diverso da zero su tutto l'asse delle frequenze)

PROBLEMA: BANDA INFINITA A FRONTE DI SEGNALI DI DURATA LIMITATA



- DURATA E BANDA AL 99% DELL'ENERGIA



$$\int_{-B}^{B} |X(f)|^2 \, df = E_X^{(B)}$$

$$B_{99} = B : \frac{\int_{-B}^{B} |X(f)|^2 \, df}{\int_{-\infty}^{+\infty} |X(f)|^2 \, df} = 0{,}99 = \frac{E_x^{(B)}}{E_x}$$

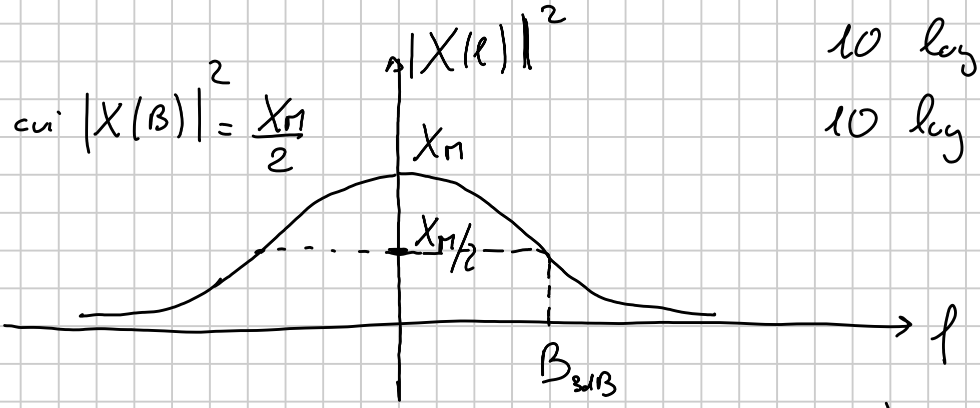
$$D_{99} = D : \frac{\int_{-D}^{D} |x(t)|^2 \, dt}{\int_{-\infty}^{+\infty} |x(t)|^2 \, dt} = 0{,}99 = \frac{E_x^{(D)}}{E_x}$$

-) BANDA E DURATA "a -3 dB"

$10 \log 2 \cong 3\,dB$

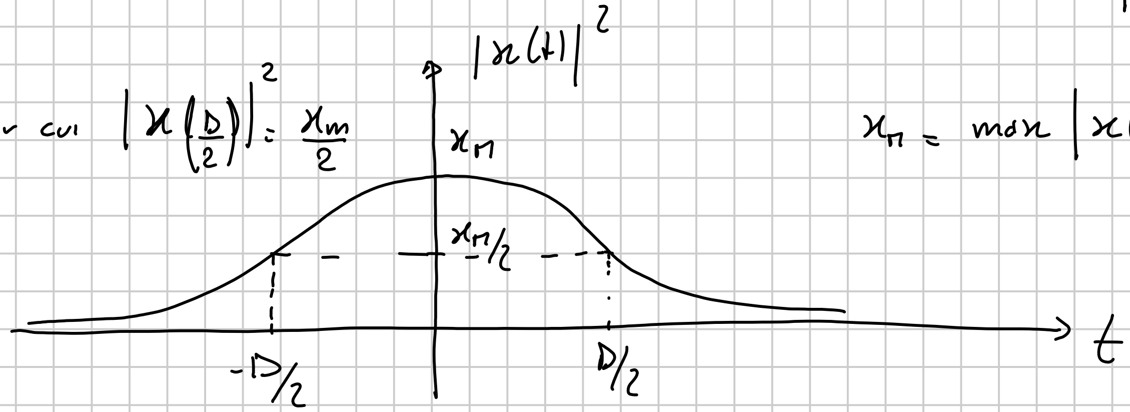
$10 \log \frac{1}{2} \cong -3\,dB$

B_{3dB}: B per cui $|X(B)|^2 = \frac{X_M}{2}$



$X_M = \max |X(f)|^2$

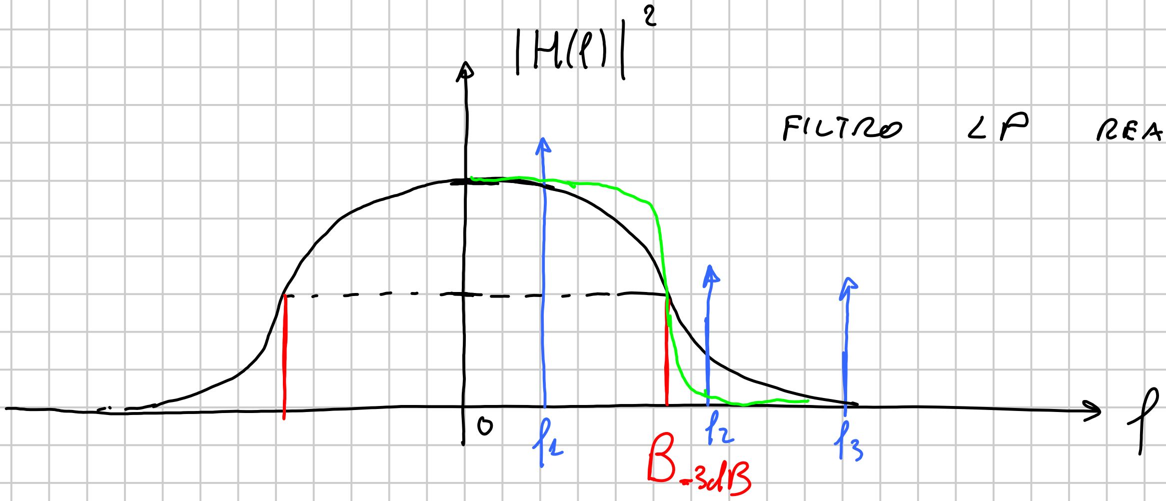
D_{3dB}: D per cui $\left|x\left(\frac{D}{2}\right)\right|^2 = \frac{x_M}{2}$



$x_M = \max |x(t)|^2$

21/03/2013

BANDA DI UN FILTRO



DISTORSIONI LINEARI

·) Risposta fedele (replica fedele di un segnale)

$$y(t) = K\,x(t-t_0)$$

$$Y(f) = K\,X(f)\,e^{-j2\pi f t_0}$$

$$|Y(f)| = |K|\,|X(f)|$$

$$\varphi(f) = \angle X(f) - 2\pi t_0 f + \angle K$$

$$\varphi(f) = \underbrace{-2\pi t_0}_{a} \cdot f$$

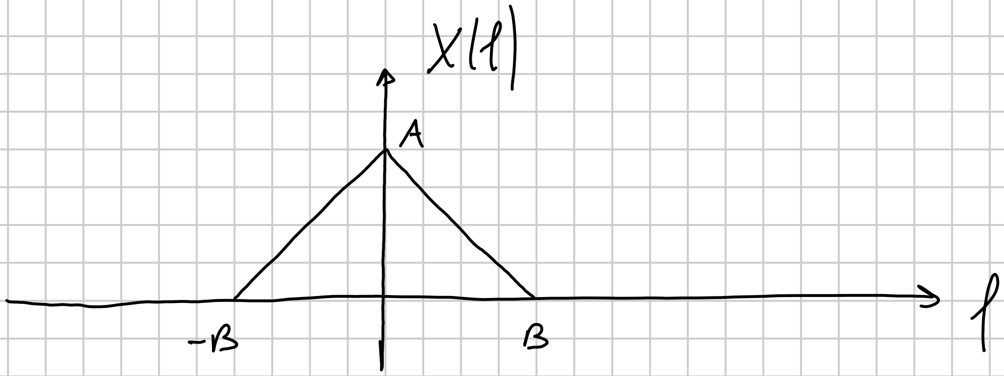
$$\Rightarrow h(t) = K\,\delta(t-t_0)$$

$$\Rightarrow H(f) = K\,e^{-j2\pi f t_0}$$

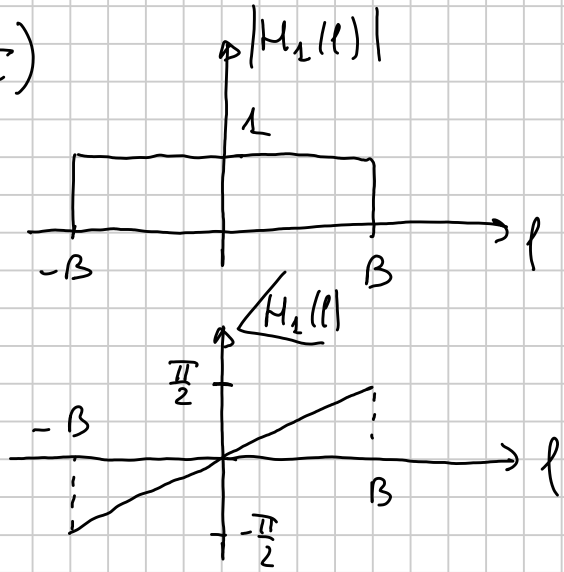
$$|H(f)| = |K|$$

$$y(t) = x(t) \otimes K\,\delta(t-t_0) = K\,x(t-t_0)$$

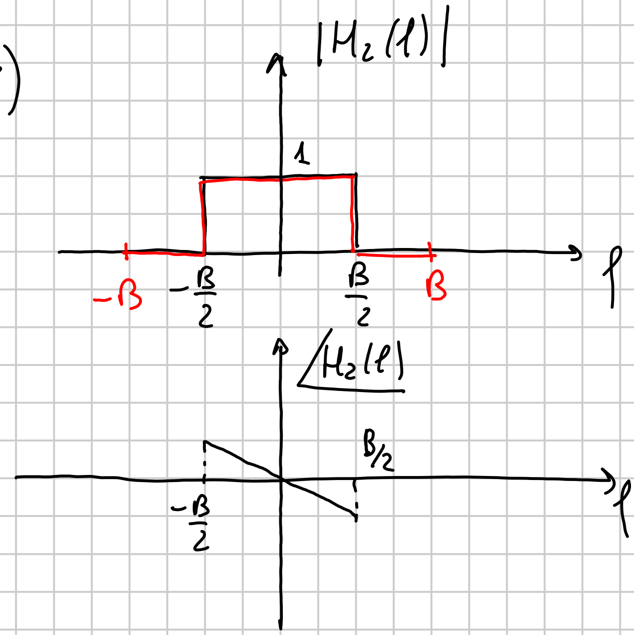
ESEMPI



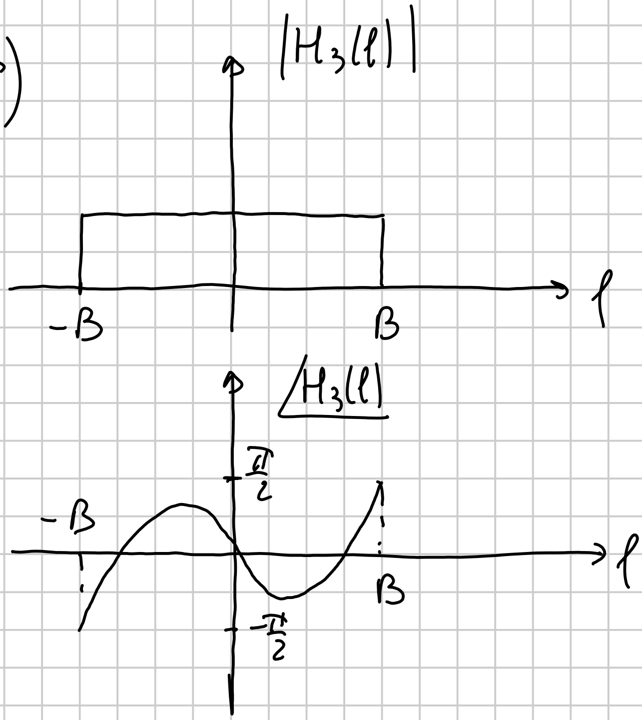
I)



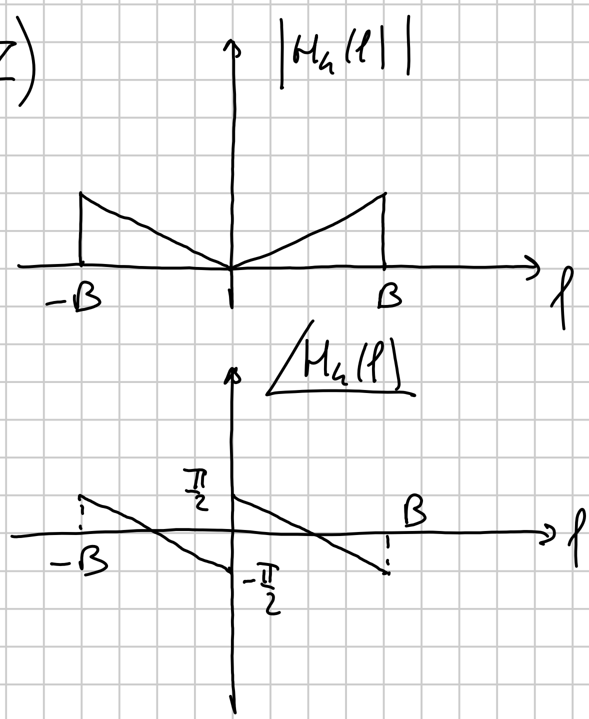
II)



III)



IV)



I)

$$Y_1(f) = X(f) \cdot H_1(f)$$

$$|Y_1(f)| = |X(f)|\,|H_1(f)| = \overset{=1}{K}\,|X(f)|$$ NON HO DIST. SUL MODULO

$$\angle Y_1(f) = \angle X(f) + \angle H_1(f) = 0 + \frac{\pi}{2B} f = \overset{a}{\frac{\pi}{2B}} f$$

$$\frac{\pi}{2B}\cancel{f} = -2\cancel{\pi}\cancel{f}t_0 \Rightarrow t_0 = -\frac{1}{4B}$$

$$y(t) = x\left(t + \frac{1}{4B}\right)$$

II) $|Y_2(f)| = |X(f)| \cdot |H_2(f)| \neq |K||X(f)|$ DIST. DI AMPIEZZA

⇓

NON È UNA REPLICA FEDELE

$$\angle Y_2(f) = \angle H_2(f)$$

III) $|Y_3(f)| = |X(f)|$ NON INTRODUCE DIST. DI AMP.

$\angle Y_3(f) = \angle H_3(f)$ INTRODUCE DIST. DI FASE

non è lineare

⇓

NON È UNA REPLICA FEDELE

IV) $|Y_4(f)| \neq |K||X(f)|$ INTRODUCE DIST. DI AMP.

$\angle Y_4(f) \neq \angle X(f) - j2\pi f t_0$ INTRODUCE DIST. DI FASE

⇓

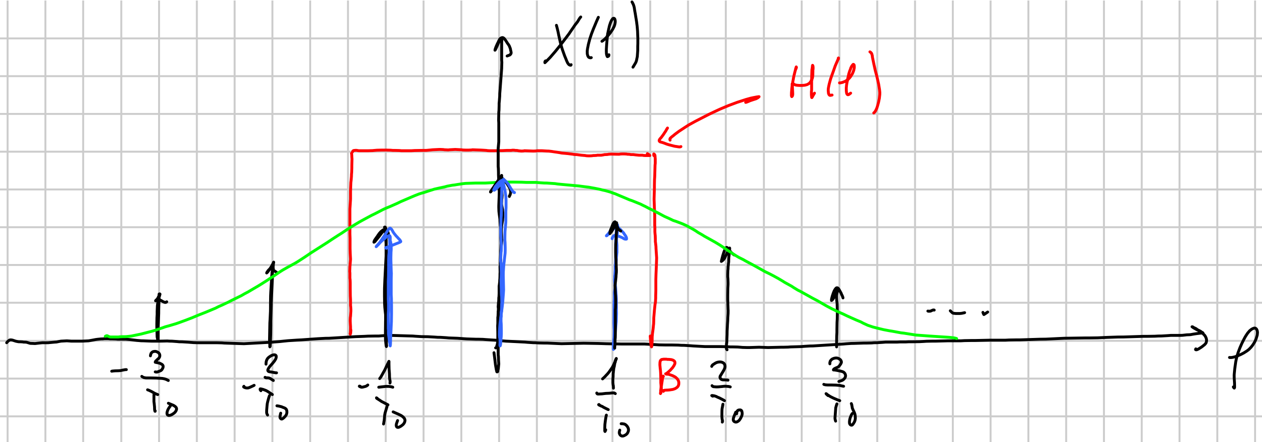
NON È UNA REPLICA FEDELE

FILTRAGGIO DI SEGNALI PERIODICI

$$y(t) = x(t) \otimes h(t)$$

$$Y(f) = X(f)\, H(f)$$

TCF di segnali sia aperiodici che periodici



$$X(f) = \sum_{n=-\infty}^{+\infty} X_n \, \delta\left(f - \frac{n}{T_0}\right)$$

$$Y(f) = \sum_{n=-\infty}^{+\infty} Y_n \, \delta\left(f - \frac{n}{T_0}\right)$$

FILTRI LINEARI E STAZIONARI E ANALISI ENERGETICA

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$C_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt$$

$$C_y(\tau) = \int_{-\infty}^{+\infty} y(t)\, y^*(t-\tau)\, dt$$

$$\int_{-\infty}^{+\infty} x(\alpha)\, h(t-\alpha)\, d\alpha \qquad\qquad \int_{-\infty}^{+\infty} \cdots\cdots d\beta$$

$$C_y(\tau) = C_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

$$S_x(f) = TCF\left[C_x(\tau)\right], \quad S_y(f) = TCF\left[C_y(\tau)\right]$$

$$S_y(\tau) = |Y(f)|^2 = |X(f)|^2 |H(f)|^2 =$$

$$= S_x(f)\, H(f)\, H^*(f) \qquad h(t) \text{ reale}$$

per simm. hermitiana

$$C_y(\tau) = C_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

$$E_y = \int_{-\infty}^{+\infty} |Y(f)|^2 \, df = \int_{-\infty}^{+\infty} S_y(f)\, df = \int_{-\infty}^{+\infty} S_x(f) |H(f)|^2 \, df$$

$$= C_y(0) = C_x(\tau) \otimes h(\tau) \otimes h(-\tau) \Big|_{\tau=0}$$

x SEGNALI PERIODICI

$$C_x(\tau) \triangleq \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, x^*(t-\tau)\, dt$$

$$C_y(\tau) = C_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

x SEGNALI APERIODICI A POTENZA FINITA

$$C_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, x^*(t-\tau)\, dt$$

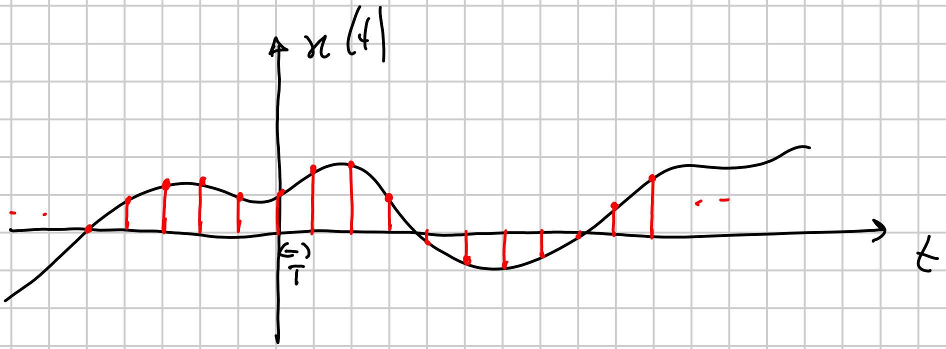
$$C_y(\tau) = C_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

25/03/2013

SEGNALI CAMPIONATI

$\underline{x(t)} \xrightarrow{T} \underline{x[n] = x(nT)}$

$x(t)$ è un segnale analogico
$x[n]$ è una sequenza



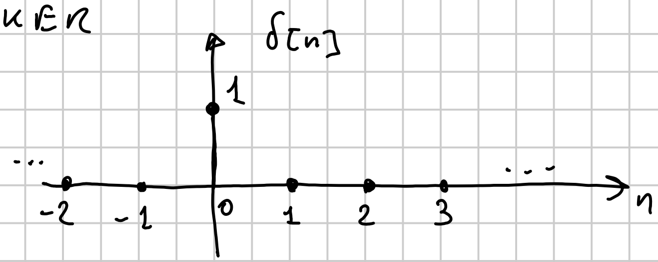
$T \triangleq$ intervallo di campionamento

$f_c \triangleq \frac{1}{T}$ freq. di campionamento

SEQUENZE NOTEVOLI

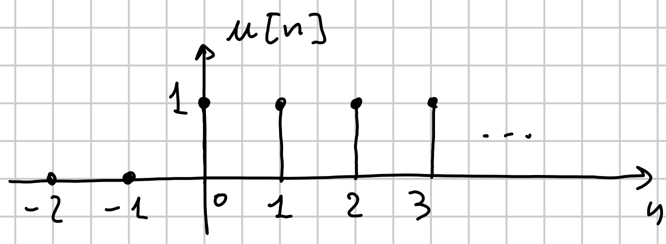
·) $\delta[n]$ DELTA DI KRONECKER

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{altrove} \end{cases}$$

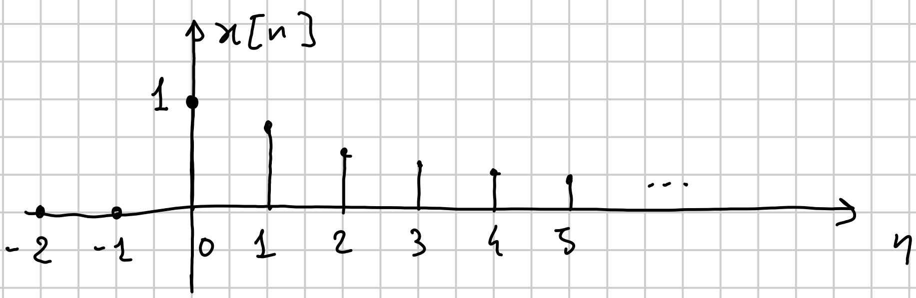


·) $u[n]$ GRADINO

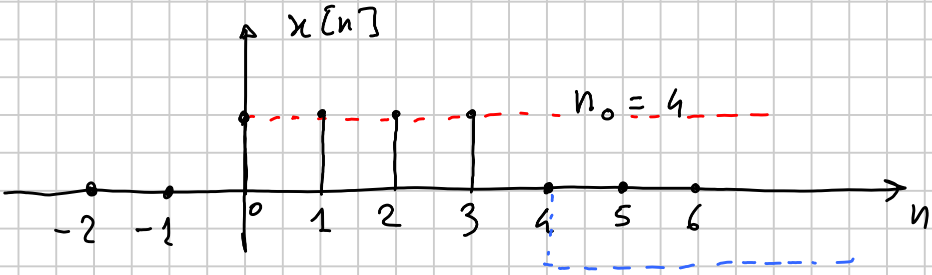
$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{altrove} \end{cases}$$



·) $x[n] = a^n \, u[n]$, $0 < a < 1$ (esponenziale)



·) $x[n] = u[n] - u[n-n_0]$ RETTANGOLO

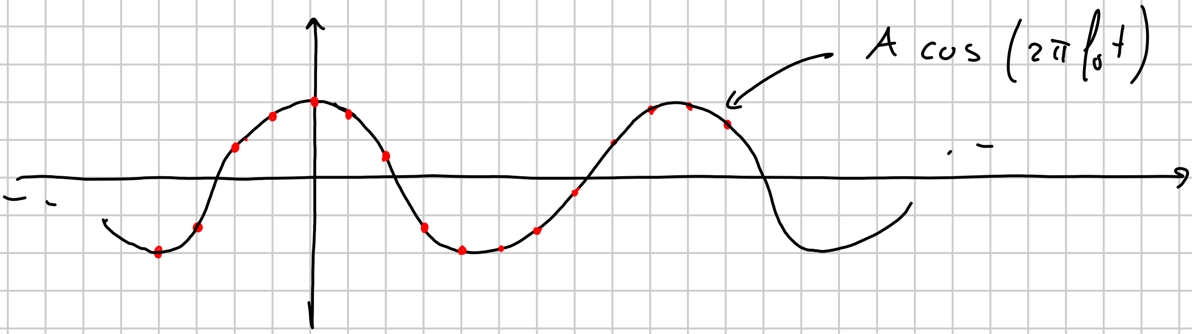


·) $x[n] = e^{j2\pi F_0 n}$ OSCILLAZIONE DISCRETA

SEQUENZA PERIODICA

$x[n] = x[n - kN_0]$, N_0 è il periodo

$k \in \mathbb{Z}$

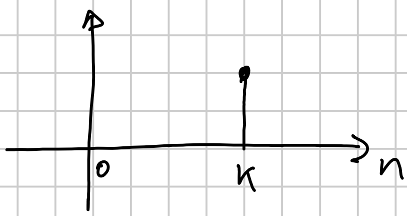


$x[n]$ è periodico se e solo se $F_0 = \frac{p}{q}$, $p, q \in \mathbb{Z}$

$F_0 \in \mathbb{Q}$

Proprietà

·) $\delta[n-k]$



·) $\delta[n] = u[n] - u[n-1]$

·) $u[n] = \sum_{k=0}^{+\infty} \delta[n-k]$

TRASFORMATA DI FOURIER DI UNA SEQUENZA

$$x[n] \overset{TFS}{\Longleftrightarrow} \bar{X}(f)$$

$$\boxed{\bar{X}(f) \triangleq \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi n f T}}$$

TRASFORMATA DI FOURIER DI UNA SEQUENZA

PROPRIETÀ

$$\bar{X}(f) = \bar{X}\left(f - \frac{K}{T}\right) \quad , \quad K \in \mathbb{Z}$$

Dim

$$\bar{X}\left(f - \frac{K}{T}\right) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi n\left(f - \frac{K}{T}\right)T} =$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi n f T} \cdot e^{j2\pi n k} \; = 1$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi n f T} = \bar{X}(f)$$

ANTITRASFORMATA DI FOURIER DI UNA SEQUENZA

$$x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, e^{j2\pi n f T}\, df$$

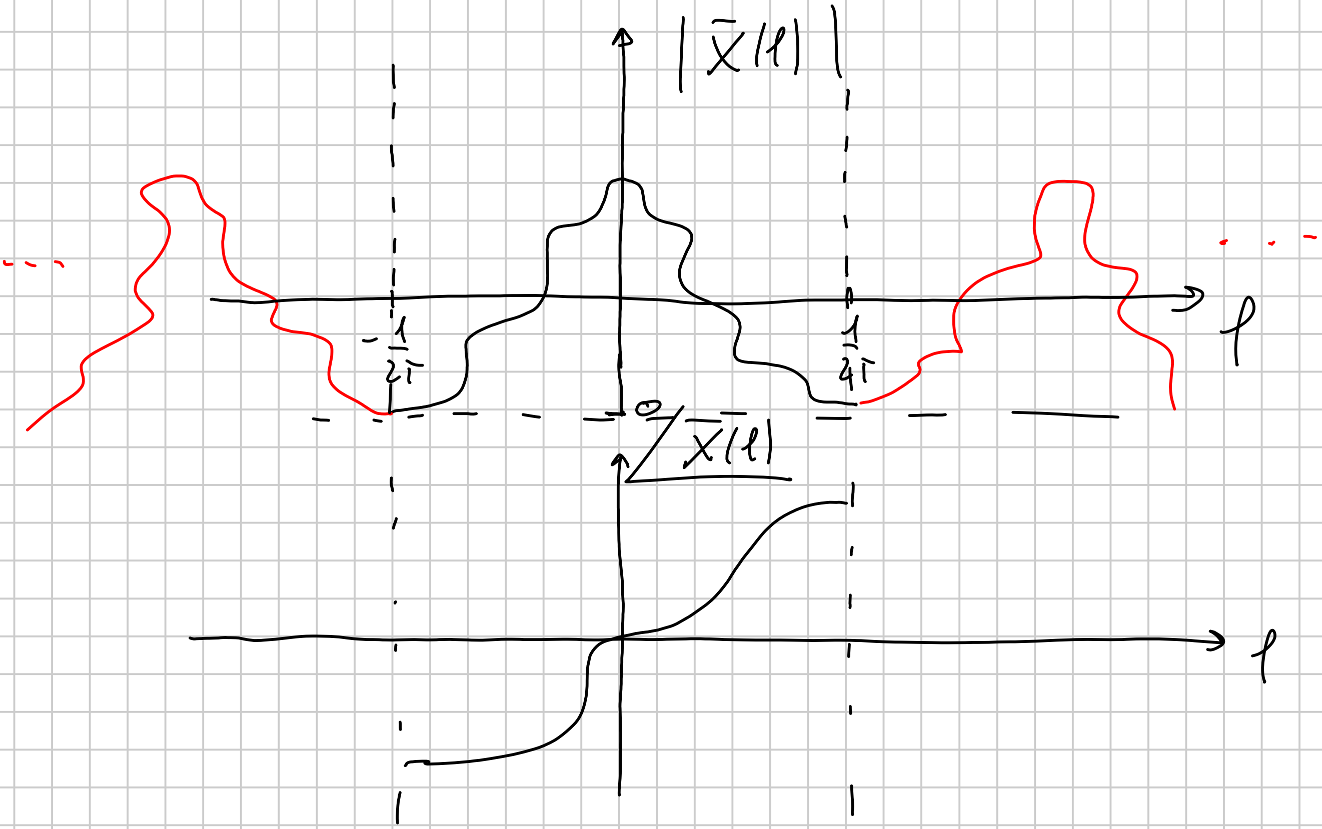
Dim.

$$T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi k f T}}_{\bar{X}(f)} e^{j2\pi n f T}\, df =$$

$$= T \sum_{k=-\infty}^{+\infty} x[k] \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{j2\pi(n-k) f T}\, df$$

$$= \cancel{T} \sum_{k=-\infty}^{+\infty} x[k] \frac{1}{\cancel{T}} \delta[k-n] = x[n] \qquad \text{c.v.d.}$$

$$TFS\left[x[n]\right] \Rightarrow \text{SPETTRO DI } x[n]$$



-) ESEMPIO : DUE OSCILLAZIONI DISCRETE

$$x_1[n] = e^{j2\pi F_0 nT}$$

$$x_2[n] = e^{j2\pi \underbrace{\left(F_0 + \frac{k}{T}\right)}_{F_1} nT} \quad , \; k \in \mathbb{Z}$$

$$x_2[n] = e^{j2\pi F_0 nT} \underbrace{e^{j2\pi kn}}_{=1} = x_1[n]$$

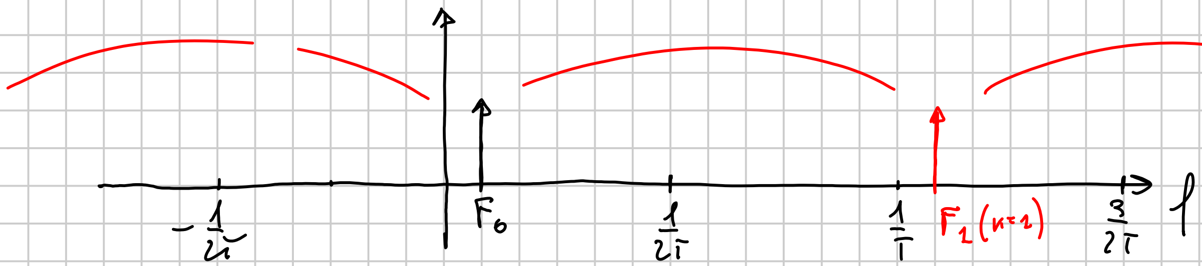
$$\bar{X}_1(f) = \sum_{n=-\infty}^{+\infty} e^{j2\pi F_0 nT} e^{-j2\pi f nT} = \sum_{n=-\infty}^{+\infty} e^{-j2\pi(f-F_0)nT}$$

$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} \delta\left(f - F_0 - \frac{n}{T}\right)$$

$$\bar{X}_2(f) = \sum_{n=-\infty}^{+\infty} e^{j2\pi\left(F_0+\frac{K}{T}\right)nT} e^{-j2\pi f nT} = \sum_{n=-\infty}^{+\infty} e^{-j2\pi\left(f-F_0-\frac{K}{T}\right)nT}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{+\infty}\delta\left(f-F_0-\frac{K}{T}-\frac{n}{T}\right) = \dots \left(K+n=n'\right)$$

$$= \frac{1}{T}\sum_{n'=-\infty}^{+\infty}\delta\left(f-F_0-\frac{n'}{T}\right) = \bar{X}_1(f)$$



CONDIZIONE SUFFICIENTE PER L'ESISTENZA DELLA TFS

$x[n]$ ammette TFS se

$$\sum_n |x[n]| < \infty$$

Dim

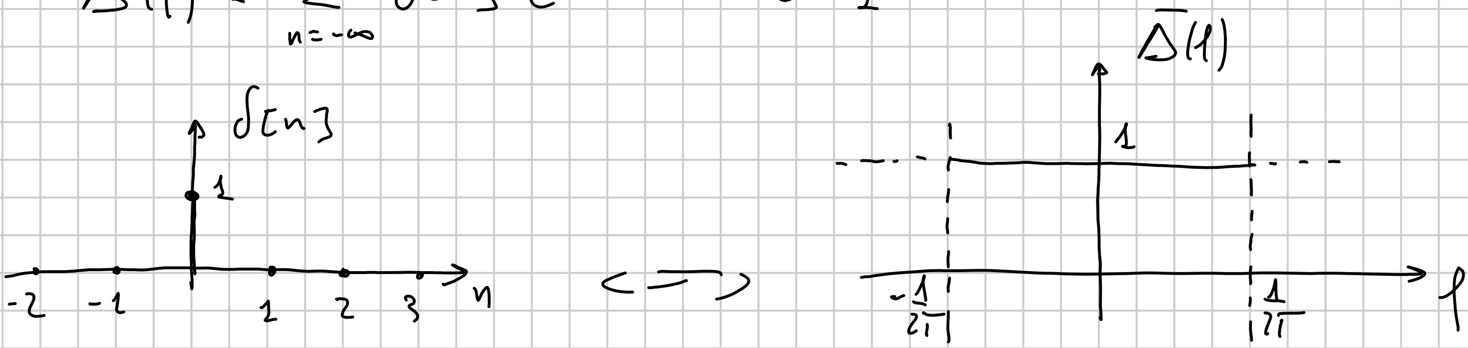
$$|\bar{X}(f)| = \left|\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f nT}\right| \leq \sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$

$\forall f$

c.v.d.

TFS DELLA $\delta[n]$

$$\bar{\Delta}(f) = \sum_{n=-\infty}^{+\infty} \delta[n]\, e^{-j2\pi n f T} = 1$$



RELAZIONE TRA $\bar{X}(f)$ E LA $X(f)$

$$\bar{X}(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi n f T} = \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j2\pi n f T}$$

$$= \sum_{n=-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} X(\alpha)\, e^{j2\pi\alpha \underbrace{nT}_{t=nT}}\, d\alpha}_{x(nT) = x(t)\big|_{t=nT}}\; e^{-j2\pi n f T}$$

$$= \int_{-\infty}^{+\infty} X(\alpha) \sum_{n=-\infty}^{+\infty} e^{-j2\pi(f-\alpha)nT}\, d\alpha =$$

$$= \int_{-\infty}^{+\infty} X(\alpha)\, \frac{1}{T} \sum_{n=-\infty}^{+\infty} \delta\left(f-\alpha-\frac{n}{T}\right) d\alpha =$$

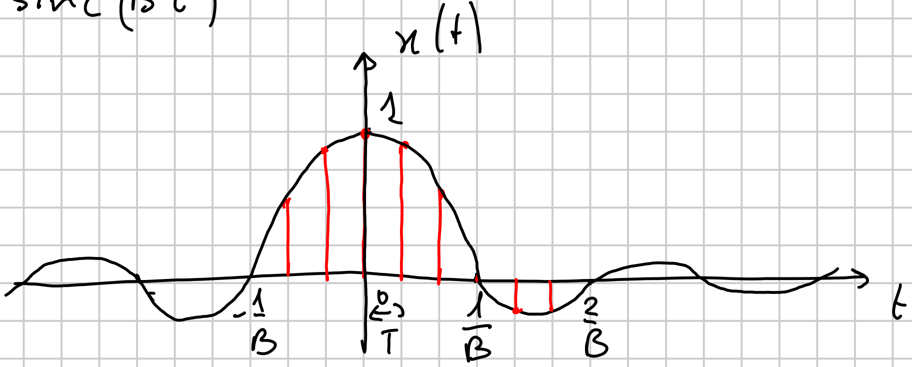
$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\alpha)\, \delta\Big[\alpha - \underbrace{\left(f-\frac{n}{T}\right)}_{f'}\Big]\, d\alpha =$$

$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\Big(\underbrace{f-\frac{n}{T}}_{f'}\Big)$$

$$\Rightarrow \boxed{\bar{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f-\frac{n}{T}\right)}$$

Esempio

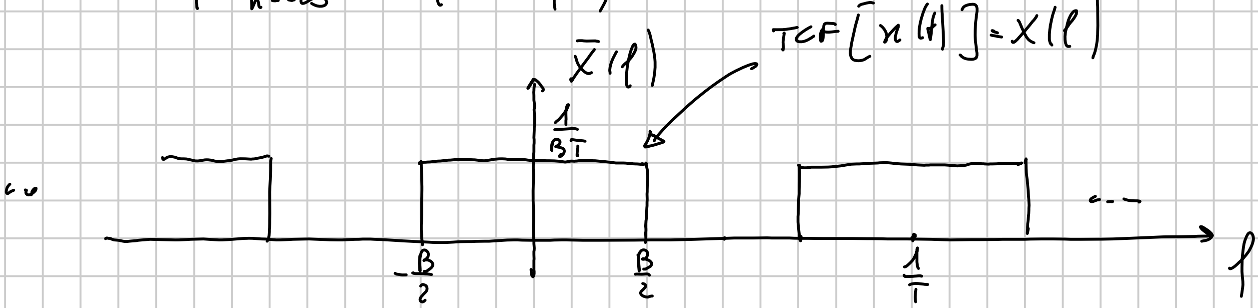
$x(t) = \text{sinc}(Bt)$



$x[n] = x(nT)$

$\bar{X}(f) = TFS\left[x[n] \right]$

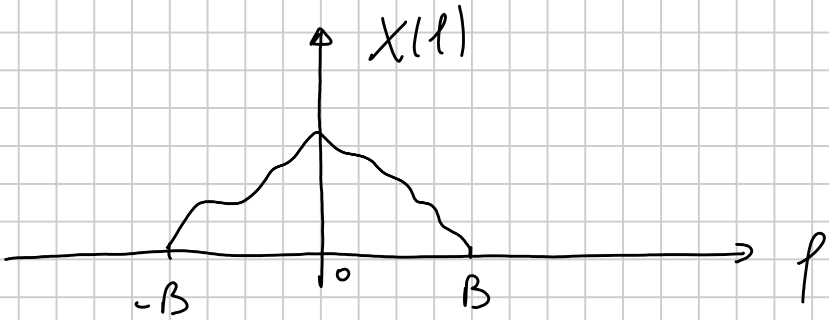
$$\bar{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right)$$



$$X(f) = \frac{1}{B} \text{rect}\left(\frac{f}{B}\right) \Rightarrow \bar{X}(f) = \frac{1}{BT} \sum_{n=-\infty}^{+\infty} \text{rect}\left(\frac{f - n/T}{B}\right)$$

CONDIZIONE DI NYQUIST

Si applica di segnali a banda rigorosamente limitata

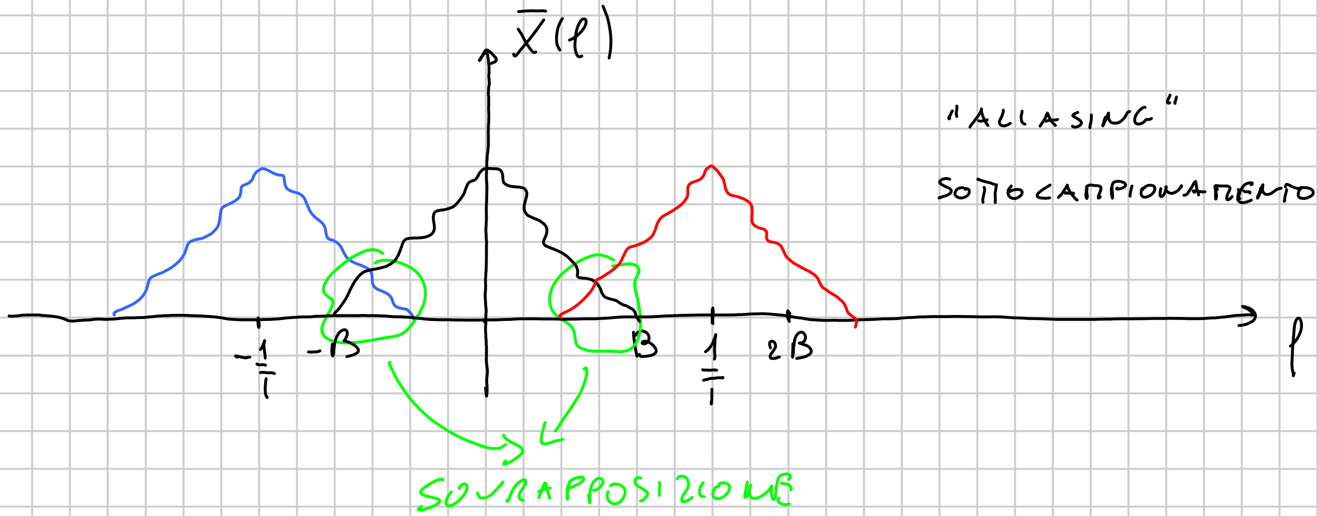


$$x(t) = x[n] = x(nT)$$

$$\bar{X}(f) = ?$$

I° caso) $\quad T > \frac{1}{2B} \Rightarrow \frac{1}{T} < 2B$

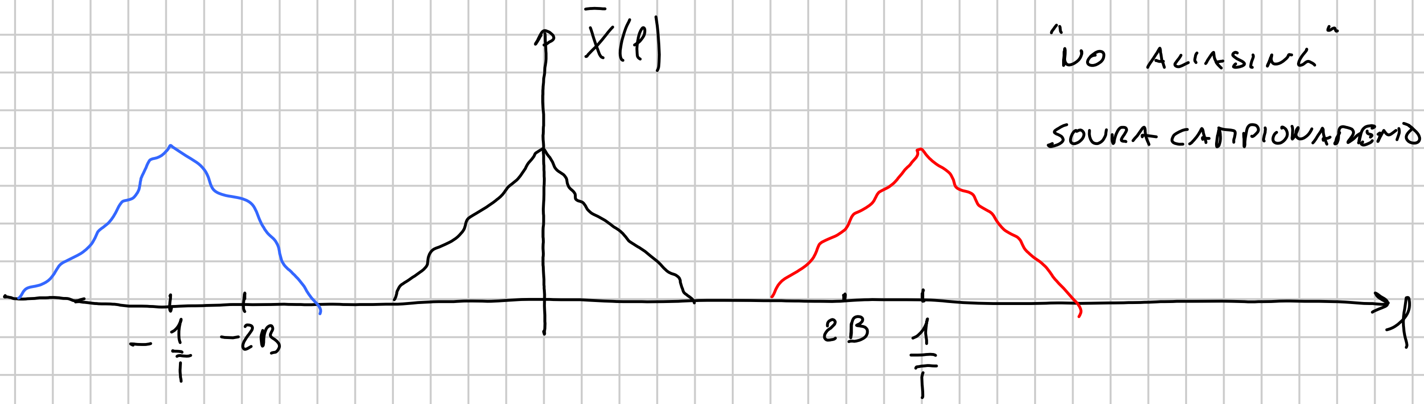
$$\bar{X}(f) = \frac{1}{T} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T}\right)$$



"ALIASING"

SOTTOCAMPIONAMENTO

II° caso $\quad T < \frac{1}{2B} \Rightarrow \frac{1}{T} > 2B$



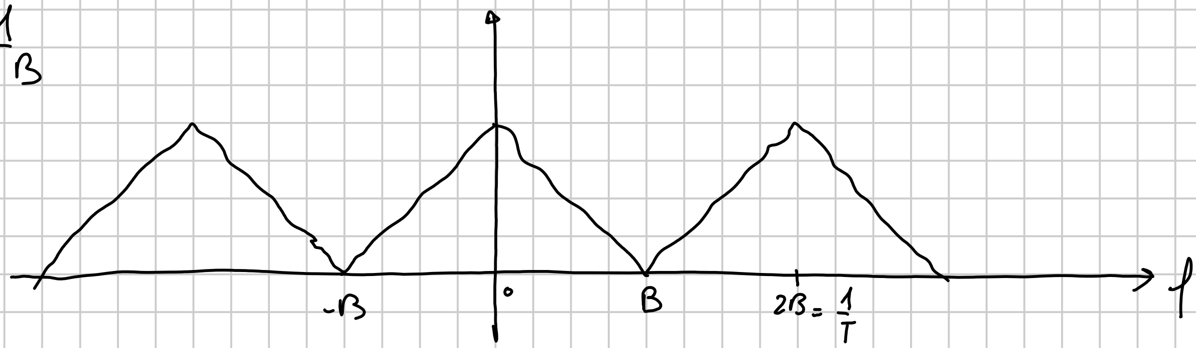
"NO ALIASING"

SOVRACAMPIONAMENTO

CONDIZIONE DI NYQUIST

Un segnale è campionato rispettando la cond. di Nyquist se $\quad T \leq \frac{1}{2B} \Rightarrow \frac{1}{T} \geq 2B$

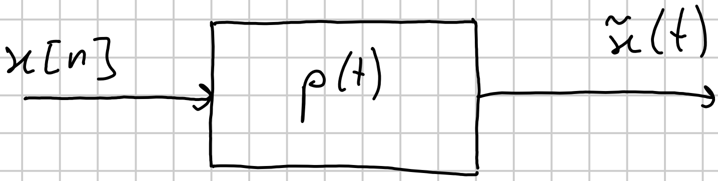
$$T = \frac{1}{2B}$$



$\frac{1}{T} = f_N = 2B$ freq di Nyquist

$$\boxed{\text{COND. DI NYQUIST} \Rightarrow f_c \geq 2B}$$

·) INTERPOLATORE



$x[n]$ seq. di ingresso

$p(t)$ funzione interpolatrice

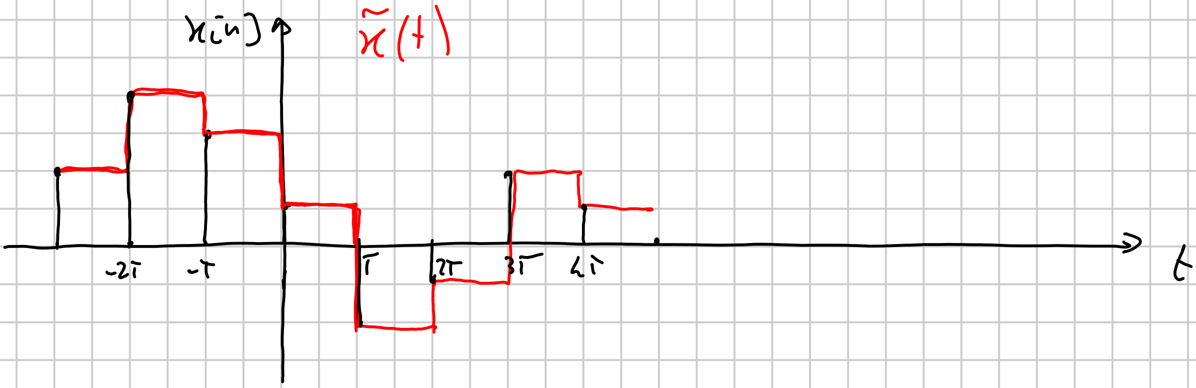
$$\boxed{\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT)}$$

N.B. $p(t)$ non è la risp. imp. di un SLS !!
L'interpolatore non è un filtro !!

INTERPOLATORE A MANTENIMENTO

$$p(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right)$$

$$\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x[n]\, \mathrm{rect}\left(\frac{t - T/2 - nT}{T}\right)$$



$$\mathrm{TCF}\left[\tilde{x}(t)\right] = \int_{-\infty}^{+\infty} \tilde{x}(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{+\infty} \underbrace{\sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT)}_{\tilde{x}(t)}\, e^{-j2\pi f t}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \int_{-\infty}^{+\infty} p(t-nT)\, e^{-j2\pi f t}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, P(f)\, e^{-j2\pi f nT} = \underbrace{\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f nT}}_{\bar{X}(f)} P(f)$$

$$\boxed{\tilde{X}(f) = \bar{X}(f)\; P(f)}$$

$\tilde{X}(f)$: $\mathrm{TCF}\left[\tilde{x}(t)\right]$ — $\bar{X}(f)$: $\mathrm{TFS}\left[x[n]\right]$ — $P(f)$: $\mathrm{TCF}\left[p(t)\right]$

$$\begin{array}{ccccc} x[n] & & p(t) & & \tilde{x}(t) \\ \downarrow \mathrm{TFS} & & \downarrow \mathrm{TCF} & & \uparrow \mathrm{TCF}^{-1} \\ \bar{X}(f) & \cdot & P(f) & = & \tilde{X}(f) \end{array}$$

$$x[n] \longrightarrow \boxed{p(t)} \longrightarrow \tilde{x}(t) \qquad \tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT)$$

$$\Updownarrow$$

$$\bar{X}(f) \longrightarrow \boxed{P(f)} \longrightarrow \tilde{X}(f) \qquad \tilde{X}(f) = \bar{X}(f)\, P(f)$$

TEOREMA DEL CAMPIONAMENTO (TEOREMA DI SHANNON)

·) $x(t)$ sia a b. r. l.

$x(t)$ può essere ricostruito dai suoi campioni se e solo se

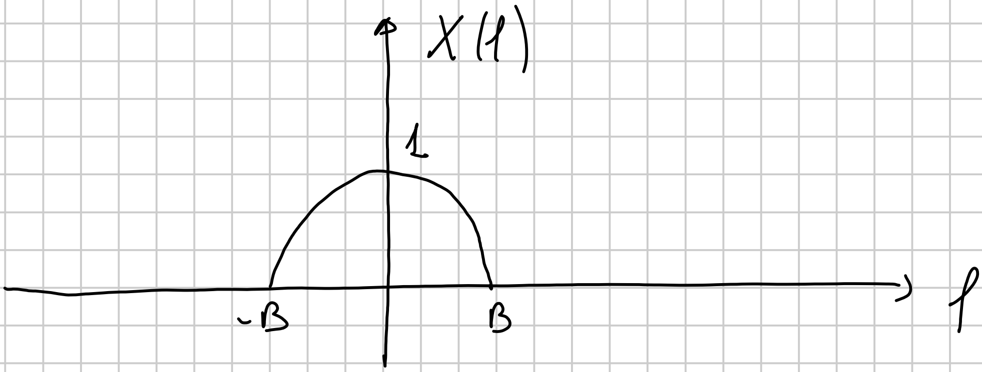
j) è soddisfatta la condizione di Nyquist

jj) scelgo $p(t) = 2B \operatorname{sinc}(2Bt)$ (INTERP. CARDINALE)

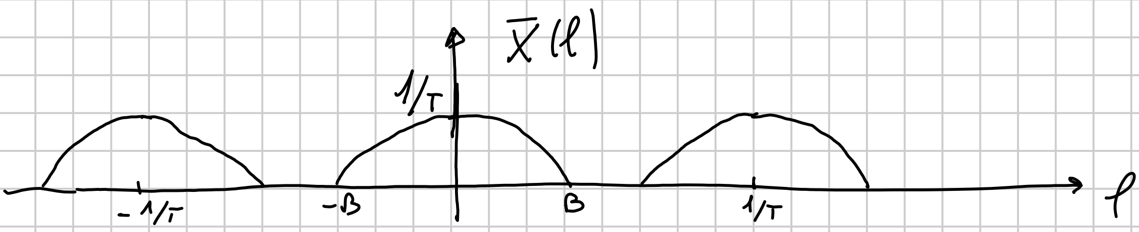
$$\Downarrow$$

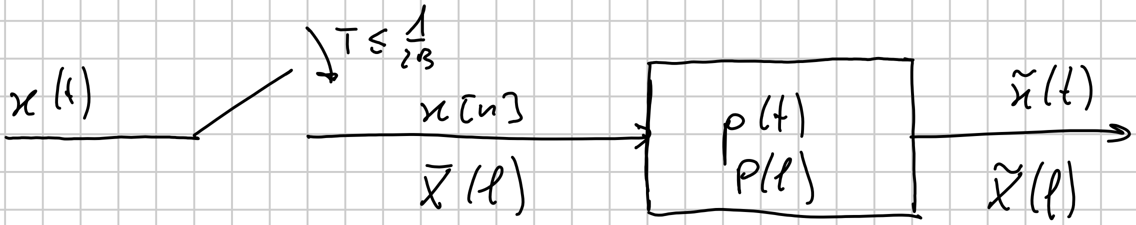
$$P(f) = \operatorname{rect}\left(\frac{f}{2B}\right)$$

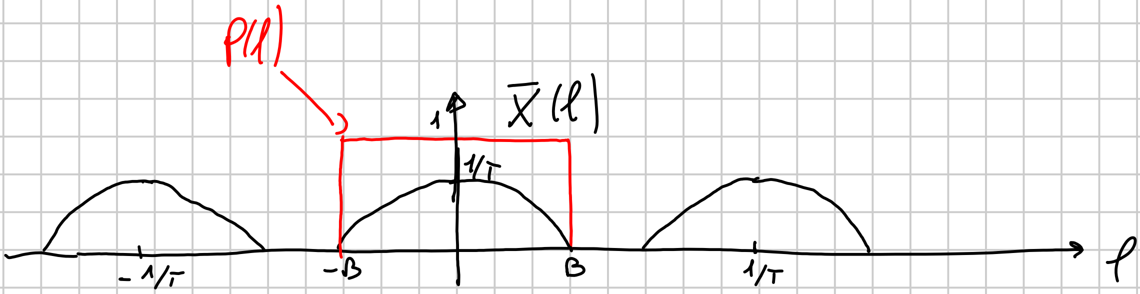
Dimostrazione (x via grafica)

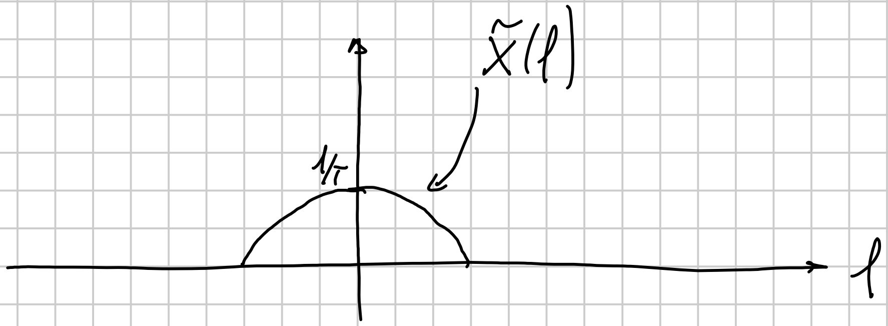


$x[n] = x(nT)$, $T \leq \frac{1}{2B}$ (COND. DI NYQUIST)





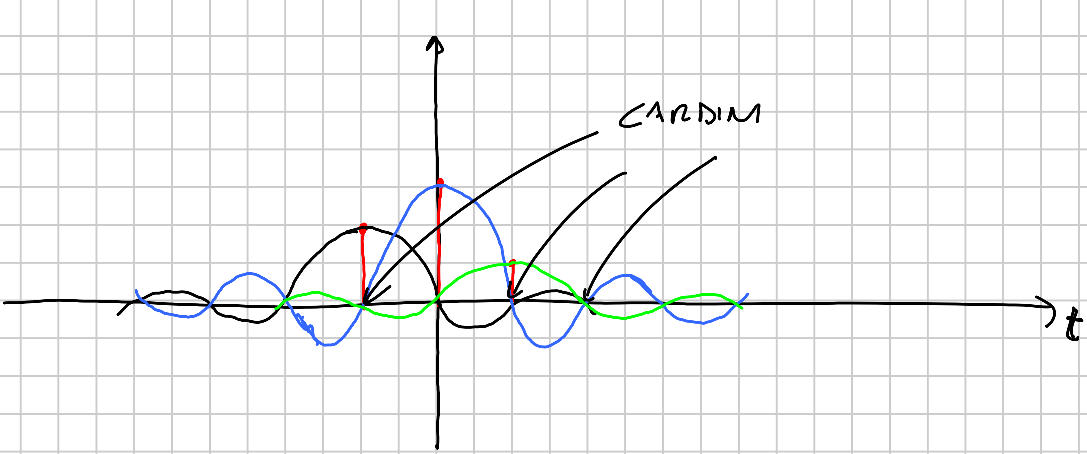




$$\tilde{X}(f) = \frac{1}{T} X(f) \implies \tilde{x}(t) = \frac{1}{T} x(t)$$

$$P'(f) = T\,P(f) \implies \tilde{x}(t) = x(t)$$

$$\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot 2B\,\mathrm{sinc}(2Bt) = \frac{1}{T} x(t)$$



CORRELAZIONE TRA SEQUENZE

$$C_{xy}[k] \triangleq \sum_n x[n]\, y^*[n-k]$$ CORRELAZIONE TRA SEQUENZE

$$C_x[k] \triangleq \sum_n x[n]\, x^*[n-k]$$ AUTOCORRELAZIONE

DENSITÀ SPETTRALE DI ENERGIA DI UNA SEQUENZA

$$S_x(f) = TFS\left[C_x[k]\right] =$$

$$= \sum_k C_x[k]\, e^{-j2\pi k f T} = \sum_k \underbrace{\sum_n x[n]\, x^*[n-k]}_{C_x[k]}\, e^{-j2\pi k f T}$$

$$= \sum_n x[n] \sum_k x^*[n-k]\, e^{-j2\pi k f T} = \quad (n-k = k')$$

$$= \sum_n x[n] \sum_{k'} x^*[k']\, e^{-j2\pi(n-k')fT} =$$

$$= \sum_n x[n] \sum_{k'} x^*[k']\, e^{j2\pi k' f T} \cdot e^{-j2\pi n f T}$$

$$= \sum_n x[n]\, e^{-j2\pi n f T} \cdot \left[\sum_{k'} x[k']\, e^{-j2\pi k' f T}\right]^*$$

$$= \bar{X}(f)\, \bar{X}^*(f) = \boxed{\left|\bar{X}(f)\right|^2 = S_x(f) = TFS\left[C_x[k]\right]}$$

TEOREMA DI PARSEVAL

$$\sum_n x[n]\, y^*[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, \bar{Y}^*(f)\, df$$

Dim.

$$\sum_n x[n]\, y^*[n] = \sum_n T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, e^{j2\pi n f T}\, df\; y^*[n]$$

$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f) \sum_n y^*[n]\, e^{j2\pi n f T}\, df$$

$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f) \left[\sum_n y[n]\, e^{-j2\pi n f T} \right]^* df$$

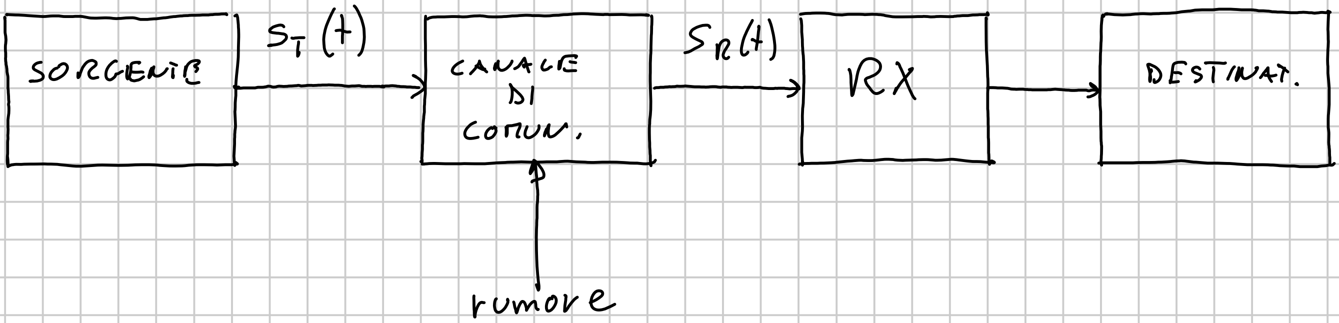
$$= T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, \bar{Y}^*(f) \qquad \text{c.v.d.}$$

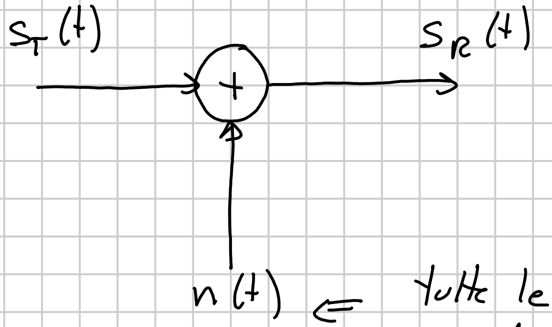
$$\Rightarrow \quad \sum_n x[n]\, x^*[n] = \sum_n |x[n]|^2 \triangleq E_x$$

$$E_x = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} |\bar{X}(f)|^2\, df$$

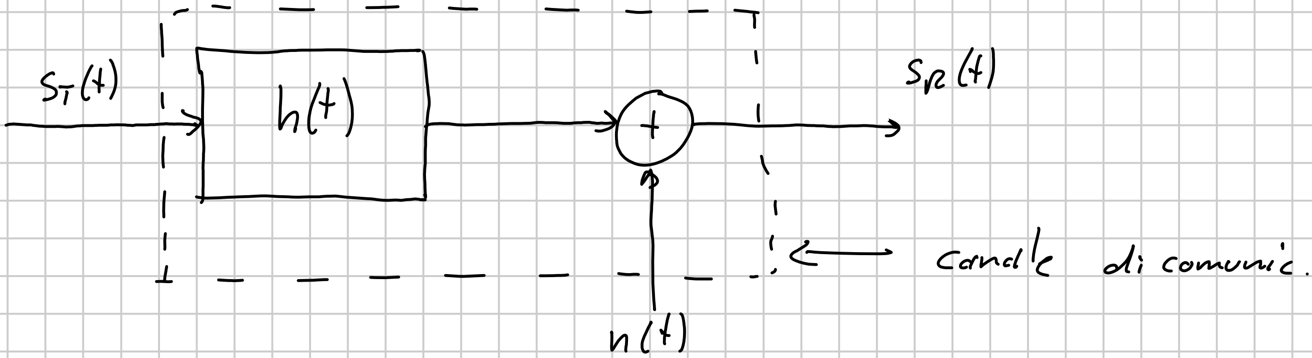
08/04/2013

RUMORE





$\Leftarrow$ Tutte le sorgenti di rumore vengono considerate in questo termine



ELEMENTI DI PROBABILITÀ

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$$

DEFINIZIONE DI EVENTO

- se A è un evento, allora anche $\bar{A}$, rispetto a Ω, è un evento
- se A e B sono eventi, anche l'unione $A \cup B$ è un evento

PROPRIETÀ

·) $A \cap B$ è un evento

·) $A \cup \bar{A} = \Omega$ evento certo

·) $A \cap \bar{A} = \phi$ evento impossibile

CARATTERIZZAZIONE DI UN ESPERIMENTO CASUALE

j) Spazio campione Ω

jj) Definizione e proprietà degli eventi

jjj) Definizione di una legge di probabilità

DEFINIZIONE DI PROBABILITÀ ASSIOMATICA (KOLMOGOROV)

j) La probabilità di un evento A è $P\{A\} \geq 0$

jj) La prob. di Ω $P\{\Omega\} = 1$

jjj) A, B mutuamente esclusivi allora

$$P\{A \cup B\} = P\{A\} + P\{B\} \qquad (A \cap B = \phi)$$

PROPRIETÀ

·) $P\{\bar{A}\} = 1 - P\{A\}$

·) $P\{\phi\} = 0$

·) $0 \leq P\{A\} \leq 1$

·) $P\{A \cup B\} = P\{A\} + P\{B\} - P\{A \cap B\}$

NOTAZIONE

$A \cup B \Rightarrow A + B$

$A \cap B \rightarrow AB$

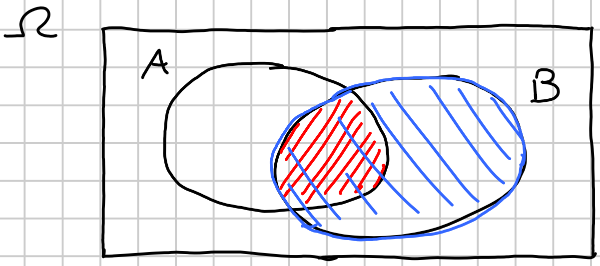
$P\{AB\}$ probabilità congiunta

$P\{A\}, P\{B\}$ probabilità marginali

PROBABILITÀ CONDIZIONATA

$$P\{A|B\} \triangleq \frac{P\{AB\}}{P\{B\}}$$

probabilità dell'evento A condizionata all'evento B



DEFINIZIONE CLASSICA (PASCAL)

$$P\{A\} \triangleq \frac{N_F(A)}{N_P}$$

$N_F(A) \triangleq$ nr. di casi favorevoli ad A

$N_P \triangleq$ nr. di casi possibili

Vale solo quando tutti i risultati dell'esperimento si verificano con la stessa probabilità

DEFINIZIONE DI PROBABILITÀ DI VON MISES (FREQUENTISTA)

$$P\{A\} = \lim_{N \to \infty} \frac{N_A}{N},$$

N_A = nr. di risultati favorevoli ad A

N = nr. di prove dell'esperimento

j) $P\{A\} \geq 0$

jj) $P\{\Omega\} = 1$

iii) $P\{A \cup B\} = \lim_{N \to \infty} \frac{N_A + N_B}{N} = \underbrace{\lim_{N \to \infty} \frac{N_A}{N}}_{P\{A\}} + \underbrace{\lim_{N \to \infty} \frac{N_B}{N}}_{P\{B\}}$

INDIPENDENZA TRA EVENTI

A, B sono indipendenti se

$$P\{A\} = P\{A|B\}$$

$$P\{A\} = P\{A|B\} = \frac{P\{AB\}}{P\{B\}}$$

$$\Downarrow$$

$$P\{AB\} = P\{A\} \cdot P\{B\}$$

TEOREMA DI BAYES

$$\boxed{P\{A|B\} = \frac{P\{B|A\}\, P\{A\}}{P\{B\}}} \qquad P\{A\},\ P\{B\} \neq 0$$

$$\Downarrow$$

$$P\{B|A\} = \frac{P\{A|B\}\, P\{B\}}{P\{A\}}$$

Dimostrazione:

$$P\{AB\} = P\{BA\}$$

$$P\{AB\} = P\{A|B\}\, P\{B\}$$

$$P\{BA\} = P\{B|A\}\, P\{A\}$$

$$\Downarrow$$

cvd

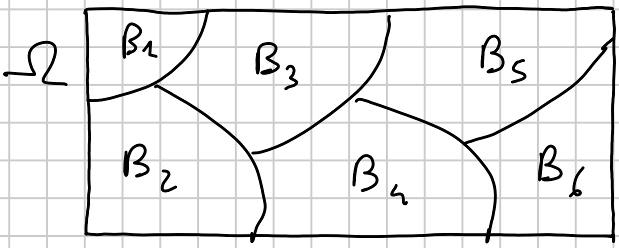
$$P\{A|B\}\, P\{B\} = P\{B|A\}\, P\{A\} \Rightarrow P\{A|B\} = \frac{P\{B|A\}\, P\{A\}}{P\{B\}}$$

PARTIZIONE DI UNO SPAZIO CAMPIONE

$B_i \quad , \quad i = 1, \dots, N$

·) $B_i B_j = \phi = B_i \cap B_j \quad \forall\, i,j$ con $i \neq j$

·) $\sum_{i=1}^{N} B_i = B_1 + B_2 + \dots + B_N = B_1 \cup B_2 \cup \dots \cup B_N = \Omega$



TEOREMA DELLA PROBABILITÀ TOTALE

Se B_i , $i = 1, \dots, N$, sono una partizione di Ω

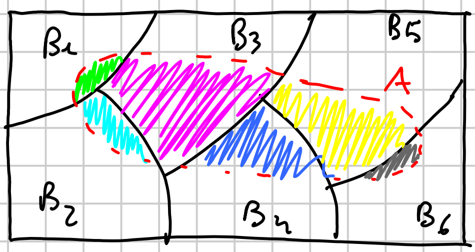
$$\boxed{P\{A\} = \sum_{i=1}^{N} P\{A \mid B_i\} P\{B_i\}}$$

Dim.

$$P\{A\} = P\{A\Omega\} = P\left\{A \sum_{i=1}^{N} B_i\right\} = P\left\{\sum_{i=1}^{N} AB_i\right\}$$

$$= \sum_{i=1}^{N} P\{AB_i\} = \sum_{i=1}^{N} P\{A \mid B_i\} P\{B_i\}$$

c.v.d.



ESPERIMENTO ALEATORIO COMPOSTO

SPAZIO CAMPIONE : PRODOTTO CARTESIANO DEGLI SPAZI CAMPIONE DEI SINGOLI ESPERIMENTI

$$\Omega = \Omega_1 \times \Omega_2 \times \Omega_3 \times \dots \times \Omega_N$$

EVENTO DELL' ESP. COMP. : PRODOTTO CARTESIANO DI EVENTI DEI SINGOLI ESP.

$$A = A_1 \times A_2 \times A_3 \times \dots \times A_N$$

PROBABILITÀ DI UN EVENTO COMPOSTO

$$P\{A\} \neq P\{A_1\} \cdot P\{A_2\} \cdot \dots P\{A_N\}$$

NEL CASO IN CUI A_i SONO INDIP.

$$P\{A_i\} = P\{A_i | A_j\} \quad \forall i, j \quad i \neq j$$

$$P\{A\} = P\{A_1\} \cdot P\{A_2\} \cdot \dots P\{A_N\}$$

PROBLEMA DELLE PROVE RIPETUTE BINARIE E INDIPENDENTI

$$\Omega = \{ \omega_0, \omega_1 \}$$

$$P\{\omega_0\} = p$$

$$P\{\omega_1\} = 1 - P\{\omega_0\} = 1 - p$$

$$A = \{ \omega_0 \text{ si presenta } k \text{ volte su } n \}$$

$P\{A\} = ?$ Bernoulli

$$P\{A\} = \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \leq k \leq n$$

$$\binom{n}{k} \triangleq \frac{n!}{k!\,(n-k)!}$$

ESEMPIO (1)

1 moneta perfetta $\Rightarrow$ $P\{\text{"testa"}\} = P\{\text{"croce"}\} = 0{,}5$

1 moneta truccata $\Rightarrow$ $P\{\text{"testa"}\} = 0{,}8$, $P\{\text{"croce"}\} = 0{,}2$

Esperimento:

·) scelgo a caso una delle due monete

·) lancio della moneta per 10 volte , si osserva che:
5 volte esce "testa" , 5 volte esce "croce"

Domanda: calcolare la probabilità di aver scelto la moneta perfetta.

Soluzione

$A = \{\text{scelta della moneta perfetta}\}$

$B = \{\text{5 volte è uscita "testa", 5 volte è uscito "croce" su 10 lanci}\}$

$P\{A|B\} = ?$

$$P\{A|B\} = \frac{P\{B|A\}\,P\{A\}}{P\{B\}}$$

$P\{A\} = 0{,}5$

$$P\{B|A\} = \binom{10}{5} 0{,}5^5 \cdot 0{,}5^5 = \frac{10!}{5!\,5!}\, 0{,}5^5 \cdot 0{,}5^5 \cong 0{,}246$$

$P\{B\}$

$A = \{\text{scelta moneta perfetta}\}$

$C = \{\text{scelta moneta truccata}\}$

$A \cap C = \phi$, $A \cup C = \Omega$, A e C sono una partizione di Ω

$$P\{B\} = \underbrace{P\{B|A\}}_{\text{nota}}\,\underbrace{P\{A\}}_{\text{nota}} + \underbrace{P\{B|C\}}_{?}\,\underbrace{P\{C\}}_{\text{nota}}$$

$$P\{B|C\} = \binom{10}{5} 0{,}8^5 \cdot 0{,}2^5 = \frac{10!}{5!\,5!} 0{,}8^5 \cdot 0{,}2^5 \simeq 0{,}0264$$

$$P\{B\} = 0{,}246 \cdot 0{,}5 + 0{,}0264 \cdot 0{,}5 \simeq 0{,}136$$

$$P\{A|B\} = \frac{0{,}246 \cdot 0{,}5}{0{,}136} \simeq 0{,}903$$

ESEMPIO 2

Classe con 30 persone. Calcolare la prob. che almeno due persone siano nate nello stesso giorno.

$A = \{$ su 30 persone almeno 2 nate nello stesso giorno $\}$

$\bar{A} = \{$ su 30 persone tutte sono nate in giorni diversi $\}$

$$P\{A\} = 1 - P\{\bar{A}\}$$

$n = 2 \Rightarrow P\{\bar{A}_2\} = \frac{364}{365}$

$n = 3 \Rightarrow P\{\bar{A}_3\} = P\{\bar{A}_2\} \cdot \frac{363}{365} =$

$n = 4 \Rightarrow P\{\bar{A}_4\} = P\{\bar{A}_3\} \cdot \frac{362}{365} = P\{\bar{A}_2\} \cdot \frac{363}{365} \cdot \frac{362}{365}$

$$= \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365}$$

$3 < n < 30 \Rightarrow P\{\bar{A}_n\} = P\{\bar{A}_{n-1}\} \cdot \dfrac{365-(n-1)}{365}$

$$= \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-(n-1)}{365}$$

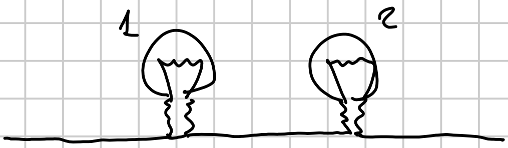
$$= \frac{(365-1)(365-2)\cdots(365-n+1)}{365^{n-1}} =$$

$$= \frac{365!}{(365-n)!\,365^{n-1}}$$

per $n = 30 \Rightarrow P\{\bar{A}\} \cong 0,27$

$\Rightarrow P\{A\} \cong 1 - 0,27 \cong 0,73$

PROBLEMA X CASA



p = prob. che una lampadina si guasti ad un certo istante

guasto di una lampadina è indipendente dal guasto dell'altra

Calcolare la probabilità che la stanza sia buia.

$A = \{$ stanza al buio $\}$

$A_1 = \{$ si guasti la lampad. 1 $\}$

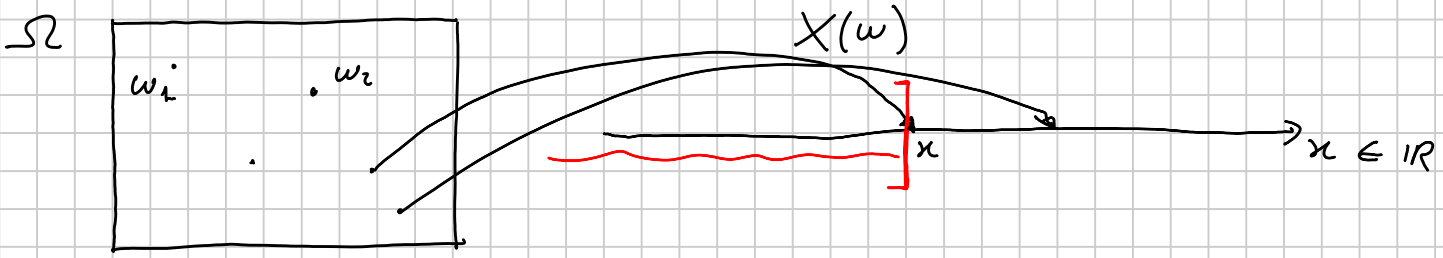
$A_2 = \{$ si guasti la lampad. 2 $\}$

$A = A_1 \bar{A}_2 + \bar{A}_1 A_2 + A_1 A_2$

$\bar{A} = \bar{A}_1 \bar{A}_2$

$$P\{A\} = 1 - P\{\bar{A}\}$$

VARIABILI ALEATORIE 10/04/2013



$$X, Y, Z$$

$$A = \{X \leq x\} \quad \text{evento}$$

$$P\{A\} = P\{X \leq x\} \quad \text{DISTRIBUZIONE DI PROBABILITÀ}$$

$$F_X(x) = P\{X \leq x\}$$

PROPRIETÀ

- $0 \leq F_X(x) \leq 1$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = 0$
- $x_2 > x_1 \Rightarrow F_X(x_2) \geq F_X(x_1)$ MONOTONA NON DECRESCENTE
- $\lim_{h \to 0^+} F_X(x+h) = F_X(x)$ CONTINUITÀ DA DESTRA
- Se ho una discontinuità in $x = \bar{x}$ di prima specie

$$P\{x = \bar{x}\} = K > 0 \qquad (K \leq 1)$$

- $P\{a < X \leq b\} = F_X(b) - F_X(a)$

VARIABILI ALEATORIE DISCRETE, CONTINUE, MISTE

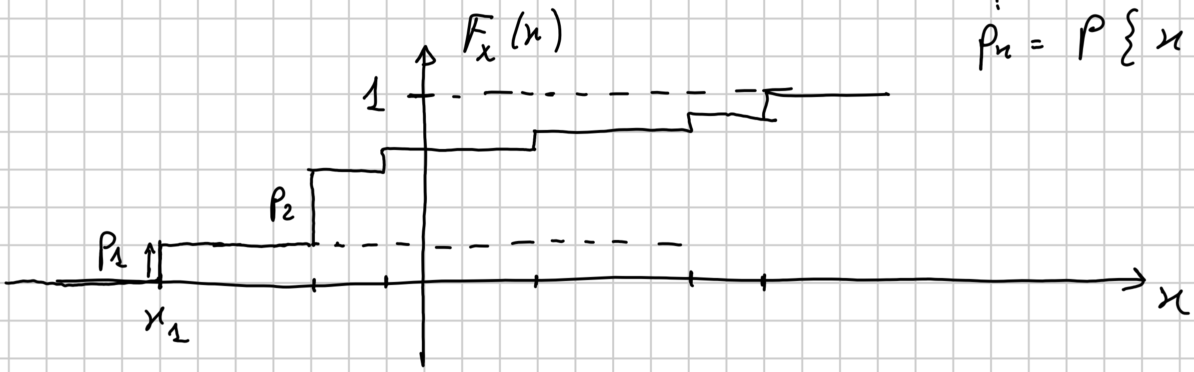
·) DISCRETE

$$F_X(x) = \sum_{k=1}^{N} p_k \, u(x - x_k)$$

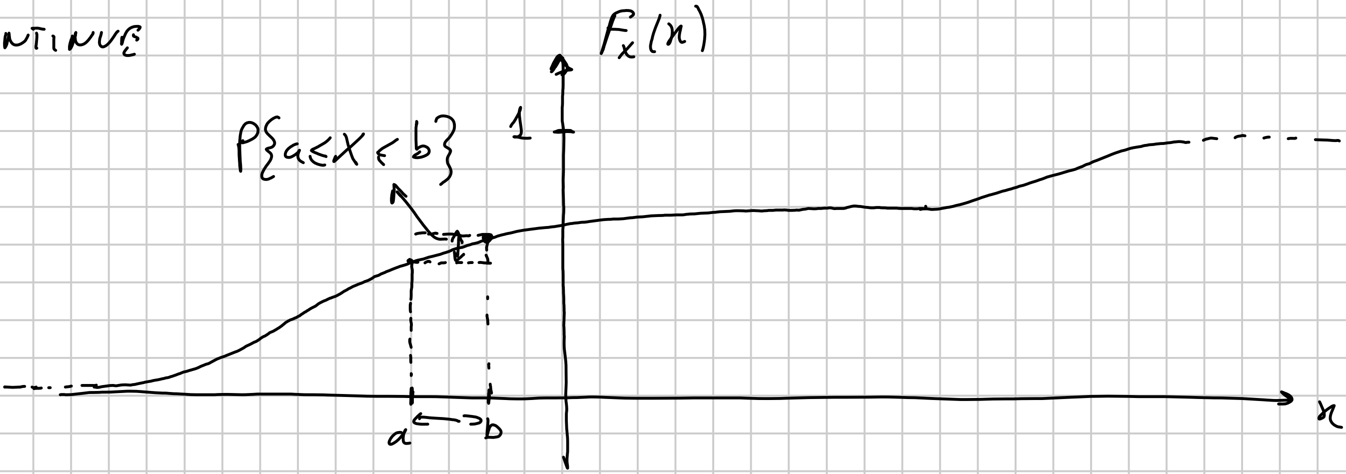
$p_1 = P\{x = x_1\}$

$\vdots$

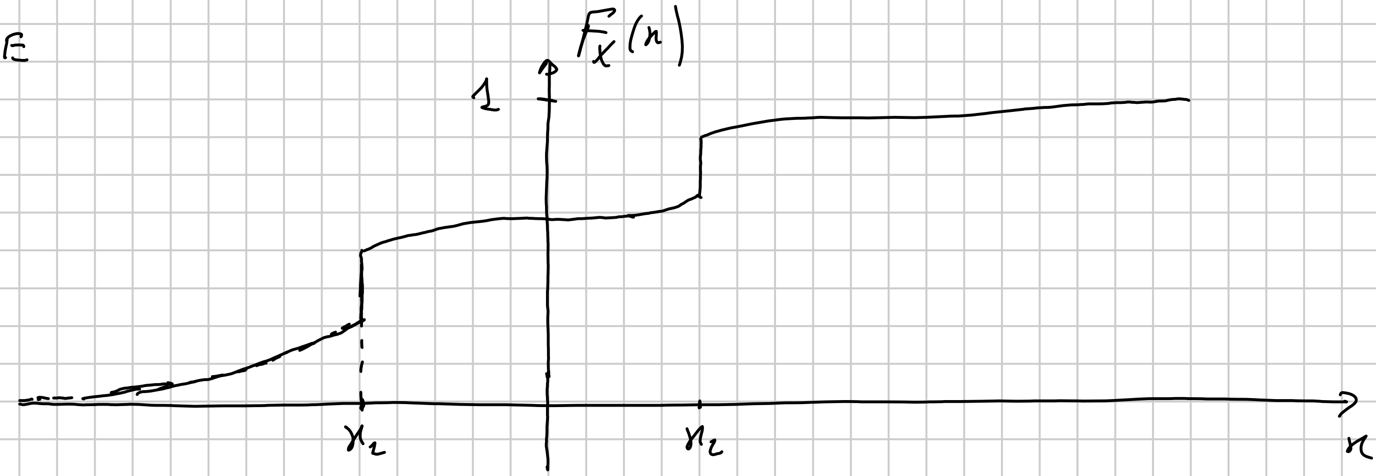
$p_n = P\{x = x_n\}$



·) CONTINUE



·) MISTE



DENSITÀ DI PROBABILITÀ DI UNA V. A.

$$\boxed{f_X(x) \triangleq \frac{d}{dx} F_X(x)}$$

DENSITÀ DI PROBABILITÀ (DDP)

$$\Rightarrow \quad F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

PROPRIETÀ

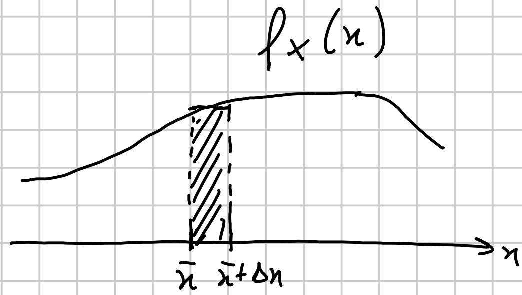
1) $f_X(x) \geq 0$

2) $P\{a \leq X \leq b\} = F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X(\alpha)\, d\alpha \; +$

$$- \int_{-\infty}^{a} f_X(\alpha)\, d\alpha = \int_{a}^{b} f_X(x)\, dx$$

3) $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

CONCETTO DI DDP

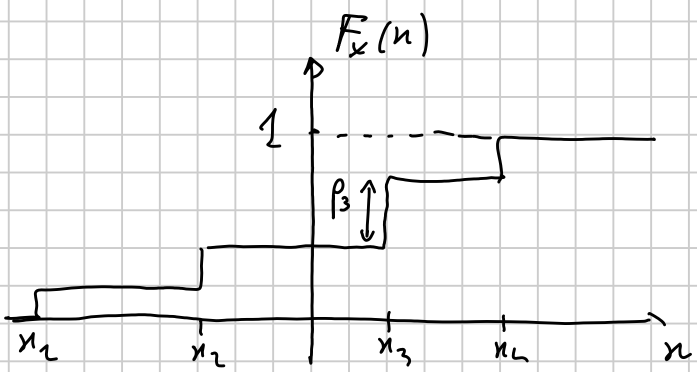


$$P\{\bar{x} \leq X \leq \bar{x} + \Delta x\} =$$

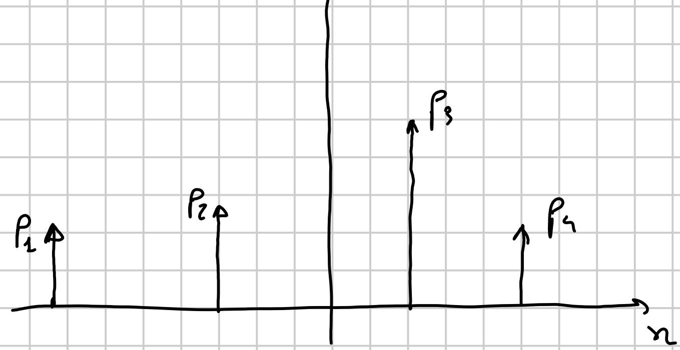
$$= \int_{\bar{x}}^{\bar{x}+\Delta x} f_X(x)\, dx \simeq f_X(\bar{x}) \cdot \Delta x$$

$$f_X(x) \simeq \frac{P\{x \leq X \leq x + \Delta x\}}{\Delta x}$$

DDP DI V.A. DISCRETE



$\Rightarrow$



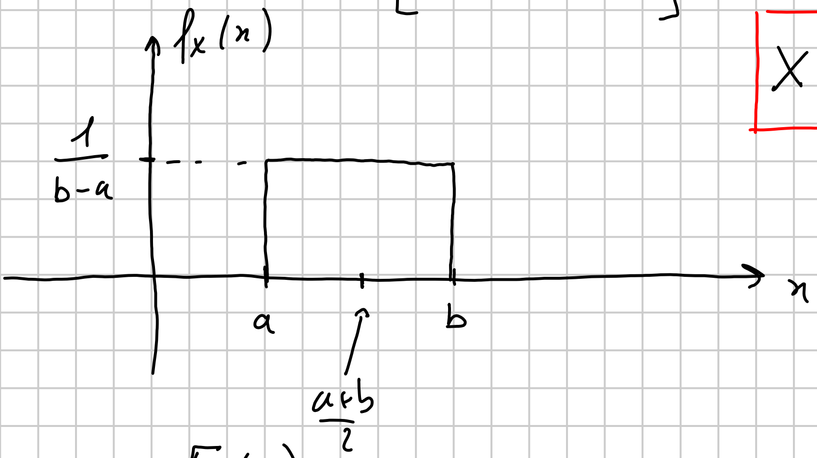
$$p_n = P\{x = x_n\}$$

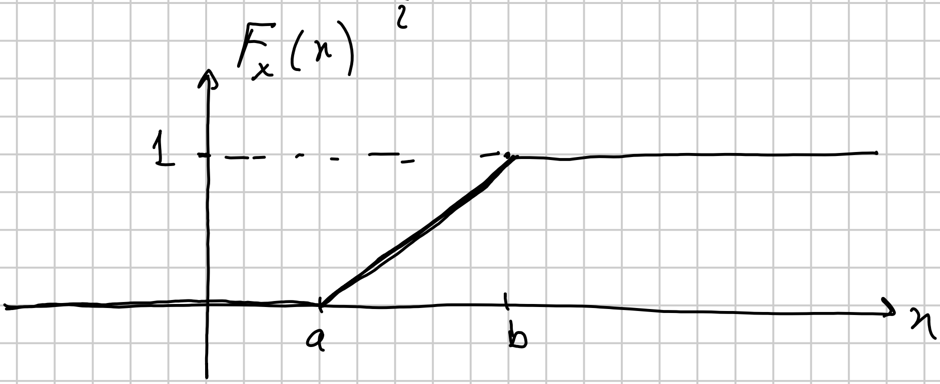
$$f_X(x) = \sum_{k=1}^{N} p_k \, \delta(x - x_k)$$ DDP. DI UNA V.A. DISCRETA

·) V.A. UNIFORMI

$$f_X(x) = \frac{1}{b-a} \operatorname{rect}\left[\frac{x - \frac{b+a}{2}}{b-a}\right]$$

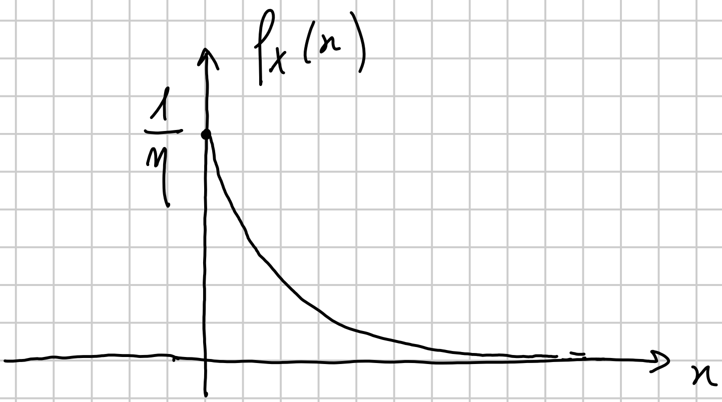
$$\boxed{X \in \mathcal{U}[a,b]}$$





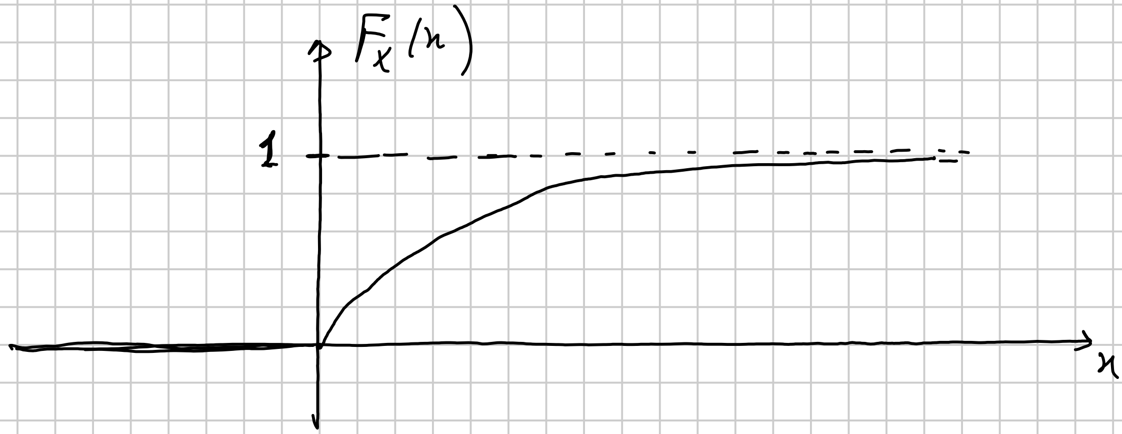
·) V.A. ESPONENZIALE UNILATERA

$$f_X(x) \triangleq \frac{1}{\eta} e^{-\frac{x}{\eta}} u(x)$$



$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\,d\alpha = \frac{1}{\eta}\int_0^x e^{-\frac{\alpha}{\eta}}\,d\alpha =$$

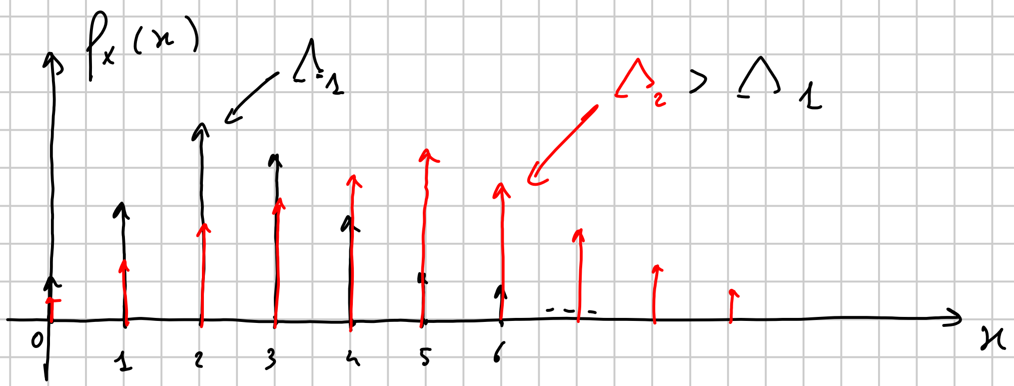
$$= \frac{1}{\cancel{\eta}}\left[-\cancel{\eta}\, e^{-\frac{\alpha}{\eta}}\right]_0^x = -e^{-\frac{x}{\eta}} + 1 = \left[1 - e^{-\frac{x}{\eta}}\right]u(x)$$



·) V.A. DI POISSON (DISCRETA)

$$f_X(x) = \sum_{k=0}^{+\infty} e^{-\Lambda}\frac{\Lambda^k}{k!}\,\delta(x-k)$$

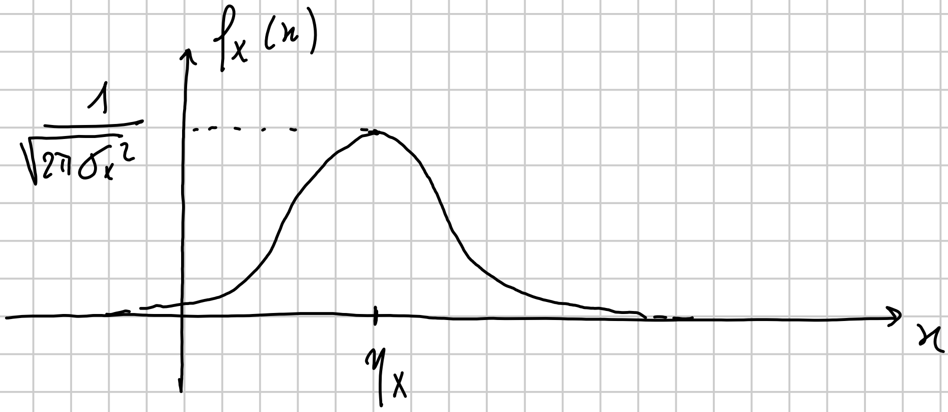
$$p_k = e^{-\Lambda}\frac{\Lambda^k}{k!} \Rightarrow f_X(x) = \sum_{k=0}^{+\infty} p_k\,\delta(x-k)$$



·) V.A. GAUSSIANE

$$X \in \mathcal{N}(\eta_X, \sigma_X^2)$$

$$f_X(x) \triangleq \frac{1}{\sqrt{2\pi\sigma_X^2}} \, e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}}$$



$$F_X(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

V.A. GAUSSIANA STANDARD

$$X \in \mathcal{N}(0,1) \quad , \eta_X = 0 \quad , \sigma_X^2 = 1$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \, e^{-\frac{x^2}{2}}$$

$F_X(x)$ si trova tabulata o in calcolatori (metodi numerici)

TRASFORMAZIONE DI UNA V.A.

X v.a.

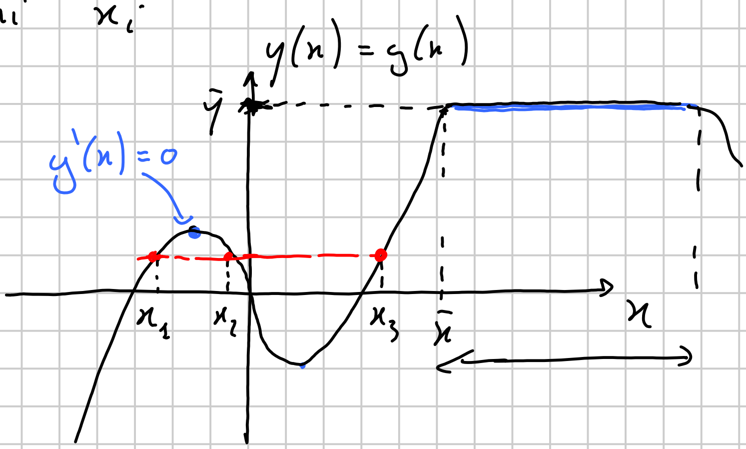
$y = g(x)$ è una trasformazione della grandezza x

TEOREMA FONDAMENTALE DELLA TRASF. DI V.A.

$$f_Y(y) = \sum_{i=1}^{N} \frac{f_X(x_i)}{|g'(x_i)|}$$

x_i sono le soluzioni di $x = g^{-1}(y)$

·) soluzioni x_i



Procedimento

·) a seconda del valore di y

·) $\{x_i\} = \emptyset \Rightarrow f_Y(y) = 0$

·) $\{x_i\} \neq \emptyset$ possono esistere un numero finito o infinito numerabile

·) quando $g'(x_i) = 0$

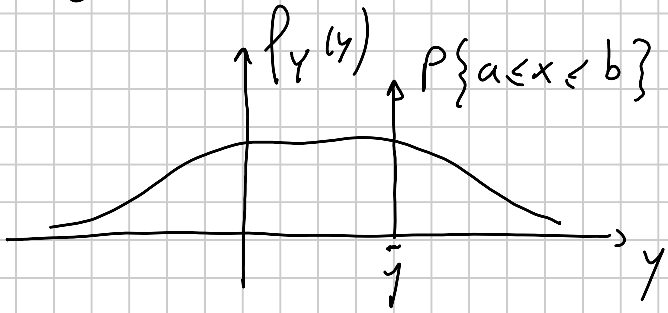
·) $f_X(x_i) = 0 \Rightarrow f_Y(y) = 0$

·) $f_X(x_i) \neq 0 \Rightarrow f_Y(y) \to \infty$

·) quando $g'(x) = 0$ per un intervallo finito

$$f_Y(\bar{y}) = P\{a \leq x \leq b\}\, \delta(y - \bar{y})$$

$$a \leq x \leq b \Rightarrow g'(x) = 0 \Rightarrow g(x) = \bar{y}$$



VALOR MEDIO

$$\boxed{\eta_x \triangleq \int_{-\infty}^{+\infty} x\, f_X(x)\, dx}$$

PER V.A. DISCRETE

$$\eta_x = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_{-\infty}^{+\infty} x \sum_k p_k\, \delta(x-x_k)\, dx$$

$$= \sum_k p_k \int_{-\infty}^{+\infty} x\, \delta(x-x_k)\, dx = \sum_k p_k x_k$$

$$\eta_x = E[X]$$ Expectation, VALORE ATTESO

TEOREMA DEL VALOR MEDIO

$X \quad , \quad Y = g(X)$

$$\eta_Y = E[Y] = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy$$

$$\boxed{\eta_Y = E[Y] = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx} = E[g(X)]$$

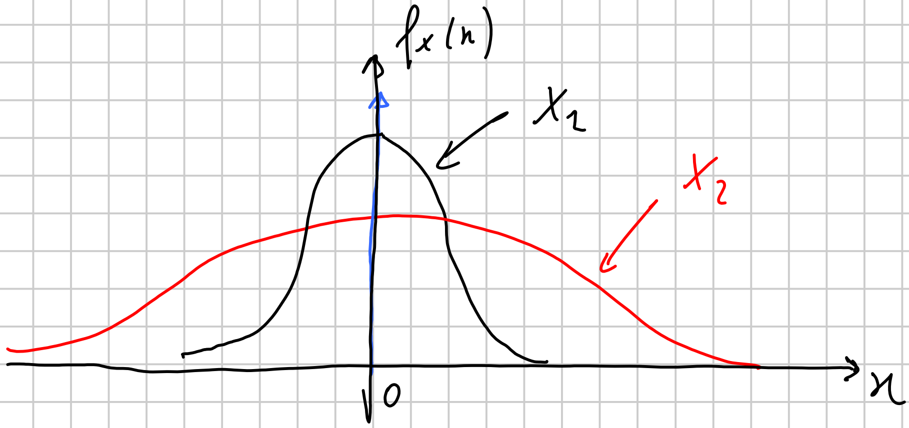
PROPRIETÀ DI LINEARITÀ

$$Y = \alpha\, g(X) + \beta\, h(X)$$

$$\eta_Y = \alpha\, E[g(X)] + \beta\, E[h(X)]$$

VARIANZA DI UNA V.A.

$$\sigma_x^2 \triangleq E\left[(X - \eta_x)^2 \right] = \int_{-\infty}^{+\infty} (x-\eta_x)^2 f_x(x)\, dx$$



$$f_x(x) = \delta(x - \bar{x}) - \delta(x - \eta_x) \Rightarrow \sigma_x^2 = 0$$

DEVIAZIONE STANDARD

$$\boxed{\sigma_x \triangleq \sqrt{\sigma_x^2}}$$

11/04/2013

VALOR QUADRATICO MEDIO

$$m_x^2 \triangleq E\left[X^2\right] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx$$

LEGAME TRA σ_x^2, m_x^2 e η_x

$$\sigma_x^2 = E\left[\left(X-\eta_x\right)^2\right] = E\left[X^2 + \eta_x^2 - 2\eta_x X\right] =$$

$$= E\left[X^2\right] + \eta_x^2 - 2\eta_x E\left[X\right] =$$

$$= m_x^2 + \eta_x^2 - 2\eta_x^2 = \boxed{m_x^2 - \eta_x^2 = \sigma_x^2}$$

INDICI CARATTERISTICI DI V.A. GAUSSIANE

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}$$

·) VALOR MEDIO

$$\eta_x \triangleq E\left[X\right] = \int_{-\infty}^{+\infty} x \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}\, dx = \left(x-\eta_x = y\right)$$

$$= \int_{-\infty}^{+\infty} \left(y+\eta_x\right)\frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{y^2}{2\sigma_x^2}}\, dy =$$

$$= \underbrace{\int_{-\infty}^{+\infty} y \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{y^2}{2\sigma_x^2}}\, dy}_{= 0} + \eta_x \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{-\frac{y^2}{2\sigma_x^2}}\, dy}_{= 1} =$$

prodotto di funz pari e funz dispari = funz. dispari

$$\eta_x = \eta_x$$

$$X \in \mathcal{N}\left(\underbrace{\eta_x}_{\text{VALOR MEDIO}}, \underbrace{\sigma_x^2}_{\text{VARIANZA}}\right)$$

- VARIANZA

$$E\left[(X-\eta_x)^2\right] = \int_{-\infty}^{+\infty} (x-\eta_x)^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}} \, dx \qquad (y = x-\eta_x)$$

$$= \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} \, dy =$$

$$= -\sigma_x^2 \int_{-\infty}^{+\infty} \underbrace{y \frac{1}{\sqrt{2\pi\sigma_x^2}}}_{f(y)} \cdot \underbrace{\left[-\frac{y}{\sigma_x^2} e^{-\frac{y^2}{2\sigma_x^2}}\right]}_{g'(y)} dy$$

$$g(y) = e^{-\frac{y^2}{2\sigma_x^2}} \Rightarrow \frac{d}{dy}\left[g(y)\right] = e^{-\frac{y^2}{2\sigma_x^2}} \left[-\frac{\cancel{2}y}{\cancel{2}\sigma_x^2}\right] = -\frac{y}{\sigma_x^2} e^{-\frac{y^2}{2\sigma_x^2}}$$

$$= \underbrace{-\sigma_x^2 \left[y \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}}\right]_{-\infty}^{+\infty}}_{= 0} + \sigma_x^2 \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{y^2}{2\sigma_x^2}} \, dy}_{= 1}$$

$$= \sigma_x^2 = E\left[(X-\eta_x)^2\right]$$

es.

$$\boxed{f_X(x) = \frac{1}{\sqrt{\pi k^2}} e^{-\frac{(x-p)^2}{k^2}}}$$

$\eta_X = p$

$2k'^2 = k^2$

$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi k'^2}} e^{-\frac{(x-p)^2}{2k'^2}}}$$

$\sigma_X^2 = k'^2 = \frac{k^2}{2}$

RELAZIONE TRA V.A. GAUSSIANE STANDARD e NON-STANDARD

$$X = \sigma_X N + \eta_X$$

X: V.A. GAUSSIANA GENERICA (NON STD) $\rightarrow \mathcal{N}(\eta_X, \sigma_X^2)$

σ_X: DEV. STD DI X

η_X: V.M. DI X

N: V.A. GAUSSIANA STANDARD $\rightarrow \mathcal{N}(0,1)$

Dim

$$X = \sigma_X N + \eta_X \qquad X = g(N)$$

$$N = \frac{X - \eta_X}{\sigma_X}$$

$$f_X(x) = \frac{f_N\left(\frac{x-\eta_X}{\sigma_X}\right)}{\sigma_X} = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{\left(\frac{x-\eta_X}{\sigma_X}\right)^2}{2}} =$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}}$$

$$\Phi(x) = \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

$$f_X \rightarrow \mathcal{N}(0,1)$$

$$P\{a \leq X \leq b\} = F(b) - F(a) = \Phi\left(\frac{b-\eta_x}{\sigma_x}\right) - \Phi\left(\frac{a-\eta_x}{\sigma_x}\right)$$

·) FUNZIONE "ERF"

$$\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-\theta^2}\, d\theta$$

·) FUNZIONE "ERFC"

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-\theta^2}\, d\theta$$

RELAZIONE TRA Φ e ERF

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

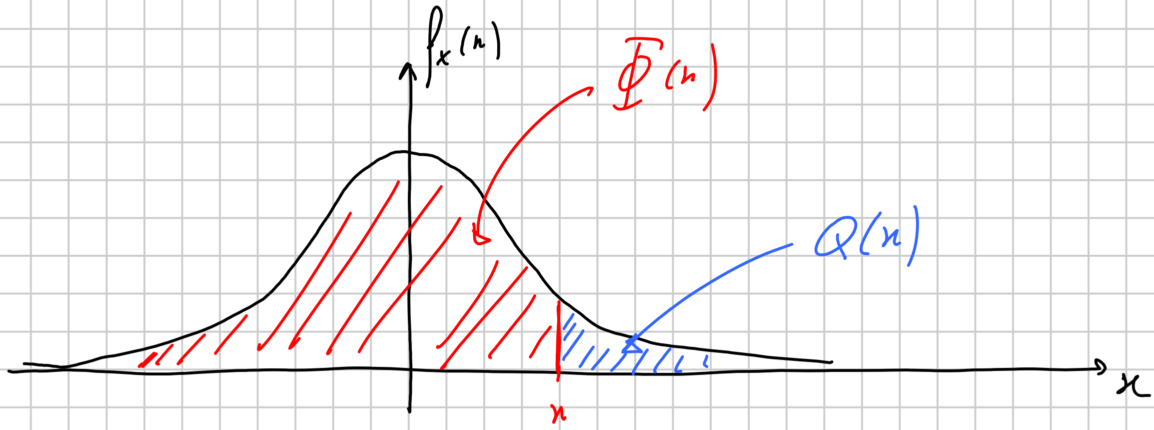
FUNZIONE "CODA", "Q" (QUEUE)

$$Q(x) = 1 - \Phi(x) = 1 - \left[\frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$$

$$= \frac{1}{2} - \frac{1}{2}\left[1 - \mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)\right]$$

$$Q(x) = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$



V.A. CONDIZIONATE

$$P\{X \le x\} = F_X(x)$$

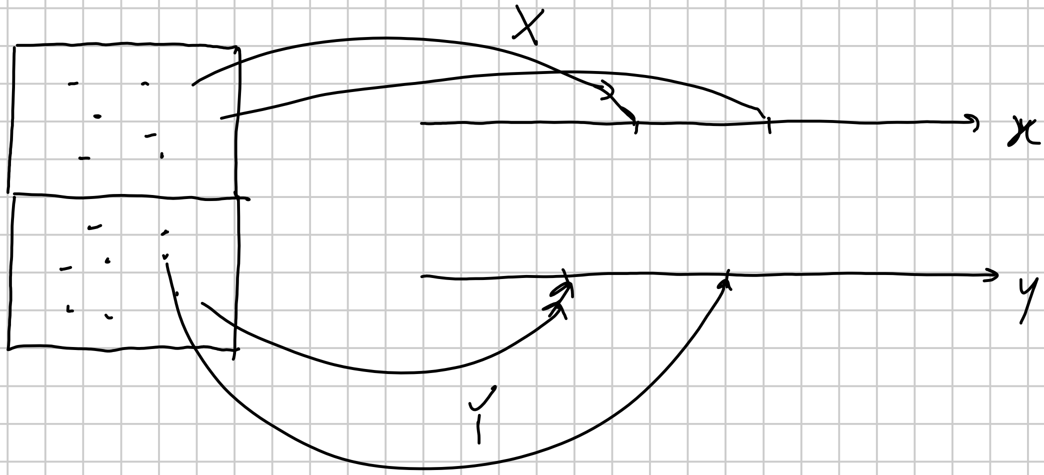
$$P\{X \le x \mid B\} = F_X(x \mid B) \triangleq \frac{P\{X \le x, B\}}{P\{B\}}$$

DISTRIBUZIONE DI PROB. COND.

DDP CONDIZIONATA

$$f_X(x \mid B) \triangleq \frac{d}{dx} F_X(x \mid B)$$

SISTEMI DI DUE V.A.



$$(X \le x, \; Y \le y)$$

EVENTO CONGIUNTO

$$\boxed{P\{X \le x, Y \le y\} \triangleq F_{XY}(x, y)}$$

DISTRIBUZIONE DI PROBABILITÀ CONGIUNTA

PROPRIETÀ

·) $0 \leq F_{XY}(x,y) \leq 1$

·) $F_{XY}(x, y_0)$ è monotona non decrescente $\forall y_0$

·) $F_{XY}(x_0, y)$ è monotona non decrescente $\forall x_0$

·) $F_{XY}(-\infty, y) = P\{X \leq -\infty, Y \leq y\} = 0$

·) $F_{XY}(x, -\infty) = P\{X \leq x, Y \leq -\infty\} = 0$

·) $F_{XY}(-\infty, -\infty) = 0$

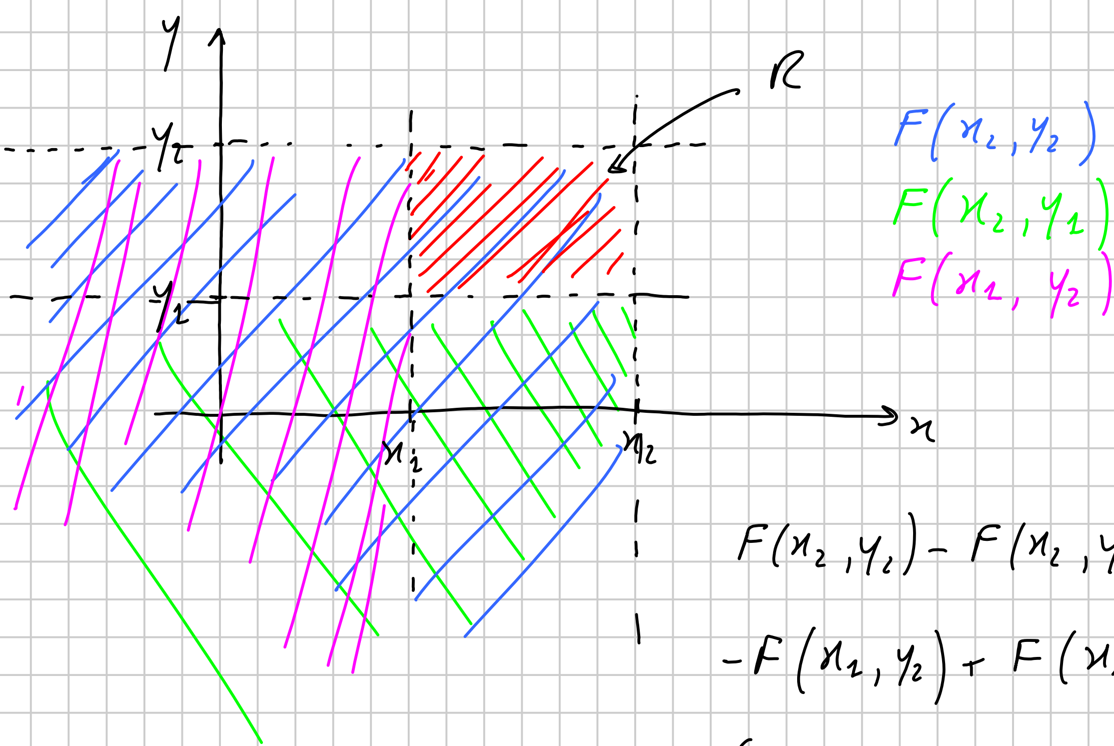
·) $F_Y(y) = F_{XY}(+\infty, y) = P\{X \leq +\infty, Y \leq y\}$

·) $F_X(x) = F_{XY}(x, +\infty)$

·) $F_{XY}(+\infty, +\infty) = 1$

·) Prob. di un evento "rettangolare"

$P\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\}$



$F(x_2, y_2) - F(x_2, y_1) +$

$-F(x_1, y_2) + F(x_1, y_1)$

$= P\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\}$

... Per un rettangolo molto piccolo

$$P\{\bar{x} \le X \le \bar{x}+\Delta x \ , \bar{y} \le Y \le \bar{y}+\Delta y\} =$$

$$= F_{XY}(\bar{x}+\Delta x, \bar{y}+\Delta y) - F_{XY}(\bar{x}+\Delta x, \bar{y}) - F_{XY}(\bar{x}, \bar{y}+\Delta y) + F_{XY}(\bar{x}, \bar{y})$$

$$= F_{XY}(\bar{x}+\Delta x, \bar{y}+\Delta y) - F_{XY}(\bar{x}+\Delta x, \bar{y}) - \left[F_{XY}(\bar{x}, \bar{y}+\Delta y) - F_{XY}(\bar{x}, \bar{y}) \right] =$$

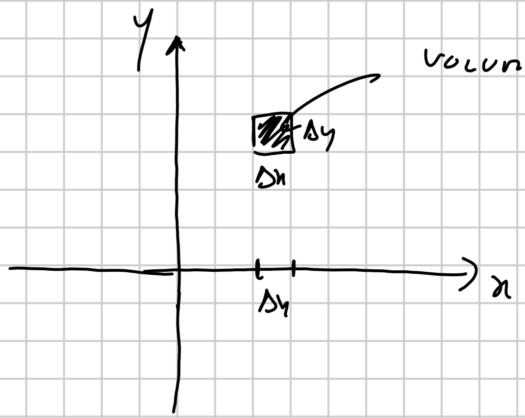
$$\simeq \frac{\partial}{\partial x} F_{XY}(\bar{x}, \bar{y}+\Delta y)\,\Delta x - \frac{\partial}{\partial x} F_{XY}(\bar{x}, \bar{y})\,\Delta x$$

$$\simeq \frac{\partial^2}{\partial x \partial y} F_{XY}(\bar{x}, \bar{y})\,\Delta x\,\Delta y$$

$$\boxed{f_{XY}(x,y) \triangleq \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)}$$

$$\Rightarrow P\{x \le X \le x+\Delta x \ , y \le Y \le y+\Delta y\} \simeq f_{XY}(x,y)\,\Delta x\,\Delta y$$

VOLUMETTO SOTTESO



PROPRIETÀ

:) $f_{XY}(x,y) \ge 0$

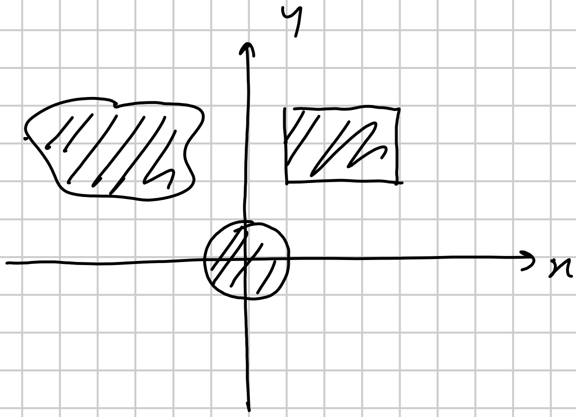
:) $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$

:) $\displaystyle f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy$

$\cdot)\ f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$

$\cdot)\ P\{(x,y) \in D\} =$

$= \iint_D f_{XY}(x,y)\,dx\,dy$

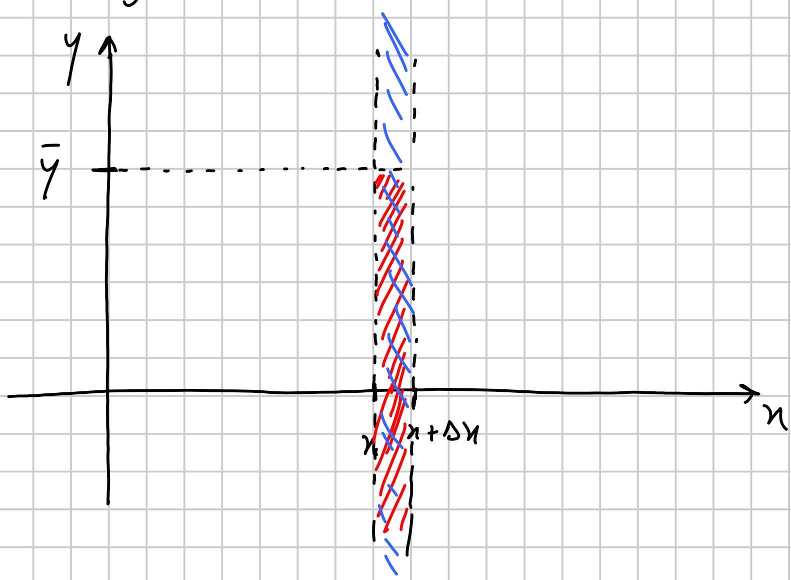


$\cdot)\ F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(\alpha,\beta)\,d\alpha\,d\beta$

DISTRIBUZIONE DI UNA V.A. Y CONDIZIONATA AD X

$$\boxed{F_{Y|X}(y|x) \triangleq \frac{\int_{-\infty}^{y} f_{XY}(x,\beta)\,d\beta}{f_X(x)}}$$

Interpretazione grafica



$F_{Y|X}(x|y) \simeq P\{Y \leq \bar{y} \mid x \leq X \leq x+\Delta x\} = \frac{P\{Y \leq \bar{y},\ x \leq X \leq x+\Delta x\}}{P\{x \leq X \leq x+\Delta x\}}$

DDP CONDIZIONATA

$$\boxed{f_{Y|X}(y|x) \triangleq \frac{d}{dy} F_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}}$$

INDIPENDENZA TRA X e Y

$$f_{Y|X}(y|x) = f_Y(y)$$

$$f_Y(y) = f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \Rightarrow f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

TRASFORMAZIONE DI UNA COPPIA DI V.A.

$$Z = g(X,Y)$$

) nota $f_{XY}(x,y)$

i) $\Rightarrow F_Z(z) = P\{Z \leq z\} = P\{g(X,Y) \leq z\}$

$$F_Z(z) = \iint_{R(z)} f_{XY}(x,y)\, dx\, dy$$

$R(z)$ è la regione di piano (x,y) dove

$g(x,y) \leq z$

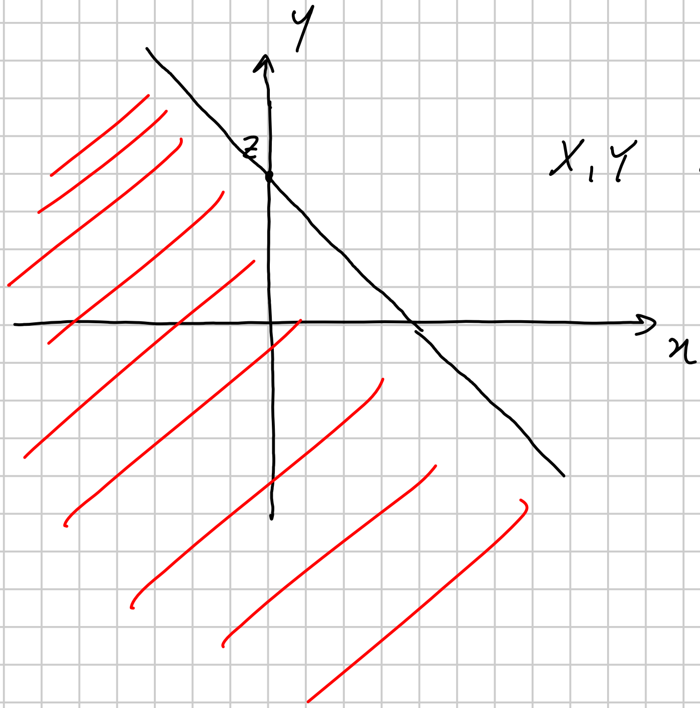
ii) $\Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z)$

ESEMPIO

$$X + Y = Z$$

$$f_Z(z) = ?$$

$$x + y \leqslant z$$



X, Y indipend.

$$F_Z(z) = \iint_{R(z)} f_{XY}(x,y)\, dx\, dy = \int_{-\infty\ (x)}^{+\infty} \int_{-\infty}^{z-x} f_{XY}(x,y)\, dx\, dy$$

$$(y' = y + x)$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{z} f_{XY}(x, y'-x)\, dx\, dy'$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{+\infty} \frac{d}{dz} \int_{-\infty}^{z} f_{XY}(x, y'-x)\, dx\, dy'$$

$$= \int_{-\infty}^{+\infty} f_{XY}(x, z-x)\, dx = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= f_X(x) \otimes f_Y(y)$$

$\Rightarrow$ Se $Z = X + Y$ dove X, Y sono ind.

$$\Rightarrow f_Z(z) = f_X(x) \otimes f_Y(y)$$

$$\Rightarrow Z = \sum_{i=1}^{N} X_i \quad , X_i \text{ sum ind.}$$

$$\Rightarrow f_Z(z) = f_{X_1}(x_1) \otimes f_{X_2}(x_2) \otimes \dots \otimes f_{X_N}(x_N)$$

CORRELAZIONE E COVARIANZA TRA V.A.

$$r_{XY} \triangleq E[XY] = \int_{-\infty}^{+\infty} x\,y\, f_{XY}(x,y)\,dx\,dy$$ CORRELAZIONE

$$C_{XY} \triangleq E\left[(X-\eta_X)(Y-\eta_Y)\right] = \int_{-\infty}^{+\infty} (x-\eta_X)(y-\eta_Y)\, f_{XY}(x,y)\,dx\,dy$$

COVARIANZA

$$C_{XY} = r_{XY} - \eta_X \eta_Y$$

INCORRELAZIONE TRA V.A. $\Rightarrow$ $C_{XY} = 0$

COVARIANZA NULLA

(NON CORRELAZIONE NULLA)

$$\underset{\downarrow\\ 0}{C_{XY}} = \underset{\downarrow\\ \neq 0}{r_{XY}} - \underset{\downarrow\\ \neq 0}{\eta_X \eta_Y}$$

COEFFICIENTE DI CORRELAZIONE

$$\boxed{\rho_{XY} \triangleq \frac{C_{XY}}{\sigma_X \sigma_Y}}$$

$$-1 \le \rho_{XY} \le 1$$

$$|\rho_{XY}| \le 1$$

$$Y = aX + b \Rightarrow \rho_{XY} = 1$$

INCORRELAZIONE E INDIPENDENZA TRA V.A.

Ip. 2 V.A. INDIPENDENTI

$$\Downarrow$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$C_{XY} = r_{XY} - \eta_X \eta_Y$$

$$r_{XY} = E[XY] = \iint_{-\infty}^{+\infty} x\, y\, f_{XY}(x,y)\, dx\, dy =$$

$$= \int_{-\infty}^{+\infty} x\, f_X(x)\, dx \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy = \eta_X \eta_Y$$

(x) (y)

$$C_{XY} = \eta_X \eta_Y - \eta_X \eta_Y = 0$$

$$\text{INDIPENDENZA} \Rightarrow \text{INCORRELAZIONE}$$

$$\not\Leftarrow$$

Esempio

Θ v.a. $\in U[0, 2\pi]$

definiamo

$$X = \cos\Theta$$

$$Y = \sin\Theta$$

Calcolare C_{XY}

$$C_{XY} = V_{XY} - \eta_X \eta_Y$$

$$V_{XY} = E[XY] = \int_{-\infty}^{+\infty} \cos\theta \cdot \sin\theta \, f_\Theta(\theta) \, d\theta$$

uso il teo. del valor medio

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\sin(2\theta)\, d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin(2\theta)\, d\theta = 0$$

$$\eta_X = E[X] = \int_{-\infty}^{+\infty} \cos\theta \, f_\Theta(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \, d\theta = 0$$

$$\eta_Y = E[Y] = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \, d\theta = 0$$

$$C_{XY} = 0 - 0 = 0 \quad \Rightarrow \quad X, Y \text{ incorrelate}$$

$$\Rightarrow X^2 + Y^2 = 1 \quad \Rightarrow \quad Y = \pm\sqrt{1-x^2}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$\Downarrow$

NON SONO INDIPENDENTI

SISTEMI DI V.A. (VETTORI ALEATORI)

$$F_{X_1, X_2, \dots, X_N}(x_1, x_2, \dots, x_N) \triangleq P\{X_1 \le x_1, X_2 \le x_2, \dots, X_N \le x_N\}$$

DISTRIB. DI PROBABILITA'

$$f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \frac{\partial^N}{\partial x_1 \partial x_2 \dots \partial x_N} F_{X_1,\dots,X_N}(x_1,\dots,x_N)$$

D.D.P.

DDP. CONDIZIONATE

$$f_{\{X_i\}|\{X_j\}}\left(\{x_i\}|\{x_j\}\right) \triangleq \frac{f_{X_1,\dots,X_N}(x_1,\dots,x_N)}{f_{\{X_j\}}(\{x_j\})}$$

ESEMPIO 5 V.A.

$$f_{X_1,X_2,X_5|X_3,X_4}\left(x_1,x_2,x_5|x_3,x_4\right) \triangleq \frac{f_{X_1\dots X_5}(x_1,\dots,x_5)}{f_{X_3,X_4}(x_3,x_4)}$$

V.A. INDIP.

$$f_{X_1,\dots,X_N}(x_1,\dots,x_N) = \prod_{i=1}^{N} f_{X_i}(x_i)$$

$$\underline{X} \triangleq \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = \left[X_1, X_2, \dots, X_N\right]^T$$

$F_{\underline{X}}(\underline{x})$, $f_{\underline{X}}(\underline{x})$ (SIMBOLICHE)

$$\underline{\eta}_X = \begin{bmatrix} \eta_{X_1} \\ \eta_{X_2} \\ \vdots \\ \eta_{X_N} \end{bmatrix}$$ VETTORE VALOR MEDIO

$$\underline{\underline{R}}_X \triangleq E\left[\underline{X}\,\underline{X}^T\right] = E\left[\begin{bmatrix} X_1 \\ \vdots \\ X_N \end{bmatrix} \cdot \left[X_1, \dots, X_N\right]\right]$$

$$
\underline{\underline{R}}_X = \begin{array}{c} \begin{array}{cccc} r_{x_1x_1} & r_{x_1x_2} & & \end{array} \\ \begin{bmatrix} E[X_1 X_1] & E[X_1 X_2] & \cdots & E[X_1 X_n] \\ E[X_2, X_1] & & & \\ \vdots & & \ddots & \\ E[X_n, X_1] & \cdots & \cdots & E[X_n, X_n] \end{bmatrix} \end{array}
$$

correl. $\left.\begin{matrix} E[X_1 X_1] \\ \vdots \\ E[X_i, X_i] \end{matrix}\right\}$ sulla diagonale

$$
\underline{\underline{C}}_X = \begin{bmatrix} c_{x_1x_1} & c_{x_1x_2} & \cdots \\ & & \\ & & \end{bmatrix} = E\left[(\underline{X} - \underline{\eta}_X)(\underline{X} - \underline{\eta}_X)^T \right]
$$

$$
\underline{\underline{C}}_X = \underline{\underline{R}}_X - \underline{\eta}_X \underline{\eta}_X^T
$$

17/04/2013

VETTORI GAUSSIANI

(V.A. CONG. GAUSSIANE)

$$\underline{X} = [X_1, X_2, \dots, X_N]^T$$

$$\boxed{f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N \det(\underline{\underline{C}}_x)}} e^{-(\underline{x}-\underline{\eta}_x)^T \underline{\underline{C}}_x^{-1} (\underline{x}-\underline{\eta}_x)}}$$

PROPRIETÀ

1) ddp di un vettore Gaussiano è completamente determinata da $\underline{\eta}_x$ e $\underline{\underline{C}}_x$

$$\underline{X} \in \mathcal{N}(\underline{\eta}_x, \underline{\underline{C}}_x)$$

2) Se le v.a. sono incorrelate tra di loro

$\Downarrow$

queste sono anche indipendenti

Dim per il caso $N=2$

$$\underline{X} = [X_1, X_2]$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sqrt{(2\pi)^2 \det(\underline{\underline{C}}_x)}} e^{-\frac{1}{2}(\underline{x}-\underline{\eta}_x)^T \underline{\underline{C}}_x^{-1} (\underline{x}-\underline{\eta}_x)}$$

$$\underline{\eta}_x = \begin{bmatrix} \eta_{x_1} \\ \eta_{x_2} \end{bmatrix} , \quad \underline{\underline{C}}_x = \begin{bmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{bmatrix} = \begin{bmatrix} \overset{\sigma_{x_1}^2}{C_{x_1x_1}} & \cancel{C_{x_1x_2}} \\ \cancel{C_{x_2x_1}} & \underset{\sigma_{x_2}^2}{C_{x_2x_2}} \end{bmatrix}$$

$$\det(\underline{\underline{C}}_x) = \sigma_{x_1}^2 \sigma_{x_2}^2 , \quad \underline{\underline{C}}_x^{-1} = \begin{bmatrix} 1/\sigma_{x_1}^2 & 0 \\ 0 & 1/\sigma_{x_2}^2 \end{bmatrix}$$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\,\sigma_{x_1}\sigma_{x_2}}\, e^{-\frac{(x_1-\eta_{x_1})^2}{2\sigma_{x_1}^2} - \frac{(x_2-\eta_{x_2})^2}{2\sigma_{x_2}^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_{x_1}^2}}\, e^{-\frac{(x_1-\eta_{x_1})^2}{2\sigma_{x_1}^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_{x_2}^2}}\, e^{-\frac{(x_2-\eta_{x_2})^2}{2\sigma_{x_2}^2}}$$

$$= f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

Per V.A. CONG. GAUSSIANE

INDIPENDENZA $\rightleftarrows$ INCORRELAZIONE

3) Prendendo un qualunque vettore ridotto di $\underline{X}$

es. $\underline{Y} = [X_1, X_2, \dots, X_k] \quad k \leq N$

$\Rightarrow$ $\underline{Y}$ è un vettore Gaussiano

$\Rightarrow$ per $k=1$ $\Rightarrow$ ddp. marginale

$\Rightarrow$ tutte le ddp marginali sono Gaussiane

4) $\underline{Y} = \underline{\underline{A}}\,\underline{X} + \underline{b}$, dove $\underline{X}$ è un vettore Gaussiano

$\Rightarrow$ $\underline{Y}$ è un vettore Gaussiano

$$\left.\begin{aligned} &\Rightarrow \underline{\eta}_Y = \underline{\underline{A}}\,\underline{\eta}_X + \underline{b} \\ &\Rightarrow \underline{\underline{C}}_Y = \underline{\underline{A}}\,\underline{\underline{C}}_X\,\underline{\underline{A}}^T \end{aligned}\right\} \Rightarrow f_{\underline{Y}}(\underline{y})$$

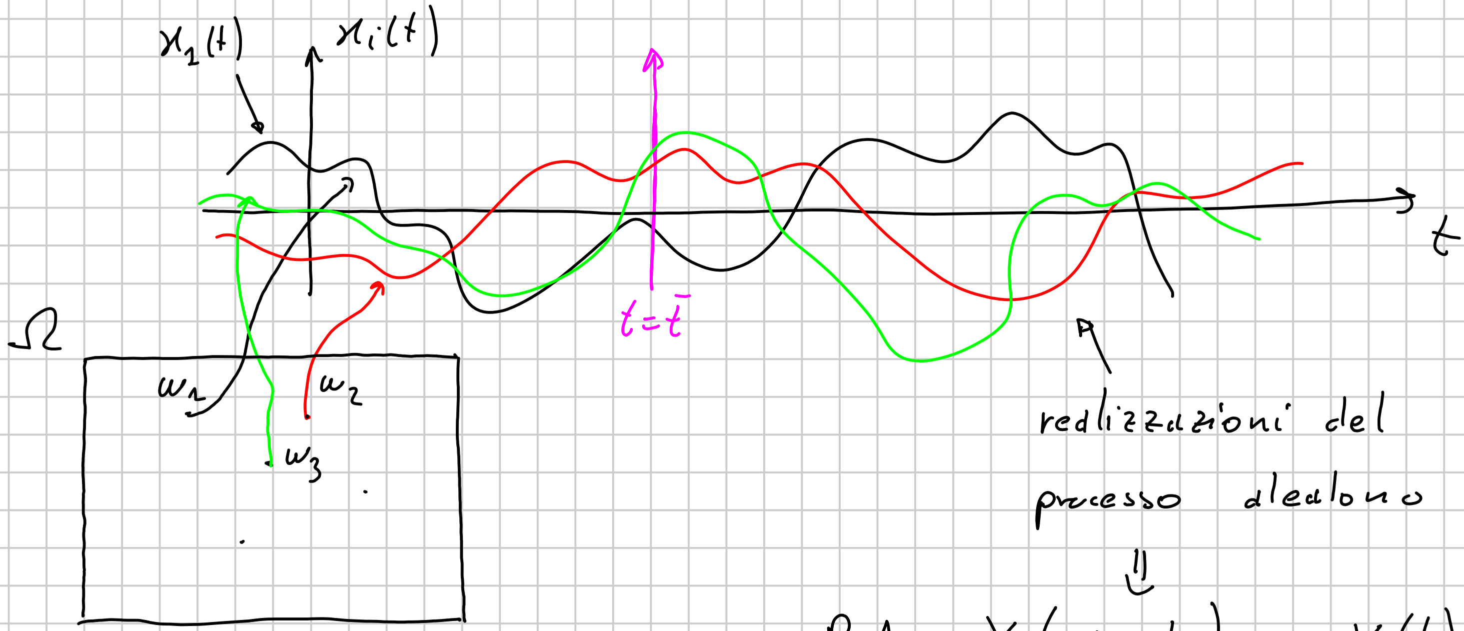
5) $\underline{Y} = \{X_i\} \mid \{X_j\}$ è Gaussiano

es $\Rightarrow$ $\underline{X} = [X_1, X_2]$ $\Rightarrow$ $Y \triangleq X_1 | X_2$

$\Downarrow$

è Gaussiana

SEGNALI ALEATORI (PROCESSI ALEATORI)



realizzazioni del processo aleatorio

$\Downarrow$

P.A. $X(w_i, t) \Rightarrow X(t)$

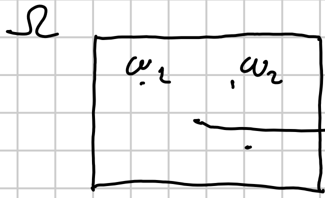
V.A. $X(w_i) \Rightarrow X$

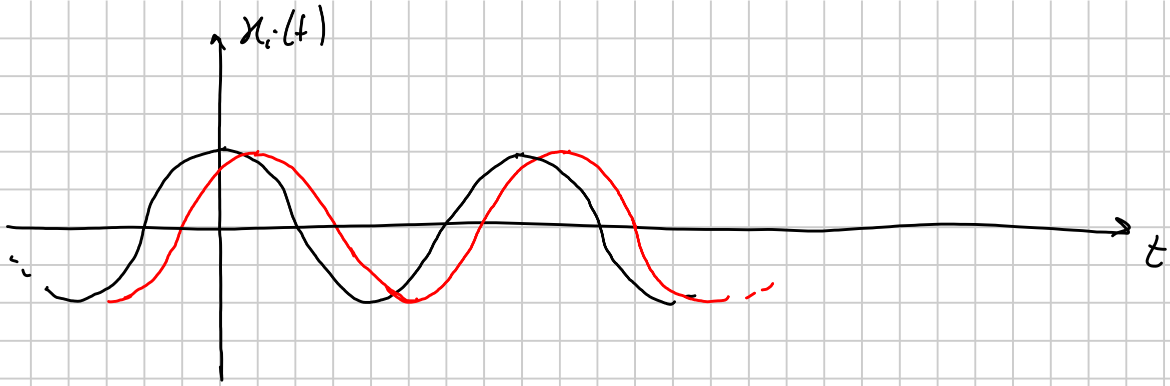
$Y = X(\bar{t}) \Rightarrow$ V.A.

P.A. PARAMETRICI

es. $X(t) = a\cos(2\pi f_0 t + \Theta)$

v.a.





$$X(t) = A \cos\left(2\pi f_0 t + \Theta\right)$$

v.a.   v.a.

CARATTERIZZAZIONE DI P.A.

Funzione <u>distribuzione di probabilità del I° ordine</u>

$$F_X(x; t_1) \triangleq P\{X(t_1) \leq x\}$$

$$P\{X(t_2) > X(t_1)\}$$

distribuzione di prob. del II° ordine

$$F_X(x_1, x_2; t_1, t_2) \triangleq P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

distribuzione di prob. di ordine N

$$F_X(x_1, \dots, x_N; t_1, \dots, t_N) \triangleq P\{X(t_1) \leq x_1, \dots, X(t_N) \leq x_N\}$$

D.D.P di ordine N

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) \triangleq \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_X(x_1, \dots, x_N; t_1, \dots, t_N)$$

VALOR MEDIO (STATISTICO)

$$\eta_X(t) \triangleq E[X(t)] = \int_{-\infty}^{+\infty} x\, f_X(x,t)\, dx$$

POTENZA MEDIA STATISTICA (VALOR QUADRATICO MEDIO)

$$P_X(t) \triangleq E[X^2(t)] = \int_{-\infty}^{+\infty} x^2 f_X(x,t)\, dx$$

VARIANZA

$$\sigma_X^2(t) \triangleq E\left[\left(X(t) - \eta_X(t)\right)^2\right] = \int_{-\infty}^{+\infty} \left(x - \eta_X(t)\right)^2 f_X(x,t)\, dx$$

$$\sigma_X^2(t) = P_X(t) - \eta_X^2(t)$$

Dim

$$\sigma_X^2(t) = E\left[\left(X(t) - \eta_X(t)\right)^2\right] = E\left[X^2(t)\right] + E\left[\eta_X^2(t)\right] +$$
$$- 2E\left[\eta_X(t)\, X(t)\right] = P_X(t) + \eta_X^2(t) - 2\eta_X^2(t) =$$
$$= P_X(t) - \eta_X^2(t)$$

AUTOCORRELAZIONE

$$R_X(t_1, t_2) \triangleq E[X(t_1)\, X(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2\, f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

AUTOCOVARIANZA

$$C_X(t_1, t_2) \triangleq E\left[\left(X(t_1) - \eta_X(t_1)\right)\left(X(t_2) - \eta_X(t_2)\right)\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(x_1 - \eta_X(t_1)\right)\left(x_2 - \eta_X(t_2)\right) f_X(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

$$\underbrace{C_X(t_1, t_2)}_{\text{AUTOCOVARIANZA}} = \underbrace{R_X(t_1, t_2)}_{\text{AUTOCORRELAZIONE}} - \underbrace{\eta_X(t_1)\eta_X(t_2)}_{\text{VALORI MEDI}}$$

PROCESSI STAZIONARI

·) PROCESSI STAZIONARI IN SENSO STRETTO (SSS)

$$f_X(x_1, \dots, x_n; t_1, \dots, t_n) = f_X(x_1, \dots, x_n; t_1 + \Delta t, \dots, t_n + \Delta t)$$

·) STATISTICHE DEL PRIMO ORDINE

$$\eta_X(t) = E\left[X(t)\right] = \int_{-\infty}^{+\infty} x\, f_X(x; t)\, dx$$

$$f_X(x; t) = f_X(x; t + \Delta t) = f_X(x)$$ non dipende da t

$$\eta_X(t) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \eta_X$$ costante

$$P_X(t) = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = P_X$$ costante

$$\sigma_X^2(t) = P_X(t) - \eta_X^2(t) = P_X - \eta_X^2$$ costante

·) STATISTICHE DEL SECONDO ORDINE

$$f_X(x_1, x_2 ; t_1, t_2) = f_X(x_1, x_2 ; t_1 + \Delta t, t_2 + \Delta t)$$

$$= f_X(x_1, x_2 ; t_1 - t_2) = f_X(x_1, x_2 ; \tau) \qquad (\tau \triangleq t_1 - t_2)$$

·) $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$

·) $C_X(t_1, t_2) = C_X(t_1 - t_2) = C_X(\tau)$

·) STATISTICHE DI ORDINE N

$$f_X(x_1, \dots, x_N ; t_1, \dots, t_N) = f_X(x_1, \dots, x_N ; t_1 - t_2, t_2 - t_3, \dots, t_{N-1} - t_N)$$

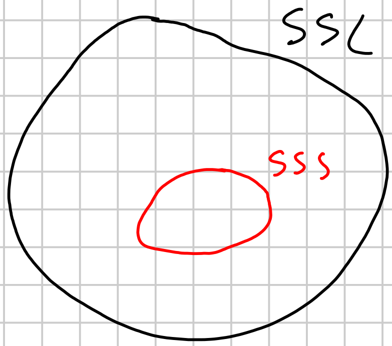
STAZIONARIETÀ DI ORDINE N

$\Downarrow \quad \cancel{\Uparrow}$

STAZIONARIETÀ DI ORDINE K < N

STAZIONARIETÀ IN SENSO LATO (SSL)

$$\left.\begin{array}{l} 1)\ \eta_X(t) = \eta_X \\ 2)\ R_X(t_1, t_2) = R_X(\tau) \end{array}\right\} \text{SSL}$$



$$\Downarrow$$

$$C_X(t_1, t_2) = C_X(\tau)$$

$$= R_X(\tau) - \eta_X^2$$

PROPRIETÀ DELLA AUTOCORRELAZIONE DI UN PROC. SSL

1) $R_X(\tau) = R_X(-\tau)$

$(t = t_1 \ , \ \tau = t_1 - t_2) \Rightarrow E[X(t_1)\, X(t_2)]$

Dim.

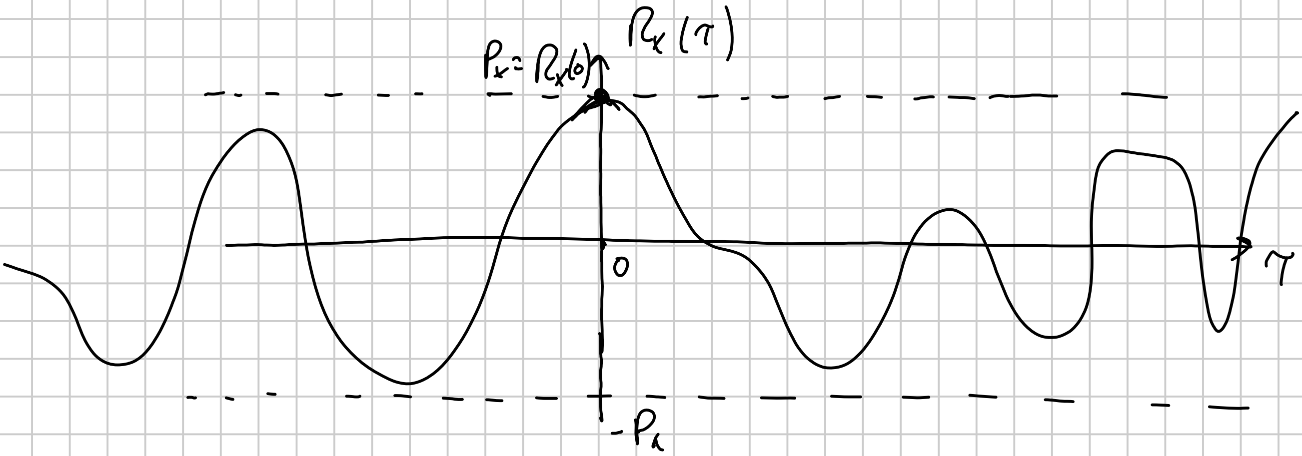
$$R_X(\tau) = E[X(t)\, X(t-\tau)] = \ldots \ (t - \tau = t')$$

$$= E[X(t'+\tau)\, X(t')] = E[X(t')\, X(t' - (-\tau))]$$

$$= R_X(-\tau)$$

2) $R_X(0) = E[X(t)\, X(t)] = E[X^2(t)] = P_X$

3) $R_X(0) \geq |R_X(\tau)|$



Dim

$$E[\{X(t) \pm X(t-\tau)\}^2] \geq 0$$

$$\underbrace{E[X^2(t)]}_{P_X} + \underbrace{E[X^2(t-\tau)]}_{P_X} \pm 2\, \underbrace{E[X(t)\, X(t-\tau)]}_{R_X(\tau)} \geq 0$$

$$\cancel{2} P_X \pm \cancel{2} R_X(\tau) \geq 0 \quad \Rightarrow R_X(0) = P_X \geq |R_X(\tau)|$$

c.v.d.

4) Se la $R_x(\tau)$ non ha componenti periodiche

$x(t)$ con comp. periodiche

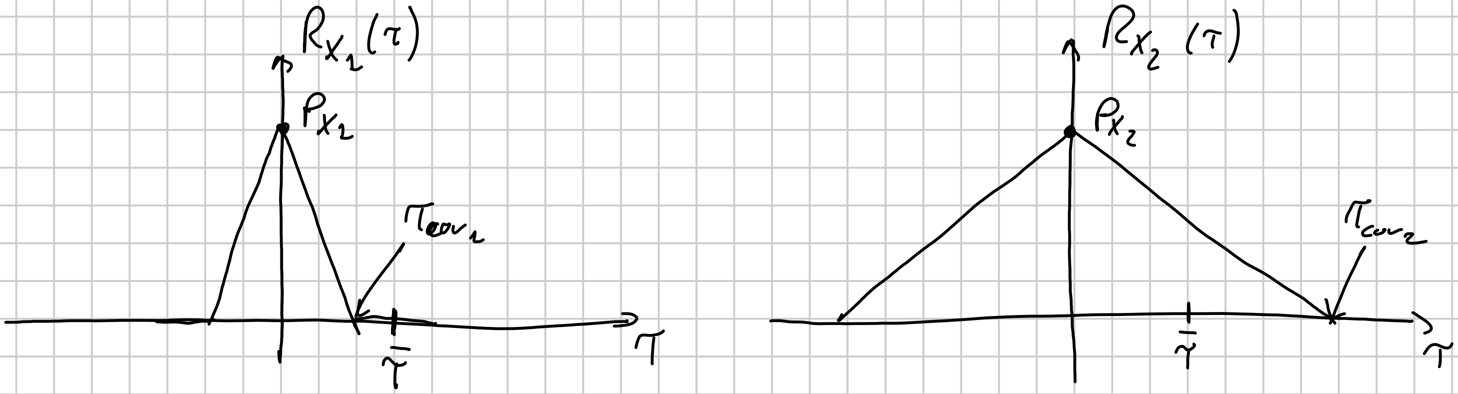
$$x(t) = x'(t) + x_p(t)$$

comp. aperiodica — componente periodica

$$\Rightarrow \lim_{\tau \to \infty} R_x(\tau) = \eta_x^2$$

Giustificaz.

$$\underbrace{\lim_{\tau \to \infty} C_x(\tau)}_{0} + \eta_x^2 \to \eta_x^2$$

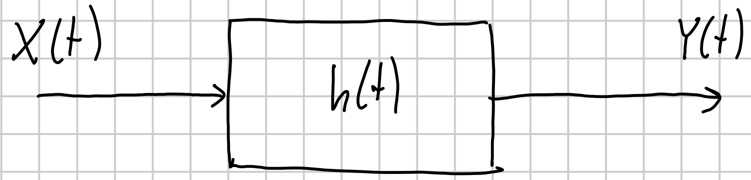


$\eta_{X_1} = \eta_{X_2} = 0 \qquad P_{X_1} = P_{X_2}$

TEMPO DI DECORRELAZIONE

$$\tau_{cor} : \quad R_x(\tau) = 0 \quad \tau \geq \tau_{cor}$$

FILTRAGGIO DI UN PROCESSO ALEATORIO



$$r(t) = s(t) + n(t) \longrightarrow \boxed{h(t)} \longrightarrow r_u(t) = s_u(t) + n_u(t)$$

$r(t)$: segnale ricevuto

$s(t)$: segnale utile

$n(t)$: rumore

$$s_u(t) = s(t) \otimes h(t)$$

$$n_u(t) = ?$$

$n(t)$ realizzazione di un processo di rumore $N(t)$

$$n(w_i, t) \longrightarrow \boxed{h(t)} \longrightarrow n_u(w_i, t)$$

$$n_u(w_i, t) = n(w_i, t) \otimes h(t)$$

$$\Downarrow$$

$$N_u(t) = N(t) \otimes h(t)$$

$$\xrightarrow[X(t)]{N(t)} \boxed{h(t)} \xrightarrow[Y(t)]{N_u(t)}$$

ddp di ordine N in ingresso $\longrightarrow$? ddp di ordine N in uscita

CALCOLO DEGLI INDICI STATISTICI DEL P.A. IN USCITA

·) VALOR MEDIO

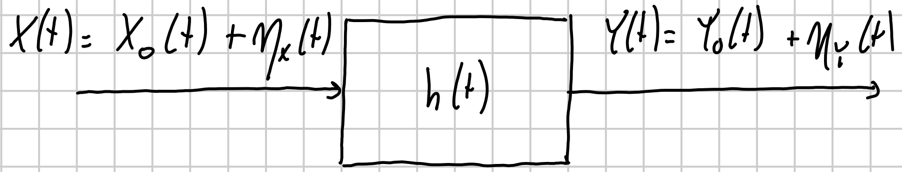
$$\eta_Y(t) = E[Y(t)] = E[X(t) \otimes h(t)] =$$

$$= E\left[\int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha\right] = \int_{-\infty}^{+\infty} E[X(\alpha)]\, h(t-\alpha)\, d\alpha$$

$$= \int_{-\infty}^{+\infty} \eta_X(\alpha)\, h(t-\alpha)\, d\alpha = \eta_X(t) \otimes h(t)$$

$$\Rightarrow \eta_X(t) = 0 \Rightarrow \eta_Y(t) = 0$$

INTERPRETAZIONE



$$Y_0(t) = X_0(t) \otimes h(t)$$
$$\eta_Y(t) = \eta_X(t) \otimes h(t)$$

$$E[Y(t)] = \underbrace{E[Y_0(t)]}_{=0} + \underbrace{E[\eta_Y(t)]}_{= \eta_Y(t) = \eta_X(t) \otimes h(t)}$$

AUTOCORR. DI UN P.A. IN USCITA DA UN SLS

$$R_Y(t_1, t_2) = E[Y(t_1)\, Y(t_2)] =$$

$$= E\left[\int_{-\infty}^{+\infty} X(\alpha)\, h(t_1-\alpha)\, d\alpha \cdot \int_{-\infty}^{+\infty} X(\beta)\, h(t_2-\beta)\, d\beta \right]$$

$$= \underset{(\alpha)}{\int_{-\infty}^{+\infty}} \underset{(\beta)}{\int_{-\infty}^{+\infty}} E[X(\alpha)\, X(\beta)]\, h(t_1-\alpha)\, h(t_2-\beta)\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} R_X(\alpha, \beta)\, h(t_1-\alpha)\, h(t_2-\beta)\, d\alpha\, d\beta$$

$$= \underset{(\beta)}{\int_{-\infty}^{+\infty}} \left[\overbrace{\underset{(\alpha)}{\int_{-\infty}^{+\infty}} R_X(\alpha, \beta)\, h(t_1-\alpha)\, d\alpha}^{R_X(t_1, \beta) \otimes h(t_1)} \right] h(t_2-\beta)\, d\beta$$

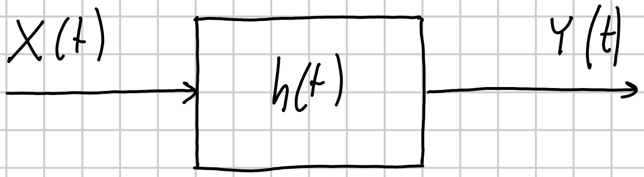
$$= \int_{-\infty}^{+\infty} \underbrace{\left[R_X(t_1, \beta) \otimes h(t_1) \right]}_{f(\beta)} h(t_2 - \beta)\, d\beta$$

(β)

$$= R_X(t_1, t_2) \otimes h(t_1) \otimes h(t_2)$$

18/04/2013

FILTRAGGIO DI UN P.A SSL



$X(t)$ è un P.A. SSL $\to$ $\eta_X(t) = \eta_X$

$\to$ $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$

$\eta_Y(t) = ?$, $R_X(t_1, t_2) = ?$

·) $\eta_Y(t) = \eta_X(t) \otimes h(t) = \int_{-\infty}^{+\infty} \eta_X \, h(t-\tau) \, d\tau = \dots \; (t - \tau = \tau')$

$= \eta_X \int_{-\infty}^{+\infty} h(\tau') \, d\tau' = \boxed{\eta_X \, H(0) = \eta_Y}$

$$H(0) = H(f)\Big|_{f=0} = \int_{-\infty}^{+\infty} h(t) \, e^{-j2\pi f t} \, dt \Big|_{f=0}$$

·) $R_Y(t_1, t_2) = R_Y(t, t-\tau) =$ $t_1 = t$, $t_2 = t - \tau$

$= E[Y(t) Y(t-\tau)] = E\Big[[X(t) \otimes h(t)] \, [X(t-\tau) \otimes Y(t-\tau)]\Big]$

$= E\left[\int_{-\infty}^{+\infty} X(\alpha) \, h(t-\alpha) \, d\alpha \int_{-\infty}^{+\infty} X(\beta) \, h(\underset{\sim}{t} - \tau - \beta) \, d\beta\right]$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E[X(\alpha) X(\beta)] \, h(t-\alpha) \, h(t - \tau - \beta) \, d\alpha \, d\beta$$

(α) (β)

$$= \int_{-\infty\,(\alpha)}^{+\infty} \int_{-\infty\,(\beta)}^{+\infty} R_X(\alpha-\beta)\, h(t-\alpha)\, h(t-\tau-\beta)\, d\alpha\, d\beta$$

$$\alpha - \beta = \alpha' \quad \Rightarrow \quad \alpha = \alpha' + \beta$$

$$= \int_{-\infty\,(\alpha')}^{+\infty} \int_{-\infty\,(\beta)}^{+\infty} R_X(\alpha')\, h(t-\alpha'-\beta)\, h(t-\tau-\beta)\, d\alpha'\, d\beta$$

$$= \int_{-\infty\,(\beta)}^{+\infty} \underbrace{\int_{-\infty\,(\alpha')}^{+\infty} R_X(\alpha')\, h(t-\beta-\alpha')\, d\alpha'}\, h(t-\tau-\beta)\, d\beta$$

$$= \int_{-\infty\,(\beta)}^{+\infty} \underbrace{\left[R_X(t-\beta) \otimes h(t-\beta)\right]}_{f(t-\beta)} h(t-\tau-\beta)\, d\beta$$

$$= \int_{-\infty}^{+\infty} f(\underbrace{t-\beta}_{\beta'})\, h\left[-\left(\tau - (\underbrace{t-\beta}_{\beta'})\right)\right] d\beta'$$

$$= f(\tau) \otimes h(-\tau) = \boxed{R_X(\tau) \otimes h(\tau) \otimes h(-\tau) = R_Y(\tau)}$$

$\Rightarrow Y(t)$ e' SSL

$$R_Y(\tau) = R_X(\tau) \otimes r_h(\tau)$$

$$r_h(\tau) = h(\tau) \otimes h(-\tau) = \int_{-\infty}^{+\infty} h(t)\, h(t-\tau)\, dt$$

AUTOCORRELAZIONE

$\xrightarrow{P.A.} E[X(t_1)X(t_2)] = E[X(t)X(t-\tau)]$

$\xrightarrow{S.D.} \int_{-\infty}^{+\infty} x(t)x(t-\tau)\,dt$

DENSITÀ SPETTRALE DI POTENZA DI UN P.A. STAZIONARIO (ALMENO IN SENSO LATO)

$X(t) \to \eta_x(t) = \eta_x$

$X(t) \to R_x(t_1,t_2) = R_x(\tau)$

Definizione per segnali deterministici

$$S_x(f) \triangleq \lim_{T\to\infty} \frac{|X_T(f)|^2}{T}$$

$$x_T(t) \triangleq x(t)\,\mathrm{rect}\left(\frac{t}{T}\right) \quad , \quad X_T(f) = TCF[x_T(t)]$$

$$S_x(\omega_i, f) \triangleq \lim_{T\to\infty} \frac{|X_T(\omega_i,f)|^2}{T}$$

$$S_{\cancel{x}}(f) \triangleq E[S_x(\omega_i,f)] = E\left[\lim_{T\to\infty} \frac{|X_T(\omega_i,f)|^2}{T}\right]$$

$$\boxed{S_x(f) \triangleq \lim_{T\to\infty} \frac{E[|X_T(\omega_i,f)|^2]}{T}}$$

TEOREMA DI WIENER - KHINTCHINE

$$\boxed{S_x(f) = \int_{-\infty}^{+\infty} R_x(\tau)\,e^{-j2\pi f\tau}\,d\tau = TCF[R_x(\tau)]}$$

PROPRIETÀ DEL DSP

1) $S_X(f)$ è reale e pari perché $R_X(\tau)$ è reale e pari

2) $P_X = \int_{-\infty}^{+\infty} S_X(f)\, df$

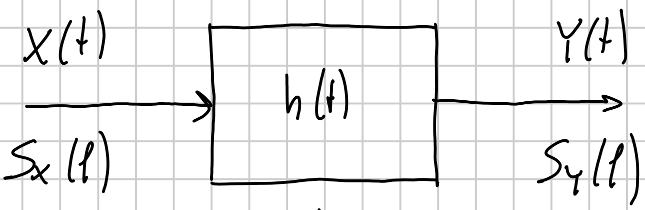
Dim

$$P_X = E\left[X^2(t)\right] = R_X(0) = E\left[X(t)\,X(t-\tau)\right]\Big|_{\tau=0}$$

$$R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\, e^{j2\pi f\tau}\, df\,\Big|_{\tau=0} = \int_{-\infty}^{+\infty} S_X(f)\, df$$

3) $S_X(f) \geq 0 \quad \forall f$

DENSITÀ SPETTRALE DI POTENZA DI UN P.A. IN USCITA AD UN SLS QUANDO IL P.A. IN INGRESSO È SSL



$X(t)$ è SSL

$Y(t)$ è SSL

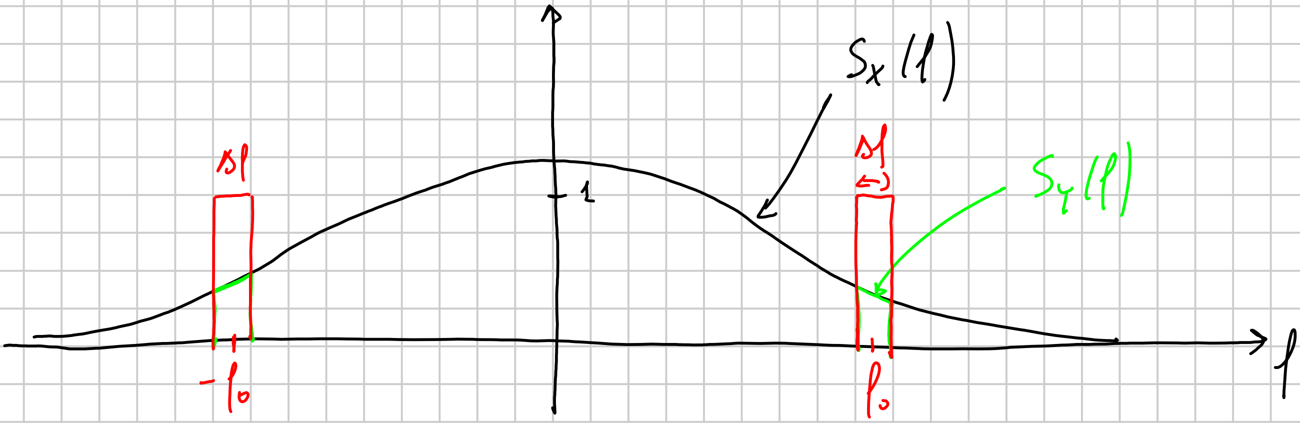
$h(t)$ è reale

$S_Y(f) = ?$

$$S_Y(f) = TCF\left[R_Y(\tau)\right] = TCF\left[R_X(\tau) \otimes h(\tau) \otimes h(-\tau)\right]$$

$$= S_X(f)\, H(f)\, H^*(f) = \boxed{S_X(f)\,|H(f)|^2 = S_Y(f)}$$

$$P_Y = \int_{-\infty}^{+\infty} S_X(f)\,|H(f)|^2\, df \geq 0$$



$$P_Y = \int_{-\infty}^{+\infty} S_Y(f)\,df = \int_{-\infty}^{+\infty} S_X(f)\,|H(f)|^2\,df =$$

$$= 2\int_{f_0-\frac{\Delta f}{2}}^{f_0+\frac{\Delta f}{2}} S_X(f)\,df \simeq 2\,S_X(f_0)\,\Delta f \geq 0 \qquad \forall f_0,\ \forall \Delta f \to 0$$

$$\Downarrow$$

$$S_X(f) \geq 0$$

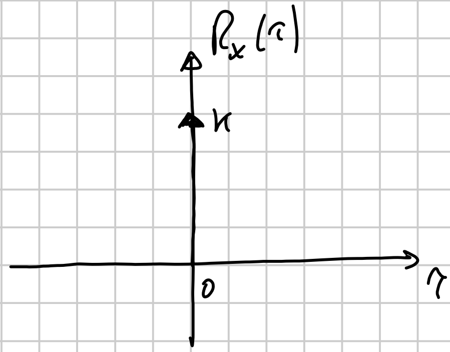
(DIM III° PROPRIETÀ DELLA DSP)

PROCESSO DI RUMORE BIANCO

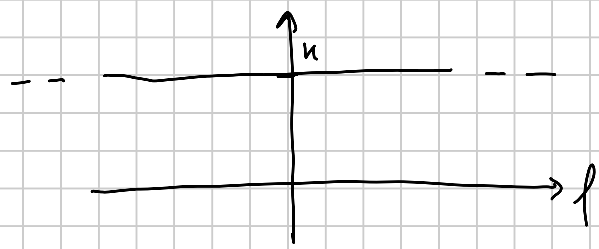
$X(t)$ SSL

j) $\eta_X = 0$

jj) $R_X(\tau) = K\,\delta(\tau)$

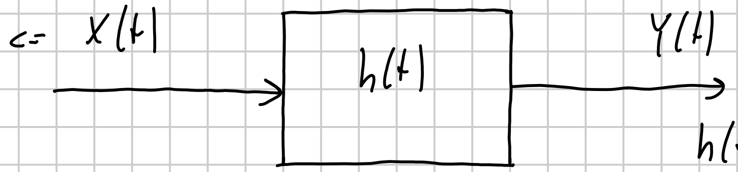


$S_X(f) = K \quad \forall f$



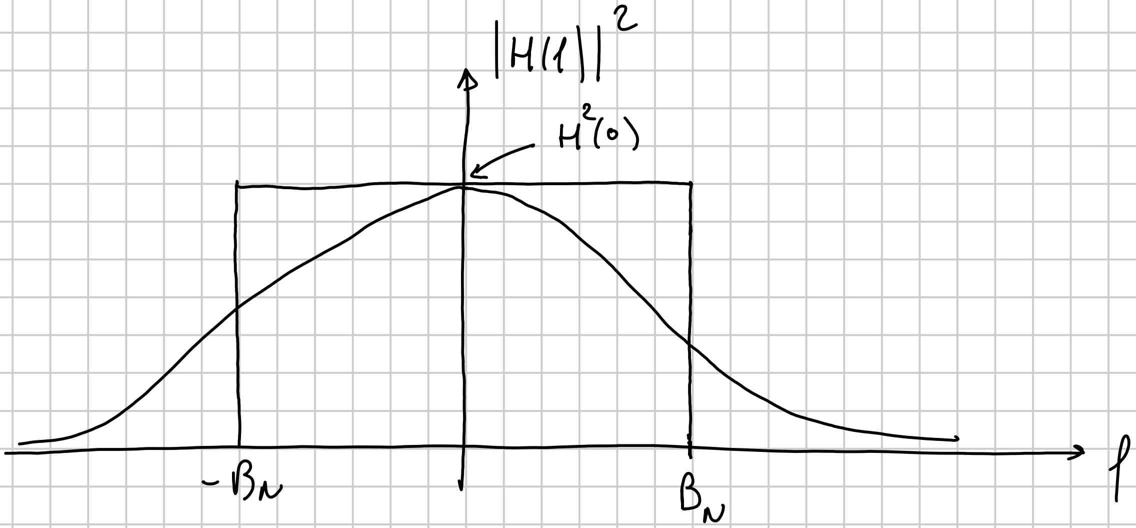
BANDA EQUIVALENTE DI RUMORE DI UN FILTRO L.S.

P.R.B



h(t) reale

$$P_Y = \int_{-\infty}^{+\infty} K\,|H(f)|^2\,df \triangleq \int_{-B_N}^{B_N} K\,H^2(0)\,df$$



$$\cancel{K}\int_{-\infty}^{+\infty} |H(f)|^2\,df = \cancel{K}H^2(0)\int_{-B_N}^{B_N} 1\,df$$

$$\cancel{2}\int_{0}^{+\infty} |H(f)|^2\,df = \cancel{2}H^2(0)\,B_N \Rightarrow \boxed{B_N = \frac{\int_0^{+\infty}|H(f)|^2\,df}{H^2(0)}}$$

PROCESSI ALEATORI GAUSSIANI

Un processo aleatorio è Gaussiano se comunque estratta una n-upla di V.A., fissando $t_1, \dots, t_n$, queste rappresentano un vettore aleatorio Gaussiano.

Ogni qualunque ddp. di ordine n sarà completamente descritta da $\underline{\eta}_X(t)$, $\underline{\underline{C}}_X(t_1,t_2)$

$$f_X(x_1,\dots,x_N;t_1,\dots,t_N) = \frac{1}{\sqrt{(2\pi)^N \det(\underline{\underline{C}}_X)}} e^{-\frac{1}{2}(\underline{x}-\underline{\eta}_X)^T \underline{\underline{C}}_X^{-1}(\underline{x}-\underline{\eta}_X)}$$

$$\underline{\eta}_X(t) = \begin{bmatrix} E[X(t_1)] \\ E[X(t_2)] \\ \vdots \\ E[X(t_N)] \end{bmatrix}$$

$$\underline{\underline{C}}_X = \begin{bmatrix} C_X(t_1,t_1) & \cdots & C_X(t_1,t_N) \\ \vdots & & \\ C_X(t_N,t_1) & \cdots & C_X(t_N,t_N) \end{bmatrix}$$

P.A. GAUSSIANO S.S.L.

$$\underline{\eta}_X(t) = \underline{\eta}_X$$

$$R_X(t_1,t_2) = R_X(t_1-t_2)$$

$$C_X(t_1,t_2) = C_X(t_1-t_2)$$

$\underline{\eta}_X$ costante

$$\underline{\underline{C}}_X = \begin{bmatrix} C_X(0) & C(t_1-t_2) & \cdots & C_X(t_1-t_N) \\ \vdots & \ddots & & \\ & & \ddots & \\ C_X(t_N-t_1) & \cdots & \cdots & C_X(0) \end{bmatrix}$$

$$\underline{X}'(t) \triangleq \left[X(t_1+\Delta t), X(t_2+\Delta t), \dots, X(t_N+\Delta t) \right]$$

$$\underline{\eta}_{X'} = \underline{\eta}_X$$

$$\underline{\underline{C}}_{X'} = \begin{bmatrix} C_X\left(t_1+\cancel{\Delta t} - (t_1+\cancel{\Delta t})\right) & C_X\left(t_1+\cancel{\Delta t} - (t_2+\cancel{\Delta t})\right) & \cdots \\ \vdots & & \\ C_X\left(t_N+\cancel{\Delta t} - (t_1+\cancel{\Delta t})\right) & \cdots & \end{bmatrix} = \underline{\underline{C}}_X$$

$$\boxed{\begin{aligned} \underline{\eta}_{X'} &= \underline{\eta}_X \\ \underline{\underline{C}}_{X'} &= \underline{\underline{C}}_X \end{aligned}}$$

$$f_{X'}(x_1, \dots, x_N; t_1+\Delta t, t_2+\Delta t, \dots, t_N+\Delta t) = f_X(x_1, \dots, x_N; t_1, \dots, t_N)$$

$$\left.\begin{matrix} \text{SSL} \\ + \\ \text{GAUSS.} \end{matrix}\right\} \Longrightarrow \text{SSS}$$

FILTRAGGIO DI P.A. GAUSSIANI

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

Se $X(t)$ è un P.A. Gaussiano e $h(t)$ è la risposta imp. di un SLS $\Rightarrow$ $Y(t)$ è un P.A. Gaussiano

Dim

$$Y(t) = \int_{-\infty}^{+\infty} X(\alpha)\, h(t-\alpha)\, d\alpha \simeq \sum_{n=-\infty}^{+\infty} X(n\Delta\alpha)\, h(t - n\Delta\alpha)$$

Se estraiamo una n-upla da $Y(t)$ allora questa rappresenta un vettore Gaussiano

$$Y(t_1) = \sum_{n=-\infty}^{+\infty} X(n\Delta\alpha)\, h(t_1 - n\Delta\alpha) = \sum_{n=-\infty}^{+\infty} \alpha(t_1, n)\, X(n)$$

$$Y(t_2) = \sum_{n} X(n\Delta\alpha)\, h(t_2 - n\Delta\alpha) = \sum_{n} \alpha(t_2, n)\, X(n)$$

$$\vdots \qquad \vdots$$

$$Y(t_N) = \sum_{n} X(n\Delta\alpha)\, h(t_N - n\Delta\alpha) = \sum_{n} \alpha(t_N, n)\, X(n)$$

$$\boxed{\underline{Y} = \underline{\underline{A}}\ \underline{X}} \Rightarrow \underline{Y} \text{ e' un vettore Gaussiano}$$

$$\underline{\underline{A}} = \begin{bmatrix} \cdots & \alpha(t_1, n) & \alpha(t_1, n+1) & \cdots \\ & \vdots & & \\ \cdots & \alpha(t_N, n) & \alpha(t_N, n+1) & \cdots \end{bmatrix} \qquad \begin{bmatrix} \vdots \\ X(n) \\ X(n+1) \\ \vdots \\ \vdots \end{bmatrix} = \underline{X}$$

$$p_Y(y_1, \ldots, y_N) = \frac{1}{\sqrt{(2\pi)^N \det(\underline{\underline{C}}_Y)}}\, e^{-\frac{1}{2}(\underline{y} - \underline{m}_y)^T \underline{\underline{C}}_Y^{-1} (\underline{y} - \underline{m}_y)}$$

$$\underline{m}_Y, \quad \underline{\underline{C}}_Y \Rightarrow m_Y(t), \quad R_Y(t_1, t_2)$$

ESERCIZIO ① – P.A. PARAMETRICO

$$X(t) = A\cos(2\pi f_0 t + \Theta)$$

A, Θ sono V.A. indipendenti

$$f_A(a) = \frac{1}{\eta} e^{-\frac{a}{\eta}} u(a)$$

$$f_\Theta(\theta) = \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi}\right)$$

·) Calcolare $\eta_x(t)$, $P_x(t)$

·) Stabilire se $X(t)$ è SSL

·) $\eta_x(t) = E[X(t)] = E[A\cos(2\pi f_0 t + \Theta)] =$

$= E[A]\, E[\cos(2\pi f_0 t + \Theta)]$

$$\int_a\int_\theta a\cos(2\pi f_0 t + \theta)\, f_{A\Theta}(a,\theta)\, da\, d\theta =$$

$$= \underbrace{\int_a a\, f_A(a)\, da}_{E[A]} \underbrace{\int_\theta \cos(2\pi f_0 t + \theta)\, f_\Theta(\theta)\, d\theta}_{E[\cos(2\pi f_0 t + \Theta)]}$$

$$\eta_x(t) = E[A]\, E[\cos(2\pi f_0 t + \Theta)]$$

$$E[A] = \int_{-\infty}^{+\infty} a \, f_A(a) \, da = \int_0^{+\infty} \frac{a}{\eta} e^{-\frac{a}{\eta}} \, da =$$

$$= \frac{1}{\eta} \left\{ \underbrace{\left[(-\eta) e^{-\frac{a}{\eta}} \cdot a \right]_0^{+\infty}}_{= 0} - \int_0^{+\infty} (-\eta) e^{-\frac{a}{\eta}} \, da \right.$$

$$= \eta \underbrace{\int_0^{+\infty} \frac{1}{\eta} e^{-a/\eta} \, da}_{1} = \eta \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\eta} e^{-\frac{a}{\eta}} u(a) \, da}_{1} = \eta$$

$$E[A] = \eta$$

$$E[\underbrace{\cos(2\pi f_0 t + \Theta)}_{Y}] = \int_{-\infty}^{+\infty} \cos(2\pi f_0 t + \theta) \frac{1}{2\pi} \operatorname{rect}\left(\frac{\theta}{2\pi}\right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\theta + \underbrace{2\pi f_0 t}_{\alpha}) \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\theta + \alpha) \, d\theta = 0$$

$$\eta_X(t) = \eta \cdot 0 = 0$$

$$\cdot) \ P_X(t) = E[X^2(t)] = E[A^2 \cos^2(2\pi f_0 t + \Theta)] =$$

$$= E\left[\frac{A^2}{2} + \frac{A^2}{2} \cos(4\pi f_0 t + 2\Theta) \right] =$$

$$= E\left[\frac{A^2}{2}\right] + E\left[\frac{A^2}{2} \cos(4\pi f_0 t + 2\Theta)\right]$$

$$E\left[\frac{A^2}{2}\right] = \frac{1}{2}\, E\left[A^2\right]$$

$$E\left[A^2\right] = \int_0^{+\infty} \frac{a^2}{\eta}\, e^{-\frac{a}{\eta}}\, da = \frac{1}{\eta}\left\{\underbrace{\left[a^2 \cdot (-\eta)\, e^{-\frac{a}{\eta}}\right]_0^{+\infty}}_{0} +\right.$$

$$\left. - \int_0^{+\infty} (-\eta)\, e^{-\frac{a}{\eta}}\, 2a\, da \right\}$$

$$= 2\int_0^{+\infty} a\, e^{-\frac{a}{\eta}}\, da = 2\eta \underbrace{\int_0^{+\infty} a\, \frac{1}{\eta}\, e^{-\frac{a}{\eta}}\, da}_{\eta} =$$

$$= 2\eta^2$$

$$E\left[\frac{A^2}{2}\right] = \eta^2$$

$$E\left[\frac{A^2}{2}\cos\left(4\pi f_0 + 2\theta\right)\right] = \underbrace{E\left[\frac{A^2}{2}\right]}_{\eta^2} \cdot E\left[\cos\left(4\pi f_0 t + 2\theta\right)\right]$$

$$E\left[\cos\left(4\pi f_0 t + 2\theta\right)\right] = \int_{-\infty}^{+\infty} \cos\left(4\pi f_0 t + 2\theta\right) \frac{1}{2\pi}\, \mathrm{rect}\left(\frac{\theta}{2\pi}\right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos\left(2\theta + \beta\right) d\theta = 0$$

$$P_x(t) = \eta^2 + \eta^2 \cdot 0 = \eta^2 = P_x$$

$$X(t) \text{ è SSL} \begin{cases} \eta_X(t) = \eta_X \quad \text{ok} \\ R_X(\tau) \end{cases}$$

$$R_X(t, t-\tau) = E[X(t)\,X(t-\tau)] =$$

$$= E\left[A\cos(2\pi f_0 t + \Theta) \cdot A\cos[2\pi f_0 (t-\tau) + \Theta]\right]$$

$$= E\left[\underbrace{\frac{A^2}{2}\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\Theta)}_{=0} + \frac{A^2}{2}\cos(2\pi f_0 \tau)\right]$$

$$\left(\cos\alpha \cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)\right)$$

$$\Rightarrow E\left[\frac{A^2}{2}\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\Theta)\right] =$$

$$= \underbrace{E\left[\frac{A^2}{2}\right]}_{\eta^2} \; \underbrace{E\left[\cos(4\pi f_0 t - 2\pi f_0 \tau + 2\Theta)\right]}_{\int_{-\infty}^{+\infty} \cos(4\pi f_0 t - 2\pi f_0 \tau + 2\Theta)\frac{1}{2\pi}\,\mathrm{rect}\left(\frac{\Theta}{2\pi}\right) d\Theta}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(2\Theta + \alpha)\, d\Theta = 0$$

$$E\left[\frac{A^2}{2}\cos(2\pi f_0 \tau)\right] = E\left[\frac{A^2}{2}\right]\cos(2\pi f_0 \tau) = \eta^2 \cos(2\pi f_0 \tau)$$

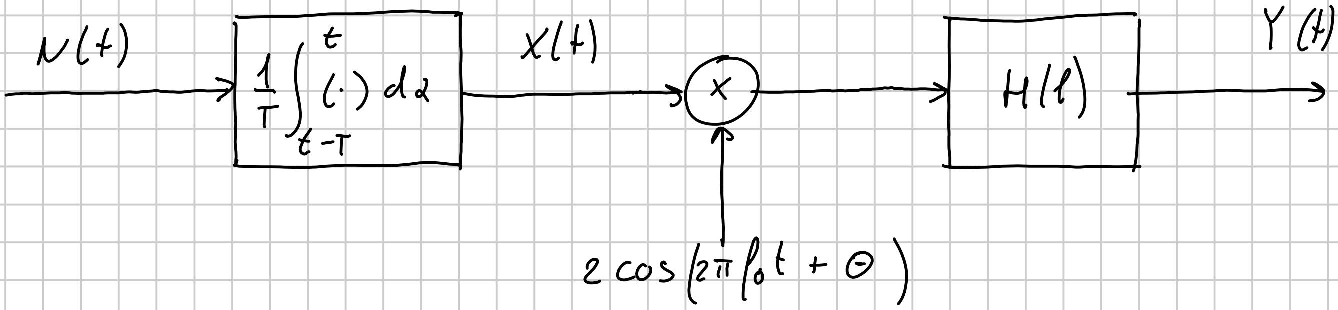
$$R_X(\tau) = \eta^2 \cos(2\pi f_0 \tau) \Rightarrow \text{SSL}$$

$$\eta_X = 0$$

$$\lim_{\tau \to \infty} R_X(\tau) \neq \eta_X^2$$

NO PERCHÈ È PRESENTE UNA COMPONENTE PERIODICA

ES. PER CASA



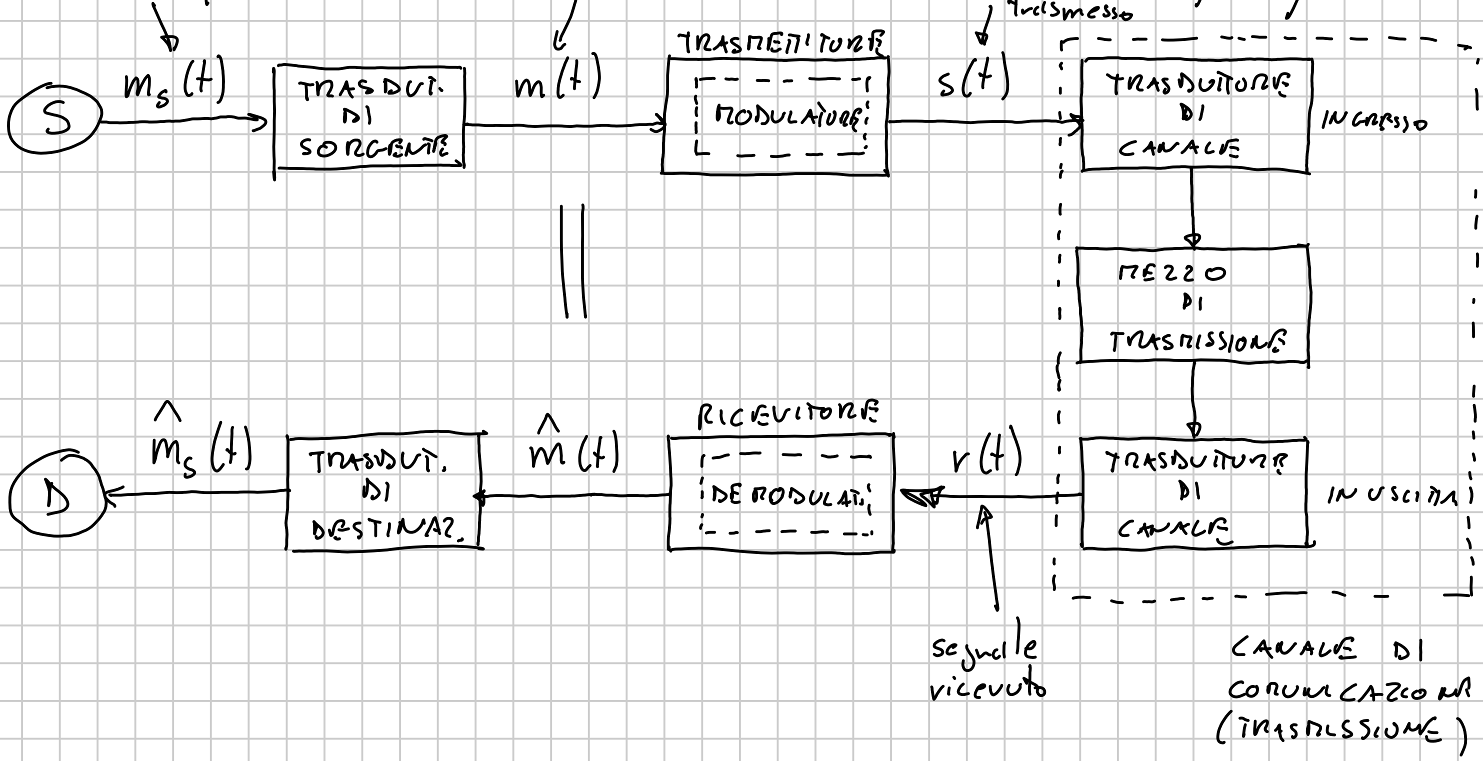
$N(t)$ è un processo GAUSSIANO SSL con $\left.\begin{array}{l} S_N(f) = K \\ m_N = 0 \end{array}\right\}$ BIANCO

Θ è una V.A. indipendente da $N(t)$

$\Theta \in U(-\pi, \pi)$

·) Calcolare la $S_Y(f)$

22/04/2013

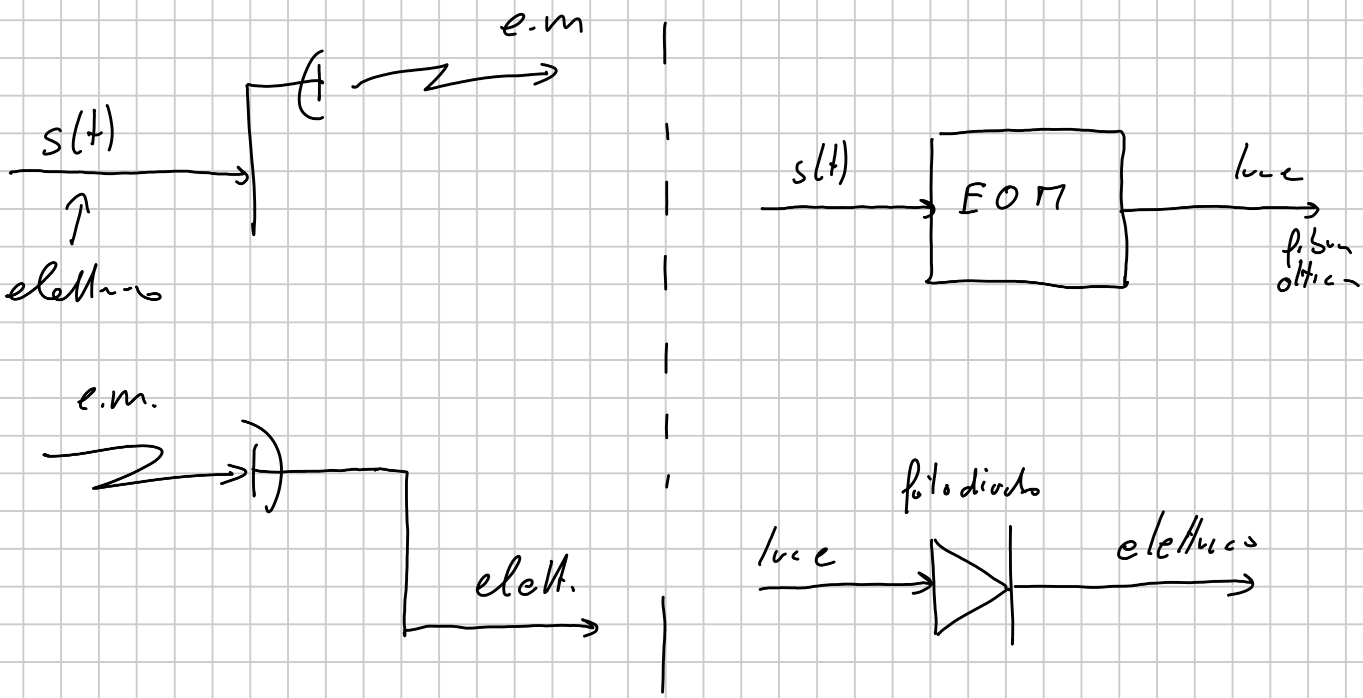


SISTEMA DI COM. IDEALE

$$\hat{m}_s(t) = m_s(t)$$

TRASMETTITORE

- SAGOMATURA
- TRASLAZ. IN FREQ (MODULAZIONI)



RICEVITORE:

nel caso ideale $r(t) = s(t)$

nel caso reale $r(t) \neq s(t)$

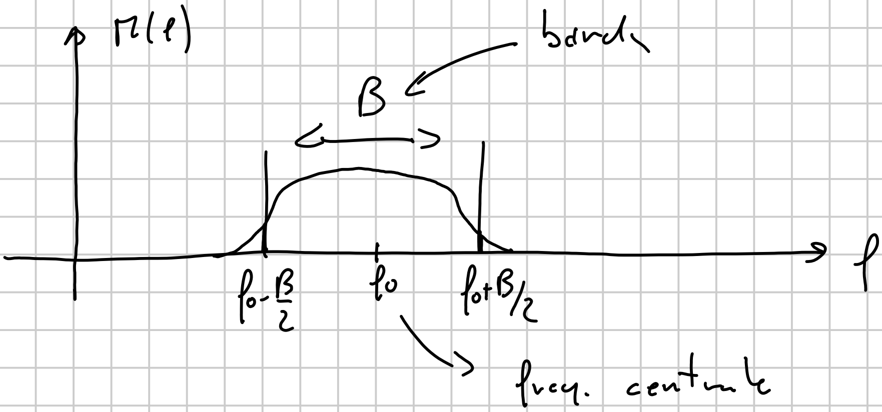
Compito del ricevitore è di ricostruire $\hat{m}(t)$: $\hat{m}(t)$ sia il più possibile simile a $m(t)$

SEGNALE IN B.B

$TCF[m(t)] = M(f)$

$B < \infty$

SEGNALE IN B.P.



SEGNALI A BANDA STRETTA o LARGA

$f_0 > 2B$ BANDA STRETTA

$f_0 < 2B$ " LARGA

ESEMPI

·) DOPPINO TELEFONICO

BANDA PASSANTE $300\ Hz \div 4\ KHz$ (ZONA LINEARE)

$$\left.\begin{aligned} f_0 &= 2.15\ KHz \\ B &= 3.7\ KHz \end{aligned}\right\}\ \text{BANDA LARGA}$$

·) DVB-T

$$\left.\begin{aligned} f_0 &= 400\ MHz \\ B &= 8\ MHz \end{aligned}\right\}\ \text{BANDA STRETTA}$$

·) CANALE RADIO

- ·) passa banda
- ·) tipicamente a banda stretta
- ·) trasduttori sono "antenne"

Dimensione dell'antenna minima x trasmissione/ricezione

$$\frac{\lambda}{10}, \quad \lambda = \frac{c}{f_0}$$

c ← vel. della luce nel vuoto

f_0 ← freq. centrale

LF (Low frequency) $30 - 300\ KHz$ ($\lambda = 1 - 10\ Km$) Radiolocalizzazione marittima/aeronautica

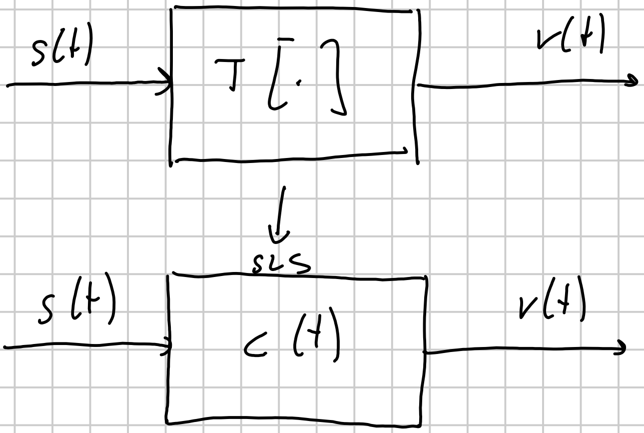
MF (Medium ") $300 - 3000\ KHz$ ($\lambda = 100 - 1000\ m$) Radionavigazione e radio diffusione

HF (High freq.) $3 - 30\ MHz$ $\lambda = (10 - 100\ m)$ Radio comunicaz. a grande dist.

VHF	(Very High freq)	30 - 300 MHz	Radio FM (DVB-T) Radio diff.
UHF	(Ultra High freq)	300 - 3000 MHz	Telefonia mobile LAN wireless
SHF	(Super High freq)	3 - 30 GHz	TV satellitare, ponti radio, ecc.

CANALE DI COMUNICAZIONE

$$r(t) \neq s(t)$$



$$r(t) = s(t) \otimes c(t)$$

$$R(f) = S(f)\, C(f)$$

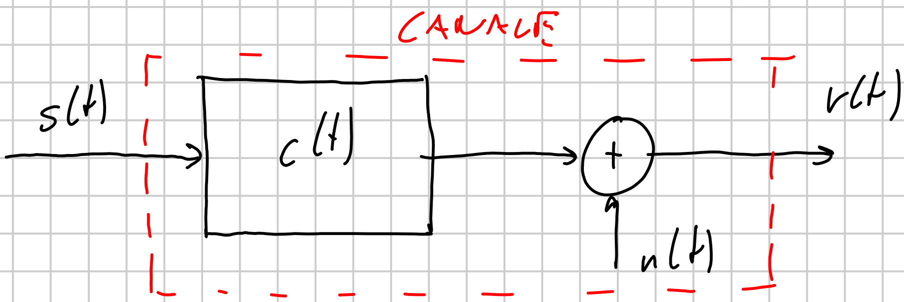
RUMORE

noise ovvero rumore

$$r(t) = s(t) \otimes c(t) + n(t)$$

$r(t)$ → processo aleatorio

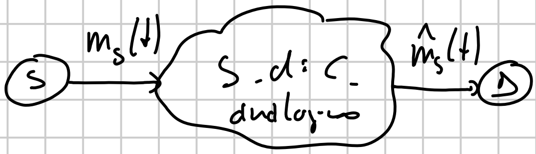
$n(t)$ → processo aleatorio



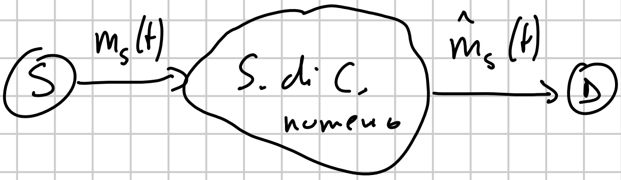
$n(t)$ comprende tutte le sorgenti di rumore: sia quelle interne che esterne al ricevitore

SISTEMA DI COMUNICAZIONE ANALOGICO

$m_s(t)$ e quindi $\hat{m}_s(t)$ sono segnali analogici

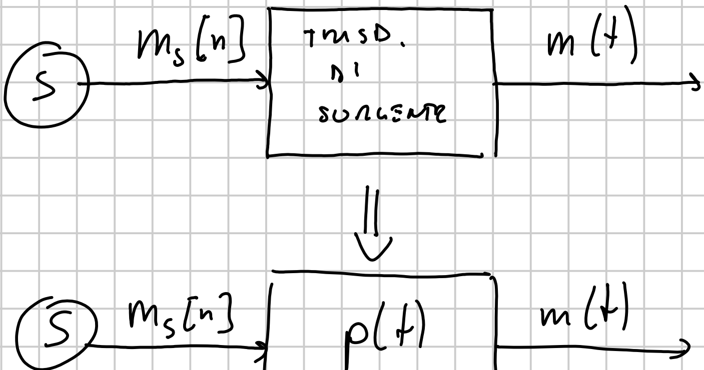


SISTEMA DI COMUNICAZIONE NUMERICO



$m_s(t)$ e quindi $\hat{m}_s(t)$ sono segnali numerici

Sarebbe più idoneo indicarli come $m_s[n]$, $\hat{m}_s[n]$



T_s

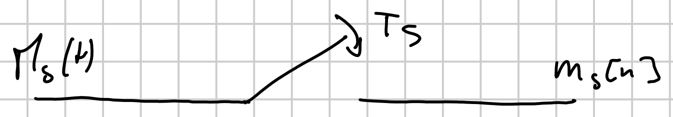
↳ periodo di segnalazione

simbolo

$A_s = \{0, 1\}$, $A_s \{\pm 1, \pm 3\}$, $A_s \{0, \pm 1\}$

$m_s[n]$ sequenza di simboli emessi dalla sorgente

$m_s[n]$ è rappresentabile come il risultato di un campionamento con $T_c = T_s$ di un Processo Aleatorio $M_s(t)$



$\hat{m}_s[n]$ sequenza aleatoria (di destinazione)

$$m(t) = \sum_{k=-\infty}^{+\infty} m_s[n]\, p(t - kT_s)$$

P.A. a tempo continuo

$p(t)$ = impulso in trasmissione

esempio: $p(t) = \mathrm{rect}\left(\frac{t}{T_s}\right)$

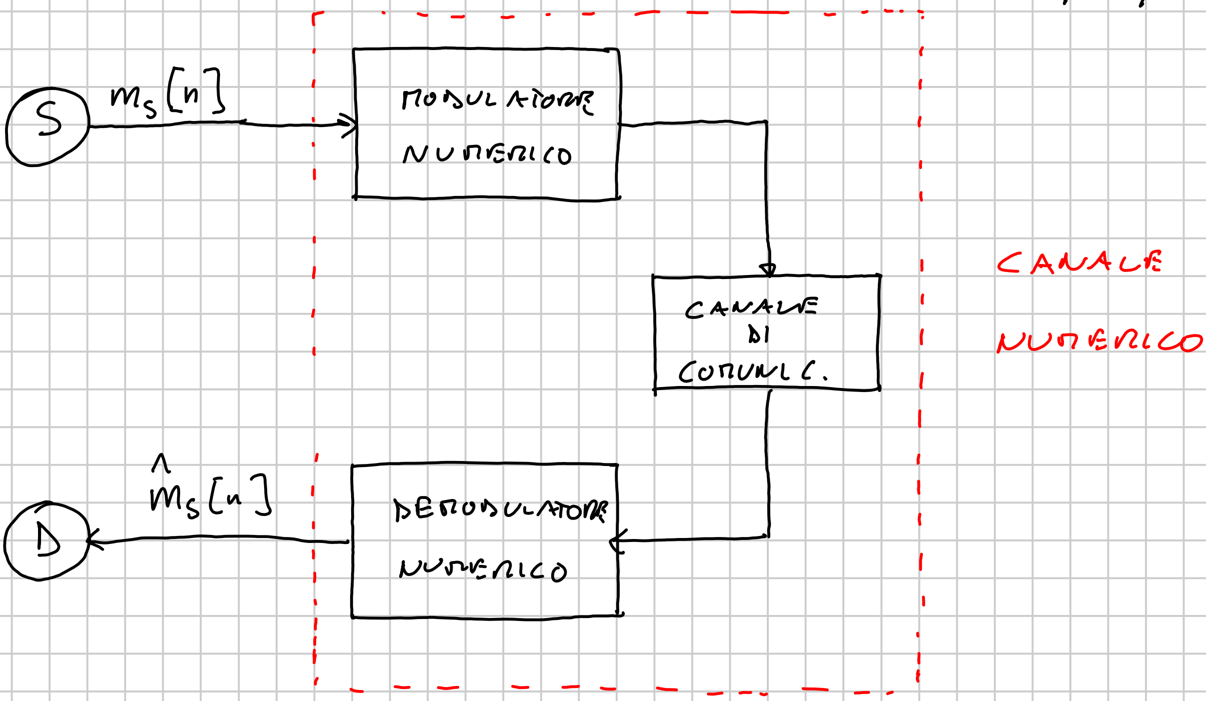
TASSO DI EROGAZIONE DELLA SORGENTE

$$R_s \triangleq \frac{1}{T_s} \qquad [\text{simboli}/\text{secondo}]$$

$$R_b = \frac{\log_2 M}{T_s}$$ TASSO DI EROG BINARIO $[\text{bit}/s]$

M = cardinalità di A_s

24/04/2013



CANALE NUMERICO È IDEALE SE

$$\hat{m}_s[n] = m_s[n] \qquad \forall n$$

NON IDEALITÀ $\Rightarrow$ $\hat{m}_s[n] \neq m_s[n]$

PROBABILITÀ DI TRANSIZIONE

$$P\{i|j\} \triangleq P\{\hat{m}_s[n] = \alpha_i \mid m_s[n] = \alpha_j\}$$

in generale dipende da "n"

Quando $P\{i|j\}$ è independente da "n"

$$\Downarrow$$

CANALE NUMERICO È STAZIONARIO

CANALE IDEALE

$$\begin{cases} P\{i|j\} = 1 & \text{se} \quad i = j \\ P\{i|j\} = 0 & \text{se} \quad i \neq j \end{cases}$$

PROBABILITÀ DI ERRORE DI SIMBOLO (M-ARIO)

$$P_E(M) \triangleq P\{\hat{m}_s[n] \neq m_s[n]\}$$

Se $P_E(M)$ è indipendente da "n"

$\Downarrow$

SISTEMA DI COMUNIC. È STAZIONARIO

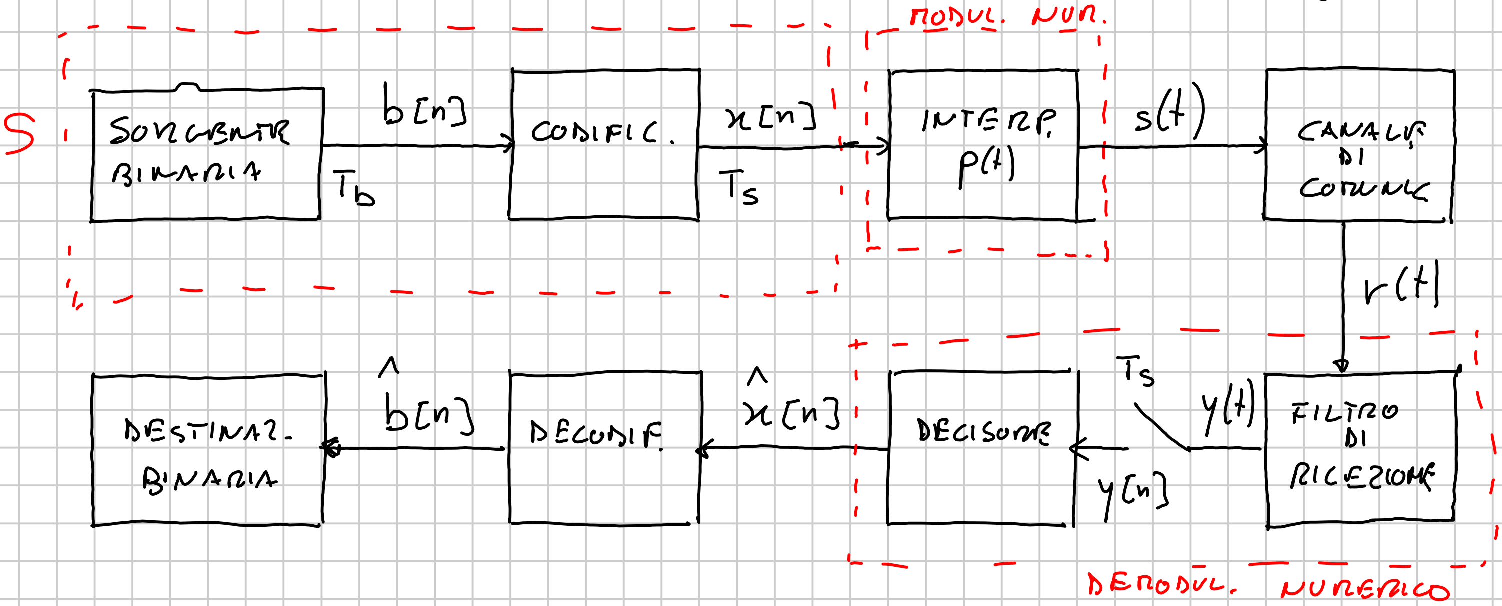
QoS (QUALITY OF SERVICE)

$$P_E(M) \leq P_{MAX}$$

Servizio fonia $\Rightarrow$ $P_{MAX} \cong 10^{-3}$

servizio dati $\Rightarrow$ $P_{MAX} \cong 10^{-6} - 10^{-8}$

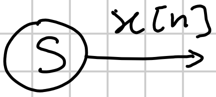
MODULAZIONI NUMERICHE IN BANDA BASE



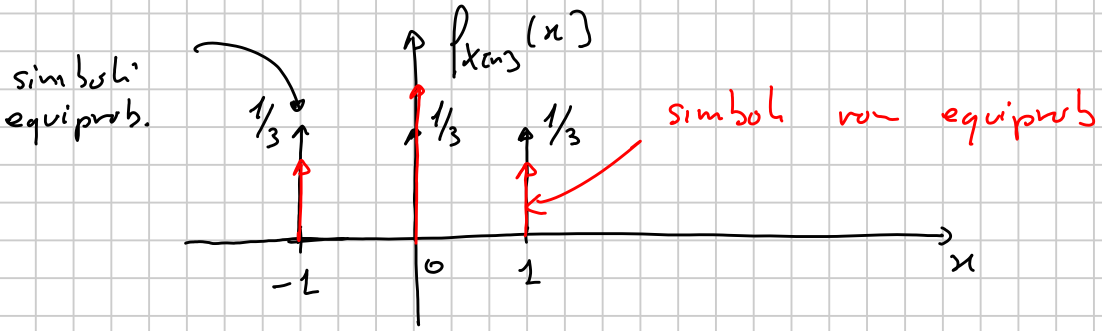
·) $s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT_s)$ $\quad$ $p(t)$ è detto impulso sagomatore

CODIFICATORE SINCRONO CON LA SORG. BIN.

$$T_s = T_b \log_2 M \Rightarrow R_s \triangleq \frac{1}{T_s} = \frac{1}{T_b \log_2 M}$$



$x[n]$ simboli appartenenti ad un processo SSL



INTERPOLATORE

$p(t)$

$$S_n(t) = x[n]\, p(t - nT_s)$$

$p(t) \Rightarrow$ banda B_p

$\Rightarrow$ Energia $E_p = \int_{-\infty}^{+\infty} |p(t)|^2 \, dt = \int_{-\infty}^{+\infty} |P(f)|^2 \, df$

$s(t)$ è un processo SSL

↳ P_s, B_T

$$E_s = \int_{-\infty}^{+\infty} |x[n]\, p(t)|^2 \, dt$$

$$E_s = P_s T_s \Rightarrow E_b = \frac{E_s}{\log_2 M} = \frac{P_s T_s}{\log_2 M}$$

DEMODULATORE NUMERICO

$\Rightarrow$ fornire una sequenza $\hat{x}[n]$ che sia il più possibile "vicina" a $x[n]$

$\Rightarrow$ minimizzare P_E, definendo prima

$$P_{E_S} = P\{\hat{x}[n] \neq x[n]\}$$

$$P_{E_b} = P\{\hat{b}[n] \neq b[n]\} \qquad \text{BEP}, \ \text{BER}$$

PRESTAZIONI

$$x[n] \longrightarrow \boxed{\begin{array}{c}\text{CANALE}\\ \text{NUMER.}\end{array}} \longrightarrow \hat{x}[n]$$

$x[n] \in A_S$, $\quad A_S = \{\alpha_1, \alpha_2, \dots, \alpha_M\}$ ALFAB. M-ARIO

$$P_E(M) \triangleq P\{\hat{x}[n] \neq x[n]\} =$$

$$= \sum_{i=1}^{M} \sum_{\substack{j=1 \\ j\neq i}}^{M} P\{\hat{x} = \alpha_i, x = \alpha_j\}$$

$$\begin{bmatrix} (\hat{x}=\alpha_1, x=\alpha_1) & (\hat{x}=\alpha_2, x=\alpha_1) & \dots & (\hat{x}=\alpha_M, x=\alpha_1) \\ (\hat{x}=\alpha_1, x=\alpha_2) & (\hat{x}=\alpha_2, x=\alpha_2) & \dots & \\ \vdots & & & \\ (\hat{x}=\alpha_1, x=\alpha_M) & \dots & \dots & (\hat{x}=\alpha_M, x=\alpha_M) \end{bmatrix}$$

$$\doteq \sum_{i=1}^{M} \sum_{\substack{j=1 \\ j\neq i}}^{M} \underbrace{P\{\hat{x}=\alpha_i \mid x=\alpha_j\}}_{\text{PROB. DI TRANS.}} \underbrace{P\{\alpha_j\}}_{\text{PROB A PRIORI}}$$

Se i simboli sono equiprob.

$$= \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1\,(j\neq i)}^{M} P\{\hat{x}=\alpha_i \mid x=\alpha_j\}$$

FORMATO DI MODULAZIONE EQUIENERGIA

$$E_S(i) = \int_{-\infty}^{+\infty} |s_i(t)|^2 \, dt$$

$s_i(t)$ è il segnale trasmesso in corrispondenza del simbolo α_i

$$E_S(i) = E_S \qquad \forall i$$

ORTOGONALITÀ

$$\int_{-\infty}^{+\infty} s_i(t) \, s_j(t) \, dt = 0 \qquad \forall i \neq j$$

EFFICIENZA DI UN FORMATO DI MODULAZIONE

·) EFFICIENZA ENERGETICA

$$\eta_P \triangleq \frac{1}{SNR} \quad , \quad SNR \triangleq \frac{P_S}{P_N}$$

Signal to Noise Ratio

η_P che garantisce una determinata BEP

EFFICIENZA SPETTRALE

fissata la BEP

$$\eta_B \triangleq \frac{R_b}{B_T} \qquad [bit/s/Hz]$$

$$\eta_B = \frac{1}{B_T T_b} = \frac{\log_2 M}{B_T T_S}$$

PULSE AMPLITUDE MODULATION (PAM)

M-PAM o PAM M-aria

M è il # dei simboli di A_s

Caratteristica:

1) $s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT_s)$

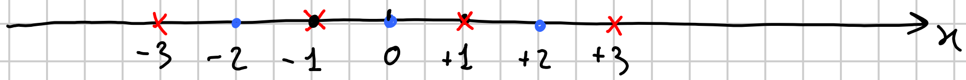
2) $M \geq 2 \Rightarrow A_s = \{\alpha_1, \dots, \alpha_M\}$

$$\boxed{\alpha_i = 2i - 1 - M}$$

$M = 2 \Rightarrow \alpha_1 = -1, \quad \alpha_2 = 1$

$M = 3 \Rightarrow \alpha_1 = -2, \quad \alpha_2 = 0, \quad \alpha_3 = 2$

$M = 4 \Rightarrow \alpha_1 = -3, \quad \alpha_2 = -1, \quad \alpha_3 = 1, \quad \alpha_4 = 3$



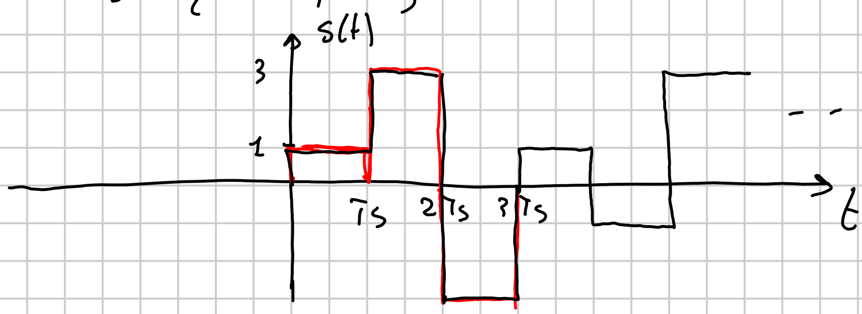
$$E_s(i) = \int_{-\infty}^{+\infty} s_i^2(t)\, dt = \int_{-\infty}^{+\infty} \alpha_i^2\, p^2(t)\, dt =$$

$$= \int_{-\infty}^{+\infty} (2i - 1 - M)^2\, p^2(t)\, dt = (2i - 1 - M)^2 E_p$$

M è solitamente una potenza di 2

4-PAM $(M = 4) \Rightarrow A_s = \{\pm 1, \pm 3\}$

$p(t) = \text{rect}\left(\frac{t - T_s/2}{T_s}\right)$



PROPRIETÀ DELLA M-PAM

1) $E\left[s(t)\right] = 0 \quad \forall t$ se i simboli α_i sono equiprob.

$$E\left[s(t)\right] = E\left[\sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT_s)\right] =$$

$$= \sum_{n=-\infty}^{+\infty} E\left[x[n]\right] p(t-nT_s) =$$

$$E\left[x[n]\right] = \int_{-\infty}^{+\infty} x\, f_x(x)\, dx = \sum_{i=1}^{M} p\{\alpha_i\}\, \alpha_i =$$

$$= \frac{1}{M}\sum_{i=1}^{M}\left(2i-1-M\right) = \frac{1}{M}\sum_{i=1}^{M} 2i - 1 - M =$$

$$= \frac{2}{M}\sum_{i=1}^{M} i - 1 - M = \frac{\cancel{2}}{\cancel{M}}\,\frac{\cancel{M}(M+1)}{\cancel{2}} - (1+M) = 0$$

$$\int_{-\infty}^{+\infty} x(t)\, dt = 0 \quad \Rightarrow \quad X(f)\Big|_{f=0} = X(0) = 0$$

$x(t)$

$X(0) = 0$

2) $S_s(f) = \dfrac{1}{T_s}\, S_x(f)\, |P(f)|^2$

DSP del segnale trasmesso

$P(f) = TCF\left[p(t)\right]$

$TCF\left[R_x[m]\right]$

funzione di autocorrelazione dei simboli $x[n]$

Dimostrazione nelle note del docente (MOODLE)

CASO PARTICOLARE

- $E\left[x[n] \right] = 0$
- $R_x[m] = \sigma_x^2 \delta[m]$ (simboli incorrelati)

$$\Rightarrow S_s(f) = \frac{\sigma_x^2}{T_S} |P(f)|^2$$

↳ DSE (segnale deterministico)

$\Rightarrow B_T$ è esattamente uguale a B_P

$$B_T = B_P$$

σ_x^2 per una M-PAM

$$\sigma_x^2 = E\left[(x-\eta_x)^2 \right] = \int_{-\infty}^{+\infty} (x-\eta_x)^2 f_x(x)\, dx$$

$$= \frac{1}{M} \sum_{i=1}^{M} (x-\eta_x)^2 = \frac{1}{M} \sum_{i=1}^{M} (2i-1-M)^2$$

$$\eta_x = E\left[x \right] = 0$$

$$\sigma_x^2 = \frac{1}{M} \sum_{i=1}^{M} \left[4i^2 + (1+M)^2 - 4i(1+M) \right] =$$

$$= \frac{1}{M} \sum_{i=1}^{M} 4i^2 + \frac{1}{M} \sum_{i=1}^{M} (1+M)^2 - \frac{1}{M} \sum_{i=1}^{M} 4i(1+M) =$$

$$= \frac{4}{M} \sum_{i=1}^{M} i^2 + (1+M)^2 - \frac{4(1+M)}{M} \sum_{i=1}^{M} i$$

$$\sum_{i=1}^{M} i = \frac{M(M+1)}{2} \quad , \quad \sum_{i=1}^{M} i^2 = \frac{M(M+1)(2M+1)}{6}$$

$$= \dots \Rightarrow \boxed{\sigma_x^2 = \frac{M^2-1}{3}}$$

3) $$P_s = \int_{-\infty}^{+\infty} S_s(f)\, df = \int_{-\infty}^{+\infty} \frac{\sigma_x^2}{T_s} |P(f)|^2 \, df =$$

$$= \frac{\sigma_x^2}{T_s} E_p = \boxed{\frac{M^2-1}{3} \frac{E_p}{T_s}}$$

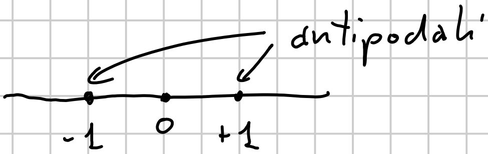
EFFICIENZA SPETTRALE DI UNA M-PAM

$$\eta_B = \frac{R_b}{B_T} = \frac{\log_2 M}{B_T T_s} = \frac{\log_2 M}{B_p T_s}$$

PAM BINARIA (BPSK = binary Phase shift keying)

- $s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT_s)$
- $A_s = \{\pm 1\}$
- $T_b = T_s$
- Equienergia

$$E_s(i) = \int_{-\infty}^{+\infty} s_i^2(t)\, dt = \int_{-\infty}^{+\infty} (\pm 1)^2 p^2(t)\, dt \quad E_p$$



- $E[s(t)] = 0$
- $S_s(f) = \frac{1}{T_b} |P(f)|^2 \qquad \sigma_x^2 = \frac{M^2-1}{3} = \frac{4-1}{3} = 1$

·) $P_s = \frac{E_p}{T_b}$

·) $B_T = B_p$

·) $\eta_B = \frac{1}{B_p T_b}$

↳ se si scelgono impulsi rettangolari $p(t) = \text{rect}\left(\frac{t - T_s/2}{T_s}\right)$

$T_s = T_b$

$$\Downarrow$$

$$B_p = \frac{1}{T_b} \Rightarrow \eta_B = 1$$

SEGNALAZIONE ON-OFF

·) $A_s = \{0, 1\}$

·) $p(t) = \text{rect}\left(\frac{t - T_b/2}{T_b}\right)$

$s_1(t) = 0$

$s_2(t) = p(t)$

$$s(t) = \sum_{n=-\infty}^{+\infty} x[n] \; \text{rect}\left(\frac{t - T_b/2 - nT_b}{T_b}\right)$$

$$E_{s_1} = \int_{-\infty}^{+\infty} s_1^2(t)\, dt = 0$$

$$E_{s_2} = \int_{-\infty}^{+\infty} s_2^2(t)\, dt = \int_{-\infty}^{+\infty} p^2(t)\, dt = T_b$$

$$E_s = E\left[\int_{-\infty}^{+\infty} \sum_n x^2[n]\, p^2(t - nT_b)\; dt\right] = \frac{1}{2} \cdot T_b = \frac{T_b}{2}$$

$$P_s = \frac{E_s}{T_b} = \frac{1}{2}$$

$\cdot)\ R_x[m] = C_x[m] + \eta_x^2 = \frac{1}{4}\delta[m] + \frac{1}{4}$

$\eta_x = E\left[x[n]\right] = \frac{1}{2}$

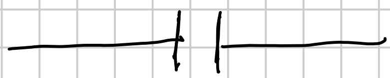
$\cdot)\ S_x(f) = TFS\left[R_x[m]\right] = \sum_{m=-\infty}^{+\infty} R_x[m]\, e^{-j2\pi f m T_b}$

$= \frac{1}{4} + \frac{1}{4}\sum_{m=-\infty}^{+\infty} e^{-j2\pi f m T_b} = \frac{1}{4} + \frac{1}{4T_b}\sum_m \delta\left(f - \frac{m}{T_b}\right)$

$\cdot)\ S_s(f) = \frac{1}{T_b} S_x(f)\,|P(f)|^2 =$

$= \frac{1}{T_b}\left[\frac{1}{4} + \frac{1}{4T_b}\sum_m \delta\left(f - \frac{m}{T_b}\right)\right] T_b^2 \operatorname{sinc}^2(T_b f)$

$= T_b\left(\frac{1}{4} + \frac{1}{4T_b}\delta(f)\right)\operatorname{sinc}^2(T_b f)$



c'è una comp. continua

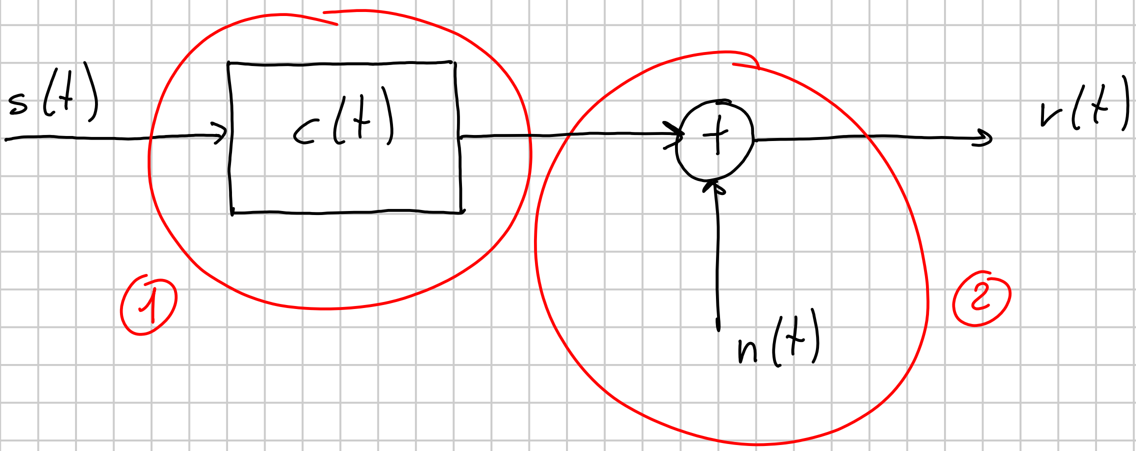
$\cdot)\ \eta_B = \frac{\log_2 2}{T_b B_p} = \frac{1}{1} = 1$

06/05/2013

PRESTAZIONI DI SISTEMI DI COMUNICAZIONE NUMERICI IN BANDA BASE

1) DISTORSIONI
2) RUMORE

} INTRODOTTE DAL CANALE

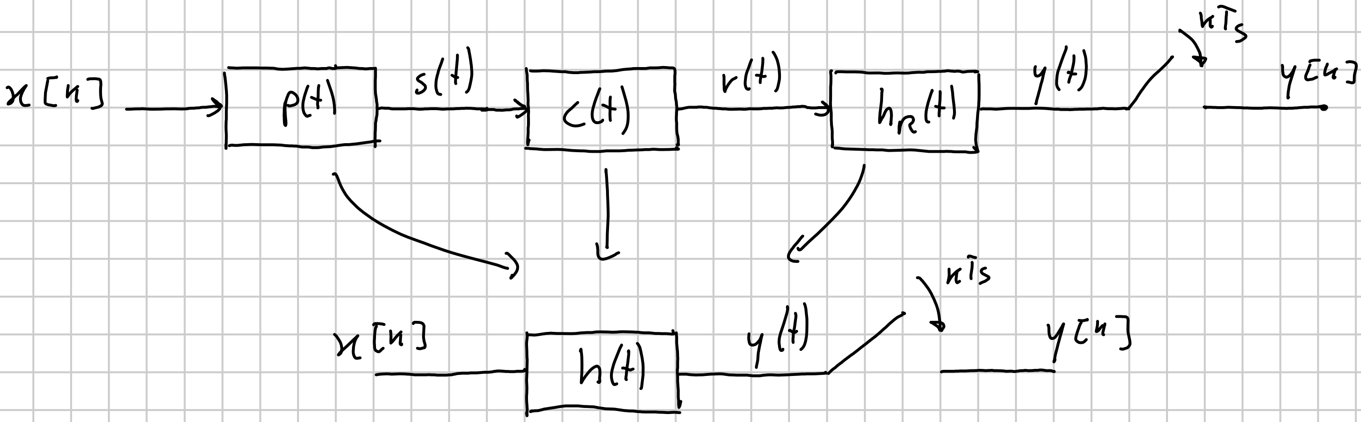


INTERFERENZA INTER-SIMBOLICA (ISI)

Inter symbolic Interference

I) $y[n] = f[x[n]]$ ASSENZA DI ISI

II) $y[n] = f[\ldots, x[n-1], x[n], x[n+1], \ldots]$ PRESENZA DI ISI



$$h(t) \triangleq p(t) \otimes c(t) \otimes h_R(t)$$

↳ interpolatore equivalente

$$H(f) = P(f)\, C(f)\, H_R(f)$$

$$Y(f) = \bar{X}(f)\, H(f) = \bar{X}(f) \underbrace{P(f)\, C(f)\, H_R(f)}_{H(f)}$$

$$y(t) = \sum_{n=-\infty}^{+\infty} x[n]\, h(t - nT_s)$$

$$y[k] = y(kT_s) = \sum_{n=-\infty}^{+\infty} x[n]\, h(kT_s - nT_s) =$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, h((k-n)T_s) =$$

$$= \underbrace{x[k]\, h(0)}_{\text{termine utile}} + \underbrace{\sum_{\substack{n=-\infty \\ n \neq k}}^{+\infty} x[n]\, h((k-n)T_s)}_{ISI}$$

1) Determinare il valore minimo di T_s per cui si può eliminare l'ISI

2) Condizioni per cui è possibile trasmettere una mod. M-PAM con assenza di ISI

⇒ CRITERIO DI NYQUIST PER LA TRASMISSIONE PRIVA DI ISI

·) $h[k] = h(kT_s) = \begin{cases} 1 & \text{per } k=0 \\ 0 & \text{per } k \neq 0 \end{cases}$ DOMINIO DEL TEMPO

·) $\sum_{k=-\infty}^{+\infty} H\left(f - \frac{k}{T_s}\right) = T_s \quad \forall f$ DOMINIO DELLA FREQ

Dim. (DOMINIO DEL TEMPO)

$$y[k] = f(x[k]) = \sum_{n=-\infty}^{+\infty} x[n]\, h((k-n)T_s) =$$

$$= x[k]\, h(0) + \sum_{\substack{n=-\infty \\ n\neq k}}^{+\infty} x[n]\, h((k-n)T_s)$$

$$h[n] = 0 \qquad n \neq 0 \qquad\qquad (k-n = n')$$

$$y[k] = x[k]\, h(0) + \sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} x[k-n']\, h(n' T_s)$$

$$= x[k]\, h[0] + \underbrace{\sum_{\substack{n'=-\infty \\ n'\neq 0}}^{+\infty} x[k-n']\, h[n']}_{=\,0} \Rightarrow \text{NO ISI}$$

Dim (DOMINIO FREQ.)

$$h[k] = \begin{cases} 1 & k=0 \\ 0 & k\neq 0 \end{cases} \Rightarrow h[k] = \delta[k]$$

$$\bar{H}(f) = 1 \qquad \forall f$$

$$\bar{H}(f) = \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right) = 1 \qquad \forall f$$

$$\Rightarrow \sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right) = T_s \qquad \forall f$$

c.v.d.

) VALORE MINIMO DI T_s

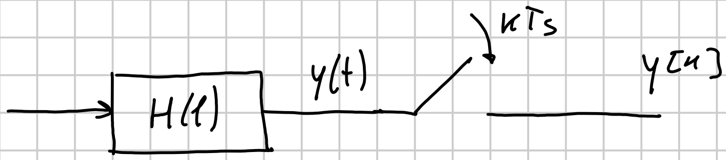
Si suppone di avere un canale $c(t)$ di banda rigorosamente limitata B_c

$$C(f) = 0 \qquad |f| > B_c$$

$$H(f) = P(f)\, C(f)\, H_R(f)$$

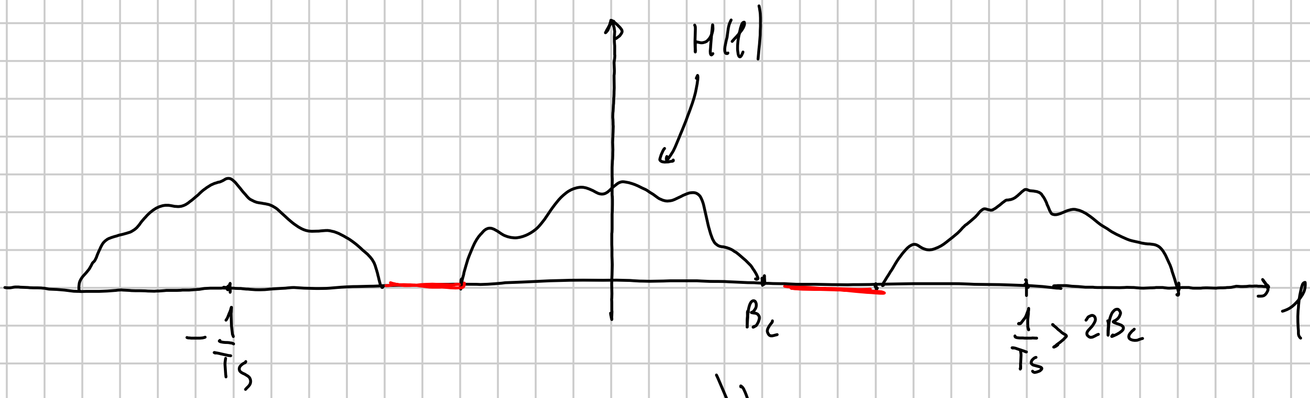
$\Downarrow$

banda rig. lim. $B_H \le B_c$



$$T_s < \frac{1}{2B_c}$$

$$\bar{H}(f) = TFS[h[n]]$$

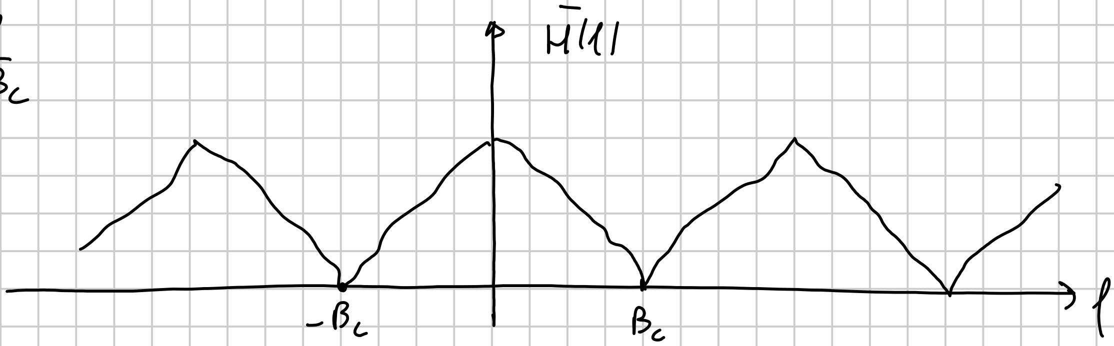


$\Downarrow$

NON PUÒ ESISTERE LA CONDIZIONE PER CUI SI PUÒ VERIFICARE

$$\bar{H}(f) = T_s \qquad \forall f \qquad \Longleftarrow$$

$$T_s = \frac{1}{2B_c}$$

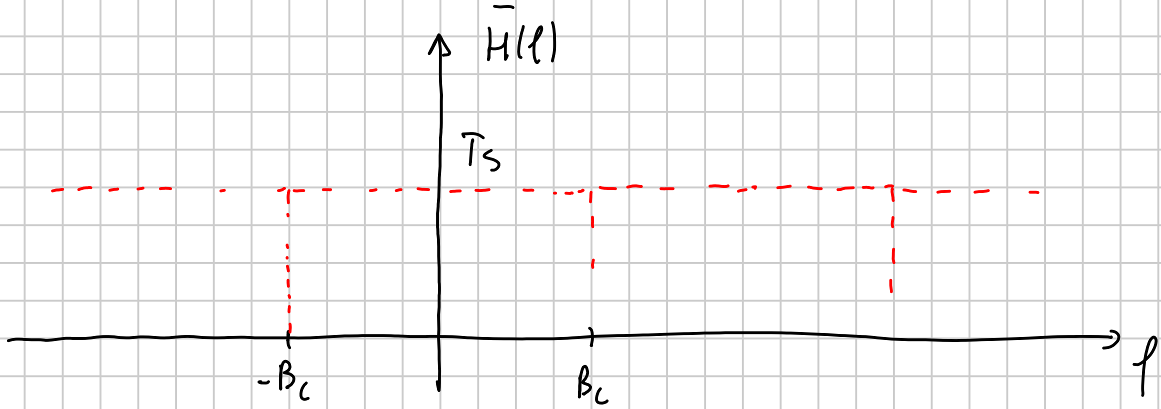


$\bar{H}(f) = T_s \quad \forall f$ si verifica se e solo se

$$H(f) = T_s \, \mathrm{rect}\left(\frac{f}{2B_c}\right) = \frac{1}{2B_c} \, \mathrm{rect}\left(\frac{f}{2B_c}\right)$$

$$\Downarrow$$

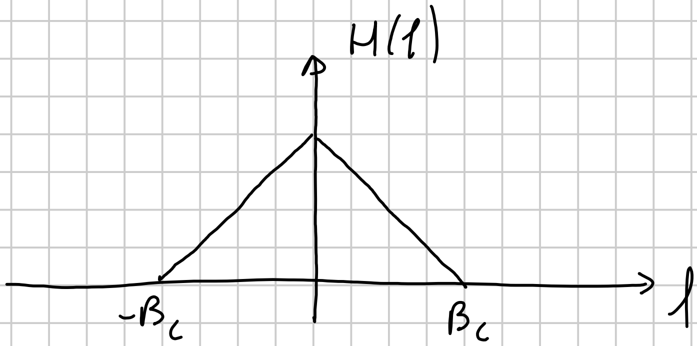
$$h(t) = \mathrm{sinc}(2B_c t)$$



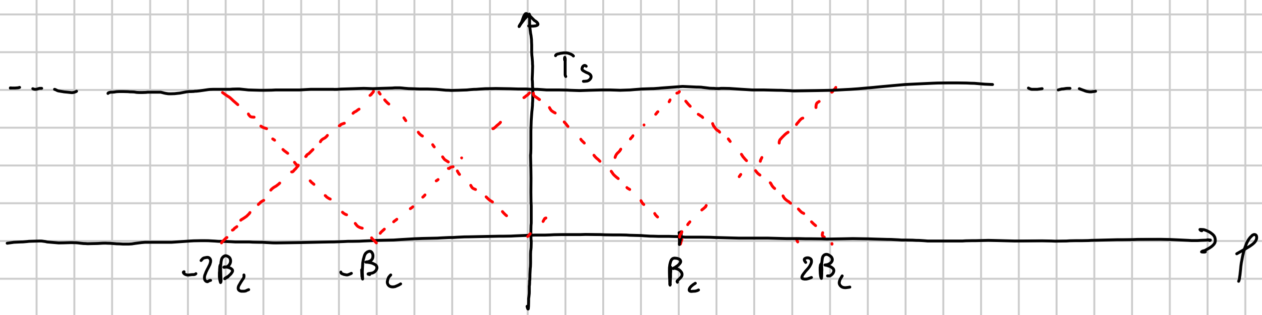
$$T_s^{(min)} = \frac{1}{2B_c}$$

Se scelgo $T_s > \frac{1}{2B_c} \Rightarrow$ soluzioni infinite

esempio



$$T_s = \frac{1}{B_c}$$



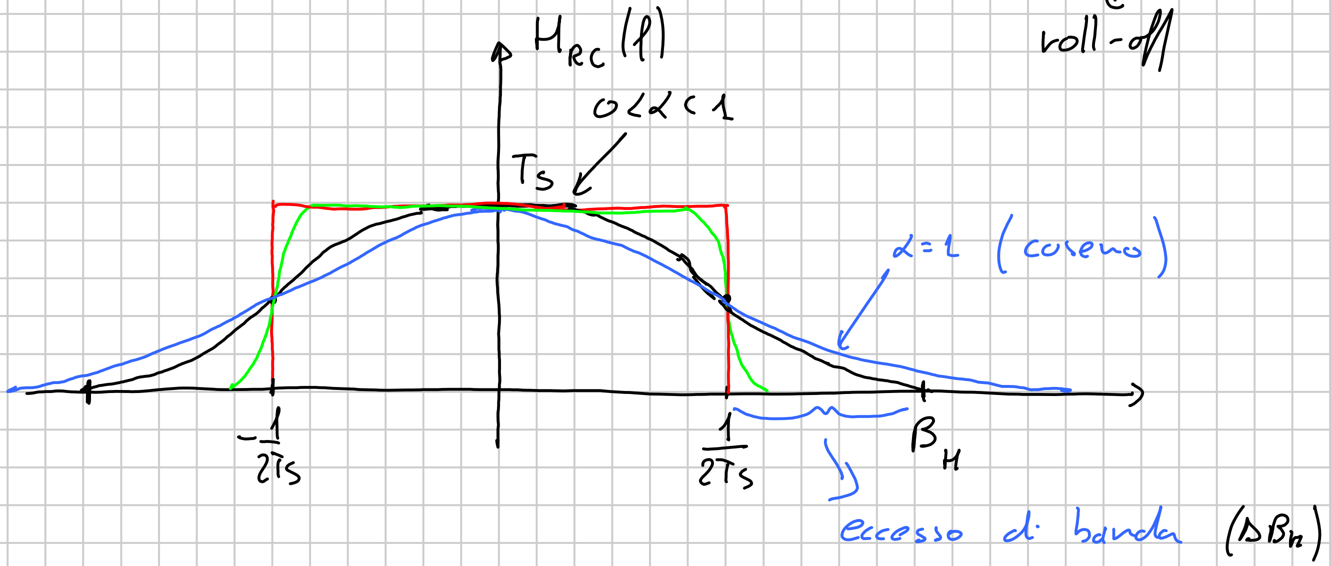
Se $C(f)$ introduce distorsioni, allora si possono progettare $P(f)$ e $H_R(f)$ tali che $H(f) = P(f)\,C(f)\,H_R(f)$ garantisca il criterio di Nyquist.

COSENO RIALZATO

$$H_{RC}(f) = \begin{cases} T_s & 0 \le |f| \le \frac{1-\alpha}{2T_s} \\ \frac{T_s}{2}\left[1 - \sin\left(\frac{\pi T_s}{\alpha}\left(|f| - \frac{1}{2T_s}\right)\right)\right] & \frac{1-\alpha}{2T_s} \le |f| \le \frac{1+\alpha}{2T_s} \\ 0 & |f| \ge \frac{1+\alpha}{2T_s} \end{cases}$$

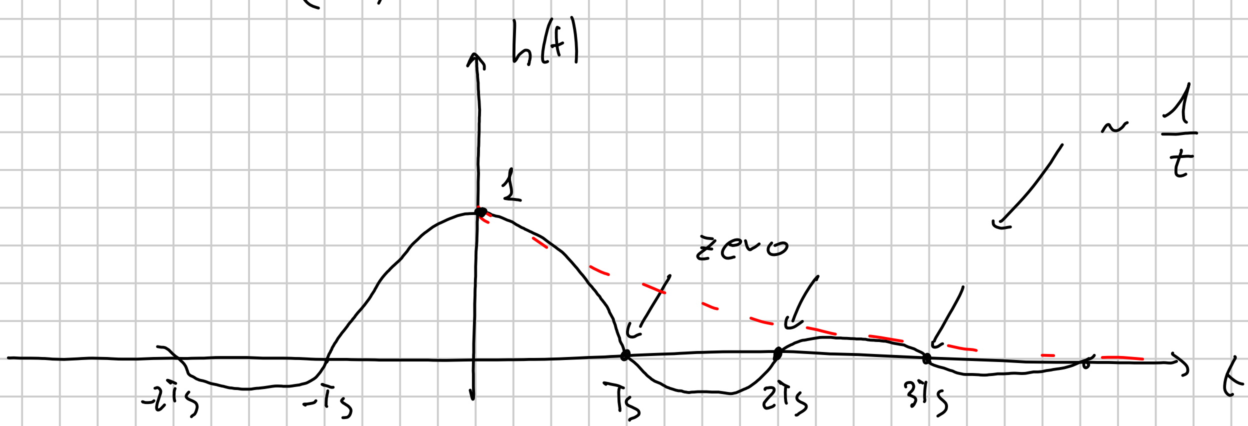
raised cosine

$(0 \le \alpha \le 1)$ roll-off



- $B_H = \frac{1+\alpha}{2T_s}$
- $h_{RC}(t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)\frac{\cos\frac{\alpha\pi t}{T_s}}{\left(1 - \frac{2\alpha t}{T_s}\right)^2} \sim \frac{1}{t^3}$

$\Rightarrow H(f) = \frac{1}{2B_c}\,\mathrm{rect}\left(\frac{f}{2B_c}\right) \Leftrightarrow h(t) = \mathrm{sinc}(2B_c t) = \mathrm{sinc}\left(\frac{t}{T_s}\right)$



Efficienza spettrale

$$\eta_B = \frac{R_b}{B_T} = \frac{\log_2 M}{B_T\, T_s} = \frac{\log_2 M}{T_s}\;\frac{2T_s}{1+\alpha} = \boxed{\frac{2\log_2 M}{1+\alpha}}$$

$$\alpha \simeq 0.4$$

ECCESSO DI BANDA

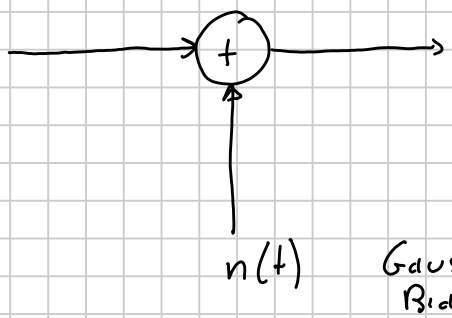
$$\Delta B_H = B_H - \frac{1}{2T_s} = \frac{\alpha}{2T_s}$$

CAPACITÀ DI CANALE

$$C = \max\{R_b\} \quad , \quad P_E(b) = P\{\hat{b}[n] \neq b[n]\} = 0$$

↳ bit/s

Nella condizione di



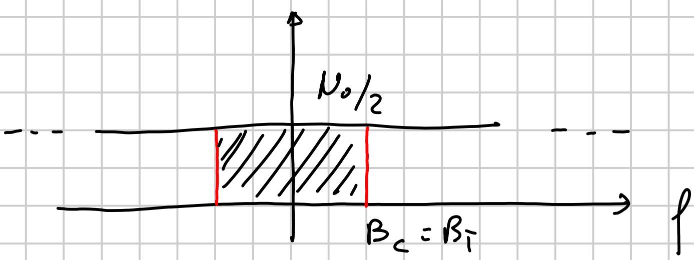
Gaussiano Bianco

AWGN → Additive, White, Gaussian, Noise

$$\boxed{C = B_T \log_2\left(1 + \frac{P_s}{\underbrace{N_0 B_T}_{P_N}}\right)}$$

$$\boxed{B_c = B_T}$$

si trasmette occupando tutta la banda disponibile



Fissiamo B_T

·) $\lim_{P_S/N_0 \to 0} C = 0$

·) $\lim_{P_S/N_0 \to \infty} C = +\infty$

Fisso P_S/N_0

·) $\lim_{B_T \to 0} C = 0$

·) $\lim_{B_T \to \infty} C = \log_2 e \, \frac{P_S}{N_0}$

$$\frac{C}{B_T} = \log_2 \left(1 + \frac{E_b}{N_0} \frac{R_b}{B_T} \right)$$

E_b = energia per bit

R_b = tasso di em. binario $\leq C$

SISTEMA DI COMUNICAZIONE NUMERICO IDEALE

$$\boxed{\begin{array}{l} \cdot)\ P_E(b) = 0 \\ \cdot)\ R_b = C \end{array}}$$

$\Downarrow$

$$\frac{R_b}{B_T} = \log_2 \left(1 + \frac{E_b}{N_0} \cdot \frac{R_b}{B_T} \right)$$

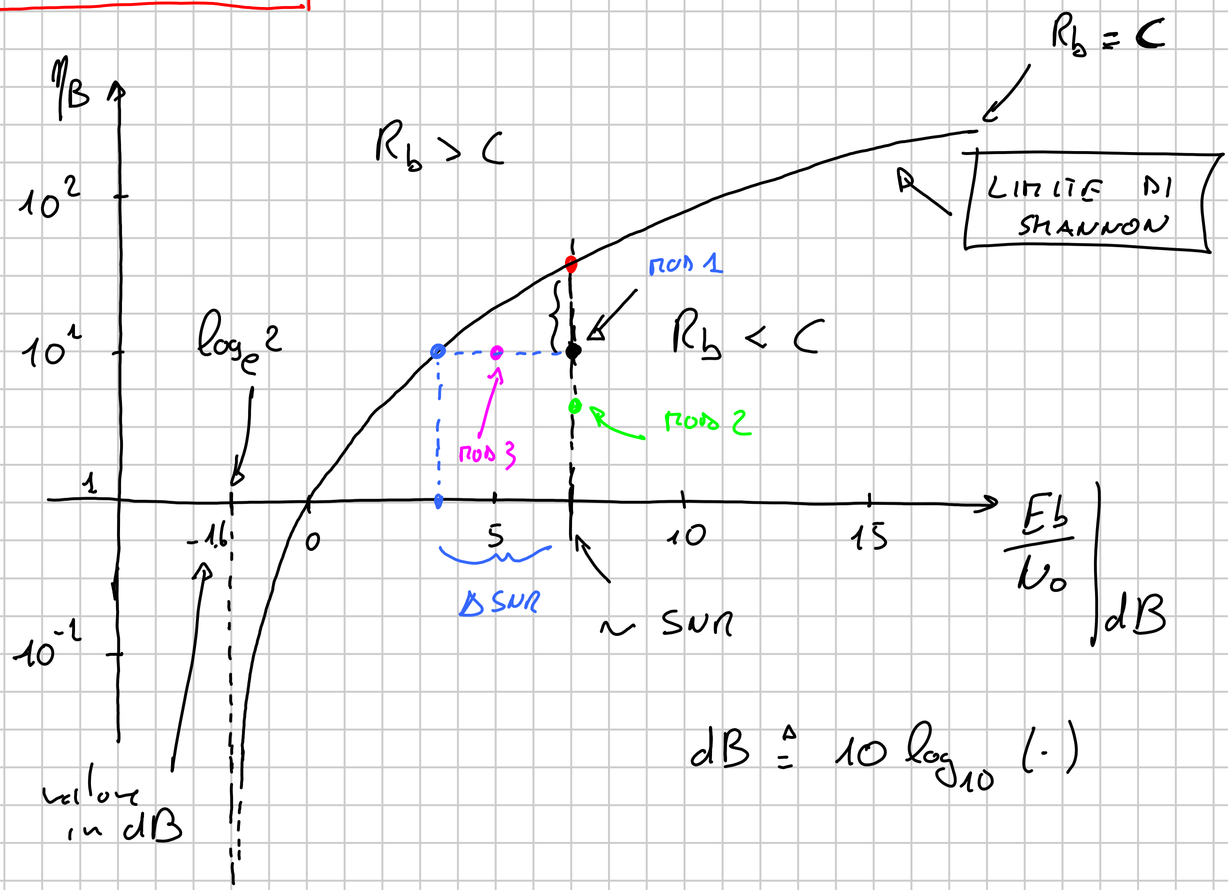
$\frac{R_b}{B_T} = \eta_B$

$\Downarrow$

$$\eta_B = \log_2 \left(1 + \frac{E_b}{N_0} \eta_B \right)$$

$$2^{\eta_B} = 1 + \frac{E_b}{N_0}\eta_B$$

$$\boxed{\frac{E_b}{N_0} = \frac{2^{\eta_B} - 1}{\eta_B}}$$



$$dB \triangleq 10 \log_{10}(\cdot)$$

- $\lim_{\eta_B \to +\infty} \frac{2^{\eta_B} - 1}{\eta_B} = +\infty$

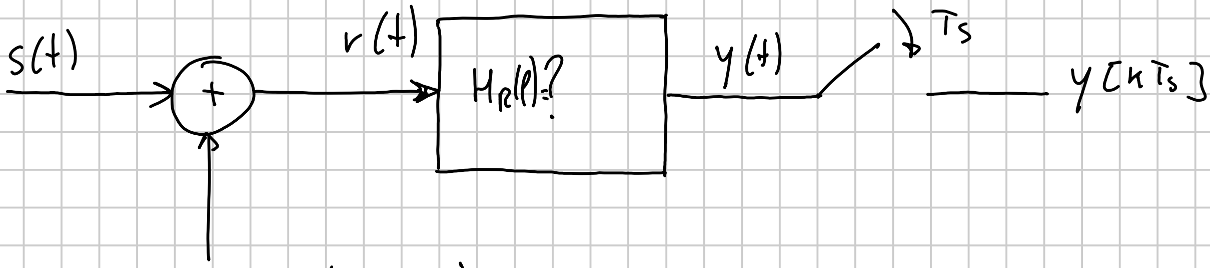
- $\lim_{\eta_B \to 0} \frac{2^{\eta_B} - 1}{\eta_B} = \log_e 2 \quad (-1.6 \text{ dB})$

RICEZIONE OTTIMA IN PRESENZA DI RUMORE BIANCO

$r(t) = s(t) + n(t)$, $n(t)$ bianco

- $c(t) = \delta(t)$ (assenza di distorsioni)

$r(t) = c(t) \otimes s(t) + n(t)$



$n(t)$ (bianco) $\Rightarrow$ $S_n(f) = \frac{N_0}{2} \quad \forall f$

CRITERIO DI OTTIMALITÀ $\Rightarrow$ MASSIMIZZAZIONE DEL SNR IN USCITA DAL FILTRO ALL'ISTANTE $t = T_s$

$$r(t) = s(t) + n(t)$$

$$y(t) = s_u(t) + n_u(t)$$

$$s_u(t) = s(t) \otimes h_R(t)$$

$$n_u(t) = n(t) \otimes h_R(t)$$

$$y(T_s) = s_u(T_s) + n_u(T_s)$$

$$SNR \triangleq \frac{s_u^2(T_s)}{E[n_u^2(T_s)]}$$

$h_R(t)$ che massimizza il SNR si chiama

FILTRO ADATTATO

$s_u(T_s) = ?$, $n_u(T_s) = ?$

$$s_u(t) = s(t) \otimes h_R(t) = \int_{-\infty}^{+\infty} s(\tau)\, h(t-\tau)\, d\tau$$

$$n_u(t) = n(t) \otimes h_R(t) = \int_{-\infty}^{+\infty} n(\tau)\, h(t-\tau)\, d\tau$$

$$s_u(T_s) = \int_{-\infty}^{+\infty} s(\tau)\, h(T_s-\tau)\, d\tau$$

$$n_u(T_s) = \int_{-\infty}^{+\infty} n(\tau)\, h(T_s-\tau)\, d\tau$$

$$s_u^2(T_s) = \left[\int_{-\infty}^{+\infty} s(\tau)\, h(T_s-\tau)\, d\tau\right]^2 = \qquad \text{(Parseval)}$$

$$= \left[\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df\right]^2$$

$$E\left[n_u^2(T_s)\right] = R_{n_u}(0) = \int_{-\infty}^{+\infty} S_{n_u}(f)\, df$$

$$\Rightarrow S_{n_u}(f) = S_n(f)\,|H_R(f)|^2$$

$$R_{n_u}(\tau) = R_n(\tau) \otimes h_R(\tau) \otimes h_R(-\tau)$$

$$SNR = \frac{\left[\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df\right]^2}{\int_{-\infty}^{+\infty} S_n(f)\,|H_R(f)|^2\, df} =$$

$$= \frac{\left[\int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df\right]^2}{\frac{N_0}{2}\int_{-\infty}^{+\infty} |H_R(f)|^2\, df} =$$

$$SNR = \frac{2}{N_0 E_{H_R}} \int_{-\infty}^{+\infty} S(f)\, H_R(f)\, e^{j2\pi f T_s}\, df$$

DISEGUAGLIANZA DI SCHWARZ $\Rightarrow H_R(f)\, e^{j2\pi f T_s} = S^*(f)$

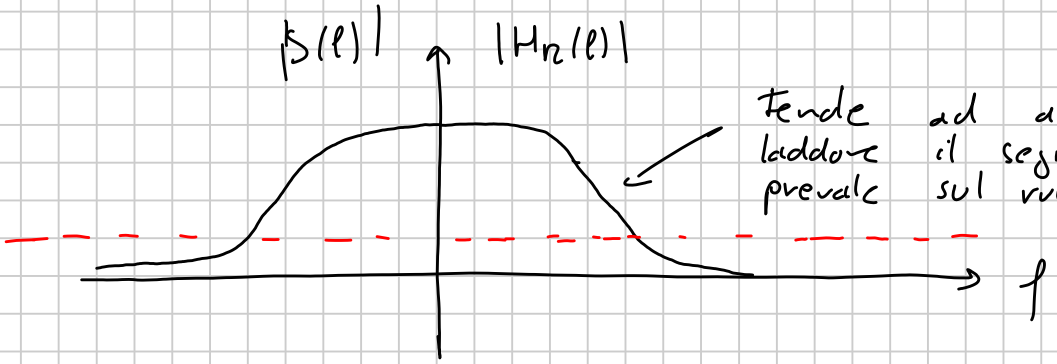
$$\Rightarrow H_R(f) = S^*(f)\, e^{-j2\pi f T_s}$$

$$\Rightarrow \boxed{h_R(t) = s(T_s - t)} \Rightarrow \text{SNR è massimo}$$

$\Uparrow$

risposta impulsiva del filtro adattato

$$|H_R(f)| = |S(f)|$$



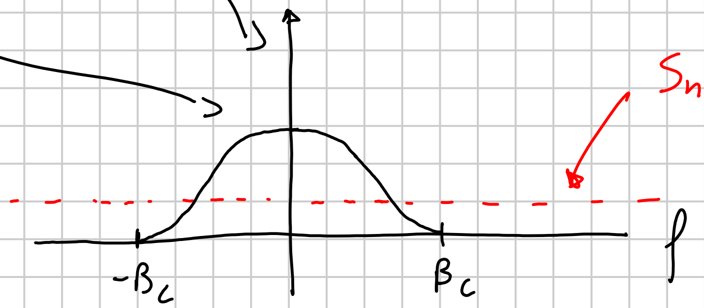
tende ad amplificare laddove il segnale utile prevale sul rumore

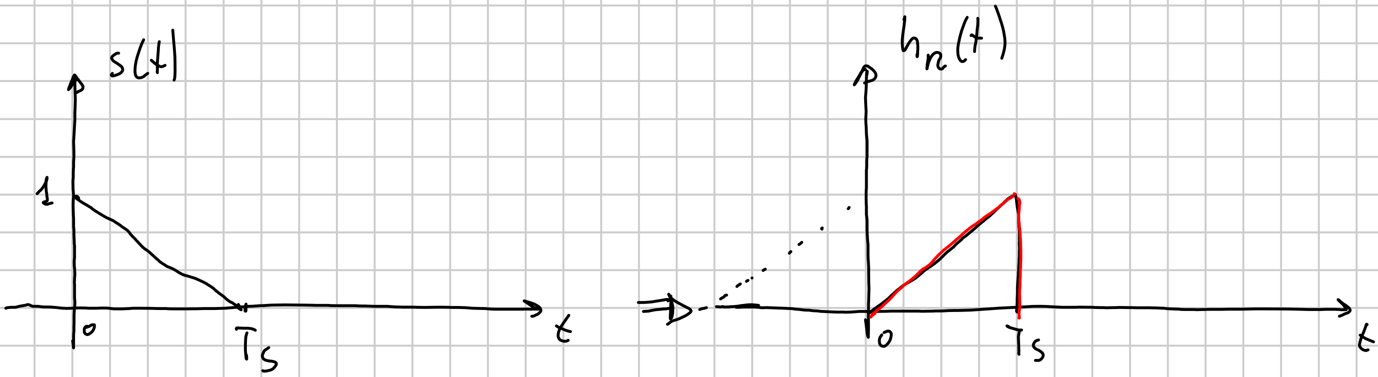
09/05/2013

$\cdot)\ h_R(t) = s(T_s - t)$

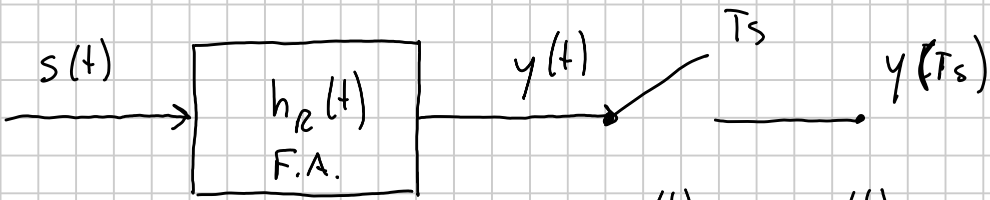
$\cdot)\ H_R(f) = S^*(f)\, e^{-j2\pi f T_s}$ } FILTRO ADATTATO

$$|H_R(f)| = |S(f)| = |P(f)|$$





Calcolo del SNR



$$y(t) = s_u(t) + n_u(t)$$

$$SNR = \frac{s_u^2(T_s)}{E[n_u^2(T_s)]} =$$

$$s_u(t) = s(t) \otimes h_R(t) =$$

$$= \int_{-\infty}^{+\infty} s(\tau)\, h_R(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} s(\tau)\, s(-T_s + t + \tau)\, d\tau$$

$$\Rightarrow s_u(T_s) = \int_{-\infty}^{+\infty} s(\tau)\, s(-T_s+T_s+\tau)\, d\tau =$$

$$= \int_{-\infty}^{+\infty} s^2(\tau)\, d\tau = E_s$$

$$\boxed{SNR = \frac{E_s^2}{\frac{N_0}{2} E_s} = \frac{2E_s}{N_0}}$$

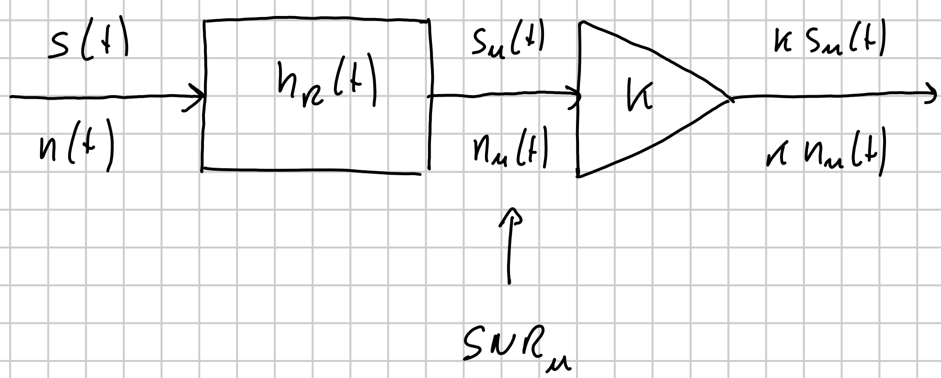
$$h_R(t) = K\, s(T_s - t)$$

$$s_u(T_s) = K E_s$$

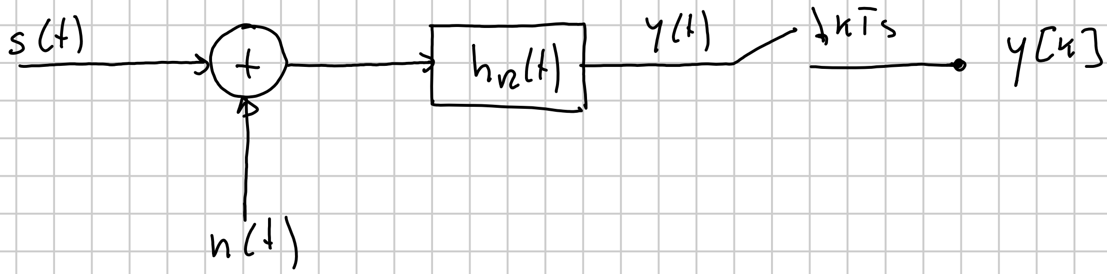
$$E\left[n_u^2(T_s)\right] = E\left[n_u^2(t)\right] = \int_{-\infty}^{+\infty} S_{n_u}(f)\, df =$$

$$= \int_{-\infty}^{+\infty} S_n(f)\, |H_R(f)|^2\, df = \frac{N_0}{2} K^2 E_s$$

$$SNR = \frac{(K E_s)^2}{\frac{N_0}{2} K^2 E_s} = \frac{2E_s}{N_0}$$



RICEVITORE CON FILTRO ADATTATO



$$s_u(t) = \int_{-\infty}^{+\infty} s(\tau)\, s[\tau - (t - T_s)]\, d\tau = C_s(t - T_s)$$

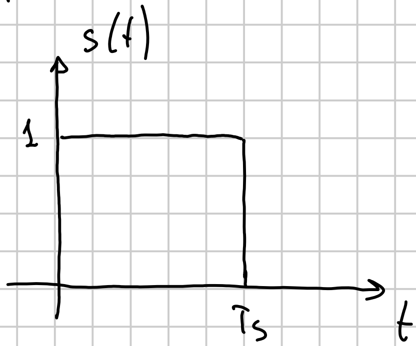
$$s(t) = \alpha\, p(t)$$

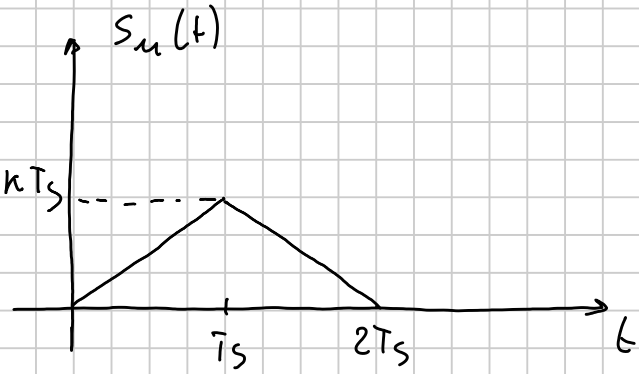
$$h_R(t) = K\, p(T_s - t)$$

$$s_u(t) = \int_{-\infty}^{+\infty} \alpha\, p(\tau)\, K\, p[\tau - (t - T_s)]\, d\tau =$$

$$= \alpha K \int_{-\infty}^{+\infty} p(\tau)\, p(\tau - (t - T_s))\, d\tau = \alpha K\, C_p(t - T_s)$$

Esempio





$n_u(t) = ?$

$n(t)$ = rumore bianco additivo Gaussiano

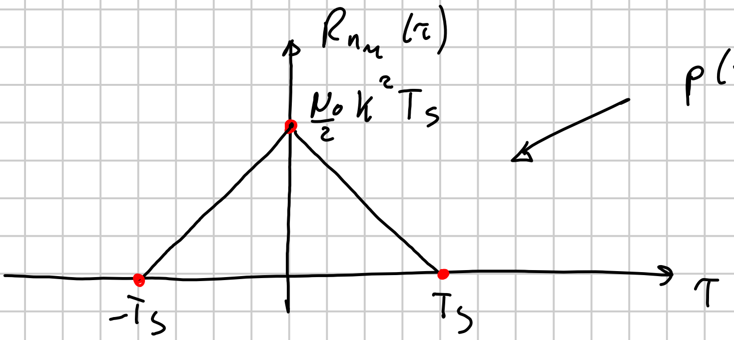
$n_u(t)$ e' un processo di rumore Gaussiano e stazionario

$$S_{n_u}(f) = S_n(f)\, |H_R(f)|^2$$

$$R_{n_u}(\tau) = R_{n_i}(\tau) \otimes h_R(\tau) \otimes h_R(-\tau)$$

$$S_n(f) = \frac{N_0}{2} \Rightarrow R_n(\tau) = \frac{N_0}{2}\, \delta(\tau)$$

$$R_{n_u}(\tau) = \frac{N_0}{2} h_R(\tau) \otimes h_R(-\tau) = \frac{N_0}{2} k^2 C_p(\tau)$$



$$p(t) = 1 \operatorname{rect}\left(\frac{t - T_s/2}{T_s}\right)$$

$$P_{n_u} = \int_{-\infty}^{+\infty} S_n(f) \, |H_R(f)|^2 \, df = \frac{N_0}{2} k^2 E_p \qquad (E_p = T_s)$$

$$R_{n_u}(0) = P_{n_u}$$

CORRELAZIONE TRA CAMPIONI DI RUMORE

$$E\left[n_u[k] \, n_u[n] \right] = 0 \qquad n \neq k$$

$$\Downarrow$$

incorrelazione tra camp di rumore

$$R_{n_u}(\tau) = E\left[n_u(t) \, n_u(t - \tau) \right]$$

$$R_{n_u}(kT_s) \Rightarrow R_{n_u}[k]$$

Incorrelazione tra campioni $\Rightarrow$ $R_{n_u}[k] = \begin{cases} R_{n_u}(0) & k = 0 \\ 0 & k \neq 0 \end{cases}$

Impulso a coseno rialzato

$$h_{RC}(t) = \operatorname{sinc}\left(\frac{t}{T_s}\right) \cdot \frac{\cos\left(\frac{\alpha \pi t}{T_s}\right)}{\left(1 - \frac{2\alpha t}{T_s}\right)^2}$$

Radice di coseno rialzato

$$P(f) = \sqrt{H_{RC}(f)} \qquad H_{RC}(f) = \text{coseno rialzato}$$

$$S_{n_u}(f) = k^2 \frac{N_0}{2} |P(f)|^2 = k^2 \frac{N_0}{2} H_{RC}(f)$$

$$R_{n_u}(\tau) = k^2 \frac{N_0}{2} \underbrace{\text{sinc}\left(\frac{\tau}{T_s}\right)}_{\text{si annulla in } \tau = kT_s} \frac{\cos\left(\frac{\alpha\pi\tau}{T_s}\right)}{\left(1 - \frac{2\alpha\tau}{T_s}\right)^2}$$

$$R_{n_u}[k] = R_{n_u}(kT_s) = \begin{cases} \neq 0 & k = 0 \\ = 0 & k \neq 0 \end{cases}$$

$\Downarrow$

incorrelazione tra campioni di rumore

$$y[n] = s_u[k] + n_u[k]$$

assenza di ISI — incorrelazione

DECISIONE SUL SIMBOLO k-ESIMO

BASATA SOLO SU $y[k]$ ("ad un sol colpo")

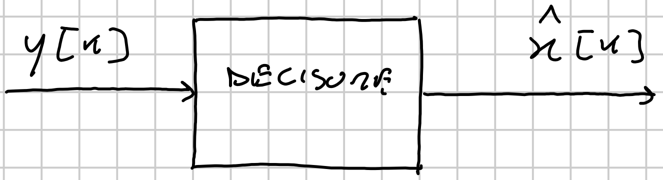
SNR per bit all'ingresso del ricevitore

$$SNR_b \triangleq \frac{E_b}{N_0}$$

$$E_b = P_s T_b = E\left[x^2[n]\right] T_b \quad , \quad S_n(f) = \frac{N_0}{2}$$

$$SNR_b = \frac{E[x^2[n]]\,T_b}{N_0} = \frac{E[x^2[k]]}{R_b\,N_0}$$

DECISORE OTTIMO E CRITERIO DELLA MASSIMA VEROSIMIGLIANZA



$$y[n] = s_n[n] + n_n[n]$$

↳ statisticamente indipendenti

MINIMIZZAZIONE DELLA PROBABILITÀ DI ERRORE

$$x \triangleq x[n] \quad , \quad y \triangleq y[n] \quad , \quad n_n \triangleq n_n[n] \quad , \quad \hat{x} \triangleq \hat{x}[n]$$

CRITERIO A MASSIMA PROBABILITÀ A POSTERIORI

(CRITERIO A MINIMA PROB. DI ERRORE)

MAP = Maximum A-posteriori Probability

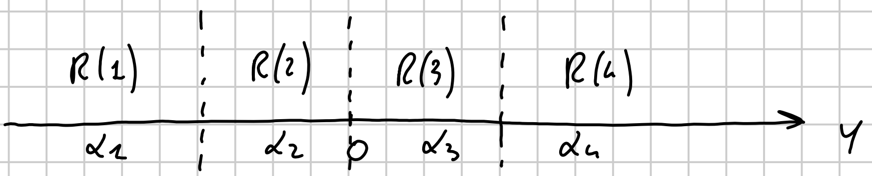
$$\boxed{\hat{x} = \max_{i=1,\dots,M} P(x = \alpha_i \mid y)}$$

Dim. equivalenza tra MAP e minima P_E

Definiamo $R(i)$ la regione di y per cui viene deciso il simbolo i-esimo

$$R(i) \triangleq \{\, y \in \mathbb{R} : \hat{x} = \alpha_i \,\}$$

esempio 4-PAM



$$P\{x=\alpha_i \mid y\} = \boxed{\frac{f_Y(y \mid x=\alpha_i)\, P(x=\alpha_i)}{f_Y(y)}}$$

BAYES

$$P\{\hat{x}=\alpha_i \mid x=\alpha_i\}$$ corretta decisione

$$P_E = 1 - \sum_{i=1}^{M} P\{\hat{x}=\alpha_i \mid x=\alpha_i\}\, P\{x=\alpha_i\}$$

$$= \sum_{i=1}^{M} \sum_{\substack{j=1 \\ j\neq i}}^{M} P\{\hat{x}=\alpha_j \mid x=\alpha_i\}\, P\{x=\alpha_i\}$$

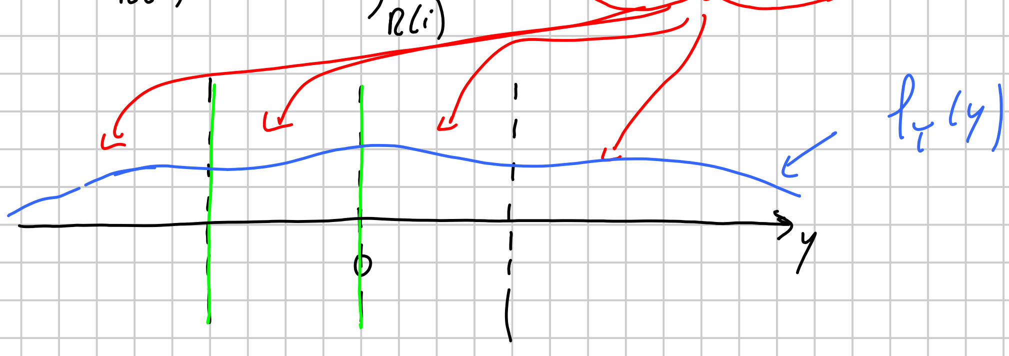
$$= 1 - \sum_{i=1}^{M} P\{x=\alpha_i\}\, P\{y \in R(i) \mid x=\alpha_i\}$$

$$= 1 - \sum_{i=1}^{M} P\{x=\alpha_i\} \int_{R(i)} f_Y(y \mid x=\alpha_i)\, dy$$

$$= 1 - \sum_{i=1}^{M} \int_{R(i)} f_Y(y)\, P(x=\alpha_i \mid y)\, dy$$

$$P(A \mid B)\, P(B) = P(B \mid A)\, P(A)$$

$$\min_{R(i)} P_E = \max_{R(i)} \sum_{i=1}^{M} \int_{R(i)} f_Y(y)\, P(x=\alpha_i \mid y)\, dy$$



$\hookrightarrow$ $P(x=a_i \mid y)$ sono massime (MAP) c.v.d.

Caso in cui $P(x=\alpha_i) = \frac{1}{M} \quad \forall i$

$$\hat{x} = \max_{i=1,..,M} \frac{1}{M} \frac{f_Y(y|x=\alpha_i)}{f_Y(y)} = \max_{i=1,..,M} \underbrace{f_Y(y|x=\alpha_i)}_{\text{FUNZIONE DI VEROSIMIGLIANZA}}$$

$\Rightarrow$ Quando i simboli trasmessi sono equiprobabili il criterio MAP $\Rightarrow$ CRITERIO A MASSIMA VEROSIMIGLIANZA

CASO DI AGWN

$$y = s_u + n_u = \alpha_i + n_u$$

$$n_u \in \mathcal{N}(\eta, \sigma^2)$$

$$\eta \rightarrow 0$$

$$\sigma^2 \rightarrow \sigma_{n_u}^2 = P_{n_u} = \int_{-\infty}^{+\infty} \frac{N_0}{2} |H_R(f)|^2 df = R_{n_u}(0)$$

$$f_{n_u}(n_u) = \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{n_u^2}{2\sigma_{n_u}^2}}$$

$$n_u = y - \alpha_i$$

$$f_Y(y|\alpha_i) = \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}} e^{-\frac{(y-\alpha_i)^2}{2\sigma_{n_u}^2}}$$

⇒ CASO DI SIMBOLI EQUIPROBABILI

$$\hat{x} = \max_{i=1,\dots,n} f_Y(y|\alpha_i) = \max_{i=1,\dots,n} f_Y(y|x=\alpha_i)$$

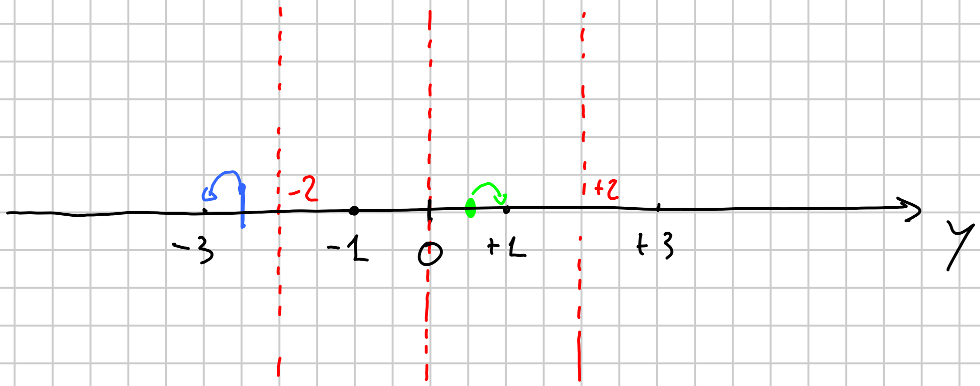
$$\max_i f_Y(y|\alpha_i) = \max_i \cancel{\frac{1}{\sqrt{2\pi\sigma_n^2}}} e^{-\frac{(y-\alpha_i)^2}{2\sigma_n^2}}$$

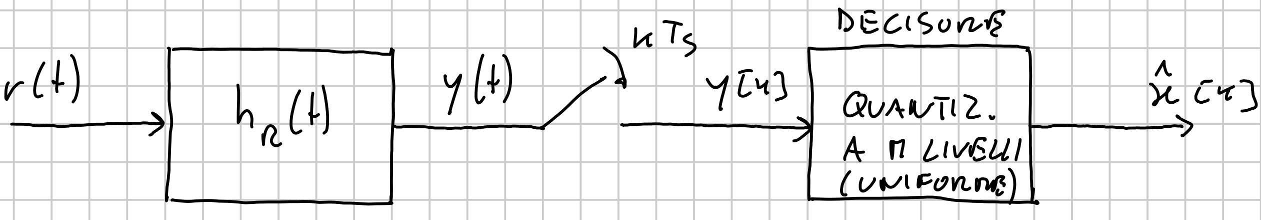
$$= \max_i e^{-\frac{(y-\alpha_i)^2}{2\sigma_n^2}} = \max_i \left[-\frac{(y-\alpha_i)^2}{2\sigma_n^2} \right]$$

$$= \min_i \underbrace{|y-\alpha_i|}$$

distanza Euclidea tra y e α_i

4 - PAM





RICEVITORE OTTIMO PER M-PAM CON AWGN

13/05/2013

PROBABILITÀ DI ERRORE

·) DI BIT $\Rightarrow P\{\hat{b}[k] \neq b[k]\} = P_E(b)$

·) DI SIMBOLO $\Rightarrow P\{\hat{x}[k] \neq x[k]\} = P_E(M)$

PAM BINARIA (2-PAM) $\Rightarrow P_E(b) = P_E(M)$

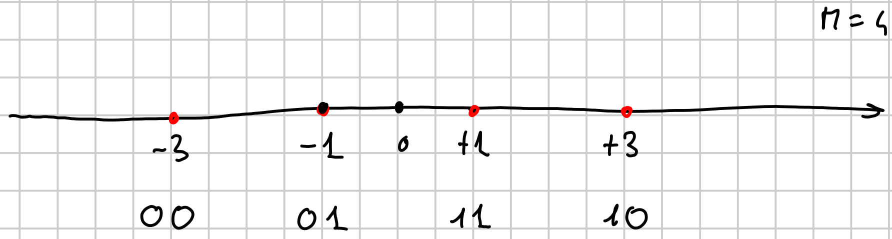
$M \neq 2 \Rightarrow P_E(b) \neq P_E(M)$

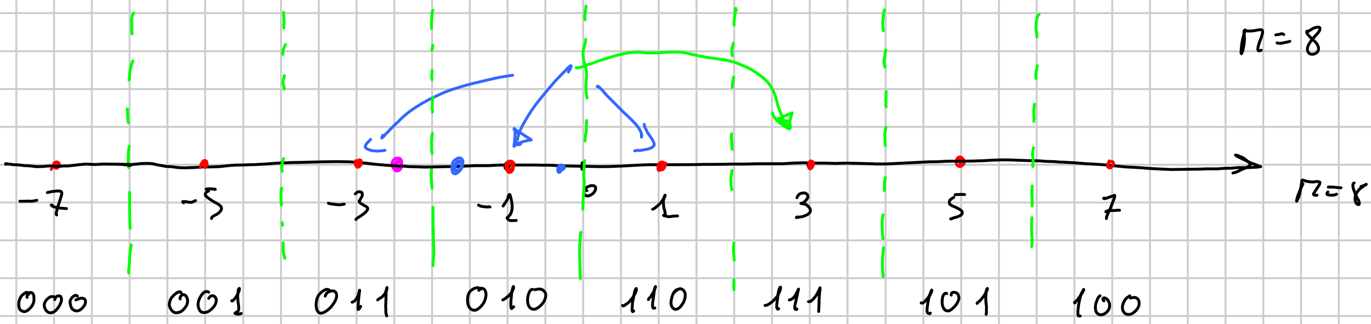
In generale vale che:

$$\frac{P_E(M)}{\log_2 M} \leq P_E(b) \leq \frac{M/2}{M-1} P_E(M)$$

CODIFICA DI GRAY

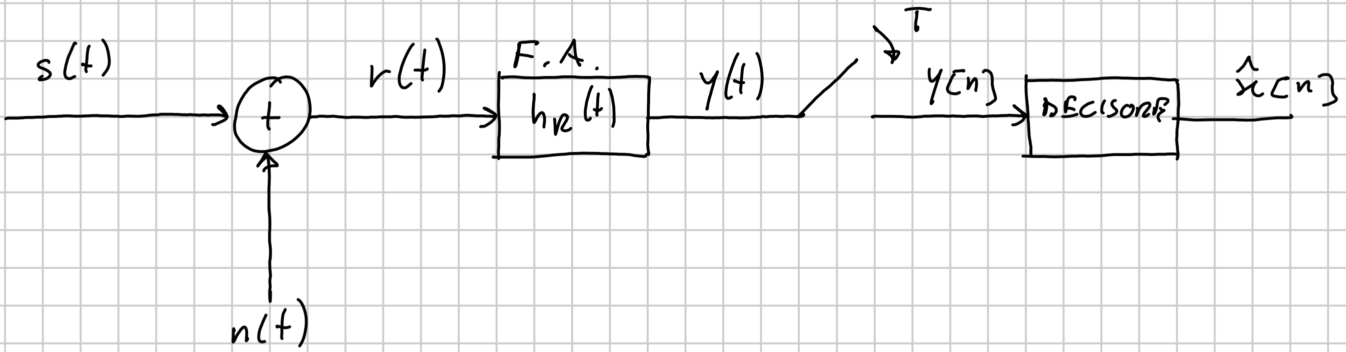
$$\boxed{\alpha_i = 2i - M - 1} \qquad i = 1, \dots, M$$





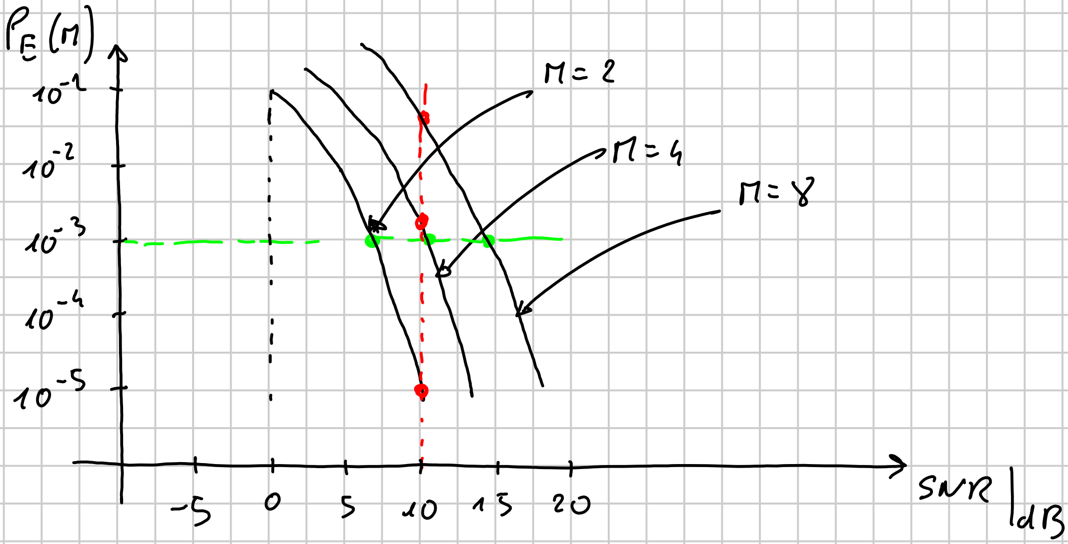
$$\left.\begin{array}{l} \cdot)\ \text{SNR} > 10\ \text{dB} \\ \cdot)\ \text{CODIFICA DI GRAY} \end{array}\right\} \quad P_E(b) \simeq \frac{P_E(M)}{\log_2 M}$$

PRESTAZIONI DI UN SIST. DI COMUNIC. CON MODULAZIONE M-PAM IN PRESENZA DI RUMORE ADDITIVO, BIANCO, GAUSS.



$$s(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT)$$

$$\boxed{P_E(M) = \frac{M-1}{M}\, \mathrm{erfc}\left(\sqrt{\frac{3 \log_2 M}{M^2-1}\, SNR}\right)}$$



TRADE-OFF PER LA SCELTA DI M:

·) M PICCOLI PER AUMENTARE L'EFFICIENZA ENERGETICA

$$\eta_E = \frac{1}{SNR}$$

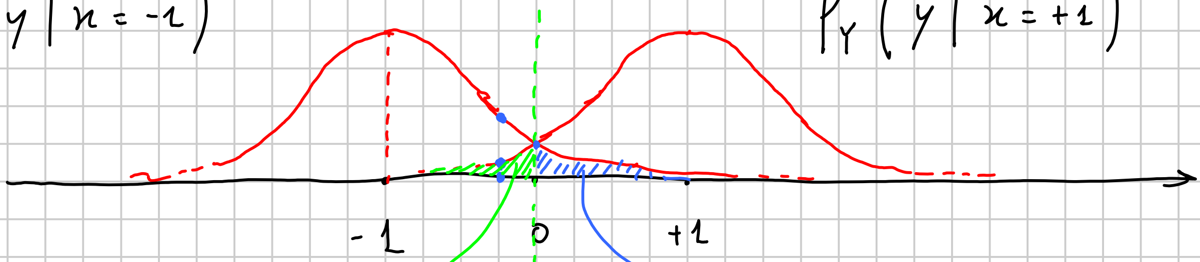
·) M GRANDI PER AUMENTARE L'EFFICIENZA SPETTRALE

$$\eta_B = \frac{R_b}{B_T} = \frac{\log_2 M}{T_s\, B_T}$$

CASO DELLA 2 - PAM

$$P_E(M) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{3\cdot 1}{3}\,SNR}\right) = \boxed{\frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right)}$$

$f_Y(y\,|\,x=-1)$ $\qquad\qquad$ $f_Y(y\,|\,x=+1)$



$$P_E(M) = \underbrace{P(\hat{x}=-1\,|\,x=1)}\,P(x=1) +$$

$$+\,\underbrace{P(\hat{x}=+1\,|\,x=-1)}\,P(x=-1) =$$

$$P(\hat{x}=-1\,,\,x=1) + P(\hat{x}=1\,,\,x=-1)$$

$$P(\hat{x}=-1\,|\,x=1) = \int_{-\infty}^{0} f_Y(y\,|\,x=1)\,dy$$

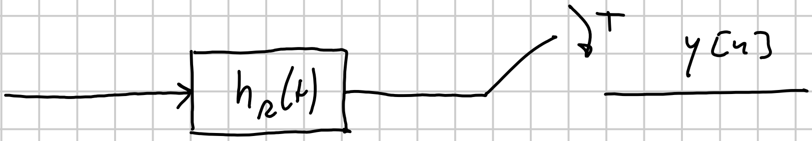
$$P(x=1) = P(x=-1) = \frac{1}{2}$$

$$P(\hat{x}=-1\,|\,x=1) = P(\hat{x}=1\,|\,x=-1) =$$

$$= \int_{-\infty}^{0} f_Y(y\,|\,x=1)\,dy = \int_{0}^{+\infty} f_Y(y\,|\,x=-1)\,dy$$

$$= \int_{0}^{+\infty} \frac{1}{\sqrt{2\pi\sigma_{n_u}^2}}\,e^{-\frac{(y+h(0))^2}{2\sigma_{n_u}^2}}\,dy = Q\left(\frac{0+h(0)}{\sqrt{\sigma_{n_u}^2}}\right)$$

$\uparrow$
λ



$$y[k] = h(0)\, x[k] + \underbrace{\sum_{\substack{n=-\infty \\ n \neq k}}^{+\infty} x[n]\, h[n-k]}_{=0}$$

$$y[k] = h(0)\, x[k]$$

$$R_{n_u}(\tau) = \frac{N_0}{2} C_{h_R}(\tau) \Rightarrow R_{n_u}(0) = \frac{N_0}{2} h(0) = E_{h_R}$$

$$SNR \triangleq \frac{h^2(0)\,(\pm 1)^2}{\frac{N_0}{2} h(0)} =$$

$$SNR = \frac{2h(0)}{N_0}$$

$$y[k] = h(0)\, \underbrace{x[k]}_{\pm 1}$$

$$y^2[k] = h^2(0)\,(\pm 1)^2$$

$$P(\hat{x}=-1 \mid x=1) = Q\left(\frac{h(0)}{\sigma_{n_u}}\right) = Q\left(\frac{h(0)}{\sqrt{\frac{N_0}{2}h(0)}}\right)$$

$$= Q\left(\sqrt{\frac{2h(0)}{N_0}}\right) = Q\left(\sqrt{SNR}\right) = \frac{1}{2}\,erfc\left(\sqrt{SNR}\right)$$

$$P_E(2) = \frac{1}{2}\cdot\frac{1}{2}\,erfc\left(\sqrt{SNR}\right) + \frac{1}{2}\cdot\frac{1}{2}\,erfc\left(\sqrt{SNR}\right) =$$

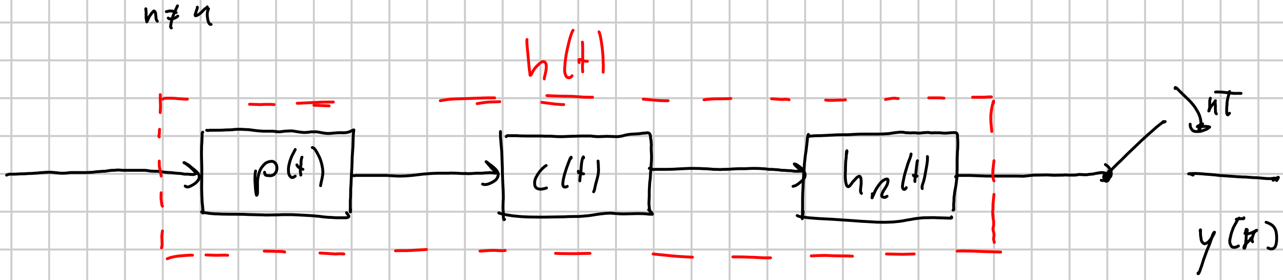
$$= \boxed{\frac{1}{2}\,erfc\left(\sqrt{SNR}\right)}$$

CASO PRESENZA DI RUMORE E ISI

j) ELIMINAZIONE DELL'ISI

↳ $\boxed{y[k] = h(0)\,x[k]} + \underset{\substack{\| \\ 0}}{I[k]}$ (ISI)

$$I[k] = \sum_{\substack{n=-\infty \\ n\neq k}}^{+\infty} x[n]\,h[n-k] = 0$$



$$h(t) \Rightarrow \begin{cases} h(0) \neq 0 \\ h(kT) = 0 \end{cases}$$

$$P(f)\,C(f)\,H_R(f) = H_{RC}(f)\,\underbrace{e^{-j2\pi fT}}$$

↳ termine di fase x la causalità

$$\begin{cases} |P(f)| = |H_R(f)| = \sqrt{\dfrac{H_{RC}(f)}{|C(f)|}} \\ \angle P(f) = \angle H_R(f) = -\pi f T - \dfrac{\angle C(f)}{2} \end{cases}$$

$$P(f) \cdot C(f) \cdot H_R(f) = \sqrt{\frac{H_{RC}(f)}{\cancel{|C(f)|}}} \cdot \cancel{|C(f)|} \cdot \sqrt{\frac{H_{RC}(f)}{\cancel{|C(f)|}}} \cdot$$

$$\cdot\, e^{-j\pi f T}\, \cancel{e^{-j\frac{\angle C(f)}{2}}} \cdot \cancel{e^{j\angle C(f)}} \cdot e^{-j\pi f T}\, \cancel{e^{-j\frac{\angle C(f)}{2}}}$$

$$= H_{RC}(f)\, e^{-j2\pi f T}$$

$$\Rightarrow h_{RC}(t) = h_{RC}(t-T) \qquad \text{(IMPULSO DI NYQUIST)}$$

CASO DI CANALE IDEALE

$$C(f) = 1$$

$$P(f) = H_R(f) = \sqrt{H_{RC}(f)}\, e^{-j\pi f T}$$

$$\boxed{SNR = \frac{E[x^2[n]]\, h^2(0)}{\frac{N_0}{2}\int_{-\infty}^{+\infty} \frac{H_{RC}(f)}{|C(f)|}\, df} \underset{C(f)=1}{=} \frac{2E_s}{N_0}}$$

CANALE NON IDEALE

CANALE IDEALE

$$h(0) = \int_{-\infty}^{+\infty} H(f)\, df$$

$$\boxed{E_s = P_s T_s = \frac{T_s (M^2-1)}{3}}$$

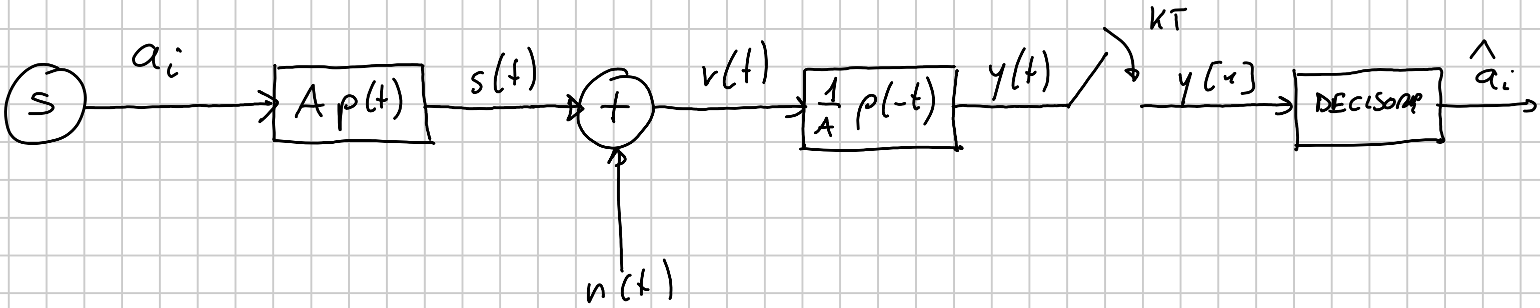
$$P(f) = \sqrt{H_{RC}(f)}$$

$$H_R(f) = \sqrt{H_{RC}(f)}\; e^{-j2\pi f T} = \sqrt{H^*_{RC}(f)}\; e^{-j2\pi f T}$$

$$p(t) = h'_{RC}(t)$$

$$h_R(t) = h'_{RC}(T - t) \qquad\qquad h_R(t) = h_{FA}(t)$$

ESERCIZIO



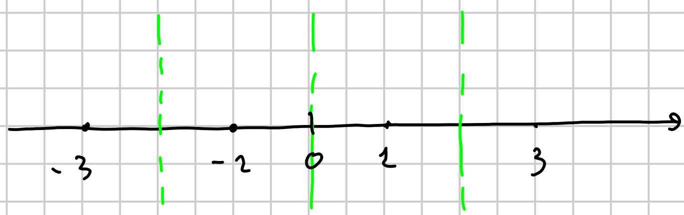
$n(t)$ = Gaussiano Bianco con $S_n(f) = \frac{N_0}{2}$

$a_i \in \{\pm 1, \pm 3\}$ indipendenti ed equiprobabili

$$P(f) = \begin{cases} \sqrt{T}\,\sqrt{1 - |fT|} \\ 0 \quad \text{altrove} \end{cases}$$

Strategia di decisione

$$\hat{a}_n = \begin{cases} 3 & \text{se} \quad y[n] \geq 2 \\ 1 & \text{se} \quad 0 \leq y[n] < 2 \\ -1 & \text{se} \quad -2 \leq y[n] < 0 \\ -3 & \text{se} \quad y[n] < -2 \end{cases}$$



Si determini:

1) $SNR = \frac{E_S}{N_0}$ dove E_S e' l'energia media per intervallo di segnalazione del segnale trasmesso

2) $P_E(M)$ in funzione di E_S/N_0

Svolgimento

$\Rightarrow$ Presenza di ISI

$\Rightarrow$ $\sigma_{n_m}^2$

$\Rightarrow$ $P_E(M)$

$$s(t) = A \sum_k x[k]\, p(t-kT)$$

$$E_S = E\left\{ \int_0^T s^2(t)\, dt \right\} = E\left\{ \int_0^T \left[A \sum_k x[k]\, p(t-kT) \right]^2 dt \right\} =$$

$$= E\left\{ \int_0^T A^2 \sum_k x[k]\, p(t-kT) \sum_n x[n]\, p(t-nT)\, dt \right\} =$$

$$= A^2 \sum_k \sum_n E\left[x[k]\, x[n] \right] \int_0^T p(t-kT)\, p(t-nT)\, dt =$$

$$\Rightarrow E\left[x[k]\, x[n] \right] = \begin{cases} E\left[x^2[k] \right] & k = n \\ 0 & k \neq 0 \end{cases} \quad \times \text{ indipend simboli}$$

$$= A^2 \sum_k E\left[x^2[k] \right] \int_0^T p^2(t-kT)\, dt =$$

$$\Rightarrow E\left[x^2[k] \right] = \frac{1}{4}(-3)^2 + \frac{1}{4}(-1)^2 + \frac{1}{4}(1)^2 + \frac{1}{4}(3)^2 = 5$$

$$= 5A^2 \sum_k \int_0^T p^2(t-kT)\, dt = 5A^2 \sum_k \int_{-kT}^{-(k-1)T} p^2(t)\, dt = 5A^2 \int_{-\infty}^{+\infty} p^2(t)\, dt = 5A^2 E_p$$

20/05/2013

MODULAZIONI NUMERICHE IN BANDA PASSANTE

SEGNALE PASSA-BANDA

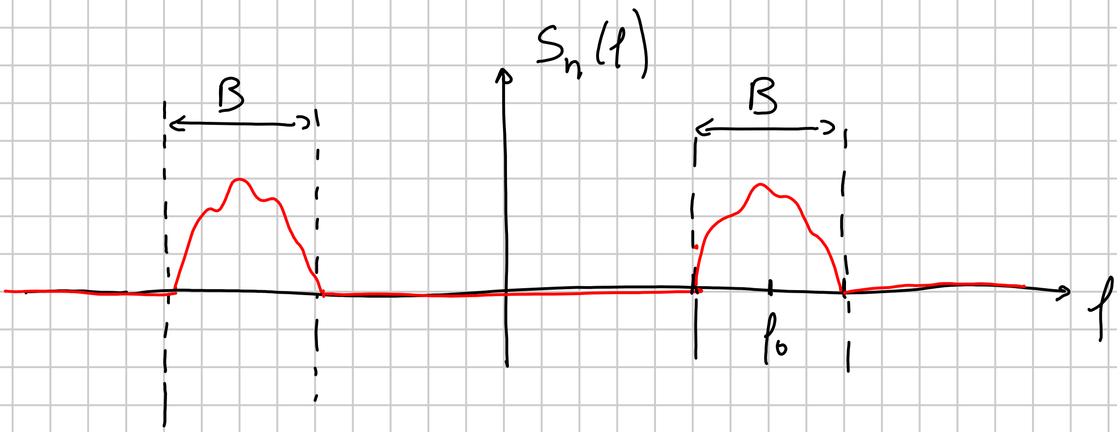
$$s(t) = \underbrace{a(t)}_{\text{inviluppo reale di } s(t)} \cos\left[2\pi f_0 t + \underbrace{\theta(t)}_{\text{fase di } s(t)}\right]$$

$$s(t) = \mathrm{Re}\left\{ a(t)\, e^{j(2\pi f_0 t + \theta(t))} \right\} =$$

$$= \mathrm{Re}\left\{ \tilde{s}(t)\, e^{j2\pi f_0 t} \right\}$$

$\tilde{s}(t) = a(t)\, e^{j\theta(t)}$ inviluppo complesso di $s(t)$

PROCESSI PASSA-BANDA



$$n(t) = \mathrm{Re}\left\{ \underbrace{\tilde{n}(t)}_{\text{inviluppo complesso}}\, e^{j2\pi f_0 t} \right\} = \frac{1}{2}\tilde{n}(t)\, e^{j2\pi f_0 t} + \frac{1}{2}\tilde{n}^*(t)\, e^{-j2\pi f_0 t}$$

$$\tilde{n}(t) = n_c(t) + j\, n_s(t)$$

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

$$\mathrm{Re}\left\{\left[n_c(t)+j\,n_s(t)\right]\left[\cos(2\pi f_0 t)+j\sin(2\pi f_0 t)\right]\right\} =$$

$$= \mathrm{Re}\left\{n_c(t)\cos(2\pi f_0 t) + j\,n_c(t)\sin(2\pi f_0 t) + j\,n_s(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)\right\}$$

AUTOCORRELAZIONE (PROCESSI SSL)

$$R_n(\tau) \triangleq E\left[n(t)\,n(t-\tau)\right]$$

$$R_{n_c}(\tau) \triangleq E\left[n_c(t)\,n_c(t-\tau)\right]$$

$$R_{n_s}(\tau) \triangleq E\left[n_s(t)\,n_s(t-\tau)\right]$$

$$R_{n_c n_s}(\tau) \triangleq E\left[n_c(t)\,n_s(t-\tau)\right] = -R_{n_s n_c}(\tau) = E\left[n_s(t)\,n_c(t-\tau)\right]$$

$$R_n(\tau) = R_{n_c}(\tau)\cos(2\pi f_0 \tau) - R_{n_s n_c}(\tau)\sin(2\pi f_0 \tau) =$$

$$= R_{n_s}(\tau)\cos(2\pi f_0 \tau) + R_{n_c n_s}(\tau)\sin(2\pi f_0 \tau)$$ (Quando $R_{n_s}(\tau) = R_{n_c}(\tau)$)

$$R_{\tilde n}(\tau) \triangleq E\left[\tilde n(t)\,\tilde n^*(t-\tau)\right] = R_{n_c}(\tau) + j\,R_{n_s n_c}(\tau)$$

$$R_n(\tau) = \mathrm{Re}\left\{R_{\tilde n}(\tau)\, e^{j2\pi f_0 t}\right\}$$

$$\Rightarrow S_n(f) = TCF\left[R_n(\tau)\right] = TCF\left[\mathrm{Re}\left\{R_{\tilde n}(\tau)\, e^{j2\pi f_0 t}\right\}\right] =$$

$$= TCF\left[\frac{1}{2}R_{\tilde n}(\tau)\, e^{j2\pi f_0 t} + \frac{1}{2}R^*_{\tilde n}(\tau)\, e^{-j2\pi f_0 t}\right]$$

$$= \frac{1}{2}S_{\tilde n}(f-f_0) + \frac{1}{2}S^*_{\tilde n}(-f-f_0)$$

$$R_{\tilde n}(\tau) = E\left[\tilde n(t)\,\tilde n^*(t-\tau)\right] = R^*_{\tilde n}(-\tau)$$

$$R_{\tilde n}(-\tau) = E\left[\tilde n(t)\,\tilde n^*(t+\tau)\right] = E\left[n^*(t')\,n(t'-\tau)\right]$$

$$= \left[E\left[n(t') \, n^*(t'-\tau) \right] \right]^* = R_{\tilde{n}}^*(\tau) \qquad \text{simm. herm.}$$

$$S_n(f) = \frac{1}{2} S_{\tilde{n}}(f - f_0) + \frac{1}{2} S_{\tilde{n}}(-f - f_0)$$

MODULAZIONI IN BANDA PASSANTE

MODULAZIONI PRIVE DI MEMORIA

$$s(t) = f(x[k]) \qquad kT_s \le t < (k+1)T_s$$

- PAM in banda passante
- PSK (Phase Shift Keying)
- QAM (Quadrature Amplitude Modulation)

⇒ PAM IN BANDA PASSANTE

$$s_m(t) = A_m \, p(t) \cos(2\pi f_0 t) \qquad m = 1, \dots, M$$

$$A_m = 2m - 1 - M$$

$p(t)$ = impulso sagomatore

$$\tilde{s}_m(t) = A_m \, p(t) \qquad \text{inviluppo complesso}$$

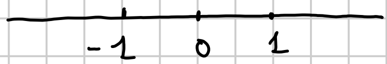
$$s_m(t) = \mathrm{Re}\left\{ \tilde{s}_m(t) \, e^{j 2\pi f_0 t} \right\} =$$

$$= \mathrm{Re}\left\{ A_m \, p(t) \, e^{j 2\pi f_0 t} \right\} =$$

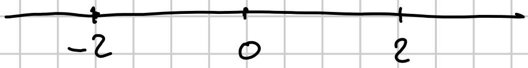
$$= \mathrm{Re}\left\{ A_m \, p(t) \left[\cos(2\pi f_0 t) + j \sin(2\pi f_0 t) \right] \right\}$$

$$= \mathrm{Re}\left\{ A_m \, p(t) \cos(2\pi f_0 t) + j A_m \, p(t) \sin(2\pi f_0 t) \right\}$$

$$= A_m \, p(t) \cos(2\pi f_0 t)$$



$M = 2$



$M = 3$

$\vdots$

$$E_m = \int s_m^2(t)\, dt = \int A_m^2\, p^2(t) \cos^2(2\pi f_0 t)\, dt =$$

$$= \frac{A_m^2}{2} \int p^2(t) \left[1 + \cos(4\pi f_0 t) \right] dt =$$

$$= \frac{A_m^2}{2} E_p + 0$$

$p(t)$ ha banda $B \ll f_0$

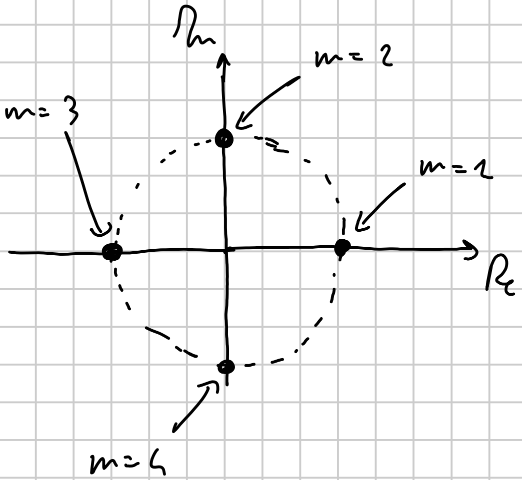
$$\frac{A_m^2}{2} \int p^2(t) \cos(4\pi f_0 t)\, dt \simeq 0$$

MODULAZIONE PSK (MODULAZIONE DI FASE)

$$s_m(t) = p(t) \cos(2\pi f_0 t + \theta_m) \qquad m = 1, \dots, M$$

$$\theta_m = \frac{2\pi}{M}(m-1)$$

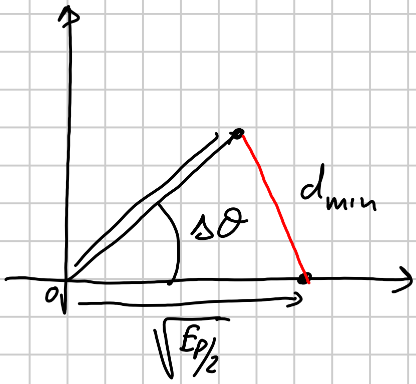
Esempio $M = 4$

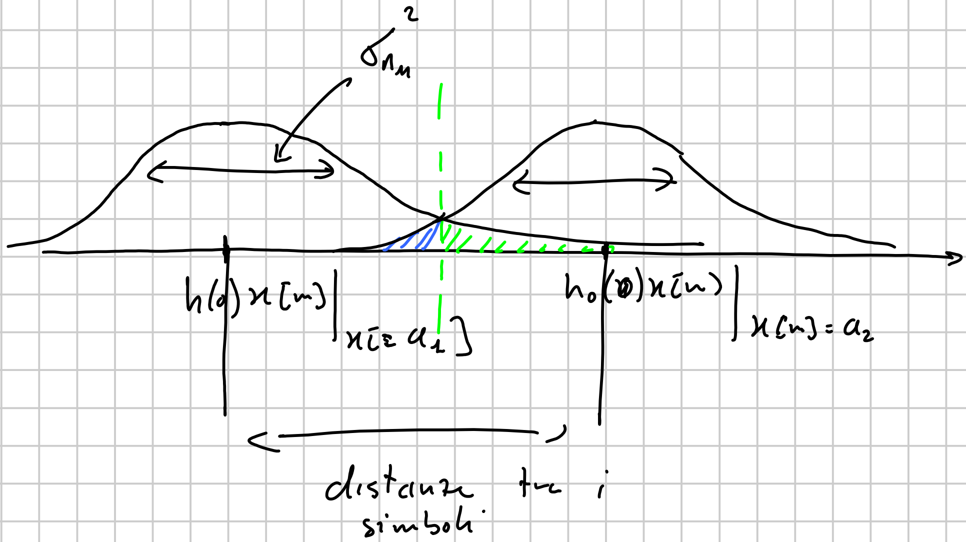


$$E_m = \frac{E_p}{2} = \int p^2(t) \cos^2(2\pi f_0 t + \theta_m)\, dt =$$

$$= \frac{1}{2} \int p^2(t) \left(\cos(4\pi f_0 t + 2\theta_m) + 1 \right) dt$$

$$\tilde{S}_m(t) = p(t)\, e^{j\theta_m}$$





$$d_{min}^2 = \frac{E_p}{2} + \frac{E_p}{2} - 2\,\frac{E_p}{2}\cos\left(\frac{2\pi}{M}\right) = E_p\left(1 - \cos\frac{2\pi}{M}\right) =$$

$$= 2E_p \sin^2\left(\frac{\pi}{M}\right)$$

aumenta l'eff spettrale

R_b aumenta se aumenta M

d_{min} = cost

aumentare l'E_p ⇒ diminuisce l'eff. energ.

$\eta_B \triangleq \dfrac{R_b}{B_T}$

TRADE-OFF

$\eta_E \triangleq \dfrac{1}{SNR}$

MODULAZIONE QAM

$$s_m(t) = \underbrace{A_m^c\, p(t)\cos(2\pi f_0 t)}_{\text{componente in fase}} - \underbrace{A_m^s\, p(t)\sin(2\pi f_0 t)}_{\text{componente in quadratura}}$$

$$x[n] \begin{cases} \to A_m^c \\ \to A_m^s \end{cases}$$

$$\tilde{s}_m(t) = A_m^c\, p(t) + j A_m^s\, p(t) = (A_m^c + j A_m^s)\, p(t)$$

$$= V_m e^{j\theta_m} p(t)$$

$$V_m = \sqrt{A_m^{c\,2} + A_m^{s\,2}} \quad , \quad \theta_m = \arctan \frac{A_m^s}{A_m^c}$$

$$E_m = \int s_m^2(t)\, dt = \int \underline{A_m^{c\,2} p^2(t)\cos^2(2\pi f_0 t)} + \underline{A_m^{s\,2} p^2(t)\sin^2(2\pi f_0 t)}$$

$$- 2 A_m^c A_m^s \cos(2\pi f_0 t)\sin(2\pi f_0 t)\, dt =$$

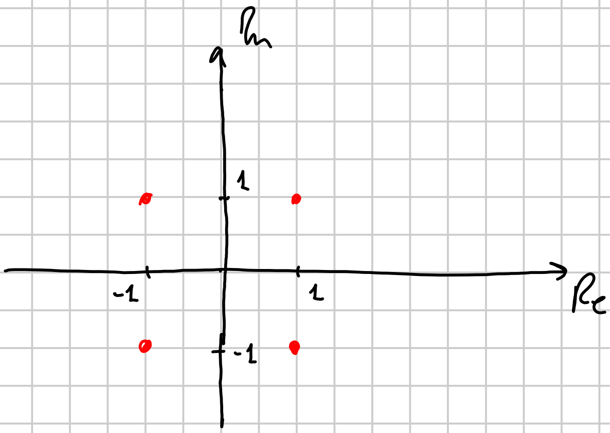
$$= \underline{\frac{A_m^{c\,2}}{2} E_p} + \underline{\frac{A_m^s}{2} E_p} = \frac{E_p}{2} V_m^2$$

$$x_m = A_m^c + j A_m^s$$

$$A_m^c = 2m - 1 - M'$$

$$A_m^s = 2m - 1 - M''$$

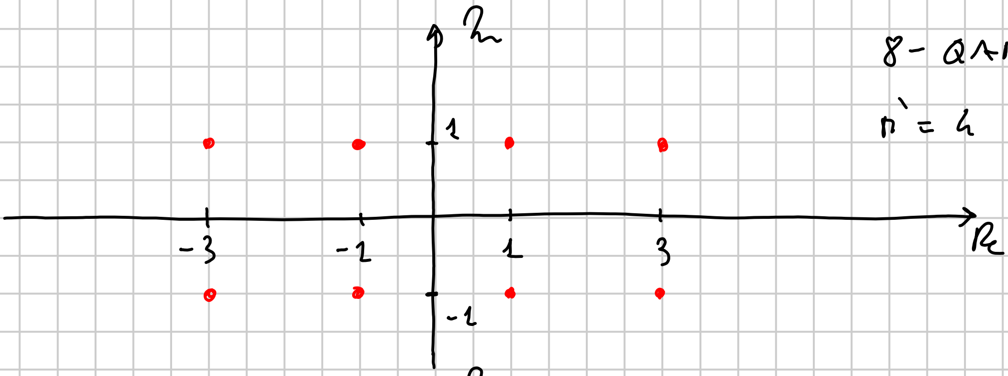
$$M' \cdot M'' = M$$



4-QAM

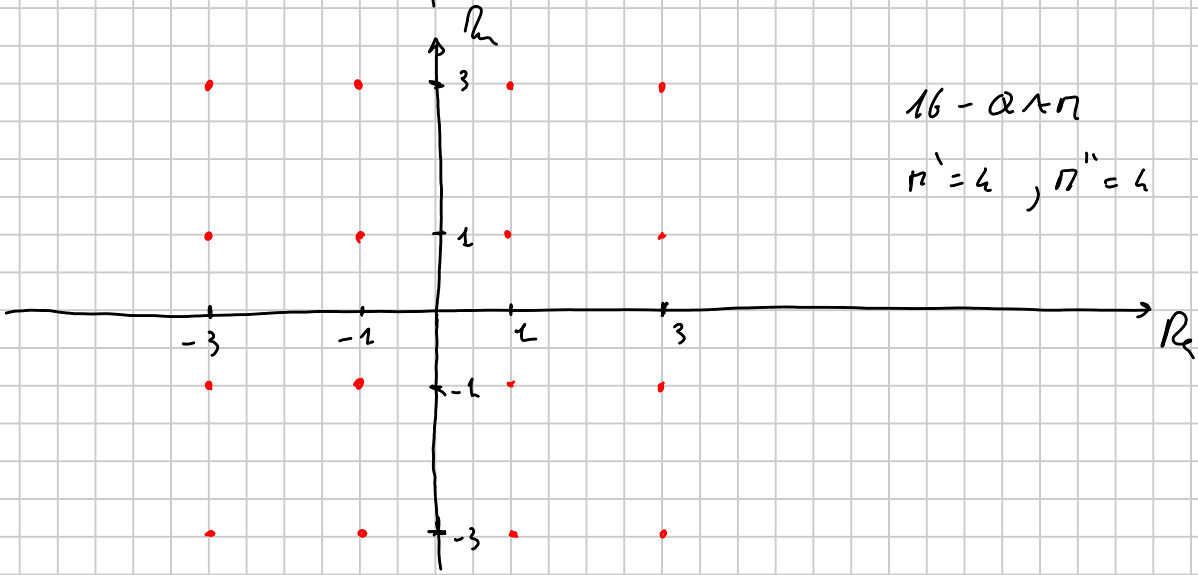
$M' = 2$

$M'' = 2$



8-QAM

$M' = 4$, $M'' = 2$

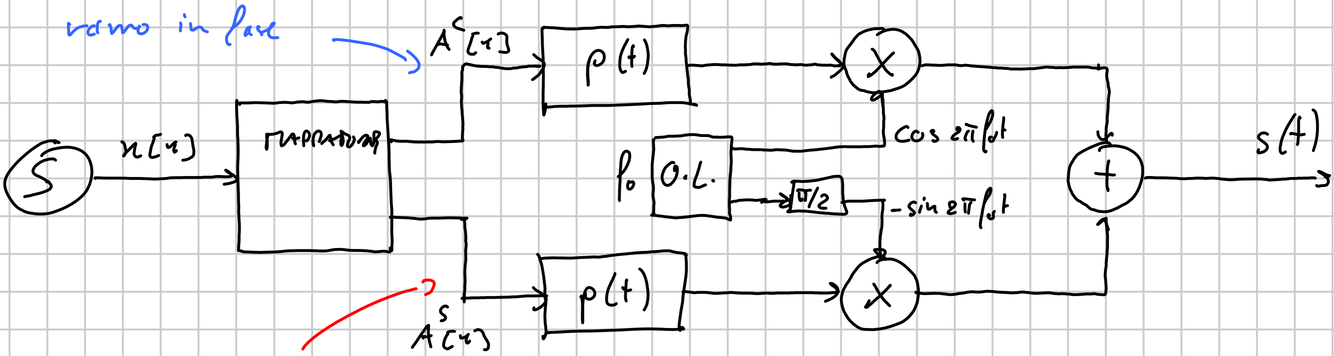


16-QAM

$M' = 4$, $M'' = 4$

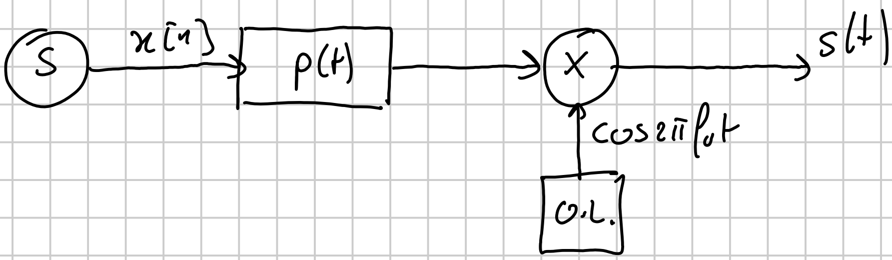
MODULATORE QAM

ramo in fase



ramo in quadratura

MODULAZIONE PAM IN BANDA PASSANTE



MODELLO EQUIVALENTE PER PAM IN B.P., PSK, QAM

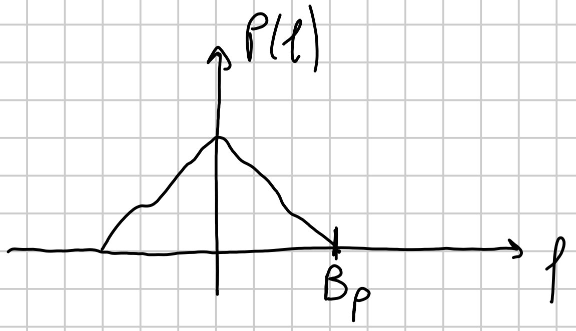
$$s(t) = \sum_{k=-\infty}^{+\infty} \tilde{x}[k]\, p(t - kT_s)$$

$$\tilde{x}[k] = \begin{cases} A_m \in A_s \text{ con elementi reali} & (PAM) \\ e^{j\theta_m}, \quad \theta_m = \frac{2\pi}{M}(m-1) & (PSK) \\ A_m^c + jA_m^s & (QAM) \end{cases}$$

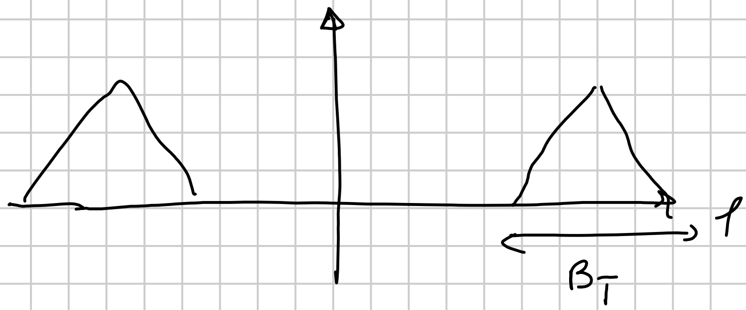
DENSITÀ SPETTRALE DI POTENZA DI UNA MODULAZIONE IN BANDA PASSANTE (PAM, PSK, QAM)

$$S_s(f) = \frac{1}{2} \frac{\sigma_{\tilde{x}}^2}{2} \left[P^2(f-f_0) + P^2(f+f_0) \right]$$

$$P(f) = TCF\left[p(t) \right]$$



⇒

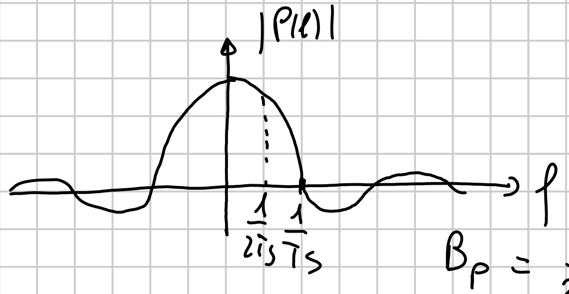


$$B_T = 2B_p$$

$$\eta_B = \frac{R_b}{2B_p} = \frac{\log_2 M}{2B_p T_s}$$

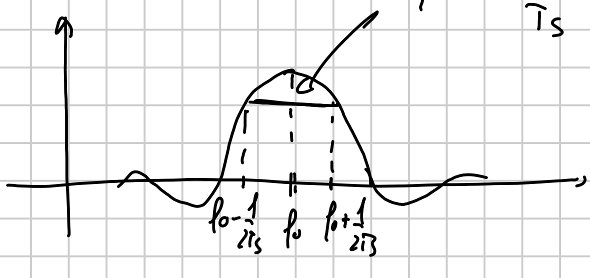
$$\Rightarrow p(t) = rect\left(\frac{t - T_s/2}{T_s}\right)$$

$$\Rightarrow P(f) = T_s \operatorname{sinc}(T_s f)\, e^{-j2\pi f \frac{T_s}{2}}$$



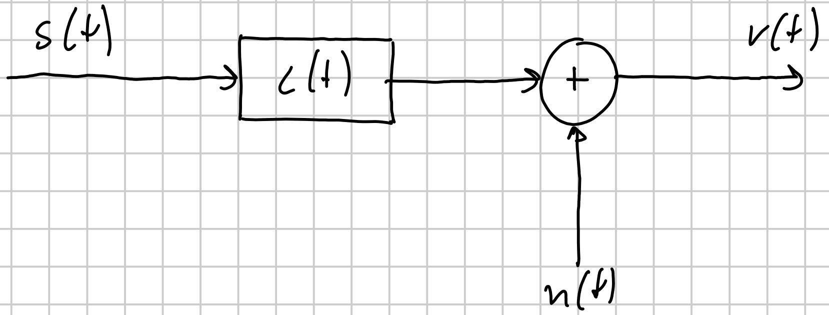
$$B_p = \frac{1}{2T_s}$$

$$B_T = \frac{1}{T_s}$$



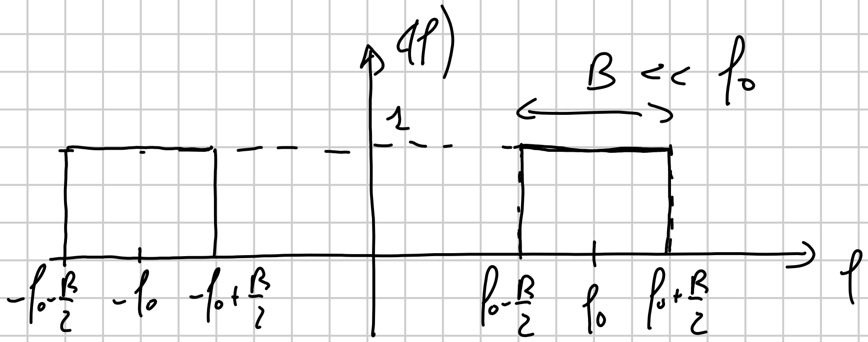
$$\eta_B = \frac{\log_2 M}{2 \frac{1}{2T_S} T_S} = \log_2 M$$

CANALE PASSA - BANDA

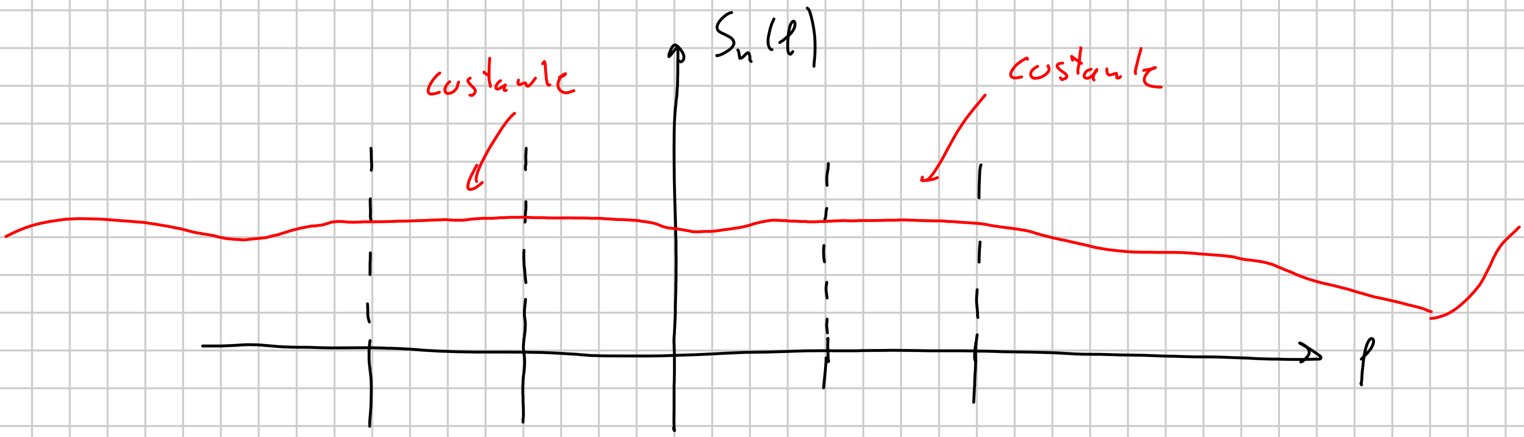


$$c(t) \Rightarrow C(f) = \begin{cases} C(f) & f_0 - \frac{B}{2} \le f \le f_0 + \frac{B}{2} \\ 0 & \text{altrove} \end{cases}$$

CANALE PASSA - BANDA IDEALE



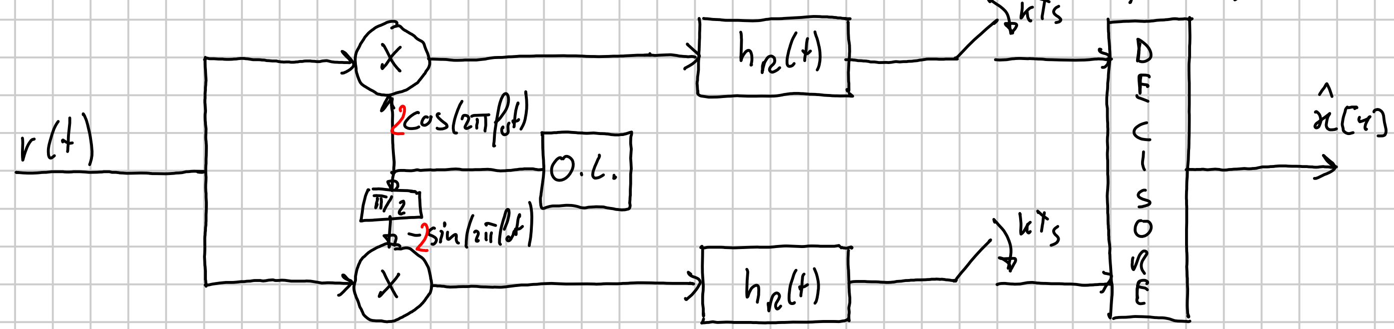
$n(t) \Rightarrow$ processo di rumore Gaussiano Bianco in banda



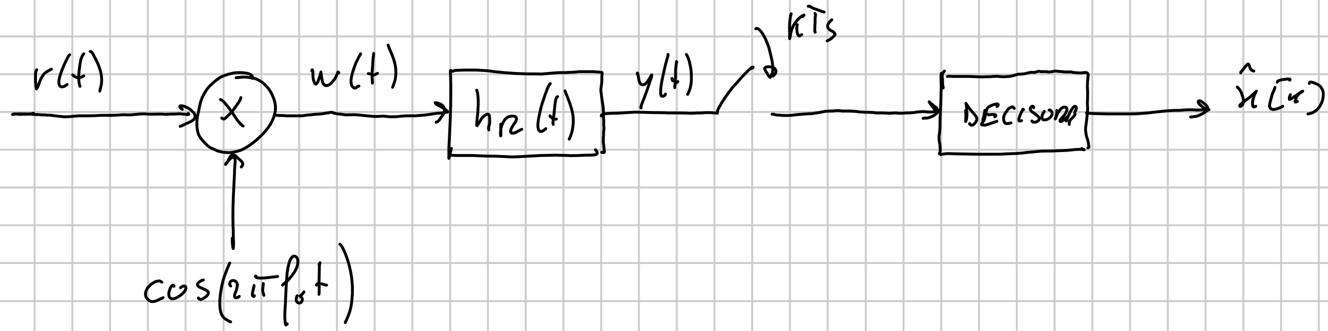
$$S_n(f) = \frac{N_0}{2}\left[\mathrm{rect}\left(\frac{f-f_0}{B}\right) + \mathrm{rect}\left(\frac{f+f_0}{B}\right)\right]$$

bianco in banda

RICEVITORE GENERALIZZATO PER PAM (B.P.), PSK, QAM



M-PAM



$$s(t) = \sum_k x[k]\, p(t-kT) \cos(2\pi f_0 t)$$

$$r(t) = s(t) \otimes c(t) + n(t)$$

$\Rightarrow$ Supponiamo che $c(t)$ sia ideale

$$r(t) = s(t) + n(t)$$

$$\Rightarrow w(t) = r(t)\cos(2\pi f_0 t) =$$

$$= \sum_k x[k]\, p(t-kT)\cos^2(2\pi f_0 t) + n(t)\cos(2\pi f_0 t) =$$

$$= \frac{1}{2}\sum_k x[k]\, p(t-kT) + \underbrace{\frac{1}{2}\sum_k x[k]\, p(t-kT)\cos(4\pi f_0 t)}_{2f_0} +$$

$$+ n(t)\cos(2\pi f_0 t)$$

$h_R(t)$ è un passa-basso

$$w(t) = \frac{1}{2}\sum_k x[k]\, p(t-kT) + n(t)\cos(2\pi f_0 t) + \text{comp a } 2f_0$$

$\Rightarrow$ Se utilizzo $P(f) = \sqrt{H_{RC}(f)}$

$\Downarrow$

elimino ISI e massimizzo l'SNR (F.A.)

$$y(t) = \cancel{\frac{1}{2}}\sum_{k=-\infty}^{+\infty} x[k]\, h(t-kT_s) + 2\,n(t)\cos(2\pi f_0 t) \otimes h_R(t)$$

$\Downarrow$

esattamente uguale alla componente del segnale utile nel caso PAM in banda base

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

$$\tilde{n}(t) = n_c(t) + j\, n_s(t)$$

Gaussiani bianchi

$$n(t)\cos(2\pi f_0 t) = \frac{1}{2} n_c(t)\left[1 + \underbrace{\cos(4\pi f_0 t)}_{2f_0}\right] - \frac{1}{2} n_s(t) \underbrace{\sin(4\pi f_0 t)}_{2f_0}$$

$$\sin\alpha\cos\alpha = \frac{1}{2}\sin(2\alpha)$$

In uscita al filtro $h_R(t)$ le componenti a $2f_0$ vengono eliminate

$$n(t)\cos(2\pi f_0 t) \otimes h_R(t) \cong n_c(t)$$

$$y(t) = \frac{1}{2}\sum_k x[k]\, h(t-kT_s) + \frac{1}{2} n_c(t)$$

Il segnale in ingresso al campionatore è formalmente identico al caso PAM in b.b.

$\Rightarrow$ Per il calcolo delle prestazioni

- Assenza di ISI
- $h(0) \Rightarrow y[n] = h(0)\, x[n] + n_u[n]$
- $n_u[n]$ è un v.a. $\mathcal{N}(0, \sigma_{n_u}^2)$

$$\downarrow$$

$$P_{n_u} = \int_{-\infty}^{+\infty} S_{n_c}(f)\, |H_R(f)|^2 \, df$$

23/05/2013

QAM

$$s(t) = \sum_k x_c[k] \cos(2\pi f_0 t)\, p(t-kT) + \qquad (I)$$
$$- \sum_k x_s[k] \sin(2\pi f_0 t)\, p(t-kT) \qquad (Q)$$

$I \triangleq$ In-phase

$Q \triangleq$ Quadrature

$x_c[n]$, $x_s[n]$

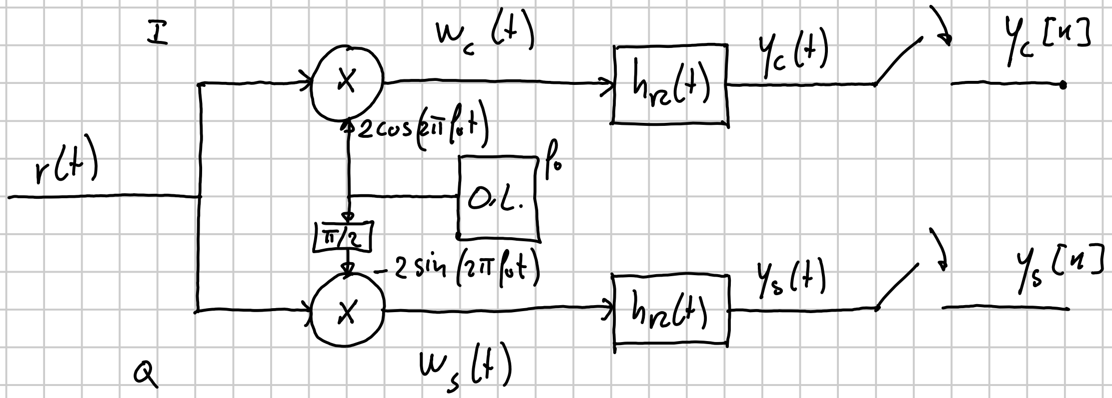
$$r(t) = s(t) \otimes c(t) + n(t)$$

·) $c(t) = \delta(t)$ canale ideale $\Rightarrow$ $C(f)$ piatta (costante) nella banda del segnale

·) $n(t)$ di tipo passa banda

$$n(t) = n_c(t)\cos(2\pi f_0 t) - n_s(t)\sin(2\pi f_0 t)$$

$$r(t) = \sum_k x_c[k]\, p(t-kT)\cos(2\pi f_0 t) + n_c(t)\cos(2\pi f_0 t) + \qquad (I)$$
$$- \left(\sum_k x_s[k]\, p(t-kT)\sin(2\pi f_0 t) + n_s(t)\sin(2\pi f_0 t)\right) \qquad (Q)$$



·) Ramo I

$$w_c(t) = r(t) \cdot 2\cos(2\pi f_0 t) =$$

$$= \left[\sum_n x_c[n]\, p(t-nT) + n_c(t)\right] \cos(2\pi f_0 t) \cdot 2\cos(2\pi f_0 t) \; +$$

$$- \left[\sum_n x_s[n]\, p(t-nT) + n_s(t)\right] \sin(2\pi f_0 t) \cdot 2\cos(2\pi f_0 t)$$

$$= \left[\sum_n x_c[n]\, p(t-nT) + n_c(t)\right] \left(1 + \cos(4\pi f_0 t)\right) \; +$$

$$- \left[\sum_n x_s[n]\, p(t-nT) + n_s(t)\right] \sin(4\pi f_0 t)$$

$w_c(t)$ ha una componente in bb. e due componenti alla freq. $2f_0$

$h_R(t)$ e' un filtro in b.b. $\Rightarrow$ elimina le componenti a $2f_0$

$$y_c(t) = \sum_n x_c[n]\, p(t-nT) \otimes h_R(t) + n_c(t) \otimes h_R(t)$$

·) sul ramo Q

$$w_s(t) = r(t) \cdot \left(-2\sin(2\pi f_0 t)\right) =$$

$$= -\left[\sum_n x_c[n]\, p(t-nT) + n_c(t)\right] \cos(2\pi f_0 t) \cdot 2\sin(2\pi f_0 t) \; +$$

$$+ \left[\sum_n x_s[n]\, p(t-nT) + n_s(t)\right] \sin(2\pi f_0 t) \cdot 2\sin(2\pi f_0 t)$$

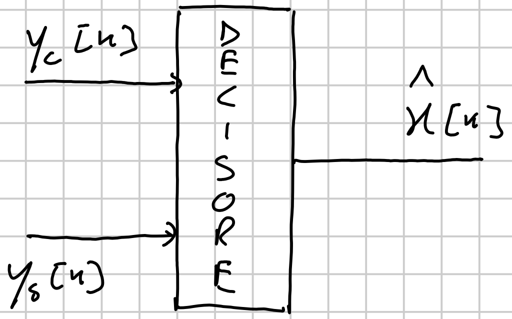
$$= -\left[\sum_n x_c[n]\, p(t-nT) + n_c(t)\right] \sin(4\pi f_0 t) \; +$$

$$+ \left[\sum_n x_s[n]\, p(t-nT) + n_s(t)\right] \left(1 - \cos(4\pi f_0 t)\right)$$

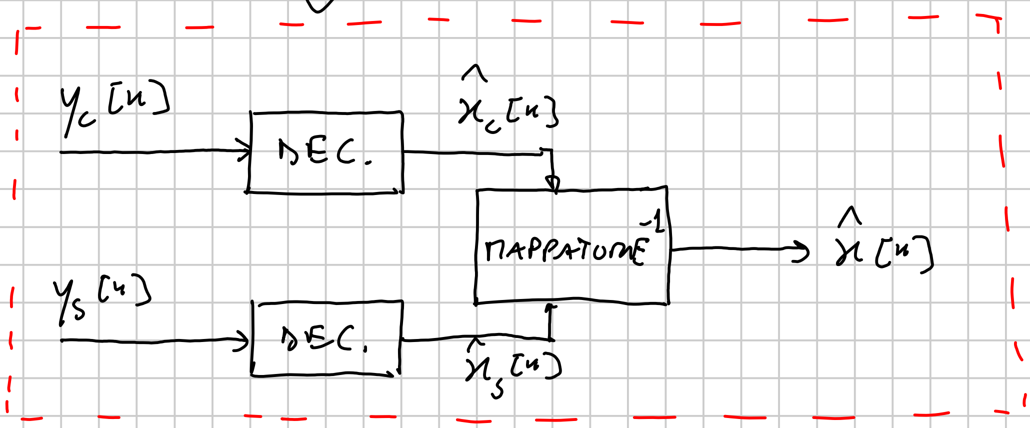
$$y_s(t) = \sum_n x_s[n]\, p(t-nT) \otimes h_R(t) + n_s(t) \otimes h_R(t)$$

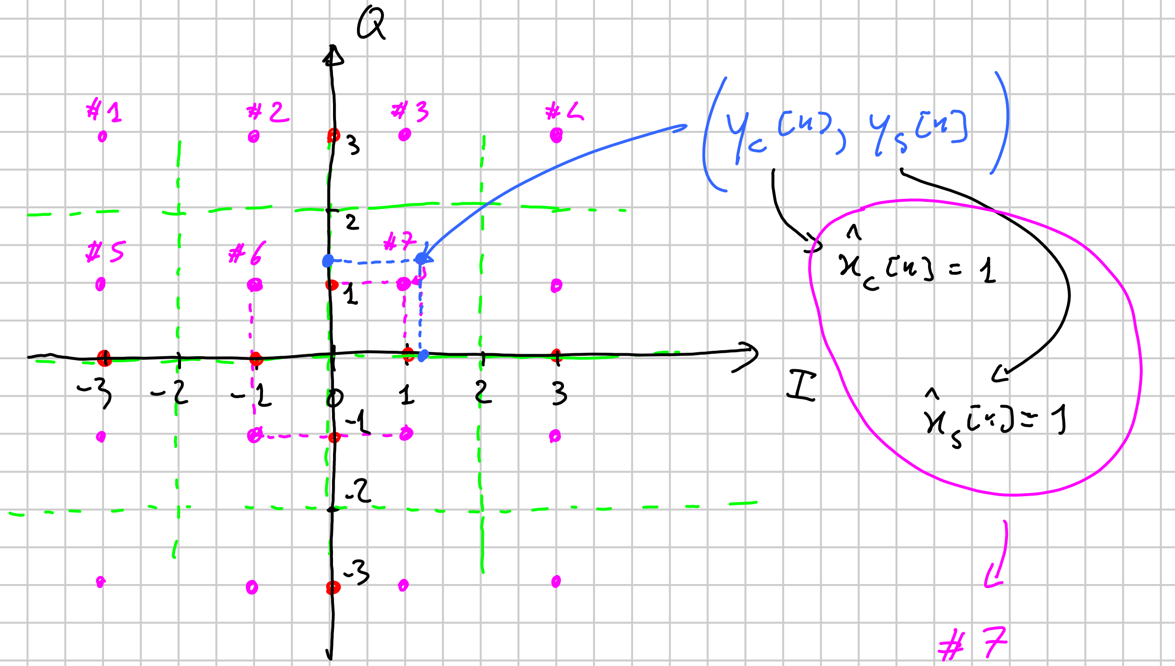
$$y_c(t) = \sum_n x_c[n]\, p(t-nT) \otimes h_R(t) + n_c(t) \otimes h_R(t) \qquad (I)$$

$$y_s(t) = \sum_n x_s[n]\, p(t-nT) \otimes h_R(t) + n_s(t) \otimes h_R(t) \qquad (Q)$$



$\Downarrow$





Prob. di errore sul simbolo

$$P_E^{(QAM)}(M) = ?$$

$$P_E^{(PAM)}(M_c) \quad , \quad P_E^{(PAM)}(M_s)$$

$$P_E^{(QAM)}(M) = P_E^{(PAM)}(M_c)\left(1 - P_E^{(PAM)}(M_s)\right) + P_E^{(PAM)}(M_s)\left(1 - P_E^{(PAM)}(M_c)\right) +$$
$$+ P_E^{(PAM)}(M_c)\, P_E^{(PAM)}(M_s)$$

$$P_E^{(PAM)}(M_c) \, , \, P_E^{(PAM)}(M_s) << 1$$

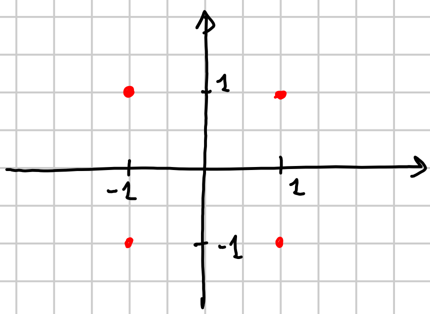
$$P_E^{(QAM)}(M) \simeq P_E^{(PAM)}(M_c) + P_E^{(PAM)}(M_s)$$

⇒ Se utilizzo una codifica di Gray

⇒ se $SNR >> 1$

$$P_b^{(QAM)} = \frac{P_E^{(QAM)}(M)}{\log_2 M}$$

4 - QAM



$$P_E^{(PAM)}(\underset{2}{\underset{\downarrow}{M_c}}) = P_E^{(PAM)}(\underset{2}{\underset{\downarrow}{M_s}}) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{SNR}\right)$$

$$P_E^{(QAM)}(M) = \operatorname{erfc}\left(\sqrt{SNR}\right)$$

$$P_E^{(QAM)}(b) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{SNR}\right) = P_E^{(PAM)}(b)$$

Stessa efficienza energetica 4-QAM e 2-PAM (BPSK)

2-PAM
$$\eta_B = \frac{1}{T_b B_T}$$

4-QAM
$$\eta_B = \frac{2}{T_b B_T}$$

4-QAM $\Rightarrow$ efficienza spettrale doppia rispetto alla 2-PAM

PROBLEMA DELLA FASE

$$r(t) = \sum_n x_c[n]\, p(t-nT) \cos(2\pi f_0 t + \theta_0) + n_c(t) \cos(2\pi f_0 t) +$$
$$- \left[\sum_n x_s[n]\, p(t-nT) \sin(2\pi f_0 t + \theta_0) + n_s(t) \sin(2\pi f_0 t) \right]$$

$$\theta_0 \in \mathcal{U}[0, 2\pi)$$

·) ramo I (trascuriamo il rumore)

$$w_c(t) = \sum_n x_c[n]\, p(t-nT) \cos(2\pi f_0 t + \theta_0) \cdot 2\cos(2\pi f_0 t) +$$
$$- \sum_n x_s[n]\, p(t-nT) \sin(2\pi f_0 t + \theta_0) \cdot 2\cos(2\pi f_0 t)$$

$$= \sum_n x_c[n]\, p(t-nT) \left[\cos\theta_0 + \cos(4\pi f_0 t + \theta_0) \right] +$$
$$- \sum_n x_s[n]\, p(t-nT) \left[\sin\theta_0 + \sin(4\pi f_0 t + \theta_0) \right] + \text{rumore}$$

$$w_c(t) \longrightarrow \boxed{h_R(t)} \longrightarrow y_c(t)$$

$$y_c(t) = \underbrace{\sum_n x_c[n]\, p(t-nT) \cos\theta_0}_{\text{comp. utile (I)}} - \underbrace{\sum_n x_s[n]\, p(t-nT) \sin\theta_0}_{\text{comp. di disturbo}} + \text{rumore}$$

·) Si può dimostrare allo stesso modo che si ha lo stesso effetto sul canale Q

CROSS - TALK

$\Rightarrow$ Bisogna stimare θ_0

$\Rightarrow$ Si trasmette una sequenza di training

Esempio :

$$x[n] \text{ noto } \begin{cases} \to 0 & \text{sul ramo in fase} \\ \to 1 & \text{sul ramo in quadratura} \end{cases}$$

$y_c[n] = ?$ $\qquad h(t) = p(t) \otimes h_R(t)$ (canale ideale)

$$y_c(t) = \sum_n x_c[n]\, h(t-nT) \cos\theta_0 + n_u^{(c)}(t)$$
$$- \sum_n x_s[n]\, h(t-nT) \sin\theta_0 - n_u^{(s)}(t)$$

$$n_u^{(c)}(t) = n_c(t) \otimes h_R(t)$$

$$n_u^{(s)}(t) = n_s(t) \otimes h_R(t)$$

$x_c[n] = 0$, $x_s[n] = 1$, assenza di ISI

$$y_c[n] = h(0)\, x_c[n] \cos\theta_0 + n_u^{(c)}[n] +$$
$$- h(0)\, x_s[n] \sin\theta_0 - n_u^{(s)}[n]$$

$$= n_u^{(c)}[n] - h(0) \sin\theta_0 - n_u^{(s)}[n]$$

$\Rightarrow$ supponiamo che SNR >> 1

$$y_c[n] \simeq -h(0) \sin\theta_0$$

$$\Rightarrow \boxed{\sin\theta_0 \simeq -\frac{y_c[n]}{h(0)}} \Rightarrow \theta_0 = -\arcsin\left[\frac{y_c[n]}{h(0)}\right]$$

$\Rightarrow$ SNR ~ 1

lunghezza della seq. di training

$$\underline{y} = [\, y[1], y[2], \dots, y[N] \,]$$

$$y[n] = -h(0)\sin\theta_0 + \underbrace{n_n^{(c)}[n] - n_n^{(s)}[n]}_{n[n]}$$

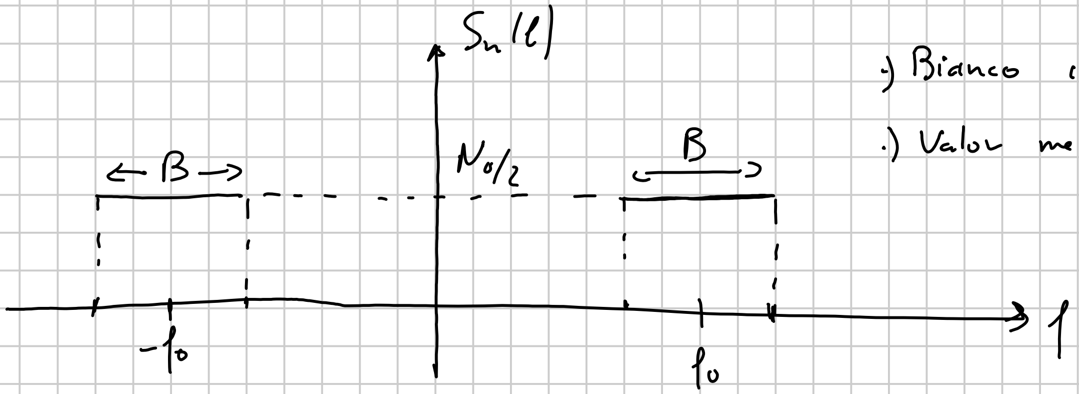
$$\hat{\sin\theta_0} = \frac{1}{N}\sum_{k=1}^{N}\frac{y[n]-n[n]}{h(0)} = \frac{1}{N}\sum_{k=1}^{N}\frac{y[n]}{h(0)} + \underbrace{\frac{1}{N}\sum_{k=1}^{N}\frac{n[n]}{h(0)}}_{\text{diminuisce all'aumentare di } N}$$

$$E_S = E\left[\int_0^T s^2(t)\,dt\right]$$

Energia media per intervallo di segnalazione

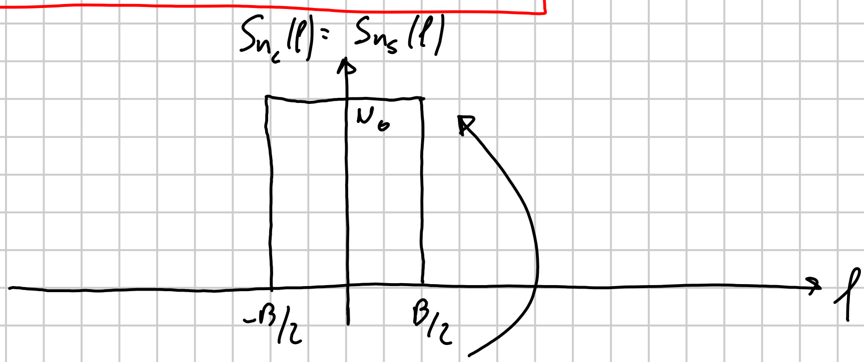
$$E_s = E\left[\int_0^T s_n^2(t)\,dt\right]$$

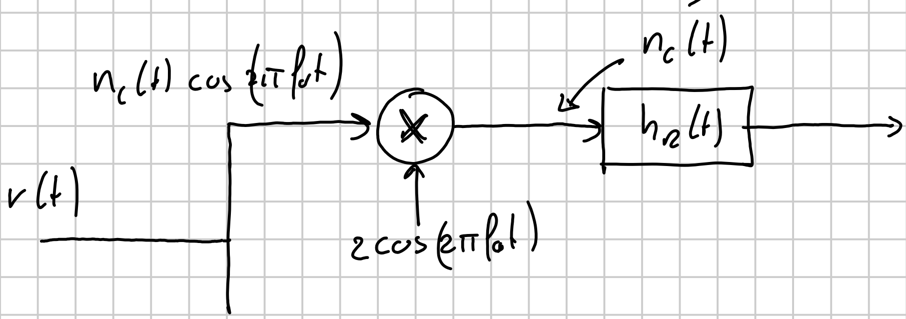
Energia media per simbolo trasmesso



- Bianco in banda
- Valor medio nullo

$$\boxed{S_{n_c}(f) = S_{n_s}(f) = N_0\,\mathrm{rect}\left(\frac{f}{B}\right)}$$





$r(t) =$ comp. utile I + comp utile Q + $n_c(t) \cos 2\pi f_0 t$ +
$+ \; n_s(t) \sin(2\pi f_0 t)$

·) ramo I

$$n_s(t) \sin(2\pi f_0 t) \; 2\cos(2\pi f_0 t) =$$

$$= n_s(t) \sin(4\pi f_0 t) \longrightarrow h_R(f) \longrightarrow \emptyset$$

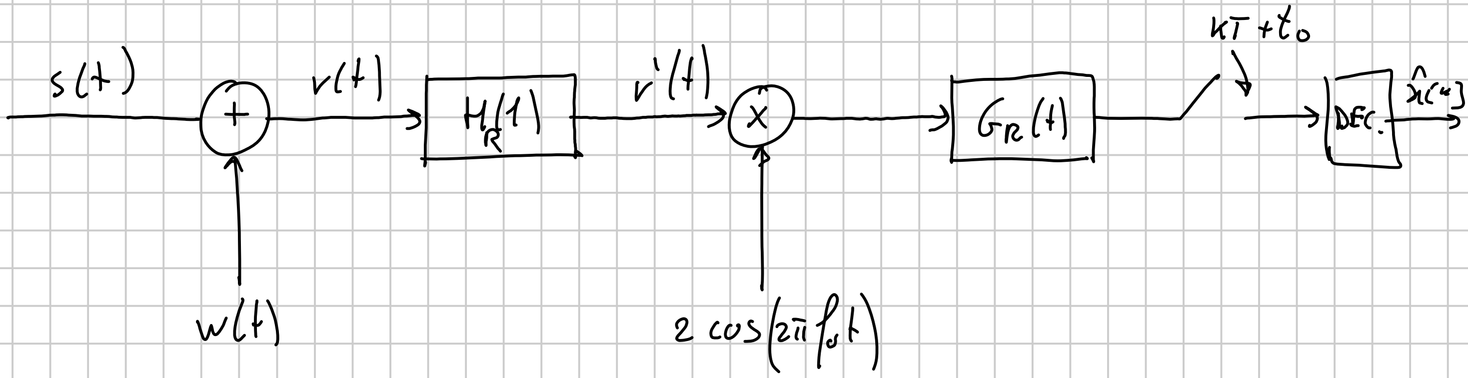
$$y_c(t) = \sum_k x_c[k] \, h(t - kT) + n_c(t)$$

$$y_s(t) = \sum_k x_s[k] \, h(t - kT) + n_s(t)$$

$$P_{n_u^{(c)}} = \int_{-\infty}^{+\infty} S_{n_c}(f) \, |H_R(f)|^2 \, df$$

$$P_{n_u^{(s)}} = \int_{-\infty}^{+\infty} S_{n_s}(f) \, |H_R(f)|^2 \, df \quad \Rightarrow \sigma_{n_u}^2$$

ESERCIZIO



$w(t)$ GAUSSIANO BIANCO IN BANDA

$g_R(t) = p(t)$

$$s(t) = \sum_k x[k] \, p(t - kT) \cos(2\pi f_0 t)$$

$t_0 = ?$

$\alpha_i \in [\pm 1]$ ind. ed equip.

$$P(f) = \begin{cases} \sqrt{T} \cos\left(\pi f T/2\right) & |fT| \leq 1 \\ 0 & \text{altrove} \end{cases}$$

$$H_R(f) = \begin{cases} e^{-j\left[2\pi (f-f_0)T + \pi/6\right]} & |f-f_0| \leq \frac{1}{T} \\ e^{-j\left[2\pi (f+f_0)T - \pi/6\right]} & |f+f_0| \leq \frac{1}{T} \\ 0 & \text{altrove} \end{cases}$$

$$P_E = ?$$

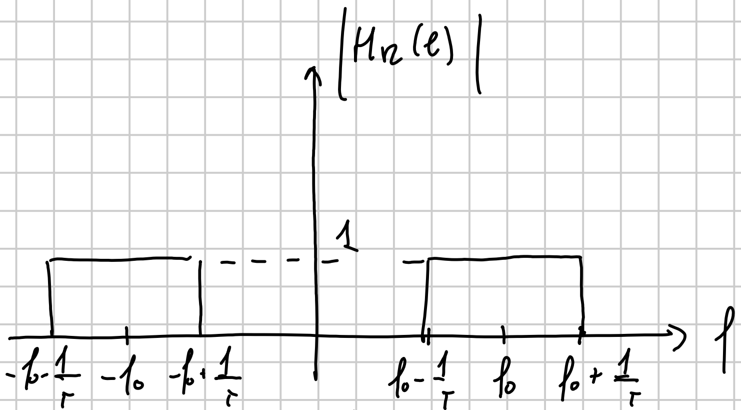
$$r'(t) = \underbrace{s(t) \otimes h_R(t)}_{\text{comp utile}} + \underbrace{w(t) \otimes h_R(t)}_{\text{comp di rumore}}$$

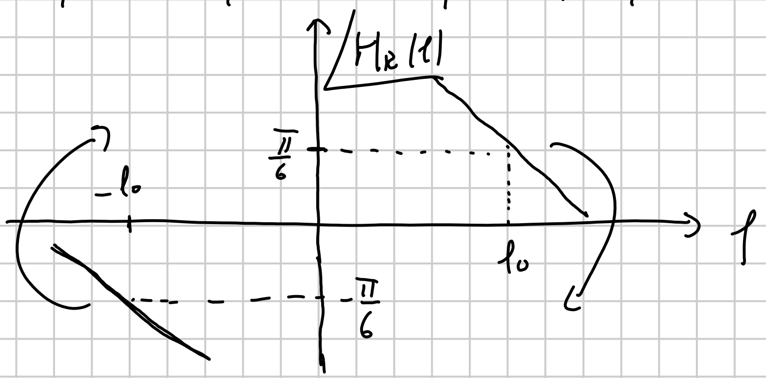
I) ASSENZA DI ISI

II) STATISTICHE DEL RUMORE (VALOR MEDIO, VARIANZA)

$$s(t) \otimes h_R(t)$$

$h_R(t)$





$$H_R(f) = \text{rect}\left(\frac{f-f_0}{1/T}\right) e^{-j\left(2\pi(f-f_0)\tau + \pi/6\right)} +$$

$$+ \text{rect}\left(\frac{f+f_0}{1/T}\right) e^{-j\left(2\pi(f+f_0)\tau - \pi/6\right)}$$

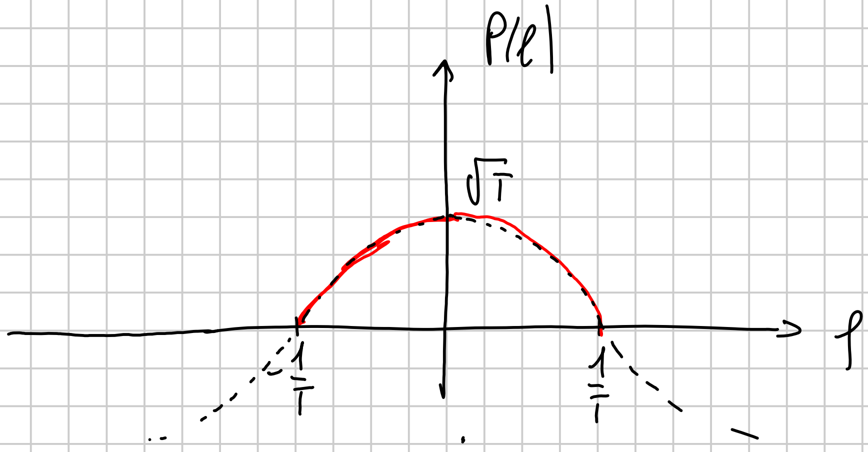
$$h_R(t) = \frac{e^{-j\pi/6}}{T} \text{sinc}\left(\frac{t}{T}\right) \otimes \delta(t-\tau)\, e^{+j2\pi f_0 \tau} \cdot e^{+j2\pi f_0 t} +$$

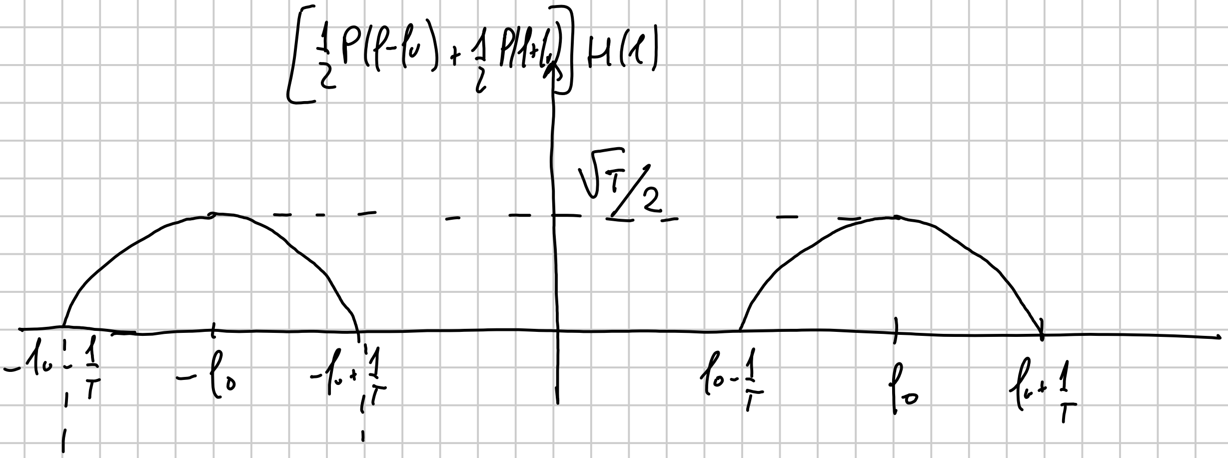
$$+ \frac{1}{T} \text{sinc}\left(\frac{t}{T}\right) \otimes \delta(t-\tau)\, e^{-j2\pi f_0 \tau} \cdot e^{-j2\pi f_0 t} \ldots$$

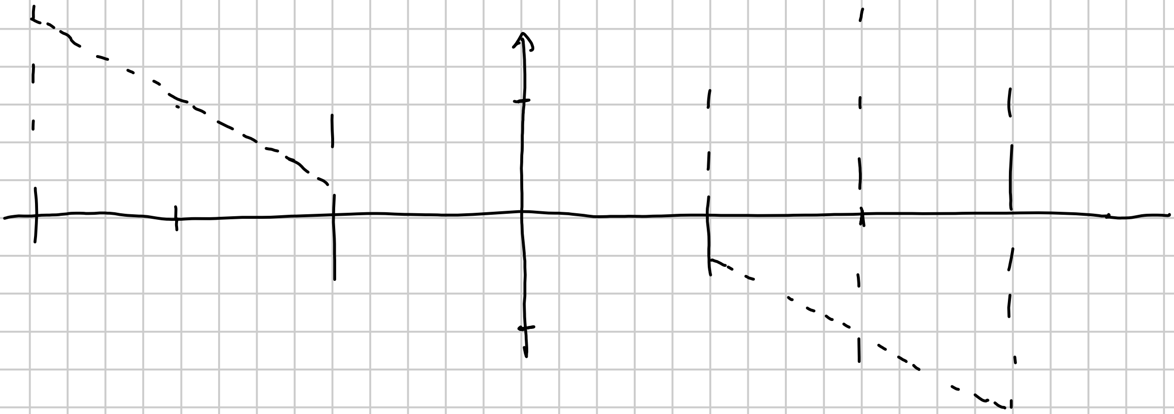
·) la strada è complicata, ragioniamo in frequenza:

$$P(f) \otimes \left[\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right] =$$

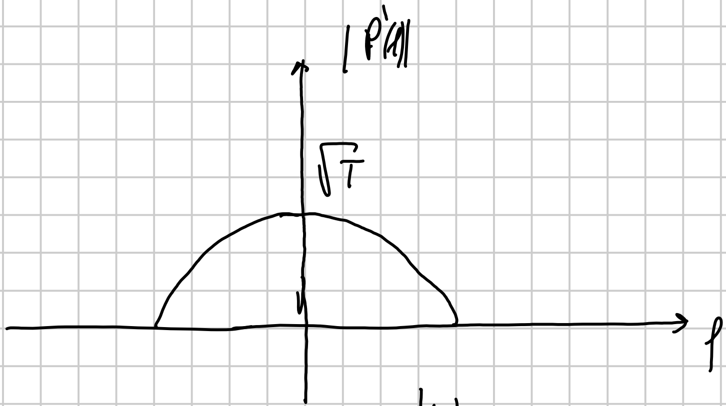
$$= \frac{1}{2}P(f-f_0) + \frac{1}{2}P(f+f_0)$$

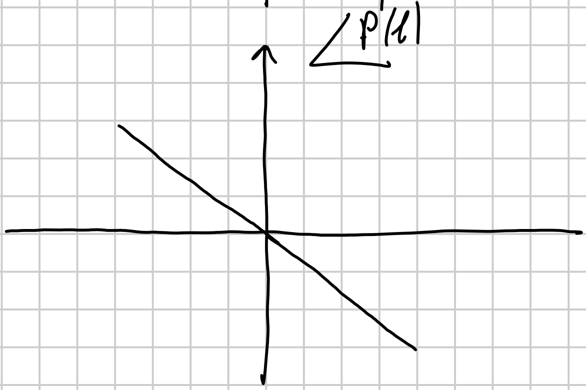






$$P''(f) = \left[\frac{1}{2} P(f-f_0) + \frac{1}{2} P(f+f_0)\right] H(f)$$



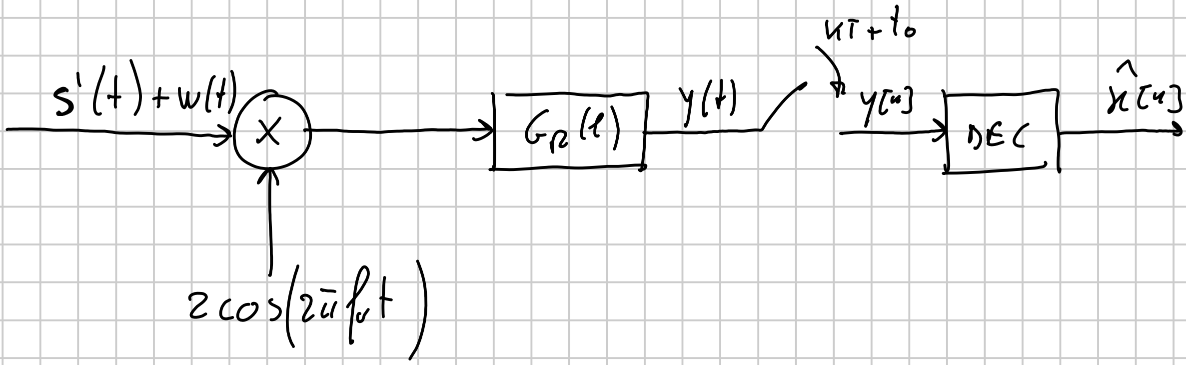


$$P'(f) = P(f)\, e^{-j 2\pi f T}$$

$$P''(f) = P'(f) \otimes \left[\frac{e^{-j\frac{\pi}{6}}}{2} \delta(f-f_0) + \frac{e^{j\frac{\pi}{6}}}{2} \delta(f+f_0)\right]$$

$$p''(t) = p'(t) \cos\left(2\pi f_0 t - \frac{\pi}{6}\right)$$

$$s'(t) = \sum_n x[n]\, p'(t) \cos\left(2\pi f_0 t - \pi/6\right)$$



$$w'(t) = w(t) \otimes h_R(t)$$

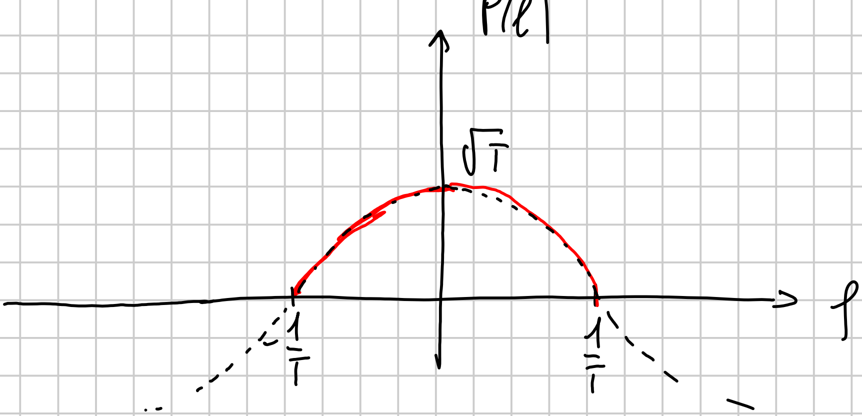
$$S_{w'}(f) = S_w(f)\,|H_R(f)|^2 = S_w(f)$$

$$p'(t) = p(t-\tau)$$

$$r(t) = \sum_k x[k]\, p(t-\tau-kT)\cos(2\pi f_0 t - \pi/6) + w(t)$$

$\Rightarrow$ $t_0 = \tau$ perché il segnale in uscita è ritardato di "τ" $\Rightarrow$ l'istante "ottimo" di campionamento è $kT + \tau$

VERIFICA SU ISI

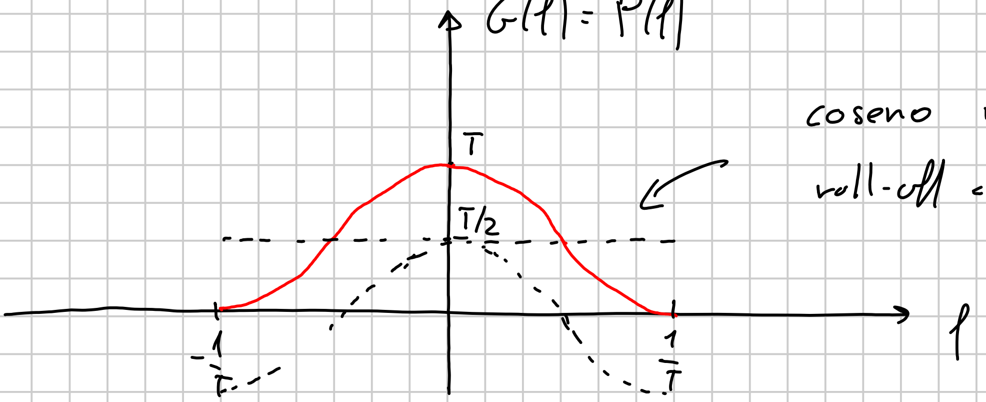


$$g_R(t) = p(-t) = p(t)$$

$$G_R(f) = P(f)$$

$$g(t) = p(t) \otimes g_R(t) \qquad \text{rect}\left(\frac{f}{2/T}\right)$$

$$G(f) = P(f)\, G_R(f) = P^2(f) = T\cos^2\left(\pi f \frac{T}{2}\right) = \frac{T}{2}\left(1 + \cos\left(2\pi f \frac{T}{2}\right)\right)\text{rect}\left(\frac{f}{2/T}\right)$$



coseno rialzato con roll-off = 1

$\Downarrow$

ASSENZA DI ISI

·) FILTRO ADATTATO

PAM BINARIA CON SIMBOLI IND. EQUIP. (SIMMETRICI)

$$P_E(a) = P_E(b) = \frac{1}{2}\,\text{erfc}\left(\sqrt{SNR}\right)$$

·) SNR in uscita al f.d.

$$SNR = \frac{E\left[2\, g^2(0)\, x^2[n]\right]}{\sigma_{n_u}^2} = \frac{2\, g^2(0)\, E\left[x^2[n]\right]}{\sigma_{n_u}^2}$$

·) $g(0)$
·) $E\left[x^2[n]\right]$
·) $\sigma_{n_u}^2$

$$g(0) = g(t)\Big|_{t=0} = \int_{-\infty}^{+\infty} P^2(f)\, df = E_p = 1$$

$$E\left[x^2[n]\right] = 1$$

$$\sigma_{n_u}^2 = P_{n_u} = \int_{-\infty}^{+\infty} N_0 \, |G_R(f)|^2 \, df = N_0 \, E_{G_R} = N_0 \, E_p = N_0$$

$$P_E(b) = \frac{1}{2} \, \mathrm{erfc}\left(\sqrt{\frac{2}{N_0}} \right)$$

$$E_{IS} \triangleq E\left[\int_0^T |s(t)|^2 dt \right] = P_S T = T \int_{-\infty}^{+\infty} S_s(f) \, df$$

$$E_S = E\left[\int_{-\infty}^{+\infty} s_n^2(t) \, dt \right]$$

esempio: PAM

$$s_n(t) = x[n] \, p(t - nT)$$

se $p(t) \neq 0 \quad 0 \leq t \leq T$

$$E\left[\int_0^T s^2(t) \, dt \right] = E\left[\int_0^T \left[\sum_n x[n] \, p(t - nT) \right]^2 dt \right] =$$

$$= E\left[\sum_n \int_{-nT}^{-(n-1)T} x^2[n] \, p^2(t - nT) \, dt \right] = E\left[\int_{-\infty}^{+\infty} x^2[n] \, p^2(t - nT) \, dt \right]$$

$$\boxed{S_s(f) = \frac{\sigma_x^2}{T} P^2(f)}$$

$$\sum_n H\left(f - \frac{k}{T}\right) = C$$

ESERCITAZIONE

13/03/2013

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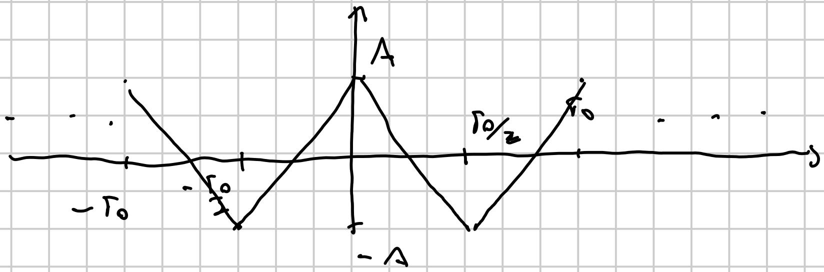


$$x(t) = \sum_k x_0(t - kT_0)$$

$$x(t) = -x\left(t - \frac{T_0}{2}\right)$$

$$x_0(t) = \left(A - \frac{4A|t|}{T_0}\right)\mathrm{rect}\left(\frac{t}{T_0}\right)$$

$x(t)$ PERIODICO, REALE, PARI, ALTERNATIVO

$$X_n = \begin{cases} 0 & n \text{ PARI} \\ \dfrac{2}{T_0}\displaystyle\int_0^{T_0/2} x(t)\, e^{-j2\pi n f_0 t}\, dt & \text{per } n \text{ DISP.} \end{cases}$$

$$X_n = \frac{2}{T_0}\int_0^{T_0/2} \underset{g}{\left(A - \frac{4At}{T_0}\right)} \underset{h'}{e^{-j2\pi n f_0 t}}\, dt =$$

$$= \frac{2}{T_0}\left[\underset{g}{\left(A - \frac{4At}{T_0}\right)} \underset{h}{\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}}\right]_0^{T_0/2} + \int_0^{T_0/2} \underset{g'}{\frac{4A}{T_0}} \underset{h}{\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}}\, dt$$

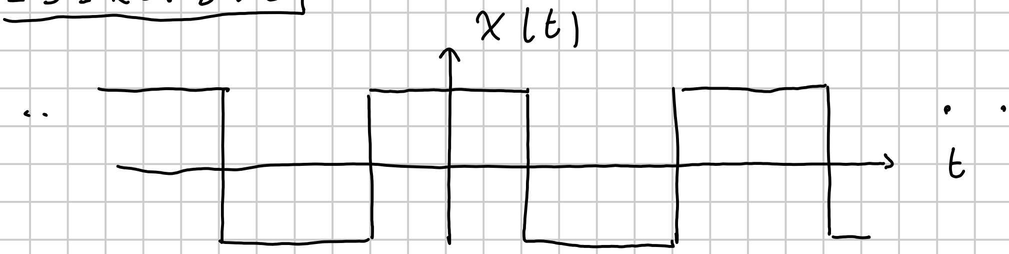
$$= \frac{\cancel{2}}{\cancel{T_0}}\left[(A - 2A)\frac{e^{-j\pi n}}{-j\cancel{2}\pi n \cancel{f_0}} - A + \frac{4A}{\cancel{T_0}}\frac{e^{-j2\pi n f_0 t}}{-j2\pi n \cancel{f_0}}\Bigg|_0^{T_0/2}\right.$$

$$= -\frac{A}{j\pi n}\left[e^{-j\pi n} - 1 - \frac{2}{j\pi n}\left(e^{-j\pi n} - 1\right)\right] =$$

$$= \frac{A}{j\pi n}\left(\underbrace{e^{-j\pi n}}_{-1} + 1\right) - \frac{2A}{\pi^2 n^2}\left(\underbrace{e^{-j\pi n}}_{-1} - 1\right) \Rightarrow$$

$$X_n = \frac{4A}{\pi^2 n^2}$$

ESERCIZIO



x(t) PARI, REALE, ALTERNATIVO $\rightleftarrows$ $X_n = X_{-n}$

$Im\{X_n\} = 0$

$X_n = 0$ n pari

$$X_n = \begin{cases} 0 & n \text{ PARI} \\ \frac{2}{T_0}\int_0^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t}\, dt & n \text{ DISP} \end{cases}$$

$$= \frac{2}{T_0}\left[\int_0^{T_0/4} A\, e^{-j2\pi n f_0 t} + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} (-A)\, e^{-j2\pi n f_0 t}\, dt\right]$$

$$= \frac{2A}{T_0}\left[\left.\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}\right|_0^{\frac{T_0}{4}} - \left.\frac{e^{-j2\pi n f_0 t}}{-j2\pi n f_0}\right|_{\frac{T_0}{4}}^{\frac{T_0}{2}}\right] =$$

$$= \frac{\cancel{2}A}{-j\cancel{T_0}\cancel{2}\pi n \cancel{f_0}}\left[e^{-j2\pi n f_0 \frac{T_0}{4}} - 1 - \left(e^{-j2\pi n f_0 \frac{T_0}{2}} - e^{-j2\pi n f_0 \frac{T_0}{4}}\right)\right] =$$

$$= j\frac{A}{\pi n}\left[e^{-j\frac{\pi n}{2}} - 1 - e^{-j\pi n} + e^{-j\frac{\pi n}{2}}\right] =$$

$$= -j\frac{A}{\pi n}\left[-2\underbrace{e^{-j\frac{\pi n}{2}}} + \cancel{(\underbrace{e^{-j\pi n}}_{-1} + 1)}\right]$$

$$e^{-j\pi \frac{n}{2}} = -j$$

$$X_n = \frac{2A}{\pi n} \underbrace{j(-j)^n}_{e^{j\frac{\pi}{2}}\left(e^{-j\frac{\pi}{2}}\right)^n = e^{j\frac{\pi}{2}} e^{-jn\frac{\pi}{2}} = e^{-j\frac{(n-1)}{2}\pi} = \left(e^{-j\pi}\right)^{\frac{n-1}{2}} = (-1)^{\frac{n-1}{2}}} \qquad n \text{ DISP.}$$

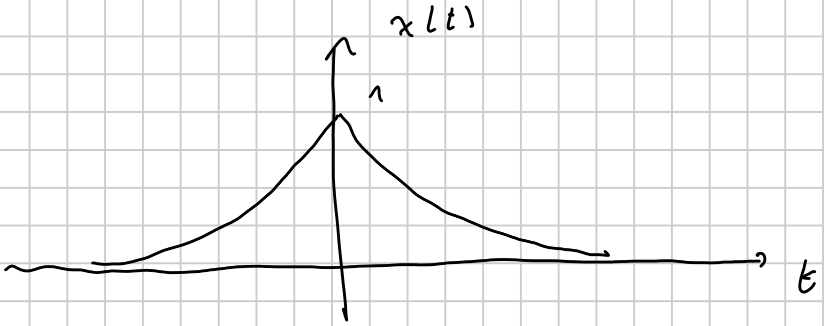
$$X_1 = \frac{2A}{\pi}(-1)^0 = \frac{2A}{\pi}$$

$$X_3 = \frac{2A}{3\pi}(-1)^1 = -\frac{2A}{3\pi}; \quad X_5 = \frac{2A}{5\pi}(-1)^2 = \frac{2A}{5\pi}$$

$$\boxed{TCF}$$

ESERCIZIO ESPONENZIALE BILATERO

$$x(t) = e^{-|t|}$$



$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-\infty}^{0} e^{t} e^{-j2\pi f t}\, dt + \int_{0}^{+\infty} e^{-t} e^{-j2\pi f t}\, dt =$$

$$= \left.\frac{e^{t(1-j2\pi f)}}{1-j2\pi f}\right|_{-\infty}^{0} + \left.\frac{e^{-t(1+j2\pi f)}}{-(1+j2\pi f)}\right|_{0}^{+\infty}$$

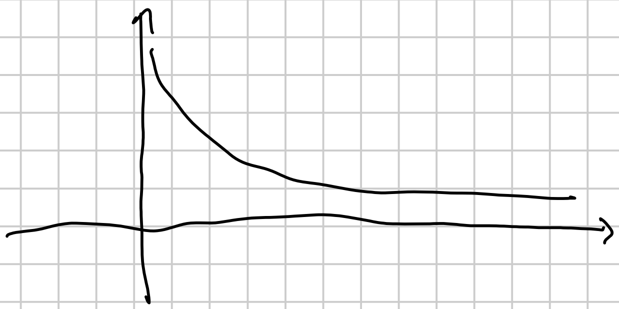
$$= \left(\frac{1}{1-j2\pi f} - 0\right) + \left(0 + \frac{1}{1+j2\pi f}\right) =$$

$$X(f) = \frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} = \frac{1+\cancel{j2\pi f}+1-\cancel{j2\pi f}}{1+4\pi^2 f^2} =$$

$$= \frac{2}{1+4\pi^2 f^2}$$

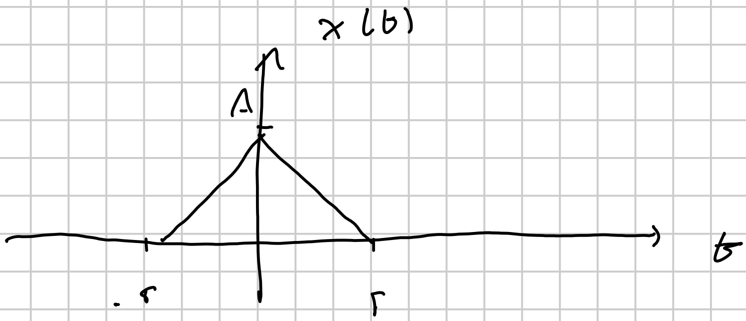
ESERCIZIO 2 | ESPONENZIALE MONOLATERO

$$x(t) = e^{-t}\, u(t)$$



$$X(f) = \int_0^{+\infty} e^{-t}\, e^{-j2\pi f t}\, dt = \frac{1}{1+j2\pi f}$$

ESERCIZIO IMPULSO TRIANGOLARE



$$x(t) = A\left(1 - \frac{|t|}{T}\right) \mathrm{rect}\left(\frac{t}{T}\right)$$

I) $$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt =$$

$$= \int_{-T}^{0} A\left(1 + \frac{t}{T}\right) e^{-j2\pi f t}\, dt + \int_{0}^{T} A\left(1 - \frac{t}{T}\right) e^{-j2\pi f t}\, dt =$$

$$\int_{-T}^{0} A\left(1 + \frac{t}{T}\right) e^{-j2\pi f t}\, dt = A\left(1 + \frac{t}{T}\right) \frac{e^{-j2\pi f t}}{(-j2\pi f)} \Bigg|_{-T}^{0} +$$

g f' g f

$$- \int_{-T}^{0} \frac{A}{T} \frac{e^{-j2\pi f t}}{(-j2\pi f)}\, dt =$$

g' f

$$= \frac{A}{-j2\pi f} - 0 - \frac{A}{T} \frac{e^{-j2\pi f t}}{(-j2\pi f)^2} \Bigg|_{-T}^{0} =$$

$$= -\frac{A}{j2\pi f} + \frac{A}{T}\left(\frac{1 - e^{j2\pi f T}}{4\pi^2 f^2}\right)$$

$$\int_{0}^{T} A\left(1 - \frac{t}{T}\right) e^{-j2\pi f t}\, dt = \frac{A}{j2\pi f} - \frac{A}{T4\pi^2 f^2}\left(e^{-j2\pi f T} - 1\right)$$

$$X(f) = -\cancel{\frac{A}{j2\pi f}} + \frac{A}{\tau 4\pi^2 f^2}\left(1 - e^{j2\pi f\tau}\right) + \cancel{\frac{A}{j2\pi f}} - \frac{A}{\tau 4\pi^2 f^2}\left(e^{-j2\pi f\tau} - 1\right) =$$

$$= \frac{A}{4\pi^2 f^2 \tau}\left[1 - e^{j2\pi f\tau} + 1 - e^{-j2\pi f\tau}\right] =$$

$$= \frac{A}{4\pi^2 f^2 \tau}\left(1 - e^{j2\pi f\tau}\right)\left(1 - e^{-j2\pi f\tau}\right) =$$

$$= \frac{A}{4\pi^2 f^2 \tau}\, \cancel{e^{+j\pi f\tau}}\left(e^{-j\pi f\tau} - e^{+j\pi f\tau}\right)\cancel{e^{-j\pi f\tau}}\left(e^{+j\pi f\tau} - e^{-j\pi f\tau}\right) =$$

$$4 = -2j \cdot 2j$$

$$= \frac{A}{\pi^2 f^2 \tau}\left(\frac{e^{-j\pi f\tau} - e^{+j\pi f\tau}}{-2j}\right)\left(\frac{e^{+j\pi f\tau} - e^{-j\pi f\tau}}{-2j}\right) =$$

$$= \frac{A}{\pi^2 f^2 \tau}\sin(\pi f\tau)\sin(\pi f\tau) =$$

$$= A\tau \frac{\sin(\pi f\tau)}{\pi f\tau}\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau\,\mathrm{sinc}^2(f\tau)$$

II)

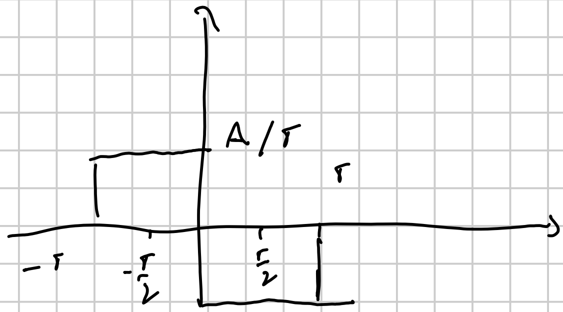
$$x(t) = \frac{A}{\tau}\left(\mathrm{rect}\left(\frac{t}{\tau}\right) \otimes \mathrm{rect}\left(\frac{t}{\tau}\right)\right)$$

$$z(t) = x(t) \otimes y(t) \rightleftharpoons Z(f) = X(f)\,Y(f)$$

$$X(f) = \frac{A}{\tau}\,\tau\,\mathrm{sinc}(f\tau)\,\tau\,\mathrm{sinc}(f\tau) = A\tau\,\mathrm{sinc}(f\tau)$$

III)

$$y(t) = \frac{\partial}{\partial t} x(t)$$



$$y(t) = \frac{d}{dt} x(t) \rightleftarrows Y(f) = j2\pi f X(f)$$

$$X(f) = \frac{Y(f)}{j2\pi f}$$

$$y(t) = \frac{A}{T} \operatorname{rect}\left(\frac{t + T/2}{T}\right) - \frac{A}{T} \operatorname{rect}\left(\frac{t - T/2}{T}\right)$$

$$y_0(t) = \operatorname{rect}\left(\frac{t}{T}\right) \qquad y(t) = \frac{A}{T}\left[y_0\left(t + \frac{T}{2}\right) - y_0\left(t - \frac{T}{2}\right)\right]$$

$$Y(f) = \frac{A}{T} Y_0(f) e^{+j2\pi f \frac{T}{2}} - \frac{A}{T} Y_0(f) e^{-j2\pi f \frac{T}{2}} =$$

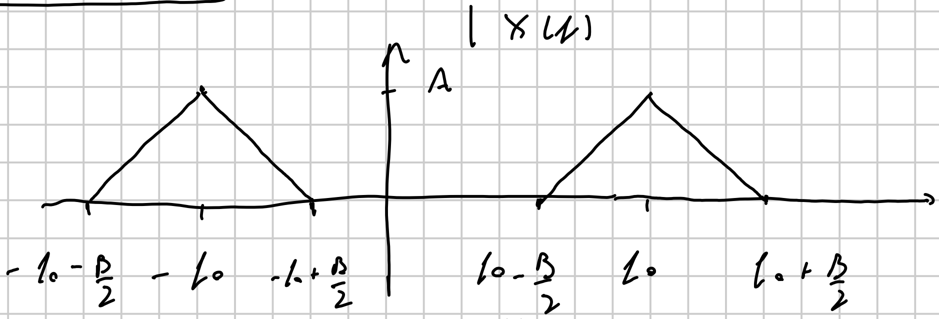
$$= \frac{A}{T} T \operatorname{sinc}(fT) \left[e^{+j\cancel{2}\pi f \frac{T}{\cancel{2}}} - e^{-j\pi f T}\right] \frac{2j}{2j}$$

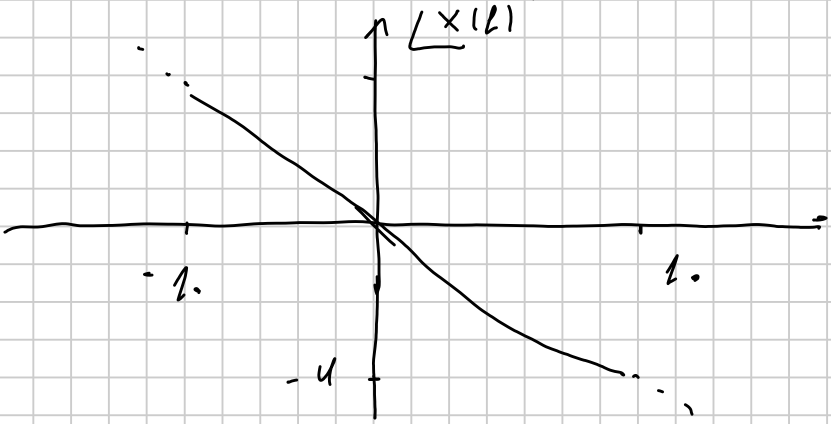
$$= A\, 2j \operatorname{sinc}(fT) \sin(\pi f T) = Y(f)$$

$$X(f) = \frac{Y(f)}{j2\pi f} = \frac{AT}{\pi f T} \operatorname{sinc}(fT) \sin(\pi f T) =$$

$$= AT \operatorname{sinc}^2(fT)$$

ESERCIZIO





$$X(f) = |X(f)| e^{j \angle X(f)}$$

$$|X(f)| = X_1(f - f_0) + X_1(f + f_0)$$

$$X_1(f) = A\left(1 - \frac{|f|}{B/2}\right) \mathrm{rect}\left(\frac{f}{B}\right)$$

$$\angle X(f) = a f + b$$

$b = 0$

$a = -\frac{\varphi}{f_0}$

$$X(f) = \underbrace{\left[X_1(f - f_0) + X_1(f + f_0)\right]}_{A(f)} e^{-j\frac{\varphi}{f_0} f}$$

$$X(f) = A(f)\, e^{-j 2\pi f t_0}$$

$t_0 = \frac{\varphi}{2 f_0 \pi}$

.) teorema del ritardo

$$x(t) = a(t - t_0)$$

.) $A(f) = X_1(f - f_0) + X_1(f + f_0)$

$$a(t) = x_1(t)\; 2\cos(2\pi f_0 t)$$

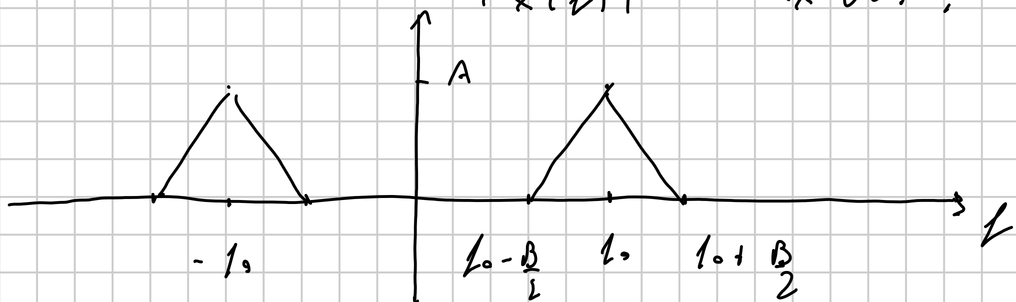
$$\boxed{\begin{aligned} y(t) &= x(t)\cos 2\pi f_0 t \\ Y(f) &= \frac{X(f-f_0) + X(f+f_0)}{2} \end{aligned}}$$

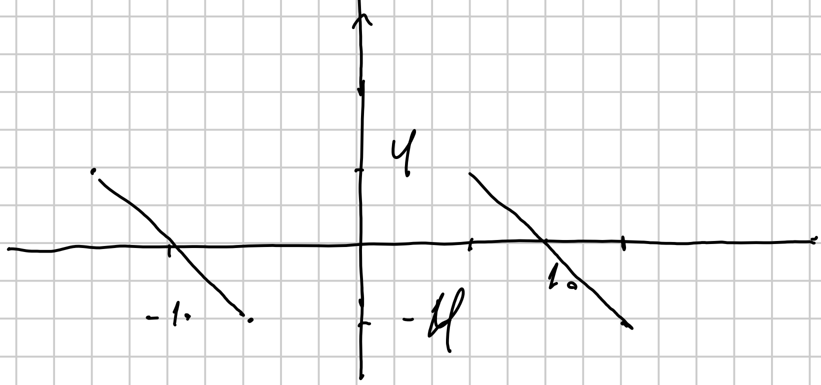
$$x_1(t) = A\,\frac{B}{2}\,\text{sinc}^2\left(t\,\frac{B}{2}\right)$$

$$x(t) = AB\cos\left(2\pi f_0 (t-t_0)\right)\,\text{sinc}^2\left(\frac{B}{2}(t-t_0)\right)$$

<u>ESERCIZIO X CASA</u>

$|X(f)|$ $\qquad$ $x(t)$?





$$X(f) = X_0(f-f_0)\,e^{j\varphi_1(f)} + X_0(f+f_0)\,e^{j\varphi_2(f)}$$

$$X_0(f) = A\left(1 - \frac{|f|}{B/2}\right)\text{rect}\left(\frac{f}{B}\right)$$

$$\varphi_1(f) = \varphi_0(f-f_0) = -\frac{2\psi}{B}(f-f_0)$$

$$\varphi_2(f) = \varphi_0(f+f_0) = -\frac{2\psi}{B}(f+f_0)$$

$$X(f) = X_0(f - f_0)\, e^{j\varphi_0(f - f_0)} + X_0(f + f_0)\, e^{j\varphi_0(f + f_0)}$$

$$Y(f) = X_0(f)\, e^{j\varphi_0(f)}$$

TEOREMA DELLA MODULAZIONE

$$x(t) = 2\cos(2\pi f_0 t)\, y(t)$$

TEOREMA DEL RITARDO

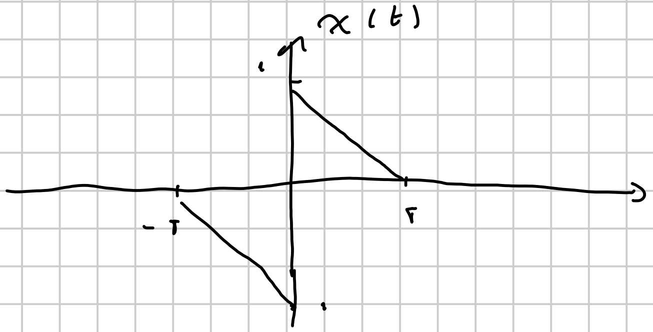
$$y(t) = x_0(t - t_0) \qquad \text{con} \qquad t_0 = \frac{\varphi}{B\pi}$$

$$x_0(t) = \frac{AB}{2}\, \text{sinc}^2\left(\frac{tB}{2}\right)$$

QUINDI

$$x(t) = AB \cos(2\pi f_0 t)\, \text{sinc}^2\left((t - t_0)\frac{B}{2}\right)$$

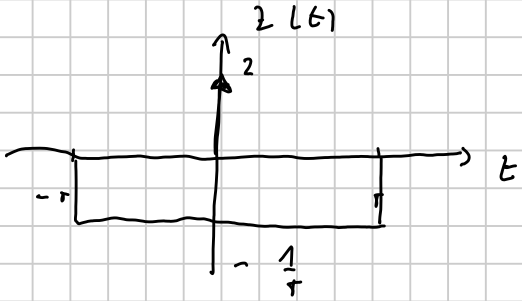
ESERCIZIO 1



$Y(f)$?

Y_k ?

$$y(t) = \sum_n x(t - 4Tn)$$



. TEO DERIVAZIONE

$$z(t) = \frac{d}{dt} x(t)$$

$$x(t) = \int_{-\infty}^{t} z(t)\, dt$$

$$z(t) = -\frac{1}{T} \operatorname{rect}\left(\frac{t}{2T}\right) + 2\,\delta(t)$$

$$Z(f) = -\frac{1}{\cancel{T}}\, 2\cancel{T} \operatorname{sinc}(f\, 2T) + 2$$

$$Z(0) = 0$$

$$Z(f) = j2\pi f\, X(f) \Rightarrow X(f) = \frac{Z(f)}{j2\pi f}$$

$$X(f) = \frac{2\left[1 - \operatorname{sinc}(f\,2T)\right]}{j2\pi f}$$

$$\boxed{T_0 = 4T}$$

$$Y(f) = \frac{1}{4T} \sum_k \frac{2\left(1 - \operatorname{sinc}\left(\frac{k}{4T} 2T\right)\right)}{j2\pi \frac{k}{4T}}\, \delta\left(f - \frac{k}{4T}\right)$$

$$Y_k = \frac{1 - \operatorname{sinc}\left(\frac{k}{2}\right)}{j\pi k}$$

ESERCIZIO 2

$$s(t) = \text{sinc}\left(2Bt - \frac{1}{2}\right) + \text{sinc}\left(2Bt + \frac{1}{2}\right)$$

$$x(t) = s(t)\,\sin(2\pi B t) \qquad E_x\,,\ P_x\ ?$$

$$y(t) = \sum_k s\left(t - \frac{2k}{B}\right) \qquad E_y\,,\ P_y\ ?$$

$$s(t) = \text{sinc}\Big(2B\big(t - \underbrace{\tfrac{1}{4B}}_{t_0}\big)\Big) + \text{sinc}\Big(2B\big(t + \underbrace{\tfrac{1}{4B}}_{t_0}\big)\Big)$$

$$S(f) = \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) e^{-j2\pi f t_0} + \frac{1}{2B}\,\text{rect}\left(\frac{f}{2B}\right) e^{+j2\pi f t_0} =$$

$$= \frac{1}{B}\,\text{rect}\left(\frac{f}{2B}\right)\left(\underbrace{\frac{e^{-j2\pi f t_0} + e^{+j2\pi f t_0}}{}}_{2}\right) =$$

$$= \frac{1}{B}\,\text{rect}\,\frac{f}{2B}\,\cos(2\pi f t_0)$$

$$X(f) = \frac{1}{2j}\Big[S(f - f_0) - S(f + f_0)\Big] =$$

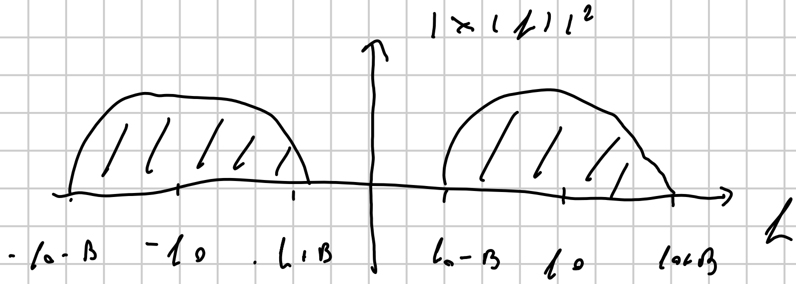
$$= \frac{1}{2jB}\Big[\text{rect}\left(\frac{f - f_0}{2B}\right)\cos\left(\pi\,\frac{(f - f_0)}{2B}\right) +$$

$$- \text{rect}\left(\frac{f + f_0}{2B}\right)\cos\left(\pi\,\frac{(f + f_0)}{2B}\right)\Big] =$$

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |X(f)|^2\,df =$$

$$|X(f)|^2 = \frac{1}{4B^2}\Big[\text{rect}\left(\frac{f - f_0}{2B}\right)\cos^2\left(\pi\,\frac{(f - f_0)}{2B}\right) +$$

$$+ \text{rect}\left(\frac{f + f_0}{2B}\right)\cos^2\left(\pi\,\frac{(f + f_0)}{2B}\right)\Big]$$



$$E_x = \frac{2}{4B^2}\int_{f_0-B}^{f_0+B} \cos^2\left(\pi\frac{(f-f_0)}{2B}\right) df =$$

$$f' = f - f_0$$

$$= \frac{2}{4B^2}\int_{-B}^{B} \cos^2\left(\frac{\pi f'}{2B}\right) df' =$$

$$= \frac{2}{4B^2}\int_{-B}^{B}\left[\frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi f'}{B}\right)\right] df' = \frac{1}{2B^2} \qquad B = \frac{1}{2B}$$

$$P_x = 0$$

$$y(t) = \sum_k x\left(t - \frac{2k}{B}\right) \qquad T_0 = \frac{2}{B}$$

$$E_y = \infty$$

$$P_y = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |y(t)|^2\, dt = \sum_k |Y_k|^2$$

$$Y_k = \frac{1}{T_0} S\left(\frac{k}{T_0}\right) = \frac{B}{2} S\left(k\frac{B}{2}\right) =$$

$$= \frac{B}{2}\frac{1}{B}\,\mathrm{rect}\left(\frac{kB/2}{2B}\right)\cos\left(\pi\frac{B/2}{2B}\right) =$$

$$= \frac{1}{2}\,\mathrm{rect}\left(\frac{k}{4}\right)\cos\left(k\frac{\pi}{4}\right)$$

$$Y_k = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{\sqrt{2}}{4} & k = \pm 1 \\ 0 & \text{altrove} \end{cases} \qquad P_y = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

ESERCIZIO 3 AM CON FASE

$$y(t) = x(t) \cos(2\pi f_0 t + \varphi)$$

$$x(t) = A B \,\mathrm{sinc}^2(B t)$$

$$Y(f) = A \left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right) \otimes \frac{1}{2}\left[e^{j\varphi} \delta(f-f_0) + e^{-j\varphi} \delta(f+f_0)\right]$$

$$= \frac{A}{2} e^{j\varphi} \left(1 - \frac{|f-f_0|}{B}\right) \mathrm{rect}\left(\frac{f-f_0}{2B}\right) +$$

$$+ \frac{A}{2} e^{-j\varphi} \left(1 - \frac{|f+f_0|}{B}\right) \mathrm{rect}\left(\frac{f+f_0}{2B}\right)$$

ESERCIZIO 4

$$x(t) = A \cos(2\pi f_1 t) + B \cos(2\pi f_2 t)$$

$$f_1 / f_2 = k \in \mathbb{R} \quad \text{(non intero)}$$

$$X(f) = \frac{A}{2}\left[\delta(f-f_1) + \delta(f+f_1)\right] + \frac{B}{2}\left[\delta(f-f_2) + \delta(f+f_2)\right]$$

SISTEMI

ESERCIZIO 1

$$y(t) = \int_{T}^{t} x(\alpha)\, d\alpha$$

1) LINEARITÀ ?
2) STAZIONARIETÀ ?
3) STABILITÀ BIBO ?
4) ISTANTANEITÀ ?

1) $x(t) = a\, x_1(t) + b\, x_2(t)$

$$y(t) = T[x(t)] = a\, T[x_1(t)] + b\, T[x_2(t)]$$

$$y(t) = \int_{T}^{t} [a\, x_1(t) + b\, x_2(t)]\, dt =$$

$$= a \int_{T}^{t} x_1(t)\, dt + b \int_{T}^{t} x_2(t)\, dt =$$

$$= a\, T[x_1(t)] + b\, T[x_2(t)]$$

LINEARE

2) $y(t - t_0) = T[x(t - t_0)]$

$$\int_{T}^{t} x(\alpha - t_0)\, d\alpha = \int_{T - t_0}^{t - t_0} x(\beta)\, d\beta =$$

$$= \underbrace{\int_{T - t_0}^{T} x(\beta)\, d\beta}_{\alpha} + \underbrace{\int_{T}^{t - t_0} x(\beta)\, d\beta}_{y(t - t_0)} =$$

NO STAZIONARIO

3) STABILITÀ BIBO

$|x(t)| \leq M \qquad |y(t)| \leq K$

$x(\alpha) = M$

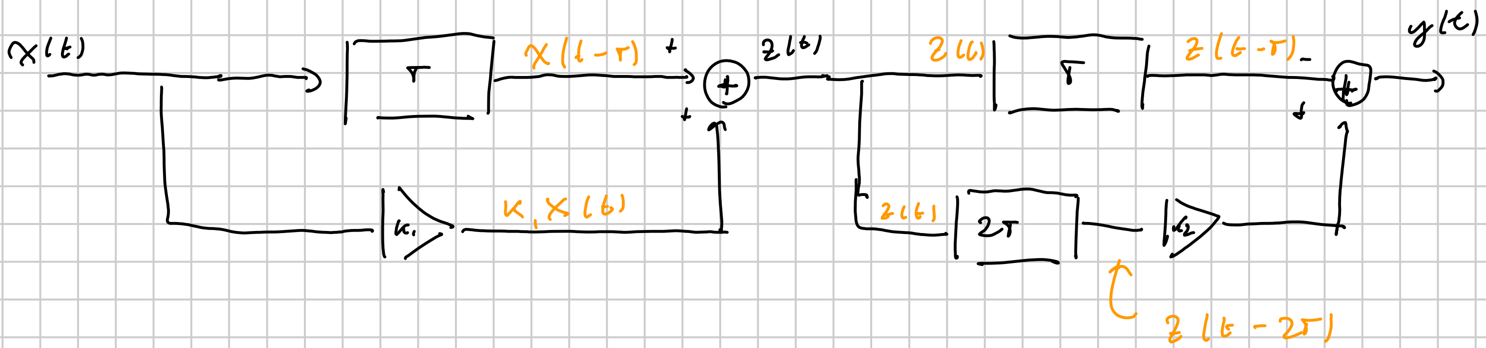
$$\int_T^t x(\alpha)\, d\alpha = \int_T^t M \, d\alpha = M(t-T) = y(t)$$

NO STABILE

4) ISTANTANEITÀ

$$y(t) = \int_T^t x(\alpha)\, d\alpha$$ È CON MEMORIA

ESERCIZIO 2



$z(t) = x(t-T) + k_1 x(t)$

$y(t) = -z(t-T) + k_2 z(t-2T)$

$$y(t) = -\left(x(t-2T) + k_1 x(t-T)\right) + k_2\left(x(t-3T) + k_1 x(t-2T)\right) =$$

$$= -k_1 x(t-2T) + x(t-2T)(k_1 k_2 - 1) + k_2 x(t-3T)$$

LINEARITÀ

$x(t) = a\, x_1(t) + b\, x_2(t)$

$$y(t) = -k_1 \left(a x_1 (t - T) + b x_2 (t - T) \right) +$$

$$+ \left(a x_1 (t - 2T) + b x_2 (t - 2T) \right) (k_1 k_2 - 1) +$$

$$+ k_2 (a x_1 (t - 3T)) + b x_2 (t - 3T) = \ldots$$

2) STAZIONARIETA'

$$x(t - t_0) \Rightarrow -k_1 x(t - t_0 - T) + x(t - t_0 - 2T)(k_1 k_2 - 1) +$$

$$+ k_2 x(t - t_0 - 3T) = y(t - t_0)$$

STAZIONARIO

3) MEMORIA

$y(t)$ dipende da $x(t - T)$, $x(t - 2T)$ $x(t - 3T)$

4) CAUSALITA' SI'

5) STABILITA'

$|x(t)| \leq M \quad \forall t$

$$|y(t)| = \left| -k_1 x(t - T) + (k_2 k_1 - 1) x(t - 2T) + k_2 x(t - 3T) \right| \leq$$

$$\leq |k_1| M + |k_1 k_2 - 1| M + |k_2| M \leq +\infty$$

STABILE

$h(t)$ risp. impulsiva

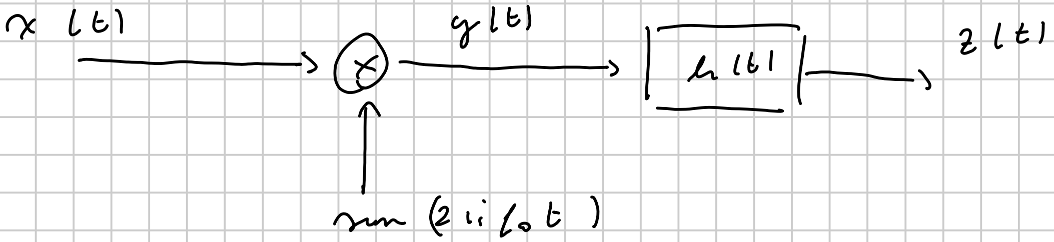
$H(f)$ risp. in frequenza

$x(t) = \delta(t)$

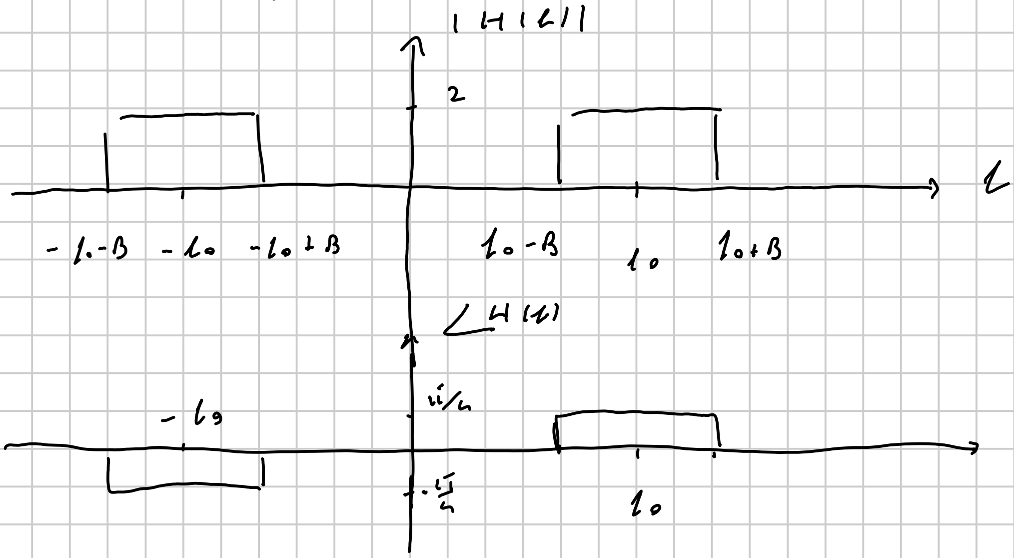
$$y(t) = h(t) = -k_1 \delta(t - T) + (k_2 k_1 - 1) \delta(t - 2T) + k_2 \delta(t - 3T)$$

$$H(f) = -k_1 e^{-j2\pi fT} + (k_2 k_1 - 1) e^{-j4\pi fT} + k_2 e^{-j6\pi fT}$$

ESERCIZIO 3



$$X(f) = A\left(1 - \frac{|f|}{B}\right) \operatorname{rect}\left(\frac{f}{2B}\right)$$

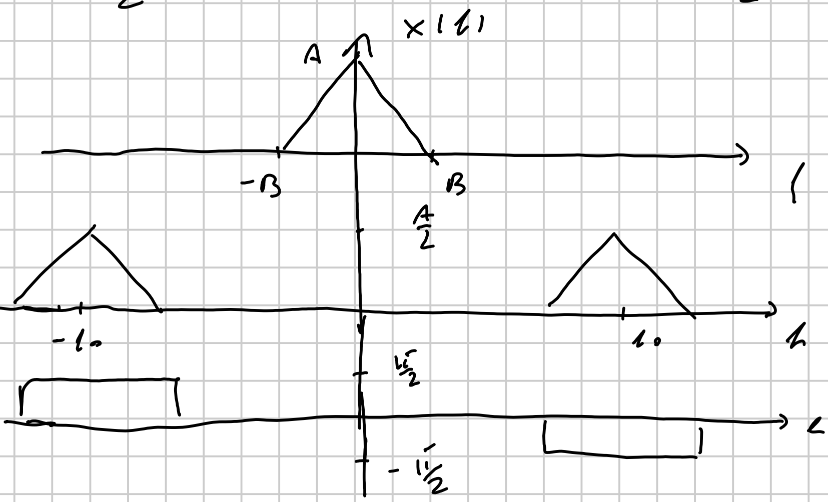


$z(t)$? $\qquad E_z$?

$$y(t) = x(t) \sin(2\pi f_0 t)$$

$$Y(f) = X(f) \otimes \frac{1}{2j}\left[\delta(f - f_0) - \delta(f + f_0)\right] =$$

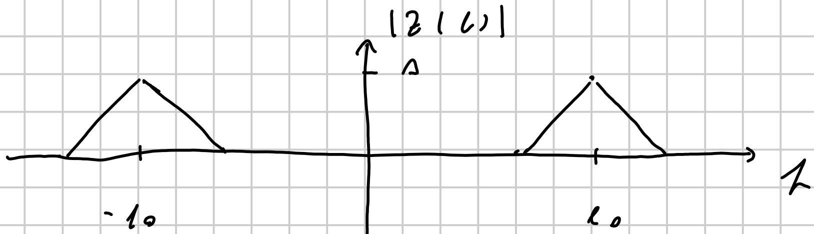
$$= \frac{e^{-j\frac{\pi}{2}}}{2} X(f - f_0) + \frac{e^{j\frac{\pi}{2}}}{2} X(f + f_0)$$

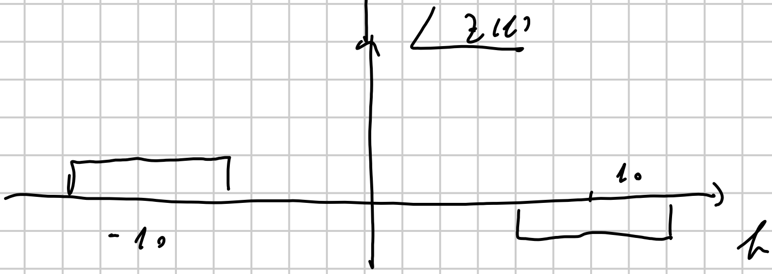


$$Z(f) = Y(f)\,H(f)$$

$$|Z(f)| = |Y(f)|\,|H(f)|$$

$$\underline{\angle Z(f)} = \underline{\angle Y(f)} + \underline{\angle H(f)}$$





$$Z(f) = A\left(1 - \frac{|f - f_0|}{B}\right)\mathrm{rect}\left(\frac{f - f_0}{2B}\right)e^{-j\frac{\pi}{4}} +$$

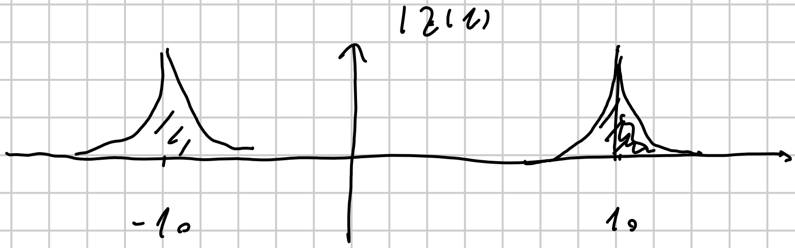
$$+ A\left(1 - \frac{|f + f_0|}{B}\right)\mathrm{rect}\left(\frac{f + f_0}{2B}\right)e^{j\frac{\pi}{4}} =$$

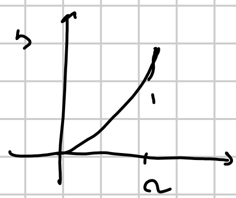
$$= A\left(1 - \frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right) \otimes \left[\delta(f - f_0)e^{-j\frac{\pi}{4}} + \delta(f + f_0)e^{j\frac{\pi}{4}}\right]$$

$$z(t) = AB\,\mathrm{sinc}^2(tB)\left[e^{+j\left(2\pi f_0 t - \frac{\pi}{4}\right)} + e^{-j\left(2\pi f_0 t - \frac{\pi}{4}\right)}\right] =$$

$$= 2AB\,\mathrm{sinc}^2(tB)\cos\left(2\pi f_0 t - \frac{\pi}{4}\right)$$

$$E_z = \int_{-\infty}^{+\infty} |z(t)|^2\,dt = \int_{-\infty}^{+\infty} |Z(f)|^2\,df$$

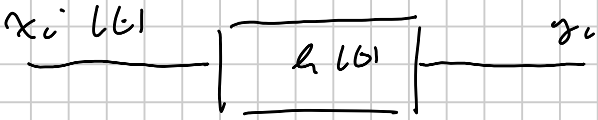




$$\int_0^a y(t)\,dt = \frac{ab}{3}$$

$$E_z = 4\int_{f_0-B}^{f_0} |Z(f)|^2 \, df = 4\,\frac{A^2 B}{3}$$

ESERCIZIO 4



$h(t) = \mathrm{sinc}\left(\frac{t}{T}\right) \mathrm{sinc}\left(\frac{t}{2T}\right)$

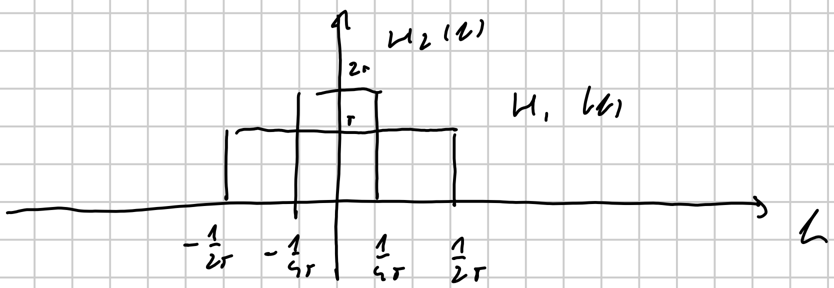
$x_1(t) = e^{-t/T}\, u(t)$

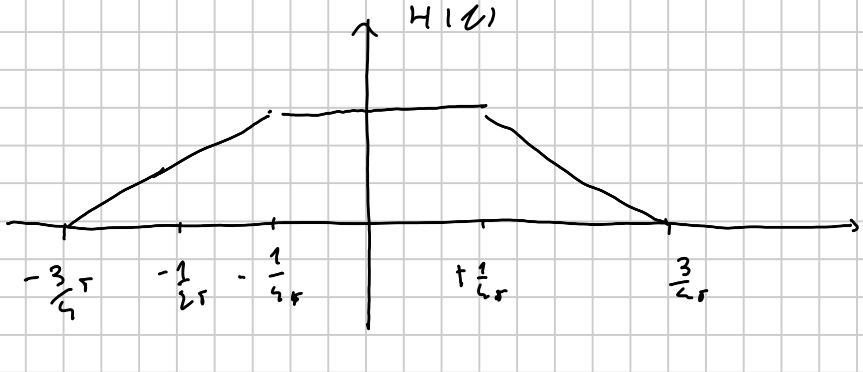
$x_2(t) = \mathrm{sinc}\left(\frac{t-T/2}{3T}\right) + \mathrm{sinc}\left(\frac{t+T/2}{3T}\right)$

$x_3(t) = \mathrm{sinc}^2\left(\frac{t}{T}\right)$

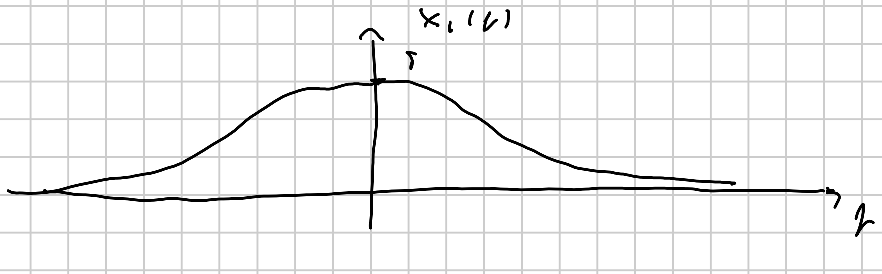
$$H(f) = \underbrace{T\,\mathrm{rect}\left(\frac{f}{1/T}\right)}_{H_1(f)} \otimes \underbrace{2T\,\mathrm{rect}\left(\frac{f}{1/2T}\right)}_{H_2(f)} =$$

$$= H_1(f) \otimes H_2(f)$$





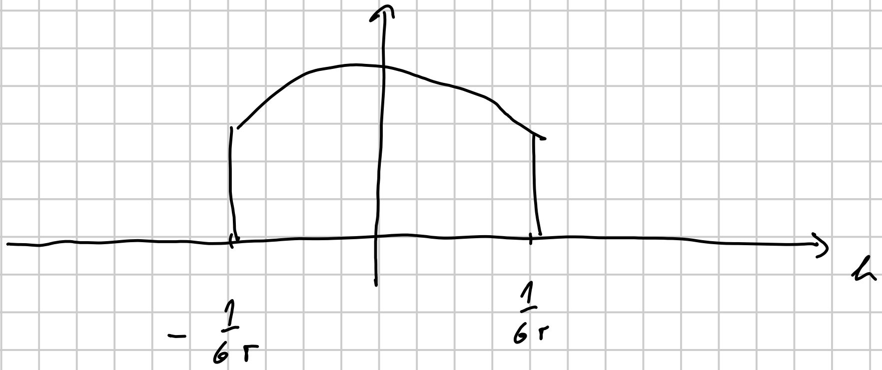
$$X_1(f) = \frac{T}{1+j2\pi f T} \Rightarrow |X_1(f)| = \frac{T}{1+4\pi^2 f^2 T^2}$$



y_1 distorta in ampiezza

$$X_2(f) = 3T \,\mathrm{rect}\left(\frac{f}{1/3T}\right) e^{-j2\pi f T/2} + 3T \,\mathrm{rect}\left(\frac{f}{1/3T}\right) e^{j2\pi f T/2} =$$

$$= 6T \,\mathrm{rect}\left(\frac{f}{1/3T}\right) \cos(\pi f T)$$



NO DISTORSIONE

$$E_{y_2} = \int_{-\infty}^{+\infty} \left| 6T^2 \,\mathrm{rect}\left(\frac{f}{1/3T}\right) \cos(\pi f T)^2 \right|^2 df =$$

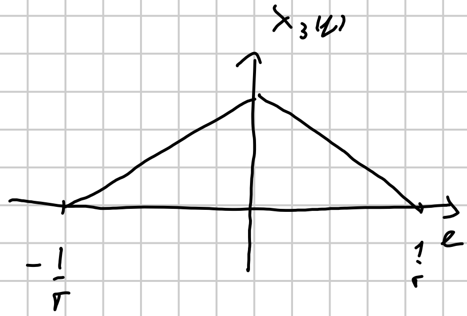
$$= 6T^2 \int_{-\frac{1}{6T}}^{\frac{1}{6T}} \cos^2(\pi f T)\, df = 6T^2 \int_{-\frac{1}{6T}}^{\frac{1}{6T}} \left[\frac{1}{2} + \frac{1}{2}\cos(2\pi f T)\right] df =$$

$$= 6T^2 \left[\frac{1}{6T} + \frac{1}{2} \sin(2\pi f t) \Big|_{-\frac{1}{6T}}^{\frac{1}{6T}} \right] =$$

$$= 6T^2 \left[\frac{1}{6T} + \frac{\sqrt{3}}{2} \right]$$

$$P_{y_2} = 0$$

$$X_3(f) = T\left(1 - \frac{|f|}{1/T}\right) \mathrm{rect}\left(\frac{f}{2/T}\right)$$



$|Y_3(f)| = |X_3(f)|\,|H(f)|$ è distorto in un mezzo

ESERCIZIO 1

$$x[n] = a^{|n|} \qquad 0 \leq a < 1$$

$$x[n] = \underbrace{a^{n} u[n]}_{y[n]} + a^{-n} u[-n] - x[0]\,\delta[n] =$$

$$= y[n] + y[-n] - x[0]\,\delta[n]$$

$$Y(f) = \sum_{n=0}^{\infty} a^{n} e^{-j2\pi f n T} = \sum_{n=0}^{+\infty} \left(a e^{-j2\pi f T}\right)^{n} =$$

$$= \frac{1}{1 - a e^{-j2\pi f T}}$$

$$\boxed{\sum_{n=0}^{\infty} q^{n} = \frac{1}{1-q} \qquad |q| < 1}$$

$$Y(-f) = Y^{*}(f)$$

$$X(f) = Y(f) + Y^{*}(f) - x[0] =$$

$$= 2\,\mathrm{Re}\{Y(f)\} - x[0] =$$

$$= \frac{1}{1 - a e^{-j2\pi f T}} + \frac{1}{1 - a e^{+j2\pi f T}} - 1 =$$

$$= \frac{1 - a e^{+j2\pi f T} + 1 - a e^{-j2\pi f T} - (1 - a e^{-j2\pi f T})(1 - a e^{+j2\pi f T})}{(1 - a e^{-j2\pi f T})(1 - a e^{+j2\pi f T})}$$

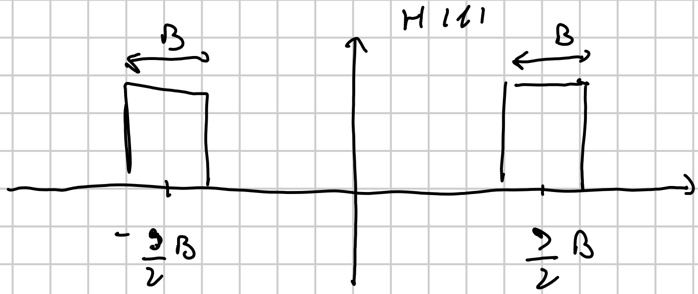
$$= \frac{1 - \cancel{a e^{+j2\pi f T}} + 1 - \cancel{a e^{-j2\pi f T}} - 1 - a^{2} + \cancel{2a \cos 2\pi f T}}{1 + a^{2} - 2a \cos(2\pi f T)} =$$

$$= \frac{1 - a^{2}}{1 - a^{2} - 2a \cos(2\pi f T)}$$

ESERCIZIO 2

$$x(t) \longrightarrow \boxed{(\cdot)^3} \xrightarrow{y(t)} \boxed{h(t)} \longrightarrow z(t)$$

$$x(t) = A \cos(3\pi B t + \theta)$$



$$y(t) = (x(t))^3 = A^3 \cos^3(3\pi B t + \theta) =$$

$$= A^3 \cos(3\pi B t + \theta) \cos^2(3\pi B t + \theta) =$$

$$= A^3 \cos(3\pi B t + \theta)\left(\frac{1}{2} + \frac{1}{2}\cos(6\pi B t + 2\theta)\right) =$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$= \frac{A^3}{2}\cos(3\pi B t + \theta) + \frac{A^3}{2}\cos(3\pi B t + \theta)\cos(6\pi B t + 2\theta) =$$

$$= \frac{A^3}{2}\cos(3\pi B t + \theta) + \frac{A^3}{2}\left(\frac{1}{2}\cos(9\pi B t + 3\theta) + \frac{1}{2}\cos(3\pi B t + \theta)\right) =$$

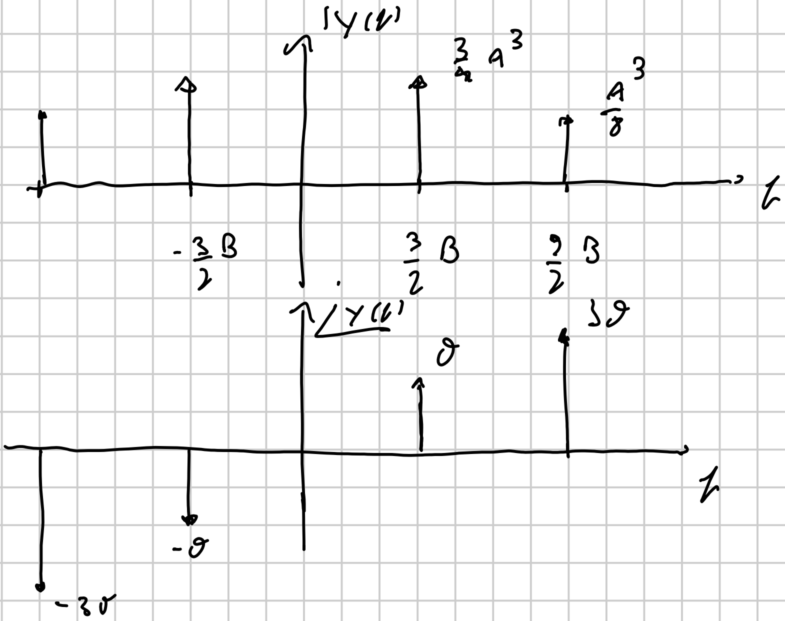
$$= \underbrace{\left(\frac{A^3}{2} + \frac{A^3}{4}\right)}_{\frac{3}{4}A^3}\cos(\underbrace{3\pi B t + \theta}_{2\pi \frac{3}{2} B t + \theta}) + \frac{A^3}{4}\cos(\underbrace{9\pi B t + 3\theta}_{2\pi \frac{9}{2} B t + 3\theta})$$

$$Y(f) = \frac{3}{4}A^3\left(\frac{\delta(f - f_{01})e^{j\theta} + \delta(f + f_{01})e^{-j\theta}}{2}\right) +$$

$$\frac{A^3}{4}\left(\frac{\delta(f - f_{02})e^{j3\theta} + \delta(f + f_{02})e^{-j3\theta}}{2}\right)$$

$f_{01} = \frac{3}{2} B$

$f_{02} = \frac{9}{2} B$



$$Z(f) = \frac{A^3}{8}\left[\delta\left(f - \frac{9}{2}B\right)e^{+j3\theta} + \delta\left(f + \frac{9}{2}B\right)e^{-j3\theta}\right]$$

$$z(t) = \frac{A^3}{4}\cos\left(2\pi \frac{9}{2}B t + 3\theta\right)$$

ESERCIZIO 3

$$y(t) = |x(t)| + \int_a^t x(\alpha)\,d\alpha$$

1) LINEARE?

2) STAZIONARIO?

3) CON MEMORIA?

4) STABILE BIBO?

5) CAUSALE?

1) $x(t) = a\,x_1(t) + b\,x_2(t)$

$$T[x(t)] = |a\,x_1(t) + b\,x_2(t)| + \int_a^t (a\,x_1(\alpha) + b\,x_2(\alpha))\,d\alpha \neq$$

$$a|x_1(t)| + b|x_2(t)| + a\int_a^t x_1(\alpha)\,d\alpha + b\int_a^t x_2(\alpha)\,d\alpha$$

NO LINEARE

2) $x(t-t_0) \Rightarrow y(t-t_0)$

$$T[x(t-t_0)] = |x(t-t_0)| + \int_a^t x(\alpha - t_0)\, d\alpha =$$

$$= |x(t-t_0)| + \int_{a-t_0}^{t-t_0} x(\alpha')\, d\alpha'$$

$$\neq$$

$$y(t-t_0) = |x(t-t_0)| + \int_a^{t-t_0} x(\alpha)\, d\alpha$$

NO STAZIONARIO

3) CON MEMORIA

4) CAUSALE

5) $|x(t)| < M \quad \forall t$

$$y(t) = |x(t)| + \int_a^t x(\alpha)\, d\alpha$$

$$|y(t)| = \left| |x(t)| + \int_a^t x(\alpha)\, d\alpha \right| \leq$$

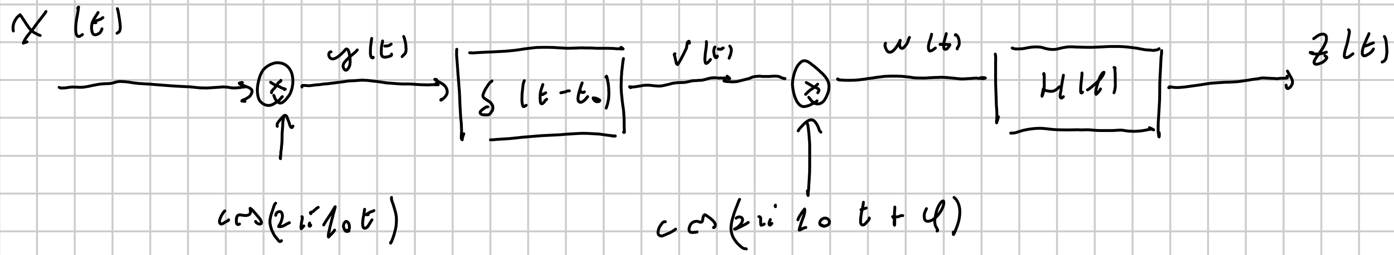
$$\leq |x(t)| + \left| \int_a^t x(\alpha)\, d\alpha \right| \leq$$

$$\leq M + \int_a^t |x(\alpha)|\, d\alpha = M + M(t-a) \not< H$$

$t \to \infty$

NON E' STABILE

ESERCIZIO 4 — SISTEMA WIRELESS



$$x(f) = \left[1 + \cos\left(\pi \frac{f}{B}\right)\right] \mathrm{rect}\left(\frac{f}{2B}\right)$$

$H(f)$ filtro LP di Banda B

1) $Y(f)$
2) $W(f)$
3) $Z(f)$
4) φ : $z(t)$ E_z max

$$y(t) = x(t) \cos(2\pi f_0 t)$$

$\Updownarrow$

$$Y(f) = \frac{X(f - f_0) + X(f + f_0)}{2} =$$

$$= \frac{1}{2}\left(1 + \cos\left(\pi \frac{f - f_0}{B}\right) \mathrm{rect}\left(\frac{f - f_0}{2B}\right)\right) + \frac{1}{2}\left(1 + \cos\left(\pi \frac{f + f_0}{B}\right) \mathrm{rect}\left(\frac{f + f_0}{2B}\right)\right)$$

$$v(t) = y(t) \otimes \delta(t - t_0)$$

$\Updownarrow$

$$V(f) = Y(f)\, e^{-j 2\pi f t_0}$$

$$w(t) = v(t) \cos(2\pi f_0 t + \varphi)$$

$$W(f) = V(f) \otimes \left(\frac{1}{2} \delta(f-f_0) e^{j\varphi} + \frac{1}{2} \delta(f+f_0) e^{-j\varphi} \right) =$$

$$= \frac{1}{2} V(f-f_0) e^{j\varphi} + \frac{1}{2} V(f+f_0) e^{-j\varphi}$$

$$W(f) = \frac{1}{2} Y(f-f_0) e^{-j2\pi(f-f_0)t_0} e^{j\varphi} +$$

$$+ \frac{1}{2} Y(f+f_0) e^{-j2\pi(f+f_0)t_0} e^{-j\varphi} =$$

$$= \frac{1}{2} e^{-j2\pi f t_0} \Bigg\{ \left[\left(\frac{1}{2} + \frac{1}{2} \cos\left(\pi \frac{(f-2f_0)}{B} \right) \right) \mathrm{rect}\left(\frac{f-2f_0}{B} \right) + \right.$$

$$\left. + \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi f}{B}\right) \right) \mathrm{rect}\left(\frac{f}{2B} \right) \right] e^{j2\pi f_0 t_0 + j\varphi} +$$

$$+ \left[\left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{\pi f}{B}\right) \right) \mathrm{rect}\left(\frac{f}{2B}\right) + \left(\frac{1}{2} + \frac{1}{2} \cos\left(\frac{f+2f_0}{2B} \right) \right) \mathrm{rect}\left(\frac{f+2f_0}{2B} \right) \right] \cdot$$

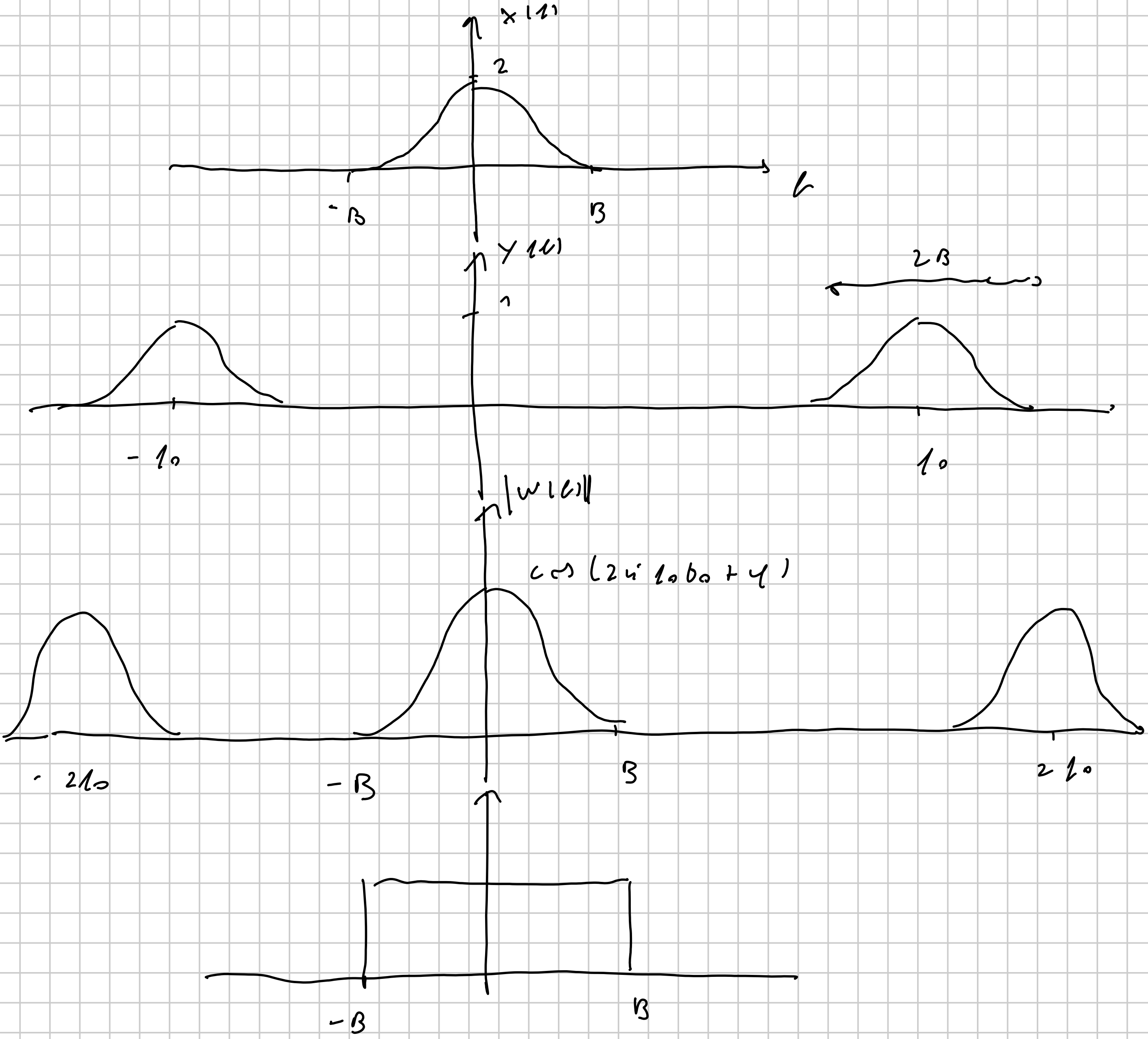
$$\cdot e^{-j2\pi f_0 t_0 - j\varphi} \Bigg\}$$

DEFINISCO

$$W_{2f_0} = \frac{1}{4} e^{-j2\pi f t_0} e^{j2\pi f_0 t_0 + j\varphi} \left(1 + \cos\left(\pi \frac{(f-2f_0)}{B} \right) \right) \mathrm{rect}\left(\frac{f-2f_0}{2B} \right)$$

$$W_{-2f_0} = \frac{1}{4} e^{-j2\pi f t_0} e^{-j2\pi f_0 t_0 - j\varphi} \left(1 + \cos\left(\pi \frac{(f+2f_0)}{B} \right) \right) \mathrm{rect}\left(\frac{f+2f_0}{2B} \right)$$

$$W(f) = W_{2f_0} + W_{-2f_0} + \frac{1}{2} e^{-j2\pi f t_0} \left(1 + \cos\left(\frac{\pi f}{B}\right) \right) \cos(2\pi f_0 t_0 + \varphi)\, \mathrm{rect}\left(\frac{f}{2B}\right)$$



$$z(f) = \frac{1}{2} e^{-j2\pi f t_0} \cos(2\pi f_0 t_0 + \varphi) \underbrace{\left(1+\cos\left(\frac{\pi f}{B}\right)\right) \mathrm{rect}\left(\frac{f}{2B}\right)}_{X(f)}$$

$$z(t) = \frac{1}{2} \cos(2\pi f_0 t + \varphi)\, x(t - t_0)$$

$$X(f) = \mathrm{rect}\left(\frac{f}{2B}\right) + \mathrm{rect}\left(\frac{f}{2B}\right) \cos\left(\frac{\pi f}{B}\right)$$

$$\Updownarrow$$

$$x(t) = 2B\, \mathrm{sinc}(t\, 2B) + B\, \mathrm{sinc}(t\, 2B) \otimes \left[\delta\left(t - \frac{1}{2B}\right) + \delta\left(t + \frac{1}{2B}\right)\right]$$

$$E_z = \int_{-\infty}^{+\infty} |z(t)|^2 \, dt = \int_{-\infty}^{+\infty} |Z(f)|^2 \, df$$

$$z(t) = \frac{1}{2} \cos(2\pi f_0 t_0 + \varphi) \, x(t - t_0)$$

$$E_z = \frac{1}{4} \cos^2 (2\pi f_0 t_0 + \varphi) \; E_x$$

$$\cos^2 (2\pi f_0 t_0 + \varphi) = 1 \implies 2\pi f_0 t_0 + \varphi = k\pi$$

$$\boxed{\varphi = -2\pi f_0 t_0 + k\pi}$$

$$E_{z\,max} = \frac{1}{4} E_x$$

ESERCIZIO 5

$$x(t) \xrightarrow{\;nT\;} x[n] \longrightarrow \boxed{p(t)} \xrightarrow{\;z(t)\;} \boxed{h(t)} \longrightarrow y(t)\ ?$$

$$x(t) = \text{sinc}\left(\frac{t}{T}\right)$$

$$p(t) = \text{rect}\left(\frac{t - T/2}{T}\right)$$

$$H(f) = \frac{\pi f T}{\sin(\pi f T)} \, \text{rect}(fT)$$

$$x[n] = \text{sinc}\left(n\frac{T}{T}\right) = \text{sinc}(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$z(t) = \sum_n x[n] \, p(t - nT) = p(t)$$

$$y(t) = z(t) \otimes h(t)$$

$$\Updownarrow$$

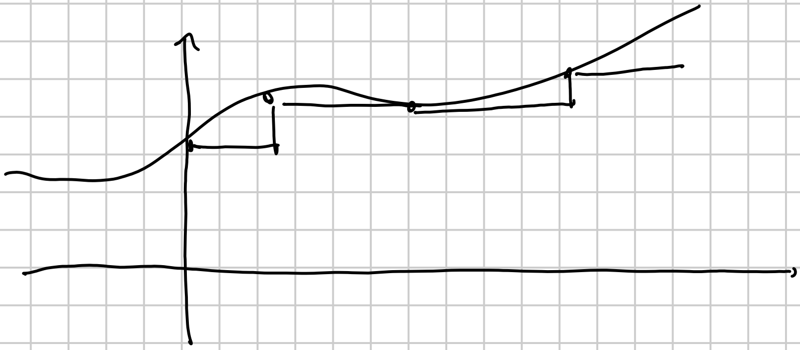
$$Y(f) = Z(f)\, H(f)$$

$$P(f) = T \,\mathrm{sinc}(fT)\, e^{-j2\pi f \frac{T}{2}} = T\,\mathrm{sinc}(fT)\, e^{-j\pi f T}$$

$$Y(f) = T\,\mathrm{sinc}(fT)\, e^{-j\pi f T} \underbrace{\frac{\pi f T}{\sin(\pi f T)}} \mathrm{rect}(fT) =$$

$$= T \frac{\cancel{\sin(\pi f T)}}{\cancel{\pi f T}} e^{-j\pi f T} \frac{\cancel{\pi f T}}{\cancel{\sin(\pi f T)}} \mathrm{rect}(fT)$$

$$y(t) = \mathrm{sinc}\left(\frac{t - T/2}{T}\right)$$



ESERCIZIO 1   15/02/10

$$z(t) = \underbrace{\mathrm{rect}\left(\frac{t - T/2}{T}\right) e^{-t/\tau}}_{z_1(t)} + \overbrace{\mathrm{sinc}\left(\frac{t}{T}\right)\cos(2\pi f_0 t + \theta)}^{z_2(t)}$$

$$\boxed{\begin{array}{l} T \gg \dfrac{1}{f_0} \\[2ex] \tau \gg T \end{array}} \leftarrow$$

$$z(f) = z_1(f) + z_2(f)$$

$$z_1(f) = \int_0^T e^{-\frac{t}{2}} e^{-j2\pi f t}\, dt = \left. \frac{e^{-\frac{t}{2} - j2\pi f t}}{-\frac{1}{2} - j2\pi f} \right|_0^T =$$

$$= \frac{e^{-\frac{T}{2} - j2\pi f T} - 1}{-\frac{1}{2} - j2\pi f}$$

$$z_2(f) = \frac{1}{T} \operatorname{rect}\left(\frac{f}{1/T}\right) \otimes \left[\delta(f - f_0) \frac{e^{j\vartheta}}{2} + \delta(f + f_0) \frac{e^{-j\vartheta}}{2} \right] =$$

$$= \frac{1}{2T} \operatorname{rect}\left(\frac{f - f_0}{1/T}\right) e^{j\vartheta} + \frac{1}{2T} \operatorname{rect}\left(\frac{f + f_0}{1/T}\right) e^{-j\vartheta}$$

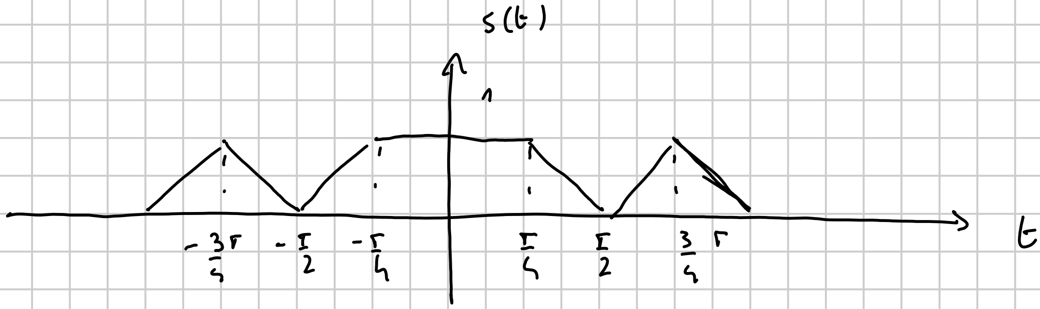
$$E_z = E_{z_1} + E_{z_2}$$

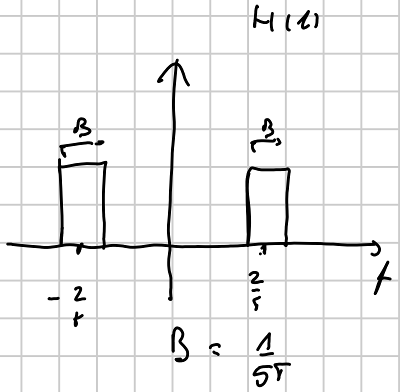
$$E_{z_2} = 2 \frac{1}{4T^2} T = \frac{1}{2T^3}$$

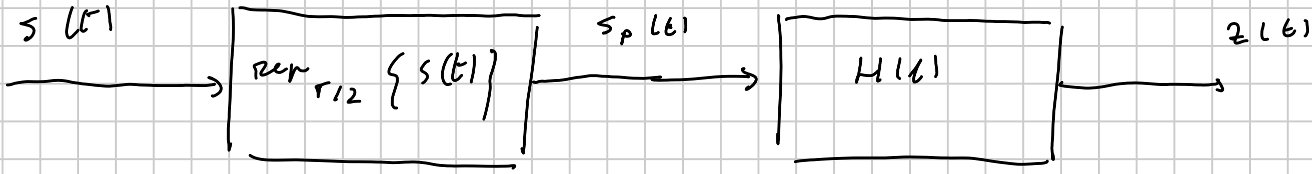
$$E_{z_1} = \int_0^T e^{-2t/2}\, dt = -\left. \frac{2}{2} e^{-2t/2} \right|_0^T =$$

$$= -\frac{2}{2} \left(e^{-2T/2} - 1 \right)$$

ESERCIZIO 20/04/2004







$$s_p(t) = \sum_n s\left(t - n\frac{T}{2}\right)$$

$$s_1(t) = 2\left(1 - \frac{|t|}{T/2}\right) rect\left(\frac{t}{T}\right) \rightleftarrows S_1(f) = \cancel{2}\frac{T}{\cancel{2}} sinc^2\left(f\frac{T}{2}\right)$$

$$s_2(t) = \left(1 - \frac{|t|}{T/4}\right) rect\left(\frac{t}{T/2}\right) \rightleftarrows S_2(f) = \frac{T}{4} sinc^2\left(f\frac{T}{4}\right)$$

$$s(t) = s_1(t) - s_2(t) + s_2\left(t - \frac{3}{4}T\right) + s_2\left(t + \frac{3}{4}T\right)$$

$$S(f) = S_1(f) - S_2(f)\left[1 - e^{-j2\pi f\frac{3}{4}T} - e^{+j2\pi f\frac{3}{4}T}\right] =$$

$$= S_1(f) - S_2(f)\left[1 - 2\cos 2\pi f\frac{3}{4}T\right]$$

$$T_0 = \frac{T}{2}$$

$$S_p(f) = \frac{1}{T_0}\sum_k S\left(\frac{k}{T_0}\right)\delta\left(f - \frac{k}{T_0}\right) = \frac{2}{T}\sum_k S\left(\frac{2k}{T}\right)\delta\left(f - \frac{2k}{T}\right)$$

$$k = \pm 1$$

$$S\left(\frac{2}{T}\right) = S\left(-\frac{2}{T}\right) = \cancel{T\, sinc^2\left(\frac{2}{T}\frac{T}{2}\right)} - \frac{T}{4} sinc^2\left(\frac{2}{T}\frac{T}{4}\right)\cdot$$

$$\cdot\left(1 - 2\cos 3\pi\right) =$$

$$= -\frac{T}{4}\cdot\frac{4}{\pi^2}(1+2) = -3\frac{T}{\pi^2}$$

$$\frac{2}{T}\, S\left(\frac{2}{T}\right) = \frac{2}{T}\, S\left(-\frac{2}{T}\right) = -\frac{6}{\pi^2}$$

$$Z(f) = -\frac{6}{\pi^2}\left[\delta\left(f-\frac{2}{T}\right)+\delta\left(f+\frac{2}{T}\right)\right]$$

$$\Updownarrow$$

$$z(t) = -\frac{12}{\pi^2}\cos\left(2\pi\frac{2}{T}t\right)$$

ESERCIZIO 2

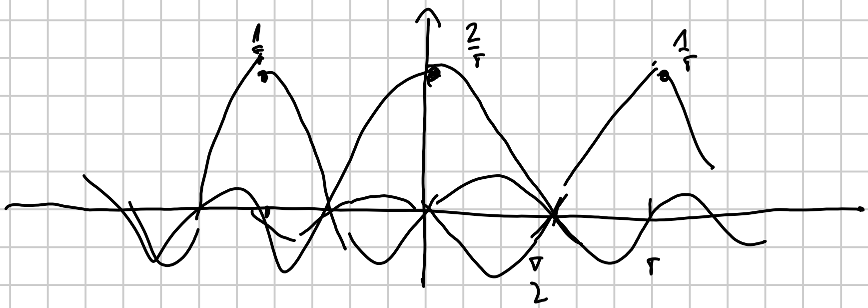
$$X(f) = \left[1+\cos(2\pi f T)\right]\mathrm{rect}\left(f\frac{T}{2}\right)$$

$$F_c = \frac{2}{T}$$

$$x(t) = \left\{\delta(t)+\frac{1}{2}\left[\delta(t-T)+\delta(t+T)\right]\right\}\otimes\frac{2}{T}\,\mathrm{sinc}\left(\frac{2t}{T}\right)$$

$$= \frac{2}{T}\,\mathrm{sinc}\left(\frac{2t}{T}\right) + \frac{1}{T}\left(\mathrm{sinc}\left(\frac{2(t-T)}{T}\right)+\mathrm{sinc}\left(\frac{2(t+T)}{T}\right)\right)$$

$$x[n] = x(nT_c) = x\left(n\frac{T}{2}\right) = \begin{cases}\frac{2}{T} & n=0\\ \frac{1}{T} & n=\pm 2\end{cases}$$



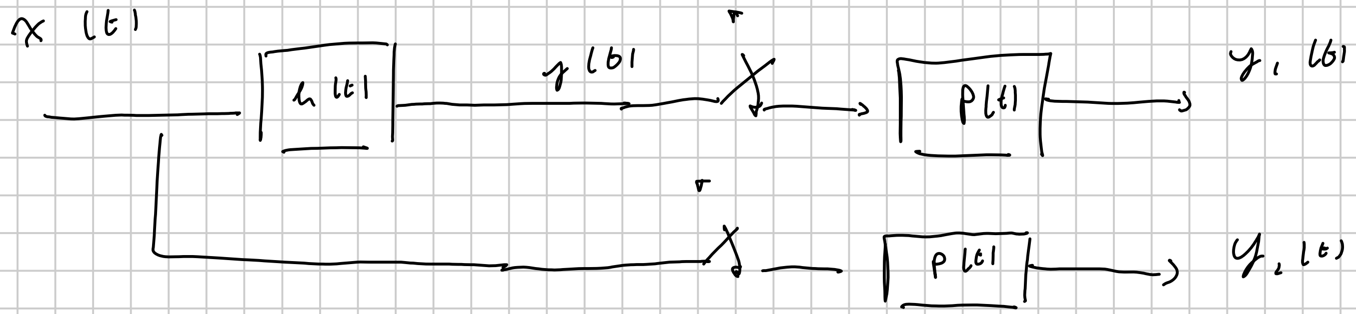
$$\bar{X}(f) = \frac{T}{2} \sum_n x\left(n\frac{T}{2}\right) e^{-j2\pi n f \frac{T}{2}} =$$

$$= \frac{T}{2}\left\{ \frac{2}{T} + \frac{1}{T}\left[e^{-j2\pi f 2\frac{T}{2}} + e^{+j2\pi f 2 \frac{T}{2}} \right]\right\} =$$

$$= 1 + \cos(2\pi f T)$$

$$X(f) = \bar{X}(f)\,\mathrm{rect}\left(f\frac{T}{2}\right)$$

ESERCIZIO 2 del 19/04/10



$x(t) = 2AB\,\mathrm{sinc}^2(2Bt)$

$h(t) = 2B\,\mathrm{sinc}(2Bt)$

$p(t)$ INTERP. CARD. DI BANDA B

$T = \frac{1}{2B}$

$y_1(t)$

$y_2(t)$

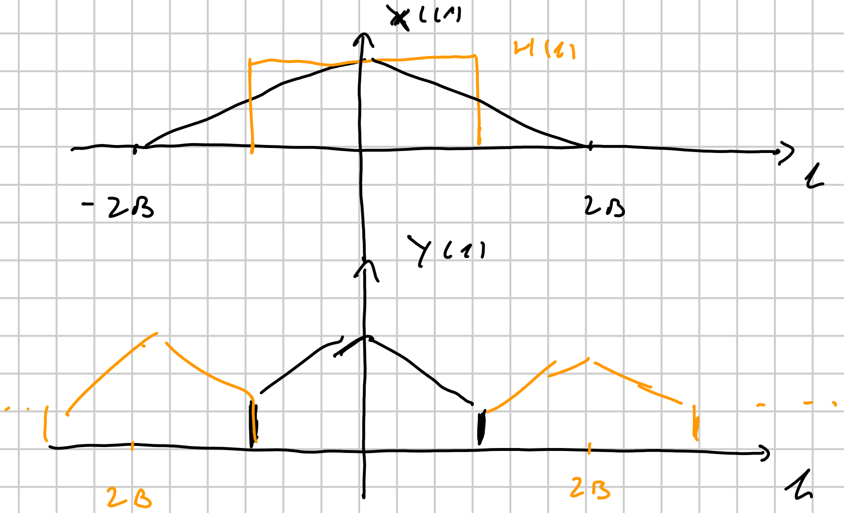
E_{y_1}, P_{y_1}

E_{y_2}, P_{y_2}

$$X(f) = A\left(1 - \frac{|f|}{2B}\right)\mathrm{rect}\left(\frac{f}{4B}\right)$$

$$H(f) = \mathrm{rect}\left(\frac{f}{2B}\right)$$

$$Y(f) = X(f)\,H(f)$$



$$\bar{Y}(f) = \frac{1}{T}\sum_n Y\left(f - \frac{n}{T}\right)$$

$$= 2B \sum_n Y(f - n\,2B)$$

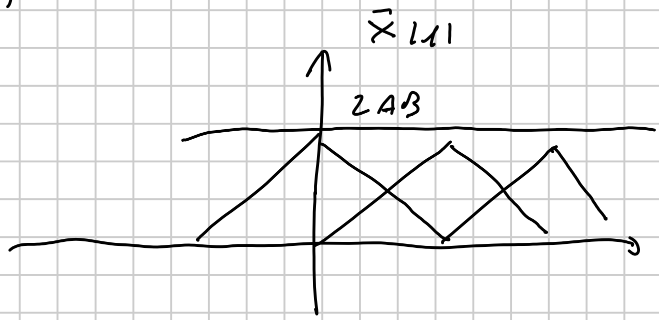
$$Y_1(f) = 2AB\left(1 - \frac{|f|}{2B}\right)\mathrm{rect}\left(\frac{f}{4B}\right)\underbrace{\mathrm{rect}\left(\frac{f}{2B}\right)}_{P(f)} =$$

$$= 2AB\left(1 - \frac{|f|}{2B}\right)\mathrm{rect}\left(\frac{f}{2B}\right) =$$

$$= AB\,\mathrm{rect}\left(\frac{f}{2B}\right) + AB\left(1 - \frac{|f|}{B}\right)\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$y_1(t) = 2AB^2\,\mathrm{sinc}(t\,2B) + AB^2\,\mathrm{sinc}^2(t\,B)$$

$$\bar{X}(f) = 2B\sum_n X(f - 2nB)$$



$$Y_2(f) = \bar{X}(f)\,P(f) = 2AB\,\mathrm{rect}\left(\frac{f}{2B}\right)$$

$$y_2(t) = 4AB^2\,\mathrm{sinc}(2Bt)$$

$$E_{y_1} = \int_{-\infty}^{+\infty} |Y_1(f)|^2 \, df = \int_{-B}^{B} (AB)^2 \, df + 2\int_{0}^{B} (AB^2)\left(1 - \frac{f}{B}\right)^2 df +$$

$$+ \int_{-B}^{B} 2(AB^2)\left(1 - \frac{|f|}{B}\right) df =$$

$$\int_{-B}^{B} (AB)^2 \, df = 2B\, A^2 B^2$$

$$2\int_{0}^{B} A^2 B^2 \left(1 - \frac{f}{B}\right)^2 df = 2A^2B^2 \int_{0}^{B} \left(1 + \frac{f^2}{B^2} - \frac{2f}{B}\right) df =$$

$$= 2A^2B^2 \left(f + \frac{f^3}{3B^2} - \frac{f^2}{B}\right)\Bigg|_{0}^{B} = 2A^2B^2 \left[\cancel{B} + \frac{B}{3} - \cancel{B}\right] =$$

$$= \frac{2}{3} A^2 B^3$$

$$4\int_{0}^{B} A^2B^2 \left(1 - \frac{f}{B}\right) df = 4A^2B^2 \left[f - \frac{f^2}{2B}\right]\Bigg|_{0}^{B} = 4A^2B^2\left(B - \frac{B}{2}\right) =$$

$$= 2A^2B^3$$

$$E_{y_1} = \frac{14}{3} A^2 B^3$$

$$E_{y_2} = \int_{-\infty}^{+\infty} |Y_2(f)|^2 = \int_{-B}^{B} 4A^2B^2 \, df = 4A^2B^2\, 2B = 8A^2B^3$$

$$P_{y_1} = P_{y_2} = 0$$

Esercitazione 28 - 03 - 2012

Esercizio #1

Una stanza illuminata da due lampadine in serie. La probabilità che una lampadina sia guasta ad un certo istante è p. Si possono guastare in maniera indipendente l'una dall'altra.

Calcolare la probabilità che la stanza sia buia.

$$\Omega = \{ (F_1, G_2) ; (G_1, F_2) ; (G_1, G_2) ; (F_1, F_2) \}$$

$$A = \{ (F_1, G_2) ; (G_1, F_2) ; (G_1, G_2) \}$$

$$Pr\{A\} = ?$$

1) $Pr\{A\} = Pr\{(F_1, G_2)\} + Pr\{(G_1, F_2)\} + Pr\{(G_1, G_2)\}$

2) $Pr\{A\} = 1 - Pr\{(F_1, F_2)\}$

3) $\alpha_1 = \{ (G_1, F_2) ; (G_1, G_2) \}$ $\qquad Pr\{\alpha_1\} = p$

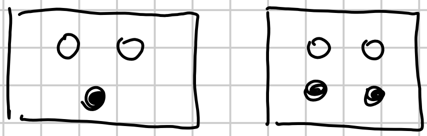
$\alpha_2 = \{ (F_1, G_2) ; (G_1, G_2) \}$ $\qquad Pr\{\alpha_2\} = p$

$$Pr\{A\} = Pr\{\alpha_1 \cup \alpha_2\} = Pr\{\alpha_1\} + Pr\{\alpha_2\} - Pr\{\alpha_1 \cap \alpha_2\} =$$

$= p + p - p^2 = 2p - p^2$

Esercizio #2

Ci sono due urne, una contiene 1 pallina nera e 2 bianche e l'altra 2 bianche e 2 nere



L'esperimento consiste nello scegliere un'urna e poi nello scegliere una pallina.

Qual è la probabilità di estrarre una pallina bianca.

$$\Omega = \{ (U_1, b_{11})\ (U_1, b_{12})\ (U_2, b_{21})\ (U_2, b_{22})\ (U_1, n_{11})\ (U_2, n_{12})\ (U_2, n_{22}) \}$$

$$B = \{ (U_1, b_{11})\ (U_1, b_{12})\ (U_2, b_{21})\ (U_2, b_{22}) \}$$

$$P_r\{B\} = \sum_i P_r\{B | U_i\} \cdot P_r\{U_i\}$$

$U_1 \cup U_2 = \Omega$

$U_1 \cap U_2 = \emptyset$

$$Pr\{B \mid U_1\} \cdot Pr\{U_1\} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

$$Pr\{B \mid U_2\} \cdot Pr\{U_2\} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$Pr\{B\} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

Calcolare la probabilità di aver scelto l'urna 1 sapendo che ho estratto una pallina bianca.

$$Pr\{U_1 \mid B\} = \frac{Pr\{B \mid U_1\} \cdot Pr\{U_1\}}{Pr\{B\}}$$

$$Pr\{B\} = \frac{7}{12}$$

$$Pr\{U_1\} = \frac{1}{2}$$

$$Pr\{B \mid U_1\} = \frac{2}{3}$$

$$Pr\{U_1 \mid B\} = \frac{\cancel{2}}{\cancel{3}} \cdot \frac{1}{\cancel{2}} \cdot \frac{\cancel{12}^{4}}{7} = \frac{4}{7}$$

Esercizio #3

Un trasmettitore trasmette un segnale binario

$$Pr\{T_b = 1\} = 0{,}3$$

$$Pr\{T_b = 0\} = 0{,}7$$

$$Pr\{E\} = 0{,}01$$ probabilità che il bit sia ricevuto erroneamente

Il ricevitore riceve uno 0 logico.
Calcolare la probabilità che sia stato trasmesso uno 0

$$Pr\{T_0 \mid R_0\} = ?$$

Bayes

$$Pr\{T_0 \mid R_0\} = \frac{Pr\{R_0 \mid T_0\}\, Pr\{T_0\}}{Pr\{R_0\}}$$

$$Pr\{R_0 \mid T_0\} = 1 - Pr\{R_0 \mid T_1\} = 1 - 0{,}01 = 0{,}99$$

$$Pr\{T_0\} = 0{,}7$$

$$Pr\{R_0\} = Pr\{R_0 \mid T_0\} \cdot Pr\{T_0\} + Pr\{R_0 \mid T_1\} \cdot Pr\{T_1\} =$$

$$T_1 \cup T_0 = \Omega \qquad T_1 \cap T_0 = \emptyset$$

$$= 0{,}99 \cdot 0{,}7 + 0{,}01 \cdot 0{,}3 = 0{,}696$$

$$\Pr\{T_0 \mid R_0\} = \frac{0{,}99 \cdot 0{,}7}{0{,}696} \cong 0{,}9956$$

Esercizio #4

Lo studente X è sottoposto ad un quiz con M possibili risposte.

Se X ha studiato risponderà correttamente, altrimenti risponderà a caso.

X ha studiato con probabilità p, supponiamo che sottoposto al quiz risponda correttamente.

Qual è la probabilità che abbia studiato davvero?

$$S = \{X \text{ ha studiato}\}$$

$$\Pr\{S\} = p$$

$$C = \{X \text{ ha risposto correttamente}\}$$

$$\Pr\{C \mid S\} = 1$$

$$\bar{S} = \{X \text{ non ha studiato}\}$$

$$Pr\{ C \mid \bar{S} \} = \frac{1}{M}$$

$$S \cap \bar{S} = \emptyset$$

$$S \cup \bar{S} = \Omega$$

$$Pr\{ S \mid C \} = ?$$

Bayes

$$Pr\{ S \mid C \} = \frac{Pr\{C \mid S\} \cdot Pr\{S\}}{Pr\{\varepsilon\}}$$

$$Pr\{ C \} = Pr\{ C \mid S \} \cdot Pr\{ S \} + Pr\{ C \mid \bar{S} \} \cdot Pr\{ \bar{S} \} =$$

$$= 1 \cdot p + \frac{1}{M} \cdot (1-p)$$

$$Pr\{ S \mid C \} = \frac{p}{p + \frac{1}{M}(1-p)}$$

Esercizio # 5

Un libro A che ha 200 pagine e un libro B che ha 300 pagine.

Questi due libri sono aperti indipendentemente da due lettori.

Calcolare la probabilità dell'evento

$$\alpha = \{ \text{pag } A > \text{pag } B \} = \{ P_A > P_B \}$$

- I risultati dell'esperimento sono coppie di pagine (P_A, P_B)

- le coppie favorevoli sono quelle in cui $P_A > P_B$

-
$$Pr\{ P_A > P_B \} = \sum_{n=1}^{300} Pr\{ P_A > P_B \mid P_B = n \} \cdot Pr\{ P_B = n \}$$

$$Pr\{ P_B = n \} = \frac{1}{300}$$

$n \le 200$ $\qquad Pr\{ P_A > P_B \mid P_B = n \} = \dfrac{200 - n}{200}$

$$Pr\{ P_A > P_B \} = \sum_{n=1}^{200} \frac{1}{300} \cdot \frac{200-n}{200} = \frac{1}{300 \cdot 200} \sum_{n=1}^{200} (200 - n) =$$

$$= \frac{1}{300 \cdot 200} \sum_{n=0}^{199} n = \frac{1}{300 \cdot 200} \left(\frac{199 \cdot 198}{2} \right)$$

Esercizio # 6

È data una V.A. $X \in \mathcal{N}(0,1)$

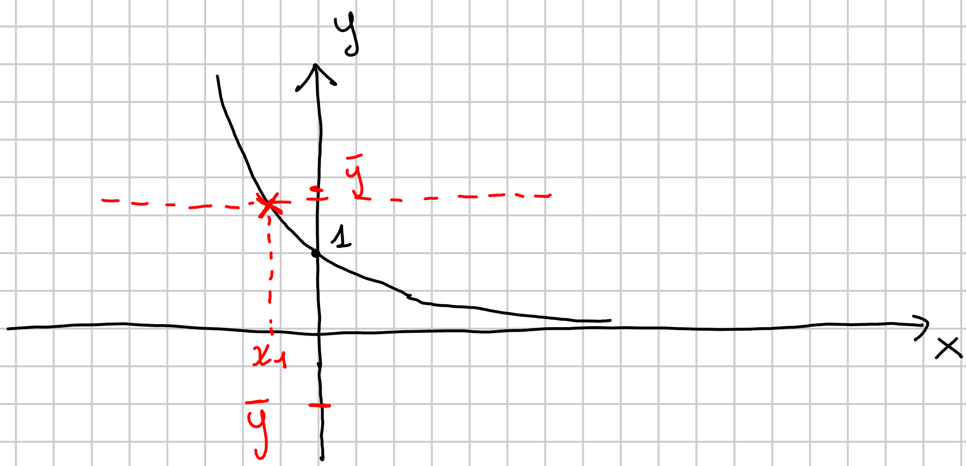
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Determinare $f_Y(y)$ e η_Y della V.A.

$$Y = e^{-X} \qquad \text{V.A. LOG NORMALE}$$

$$f_Y(y) = \sum_{i=1}^{m} \frac{f_X(x_i)}{|g'(x_i)|}$$

$$y = g(x_i)$$



Se $\bar{y} < 0$ non ci sono soluzioni $\Rightarrow f_Y(y) = 0 \quad \bar{y} < 0$

Se $\bar{y} > 0$ $\exists$ 1 sola soluzione $x_1 = -\ln(\bar{y})$

$$f_X(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln^2 y}{2}}$$

$$g'(x_1) = -e^{-x_1} = -y$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y^2}} e^{-\frac{\ln^2 y}{2}} u(y)$$

$$\eta_y = E\{Y\} = \int_{-\infty}^{+\infty} y \cdot f_Y(y)\, dy$$

$$\eta_y = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx$$

$$\eta_y = \int_{-\infty}^{+\infty} e^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\frac{x^2}{2}+x\right)} dx =$$

$$\frac{x^2}{2} + x = \frac{1}{2}\left[(x+1)^2 - 1\right]$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{+\frac{1}{2}} \cdot e^{-\frac{1}{2}(x+1)^2} dx =$$

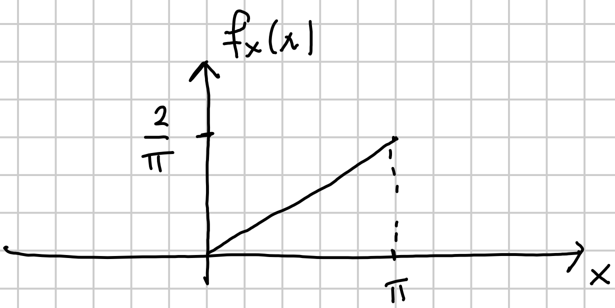
$$= e^{1/2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+1)^2}{2}} dx}_{\mathcal{N}(-1,\,1)} = e^{1/2}$$

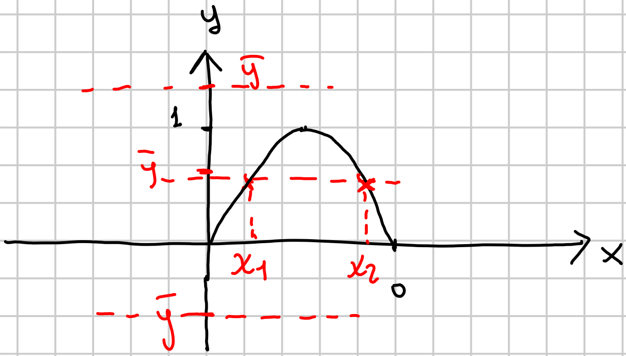
Esercizio #6

Sia X una v.A. tale che

$$f_X(x) = \frac{2x}{\pi^2} \operatorname{rect}\left(\frac{x-\pi/2}{\pi}\right)$$

Sia $Y = \sin(X)$ Trovare $f_Y(y)$ e η_Y





) se $\bar{y} > 1$ e $\bar{y} < 0$ non esistono soluzione dell'equazione

$y = g(x) \Rightarrow f_Y(y) = 0 \quad y > 1$ e $y < 0$

) se $0 \leq \bar{y} \leq 1$

$x_1 = \arcsin(y)$

$x_2 = \pi - x_1$

) $g(x) = \sin(x)$

$g'(x) = \cos(x)$

$g'(x_1) = \cos(x_1)$

$g'(x_2) = \cos(\pi - x_1) = -\cos(x_1)$

$$f_X(x_1) = \frac{2x_1}{\pi^2}$$

$$f_X(x_2) = \frac{2(\pi - x_1)}{\pi^2}$$

$$f_Y(y) = \frac{2x_1}{\pi^2} \cdot \frac{1}{\cos(x_1)} + \frac{2(\pi - x_1)}{\pi^2 \cdot \cos(x_1)} = \frac{2\cancel{\pi}}{\pi^2 \cdot \cos x_1} = \frac{2}{\pi \cos x_1} =$$

$$= \frac{2}{\pi \cos(\arcsin(y))}$$

$$\eta_y = \int_{-\infty}^{+\infty} y \cdot f_y(y)\, dy$$

$$\eta_y = \int_{-\infty}^{+\infty} g(x) \cdot f_x(x)\, dx$$

$$\eta_y = \int_0^1 y \cdot \frac{2}{\pi \cos(\operatorname{asin}(y))}\, dy$$

$z = \operatorname{asin}(y)$

$y = \sin z$

$dy = \cos z \cdot dz$

$y = 0 \Rightarrow z = 0$

$y = 1 \Rightarrow z = \pi/2$

$$= \int_0^{\pi/2} \sin z \frac{2}{\pi \cdot \cancel{\cos z}} \cdot \cancel{\cos z}\, dz =$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \sin z \quad dz = -\frac{2}{\pi} \int_0^{\pi/2} -\sin(z)\, dz = -\frac{2}{\pi} \cos(z) \Big|_0^{\pi/2}$$

$$= -\frac{2}{\pi} \Big(0 - (1) \Big) = \frac{2}{\pi}$$

Esercizio #7

E' data una V.A. $X \in \mathcal{N}(2,2)$

Si costruisce una variabile

$$Y = \begin{cases} 1 & \text{se } X > 0 \\ -1 & \text{se } X < 0 \end{cases}$$

Calcolare η_y e σ_y^2

$$\eta_y = \sum_{i=1}^{n} y_i \cdot P_i = 1 \cdot P_1 - 1 \cdot P_{-1}$$

$$P_1 = P_r\{Y = 1\}$$

$$P_{-1} = P_r\{Y = -1\}$$

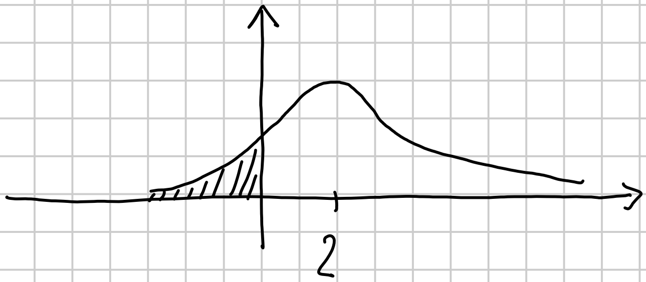
$$P_1 = \int_0^{+\infty} f_x(x)\, dy = P_r\{X > 0\}$$

$$P_{-1} = 1 - P_1$$

$$\eta_y = \int_{-\infty}^{+\infty} g(x) \cdot f_x(x)\, dx =$$

$y = g(x) = \operatorname{sgn}(x)$

$$\eta_y = -\int_{-\infty}^{0} f_x(x)\,dx + \int_{0}^{+\infty} f_x(x)\,dx$$



$$\int_{-\infty}^{0} f_x(x)\,dx = \Phi\left(\frac{0-2}{\sqrt{2}}\right) = \Phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\int_{0}^{+\infty} f_x(x)\,dx = 1 - \Phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\eta_y = -\Phi\left(-\frac{2}{\sqrt{2}}\right) + 1 - \Phi\left(-\frac{2}{\sqrt{2}}\right) = 1 - 2\,\Phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\sigma_y^2 = E\left\{(y-\eta_y)^2\right\} = \int_{-\infty}^{+\infty} (y-\eta_y)^2 f_y(y)\,dy =$$

$$= \sum_i (y - \eta_y)^2 \cdot p_i = (1 - \eta_1)^2 \cdot p_1 + (-1 - \eta_1)^2 \cdot p_{-1}$$

TRASFORMAZIONE DI UN VETTORE ALEATORIO

$\underline{X}$

$\underline{Y} = g(\underline{X})$ $\quad\quad\quad\quad$ $\underline{x}_i = g^{-1}(\underline{y})$

$f_{\underline{Y}}(\underline{y})$?

$$f_{\underline{Y}}(\underline{y}) = \sum_i \frac{f_{\underline{X}}(\underline{x}_i)}{|\det(J(\underline{x}_i))|}$$

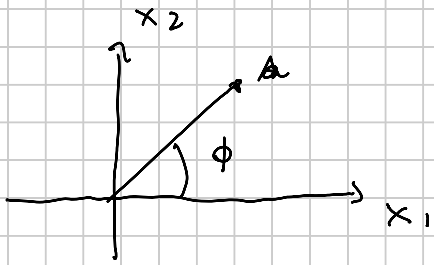
$$\underline{J}(\underline{x}_i) = \begin{bmatrix} \frac{\partial g_1(\underline{x})}{\partial x_1} & \frac{\partial g_1(\underline{x})}{\partial x_2} & \cdots & \frac{\partial g_1(\underline{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_n(\underline{x})}{\partial x_1} & \frac{\partial g_2(\underline{x})}{\partial x_2} & \cdots & \frac{\partial g_n(\underline{x})}{\partial x_n} \end{bmatrix}$$

TRASF. DA 2 V.A. a 2 V.A.

$$f_{Y_1,Y_2}(y_1, y_2) = \sum_i \frac{f_{X_1,X_2}(x_{1i}, x_{2i})}{\left| \begin{vmatrix} \frac{\partial g_1(\underline{x})}{\partial x_1} & \frac{\partial g_1(\underline{x})}{\partial x_2} \\ \frac{\partial g_2(\underline{x})}{\partial x_2} & \frac{\partial g_2(\underline{x})}{\partial x_2} \end{vmatrix} \right|_{\underline{x}=\underline{x}_i}}$$

$\left\{ x_{1,i} \quad x_{2,i} \right\}$ soluzioni: $\underline{x} = g^{-1}(\underline{y})$

ESEMPIO



X_1 , X_2 indipendenti

X_1 , x_2 $\in \mathcal{N}(0, \sigma^2)$

$$\begin{cases} A = \sqrt{x_1^2 + x_2^2} \\ \phi = tg^{-1}\left(\frac{x_2}{x_1}\right) \end{cases} \qquad \begin{cases} X_1 = A \cos\phi \\ X_2 = A \sin\phi \end{cases}$$

$$f_{A\phi}(a, \varphi) = \sum_i \frac{f_{x_1 x_2}(x_{1i}, x_{2i})}{\left| \det\left(J(x_{1i}, x_{2i})\right)\right|}$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial g_1(x)}{\partial x_1} & \frac{\partial g_1(x)}{\partial x_2} \\ \frac{\partial g_2(x)}{\partial x_1} & \frac{\partial g_2(x)}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial g_1(\underline{x})}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1^2 + x_2^2)^{\frac{1}{2}} = \frac{1}{2}(x_1^2 + x_2^2)^{-\frac{1}{2}} \, 2x_1 =$$

$$= \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_1(\underline{x})}{\partial x_2} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$$

$$\frac{\partial g_2(\underline{x})}{\partial x_1} = \frac{1}{1 + \frac{x_2^2}{x_1^2}} \cdot x_2 \, (-x_1)^{-2}$$

$$\frac{\partial g_2(\underline{x})}{\partial x_2} = \frac{1}{1 + \frac{x_2^2}{x_1^2}} \cdot \frac{1}{x_1} = \frac{x_1}{x_1^2 + x_2^2}$$

$$\frac{\partial g_1(x_{1,1})}{\partial x_1} = \frac{a \cos\varphi}{a} = \cos\varphi$$

$$\frac{\partial g_1(x_{1,1})}{\partial x_2} = \frac{a \sin\varphi}{a} = \sin\varphi$$

$$\frac{\partial g_2(x_{1,1})}{\partial x_1} = -\frac{a \sin\varphi}{a^2} = -\frac{\sin\varphi}{a}$$

$$\frac{\partial g_2(x_{1,1})}{\partial x_2} = \frac{\cos\varphi}{a}$$

$$\underline{\underline{J}} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\frac{\sin\varphi}{a} & \frac{\cos\varphi}{a} \end{bmatrix}$$

$$\left|\det(\underline{\underline{J}})\right| = \left|\frac{\cos^2\varphi}{a} + \frac{\sin^2\varphi}{a}\right| = \left|\frac{1}{a}\right| = \frac{1}{a}$$

$$f_{x_1 x_2}(x_1, x_2) = \frac{1}{\sqrt{(2\pi)^2 \sigma^2 \sigma^2}} e^{-\frac{x_1^2 + x_2^2}{2\sigma^2}}$$

$$f_{A\Phi} = \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)$$

$$f_A(a) = \int_{-\infty}^{+\infty} f_{A\Phi}(a,\varphi)\, d\varphi = \int_{-\pi}^{\pi} \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)\, d\varphi =$$

$$= \frac{a}{\sigma^2} e^{-\frac{a^2}{2\sigma^2}} u(a)$$

d.d.p. di RAYLEIGH

$$f_\phi(u) = \int_0^{+\infty} \frac{a}{2\pi\sigma^2} e^{-\frac{a^2}{2\sigma^2}} \, da =$$

$$a^2 = b$$

$$= \int_0^{+\infty} \frac{1}{4\pi\sigma^2} e^{-\frac{b}{2\sigma^2}} \, db =$$

$$db = 2a \, da$$

$$= \frac{1}{4\pi\sigma^2} \left[-2\sigma^2 e^{-\frac{b}{2\sigma^2}} \right]_0^{+\infty} = -\frac{1}{2\pi} e^{-\frac{b}{2\sigma^2}} \Big|_0^{+\infty} =$$

$$= \frac{1}{2\pi} \operatorname{rect}\left(\frac{u}{2\pi} \right)$$

ESERCIZIO 1

$X \in \mathcal{U}[0,1]$

$Y \in \mathcal{U}[0,1]$

$$\begin{cases} Z = X - Y \\ V = X + Y \end{cases}$$

$$A = \{ V \leq 1 \}$$

$f_{Z|A}(z|A)$?

$$f_{Z|A}(z|A) \triangleq \frac{d}{dz} F_{Z|A}(z|A)$$

$$F_{Z|A}(z|A) \triangleq \frac{\Pr\{Z \leq z, A\}}{\Pr\{A\}}$$

$$f_{ZV}(z,v) = \sum_i \frac{f_{XY}(x_i, y_i)}{|\det(J(x_i, y_i))|}$$

x_i, y_i soluzioni del sistema inverso

$$\begin{cases} X = \frac{Z+V}{2} \\ Y = \frac{V-Z}{2} \end{cases}$$

$$\underline{\underline{S}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow |\det(S)| = 2$$

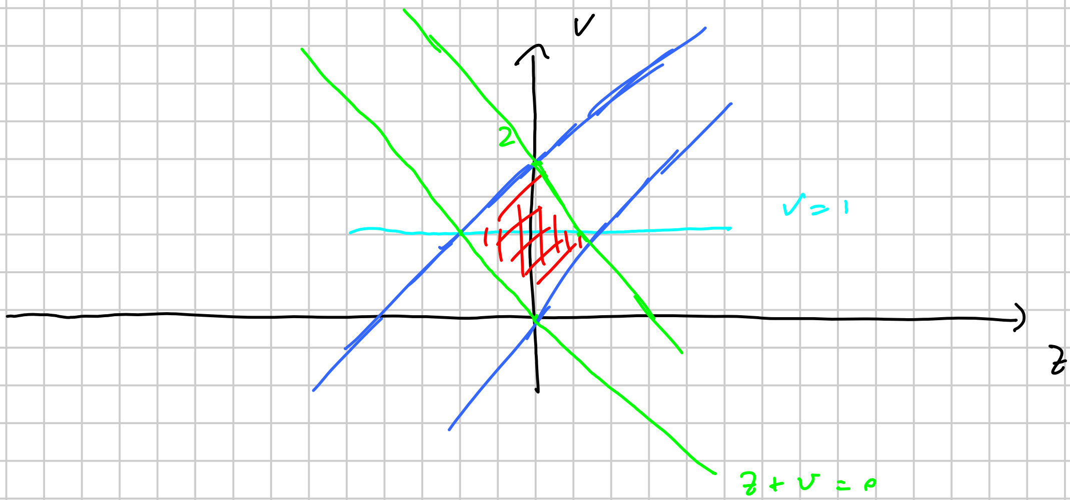
$$f_{XY}(x,y) \overset{\text{IND.}}{=} f_X(x) f_Y(y) = \text{rect}\left(\frac{x-\frac{1}{2}}{1}\right) \text{rect}\left(\frac{y-\frac{1}{2}}{1}\right)$$

$$f_{ZV}(z,v) = \frac{1}{2} \text{rect}\left(\frac{\frac{z+v}{2} - \frac{1}{2}}{1}\right) \text{rect}\left(\frac{\frac{v-z}{2} - \frac{1}{2}}{1}\right) =$$

$$= \frac{1}{2} \text{rect}\left(\frac{z+v}{2} - 1\right) \text{rect}\left(\frac{v-z}{2} - 1\right)$$

$$-1 \leq z+v-1 \leq 1 \Rightarrow 0 \leq z+v \leq 2$$

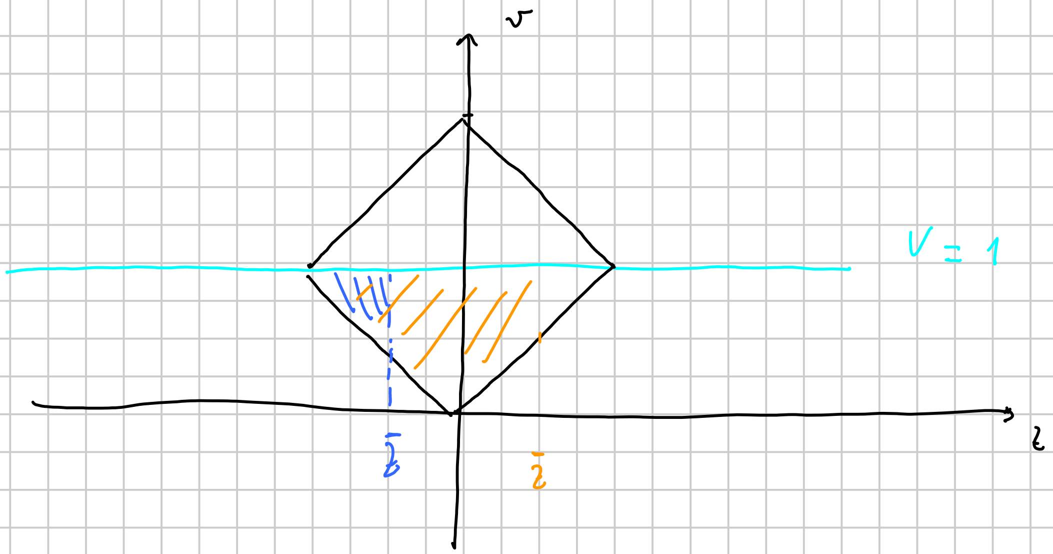
$$-1 \leq v-z-1 \leq 1 \Rightarrow 0 \leq v-z \leq 2$$



$$P_r\{A\} = P_r\{V \leq 1\} = \int_{z=-\infty}^{+\infty} \int_{v=-\infty}^{1} f_{ZV}(z,v) = F_V(1) = \frac{1}{2}$$

$$P_r\{Z \leq z, A\} = \int_{-\infty}^{z} \int_{-\infty}^{1} f_{ZV}(z,v)\, dz\, dv =$$

$$= \int_{-\infty}^{z} \int_{-\infty}^{1} \frac{1}{2} \text{rect}\left(\frac{z+v}{2} - 1\right) \text{rect}\left(\frac{v-z}{2} - 1\right) dz\, dv =$$



$-1 \leq z \leq 0$

$$\Pr\{Z \leq z, V \leq 1\} = \left(\frac{1+z}{2}\right)^2 \cdot \frac{1}{2}$$

$0 \leq z \leq 1$

$$\Pr\{Z \leq z, V \leq 1\} =$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2} - \left(\frac{(1-z)^2}{2}\right)\right) = \frac{1}{2}\left[1 - \frac{(1-z)^2}{2}\right]$$

$$F_{Z|A}(z|A) = \begin{cases} \frac{(1+z)^2}{2} & -1 \leq z \leq 0 \\ 1 - \frac{(1-z)^2}{2} & 0 \leq z \leq 1 \end{cases}$$

$$f_{Z|A}(z|A) = \frac{d}{dz} F_{Z|A}(z|A) = \begin{cases} 1+z & -1 \leq z \leq 0 \\ 1-z & 0 \leq z \leq 1 \end{cases}$$

$$f_{Z|A}(z|A) = 1 - |z| \qquad -1 \leq z \leq 1$$

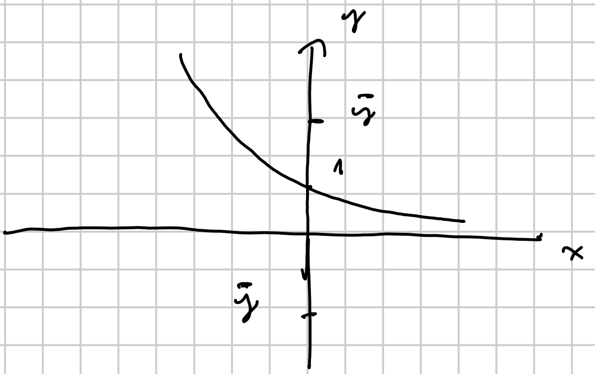
ESERCIZIO

$X \in \mathcal{N}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$Y = e^{-X}$ $\qquad f_Y(y)$? $\qquad \eta_Y$?

$$f_Y(y) = \sum_i \frac{f_X(x_i)}{|g'(x_i)|}$$



$\bar{y} \leq 0 \Rightarrow f_Y(y) = 0$

$\bar{y} > 0 \quad \exists$ 1 soluzione

$x_1 = -\ln(\bar{y})$

$$f_X(x_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\ln^2 y}{2}}$$

$$g'(x_1) = -e^{-x_1} = -y$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi y^2}} \exp\left\{-\frac{\ln^2 y}{2}\right\} u(y)$$

$$\eta_Y = E\{Y\} = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy =$$

$$= \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx = \int_{-\infty}^{+\infty} e^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\left(\frac{x^2}{2}+x\right)}\, dx =$$

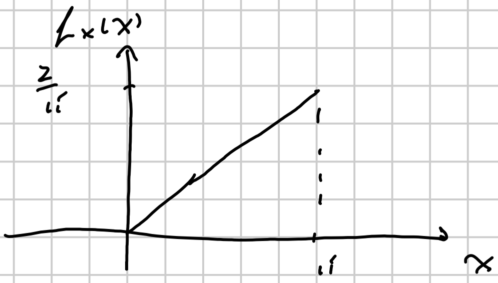
$$\frac{x^2}{2}+x = \frac{1}{2}\left((x+1)^2-1\right)$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{\frac{1}{2}}\, e^{-\frac{1}{2}(x+1)^2}\, dx =$$

$$= e^{+\frac{1}{2}} \underbrace{\underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(x+1)^2}\, dx}_{\mathcal{N}(-1,\,1)}}_{1} = e^{+\frac{1}{2}}$$

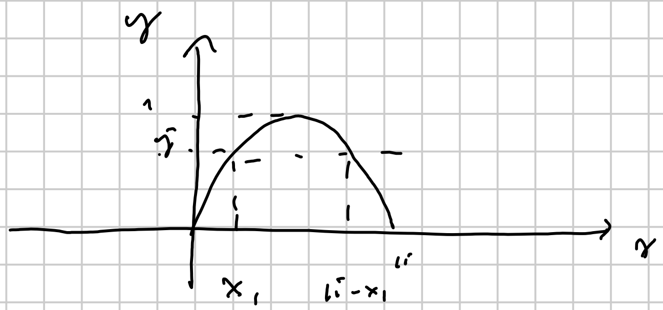
ESERCIZIO 2

X v.a. t.c. $f_X(x) = \frac{2x}{\pi^2}\, \mathrm{rect}\left(\frac{x-\pi/2}{\pi}\right)$



$Y = \sin(X)$

$f_Y(y)$? η_Y ?



Se $\bar{y} > 1$ e $\bar{y} < 0$ no soluzioni, 8c $\quad 0 \le y \le 1$

$$X_1 = \operatorname{asin}(y)$$

$$x_2 = \pi - x_1$$

$$g(x) = \sin(x)$$

$$g'(x) = \cos(x)$$

$$g'(x_1) = \cos(x_1)$$

$$g'(x_2) = \cos(x_2) = \cos(\pi - x_1) = -\cos(x_1)$$

$$f_x(x_1) = \frac{2x_1}{\pi^2} \qquad f_x(x_2) = 2\frac{(\pi - x_1)}{\pi^2}$$

$$f_Y(y) = \frac{2x_1}{\pi^2} \cdot \frac{1}{\cos(x_1)} + \frac{2(\pi - x_1)}{\pi^2 \cos(x_1)} = \frac{2}{\pi \cos(x_1)} =$$

$$= \frac{2}{\pi \cos(\operatorname{asin}(y))} \cdot \operatorname{rect}\left(\frac{y - \frac{1}{2}}{1}\right)$$

$$\eta_Y = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy = \int_{-\infty}^{+\infty} g(x)\, f_x(x)\, dx =$$

$$= \int_0^1 y \cdot \frac{2}{\pi \cos(\operatorname{asin}(y))}\, dy =$$

$$z = \operatorname{asin}(y)$$

$$y = \sin(z)$$

$$dy = \cos z\, dz$$

$$y = 0 \Rightarrow z = 0$$

$$y = 1 \Rightarrow z = \frac{\pi}{2}$$

$$= \int_0^{\pi/2} \sin(z) \frac{2}{\pi \cos(z)} \cos z \, dz =$$

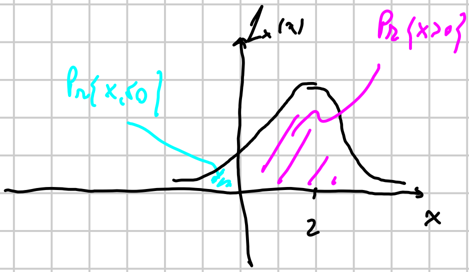
$$= \frac{2}{\pi} \int_0^{\pi/2} \sin(z) \, dz = -\frac{2}{\pi} \int_0^{\pi/2} -\sin z \, dz = -\frac{2}{\pi} \cos(z) \Big|_0^{\pi/2} =$$

$$= \frac{2}{\pi}$$

ESERCIZIO

$$X \sim \mathcal{N}(2, 2)$$

$$Y = \begin{cases} 1 & \text{se } X > 0 \\ -1 & \text{se } X \leq 0 \end{cases} \qquad \eta_Y \qquad \sigma^2_Y$$



$$\eta_Y = 1 \, p_1 - 1 \, p_{-1}$$

$$p_1 = Pr\{Y = 1\} \qquad p_{-1} = Pr\{Y = -1\}$$

$$p_1 = \int_0^{+\infty} f_x(x) \, dx = Pr\{X > 0\}$$

$$\eta_Y = \int_{-\infty}^{+\infty} g(x) \, f_x(x) \, dx$$

$\hookrightarrow \quad g(x) = \operatorname{sgn}(x)$

$$= -\int_{-\infty}^{0} f_x(x) \, dx + \int_0^{+\infty} f_x(x) \, dx$$

$$\int_{-\infty}^{0} f_x(x)\,dx = \phi\left(\frac{0-2}{\sqrt{2}}\right) = \phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\int_{0}^{+\infty} f_x(x)\,dx = 1 - \phi\left(-\frac{2}{\sqrt{2}}\right)$$

$$\eta_Y = -\phi\left(-\frac{2}{\sqrt{2}}\right) + 1 - \phi\left(-\frac{2}{\sqrt{2}}\right) = 1 - 2\phi\left(-\frac{2}{\sqrt{2}}\right)$$

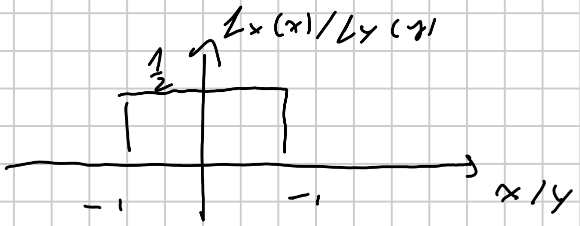
$$\sigma_Y^2 = E\left\{(y-\eta_Y)^2\right\} = \int_{-\infty}^{+\infty} (y-\eta_Y)^2 f_Y(y)\,dy =$$

$$= (1-\eta_Y)^2 P_1 + (-1-\eta_Y)^2 P_{-1}$$

ESERCIZIO

$X \in \mathcal{U}(-1,1)$

$Y \in \mathcal{U}(-1,1)$



$\alpha^2 + 2X\alpha + Y = 0$ abbia radici reali

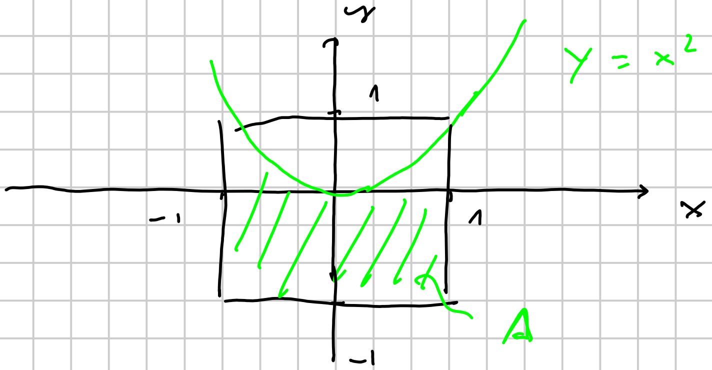
$\Delta = b^2 - 4ac = \underline{4X^2 - 4Y} \geq 0$

$X^2 - Y \geq 0$

$P_r\{Y \leq X^2\}$?

$Y \leq X^2$

$$f_{XY}(x,y) \overset{\text{IND.}}{=} f_X(x) \cdot f_Y(y)$$



$$f_{XY}(x,y) = \frac{1}{4}\,\text{rect}\left(\frac{x}{2}\right)\text{rect}\left(\frac{y}{2}\right)$$

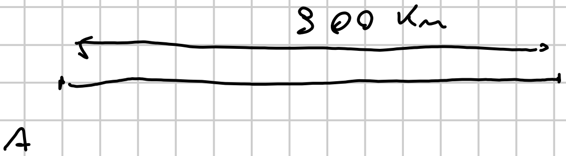
$$\Pr\{Y \le x^2\} = \iint_A f_{XY}(x,y)\,dx\,dy =$$

$$= \frac{1}{4}\iint_A dx\,dy = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{x^2} dx\,dy = \frac{1}{4}\int_{-1}^{1} y\Big|_{-1}^{x^2} dx =$$

$$= \frac{1}{4}\int_{-1}^{1}(x^2+1)\,dx = \frac{1}{2}\int_{0}^{1}(x^2+1)\,dx =$$

$$= \frac{1}{2}\left(\frac{x^3}{3} + x\right)\Big|_0^1 = \frac{1}{2} + \left(\frac{1}{3} + 1\right) = \frac{2}{3}$$

ESERCIZIO

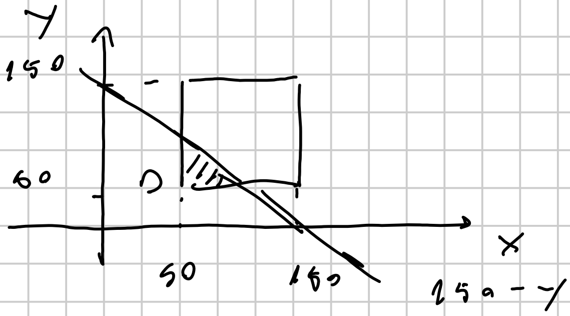


$X, Y \in \mathcal{U}[50 : 150]$ km/h

$\Pr\{$CHE I 2 TRENI SI INCONTRINO DOPO 6 ORE$\}$

$$900 \ge 6X + 6Y \Rightarrow X + Y \le 150$$

$$X \le 150 - Y$$



$$\Pr\{ X \le 150 - Y \} = \iint_D f_{XY}(x,y)\, dx\, dy =$$

$$= \frac{1}{100^2} \iint_D dx\, dy = \frac{1}{100^2} \, \frac{50^2}{2} = 0{,}125 = 12{,}5\,\%$$

ESERCIZIO 1

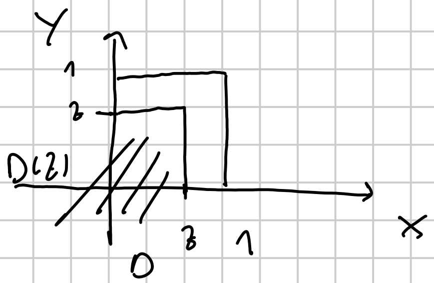
$X, Y \in \mathcal{U}(0,1)$

$Z = \max(X, Y)$ $\qquad f_Z(z)$

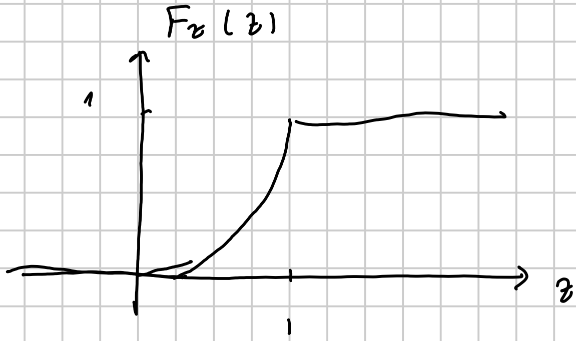
$$F_Z(z) = \Pr\{ Z \le z \} =$$

$$= \Pr\{ \max(X,Y) \le z \} =$$

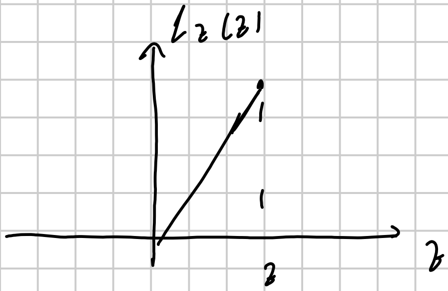
$$= \iint_{D(z)} f_{XY}(x,y)\, dx\, dy =$$



$$= \begin{cases} 0 & \text{se } z < 0 \\ z^2 & \text{se } 0 \le z < 1 \\ 1 & \text{se } z \ge 1 \end{cases}$$



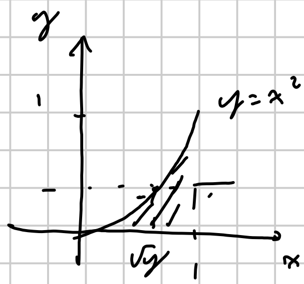
$$f_z(z) = \frac{d}{dz} F_z(z) = \begin{cases} 2z & \text{se } 0 \le z \le 1 \\ 0 & \text{altrimenti} \end{cases}$$



ESERCIZIO

X, Y v.a.

$$f_{XY}(x,y) = \begin{cases} Ax & (x,y) \in D \\ 0 & \text{altrove} \end{cases}$$



A? $\quad f_X(x)$, $f_Y(y)$?

$f_{X|Y}(x|y)$?

X e Y INDIPENDENTI?

A)

$$\iint_D f_{XY}(x,y)\, dx\, dy = 1 \qquad D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le x^2\}$$

$$\int_0^1 \int_0^{x^2} A x \, dx \, dy = 1$$

$$A \int_0^1 x x^2 \, dx = A \int_0^1 x^3 \, dx = A \left. \frac{x^4}{4} \right|_0^1 = \frac{A}{4} \Rightarrow \boxed{A = 4}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dy = \int_0^{x^2} 4x \, dy = 4x^3$$

$$0 < x < 1$$

$$= 4x^3 \, \mathrm{rect}\left(\frac{x - \frac{1}{2}}{1} \right)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y) \, dx \int_{\sqrt{y}}^1 4x \, dx = \cancel{4}^{2} \left. \frac{x^2}{\cancel{2}} \right|_{\sqrt{y}}^1 = 2 - 2y$$

NON SONO IND. PERCHE' $\boxed{f_{XY}(x, y) \neq f_X(x) f_Y(y)}$

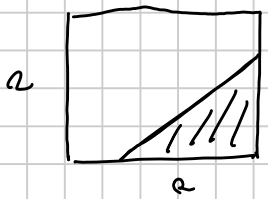
$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{4x}{2(1-y)} \, \mathrm{rect}\left(\frac{x - \frac{1}{2}}{1} \right)$$

SE VALE IND. $f_{X|Y}(x|y) = f_X(x)$

$$P_r \{ X < 0{,}5 \mid Y = 0{,}5 \} = P_r \{ B \}$$

$$f_{X|Y}(x | y = 0{,}5) = 4x \qquad P_r \{ B \} = \int_0^{\frac{1}{2}} 4x \, dx = \left. 4 \frac{x^2}{2} \right|_0^{\frac{1}{2}} = \frac{1}{2}$$

ESERCIZIO 1



$$A_T < \frac{1}{8} A_Q$$

$$A_Q = a^2 \Rightarrow P_r\left\{ A_T < \frac{a^2}{8} \right\}$$

X, Y ind $\in \mathcal{U}[0, a]$

$X = a X_1$

$Y = a Y_1$

$X_1, Y_1 \in \mathcal{U}[0,1]$

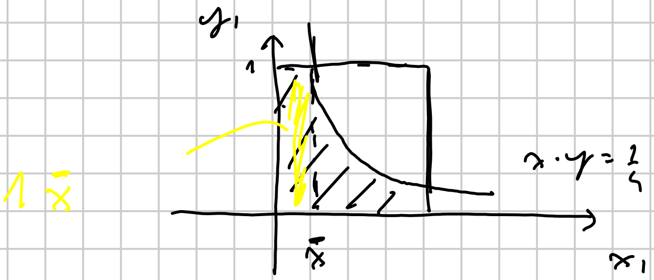
$$A_T = \frac{XY}{2} = a^2 \frac{X_1 Y_1}{2}$$

$$P_r\left\{ A_T < \frac{1}{8} A_Q \right\} = P_r\left\{ \cancel{a^2} \frac{X_1 Y_1}{2} < \frac{1}{8} \cancel{a^2} \right\} =$$

$$= P_r\left\{ \frac{X_1 Y_1}{2} < \frac{1}{8} \right\} = P_r\{ \underbrace{X_1 Y_1}_{Z} < \frac{1}{4} \}$$

$$P_r\left\{ Z < \frac{1}{4} \right\} = F_Z\left(\frac{1}{4}\right)$$

$$f_{X_1 Y_1}(x_1, y_1) = \mathrm{rect}\left(\frac{x_1 - \frac{1}{2}}{1}\right) \mathrm{rect}\left(\frac{y_1 - \frac{1}{2}}{1}\right)$$



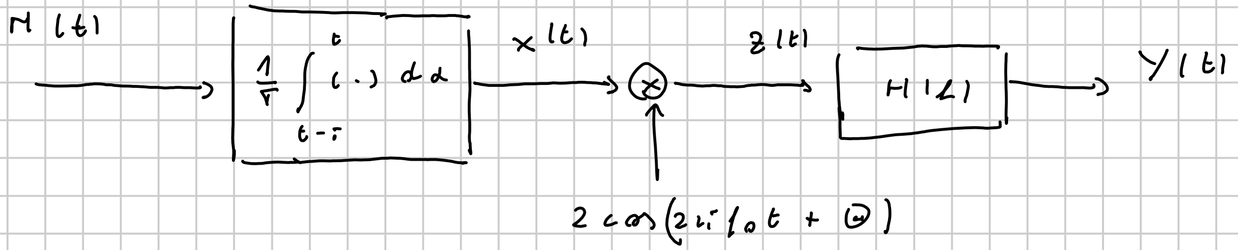
$$\bar{y} = 1 \Rightarrow \bar{x} = \frac{1}{4}$$

$$A = 1\,\bar{x} + \int_{\bar{x}}^{1} \frac{1}{4x}\,dx = \bar{x} + \frac{1}{4}\ln(x)\Big|_{\bar{x}}^{1} =$$

$$= \bar{x} + \frac{1}{4}\left[\ln(1) - \ln(\bar{x})\right]$$

$$P_r\left\{ Z \le \frac{1}{4} \right\} = P_r\left\{ A_r < \frac{1}{8} A_e \right\} = \frac{1}{4}\left[1 - \ln\left(\frac{1}{4}\right)\right]$$

ESERCIZIO 11



$n(t)$ è un processo SSL con $S_n(f) = \xi$

(Gaussiano bianco)

Θ È UNA v.a. INDIP. DA $n(t)$ $\mathcal{U}[-\pi, \pi]$

$S_y(f)$

$X(t)$ è Gaussiano e SSL

$\eta_x = \eta_n H(0) = 0$

$R_x(\tau) = R_n(\tau) \otimes h_1(\tau) \otimes h_1(-\tau)$

$\Updownarrow$

$S_x(f) = S_n(f)\, |H_1(f)|^2$

$h_1(t) = \frac{1}{T} \operatorname{rect}\left(\frac{t - T/2}{T}\right)$

$\Updownarrow$

$H_1(f) = \operatorname{sinc}(fT)\, e^{-j2\pi f \frac{T}{2}} \Rightarrow |H_1(f)|^2 = \operatorname{sinc}^2(fT)$

$R_x(\tau) = \underbrace{\xi\, \delta(\tau)}_{R_n(\tau)} \otimes \underbrace{h_1(\tau) \otimes h_1(-\tau)}_{R_h(\tau)} = \xi\, R_h(\tau) =$

$S_n(f) = \xi$

$$= \mathfrak{F}\ \frac{1}{T}\left(1 - \frac{|\tau|}{T}\right) \operatorname{rect}\left(\frac{\tau}{2T}\right) = R_x(\tau)$$

$$s(t) = 2\cos(2\pi f_0 t + \Theta) \qquad \Theta \in U(-\pi, \pi)$$

$$\eta_s = 0$$

$$R_s(\tau) = 2\cos(2\pi f_0 \tau)$$

$$z(t) = x(t) \cdot s(t)$$

IND

$$\eta_z(t) = E\{z(t)\} = 2E\{x(t)\cos(2\pi f_0 t + \Theta)\} =$$

$$= 2E\{x(t)\}\ E\{\cos(2\pi f_0 t + \Theta)\} = 0$$

$$R_z(t_1, t_2) = R_z(t, t-\tau) = E\{z(t)\ z(t-\tau)\} =$$

$$= 4E\{x(t)\ x(t-\tau)\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0(t-\tau) + \Theta)\} \overset{IND}{=}$$

$$= 4E\{x(t)\ x(t-\tau)\}\ E\{\cos(2\pi f_0 t + \Theta)\cos(2\pi f_0(t-\tau) + \Theta)\} =$$

$$= 4R_x(\tau)\ R_s(\tau) = 8R_x(\tau)\cos(2\pi f_0 \tau)$$

$$S_y(f) = S_z(f)\ |H(f)|^2$$

$$S_z(f) = 4\mathfrak{F}\left\{\operatorname{sinc}^2((f+f_0)T) + \operatorname{sinc}^2((f-f_0)T)\right\}$$

LE 2 REPLICHE CENTRATE A $\pm f_0$ POSSONO CONSIDERARSI DISGIUNTE PERCHÈ $\boxed{f_0 \gg \frac{1}{T}}$

$$S_Y(f) \cong \frac{1}{4}\left[\operatorname{sinc}^2((f+f_0)T)\,\operatorname{rect}\left(\frac{f+f_0}{2/T}\right) + \operatorname{sinc}^2((f-f_0)T)\,\operatorname{rect}\left(\frac{f-f_0}{2/T}\right)\right]$$

È POSSIBILE RICAVARE LA ddp DEL 1° ORDINE DI $Y(t)$, $f_Y(y;t)$?

CONDIZIONATAMENTE A Θ, IL PROCESSO $Y(t)$ È GAUSSIANO QUINDI $f_{Y|\Theta}(y|\theta;t)$ È LA D.D.P. DI UNA GAUSSIANA.

$f_Y(y|\theta;t)$ è la ddp di una gaussiana

$f_\Theta(\theta)$ È NOTA $\Theta \sim \mathcal{U}[-\pi,\pi]$

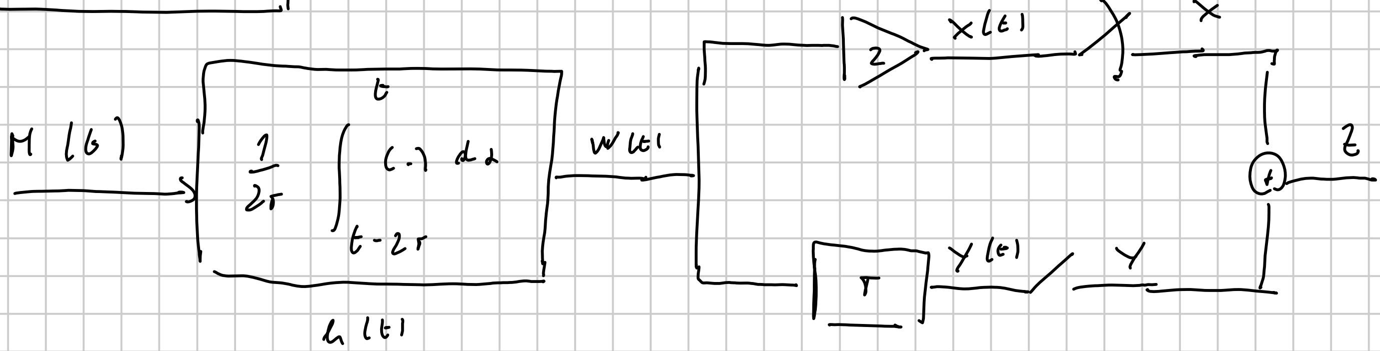
POSSO CALCOLARE LA d.d.p. CONGIUNTA

$$f_{Y\Theta}(y,\theta;t) = f_{Y|\Theta}(y|\theta;t)\cdot f_\Theta(\theta)$$

E DALLA CONGIUNTA POSSO RICAVARE LA MARGINALE

$$f_Y(y;t) = \int_{-\infty}^{+\infty} f_{Y\Theta}(y,\theta;t)\, d\theta$$

ESERCIZIO 2



$n(t)$ GAUSSIANO BIANCO (SSL) $S_n(f) = 2T$

$h(t) = \frac{1}{2T}\,\mathrm{rect}\left(\frac{t-T}{2T}\right)$

$X(t) = 2\,w(t) \Rightarrow X(0) = X = 2w(0)$

$Y(t) = w(t-T) \Rightarrow Y(0) = Y = w(-T)$

$Z = X + Y$

$f_Z(z)$

X e Y SONO V.A. CONG. GAUSSIANE

$Z = X + Y \qquad (\eta_Z, \sigma_Z^2)$

$\eta_w(t) = \underset{\downarrow \atop 0}{\eta_n(t)} \cdot H(0) = 0$

$R_w(\tau) = \left(1 - \frac{|\tau|}{2T}\right)\mathrm{rect}\left(\frac{\tau}{4T}\right)$

$$E\{z\} = E\{x+y\} = E\{x\} + E\{y\} = 0$$

$$\sigma_z^2 = E\{(z-\eta_z)^2\} = E\{z^2\} =$$

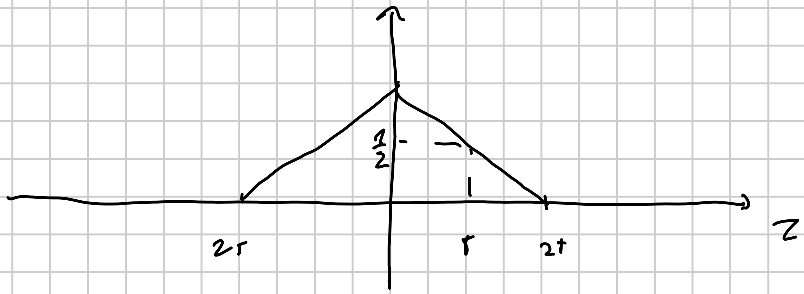
$$= E\{(2w(0) + w(-\tau))^2\} =$$

$$= E\{4w^2(0) + w^2(-\tau) + 4w(0)w(-\tau)\} =$$

$$= 4E\{w^2(0)\} + E\{w^2(-\tau)\} + 4E\{w(0)w(-\tau)\} =$$

$$= 4R_w(0) + R_w(0) + 4R_w(\tau) =$$

$$= 4 + 1 + 4 \cdot \frac{1}{2} = 7$$



$\eta_z = 0$ $\sigma_z^2 = 7$

$$f_z(z) = \frac{1}{\sqrt{2\pi 7}} e^{-\frac{z^2}{2\cdot 7}} = \frac{1}{\sqrt{14\pi}} e^{-\frac{z^2}{14}}$$

ESERCIZIO 3

$x(t)$ PROCESSO ALEATORIO GAUSSIANO SSL

$$R_x(\tau) = \delta(\tau) + 1$$

$$|H(f)| = \sqrt{\left(1 - \frac{|f|}{2B}\right) \mathrm{rect}\left(\frac{f}{4B}\right)}$$

$$\xrightarrow{x(t)} \boxed{H(f)} \longrightarrow Y(t)$$

$R_y(\tau)$ $\quad P_y$?

$$S_x(f) = \mathcal{F}^{-1}\{R_x(\tau)\} = 1 + \delta(f)$$

$$R_y(\tau) = R_x(\tau) \otimes h(\tau) \otimes h(-\tau)$$

$$\Updownarrow$$

$$S_y(f) = S_x(f)\,|H(f)|^2 = (1 + \delta(f))\left(1 - \frac{|f|}{2B}\right) \mathrm{rect}\left(\frac{f}{4B}\right)$$

$$P_y = \int_{-\infty}^{+\infty} S_y(f) = 1 + \frac{4B \cdot 1}{2} = 1 + 2B$$

$$R_y(\tau) = \frac{1}{r} \, \mathrm{sinc}^2(2B\tau) + 1$$

$Y(t_1)$ e $Y(t_2)$ v.a. estratte da $Y(t)$

SOTTO QUALI CONDIZIONI DI t_1 e t_2 $Y(t_1) = y_1$ E $Y(t_2) = y_2$ SONO INDIPENDENTI?

$$E^2\{Y\} = \eta_Y^2 = \lim_{\tau \to \infty} R_Y(\tau) = 1$$

$$\Downarrow$$

$$\boxed{\eta_Y = 1}$$

$$C_{Y_1 Y_2} = E\{(Y_1 - \eta_{Y_1})(Y_2 - \eta_{Y_2})\} =$$

$$= R_Y(\tau) - \eta_{Y_1}\eta_{Y_2} = \frac{1}{T} \operatorname{sinc}^2(2B\tau) + \cancel{1} - \cancel{1} =$$

$$= 2B \operatorname{sinc}^2(2B\tau) \overset{?}{=} 0$$

$$\tau = \frac{k}{2B} \qquad \tau \neq 0$$

$$t_2 = t_1 + \frac{k}{2B}$$

INCORRELATE $\Rightarrow$ INDIPENDENTI

Esercizio 3 FRAA

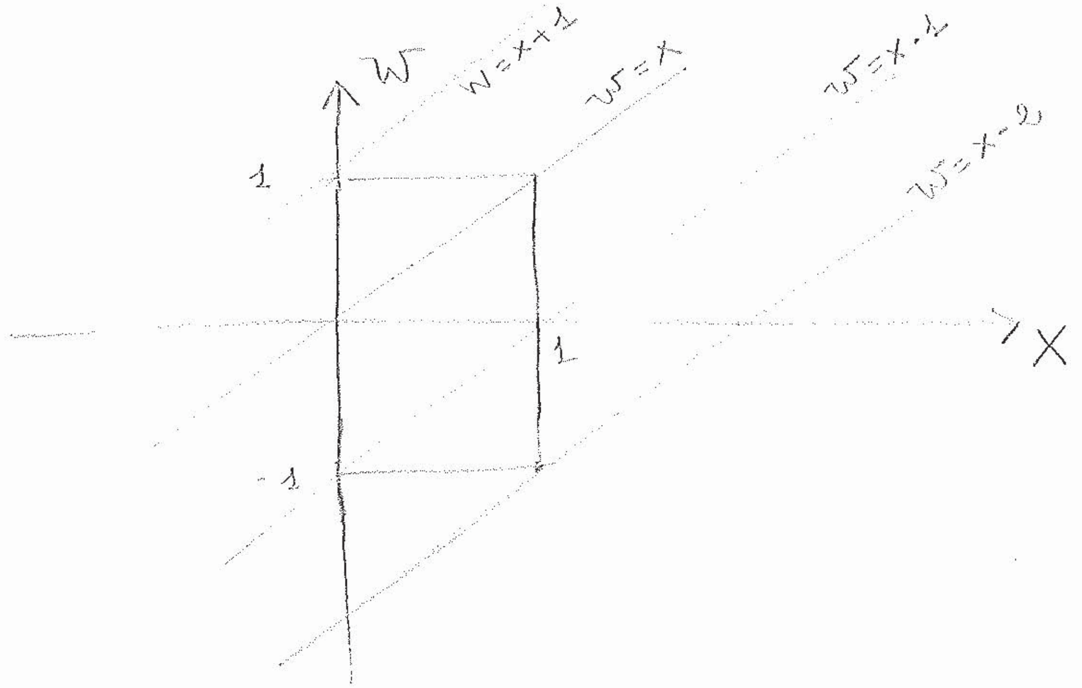
$X \in \mathcal{U}(0,1) \qquad f_X(x) = \text{rect}\left(\frac{x-1/2}{1}\right)$

$Y \in \mathcal{U}(0,1) \qquad f_Y(y) = \text{rect}\left(\frac{y-1/2}{1}\right)$

$Z = X - 2Y + 1$

$W = 2Y - 1 \in \mathcal{U}(-1,1) \qquad f_W(w) = \frac{1}{2}\,\text{rect}\left(\frac{w}{2}\right)$

$Z = X - W$ e X e W sono ancora indipendenti



$F_Z(z) = \Pr\{Z \le z\} = \Pr\{X - W \le z\} =$

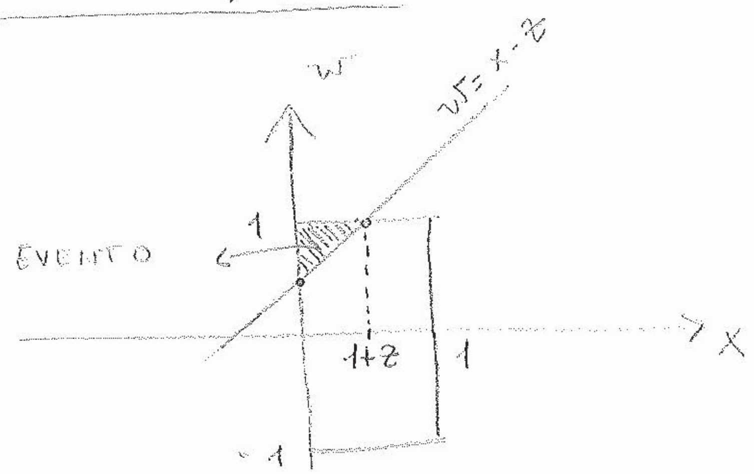
$= \Pr\{W \ge X - z\}$

Se $z \le -1 \qquad F_Z(z) = 0$

Se $z \ge 2 \qquad F_Z(z) = 1$

Se $\quad 0 \geq z \geq -1$

$0 = x - z$

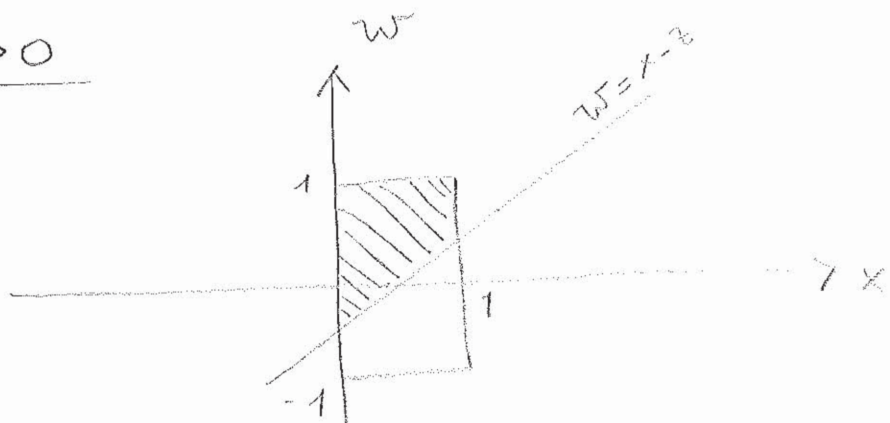


$$\Pr\{Z \leq z\} = \int_{x=0}^{1+z} \int_{w=x-z}^{1} f_{XW}(x,w)\,dx\,dw =$$

$$= \int_0^{1+z} dx \int_{x-z}^{1} \frac{1}{2}\,dy = \frac{1}{2}\int_0^{1+z} (1-x+z)\,dx = \frac{1}{2}\left[(1+z)x - \frac{x^2}{2}\right]_0^{1+z} =$$

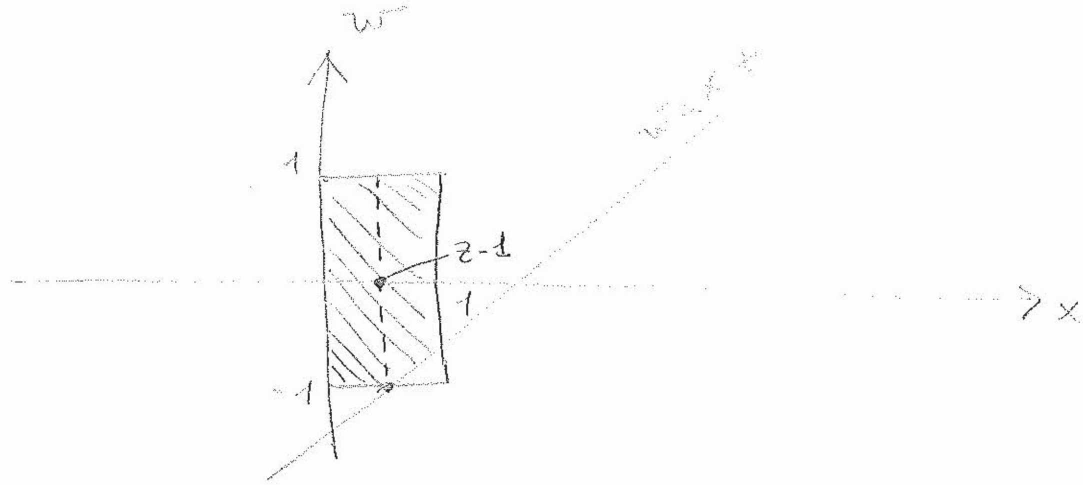
$$= \frac{z^2}{4} + \frac{z}{2} + \frac{1}{4}$$

Se $\quad 1 \geq z > 0$



$$\Pr\{Z \leq z\} = \int_{x=0}^{1} \int_{w=x-z}^{1} \frac{1}{2}\,dw\,dx = \frac{z}{2} + \frac{1}{4}$$

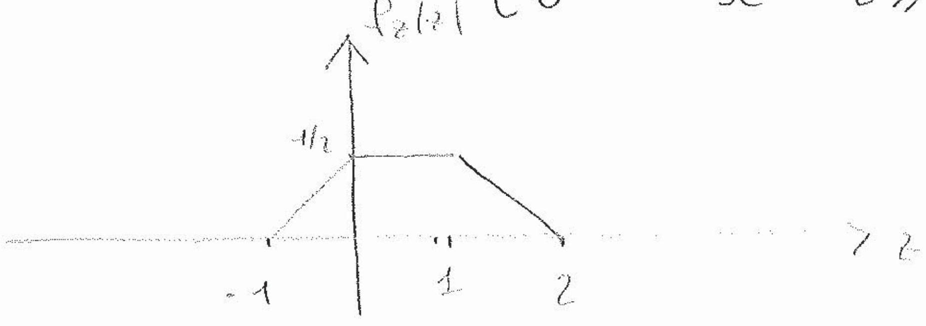
Se $2 \geq z > 1$



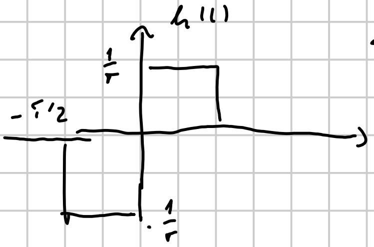
$$P_r\{Z \leq z\} = \frac{(z-1)^2}{2} + \int_{x=z-1}^{1}\int_{w=1-z}^{1} \frac{1}{2}\, dx\, dw = -\frac{z^2}{4} + z$$

$$F_Z(z) = \begin{cases} 0 & \text{se } z \leq -1 \\ \frac{z^2}{4} + \frac{z}{2} + \frac{1}{4} & \text{se } 0 \geq z > -1 \\ \frac{z}{2} + \frac{1}{4} & \text{se } 1 \geq z > 0 \\ 1 & \text{se } z \geq 2 \end{cases}$$

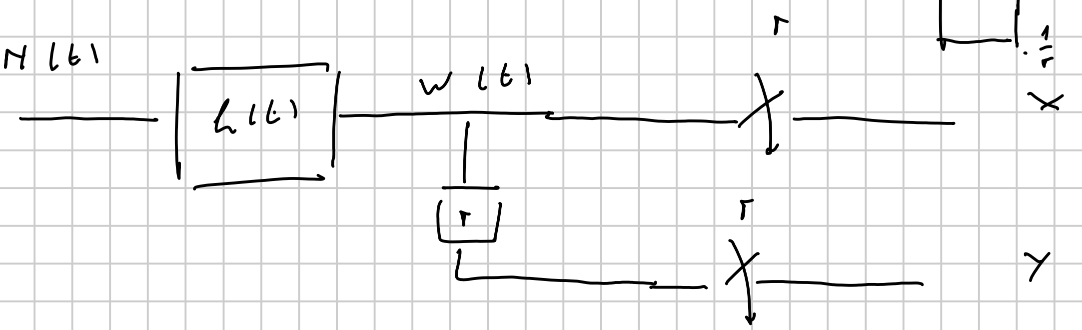
$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z) = \begin{cases} 0 & \text{se } z \leq -1 \\ \frac{z}{2} + \frac{1}{2} & \text{se } 0 \geq z > -1 \\ \frac{1}{2} & \text{se } 1 \geq z > 0 \\ -\frac{z}{2} + 1 & \text{se } 2 \geq z > 1 \\ 0 & \text{se } z \geq 2 \end{cases}$$



ESERCIZIO 2



$S_N(f) = \frac{N_0}{2}$



$f_{XY}(x,y)$?

$X = w(T)$ $\qquad \eta_x \; ; \; \eta_y \; ; \; \sigma_x^2 \; ; \; \sigma_y^2 \; ; \; cov(x,y)$

$Y = w(0)$

$$E\{w(t)\} = E\{N(t) \; H(0)\} = 0 \implies \eta_x = \eta_y = 0$$

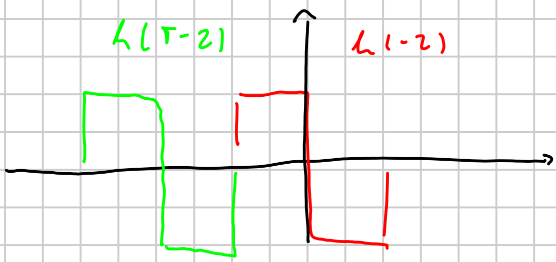
$$E\{x^2\} = E\{y^2\} = \int_{-\infty}^{+\infty} S_N(f) \, |H(f)|^2 \, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H(f)|^2 \, df =$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h^2(t) \, dt = \frac{N_0}{2} \frac{1}{T^2} T = \frac{N_0}{2T} = \sigma_x^2 = \sigma_y^2 = \sigma^2$$

$$cov(X,Y) = E\{XY\} = E\{w(0) \; w(T)\} = R_w(T) =$$

$$= \underbrace{\frac{N_0}{2} \delta(z)}_{S_N(f)} \otimes h(-z) \otimes h(z) \Big|_{z=T} =$$

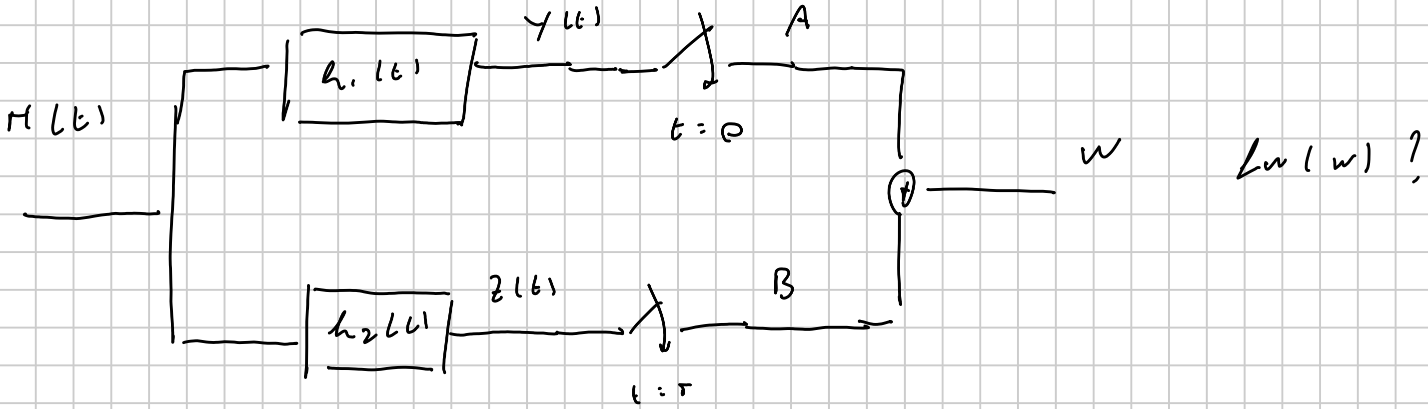
$$= \frac{N_0}{2} h(-z) \otimes h(z) \Big|_{z=T} = \frac{N_0}{2} \int_{-\infty}^{+\infty} h(-z) \, h(T-z) \, dz = 0$$

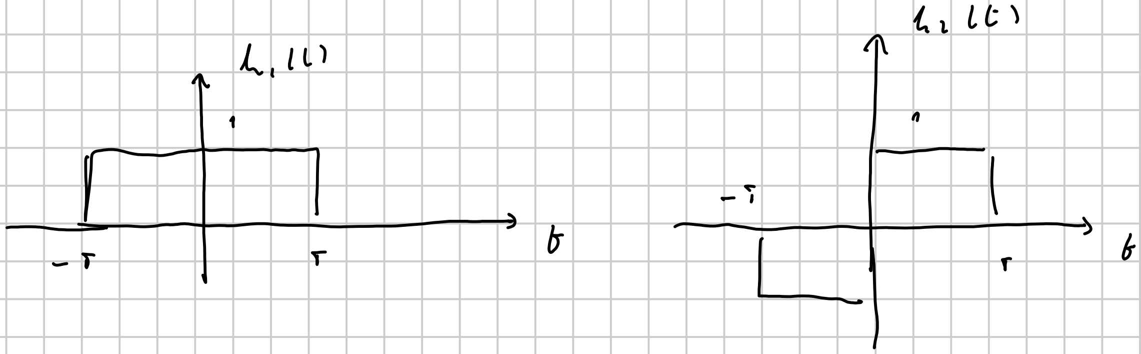


$cov(X,Y) = 0$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

ESERCIZIO 3





n(t) GAUSSIANO BIANCO

$E\{n(t)\} = 0$ $\qquad R_n(\tau) = \frac{N_0}{2}\delta(\tau)$ $\qquad S_n(f) = \frac{N_0}{2}$

Y(t) e Z(t) SONO PROCESSI ALEATORI GAUSSIANI S.S.L.

A e B SONO V.A. CONG. GAUSSIANE

$$A = \int_{-\infty}^{+\infty} n(t)\, h_1(t-\tau)\, d\tau \Big|_{t=0}$$

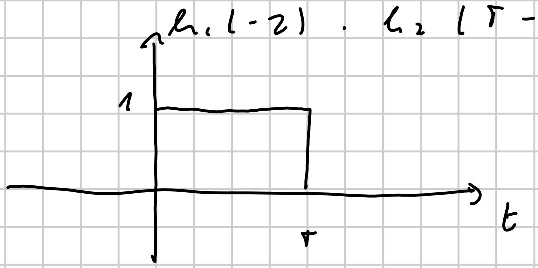
$$E\{A\} = \int_{-\infty}^{+\infty} E\{n(\tau)\}\, h_1(-\tau) = 0$$

$$B = \int_{-\infty}^{+\infty} n(\tau)\, h_2(t-\tau)\, d\tau \Big|_{t=T}$$

$$E\{B\} = 0$$

$$E\{AB\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{E\{n(\tau_1)\, n(\tau_2)\}}_{R_n(\tau_1-\tau_2) = \frac{N_0}{2}\delta(\tau_1-\tau_2)} h_1(-\tau_1)\, h_2(T-\tau_2)\, d\tau_1 d\tau_2$$

$$= \int_{-\infty}^{+\infty} \frac{N_0}{2}\, h_1(-\tau)\, h_2(T-\tau)\, d\tau$$



$$E\{AB\} = \frac{N_0}{2} T \neq \eta_A\, \eta_B$$

A e B SONO CORRELATE

$$A \in N(0, \sigma_A^2)$$

$$B \in N(0, \sigma_B^2)$$

$$\mathrm{cov}\{AB\} = \frac{N_0}{2} T$$

$$E\{A^2\} = E\{Y^2(0)\} = \int_{-\infty}^{+\infty} \frac{N_0}{2} |H_1(f)|^2 df =$$

$$= \frac{N_0}{2}\int_{-\infty}^{+\infty} h_1^2(t)\, dt = \frac{N_0}{2} \cdot 2T = N_0 T = \sigma_A^2$$

$$E\{B^2\} = \frac{N_0}{2}\int_{-\infty}^{+\infty} h_2^2(t)\, dt = N_0 T$$

$$\rho = \frac{\mathrm{cov}(A,B)}{\sigma_A\, \sigma_B} = \frac{N_0 T/2}{N_0 T} = \frac{1}{2}$$

$$w \in \mathcal{N}(\eta_w, \sigma_w^2)$$

$$w = A + B$$

$$\eta_w = E\{A + B\} = E\{A\} + E\{B\} = 0$$

$$\sigma_w^2 = E\{(w - \eta_w)^2\} = E\{w^2\} =$$

$$= E\{(A + B)^2\} = E\{A^2\} + E\{B^2\} + 2E\{AB\} =$$

$$= N_0 T + N_0 T + 2\frac{N_0 T}{2} = 3 N_0 T$$

$$f_w(w) = \frac{1}{\sqrt{2\pi\, 3N_0 T}}\, e^{-\frac{w^2}{6N_0 T}}$$

ESERCIZIO 4

X e Y v.a. i.i.d. $\in \mathcal{N}(0,1)$

$$\begin{cases} V = 5X + 3Y \\ Z = -3X + 5Y \end{cases}$$

Z e V SONO INCORRELATE $\Rightarrow$ INDIPENDENTI

$$C_{ZV} \triangleq R_{vz} - \eta_Z \eta_V = R_{vz}$$

$$\eta_Z = E\{-3X + 5Y\} = -3\eta_X + 5\eta_Y = 0$$

$$\eta_V = 5\eta_X + 3\eta_Y = 0$$

$$C_{ZV} = E\{ZV\} = E\{(5X + 3Y)(-3X + 5Y)\} =$$

$$= E\{-15X^2 + 25XY - 9XY + 15Y^2\} =$$

$$= -15\,E\{X^2\} + 16\,E\{XY\} + 15\,E\{Y^2\}$$

$$E\{X^2\} = \sigma_X^2 + \eta_X^2 = 1$$

$$E\{Y^2\} = \sigma_Y^2 + \eta_Y^2 = 1$$

$$E\{XY\} = R_{XY} = C_{XY} = 0$$

X e Y INDIPENDENTI

$$C_{ZV} = -15 + 15 = 0 \Rightarrow$$ INCORRELATE e QUINDI INDIPENDENTI

$$f_{ZV}(z,v) = f_Z(z)\, f_V(v)$$

$$\begin{cases} V = 5X + 3Y \\ \\ Z = -3X + 5Y \end{cases} \qquad \begin{aligned} \bar{x} &= \frac{5v - 3z}{34} \\ \bar{y} &= \frac{3v + 5z}{34} \end{aligned}$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} 5 & +3 \\ -3 & 5 \end{bmatrix}$$

$$|\underline{\underline{J}}| = 34$$

$$f_{ZV}(v,z) = \frac{1}{34} f_{XY}(x,y)\Big|_{\substack{x=\bar{x} \\ y=\bar{y}}} =$$

$$= \underbrace{\frac{1}{34}}\ \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{f_X(x)}\ \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{f_Y(y)}\Big|_{\substack{x=\bar{x} \\ y=\bar{y}}} =$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

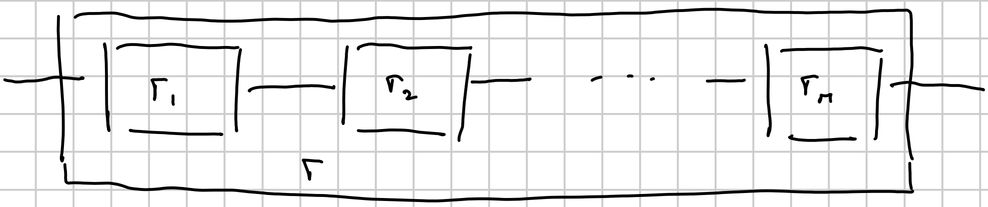
$$= \frac{1}{34 \cdot 2\pi} e^{-\frac{1}{34^2}\frac{1}{2}\left(25v^2 + 9z^2 \cancel{-30vz} + 9v^2 + 25z^2 \cancel{+30vz}\right)} =$$

$$= \frac{1}{2\pi\, 34} e^{-\frac{1}{2}\left(\frac{\cancel{34}v^2}{34^{\cancel{2}}} + \frac{\cancel{34}z^2}{34^{\cancel{2}}}\right)} =$$

$$= \frac{1}{\sqrt{2\pi\, 34}} e^{-\frac{1}{2}\frac{v^2}{34}} \quad \frac{1}{\sqrt{2\pi\, 34}} e^{-\frac{1}{2}\frac{z^2}{34}}$$

V e Z $\Rightarrow$ INDIPENDENTI.

ESERCIZIO 5



T_i = TEMPO DI VITA DELL i-esima LAMPADINA

$E\{T\}$ $\quad T = \min\{T_1, T_2, \ldots, T_n\}$ $\quad E\{T\}$

$$f_{T_i}(t_i) = \frac{1}{\eta} e^{-t_i/\eta} \qquad t_i \geq 0$$

$$E\{T_i\} = \eta$$

Pr{ i-esima è accesa al tempo t } =

$$= Pr\{T_i > t\} = \int_t^{+\infty} \frac{1}{\eta} e^{-t_i/\eta}\, dt_i = \left[-e^{-t_i/\eta}\right]_t^{+\infty} =$$

$$= e^{-t/\eta}$$

$$Pr\{T > t\} = Pr\left\{\bigcap_{i=1}^{n} T_i > t\right\} = \prod_{i=1}^{n} Pr\{T_i > t\} =$$

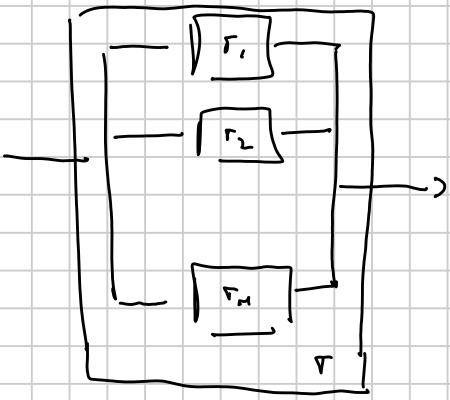
$$= e^{-nt/\eta}$$

$$F_T(t) = Pr\{T \leq t\} = 1 - Pr\{T > t\} =$$

$$= 1 - e^{-nt/\eta}$$

$$f_T(t) = \frac{n}{\eta} e^{-nt/\eta} \qquad E\{T\} = \frac{\eta}{n} < \eta$$

PARALLELO



$$T = \max\{ T_1, \ldots, T_M \}$$

$$E\{T\}$$

$$P_r\{ T_i \le t \} = 1 - e^{-t/\eta}$$

$$P_r\{ T \le t \} = P_r\left\{ \bigcap_{i=1}^{M} T_i \le t_i \right\} = \prod_{i=1}^{M} P_r\{ T_i \le t \} =$$

$$= \left(1 - e^{-t/\eta}\right)^M = F_T(t)$$

$$f_T(t) = \frac{\partial}{\partial t} F_T(t) = \frac{M}{\eta}\left(1 - e^{-t/\eta}\right)^{M-1} e^{-t/\eta} \quad \text{per } t > 0$$

$$E\{T\} = \int_{-\infty}^{+\infty} t \, f_T(t) \, dt$$

$$M = 2$$

$$f_T(t) = \frac{2}{\eta}\left(e^{-t/\eta} - e^{-2t/\eta}\right) = 2 \cdot \frac{1}{\eta} e^{-t/\eta} - \frac{1}{\eta/2} e^{-t/\eta/2}$$

$$E\{T\} = 2\eta - \frac{\eta}{2} = \frac{3}{2}\eta > \eta$$

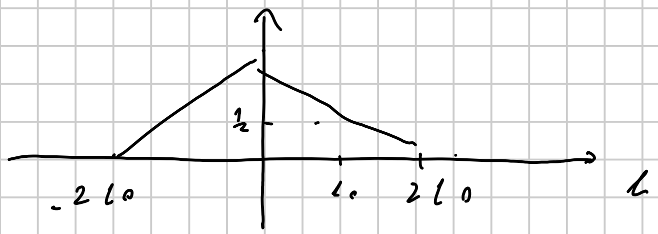
ESERCIZIO 6 | Processi parametrici

$$x(t) = A \cos(2\pi f_0 t) + B \sin(2\pi f_0 t)$$

$$A, B \in \mathcal{N}(0, \sigma^2) \qquad \sigma^2 = 4$$

$$x(t) \longrightarrow \boxed{H(f)} \longrightarrow y(t)$$

$$f_y(y;t)$$



$$x_n(t) \Rightarrow X_n(f) = \frac{A}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) + \frac{B}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$$

$$Y_n(f) = X_n(f)\, H(f) = \frac{1}{2} X_n(f) \Rightarrow$$

$$y_n(t) = \frac{1}{2} x_n(t)$$

$$y(t) = \frac{1}{2} x(t)$$

$$Y = y(\bar{t}) = \frac{A}{2}\cos(2\pi f_0 \bar{t}) + \frac{B}{2}\sin(2\pi f_0 \bar{t}) =$$

$$= K_1 \frac{A}{2} + K_2 \frac{B}{2} = \qquad K_1 = \cos(2\pi f_0 \bar{t})$$

$$K_2 = \sin(2\pi f_0 \bar{t})$$

$$= Y_A + Y_B$$

Y è V.A. GAUSSIANA

A e B IND. $\Rightarrow$ Y_A e Y_B INDIPENDENTI $\Rightarrow$ INCORRELATE

$$\eta_{Y_A} = E\left\{ K_1 \frac{A}{2} \right\} = 0$$

$$\eta_{Y_B} = E\left\{ K_2 \frac{B}{2} \right\} = 0$$

$$\sigma^2_A = E\left\{ \left(\frac{K_1 A}{2} \right)^2 \right\} = \frac{K_1^2}{4} \sigma^2$$

$$\sigma^2_B = E\left\{ \left(\frac{K_2 B}{2} \right)^2 \right\} = \frac{K_2^2}{4} \sigma^2$$

$$\eta_Y = 0$$

$$\sigma^2_Y = E\left\{ (Y_A + Y_B)^2 \right\} = E\left\{ Y_A^2 \right\} + E\left\{ Y_B^2 \right\} + \underbrace{2E\left\{ Y_A Y_B \right\}}_{0} =$$

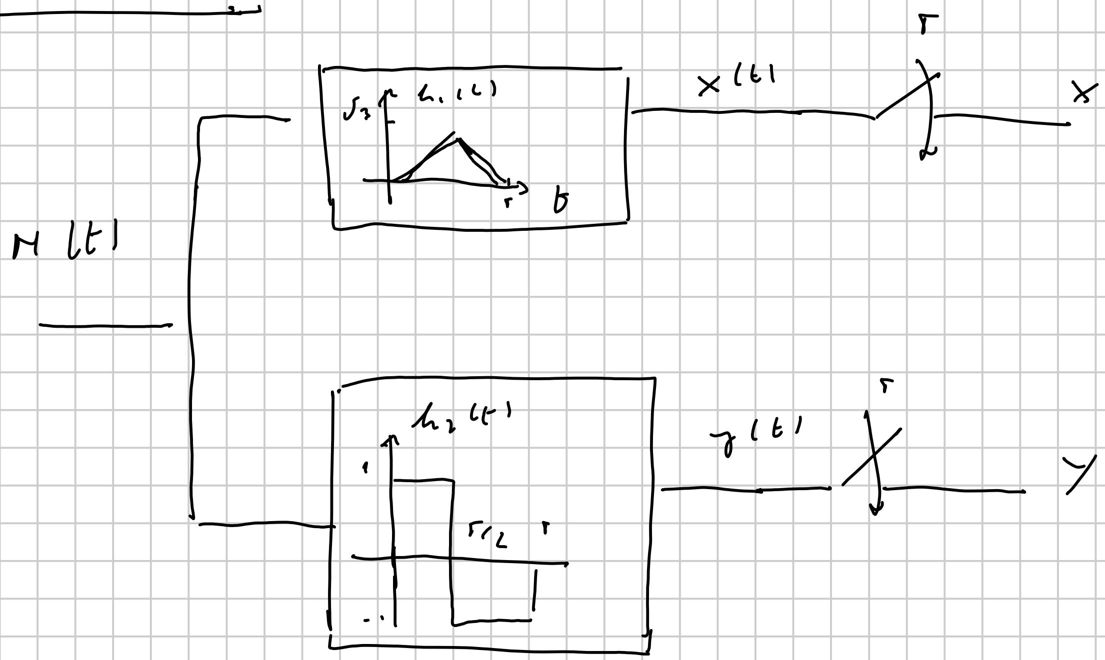
$$= \frac{1}{4} \underbrace{\left(K_1^2 + K_2^2 \right)}_{1} \sigma^2 =$$

$$= \frac{1}{4} \left(\cos^2 (2\pi f_0 \bar{t}) + \sin^2 (2\pi f_0 \bar{t}) \right) \sigma^2 = 1$$

$$Y \in \mathcal{N}(0, 1)$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

ESERCIZIO 7



$n(t)$ GAUSSIANO BIANCO $S_n(f) = \frac{N_0}{2}$

- $P_r\{X > Y\}$

$E\{XY\}$

$$x(t) = \int_{-\infty}^{+\infty} n(\tau)\, h_1(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} n(\tau)\, h_2(t-\tau)\, d\tau$$

$X = x(T)$

$Y = y(T)$

$\eta_n(t) = 0$

$\eta_x(t) = \eta_n(t) \otimes h_1(t) = 0$

$\eta_Y(t) = 0$

$$S_x(f) = S_n(f)\,|H_1(f)|^2$$

$$P_x = \int_{-\infty}^{+\infty} S_n(f)\,|H_1(f)|^2\,df = \frac{N_0}{2}\int_{-\infty}^{+\infty} h_1^2(t)\,dt = \frac{N_0}{2}\cdot 3\,\frac{2}{3}\,\frac{T}{2} = \frac{N_0 T}{2}$$

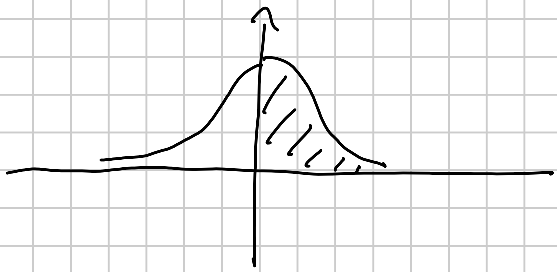
$$= \sigma_x^2$$

$$P_y = \sigma_y^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |h_2(t)|^2\,dt = N_0\frac{T}{2}$$

$$X \text{ e } Y \in \mathcal{N}\left(0, \frac{N_0 T}{2}\right)$$

$$P_r\{X > Y\} = P_r\{X - Y > 0\} = \frac{1}{2}$$

$$Z = X - Y \in \mathcal{N}(0, \sigma_z^2)$$

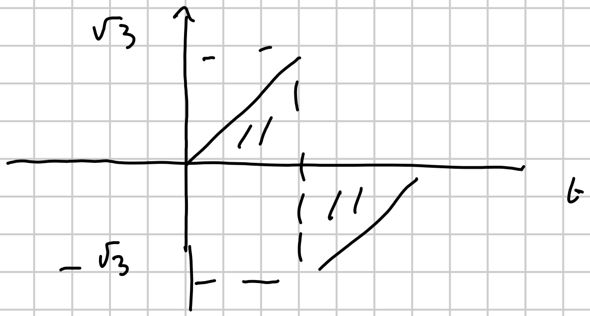


$$E\{XY\} = E\left\{\underbrace{\int_{-\infty}^{+\infty} n(z)\,h_1(T-z)\,dz}_{X}\;\underbrace{\int_{-\infty}^{+\infty} n(\alpha)\,h_2(T-\alpha)\,d\alpha}_{Y}\right\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{E\{n(z)\,n(\alpha)\}}\,h_1(T-z)\,h_2(T-\alpha)\,dz\,d\alpha =$$

$$\frac{N_0}{2}\,\delta(z-\alpha) \neq 0 \quad \text{se} \quad z = \alpha$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} h_1(\tau - \lambda)\, h_2(\tau - \lambda)\, d\lambda = 0$$



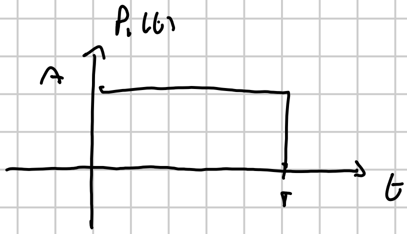
ESERCIZIO 1

$$s(t) = \sum_n x[n]\, p(t - nT)$$

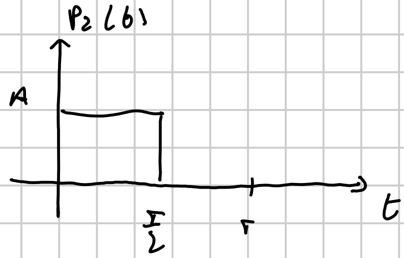
$x[n] \in A_s = \{0, 1\}$ SEGNALAZIONE ON-OFF

Calcolare $S_s(f)$ e P_s nel caso in cui

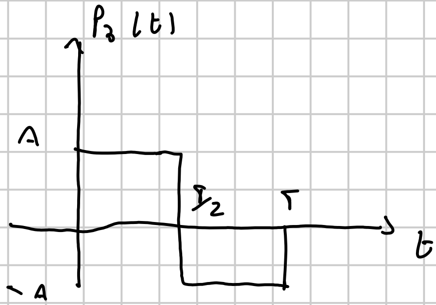
1) $p(t) = \begin{cases} A & 0 \le t \le T \\ 0 & \text{altrove} \end{cases}$



2) $p(t) = \begin{cases} A & 0 \le t \le \frac{T}{2} \\ 0 & \text{altrove} \end{cases}$



3) $p(t) = \begin{cases} A & 0 \le t \le \frac{T}{2} \\ -A & \frac{T}{2} \le t \le T \\ 0 & \text{altrove} \end{cases}$



$$S_s(f) = \frac{1}{T} S_x(f) \, |P(f)|^2$$

$$S_x(f) = TFS\left[R_x[m] \right]$$

$$R_x[m] = E\{x[i+m]\, x[i]\} = \begin{cases} E\{x^2[i]\} = \frac{1}{2} & m = 0 \\ E\{x^2[i]\} = \frac{1}{4} & m \neq 0 \end{cases}$$

$$R_x[m] = \frac{1}{4} + \frac{1}{4}\delta[m]$$

$$S_x(f) = \frac{1}{4} + \sum_k \frac{1}{4T} \delta\left(f - \frac{k}{T}\right)$$

$$S_s(f) = \frac{1}{T}\left(\frac{1}{4} + \sum_k \frac{1}{4T} \delta\left(f - \frac{k}{T}\right) \right) A^2 T^2 \, \mathrm{sinc}^2(fT) =$$

$$= \frac{A^2 T}{4}\, \mathrm{sinc}^2(fT) + A^2 T \sum_k \frac{1}{4T}\, \delta\left(f - \frac{k}{T}\right) \mathrm{sinc}^2\left(\frac{k}{T} T\right) =$$

$$= \frac{A^2 T}{4}\, \mathrm{sinc}^2(fT) + \frac{A^2}{4}\, \delta(f)$$

$$P_s = \frac{A^2}{4} + \int_{-\infty}^{+\infty} \frac{A^2 T}{4}\, \mathrm{sinc}^2(fT)\, df = \frac{A^2}{4} + \frac{A^2 T}{4} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\frac{1}{T}\, \mathrm{rect}\left(\frac{t}{T}\right)\right)^2 dt =$$

$$= \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

2)

$$|P_2(f)|^2 = \frac{A^2 T^2}{4}\, \mathrm{sinc}^2\left(f \frac{T}{2}\right)$$

$$S_s(f) = \frac{1}{T}\left(\frac{1}{4} + \frac{1}{4T} \sum_k \delta\left(f - \frac{k}{T}\right)\right) \frac{A^2 T^2}{4}\, \mathrm{sinc}^2\left(f \frac{T}{2}\right) =$$

$$= \frac{A^2 T}{16}\, \mathrm{sinc}^2\left(f \frac{T}{2}\right) + \frac{A^2}{16} \sum_k \mathrm{sinc}^2\left(\frac{k}{\cancel{T}} \frac{\cancel{T}}{2}\right) \delta\left(f - \frac{k}{T}\right)$$

$$P_s = \int_{-\infty}^{+\infty} S_s(f)\, df = \frac{A^2 T}{4} \int_{-\infty}^{+\infty} \mathrm{sinc}^2\left(f \frac{T}{2}\right) df + \frac{A^2}{16} \sum_k \mathrm{sinc}^2\left(\frac{k}{2}\right) =$$

$$= \frac{A^2}{8} + \frac{A^2}{8} = \frac{A^2}{4}$$

3)

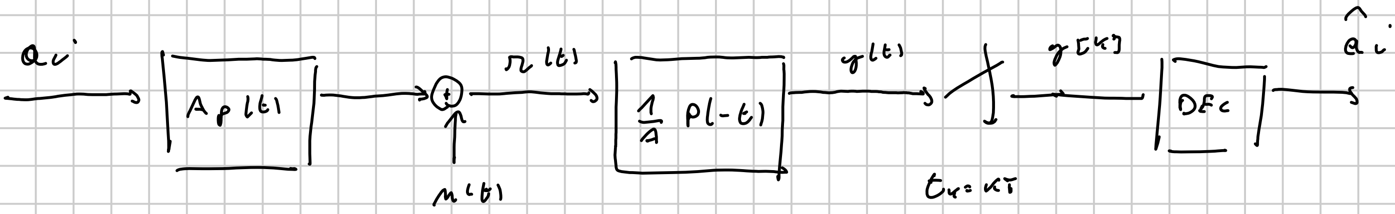
$$|P_3(f)|^2 = A^2 T^2\, \mathrm{sinc}^2\left(f \frac{T}{2}\right) \sin^2\left(\pi f \frac{T}{2}\right)$$

$$S_s(f) = A^2 T \left(\frac{1}{4} + \frac{1}{4T} \sum_k \delta\left(f - \frac{k}{T}\right)\right) \mathrm{sinc}^2\left(f \frac{T}{2}\right) \sin^2\left(\pi f \frac{T}{2}\right) =$$

$$= \frac{A^2 T}{4}\, \mathrm{sinc}^2\left(f \frac{T}{2}\right) \sin^2\left(\pi f \frac{T}{2}\right) + \frac{A^2}{4} \sum_\tau \mathrm{sinc}^2\left(\frac{k}{2}\right) \sin^2\left(\frac{\pi k}{2}\right) \delta\left(f - \frac{k}{T}\right)$$

$$P_s = \int_{-\infty}^{+\infty} S_s(f)\, df = \frac{A^2}{4} + \frac{A^2}{4} \sum_k \mathrm{sinc}^2\left(\frac{k}{2}\right) \sin^2\left(k \frac{\pi}{2}\right) = \frac{A^2}{4} + \frac{A^2}{4} = \frac{A^2}{2}$$

ESERCIZIO PAM 4 LIVELLI



$$r(t) = \sum_i a_i \, A \, p(t - iT) + n(t)$$

$$a_i \in \{ \pm 3 \, ; \pm 1 \}$$

$n(t)$ AWGN $\qquad S_n(f) = \frac{N_0}{2} \quad \forall f$

$$P(f) = \begin{cases} \sqrt{T}\,\sqrt{1 - |fT|} & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$

$$\hat{a}_k = \begin{cases} +3 & \text{se } y[k] \ge 2 \\ 1 & \text{se } 0 \le y[k] < 2 \\ -1 & \text{se } -2 \le y[k] < 0 \\ -3 & \text{se } y[k] < -2 \end{cases}$$

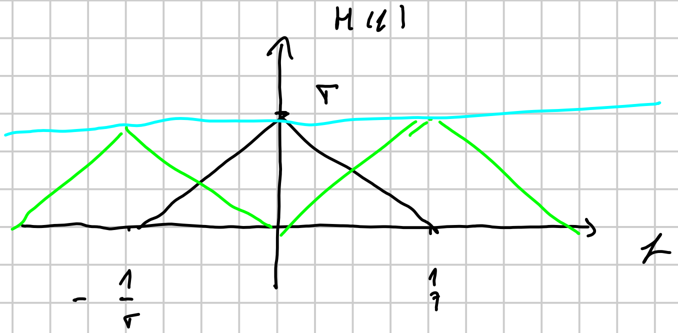
$SNR = \frac{E_s}{N_0}$?

$P_E(b)$?

$$y(t) = \sum_i a_i \, h(t - iT) + n_u(t)$$

$$h(t) = p(t) \otimes p(-t)$$

$$H(f) = P(f) \cdot P^*(f) = |P(f)|^2 = \begin{cases} T\,(1 - |fT|) & |fT| \le 1 \\ 0 & \text{altrove} \end{cases}$$



$$\sum_k H\left(f - \frac{k}{T}\right) = T \Rightarrow h(n) = 1$$

$$n_n(t) = n(t) \otimes h_R(t) = \frac{1}{A}\, n(t) \otimes p(-t)$$

$$\sigma^2_{nn} = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H_R(f)|^2\, df = \frac{N_0}{2A^2}\int_{-\infty}^{+\infty} \underbrace{|P(f)|^2}_{H(f)}\, df =$$

$$= \frac{N_0}{2A^2}\, h(0) = \frac{N_0}{2A^2}$$

$$E_s = E\left\{\int_0^T s^2(t)\, dt\right\} = A^2 E\left\{\int_0^T \sum_i \sum_m a_i a_m p(t-iT)p(t-mT)\,dt\right\} =$$

$$= A^2 \sum_i \sum_m E\{a_i a_m\} \int_0^T p(t-iT)\, p(t-mT)\, dt =$$

$$E\{a_i a_m\} = \begin{cases} E\{a_i^2\} & \text{se } i = m \\ E\{a_i\}E\{a_m\} & \text{se } i \neq m \end{cases} =$$

$$E\{a_i\} = \frac{1}{4}(-3) + \frac{1}{4}(-1) + \frac{1}{4}(1) + \frac{1}{4}(3) = 0$$

$$E\{a_i^2\} = \frac{1}{4}(-3)^2 + \frac{1}{4}(-1)^2 + \frac{1}{4}(1)^2 + \frac{1}{4}(3)^2 = 5$$

$$E\{a_i a_m\} = \begin{cases} \sigma & i = m \\ 0 & \text{altrimenti} \end{cases}$$

$$E_s = \sigma A^2 \sum_i \int_0^T p^2(t - iT) = \sigma A^2 \int_{-\infty}^{+\infty} p^2(t)\, dt =$$

$$= \sigma A^2 \int_{-\infty}^{+\infty} p^2(t)\, dt = \sigma A^2$$

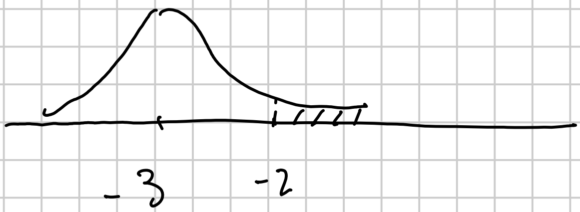
$$P_e(b) = \frac{1}{4} P_r\{e \mid a_k = -3\} + \frac{1}{4} P_r\{e \mid a_k = -1\} +$$

$$+ \frac{1}{4} P_r\{e \mid a_k = 1\} + \frac{1}{4} P_r\{e \mid a_k = 3\}$$

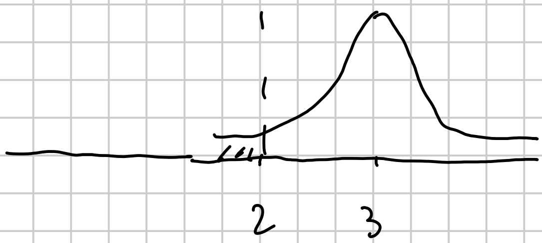
$$P_r\{e \mid a_k = -3\}$$

$$y_k \mid a_k = -3 \in \mathcal{N}(-3, \sigma^2_{nn})$$

$$P_r\{-3 + n_k > 2\} = Q\left(\frac{\lambda - \eta}{\sigma_{nn}}\right) = Q\left(\frac{-2 + 3}{\sigma_{nn}}\right) =$$



$$= Q\left(\frac{1}{\sigma_{nn}}\right)$$



$$\Pr\{e \mid a_k = 1\} = \Pr\{e \mid a_k = -1\} =$$

$$= \Pr\{-1 + n_k > 0 \,,\; -1 + n_k \leq -2\} =$$

η : v.a

λ : soglia

$$= \Pr\{-1 + n_k > 0\} + \Pr\{-1 + n_k \leq -2\} =$$

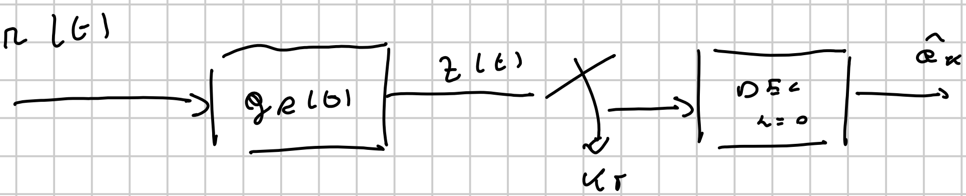
$$= Q\left(\frac{0+1}{\sigma_{nn}}\right) + Q\left(\frac{2-1}{\sigma_{nn}}\right) = 2\,Q\left(\frac{1}{\sigma_{nn}}\right)$$

$$P_e = \frac{1}{4}\; 6\, Q\left(\frac{1}{\sigma_{nn}}\right) = \frac{3}{2}\, Q\left(\frac{1}{\sigma_{nn}}\right)$$

$$\sigma^2_{nn} = \frac{N_0}{2A^2} = \frac{5}{5}\,\frac{N_0}{2A^2} = \frac{5}{2}\,\frac{N_0}{E_s}$$

$$P_e(b) = \frac{3}{2}\, Q\left(\sqrt{\frac{2}{5}\,\frac{E_s}{N_0}}\right)$$

ESERCIZIO 1



$$r(t) = \sum_i a_i \, g_T(t - iT) + w(t)$$

$w(t)$ AWGN $\qquad S_w(f) = \frac{N_0}{2}$

$a_i \in \{-1, 1\}$ ind. equiprobabili

$$g_T(t) = e^{-|t|/T} \, \mathrm{rect}\left(\frac{t}{T}\right)$$

$$g_R(t) = g_T(t)$$

1) Energia media di $r(t)$

2) Verificare assenza ISI

3) Potenza media di rumore dopo $g_R(t)$

4) BER ($P_E(b)$)

$$E_s = E\left\{ \int_0^T r^2(t)\, dt \right\} = E\left\{ \int_0^T \sum_i a_i \, g_T(t - iT) \sum_n a_n \, g_T(t - nT)\, dt \right\} =$$

$$= \int_0^T \sum_i \sum_n E\{a_i a_n\} \, g_T(t - iT) \, g_T(t - nT) \, dt$$

$$E\{a_i a_n\} = \begin{cases} E\{a_i^2\} = 1 & i = n \\ E^2\{a_i\} = 0 & i \neq n \end{cases}$$

$$E\{a_i^2\} = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$$

$$E\{a_i\} = \frac{1}{2}\,1 + \frac{1}{2}(-1) = 0$$

$$= \sum_i \int_0^T g_T^2(t - iT)\,dt = \int_{-\infty}^{+\infty} g_T^2(t)\,dt =$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{2|t|/T} = 2\int_0^{\frac{T}{2}} e^{-2t/T}\,dt =$$

$$= 2\left(-\frac{2}{T}\right) e^{-2t/T}\Big|_0^{\frac{T}{2}} = \frac{2}{T}\left(1 - \frac{1}{e}\right) = E_s$$

2)

$$z(t) = r(t) \otimes g_R(t) = \sum_i a_i\, g(t - iT) + n(t)$$

$$g(t) = g_T(t) \otimes g_R(t)$$

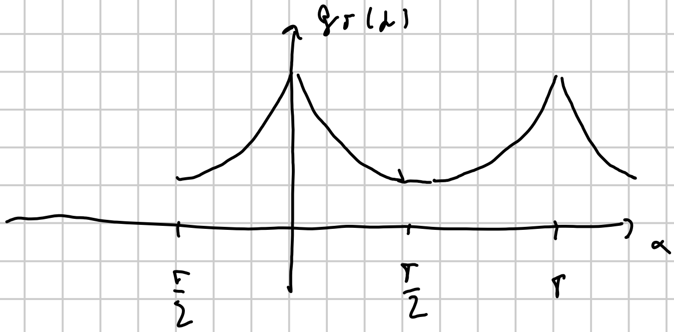
$$n(t) = w(t) \otimes g_R(t)$$

NO ISI SE $g(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$

$$g(t) = \int_{-\infty}^{+\infty} g_T(\alpha)\, g_T(t - \alpha)\, d\alpha$$

$$g(0) = \int_{-\infty}^{+\infty} g(\alpha)\, g(-\alpha)\, d\alpha = \int_{-\infty}^{+\infty} g_T^2(\alpha)\, d\alpha$$

$$g(kT) = \int_{-\infty}^{+\infty} g_T(\alpha)\, g_T(kT - \alpha)\, d\alpha = 0$$



MO ISI

3) σ^2_{nn} ?

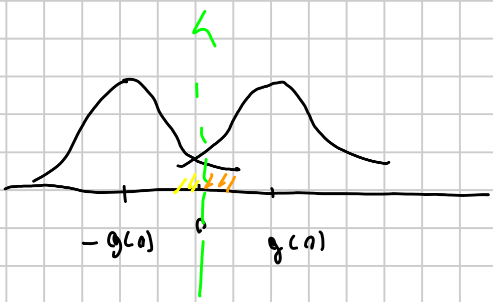
$$\sigma^2_{nn} = \frac{N_0}{2}\int_{-\infty}^{+\infty} |G_R(f)|^2\, df = \frac{N_0}{2}\, g(0)$$

4)

$$z_k = a_k\, g(0) + n_k$$

$$P(e) = \frac{1}{2} P_r\{e \mid a_k = -1\} + \frac{1}{2} P_r\{e \mid a_k = 1\} =$$

$$= P_r\{g(0) + n_k < 0 \mid a_k = 1\} = Q\left(\frac{g(0)}{\sqrt{\frac{N_0}{2} g(0)}}\right) =$$



$$= Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{SNR}\right)$$

ESERCIZIO 1

$$x(t) \longrightarrow \boxed{G_R(f)} \xrightarrow{z(t)} \underset{kT}{\times} \xrightarrow{z_k} \boxed{DEC} \longrightarrow \hat{a}_k$$

$$x(t) = \sum_i a_i \, g_T(t - iT) + w(t)$$

$$G_T(f) = \sqrt{T} \cos\left(\pi f \frac{T}{2}\right) \mathrm{rect}\left(f \frac{T}{2}\right)$$

$a_i \in \{-1, 1\}$ ind. eq.

$w(t)$ GAUSSIANO con $S_w(f) = \frac{N_0}{2} \mathrm{rect}\left(f \frac{T}{8}\right)$

⚠ NO AWGN

1) $G_R(f)$ OTTIMO?

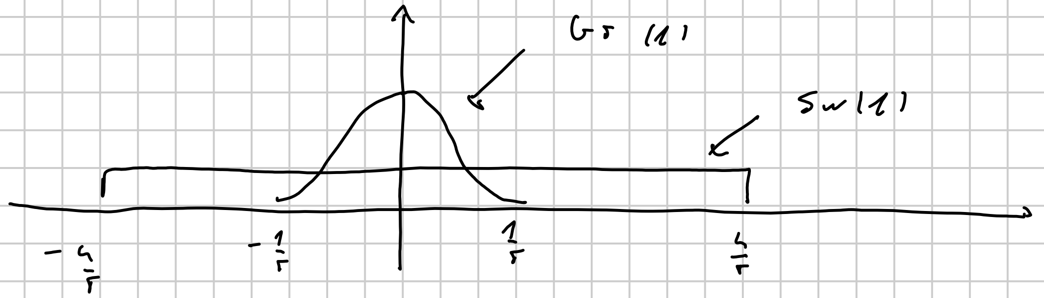
↳ NO ISI

MAX SNR

2) $P(e \mid a = -1)$

$$G_T(f) = \sqrt{T} \cos\left(\pi f \frac{T}{2}\right) \mathrm{rect}\left(\frac{f}{2/T}\right)$$

$$S_w(f) = \frac{N_0}{2} \mathrm{rect}\left(\frac{f}{8/T}\right)$$



IL RUMORE È BIANCO NELLA BANDA DEL SISTEMA

IL FILTRO OTTIMO È IL FILTRO ADATTATO

$$g_R(t) = g_T(-t) = g_T(t)$$

$$\downarrow$$

$G_T(f)$ REALE E PARI

$$G_R(f) = G_T^*(f) = G_T(f)$$

$$z(t) = \sum_i a_i \, g(t - iT) + n(t)$$

$$g(t) = g_T(t) \otimes g_R(t) = g_T(t) \otimes g_T(t)$$

$$n(t) = w(t) \otimes g_R(t)$$

$$G(f) = T \cos^2\left(\pi f \frac{T}{2}\right) \operatorname{rect}\left(f \frac{T}{2}\right) =$$

$$= \frac{T}{2}\left(1 + \cos(\pi f T)\right) \operatorname{rect}\left(f \frac{T}{2}\right)$$

$$g(kT) = \begin{cases} g(0) = 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

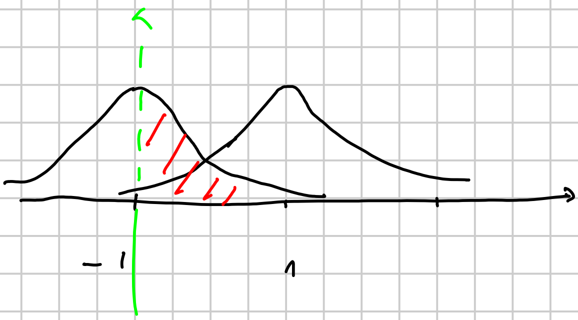
$$S_n(f) = S_w(f)\,|G_T(f)|^2 = \frac{N_0}{2}\,|G_T(f)|^2$$

$$\sigma_{nn}^2 = \frac{N_0}{2}\int_{-\infty}^{+\infty} |G_T(f)|^2 \, df = \frac{N_0}{2}\, g(0) = \frac{N_0}{2}$$

$$z_k = a_k + n_k$$

$$z_k \mid a_k = 1 \in \mathcal{N}(1, \sigma_n^2)$$

$$z_k \mid a_k = -1 \in \mathcal{N}(-1, \sigma_n^2)$$

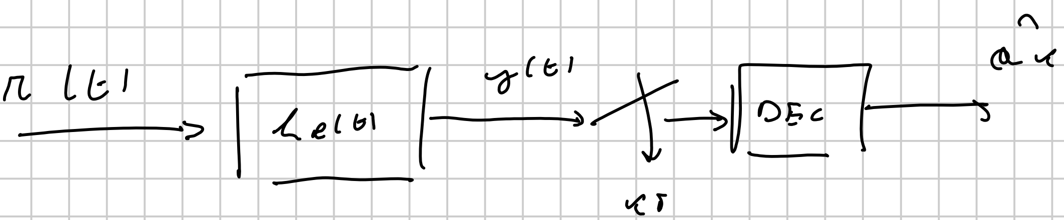


$$\Pr\{e \mid a_k = 1\} = \Pr\{z_k < \lambda \mid a_k = 1\} =$$

$$= \Pr\{1 + n_k < \lambda \mid a_k = 1\} = Q\left(\frac{1-\lambda}{\sigma_n}\right)$$

$$\Pr\{e \mid a_k = -1\} = \frac{1}{2} = Q(0)$$

ESERCIZIO x CASA



$a_i \in \{-e \,;\, 2e\}$ indipendenti equiprobabili

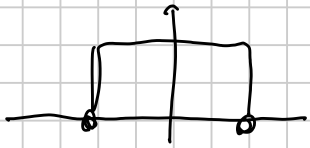
$$r(t) = \sum_i a_i \, p(t - iT) + n(t)$$

$n(t)$ AWGN $S_n(f) = \frac{N_0}{2}$

$P(f) = T\,|f|\,e^{-|f|}\,\text{rect}\left(f\frac{T}{2}\right)$

$\text{rect}(x) = 0$

$x = \pm\frac{1}{2}$



$H_R(f) = e^{|f|}\,\text{rect}\left(f\frac{T}{2}\right)$

$$\hat{a}_n = \begin{cases} -e & x_n < \lambda \\ 2e & x_n > \lambda \end{cases}$$

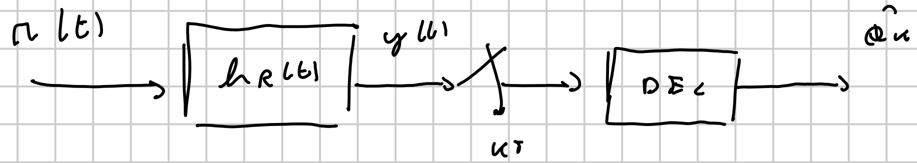
$\lambda = \frac{3}{2}e$

E_s ?

$P_r\{e\}$

λ ott. (SOGLIA CHE MINIMIZZA LA $P_r\{e\}$)

ESERCIZIO 1



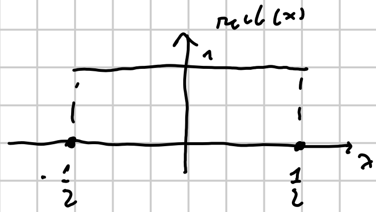
$a_i \in \{-e \,;\, 2e\}$ indipendenti ed equiprobabili

$$r(t) = \sum_i a_i \, p(t - iT) + n(t) = s(t) + n(t)$$

$$S_n(f) = \frac{N_0}{2} \quad \forall f$$

$$P(f) = T \, |f| \, e^{-|f|} \, \mathrm{rect}\left(f \frac{T}{2}\right)$$

$\mathrm{rect}(x) = 0$ se $x = \pm \frac{1}{2}$



$$H_R(f) = e^{|f|} \, \mathrm{rect}\left(f \frac{T}{2}\right)$$

$$\hat{a}_k = \begin{cases} -e & \text{se } x_k \le \lambda \\ 2e & \text{se } x_k > \lambda \end{cases}$$

$$\lambda = \frac{3}{2} e$$

1) E_s ?

2) $P_E(b)$?

3) λ_{ott} ?

1) $E_s = P_r \, T$

$$P_r = \int_{-\infty}^{+\infty} S_s(f) \, df$$

$$S_s(f) = \frac{1}{T} S_a(f) \, |P(f)|^2$$

$$S_a(f) \hat{=} \sum_{m} R_a[m]\, e^{-j 2\pi f m T}$$

$$R_a[m] = E\{a_i\, a_{i+m}\} = \begin{cases} E\{a_i^{2}\} & m = 0 \\ E\{a_{i+m}\}\, E\{a_i\} & m \neq 0 \end{cases}$$

$$E\{a_i\} = \frac{1}{2}(-e) + \frac{1}{2}(2e) = \frac{e}{2}$$

$$E\{a_i^{2}\} = \frac{1}{2}(-e)^2 + \frac{1}{2}(2e)^2 = \frac{5}{2} e^2$$

$$R_a[m] = \begin{cases} \frac{5}{2} e^2 & m = 0 \\ \frac{e^2}{4} & m \neq 0 \end{cases} = \frac{5}{2} e^2\, \delta[m] + \frac{1}{4} e^2$$

$$S_a(f) = \frac{5}{2} e^2 + \frac{e^2}{4T} \sum_{k} \delta\left(f - \frac{k}{T}\right)$$

$$S_s(f) = \frac{1}{T} |P(f)|^2 \left[\frac{5}{2} e^2 + \frac{e^2}{4T} \sum_{k} \delta\left(f - \frac{k}{T}\right) \right]$$

$$E_s = \int_{-\infty}^{+\infty} |P(f)|^2 \left[\frac{5}{2} e^2 + \frac{e^2}{4T} \sum_{k} \delta\left(f - \frac{k}{T}\right) \right] df =$$

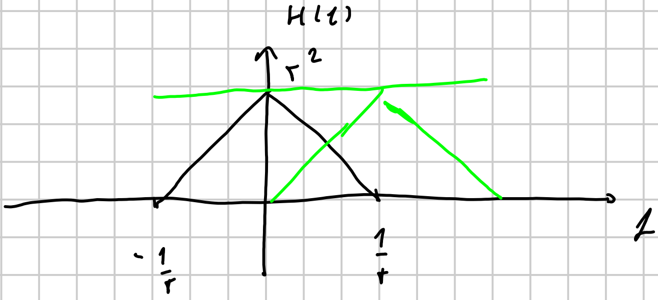
$$= \frac{5}{2} e^2 \underbrace{\int_{-\infty}^{+\infty} |P(f)|^2\, df}_{1} + \frac{e^2}{4T} \sum_{k} \underbrace{\int_{-\infty}^{+\infty} |P(f)|^2\, \delta\left(f - \frac{k}{T}\right) df}_{2} =$$

(1) $$\int_{-\infty}^{+\infty} |P(f)|^2\, df = \frac{1}{4}\left[1 - e^{-2/T}\left(1 + \frac{2}{T} + \frac{2}{T^2}\right)\right] = E_p$$

(2) $$\int_{-\infty}^{+\infty} |P(f)|^2\, \delta\left(f - \frac{k}{T}\right) df = |P(0)|^2 = 0$$

$$E_s = \frac{9}{5} a^2 E_p$$

$$H(f) = P(f)\, H_R(f) = T^2 |f| \,\mathrm{rect}\left(f \frac{T}{2}\right)$$



$$\sum_k H\left(f - \frac{k}{T}\right) = T^2$$

$$\Downarrow$$

$$h(0) = T$$

NO ISI

$$y_k = h(0)\, a_k + n_k = T a_k + n_k$$

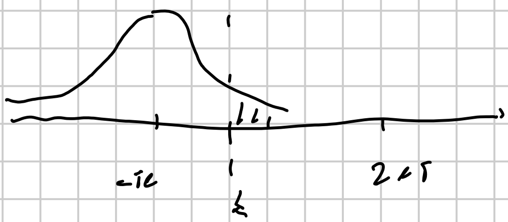
$$n_n(t) = n(t) \otimes h_R(t)$$

$$S_{nn}(f) = \frac{N_0}{2} |H_R(f)|^2$$

$$\sigma_{nn}^2 = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2 \, df = \frac{N_0}{2}\, 2 \int_0^{+\infty} e^{2f} \, df = \frac{N_0}{2} \left(e^{2/T} - 1\right)$$

$$P_E(b) = \frac{1}{2} \Pr\{e \mid a_k = -a\} + \frac{1}{2} \Pr\{e \mid a_k = 2a\}$$

$$\Pr\{e \mid a_k = -a\} = Q\left(\frac{\lambda + aT}{\sigma_{nn}}\right)$$



$$P_r\{e \mid a_k = 2e\} = Q\left(\frac{h + 2e\sigma}{\sigma_{nn}}\right)$$

$$P_E(h) = \frac{1}{2} Q\left(\frac{-h + 2e\sigma}{\sigma_{nn}}\right) + \frac{1}{2} Q\left(\frac{h + e\sigma}{\sigma_{nn}}\right)$$

$$h = \frac{3}{2} e$$

3) h att ?

$$\frac{d P_E}{d h} = 0$$

$$Q'(x) \hat{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\frac{d P_E(h)}{d h} = -\frac{1}{2} \cancel{\left(+\frac{1}{\sqrt{2\pi}}\right)} e^{-\frac{(-h + 2e\sigma)^2}{2\sigma^2_{nn}}} +$$

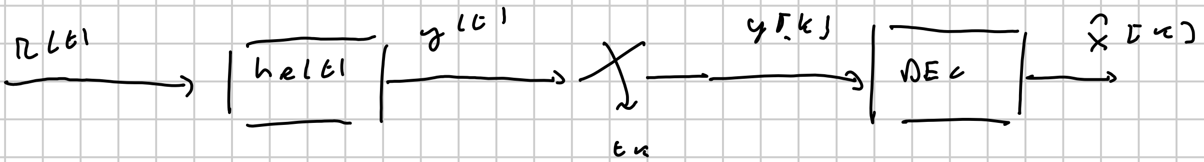
$$+ \frac{1}{2} \cancel{\left(\frac{1}{\sqrt{2\pi}}\right)} e^{-\frac{(h + e\sigma)^2}{2\sigma^2_{nn}}} = 0$$

$$e^{-\frac{(-h + 2e\sigma)^2}{2\sigma^2_{nn}}} = e^{-\frac{(h + e\sigma)^2}{2\sigma^2_{nn}}}$$

$$(-h + 2e\sigma)^2 = (h + e\sigma)^2$$

$$\boxed{h = \frac{e}{2}\sigma}$$

ESERCIZIO 1



$x[i] \in \{0, 2\}$

$w(t)$ AWGN $\qquad S_w(f) = \frac{N_0}{2}$

$p(t) = 2B \,\text{sinc}(2Bt) + B \,\text{sinc}(Bt)$

$H_R(f) = \text{rect}\left(\frac{f}{2B}\right)$

$$\hat{x}_k = \begin{cases} 0 & \text{se } y[k] \le \lambda \\ 2 & \text{se } y[k] > \lambda \end{cases} \qquad \lambda = 1$$

1) E_s

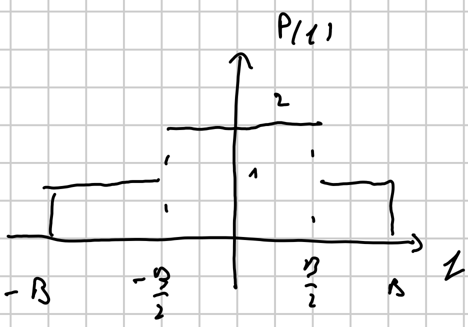
2) ISTANTE DI CAP. OTTIMO (NO ISI)

3) $P_E(b)$

$p(t) = 2B \,\text{sinc}(2Bt) + B \,\text{sinc}(Bt)$

$P(f) = \text{rect}\left(\frac{f}{2B}\right) + \text{rect}\left(\frac{f}{B}\right)$

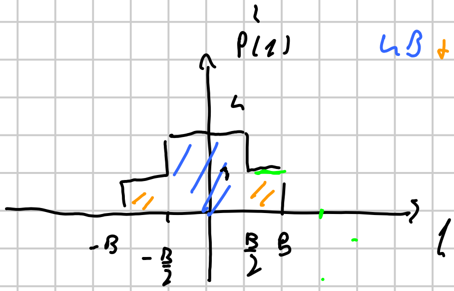
$H_R(f) = \text{rect}\left(\frac{f}{2B}\right)$



1)

$$E_s = \left[\frac{1}{2}(2)^2 + \frac{1}{2}(0)^2\right] E_p = 2E_p$$

$$E_p = \int_{-\infty}^{+\infty} |P(f)|^2 \, df = 5B$$

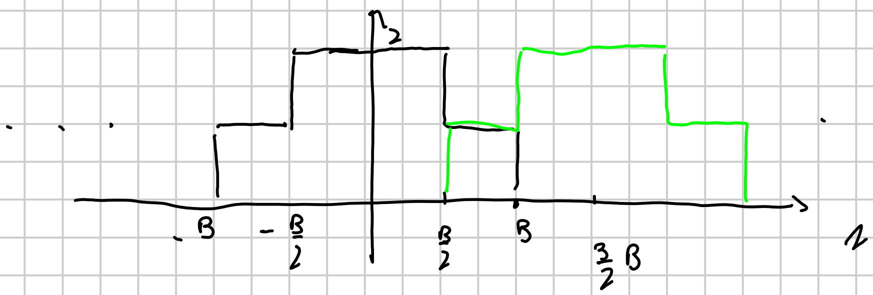


$4B + B = 5B$

$$E_s = 10B$$

2) NO ISI ⇔ COND. NYQ.

$$\sum_m H\left(f - \frac{m}{T_c}\right) = T_c \, h(0)$$



$\boxed{T_c = \frac{2}{3B}}$ ⇒ $t_k = k \frac{2}{3B}$

VERIFICHIAMO NYQ NEL TEMPO

$$h(t) = p(t) = 2B \, \mathrm{sinc}(2Bt) + B \, \mathrm{sinc}(Bt)$$

$$h[n] = 2B \, \mathrm{sinc}\left(2B \frac{2}{3B} n\right) + B \, \mathrm{sinc}\left(B \frac{2}{3B} n\right) =$$

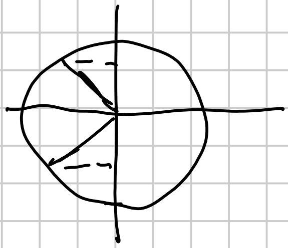
$$= \begin{cases} h[0] & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$h[n] = 2B \frac{\sin\left(\frac{4}{3} \pi n\right)}{\frac{4}{3} \pi n} + B \frac{\sin\left(\frac{2}{3} \pi n\right)}{\frac{2}{3} \pi n} =$$

$$= \frac{3}{2}\,\frac{B}{\pi i}\left[\sin\left(\frac{4}{3}\pi i\right) + \sin\left(\frac{2}{3}\pi i\right)\right]$$

$$h(n) = 3B$$

$$h[m]\Big|_{m \neq 0} = 0 \qquad \sin\left(\frac{4}{3} \pi i\right) = -\sin\left(\frac{2}{3}\pi i\right)$$



3)

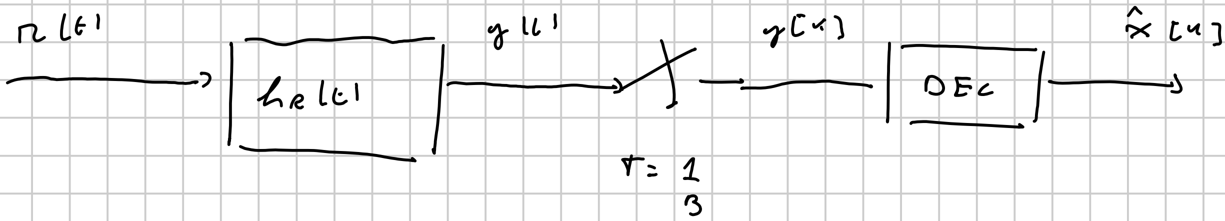
$$S_{nn}(f) = \frac{N_0}{2}\int_{-\infty}^{+\infty} |H_R(f)|^2\, df = N_0 B$$

$$h(0) = 3B$$

$$y[k] = 3B\, x[k] + n[k]$$

$$P_b(e) = \underbrace{\frac{1}{2}\, Q\left(\frac{1}{\sqrt{N_0 B}}\right)}_{P(e \mid a_k = 0)} + \underbrace{\frac{1}{2}\, Q\left(\frac{\cancel{B}B - 1}{\sqrt{N_0 B}}\right)}_{P(e \mid a_k = 2)}$$

ESERCIZIO



$$s(t) = \sum_k x[k]\, p(t - kT)$$

$x[k] \in \{-2\ ;\ +3\}$ ind. equip.

$$p(t) = 2B\ \mathrm{sinc}(2Bt)$$

$$\boxed{c(t) = \mathrm{sinc}^2(Bt)} \qquad C(f) \neq 1$$

$$S_n(f) = \frac{N_0}{2} \qquad \text{AWGN}$$

$$h_R(t) = 2B\ \mathrm{sinc}(2Bt)$$

1) E_s

2) P_{nu} (Potenza media di rumore dopo $h_R(t)$)

3) $P_E(b)$

1) $$E_s = \frac{1}{2} E_{s_1} + \frac{1}{2} E_{s_2}$$

$$E_{s_1} = \int_{-\infty}^{+\infty} s_1^2(t)\, dt = \int_{-\infty}^{+\infty} \left(-2\, p(t)\right)^2$$

$$E_{s_2} = \int_{-\infty}^{+\infty} \left(+3\, p(t)\right)^2 dt$$

$$E_p = \int_{-\infty}^{+\infty} p^2(t)\, dt = \int_{-\infty}^{+\infty} P^2(f)\, df = 2B$$

$$E_s = \frac{1}{2} 4 E_p + \frac{1}{2} 9 E_p = 13 B$$

2) $y(t) = r(t) \otimes h_R(t) =$

$$= \sum_k x[k]\, h(t - kT) + w(t)$$

$$h(t) = p(t) \otimes c(t) \otimes h_R(t)$$

$$P_{nn} = \sigma^2_{nn} = \int_{-\infty}^{+\infty} S_w(f)\, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2\, df = \frac{N_0}{2} 2B = N_0 B$$

3) $P_E(b)$

Verifica Nyq.

$$h(t) = p(t) \otimes c(t) \otimes h_R(t)$$

$$\Updownarrow$$

$$H(f) = P(f)\, C(f)\, H_R(f) = \underbrace{\mathrm{rect}\left(\frac{f}{2B}\right)}_{P(f)} \underbrace{\frac{1}{B}\left[\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right)\right]}_{C(f)} \cdot$$

$$\cdot \underbrace{\mathrm{rect}\left(\frac{f}{2B}\right)}_{H_R(f)} =$$

$$= \frac{1}{B}\left(1 - \frac{|f|}{B}\right) \mathrm{rect}\left(\frac{f}{2B}\right)$$

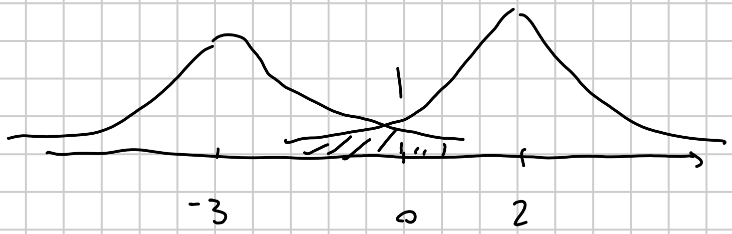
$$\sum_k H(f - kB) = \frac{1}{B} = T \quad \Rightarrow \quad h(0) = 1$$

$y_k = h(n)\, x[k] + w[k] = x[k] + w[k]$

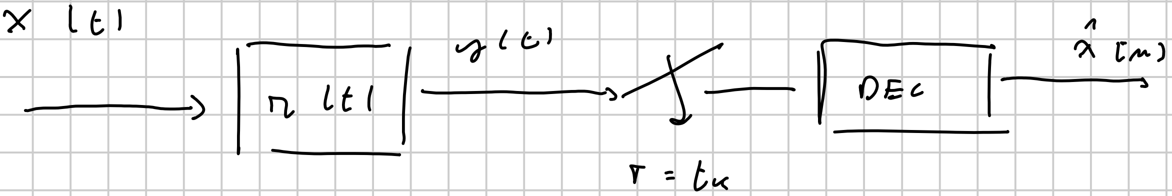
$h = 0$

$$P_E(n) = \frac{1}{2} P_r\{\hat{x}[k] = -2 \mid x[k] = 3\} + \frac{1}{2} P_r\{\hat{x}[k] = 3 \mid x[k] = 2\}$$

$$= \frac{1}{2} Q\left(\frac{3}{\sqrt{N_0 T}}\right) + \frac{1}{2} Q\left(\frac{2}{\sqrt{N_0 T}}\right)$$

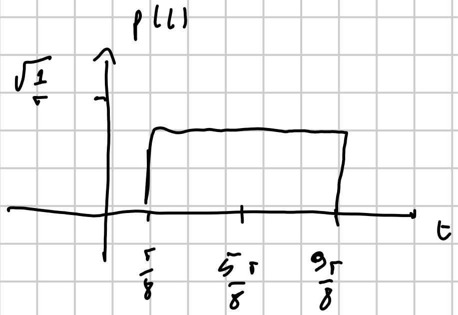


ESERCIZIO



$x[k] \in \{0, +1\}$

$$p(t) = \begin{cases} \sqrt{\frac{1}{T}} & \frac{T}{8} < t < \frac{T}{8} + T \\ 0 & \text{altrove} \end{cases}$$



$S_n(f) = \frac{N_0}{2}$ AWGN $h = \frac{1}{4}$

1) E_s

2) $r(t) = p(t_0 - t)$ determinare t_0 $r(t)$ causale

3) P_{nn} in uscita da $r(t)$

4) $t_{k\,ott}$

1)

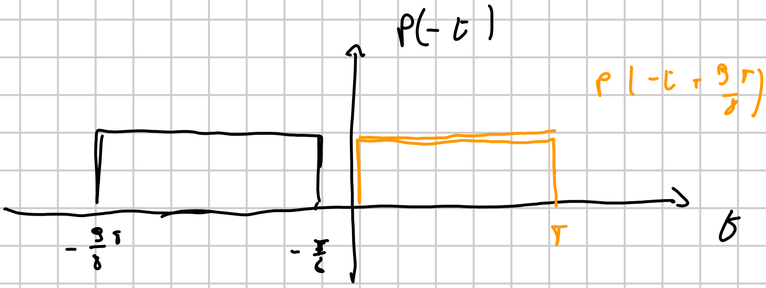
$$E_s = \frac{1}{2} E_{s_0} + \frac{1}{2} E_{s_1} = 0 + \frac{1}{2}(1) = \frac{1}{2}$$

2)

$$r(t) = A\, p(t_0 - t)$$

$$p(t) = \sqrt{\frac{1}{T}}\ \mathrm{rect}\left(\frac{t - 5T/8}{T} \right)$$

$p(-t)$



$$r(t) = p\left(-t + \frac{9T}{8}\right)$$

$$\sigma^2_{nn} = \frac{N_0}{2} \int_{-\infty}^{+\infty} |R(f)|^2\, df = \frac{N_0}{2} \int_{-\infty}^{+\infty} r^2(t) = A^2 \frac{N_0}{2}$$

4)

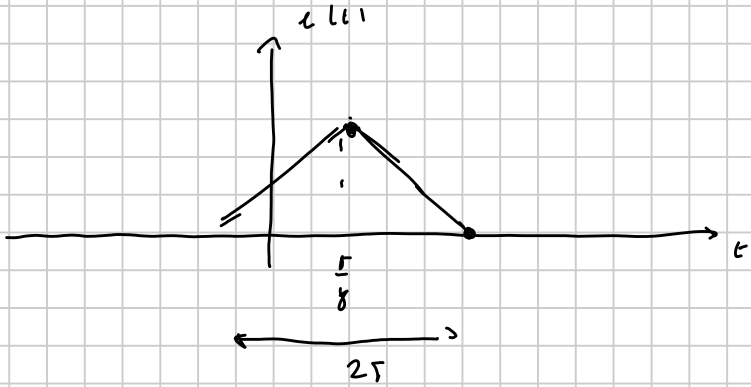
$$H(f) = P(f)\, R(f)$$

$$P(f) = \sqrt{\frac{1}{T}}\ T\, \mathrm{sinc}(fT)\, e^{-j 2\pi f 5\frac{T}{8}}$$

$$R(f) = A\, P(f)\, e^{j2\pi f\, 3T/8} = A\sqrt{T}\,\mathrm{sinc}(fT)\, e^{j2\pi f \frac{T}{2}}$$

$$H(f) = A\,T\,\mathrm{sinc}^2(fT)\, e^{-j2\pi f \frac{T}{8}}$$

$$h(t) = A\left(1 - \frac{|t - T/8|}{T}\right)\mathrm{rect}\left(\frac{t - T/8}{2T}\right)$$



$$t_k = \frac{T}{8} + kT$$

$P_E(b)$

$$y[k] = A\,x[k] + n_k$$

$$P(e \mid x[k]=0) = Q\left(\frac{1/4}{\sqrt{A^2 N_0/2}}\right) = Q\left(\sqrt{\frac{1}{8A^2N_0}}\right)$$

$$P(e \mid x[k]=1) = Q\left(\sqrt{\frac{(A - 1/4)^2}{A^2N_0/2}}\right)$$

A t.c. $P(e)$ minima

$$P\{e \mid x[k]=0\} = P\{e \mid x[k]=1\}$$

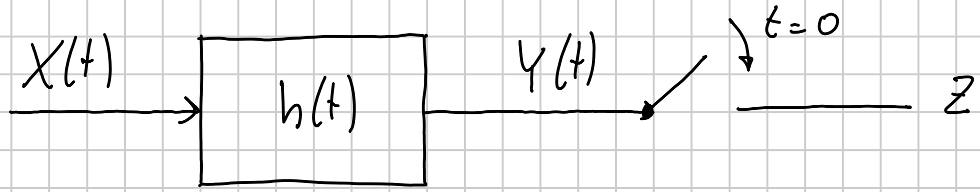
$A : \frac{1}{2}$

29/05/2013

$$X(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) + B\,\mathrm{rect}\left(\frac{t-T/2}{T}\right)$$

A, B sono V.A. indip.

$A \in \mathcal{U}[0,1]$, $B \in \mathcal{U}[0,2]$



$$h(t) = \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right)$$

·) Calcolare la $f_Z(z)$

$$Y(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) \otimes \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right) + \underbrace{B\,\mathrm{rect}\left(\frac{t-T/2}{T}\right)} \otimes \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right)$$

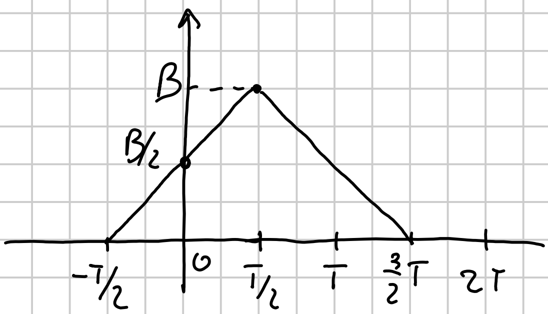
$$= A\,\frac{1}{\cancel{T}} \cdot \cancel{T}\left(1 - \frac{|t|}{T}\right)\mathrm{rect}\left(\frac{t}{2T}\right) +$$

$$+ B \cdot \frac{1}{\cancel{T}} \cdot \cancel{T}\left(1 - \frac{|t-T/2|}{T}\right)\mathrm{rect}\left(\frac{t-T/2}{2T}\right)$$

$$\left[B\,\mathrm{rect}\left(\frac{t}{T}\right) \otimes \delta(t - T/2)\right] \otimes \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$\left[B\,\mathrm{rect}\left(\frac{t}{T}\right) \otimes \frac{1}{T}\,\mathrm{rect}\left(\frac{t}{T}\right)\right] \otimes \delta(t - T/2)$$

$$Z = Y(0) = A + \frac{B}{2}$$

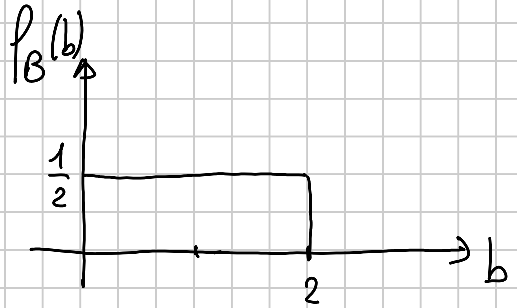


$$Z = A + \frac{B}{2}$$

$Z = X + Y$, X, Y sono v.A. indipendenti

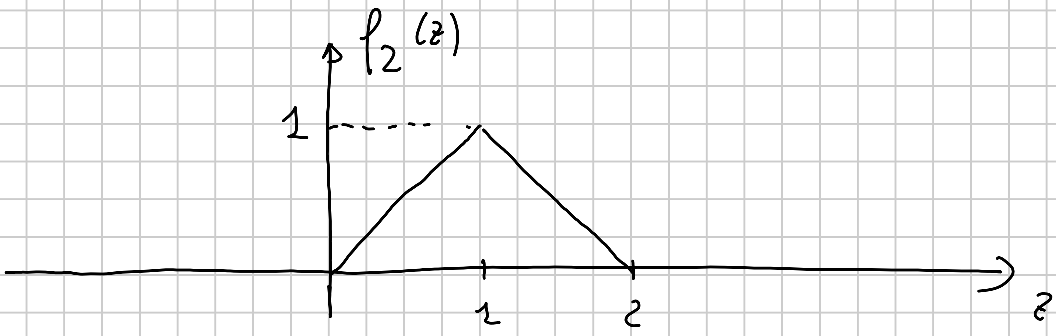
$$f_Z(z) = f_X(z) \otimes f_Y(z)$$

$$C = \frac{B}{2} \Rightarrow f_C(c) = \frac{f_B(b)}{|g'(b)|}\Bigg|_{b=g^{-1}(c)} = \frac{\cancel{\tfrac{1}{2}}\,\mathrm{rect}\left(\frac{b-1}{2}\right)}{\cancel{\tfrac{1}{2}}}\Bigg|_{b=g^{-1}(c)}$$



$$f_C(c) = \mathrm{rect}\left(\frac{2c-1}{2}\right) = \mathrm{rect}\left(\frac{c-1/2}{1}\right) = f_A(a)$$

$$f_Z(z) = \mathrm{rect}\left(\frac{z-1/2}{1}\right) \otimes \mathrm{rect}\left(\frac{z-1/2}{1}\right) = \left(1 - \frac{|z-1|}{1}\right)\mathrm{rect}\left(\frac{z-1}{2}\right)$$

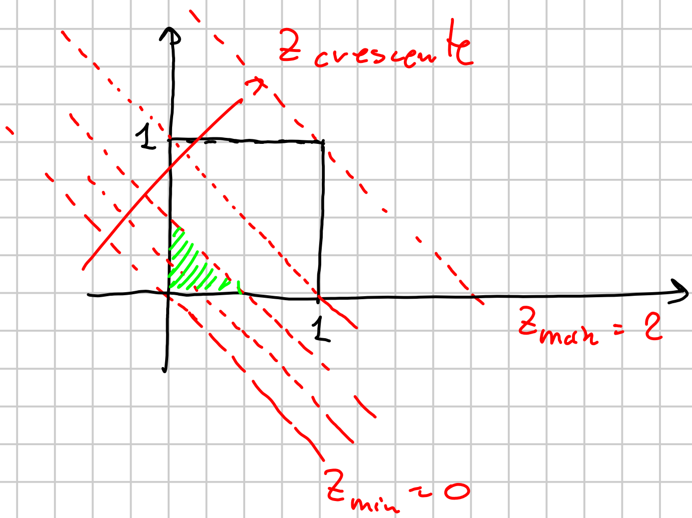


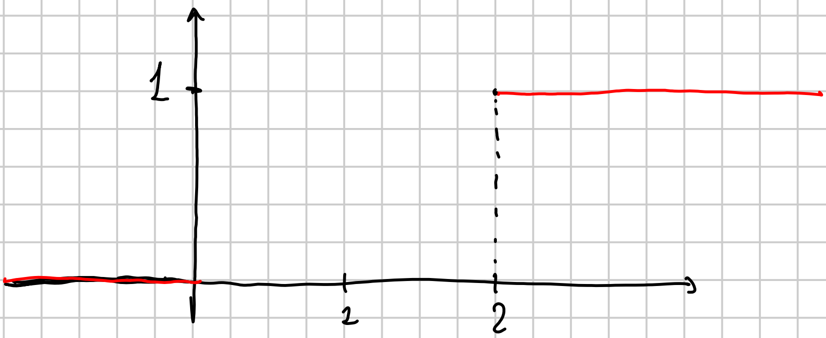
SOL. ALTERNATIVA

$$Z = A + \frac{B}{2}$$

$$F_Z(z) = P\left\{ Z \le z \right\} = P\left\{ A + \frac{B}{2} \le z \right\}$$

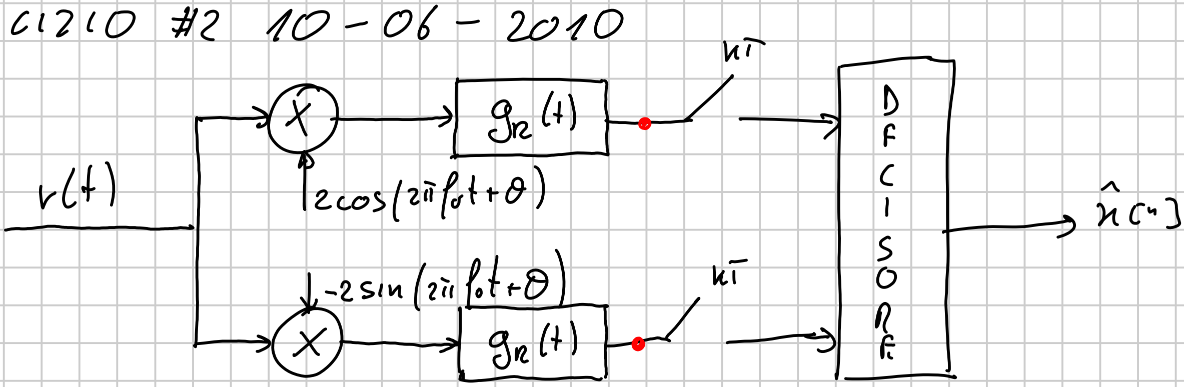
$$= P\left\{ A + C \le z \right\}$$



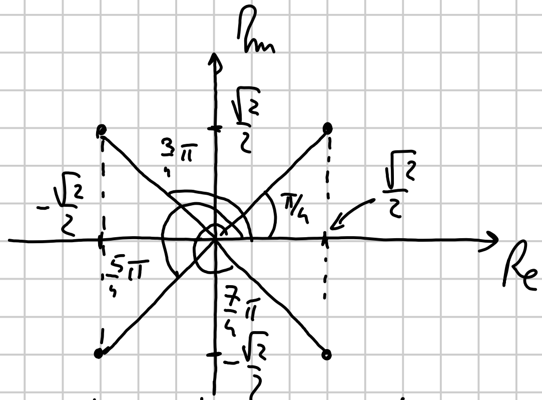


$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

ESERCIZIO #2 10-06-2010



$f_0 \gg B$



4 - QPSK

4 - QAM

$$a_i \in A_s^{(c)} \left\{ \pm \frac{\sqrt{2}}{2} \right\}$$

$$b_i \in A_s^{(s)} \left\{ \pm \frac{\sqrt{2}}{2} \right\}$$

simboli ind. ed equiprob.

$$r(t) = \sum_i a_i \, g_T(t-iT) \cos(2\pi f_0 t) - \sum_i b_i \, g_T(t-iT) \sin(2\pi f_0 t) + w(t)$$

$w(t)$ Gaussiano e bianco nella banda del segnale

Ip:

1) $g_T(t) = \left(\frac{t + T/2}{T}\right) \mathrm{rect}\left(\frac{t}{T}\right)$

2) $c(t)$ e' ideale

3) $g_R(t) = -A \left(\frac{t - T/2}{T}\right) \mathrm{rect}\left(\frac{t}{T}\right)$

Domande

1) $E_S = E\left[\int_{-\infty}^{+\infty} s_i^2(t) \, dt \right]$

$$s_i(t) = a_i \, g_T(t-iT) \cos(2\pi f_0 t) - b_i \, g_T(t-iT) \sin(2\pi f_0 t)$$

$$E_s = E\Big[\int_{-\infty}^{+\infty} a_i^2 g_T^2(t-iT) \cos^2(2\pi f_0 t) \quad +$$

$$+ \; b_i^2 g_T^2(t-iT) \sin^2(2\pi f_0 t) \quad +$$

$$- \; 2 \, a_i b_i \, g_T^2(t-iT) \cos(2\pi f_0 t) \sin(2\pi f_0 t) \, dt \Big]$$

$$= E\left[a_i^2\right] \int_{-\infty}^{+\infty} g_T^2(t-iT)\left[\frac{1}{2}+\frac{1}{2}\cos(4\pi f_0 t)\right] dt \; +$$

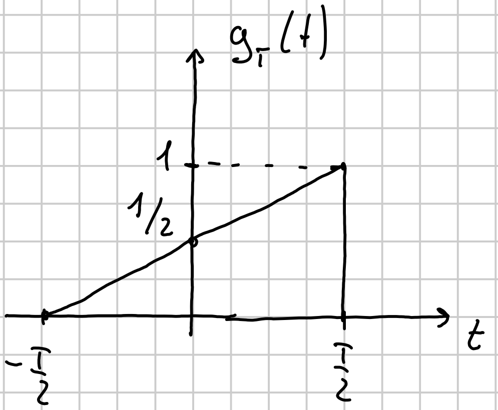
$$+ \; E\left[b_i^2\right] \int_{-\infty}^{+\infty} g_T^2(t-iT)\left[\frac{1}{2}-\frac{1}{2}\cos(4\pi f_0 t)\right] dt \; +$$

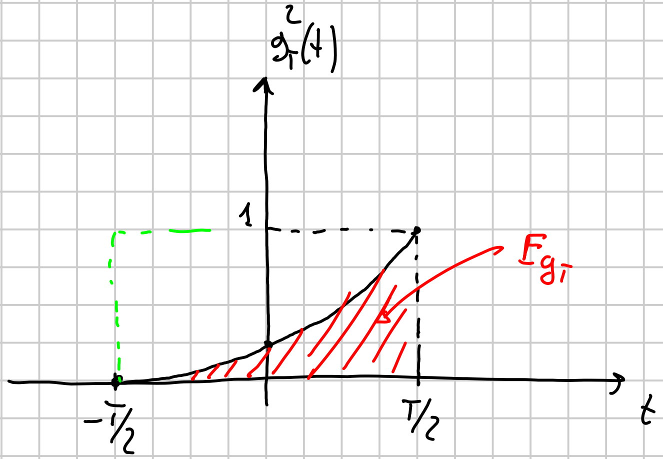
$$- \; 2\underbrace{E\left[a_i \cdot b_i\right]}_{=0} \int_{-\infty}^{+\infty} g_T^2(t-iT)\, \frac{1}{2}\sin(4\pi f_0 t)\; dt$$

$$E_S \simeq \frac{1}{2}\cdot\frac{1}{2} E_{g_T} + \frac{1}{2}\cdot\frac{1}{2} E_{g_T} = \frac{E_{g_T}}{2} = \boxed{\frac{T}{6}}$$

$$E\left[a_i \cdot b_i\right] = \underbrace{E\left[a_i\right]}_{=0} \underbrace{E\left[b_i\right]}_{=0} = 0$$

$$E_{g_T} = \int_{-\infty}^{+\infty} g_T^2(t)\, dt = \frac{T}{3}$$

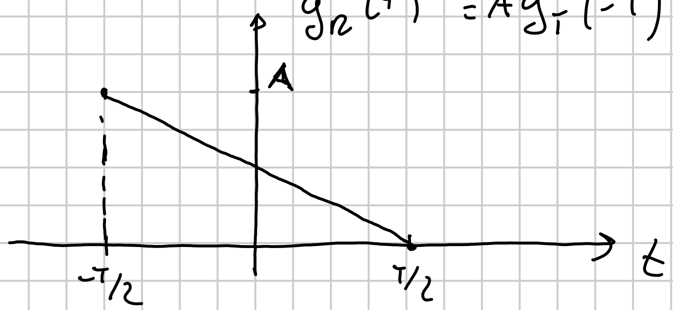




2) $h(t) = g_T(t) \otimes c(t) \otimes g_R(t) = g_T(t) \otimes g_R(t)$

$$g_R(t) = A\left(\frac{T/2 - t}{T}\right) \text{rect}\left(\frac{t}{T}\right)$$

$g_R(t) = A\, g_T(-t)$ F.A. !!



$h(0) = 1$

$$h(t)\Big|_{t=0} = \int_{-\infty}^{+\infty} g_T(\tau)\, g_R(t-\tau)\, d\tau \Big|_{t=0} =$$

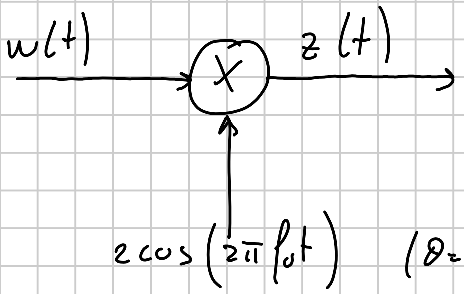
$$= A \int_{-\infty}^{+\infty} g_T(\tau)\, g_T(\tau - t)\, d\tau \Big|_{t=0} = A \int_{-\infty}^{+\infty} g_T^2(\tau)\, d\tau =$$

$$= A\, E_{g_T} = A \frac{T}{3} = 1 \Rightarrow \boxed{A = \frac{3}{T}}$$

3) $P_{n_M}^{(c)}$, $P_{n_M}^{(s)}$ nei casi $\theta = 0$, $\theta = \frac{\pi}{12}$

$$w(t) = w_c(t)\cos(2\pi f_0 t) - w_s(t)\sin(2\pi f_0 t)$$

$$S_{w_c}(f) = S_{w_s}(f) = N_0$$



$(\theta = 0)$

$$z_c(t) = \left[w_c(t)\cos(2\pi f_0 t) - w_s(t)\sin(2\pi f_0 t)\right] \cdot 2 \cdot \cos(2\pi f_0 t)$$

$$= w_c(t)\left[1 + \cos(4\pi f_0 t)\right] - w_s(t)\sin(4\pi f_0 t)$$

$$n_c(t) = z_c(t) \otimes g_R(t)$$

$$S_{n_c}(f) = N_0 \left| G_R(f) \right|^2$$

$$P_{n_c} = N_0 E_{g_R} = N_0 A^2 E_{g_T} = N_0 \frac{9}{T^2} \cdot \frac{T}{3} = \frac{3N_0}{T}$$

$$z_s(t) = \left[w_c(t)\cos(2\pi f_0 t) - w_s(t)\sin(2\pi f_0 t)\right]\left[-2\sin(2\pi f_0 t)\right] =$$

$$= -w_c(t)\sin(4\pi f_0 t) + w_s(t)\left[1 - \cos(4\pi f_0 t)\right]$$

$$n_s(t) = z_s(t) \otimes g_R(t)$$

$$S_{n_s}(f) = N_0 \left|G_R(f)\right|^2 \Rightarrow P_{n_s} = \frac{3N_0}{T}$$

CASO $\theta = \frac{\pi}{12}$

$$z_c(t) = \left[w_c(t)\cos(2\pi f_0 t) - w_s(t)\sin(2\pi f_0 t)\right] 2\cos(2\pi f_0 t + \theta)$$

$$= w_c(t)\left[\cos(4\pi f_0 t + \theta) + \cos\theta\right] +$$

$$- w_s(t)\left[\sin(4\pi f_0 t + \theta) + \sin(-\theta)\right]$$

$$\Rightarrow n_c(t) = z_c(t) \otimes g_R(t)$$

$$n_c(t) = \left[w_c(t)\cos\theta + w_s(t)\sin\theta\right] \otimes g_R(t)$$

$$= \cos\theta \; w_c(t) \otimes g_R(t) + \sin\theta \; w_s(t) \otimes g_R(t)$$

$$S_{n_c}(f) = \cos^2\theta \; N_0 \left|G_R(f)\right|^2 + \sin^2\theta \; N_0 \left|G_R(f)\right|^2$$

$$= N_0 \left|G_R(f)\right|^2 \Rightarrow P_{n_c} = \frac{3N_0}{T}$$

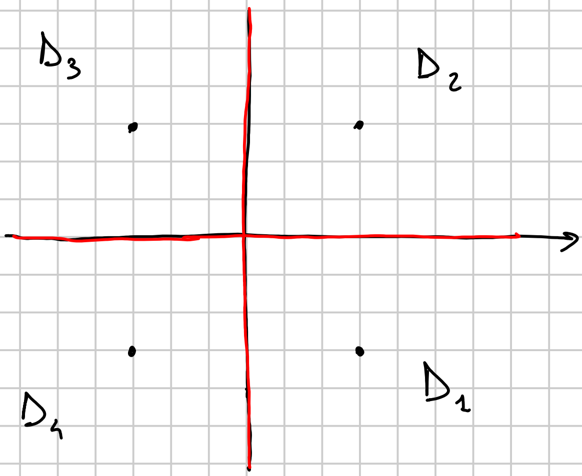
sul ramo in quadratura e' lo stesso ...

$$P_{n_s} = \frac{3N_0}{T}$$

Si deduce che la presenza di una fase nell'O.L. non cambia le statistiche di rumore.

4) $P_E^{QAM}(M)$ nelle ipotesi

- $\theta = 0$
- strategia di decisione sia la seguente



- Assicuriamoci che:
  - ASSENZA DI CROSS-TALK
  - ASSENZA DI ISI
  - F.A. (GIÀ NOTO)

$\Rightarrow$ 4-QAM $\Rightarrow$ DOPPIA 2-PAM

$$\Rightarrow P_E^{QAM}(4) \simeq 2\, P_E^{PAM}(2)$$

$$P_E^{QAM}(4) = P_E^{PAM}(M_c)\left(1 - P_E^{PAM}(M_s)\right) + P_E^{PAM}(M_s)\left(1 - P_E^{PAM}(M_c)\right) + $$
$$+ P_E^{PAM}(M_c)\, P_E^{PAM}(M_s) \qquad P_E \ll 1$$

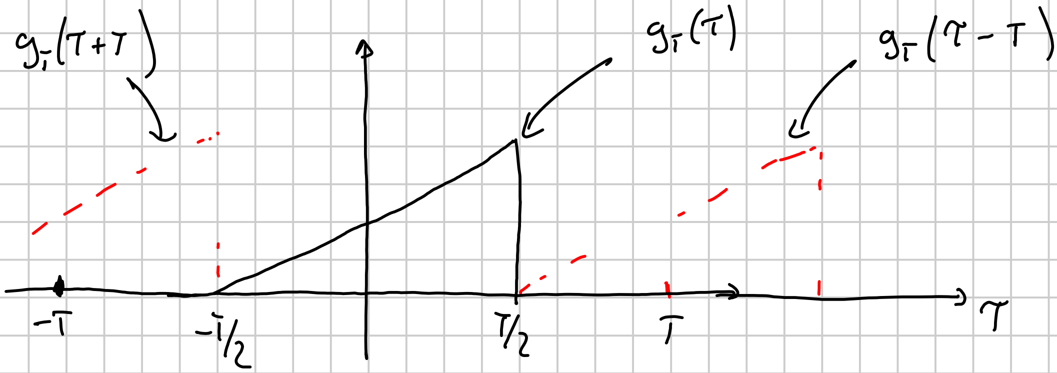
$\vartheta = 0 \Rightarrow$ ASSENZA DI CROSS-TALK

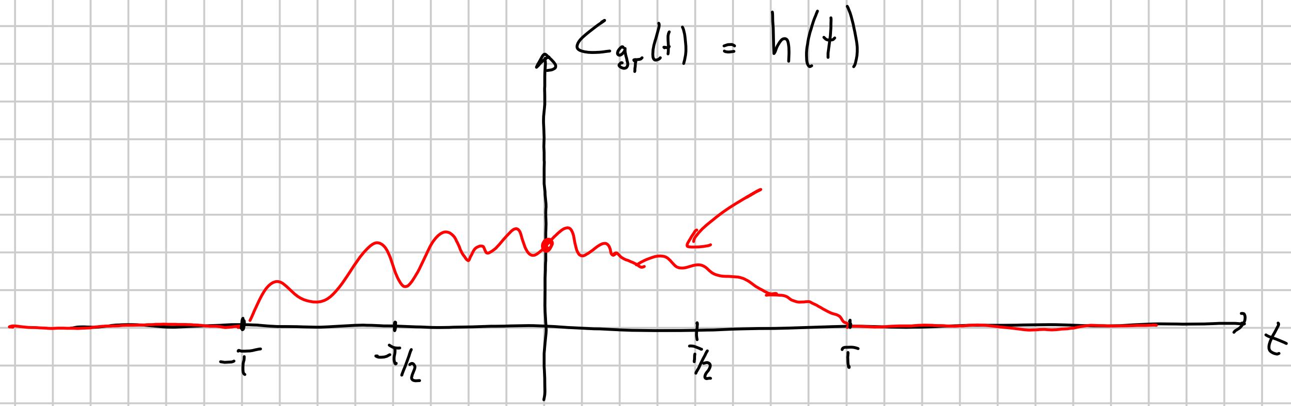
$\Rightarrow$ ASSENZA DI ISI

$$g_T(t) = \left(\frac{t + T/2}{T}\right) \mathrm{rect}\left(\frac{t}{T}\right)$$

$$h(t) = g_T(t) \otimes g_R(t) = A\, g_T(t) \otimes g_T(-t) =$$

$$= A\, C_{g_T}(t)$$





$$h[n] = h(nT) = \begin{cases} h(0) & n = 0 \\ 0 & \text{altrove} \end{cases} \Rightarrow \text{ASSENZA DI ISI}$$

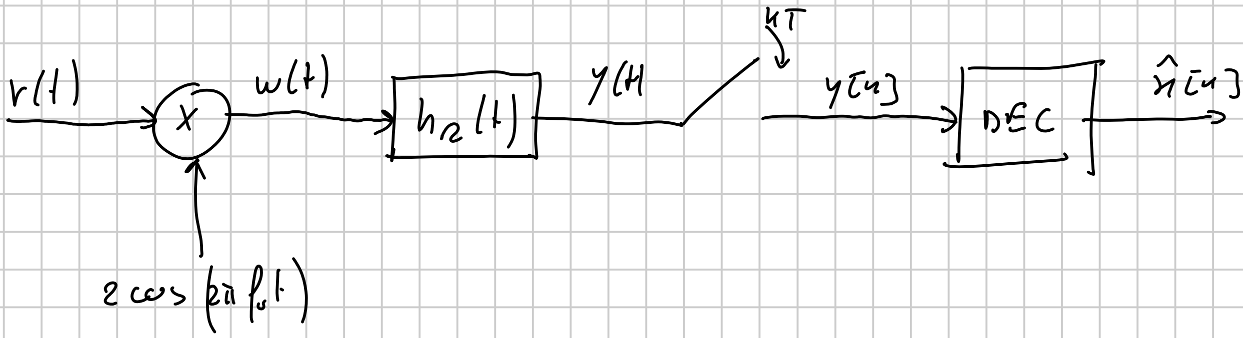
$$P_E^{PAM}(2) = P_E^{PAM}(6) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{SNR}\right) = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{T}{6N_0}}\right)$$

$$SNR_c = \frac{h^2(0) \cdot E\left[x^2[n]\right]}{\sigma_{n_c}^2} = SNR_s = \frac{1 \cdot 1/2}{\frac{3N_0}{T}} = \frac{T}{6N_0}$$

$$P_E^{QAM}(n) \simeq \mathrm{erfc}\left(\sqrt{\frac{T}{6N_0}}\right)$$

$$P_E^{QAM}(n) = \underbrace{2 \cdot \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{T}{6N_0}}\right)\left(1 - \underbrace{\mathrm{erfc}}_{\simeq 1}\left(\sqrt{\frac{T}{6N_0}}\right)\right)} + \frac{1}{4}\,\mathrm{erfc}^2\left(\sqrt{\frac{T}{6N_0}}\right)$$

ES 2 07/02/2013



$$r(t) = s(t) + n(t)$$

$$s(t) = \sum_n x[n]\, p(t - nT)\, \cos(2\pi f_0 t - \pi/3)$$

$x[k] \in A_s = \{-2, 1\}$ ind. equip.

$$p(t) = 2B\,\mathrm{sinc}(2Bt) + B\,\mathrm{sinc}\left(2B\left(t - \frac{1}{2B}\right)\right) +$$
$$+ B\,\mathrm{sinc}\left(2B\left(t + \frac{1}{2B}\right)\right), \quad f_0 \gg B, \quad T = \frac{1}{B}$$

$c(t)$ ideale

$n(t)$ è bianco in banda con DSP $\frac{N_0}{2}$

$h_R(t)$ è passa-basso ideale di banda B

$\lambda = 0$

1) $E_S = E\left[\int_{-\infty}^{+\infty} x^2[n]\, p^2(t-nT)\cos^2\left(2\pi f_0 t - \pi/3\right) dt\right]$

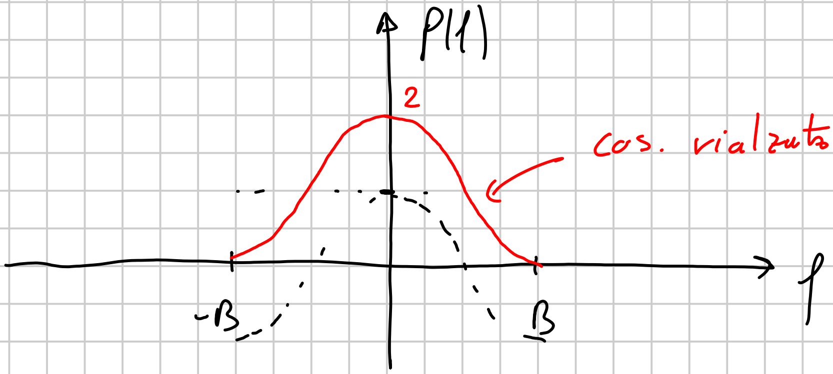
$= E\left[x^2[n]\right]\int_{-\infty}^{+\infty} p^2(t-nT)\cos^2\left(2\pi f_0 t - \pi/3\right) dt$

$\simeq \left(\frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 1\right)\frac{1}{2} E_p = \frac{1}{2}\cdot 3B = \frac{3}{2}B$

$E_p = ?$

$P(f) = \mathrm{rect}\left(\frac{f}{2B}\right) + \frac{1}{2}\mathrm{rect}\left(\frac{f}{2B}\right)\left[e^{-j\frac{2\pi f}{2B}} + e^{j2\pi f/2B}\right]$

$= \mathrm{rect}\left(\frac{f}{2B}\right)\left[1 + \cos\left(2\pi f/2B\right)\right]$



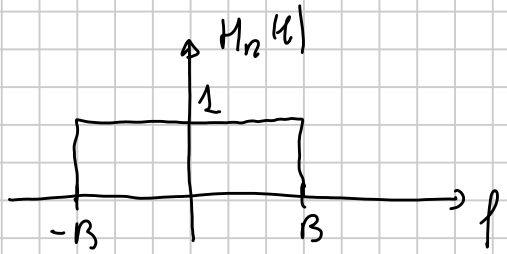
$E_p = \int_{-\infty}^{+\infty} P^2(f)\, df$

$P^2(f) = \left[1 + \cos\left(2\pi f/2B\right)\right]^2 \mathrm{rect}\left(\frac{f}{2B}\right)$

$= \left[1 + \left(\frac{1}{2} + \frac{1}{2}\cos\left(2\pi f/B\right)\right) + 2\cos\left(2\pi f/2B\right)\right]\mathrm{rect}\left(\frac{f}{2B}\right)$

$$E_p = \int_{-\infty}^{+\infty} P^2(f)\, df = 2B + B = 3B$$

2) $$P_{n_u} = \int_{-\infty}^{+\infty} N_0 |H_R(f)|^2 \, df = N_0 E_{h_R} = 2B N_0$$



3) $P_E(b)$

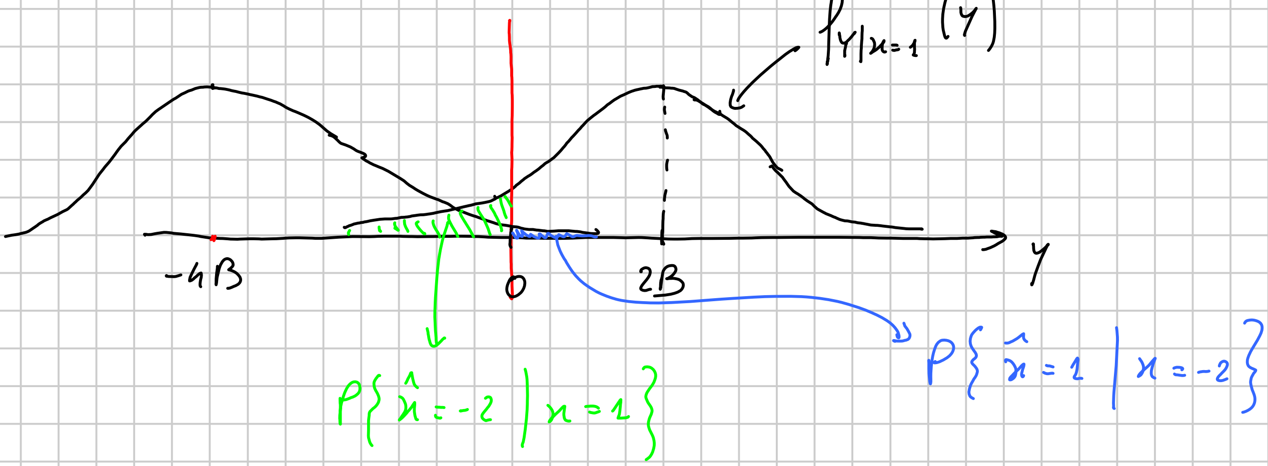
$H(f) = P(f) H_R(f) = P(f)$ => coseno rialzato => ASSENZA DI ISI

.) NON HO F.A.

$$P_E(2) = P\{\hat{x} = -2 \mid x = 1\} \underbrace{P\{x=1\}}_{1/2} + P\{\hat{x} = 1 \mid x = -2\} \underbrace{P\{x=-2\}}_{1/2}$$

$$y \mid x=1 = h(0) \cdot 1 + n_u = 2B + n_u$$

$$h(0) = \int_{-\infty}^{+\infty} P(f)\, df = 2B$$



$$P\{\hat{x}=-2 \mid x=1\} = Q\left(\frac{2B}{\sqrt{2BN_0}}\right) = Q\left(\sqrt{\frac{2B}{N_0}}\right)$$

$$P\{\hat{x}=1 \mid x=-2\} = Q\left(\frac{4B}{\sqrt{2BN_0}}\right) = Q\left(\sqrt{\frac{8B}{N_0}}\right)$$

$$P_E(b) = \frac{1}{2} Q\left(\sqrt{\frac{2B}{N_0}}\right) + \frac{1}{2} Q\left(\sqrt{\frac{8B}{N_0}}\right)$$