Segnali analogici

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$$P_X = |X(t)|^2$$

$$E_X = \int_{-\infty}^{+\infty} P_X(t) = \int_{-\infty}^{+\infty} |X(t)|^2 \, dt$$

- segnale troncato

$$x(t) \triangleq \begin{cases} X(t) & -T/2 < x < T/2 \\ 0 & \text{altrove} \end{cases}$$

- POTENZA MEDIA

$$P_{XT} \triangleq \frac{E_{XT}}{T}$$

$$P_X \triangleq \lim_{T \to \infty} \frac{E_{XT}}{T} = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} |X(t)|^2 \, dt$$

- se $X(t)$ ha $E_X < \infty \Rightarrow P_X = \emptyset$

Dim: $E_X < \infty \Rightarrow P_X = \lim_{T \to \infty} \frac{E_{XT}}{T} = \emptyset$

- se $X(t)$ ha $P_X = k < \infty \Rightarrow E_X = \infty$

Dim: $E_{XT} = \lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} |X(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(t)|^2 \, dt = E_X = \infty$

- VALORE EFFICACE

$$X_{eff} = \sqrt{P_X}$$

- VALOR MEDIA

$$X_m = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} X(t) \, dt$$

$$P_X = \emptyset \begin{cases} \nearrow X_{eff} \\ \searrow X_m \end{cases}$$

① $P_X = \emptyset \Rightarrow X_M = \emptyset$

$x(t) = X_m + x'(t)$ a media

$$x(t) = X_m + x'(t)$$

v. medio — a media nulla

$$x'(t) = x(t) - X_m$$

$$X'_m = \underbrace{\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} x(t)\,dt}_{X_m} - \underbrace{\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X_m\,dt}_{X_m} = \emptyset$$

$$P_x = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} |x(t)|^2 dt = \emptyset$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} |X_m + x'(t)|^2 dt$$

$$|z|^2 = z \cdot z^*$$
$$= c\cancel{e^{j\varphi}} \cdot c\,\cancel{e^{-j\varphi}}$$
$$= c^2$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} (X_m + x'(t))(X_m^* + x'(t)^*)$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} \Big(|X_m|^2 + |x'(t)|^2 + \underbrace{X_m x'^*(t)}_{z} + \underbrace{X_m^* x'(t)}_{z^*}\Big) dt$$

$$z + z^* = a + jb + a - jb = 2\,\mathrm{Re}\{z\}$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} \Big(|X_m|^2 + |x'(t)|^2 + 2\,\mathrm{Re}\{X_m x'^*(t)\}\Big) dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} |X_m|^2 dt + \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |x'(t)|^2 dt + \underbrace{2\,\mathrm{Re}\left\{\lim_{T\to\infty} \frac{1}{T} X_m \int_{-\frac{T}{2}}^{T/2} x'^*(t)\,dt\right\}}_{\emptyset}$$

$$P_x = \underset{\emptyset}{\underset{\parallel}{|X_m|^2}} + \underset{\emptyset}{\underset{\parallel}{P_{x'}}} = 0$$

$$\Downarrow$$

$$X_m = \emptyset$$

Segnali periodici

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- $E_x = \infty$ $\forall$ segnale periodico

- $P_x = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2 \, dt$

- VALOR MEDIO

$$X_m = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \, dt$$

TSF:

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \, e^{-j2\pi n f_0 t} \, dt$$

ATSF:

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \cdot e^{j2\pi n f_0 t}$$

BIUNIVOCITA': TSF[ATSF [X_n]]

$$X_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{n=-\infty}^{+\infty} X_n \, e^{j2\pi n f_0 t} \, e^{-j2\pi m f_0 t} \, dt$$

$$= \sum_{n=-\infty}^{+\infty} X_n \underbrace{\frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} e^{j2\pi (n-m) f_0 t} \, dt}$$

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \cos[2\pi(n-m) f_0 t]\, dt + j \underbrace{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin[2\pi(n-m) f_0 t]\, dt}_{= \emptyset}$$

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

$$\begin{cases} 0 & n \neq m \\ T_0 & n = m \end{cases}$$

$$= \sum_{n=-\infty}^{+\infty} X_n \cdot \frac{1}{T_0} \cdot T_0 \; \delta[n-m]$$

$$= X_m$$

PROPRIETA' DELLA TSF

- LINEARITA'

$$\begin{cases} z(t) = a x(t) + b y(t) \\ x(t) \overset{TSF}{\rightleftarrows} X_n \\ y(t) \overset{TSF}{\rightleftarrows} Y_n \end{cases}$$

Tn: $Z_n = TSF[z(t)] = a X_n + b Y_n$

Dim:

$$Z_n = TSF[z(t)] = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} z(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underbrace{[a x(t) + b y(t)]}_{z(t)} e^{-j2\pi n f_0 t}\, dt$$

To

$$\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underbrace{[a x(t) + b y(t)]}_{z(t)} e^{-j2\pi n f_0 t} dt$$

$$= a\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt + b\,\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi n f_0 t} dt$$

$$= a X_n + b Y_n$$

- SIMMETRIA HERMITIANA

Hp: $x(t)$ e' reale $\Rightarrow x(t) = x^*(t)$

Tn: $X_{-n} = X_n^*$

Dim:

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi (-n) f_0 t} dt$$

$$= \left\{\left[\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t} dt\right]^*\right\} = \left\{\left(\frac{1}{T_0}\right)^*\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x^*(t)\, e^{-j2\pi f_0 t} dt\right\}^*$$

$$= \left\{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{-j2\pi n f_0 t} dt\right\}^* = X_n^* = X_{-n}$$

$x(t)$ reale $\Rightarrow X_{-n} = X_n^*$

- $|X_{-n}| = |X_n|$ pari
- $\angle X_{-n} = -\angle X_n$ dispari

- $x(t)$ reale e pari

$\Downarrow$

Ip: $\begin{cases} x(t) = x^*(t) \\ x(-t) = x(t) \end{cases}$

X_n reale e pari

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi n f_0 t}\, dt \qquad t' = -t$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(-t')\, e^{-j2\pi n f_0 t'}\, (-dt')$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} x(t')\, e^{-j2\pi f_0 t'}\, dt' = X_n$$

$$\underbrace{X_n^* = \overbrace{X_{-n} = X_n}^{\text{pari}}}_{\text{reale}}$$

- $x(t)$ reale e dispari

$\Downarrow$

X_n immaginaria e dispari

$$\text{Ip: } \begin{cases} x(t) = x^*(t) \\ x(-t) = -x(t) \end{cases}$$

Dim:

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, e^{j2\pi f_0 n t}\, dt \qquad t' = -t$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} x(-t')\, e^{-j2\pi n f_0 t'}\, dt = -\underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t')\, e^{-j2\pi n f_0 t'}\, dt'}_{X_n}$$

$$= -X_n$$

$$\underbrace{X_n^* = \overbrace{X_{-n} = -X_n}^{\text{dispari}}}_{\text{immaginario}}$$

Immaginario

TCF

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$$X(f) \triangleq \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$x(t) \triangleq \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

PROPRIETA':

- HERMITIANA:

Ip: $x(t)$ e' reale

Th: $X(f)$ e' hermitiana $\Rightarrow X(-f) = X^*(f)$

$|X(f)| = |X(-f)|$

$\angle X(f) = -\angle X(f)$

Dimostrazione uguale a TSF

- PARITA'

Ip: $x(t)$ e' reale e pari

Th: $X(f)$ e' reale e pari

- DISPARITA'

Ip: $x(t)$ e' reale e dispari

Th: $X(f)$ e' immaginaria e dispari

TEOREMI SULLA TCF

① LINEARITA'

Ip: $x(t) = a\, x_1(t) + b\, x_2(t)$

$x_1(t) \rightleftarrows X_1(f)$

$x_2(t) \rightleftarrows X_2(f)$

Th: $X(f) = a\, X_1(f) + b\, X_2(f)$

DUALITA'

Ip: $x(t) \rightleftarrows X(f)$

Th: $X(t) \rightleftarrows X(-f)$

Dim:

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$X(t) = \int_{-\infty}^{+\infty} x(f)\, e^{-j2\pi f t}\, df$$

$f' = -f$

$$X(t) = \underbrace{\int_{-\infty}^{+\infty} X(-f')\, e^{j2\pi f' t}\, df'}_{ATCF[X(-f)]}$$

$$X(t) = ATCF\left[X(-f)\right]$$

$$\Downarrow$$

$$X(-f) = TCF\left[X(t)\right]$$

$$X(t) \overset{TCF}{\rightleftarrows} x(-f)$$

- RITARDO

Ip: $x(t) \rightleftarrows X(f)$

$y(t) = x(t - t_0)$

Th: $Y(f) = TCF\left[y(t)\right] = X(f) \cdot e^{-j2\pi f t_0}$

Dim:

$$Y(f) = \int_{-\infty}^{+\infty} y(t) \cdot e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(t - t_0)\, e^{-j2\pi f t}\, dt$$

$t' = t - t_0$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f (t' + t_0)}\, dt'$$

$$= \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, dt'\; e^{-j2\pi f t_0} = X(f)\, e^{-j2\pi f t_0}$$

TEOREMA DELLA MODULAZIONE

$$I_p: \begin{cases} y(t) = x(t)\cos(2\pi f_0 t) \\ x(t) \rightleftarrows X(f) \end{cases}$$

$$Th: \; Y(f) = \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)$$

Dim:

$$Y(f) = \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(t)\cos(2\pi f_0 t)\, e^{-j2\pi f t}\, dt$$

$$\int_{-\infty}^{+\infty} x(t) \; \frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2} \cdot e^{-j2\pi f t}\, dt$$

$$= \underbrace{\int_{-\infty}^{+\infty} \frac{1}{2} x(t)\, e^{-j2\pi (f - f_0) t}\, dt}_{TCF[x(t)]\big|_{f - f_0}} + \frac{1}{2} \underbrace{\int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f + f_0) t}\, dt}_{TCF[x(t)]\big|_{f + f_0}}$$

$$= \frac{1}{2} X(f - f_0) + \frac{1}{2} X(f + f_0)$$

MODULAZIONE CON SENO

$$I_p: \begin{cases} x(t) \rightleftarrows X(f) \\ y(t) = x(t)\sin(2\pi f_0 t) \end{cases}$$

$$Th: \; Y(f) = \frac{1}{2j} X(f - f_0) - \frac{1}{2j} X(f + f_0)$$

Dim Omessa

MODULAZIONE CON COSINUSOIDE

$$I_p: \begin{cases} x(t) \rightleftarrows X(f) \\ y(t) = x(t)\cos(2\pi f_0 t + \varphi) \end{cases}$$

$$Th: \; Y(f) = \frac{e^{j\varphi}}{2} X(f - f_0) + \frac{e^{-j\varphi}}{2} X(f + f_0)$$

Dim:

$$Y(f) = \int_{-\infty}^{+\infty} x(t) \; \frac{e^{j(2\pi f_0 t + \varphi)} + e^{-j(2\pi f_0 t + \varphi)}}{2} \; e^{-j2\pi f t}\, dt$$

$$= \frac{e^{j\varphi}}{2} \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f - f_0) t}\, dt + \frac{e^{-j\varphi}}{2} \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f + f_0) t}\, dt$$

$$\frac{C}{2}\int_{-\infty} x(t)\,e \quad dt + \frac{C}{2}\int_{-\infty} x(t)\,e \quad dt$$

$$= \frac{e^{j\varphi}}{2} X(f-f_0) + \frac{e^{-j\varphi}}{2} X(f+f_0)$$

MODULAZIONE CON ESP. COMPLESSO

Ip: $\begin{cases} x(t) \rightleftarrows X(f) \\ y(t) = x(t)\, e^{j2\pi f_0 t} \end{cases}$

Th: $Y(f) = X(f-f_0)$

Dim:

$$Y(f) = \int_{-\infty}^{+\infty} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi (f-f_0)t}\, dt = X(f-f_0)$$

DUALITA'

RIT: $x(t-t_0) \rightleftarrows X(f)\, e^{-j2\pi f t_0}$

MOD: $x(t)\, e^{j2\pi f_0 t} \rightleftarrows X(f-f_0)$

TEOREMA DELLA DERIVAZIONE

Ip: $\begin{cases} x(t) \rightleftarrows X(f) \\ y(t) = \frac{d}{dt} x(t) \end{cases}$

Th: $Y(f) = j2\pi f\, X(f)$

Dim:

$$y(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \int_{-\infty}^{+\infty} X(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-\infty}^{+\infty} X(f) \frac{d}{dt} e^{j2\pi f t}\, df = \int_{-\infty}^{+\infty} X(f)\, j2\pi f\, e^{j2\pi f t}\, df$$

$$y(t) = \int_{-\infty}^{+\infty} j2\pi f\, X(f)\, e^{j2\pi f t}\, df = \int_{-\infty}^{+\infty} Y(f)\, e^{j2\pi f t}\, df$$

$$\Downarrow$$

$$Y(f) = j2\pi f\, X(f)$$

TEOREMA DELL' INTEGRAZIONE

Ip $\begin{cases} x(t) \rightleftarrows X(f) \\ y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha \\ \int_{-\infty}^{+\infty} x(t)\, dt = 0 \iff X(0) = 0 \end{cases}$

th: $$Y(f) = \frac{X(f)}{j2\pi f}$$

Dim:

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha \Rightarrow x(t) = \frac{d}{dt} y(t)$$

th. derivazione

$$\Rightarrow \quad X(f) = j2\pi f\, Y(f) \Rightarrow Y(f) = \frac{X(f)}{j2\pi f}$$

DERIVAZIONE IN FREQUENZA

Ip: $\begin{cases} x(t) \rightleftarrows X(f) \\ Y(f) = \frac{d}{df} X(f) \end{cases}$

Th: $y(t) = -j2\pi t\, x(t)$

segno opposto *

Dim identica al tempo...

INTEGRAZIONE IN FREQUENZA

$\begin{cases} x(t) \rightleftarrows X(f) \\ Y(f) = \int_{-\infty}^{f} X(\alpha)\, d\alpha \\ \int_{-\infty}^{+\infty} X(\alpha)\, d\alpha = 0 \iff x(0) = 0 \end{cases}$

*

Th: $$y(t) = - \frac{x(t)}{j2\pi t}$$

Dim equivalente.

CONVOLUZIONE

$\int^{+\infty}$

CONVOLUZIONE

$$x(t) \otimes y(t) \triangleq \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau$$

TEOREMA

$$\text{Ip} \begin{cases} x(t) \rightleftharpoons X(f) \\ y(t) \rightleftharpoons Y(f) \\ z(t) = x(t) \otimes y(t) \end{cases}$$

th: $Z(f) = X(f)\, Y(f)$

Dim:

$$Z(f) = \int_{-\infty}^{+\infty} z(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau\, e^{j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(\tau) \int_{-\infty}^{+\infty} y(t-\tau)\, e^{-j2\pi f t}\, dt\, d\tau = \int_{-\infty}^{+\infty} x(\tau)\, \underset{\substack{\uparrow \\ \text{th. ritardo}}}{Y(f)}\, e^{-j2\pi f \tau}\, d\tau$$

$$= Y(f) \int_{-\infty}^{+\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau = Y(f) \cdot X(f)$$

PROPRIETA' DELLA CONVOLUZIONE:

- **COMMUTATIVA**

$$x(t) \otimes y(t) = y(t) \otimes x(t)$$

$$\downarrow$$

$$\int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau)\, d\tau \qquad \begin{aligned} t-\tau &= \tau' \\ \tau &= t-\tau' \end{aligned}$$

$$= \int_{-\infty}^{+\infty} x(t-\tau')\, y(\tau')\, d\tau' = y(t) \otimes x(t)$$

- **DISTRIBUTIVA**

$$x(t) \otimes \big(y(t) + z(t)\big) = x(t) \otimes y(t) + x(t) \otimes z(t)$$

$$\downarrow$$

$$\int_{-\infty}^{+\infty} x(\tau) \left[y(t-\tau) + z(t-\tau) \right] = \int_{-\infty}^{+\infty} x(\tau)\, y(t-\tau) + \int_{-\infty}^{+\infty} x(\tau)\, z(t-\tau)$$

$$= x(t) \otimes y(t) + x(t) \otimes z(t)$$

- **ASSOCIATIVA**

$$\big(x(t) \otimes y(t)\big) \otimes z(t) = x(t) \otimes \big(y(t) \otimes z(t)\big)$$

$$\Rightarrow X(f)\big(Y(f) \cdot Z(f)\big) = \big(X(f) \cdot Y(f)\big) Z(f)$$

TEOREMA DEL PRODOTTO

$$Ip: \begin{cases} x(t) \rightleftarrows X(f) \\ y(t) \rightleftarrows Y(f) \\ z(t) = x(t)\, y(t) \end{cases}$$

$$Th: \quad Z(f) = X(f) \otimes Y(f)$$

Dim:

$$Z(f) = \int_{-\infty}^{+\infty} z(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} x(t)\, y(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\alpha)\, e^{+j2\pi \alpha t}\, d\alpha \; y(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} X(\alpha) \underbrace{\int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi (f-\alpha) t}\, dt}_{Y(f-\alpha)}\, d\alpha$$

$$= \int_{-\infty}^{+\infty} X(\alpha)\, Y(f-\alpha)\, d\alpha = X(f) \otimes Y(f)$$

IMPULSO DI DIRAC

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t) \qquad \delta_\varepsilon(t) = \frac{1}{\varepsilon}\, \mathrm{rect}\left(\frac{t}{\varepsilon}\right)$$

PROPRIETA':

$$\delta(t) = \frac{d}{dt} u(t) \rightleftarrows u(t) = \int_{-\infty}^{t} \delta(\alpha)\, d\alpha$$

1) $\displaystyle\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$

$$\int_{-\infty}^{+\infty} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, \mathrm{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \mathrm{rect}\left(\frac{t}{\varepsilon}\right) dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \cdot \cancel{\varepsilon} = 1$$

2) PROPRIETA' CAMPIONATRICE

$$\int_{-\infty}^{+\infty} x(t)\, \delta(t)\, dt = x(0)$$

$x(t)$ continua in $t = 0$

Dim:

$$\int_{-\infty}^{+\infty} x(t) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, \mathrm{rect}\left(\frac{t}{\varepsilon}\right) dt$$

ε

$$\int_{-\infty}^{+\infty} x(t) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} x(t) \operatorname{rect}\left(\frac{t}{\varepsilon}\right) dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \underbrace{\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} x(t)\, dt}_{\text{MEDIA}}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \cdot \cancel{\varepsilon}\, x(\bar{t}) = x(0)$$

③ PARITA'

$$\delta(t) = \delta(-t)$$

$$\int_{-\infty}^{+\infty} x(t)\delta(t) = \int_{-\infty}^{+\infty} x(t)\,\delta(-t)$$

Dim: $t' = -t$

$$\int_{-\infty}^{+\infty} x(-t)\,\delta(t)\, dt = x(-0) = x(0)$$

④ $$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\, dt = x(t_0)$$

$$t' = t - t_0$$

$$\int_{-\infty}^{+\infty} x(t'+t_0)\,\delta(t')\, dt = x(t'+t_0)\Big|_{t'=0} = x(t_0)$$

⑤ $$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

Dim:

$$\int_{-\infty}^{+\infty} x(t_0)\,\delta(t-t_0)\, dt = x(t_0) \int_{-\infty}^{+\infty} \delta(t-t_0)\, dt \qquad t - t_0 = t'$$

$$= x(t_0) \underbrace{\int_{-\infty}^{+\infty} \delta(t')\, dt'}_{1} = x(t_0)$$

⑥ $$x(t) \otimes \delta(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\delta(\tau - t)\, d\tau = x(t) = x(t) \otimes \delta(t)$$

⑦ $$x(t) \otimes \delta(t-t_0) = x(t-t_0)$$

$$= \int_{-\infty}^{+\infty} x(\tau)\,\delta(\underbrace{t-t_0}_{t'}-\tau)\, d\tau = x(t') = x(t-t_0)$$

ENERGIA $\delta(t)$

$$\int_{-\infty}^{+\infty} \delta^2(t)\, dt = \int_{-\infty}^{+\infty} \left[\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \operatorname{rect}\left(\frac{t}{\varepsilon}\right)\right]^2 dt$$

$$= \lim_{\varepsilon \to 0} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left(\frac{1}{\varepsilon}\right)^2 = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\varepsilon^2} = \infty$$

TCF $\delta(t)$

$$\Delta(f) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j2\pi f t}\, dt = 1$$

$$\delta(t) \rightleftharpoons 1 \quad \forall f$$

RAPPORTO TRA IMPULSO E GRADINO

$$\delta(t) = \frac{d}{dt} u(t) \quad \rightleftharpoons \quad u(t) = \int_{-\infty}^{t} \delta(t)$$

Dim: $\delta_\varepsilon(t) = \frac{d}{dt} u_\varepsilon(t)$

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{d}{dt} u_\varepsilon(t) = \frac{d}{dt} \lim_{\varepsilon \to 0} u_\varepsilon(t) = \frac{d}{dt} u(t)$$

- $\int_{-\infty}^{t} x(\alpha)\, d\alpha = x(t) \otimes u(t)$
- $U(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

TEOREMA DELL'INTEGRAZIONE COMPLETO

Ip $\begin{cases} x(t) \rightleftharpoons X(f) \\ y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha \end{cases}$

Th: $Y(f) = \text{TCF}[X(f)] = \frac{X(0)}{2}\delta(f) + \frac{X(f)}{j2\pi f}$

Dim:

$$y(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha = x(t) \otimes u(t)$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, u(t-\tau)\, dt$$

$$y(t) = x(t) \otimes u(t)$$

$$\Downarrow$$

$$Y(f) = X(f) \cdot U(f)$$

$$= X(f)\left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right]$$

$$= \frac{1}{2} X(0)\,\delta(f) + \frac{X(f)}{j2\pi f}$$

DUALITA'

$$\delta(t \pm t_0) \rightleftharpoons e^{\pm j2\pi f t_0}$$

$$e^{\pm j2\pi f_0 t} \rightleftharpoons \delta(f \mp f_0)$$

BIUNIVOCITA' TCF

$e \rightleftarrows \delta(f \mp f_0)$

BIUNIVOCITA' TCF

$$X(f) = TCF[x(t)]$$

$$x(t) = \underbrace{\int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} x(\alpha)\, e^{-j2\pi f\alpha}}_{X(f)}\, e^{j2\pi f t}\, df\, d\alpha}_{x(t)}$$

$$= \int_{-\infty}^{+\infty} x(\alpha) \int_{-\infty}^{+\infty} e^{j2\pi f(t-\alpha)}\, df\, d\alpha$$

$$= \int_{-\infty}^{+\infty} x(\alpha)\, \delta(t-\alpha)\, d\alpha = x(t) \otimes \delta(t) = x(t)$$

ANALISI ENERGETICA SEGNALI APERIODICI

$$E_x \triangleq \int_{-\infty}^{+\infty} |x(t)|^2\, dt$$

CORRELAZIONE:

$$C_{xy}(\tau) \triangleq \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt$$

AUTOCORRELAZIONE:

$$C_x(\tau) \triangleq \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt$$

PROPRIETA':

① $C_x(0) = C_x(\tau)\Big|_{\tau=0} = \int_{-\infty}^{+\infty} x(t)\, x^*(t)\, dt$

$$= \int_{-\infty}^{+\infty} |x(t)|^2\, dt = E_x$$

② SIMMETRIA HERMITIANA

$$C_x^*(\tau) = C_x(-\tau)$$

$$C_x(-\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t+\tau)\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, x^*(t+\tau)\, dt$$

$t' = t + \tau \qquad t = t' - \tau$

$$= \int_{-\infty}^{+\infty} x(t'-\tau)\, x^*(t')\, dt' = \int_{-\infty}^{+\infty} x^*(t')\, x(t'-\tau)\, dt'$$

$$= \Big[\underbrace{\int_{-\infty}^{+\infty} x(t')\, x^*(t'-\tau)\, dt'}_{C_x(\tau)}\Big]^* = C_x^*(\tau) = C_x(-\tau)$$

③ TCF $\quad S_x(f) \rightleftarrows C_x(\tau)$

$$S_x(f) = \int_{-\infty}^{+\infty} C_x(\tau)\, e^{-j2\pi f t} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt$$

$$= \int_{-\infty}^{+\infty} x(t) \int_{-\infty}^{+\infty} x^*(t-\tau)\, d\tau\, dt \qquad t - \tau = \tau'$$

$$= \int_{-\infty}^{+\infty} x(t) \int_{-\infty}^{+\infty} x^*(\tau')\, e^{-j2\pi f(t-\tau')}\, d\tau'\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t} \Big[\int_{-\infty}^{+\infty} x(\tau')\, e^{-j2\pi f \tau'}\, d\tau'\Big]^*$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t} \left[\int_{-\infty}^{+\infty} x(\tau')\, e^{-j2\pi f \tau'} d\tau' \right]^*$$

$$= X(f) \cdot X^*(f) = |X(f)|^2 = S_x(f)$$

$$C_x(\tau) = S_x(f) = |X(f)|^2$$

PROPRIETA':

$$\int_{-\infty}^{+\infty} S_x(f)\, df = \int_{-\infty}^{+\infty} S_x(f)\, e^{j2\pi f \tau} \Big|_{\tau=0} = C_x(\tau) \Big|_{\tau=0}$$

$$= C_x(0) = E_x$$

$$\Rightarrow E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

TEOREMA DI PARSEVAL (P. 157 Appunti Rhamani)

$$\int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{+\infty} X(f)\, Y^*(f)\, df$$

Dim:

$$C_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt$$

$$S_{xy}(f) = TCF\left[C_{xy}(\tau)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, e^{-j2\pi f t}\, dt\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(t) \int_{-\infty}^{+\infty} y^*(t-\tau)\, e^{-j2\pi f t}\, dt\, d\tau$$

$t-\tau = \tau'$

$$= \int_{-\infty}^{+\infty} x(t) \int_{-\infty}^{+\infty} y^*(\tau')\, e^{-j2\pi (t-\tau')}\, d\tau'\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \left[\int_{-\infty}^{+\infty} y(\tau')\, e^{-j2\pi f \tau'} d\tau' \right]^*$$

$$= X(f) \cdot Y^*(f)$$

PERIODICIZZAZIONE SEGNALI APERIODICI

$x(t)$ APERIODICO

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$ e' periodico di periodo T_0

$$Y_n = TSF\left[y(t)\right] = \frac{1}{T_0} \int_{-T_0/2}^{\frac{T_0}{2}} y(t)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{n=-\infty}^{+\infty} x(t - kT_0)\, e^{-j2\pi n f_0 t}\, dt$$

$$= \frac{1}{T_0} \sum_k \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t-kT_0)\, e^{-j2\pi n f_0 t}\, dt$$

$t - kT_0 = t'$

$$= \frac{1}{T_0} \underbrace{\sum_k \int_{-\frac{T_0}{2}-kT_0}^{\frac{T_0}{2}-kT_0}} x(t')\, e^{-j2\pi n f_0 (t'+kT_0)}\, dt'$$

$$= \frac{1}{T_0} \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi \underbrace{n f_0}_{f'} t'} \cdot \underbrace{e^{-j2\pi n f_0 k T_0}}_{1}$$

$$= \frac{1}{T_0} X(f') = X(n f_0) = \boxed{\frac{1}{T_0} X\left(\frac{n}{T_0}\right) = Y_n}$$

$$= \frac{1}{T_0} X(f') = X(n f_0) = \boxed{\frac{1}{T_0} X\left(\frac{n}{T_0}\right) = Y_n}$$

1° FORMULA POISSON

$$\sum_{n=-\infty}^{+\infty} x(t - nT_0) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X\left(\frac{n}{T_0}\right) e^{j2\pi n \frac{t}{T_0}}$$

2° FORMULA DI POISSON

$$\underbrace{\sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f n T_0}}_{= X[n]} = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X\left(f - \frac{n}{T_0}\right)$$

TCF DI SEGNALI PERIODICI

$$y(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

$$Y(f) = \int_{-\infty}^{+\infty} y(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{+\infty} \sum_n x(t - nT_0)\, e^{-j2\pi f t}\, dt$$

$$= \sum_n \int_{-\infty}^{+\infty} x(t - nT_0)\, e^{-j2\pi f t}\, dt \qquad t - nT_0 = t'$$

$$= \sum_n \int_{-\infty}^{+\infty} x(t')\, e^{-j2\pi f t'}\, e^{-j2\pi f n T_0}\, dt'$$

$$= \sum_n X(f)\, e^{-j2\pi n f T_0} = X(f) \sum_n e^{-j2\pi n f T_0} = X(f) \frac{1}{T_0} \sum_n \delta\left(f - \frac{n}{T_0}\right)$$

$$= \frac{1}{T_0} \sum_n X\left(\frac{n}{T_0}\right) \delta\left(f - \frac{n}{T_0}\right) = \boxed{\sum_{n=-\infty}^{+\infty} Y_n\, \delta\left(f - \frac{n}{T_0}\right) = Y(f)}$$

$$Y_n = \frac{1}{T_0} X\left(\frac{n}{T_0}\right)$$

Sistemi

lunedì 20 giugno 2022 16:50

$$y(t) = T[x(t)]$$

PROPRIETA' DEI SISTEMI

- **LINEARITA'**

se $x(t) = a\,x_1(t) + b\,x_2(t)$

$\Rightarrow y(t) = a\,T[x_1(t)] + b\,T[x_2(t)]$

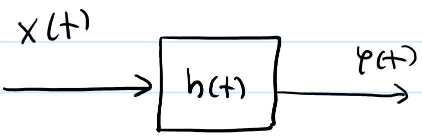
- **STAZIONARIETA'**

se $y(t) = T[x(t)]$

allora $y(t - t_0) = T[x(t - t_0)]$

SISTEMI LINEARI E STAZIONARI (SLS)

$h(t)$ caratterizza completamente i sistemi SLS



$$y(t) = T[x(t)] = x(t) \otimes h(t)$$

Dim:

$y(t) = T[x(t)] = T[x(t) \otimes \delta(t)]$ poiche' $h(t) \triangleq T[\delta(t)]$

risposta all'impulso

$= T\left[\int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right]$

$= \int_{-\infty}^{+\infty} x(\tau)\,T[\delta(t-\tau)]\,d\tau$

$= \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) \otimes h(t)$

③ **CAUSALITA'**

$y(t) = T[x(\alpha) ; \alpha \leq t]$

l'uscita $y(t)$ dipende dall'ingresso a istanti $\leq t$

④ STABILITA' BIBO

$|x(t)| \leq M < \infty \Rightarrow |y(t)| \leq N < \infty \quad \forall t$

⑤ MEMORIA

Un sistema e' senza memoria se:

$y(t) = T[x(\alpha) ; \alpha = t]$

dipende solo da t

⑥ INVERTIBILITA'

Se $y(t) = T[x(t)]$

allora $x(t) = T^{-1}[y(t)]$

RISPOSTA IN FREQUENZA

① $H(f) = TCF[h(t)]$

② $H(f) = \dfrac{Y(f)}{X(f)}$ ③ $Y(f) = \dfrac{y(t)}{x(t)}\Bigg|_{x(t) = e^{j2\pi f t}}$

Dim ③:

$$y(t) = x(t) \otimes h(t) \Big|_{x(t) = e^{j2\pi f t}} = \int_{-\infty}^{+\infty} e^{j2\pi f \tau} h(t-\tau)\, d\tau \qquad t - \tau = \tau'$$

$$= \int_{-\infty}^{+\infty} e^{j2\pi f (t-\tau')} h(\tau')\, d\tau'$$

$$= e^{j2\pi f t}\int_{-\infty}^{+\infty} h(\tau')\, e^{-j2\pi f \tau'}\, d\tau' = e^{-j2\pi f t}\, H(f) = y(t)$$

$$\Rightarrow H(f) = \frac{y(t)}{x(t)}\Bigg|_{x(t)=e^{j2\pi f t}}$$

Dim ②

$$y(t) = x(t) \otimes h(t)$$

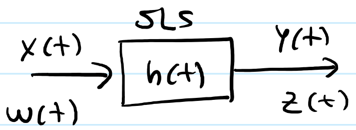
$$\Downarrow \mathcal{F}$$

$$Y(f) = X(f)\cdot H(f) \Rightarrow H(f) = \frac{Y(f)}{X(f)}$$

PROPRIETA' DELL' INTEGRAZIONE PER SLS

$$y(t) = x(t) \otimes h(t)$$

$$w(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha$$



$$z(t) = \int_{-\infty}^{t} y(\alpha)\, d\alpha$$

Dim:

$$z(t) = w(t) \otimes h(t)$$

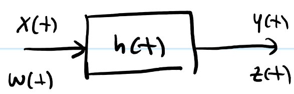
$$= \underbrace{x(t) \otimes u(t)} \otimes h(t)$$

$$x(t) \otimes u(t) = \int_{-\infty}^{t} x(\alpha)\, d\alpha$$

$$= \underbrace{[x(t) \otimes h(t)]}_{y(t)} \otimes u(t) = \int_{-\infty}^{t} y(\alpha)\, u(\alpha)$$

PROPRIETA' DELLA DERIVAZIONE

$$w(t) = \frac{d}{dt}x(t)$$



$$z(t) = \frac{d}{dt}y(t)$$

Dim:

dt

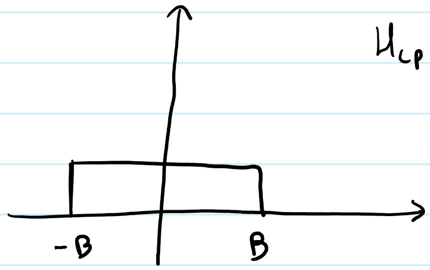
Dim:

$$y(t) = \int_{-\infty}^{t} z(\alpha)\, d\alpha \qquad x(t) = \int_{-\infty}^{t} w(\alpha)\, d\alpha$$

$\Rightarrow$ th. precedente ...

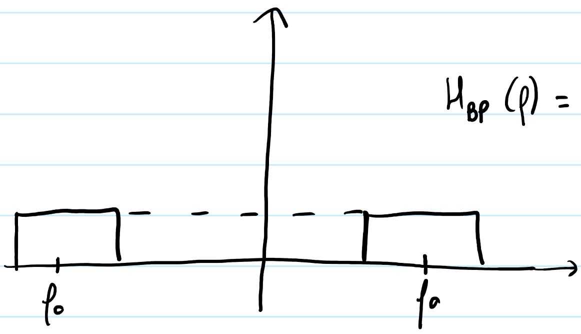
FILTRI IDEALI

- PASSA - BASSO



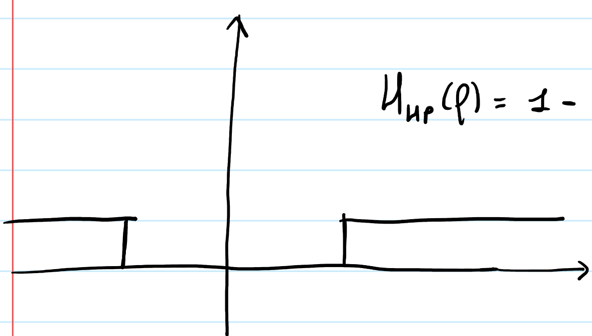
$$H_{LP}(f) \triangleq \text{rect}\left(\frac{f}{2B}\right)$$

- PASSA - BANDA



$$H_{BP}(f) = \text{rect}\left(\frac{f - f_0}{B}\right) + \text{rect}\left(\frac{f + f_0}{B}\right)$$

- PASSA - ALTO

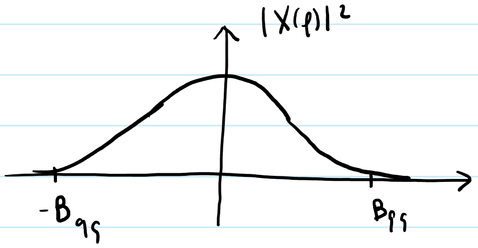


$$H_{HP}(f) = 1 - \text{rect}\left(\frac{f}{2B}\right)$$

DEFINIZIONI DI BANDA 10

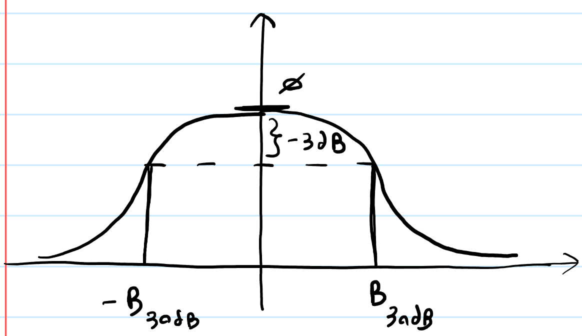
- BANDA AL 99% DELL'ENERGIA

- **BANDA AL 99% DELL' ENERGIA**



$$B_{99} : \frac{\int_{-B_{99}}^{B_{99}} |X(f)|^2 df}{E_X} = 0.99$$

- **BANDA A -3dB**



DISTORSIONI LINEARI

- **REPLICA FEDELE:**

$$y(t) = K x(t - t_0) \qquad K \in \mathbb{C}$$
$$t_0 \in \mathbb{R}$$
$$Y(f) = K X(f) e^{-j2\pi f t_0}$$

- **FILTRO FEDELE**

$$h(t) = k\,\delta(t - t_0) \qquad [\, x(t) \otimes \delta(t-t_0) = x(t-t_0) \,]$$
$$H(f) = k e^{-j2\pi f t_0}$$

DISTORSIONE DI FASE:

$$Y(f) \neq K |X(f)|$$

DISTORSIONE DI AMPIEZZA:

$\angle Y(f) \neq af + b$

$S_y(f) = TCF[C_y(\tau)] = |Y(f)|^2$

$= Y(f)\, Y^*(f) = X(f) H(f) \; X^*(f) H^*(f)$

$= X(f) X^*(f) \cdot H(f) H^*(f)$

$= S_x(f) \cdot |H(f)|^2 = S_y(f)$

$C_y(\tau) = ATCF[S_y(f)]$

$= ATCF[S_x(f) \cdot H(f) \cdot H^*(f)]$

$= C_x(\tau) \otimes h(\tau) \otimes h(-\tau) = C_y(\tau)$

Campionamento e sequenze

martedì 21 giugno 2022 16:53

$x(t)$ —/— T_c — $x[n] = x(nT)$

TFS:

$$\bar{X}(f) \triangleq \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f n T}$$

$$\Downarrow\Uparrow$$

$$x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{X}(f)\, e^{j2\pi f n T}$$

PROPRIETA':

- $\bar{X}(f)$ e' periodica di periodo $\frac{1}{T}$

$$\bar{X}(f) = \bar{X}\left(f - \frac{n}{T}\right)$$

$$= \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi\left(f-\frac{n}{T}\right)kT}$$

$$= \underbrace{\sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi f k T}}_{\bar{X}(f)} \cdot \underbrace{e^{j2\pi \frac{n}{\cancel{T}} k \cancel{T}}}_{=1}$$

$$= \bar{X}(f)$$

- BIUNIVOCITA'

$$x[n] = T \int_{-\frac{1}{2T}}^{1/2T} \bar{X}(f)\, e^{j2\pi f n T}\, df$$

$$\bar{X}(f) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j2\pi f k T}$$

$$x[n] = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \sum_k x[k]\, e^{-j2\pi f k T}\, e^{j2\pi f n T}\, df$$

$$= T \sum_k x[k] \int_{-\frac{1}{2T}}^{\frac{1}{2T}} e^{j2\pi f (n-k) T}\, df$$

$$= T \sum_k x[n] \int_{-\frac{1}{2T}}^{1/2T} \cos(2\pi f (n-k) T) + j \int_{-\frac{1}{2T}}^{1/2T} \sin(2\pi f (n-k) T)\, df$$

$$= T\sum_k x[n]\underbrace{\int_{-\frac{1}{2T}}^{\frac{1}{2T}}\cos(2\pi f(n-k)T)}_{\begin{cases}1/T & n=k\\ 0 & n\neq k\end{cases}} + j\underbrace{\int_{-\frac{1}{2T}}^{\frac{1}{2T}}\sin(2\pi f(n-k)T)\,df}_{\emptyset\ \forall n,k}$$

$$\Downarrow$$

$$\frac{1}{T}\delta[n-k]$$

$$x[n] = T\sum_k \frac{1}{T}\delta[n-k]\cdot x[k]$$

$$= \sum_k x[k]\,\delta[n-k] = x[n]$$

COND. SUFF PER L'ESISTENZA DELLA TFS

$$\sum_{k=-\infty}^{+\infty}|x[n]| = k < \infty$$

Dim:

$$|\bar{X}(f)| = \left|\sum_n x[n]\,e^{-j2\pi f nT}\right|$$

$$\leq \sum_n |x[n]|\,\left|e^{-j2\pi f nT}\right| = k < \infty$$

$$\Downarrow$$

$$|\bar{X}(f)| < \infty \qquad \text{Converge}$$

RELAZIONE TRA TCF E TFS

$$\bar{X}(f) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X\left(f-\frac{n}{T}\right)$$

Dim:

$$\bar{X}(f) = \sum_n x[n]\,e^{-j2\pi f nT} = \sum_n x(nT)\,e^{-j2\pi f nT}$$

$$x(nT) = \int_{-\infty}^{+\infty} X(\alpha)\,e^{j2\pi\alpha nT}$$

$$= \sum_n \underbrace{\int_{-\infty}^{+\infty} X(\alpha)\,e^{j2\pi\alpha nT}\,d\alpha}_{x(nT)}\ e^{-j2\pi f nT}$$

$$= \int_{-\infty}^{+\infty} X(\alpha)\underbrace{\sum_n e^{-j2\pi(f-\alpha)nT}}\,d\alpha$$

$$= \int_{-\infty} X(\alpha) \underbrace{\sum_n e^{-j2\pi(f-\alpha)nT}}_{\substack{\| \, 2^{a} \text{ poisson} \\ \frac{1}{T}\sum_n \delta(f-\alpha-\frac{n}{T})}} d\alpha$$

$$= \frac{1}{T} \int_{-\infty}^{+\infty} X(\alpha)\, \delta(\underbrace{f-\tfrac{n}{T}}_{f'} - \alpha)\, d\alpha$$

$$= \frac{1}{T} \sum_n X(f) \otimes \delta(f - \tfrac{n}{T})$$

$$f'$$

$$\Rightarrow \bar{X}(f) = \frac{1}{T} \sum_n X(f - \tfrac{n}{T})$$

CORRELAZIONE TRA SEQUENZE

$$R_{xy}[m] = \sum_{n=-\infty}^{+\infty} x[n]\, y^*[n-m]$$

$$R_x[m] = \sum_{n=-\infty}^{+\infty} x[n]\, x^*[n-m] \qquad \text{autocorrelazione}$$

$$\bar{S}_x(f) = TFS[R_x[m]] = \sum_m R_x[m]\, e^{-j2\pi f m T}$$

$$= \sum_n \sum_m x[n]\, x^*[n-m]\, e^{-j2\pi m f T} \qquad n - m = m'$$

$$= \sum_n x[n] \sum_{m'} x^*[m']\, e^{-j2\pi(n-m')T}$$

$$= \underbrace{\sum_n x[n]\, e^{-j2\pi nTf}}_{\bar{X}(f)} \underbrace{\left[\sum_{m'} x^*[m']\, e^{-j2\pi f m' T}\right]^*}_{\bar{X}(f)^*}$$

$$= \bar{X}(f)\, \bar{X}(f)^* = |X(f)|^2 = \bar{S}_x(f)$$

$$R_x[m]\Big|_{m=\emptyset} = \sum_{n=-\infty}^{+\infty} x[n]\, x^*[n-m]\Big|_{m=0} = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = E_x$$

$$E_x = R_x[m]\Big|_{m=0} = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{S}_x(f)\, e^{j2\pi f m T}\, df\Big|_{m=0} = T \int_{-\frac{1}{2T}}^{\frac{1}{2T}} \bar{S}_x(f)\, df$$

CONDIZIONE DI NYQUIST

$$T \leq \frac{1}{2B}$$

← banda del segnale a banda rigorosamente

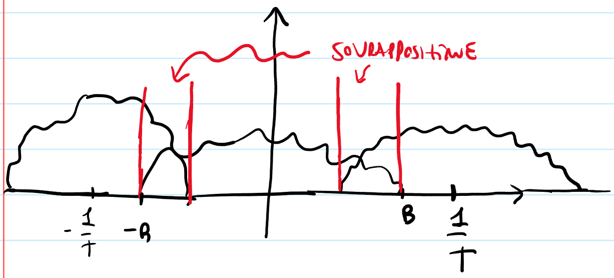
rigorosamente
limitata

Dim:

2° casi $\begin{cases} T > \frac{1}{2B} \\ T < \frac{1}{2B} \end{cases}$

1° caso: $T > \frac{1}{2B}$

$\bar{X}(f) = TFS[x[n]] = \frac{1}{T} \sum_n X(f - \frac{n}{T})$



$T > \frac{1}{2B} \Rightarrow$ ALIASING

2° CASO: $T < \frac{1}{2B}$

INTERPOLAZIONE

$$\tilde{x}(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t - nT)$$

TEOREMA:

$$\tilde{X}(f) = TCF[\tilde{x}(t)] = \int_{-\infty}^{+\infty} \tilde{x}(t)\, e^{-j2\pi f t}\, dt$$

$$\Big(\tilde{x}(t) = \sum_n x[n]\, p(t - nT)\, e^{-j2\pi f t}\, dt$$

$$\tilde{X}(t) = \sum_n x[n]\, p(t-nT)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \sum_n x[n]\, p(t-nT)\, e^{-j2\pi f t}\, dt$$

$$= \sum_n x[n] \int_{-\infty}^{+\infty} p(t-nT)\, e^{-j2\pi f t}\, dt \qquad t-nT = t'$$

$$= \sum_n x[n] \int_{-\infty}^{+\infty} p(t')\, e^{-j2\pi f (t'+nT)}\, dt'$$

$$= \underbrace{\sum_n x[n]\, e^{-j2\pi f nT}}_{\bar{X}(f)} \underbrace{\int_{-\infty}^{+\infty} p(t')\, e^{-j2\pi f t'}\, dt'}_{P(f)}$$

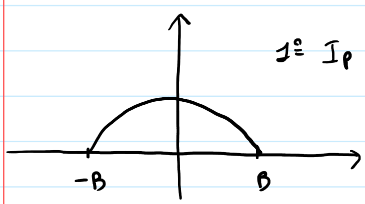
$$\tilde{X}(f) = \bar{X}(f) \cdot P(f)$$

TEOREMA DEL CAMPIONAMENTO

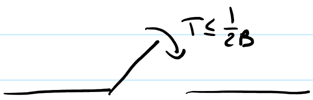
Ip:
- $x(t)$ a banda rigorosamente limitata
- $T \leq 1/2B$ (cond Nyquist)
- $p(t)$ e' un interpolatore cardinale $\begin{cases} p(t) = \mathrm{sinc}(2Bt) \\ P(f) = \frac{1}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right) \end{cases}$

Ts: si puo' ricostruire esattamente il segnale $x(t)$ a partire dai suoi campioni

Dim:

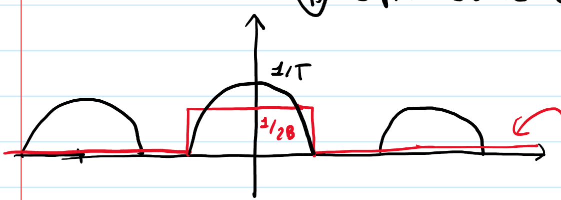


1° Ip : Banda rigorosamente limitata

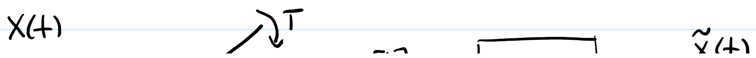


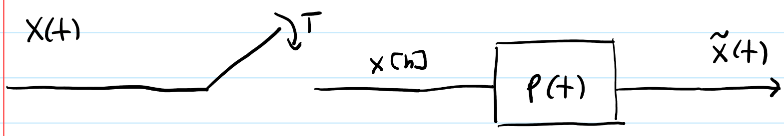
2° Ip: Nyquist soddisfatta

$\Downarrow$ copie senza aliasing



interpolatore cardinale : seleziona una copia, che e' la X originale con un fattore moltiplicativo





è la x originale con un fattore moltiplicativo

$$\tilde{X}(f) = \bar{X}(f)\, P(f) = \frac{1}{2BT} X(f)$$

$$\tilde{X}(f) = K\, X(f)$$

$$\Updownarrow \text{TCF}$$

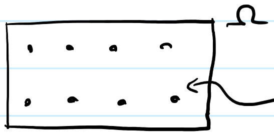
$$\tilde{x}(t) = K\, x(t)$$

Probabilità

martedì 21 giugno 2022 18:15

TEORIA DELLA PROBABILITA'

- SPAZIO CAMPIONE



tutte le possibili soluzioni dell'esperimento casuale

$\Omega = \{\omega_1, \omega_2 .. \omega_n\}$

- EVENTO: e' un sottinsieme dello spazio campione secondo una certa regola

PROPRIETA' DI UN EVENTO:

- Se A e' un evento, allora anche $\bar{A}$ e' un evento

  $\bar{A}$ e' il complemento di A

- se A e B sono eventi, allora anche $A \cup B$ e' un evento.

PROPRIETA' DERIVATE:

- $A \cap B$ e' un evento
- $A \cup \bar{A} = \Omega$ evento certo
- $A \cap \bar{A} = \emptyset$ evento impossibile

PROBABILITA'

DEFINIZIONE ASSIOMATICA (KOLMOGOROV)

- $P(A) \geq 0$
- $P(\Omega) = 1$

- Se A e B sono due eventi mutuamente esclusi $(A \cap B = \emptyset)$

$$P(A \cup B) = P(A) + P(B)$$

PROPRIETA':

- $P(\bar{A}) = 1 - P(A)$
- $P(\emptyset) = 0$
- $0 \leq P(A) \leq 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

PROBABILITA' CONDIZIONATA

$$P(A|B) \triangleq \frac{P(AB)}{P(B)}$$

- DEFINIZIONE CLASSICA DI PROBABILITA' (PASCAL)

$$P(A) \triangleq \frac{N_f(A)}{N}$$

$N_f(A)$ ← casi favorevoli

N ← casi totali

- INDIPENDENZA

A e' indipendente da B se

$$P(A|B) = P(A)$$

- TEOREMA DI BAYES

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Dim:

$$P(AB) = P(BA)$$

$$P(AB) = P(BA)$$

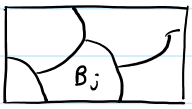
$$P(A|B)\,P(B) = P(B|A)\,P(A)$$

$$\Downarrow$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

TEOREMA DELLA PROBABILITA' TOTALE

PARTIZIONE:



$\{B_i\}$ e' una partizione di Ω se:

- $B_1 \cup B_2 \cup .. B_N = \Omega$
- $B_i \cap B_J = \emptyset \quad \forall i, J$

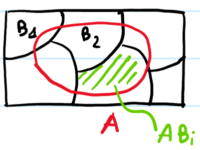
TEOREMA

$$P(A) = \sum_{i=1}^{N} P(A|B_i)\,P(B_i)$$

con $\{B_i\}$ partizione di Ω

Dim:

$$P(A) = P(A\,\Omega) = P\left(A \sum_{i=1}^{N} B_i\right) = P\left(\sum_{i=1}^{N} AB_i\right)$$



$$= P(AB_1 \cup AB_2 \cup .. AB_N)$$

$$= P(AB_1) + P(AB_2) + .. P(AB_N)$$

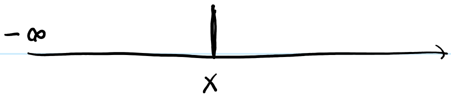
$$= \sum_{i=1}^{N} P(AB_i) = \sum_{i=1}^{N} P(A|B_i)\,P(B_i)$$

Variabili aleatorie

mercoledì 22 giugno 2022 18:43

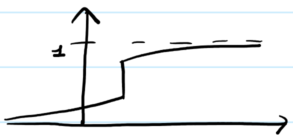
DISTRIBUZIONE DI PROBABILITA'

$$F_X(x) \triangleq P(X \leq x) \qquad X \leq x \text{ e' un evento}$$



PROPRIETA':

- $0 \leq F_X(x) \leq 1$
- $\lim_{x \to \infty} F_X(x) = 1$
- $\lim_{x \to -\infty} F_X(x) = \emptyset$
- $F_X(x)$ e' monotona non decrescente
  $[P(x \leq 5) < P(x \leq 6)]$
- $F_X(x)$ puo' avere discontinuita' di prima specie



- $P(a \leq X \leq b) = F_X(b) - F_X(a)$

DENSITA' DI PROBABILITA'

$$f_X(x) \triangleq \frac{d}{dx} F_X(x)$$

$$F_X(x) \triangleq \int_{-\infty}^{x} f_X(\alpha)\, d\alpha$$

PROPRIETA':

- $f_X(x) \geq 0$
- $P(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx$
- $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

V. A. UNIFORMI

$$f_X(x) = \frac{1}{b-a} \operatorname{rect}\left(\frac{x - \frac{a+b}{2}}{b-a}\right)$$

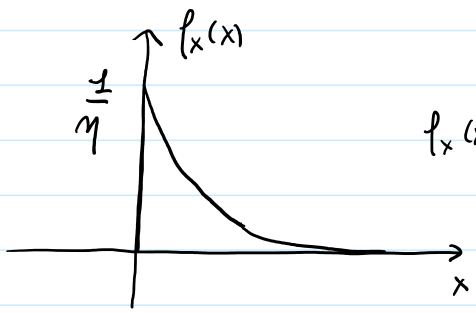
$b - a$ = intervallo

$\frac{a+b}{2}$ = valore medio

Es $x \in [0, 10] \rightarrow b - a = 10$

$\frac{a+b}{2} = 5$

V. A. ESPONENZIALI

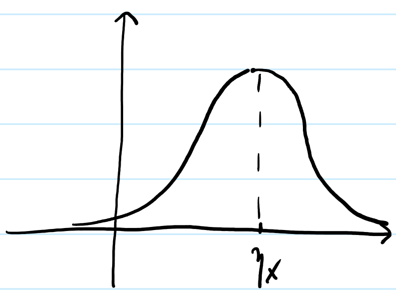


$$f_X(x) = \frac{1}{\eta} e^{-\frac{x}{\eta}} u(x)$$

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x}{\eta}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

V. A. GAUSSIANA

$$f_X(x) \triangleq \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\eta_x)^2}{2\sigma_x^2}}$$



$$X \in N(\eta_x, \sigma_x^2)$$

GAUSSIANA STANDARD: $X \in N(0,1)$

TH. FONDAMENTALE PER LA TRASFORMAZIONE DI UNA V.A.

$$Y = T[X] = g(x)$$

$$f_Y(y) = \sum_i \frac{f_X(x_i)}{|g'(x_i)|} \qquad \text{con } g'(x) = \frac{d}{dx} g(x)$$

con x_i soluzioni dell' inversa $[x = g(y)]$

INDICI CARATTERISTICI DI UNA D.D.P.

- VALORE MEDIO

$$\eta_X = E[X] = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$$

- TEOREMA DEL VALOR MEDIO

se $Y = g(x)$

$$\eta_Y = E[Y] = \int_{-\infty}^{+\infty} y \, f_Y(y) \, dy = \int_{-\infty}^{+\infty} g(x) \, f_X(x) \, dx$$

$$= E[g(x)]$$

$$\Rightarrow \quad E[Y] = E[g(x)]$$

- LINEARITA'

$$Y = \alpha g(x) + \beta h(x)$$

$$\Downarrow$$

$$\eta_Y = \alpha E[g(x)] + \beta E[h(x)]$$

- VARIANZA

- VARIANZA

$$\sigma_X^2 = E\left[(X-\eta_X)^2\right] = \int_{-\infty}^{+\infty} (x-\eta_X)^2 f_X(x)\,dx$$

- DEVIAZIONE STANDARD

$$\sigma_X = \sqrt{\sigma_X^2}$$

- VALOR QUADRATICO MEDIO

$$m_X^2 \triangleq E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

RELAZIONE TRA $\eta_X, \sigma_X^2, m_X^2$

$$\sigma_X^2 = E\left[(X-\eta_X)^2\right] = E\left[X^2 + \eta_X^2 - 2X\eta_X\right]$$

$$= E[X^2] + E[\eta_X^2] - 2E[X\eta_X]$$

$$= m_X^2 + \eta_X^2 - 2\eta_X E[X]$$

$$= m_X^2 + \eta_X^2 - 2\eta_X^2$$

$$= m_X^2 - \eta_X^2 = \sigma_X^2$$

DIM CORRETTEZZA V.A. GAUSSIANE

$$X \in N(\eta_X, \sigma_X^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\eta_X)^2}{2\sigma_X^2}} dx$$

Dim η_X:

$$\eta_X = E[X] = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-(x-\eta_X)^2}{2\sigma_X^2}} dx \qquad y = x - \eta_X$$

$$\eta_x = E[X] = \int_{-\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{2\sigma_x^2} dx \qquad y = x - \eta_x$$

$$= \int_{-\infty}^{+\infty} (y + \eta_x) \frac{1}{\sqrt{2\sigma_x^2 \pi}} e^{-y^2/2\sigma_x^2} dy$$

$$= y \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\sigma_x^2}} e^{-y^2/2\sigma_x^2}}_{=\emptyset} + \eta_x \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\sigma_x^2}} e^{-y^2/2\sigma_x^2}}_{=1}$$

poiche' e' una funz dispari

$$= \eta_x = E[X]$$

Dim σ_x^2:

$$E[(x-\eta_x)^2] = \int_{-\infty}^{+\infty} (x-\eta_x)^2 \cdot \frac{1}{\sqrt{2\sigma_x^2}} e^{\frac{-(x-\eta_x)^2}{2\sigma_x^2}} \qquad x - \eta_x = y$$

$$= \int_{-\infty}^{+\infty} y^2 \frac{1}{\sqrt{2\sigma_x^2}} e^{-y^2/2\sigma_x^2} dy$$

$$= -\sigma_x^2 \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\sigma_x^2}} y \left(-\frac{y}{\sigma_x^2} e^{-y^2/2\sigma_x^2}\right) dy$$

$$\left(\frac{d}{dy}\left[e^{-y^2/2\sigma_x^2}\right] = -\frac{1}{\cancel{2}\sigma_x^2} e^{\frac{-y^2}{2\sigma_x^2}} \cdot \cancel{2}y\right)$$

$$= -\sigma_x^2 \left\{ \underbrace{\frac{1}{\sqrt{2\pi\sigma_x^2}} y\, e^{-y^2/2\sigma_x^2} \Big|_{-\infty}^{+\infty}}_{=\emptyset} - \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\sigma_x^2\pi}} e^{-y^2/2\sigma_x^2}}_{1} \right\}$$

$$= \sigma_x^2 = E[(x-\eta_x)^2]$$

TRASFORMAZIONE DI UNA V.A.

$$N \in \mathcal{N}(0,1)$$

INVERSA

$$X = \sigma_x^2 N + \eta_x \qquad N = \frac{X - \eta_x}{\sigma_x^2}$$

$$f_x(x) = \left. \frac{f_N(n)}{|g'(n)|} \right|_{n = \frac{x - \eta_x}{\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{x-\eta_x}{2\sigma_x^2}\right)} \Rightarrow \mathcal{N}\left(\eta_x, \sigma_x^2\right)$$

Stazionarietà

giovedì 23 giugno 2022 11:07

STAZIONARIETA' IN SENSO STRETTO

$$f_X(x_1, x_2 .. x_N, t_1, t_2, .. t_N) = f_X(x_1 .. x_N, t_1 + \Delta t \dots t_N + \Delta t)$$

$\Downarrow$

Invarianza rispetto al tempo

1° ORDINE:

$$f_X(x, t_1) = f_X(x, t_1 + \Delta t) = f_X(x)$$

2° ORDINE:

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2, t_1 + \Delta T, t_2 + \Delta T)$$

$$= f_X(x_1, x_2, t_1 - t_2)$$

INDICI:

- $$\eta_X(t) = E[X(t)] = \int_{-\infty}^{+\infty} x\, f_X(x,t)\, dx$$

$$= \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \eta_X \quad \text{costante}$$

- $$P_X(t) \triangleq E[X^2(t)] = \int_{-\infty}^{+\infty} x^2 f(x,t)\, dx$$

$$= \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = P_X$$

$$\sigma^2(t) = \sigma_X^2$$

- $\sigma_X^2(t) = \sigma_X^2$

2° ORDINE:

- $R_X(t_1, t_2) = E[X(t_1)X(t_2)]$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_X(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_X(x_1, x_2, t_1 - t_2)\, dx_1\, dx_2$$

$$= R_X(t_1 - t_2) = R_X(\tau) \qquad \tau \triangleq t_1 - t_2$$

- $C_X(t_1, t_2) = E[(X(t_1) - \eta_X(t_1))(X(t_2) - \eta_X(t_2))]$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x_1 - \eta_X(t_1))(x_2 - \eta_X(t_2)) f_X(x_1, x_2, t_1, t_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x_1 - \eta_X)(x_2 - \eta_X) f_X(x_1, x_2, t_1 - t_2)\, dx_1\, dx_2$$

$$= C_X(t_1 - t_2) = C_X(\tau)$$

$C_X(\tau) = R_X(\tau) - \eta_X^2$ per processi stazionari

STAZIONARIETA' IN SENSO LATO (SSL)

Un processo e' stazionario in senso lato se:

- $\eta_X(t) = \eta_X$ costante
- $R_X(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$

$$\Downarrow$$

$C_X(t_1, t_2) = C_X(\tau) = R_X(\tau) - \eta^2$

$$\Downarrow$$

$$C_X(t_1,t_2) = C_X(\tau) = R_X(\tau) - \eta_X^2$$

PROPRIETA' PROCESSI SSL:

- $R_X(\tau) = R_X(-\tau)$

Dim:

$R_X(\tau) = E\left[X(t)\,X(t-\tau)\right]$ $\quad t = t_1$, $\tau = t_1 - t_2$

$= E\left[X(t_1)\,X(t_1-(t_1-t_2))\right]$

$= E\left[X(t_1)\,X(t_2)\right]$

$R_X(-\tau) = E\left[X(t)\,X(t+\tau)\right]$ $\quad t+\tau = t'$

$= E\left[X(t'-\tau)\,X(t')\right] = R_X(\tau)$

- $R_X(0) = E\left[X(t)X(t-\tau)\right]\Big|_{\tau=0}$

$= E\left[X(t)^2\right] = P_X$ costante poiché SSL

- $R_X(0) \geq |R_X(\tau)|$

Dim:

$E\left[\{X(t) \pm X(t-\tau)\}^2\right] \geq 0$

$E\left[X^2(t) + X^2(t-\tau) \pm 2X(t)X(t-\tau)\right] \geq 0$

$$\underbrace{E\left[X^2(t)\right]}_{= P_X} + \underbrace{E\left[X^2(t-\tau)\right]}_{= P_X} \pm 2\underbrace{E\left[X(t)X(t-\tau)\right]}_{R_X} \geq 0$$

$$\underset{\displaystyle P_x}{\parallel} \qquad \underset{\displaystyle P_x}{\parallel} \qquad \underbrace{\qquad\qquad}_{R_x}$$

$$\cancel{2} P_x \geq \cancel{2} |R_x(\tau)|$$

$$\underset{\displaystyle R_x(0)}{\parallel}$$

- Se $R_x(\tau)$ non ha componenti periodiche:

$$\lim_{\tau \to \infty} R_x(\tau) = \eta_x^2$$

Dim:

$$R_x(\tau) = C_x(\tau) + \eta_x^2$$

$$\lim_{\tau \to \infty} C_x(\tau) = \emptyset$$

poiché due variabili estratte a distanza di tempo infinita si assumono incorrelate.

FILTRAGGIO DI UN PROCESSO ALEATORIO

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

INDICI:

- VALORE MEDIO:

$$\eta_y(t) = E[y(t)] = E[x(t) \otimes h(t)]$$

$$= E\left[\int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{+\infty} E[x(\tau)]\, h(t-\tau)\, d\tau$$

$$= \int^{+\infty} \eta_x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} \eta_x(\tau)\, h(t-\tau)\, d\tau$$

$$= \eta_x(t) \otimes h(t) = \eta_y(t)$$

- **AUTOCORRELAZIONE**

$$R_y(t_1,t_2) \triangleq E[\,Y(t_1)\,Y(t_2)\,]$$

$$= E\left[\int_{-\infty}^{+\infty} X(\tau_1)\, h(t_1-\tau_1)\, d\tau_1 \int_{-\infty}^{+\infty} X(\tau_2)\, h(t_2-\tau_2)\, d\tau_2\right]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[\,X(\tau_1)\,X(\tau_2)\,]\, h(t_1-\tau_1)\, h(t_2-\tau_2)\, d\tau_2$$

$$= \int_{-\infty}^{+\infty} \underbrace{\int_{-\infty}^{+\infty} R_x(\tau_1,\tau_2)\, h(t_1-\tau_1)\, d\tau_1}\; h(t_2-\tau_2)\, d\tau_2$$

$$= \int_{-\infty}^{+\infty} \underbrace{R_x(t_1,\tau_2) \otimes h(t_1)}_{R_x'(\tau_2)}\; h(t_2-\tau_2)\, d\tau_2$$

$$= R_x'(t_2) \otimes h(t_2) = R_x(t_1,t_2) \otimes h(t_1) \otimes h(t_2)$$

$$\Downarrow$$

$$R_y(t_1,t_2) = R_x(t_1,t_2) \otimes h(t_1) \otimes h(t_2)$$

PROCESSI SSL

- $\eta_y(t) = \eta_x(t) \otimes h(t) = \eta_x \otimes h(t)$

$$= \int_{-\infty}^{+\infty} \eta_x\, h(t-\tau)\, d\tau = \eta_x \int_{-\infty}^{+\infty} h(t-\tau)\, d\tau \qquad t-\tau=\tau'$$

$$= \eta_x \int_{-\infty}^{+\infty} h(\tau')\, d\tau' = \boxed{\eta_x \cdot H(0) = \eta_y}$$ costante

$$\Uparrow$$

$$H(f)\Big|_{f=0} = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt\Big|_{f=0} = \int_{-\infty}^{+\infty} h(t)\, dt$$

- $R_Y(t_1,t_2) = R_Y(\tau) = R_X(\tau) \otimes h(\tau) \otimes h(-\tau)$

DENSITA' SPETTRALE DI POTENZA

$$S_X(f) \triangleq \lim_{T\to\infty} E\left[\frac{|X_T(f)|^2}{T}\right]$$

TEOREMA DI WIENER-KHINTCHINE

$$S_X(f) = TCF\left[R_X(\tau)\right]$$

PROPRIETA':

- $S_X(f)$ e' reale e pari per processi SSL reali, poiche' $R_X(\tau)$ e' reale e pari

- $P_X = E\left[X^2(t)\right] = R_X(0) = \int_{-\infty}^{+\infty} S_X(f)\, df$

- $S_X(f) \geq 0 \quad \forall f$

FILTRAGGIO

$$S_Y(f) = TCF\left[R_Y(f)\right]$$

$$= TCF\left[R_X(\tau) \otimes h(\tau) \otimes h(-\tau)\right]$$

$$= S_X(f)\, H(f)\, H^*(f) = S_X(f)\cdot|H(f)|^2 = S_Y(f)$$

PROCESSO DI RUMORE BIANCO

- $\eta_X = \emptyset$
- $R_X(\tau) = k\,\delta(\tau) \Rightarrow S_X(f) = k$

solitamente

$$\left[S_X(f) = k \, rect\left(\frac{f}{B}\right) \right]$$

PROCESSI ALEATORI GAUSSIANI

$$X(t_0) \triangleq X_0 \in \mathcal{N}\left(\eta_{X_0}(t_0), \sigma^2_{X_0}(t_0)\right)$$

$$f_{X_0}(x_0, t_0) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-(x_0 - \eta_X(t_0))^2}{2\sigma_X^2}}$$

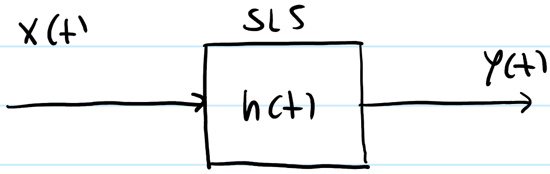
- PROCESSO GAUSSIANO E BIANCO

- $\eta_X = 0$
- $R_X(\tau) = k\,\delta(t) \rightarrow S_X(f) = k$

Solitamente

$$\left[S_X(f) = \frac{N_0}{2} \right]$$

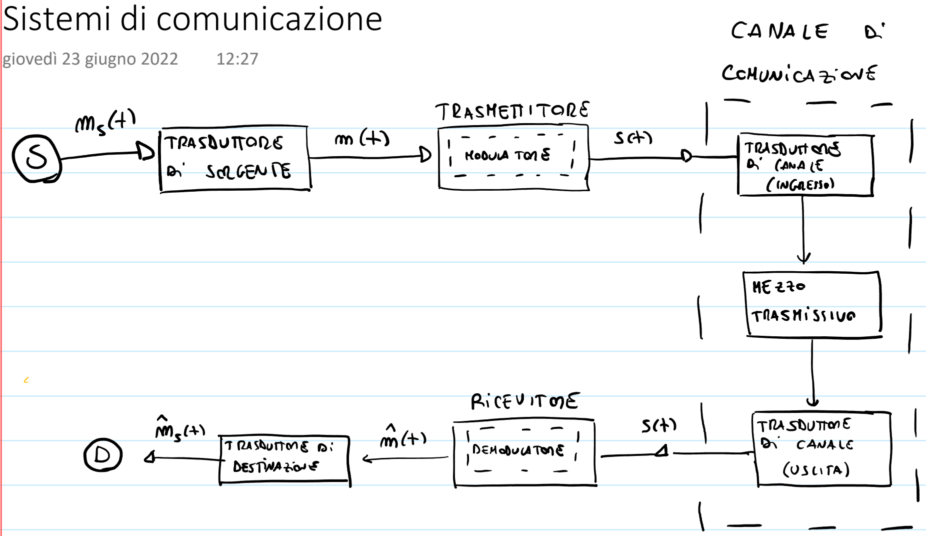
FILTRAGGIO



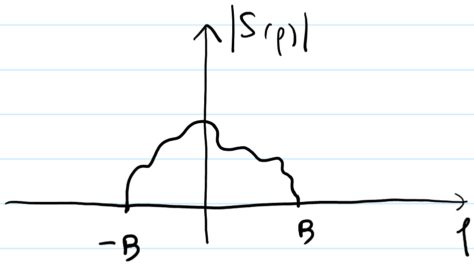
Se x(t) e' gaussiano, allora anche y(t) e' gaussiano.

Sistemi di comunicazione

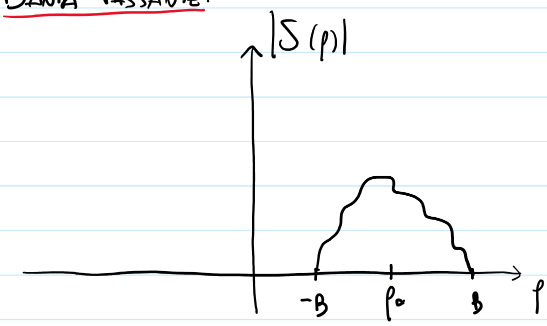
giovedì 23 giugno 2022 12:27



BANDA BASE:



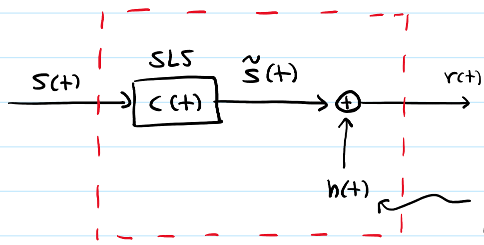
BANDA PASSANTE:



Banda larga: $f_0 \leq 2B$

Banda stretta: $f_0 >> 2B$

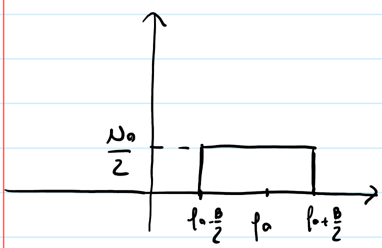
CANALE DI COMUNICAZIONE



Gaussiano Bianco:

$$\begin{cases} \eta_n = \emptyset \\ S_n(f) = \dfrac{N_0}{2} \quad \left(R_n(\tau) = \dfrac{N_0}{2}\delta(\tau) \right) \end{cases}$$

RUMORE BIANCO IN BANDA:

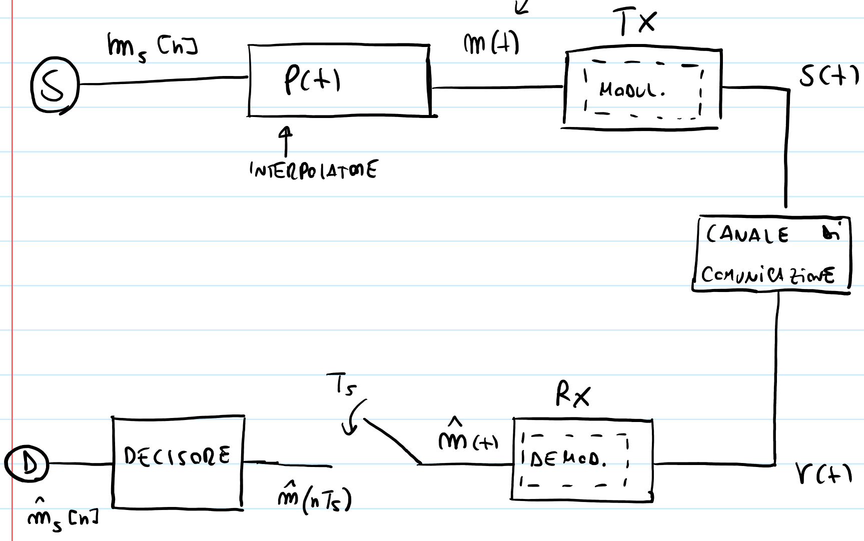


$$n(t): \begin{cases} \eta_n = \emptyset \\ S_n(f) = \dfrac{N_0}{2}\left[\mathrm{rect}\left(\dfrac{f-f_0}{B}\right) + \mathrm{rect}\left(\dfrac{f+f_0}{B}\right)\right] \end{cases}$$

SISTEMA DI COMUNICAZIONE NUMERICO

$m_s[n]$, $\hat{m}_s[n]$ numeri

$m(t) = \sum_n m_s[n]\, p(t - nT_s)$ con T_s = periodo di segnalazione della sorgente



$R_s \triangleq \frac{1}{T_s}$: tasso di erogazione dei simboli

PROBABILITA' DI TRANSIZIONE

$$P\{i|j\} \triangleq P\{\hat{m}_s = \alpha_i \mid m_s = \alpha_j\} \qquad \alpha_i, \alpha_j \in A_s$$

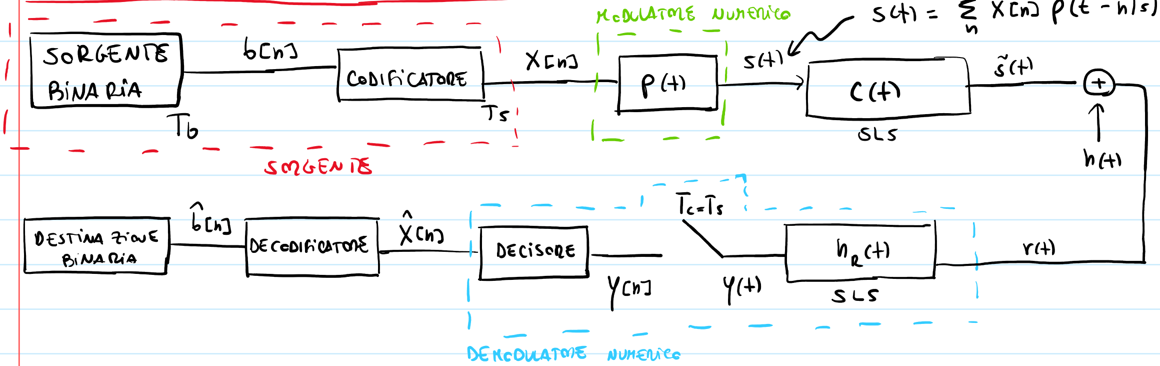
PROBABILITA' DI ERRORE SUL SIMBOLO:

$$P_E(M) \triangleq P\{\hat{m}_s[n] \neq m_s[n]\}$$

num simboli in A_s

MODULAZIONI NUMERICHE IN BANDA BASE

$s(t) = \sum_n x[n]\, p(t - nT_s)$



$$s(t) = \sum_n x[n]\, p(t - nT_s)$$

è un segnale aleatorio, poiché generato da una sequenza di simboli aleatori.

PROBABILITA' DI ERRORE:

- sul bit $\Rightarrow P(\hat{b}[n] \neq b[n])$
- sul simbolo $\Rightarrow P(\hat{x}[n] \neq x[n])$

- sul bit $\Rightarrow P(\hat{b}[n] \neq b[n])$
- sul simbolo $\Rightarrow P(\hat{x}[n] \neq x[n])$

$$\underbrace{BER}_{\text{bit error rate}} = P_E(b) \cdot \underbrace{R_b}_{\text{bit rate}}$$

FORMULA:

$$P_E(M) = P(\hat{X}[n] \neq X[n]) = \sum_{i=1}^{M} \sum_{\substack{J=1 \\ J\neq i}}^{M} P(\hat{X}[n] = \alpha_i \mid X[n] = \alpha_J)\, P(X[n] = \alpha_J)$$

SIMBOLI EQUIPROBABILI:

$$P_E(M) = \frac{1}{M} \sum_{i=1}^{M} \sum_{\substack{J=1 \\ J\neq i}}^{M} P(i \mid J) \qquad \left[P(i \mid J) = \frac{P(i,J)}{P(J)} \right]$$

ENERGIA:

$$\alpha_i \Rightarrow E_{S_i} = E_S \Big|_{X[n] = \alpha_i} = \int_{-\infty}^{+\infty} |s_i(t)|^2 dt$$

EQUIENERGIA $\Rightarrow E_{S_i} = E_S \quad \forall i$

PERFORMANCE

- EFFICENZA ENERGETICA

$$\eta_P \triangleq \frac{1}{SNR} \qquad SNR \triangleq \frac{P_S}{P_n}$$

P_S: Potenza utile del segnale

P_n: Potenza di rumore

- EFFICENZA SPETTRALE

$$\eta_B \triangleq \frac{R_b}{B_T}$$

PULSE AMPLITUDE MODULATION (PAM) (20-21)

$$x[n] \longrightarrow \boxed{p(t)} \xrightarrow{s(t)} \quad \Leftrightarrow \quad s(t) = \sum_n x[n]\, p(t - nT_s)$$

PROPRIETA':

- M simboli equiprobabili $\Leftarrow$ STANDARD

$$\Downarrow$$

$$\alpha_i = 2i - 1 - M \qquad \forall i = 1..M$$

- $E[s(t)] = E\left[\sum_n x[n]\, p(t - nT_s)\right]$

$= \sum_n E[x[n]]\, p(t - nT_s)$

$$E[x[n]] = \underbrace{\sum_{i=1}^{N} P[x[n] = \alpha_i]\,\alpha_i}_{\text{valore medio per V.A. discrete}} = \frac{1}{M} \underbrace{\sum_{i=1}^{M} \alpha_i}_{\emptyset} = \emptyset$$

$E[s(t)] = \emptyset$

- $S_s(f) = \frac{1}{T_s}\, \bar{S}_x(f)\, |P(f)|^2$

DSP

$\bar{S}_x(f) = TFS[R_x[m]]$

$R_x[m] = E[x[n]\, x[n-m]]$

$R_x[m] = C_x[m] + \eta_x^2[m]$

Dato che nelle PAM $\eta_x[m] = \emptyset$:

$R_x[m] = C_x[m]$

$C_x[m] = E[(x[n] - \eta_x)(x[n-m] - \eta_x)]$

$m = 0 \Rightarrow E[(x[n] - \eta_x)^2] = \sigma_x^2$

questo perche' 2 V.A. indipendenti sono incorrelate $(C_x[m] = \emptyset)$

$$C_x[m] = \begin{cases} \sigma_x^2 & m = 0 \\ 0 & m \neq 0 \end{cases} \Rightarrow C_x[m] = \sigma_x^2\, \delta[m]$$

$R_x[m] = C_x[m] = \sigma_x^2\, \delta[m]$

$\bar{S}_x(f) = TFS[R_x[m]] = \sigma_x^2$

$$\boxed{S_s(f) = \frac{\sigma_x^2}{T_s} |P(f)|^2} \leftarrow \text{solo per PAM}$$

$P_s = \int_{-\infty}^{+\infty} S_s(f)\, df$

ENERGIA MEDIA PER SIMBOLO TRASMESSO

$\alpha_i \Rightarrow E_{s_i} = \int_{-\infty}^{+\infty} (\alpha_i)^2\, p^2(t - nT_s)\, dt = \alpha_i^2\, E_p$

$$\alpha_i \Rightarrow E_{s_i} = \int_{-\infty}^{+\infty} (\alpha_i)^2 \, p^2(t-nT_s) \, dt = \alpha_i^2 E_p$$

$$E_s = E\left[\int_{-\infty}^{+\infty} X^2[n] \, p^2(t-nT_s) \, dt \right]$$

media statistica calcolata su tutti i simboli dell'alfabeto

$$\int_{-\infty}^{+\infty} E[X[n]^2] \, p^2(t-nT_s) \, dt$$

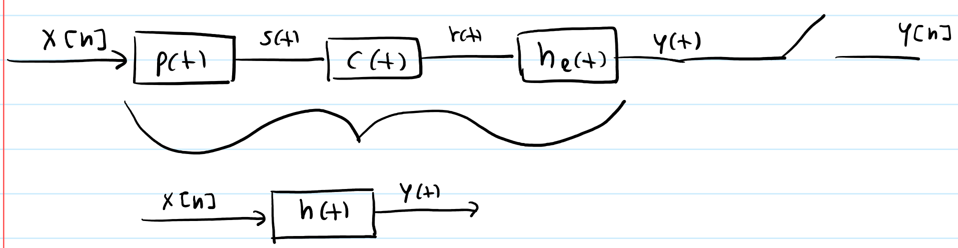
$$= E[X[n]^2] \int_{-\infty}^{+\infty} p^2(t-nT_s) \, dt$$

$$= E[X[n]^2] \, E_p$$

$$E[X[n]^2] = \sum_{i=1}^{M} P(\alpha_i) \, \alpha_i^2$$

INTERFERENZA INTER-SIMBOLICA

$$y[n] = \begin{cases} f[x[n]] & \text{assenza di ISI} \\ f[\ldots, x[n-1], x[n], x[n+1] \ldots] \end{cases}$$



$$h(t) = p(t) \otimes c(t) \otimes h_e(t) \quad \Big| \quad H(f) = P(f) \cdot C(f) \cdot H_e(f)$$

$$Y(f) = R(f) \, H_e(f) = \underbrace{S(f) \, C(f)}_{R(f)} \, H_e(f)$$

$$= \underbrace{X(f) \, P(f)}_{S(f)} \, C(f) \, H_e(f) = \bar{X}(f) \cdot H(f)$$

$$y(t) = \sum_{n=-\infty}^{+\infty} x[n] \, h(t-nT_s)$$

$$y[k] = y(kT_s) = \sum_{n=-\infty}^{+\infty} x[n] \, h((k-n)T_s)$$

$$= x[k] \, h(0) + \sum x[n] \, h((k-n)T_s)$$

$$= \underbrace{x[k]\,h(0)} + \underbrace{\sum_{\substack{n=-\infty \\ n\neq k}}^{+\infty} x[n]\,h((k-n)T_s)}_{ISI}$$

componente che dipende dal simbolo k-esimo

CRITERIO DI NYQUIST

- **NEL TEMPO:**

$$h(kT_s) = \begin{cases} c & k=0 \\ 0 & k\neq 0 \end{cases} = c\,\delta[k]$$

$$y[k] = x[k]\,h(0) + \sum_{\substack{n \\ n\neq k}} x[k]\,\underbrace{h((k-n)T_s)}_{\emptyset}$$

$$= c \cdot x[k]$$

- **IN FREQUENZA:**

$$h(kT_s) = c\,\delta[k]$$

$$\Downarrow$$

$$\bar{H}(f) = c = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} H\left(f-\frac{n}{T_s}\right)$$

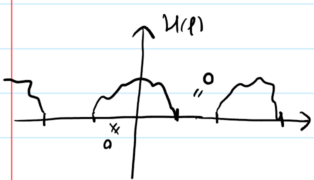
$$\Downarrow$$

$$\boxed{\frac{1}{T_s}\sum_{n=-\infty}^{+\infty} H\left(f-\frac{n}{T_s}\right) = c'}$$

$$\boxed{T_s < \frac{1}{2B_c}}$$

condizione per cui non si può eliminare l'ISI

- $T < \frac{1}{2B_c}$

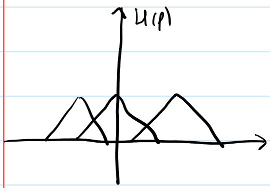


H(f) non può essere calante ⇒ ISI

- $T = \frac{1}{2B_c}$

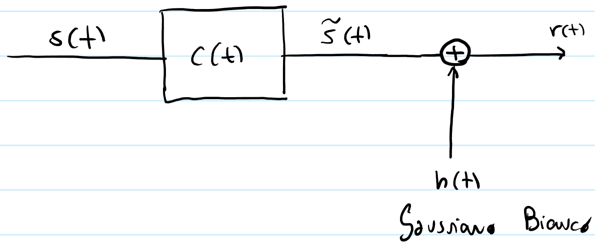
Caso non ottimale, $\sum_n H(f-\frac{n}{T})$ calante se $H(f)$ e' una rect $\Rightarrow$ No ISI

- $T > \frac{1}{2B}$



Caso migliore, si puo' ottenere $\sum_n H(f-\frac{n}{T}) = k$ con diverse funzioni H $\Rightarrow$ No ISI

PRESENZA DI RUMORE



$$r(t) = s(t) \otimes c(t) + n(t)$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, \tilde{p}(t-nT_s) + n(t)$$

$$\tilde{p}(t) = p(t) \otimes c(t)$$

Dim:

$$\tilde{s}(t) = s(t) \otimes c(t)$$

$$= \sum_n x[n]\, p(t-nT_s) \otimes c(t)$$

$$= \int_{-\infty}^{+\infty} \sum_n x[n]\, p(\tau-nT_s)\, c(t-\tau)\, d\tau$$

$$= \sum_n x[n] \int_{-\infty}^{+\infty} p(\tau-nT_s)\, c(t-\tau)\, d\tau \qquad \tau' = \tau - nT_s$$

$$= \sum_n x[n] \int_{-\infty}^{+\infty} p(\tau')\, c\left[(t-nT_s)-\tau'\right] d\tau'$$

$$= \sum_n x[n] \Big[\underbrace{p(t) \otimes c(t)}_{\tilde{p}(t)}\Big]\Big|_{t=t-nT_s}$$

$$\tilde{p}(t) = p(t) \otimes c(t)$$

$$= \underbrace{\sum_n x[n]\, \tilde{p}(t-nT_s)}_{\text{Componente utile}} + \underbrace{n(t)}_{\text{Rumore}}$$

FILTRO ADATTATO

minimizza l'effetto del rumore

$$SNR \triangleq \frac{P_S}{P_N}$$

$$h_{FA}(t) = K \tilde{p}(T_s - t)$$

$\Downarrow \mathcal{F}$

$$H_{FA}(f) = K \tilde{P}^*(f) e^{-j2\pi f T_s}$$

CRITERIO MAP (MAXIMUM A POSTERIORI PROBABILITY)

$$\hat{x} = \max_{i=1,\dots M} P(x = \alpha_i | y)$$

se simboli tutti equiprobabili:

CRITERIO DELLA MASSIMA VEROSIMIGLIANZA

$$\hat{x} = \max_{i=1,\dots M} f_y(y | x = \alpha_i)$$

Dim:

$$y[n] = x[n] h(0) + n_u[n]$$

$$y[n | x = \alpha_i] = \alpha_i h(0) + n_u[n]$$

$\downarrow$

V.A. $\in \mathcal{N}(0, \sigma_{nu}^2)$

$$\sigma_{nu}^2 = E[n_u^2] = P_{nu} = \int_{-\infty}^{+\infty} S_{nu}(f)\, df$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} |H_R(f)|^2\, df$$

$$= \frac{N_0}{2} E_{HR}$$

$$y[n | x = \alpha_i] = \alpha_i h(0) + n_u \in \mathcal{N}\left(\alpha_i h(0), \frac{N_0}{2} E_{HR}\right)$$

PAM IN BANDA BASE

FORMULE:

$$h(0) = p(t) \otimes c(t) \otimes h_R(t) \Big|_{t=0}$$

$$\sigma_{nu}^2 = P_{nu} = \frac{N_0}{2} E_{HR}$$

$$E_s = E_s \, E[x^2]$$

$$f_y(y|x=\alpha_i) \in \mathcal{N}\left(h(o)\alpha_i, \frac{N_o}{2} E_{hR}\right)$$

$$\updownarrow$$

$$f_y(y|x=\alpha_i) = \frac{1}{\sqrt{2\pi \frac{N_o}{2} E_{hR}}} e^{-\frac{(y-h(o)\alpha_i)^2}{2\frac{N_o}{2}E_{hR}}}$$

CRITERIO:

$$\text{MIN } P_E = \max_i f_y(y|x=\alpha_i)$$

Dim:

$$\hat{X} = \arg\max_{\alpha_i} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(y-h(o)\alpha_i)}{2\sigma_x^2}}$$

$$= \arg\max_{\alpha_i} e^{-\frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2}}$$

$$= \arg\min_{\alpha_i} \left[-\frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2}\right]$$

$$= \arg\min_{\alpha_i} \frac{(y-h(o)\alpha_i)^2}{2\sigma_{nu}^2}$$

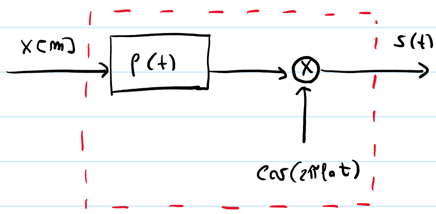
$$= \arg\min_{\alpha_i} (y-h(o)\alpha_i)^2$$

$$= \arg\min_{\alpha_i} |y-h(o)\alpha_i| = \hat{X}$$

$\Leftarrow$ si sceglie la v.A. più vicina a y osservata

PAM IN BANDA PASSANTE

$$S(t) = \sum_{n=-\infty}^{+\infty} x[n]\, p(t-nT_s) \cos(2\pi f_o t)$$



MODULATORE NUMERICO

Uguale a PAM. in B.B, ma con un coseno a modulare

- ENERGIA MEDIA PER SIMBOLO TRASMESSO (E_s)

$$S_n(t) = x[n]\, p(t-nT_s) \cos(2\pi f_o t)$$

$$E_s = E\left[\int_{-\infty}^{+\infty} S_n^2(t)\right] = E\left[\int_{-\infty}^{+\infty} x^2[n]\, p^2(t-nT_s) \cos^2(2\pi f_o t)\, dt\right]$$

media

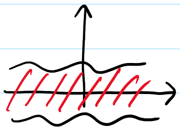
$$E_{Sn} = \int_{-\infty}^{+\infty} S_n^2(t)$$

$$= \int_{-\infty}^{+\infty} E[x^2[n]]\, p^2(t-nT_s)\cos^2(2\pi f_0 t)\, dt$$

$$= E[x^2[n]] \int_{-\infty}^{+\infty} p^2(t-nT_s)\left[\frac{1}{2}+\frac{1}{2}\cos(4\pi f_0 t)\right] dt \quad \leftarrow \cos^2(\alpha) = \frac{1}{2}+\frac{1}{2}\cos(2\alpha)$$

$$= \frac{1}{2}E[x^2[n]]\,E_p + \frac{1}{2}E[x^2[n]] \underbrace{\int_{-\infty}^{+\infty} p^2(t-nT_s)\cos(4\pi f_0 t)}_{\emptyset}$$

Poiche' e' speculare



$$\int_{-\infty}^{+\infty} \cdot = \emptyset$$

$$E_s = \frac{1}{2}E_p\, E[x[n]^2]$$

QAM

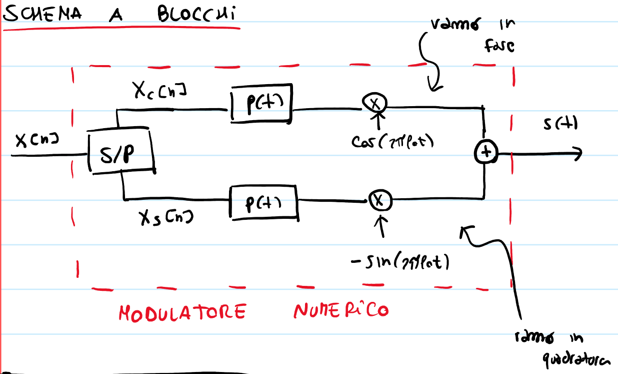
$$s(t) = \sum_n \Big(X_c[n]\, p(t-nT_s)\cos(2\pi f_0 t)$$

$$- X_s[n]\, p(t-nT_s)\sin(2\pi f_0 t)\Big)$$

$$X_c[n] \in A_s^{(c)} \qquad X_s[n] \in A_s^{(s)}$$

STANDARD: $\alpha_i = 2i - 1 - M$

SCHEMA A BLOCCHI



• ENERGIA MEDIA PER SIMBOLO TRASMESSO

$$E_s = E\left[\int_{-\infty}^{+\infty} \Big(X_c[n]\, p(t-nT_s)\cos(2\pi f_0 t)\right.$$

$$\left. - X_s[n]\, p(t-nT_s)\cos(2\pi f_0 t)\Big)^2 dt\right]$$

$$= \int_{-\infty}^{+\infty} E[X_c[n]^2]\, p^2(t-nT_s)\cos^2(2\pi f_0 t)$$

$$+ \int_{-\infty}^{+\infty} E[X_s[n]^2]\, p^2(t-nT_s)\sin^2(2\pi f_0 t)$$

$$-2\int_{-\infty}^{+\infty} \underbrace{E[X_c[n]\,X_s[n]]}_{=\ \emptyset \text{ poiche'}}\, p^2(t-nT_s)\sin(2\pi f_0 t)\cos(2\pi f_0 t)\, dt$$

X_c e X_s sono indipendenti

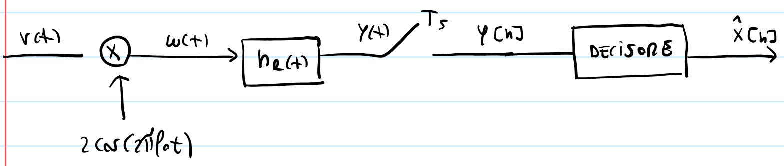
$$= \frac{1}{2} E[X_c[n]^2] E_p + \frac{1}{2} E[X_s[n]^2] E_p$$

$$= \frac{1}{2} \left[E[X_c[n]^2] + E[X_s[n]^2] \right] E_p$$

DENSITA' SPETTRALE DI POTENZA

$$S_s(f) = \frac{1}{T} \frac{\bar{S}_x(f)}{2} \left[P^2(f-f_0) + P^2(f+f_0) \right]$$

RICEVITORE PAM B.P



$$r(t) = s(t) \otimes c(t) + n(t)$$

$$= \tilde{s}(t) + n(t) \qquad \text{con } \tilde{s}(t) = s(t) \otimes c(t)$$

$\Rightarrow$ CASO SENZA RUMORE

$$\tilde{s}(t) = \int_{-\infty}^{+\infty} \sum_n x[n] \, p(\tau - nT_s) \cos(2\pi f_0 \tau) \, c(t-\tau) \, d\tau$$

$$c(t) \triangleq 2\tilde{c}(t)\cos(2\pi f_0 t)$$

$$= 2\sum_n x[n] \int_{-\infty}^{+\infty} p(\tau - nT_s) \, \tilde{c}(t-\tau) \, \cos(2\pi f_0 t) + \cos(2\pi f_0 (t-\tau)) \, d\tau$$

$$= \sum_n x[n] \int_{-\infty}^{+\infty} p(\tau - nT_s) \tilde{c}(t-\tau) \left[\cos(2\pi f_0 t) + \cos(4\pi f_0 \tau - 2\pi f_0 t) \right]$$

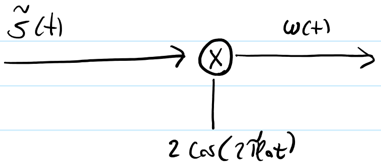
$$= \sum_n x[n] \cos(2\pi f_0 t) \int_{-\infty}^{+\infty} p(\tau - nT_s) \, \tilde{c}(t-\tau) \, d\tau$$

$$+ \sum_n x[n] \underbrace{\int_{-\infty}^{+\infty} p(\tau - nT_s) \, \tilde{c}(t-\tau) \cos(4\pi f_0 \tau - \varphi) \, d\tau}_{\simeq \emptyset \text{ poichè spoculare}}$$

$$\Rightarrow \tilde{s}(t) = \sum_n x[n] \underbrace{\int_{-\infty}^{+\infty} p(\tau - nT_s) \, \tilde{c}(t-\tau) \, d\tau}_{\substack{p(t-nT_s) \otimes \tilde{c}(t) \\ \| \\ \tilde{p}(t-nT_s)}} \cos(2\pi f_0 t)$$

$$\tilde{s}(t) = \sum_n x[n] \, \tilde{p}(t-nT_s) \cos(2\pi f_0 t)$$

$\tilde{s}(t)$ $\qquad$ $w(t)$



$$w(t) = 2\,\tilde{s}(t)\cos(2\pi f_0 t)$$

$$= \sum_n x[n]\,\tilde{p}(t-nT_s)\,2\cos^2(2\pi f_0 t)$$

$$= \underbrace{\sum_n x[n]\,\tilde{p}(t-nT_s)}_{\text{Componente in b.b.}} + \cancel{\underbrace{\sum_n x[n]\,\tilde{p}(t-nT_s)\cos(4\pi f_0 t)}_{\text{Componente a } 2f_0}}$$

Il filtro $h_R(t)$ taglie le componenti a $2f_0$:

$$y(t) = \sum_n x[n]\,\tilde{p}(t-nT_s) \qquad \text{con} \qquad \tilde{p}(t) = p(t) \otimes \tilde{c}(t)$$

$$y(t) = \sum_n x[n]\,h(t-nT_s)$$

$$h(t) = p(t) \otimes \tilde{c}(t) \otimes h_R(t)$$

$$r(t) = \sum_{n=-\infty}^{+\infty} x[n]\,\tilde{p}(t-nT_s) + n(t)$$

$$\tilde{p}(t) = p(t) \otimes \tilde{c}(t)$$

$$c(t) = 2\cos(2\pi f_0 t)\,\tilde{c}(t)$$

Quindi: $y(t)$ è uguale alla b.b. sostituendo $\tilde{c}(t)$ a $c(t)$

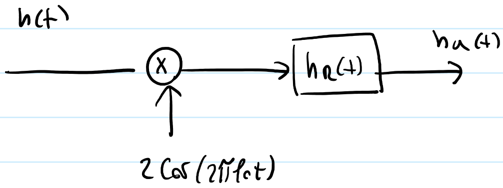
$$y[n] = y(nT_s)$$

$$y[n] = \sum_k x[k]\,h(nT_s - kT_s)$$

$$= \sum_k x[k]\,h((n-k)T_s)$$

$$= h(0)\,x[k] + \cancel{\underbrace{\sum_{n\neq k} x[k]\,h((n-k)T_s)}_{\text{ASSENZA ISI}}} = y[n]$$

COMPONENTE DI RUMORE



$n(t)$ BIANCO IN BANDA: $\begin{cases} \eta_n = 0 \\ S_n = \frac{N_0}{2}\left(rect\left(\frac{f-f_0}{B}\right) + rect\left(\frac{f+f_0}{B}\right)\right) \end{cases}$

$R_N(\tau) \rightleftarrows S_N(f)$

$R_N(\tau) = N_0 B \ \mathrm{sinc}(B\tau)\cos(2\pi f_0 \tau)$

$W(t) = n(t) \cdot 2\cos(2\pi f_0 t)$

$R_W(t_1,t_2) = E[W(t_1)W(t_2)]$

$= 4\underbrace{E[n(t_1)n(t_2)]}_{R_N(\tau)}\cos(2\pi f_0 t_1)\cos(2\pi f_0 t_2)$

$= 4 N_0 B \ \mathrm{sinc}[B(t_1-t_2)]\cos[2\pi f_0(t_1-t_2)] \cdot$

$\left[\frac{1}{2}\cos[2\pi f_0(t_1+t_2)] + \frac{1}{2}\cos[2\pi f_0(t_1-t_2)]\right]$

$= 2N_0 B \ \mathrm{sinc}[B(t_1-t_2)]\left[\cancel{\frac{1}{2}\cos(4\pi f_0 t_1) + \frac{1}{2}\cos(4\pi f_0 t_2)} + \frac{1}{2} \cancel{+ \frac{1}{2}\cos(4\pi f_0(t_1-t_2))}\right]$

Componenti a $2f_0$ tagliate dal filtro

$R_W(t_1,t_2) = 2N_0 B \ \mathrm{sinc}[B(t_1-t_2)]$

$R_{n_u}(t_1,t_2) = R_W(t_1,t_2) \otimes h_R(t_1) \otimes h_R(t_2)$

$= N_0 B \ \mathrm{sinc}(B\underbrace{(t_1-t_2)}_{SSL \Rightarrow \tau}) \otimes h_R(t_1) \otimes h_R(t_2)$

$R_{n_u}(\tau) = N_0 B \ \mathrm{sinc}(B\tau) \otimes h(\tau) \otimes h(-\tau)$

$S_{n_u}(f) = S_W(f) \cdot |H_R(f)|^2$

$W(t) \Longrightarrow R_W(\tau) = N_0 B \ \mathrm{sinc}(B\tau)$

$S_W(f) = N_0 \ rect\left(\frac{f}{B}\right)$

$S_{n_u}(f) = N_0 \ rect\left(\frac{f}{B}\right)|H_R(f)|^2$

$\Downarrow$ se Banda Base ha banda B:

$$S_{nu}(f) = N_0 \, \mathrm{rect}\left(\frac{f}{2B}\right) |H_e(f)|^2 = \underline{N_0 |H_e(f)|^2}$$

poiché in banda base raddoppia la banda

POTENZA DI RUMORE (BANDA B)

$$P_{nu} = \int_{-\infty}^{+\infty} S_{nu}(f)\, df = \int_{-B}^{B} N_0 |H_e(f)|^2 df$$

$$= \underline{N_0 \cdot E_{he} = P_{nu}} \Leftarrow \text{doppia rispetto a B.B} \quad \left[\overset{B.B}{P_{nu} = \frac{N_0}{2} E_{he}} \right]$$

$$E[w(t)] = E[n(t) \cdot 2\cos(2\pi f_0 t)]$$

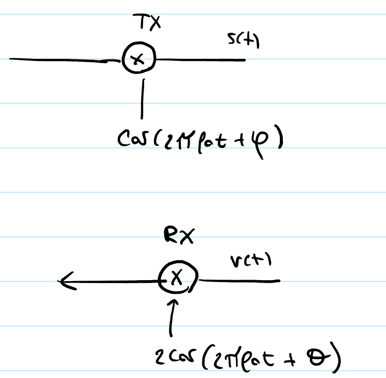
$$= \underset{\substack{\| \\ \emptyset}}{E[n(t)]} \cdot 2\cos(2\pi f_0 t) = \emptyset$$

poiché $n(t)$ gaussiano

$$\eta_{nu}(t) = \eta_w(t) \otimes h_e(t) = \emptyset \Rightarrow n_u[n] \in \mathcal{N}(0, P_{nu})$$

$$\underline{y[n] = h(0)\, x[n] + n_u[n]}$$

SINCRONISMO DI FASE



$\Rightarrow$ In genere $\theta \neq \varphi$

Se $\varphi \neq \theta \Rightarrow$ DISALLINEAMENTO DI FASE

$$s(t) = \sum_n x[n]\, p(t - nT_s) \cos(2\pi f_0 t + \varphi)$$

$$\tilde{s}(t) = s(t) \otimes c(t)$$
$$c(t) = 2\,\tilde{c}(t) \cos(2\pi f_0 t)$$

$$\tilde{s}(t) = \int_{-\infty}^{+\infty} \sum_n x[n]\, p(\tau - nT_s) \cos(2\pi f_0 \tau + \varphi) \cdot 2\,\tilde{c}(t-\tau) \cos(2\pi f_0 (t-\tau))\, d\tau$$

$$\tilde{s}(t) = \int_{-\infty}^{+\infty} \sum_n x[n]\, p(\tau - nT_s) \cos(2\pi f_0 \tau + \varphi) \cdot 2\, \tilde{c}(t-\tau) \cos(2\pi f_0 (t-\tau))\, d\tau$$

$$= \sum_n x[n] \int_{-\infty}^{+\infty} p(\tau - nT_s)\, \tilde{c}(t-\tau) \left[\cos(2\pi f_0 t + \varphi) + \cos(4\pi f_0 \tau - 2\pi f_0 t + \varphi) \right] d\tau$$

$$= \sum_n x[n] \cos(2\pi f_0 t + \varphi) \underbrace{\int_{-\infty}^{+\infty} p(\tau - nT_s)\, \tilde{c}(t-\tau)\, d\tau}_{p(t-nT_s) \otimes \tilde{c}(t) \;=\; \tilde{p}(t-nT_s)}$$

$$+ \sum_n x[n] \underbrace{\int_{-\infty}^{+\infty} p(\tau - nT_s)\, \tilde{c}(t-\tau) \cos(4\pi f_0 \tau + k)\, d\tau}_{\simeq \emptyset}$$

$$\Downarrow$$

$$\tilde{s}(t) = \sum x[n]\, \tilde{p}(t - nT_s) \cos(2\pi f_0 t + \varphi)$$

$$w(t) = \tilde{s}(t) \cdot 2 \cos(2\pi f_0 t + \theta)$$

$$= \sum_n x[n]\, \tilde{p}(t - nT_s) \left[\underbrace{\cancel{\cos(4\pi f_0 t + \varphi + \theta)}}_{\text{Comp a } 2f_0} + \cos(\varphi - \theta) \right]$$

$$= \sum_n x[n]\, \tilde{p}(t - nT_s) \underline{\cos(\varphi - \theta)}$$

$$\Updownarrow$$

$$y[n] = h(0)\, x[n]\, \underline{\cos(\varphi - \theta)}$$

$$+ \underbrace{\sum_{n \neq k} x[k]\, h((n-k)T_s) \cos(\varphi - \theta)}_{ISI}$$

$\Longrightarrow$ il disallineamento di fase inficia le prestazioni, poiché $\cos(\varphi - \theta) \leq 1$

indipendente dall'ISI

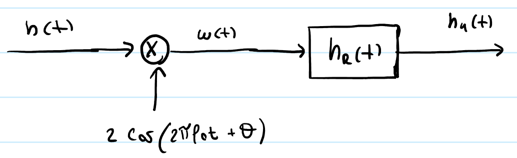
Quindi in presenza di disallineamento di fase (e assenza di ISI):

$$y[n] = \underline{h'(0)}\, x[n] + n_u$$

$$\uparrow$$

$$h'(0) = h(0) \cos(\varphi - \theta)$$

RUMORE CON DISALLINEAMENTO DI FASE



$$w(t) = 2\, n(t) \cos(2\pi f_0 t + \theta)$$

$$R_w(t_1, t_2) = E[w(t_1) w(t_2)]$$

$$= 4 E[n(t_1) n(t_2)] \underbrace{\cos(2\pi f_0 t_1 + \theta) \cos(2\pi f_0 t_2 + \theta)}_{\frac{1}{2} \cos(2\pi f_0 (t_1 + t_2) + 2\theta) + \frac{1}{2} \cos(2\pi f_0 (t_1 - t_2))}$$

$$E[n(t_1)\,n(t_2)] = R_n(t_1,t_2) = 2B\,\mathrm{sinc}\left(2B(t_1-t_2)\right)\cos\left[2\pi f_0(t_1-t_2)\right]$$

le componenti del coseno vengono dunque filtrate, poiché a $2f_0$:

$$S_{n_u}(f) = N_0\,\mathrm{rect}\left(\frac{f}{B}\right)|H_R(f)|^2 \longleftarrow$$ INDIPENDENTE DA θ

$$\Downarrow$$

$$P_{n_u} = N_0 \cdot E_{H_R}$$

$$\Rightarrow \underbrace{y[n] = h'(0)\,x[n]}_{\text{DIPENDE DA } \theta} + \underbrace{n_u[n]}_{\text{NON DIPENDE DA } \theta}$$

$$\Rightarrow \text{MAX SNR} \;\Big|\; \text{MIN } P_E(b):$$

$$\boxed{\varphi = \theta}$$

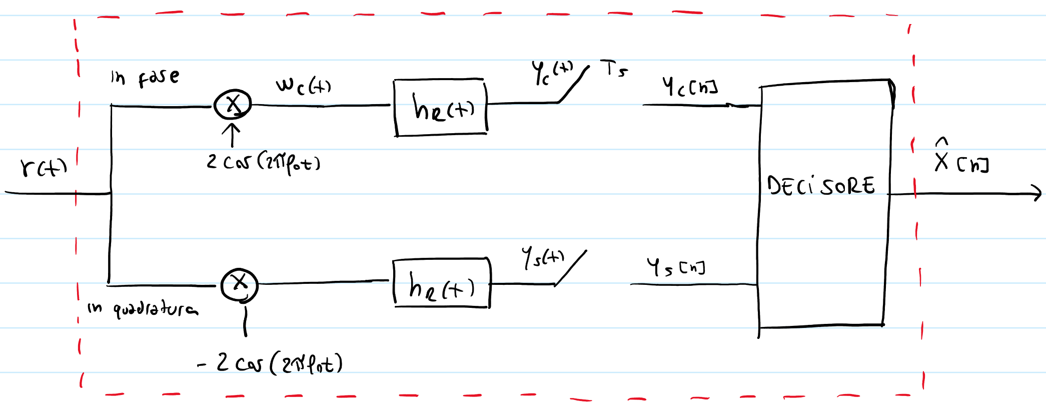
QAM

$$s(t) = \sum_n \Big[x_c[n]\,p(t-nT_s)\cos(2\pi f_0 t) - x_s[n]\,p(t-nT_s)\sin(2\pi f_0 t)\Big]$$

$$r(t) = \underbrace{s(t) \otimes c(t)}_{\tilde{s}(t)} + n(t)$$

$$\tilde{p}(t) = p(t) \otimes \tilde{c}(t)$$

$$r(t) = \sum_n x_c[n]\,\tilde{p}(t-nT_s)\cos(2\pi f_0 t) - x_s[n]\,\tilde{p}(t-nT_s)\sin(2\pi f_0 t) + n(t)$$



DEMODULATORE NUMERICO QAM

- RAMO IN FASE:

$$W_c(t) = \sum_n x_c[n]\tilde{p}(t-nT_s)\cos(2\pi f_0 t)\cdot 2\cos(2\pi f_0 t)$$
$$-\sum_n x_s[n]\tilde{p}(t-nT_s)\sin(2\pi f_0 t)\cdot 2\cos(2\pi f_0 t)$$

$$= \sum_n x_c[n]\tilde{p}(t-nT_s)\underline{[1+\cos(4\pi f_0 t)]} \quad \cos^2\alpha = \frac{1}{2}+\frac{1}{2}\cos(2\alpha)$$
$$-\sum_n x_s[n]\tilde{p}(t-nT_s)\underline{\sin(4\pi f_0 t)}$$

$$2\cos\alpha\sin\alpha = \sin(2\alpha)$$

il filtro $h_e(t)$ taglia le frequenze a $2f_0$:

$$W_c(t) = \sum x_c[n]\tilde{p}(t-nT_s) + \dots$$

$$y(t) = W_c(t)\otimes h_e(t) = \sum_n x_c[n]\tilde{p}(t-nT_s)\otimes h_e(t)$$

$$= \underline{\sum_n x[n]\,h(t-nT_s) = y(t)} \Longleftarrow$$ $y(t)$ dipende solo dai simboli in fase

$$h(t) = \tilde{p}(t)\otimes h_R(t) = p(t)\otimes\tilde{c}(t)\otimes h_e(t)$$

- RAMO IN QUADRATURA

$$W_s(t) = \sum_n x_c[n]\tilde{p}(t-nT_s)\cos(2\pi f_0 t)[-2\sin(2\pi f_0 t)]$$
$$-\sum_n x_s[n]\tilde{p}(t-nT_s)\sin(2\pi f_0 t)[-2\sin(2\pi f_0 t)]$$

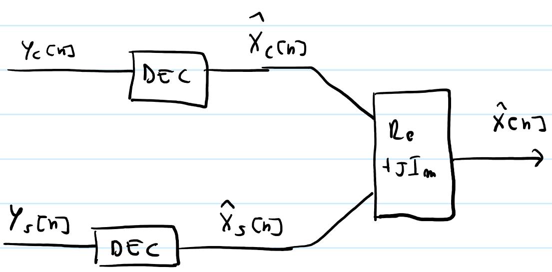
$$= -\sum_n x_c[n]\tilde{p}(t-nT_s)\sin(4\pi f_0 t)$$
$$+\sum_n x_s[n]\tilde{p}(t-nT_s)\underline{(1-\cos(4\pi f_0 t))} \quad \sin^2(\alpha) = \frac{1}{2}-\frac{1}{2}\cos(2\alpha)$$

$$y(t) = W_s(t)\otimes h_e(t) = \underline{\sum_n x_s\,h(t-nT_s)}$$

$\Uparrow$ non dipende da simboli in fase

$$h(t) = \tilde{p}(t)\otimes h_e(t)$$
$$= p(t)\otimes\tilde{c}(t)\otimes h_e(t)$$

IL DECISORE



$$P_E = P_E^c(1-P_E^s) + P_E^s(1-P_E^c) + P_E^s P_E^c$$

$$= \underline{1-(1-P_E^c)(1-P_E^s)}$$

PROB DI ERRORE SUL SIMBOLO

RUMORE

$$P_{n_s} = P_{n_c} = \underline{P_{n_u} = N_0 \cdot E_{HR}}$$

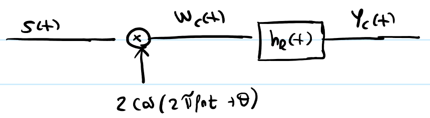
in assenza di ISI e con rumore bianco in banda e gaussiano.

SINCRONISMO DI FASE

$$\tilde{s}(t) = \sum_n x_c[n]\,\tilde{p}(t-nT_s)\cos(2\pi f_0 t+\varphi)$$

$$-\sum_n x_s[n]\,\tilde{p}(t-nT_s)\sin(2\pi f_0 t+\varphi)$$

IN FASE:



$$w_c(t) = \sum_n x_c[n]\,\tilde{p}(t-nT_s)\cos(2\pi f_0 t+\varphi)\cdot 2\cos(2\pi f_0 t+\theta)$$

$$-\sum_n x_c[n]\,\tilde{p}(t-nT_s)\sin(2\pi f_0 t+\varphi)\,2\cos(2\pi f_0 t+\theta)$$

$$= \sum_n x_c[n]\,\tilde{p}(t-nT_s)\left[\cos(4\pi f_0 t+\varphi+\theta)+\cos(\varphi-\theta)\right]$$
$$-\sum_n x_s[n]\,\tilde{p}(t-nT_s)\left[\sin(4\pi f_0 t+\varphi+\theta)+\sin(\varphi-\theta)\right]$$

$$= \sum_n x_c[n]\,\tilde{p}(t-nT_s)\cos(\varphi-\theta) + \sum_n x_s[n]\,\tilde{p}(t-nT_s)\sin(\varphi-\theta) + \underbrace{\ldots}_{\text{componenti a } 2f_0}$$

$$y(t) = w_c(t)\otimes h_e(t)$$

$$= \underline{\sum_n x_c[n]\,\tilde{p}(t-nT_s)\cos(\varphi-\theta)}$$

componente utile attenuata ($\cos(\varphi-\theta)\leq 1$)

$$\underbrace{+\sum_n x_s[n]\,\tilde{p}(t-nT_s)\sin(\varphi-\theta)}$$

componente di cross-talk

Danni ancora maggiori nella QAM.
Si può dimostrare che la stessa cosa accade nel ramo in quadratura.

$\Longrightarrow$ NO CROSS-TALK $\rightarrow$ $\underline{\varphi=\theta}$
