YSU Statistical ML, Fall 2019

Lecture 02

Michael Poghosyan
YSU, AUA
michael@ysu.am, mpoghosyan@aua.am

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Contents

- Conditional Distributions
- Bayes Predictor for the Classification Problem

Last Lecture Recap

Recall that our Learning Problem was stated as: We are given

- A Dataset of Observations $(\mathbf{x}_k, y_k)$, $k = 1, .., n$, coming as a realization of $(\mathbf{X}_k, Y_k)$ from an unknown Distribution $\mathcal{F}$;
- A Loss Function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}$;
- A Predictive Model (set of Functions) $\mathcal{G}$

and we want to find $g^* \in \mathcal{G}$ such that

$$g^* \in \underset{g \in \mathcal{G}}{argmin}\, Risk(g) = \underset{g \in \mathcal{G}}{argmin}\, \mathbb{E}\Big(\ell(Y, g(\mathbf{X}))\Big).$$

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Example: Binary Classification Problem

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Say, we can have X, Y on the same Probability Space with X being continuous and $Y \sim Bernoulli(0.5)$. Then we can interpret $X|Y = y$ as the Distribution of X, given that $Y = y$, e.g., we can have

$$X|Y = 0 \sim \mathcal{N}(0, 1) \qquad \textit{and} \qquad X|Y = 1 \sim \mathcal{N}(2, 1)$$

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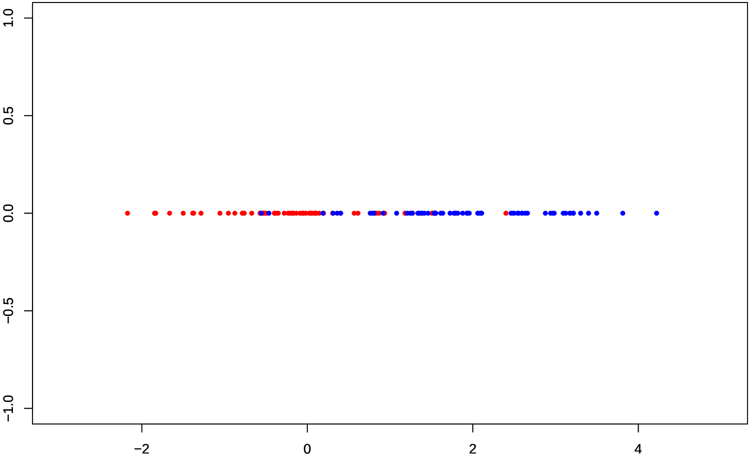
Here the Distribution of Y, in our case $\mathit{Bernoulli}(0.5)$ is called the Prior Class Distribution.

Example



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On important property of the Conditional Expectation is the following:

Theorem: If X and Y are r.v. with finite Expectation and Variance, then for any function $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\Big((X - \mathbb{E}(X|Y))^2\Big) \leq \mathbb{E}\Big((X - g(Y))^2\Big).$$

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In other words, if we have r.v.s X and Y, and we want to find a r.v. of the form $g(Y)$ which is the closest one to X in the MSE sense, then the best one is $\mathbb{E}(X|Y)$.

Bayes Predictor in the Binary Classification Problem

Let us go bact to our Binary Classification case with $\mathcal{Y} = \{0, 1\}$. Recall that, in the Binary Classification case, the the problem of finding the Bayes Predictor reduces to

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so

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So we need to find the maximum of $\mathbb{P}(Y = g(X))$ over all g.

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Notes

Let us summarize: the following Predictor

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So in this case Bayes Classifier is not unique.

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Then the condition $\eta(x) \geq \frac{1}{2}$ can be written in the form

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Generalization for K Classes

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And again, we can write this as

$$g^*(x) = \underset{k=1,...,K}{argmax}\; \mathbb{P}(X = x | Y = k) \cdot \mathbb{P}(Y = k).$$

Question, K-class Classification Problem

What will happen, if we will take the following Loss Function:

$$\ell(k, m) = |k - m| \ ?$$