4.5 COUNTING RULES

The probability of an outcome of interest occurring is defined as the number of ways the outcome of interest occurs, divided by the total number of possible outcomes. In many instances, because of the large number of possibilities, it is not feasible to actually list each of the outcomes. In these circumstances, rules for counting the number of outcomes without physically constructing the complete listing have been developed. In this section five different counting rules will be discussed.

Suppose a coin is flipped 10 times. What is the number of different possible outcomes (the sequences of heads and tails)?

COUNTING RULE 1

If any one of k different mutually exclusive and collectively exhaustive events can occur on each of n trials, the number of possible outcomes is equal to

$$k^n \tag{4.11}$$

If a coin (having two sides) is tossed 10 times, using Equation (4.11), the number of outcomes is $2^{10} = 1{,}024$. If a die (having six sides) is rolled twice, using Equation (4.11), the number of different outcomes is $6^2 = 36$.

The second counting rule is a more general version of the first, and allows for the number of possible events to differ from trial to trial. For example, a state motor vehicle department would like to know how many license plate numbers are available if the license plates consist of three letters followed by three digits. The fact that three values are letters (each having 26 possible outcomes) and three positions are digits (each having 10 outcomes) leads to the second rule of counting.

COUNTING RULE 2

If there are k_1 events on the first trial, k_2 events on the second trial, . . . and k_n events on the nth trial, then the number of possible outcomes is

$$(k_1)(k_2) \cdots (k_n) \tag{4.12}$$

Thus, using Equation (4.12), if a license plate consists of three letters followed by three digits, the total number of possible outcomes is $(26)(26)(26)(10)(10)(10) = 17{,}576{,}000$. Taking another example, if a restaurant menu has a price-fixed complete dinner that consists of an appetizer, entrée, beverage, and dessert and there is a choice of five appetizers, ten entrées, three beverages, and six desserts, using Equation (4.12), the total number of possible dinners is $(5)(10)(3)(6) = 900$.

The third counting rule involves the computation of the number of ways that a set of objects can be arranged in order. If a set of six textbooks is to be placed on a shelf, in how many ways can the six books be arranged? To begin, realize that any of the six books could occupy the first position on the shelf. Once the first position is filled, there are five books to choose from in filling the second. This assignment procedure is continued until all the positions are occupied.

COUNTING RULE 3

The number of ways that all n objects can be arranged in order is

$$n! = (n)(n-1) \cdots (1) \tag{4.13}$$

where $n!$ is called *n factorial*, $0!$ is defined as 1 and $1!$ equals 1.

The number of ways that six books can be arranged is

$$n! = 6! = (6)(5)(4)(3)(2)(1) = 720$$

In many instances you need to know the number of ways in which a subset of the entire group of objects can be arranged in *order*. Each possible arrangement is called a **permutation**. For example, modifying the preceding problem, if six textbooks are involved, but there is room for only four books on the shelf, how many ways can these books be arranged on the shelf?

COUNTING RULE 4

Permutations: The number of ways of arranging X objects selected from n objects in order is

$$\frac{n!}{(n-X)!} \tag{4.14}$$

Therefore, using Equation (4.14), the number of ordered arrangements of four books selected from six books is equal to

$$\frac{n!}{(n-X)!} = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{(6)(5)(4)(3)(2)(1)}{(2)(1)} = 360$$

Finally, in many situations you are not interested in the *order* of the outcomes but only in the number of ways that X objects can be selected out of n objects, *irrespective of order*. This rule is called the rule of **combinations**.

COUNTING RULE 5

Combinations: The number of ways of selecting X objects out of n objects, irrespective of order, is equal to

$$\binom{n}{X} = \frac{n!}{X!(n-X)!} \tag{4.15}$$

Comparing this rule to the previous one, you see that it differs only in the inclusion of a term $X!$ in the denominator. Note that when permutations were used, all of the arrangements of the X objects are distinguishable. With combinations, the $X!$ possible arrangements of objects are irrelevant. Therefore, using Equation (4.15), the number of combinations of four books selected from six books is expressed by

$$\binom{n}{X} = \frac{n!}{X!(n-X)!} = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{(6)(5)(4)(3)(2)(1)}{(4)(3)(2)(1)(2)(1)} = 15$$

PROBLEMS FOR SECTION 4.5

Applying the Concepts

4.54 If there are ten multiple-choice questions on an exam, each having three possible answers, how many different possibilities are there in terms of the sequence of correct answers?

4.55 A lock on a bank vault consists of three dials, each with 30 positions. In order for the vault to open when closed, each of the three dials must be in the correct position.

a. How many different possible "dial combinations" are there for this lock?
b. What is the probability that if you randomly select a position on each dial, you will be able to open the bank vault?
c. Explain why "dial combinations" are not mathematical combinations expressed by Equation (4.15).

4.56 **a.** If a coin is tossed seven times, how many different outcomes are possible?

b. If a die is tossed seven times, how many different outcomes are possible?

c. Discuss the differences in your answers to (a) and (b).

4.57 A particular brand of women's jeans can be ordered in seven different sizes, three different colors, and three different styles. How many different jeans would have to be ordered if a store wanted to have one pair of each type?

4.58 If each letter is used once, how many different four-letter "words" can be made from the letters E, L, O, and V?

4.59 There are five teams in the Western Division of the National League: Arizona, Los Angeles, San Francisco, San Diego, and Colorado. How many different orders of finish are there for these five teams? Do you *really* believe that all these orders are equally likely? Discuss.

4.60 Referring to Problem 4.59, how many different orders of finish are possible for the first four positions?

4.61 A gardener has six rows available in his vegetable garden to place tomatoes, eggplant, peppers, cucumbers, beans, and lettuce. Each vegetable will be allowed one and only one row. How many ways are there to position these vegetables in his garden?

4.62 The Big Triple at the local racetrack consists of picking the correct order of finish of the first three horses in the ninth race. If there are 12 horses entered in today's ninth race, how many Big Triple outcomes are there?

4.63 The Quinella at the local racetrack consists of picking the horses that will place first and second in a race *irrespective* of order. If eight horses are entered in a race, how many Quinella combinations are there?

4.64 A student has seven books that she would like to place in an attaché case. However, only four books can fit into the attaché case. Regardless of the arrangement, how many ways are there of placing four books into the attaché case?

4.65 A daily lottery is to be conducted in which two winning numbers are to be selected out of 100 numbers. How many different combinations of winning numbers are possible?

4.66 A reading list for a management course contains 20 articles. How many ways are there to choose three articles from this list?