11.6 χ^2 TEST OF HYPOTHESIS FOR THE VARIANCE OR STANDARD DEVIATION

When analyzing numerical data, sometimes it is important to draw conclusions about the variability as well as the average of a characteristic of interest. For example, recall that in the cereal-filling process described in section 8.2, the standard deviation σ was assumed to be equal to 15 grams. Suppose you are interested now in determining whether there is evidence that the standard deviation has changed from the previously specified level of 15 grams.

In attempting to draw conclusions about the variability in the population, you first must determine what test statistic can be used to represent the distribution of the variability in the sample data. If the variable is assumed to be normally distributed, the **χ^2-test statistic** given in Equation (11.9) is used for testing whether or not the population variance or standard deviation is equal to a specified value.

χ^2 TEST FOR THE VARIANCE OR STANDARD DEVIATION

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} \tag{11.9}$$

where

n = sample size
S^2 = sample variance
σ^2 = hypothesized population variance

the test statistic χ^2 follows a chi-square distribution with $n - 1$ degrees of freedom.

The **chi-square distribution** is a right-skewed distribution whose shape depends solely on its number of degrees of freedom. The mean of a chi-square distribution is equal to its degrees of freedom and the variance is twice the degrees of freedom. Table E.4 contains various upper-tail areas for chi-square distributions pertaining to different degrees of freedom. A portion of this table is displayed as Table 11.16.

TABLE 11.16

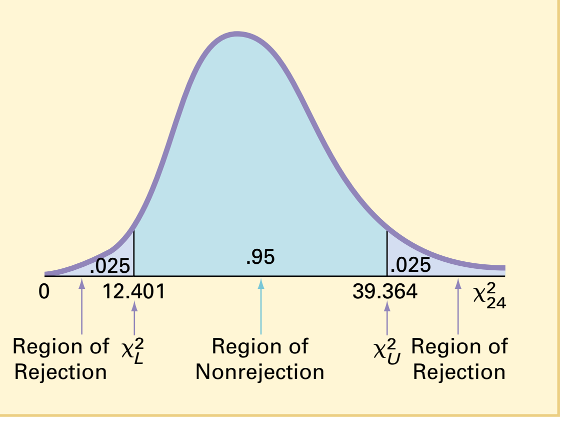
Obtaining the critical values from the chi-square distribution with 24 degrees of freedom

Degrees of Freedom	Upper-Tail Areas							
	.995	.99	$\boxed{.975}$	.95	.90	.10	.05	$\boxed{.025}$
1	. . .	. . .	0.001	0.004	0.016	2.706	3.841	5.024
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076
$\boxed{24}$	9.886	10.856	$\boxed{12.401}$	13.848	15.659	33.196	36.415	$\boxed{39.364}$
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646

Source: Extracted from Table E.4.

The value at the top of each column indicates the area in the upper portion (or right side) of a particular chi-square distribution. As examples, with 24 degrees of freedom, the critical value of the χ^2 test statistic corresponding to an upper-tail area of 0.025 is 39.364, while the critical value corresponding to an upper-tail area of 0.975 (that is, a lower-tail area of 0.025) is 12.401. These are seen in Figure 11.15. This means that for 24 degrees of freedom, the probability of equaling or exceeding the critical value of 12.401 is 0.975, while the probability of equaling or exceeding the critical value of 39.364 is 0.025. Thus, the probability that a χ^2 test statistic falls between the critical values of 12.401 and 39.364 is 0.95. Therefore, once you determine the level of significance and the degrees of freedom, any critical value of the χ^2 test statistic can be found for a particular chi-square distribution.

FIGURE 11.15
Determining the lower and upper critical values of a chi-square distribution with 24 degrees of freedom corresponding to a 0.05 level of significance for a two-tail test of hypothesis about a population variance or standard deviation



To apply the test of hypothesis, return to the cereal-filling example. You are interested in determining whether there is evidence that the standard deviation has changed from the previously specified level of 15 grams. Thus, a two-tail test is appropriate and the null and alternative hypotheses can be stated as follows:

$$H_0: \sigma = 15 \text{ grams. (or } \sigma^2 = 225 \text{ grams squared)}$$
$$H_1: \sigma \neq 15 \text{ grams. (or } \sigma^2 \neq 225 \text{ grams squared)}$$

If a sample of 25 cereal boxes is selected, the null hypothesis is rejected if the χ^2 test statistic falls into either the lower or upper tail of a chi-square distribution with 25 – 1 = 24 degrees of freedom as shown in Figure 11.9. From Equation (8.5), the χ^2 test statistic falls into the lower tail of the chi-square distribution if the sample standard deviation (S) is sufficiently smaller than the hypothesized σ of 15 grams, and it falls into the upper tail if S is sufficiently larger than 15 grams. From Table 11.15 (extracted from Table E.4, the table of the chi-square distribution) and Figure 11.15, observe that if a level of significance of 0.05 is selected, the lower (χ^2_L) and upper (χ^2_U) critical values are 12.401 and 39.364, respectively. Therefore, the decision rule is

$$\text{Reject } H_0 \text{ if } \chi^2 > \chi^2_U = 39.364 \text{ or if } \chi^2 < \chi^2_L = 12.401;$$
$$\text{otherwise do not reject } H_0.$$

Suppose that from the sample of 25 cereal boxes, the standard deviation (S) is computed to be 17.7 grams. To test the null hypothesis at the 0.05 level of significance using Equation (8.5),

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2} = \frac{(25-1)(17.7)^2}{(15)^2} = 33.42$$

Note that 33.42, the computed value of the χ^2 test statistic, falls between the lower- and upper-tail critical values of 12.401 and 39.364. Since $\chi^2_L = 12.401 < \chi^2 = 33.42 < \chi^2_U = 39.364$, or since the p-value = 0.0956 > 0.05 (see Figure 11.16), you do not reject H_0. You conclude that there is insufficient evidence that the process population standard deviation is different from 15 grams.

FIGURE 11.16

Microsoft Excel output for the test of hypothesis for a variance for the cereal-filling process

	A		B
1	**Cereal-Filling Variance Hypothesis**		
2			
3	**Data**		
4	**Null Hypothesis**	σ^2=	225
5	**Level of Significance**		0.05
6	**Sample Size**		25
7	**Sample Standard Deviation**		17.7
8			
9	Intermediate Calculations		
10	Degrees of Freedom		24
11	Half Area		0.025
12	Chi-Square Statistic		33.4176
13			
14	**Two-Tail Test**		
15	Lower Critical Value		12.4011
16	Upper Critical Value		39.3641
17	***p*-Value**		**0.0956**
18	**Do not reject the null hypothesis**		

The χ^2 test for the variance or standard deviation is considered a *classical parametric* procedure that makes assumptions that must hold if the results obtained are to be valid. Assumptions for the χ^2 test for the variance or standard deviation are presented in the following box.

CHECKING THE ASSUMPTIONS OF THE χ^2 TEST FOR THE VARIANCE OR STANDARD DEVIATION

In testing a hypothesis about a population variance or standard deviation, the data in the population are assumed to be normally distributed. Unfortunately, this χ^2 test statistic is quite sensitive to departures from this assumption (i.e., it is not a *robust* test), so that if the population is not normally distributed, particularly for small sample sizes, the accuracy of the test can be seriously affected.

Excel Strategies Performing the χ^2 Test of Hypothesis for the Variance

You implement a worksheet that features the **CHIINV** and **CHIDIST** functions to perform a χ^2 test of hypothesis for the variance. PHStat2 can construct this worksheet for you in a single step.

For example, to perform a χ^2 test of hypothesis for the cereal-filling process, shown in Figure 11.16 above, open to an empty worksheet and use one of these strategies:

CHAPTER 11.xls

*Retrieve the Figure 11.16 worksheet of the **Chapter 11.xls** workbook to view the χ^2 test of hypothesis for the variance shown above.*

Using Excel with PHStat2

Use the PHStat 2 **Chi-Square Test for the Variance** procedure to perform a χ^2 test of hypothesis for the variance.

Select **PHStat | One-Sample Tests | Chi-Square Test for the Variance**.

In the Chi-Square Test for the Variance dialog box (shown at right):

Enter **225** in the **Null Hypothesis** edit box.

Enter **0.05** in the **Level of Significance** edit box.

Enter **25** in the **Sample Size** edit box.

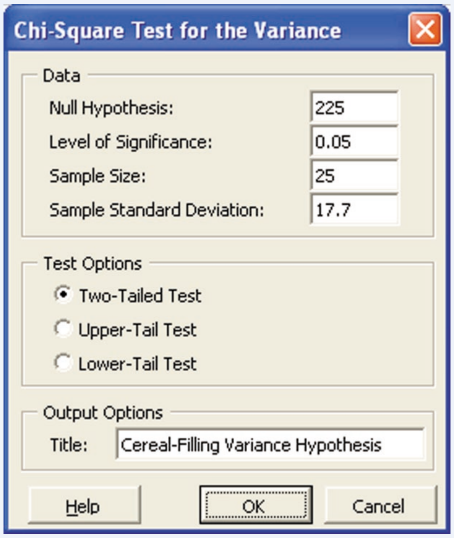
Enter **17.7** in the **Sample Standard Deviation** edit box.

Select the **Two-Tail Test** option button.

Enter a title in the **Title** edit box.

Click the **OK** button.

*For problems that require one-tailed test, select the **Upper-Tail Test** or **Lower-Tail Test** option button, as appropriate, when completing this dialog box.*



Using Excel

To perform a χ^2 test of hypothesis for the variance, implement a worksheet that features the **CHIINV** and **CHIDIST** functions. These functions are entered as **CHIINV(1 — level of significance, degrees of freedom)** and **CHIDIST(chi-square statistic, degrees of freedom)**.

Table EH11.13 contains the design for a **Chi-Square Test** worksheet that performs a χ^2 test of hypothesis for the variance for the cereal-filling process, shown in Figure 11.16. This design uses the CHIINV function to determine the lower and upper critical values and uses the CHIDIST function to determine the *p*-value from the χ^2 statistic calculated in cell B12. This design displays a message advising whether or not to reject the null hypothesis in cell A18 by using the IF function to compare the *p*-value in cell B17 to the level of significance in cell B5.

When implementing this design, enter the formulas for cells B17 and A18, typeset in Table EH11.13 as three lines, as single continuous lines.

For problems that require one-tailed tests, use the alternative entries for rows 14 through 17 shown in either Table EH11.14L or Table EH11.14U. (Row 18 is left empty for both one-tail tests.) These alternatives use the functions found in rows 14 through 18 in Table EH8.1 in similar ways to determine whether or not to reject the null hypothesis. When implementing one of these alternatives, enter the formula containing the IF function in cell A17 typeset as three lines (in the table displayed) as one continuous line.

Table EH11.13
Chi-Square Test worksheet design

	A	B
1	Cereal-Filling Variance Hypothesis	
2		
3	Data	
4	Null Hypothesis σ^2=	225
5	Level of Significance	0.05
6	Sample Size	25
7	Sample Standard Deviation	17.7
8		
9	Intermediate Calculations	
10	Degrees of Freedom	=B6 — 1
11	Half Area	=B5/2
12	Chi-Square Statistic	=B10 * B7 ^ 2/B4
13		
14	Two-Tailed Test	
15	Lower Critical Value	=CHIINV(1 — B11, B10)
16	Upper Critical Value	=CHIINV(B11, B10)
17	*p*-Value	=IF(B12 — B15 < 0, 1 — CHIDIST(B12, B10), CHIDIST(B12, B10))
18	=IF(B17 < B5/2, "Reject the null hypothesis", "Do not reject the null hypothesis")	

Table EH11.14L
Lower-tail alternate entries for *Z* Test worksheet design

	A	B
14	Lower-Tail Test	
15	Lower Critical Value	=CHIINV(1 — B5, B10)
16	*p*-Value	=1 — CHIDIST(B12, B10)
17	=IF(B16 < B5, "Reject the null hypothesis", "Do not reject the null hypothesis")	

Table EH11.14U
Upper-tail alternate entries for *Z* Test worksheet design

	A	B
14	Upper-Tail Test	
15	Upper Critical Value	=CHIINV(B5, B10)
16	*p*-Value	=CHIDIST(B12, B10)
17	=IF(B16 < B5, "Reject the null hypothesis", "Do not reject the null hypothesis")	

PROBLEMS FOR SECTION 11.6

Learning the Basics

11.63 Determine the critical value of χ^2 in each of the following circumstances:

a. Upper-tail area = 0.01, $n = 16$
b. Upper-tail area = 0.025, $n = 11$
c. Upper-tail area = 0.05, $n = 8$
d. Upper-tail area = 0.95, $n = 28$
e. Upper-tail area = 0.975, $n = 21$
f. Upper-tail area = 0.99, $n = 5$

11.64 Determine the critical value of χ^2 in each of the following circumstances:

a. Lower-tail area = 0.01, $n = 16$
b. Lower-tail area = 0.025, $n = 11$
c. Lower-tail area = 0.05, $n = 8$
d. Lower-tail area = 0.95, $n = 28$
e. Lower-tail area = 0.975, $n = 21$
f. Lower-tail area = 0.99, $n = 5$

11.65 Determine the lower- and upper-tail critical values of χ^2 for each of the following two-tail tests:

a. $\alpha = 0.01$, $n = 26$
b. $\alpha = 0.05$, $n = 17$
c. $\alpha = 0.10$, $n = 14$

11.66 In a sample of size $n = 16$ selected from an underlying normal population, the sample standard deviation $S = 10$. What is the value of the χ^2 test statistic if you are testing the null hypothesis H_0 that $\sigma = 12$?

11.67 In problem 11.66, how many degrees of freedom are there in the one-sample χ^2 test?

11.68 In problems 11.66 and 11.67, what are the critical values from Table E.4 if the level of significance α is chosen to be 0.05 and the alternative hypothesis H_1 is as follows:

a. $\sigma \neq 12$?
b. $\sigma < 12$?

11.69 In problems 11.66, 11.67, and 11.68, what is your statistical decision if your alternative hypothesis H_1 is

a. $\sigma \neq 12$?
b. $\sigma < 12$?

11.70 If, in a sample of size $n = 16$ selected from a very left-skewed population, the sample standard deviation is $S = 24$, would you use the one-sample χ^2 test in order to test the null hypothesis H_0 that $\sigma = 20$? Discuss.

Applying the Concepts

Note: Problems 11.71–11.76 can be solved manually or by using Microsoft Excel.

11.71 A manufacturer of candy must monitor the temperature at which the candies are baked. Too much variation will cause inconsistency in the taste of the candy. Past records show that the standard deviation of the temperature has been 1.2°F. A random sample of 30 batches of candy is selected and the sample standard deviation of the temperature is 2.1°F.

a. At the 0.05 level of significance, is there evidence that the population standard deviation has increased above 1.2°F?
b. What assumptions are being made in order to perform this test?
c. Compute the *p*-value in (a) and interpret its meaning.

11.72 A market researcher for an automobile dealer intends to conduct a nationwide survey concerning car repairs. Among the questions to be included in the survey is the following: "What was the cost of all repairs performed on your car last year?" In order to determine the sample size necessary, he needs to obtain an estimate of the standard deviation. Using his past experience and judgment, he estimates that the standard deviation of the amount of repairs is \$200. Suppose that a pilot study of 25 auto owners selected at random indicates a sample standard deviation of \$237.52.

a. At the 0.05 level of significance, is there evidence that the population standard deviation is different from \$200?
b. What assumptions are made in order to perform this test?
c. Compute the *p*-value in part (a) and interpret its meaning.

11.73 The marketing manager of a branch office of a local telephone operating company wants to study characteristics of residential customers served by her office. In particular, she wants to estimate the average monthly cost of calls within the local calling region. In order to determine the sample size necessary, an estimate of the standard deviation must be made. On the basis of her past experience and judgment, she estimates that the standard deviation is equal to \$12. Suppose that a pilot study of 15 residential customers indicates a sample standard deviation of \$9.25.

a. At the 0.10 level of significance, is there evidence that the population standard deviation is different from \$12?
b. What assumptions are made in order to perform this test?
c. Compute the *p*-value in part (a) and interpret its meaning.

11.74 A manufacturer of doorknobs has a production process that is designed to provide a doorknob with a target diameter of 2.5 inches. In the past, the standard deviation of the diameter has been 0.035 inch. In an effort to reduce the variation in the process, various studies have resulted in a redesigned process. A sample of 25 doorknobs produced under the new process indicates a sample standard deviation of 0.025 inch.

a. At the 0.05 level of significance, is there evidence that the population standard deviation is less than 0.035 inch in the new process?
b. What assumptions are made in order to perform this test?
c. Compute the *p*-value in part (a) and interpret its meaning.

11.75 A machine used for packaging seedless golden raisins is set so that the standard deviation in the weight of raisins packaged per box is 0.25 ounce. The operations manager wishes to test the machine setting and selects a sample of 30 consecutive raisin packages filled during the production process. Their weights are recorded as follows: **RAISINS.xls**

15.2	15.3	15.1	15.7	15.3	15.0	15.1	14.3	14.6	14.5
15.0	15.2	15.4	15.6	15.7	15.4	15.3	14.9	14.8	14.6
14.3	14.4	15.5	15.4	15.2	15.5	15.6	15.1	15.3	15.1

a. At the 0.05 level of significance is there evidence that the population standard deviation differs from 0.25 ounce?
b. What assumptions are made in order to perform this test?
c. Obtain the *p*-value in part (a) and interpret its meaning.

11.76 A manufacturer claims that the standard deviation in capacity of a certain type of battery the company produces is 2.5 ampere-hours. An independent consumer protection agency wishes to test the credibility of the manufacturer's claim and measures the capacity of a random sample of 20 batteries from a recently produced batch. The results, in ampere-hours, are as follows: **AMPHRS.xls**

137.4	140.0	138.8	139.1	144.4	139.2	141.8	137.3	133.5	138.2
141.1	139.7	136.7	136.3	135.6	138.0	140.9	140.6	136.7	134.1

a. At the 0.05 level of significance is there evidence that the population standard deviation in battery capacity exceeds 2.5 ampere-hours?
b. What assumptions are made in order to perform this test?
c. Obtain the *p*-value in part (a) and interpret its meaning.