

5.6 USING THE POISSON DISTRIBUTION TO APPROXIMATE THE BINOMIAL DISTRIBUTION

For those situations in which n is large and p is very small, the Poisson distribution can be used to approximate the binomial distribution. The larger the n and the smaller the p , the better is the approximation. The following mathematical expression for the Poisson model is used to approximate the true (binomial) result:

$$P(X) \cong \frac{e^{-np} (np)^x}{X!} \quad (5.18)$$

where

$P(X)$ = probability of X successes given the parameters n and p

n = sample size

p = true probability of success

e = mathematical constant approximated by 2.71828

X = number of successes in the sample ($X = 0, 1, 2, \dots, n$)

The Poisson random variable theoretically ranges from 0 to ∞ . However, when used as an approximation to the binomial distribution, the Poisson random variable—the number of successes out of n observations—cannot be greater than the sample size n . With large n and small p , Equation (5.18) implies that the probability of observing a large number of successes becomes small and approaches zero quite rapidly.

As mentioned previously, in the Poisson distribution, the mean μ and the variance σ^2 are each equal to λ . Thus, when using the Poisson distribution to approximate the binomial distribution, Equation (5.19) is used to approximate the mean.

$$\mu = E(X) = \lambda = np \quad (5.19)$$

Equation (5.20) is used to approximate the standard deviation.

$$\sigma = \sqrt{\lambda} = \sqrt{np} \quad (5.20)$$

Note that the standard deviation given by Equation (5.20) agrees with that given for the binomial model [Equation (5.13)] when p is close to zero so that $(1 - p)$ is close to one.

Suppose 8% of the tires manufactured at a particular plant are defective. To illustrate the use of the Poisson approximation for the binomial, the probability of obtaining exactly one defective tire from a sample of 20 using Equation (5.18) is calculated as follows:

$$P(X = 1) \cong \frac{e^{-(20)(.08)} [(20)(.08)]^1}{1!} = \frac{e^{-1.6} (1.6)^1}{1!} = 0.3230$$

Alternatively, tables of the Poisson distribution (Table E.7) can be employed. Referring to these tables, the only values necessary are the parameter λ and the desired number of successes X . Since in the above example $\lambda = 1.6$ and $X = 1$, using Table E.7

$$P(X = 1) = .3230$$

This is shown in Table 5.5 (which is a portion of Table E.7).

TABLE 5.5
Obtaining a Poisson
probability

	λ									
X	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	.3329	.3012	.2725	.2466	.2231	.2019	.1827	.1653	.1496	.1353
1	.3662	.3614	.3543	.3452	.3347	.3230	.3106	.2975	.2842	.2707
2	.2014	.2169	.2303	.2417	.2510	.2584	.2640	.2678	.2700	.2707
3	.0738	.0867	.0998	.1128	.1255	.1378	.1496	.1607	.1710	.1804
4	.0203	.0260	.0324	.0395	.0471	.0551	.0636	.0723	.0812	.0902

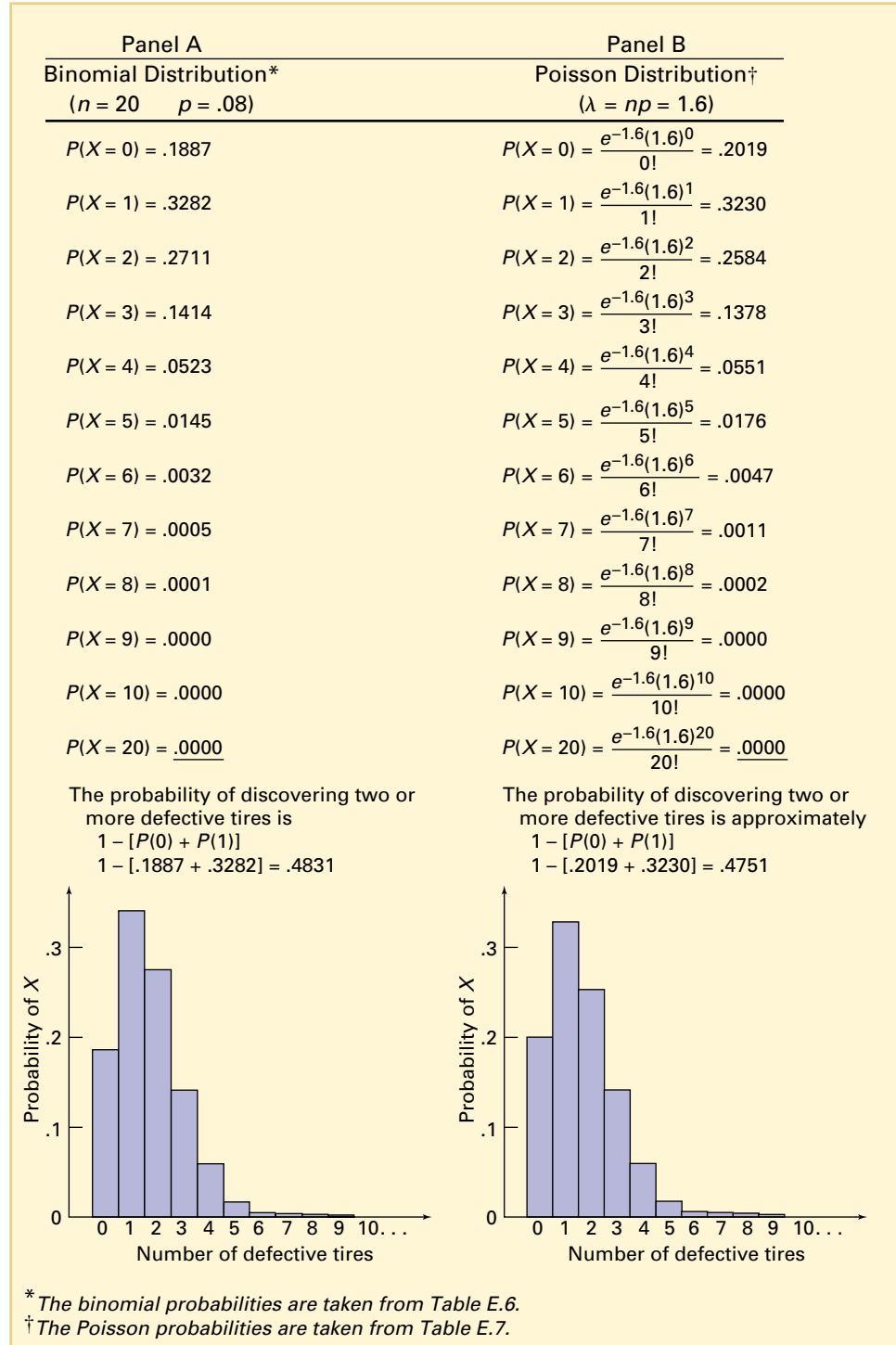
Source: Extracted from Table E.7.

Had the true distribution, the binomial, been used instead of the approximation,

$$p(X = 1) = \binom{20}{1} (.08)^1 (.92)^{19} = .3282$$

This computation, though, is tedious. Clearly, with binomial tables such as Table E.6 or Microsoft Excel available, you could find the binomial probability directly for $n = 20$, $p = .08$, and $X = 1$ and not bother calculating it or using the Poisson approximation.

To summarize, Figure 5.6 on page CD5-3 compares the binomial distribution (panel A) and its Poisson approximation (panel B) for the number of defective tires in a sample of 20. The similarities of the two results are clearly evident, thus demonstrating the usefulness of the Poisson approximation even when p is as large as .08.

FIGURE 5.6Binomial distribution
and its Poisson
approximation

PROBLEMS FOR SECTION 5.6

Learning the Basics

5.58 When should the Poisson distribution be used to approximate the binomial distribution?

5.59 When given the parameters of a binomial distribution, n and p , what are the mean and variance of the Poisson distribution used for approximating the binomial?

5.60 Given a binomial distribution with $n = 100$ and $p = 0.01$, use the Poisson distribution to calculate the following:

- a. $P(X = 0)$
- b. $P(X = 1)$
- c. $P(X = 2)$
- d. $P(X \leq 2)$
- e. $P(X > 2)$

5.61 Given a binomial distribution with $n = 50$ and $p = 0.004$, use the Poisson distribution to calculate the following:

- a. $P(X = 0)$
- b. $P(X = 1)$
- c. $P(X = 2)$
- d. $P(X \leq 2)$
- e. $P(X > 2)$

Applying the Concepts

5.62 Based upon past experience, 1% of the telephone bills mailed to households are incorrect. If a sample of 20 bills is

selected, find the probability that at least one bill will be incorrect. Do this using two probability distributions (the binomial and the Poisson) and briefly compare and explain your results.

5.63 A computer manufacturing company conducts acceptance sampling for incoming computer chips. After receiving a huge shipment of computer chips, the company randomly selects 800 chips. If three or fewer nonconforming chips are found, the entire lot is accepted without inspecting the remaining chips in the lot. If four or more chips are nonconforming, every chip in the entire lot is carefully inspected at the supplier's expense. Assume that the true proportion of nonconforming computer chips being supplied is 0.001. What is the probability the lot will be accepted?

5.64 Last month your company sold 10,000 new watches. Past experience indicates that the probability that a new watch will need repair during its warranty period is 0.002. Compute the probability that:

- a. zero watches will need warranty work.
- b. no more than 5 watches will need warranty work.
- c. no more than 10 watches will need warranty work.
- d. no more than 20 watches will need warranty work.