Unit-4

# Partial Differential Equations (PDEs)



PDEs are models of various physical and geometrical problems, arising when the unknown functions depend on two or more variables. Most problems in dynamics, elasticity, heat transfer, electromagnetic theory and quantum mechanics require PDEs.

Defⁿ: A Partial differential equation (PDE) is an equation involving one or more partial derivatives of an (unknown function) say $u$, that depends on two or more variables, time $t$ and one or more variables in space.

The order of the highest derivative is called the order of the PDE.

Defⁿ. A PDE is linear if it is of the first degree in the unknown function $u$ and its partial derivatives. Otherwise we call it nonlinear.

## Important Second Order PDEs

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One-dimensional wave equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ One-dimensional heat equation

$$u_{xx} + u_{yy} = 0$$ Laplace equation

$$u_{xx} + u_{yy} = f(x,y)$$ Poisson equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$ Two dimensional wave equation

Def$^n$: Solution of PDE: A function that has all the partial derivatives appearing in the PDE in domain D and satisfies the PDE everywhere in R.

Fundamental Theorem on Superposition: If $u_1$ & $u_2$ are solutions of a homogeneous linear PDE in some region R, then

$$u = c_1 u_1 + c_2 u_2$$

with any constants $c_1$ & $c_2$ is also a solution of that PDE in region R.

(Homogeneous PDE: If each of terms of PDE contains either u or one of its partial derivatives)

eg. $u_{xx} - u = 0$

No derivatives of y. So it is an ODE

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$u(x,y) = c_1 e^x + c_2 e^{-x}$$

$c_1$ & $c_2$ may be functions of y, so we have

$$u(x,y) = A(y) e^x + B(y) e^{-x}$$

eg. 2 $u_{xy} = -u_x$

Sol: $u_x = p$ ① $\Rightarrow p_y = -p$

$$\frac{p_y}{p} = -1$$

$$\Rightarrow \ln p = -y + \tilde{c}(x)$$

$$\Rightarrow p = c(x) e^{-y}$$

Integrating ① w.r.t x

$$u(x,y) = f(x) e^{-y} + g(y),$$ where $f(x) = \int c(x) dx$

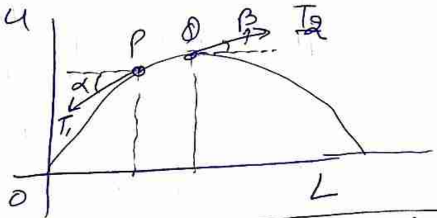
## Modeling : Vibrating String . Wave Equation

To derive the equation modeling small transverse vibrations of an elastic string, such as violin string.

Place the string along the $x$-axis, stretch it to length $L$ and fasten it at the ends $x = 0$ & $x = L$.

We then distort the string and at some instant say $t = 0$, we release it and allow it to vibrate.

The problem is to determine the vibrations of the string that is, to find its deflection $u(x,t)$ at any point $x$ and at any time $t > 0$.




Deflected string at fixed time $t$.

### Physical Assumptions

1.) The mass of the string per unit length is constant. The string is perfectly elastic and does not offer any resistance to bending.

2) Due to large tension, gravitational force acting on the string is neglected.

3) String performs small transverse motions in a vertical plane so that the deflection and the slope at every point of the string always remain small in absolute value.

### Derivation of the PDE (Wave equation) from Forces

Let $T_1$ and $T_2$ be the tension at the end points P & Q of that portion. Since the points of the string move

vertically, there is no motion in the horizontal direction)
Hence, the horizontal components of the tension must be constant.

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant.} \quad \text{—— (1)}$$

In vertical direction, we have 2 forces :- $T_1 \sin\alpha$ & $T_2 \sin\beta$

By Newton's second law, the resultant of these 2 forces is equal to the mass $\rho \Delta x$ of the portion times the acceleration $\frac{\partial^2 y}{\partial t^2}$, evaluated at some point between $x$ & $x+\Delta x$.

$\rho \rightarrow$ mass of the undeflected string per unit length

$\Delta x \rightarrow$ length of the portion of the undeflected string.

Hence, $T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 y}{\partial t^2}$

Divide by $T_2 \cos\beta = T_1 \cos\alpha = T$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha \quad \text{—— (2)}$$

$$= \frac{\rho \Delta x}{T} \frac{\partial^2 y}{\partial t^2}$$

$\tan\alpha$ & $\tan\beta$ are ~~slop~~ slopes of the string at $x$ & $x+\Delta x$.

$$\tan\alpha = \left.\frac{\partial y}{\partial x}\right|_x \quad \& \quad \tan\beta = \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$$

Dividing 2 by $\Delta x$, we have

$$\frac{1}{\Delta x}\left[\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right] = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$

let $\Delta x \to 0$, we obtain

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}} \qquad c^2 = \frac{T}{\rho}$$

This is called the one-dimensional wave eqution.

Solution by Seperating Variables. Use of Fourier Series

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)} \qquad c^2 = \frac{T}{\rho}$$

with boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \forall\, t \geq 0 \quad \text{—— (2)}$$

Furthermore, the motion of string will depend on its initial deflection, call $f(x)$ and on initial velocity call it $g(x)$ at $t = 0$. We have 2 initial conditions

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \quad 0 \leq x \leq L \quad \text{—— (3)}$$

To find the solution, the following steps are followed

**Step I:** By method of seperating variables, setting

$$u(x,t) = F(x)\,G(t),$$

we obtain two ODEs, one for $F(x)$ and the other one for $G(t)$.

**Step II:** We determine solutions of these ODEs that satisfy the boundary conditions (2)

**Step III:** Finally, using Fourier Series, we compare the solⁿ gained in Step 2 to obtain a solution of (1) satisfying (2) & (3).

## Solution by Seperating Variables. (1)

The model of a vibrating elastic string consists of one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—(1)} \quad c^2 = \frac{T}{\rho}$$

for the unknown deflection $u(x,t)$ of the string, with boundary conditions

$$u(0,t) = 0 \quad , \quad u(L,t) = 0 \quad \forall\, t \geqslant 0 \quad \text{—(2)}$$

with initial conditions

$$u(x,0) = f(x) \qquad u_t(x,0) = g(x) \quad \text{—(3)}$$

(where $u_t = \frac{\partial u}{\partial t}$)

**Step I:** Two ODEs from the wave equation

Let $u(x,t) = F(x)\,G(t)$ —(4) be solution of (1)

then $\frac{\partial^2 u}{\partial t^2} = F.\, G''(t)$

$\frac{\partial^2 u}{\partial x^2} = F''(x)\, G(t)$

Substitute in (1), we get

$$F(x)\, G''(t) = c^2 F''(x)\, G(t)$$

Dividing by $c^2 FG$ and simplifying

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

$$\Rightarrow \quad F'' - kF = 0 \quad \text{—(5)}$$

$$G'' - c^2 k G = 0 \quad \text{—(6)}$$

$\sin \frac{n\pi}{L} x.$

which are the two ordinary differential equations

**Step 2**: Satisfying the Boundary Conditions (2)

To determine solutions of (5) & (6) so that $u = FG$ satisfies boundary conditions (2). i.e

$$u(0,t) = F(0)\,G(t) = 0 \qquad \text{--- (7)}$$
$$u(L,t) = F(L)\,G(t) = 0 \qquad \forall t$$

If $G \equiv 0$, then $u = FG \equiv 0$. It is of no interest

So $G \not\equiv 0$, then by (7)

$$F(0) = 0, \quad F(L) = 0 \qquad \text{--- (8)}$$

if $k = 0$, then general solution of (5) is

$$F = ax + b$$

from (8), we obtain $a = 0$ $b = 0$ then $F \equiv 0$ & $u \equiv 0$

(No interest)

if $k = \mu^2$ (positive), then general solⁿ of (5) is

$$F(x) = Ae^{\mu x} + Be^{-\mu x}$$

and from (8), we obtain $F \equiv 0$ (which is of not our interest)

If $k = -p^2$ (negative), then (5) becomes

$$F'' + p^2 F = 0$$

and general solution is given by

$$F(x) = A\cos px + B\sin px$$

and from (8), we obtain

$$F(0) = A = 0 \quad \& \quad F(L) = B\sin pL = 0$$

$B \neq 0 \Rightarrow \sin pL = 0.$

$$pL = n\pi$$
$$\rightarrow p = \frac{n\pi}{L}$$

if $B = 1$, we obtain $F(x) = F_n(x)$, where

$$\boxed{F_n(x) = \sin \frac{n\pi}{L} x} \qquad (n = 1, 2, \ldots)$$

Similarly, solving (6) with $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

$$G'' + \lambda_n^2 G = 0 \qquad \text{where } \lambda_n = cp = \frac{cn\pi}{L}$$

The general solution is

$$\boxed{G_n(t) = B_n \cos \lambda_n(t) + B_n^* \sin \lambda_n(t)}$$

Hence, solutions of (1) satisfying (2) are

$$u_n(x,t) = F_n(x) G_n(t) = G_n(t) F_n(x)$$

$$\boxed{u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x}$$

These functions are called eigenfunctions and the values $\lambda_n = \frac{cn\pi}{L}$ are called eigenvalues of the vibrating string. The set $\{\lambda_1, \lambda_2, \ldots\}$ is called the spectrum.

Solution of the Entire Problem. Fourier Series

Consider the infinite series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x.$$

Satisfying initial conditions $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L} x = f(x)$$

which is the fourier sine series, hence

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx}$$

Satisfying initial conditions $u_t(x,0) = g(x)$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \left(-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t\right) \left.\sin \frac{n\pi}{L} x\right|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x)$$

$$\boxed{B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx}$$

Case: if initial velocity $g(x)$ is identically zero

Then $B_n^* = 0$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi}{L} x, \quad \lambda_n = \frac{cn\pi}{L}$$

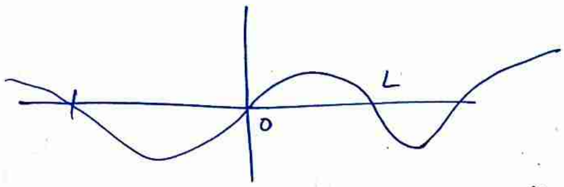
Using $\cos \frac{cn\pi}{L} t \sin \frac{n\pi}{L} x = \frac{1}{2}\left[\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}\right]$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[f^*(x-ct) + f^*(x+ct)\right]}$$

where $f^*(x-ct) = \sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\}$

$f^*(x+ct) = \sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$

where $f^*$ is the odd periodic extension of with period ③
2L.



Odd periodic extension of $f(x)$

eg Solve the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad , \quad 0 < x < l \ , t > 0$$

$$y(0,t) = y(l,t) = 0$$
$$y(x,0) = f(x)$$
$$y_t(x,0) = 0$$

Sol: Let $y(x,t) = (A\cos px + B\sin px)(C\cos ptc + D\sin cpt)$

be the solution of given wave equation

$y(0,t) = 0$

$\Rightarrow \quad 0 = A(\cos cpt + D\sin cpt)$

$\Rightarrow A = 0$

Hence, $y(x,t) = B\sin px\,(C\cos cpt + D\sin cpt)$

$y(x,t) = \sin px\,(C'\cos cpt + D'\sin cpt)$

$y(l,t) = 0$, we obtain

$0 = \sin pl\,(C'\cos cpt + D'\sin cpt)$

$\Rightarrow \sin pl = 0 \Rightarrow pl = n\pi$

$\Rightarrow p = \frac{n\pi}{l}$

Hence,

$$y(x,t) = \sin\frac{n\pi}{L}x\left(C'\cos\frac{n\pi}{L}ct + D'\sin\frac{n\pi}{L}ct\right)$$

The general solution is given by

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{L}\right)$$

at $t=0$, $y(x,t) = f(x)$  $\frac{\partial y}{\partial t} = 0$

$$\frac{\partial y(x,0)}{\partial t} = 0 \Rightarrow 0 = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} D_n \frac{n\pi c}{l}$$

$$\Rightarrow D_n = 0$$

$$\Rightarrow y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} C_n \cdot \cos\frac{n\pi ct}{l}$$

$$y(x,0) = f(x)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} f(x) \sin\frac{n\pi x}{l} C_n$$

which is fourier sine series, hence

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \quad \text{—(1)}$$

Hence, the general sol$^n$ is given by

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}}$$

where $C_n$ is given by (1)

**Exercise:** Solve $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

$$y(0,t) = y(l,t) = 0$$

$$y(x,0) = 0 \quad , \quad y_t(x,0) = \mu x(l-x)$$

$$0 < x < l.$$

**Ans.** $$y(x,t) = \frac{8\mu l^3}{\pi^4 c} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi c t}{l} \cdot \sin\frac{(2n-1)\pi x}{l}.$$

**D'Alembert's Solution of the Wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 = \frac{T}{\rho} \quad \text{—} (1)$$

$$\boxed{u(x,t) = \phi(x+ct) + \psi(x-ct)}$$

This is known as D'Alembert's solution of (1)

**Now**, $u(x,0) = f(x)$ $\qquad u_t(x,0) = g(x)$

$$u_t(x,t) = c\,\phi'(x+ct) - c\,\psi'(x-ct)$$

$$u(x,0) = \phi(x) + \psi(x) = f(x) \quad \text{—} (2)$$

$$u_t(x,0) = c\phi'(x) + c\psi'(x) = g(x) \quad \text{—} (3)$$

Dividing (3) by $c$ & integrating w.r.t $x$, we obtain

$$\phi(x) - \psi(x) = k(x_0) + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$$

$$k(x_0) = \phi(x_0) - \psi(x_0)$$

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds + \frac{1}{2}k(x_0)$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_{x_0}^{x} g(s)\,ds - \frac{1}{2} k(x_0)$$

Hence, sol$^n$ is given by

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

Characteristics: Types and Normal Forms of PDEs

Consider the PDE of the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x,y,u,u_x,u_y)$$

| Type | Condition | Example |
|---|---|---|
| 1) Hyperbolic | $AC-B^2 < 0$ | Wave equation |
| 2) Parabolic | $AC-B^2 = 0$ | Heat equation |
| 3) Elliptic | $AC-B^2 > 0$ | Laplace equation |

## Heat Equation

Consider the temperature in a long thin metal bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis and is perfectly insulated laterally, so that the ~~Laplacian~~ heat flows in the x-direction only. The one-dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Boundary conditions $\quad u(0,t) = 0, \quad u(L,t) = 0 \quad \forall t \geq 0$

Initial Condition $\quad u(x,0) = f(x)$

eg Using method of seperation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad , \quad u(x,0) = 6e^{-3x}$$

Solⁿ Let $u(x,t) = X(x)\,T(t)$ be the solⁿ of given PDE

$$X'T = 2XT' + XT$$

$$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$$X' - X - 2kX = 0$$

$$\frac{X'}{X} = 2k$$

Integrating, we get

$$\log X = (1+2k)x + \log c$$

$$\Rightarrow X = ce^{(1+2k)}$$

$$\frac{T'}{T} = k$$

$$\Rightarrow \log T = kt + \log c'$$

$$\Rightarrow T = c'e^{kt}$$

$$u(x,t) = XT = cc'e^{(1+2k)x}e^{kt}$$

$$u(x,0) = 6e^{-3x} = cc'e^{(1+2k)x}$$

$$\Rightarrow cc' = 6 \qquad 1+2k = -3 \qquad \Rightarrow k = -2$$

$$\therefore u = 6e^{-3x}e^{-2t}$$

$$\Rightarrow \boxed{u = 6e^{-(3x+2t)}}$$

eg. Solve $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$

$$y(0,t) = 0 \quad , \quad y(l,t) = 0$$

$$y(x,0) = \mu x(l-x)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$$

Sol:- $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

let $y(x,t) = F(x)G(t)$ be the sol:- of (1)

then we have

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

$$y(0,t) = 0 \Rightarrow c_1 = 0$$

$$y(l,t) = 0 \Rightarrow p = \frac{n\pi}{l}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$y(x,0) = \mu x(l-x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $\Rightarrow$ $$b_n = \frac{2}{l}\int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx$$

(Solve by Integrating by parts)

$$b_n = \frac{4\mu l^2}{n^3\pi^3}[1-(-1)^n)$$

Here, $$y(x,t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi x}{l} \cdot \cos \frac{(2m-1)\pi ct}{l}$$

eg. Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero & initial deflection $f(x) = k(\sin x - \sin 2x)$ (2)

Sol: By D'alembert's method,

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

$$= \frac{1}{2}[k\{\sin(x+ct) - \sin 2(x+ct)\} + k\{\sin(x-ct) - \sin 2(x-ct)\}]$$

$$y(x,t) = k(\sin x \cos ct) - k \sin 2x \cos 2ct$$

$$y(x,0) = k(\sin x - \sin 2x) = f(x)$$

## Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{———} (1)$$

let $u(x,t) = X(x)\,T(t)$ be sol$^n$ of (1)

$$XT' = c^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{c^2 T} = k \text{ (say)}$$

$$X'' - kX = 0 \qquad T' - kc^2 T = 0$$

$k = +ve = p^2$ we have

$$X = c_1 e^{px} + c_2 e^{-px}, \quad T = c_3 e^{c^2 p^2 t}$$

$k$ is $-ve = -p^2$ (say) $\quad X = c_4 \cos px + c_5 \sin px$

$$T = c_6 e^{-c^2 p^2 t}$$

$k$ is zero. $\quad X = c_7 x + c_8 \quad T = c_9$

The various possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px}) c_3 e^{c^2p^2t}$$

$$u = (c_4 \cos px + c_5 \sin px) c_6 e^{-c^2p^2t}$$

$$u = (c_7 x + c_8) c_9$$

$u$ should decrease with increase of time, so

$$\boxed{u = (c_1 \cos px + c_2 \sin px) e^{-c^2p^2t}}$$

is the solution of heat equation

eg. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ — ①

$$u(x,0) = 3 \sin n\pi x$$

$$u(0,t) = 0 \quad \& \quad u(1,t) = 0, \quad 0<x<l$$

Sol. $u(x,t) = (c_1 \cos px + c_2 \sin px) e^{-p^2t}$

$$u(0,t) = c_1 e^{-p^2t} = 0 \Rightarrow c_1 = 0$$

$$\Rightarrow u(x,t) = c_2 \sin px \, e^{-p^2t}$$

$$u(1,t) = 0 = c_2 \sin p \, e^{-p^2t}$$

$$p = n\pi$$

$\therefore \quad u(x,t) = b_n e^{-(n\pi)^2t} \sin n\pi x \quad$ where $b_n = c_2$

Hence, the general sol$^n$ of ① is $u(x,t) = \sum b_n e^{-n^2\pi^2t} \sin n\pi x$

$$u(0,t) = 3 \sin n\pi x = \sum b_n e^{-n^2\pi^2t} \sin n\pi x$$

$$\Rightarrow b_n = 3$$

$$\Rightarrow \quad u(x,t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2t} \sin n\pi x.$$

# Steady Two-dimensional Heat Problems

Heat equation in 2 dimension : $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ —— (1)

If $\frac{\partial u}{\partial t} = 0$ , then (1) reduces to Laplace eqⁿ

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (1)}$$

Let $u(x,y) = X(x)\,Y(y)$ be a solⁿ of (1)

Substituting, we get

$$X''Y + XY'' = 0$$

$$\frac{1}{X}X'' = -\frac{1}{Y}Y'' = k \text{ (say)}$$

$$\rightarrow X'' - kX = 0 \qquad Y'' + kY = 0$$

if $k$ is +ve, $(=p^2)$ $\quad X = c_1 e^{px} + c_2 e^{-px}$

$$Y = c_3 \cos py + c_4 \sin py$$

if $k$ is negative, $(=-p^2)$ $\quad X = c_5 \cos px + c_6 \sin px$

$$Y = c_7 e^{py} + c_8 e^{-py}$$

if $k$ is zero, $\quad X = c_9 x + c_{10} \qquad Y = c_{11} y + c_{12}$

The various possible solⁿ of (1) are

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py})$$

$$u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12})$$

eg. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(\pi, y) = 0$

$u(x, \infty) = 0 \quad 0 < x < \pi$

$u(x, 0) = u_0 \quad 0 < x < \pi$

Three possible solution (as derived earlier) are

$$u = (C_1 e^{px} + C_2 e^{-px})(C_3 \cos py + C_4 \sin py) \quad — (1)$$

$$u = (C_5 \cos px + C_6 \sin px)(C_7 e^{py} + C_8 e^{-py}) \quad — (2)$$

$$u = (C_9 x + C_{10})(C_{11} y + C_{12}) \quad — (3)$$

(2) cannot satisfy $u(0, y) = 0 \;\forall y$

(3) cannot satisfy $u(x, \infty) = 0$ in $0 < x < \pi$

The only possible sol$^n$ is

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py})$$

$$u(0, y) = 0 = C_1 (C_3 e^{py} + C_4 e^{-py})$$

$$\Rightarrow C_1 = 0$$

Hence, $u(x, y) = C_2 \sin px (C_3 e^{py} + C_4 e^{-py})$

$$u(\pi, y) = C_2 \sin p\pi (C_3 e^{py} + C_4 e^{-py}) = 0$$

$$\sin p\pi = 0 \Rightarrow p\pi = n\pi$$

$$\Rightarrow p = n$$

Now if $u = 0$ as $y \to \infty$ $\Rightarrow C_3 = 0$

$\therefore$ $u(x, y) = b_n \sin nx \, e^{-ny}$, $b_n = C_2 C_4$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \, e^{-ny}$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} u_0 \sin nx \, dx = \frac{2u_0}{n\pi} [1 - (-1)^n]$$

$b_n = 0$ if $n$ is even & $b_n = \frac{4u_0}{n\pi}$, if $n$ is odd.

## ~~Eq~~ Polar form of Laplace Equation.

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{—(1)}$$

Let $u(r,\theta) = R(r) \cdot \phi(\theta)$ be solⁿ of (1)
then solving as before, we get

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta)$$

$$u = [c_5 \cos(p \log r) + c_6 \sin(p \log r)] (c_7 e^{p\theta} + c_8 e^{-p\theta})$$

$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12})$$

## Two Dimensional Wave Equation

$$\text{(1)} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right], \quad c^2 = \frac{T}{P}$$

Let $u = X(x)\, Y(y)\, T(t)$ be solⁿ of (1)

$\cancel{f(x,y) = \sum \sum}$

Solⁿ is given by

$$u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{b} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt)$$

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Solution by Fourier Integrals & Transforms

eg. Solve $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ $(x>0, t>0)$ —— ①

$u(x,0) = f(x)$

$u(0,t) = 0$

Taking Fourier sine transform in ①, we have

(Denoting $F_s(u(x,t))$ by $\bar{u}_s$)

$$\frac{d\bar{u}_s}{dt} = c^2 [s\,u(0,t) - s^2 \bar{u}_s)$$

$$\frac{d\bar{u}_s}{dt} + c^2 s^2 \bar{u}_s = 0 \quad \text{—— ②}$$

Also $\bar{u}_s = \bar{f}(s)$ at $t = 0$ —— ③

Solving ② & ③, we get

$$\bar{u}_s = \bar{f}_s(s)\, e^{-c^2 s^2 t}$$

$$\Rightarrow f_s(s) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(s)\, e^{-c^2 s^2 t} \sin xs \, ds.$$