Defn:- If V & W are vector spaces & ~~then~~ $T: V \to W$ is linear & then $N(T)$ & $R(T)$ are finite dimensional. Then dimension of $N(T)$ is called nullity of T denoted by nullity(T) & dimension of $R(T)$ is called rank of T denoted by rank(T).

THM 2.3 **DIMENSION THEOREM.**

Let V & W be two vector spaces. $T: V \to W$ be linear. If V is finite dimensional. Then nullity(T) + rank(T) = dim(V)

Proof:- Suppose $\dim(V) = n$ and
$\dim(N(T)) = k$.

Let $\{v_1, v_2, \ldots v_k\}$ be a basis for $N(T)$

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As $N(T)$ is a subspace of V.
$\therefore$ we may extend $\{v_1, v_2, \ldots v_k\}$ to a basis for $\beta = \{v_1, v_2, \ldots v_n\}$ of V

Claim: $S = \{T(v_{k+1}), T(v_{k+2}), \ldots T(v_n)\}$ is a basis for $R(T)$.

As $T(v_i) = 0 \quad \forall \; 1 \leq i \leq k$ $\quad \left[\because v_i \in N(T) \text{ for } i=1, \ldots k \; \therefore T(v_i) = 0\right]$

Using thm 2.2,

$$R(T) = \operatorname{span}\{T(v_1), T(v_2), \ldots T(v_k), T(v_{k+1}) \ldots T(v_n)\}$$
$$= \operatorname{span}\{T(v_{k+1}), T(v_{k+2}), \ldots T(v_n)\}.$$
$$= \operatorname{span}(S).$$

Now, To prove S is linearly independent.

Let $\sum_{i=k+1}^{n} b_i T(v_i) = 0$, $b_i \in F$, $i = k+1, \ldots n$

$\Rightarrow$ ~~$T\left(\sum_{i=k+1}^{n} b_i T(v_i)\right) = T(0)$~~

$\Rightarrow b_{k+1} T(v_{k+1}) + b_{k+2} T(v_{k+2}) + \ldots b_n T(v_n) = 0$

$\Rightarrow T(b_{k+1} v_{k+1} + \ldots b_n v_n) = 0$ $\quad [\because T \text{ is linear}]$

$\Rightarrow T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$

$\Rightarrow \sum_{i=k+1}^{n} b_i v_i \in N(T)$

$\therefore \exists$ scalars $c_1, c_2, \ldots c_k \in F$ s.t.

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$$

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$\Rightarrow \sum_{i=1}^{k} -c_i v_i + \sum_{i=k+1}^{n} b_i v_i = 0$

Since β is a basis for V.

$\therefore$ we have $b_i = 0 \ \forall i$.

$\Rightarrow$ S is linear independent.

$\Rightarrow$ S is a basis for $R(T)$.

$\Rightarrow \dim(R(T)) = n-k = \text{rank}(T)$

$\dim(N(T)) = k = \text{nullity}(T)$.

Hence, $\text{nullity}(T) + \text{rank}(T)$

$= k + (n-k)$

$= n = \dim(V)$

Thm-2.4 V & W are vector space. $T: V \to W$ is linear. T is one-one if & only if $N(T) = \{0\}$.

Proof: Suppose T is one-one.

T.P $N(T) = \{0\}$.

Let $x \in N(T)$

$\Rightarrow T(x) = 0_W = T(0_V)$

$\Rightarrow x = 0_V$ [$\because$ T is one-one]

$\Rightarrow N(T) = \{0_V\}$. [$\because x \in N(T)$ was arbitrary]

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Conversely:- Suppose that $N(T) = \{0\}$.

T.P T is one-one

Let $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0_W$

$\Rightarrow T(x-y) = 0_W$ ($\because$ T is linear).

$\Rightarrow x-y \in N(T) = \{0\}$

$\Rightarrow x-y = 0$

$\Rightarrow x = y$.

Thus $T(x) = T(y)$

$\Rightarrow x = y$.

Hence, T is one-one.

Thm 2.5 Let V & W be vector spaces of equal finite dimension and let $T: V \to W$ be linear transformation. Then, the following statements are equivalent:-

(i) T is one-one

(ii) T is onto.

(iii) $\text{rank}(T) = \dim V$

Proof:- By dim thm,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

If (i) holds $\Rightarrow$ T is one-one

$\Leftrightarrow N(T) = \{0\}$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow \text{nullity}(T) = 0$

$\Rightarrow$ Hence,

$$\text{rank}(T) = \dim(V)$$

$\dim(V) = \text{rank}(T)$

$\Leftrightarrow \dim(W) = \text{rank}(T) = \dim(R(T))$

$\Leftrightarrow \dim(R(T)) = \dim W$

Also $R(T) \subseteq W$

$\Leftrightarrow W = R(T)$

co-domain range

$\Leftrightarrow T$ is onto

Hence, the result ~~is~~ is both ways true.