Groups:

Def^{n} → Consider a set G and an operation $\overset{\oplus}{\otimes} : G \times G \to G$ defined on G. Then $G := (G, \overset{\oplus}{\otimes})$ is called a Group if the following hold:

1. Closure of G under $\overset{\oplus}{\otimes}$: $\forall x, y \in G : x \overset{\oplus}{\otimes} y \in G$
2. Associativity: $\forall x, y, z \in G : (x \overset{\oplus}{\otimes} y) \overset{\oplus}{\otimes} z = x \overset{\oplus}{\otimes} (y \overset{\oplus}{\otimes} z)$
3. Identity element: $\exists\, e \in G \; \forall x \in G : x \overset{\oplus}{\otimes} e = x$ and $e \overset{\oplus}{\otimes} x = x$
4. Inverse element: $\forall x \in G \; \exists\, y \in G : \underset{x \oplus (-x) = e}{x \overset{\oplus}{\otimes} y = e}$ and $\underset{-x \oplus (x) = e}{y \overset{\oplus}{\otimes} x = e}$, where e is the identity element. we often write x^{-1} to denote the inverse element of x.

If there exist $\forall x, y \in G : x \overset{\oplus}{\otimes} y = y \overset{\oplus}{\otimes} x$, then $G = (G, \overset{\oplus}{\otimes})$ is an abelian Group (Commutative).

Ex: • $(\mathbb{Z}, +)$ is an abelian Group.

* $(\mathbb{N}_0, +)$ is not a group. Although $(\mathbb{N}_0, +)$ possesses a neutral (identity) element (0), the inverse elements are missing.
* $(\mathbb{Z}, \cdot)$ is not a group. Although $(\mathbb{Z}, \cdot)$ contains an identity element 1, the inverse elements are missing.
* $(\mathbb{R}, \cdot)$ is not a group, since 0 does not possess an inverse element

Vector Spaces:

Vector spaces are the basic settings in which linear algebra happens.

Definition →
A real valued vector space $V = (V, +, \cdot)$ is a set V with two operations

$$+ : V \times V \to V$$
$$\cdot : \mathbb{R} \times V \to V$$

where

1. $(V, +)$ is an abelian Group
2. Distributivity:
   (i) $\forall \lambda \in \mathbb{R}, x, y \in V : \lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$
   (ii) $\forall \lambda, \psi \in \mathbb{R}, x \in V : (\lambda + \psi) \cdot x = \lambda \cdot x + \psi \cdot x$
3. Associativity: $\forall \lambda, \psi \in \mathbb{R}, x \in V : \lambda \cdot (\psi \cdot x) = (\lambda \psi) \cdot x$
4. ~~Identity element w.r.t.~~ There exists a multiplicative identity (1) in $\mathbb{R}$ such that $1 \cdot x = x \; \forall x \in V$.

The elements $x \in V$ are called vectors. The identity element of $(V, +)$ is the zero vector $0 = [0, \cdots, 0]^T$ and the operation $\oplus$ is called vector addition. The elements $\lambda \in \mathbb{R}$ are called scalars and the operation $\odot$ is a multiplication by scalars.

Ex 1. $\mathbb{R}^n = \{ (x_1, \cdots, x_n), ~~\in \mathbb{R}~~ (y_1, y_2, \cdots y_n) : x_n, y_n \in \mathbb{R}^n \}$

$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots y_n) := (x_1+y_1, x_2+y_2, \cdots, x_n+y_n) \in \mathbb{R}^n$ (closed under addition)

For c, $c(x_1, x_2, \cdots x_n) = (cx_1, cx_2, \cdots, cx_n) \in \mathbb{R}^n$ (scalar multiplication)

Is multiplication linear?

$(a+b)(x_1, x_2, \cdots, x_n) = a(x_1, \cdots, x_n) + b(x_1, x_2, \cdots x_n)$

To show. $(a+b)(x_1, x_2, \cdots x_n) = ((a+b)x_1, \cdots, (a+b)x_n)$

$= (ax_1 + bx_1, \cdots, ax_n + bx_n)$

$= (ax_1, \cdots, ax_n) + (bx_1, \cdots, bx_n)$

$= a(x_1, \cdots, x_n) + b(x_1, \cdots, x_n)$ //

Ex 2. The scalars $\mathbb{R}$ is a vector space with vector addition as the usual addition of real nos and the scalar multiplication of real nos.

Ex 3: Let $V = \{0\}$.

Vector addition : $0 + 0 = 0$

Scalar multiplication: $c \cdot 0 = 0$

Check that the properties are satisfied. This vector space is called as the zero vector space.

Ex. 1. Let $R^n = \{(a_1, a_2, \cdots, a_n) : a_i \in R, 1 \le i \le n\}$.

be the set of all n-tuples of real nos.

We introduce two operations as follows:

1. $(a_1, a_2, \cdots, a_n) + (b_1, b_2, \cdots, b_n) = (a_1+b_1, a_2+b_2+ \cdots a_n+b_n)$

$\forall\, u = (a_1, a_2, \cdots a_n),\ v = (b_1, b_2, \cdots b_n) \in R^n$

clearly $u+v \in R^n\ \forall\, u, v \in R^n$

2. $\alpha(a_1, a_2, \cdots, a_n) = \alpha a_1, \alpha a_2, \cdots, \alpha a_n)$

$\forall\, \alpha \in R,\ u = (a_1, a_2, \cdots, a_n) \in R^n$

clearly, $\alpha u \in R^n\ \forall\, \alpha \in R,\ u \in R^n$

Now

A1. Let $u = (a_1, a_2, \cdots, a_n)$

$v = (b_1, b_2, \cdots, b_n)$ and

$w = (c_1, c_2, \cdots, c_n) \in R^n$

$(u+v)+w = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n) + (c_1, c_2, \cdots c_n)$

$= ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \cdots (a_n+b_n)+c_n)$

$= (a_1+(b_1+c_1), a_2+(b_2+c_2), \cdots, a_n+(b_n+c_n))]$

$= (a_1, a_2, \cdots, a_n) + (b_1+c_1, b_2+c_2, \cdots, b_n+c_n)$

$= u+(v+w)$

A2. $\exists\ 0 = (0, 0, \cdots 0) \in R^n$ such that

$(a_1, a_2, \cdots, a_n) + (0, 0, \cdots 0) = (a_1, a_2, \cdots a_n)$

$\forall\, (a_1, a_2, \cdots, a_n) \in R^n$

A3. $\forall\, u = (a_1, a_2, \cdots, a_n) \in R^n$ there exists

$-u = (-a_1, -a_2, \cdots, -a_n) \in R^n$

such that $u+(-u) = (a_1, a_2, \cdots, a_n) + (-a_1, -a_2, \cdots, -a_n)$

$= (0, 0, \cdots 0)$

$= 0$

A4. $\forall\, u = (a_1, a_2, \cdots, a_n); v = (b_1, b_2, \cdots, b_n) \in R^n$

$u+v = (a_1+b_1, a_2+b_2, \cdots, a_n+b_n)$

$= (b_1+a_1; b_2+a_2, \cdots, b_n+a_n)$

$= v + u$

(contd)

By virtue of above discussion R^n is abelian group w.r.t. addition [Q (contd)]
defined above.

Now, if $\alpha, \beta \in \mathbb{R}$, $u = (a_1, a_2, \ldots, a_n)$, $v = (b_1, b_2, \ldots, b_n) \in R^n$, then

M1. $\alpha(u+v) = \alpha(a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$

$= (\alpha(a_1+b_1), \alpha(a_2+b_2), \ldots, \alpha(a_n+b_n))$

$= (\alpha a_1+\alpha b_1, \alpha a_2+\alpha b_2, \ldots, \alpha a_n+\alpha b_n)$

$= (\alpha a_1, \alpha a_2, \ldots, \alpha a_n) + (\alpha b_1, \alpha b_2, \ldots, \alpha b_n)$

$= \alpha(a_1, a_2, \ldots, a_n) + \alpha(b_1, b_2, \ldots, b_n)$

$= \alpha u + \alpha v$.

M2. $(\alpha+\beta)u = ((\alpha+\beta)a_1, (\alpha+\beta)a_2, \ldots, (\alpha+\beta)a_n)$

$= (\alpha a_1+\beta a_1, \alpha a_2+\beta a_2, \ldots \alpha a_n+\beta a_n)$

$= (\alpha a_1, \alpha a_2, \ldots, \alpha a_n) + (\beta a_1, \beta a_2, \ldots, \beta a_n)$

$= \alpha u + \beta u$

M3. $\alpha(\beta u) = \alpha(\beta a_1, \beta a_2, \ldots, \beta a_n)$

$= (\alpha(\beta a_1), \alpha(\beta a_2), \ldots, \alpha(\beta a_n))$

$= ((\alpha\beta)a_1, (\alpha\beta)a_2, \ldots, (\alpha\beta)a_n)$

$= (\alpha\beta)u$

M4. $1u = 1(a_1, a_2, \ldots, a_n)$

$= (1a_1, 1a_2, \ldots, 1a_n)$

$= u$

So by defn R^n is a vector space over $\mathbb{R}$.

Ex: Let $R^2 = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}$

Define $(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$

$\alpha(a_1, a_2) = (\alpha a_1, a_2) \; \forall \; (a_1, a_2), (b_1, b_2) \in R^2, \alpha \in \mathbb{R}$

Let $\alpha, \beta \in \mathbb{R}$ and $u = (a_1, a_2) \in R^2$. Then

$(\alpha+\beta)u = (\alpha+\beta)(a_1, a_2)$

$= ((\alpha+\beta)a_1, a_2) = (\alpha a_1+\beta a_1, a_2)$

$= (\alpha a_1, a_2) + (\beta a_1, a_2)$

$= \alpha(a_1, a_2) + \beta(a_1, a_2) \neq \alpha u + \beta u$

Hence R^2 is not a vector space over R with compositions defined above.

Ex. The set $\mathbb{C}$ of complex nos. is a vector space over over $\mathbb{R}$ of real nos.

Soln. We know that $(\mathbb{C}, +)$ is an abelian group.

Let $a, b \in \mathbb{R}$ and $x, y, z \in \mathbb{C}$. Then $ax \in \mathbb{C}$ and

| $x = a+ib,\ y = c+id$ $z = e+if$.

1. $a(x+y) = ax+ay$
2. $(a+b)x = ax+by$
3. $a(bx) = (ab)x$
4. $1 \cdot x = x$

Hence $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Ex. The set $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ of all polynomials of degree 2 over $\mathbb{R}$ is a vector space over $\mathbb{R}$ w.r.t. the compositions:

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 \quad \text{---(1)}$$

$$a(a_0 + a_1x + a_2x^2) = aa_0 + aa_1x + aa_2x^2 \quad \text{---(2)}$$

Soln. It is easy to verify that $(V, +)$ is an abelian group whose additive identity is $0 = 0 + 0x + 0x^2$ and the additive inverse of $a_0 + a_1x + a_2x^2 \in V$ is $(-a_0) + (-a_1)x + (-a_2)x^2 \in V$

Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in V$, By (2), $\alpha f \in V$ and

1. $\alpha(f+g) = \alpha f + \alpha g$
2. $(\alpha+\beta)f = \alpha f + \beta f$
3. $\alpha(\beta f) = (\alpha\beta)f$.
4. $1f = f$.

To verify the 1st property, let

$$f = a_0 + a_1x + a_2x^2,\ g = b_0 + b_1x + b_2x^2 \in V$$

Then
$$\alpha(f+g) = \alpha[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2], \text{ by (1)}$$
$$= \alpha(a_0+b_0) + (\alpha a_1 + \alpha b_1)x + (\alpha a_2 + \alpha b_2)x^2$$
$$= (\alpha a_0 + \alpha a_1x + \alpha a_2x^2) + (\alpha b_0 + \alpha b_1x + \alpha b_2x^2)$$
$$= \alpha f + \alpha g.$$

Similarly, we can verify other three properties

Hence V is a vector space over $\mathbb{R}$.

Ex. The set $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ w.r.t. matrix addition and multiplication of a matrix by a scalar.

Soln. We know M_2 is an abelian group w.r.t. matrix addition.

Further $\alpha, \beta \in R$ and $A, B \in M_2$. Then $\alpha A \in M_2$ and

1. $\alpha(A+B) = \alpha A + \alpha B$
2. $(\alpha+\beta)A = \alpha A + \beta A$
3. $\alpha(\beta A) = (\alpha\beta)A$
4. $1A = A$.

Hence M_2 is a vector space over $\mathbb{R}$.

Ex. Prove that the set $M_{m\times n}$ of all $m\times n$ matrices (where m and n are fixed positive integers) over the set R of real numbers is a vector space over $\mathbb{R}$ w.r.t. matrix addition and multiplication of matrices by a scalar (real no.)

Soln. Here we find that

1. $(M_{m\times n}, +)$ is a commutative group (abelian group w.r.t. matrix addition.
2. If $A \in M_{m\times n}$ and $a \in R$, then the matrix $aA \in M_{m\times n}$
3. Further we find that if $a, b \in R$ and $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n} \in M_{m\times n}$, then
4. $a(A+B) = a([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n})$

$= a([a_{ij} + b_{ij}]_{m\times n})$

$= [a a_{ij} + a b_{ij}]_{m\times n}$

$= a[a_{ij}]_{m\times n} + a[b_{ij}]_{m\times n} = aA + aB$

5. $(a+b)A = (a+b)[a_{ij}]_{m\times n} = [a a_{ij} + b a_{ij}]_{m\times n}$

$= [a a_{ij}]_{m\times n} + [b a_{ij}]_{m\times n}$

$= a[a_{ij}]_{m\times n} + b[a_{ij}]_{m\times n} = aA + bA$

6. $a(bA) = a(b[a_{ij}]_{m\times n}) = a[b a_{ij}]_{m\times n} = [a b a_{ij}]_{m\times n} = (ab)[a_{ij}]_{m\times n} = (ab)A$
7. $1A = 1[a_{ij}]_{m\times n} = [1 a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = A$, where 1 is the multiplicative identity of $\mathbb{R}$

Hence all the properties of vector space are satisfied and $M_{m\times n}$ is a vector space over $\mathbb{R}$.

Ex. If $V = \{(a,b) : a, b \in R\}$. Show that R^2 is a vector space over $\mathbb{R}$ defined as

$$(a, b) + (c, d) = (a+c, b+d) \quad \text{---(I)}$$
$$k(a, b) = (ka, kb) \quad \text{---(II)}$$
$$\forall\ a, b, c, d, k \in \mathbb{R}$$

Soln. Here we find that

1. $(V, +)$ is an abelian group, since

(i) Sum (as defined by (I)) of two ordered pairs in V is an ordered pair in V.

i.e, V is closed w.r.t. +.

(ii) The operation + is associative in V, since the addition of real nos obeys associative law.

(iii) There exists the additive identity $(0,0)$ in V, such that

$$(0,0) + (a,b) = (a,b) = (a,b) + (0,0) \ \forall\ (a,b) \in V$$

(iv) For each $(a,b) \in V$ there exists its additive inverse $(-a, -b) \in V$, such that

$$(a,b) + (-a,-b) = (0,0) = (-a,-b) + (a,b)$$

(v) The operation + is commutative in V as the addition of real nos obeys Commutative law.

2. $k[(a,b) + (c,d)] = k[a+c, b+d]$ from (I)

$= [k(a+c), k(b+d)]$, from (II)

$= (ka+kc, kb+kd)$ (by distributivity in $\mathbb{R}$)

$= (ka, kb) + (kc, kd)$ from I

$= k(a,b) + k(c,d)$ from II

3. $(k+m)(a,b) = [(k+m)a, (k+m)b]$, from II

$= (ka+ma, kb+mb)$, (by distributivity in $\mathbb{R}$)

$= (ka, kb) + (ma, mb)$, from I

$= k(a,b) + m(a,b)$, from II

4. $k[m(a,b)] = k(ma, mb)$, from II

$= (kma, kmb)$, from II

$= km(a,b)$, from II

5. $1(a,b) = (1a, 1b)$, from II

$= (a,b)$

Hence all the properties of vector space are satisfied and so V is a vector space.

Ex If $V=\{(a,b): a,b\in\mathbb{R}\}$, Show that $\mathbb{R}^2$ is not a vector space under scalar addition and scalar multiplication defined as below:

$$(i)\ (a,b)+(c,d)=(a+c,\ b+d)$$
$$k(a,b)=(0,kb)$$
$$(ii)\ (a,b)+(c,d)=(a+c,\ b+d)$$
$$k(a,b)=(ka,b)$$

Soln: 1. V is not a vector space under composition defined in (i) because

$$1(a,b)=(0,1b)=(0,b)\neq(a,b)$$ i.e, property for vector space is not satisfied.

2. V is not a vector space under the composition defined in (ii) because

$$(k_1+k_2)(a,b)=[(k_1+k_2)a,\ b] \quad , \because k(a,b)=(ka,b) \text{ (from(ii))}$$
$$=(k_1a+k_2a,\ 0+b)$$
$$=(k_1a,\ 0)+(k_2a,\ b), \quad \because (a+c,\ b+d)=(a,b)+(c,d)$$
$$=k_1(a,0)+k_2(a,b), \quad \because (ka,b)=k(a,b) \text{ (from(ii))}$$
$$\neq k_1(a,b)+k_2(a,b)$$

Ex: If $V=\{(a,b): a,b\in\mathbb{R}\}$ in which addition and scalar multiplication are defined as below, is not a vector space over $\mathbb{R}$:-

$$(i)\ (a,b)+(c,d)=(a+d,\ b+c)$$
$$k(a,b)=(ka,kb)$$
$$(ii)\ (a,b)+(c,d)=(ac,\ bd)$$
$$k(a,b)=(ka,kb)$$

Soln: (i) Here we find that

$$(k_1+k_2)(a,b)=[(k_1+k_2)a,\ (k_1+k_2)b], \quad \because k(a,b)=(ka,kb)$$
$$=(k_1a+k_2a,\ k_1b+k_2b)$$
$$=(k_1a,\ k_1b)+(k_2b,\ k_2a) \quad \because (a,b)+(c,d)=(a+d,b+c)$$
$$=k_1(a,b)+k_2(b,a) \quad \because k(a,b)=(ka,kb)$$
$$\neq k_1(a,b)+k_2(a,b)$$

Hence the property is not satisfied. Hence V is not a vector space under composition defined under (i).

(ii) Here we find that

$$k[(a,b)+(c,d)] = k(ac,\ bd), \quad \because (a,b)+(c,d) = (ac,bd)$$

$$= (kac,\ kbd), \quad \because k(a,b) = (ka,\ kb) \quad \text{—(1)}$$

Also $k(a,b)+k(c,d) = (ka,kb)+(kc,kd)$

$\because k(a,b) = (ka,kb)$

$$= (ka,kb)+(kc,kd) \quad \text{—(2)}$$

$\therefore$ From (1) & (2), we find that

$$k[(a,b)+(c,d)] \neq k(a,b)+k(c,d)$$

i.e, the property for vector space is not satisfied, hence V is not a vector space under composition defined in (2).

Eg3 Let S be a ~~field~~ non empty set, $F \to$ field and let $F(S, F)$ be set of all functions from S to F, then,

$$F(S, F) = \{ f : S \to F \}$$

vector space +, scalar multiplication

$$f, g \in F(S, F), c \in F$$

$$(f+g)(s) = f(s) + g(s)$$

$$cf(s) = c(f(s))$$

All properties clearly holds.

Eg4: Consider the set of all polynomials with coeff from the field F, then this set $P(F)$ forms the vector space, ~~addition~~ under opert^n of addition & scalar multiplication

Eg5 Let F be any field, a sequence in F, is a function $\sigma : \mathbb{Z}^+ \to F$ such that $\sigma(n) = a_n$ $n = 1, 2 \ldots$, where seq is denoted by $\{a_n\}$.

Let V consist of all these sequences $\{a_n\}$ in F

$$V = \{ \{a_n\} : a_n \in F \}$$

such that the sequence has only a finite no. of non-zero terms.

$$a_n \neq 0 \text{ for finite terms}$$

Hence $\underline{a_n \to 0}$.

We define operation of addition of & scalar multiplication on V.

If $\{a_n\}, \{b_n\} \in V$, $t \in F$ then

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

$$t\{a_0\} = \{ta_n\}$$

[Clearly holds true].

Example 1.2.6. *Show that the set* $\mathbf{Q}(\sqrt{2}) = \{ a + b\sqrt{2} : a, b \in \mathbf{Q} \}$ *is a vector space over* $\mathbf{Q}$ *w.r.t. the compositions :*

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \quad \ldots(i)$$

$$\alpha(a + b\sqrt{2}) = \alpha a + \alpha b\sqrt{2}, \alpha \in \mathbf{Q}. \quad \ldots(ii)$$

Solution. We know $\{ \mathbf{Q}(\sqrt{2}), + \}$ is abelian group.

[See Illustration 7, p.2]

Properties of Scalar Multiplication.

Let $\alpha, \beta \in \mathbf{Q}$ and $u = a + b\sqrt{2}$, $v = c + d\sqrt{2} \in \mathbf{Q}(\sqrt{2})$.

From (*ii*), $\alpha u = \alpha a + \alpha b\sqrt{2} \in \mathbf{Q}(\sqrt{2})$.

1. $\alpha(u + v) = \alpha\{ (a + c) + (b + d)\sqrt{2} \}$, by (*i*)

$= \{ \alpha(a + c) + \alpha(b + d)\sqrt{2} \}$, by (*ii*)

$= \alpha a + \alpha c + (\alpha b + \alpha d)\sqrt{2}$, by distributive law in $\mathbf{Q}$

$= (\alpha a + \alpha b\sqrt{2}) + (\alpha c + \alpha d\sqrt{2})$.

$\therefore$ $\alpha(u + v) = \alpha u + \alpha v$.

2. $(\alpha + \beta)u = ((\alpha + \beta)a + (\alpha + \beta)b\sqrt{2})$, by (*ii*)

$= \alpha a + \beta a + (\alpha b + \beta b)\sqrt{2}$, by distributive law in $\mathbf{Q}$

$= (\alpha a + \alpha b\sqrt{2}) + (\beta a + \beta b\sqrt{2})$.

$\therefore$ $(\alpha + \beta)u = \alpha u + \beta u$, by (*ii*).

3. $\alpha(\beta u) = \alpha(\beta a) + \alpha(\beta b\sqrt{2})$, by (*ii*)

$= (\alpha\beta)a + (\alpha\beta)b\sqrt{2}$, by associative law in $\mathbf{Q}$

$\therefore$ $\alpha(\beta u) = (\alpha\beta)u$, by (*ii*).

4. $1u = 1(a + b\sqrt{2}) = a + b\sqrt{2} = u \ \forall \ u \in \mathbf{Q}(\sqrt{2})$.

Hence $\mathbf{Q}(\sqrt{2})$ is a vector space over $\mathbf{Q}$.

Example 1.2.7. *Let V be the set of all real valued continuous functions defined on the closed interval* $[0, 1]$ *and* $\mathbf{R}$ *be the field of real numbers. For any $f, g \in V$ and $\alpha \in \mathbf{R}$, define $f + g$ and αf by*

(*i*) $(f + g)(x) = f(x) + g(x)$, (*ii*) $(\alpha f)(x) = \alpha f(x)$, $x \in [0, 1]$.

Show that V is a vector space over $\mathbf{R}$.

Solution. Firstly, we show that $(V, +)$ is an abelian group.

Let $f, g, h \in V$.

1. Since $f : [0, 1] \to \mathbf{R}$ and $g : [0, 1] \to \mathbf{R}$ are continuous,

$\therefore \quad f + g : [0, 1] \to \mathbf{R}$ is also continuous and so $f + g \in V$.

2. $f + g = g + f$, since for any $x \in [0, 1]$,

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

(Note that + is commutative in $\mathbf{R}$)

3. $(f + g) + h = f + (g + h)$, since for any $x \in [0, 1]$

$$\begin{aligned}[(f + g) + h](x) &= (f + g)(x) + h(x) \\ &= \{ f(x) + g(x) \} + h(x) \\ &= f(x) + \{ g(x) + h(x) \}, \text{ by associative law in } (\mathbf{R}, +) \\ &= f(x) + (g + h)(x) \\ &= \{ f + (g + h) \}(x).\end{aligned}$$

4. The zero function $0 : [0, 1] \to \mathbf{R}$ defined as

$0(x) = 0 \quad \forall \ x \in [0, 1]$ satisfies $f + 0 = 0 + f = f$.

5. For $f \in V$, we define $-f : [0, 1] \to \mathbf{R}$ as $(-f)(x) = -f(x) \quad \forall$ $x \in [0, 1]$. Then $f + (-f) = (-f) + f = 0$, since

$$\{ f + (-f) \}(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x).$$

II. *Properties of Scalar Multiplication.*

Let $\alpha, \beta \in \mathbf{R}$ and $f, g \in V$.

Since f is a continuous function on $[0, 1]$ and α is a real constant, therefore, αf is a continuous function on $[0, 1]$ and as such $\alpha f \in V$. Let $x \in [0, 1]$. Then $f(x), g(x) \in \mathbf{R}$.

1. $$\begin{aligned}\{ \alpha(f + g) \}(x) &= \alpha \{ (f + g)(x) \}, \text{ by } (ii) \\ &= \alpha \{ f(x) + g(x) \}, \text{ by } (i) \\ &= \alpha f(x) + \alpha g(x), \text{ by distributive law in } \mathbf{R}. \\ &= (\alpha f)(x) + (\alpha g)(x), \text{ by } (ii) \\ &= (\alpha f + \alpha g)(x), \text{ by } (i).\end{aligned}$$

$\therefore \quad \alpha(f + g) = \alpha f + \alpha g.$

2. $$\begin{aligned}\{ (\alpha + \beta) f \}(x) &= (\alpha + \beta) f(x), \text{ by } (ii) \\ &= \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x), \text{ by } (ii) \\ &= (\alpha f + \beta f)(x), \text{ by } (i).\end{aligned}$$

$\therefore \quad (\alpha + \beta) f = \alpha f + \beta f.$

3. $$\begin{aligned}\{ \alpha(\beta f) \}(x) &= \alpha \{ (\beta f)(x), \text{ by } (ii) \\ &= \alpha \{ \beta f(x) \}, \text{ by } (ii) \\ &= (\alpha\beta) f(x), \text{ by associative law in } \mathbf{R} \\ &= \{ (\alpha\beta) f \}(x), \text{ by } (ii)\end{aligned}$$

$\therefore \quad \alpha(\beta f) = (\alpha\beta) f.$

4. $$\begin{aligned}(1f)(x) &= 1 f(x), \text{ by } (ii) \\ &= f(x) \text{ and so } 1f = f \quad \forall \ f \in V.\end{aligned}$$

Hence V is a vector space over $\mathbf{R}$.