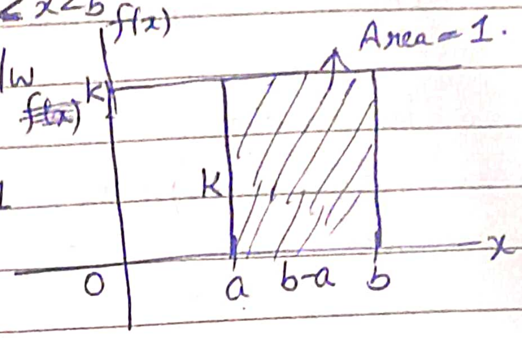
## Continuous distributions:→

Uniform distⁿ:→ A r.v. $X$ is said to be uniform distⁿ or rectangular distⁿ over the finite interval $(a,b)$ if its P.d.f. is given by

$$f(x) = \begin{cases} k ; & a \le x < b \\ 0 ; & o/w \end{cases}$$




$\because$ Total prob $= 1$

$\Rightarrow$ Area b/w $a$ & $b$ $= 1$

$\Rightarrow$ $k \times b-a = 1$

$\Rightarrow$ $k = \frac{1}{b-a}$

Or By using defⁿ of P.d.f

$$\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{a}^{b} k\, dx = 1$$

$$\Rightarrow k(b-a) = 1$$

$$\Rightarrow k = \frac{1}{b-a}$$

$\therefore$ $X \sim U(a,b)$ if its P.d.f is $f(x) = \begin{cases} \frac{1}{b-a} ; & a < x < b \\ 0 ; & o/w. \end{cases}$

Distribution function:→ let $F$ be the distⁿ fun of $U(a,b)$

i.e. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$

When $x \le a$; $\int_{-\infty}^{x} f(x)dx = 0$

when $a < x < b$, $F(x) = \int_{-\infty}^{x} f(x)dx$

$$= \int_{-\infty}^{a} f(x)dx + \int_{a}^{x} f(x)dx$$

$$= 0 + \int_{a}^{x} \frac{1}{b-a}dx = \frac{x-a}{b-a}$$

when $x \geq b$ ; $F(x) = \int_{-\infty}^{\infty} f(x)\,dx = 1$

So $F(x) = \begin{cases} 0 & ; \ x \leq a \\ \frac{x-a}{b-a} & ; \ a < x < b \\ 1 & ; \ x \geq b. \end{cases}$

Ex If a R.v. X has P.d.f

$$f(x) = \begin{cases} \frac{1}{4} & ; \ |x| < 2 \\ 0 & ; \ o/w. \end{cases}$$

$$= \begin{cases} \frac{1}{4} & ; \ -2 < x < 2 \\ 0 & ; \ o/w \end{cases} \rightarrow \text{Uniform dist}^n$$

Find (1) $P(X<1) = \int_{-\infty}^{1} f(x)\,dx = \int_{-\infty}^{-2} f(x)\,dx + \int_{-2}^{1} f(x)\,dx$

$$= 0 + \int_{-2}^{1} \tfrac{1}{4}\,dx$$

$$= \frac{1}{4}(1+2) = \frac{3}{4}$$

$$\boxed{P(X<1) = P(X\leq 1)}$$

(2) $P(|X|>1) = 1 - P(|X| \leq 1)$

$$= 1 - P(-1 \leq X \leq 1)$$

$$= 1 - \int_{-1}^{1} f(x)\,dx$$

$$= 1 - \frac{1}{4}[1+1] = 1 - \frac{2}{4} = \frac{2}{4} = \frac{1}{2}$$

On the other hand CDF = $F(x) = \begin{cases} 0 & ; \ x \leq -2 \\ \frac{x+2}{4} & ; \ -2 < x < 2 \\ 1 & ; \ x \geq 2. \end{cases}$

So $P(|X| \leq 1) = P(-1 \leq X \leq 1)$

$$= F(1) - F(-1)$$

$$= \frac{3}{4} - \frac{1}{4} = \frac{2}{4}$$

**Que** If $X \sim U[-2,2]$; Find $P(X<0)$ and $P\left(|X-1| \geq \frac{1}{2}\right)$

**Sol$^n$** → P.d.f is $f(x) = \begin{cases} \frac{1}{4}; & -2 \leq x \leq 2 \\ 0; & 0/w. \end{cases}$

$$F(x) = \begin{cases} 0 & ; \quad x \leq -2 \\ \frac{x+2}{4} & ; \quad -2 < x < 2 \\ 1 & ; \quad x \geq 2 \end{cases}$$

(1) $P(X<0) = \underset{\text{By PdF}}{\int_{-\infty}^{0} f(x)\,dx} = \int_{-\infty}^{-2} f(x)\,dx + \int_{-2}^{0} f(x)\,dx$

$$= 0 + \frac{1}{4}(0+2) = \frac{1}{2}$$

**By CDF** $P(X<0) = P(X \leq 0) = F(0) = \frac{2}{4} = \frac{1}{2}$

(2) $P\left(|X-1| \geq \frac{1}{2}\right) = 1 - P\left(|X-1| < \frac{1}{2}\right)$

$$= 1 - P\left(-\frac{1}{2} < (X-1) < \frac{1}{2}\right)$$

$$= 1 - P\left(\frac{1}{2} < X < \frac{3}{2}\right)$$

$$= 1 - \int_{1/2}^{3/2} f(x)\,dx = 1 - \frac{1}{4}\left(\frac{3}{2} - \frac{1}{2}\right)$$

$$= 3/4.$$

**By CDF** $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right) = \frac{7}{8} - \frac{5}{8} = \frac{1}{4}$

$$\therefore 1 - P\left(\frac{1}{2} < X < \frac{3}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}.$$

**Que** The no. of Pc's sold daily at ABC Computer Shop is Uniformly distributed with a min of 2000 pc's and a max of 5000 pc's.

(1) What is the Prob. that daily sales will fall b/w 2500 & 3000 pc's.
(2) What is the Prob. that it will sell atleast 4000 pc's.
(3) What is the Prob. that it will sell Exactly 2500 Pc's.

Solⁿ Let X — no. of Pc's sold

$X \sim U[2000, 5000]$

$$\therefore f(x) = \begin{cases} \frac{1}{3000} ; & 2000 \leq x \leq 5000 \\ 0 ; & o/w. \end{cases}$$

$$F(x) = \begin{cases} 0 ; & \text{if } x \leq 2000 \\ \frac{x-2000}{3000} ; & \text{If } 2000 < x < 5000 \\ 1 ; & \text{If } x \geq 5000 \end{cases}$$

(1) $P(2500 < x < 3000) = F(3000) - F(2500)$

$$= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

(2) $P(X \geq 4000) = 1 - P(X < 4000)$

$$= 1 - F(4000) = 1 - \frac{2}{3} = \frac{1}{3}$$

(3) $P(X = 2500) = 0$ ($\because$ In cts Prob. distⁿ; Prob. at a single point is always zero.)

HW Que The CDF of X is given by

$$F(x) = \begin{cases} 0 ; & x \leq -\pi \\ \frac{x+\pi}{2\pi} ; & -\pi < x < \pi \\ 1 ; & x > \pi \end{cases}$$

(1) Find Pdf of X (2) $P\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

Mean $E(x) = \int_a^b x f(x) = \frac{1}{b-a} \int_a^b x\,dx = \left(\frac{x^2}{2}\right)_a^b \cdot \frac{1}{b-a}$

$$= \frac{b+a}{2}$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{(b-a)^2}{12}$$

## Exponential Distribution:→

A Continuous r.v. X is said to follow Exponential distⁿ with parameter $\lambda$ if X has the P.d.f.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} ; & x \geq 0 \\ 0 ; & o/w. \end{cases}$$

$X \sim Exp(\lambda)$ – $\lambda$ parameter.–(Rate Parameter)

**Note:→** Exp. distⁿ is mainly used to model the time b/w Independent events that occur at a Constant average rate.

**CDF** $F(x) = P(X \leq x)$

If $x < 0$ then $F(x) = \int_{-\infty}^{0} f(x) dx = 0$

If $x \geq 0$ then $F(x) = \int_{-\infty}^{0} f(x) dx + \int_{0}^{x} f(x) dx$

$= 0 + \int_{0}^{x} \lambda e^{-\lambda x} dx$

$= \lambda \left( \frac{e^{-\lambda x}}{-\lambda} \right)_0^x = -(e^{-\lambda x} - 1)$

$= 1 - e^{-\lambda x}$

$$F(x) = \begin{cases} 0 & \text{If } x < 0 \\ 1 - e^{-\lambda x} & \text{If } x \geq 0. \end{cases}$$

**Mean** $E(X) = \int_{0}^{\infty} f(x) \, x \, dx = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx$

$= \lambda \left[ x . \frac{e^{-\lambda x}}{-\lambda} - \int (1) . \frac{e^{-\lambda x}}{-\lambda} dx \right]_0^{\infty}$

$= \left[ -x e^{-\lambda x} + \frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty}$

$= \frac{1}{\lambda}$

Similarly $E(X^2) = \frac{2}{\lambda^2}$ $\Rightarrow V(X) = E(X^2) - (E(X))^2$

$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

Que Let $X$ be a r.v. with P.d.f

$$f(x) = \begin{cases} a e^{-x/3} & ; x > 0 \\ 0 & ; o/w \end{cases}$$

Find (i) $a$ (ii) $P(X>3)$ (iii) $P(1<X<4)$.

Solⁿ → $f(x)$ is a P.d.f $\Rightarrow \int_0^\infty f(x)\,dx = 1$

$$\Rightarrow \int_0^\infty a e^{-x/3}\,dx = 1 \Rightarrow \left(\frac{a e^{-x/3}}{(-1/3)}\right)_0^\infty = 1$$

$$= -3a(0-1) = 1 \Rightarrow a = 1/3$$

$$P(X>3) = \int_3^\infty f(x)\,dx = \frac{1}{3}\int_3^\infty e^{-x/3} = -\left(e^{-x/3}\right)_3^\infty = -(0-e^{-1}) = \frac{1}{e}$$

$$P(1<X<4) = \int_1^4 f(x)\,dx = \frac{1}{3}\int_1^4 e^{-x/3} = \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_1^4$$

$$= -\left[e^{-4/3} - e^{-1/3}\right] = e^{-1/3} - e^{-4/3}$$

Or, We can use CDF also

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-1/3 x} & x \geq 0 \end{cases}$$

$$P(X>3) = 1 - P(X\leq 3) = 1 - F(3) = 1 - (1-e^{-1}) = e^{-1} = \frac{1}{e}$$

$$P(1<X<4) = F(4) - F(1) = \cancel{1} - e^{-4/3} - \cancel{1} + e^{-1/3} = e^{-1/3} - e^{-4/3}$$

Que The time (In hrs.) required to repair a machine is exp. distributed with $\lambda = \frac{1}{3}$. Find the prob. that the repair time exceeds 3 hrs.

Solⁿ → $P(X>3) = \int_3^\infty \frac{1}{3} e^{-1/3 x} = \frac{1}{e}$

$$\because f(x) = \begin{cases} \frac{1}{3} e^{-x/3} & ; x > 0 \\ 0 & ; o/w \end{cases}$$

**Memoryless Property:→** If $X$ is an exponential distⁿ then $P(X>m | X>n) = ~~P(x>m~~ P(X>m-n)$ for any $m,n \geq 0$.

**Proof:→**

$$P(X>k) = \lambda\int_k^{\infty} e^{-\lambda x}\,dx$$

$$= \lambda\left(\frac{e^{-\lambda x}}{-\lambda}\right)_k^{\infty} = e^{-k\lambda}$$

$$\therefore P(X>m | X>n) = \frac{P(X>m \cap X>n)}{P(X>n)}$$

$X>n$

$X>m$

$n \quad m$

$$= \frac{P(X>m)}{P(X>n)}$$

$$= \frac{e^{-m\lambda}}{e^{-n\lambda}} = e^{-(m-n)\lambda}$$

$$= P(X>m-n)$$

**Que** A fast food chain find that the avg time customer has to wait for service is 45 sec. If the waiting time can be treated as an Exp. ~~a~~ random variable, what is the prob. that a customer will have to wait more than 5 min given that he already waited for more than 2 minutes?

**Solⁿ**

$$P(X>5 | X>2) = ?$$

$X \sim Exp(\lambda)$

where $1/\lambda = 45$ sec

$$\Rightarrow \boxed{\lambda = \frac{1}{45}}$$

$$P(X>5 | X>2) = P(X>300 | X>120)$$

$$= P(X>180)$$

$$= 1 - P(X \leq 180)$$

$$= 1 - F(180)$$

$$= 1 - (1 - e^{-180\lambda})$$

$$= e^{-180 \times 1/45} = e^{-4}$$

Que The Life length (in months) of an Electric Component follows an Exp. dist$^n$ with parameter $= \frac{1}{2}$. What is the prob. that the Component Survives atleast 10 months given that already it had Survived for more than 9 months.

Sol$^n$ $X \sim Exp(\lambda)$

where $\lambda = \frac{1}{2}$

$$P(X>10 \mid X>9) = P(X>1)$$
$$= 1-P(X\le 1) = 1-(1-e^{-\lambda})$$
$$= e^{-\lambda} = e^{-1/2}$$

HW

Que The Length of time a person speaks over phone follows Exp. dist$^n$ with mean 6. What is the prob. that the person will talk for

(i) more than 8 min. (ii) b/w 4 & 8 min.

Que The daily Consumption of milk in excess of 20000 gallons is appr. exp. dist$^n$ with mean 3000. The city has a daily stock of 35000 gallons. What is the prob. that of two days selected at random, the stock is Insufficient for both days.

Sol$^n$ → $X$ - no. of days when Stock is Insufficient.

$X \sim Exp(\lambda)$ ; $\lambda = \frac{1}{3000}$

$$P(X>35000 \mid X>20000) = P(X>15000) = e^{-5}. \text{ (solve it)}$$

Now 2 days ~~are~~ Selected are random & Independent.

So reqd. Prob is $P(\text{one day} \cap \text{2nd day}) = (e^{-5})^2 = e^{-10}$.

→ $\frac{Var}{Mean} = \frac{1}{\lambda} \Rightarrow Var = \frac{1}{\lambda} Mean$

If $\lambda<1$ Then Var > Mean     If $\lambda>1$ Then Var < Mean

If $\lambda=1$ Then Var = Mean

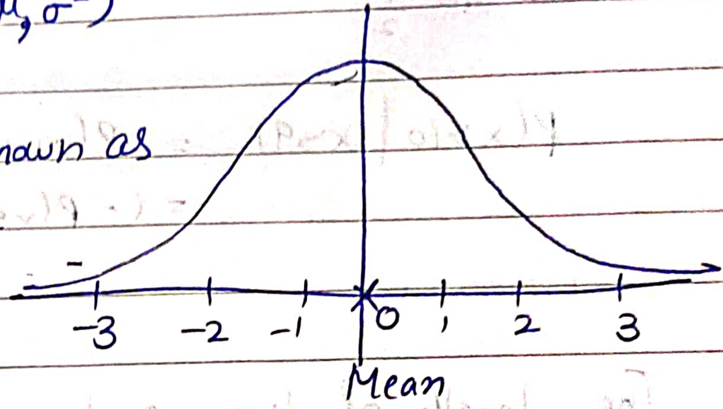
## Normal Distribution:→

A r.v. X is said to follow normal distⁿ with mean $\mu$ and variance $\sigma^2$ if its Pdf is given by

$$f(x) = \frac{1}{(\sqrt{2\pi})\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \quad -\infty < x < \infty.$$

$$X \sim N(\mu, \sigma^2)$$

→ Normal distⁿ is also known as Guassion Distⁿ or bell shaped curve distⁿ.




**Properties:→**

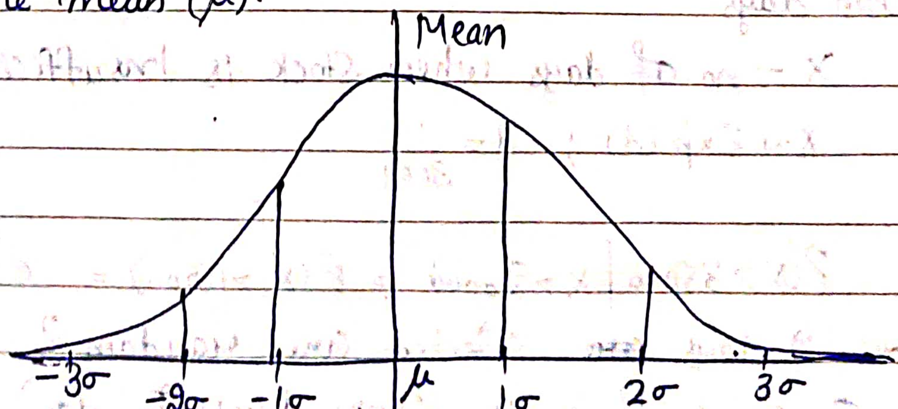
(1) Mean, Median, and Mode are all equal.

(2) The curve is symmetric about the mean.

(3) Total area under curve is 1. i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(4) Area to the left and area to the right about the mean are same i.e. 0.5

**Area property:→** The rule which tell you what % of your data falls within a certain no. of standard deviation ($\sigma$) from the mean ($\mu$).




(a) 68% of the data falls with $\mu$ and $\sigma$.

(b) 95% of the data falls with $\mu$ and $2\sigma$

(c) 99.7% of the data falls within $\mu$ and $3\sigma$.

i.e. $P(\mu-\sigma < X < \mu+\sigma) = 0.68$
$P(\mu-2\sigma < X < \mu+2\sigma) = 0.95$
$P(\mu-3\sigma < X < \mu+3\sigma) = 0.997$

→ If $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X-\mu}{\sigma}$ is called Standard normal variate.

$$E(z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu)$$

[E(Const) = Const]

$$= \frac{1}{\sigma}\left[E(X) - E(\mu)\right] = \frac{1}{\sigma}(\mu-\mu) = 0$$

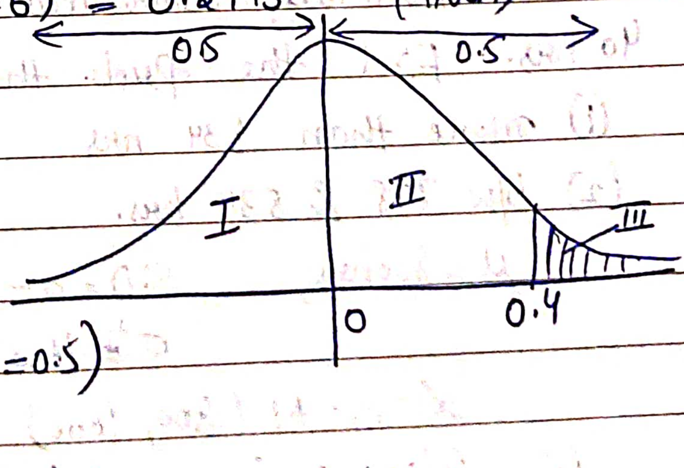
$$V(z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} V(X-\mu)$$

$$= \frac{1}{\sigma^2}(\text{Var}\,X - \text{Var}\,\mu)$$

$$= \frac{1}{\sigma^2}(\sigma^2 - 0) = \sigma^2 \quad (\text{Var}(a) = 0)$$

$\therefore Z \sim N(0,1)$ If $X \sim N(\mu, \sigma^2)$

Que If $P(0 < Z < 2) = 0.4772$; $P(Z < 0.4) = 0.6554$; $P(Z < -0.6) = 0.2743$ (Given)



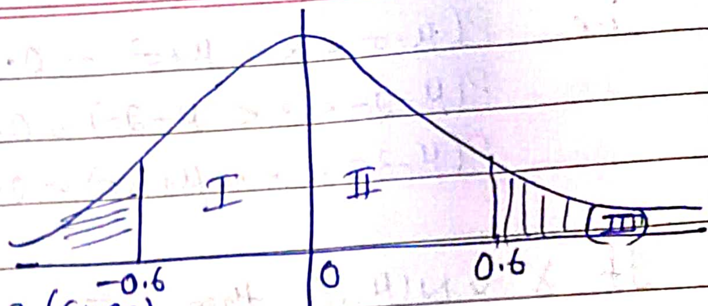

Now, we know
$II + III = 0.5$
($\because$ Area on the left = Area on the right = 0.5)
and $I = 0.5$

$P(Z < 0.4) = 0.6554$
$I + II = 0.6554 \Rightarrow II = 0.1554$.
So $III = 0.5 - 0.1554 = 0.3446$

Similarly

Find the area in (II).

i.e $P(0<z<0.6)$



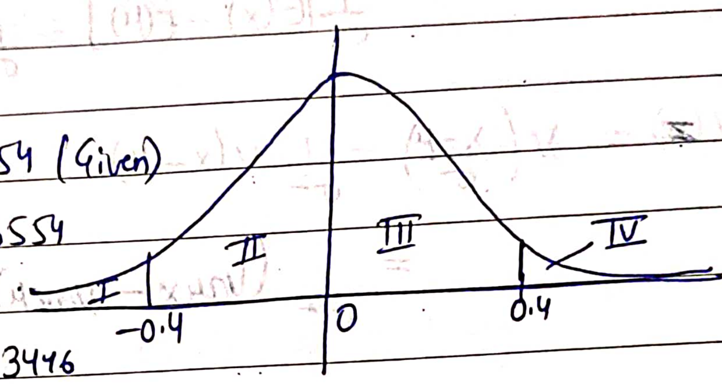

$P(z<-0.6) = 0.2743$ (Given)

$\Rightarrow P(z>0.6) = 0.2743$

$\Rightarrow$ Area in (II) $= P(0<z<0.6) = 0.5 - 0.2743$

$= 0.2257$

Find $P(-0.4<z<0)$




$P(Z<0.4) = 0.6554$ (Given)

$\Rightarrow P(Z>0.4) = 1 - 0.6554$

$= 0.3446$

$\Rightarrow P(z<-0.4) = 0.3446$

$\Rightarrow P(-0.4<z<0) = 0.5 - 0.3446$

$= 0.1554$

<u>Que</u> A Produceds light bulbs that have life, before burnout, that is normally distributed with mean 800 hrs and S.D. 40 hrs. Find the Prob. that a bulb burns

(i) more than 834 hrs

(2) b/w 778 & 834 hrs.

<u>Sol^n</u> $\mu = 800$ hrs, $S.D = \sigma = 40$

$\sigma^2 = 1600$ hrs.

$X \sim N(800, 1600)$

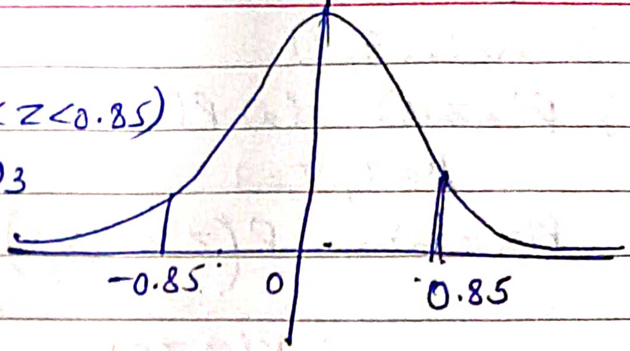
(i) $P(X>834) = P\left(\frac{x-\mu}{\sigma} \geq \frac{834-800}{40}\right)$

$= P\left(z > \frac{34}{40}\right) = P(z>0.85)$

<u>Given</u> $P(z<-0.85) = 0.1977$ ; $P(-0.55<z<0) = 0.2088$

$P(0<z<0.85) = 0.3023$ ; $P(z<0.85) = 0.8023$

$$P(Z>0.85) = 0.5 - P(0<Z<0.85)$$
$$= 0.5 - 0.3023$$
$$= 0.1977$$




$$P(778 < X < 834) = P\left(\frac{778-800}{40} < Z < \frac{834-800}{40}\right)$$
$$= P(-0.55 < Z < 0.85)$$
$$= P(-0.55 < Z < 0) + P(0 < Z < 0.85)$$
$$= 0.2088 + 0.3023$$
$$= 0.5111$$

<u>Que</u> $X \sim N(150, 2500)$

(1) $P(X > 200)$ (2) $P(120 < X < 170)$

(3) $P(X < 75)$

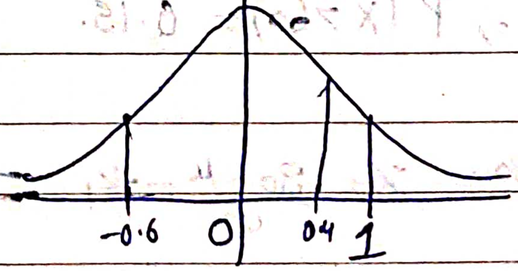
Given $P(0<Z<1) = 0.3413$; $P(Z<0.4) = 0.6554$;

$P(Z<-0.6) = 0.2743$;

$P(0<Z<1.5) = 0.4332$

(1) $P(X>200) = P\left(Z > \frac{50}{50}\right) = P(Z>1)$

$$= 0.5 - P(0<Z<1)$$
$$= 0.5 - 0.3413$$
$$= 0.1587$$




(2) $P(120 < X < 170) = P(-0.6 < Z < 0.4)$

$$= P(Z<0.4) - P(Z<-0.6)$$
$$= 0.6554 - 0.2743$$
$$= 0.3811$$

(3) $P(X<75) = 0.0668$ Ans

Que $X \sim N(9, 9)$

(i) $P(X \geq 15)$ (2) $P(X \leq 15)$ (3) $P(0 \leq X \leq 9)$

$\mu = 9, \sigma = 3$

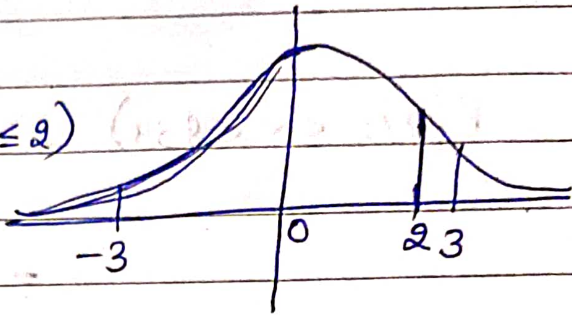
$P(X \geq 15) = P\left(Z \geq \frac{15-9}{3}\right)$

$= P(Z \geq 2) =$

$0.5 - P(0 \leq Z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228.$




(2) $P(X \leq 15) = 1 - P(X \geq 15) = 0.9772.$

(3) $P(0 \leq X \leq 9) = P\left(\frac{-9}{3} \leq Z \leq 0\right)$

$= P(-3 \leq Z \leq 0)$

$= P(0 \leq Z \leq 3)$

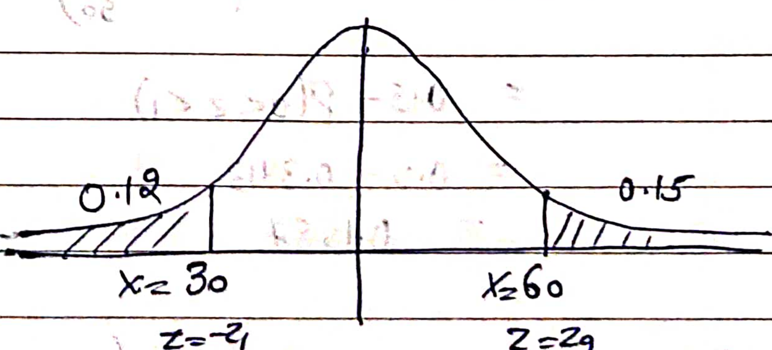
$= 0.4987.$

Que In normal dist$^n$, 12% of the items are under 30 and 85% are under 60. Find mean & std dev of the dist$^n$

Sol$^n$ let $X \sim N(\mu, \sigma^2)$

$P(X < 30) = 0.12$; $P(X < 60) = 0.85.$

$\Rightarrow P(X > 60) = 0.15.$




$X = 30$; $Z = \frac{30-\mu}{\sigma} = -Z_1$

$X = 60$; $Z = \frac{60-\mu}{\sigma} = Z_2$

$P(-Z_1 < Z < 0) = P(0 < Z < Z_1) = 0.38$

$P(0 < Z < Z_2) = 0.35$

$\Rightarrow Z_1 = 1.18$; $Z_2 = 1.04$

$\therefore \frac{30-\mu}{\sigma} = -1.18$

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$\Rightarrow 30-\mu = -1.18\sigma$

and $60-\mu = 1.04\sigma$

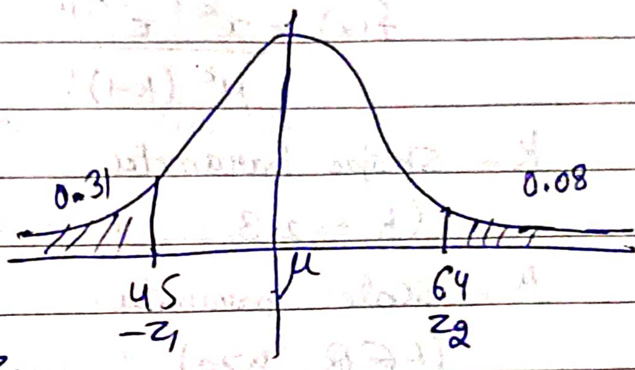
$\Rightarrow \sigma = 13.56$ & $\mu = 45.93$.

**Que** In a normal distⁿ 31% of the items are under 45 & 8% are over 64. Find mean & std dev.

**Solⁿ** let $X \sim N(\mu, \sigma^2)$

$P(X<45) = 0.31$

$P(X>64) = 0.08$




$X=45$; $\quad z = \frac{45-\mu}{\sigma} = -z_1$

$X=64$; $\quad z = \frac{64-\mu}{\sigma} = z_2$

$P(-z_1<z<0) = P(0<z<z_1) = 0.5-0.31 = 0.19$

$\Rightarrow z_1 = 0.5$

$P(0<z<z_2) = 0.5-0.08 = 0.42$

$\Rightarrow z_2 = 1.4$

$\therefore \frac{45-\mu}{\sigma} = -0.5 \Rightarrow 45-\mu = -0.5\sigma$

& $\frac{64-\mu}{\sigma} = 1.4 \Rightarrow 64-\mu = 1.4\sigma$

$\Rightarrow -19 = -1.9\sigma$

$\Rightarrow \boxed{\sigma = 10}$

and $45-\mu = -0.5\times 10$

$\Rightarrow \boxed{\mu = 50}$

Erlang Distribution:⇒ (To model the no. of telephone calls that an operator at a switching station may receive at once.)

A cts R.V. X has an Erlang distⁿ with parameter k & $\mu$ if its Pdf is

$$f(x) = \frac{x^{k-1} e^{-x/\mu}}{\mu^k (k-1)!} ; x > 0.$$

k – shape parameter
($k = 1, 2, 3, \ldots$)

$\mu$ – scale parameter
($\mu \in \mathbb{R}, \mu > 0$)

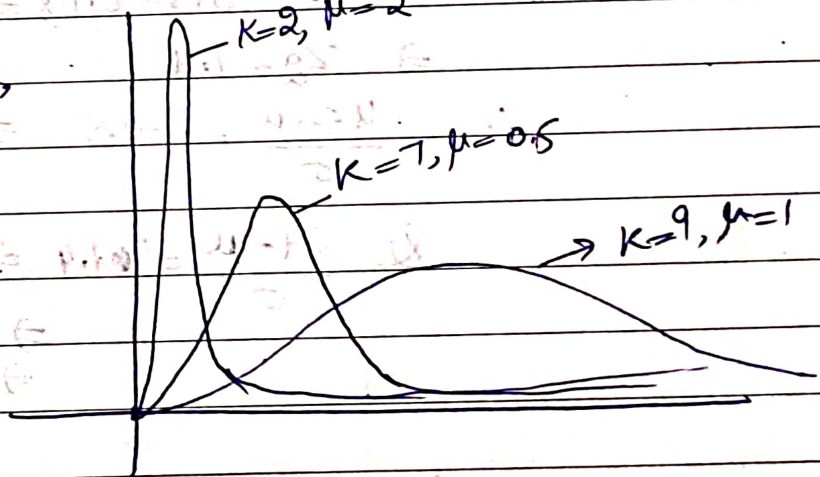
Scale parameter $\mu$ can be written as

$\mu = \frac{1}{\lambda}$, where $\lambda$ is the rate parameter.

$$\therefore \quad f(x) = \frac{x^{k-1} e^{-\lambda x} \cdot \lambda^k}{(k-1)!} ; x > 0.$$

→ As we are increasing k, curve's peak point is decreasing.

→ $\mu$ defines the spread of the curve




→ If $k = 1$, then $f(x) = \frac{e^{-x/\mu}}{\mu} = \frac{1}{\mu} e^{-x/\mu}$ → exponential distⁿ.

→ So Erlang distⁿ is generalization of Exp. distⁿ.

→ The Exp distⁿ models the time intervals to the 1st event while Erlang distⁿ models the time interval to the kth event.

→ If $X_1, X_2 \ldots X_k$ follows Exponential distⁿ, then their Sum

$$\underbrace{X_1 + X_2 + \ldots + X_k}_{k \text{ Events}} \approx \text{Erlang dist}^n.$$

or $X_i \sim Exp(\lambda)$ Then $\sum_{i=1}^{k} X_i \sim Erlang(K, \lambda)$

→ If $X \sim Erlang(K, \mu)$

Then $aX \sim Erlang(K, a\mu)$, with $a \in \mathbb{R}$

↓

Shape same but Scale changes.

→ If $X \sim Erlang(K_1, \mu)$ & $Y \sim Erlang(K_2, \mu)$

Then $X + Y \sim Erlang(K_1 + K_2, \mu)$

If X and Y are Independent.

→ $$E(X^r) = \int_0^\infty x^r f(x)\, dx$$

$$= \int_0^\infty x^r \cdot \frac{x^{K-1} e^{-x/\mu}}{\mu^K (K-1)!}\, dx$$

$$= \frac{1}{\mu^K (K-1)!} \int_0^\infty x^{K+r-1} \cdot e^{-x/\mu}\, dx$$

Put $\frac{x}{\mu} = y$; $\Rightarrow dx = \mu\, dy$

$$= \frac{1}{\mu^K (K-1)!} \int_0^\infty \mu^{K+r-1}\, y^{K+r-1} \cdot e^{-y} \cdot \mu\, dy$$

$$= \frac{\mu^r}{(K-1)!} \int_0^\infty y^{K+r-1} \cdot e^{-y}\, dy$$

$$= \frac{\mu^r}{(K-1)!} \Gamma(K+r) \qquad \left[ \because \int_0^\infty e^{-x} x^{n-1} = \Gamma(n) \right]$$

$$E(X) = \frac{\mu^1}{(K-1)!} \Gamma(K+1) = \frac{\mu}{(K-1)!} K! = \mu K$$

$$E(X^2) = \frac{\mu^2}{(K-1)!} \Gamma(K+2) = \frac{(K+1)!\, \mu^2}{(K-1)!} = (K+1) K \mu^2$$

$\Rightarrow$ Mean $= \mu k$

$V(x) = E(x^2) - (E(x))^2$

$= \mu^2 k(k+1) - \mu^2 k^2$

$= \cancel{\mu^2 k^2} + \mu^2 k - \cancel{\mu^2 k^2} = \mu^2 k$

If $\mu = \frac{1}{\lambda}$ $\rightarrow$ Then $V(x) = \frac{k}{\lambda^2}$ & $E(x) = \frac{k}{\lambda}$

for $k=1$; $E(x) = \frac{1}{\lambda}$; $V(x) = \frac{1}{\lambda^2}$ $\rightarrow$ Exp. distn.

$\rightarrow$ Var $= \mu$ Mean

$\Rightarrow$ If $\mu > 1$ Then Var $>$ Mean

If $\mu = 1$ Then Var $=$ Mean

If $\mu < 1$ Then Var $<$ Mean

$\mu$ – Scale parameter.

Var $= \frac{1}{\lambda}$ Mean

If $\lambda < 1$ Then Var $>$ Mean

If $\lambda = 1$ Then Var $=$ Mean

If $\lambda > 1$ Then Var $<$ Mean

$\lambda$ – Rate Parameter.