Unit IV

Interpolation →

The value of 'x' for which the value of 'y' is to be estimated is an intermediate value in the given set of values of 'x' then the method of determining y is called **Interpolation**.

Interpolation methods →

1. Newton's forward Interpolation } For equal intervals
2. Newton's Backward Interpolation }
3. Binomial method (For missing interval).
4. Lagrange's Interpolation. } For unequal intervals.

Newton's Forward Formula for Interpolation →

Let $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ be a set of equidistant values of the variable x.

$$\therefore\ x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = h$$

Let $u = \dfrac{x - x_0}{h}$

The Newton's forward difference formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

$$\cdots + \frac{u(u-1)(u-2)\cdots(u-n+1)}{n!}\Delta^n y_0$$

Note: This formula can be used to interpolate the value of y corresponding to 'x' near the beginning value of x in the table.

Newton's Forward Interpolation

Q: Form a difference table and interpolate the value of $f(x)$ when $x=4$, given

x	3	5	7	9
$y = f(x)$	180	150	120	90

Soln: Here interval of x is at equal distance so we will use Newton's Forward or Newton's Backward interpolation.
Here value $x=4$ is to be interpolated which lies b/w 3 & 5 which is at the beginning of the table so we will use Forward interpolation.

[Note: If the interval of x is not at equal distance then Lagrange's method has to be used.].

Here $x = 4$, $x_0 = 3$, $x_1 = 5$, $x_2 = 7$, $x_3 = 9$

$y_0 = 180$, $y_1 = 150$, $y_2 = 120$, $y_3 = 90$

h = length of the interval.

$\therefore h = x_1 - x_0 = 5 - 3 = 2$, $\therefore \boxed{h = 2}$

$$u = \frac{x - x_0}{h} = \frac{4-3}{2} = \frac{1}{2} = 0.5$$

$\therefore \boxed{u = 0.5}$

Difference Table :-

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
3	180 (y_0)	-30 (Δy_0)	0 ($\Delta^2 y_0$)	0 ($\Delta^3 y_0$)
5	150 (y_1)	-30 (Δy_1)	0 ($\Delta^2 y_1$)	
7	120 (y_2)	-30 (Δy_2)		
9	90 (y_3)			

$\Delta y \to$ Difference

$\Delta y_0 = y_1 - y_0$
$= 150 - 180 = -30$
$\Delta y_1 = y_2 - y_1 = 120 - 150$
$= -30$
$\Delta y_2 = y_3 - y_2 = 90 - 120$
$= -30$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$
$= -30 - (-30)$
$= -30 + 30 = 0$
$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$
$= -30 - (-30)$
$= -30 + 30 = 0$

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$= 0 - 0 = 0$

The Newton's forward difference formula is,

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$\therefore y = 180 + \frac{0.5}{1!}(-30) + \frac{(0.5)(0.5-1)}{2!}(0) + 0$$

$= 180 - 15 = 165$

$\therefore \boxed{y = 165}$

Q. Find by suitable interpolation formula the value of $f(2.5)$ from the following data.

x	2	3	4	5
$f(x)$	14.5	16.3	17.5	18

Soln. Here $x = 2.5$

Value of x is placed at a fixed distance i.e, at equal interval so we have to use either for Newton's Forward or Backward interpolation. Now $x = 2.5$ which lies b/w 2 and 3 and lies in the beginning of the interval so we will use forward interpolation method

Now $x = 2.5,\ x_0 = 2,\ x_1 = 3,\ x_2 = 4,\ x_3 = 5$

$y_0 = 14.5,\ y_1 = 16.3,\ y_2 = 17.5,\ y_3 = 18$

h = length of the interval

$\therefore h = x_1 - x_0 = 3 - 2 = 1.$

i.e, $\boxed{h = 1}$

$$u = \frac{x - x_0}{h} = \frac{2.5 - 2}{1} = 0.5$$

$\therefore \boxed{u = 0.5}$

Difference Table.

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
2	14.5 y_0			
	$(16.3-14.5)$	1.8 Δy_0		
3	16.3	$(1.2-1.8)$	-0.6 $\Delta^2 y_0$	
	$(17.5-16.3)$	1.2	$(-0.7-(-0.6))$	-0.1 $\Delta^3 y_0$
4	17.5	$(0.5-1.2)$	-0.7	
	$(18-17.5)$	0.5		
5	18			

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 14.5 + \frac{(0.5)}{1!}(1.8) + \frac{(0.5)(0.5-1)}{2!}(-0.6) + \frac{(0.5)(0.5-1)(0.5-2)(-0.1)}{3!}$$

$u - 1 = 0.5 - 1 = -0.5$

$u - 2 = 0.5 - 2 = -1.5$

$$y = 14.5 + 0.9 + \frac{0.15}{2} - \frac{0.0375}{6}$$

$$= 14.5 + 0.9 + 0.075 - 0.0063 = 15.4687$$

$f(2.5) = 15.4687.$

Q. Using an appropriate formula for interpolation, estimate the number of students who obtained marks between 40 and 45 from the following table.

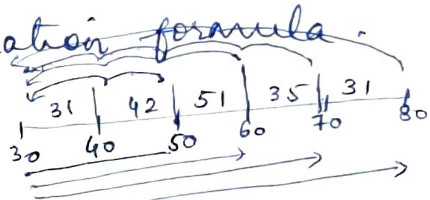
MARKS	30 − 40	40 − 50	50 − 60	60 − 70	70 − 80
NO. OF STUDENTS	31	42	51	35	31

Soln. Length of interval is 10 i.e. at equal interval

$\therefore \boxed{h = 10}$

So we will use either Newton's Forward interpolation or Backward interpolation formula.

Here we have to find the no. of students who obtained marks b/w 40 and 45 which lies in the interval 40 − 50 which is forward of the table so we will apply Newton's Forward interpolation formula.



Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	31 y_0				
		42 Δy_0			
Below 50	73		9 $\Delta^2 y_0$		
		51		$(-16-9) \to -25$ $\Delta^3 y_0$	
Below 60	124		−16		$12-(-25) \to 37$ $\Delta^4 y_0$
		35		$(-4-(-16)) \to 12$	
Below 70	159		−4		
		31			
Below 80	190				

Here x is below 50 so $x = 45$ and x_0 is the starting value so $x_0 = 40$

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = \frac{5}{10} = 0.5$$

$\boxed{u = 0.5}$

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$y = 31 + \frac{(0.5)(42)}{1!} + \frac{(0.5)(-0.5)}{2!}(9) + \frac{(0.5)(-0.5)(-1.5)}{3!}(-25)$$
$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)(37)}{4!}$$

$u-1$
$= 0.5-1$
$= -0.5$

$u-2$
$= 0.5-2$
$= -1.5$

$u-3$
$= 0.5-3$
$= -2.5$

$$y = 31 + 21 - \frac{2.25}{2} - \frac{9.3750}{6} - \frac{34.6875}{24}$$

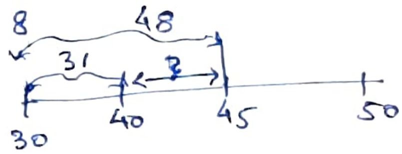
$$y = 31 + 21 - 1.1250 - 1.5625 - 1.4453$$

$$y = 47.8672$$

$$\boxed{y = 48}$$

∴ No. of students who secured below 45 = 48

we have to find b/w 40 to 45, so



∴ No. of students who secured marks b/w 40-45
$= 48 - 31 = 17$

Newton's Backward Interpolation

Q. Using Newton's Backward Interpolation formula, Find the annual premium formula, Find the annual premium at the age of 33 from the following data.

Age in years (x)	24	28	32	36	40
Annual Premium in Rs. (y)	28.06	30.19	32.75	34.94	40

Soln. → Premium at the age of 33 is at the backward of the table so we will apply Newton's Backward Interpolation formula.

Here $x = 33$, $x_0 = 24$, $x_1 = 28$, $x_2 = 32$, $x_3 = 36$, $x_4 = 40$

$y_0 = 28.06$, $y_1 = 30.19$, $y_2 = 32.75$, $y_3 = 34.94$, $y_4 = 40$

Length of age is 4yrs which is at equal interval.

$\therefore \boxed{h = 4}$

Newton's Backward Interpolation formula;

$$y = y_n + \frac{v}{1!}\Delta y_{n-1} + \frac{v(v+1)}{2!}\Delta^2 y_{n-2} + \frac{v(v+1)(v+2)}{3!}\Delta^3 y_{n-3}$$

$$+ \frac{v(v+1)(v+2)(v+3)}{4!}\Delta^4 y_{n-4} + \cdots + \frac{v(v+1)(v+2)\cdots(v+n-1)}{n!}$$

where $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ be a set of equidistant values of the argument x and $y_0, y_1, y_2, \ldots, y_{n-1}, y_n$ be the value of the function $y = f(x)$

Let $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots x_n - x_{n-1} = h$

and $v = \frac{x - x_n}{h}$, $h = 4$, $x = 33$, $x_n = 40$ (Here $n = 4$)

$$v = \frac{33-40}{4} = \frac{-7}{4} = -1.75$$

$\boxed{V = -1.75}$

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
24	28.06				
		2.13			
28	30.19		0.43		
		2.56		−0.8	
32	32.75		−0.37		4.04
		2.19			$\Delta^4 y_{n-4}$
36	34.94			3.24	
			2.87	$\Delta^3 y_{n-3}$	
40	40	5.06	$\Delta^2 y_{n-2}$		
	y_n	Δy_{n-1}			

$$y = y_n + \frac{u}{1!}\Delta y_{n-1} + \frac{u(u+1)}{2!}\Delta^2 y_{n-2} + \frac{u(u+1)(u+2)}{3!}\Delta^3 y_{n-3} + \frac{u(u+1)(u+2)(u+3)}{4!}\Delta^4 y_{n-4}$$

$$y = 40 + \frac{(-1.75)}{1!}(5.06) + \frac{(-1.75)(-1.75+1)}{2!}(2.87)$$

$$+ \frac{(-1.75)(-1.75+1)(-1.75+2)}{3!}(3.24)$$

$$+ \frac{(-1.75)(-1.75+1)(-1.75+2)(-1.75+3)}{4!}(4.04)$$

$$y = 40 + \frac{(-1.75)}{1}(5.06) + \frac{(-1.75)(-0.75)}{2}(2.87)$$

$$+ \frac{(-1.75)(-0.75)(0.25)}{6}(3.24) + \frac{(-1.75)(-0.75)(0.25)(1.25)}{24}(4.04)$$

$$y = 40 - 8.855 + 1.8834 + 0.1772 + 0.0690$$

$$y = 33.2746$$

$\therefore$ The annual premium at the age of 33 = Rs 33.2746

Lagrange's Interpolation Formula :- For unequal interval

$$y = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\cdots(x_2-x_n)} y_2 + \cdots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} y_n$$

Q. The following table gives the normal weight of a baby during the six months of life.

AGE IN MONTHS	0	2	3	5	6
WEIGHT IN LBS	5	7	8	10	12

estimate the weight of the baby at the age of 4 months.

Soln. Here $\boxed{x = 4}$

$x_0 = 0,\ x_1 = 2,\ x_2 = 3,\ x_3 = 5,\ x_4 = 6$ (values are not at equal interval), so we apply Lagrange's interpolation formula

$y_0 = 5,\ y_1 = 7,\ y_2 = 8,\ y_3 = 10,\ y_4 = 12$

Substituting these values in Lagrange's interpolation formula, we have

$$y = \frac{(4-2)(4-3)(4-5)(4-6)}{(0-2)(0-3)(0-5)(0-6)}(5)$$

$$+ \frac{(4-0)(4-3)(4-5)(4-6)}{(2-0)(2-3)(2-5)(2-6)}(7)$$

$$+ \frac{(4-0)(4-2)(4-5)(4-6)}{(3-0)(3-2)(3-5)(3-6)}(8)$$

$$+ \frac{(4-0)(4-2)(4-3)(4-6)}{(5-0)(5-2)(5-3)(5-6)}(10)$$

$$+ \frac{(4-0)(4-2)(4-3)(4-5)}{(6-0)(6-2)(6-3)(6-5)}(12)$$

$$y = \frac{(2)(1)(-1)(-2)}{(-2)(-3)(-5)(-6)}(5) + \frac{(4)(1)(-1)(-2)}{(2)(-1)(-3)(-4)}(7)$$

$$+ \frac{(4)(2)(-1)(-2)}{(3)(1)(-2)(-3)}(8) + \frac{(4)(2)(1)(-2)}{(5)(3)(2)(-1)}(10) + \frac{(4)(2)(1)(-1)}{(6)(4)(3)(1)}(12)$$

$$y = \frac{20}{180} - \frac{56}{24} + \frac{128}{18} + \frac{160}{30} - \frac{96}{72}$$

$$y = 0.111 - 2.333 + 7.111 + 5.333 - 1.333$$

$$y = 8.889 \text{ lbs}$$

So, ∴ the weight of the baby at the age of 4 months = 8.89 lbs.

~~Divided Difference~~ Interpolation with unequal Interval

When the values of the variable are not equidistant, then the following two formulae are mainly used :-

1. Newton's Divided Difference Formula
2. Lagrange's Interpolation formula.

Divided Difference Table

x	$f(x)=y$	$\Delta f(x)=\Delta y$	$\Delta^2 f(x)=\Delta^2 y$	$\Delta^3 f(x)=\Delta^3 y$	$\Delta^4 f(x)=\Delta^4 y$
x_0	$f(x_0)=y_0$	$\Delta y_0 = \frac{y_1-y_0}{x_1-x_0}$	$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2-x_0}$	$\Delta^3 y_0 = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3-x_0}$	$\Delta^4 y_0 = \frac{\Delta^3 y_1 - \Delta^3 y_0}{x_4-x_0}$
x_1	$f(x_1)=y_1$	$\Delta y_1 = \frac{y_2-y_1}{x_2-x_1}$	$\Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3-x_1}$	$\Delta^3 y_1 = \frac{\Delta^2 y_2 - \Delta^2 y_1}{x_4-x_1}$	
x_2	$f(x_2)=y_2$	$\Delta y_2 = \frac{y_3-y_2}{x_3-x_2}$	$\Delta^2 y_2 = \frac{\Delta y_3 - \Delta y_2}{x_4-x_2}$		
x_3	$f(x_3)=y_3$	$\Delta y_3 = \frac{y_4-y_3}{x_4-x_3}$			
x_4	$f(x_4)=y_4$				

$$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{y_1-y_0}{x_1-x_0}$$

$$f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_1} = \frac{\Delta y_1 - \Delta y_0}{x_2-x_0}, \quad f(x_0,x_1,x_2,x_3) = \frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0} = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3-x_0}$$

Newton's Divided Difference Formula →

$x = x_0, x_1, x_2, \ldots, x_n$
$y = y_0, y_1, y_2, \ldots, y_n$; $x \to$ given, $y = f(x)$.

$$y = f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$
$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0 + \ldots$$

Q. use Newton's divided difference formula to find $f(x)$, given

x	0	2	3	6
$f(x)$	648	704	729	792

Also find $f(4)$ and $f'(4)$

Soln: The divided difference table for given data

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
$x_0 = 0$	$y_0 = 648$	$\Delta y_0 = \frac{704-648}{2-0} = 28$		
			$\Delta^2 y_0 = \frac{25-28}{3-0} = -1$	
$x_1 = 2$	$y_1 = 704$	$\Delta y_1 = \frac{729-704}{3-2} = 25$		$\Delta^3 y_0 = 0$
$x_2 = 3$	$y_2 = 729$		$\Delta^2 y_1 = \frac{21-25}{6-2} = -1$	
		$\Delta y_2 = \frac{792-729}{6-3} = 21$		
$x_3 = 6$	$y_3 = 792$			

By Newton's divided difference interpolation formula

$$f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \ldots$$
$$= 648 + (x-0)28 + (x-0)(x-2)(-1)$$
$$\therefore f(x) = -x^2 + 30x + 648$$

Now, $f(4) = -4^2 + 30\times 4 + 648 = -16 + 120 + 648 = -16 + 768 = 752$

$f'(x) = -2x + 30$

$f'(4) = -2\times 4 + 30 = -8 + 30 = 22$.

$\therefore \boxed{f(4) = 752}$ and $\boxed{f'(4) = 22}$ i.e. $\left(\frac{dy}{dx}\right)_{x=4} = 22$ //.

Q. The following values of the function $f(x)$ for values of x are given $f(1)=4$, $f(2)=5$, $f(7)=5$, $f(8)=4$, Find the value of $f(6)$.

Soln.

x	$f(x)$	$\Delta y = \Delta f(x)$	$\Delta^2 y = \Delta^2 f(x)$	$\Delta^3 y = \Delta^3 f(x)$
$x_0 = 1$	$4 = y_0$	$\Delta y_0 = \frac{5-4}{2-1} = 1$	$\Delta^2 y_0 = \frac{0-1}{7-1} = -\frac{1}{6}$	
$x_1 = 2$	$5 = y_1$	$\Delta y_1 = \frac{5-5}{7-2} = 0$		$\Delta^3 y_0 = -\frac{1}{6} + \frac{1}{6} = 0$
$x_2 = 7$	$5 = y_2$	$\Delta y_2 = \frac{4-5}{8-7} = -1$	$\Delta^2 y_1 = \frac{-1}{8-2} = -\frac{1}{6}$	
$x_3 = 8$	$4 = y_3$			

$$f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0$$

$$= 4 + (x-1) + (x-1)(x-2)\left(-\frac{1}{6}\right).$$

$$= 4 + x - 1 + (x^2 - 3x + 2)\left(-\frac{1}{6}\right)$$

$$f(x) = \frac{24 + 6x - 6 - x^2 + 3x - 2}{6} = \frac{-x^2 + 9x + 16}{6}$$

$$f(6) = \frac{-6^2 + 9 \times 6 + 16}{6} = \frac{-36 + 54 + 16}{6} = \frac{-36 + 70}{6} = \frac{34}{6} = \frac{17}{3}$$

$$\boxed{f(6) = \frac{17}{3}}$$

Q. Find a polynomial satisfied by $(-4, 1245)$, $(-1, 33)$, $(0, 5)$, $(2, 9)$ by the use of Newton's divided difference formula.

Soln.

x	$f(x) = y$	$\Delta f(x) = \Delta y$	$\Delta^2 f(x) = \Delta^2 y$	$\Delta^3 f(x) = \Delta^3 y$
$x_0 = -4$	$1245 = y_0$			
		$-404 = \Delta y_0$		
$x_1 = -1$	$33 = y_1$		$94 = \Delta^2 y_0$	
		$-28 = \Delta y_1$		$-14 = \Delta^3 y_0$
$x_2 = 0$	$5 = y_2$		$10 = \Delta^2 y_1$	
		$2 = \Delta y_2$		
$x_3 = 2$	$9 = y_3$			

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \cdots$$

$$y = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)(x-0)(-14)$$

$$= 1245 - 404x - 1616 + (x^2+5x+4)(94) + (x^3+5x^2+4x)(-14)$$

$$= -14x^3 + x^2(94-70) + x(-404+470-56) + 1245 - 1616 + 376$$

$$\boxed{y = -14x^3 + 24x^2 + 10x + 5}$$

Numerical Integration

The area bounded by the curve $f(x)$ and x-axis between limit a and b is denoted by $I = \int_a^b f(x)\,dx$

In numerical integration, divide the interval (a, b) into n- equal interval with step size h i.e, $(a, b) = (a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b)$

Here $x_1 - x_0 = h$, $x_2 - x_1 = h$, $\cdots$ $x_n - x_{n-1} = h$

and $b - a = nh \Rightarrow h = \frac{b-a}{n}$

Trapezoidal Rule →

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{1}{2}(y_0+y_n) + y_1 + y_2 + \cdots + y_{n-1}\right] = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

where $y_0 = y(x_0)$

$y_1 = y(x_1)$

$y_2 = y(x_2)$

$\vdots$

$y_n = y(x_n)$

Note :

1. y has been assumed a linear fn.
2. Trapezoidal rule can be applied to any number sub interval odd or even.

Q. use trapezoidal rule to evaluate the following integral and after finding the true value of the integral, compare the error $\int_4^{5.2} \log_e x \, dx$

Soln- Dividing the range (4, 5.2) of integration in six equal parts, $\frac{5.2-4}{6} = \frac{1.2}{6} = .2 = h$ (we have used six interval bcz it is used in all three methods, trapezoidal & simpson)

At each point of the division, the value of the function

x	$y = \log_e x$
$x_0 = 4$	$y_0 = \log(4) = 1.3862944$
$x_1 = 4.2$	$y_1 = \log(4.2) = 1.4350845$
$x_2 = 4.4$	$y_2 = \log(4.4) = 1.4816045$
$x_3 = 4.6$	$y_3 = \log(4.6) = 1.5260563$
$x_4 = 4.8$	$y_4 = \log(4.8) = 1.5686159$
$x_5 = 5$	$y_5 = \log(5) = 1.6094379$
$x_6 = 5.2$	$y_6 = \log(5.2) = 1.6486586$

By trapezoidal rule :

$$\int_4^{5.2} \log_e x \, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{.2}{2}\left[(1.3862944 + 1.6486586) + 2(1.4350845 + 1.4816045 + 1.5260563 + 1.5686159 + 1.6094379)\right]$$

$$= \frac{.2}{2}\left[3.034953 + 2(7.6207991)\right]$$

$$= .1(18.27655) = 1.8276551 = I_A \text{ (approx value)}$$

By general integration we get (by integration by parts)

$$\int_4^{5.2} \log_e x \, dx = \left[\log_e x \int 1 dx - \int \frac{d}{dx}(\log_e x) \int dx \right) dx\Big]_4^{5.2}$$

$$= \left[x \log_e x - \int \frac{1}{x} x \, dx\right]_4$$

$$= \left[x \log_e x - x\right]_4^{5.2} = (5.2 \log 5.2 - 5.2) - (4 \log_e 4 - 4)$$

$$= 1.827847 \text{ (Exact value)} = I_E$$

$\therefore$ Error $= I_E - I_A$

$= 1.827847 - 1.8276551 = 0.0001924$

Q. Compute the value of the foll. integral by Trapezoidal rule

$$\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx$$

Soln. → For integration, divide the whole range (0.2, 1.4) into six equal parts $\frac{1.4-0.2}{6} = 0.2 = h$

x	$\sin x$	$\log_e x$	e^x	$y = \sin x - \log_e x + e^x$
0.2	.19867	−1.6095	1.2214	$y_0 = 3.0296$
0.4	.3894	−0.9163	1.4918	$y_1 = 2.7975$
0.6	.5646	−0.5108	1.8221	$y_2 = 2.8975$
0.8	.7174	−0.2232	2.2255	$y_3 = 3.1661$
1.0	.8415	0.0000	2.7183	$y_4 = 3.5598$
1.2	.9320	0.1823	3.3201	$y_5 = 4.0698$
1.4	.9855	0.3365	4.0552	$y_6 = 4.7042$

$$\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{0.2}{2}\left[7.7338 + 2(16.4907)\right] = 4.07152$$
$$= I_A$$

Next the real value

$$= \int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx$$

$$= \left[-\cos x - (x\log_e x - x) + e^x\right]_{0.2}^{1.4}$$

$$= -\left[\cos 1.4 - \cos 0.2\right] - \left[1.4\log 1.4 - 0.2\log 0.2\right] + (1.4 - 0.2) + (e^{1.4} - e^{0.2})$$

$$= \left[0.17 - 0.9801\right] - \left[1.4 \times 0.3365 - (.2)(-1.6095)\right] + (1.2) + (4.0552 - 1.2714) \quad \text{--- (2)}$$

$$= 4.0509 \quad = I_E$$

∴ Error $= |I_E - I_A|$ From (1) & (2) the error $= 4.07152 - 4.0509 = 0.0206$.

Q. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by Trapezoidal rule. [Ans : $I_A = 1.4108$
$I_E = 1.40564$]

Error $= |I_E - I_A|$
$= 1.4108 - 1.40564$
$= 0.0052$

Numerical Integration Simpson's $\frac{1}{3}$ rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots y_{n-1}) + 2(y_2+y_4+y_6+\cdots y_{n-2})\right]$$

Q. Use Simpson's $\frac{1}{3}$ rule to evaluate $\int_1^{1.3} y\,dx$ from the values of x and y tabulated as under

x	y
$1.00\ x_0$	$1.00000\ y_0$
$1.05\ x_1$	$1.02470\ y_1$
$1.10\ x_2$	$1.04381\ y_2$
$1.15\ x_3$	$1.07338\ y_3$
$1.20\ x_4$	$1.09544\ y_4$
$1.25\ x_5$	$1.11803\ y_5$
$1.30\ x_6$	$1.14017\ y_6$

Soln. we take $h = \frac{1.3-1}{6} = \frac{0.3}{6} = \frac{0.1}{2} = 0.05$ (As h is not given so we'll take interval as 6)

Now
$$\int_1^{1.3} y\,dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{0.05}{3}\left[(1+1.14017)+4(1.02470+1.07338+1.11803) + 2(1.04381+1.09544)\right]$$

$$= \frac{1}{60}\left[2.14017+12.86044+4.2885\right]$$

$$= 0.321485$$

Q. Given following data find the value of the following integral using Simpson's 1/3 rule and compare it with the actual value

$\int_0^4 e^x dx$. $[e^0 = 1, e^1 = 2.72, e^2 = 7.39, e^3 = 20.09, e^4 = 54.60]$

Soln. For integration, divide the whole range (0, 4) into four equal parts.

$h = \frac{4-0}{4} = 1$ & $y_0 = e^0 = 1$, $y_1 = e^1 = 2.72$, $y_2 = e^2 = 7.39$, $y_3 = e^3 = 20.09$, $y_4 = e^4 = 54.60$.

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$

By Simpson's 1/3 rule

$$\int_0^4 e^x dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

Now
$$\int_0^4 e^x dx = \frac{1}{3}\left[(1 + 54.60) + 4(2.72 + 20.09) + 2(7.39)\right]$$

$$= \frac{1}{3}\left[1 + 5.60 + 91.24 + 14.78\right]$$

$$= \frac{1}{3}\left[161.62\right] = 53.873333 \text{ (Approx value)}$$
$$= I_A$$

Real value of the definite integral

$$\int_0^4 e^x dx = \left[e^x\right]_0^4 = e^4 - e^0 = e^4 - 1 = 54.60 - 1$$
$$= 53.60 \text{ (Exact value)}$$
$$= I_E$$

$\therefore$ Error $= |53.60 - 53.873333|$

$= 0.27333$

Q. Use Simpson's 1/3 rule to evaluate $\int_0^1 \frac{dx}{1+x^2}$. Hence obtain the approximate value of π.

Soln.

	x	$y = \frac{1}{1+x^2}$
x_0	$x_0 = 0$	$y_0 = 1.0000$
x_1	$x_0 + h = 1/6$	$y_1 = (36/37) = 0.97297$
x_2	$x_0 + 2h = 2/6$	$y_2 = (36/40) = 0.90000$
x_3	$x_0 + 3h = 3/6$	$y_3 = (36/45) = 0.80000$
x_4	$x_0 + 4h = 4/6$	$y_4 = (36/52) = 0.69231$
x_5	$x_0 + 5h = 5/6$	$y_5 = (36/61) = 0.59016$
x_6	$x_0 + 6h = 1$	$y_6 = (1/2) = 0.50000$

we get
$h = \frac{1-0}{6} = \frac{1}{6}$

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

Now $$\int_0^1 \frac{dx}{1+x^2} = \frac{1}{3\times 6}\left[(1+0.5)+4(0.97297+0.8+0.59016)+2(0.9+0.69231)\right]$$

$$= 0.7853$$

Real value of definite integral

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}$$

~~Equipairing from bo~~ Equating from both the values

$$\therefore \frac{\pi}{4} = 0.7853$$

$$\pi = 4(0.7853) = \underline{\underline{3.1415}}$$

Simpson's 3/8 Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-1})+2(y_3+y_6+\cdots y_{n-3})\right]$$

Q. Use Simpson's 3/8 rule to evaluate $\int_4^{5.2} \log_e x\,dx$

Soln. → Dividing the range (4, 5.2) of integration in 6 equal parts.

$$h = \frac{5.2-4}{6} = \frac{1.2}{6} = .2$$

At each point of the division, the value of the function $f(x)=\log_e x$ is given in the following table.

x	$y=\log_e x$
$x_0=4$	$y_0=1.3862944$
$x_0+h=4.2$	$y_1=1.4350845$
$x_0+2h=4.4$	$y_2=1.4816045$
$x_0+3h=4.6$	$y_3=1.5260563$
$x_0+4h=4.8$	$y_4=1.5686159$
$x_0+5h=5.0$	$y_5=1.6094379$
$x_0+6h=5.2$	$y_6=1.6486586$

$$\int_4^{5.2} \log_e x\,dx = \frac{3(0.2)}{8}\left[(1.3862+1.6486)+3(1.4350+1.4816+1.5686+1.6094)+2(1.5260)\right]$$

$$= \underline{\underline{1.8278}} \text{ (Approx)}$$

Q. Use Simpson's 3/8 rule to evaluate $\int_0^6 \frac{dx}{1+x^2}$

Soln.

$h = \frac{6-0}{6} = 1$ (Dividing the range into 6 equal parts)

x	$y = \frac{1}{1+x^2}$
$x_0 = 0$	$y_0 = 1.00$
$x_1 = 1$	$y_1 = 0.5$
$x_2 = 2$	$y_2 = 0.2$
$x_3 = 3$	$y_3 = 0.100$
$x_4 = 4$	$y_4 = 0.05824$
$x_5 = 5$	$y_5 = 0.038462$
$x_6 = 6$	$y_6 = 0.027027$

$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)\right.$

$\left.+ 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$

$= \frac{3}{8}\left[1+0.0270) + 3(0.5+0.2\right.$

$+ 0.0582 + 0.0384)$

$\left.+ 2(0.1)\right]$

$= 1.3571$ (Approx value)

$\int_0^6 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^6 = \tan^{-1}6 - \tan^{-1}0$

$= \tan^{-1}6 = 1.4056$ (Exact value)

$\therefore$ Error $= |$ Exact value $-$ Approx value $|$

$= |1.4056 - 1.3571|$

$= |0.04854|$

$= 0.04854$

Numerical Differentiation

Numerical Differentiation is the process of calculating the derivatives of a function at some particular value of the independent variable by means of a set of given values of that function.

Consider the function $y = f(x)$

which is tabulated for the values $x = x_0 + ph$; $p = 0, 1, 2, \ldots$

Derivative using Newton's Forward Difference Formula :-

we know that NFF is

$$f(x_0+ph) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0) + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f(x_0) + \cdots$$

$$f(x_0+ph) = f(x_0) + p\Delta f(x_0) + \frac{p^2-p}{2}\Delta^2 f(x_0) + \frac{p^3-3p^2+2p}{6}\Delta^3 f(x_0) + \frac{p^4-6p^3+11p^2-6p}{24}\Delta^4 f(x_0) + \cdots$$

Differentiating w.r.t. p, we get

$$f'(x_0+ph).h = 0 + 1\Delta f(x_0) + \frac{2p-1}{2}\Delta^2 f(x_0) + \frac{3p^2-6p+2}{6}\Delta^3 f(x_0) + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 f(x_0) + \cdots$$

$$f''(x_0+ph).h^2 = \Delta^2 f(x_0) + (p-1)\Delta^3 f(x_0) + \frac{12p^2-36p+22}{24}\Delta^4 f(x_0) + \cdots$$

If $p = 0$, $\because$ $x = x_0 + ph$ (if $x = x_0$)

$$f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0) - \frac{1}{4}\Delta^4 f(x_0) + \cdots\right]$$

$y_0 = f(x_0)$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12}\Delta^4 f(x_0) - \cdots\right)$$

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 f(x_0) - \frac{3}{2}\Delta^4 f(x_0) + \cdots\right]$$

Derivative using Newton's Backward Difference Formula

Consider the fn $y = f(x)$
which is tabulated for the values $x = x_n + ph$; $p = 0, 1, 2, \ldots$

we know that NBF is

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n)$$
$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 f(x_n) + \cdots$$

$$f(x_n + ph) = f(x_n) + p\nabla f(x_n) + \frac{p^2+p}{2}\nabla^2 f(x_n) + \frac{p^3+3p^2+2p}{6}\nabla^3 f(x_n)$$
$$+ \frac{p^4+6p^3+11p^2+6p}{24}\nabla^4 f(x_n) + \cdots$$

Let $p = 0$

$$f'(x_n) = \frac{1}{h}\left[\nabla f(x_n) + \frac{1}{2}\nabla^2 f(x_n) + \frac{1}{3}\nabla^3 f(x_n) + \frac{1}{4}\nabla^4 f(x_n) + \cdots\right]$$

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 f(x_n) + \nabla^3 f(x_n) + \frac{11}{12}\nabla^4 f(x_n) + \cdots\right]$$

$$f'''(x_n) = \frac{1}{h^3}\left[\nabla^3 f(x_n) + \frac{3}{2}\nabla^4 f(x_n) + \cdots\right]$$

Derivative using
Newton's divided difference formula is →

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \cdots$$

$$f'(x) = \Delta f(x_0) + (2x - x_0 - x_1)\Delta^2 f(x_0) + (3x^2 - 2x(x_0+x_1+x_2) + x_0x_1 + x_1x_2 + x_2x_0)\Delta^3 f(x_0) + \cdots$$

~~Put x_0~~

Q. Given that

x	1.0	1.1	1.2	1.3
y	0.841	0.891	0.932	0.963

Find $\frac{dy}{dx}$ at $x = 1.0$

Soln. →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	0.841 (y_0)			
		0.050 (Δy_0)		
1.1	0.891		-0.009 ($\Delta^2 y_0$)	
		0.041		-0.001 ($\Delta^3 y_0$)
1.2	0.932		-0.01	
		0.031		
1.3	0.963			

Here $x = 1$, $x_0 = 1$, $p = \frac{x - x_0}{h} = \frac{1-1}{0.1} = 0$, $p = 0$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{0.1}\left[0.05 - \frac{1}{2}(-0.009) + \frac{1}{3}(-0.001)\right]$$

$= 0.5417$ at $x = 1$ when $p = 0$.

Q. Find the first and second derivatives of the function given below at the point $x = 1.2$

x	1	2	3	4	5
y	0	1	5	6	8

Soln. →

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	0 (y_0)				
		1 (Δy_0)			
2	1		3 ($\Delta^2 y_0$)		
		4		-6 ($\Delta^3 y_0$)	
3	5		-3		10 ($\Delta^4 y_0$)
		1		4	
4	6		1		
		2			
5	8				

$x = 1.2$, $x_0 = 1$, $p = \frac{1.2 - 1}{1} = 0.2 \neq 0$, $h = 1$.

when $p \neq 0$

$$f'(x) = \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[1 + \frac{2(0.2)-1}{2}(3) + \frac{3(0.2)^2-6(0.2)+2}{6}(-6) + \frac{4(0.2)^3-18(0.2)^2+22(0.2)}{24}\right]$$

$$f'(1.2) = -1.773$$

$$f''(x) = \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{12p^2-36p+22}{24}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{1}\left[3 + ((0.2)-1)(-6) + \frac{12(0.2)^2 - 36(0.2) + 22(10)}{24}\right]$$

$$= 14.166$$

Q. From the foll. table find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $x = 2.03$

x	1.96	1.98	2.00	2.02	2.04
y	0.7825	0.7739	0.7651	0.7563	0.7473

Soln. →

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.96	0.7825				
		−0.0086			
1.98	0.7739		−0.0002		
		−0.0088		0.0002 $\nabla^3 y_n$	
2.00	0.7651		0 $\nabla^2 y_n$		−0.0004
		−0.0088 ∇y_n		−0.0002	
x_n = 2.02	0.7563 y_n		−0.0002		
		−0.0090			
2.04	0.7473				

$$p = \frac{x - x_n}{h} = \frac{2.03 - 2.02}{0.02} = \frac{0.01}{0.02} = 0.5 \neq 0$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{0.02}\left[-0.0088 + 0 + \frac{3(0.5)^2+6(0.5)+2}{6}(0.0002)\right]$$

$$= -0.4304 \quad \text{at } x = 2.03.$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \cdots\right]$$

$$= \frac{1}{(0.02)^2}\left[0 + ((0.5)+1)(0.0002)\right] = 0.75 \quad \text{at } x = 2.03.$$

Newton's divided difference:

Q. Find $f'(10)$ from the following data:

x	3	5	11	27	34
y	−13	23	899	17315	35606

Soln: Here values of x are not equally spaced, so we have to use Newton's divided difference formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	−13				
		18			
5	23		16		
		146		1	
11	899		40		0
		1026		1	
27	17315		69		
		2613			
34	35606				

Newton's divided difference formula is

$$y = f(x) = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + \cdots$$

$$y' = f'(x) = \Delta y_0 + (2x - x_0 - x_1)\Delta^2 y_0 + (3x^2 - 2x(x_0+x_1+x_2) + x_0x_1 + x_1x_2 + x_2x_0)\Delta^3 y_0 + \cdots$$

Here $x_0 = 3,\ x_1 = 5,\ x_2 = 11,\ x = 10$

$$f'(10) = 18 + (20-3-5)(16) + (300 - 20(19) + 15 + 55 + 33)$$

$$f'(10) = \underline{\underline{233}}$$