Eigenvalue Problems:→

Let A be n×n square matrix. A may be singular or non-singular. Consider the hom. system of equations

$$Ax = \lambda X \Rightarrow (A - \lambda I)X = 0 \quad -(1)$$

(1) always has a unique or trivial solution. The values of λ for which (1) has a non-trivial solution, are called the eigenvalues or characteristic values of A. The corresponding non-trivial solution X are called eigen vectors or characteristic vectors of A.

- The problem of determining the eigenvalues and eigenvectors of a square matrix A is called an Eigenvalue problem.

- If the hom. system (1) has a non-trivial solution, then the Rank of $(A-\lambda I) < n$

$\Rightarrow (A-\lambda I)$ is a singular matrix

$\Rightarrow \det(A-\lambda I) = 0.$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

Expanding the determinant gives the polynomial of degree n in λ;

$$P_n(\lambda) = |A - \lambda I| = 0 \quad -(2)$$

is called characteristic equation of matrix A.

The roots of (2) are called the eigenvalues.

- Sum of eigenvalues = Trace(A)
- Product of eigenvalues = $|A|$
- If any one of the eigenvalue is zero then $|A| = 0 \Rightarrow$ A is singular matrix.
- If A is singular $\Rightarrow |A| = 0 \Rightarrow$ one of the eigenvalues is zero.

- If A is diagonal or upper triangular or lower triangular matrix, then the diagonal elements of A are the eigenvalues of A.

- Let λ be an eigenvalues of A and x be the corresponding eigen vector. Then

(1) αA has eigenvalue $\alpha\lambda$ and the corr. eigenvector is x.

$$AX = \lambda X \Rightarrow \alpha AX = \alpha\lambda X$$
$$\Rightarrow (\alpha A)X = (\alpha\lambda)X$$

(2) A^m has eigenvalue λ^m and the corr. eigenvector is X for any positive Integer m.

$$AX = \lambda X$$
$$A(AX) = A(\lambda X) = \lambda(AX) = \lambda(\lambda X) = \lambda^2 X$$
$$\Rightarrow A^2 X = \lambda^2 X$$

Similarly $A^m X = \lambda^m X$.

(3) A^{-1} has the eigenvalue $(1/\lambda)$ and the corresponding eigen vector is X; Provided A^{-1} exists.

$$\because AX = \lambda X$$
$$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$$
$$\Rightarrow (A^{-1}A)X = \lambda(A^{-1}X)$$
$$\Rightarrow IX = \lambda(A^{-1}X)$$
$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X \quad \text{or} \quad A^{-1}X = \left(\frac{1}{\lambda}\right)X.$$

(4) A and A^T have the same eigenvalues.

(5) For a real matrix A, If $\alpha + i\beta$ is an eigen value then $\alpha - i\beta$ is also an eigen value of A.

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Que Find the eigenvalues and the corresponding eigenvectors of the following matrices.

(1) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ $\qquad (A - \lambda I)x = 0$ — System

Characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda) - 12 = 0$

$\Rightarrow 2 - \lambda - 2\lambda + \lambda^2 - 12 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 10 = 0$

$\Rightarrow \lambda^2 - 5\lambda + 2\lambda - 10 = 0$

$\Rightarrow \lambda(\lambda-5) + 2(\lambda-5) = 0$

$\Rightarrow \lambda = -2, 5.$ — Two eigen values.

Corresponding to eigen value $\lambda = -2$

$(A + 2I)x = 0$

$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow 3x + 4y = 0$

$\Rightarrow x = \frac{-4y}{3}$

Let $y = 1 \Rightarrow x = -4/3$

$\therefore$ or $y = 3, \Rightarrow x = -4.$

So Eigen vector is $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$.

$\lambda = 5$

$(A - 5I)x = 0$

$\begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -4x + 4y = 0$

$\Rightarrow x = y$ $\qquad$ let $x = y = 1$

So Eigen vector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Que:→ $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$(A-\lambda I)X = 0 \rightarrow$ System of hom. equation

$|A-\lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 + 1 = 0$

$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$

$\Rightarrow \lambda = 1 \pm i$

Corr. to $\lambda = 1+i$

$(A-\lambda I)X = 0$

$\Rightarrow \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - (1+i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -ix + y = 0 \quad \Rightarrow x = +\frac{1}{i} y = (-iy)$

and $-x - iy = 0$

$y = 1, \quad x = -i$

So Eigen vector is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Corr. to $\lambda = 1-i$

$(A-\lambda I)X = 0$

$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$ix + y = 0 \Rightarrow y = -ix$

$x = 1; \; y = -i$

So eigenvector is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ or $\begin{bmatrix} i \\ 1 \end{bmatrix}$

→ Eigenvectors corresponding to distinct eigen values are linearly Independent.

Que (1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

(3) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(2) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(4) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Cayley – Hamilton Thm:→ Every matrix 'A' satisfies its own characteristic equation.

— Verify Cayley Hamilton Thm for $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

Also obtain A^{-1} and A^3.

Sol^n $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

The char. equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ -1 & 1-\lambda & 2 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 - \lambda + 3 = 0$

$A^2 = \begin{bmatrix} -1 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 6 & 5 \end{bmatrix}$; $A^3 = \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$

$-A^3 + 3A^2 - A + 3I = 0$

So $A^3 = 3A^2 - A + 3I$

$= \begin{bmatrix} -1 & 10 & 12 \\ 1 & 11 & 10 \\ -1 & 16 & 17 \end{bmatrix}$

and $A^{-1} = \frac{1}{3}[A^2 - 3A + I]$

$$= \frac{1}{3}\begin{bmatrix} -3 & -2 & 4 \\ 3 & 1 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

→ Eigen values of A are the roots of
$\lambda^3 - 3\lambda^2 + \lambda - 3 = 0$
$\Rightarrow \lambda = 3, i, -i$
$\Rightarrow$ Eigen values of A^2 are λ^2
$\Rightarrow 9, i^2, (-i)^2$
$= 9, -1, -1.$

— Spectral radius of a matrix A is $\rho(A) =$
Largest eigenvalue in magnitude.
$= \max_i |\lambda_i|$

Eg. Spectral radius of A used in above Question is 3.

→ Algebraic multiplicity of an eigen value is the no. of times the eigen value repeats itself.

Eg. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ → upper triangular matrix
$\Rightarrow \lambda = 1, 1, 1$ (diagonal Entries)

$|A - \lambda I| = 0$ — Char. Equation
Char. Equ. is $(1-\lambda)^3 = 0$
$\lambda = 1$ repeats 3 times
$\Rightarrow$ A.M. of $\lambda = 1$ is 3.

To find the eigen vector corresponding to $\lambda = 1$,

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$(A-I)X = 0$

$$\left(\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix} - \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)X = 0$$

$$\Rightarrow \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$$

$\Rightarrow$ $y = 0$

$z = 0$ and x arbitrary

$\Rightarrow$ $(1,0,0)$ if $x=1$ will be an Eigen vector.

$\Rightarrow$ Corresponding to $\lambda=1$, we have only one L.I. Eigen vector.

(2) $A = \begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix} \rightarrow$ Again upper Triangular matrix

$\Rightarrow \lambda = 1,1,1$

$\Rightarrow$ A.M. of $\lambda=1$ is 3.

To find Eigen Vector Corr. to $\lambda=1$

$(A-I)X = 0$

$$\left[\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow$ $y=0$, x and z are arbitrary

Choose $x=0$, $z=1$

$(0,0,1)$

Choose $z=0$, $x=1$

$(1,0,0)$

$\Rightarrow$ $(0,0,1)$ & $(1,0,0)$ $\rightarrow$ Two L.I. Eigen vectors Corr. to $\lambda = 1$.

(3) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow$ diagonal Matrix
$\Rightarrow \lambda = 1, 1, 1$
A.M. of $\lambda = 1$ is 3.

To find Eigen vector Corr. to $\lambda = 1$;
$(A - I)X = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ x, y, z all are arbitrary.
$\Rightarrow x = 1, y = 0, z = 0$
$x = 0, y = 1, z = 0$
$x = 0, y = 0, z = 1$
$\Rightarrow (1,0,0)\ (0,1,0)\ (0,0,1) \rightarrow$ 3 L.I. Eigen vectors.

Hence, If A.M. of any λ is > 1 i.e. λ is repeating more than once, then how many L.I. vectors are there corresponding to that particular λ?

Ans $n - \text{Rank}(A - \lambda I)$

In (1), $\text{Rank}(A - \lambda I) = 2$
$\Rightarrow 3 - 2 = 1$ L.I Eigen vector

In (2); $\text{Rank}(A - \lambda I) = 1$
$\Rightarrow 3 - 1 = 2$ L.I. Eigen vector

In (3); $\text{Rank}(A - \lambda I) = 0$
$\Rightarrow 3 - 0 = 3$ L.I. Eigen vectors.

$\rightarrow$ The no. of L.I. Eigen vectors corresponding to λ ~~are~~ is called the geometric multiplicity (G.M.) of λ.

Diagonalizable Matrix: → A Square matrix of order n is diagonalizable iff it has n L.I. Eigen vectors.

→ If $A_{n\times n}$ matrix and has all the n eigen values as distinct
i.e. $\lambda_1, \lambda_2 -- \lambda_n$ are n eigen values of A and $\lambda_i \neq \lambda_j \ \forall i,j$
$\Rightarrow$ A has n L.I. Eigen vector Corresponding to each λ
$\Rightarrow$ A is diagonalizable matrix.

Thus If A has distinct Eigenvalues $\Rightarrow$ A is diagonalizable.
But Converse need not to be true.
i.e. If A is diagonalizable then A can have repeated eigenvalues as well.
E.g. In (3) above, $\lambda = 1,1,1 \rightarrow$ A.M. is 3
But we have 3 L.I. Eigen vectors Corr. to $\lambda = 1$
$\Rightarrow$ A is diagonalizable

→ Every diagonal matrix is diagonalizable matrix, but Converse need not to be true.
i.e. A diagonalizable matrix need not to be a diagonal matrix.

E.g. $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow$ Not diagonal matrix.

But $|A - \lambda I| = 0$
$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$
$\Rightarrow \lambda = 1, 2, 3$
$\Rightarrow$ All eigen values are distinct
$\Rightarrow$ A is diagonalizable.

→ A is diagonalizable if A is similar to a diagonal matrix.
$\Rightarrow \exists$ an Invertible matrix P St. $P^{-1}AP = D$
or $A = PDP^{-1}$.

Note that this P matrix can be obtained by using the L.I. Eigenvectors corr. to eigenvalues of A.

Ex The eigenvectors of 3x3 matrix A corr. to eigenvalues 1, 1, 3 are $(1,0,-1)^T$, $(0,1,-1)^T$, $(1,1,0)^T$ respectively. Find A.

Sol^n →

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \rightarrow \text{Matrix Consisting of eigenvectors.}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{Matrix Consisting of Eigenvalues.}$$

We find $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$$A = PDP^{-1}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Que Find the matrix A if

(1) Eigenvalues are 2, 2, 4 and Eigenvectors are $(-2,1,0)^T$, $(-1,0,1)^T$, $(1,0,1)^T$

Ans $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

(2) Eigenvalues are 1, −1, 2 and Eigenvectors are $(1,1,0)^T$; $(1,0,1)^T$; $(3,1,1)^T$.

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Ans:- $\begin{bmatrix} 6 & -5 & -7 \\ 1 & 0 & -1 \\ 3 & -3 & -4 \end{bmatrix}$

(3) Eigenvalues are 1, 1, 1 and Eigenvectors are
$(-1,1,1)^T$, $(1,-1,1)^T$, $(1,1,-1)^T$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ Symmetric matrix:→ If $A = A^T$
i.e. $a_{ij} = a_{ji} \ \forall i, j$

– All eigenvalues of A and A^T are same.
– The eigenvalues of a Symmetric matrix are real.

Skew Symmetric matrix:→ If $A = -A^T$ or $A^T = -A$.
i.e. $a_{ij} = -a_{ji} \ \forall i, j$
i.e. $a_{ii} = -a_{ii} \ \forall i$
$\Rightarrow 2a_{ii} = 0$
$\Rightarrow a_{ii} = 0$
⇒ all the diagonal entries are zero.

→ The eigenvalues of a Skew symmetric matrix are Zero or Pure Imaginary.
i.e. Skew symmetric matrix Cannot have real eigenvalues.

Orthogonal Matrix:→ A real matrix A is orthogonal if
$A^T A = I$

E.g. $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal matrix

→ The eigen values of an orthogonal matrix are of magnitude 1 and are real or complex conjugate pairs.
i.e. If λ is an eigenvalue of an orthogonal matrix A Then $|\lambda| = 1$.
$\Rightarrow \lambda$ could be 1 or -1 or $i, -i$

→ If A is an orthogonal matrix then $|A| = \pm 1$.
$\left[\because |\lambda| = 1 \ \forall \ \lambda \text{ Eigenvalues of } A. \right.$
$\left. \text{and product of eigenvalues} = |A| \right]$