Q. Which of the foll. are linear transformations?

(a) $T(x, y, z) = (x-y, x^2, 2z)$

(b) $T(x, y) = (x-y, 2x+2)$

Soln: (a) Let $(x_1, y_1, z_1)$; $(x_2, y_2, z_2) \in R^3$ and $\alpha \in R$

Now, $T((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= T(x_1+x_2, y_1+y_2, z_1+z_2)$

$= ((x_1+x_2) - (y_1+y_2), (x_1+x_2)^2, 2(z_1+z_2))$

$= (x_1 - y_1 + x_2 - y_2, x_1^2 + x_2^2 + 2x_1x_2, 2z_1 + 2z_2)$

$= (x_1 - y_1, x_1^2, 2z_1) + (x_2 - y_2, x_2^2, 2z_2) + (0, 2x_1x_2, 0)$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) + (0, 2x_1x_2, 0)$.

$\neq T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$\therefore T$ is not L.T.

(b) Let $(x_1, y_1), (x_2, y_2) \in R^2$

$T((x_1, y_1) + (x_2, y_2)) = L(x_1+x_2, y_1+y_2)$

$= (x_1+x_2-y_1-y_2, 2(x_1+x_2)+2)$

$= (x_1 - y_1 + x_2 - y_2, 2x_1 + 2 + 2x_2)$

$= (x_1 - y_1, 2x_1 + 2) + (x_2 - y_2, 2x_2)$

$= T(x_1, y_1) + T(x_2, y_2) + (0, -2)$

$\therefore T((x_1, y_1) + (x_2, y_2)) \neq T(x_1, y_1) + L(x_2, y_2)$

$\therefore T$ is not linear transformation.

Q. Show that $f: R^4 \to R^4$ given
$$f(x, y, z, t) = (2x, 3y, 0, 0)$$
is a linear transformation. Find its rank and nullity.

Soln: Let $v_1 = (x_1, y_1, z_1, t_1)$, $v_2 = (x_2, y_2, z_2, t_2) \in R^4$
and $\alpha, \beta \in R$

Then $\alpha v_1 + \beta v_2 = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2, \alpha t_1 + \beta t_2)$

As given, $f(\alpha v_1 + \beta v_2) = (2(\alpha x_1 + \beta x_2), 3(\alpha y_1 + \beta y_2), 0, 0)$
$$= \alpha(2x_1, 3y_1, 0, 0) + \beta(2x_2, 3y_2, 0, 0)$$
$$= \alpha f(v_1) + \beta f(v_2)$$
Hence $f$ is a linear transformation.

We have $\ker f = \{(x, y, z, t) \in R^4 : f(x, y, z, t) = (0, 0, 0, 0)\}$

$\therefore (x, y, z, t) \in \ker f \Leftrightarrow f(x, y, z, t) = (0, 0, 0, 0)$
$$\Leftrightarrow (2x, 3y, 0, 0) = (0, 0, 0, 0)$$
$$\Leftrightarrow x = 0, y = 0$$
$$\Leftrightarrow (x, y, z, t) = (0, 0, z, t)$$
$$\ker f = \{(0, 0, z, t) : z, t \in R\}$$

Since $(0, 0, z, t) = z(0, 0, 1, 0) + t(0, 0, 0, 1)$ so $\ker f$ is spanned by the set $S = \{e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\}$.

It is easy to verify that $e_3, e_4$ are L.I.

Thus $S$ is a basis of $\ker f$ and so $\dim \ker f = 2$

Hence nullity $f = 2$

By ~~the~~ rank-nullity theorem,
$$\text{rank} f + \text{nullity} f = \dim R^4 = 4$$
Hence $\text{rank} f = 4 - 2 = 2$.

Q. Find the range, rank and nullity of the linear transformation $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$ — (1)

Soln: Let $(a, b) \in$ Range $T$ be arbitrary. Then
$(a, b) = T(x_1, x_2, x_3)$, for some $(x_1, x_2, x_3) \in R^3$.

or $(a, b) = (x_1 + x_2, 2x_3 - x_1) = (x_1 + x_2 + 0x_3, -x_1 + 0x_2 + 2x_3)$

or $(a, b) = x_1(1, -1) + x_2(1, 0) + 2x_3(0, 1)$ — (2)

we see that $S = \{(1,0),(0,1)\}$ is L.I and

$(1,-1) \in L(S)$ $[\because (1,-1) = (1,0) - 1(0,1)]$

Hence by (2), Range $T = L(S) = ((1,0),(0,1))$

$\therefore$ rank $T$ = dim Range $T = 2$

Let $(a,b,c) \in \ker T$ be arbitrary. Then

$T(a,b,c) = (0,0)$ i.e $(a+b, 2c-a) = (0,0)$, by (1)

$\Rightarrow a+b=0,\ 2c-a=0 \Rightarrow a=2, b=-2, c=1$

We see that $\ker T$ is spanned by $(2,-2,1)$

Hence $\ker T = \{(2,-2,1)\}$ and dim $\ker T = 1$ i.e, nullity $T = 1$.

Q. Find the range, rank, kernel and nullity of the linear transf $T: R^2 \to R^3$ defined by

$$T(x_1, x_2) = (x_1 + x_2,\ x_1 - x_2,\ x_2) \quad \text{---}(1)$$

Soln: Let $(a,b,c) \in$ Range $T$ be arbitrary. Then

$(a,b,c) = T(x_1,x_2)$, for some $(x_1,x_2) \in R^2$.

or $(a,b,c) = (x_1+x_2, x_1-x_2, x_2)$, by (1)

$= x_1(1,1,0) + x_2(1,-1,1)$ ; $x_1, x_2 \in R$

This shows that Range $T$ is spanned by $S = \{(1,1,0),(1,-1,1)\}$

Further $(1,1,0),(1,-1,1)$ are L.I. Since

$\alpha(1,1,0) + \beta(1,-1,1) = (0,0,0)$ ; $\alpha, \beta \in R$

$\Rightarrow \alpha+\beta=0,\ \alpha-\beta=0,\ \beta=0 \Rightarrow \alpha=0, \beta=0$

Hence $S$ is a basis of Range $T$ and so dim Range $T = 2$

i.e, Range $T = L(S) = ((1,1,0),(1,-1,1))$ and rank $T = 2$

Let $(a,b) \in \ker T$ be arbitrary. Then $T(a,b) = (0,0,0)$

or $(a+b, a-b, b) = (0,0,0)$ by (1)

$\Rightarrow a+b=0,\ a-b=0,\ b=0 \Rightarrow a=0, b=0$

$\therefore \ker T = \{(0,0)\}$ and dim $\ker T = 0$ i.e, nullity $T = 0$.

Q. Let $T: R^3 \to R^3$ be defined as

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

verify rank nullity theorem for T.

Soln. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker T$

$$\Rightarrow T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x + z = 0$$
$$y + z = 0$$

$\Rightarrow x = -z, y = -z$

$\therefore x = -a, y = -a, z = a$ is a soln.

$$\therefore \ker T = \left\{ a\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : a \in R \right\}$$

As $\ker T \neq \{0\}$

$\therefore$ T is not one-one

$\dim(\ker T) =$ nullity of $T = 1$

Let $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in R^3$

Suppose there exists $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3$

such that $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = u$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b-a \\ c-2a \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b-a \\ c-2a-b+a \end{pmatrix}$$

The soln exists only if

$c - 2a - b + a = 0$ (T is onto iff $r(T) = \dim V$)

or $c = a + b$

$\therefore$ T is not onto and any element of Range T is of type $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ ($\because \dim T = 1$, $\text{rank}\, T = 3$) $\dim T \neq \text{rank}\, T$)

$$\text{Range}\, T = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} : a, b \in R \right\}$$

$$\text{A basis of Range}\, T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$\therefore \dim(\text{Range}\, T) = \text{Rank of}\ T = 2$

Now Nullity of $T$ + Rank of $T = 1 + 2 = 3 = \dim R^3$.