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Assignment - 04

Q1) In the given question, $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$

Point $= (2, 0, 3)$

Now, $(\text{grad } \vec{V}) = (2y + z^2)\hat{i} + (2xy + 2zy)\hat{j} + (y^2 + 2zx)\hat{k}$

$(\text{grad } \vec{V})$ at $(2, 0, 3) = 9\hat{i} + 12\hat{k}$

Normal of $x^2 + y^2 + z^2 = 14$

$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 14) \Rightarrow 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Normal of sphere at $(3, 2, 1) = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Unit Normal of the sphere $= \dfrac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{6^2 + 4^2 + 2^2}}$

$\dfrac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2\sqrt{14}} = \dfrac{3i + 2j + 1k}{\sqrt{14}}$

Now, directional derivative $= (\text{grad } \vec{V}) \cdot (\text{unit normal})$

$= (9\hat{i} + 12\hat{k})\dfrac{(3i + 2j + \hat{k})}{\sqrt{14}}$

$= \dfrac{39}{\sqrt{14}}$

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Q2) Gauss theorem is

$$\iint_S f.n\,ds = \iiint \nabla.f\,dv$$

$$RHS = \iiint \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\cdot\left((x^3-yz)\vec{i} - 2x^2y\vec{j} + a\vec{k}\right)dv$$

$$\iiint (3x^2-2x^2)\,dv \Rightarrow \int_0^a\int_0^a\int_0^a x^2\,dx\,dy\,dz$$

$$\int_0^a\int_0^a \left(\frac{x^3}{3}\right)_0^a dy\,dz \Rightarrow \frac{a^3}{3}\int_0^a (y)_0^a\,dz = \frac{a^4}{3}\int_0^a dz$$

$$\Rightarrow \frac{a^4}{3}(z)_0^a \Rightarrow a^5/3$$

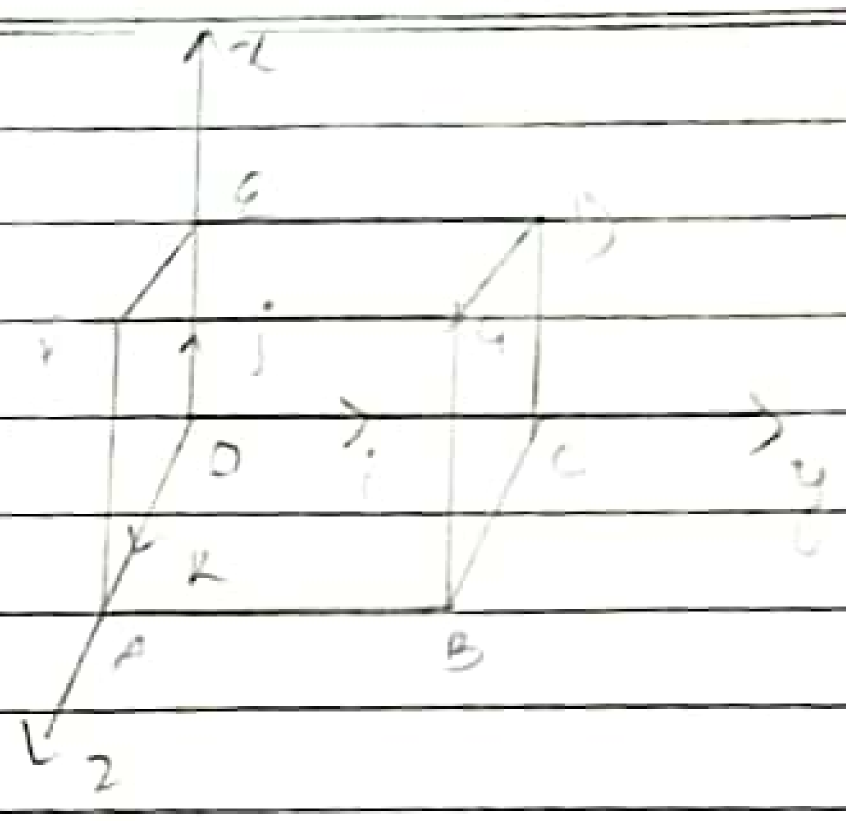
$$LHS = \iint_S f.n\,ds$$

S consists of all six faces

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Surface eqn $\vec{n}$

OABC (S_1). $z=0$ $-\vec{k}$



Surface	eqn	$\vec{n}$	ds
OABC (S_1)	$z=0$	$-\vec{k}$	$dx\,dy$
DEFG (S_2)	$z=a$	$\vec{k}$	$dx\,dy$
OAFE (S_3)	$x=0$	$-\vec{i}$	$dy\,dz$
BCDG (S_4)	$x=a$	$\vec{i}$	$dy\,dz$
OCDE (S_5)	$y=0$	$-\vec{j}$	$dx\,dz$
ABGF (S_6)	$y=a$	$\vec{j}$	$dx\,dz$

$$f=(x^3-yz)i-2x^2y\,j+2\vec{k}$$

$$S_1=\iint_{S_1}(-2)\,dx\,dy=-2\int_0^a\int_0^a dy\,dx=-2\int_0^a (x)_0^a\,dy=-2a\int_0^a dy$$

$$=-2a(y)_0^a=-2a^2$$

$$S_2=\iint_{S_2}2\,dx\,dy=2a^2$$

$$S_3=\iint_{S_3}f\cdot n\,ds=\int_0^a\int_0^a(-x^3+yz)\,dy\,dz=\int_0^a\int_0^a(yz)\,dy\,dz$$

$$=\int_0^a\int_0^a\left(\frac{y^2}{2}\right)z\,dz=\frac{a^2}{2}\int_0^a z\,dz=\frac{a^2}{2}\left(\frac{z^2}{2}\right)_0^a=\frac{a^4}{4}$$

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$$S_4 = \iint_{S_4} f\cdot n\,ds = \int_0^a\int_0^a (a^3 - yz)\,dy\,dz = \int_0^a \left(a^3 y - \frac{y^2 z}{2}\right)_0^a dz =$$

$$= \int_0^a \left(a^4 - \frac{a^2}{2} z\right) dz = \left(a^4 z - \frac{a^2}{2}\frac{z^2}{2}\right)_0^a = a^5 - \frac{a^4}{4}$$

$$S_5 = \iint_{S_5} 2x^2 y\,dx\,dz = \iint 0\,dx\,dz = 0$$

$$S_6 = \iint(-2x^2 y)\,dx\,dz = -2a\int_0^a\int_0^a x^2\,dx\,dz = -2a\int_0^a \left(\frac{x^3}{3}\right)_0^a dz$$

$$= -\frac{2a^4}{3}\int_0^a dz = \frac{-2a^4}{3}(z)_0^a = \frac{-2a^5}{3}$$

$\Rightarrow S_1 + S_2 + S_3 + S_4 + S_5 + S_6$

$\Rightarrow -2a^2 + 2a^2 + \frac{a^4}{4} + a^5 - \frac{a^4}{4} + 0 - \frac{2a^5}{3}$

$\Rightarrow \frac{a^5}{3}$

$\therefore$ LHS = RHS

The divergence theorem for given $\vec{F}$ is verified.

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Q3) S is the surface $x^2+y^2+z^2=a^2$ having lying above the xy plane & bounded by the circle
$C: x^2+y^2=1$

~~Or curve C, x =~~ The Stoke's Theorem is
$$\iint_S \text{curl}\, \vec{F}\, ds = \int_C F\, dr$$

Vector field $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$
Surface, $S = x^2+y^2+z^2 = K$

Now, evaluating the LHS,

$$\text{curl}\, \vec{F} = \nabla \times F$$
$$= i\left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right) - j\left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial z}\right) + k\left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)$$
$$= -i - j - K$$

Now integrating the curl F we get

$$\iint_S \text{curl}\, F\, ds = \iint_S (-i-j+k)\, \frac{\nabla F}{\|\nabla F\|}\, \|\nabla F\|\, dA$$

$$= \iint_S (-i-j-K)\, \nabla F\, dA \quad - \quad 1$$

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from the surface

$f = x^2 + y^2 + z^2 - 1$

$\nabla F = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$

Substitute above in equation (1) we get,

$$\iint_S \text{curl } F\, ds = \iint_S (-i\ j - k)(2x\vec{i} + 2y\vec{j} + 2z\vec{k})\, dA$$

$$= \iint_S (-2x - 2y + 2z)\, dA \quad - (2)$$

As we know, $S(s,t) = 2\cos t \sin s,\ 2\cos t \cos s,\ 2\sin t$

The eqn (2) will become

$$\iint_S \text{curl} F ds = \iint_S (-2x - 2y - 2z)\, dA$$

$$= \iint_S (-2 \cdot 2\cos t \sin s - 2 \cdot 2\cos t \cos s - 2 \cdot 2\sin t)\, ds\, dt$$

$$= -4\int_0^{2\pi}\int_0^{2\pi} (\cos t \sin s + \cos t \cos s + \sin t)\, dt\, ds$$

$$-4\iint \left(\sin t \sin s + \cos t \cos s - \cos t\right)_0^{2\pi} ds$$

$$= -4\int \left[\sin(0) + \cos(0) - (1-1)\right] ds$$

$$= -4(0) = 0$$

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Evaluating RHS

As we know, $r(t) = 2\cos t, \sqrt{2}\sin t, \sqrt{2}\sin t$

Differentiating $r(t)$, we get,

$r'(t) = -2\sin t, \sqrt{2}\cos t, \sqrt{2}\cos t$

$F(\vec{r}(t)) = \sqrt{2}\sin t\,\vec{i} + \sqrt{2}\sin t\,\vec{j} + 2\cos t\,\hat{j}$

Now find dot product of $F(r(t))$ & $r(t)$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 2\sqrt{2}(\cos^2 t - \sin^2 t) + 2\sin t\cos t$

$= 2\sqrt{2}(\cos 2t + \sin 2t)$

taking integral we get

$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \int_0^{2\pi} (2\sqrt{2}\cos 2t + \sin 2t)\,dt$

$= \left[2\sqrt{2}\frac{\sin 2t}{2} - \frac{\cos 2t}{2} \right]_0^{2\pi}$

$= \sqrt{2}(0) - \frac{1}{2}(1-1) = 0$

$\therefore$ LHS = RHS

Hence the stokes theorem for given $\vec{F}$ is verified

(iv) The point of intersection of $y = x$ & $y = x^2$
$x = x^2 \Rightarrow x - x^2 = 0 \Rightarrow x(1-x) = 0$
$x = 0, 1$

$\therefore y = (0, 1)$ & hence $(0,0)$ & $(1,1)$ are the point of intersections.

Green's theorem in a plane

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

The line integral

$$\int_C (2y^2\,dx + 3x\,dy)$$

$$\int_{OA} (2y^2\,dx + 3x\,dy) + \int_{AO} 2y^2\,dx + 3x\,dy$$

$I_1 + I_2$

Along OA, we have $y = x^2$
$\therefore dy = 2x\,dx$ & x varies from 0 to 1

$$I_1 = \int_{x=0}^{1} \cancel{2y^2} \cdot 2x^4\,dx + 3x\,2x\,(dx)$$

$$= \int_0^1 2x^4\,dx + \int_0^1 6x^2\,dx \Rightarrow 2\left[\frac{x^5}{5}\right]_0^1 + 6\left[\frac{x^3}{3}\right]$$

$\frac{2}{5} + \frac{6}{3} = \frac{2}{5} + 2 = \frac{12}{5} = I_1$

Along C_2, we have $y = x$

$\therefore dy = dx$

x varies from 1 to 0

$$I_2 = \int_0^1 2y^2\,dx + 3x\,dy = \int_0^1 2x^2\,dx + 3x\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^1 + 3\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{2}{3} + \frac{3}{2} = \frac{4+9}{6} = \frac{13}{6}$$

$$\therefore LHS = I_1 + I_2 = \frac{13}{6} + \frac{12}{5} = \frac{65+72}{30} = \frac{137}{30}$$

Now, $RHS = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

where $N = \cancel{x^2}\ 3x$ & $M = 2y^2$

$\frac{\partial N}{\partial x} = 3$ $\quad\quad$ $\frac{\partial M}{\partial y} = 4y$

R is the region bounded by $y = x^2$ & $y = x$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

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$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (3-4y)\, dx\, dy$$

$$= \int_{x=0}^{1} \left[3x - 4xy\right]_{y=x^2}^{x} dy$$

$$= \int_{x=0}^{1} (3x - 4x^2) - (3x - 4x^2)$$

$$= \int_{x=0}^{1} \left(3y - 2y^2\right)_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} \left[(3x - 2x^2) - (3x^2 - 2x^4)\right] dx$$

$$= \frac{137}{30}$$

$\therefore$ LHS = RHS

Hence Green's theorem for given $\vec{F}$ is verified.

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Q5) We have, $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$

$$\vec{F}\cdot d\vec{r} = (3x^2\hat{i} + (2xz-y)\hat{j} + z\hat{k})(dx\hat{i} + dy\hat{j} + dz\hat{k})$$
$$= 3x^2dx\ (2xz - y)dy + z\,dz$$

Here, $x^2 = 4y$ or $y = \frac{x^2}{4}$

$3x^3 = 8z$ or $\frac{3x^3}{8} = z$

For work done, $= \int_c \vec{F}\cdot d\vec{r} = \int 3x^2dx + \left[2x\left(\frac{3x^3}{8_4}\right) - \frac{x^2}{4}\right]dy$

$+ \frac{3x^3}{8}dz$

$$= \int_0^2 \left[3x^2 + 3x^4 - \frac{x^2}{4} + \frac{3x^3}{8}\right]dx$$

$$= 3\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^5}{5}\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2 + \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2$$

$$= \frac{\cancel{3}\times 2^3}{\cancel{3}} + \frac{3\times 2^5}{5} - \frac{1}{\cancel{4}}\times\frac{2^{\cancel{3}}}{3} + \frac{3}{\cancel{8}_2}\times\frac{\cancel{2^4}}{\cancel{4}}$$

$$= 2^3 + \frac{3\times 32}{5} - \frac{1}{3} + \frac{3}{2}$$

$$= 8 + \frac{96}{5} - \frac{1}{3} + \frac{1}{2} \text{ or } \frac{921}{30}\ J$$

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Q6) We have, $\vec{F} = \int_c (x^2 + xy)\,dx + (x^2+y^2)\,dy$

Comparing with standard eqn $M\,dx + N\,dy$:

$M = x^2 + xy$

$N = x^2 + y^2$

$\oint_c M\,dx + N\,dy =$

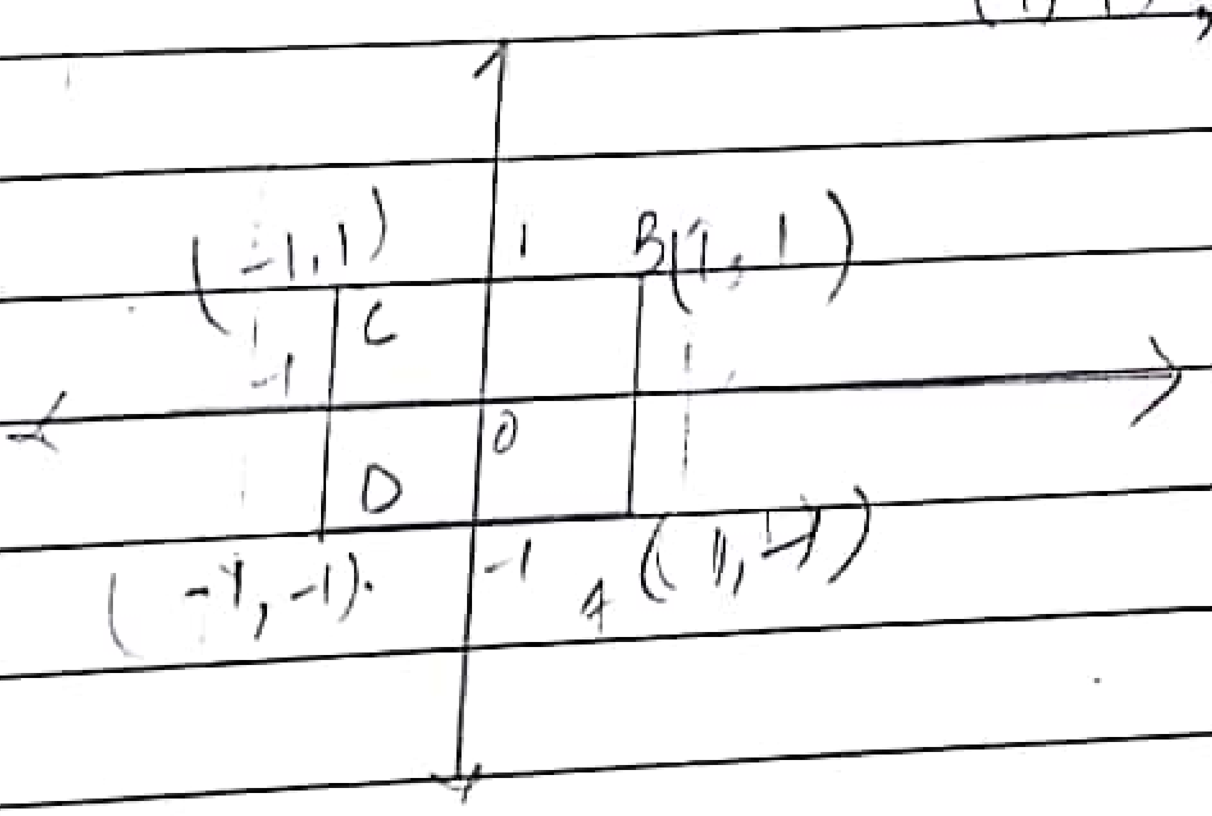
$\frac{\partial M}{\partial y} = x$

$\frac{\partial N}{\partial x} = 2x$

Green's theorem, $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \cancel{dydy}\; dy\,dx$

$\Rightarrow \iint_R (2x - x)\,dy\,dx$

The squared enclosed by C with the vertices $(1,-1), (1,1), (-1,1), (-1,-1)$



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$$I_D = \int_{-1}^{1}\int_{-1}^{1} (2x - x)\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\, dy\, dx = 0$$

$\because$ It is an odd function
the value will be 0. (zero)

Q7) Here we have, $\vec{F} = mx\hat{i} - 5y\hat{j} + 2z\hat{k}$
$\because$ $\vec{F}$ is a solenoidal vector.
$\nabla \cdot \vec{F} = 0$

$$\left(\frac{\partial}{\partial x}\hat{i} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (mx\hat{i} - 5y\hat{j} + 2z\hat{k}) = 0$$

$m - 5 + 2 = 0$
$m - 3 = 0 \Rightarrow \boxed{m = 3}$

Q8) We have, $\vec{F} = 2x\hat{i} + 4y\hat{j} + 8z\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x & 4y & 8z \end{vmatrix}$$

$$i(0-0) - j(0-0) + k(0-0) = 0$$

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Hence the given field vector is irrotational

Now, if ϕ, is the scalar potential then
$F = \vec{\nabla}\phi$

$$(2x\hat{i} + 4y\hat{j} + 8z\hat{k}) = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x} = 2x$$

$$\frac{\partial\phi}{\partial y} = 4y$$

$$\frac{\partial\phi}{\partial z} = 8z$$

But $\cancel{d\theta}$ $\partial\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$

$$\partial\phi = (2x)dx + (4y)dy + (8z)dz$$

By Integration

$$\phi = \cancel{2}\frac{x^2}{\cancel{2}} + \frac{\overset{2}{\cancel{4}}y^2}{\cancel{2}} + \frac{\overset{4}{\cancel{8}}z^2}{\cancel{2}}$$

$$\phi = x^2 + 2y^2 + 4z^2$$