3. CONVOLUTION THEOREM

The function $\int_0^t f_1(u) f_2(t-u)\,du$ is called the CONVOLUTION of the functions f_1 and f_2 and is denoted by $f_1 * f_2$.

It is easy to verify that $f_1 * f_2 = f_2 * f_1$

Let $f_1(t)$ and $f_2(t)$ be two functions of t and $L(f_1(t)) = \bar{f}_1(s)$ and $L(f_2(t)) = \bar{f}_2(s)$ then the convolution theorem states that

$$L^{-1}\left[\bar{f}_1(s)\,\bar{f}_2(s)\right] = \int_0^t f_1(u)\, f_2(t-u)\,du$$

$$= \int_0^t f_2(u)\, f_1(t-u)\,du.$$

Ex Use convolution to find

① $L^{-1}\left(\dfrac{1}{(s^2+a^2)^2}\right)$

② $L^{-1}\left(\dfrac{s}{(s^2+a^2)^3}\right)$

③ $L^{-1}\left(\dfrac{1}{s^2(s+1)^2}\right)$

④ $L^{-1}\left(\dfrac{1}{s(s+1)(s+2)}\right)$

Solution. ① Since $L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{\sin at}{a}$

Using convolution theorem here,

$$L^{-1}\left(\frac{1}{(s^2+a^2)^2}\right) = L^{-1}\left(\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right)$$

$$= \int_0^t \frac{\sin au}{a}\cdot\frac{\sin a(t-u)}{a}\,du.$$

$\cos(A-B)-\cos(A+B)$
$=2\sin A\sin B.$

$$= \frac{1}{2a^2}\int_0^t \left(\cos a(2u-t) - \cos at\right) du$$

$$= \frac{1}{2a^2}\left[\frac{\sin a(2u-t)}{2a} - u\cos at\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{a} - t\cos at\right].$$

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② Since $L\left(\frac{t\sin at}{2a}\right) = \frac{s}{(s^2+a^2)^2}$ and $L(\sin at) = \frac{a}{s^2+a^2}$

Applying convolution theorem,

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\cdot\frac{1}{s^2+a^2}\right) = \int_0^t \frac{u\sin au}{2a}\,\frac{1}{a}\sin a(t-u)\,du.$$

$$= \frac{1}{4a^2}\int_0^t u\left[\cos(2au-at) - \cos at\right] du.$$

$$= \frac{1}{4a^2}\left[u\cdot\frac{\sin(2au-at)}{2a}\Big|_0^t - \int_0^t 1\cdot\frac{\sin(2au-at)}{2a}\,du\right] - \frac{t^2}{8a^2}\cos at$$

$$= \frac{1}{4a^2}\left[\frac{t\sin at}{2a} + \frac{\cos(2au-at)}{4a^2}\Big|_0^t\right] - \frac{t^2}{8a^2}\cos at$$

$$= \frac{t\sin at}{8a^3} + \underbrace{\frac{\cos at - \cos at}{16a^4}}_{=0} - \frac{t^2}{8a^2}\cos at.$$

$$= \frac{t}{8a^3}\left(\sin at - at\cos at\right).$$

(3) Since $L(t) = \frac{1}{s^2}$ and $L(te^{-t}) = \frac{1}{(s+1)^2}$

By Convolution theorem,

$$L^{-1}\left(\frac{1}{s^2(s+1)^2}\right) = \int_0^t (t-u)\,u e^{-u}\,du$$

$$= \int_0^t (ut-u^2)e^{-u}\,du.$$

$$= \left[-(ut-u^2)e^{-u}\right]_0^t + \int_0^t (t-2u)e^{-u}\,du$$

$$= \int_0^t (t-2u)e^{-u}\,du.$$

$$= \left[-(t-2u)e^{-u}\right]_0^t + \int_0^t (-2)e^{-u}\,du.$$

$$= te^{-t} + t + \left[2e^{-u}\right]_0^t$$

$$= te^{-t} + t + 2e^{-t} - 2$$

$$= (t+2)e^{-t} + t - 2.$$

(4) Applying convolution theorem,

$$L^{-1}\left(\frac{1}{(s+1)(s+2)}\right) = \int_0^t e^{-u} \cdot e^{-2(t-u)}\, du$$

$$= \int_0^t e^{(u-2t)}\, du = \left[e^{u-2t}\right]_0^t$$

$$= e^{-t} - e^{-2t}$$

Now, $$L^{-1}\left(\frac{1}{s}\left(\frac{1}{(s+1)(s+2)}\right)\right) = \int_0^t (e^{-u} - e^{-2u})\, du.$$

$$= \left[-e^{-u} + \frac{e^{-2u}}{2}\right]_0^t = -e^{-t} + \frac{e^{-2t}}{2} + 1 - \frac{1}{2}$$

$$= \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2}.$$

Ex Find $f_1 * f_2$ if $f_1(t) = t$ and $f_2(t) = e^t$

Solution $$f_1 * f_2 = \int_0^t u \cdot e^{(t-u)}\, du$$

$$= -ue^{(t-u)} - \int \frac{e^{+(t-u)}}{-1}\, du$$

$$= \left[-ue^{(t-u)} - e^{(t-u)}\right]_0^t = -t - 1 + e^t$$

Ex Applying the convolution, find the inverse Laplace transform of the following

(1) $\dfrac{1}{s(s-4)^2}$ (2) $\dfrac{1}{(s+1)(s+9)^2}$ (3) $\dfrac{1}{(s^2+1)(s^2+9)}$

Solution (1) $L^{-1}\left(\dfrac{1}{s(s-4)^2}\right) = L^{-1}\left(\dfrac{1}{s}\cdot\dfrac{1}{(s-4)^2}\right)$

Now $L^{-1}\left(\dfrac{1}{s}\right) = 1$

$$L^{-1}\left(\dfrac{1}{(s-4)^2}\right) = e^{4t}\cdot t$$

By convolution theorem,

$$L^{-1}\left(\dfrac{1}{s(s-4)^2}\right) = \int_0^t u e^{4u}\cdot 1\, du$$

$$= \dfrac{ue^{4u}}{4} - \int_0^t 1\cdot\dfrac{e^{4u}}{4}\, du$$

$$= \left[\dfrac{ue^{4u}}{4} - \dfrac{e^{4u}}{16}\right]_0^t$$

$$= \dfrac{te^{4t}}{4} - \dfrac{e^{4t}}{16} + \dfrac{1}{16}$$

$$= \dfrac{1}{16}\left[1 + (4t-1)e^{4t}\right]$$

(2) $$L^{-1}\left(\frac{1}{(s+1)(s+9)^2}\right) = L^{-1}\left(\frac{1}{(s+1)}\cdot\frac{1}{(s+9)^2}\right)$$

Now $$L^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

$$L^{-1}\left(\frac{1}{(s+9)^2}\right) = e^{-9t}\cdot t$$

By convolution theorem,

$$L^{-1}\left(\frac{1}{(s+1)(s+9)^2}\right) = \int_0^t u e^{-9u}\cdot e^{-(t-u)}\,du$$

$$= e^{-t}\int_0^t u e^{-8u}\,du$$

$$= e^{-t}\left[\frac{ue^{-8u}}{-8} - \frac{e^{-8u}}{64}\right]_0^t$$

$$= e^{-t}\left[\frac{te^{-8t}}{-8} - \frac{e^{-8t}}{64} + \frac{1}{64}\right]$$

$$= \frac{e^{-t}}{64}\left[1-(8t+1)e^{-8t}\right].$$

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(3). $$L^{-1}\left(\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right)$$

Now, $$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{\sin 3t}{3}$$

By convolution theorem,

$$L^{-1}\left(\frac{1}{(s^2+1)(s^2+9)}\right) = \frac{1}{3}\int_0^t \sin u \sin 3(t-u)\, du$$

$$= \frac{-1}{6}\int_0^t \cos(3t-2u) - \cos(4u-3t)\, du$$

$$= \frac{-1}{6}\left[\frac{\sin(3t-2u)}{-2} - \frac{\sin(4u-3t)}{4}\right]_0^t$$

$$= \frac{1}{12}\left[\sin t - \sin 3t\right] + \frac{1}{24}\left[\sin t + \sin 3t\right]$$

$$= \frac{\sin t}{8} - \frac{\sin 3t}{24}$$