DISCRETE MATHEMATICS

Discrete Mathematics is not a type of continuous mathematics like Calculus, which we spend most of our school years studying. Continuous mathematics studies objects (e.g. real numbers) that vary smoothly. Meanwhile, discrete mathematics studies objects such as integers, graphs & statements in logic that do not vary smoothly.

Why do we need to start studying D.M.?

→ Computer Science is almost built on DM. So understanding DM will help you understand all the fundamentals of computing. It will let you think ~~as~~ like a computer & know how everything works.

→ If you had studied DM. you would have understood algorithms more easily. Since it covers probability, trees, graphs, logic, mathematical thinking it has a vital role in learning algorithms.

→ All of a computer's data is represented as bits (zeros & ones). Computers make calculations by modifying these bits in accordance with the laws of Boolean Algebra. which form the basis of all digital circuits (graph representation). Low-level programming languages rely directly on logical operators (and, not and or). Software Developers using high-level languages will often work to optimize their code by minimizing the no. of low-level operations and may even operate directly on bits. Boolean logic are also use to control program flow, i.e., which instructions are executed under certain conditions.

→ Propositional logic can be used to reason about ~~their~~ correctness of the desired result of the code.

→ Graphs are powerful data structures which are used to model relationships & answer questions about said data : for ex., your navigation app uses a graph search algo to find the fastest route from your house to your workplace.

UNIT-I

Set :-
A set is a collection of well defined distinct objects. The object in a set is called the elements or member of the set. We use capital letters such as A, B, C... to represent set & lower case letters to represent elements of the set.

For a set A we write $x \in A$ if x is an element of A, while $y \notin A$ means y is not a member of A.

Roaster form :- $A = \{1, 3, 5, 7, 9\}$

Set Builder form :- $A = \{x \mid x \text{ is a positive odd integer} < 10\}$

Subset :-
We say A is a subset of B & write $A \subseteq B$ or $B \supseteq A$, if every element of A is an element of B. We could say that B is a superset of A.

e.g. $A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5, \ldots\}$

$\Rightarrow A \subseteq B$.

Power set :-
The power set of a set A is the set of all subsets of A. It is usually denoted by P(A).

Note :- If a set has n elements then its power set P(A) will have 2^n elements.

e.g. $A = \{1, 2, 3\}$

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$

Operations on set :-

Union :- If A & B are two non-empty sets then their union is defined as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Intersection :- For two nonempty sets A & B, their intersection is defined as

$$A \cap B = \{x \mid x \in A \ \& \ x \in B\}$$

e.g. $A = \{a, b, c, d\}$

$B = \{c, d, e, f\}$

$A \cup B = \{a, b, c, d, e, f\}$

$A \cap B = \{c, d\}$

Disjoint sets :- If $A \cap B = \phi$, i.e., sets A & B have no element in common then A & B are called disjoint sets.

e.g. $A = \{1, 2, 3, 4, 5\}$

$B = \{4, 5, 6, 7\}$ then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

$C = \{7, 8\}$ $\quad A \cap B = \{4, 5\}$

$B \cap C = \{7\}$

$A \cap C = \phi$

$\Rightarrow$ A & C are disjoint sets

Note :- $A \cap B \subseteq A \subseteq A \cup B$

llly $A \cap B \subseteq B \subseteq A \cup B$.

Difference of sets :- for two non-empty sets A & B, we have $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

e.g. :- $A = \{a, b, c, d\}$

$B = \{c, d, e, f\}$

$A - B = \{a, b\}$, $B - A = \{e, f\}$.

Complement of a set :- If U is the universal set (the super set which contains all sets in consideration) and A is any non-empty set then the complement of A is denoted by $\bar{A}$ or A^c or A' with respect to U, given by $A' = \bar{A} = A^c = \{x \mid x \in U \ \& \ x \notin A\}$

e.g. $U = \mathbb{R}$ (set of real no.)

$A = \{2, -5, 9\}$

$\bar{A} = U - A = \mathbb{R} - \{2, -5, 9\}$

Laws of Set Theory :—

(a) Law of double complement :— $(\bar{\bar{A}}) = A$

(b) De Morgan's law :— $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$ &
$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

(c) Commutative Law :- $A \cup B = B \cup A$ &
$A \cap B = B \cap A$.

(d) Associative Law :- $A \cup (B \cup C) = (A \cup B) \cup C$
& $A \cap (B \cap C) = (A \cap B) \cap C$

(e) Distributive Law :- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
& $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) Idempotent Law :- $A \cup A = A$ &
$A \cap A = A$

(5) Identity Law :- $A \cup \phi = A$ & $A \cap U = A$

(6) Inverse Law :- $A \cup \bar{A} = U$ & $A \cap \bar{A} = \phi$

(7) Domination Law :- $A \cup U = U$ & $A \cap \phi = \phi$

(8) Absorption Law :- $A \cup (A \cap B) = A$ &
$A \cap (A \cup B) = A$.

Symmetric Difference of sets :- If A & B two non-empty sets, then their symmetric difference, denoted by $A \oplus B$ (or $A \Delta B$) is defined as

$$A \oplus B = (A - B) \cup (B - A)$$
$$= (A \cup B) - (A \cap B)$$
$$= A \Delta B$$

or $A \oplus B = \{x \mid x \in (A \cup B) \;\&\; x \notin (A \cap B)\}$

e.g. $A = \{a, b, d, c\}$ & $B = \{c, d, f, g\}$ then $A \oplus B = \{a, b, f, g\}$

e.g. $A = \{1,2,3,4,5\}$, $B = \{3,4,5,6,7\}$, $C = \{7,8,9\}$

$A \Delta B = \{1,2,6,7\}$

$B \Delta C = \{3,4,5,6,8,9\}$.

Principle of Inclusion & Exclusion :-

If A & B be sets with cardinalities $|A|$ & $|B|$ then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Pf. The no. of common elements in A & B is $|A \cap B|$

Each of these elements is counted twice in $|A| + |B|$, once in $|A|$ & once in $|B|$

This should be adjusted by subtracting the term $|A \cap B|$ from $|A| + |B|$

Hence

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Note :- Using distributive law, we can extend the above result for three sets A, B, C so that

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

for $|A \cup B \cup C| = |(A \cup B) \cup C|$

$= |A \cup B| + |C| - |(A \cup B) \cap C)$

$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$

$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

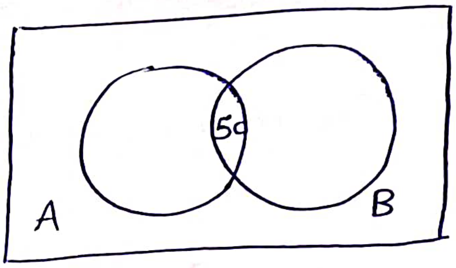
Duality :- If S be any identity involving sets and operations (e.g. complement, intersection & union etc.) and a new set S^* is obtained by replacing $\cap$ by $\cup$, $\cup$ by $\cap$, ϕ by U & U by ϕ in S, then the statement is true & is called the dual of the statement.

e.g. The dual of $A \cap (B \cup A) = A$ is
$A \cup (B \cap A) = A$

Ex. In a survey conducted on 250 persons, it has found that 180 drink tea & 70 drink coffee & 50 take both. How many drink atleast one & how many drink neither?

As

$A \rightarrow$ tea drinkers
$B \rightarrow$ coffee "



$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 180 + 70 - 50 = 200$$

~~only drink tea~~

Atleast one $\rightarrow$ 200
Total = 280
Neither = 250 − 200 = 50.

Ex. Among 100 students, 32 study Maths, 20 study Physics, 45 study Biology, 15 study Maths & Bio, 7 study Maths & Physics, 10 study Physics & Bio & 30 do not study any of the three subjects.

(i) Find the number of students studying all three subjects.

(ii) Find the no. of students studying exactly one of the three subjects.

$\underline{\text{Sol}^n}$ $\quad n(M) = 32$, $n(P) = 20$, $n(B) = 45$

Given,

$n(M \cap B) = 15$, $n(M \cap P) = 7$, $n(P \cap B) = 10$

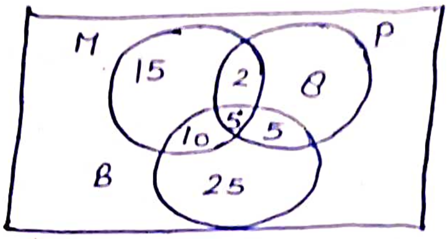
30 students do not study any of three subjects.

Now,

$$n(M \cup P \cup B) = n(M) + n(P) + n(B) - n(M \cap P) - n(P \cap B) - n(M \cap B) + n(M \cap P \cap B)$$

$$70 = 32 + 20 + 45 - 15 - 15 - 7 - 10 + n(M \cap P \cap B)$$

$$\Rightarrow n(M \cap P \cap B) = 5.$$



No. of students studying exactly one of the three subjects

$$= 15 + 8 + 25 = 48.$$

\# $\underline{\text{Cartesian Product of sets}}$:- If A & B are two non-empty sets, their Cartesian Product, denoted by $A \times B$, is defined as

$$A \times B = \{(a,b) \mid a \in A \ \& \ b \in B\}$$

where (a,b) is ordered pair. (a,b) & (b,a) are different.

$\underline{\text{e.g.}}$ $\quad A = \{1, a\}$, $B = \{1, 2, 3\}$

$$A \times B = \{(1,1), (1,2), (1,3), (a,1), (a,2), (a,3)\}$$

$$B \times A = \{(1,1), (1,a), (2,1), (2,a), (3,1), (3,a)\}.$$

$\underline{\text{Q.}}$ Let A, B, C be any three sets then prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$\underline{\text{Sol}^n}$ $\quad$ Suppose $(x, y) \in A \times (B \cap C)$

$$\Rightarrow x \in A \ \& \ y \in B \cap C$$

$$\Rightarrow x \in A \ \& \ (y \in B \ \& \ y \in C)$$

$$\Rightarrow (x \in A \ \& \ y \in B) \ \& \ (x \in A \ \& \ y \in C)$$

$\Rightarrow (x,y) \in A \times B$ & $(x,y) \in A \times C$

$\Rightarrow (x,y) \in (A \times B) \cap (A \times C)$

$\Rightarrow A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. — ①

Again suppose.

$(x,y) \in (A \times B) \cap (A \times C)$

$\Rightarrow (x,y) \in A \times B$ & $(x,y) \in A \times C$

$\Rightarrow (x \in A$ & $y \in B)$ & $(x \in A$ & $y \in C)$

$\Rightarrow (x \in A)$ & $(y \in B$ & $y \in C)$

$\Rightarrow x \in A$ & $(y \in B \cap C)$

$\Rightarrow (x,y) \in A \times (B \cap C)$

$\Rightarrow (A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$. — ②

from ① & ②

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

Q. If A, B, C, D are any four sets then prove that

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$$

Q. If A & B are any two sets then prove that

$$A \cap (B - A) = \phi$$

Ex. How many integers between 1 & 460 are divisible by 3 but not by ~~8~~ 5?

S. No. of integers b/w 1 & 460 which are divisible by 3 $\rightarrow \left[\frac{460}{3}\right] = 156$

No. of integers b/w 1 & 460 which are divisible by 3 & 5 $\longrightarrow \left[\frac{460}{3 \times 5}\right] = 31$

Hence the no. of integers b/w 1 & 460 divisible by 3 but not 5 $\longrightarrow 156 - 31 = \underline{125}$

Relations

If A & B are two non-empty sets then the relation R from A to B is the subset of the product set $A \times B$, defined by

$$R = \{(a,b) \mid a \in A, b \in B \text{ where } aRb\}$$

e.g. $A = \{1, 2, 4, 9\}$

$B = \{3, 5, 11\}$

$R_1 \equiv$ 'is greater than'

$R_2 \equiv$ 'is less than'

$R_1 = \{(4,3), (9,3), (9,5)\}$, $R_2 = \{(1,3), (1,5), (1,11), (2,3), (2,5), (2,11), (4,5), (4,11), (9,11)\}$

Types of Relations :-

① Reflexive :- A relation R on a set A is called reflexive if aRa, $\forall a \in A$, that is, every element is related to itself.

e.g. $A = \{2, 3, 4\}$.

$R \equiv$ 'is equal to'

$R = \{(2,2), (3,3), (4,4)\}$

② Symmetric :- If R is a relation on a set R such that $(a,b) \in R \Rightarrow (b,a) \in R$, $\forall a, b \in R$, then R is called symmetric.

e.g. $A = \{1, 2\}$

$R = \{(1,1), (2,2), (1,2), (2,1)\} \longrightarrow$ Symm. relation

③ Asymmetric :- A relation which is not symmetric is called asymmetric. Thus, if xRy does not imply yRx then R is asymmetric relation.

④ Antisymmetric :- If A is a non-empty set & R is a relation on A such that $(a,b) \in R$ & $(b,a) \in R$ implies that $a = b$, then R is called antisymmetric relⁿ on A.

e.g. Relation R defined on Z as aRb if $a \le b$ is antisymmetric relⁿ, since $a \le b$ & $b \le a \Rightarrow a = b$.

⑤ Transitive :- A relation R defined on a set A is called transitive if $(a,b) \in R$ and $(b,c) \in R$ implies that $(a,c) \in R$.

e.g. $A = \{1, 2, 7, 9\}$

$R = \{(1,7), (1,1), (7,1), (7,9), (9,1), (1,9)\}$ is transitive.

⑥ Equivalence Relation :- A relation which is reflexive, symmetric and transitive in a set S is called Equivalence relation on S.

e.g. $S = \{1, 2, 3, \ldots 12\}$

$R = \{(x,y) \mid (x,y) \in R$ is $(x-y)$ is a multiple of $5\}$

Ref. :- for any $a \in S$

$(a,a) \in R$

$\because a - a = 0$ which is divisible by 5.

Symm :- If $(a,b) \in R$

$\Rightarrow a-b$ is a multiple of 5

$\Rightarrow -(b-a)$ " " "

$\Rightarrow (b-a)$ is also multiple of 5

Hence $(b,a) \in R$

$\Rightarrow R$ is symmetric.

Transitive :- If (a,b) & $(b,c) \in R$

$\Rightarrow (a-b)$ & $(b-c)$ are ~~a~~ multiples of 5

$\Rightarrow (a - \cancel{b} + \cancel{b} - c)$ also ~~the~~ multiple of 5

$\Rightarrow (a-c)$ is also multiple of 5

$\Rightarrow (a,c) \in R$

$\Rightarrow R$ is transitive.

$\Rightarrow R$ is an equivalence relation on S.

Equivalence Class :- If R is an equivalence relation on a non-empty set S, then the equivalence class of an element $a \in S$, denoted by $[a]$, is defined as

$$[a] = \{x : aRx, x \in S\}.$$

e.g. find the equivalence classes of 1, 2, 5 in above ex.

$$[1] = \{1, 6, 11\} = [6] = [11]$$
$$[2] = \{2, 7, 12\} = [7] = [12]$$
$$[5] = \{5, 10\} = [10].$$

⑦ Partial Order Relation :- Let R be a relation on a set A then R is said to partial order relation on it, if it is reflexive, antisymmetric & transitive.

It is denoted by $\leq$.

ⓐ Ref - $x \leq x \ \forall x \in A$

ⓑ AntiSymm - $x \leq y$ & $y \leq x \Rightarrow x = y$

ⓒ Transitivity - $x \leq y$, $y \leq z \Rightarrow x \leq z$.

Q. Let $A = \{2, 3, 4\}$ & $B = \{3, 4, 5, 6, 7\}$. Assume a relation R from A to B such that $(x, y) \in R$ when x divides y. Determine its domain & range.

S. $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$

Domain $= \{2, 3, 4\}$

Range $= \{4, 6, 3\}$

Q. Show that the relation $R = \{(a, b) \mid a, b \in \mathbb{Z}$ & $a-b$ is divisible by 3$\}$ is an equivalence relation.

S. $R = \{(a, b) \mid a, b \in \mathbb{Z}$ & $(a-b)$ is divisible by 3$\}$.

① Reflexive :- ~~$\forall a \in \mathbb{Z}$ $a - a = 0$ whi~~ (Do it yourself)

Q. Let R be a transitive and reflexive relation on A. Let T be a relation on A s.t. (a, b) is in T iff both (a, b) & (b, a) are in R. Show that T is an equivalence relation.

Sol^n ① Reflexive :- $\forall a \in A$

according to the def. of R.

(a, a) is in T iff both (a, a) & (a, a) are in R.

$\Rightarrow (a,a) \in \cancel{R}\, T$

$\Rightarrow$ Reflexive

② Symmetric :- for $a, b \in A$

(a, b) is in T iff both (a, b) & (b, a) are in R ~~& R~~ $\Rightarrow$ ~~I~~ for $(b,a) \in T$ iff (b, a) & (a, b) in R

$\Rightarrow$ if $(a,b) \in \cancel{R}\, T$ then $(b,a) \in \cancel{R}\, T$

Hence symmetric.

③ Transitive :- If (a, b) & $(b,c) \in T$

$(a,b) \in T \Updownarrow (a,b)$ & $(b,a) \in R$

$(b,c) \in T \Updownarrow (b,c)$ & $(c,b) \in R$

$\Rightarrow (a,b), (b,c) \in R$

& R is transitive

$\Rightarrow (a,c) \in R$

$\Rightarrow (c,b)$ & $(b,a) \in R$

& R is transitive

$\Rightarrow (c,a) \in R$

Acc. to def. (a, c) & $(c,a) \in R$ then $(a,c) \in T$.

Hence transitive

$\Rightarrow$ T is an equivalence relation

Q. If R is a relation on the set of integers such that $(a,b) \in R$ iff $3a+4b = 7n$ for some integer n. Prove that R is an equivalence relation.

Ex. Let $A = \{1, 2, 3\}$, $R = \{(1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$ this rel^n is partial order relation.

Partial order set (POSET) :-
The set A, with the partial order relation R is called POSET.

Ex. In the set of real numbers, the relation of 'less than or equal to" is partial order relation. So, $(R, \leq)$ is POSET (Show).

Sol^n (i) Reflexive :- $\forall\ a \in \mathbb{R}$

for $a \leq a$ (satisfies)

$\Rightarrow (a,a) \in R$

$\Rightarrow$ Reflexive

(ii) Anti Symmetric :- for $a, b \in \mathbb{R}$

if $(a,b) \in R$ then $a \leq b$

but $b \nleq a$ so $(b,a) \notin R$

but if $a = b$ then

if $(a,b) \in R$ then $(b,a) \in R$

$\Rightarrow$ Antisymmetric

(iii) Transitive :- for (a,b) & $(b,c) \in R$

$\Rightarrow a \leq b$ & $b \leq c$

$\Rightarrow a \leq b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a,c) \in R$

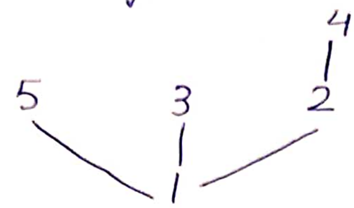
$\Rightarrow$ Transitive.

$\Rightarrow (R, \leq)$ is POSET.

LATTICE :-

Maximal Element :- In a POSET, if an element is not related to any other element.

e.g.



3, 4 & 5 are Maximal elements

Minimal Element :- In a POSET, if no element is related to an element

e.g. 1 is minimal element in above example.

Theorem :- A finite non-empty POSET (P, R) has at least one maximal element & at least one minimal element.

Pf. (P, R) be a finite non-empty POSET

let $P_1 \in R$ & ~~P_1 is not~~ let (P, R) has no maximal element

$\Rightarrow \exists P_2 \in P$ such that

$P_1 R P_2$

$\because$ No maximal element in (P, R)

then $\exists P_3 \in P$ such that

$P_2 R P_3$

and so on.

But (P, R) is a finite set. Hence there will an element $P_n \in P$ s.t. there will be No $Q \in P$ s.t.

$P_n R Q$.

$\Rightarrow P_n$ is the maximal element

$\Rightarrow$ (P, R) has atleast one maximal elt.

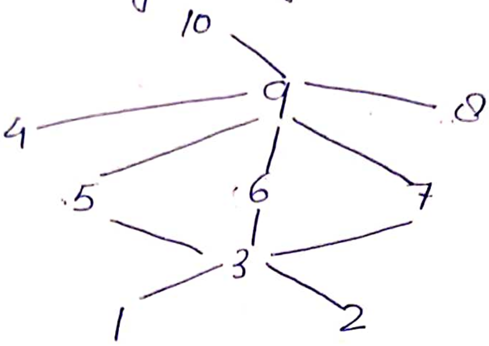
Same goes for minimal elt.

Lower Bound and Upper Bound :-

Upper Bound :- Let (P, R) be a POSET and B be a subset of P. An element $x \in A$ is a upper bound of B if y is related to x for every $y \in B$.

Lower Bound :- An elt $x \in A$ is a lower bound of B if x is related to y, $\forall y \in B$.

Ex.



Let $B = \{5, 6, 7\}$

$U(B)$ = upper bound of $B = \{9, 10\}$

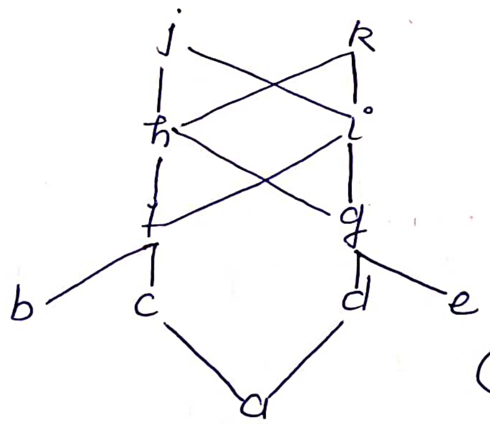
$L(B)$ lower " " " $= \{1, 2, 3\}$

Let $B = \{5, 6, 8\}$

$U(B) = \{9, 10\}$, $L\{B\}$ = None

Lattice :- A POSET (P, R) is said to be a lattice if every two elements in the set L has a unique least upper bound and unique greatest lower bound.

e.g.



1

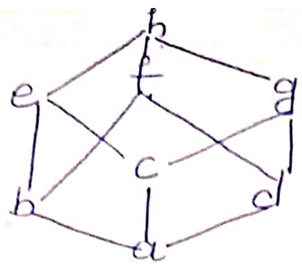
let $B = \{h, i\}$

$U(B) = \{j, k\}$.

$L(B) = \{f, g, b, c, d, e, a\}$

greatest lower bound $= \{f, g\}$, least upper bound $= \{j, k\}$

Hence 1 is not lattice.



let $B = \{e, g\}$

$L(B) = \{a, b, c, d\}$

$U(B) = \{h\}$

$\Rightarrow$ greatest $L(B) = \{c\}$

lowest $U(B) = \{h\}$

$\Rightarrow$ Lattice.

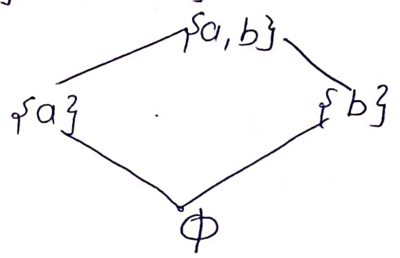
Q. Let S be any non-empty finite set & P(S) be its power set. Show that $(P(S), \subseteq)$ is lattice.

S. let $S = \{a, b\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\phi \subseteq \{a\}, \{b\}, \{a, b\}$

$\{b\}, \{a\} \subseteq \{a, b\}$



for any two elements lub & glb are unique

Hence P(S) is lattice

PIEGON HOLE PRINCIPLE :-

If $(N+1)$ or more objects are placed into N boxes then there is atleast one box containing two or more objects.

Ex. If 6 colour are used to paint 37 home. Show that at least 7 home of them will be of same colour

Solⁿ

$$\frac{37}{6} \approx 6$$

⇒ 6 home each of 6 colour

but remainder 1 will be a colour from the 6

⇒ 7 home may have a same cycle.

Generalized Pigeon Hole Principle :-

If n pigeon hole are occupied by $kn+1$ or more pigeons then atleast one pigeon hole is occupied by $(k+1)$ or more pigeon.

Ex. Find the minimum no. of teachers in a college to be sure that four of them are born in the same month.

Sol

$n = 12$

$k+1 = 4 \Rightarrow k = 3$

$kn+1 = 3 \times 12 + 1 = 37$

⇒ 37 teachers.

Ex. A box contain 10 blues balls, 20 red balls, 8 green balls, 15 yellow balls and 25 white balls. How many balls must we choose to ensure that we have 12 balls of the same colour.

Sol

$k+1 = 12 \Rightarrow k = 11$

$n = 5$ different colour balls

$kn+1 = 56.$

Ex. Prove that among 1,00,000 people there are two who are born on same time.

Sol in an hour, $\frac{1,00,000}{24} = 4166.66$

$\lceil 4166.66 \rceil \rightarrow$ ceiling ~~greatest integer~~ funⁿ $= 4167$

in a minute $\rightarrow \lceil \frac{4167}{60} \rceil = 70$ & In a second $\rightarrow \lceil \frac{70}{60} \rceil = \boxed{2}$

The Mathematical Logic :-

logic is the study of reasoning whether the reasoning is correct or not. It provides the rules & technique for determining the validity of argument or reasoning. It has practical applications to the design of computing machines, to the comupter programming languges & AI in particular

Logical Constants :- There are two logical constants 'True' denoted by 'T' (or '1') and 'False' denoted by 'F' (or '0')

Proposition :- Proposition is a declarative statement which is either universally true or universally false but not both.

e.g. 'New Delhi is the captial of India' → Proposition.
'Eight is prime' → proposition.
'3 > 2' → proposition
'How bad it is' → Not a proposition
'Watch this movie' → " " "
'Happy Birthday' → " " "

Logical Connectives :- While talking to someone we normally use simple sentences but more often we use complicated statements which are formed by joining two or more sentences by the connectors like 'and', 'or', 'if', 'then' etc. These words or phrases are called logical connectives. Simple statements connected by such logical connectives are known as compound statements.

Various connectives are as follows :-

① Disjunction ('∨') or (Inclusive OR) :- The disjunction of two propositions p & q is a compound statement written as '$p \vee q$'. ~~or as~~

Truth table for '$p \vee q$'

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(either this, or that, or both)

e.g. ① p: Kanpur is in U.P.
or q: $3+5=8$
$p \vee q$: is true

② p: Kanpur is in Orissa
or q: $3+5=9$
$p \vee q$: is false

② Conjunction ('∧') or 'AND' :- The conjunction of two propositions p & q is a new compound statement denoted by '$p \wedge q$' or 'p & q'.

Truth table for '$p \wedge q$'

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

e.g. p: Agra is in Kerala
& q: ~~q~~ $2+3=5$
$p \wedge q$: is false

③ Exclusive OR (XOR) :– '⊕' :– It is denoted by $p \oplus q$. If p & q are true, $p \oplus q$ is false & see others in truth table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

(either this, or that, not both)

④ Negation ('Not' or '~') :– If p is a proposition, its negation is denoted by '~p' or by 'not p'. Thus, if p is true then ~p is false & if p is false then ~p is true.

e.g. p: Ramesh is a good man.
~p: Ramesh is not a good man.

Truth table

p	$\sim p$
T	F
F	T

Laws of Algebra of Propositions :–

① Idempotent Law :–

ⓐ $p \vee p \equiv p$ ⓑ $p \wedge p \equiv p$

② Associative Law :–

ⓐ $(p \vee q) \vee r \equiv p \vee (q \vee r)$

ⓑ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

③ Commutative law :–

ⓐ $p \vee q \equiv q \vee p$

ⓑ $p \wedge q \equiv q \wedge p$

(4) Distributive Law :- (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

(5) Identity Law :- (a) $p \vee p \equiv p$ & $p \vee T \equiv T$

(b) $p \wedge p \equiv p$ & $p \wedge F \equiv F$.

(6) Complement Law :- (a) $p \vee \sim p \equiv T$, (b) $p \wedge \sim p \equiv F$

(7) Involution Law :- $\sim\sim p \equiv p$

(8) De Morgan's Law :- (a) $\sim (p \vee q) = \sim p \wedge \sim q$

(b) $\sim (p \wedge q) = \sim p \vee \sim q$

(9) Absorption Law :- (a) $p \vee (p \wedge q) \equiv p$

(b) $p \wedge (p \vee q) = p$.

Remark :- In order to avoid an ~~a~~ excessive number of ~~para~~ parentheses, we sometimes adopt an order of precedence for the logical connectives. Specifically

$\sim$ has precedence over $\wedge$ which has precedence over $\vee$

for example -

$\sim p \wedge q$ means $(\sim p) \wedge q$ and not $\sim (p \wedge q)$.

~~Tautogi~~ Tautologies :- Some propositions $P(p, q, \ldots)$ contain only T in the last column of their truth tables or, in other words, they are true for any truth values of their variables. Such propositions are called tautologies.

Contradictions :- $P(p, q, \ldots)$ is called a contradiction if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables.

e.g. "p or not p", i.e. $p \vee \sim p$ is a tautology.
& "p and not p", i.e. $p \wedge \sim p$ is a contradiction.

p	~p	p V ~p
T	F	T
F	T	T

Tautology

p	~p	p ∧ ~p
T	F	F
F	T	F

contradiction

Note :- The negation of a tautology is a contradiction since it is always false, and the negation of a contradiction is a tautology since it is always true.

Theorem :(Principle of Substitution) :-

If $P(p, q, \ldots)$ is a tautology, then $P(P_1, P_2, \ldots)$ is a tautology for any propositions $P_1, P_2, \ldots$.

Logical Equivalence :- Two propositions $P(p, q, \ldots)$ & $Q(p, q, \ldots)$ are said to be logically equivalent, or simply equivalent or equal, denoted by

$$P(p, q, \ldots) \equiv Q(p, q, \ldots)$$

if they have identical truth tables.

e.g. $\sim(p \wedge q) \equiv \sim p \vee \sim q$

L.H.S. $\rightarrow \sim(p \wedge q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

R.H.S. $\rightarrow \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

$$\Rightarrow \sim(p \wedge q) \equiv \sim p \vee \sim q.$$

Conditional & Biconditional Statements :-

Many statements, particularly in Mathematics, are of the form "If p then q." Such statements ~~as~~ are called conditional statements ~~are a~~ and are denoted by.

$$p \rightarrow q$$

The conditional $p \rightarrow q$ is frequentely read "p implies q" or "p only if q."

Another common statement is of the form "p if and only if q." Such statements are called biconditional statements and are denoted by

$$p \leftrightarrow q.$$

Truth table of '$p \to q$'

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth table of '$p \leftrightarrow q$'

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note :- $p \to q \equiv \sim p \vee q$

Arguments :- An argument is an assertion that a given set of propositions $P_1, P_2, \ldots P_n$, called premises, yields another proposition Q, called the conclusion. Such an argument is denoted by

$$P_1, P_2, \ldots P_n \vdash Q$$

Validity / Valid Argument :- An argument $P_1, P_2, \ldots P_n \vdash Q$ is said to be valid if Q is true whenever all the premises $P_1, P_2, \ldots P_n$ are true.

Fallacy :- An argument which is not valid is called fallacy.

e.g. $p, p \to q \vdash q$ (law of detachment)

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In row 1, p is true & $p \to q$ is true.
q is also true which is the conclusion.
There is no other row for check validity.
Hence the given argument is valid.

e.g. $p \to q, q \vdash p$

In row 1, $p \to q$ & $q \longrightarrow T$, p is also true
In row 3, $p \to q$ & $q \longrightarrow T$ but p is false
$\Rightarrow$ given argument is fallacy.

Theorem :- The argument $P_1, P_2, \dots P_n \vdash Q$ is valid iff the proposition $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \to Q$ is a tautology.

Ex "If p implies q & q implies r, then p implies r

$$[(p \to q) \wedge (q \to r)] \longrightarrow (p \to r) \equiv P$$

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p\to q)\wedge(q \to r)$	P
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	~~F~~ T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	F
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Check

$p \to q, q \to r \vdash p \to r$ (law of syllogism)

row-1 → valid
row-5 → "
row-7 → "
row-8 → "

The above argument is valid.

from above theorem we can say that

if P is tautology then $(p \to q), (q \to r) \vdash (p \to r)$ is valid &

if $(p \to q), (q \to r) \vdash (p \to r)$ is valid then P is tautology.

Ex.

S_1: If a man is a bachelor, he is unhappy
S_2: If a man is unhappy, he dies young.

S: Bachelors die young.

S_1 is $p \to q$
S_2 is $q \to r$

where p: He is bachelor
q: He is unhappy
r: He dies young

$(p \to q), (q \to r) \vdash (p \to r)$

from above example, this argument is valid.

Propositional functions & Quantifiers :-

Predicates or propositional funⁿ

Propositional logic is not enough to express the meaning of all statements in Mathematics & natural language.

→ A predicate $p(x)$ is a sentence that contains a finite number of variables & becomes a proposition when specific values are substituted for the variable

where $p(x)$ is a propositional fu^n
& x is a predicate variable.

Ex. find the truth set (domain) for each $p(x)$ defined on the set $\mathbb{N}$ of +ve integers.

(a) $p(x)$: "$x+2>7$"

Truth set : $\{6, 7, 8, \dots\}$.

(b) $p(x)$: "$x+5<3$"

Truth set : ϕ

(c) $p(x)$: "$x+5>1$"

truth set: $\mathbb{N}$

Remark :- The above example shows that if $p(x)$ is propositional function defined on a set A then $p(x)$ could be true for all $x \in A$, for some $x \in A$, or for no $x \in A$. These type of words (some/~~or~~/all/no) indicate how frequently a certain statement is true, are called quantifiers. There are two types of them —

(a) Universal Quantifiers
(b) Existential "

① Universal Quantifier :-

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\forall x \in A)\, p(x) \quad \text{or} \quad \forall x\, p(x)$$

which reads "For every x in A, $p(x)$ is true statement" or, simply, "For all x, $p(x)$."

The symbol "$\forall$" is called the universal quantifier.

e.g. If $\{x \mid x \in A, p(x)\} = A$
then $\forall x\, p(x)$ is true;
otherwise $\forall x\, p(x)$ is false.

Ex. The proposition
$(\forall n \in \mathbb{N})(n+4>3)$ is true since

$$\{n \mid n+4>3\} = \{1, 2, 3, \cdots\} = \mathbb{N}$$

Ex. The proposition
$(\forall n \in \mathbb{N})(n+2>8)$ is false since

$$\{n \mid n+2>8\} = \{7, 8, \cdots\} \neq \mathbb{N}$$

Existential Quantifier :-

Let $p(x)$ be a propositional function defined on a set A. Consider the expression

$$(\exists x \in A)\, p(x) \quad \text{or} \quad \exists x, p(x)$$

which reads "There exists an x in A such that $p(x)$ is true statement" or, simply, "for some x, $p(x)$."

The symbol "$\exists$" is called the existential quantifier.

e.g. If $\{x \mid p(x)\} \neq \phi$ then $\exists x\, p(x)$ is true; otherwise $\exists x\, p(x)$ is false.

Ex. (a) $(\exists n \in \mathbb{N})(n+4<7)$ is true since
$\{n \mid n+4<7\} = \{1,2\} \neq \phi$

(b) $(\exists n \in \mathbb{N})(n+6<4)$ is false since
$\{n \mid n+6<4\} = \phi$

Negation of quantified statements :-

Ex. "All math majors are male"

Negation :- "It is not the case that all math majors are male."

or

"there exists at least one math major ~~this~~ who is female"

In symbols, $M \to$ Math major

$\sim(\forall x \in M)(x \text{ is male}) \equiv (\exists x \in M)(x \text{ is not male})$

or

$\sim(\forall x \in M)\, p(x) \equiv (\exists x \in M)\, \cancel{p} \sim p(x).$

De Morgan's law :-

$$\sim(\forall x \in A)\, p(x) \equiv (\exists x \in A) \sim p(x)$$

&

$$\sim(\exists x \in A)\, p(x) \equiv (\forall x \in A) \sim p(x)$$

NORMAL FORM :-

Let $A(p_1, p_2, \ldots p_n)$ be a statement formula then the construction of truth tables may not be practical always So, we consider alternate procedure known as reduction to normal form

① Dis~~con~~junction Normal form :- A statement form which consist of disjunction b/w conjunction is called DNF.

Ex. (i) $(p \wedge q) \vee r$

(ii) $(p \wedge \sim q) \vee (\sim p \wedge r) \vee (r \wedge \sim q)$

Ex. Obtain the DNF of the form $(p \rightarrow q) \wedge (\sim p \wedge q)$

S. We know that

$$p \rightarrow q \Leftrightarrow \sim p \vee q$$

so the statement

$$(p \rightarrow q) \wedge (\sim p \wedge q)$$

can be written as

$$(\sim p \vee q) \wedge (\sim p \wedge q)$$

By distributive law

$$(\sim p \wedge (\sim p \wedge q)) \vee (q \wedge (\sim p \wedge q))$$

$$\Rightarrow (\sim p \wedge q) \vee (q \wedge \sim p).$$

② Conjunction Normal form :- A statement form which consists of conjunction b/w disjunction is called CNF.

Ex. (i) $p \wedge q$

(ii) $(\sim p \vee q) \wedge (\sim p \vee r)$

Ex. Obtain CNF of the form $(p\wedge q)\vee(\sim p\wedge q\wedge r)$.

Sol $(p\wedge q)\vee(\sim p\wedge q\wedge r)$

By distributive law

$p\vee(\sim p\wedge q\wedge r)\wedge q\vee(\sim p\wedge q\wedge r)$

$[(p\vee\sim p)\wedge(p\vee q)\wedge(p\vee r)]\wedge[(q\vee\sim p)\wedge(q\vee q)\wedge(q\vee r)]$

$[(p\vee q)\wedge(p\vee r)]\wedge[(q\vee\sim p)\wedge(q\vee r)\wedge q]$

Ex. Obtain DNF of $p\vee(\sim p\rightarrow(q\vee(q\rightarrow\sim r))$

Sol $p\vee(\sim p\rightarrow(q\vee(\sim q\vee\sim r))$

$p\vee(\sim p\rightarrow((q\vee\sim q)\vee(q\vee\sim r))$

$p\vee(\sim p\rightarrow(q\vee\sim r))$

$p\vee(\sim\sim p\vee(q\vee\sim r))$

$p\vee(p\vee(q\vee\sim r))$

$p\vee((p\vee q)\vee(p\vee\sim r))$

$p\vee(p\vee q\vee\sim r)$

$p\vee p\vee p\vee q\vee p\vee\sim r$

$\boxed{p\vee q\vee\sim r}$

Ex. Obtain CNF of $(p\rightarrow q)\wedge(q\vee(p\wedge r))$ and determine whether or not it is tautology.

Sol $(p\rightarrow q)\wedge(q\vee(p\wedge r))$

$\Rightarrow(\sim p\vee q)\wedge(q\vee p\wedge q\vee r)$

$\Rightarrow(\sim p\vee q)\wedge(q\vee p)\wedge(q\vee r)$.

let statement is P

p	q	r	~p	~p∨q	q∨p	q∨r	(~p∨q)∧(q∨p)	P
T	T	T	F	T	T	T	T	T
T	T	F	F	T	T	T	T	T
T	F	T	F	F	T	T	F	F
F	T	T	T	T	T	T	T	T
T	F	F	F	F	T	F	F	F
F	T	F	T	T	T	T	T	T
F	F	T	T	T	F	T	F	F
F	F	F	T	T	F	F	F	F

$\Rightarrow$ P is not a Tautology.

Q. Find CNF of $p \wedge (p \rightarrow q)$ $\qquad [p \rightarrow q \equiv \sim p \vee q]$.

$p \wedge (\sim p \vee q)$ ⟶ distributive law

complement law ⟵ $(p \wedge \sim p) \vee (p \wedge q)$

$\Rightarrow (p \wedge q)$

Q. find CNF of $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$

$(\sim p \rightarrow r) \wedge (p \rightarrow q) \wedge (q \rightarrow p)$

Involution law ⟵ $(\sim\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

$(p \vee q) \wedge (\sim p \vee q) \wedge (\sim q \vee p)$

Q. Obtain DNF of $\sim(p \vee q) \rightarrow (p \wedge q)$

Q. Test the validity of the argument

If it rain. Ram will be sick

It did not rain

∴ Ram was not sick.

Sol

p = ~~I~~ It rain

q = Ram will ~~##~~ be sick

$\sim p$ = It didn't rain

$\sim q$ = Ram was not sick

the statement $[(p \rightarrow q) \wedge \sim p] \rightarrow \sim q \equiv P$

p	q	$p \to q$	$\sim p$	$(p \to q) \wedge \sim p$	$\sim q$	p
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

$\Rightarrow$ The argument is not valid

(2) Investigate the validity of the following argument.

$$\begin{array}{c} p \to r \\ \sim p \to q \\ q \to s \\ \hline \therefore \sim p \to s \end{array}$$

or

$$p \to r,\ \sim p \to q,\ q \to s \vdash \sim p \to s$$

(3) Examine the validity of argument "If prices are high then wages are high. Prices are high or there are price controls. If there are price control then there is not an inflation, there is an inflation. Therefore wages are high.

Proof techniques :-

~~① Direct Proof~~ :-

A proof of a theorem is a finite sequence of logically valid stepts that demonstrate that the premises of a theorem imply its conclusion.

There are 4 basic proof techniques used in Mathematics

ⓐ Direct Proof
ⓑ Proof by Contradiction
ⓒ Proof by Induction
ⓓ Proof by Contrapositive

ⓐ Direct Proof :-

The basic idea of a direct proof of $p \to q$ is :

(1) Assume that p is true.
(2) Use p to show that q is true (using definitions & properties).

Ex. Prove "The sum of any two consecutive numbers is odd."

Solⁿ We can write the statement as –
~~"If the sum~~ "If a & b are consecutive numbers then a+b is odd."

Assume a, b are consecutive.
Hence, we can write

$$b = a + 1$$

We want to prove a+b is odd.

$$\text{so,} \quad a+b = a+(a+1)$$
$$= 2a+1$$

if a is an integer then $(2a+1)$ is odd (By definition)

Hence a+b is odd. $\square$

Ex. Prove "If a number is divisible by 6, then it is also divisible by 3".

Pf. Assume x is divisible by 6

this can be written as

$$x = k \times 6 \quad , k \in \mathbb{Z}$$

$$x = k \times 2 \times 3$$

$$x = (2 \times k) \times 3$$

let $2 \times k = m$

$$\Rightarrow x = m \times 3$$

$\Rightarrow x$ is divisible by 3. $\square$

Note :- A useful technique in constructing direct proofs is working backwards. You examine your conclusion (q) & try to determine what statements would imply it. You can then ask the same question about that statement, etc. Combined with working forwards you can work towards the middle.

Proof by contradiction :-

The idea here is that a proposition is either true or false, but not both. We get a contradiction when we can show a statement is both true and false, showing our initial assumptions are inconsistent.

We use this to show $p \rightarrow q$ by assuming both p and $\sim q$ are simultaneously true & deriving a contradiction. When we reach this contradiction it means one of our assumptions can't be correct. Hopefully we have either proved or assumed p to be true so that by default $\sim q$ is false

$\Rightarrow q$ is true by contradiction.

$\square$.

Ex If a & b are consecutive integers then $(a+b)$ is odd.

Sol^n Assume: a & b are consecutive integers & assume that $(a+b)$ is not odd.

If $(a+b)$ is not odd then no integer k such that

$$(a+b) = 2k+1.$$

But

$$a+b = a+a+1$$
$$= 2a+1.$$

Shown that $(a+b) \neq 2k+1$

but $(a+b) = 2a+1$

these two statements contradict each other.

Hence what we assumed is wrong.

$\Rightarrow$ By default, $(a+b)$ is odd. $\square$

Ex. Prove that $\sqrt{2}$ is irrational.

Pf Suppose $\sqrt{2}$ is rational

$$\Rightarrow \sqrt{2} = \frac{a}{b}$$

for some integers a & b such that they are co-prime numbers

Now, squaring on both sides,

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2b^2 \quad \text{—— (1)}$$

this shows a^2 is a multiple of 2.

$\Rightarrow$ a is also a multiple of 2.

let $a = 2k$, $k \in \mathbb{Z}$

from (1) $(2k)^2 = 2b^2$

$\Rightarrow b^2 = 2k^2$

this shows b^2 is a multiple of 2

$\Rightarrow$ b is also a multiple of 2

$\Rightarrow b = 2l \quad , l \in \mathbb{Z}$

Here, both a & b are multiple of 2 but in the beginning we took them coprime numbers.

So it's a contradiction

$\Rightarrow$ Our assumption is wrong

$\Rightarrow \sqrt{2} \neq p/q$

$\Rightarrow \sqrt{2}$ is irrational.

Proof by Induction :- It is a method to show an infinite number of facts by showing some specific case holds, and then using the assumption that the proposition is true for some value of 'n', that the proposition is also true for 'n+1';

Steps :- Prove $P(x)$

1. $p(x)$ is true for $x=1$ (some basic case)
2. Assume $p(x)$ is true for some x, show that this implies that $P(x+1)$ is true.
3. By PMI (Principle of Mathematical Induction), it follows that the $p(x)$ is true $\forall x$ greater or equal to the basic case.

Ex. $p(x)$: x & $x+1$ are consecutive numbers then $(2x+1)$ ~~(a+b)~~ is odd.

Pf. Consider $p(1)$;

the sum $1+2=3$ is odd

Now assume that $P(x)$ is true for some x then

$x+(x+1)$ is odd

Now we'll show that $p(x)$ is true for $x+1$

so $P(x+1)$: $x+1+x+2$

$= x+(x+1)+2$

$\because$ $x+(x+1)$ is odd then by adding 2 to this value again gives an odd number

$\Rightarrow$ ~~2x~~ $P(x+1)$ is true when $P(x)$ is true.

Hence $P(x)$ is true. $\forall x \in \mathbb{N}$. $\square$

<u>Note</u> :- <u>Contrapositive</u> :-

If $p \to q$

then $\sim q \to \sim p$ is contrapositive of $p \to q$.

& $\boxed{p \to q \equiv \sim q \to \sim p}$

<u>Proof by contrapositive</u> :- It can be useful when a direct proof is proving to be difficult or it can simply provide a different way to think about the problem.

<u>Statement</u>:- If a and b are consecutive integers then the sum $a+b$ is odd.

<u>Contrapositive Statement</u> :- If the sum $(a+b)$ is not odd then a & b are not consecutive integers.

<u>Pf</u> Assume that $a+b$ is not odd,

$\nexists k \in \mathbb{Z}$ s.t. $a+b = 2k+1$

So, $a+b = k+(k+1)$ doesn't hold

for any integer k.

But $k+1$ is the successor of k,

$\Rightarrow a$ & b are not consecutive.

$\square$

Ex. If $7x+9$ is even then x is odd, $x \in \mathbb{Z}$.

Pf Contrapositive

If x is not odd then $7x+9$ is not even

If x is not odd

$\Rightarrow x$ is even

let $x = 2a$, $a \in \mathbb{Z}$

$$7x+9 = 7(2a)+9$$
$$= 14a+9$$
$$= 2(7a+4)+1$$

let $7a+4 = b$

$\Rightarrow 7x+9 = 2b+1$. $b \in \mathbb{Z}$ & $b = 7a+4$

$(2b+1)$ is odd

$\Rightarrow 7x+9$ is not even

H.P. $\square$.

Q. Suppose $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then x is odd.

Principle of well ordering :- Every nonempty set of nonnegative integers has a least element.

e.g. $\{1, 2, 3, 4 \ldots\} \rightarrow$ well ordered set.

because it has a smallest element, which is 2.

Principle of Complete Induction :-

Let $P(n)$ be a statement defined on +ve integers. $n \in \mathbb{N}$ s.t.

1) $P(m)$ is true for some $m \in \mathbb{N}$
2) Whenever $P(m), P(m+1), \dots P(k)$ are true then $P(k+1)$ is true, where $k \geq m$.

Q. Prove that any integer $n \geq 2$ is either a prime or a product of prime.

Sol^n $P(n)$: given statement

(i) $n = 2$, $P(2) = 2 = 2 \times 1 \rightarrow$ prime
$P(3) = 3 = 3 \times 1 \longrightarrow$ prime
$P(4) = 4 = 2 \times 2 \longrightarrow$ product of prime

(ii) Assume $P(n)$ is true for $2 \leq n \leq k$.

Now, for $n = k+1$

(a) If $(k+1)$ is prime, the $P(k+1)$ is true.

(b) If $(k+1)$ is not a prime no.

then

$$k+1 = u \times v$$

$$2 \leq u \leq k, \; 2 \leq v, k$$

$\Rightarrow$ $P(u)$ & $P(v)$ statements are true, which states that u, v are prime or product of prime

$\Rightarrow$ $k+1 = u \times v$ is a product of prime no.

$\square$.