1.3

SUBSPACES

Let V be vectorspace over field F, Let W be subset of V, Then W is subspace of V if V is vectorspace over F, with operation of addition and scalar multipl^n defined on V.

$\{0\}$ and V are the subspaces for vector space V.

$x+y \in W \quad \forall\ x, y \in W$

$cx \in W \quad c \in F,\ x \in W$

W has zero vector

Each vector in W has an additive inverse in W.

Theorem Let V be vector space, W be subset of V then W is subspace of V iff the foll. three condition hold true.

(a) $0 \in W$

(b) $x+y \in W \quad \forall\ x, y \in W$

(c) $cx \in W \quad c \in F,\ x \in W.$

for operation defined in V

Proof Suppose W is subspace of V, Then (b) and (c) must hold, because W is vector space

TP (a) holds.

Since ~~same~~ W is a vectorspace over F.

(∴) By VS3 $\exists\ 0' \in W$ such that

$x + 0' = x \longrightarrow (1) \quad \forall\ x \in W$

(for $x \in V$)

Also $x + 0 = x$

$x + 0' = x + 0$

$0' = 0$

Hence $0' \in W$.

Hence, statement is true.

Converse

If W is a subspace of V, ie. W is a subspace itself

By (b) and (c) addition and scalar multiplication is well defined in W

As W is a subset of V; $\therefore$ (VS 1), (VS2), (VS 5-VS8) all hold true.

Also by (a) $0 \in W$, ie additive identity exists.
Now To prove (VS4).

Let $a \in W$, $-1 \in F$

$(-1) a \in W$ [scalar mult.]

$-a \in W$.

Also $a - a = 0$.
Hence inverses exists $\forall$ elements in W.
Hence W is a subspace of V.

Examples of subspaces:

Consider $n \times n$ matrices in $M_{n\times n}(F)$. Let W be the set of all symm. matrices in $M_{n\times n}(F)$ then W is a subspace of $M_{n\times n}(F)$.

Consider the $n \times n$ zero matrix.

Clearly $0^t = 0$
$\Rightarrow$ $0 \in W$.

Now let $A, B \in W$ $\Rightarrow A^t = A$, $B^t = B$
As A, B are $n \times n$ square matrices; $A + B$ is $n \times n$ square matrice

$\therefore$ $A+B$ is also a $(n \times n)$ square matrice.
Consider $(A+B)^T = A^T + B^T = A+B \Rightarrow A+B \in W$
$\Rightarrow$ ~~Ad~~ W is closed under addition
Also Consider
$(CA)^T = CA^T = CA \in W$.
Hence W is closed under scalar multiplic^n
Hence, By Thrm 1.3, W is subspace of $M_{n\times n}(F)$.

(2) $P(F) =$
Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$
If $a_i = 0 \;\; \forall \; i = 0,1,2 \dots n$.
Then $f(x)$ is a zero polynomial
By convent^n $\deg f(x) = -1$

Consider the vectorspace $P(F)$ of all polyno- over the field 'F' and let n be non-neg. integer and $P_n(F)$ be set of all poly. in ~~P(F)~~ $P(F)$ having degree less than or equal to n.

We will prove that $P_n(F)$ is subspace of $P(F)$.

Consider zero poly., Denote it by 0. Then by conven^n $\deg 0 = -1 < n$, $n \in \mathbb{Z}^+$.
Hence $0 \in P_n(F)$
Let $f(x), g(x) \in P_n(F)$
$\deg(f(x))$ and $\deg(g(x)) < n$
$f(x) + g(x) = h(x) \Rightarrow \deg(h(x)) < n$
Hence $h(x) \in P_n(F)$.

Coz,
~~Also~~ sum of two poly with degree $\leq n$ is an another polynomial of degree $\leq n$
Hence, operation of addition is closed.

Also, product of scalar & poly. of degree $\leq n$ is a ~~ga~~ polynomial. of degree $\leq n$.
Hence, $P_n(F)$ is closed under scalar multiplic^n.

Hence, $P_n(F)$ is subspace of $P(F)$.

Eg(3)
Let $F(\mathbb{R},\mathbb{R})$ be set of all real valued functions
Let $C(\mathbb{R})$ denote the set of all continuous real valued function defined on $\mathbb{R}$.
Then $C(\mathbb{R})$ is subset of $F(\mathbb{R},\mathbb{R})$ & in particular it forms a subspace in $F(\mathbb{R},\mathbb{R})$

Clearly, the zero of $F(\mathbb{R},\mathbb{R})$ is $f(t)=0 \ \forall t$
as a constant fun^n is always continuous fun^n.
$\therefore$

$$f \in C(\mathbb{R})$$

Also, sum of 2 continuous function is continuous & product of scalar & continuous function is conti

$C(\mathbb{R})$ is closed under addition & scalar multi.
Hence,

$C(\mathbb{R})$ is subspace of $F(\mathbb{R},\mathbb{R})$

Example 4 : Consider vector space $M_{n\times n}(F)$ of $n \times n$ square matrices over the field F.

Let X be the set of diagonal matrices in $n \times n$.

$$X = \{ A = [a_{ij}] \in M_{n\times n}(F) ;\ a_{ij} = 0,\ i \neq j \}$$

Then X is subspace of $M_{n\times n}(F)$.

(a) Clearly the zero matrix ~~is~~ a diagonal matrix. Hence $0 \in X$.

(b), (c) Also sum of 2 diagonal matrices is a diagonal matrice and product of a scalar and diagonal matrice is a diagonal matrice.
Hence, X is subspace of $M_{n\times n}(F)$.

Example-5 For a $n \times n$ matrix A, trace of A denoted by trA is the sum of its principal diagonal ~~axis~~ entries.

ie $A = [a_{ij}]$

$$trA = a_{11} + a_{22} + \dots a_{nn}$$

For vector space $M_{n\times n}(F)$. Consider the set of $n \times n$ matrices having trace $= 0$. then set of such matrices form subspace of $M_{n\times n}(F)$.

Let $S = \{ A = [a_{ij}] \in M_{n \times n}(F) : \sum_{i=1}^{n} a_{ii} = 0 \}$

Clearly,
0 matrix $\in S$
Let $A, B \in S$
$\ni A = [a_{ij}]$; $\sum a_{ii} = 0$
& $B = [b_{ij}]$; $\sum b_{ii} = 0$

$tr(A+B) = tr(A) + tr(B) = 0$
$tr\, c(A) = c\, tr(A) = c \cdot 0 = 0$

Hence, S forms subspace of $M_{n \times n}(F)$.

Eg6 Consider $M_{m \times n}(R)$, having non negative entries is not subspace of $M_{m \times n}(R)$, because it is not closed under scalar multiplicat^n

for $c = -1$ & $X \in M_{m \times n}(R)$
$cX \notin M_{m \times n}(R)$
As cX has negative entries.

Hence $M_{m \times n}(R)$ with non negative entries is not a subspace of $M_{m \times n}(R)$.

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Theorem 1.4
Any intersection of subspaces of vectorspace V is a subspace of V.

Pf. Let C be the collection of subspaces of V.
Let W denote the intersection of subspaces in C.

$$C = \{ V_i ,\ i \in I \ V_i \text{ is a subspace of } V \ \forall i \in I \}$$

& $W = \bigcap_{i \in I} V_i$

Since $0 \in V_i \quad \forall i \in I$ [V_i is subspace of V]
$\Rightarrow 0 \in \bigcap V_i$
$\Rightarrow 0 \in W$.

Let $x, y \in W$ and $c \in F$
ie $x, y \in \bigcap V_i$
$\Rightarrow x, y \in V_i \quad \forall i \in I$
Also $x+y \in V_i \quad \forall i$ [$\because V_i$ is subspace]
$\Rightarrow x+y \in \bigcap V_i$
$\Rightarrow x+y \in W$, Hence W is closed under $(+)$.

As $x \in W \Rightarrow x \in \bigcap V_i \Rightarrow x \in V_i \ \forall i$
Now for $c \in F$
$cx \in V_i \quad \forall i$ [V_i is subspace]
$\Rightarrow cx \in \bigcap V_i$
$cx \in W$. Hence, W is closed under scalar multipl^n

Hence W is subspace of V.

REMARK :
1. Union of 2 subspaces may/may not be subspace of V.
2. Union of 2 subspaces of V is a subspace of V iff one of the subspace contains the other

Ex. Let $W = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$

Sol. Let $u = (a_1, a_2, 0)$, $v = (b_1, b_2, 0) \in W$ and $\alpha \in \mathbb{R}$

$u + v = (a_1 + b_1, a_2 + b_2, 0) \in W$

$\alpha u = (\alpha a_1, \alpha a_2, 0) \in W$

$\therefore W$ is a subspace of $\mathbb{R}^3$.

Ex. Let $W = \{(a_1, a_2, 1) : a_1, a_2 \in \mathbb{R}\}$

Let $u = (a_1, a_2, 1)$, $v = (b_1, b_2, 1) \in W$, $\alpha \in \mathbb{R}$

$u + v = (a_1 + b_1, a_2 + b_2, 1 + 1)$

$= (a_1 + b_1, a_2 + b_2, 2) \notin W$

$\alpha u = (\alpha a_1, \alpha a_2, \alpha) \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^3$.

Ex. Let $W = \{\alpha(1,1,1) : \alpha \in \mathbb{R}\} = \{(\alpha, \alpha, \alpha) : \alpha \in \mathbb{R}\}$

Let $u = (a, a, a)$, $v = (b, b, b) \in W$, $\alpha \in \mathbb{R}$

$u + v = (a + b, a + b, a + b) \in W$ as $a + b \in \mathbb{R}$

$\alpha u = (\alpha a, \alpha a, \alpha a) \in W$ as $\alpha a \in \mathbb{R}$

$\therefore W$ is a subspace of $\mathbb{R}^3$.

Ex. Let $W = \{(a_1, a_2) : a_1 + a_2 = 0, a_1, a_2 \in \mathbb{R}\}$.

Let $u = (a_1, a_2)$, $v = (b_1, b_2) \in W$ and $\alpha \in \mathbb{R}$

$u + v = (a_1 + b_1, a_2 + b_2)$

Now, $(a_1 + b_1) + (a_2 + b_2) = a_1 + a_2 + b_1 + b_2$

$= 0 + 0$ as $a_1 + a_2 = 0$, $b_1 + b_2 = 0$

$= 0$

$\therefore u + v \in W$

$\alpha u = (\alpha a_1, \alpha a_2)$

Now, $\alpha a_1 + \alpha a_2 = \alpha(a_1 + a_2) = \alpha \cdot 0 = 0$

$\therefore \alpha u \in W$

Hence W is a subspace of $\mathbb{R}^2$.