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Differentiability:→

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

Then f is diff. at point $(a,b) \in D$ If

$$f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\, \phi(h,k)$$

where A, B are real constants and $\lim_{(h,k)\to(0,0)} \phi(h,k) = 0$.

i.e. $$\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - (Ah+Bk)}{\sqrt{h^2+k^2}} = 0.$$

→ If $h=0$ Then

$$f(a,b+k) - f(a,b) = Bk + |k|\,\phi(0,k)$$

$$\Rightarrow \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = B + \lim_{k\to 0} \phi(0,k)$$

$$\Rightarrow B = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k} = f_y(a,b)$$

Similarly, If $k=0$ Then

$$f(a+h, b) - f(a,b) = Ah + |h|\,\phi(h,0)$$

$$\Rightarrow \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} = A$$

$$\Rightarrow \boxed{A = f_x(a,b)}$$

and $\boxed{B = f_y(a,b)}$

→ If f is differentiable at point $(a,b) \Rightarrow f$ is Continuous at pt (a,b).

→ If f is diff. at point $(a,b) \Rightarrow f_x(a,b)$ and $f_y(a,b)$ Exist.

$\boxed{\text{i.e. Differentiability at } (a,b) \Rightarrow \text{Partial derivatives at } (a,b).}$

But Converse need not to be true.

→ If $f_x(a,b)$ and $f_y(a,b)$ Exist and both f_x and f_y are Continuous at $(a,b) \Rightarrow f$ is diff at (a,b).

Ques $f(x,y) = \begin{cases} x\sin\left(\frac{1}{x}\right) + y\sin\left(\frac{1}{y}\right) & ; x \neq 0, y \neq 0 \\ x\sin\left(\frac{1}{x}\right) & ; x \neq 0, y = 0 \\ y\sin\left(\frac{1}{y}\right) & ; x = 0, y \neq 0 \\ 0 & ; x = 0, y = 0. \end{cases}$

Sol^n f is continuous at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\sin\left(\frac{1}{h}\right)}{\cancel{h}} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not Exist

$\Rightarrow f$ is not differentiable at (a,b).

Ques $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

Soln

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta)$$

$$= \lim_{r\to 0} r(\cos\theta \sin\theta) \to 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$\Rightarrow f$ is cts at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0.$$

$$f(h,k) - f(0,0) = A\cdot h + B\cdot k + \sqrt{h^2+k^2}\cdot \phi(h,k)$$

$$\Rightarrow \frac{hk}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\ \phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{hk}{h^2+k^2} \ ; \lim_{(h,k)\to(0,0)} \frac{hk}{h^2+k^2}$$ does not Exist

$\Rightarrow f$ is not diff. at $(0,0)$.

Ex $f(x,y) = |x| + |y|$; $\forall (x,y) \in R^2$.

Check at point $(0,0)$.

Sol^n $f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} |r|\left(|\cos\theta| + |\sin\theta|\right)$$

$$= 0 = f(0,0)$$

$\Rightarrow f$ is cts at $(0,0)$.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

limit does not Exist

$\Rightarrow f$ is not diff at $(0,0)$.

Ex $f(x,y) = \sqrt{|xy|}$; $\forall (x,y) \in R^2$

Sol^n f is cont. at $(0,0)$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = 0$$

$$f(h,k) - f(0,0) = \sqrt{h^2+k^2}\,\phi(h,k)$$

$$\Rightarrow \phi(h,k) = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}}$$

$h = r\cos\theta$, $k = r\sin\theta$

$$\lim_{r\to 0} \phi(r,\theta) = \frac{\cancel{|r|}\sqrt{|\cos\theta\sin\theta|}}{\cancel{|r|}}$$

$\hookrightarrow$ Limit does not Exist.

$\Rightarrow f$ is not diff. at $(0,0)$.

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H.W

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0; & x^2+y^2 = 0. \end{cases}$$

S.T. $f(x,y)$ is cont. at $(0,0)$ but not diff. at $(0,0)$.

Que

$$f(x,y) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) + y^2\sin\left(\frac{1}{y}\right); & x\neq 0, y\neq 0 \\ x^2\sin\left(\frac{1}{x}\right) & ; x\neq 0, y=0 \\ y^2\sin\left(\frac{1}{y}\right) & ; x=0, y\neq 0 \\ 0 & ; x=0, y=0. \end{cases}$$

Find $f_x(0,0)$, $f_y(0,0)$ and check If f is diff at $(0,0)$ or not.

→ **Exact differential / Total differential** ⇒

~~Let $z = f(x,$~~

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

$z = f(x,y)$.

Let dx and dy represent changes in x and y respectively.

If the partial derivatives f_x and f_y Exist

Then the total differential of z is

$$\boxed{dz = f_x(x,y)\,dx + f_y(x,y)\,dy}$$

Or

$$\boxed{dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy}$$

Ex

Let $z = x^4e^{3y}$

Find dz

Soln

$$\frac{\partial z}{\partial x} = 4x^3e^{3y}$$

$$\frac{\partial z}{\partial y} = 3x^4e^{3y}$$

$$dz = (4x^3e^{3y})dx + (3x^4e^{3y})dy$$

$$\boxed{dz = x^3e^{3y}(4dx + 3x\,dy)}$$

Maxima / Minima at a point :→

Def^n :→ Positive definite Matrix :→ Matrix A is called Positive definite If all the principal Minors of A are Positive.

i.e. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$

then $M_{11} = a > 0$

$M_{22} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc > 0$

Negative definite Matrix :→ Matrix A is called negative definite If all the principal Minors are alternatively negative and positive starting with the negative sign.

i.e. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $M_{11} = a < 0$

$M_{22} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc > 0.$

So $(-1)^i M_{ii} > 0.$

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

→ Then $(a,b) \in D$ is called point of local Maxima if $\exists\ \delta > 0$ s.t.

$$f(x,y) \leq f(a,b) \quad \forall\ (x,y) \in N_\delta(a,b)$$

→ $(a,b) \in D$ is called point of local Minima if $\exists\ \delta > 0$ s.t.

$$f(x,y) \geq f(a,b) \quad \forall\ (x,y) \in N_\delta(a,b).$$

→ Maximum or Minimum value of function at point (a,b) is called an Extreme value.

→ Critical Points:→ Let f has first order partial derivatives at point (a,b). Then $(a,b) \in D$ is called a critical point if $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

→ Saddle points:→ If $(a,b) \in D$ is critical point but it is neither point of Minima nor point of Maxima then (a,b) is called saddle point.

Methodology:→

If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be a function s.t. $f_x, f_y, f_{xx}, f_{yy}, f_{xy}; f_{yx}$ exist and continuous at $(a,b) \in D$.

($f_{xy} = f_{yx}$ in this case)

Then (a,b) is point of Extrema

$\Rightarrow f_x(a,b) = 0$ and $f_y(a,b) = 0$.

→ $$J_{(a,b)} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(a,b)}$$

→ If J matrix is Positive definite Then (a,b) is Called Point of Minima.

→ If J matrix is negative definite Then (a,b) is called Point of maxima.

→ If $|J_{(a,b)}| < 0$ Then (a,b) is Called Saddle point. ie. neither point of minima nor maxima.

→ If $|J_{(a,b)}| = 0$ Then may or may not have Extrema at point (a,b).

Que $f(x,y) = x^3 + y^2 - 12x - 6y + 40$

$f_x = 3x^2 - 12 = 0 \Rightarrow x = \pm 2$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

$(2,3), (-2,3)$

↓ ↓ Saddle point

Que $f(x,y) = x^2 + xy + y^2 - 3x - 6y + 11$

$f_x = 0$

$f_y = 0$

→ Let A function $f(x,y)$ be defined in region $\mathbb{R}$. The minimum and maximum values attained by a function over the entire region including the boundary are called the absolute (global) minimum and absolute (global) maximum values respectively.

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Q $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x = 3x^2 - 3 = 0$

$\Rightarrow x = \pm 1$

$f_y = 3y^2 - 12 = 0$

$\Rightarrow y = \pm 2$

$(1,2)\ (1,-2)\ (-1,2)\ (-1,-2)$

at $(1,2)$

$f_{xx} = 6x = 6$

$f_{xy} = 0 = f_{yx}$

$f_{yy} = 6y = 12$

$$J = \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix}$$

$6 > 0,\quad |J| > 0$

$\Rightarrow (1,2)$ is point of Minima.

at $(1,-2)$

$f_{xx} = 6$

$f_{yy} = -12$

$$J = \begin{bmatrix} 6 & 0 \\ 0 & -12 \end{bmatrix}$$

$6 > 0,\quad |J| = -72 < 0$

↳ Saddle point

at $(-1,2)$

$f_{xx} = -6,\quad f_{yy} = 12$

$$J = \begin{bmatrix} -6 & 0 \\ 0 & +2 \end{bmatrix}$$

$-6 < 0,\quad |J| = -72 < 0$

Saddle pt.

at $(-1,-2)$

$f_{xx} = -6,\ f_{yy} = -12$

$$J = \begin{bmatrix} -6 & 0 \\ 0 & -12 \end{bmatrix}$$

$-6 < 0,\ |J| > 0 \Rightarrow J$ is negative definite

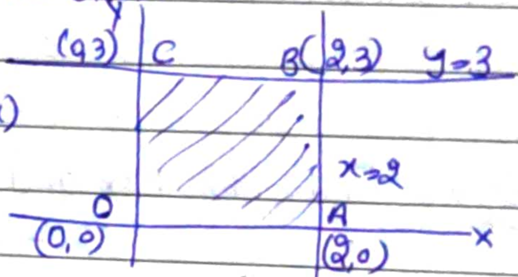
$\Rightarrow$ $(-1,-2)$ is pt of minima.

Que:→ Find the absolute max and min values of

$$f(x,y) = 4x^2 + 9y^2 - 8x - 12y + 4$$

over the rectangle in the I^{st} Quadrant bounded by the lines $x=2$, $y=3$ and the co-ordinate axes.

Solⁿ:→ on the line OA, $y=0$



$\Rightarrow f(x,y) = f(x,0) = 4x^2 - 8x + 4 = g(x)$

$g'(x) = 0 \Rightarrow 8x - 8 = 0 \Rightarrow x = 1$

$g''(x) = 8 > 0$

$\Rightarrow$ fun. $g(x)$ has minimum at $x=1$

and $g(1) = 0$.

$\therefore$ at $f(0,0) = 4 = g(0)$; $f(2,0) = g(2) = 16 - 16 + 4 = 4$.

On the line OC, $x=0$

$f(x,y) = f(0,y) = 9y^2 - 12y + 4 = h(y)$

$h'(y) = 0 \Rightarrow 18y - 12 = 0 \Rightarrow y = 2/3$

$h''(y) = 18 > 0$

$\Rightarrow h(y)$ has min at $y = 2/3$

and $h(2/3) = 4 - 8 + 4 = 0$

$f(0,0) = h(0) = 4$ and $f(0,3) = h(3) = 81 - 36 + 4 = 49$.

on the line AB, $x=2$

$f(x,y) = f(2,y) = 9y^2 - 12y + 4 = h(y)$ → Same as above.

$f(2,3) = f(0,3) = 49$.

on the line BC, $y=3$, $f(x,3) = 4x^2 - 8x + 49$.