Introduction to Robotics

UNIT II

Kinematics of Robots and Differential Motions and velocities

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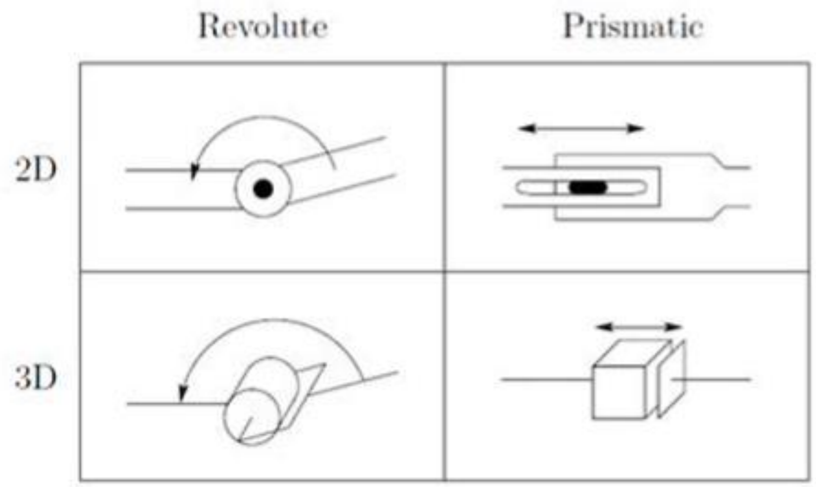
Introduction to Concepts in Robotics

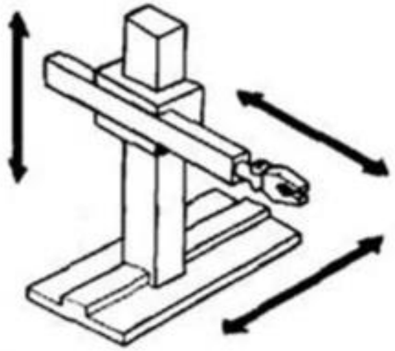
In this lecture, you will learn:

- Robot classification
- Links and Joints
- Redundant manipulator
- Workspace
- Robot Frames
- Kinematics
  - Forward kinematics
  - Inverse kinematics

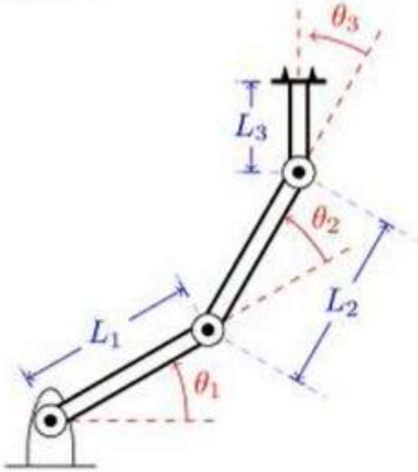
Industrial Robotic Systems

- Manipulator
  - Links
  - Joints
    - Rotary (revolute)
    - Linear (prismatic)





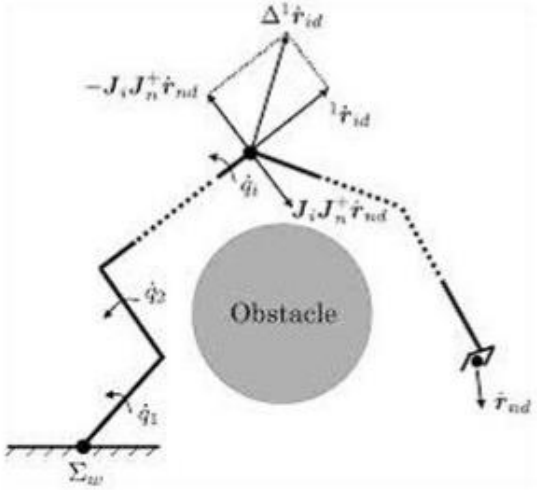
Cartesian robot

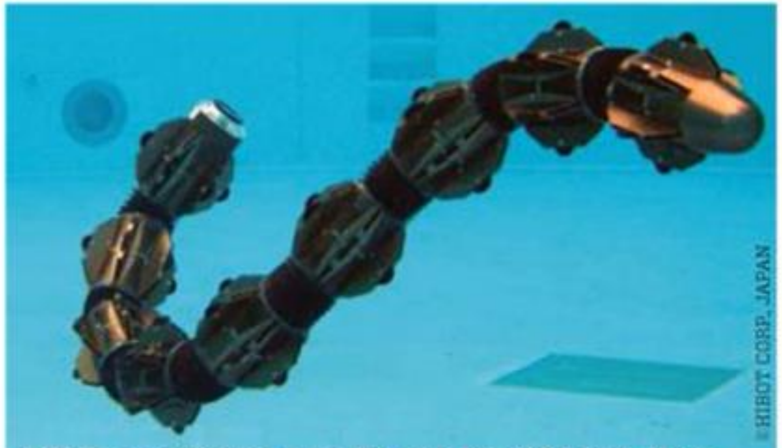


Articulated robot

Degree Of Freedom

- Rigid object in 3D space has 6 DOFs: three for positioning and three for orientation.
- Manipulator needs at least 6 joints to position an end-effector with arbitrary orientation.
- Redundant manipulator > 6 DOFs (human arm: 7DOFs)
  - Redundancy can be used to secondary tasks including energy minimization, singularity avoidance, and obstacle avoidance



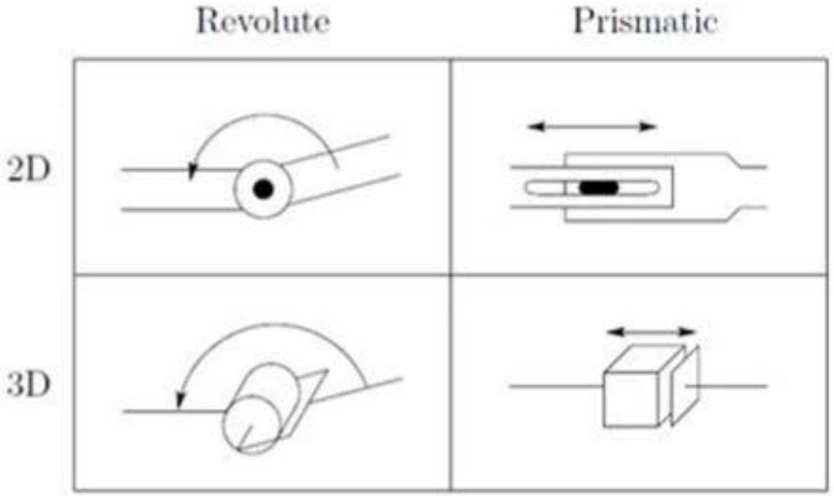


ACM-R5, an amphibious snake-like robot from HiBot Corp., Japan.

Classifications of Robotic Manipulators

- Power source
  - Electric (AC/DC motors)
    - Cheaper, cleaner, and quieter (popular)
  - Hydraulic
    - Large payload
    - Maintenance issue: leaking
  - Pneumatic
    - Inexpensive and simple, but cannot be precisely controlled

- Arm Geometry (Joints)
  - Serial link robots
    - Articulated (RRR)
    - Spherical (RRP)
    - SCARA (RRP)
    - Cylindrical (RPP)
    - Cartesian (PPP)
  - Parallel robot
    - Closed chain

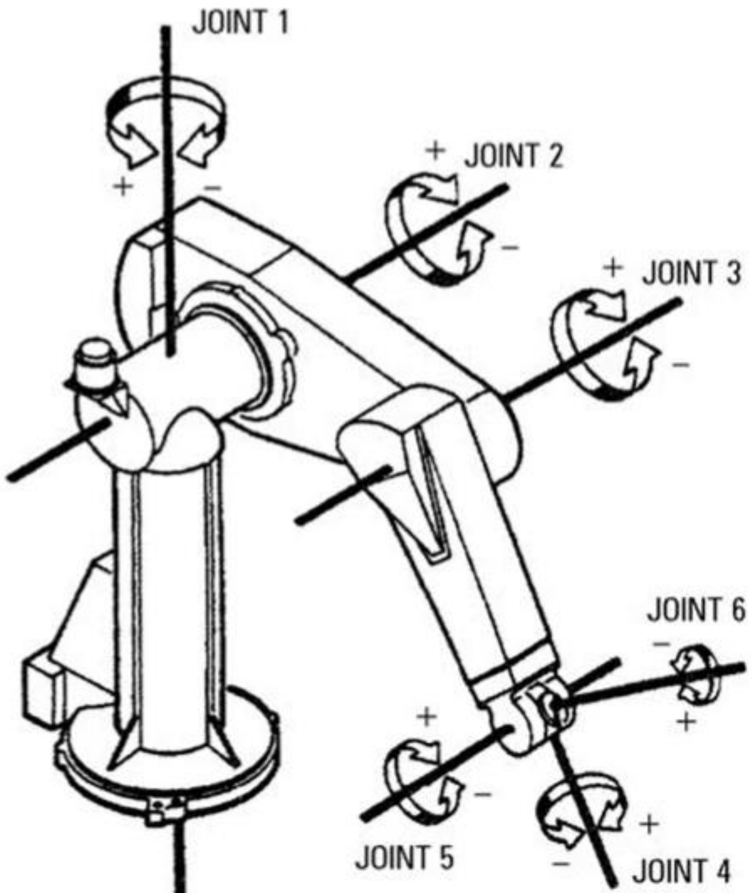


R: Revolute

P: Prismatic

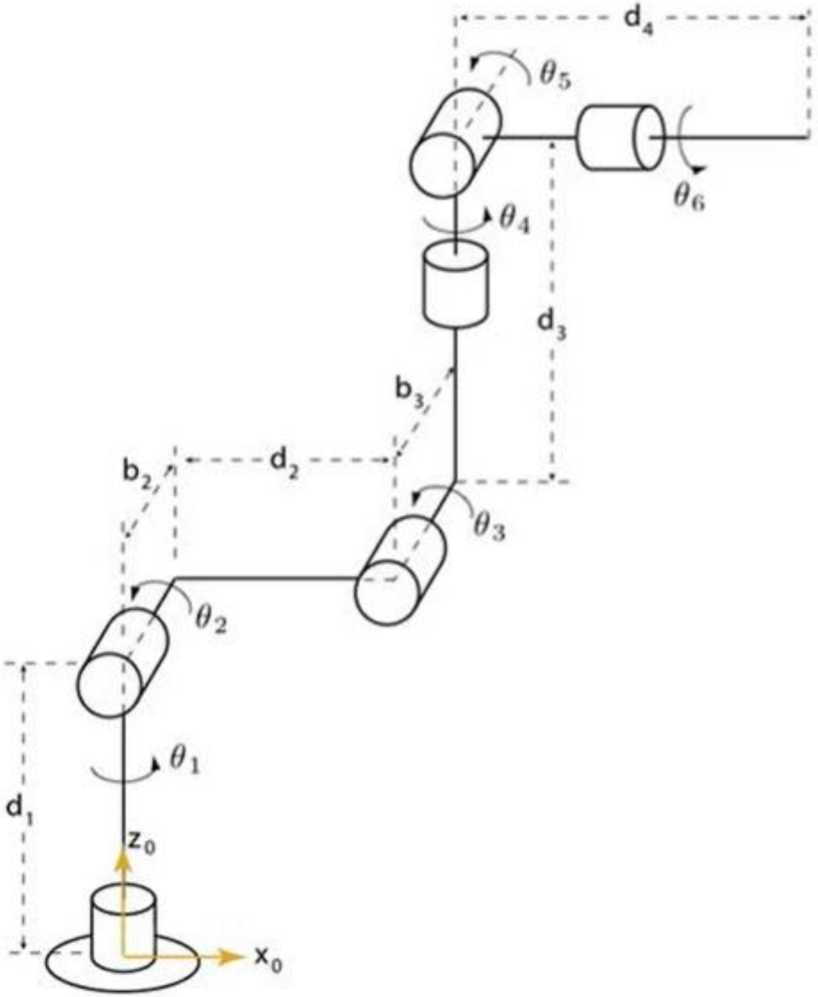
Articulated Manipulator (RRR)

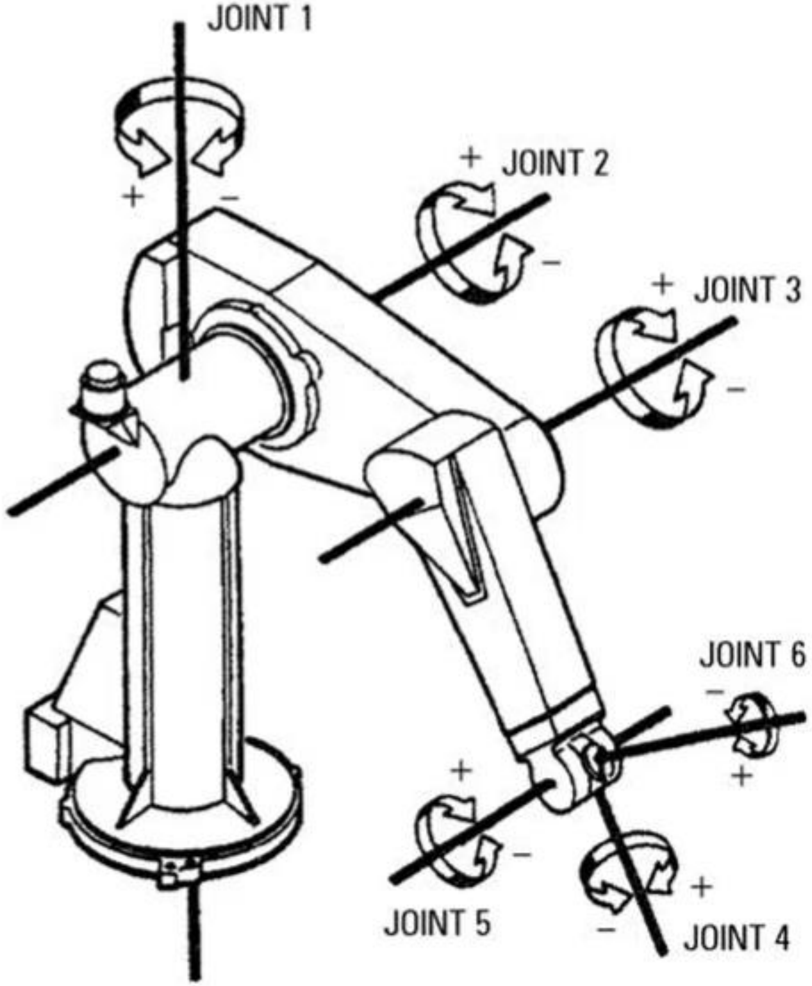
- Number of joints determines the number of DOFs.
- 6 joints = 6 DOFs
  - J1: waist
  - J2: shoulder
  - J3: elbow
  - J4: wrist rotation
  - J5: wrist bend
  - J6: Flange rotation



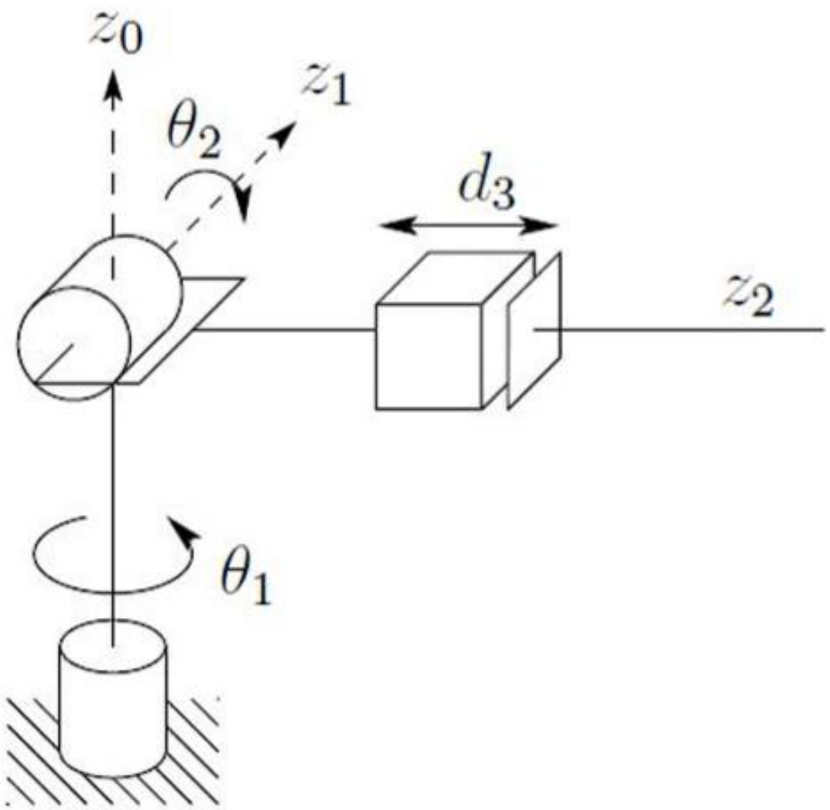
Puma 560

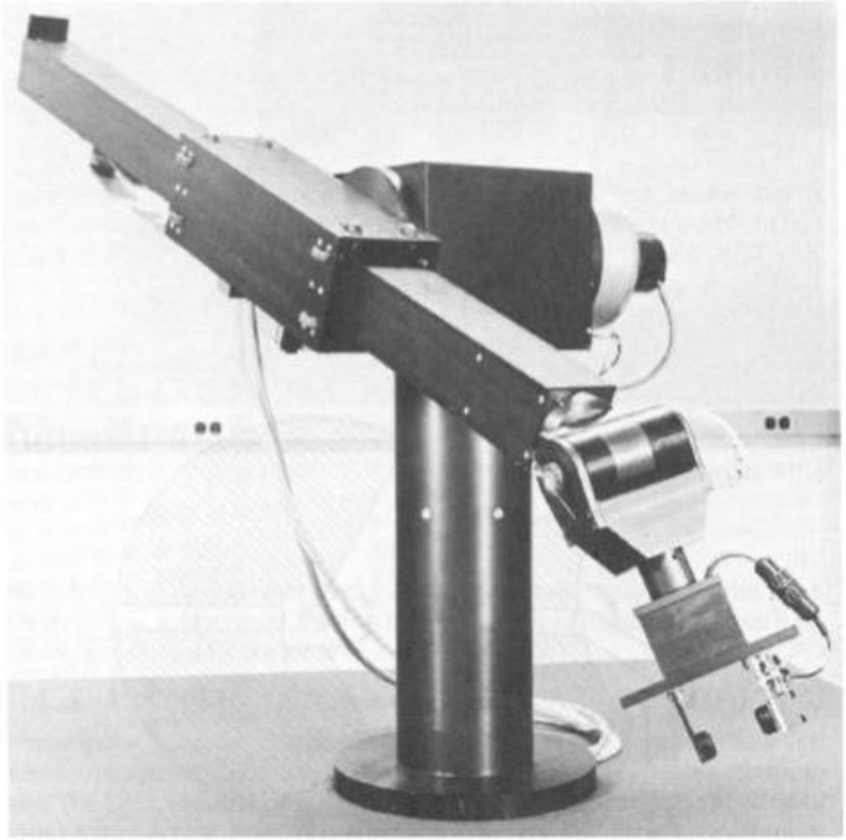
Links and Joints (RRR)





Spherical Manipulator (RRP)

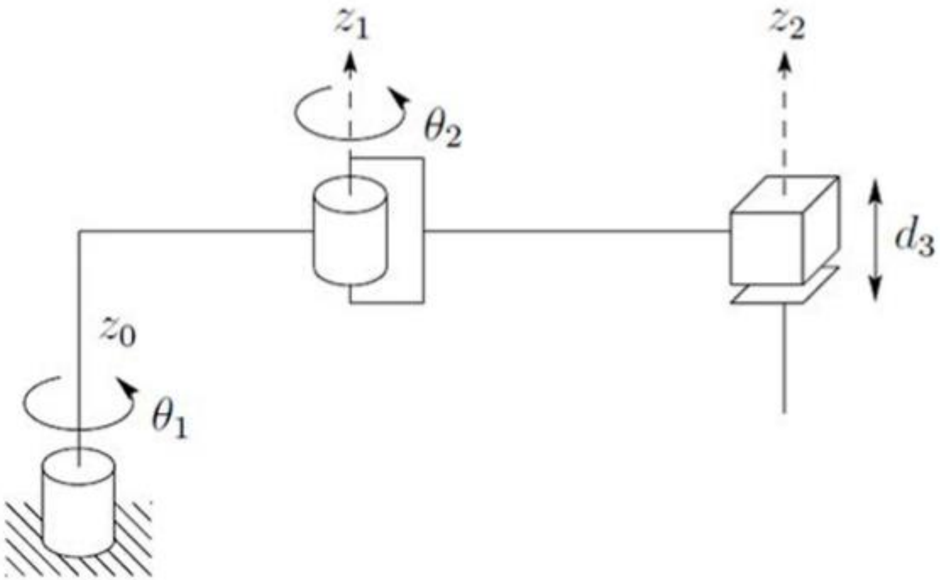




Stanford arm

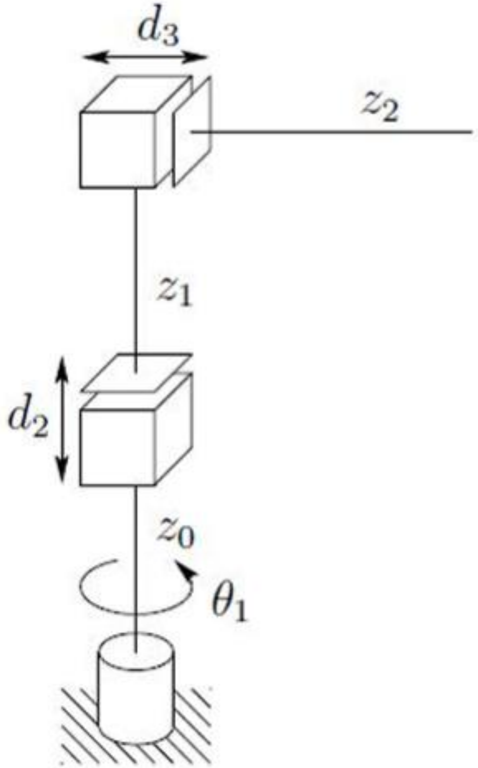
SCARA (RRP)

- SCARA: Selective Compliant Articulated Robot for Assembly



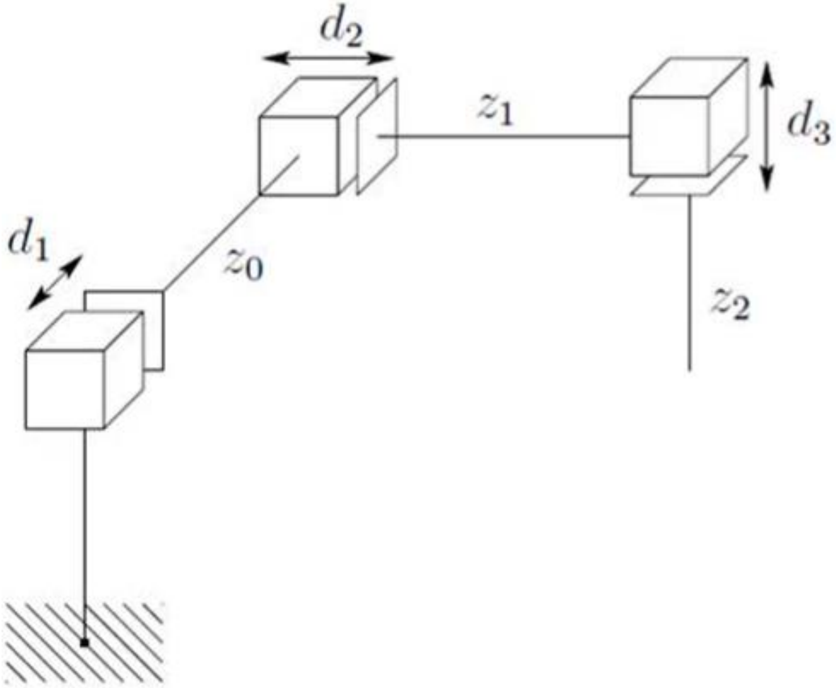
Epson E2L653S

Cylindrical Manipulator (RPP)



Seiko RT3300

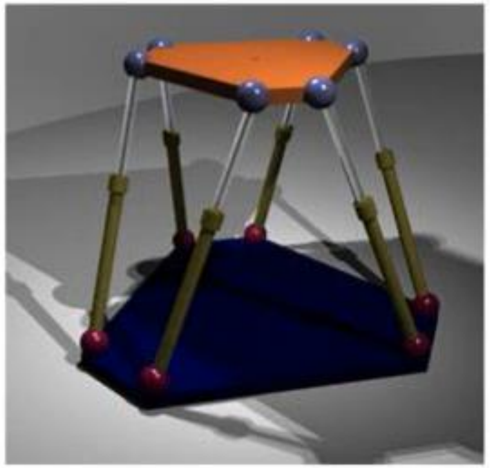
Cartesian Manipulator (PPP)



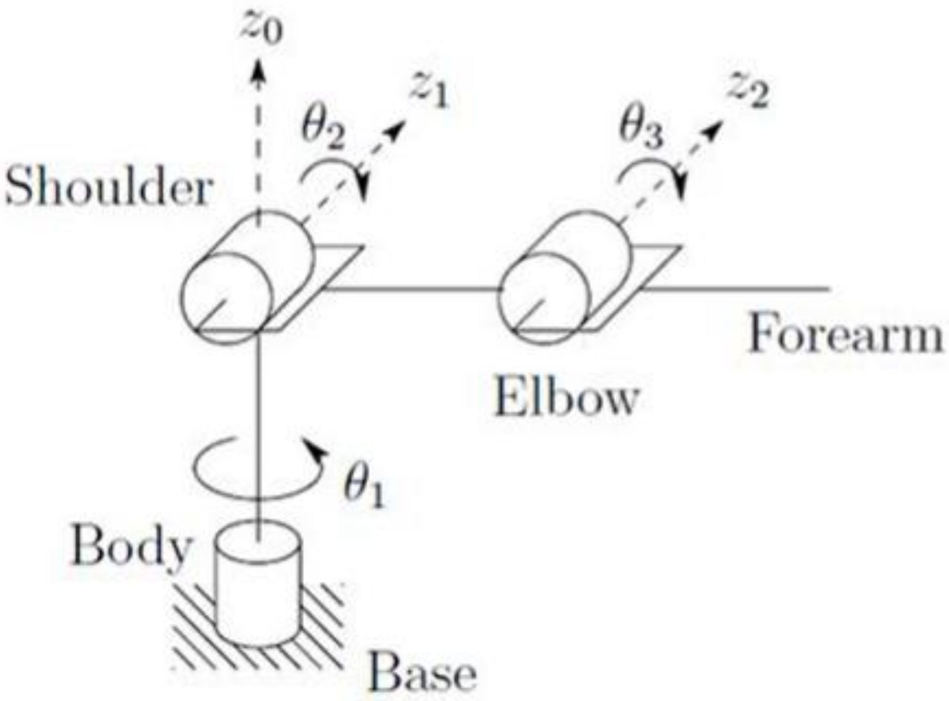
Epson Cartesian robot

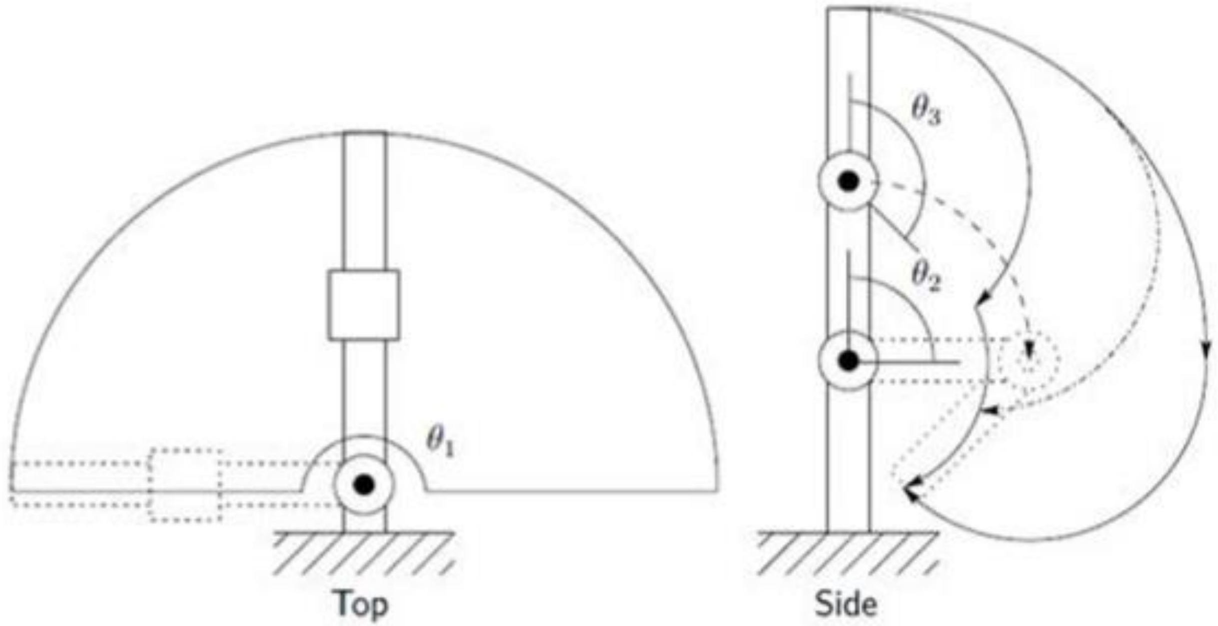
Parallel Manipulators

- Closed chains
- Prismatic actuators with spherical joints
- 6DOF Stewart platform: 6 linear actuators
- Precise positioning
- Large payload, small workspace
- Forward kinematics is hard to solve due to constraints and has multiple solutions.



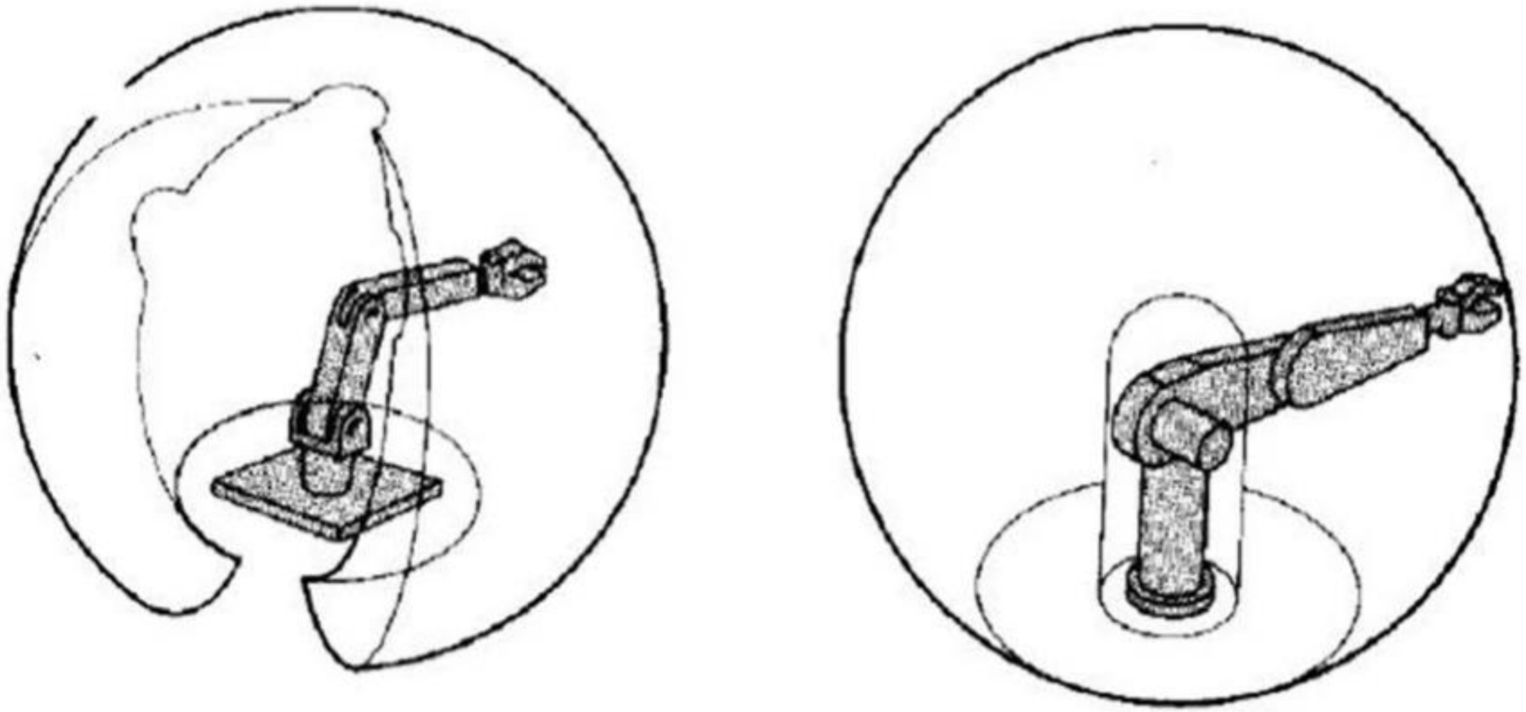
- Articulated manipulator (RRR)





Workspace

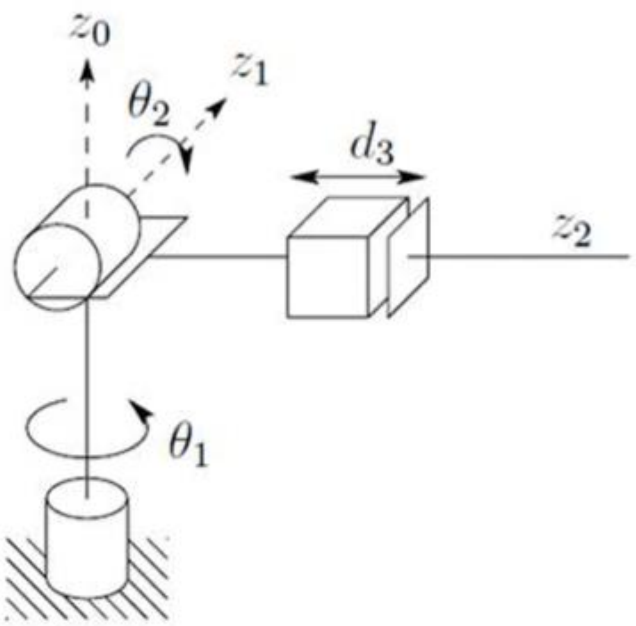
- RRR

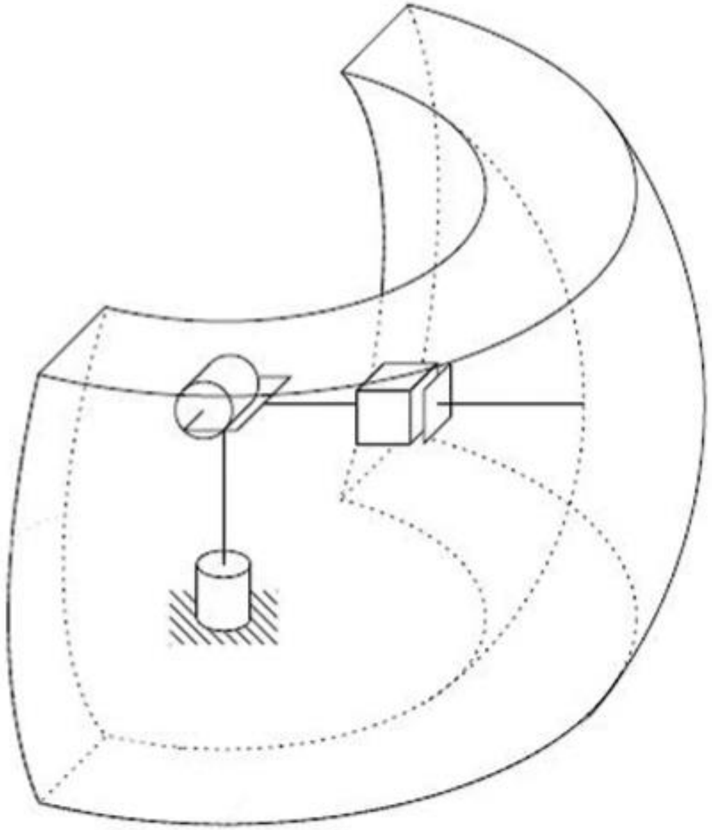


3D view of workspace

Workspace

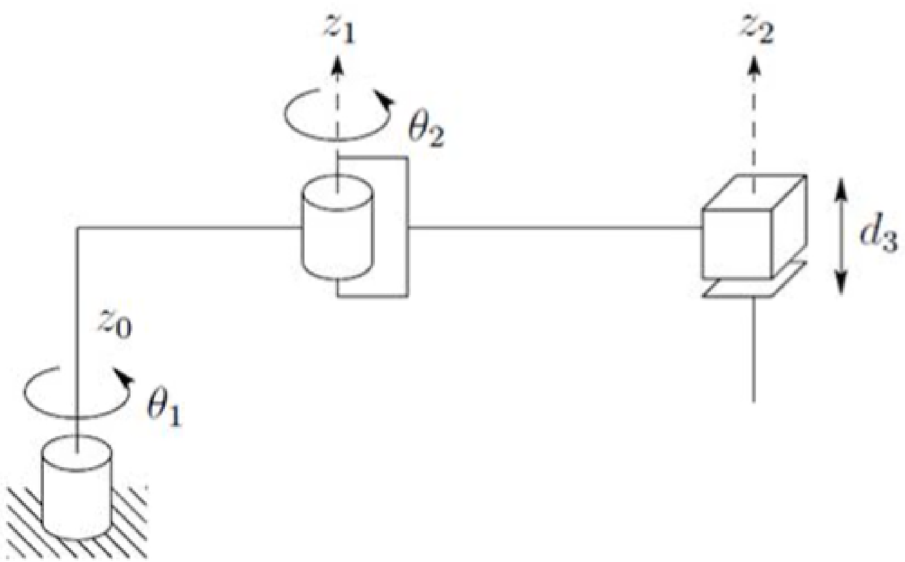
- Spherical manipulator

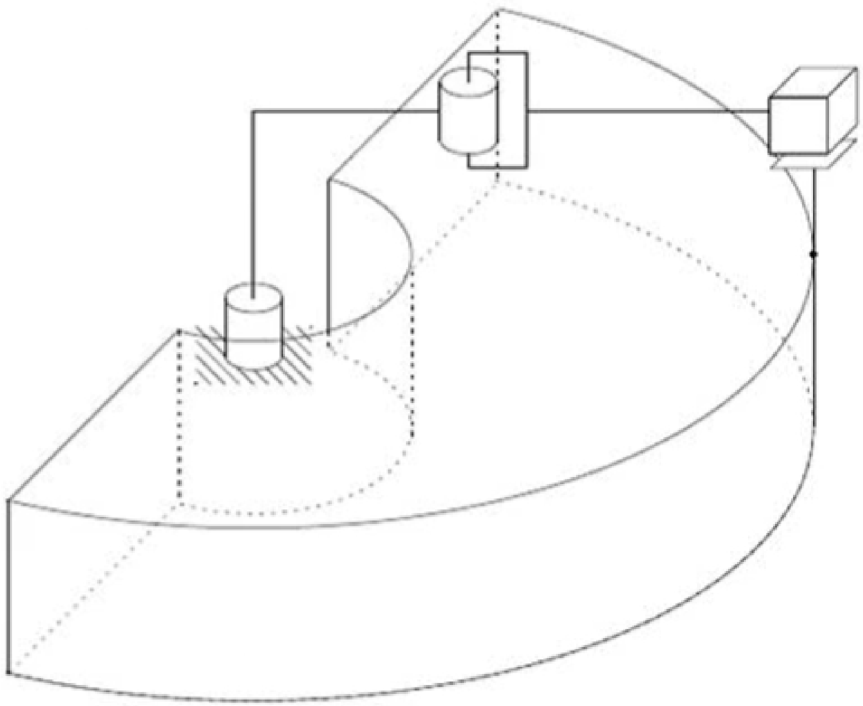




Workspace

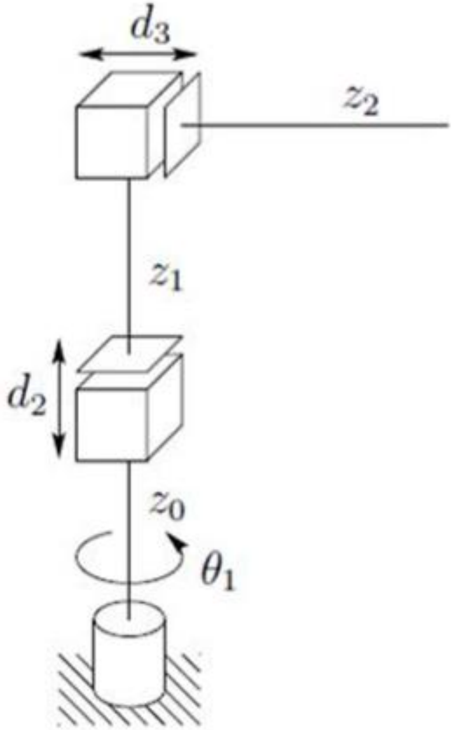
- SCARA

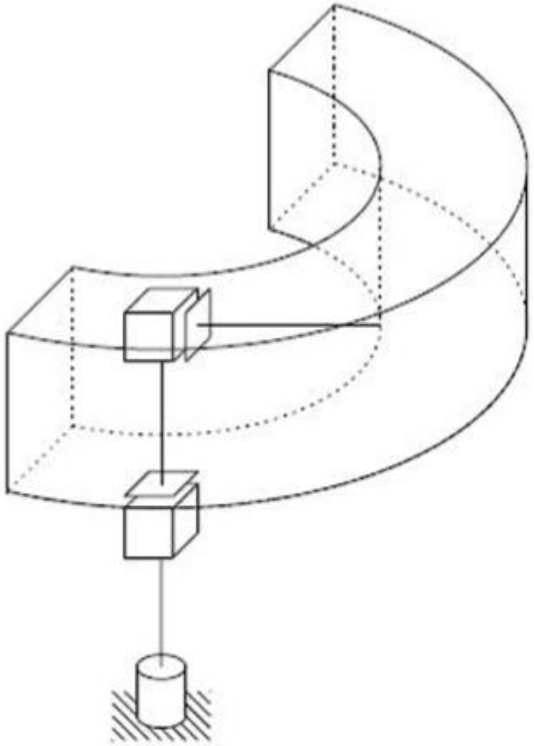




Workspace

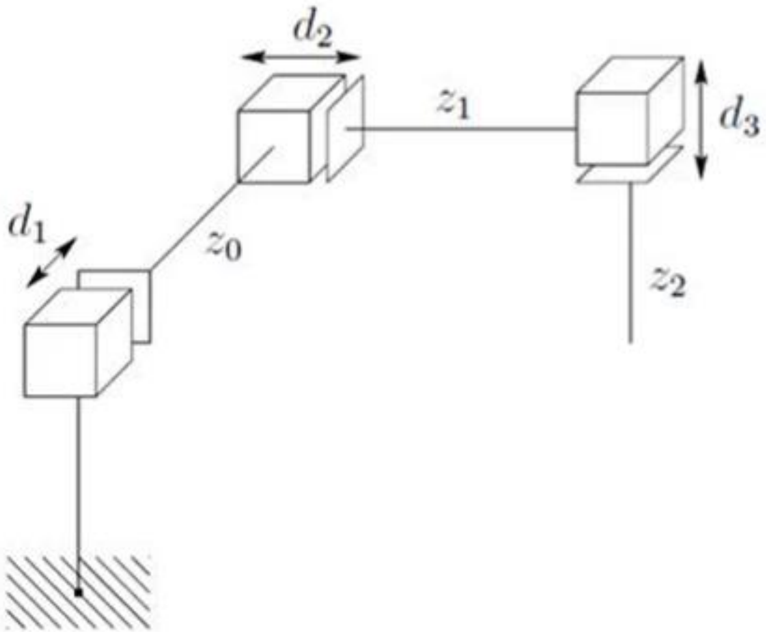
- Cylindrical manipulator

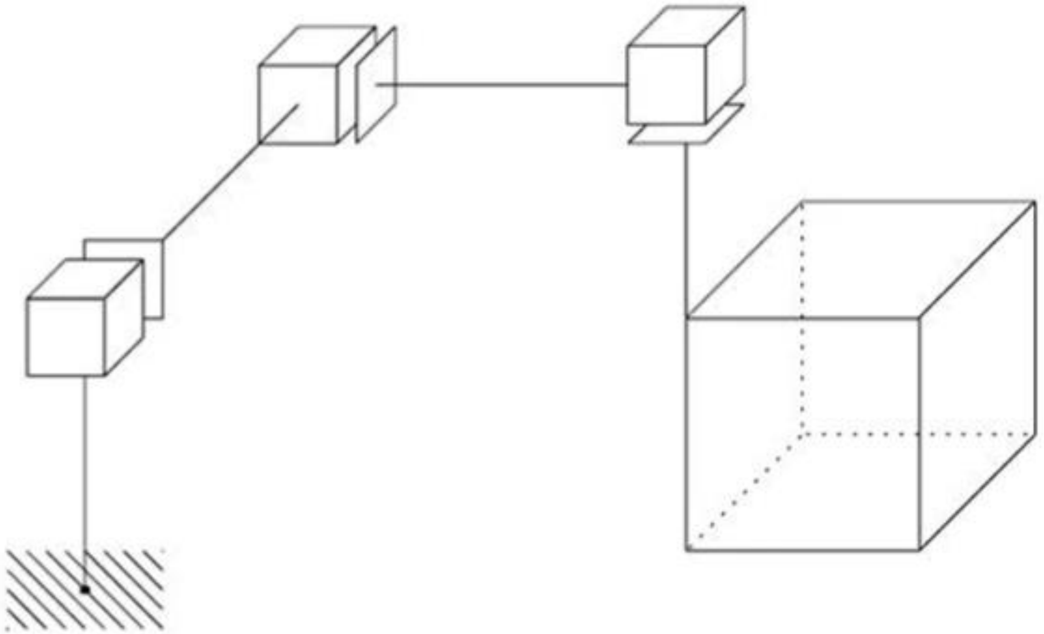




Workspace

- Cartesian manipulator





Properties of Manipulators

- The most important considerations for the application of an industrial robot are:
  - Manipulator performance
  - System integration
  - Reconfigurability/modularity

- Manipulator performance is defined as:
  - Reach (size of workspace), and dexterity (angular displacement of individual joints). Some robots can have unusable workspace due to dead-zones, singular poses, wrist-wrap poses.
  - Payload (weight that can be carried). Inertial loading for rotational wrist axes can be specified for extreme velocity and reach conditions.
  - Quickness (how fast it can move). Critical in determining robot throughput but rarely specified. Maximum speeds of joints are usually specified, but average speeds while carrying payloads in a working cycle is of interest.
  - Duty-cycle (how fast it can repeat motions without breaking down).

Properties of Manipulators

- Precision is defined by using 3 metrics: resolution, repeatability and accuracy.
  - These concepts are usually static, and dynamic precision is usually not specified.
- Accuracy is defined as how close the manipulator can come to a given point within its workspace.
  - Accuracy varies with the location of the point
- Repeatability is how close the manipulator returns to the same point in space.
  - Most present day manipulators are highly repeatable but not very accurate.
  - Repeatability for the manipulator is also defined as the ability to return to a so called “taught” position.
- Resolution is defined as the minimum motion increment that the manipulator can perform and detect.
  - example: a robot controller has 12-bit storage capacity, the full range of the robot = 1.0 cm for one joint
  - spatial resolution = $1.0\text{cm}/2^{12} = 1.0\text{ cm}/4096 = 2.44\ \mu\text{m}$

Basic Concepts

- Kinematics deals with positions and its derivatives (velocity/acceleration). Kinematics is the science of motion based on geometric description, regardless of the forces which cause it.

- The number of **DOFs** of the manipulator equals the number of independent position variables that would have to be specified in order to locate all parts of the mechanism. It equals the number of joints in an open kinematic chain.

- **Forward Kinematics** refers to the problem of computing the position and orientation of the end-effector relative to the base frame given a set of joint angles.

- **Cartesian space** (or task space, operational space) is the usual 3D Euclidian space for position and orientation (6 DOFs). The **joint space** (or configuration space) is the space in which the manipulator is described by it's joint angles.

- **Inverse kinematics** is the problem of inverse mapping between end-effector positions and orientation and the joint angles. We need to map locations in task space to the robot's internal joint space. Early robots lacked this algorithm and they were simply "taught" joint spaces by moving the end-effector (by hand) to the desired position. The inverse kinematics problem is considerably harder than forward kinematics because it involves solving a non-linear equation which may not have a closed form solution. Also, no solution, or multiple solutions may exist.

Coordinate Frame

- Two-link planar robot
- Base coordinate frame x_0y_0: Base of robot
- Frame x_1y_1 : Link1
- Frame x_2y_2 : Link2

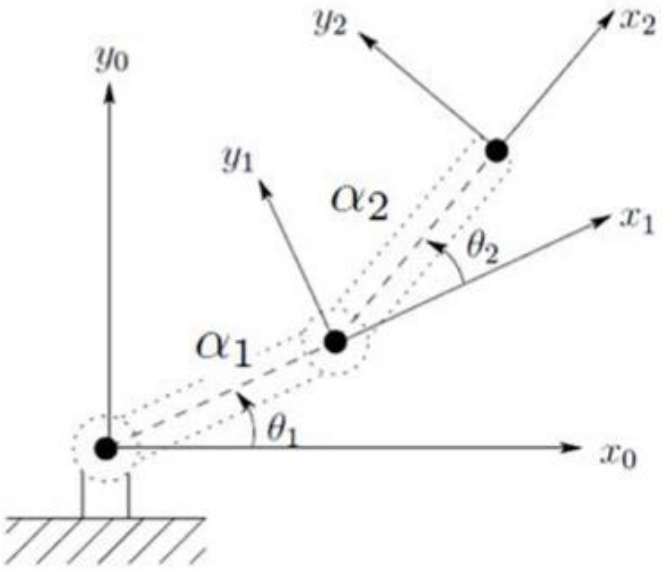


Fig. 1.24 Coordinate frames for two-link planar robot.

Forward Kinematics

- Computes the position and orientation of an end-effector in terms of the joint angles.

Position

$$
\begin{aligned}
x &= x_2 = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) \\
y &= y_2 = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)
\end{aligned}
$$

Orientation

$$
\begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix}
$$

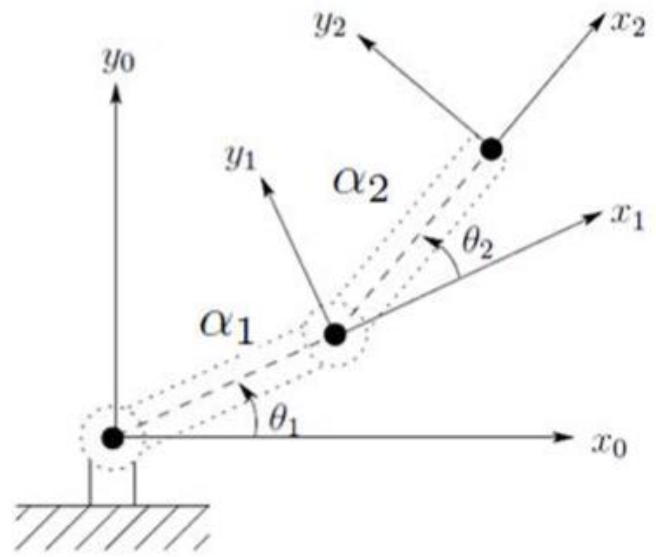


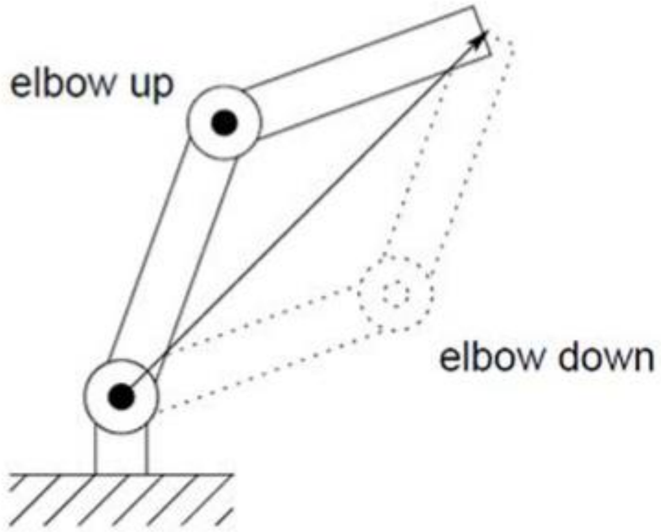
Fig. 1.24 Coordinate frames for two-link planar robot.

Inverse Kinematics

- Computes the joint angles (θ_1, θ_2) given the end-effector position (x_2, y_2)

$$
\begin{aligned}
x_2 &= \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) \\
y_2 &= \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)
\end{aligned}
$$

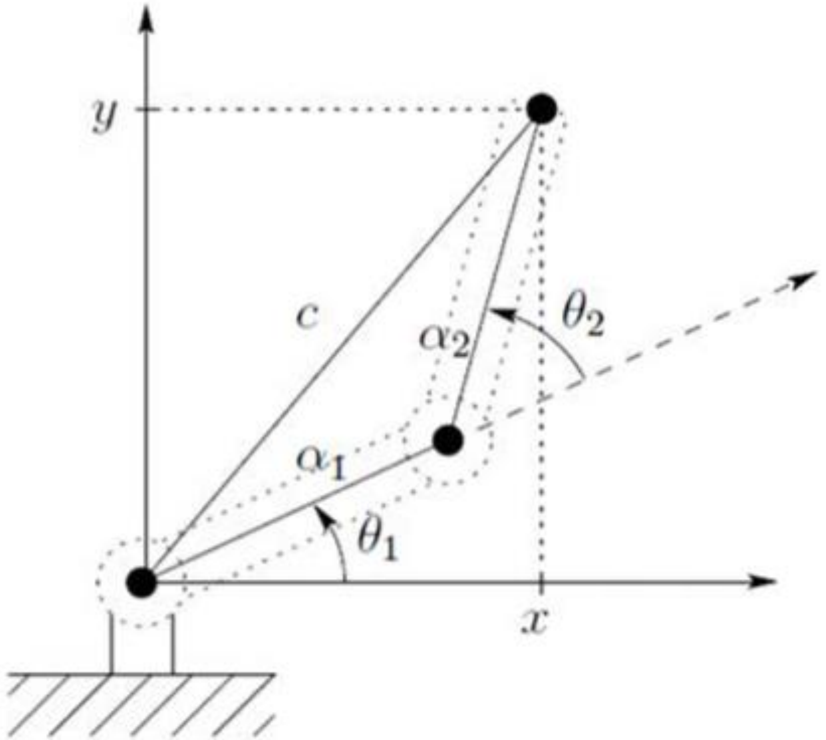
- It is not easy to find a solution because equations are nonlinear.
- Solution is not unique.
  - Two solutions



Inverse Kinematics

$$\cos\theta_2 = \frac{x^2+y^2-\alpha_1^2-\alpha_2^2}{2\alpha_1\alpha_2} := D$$

$$\theta_2 = \cos^{-1}(D)$$



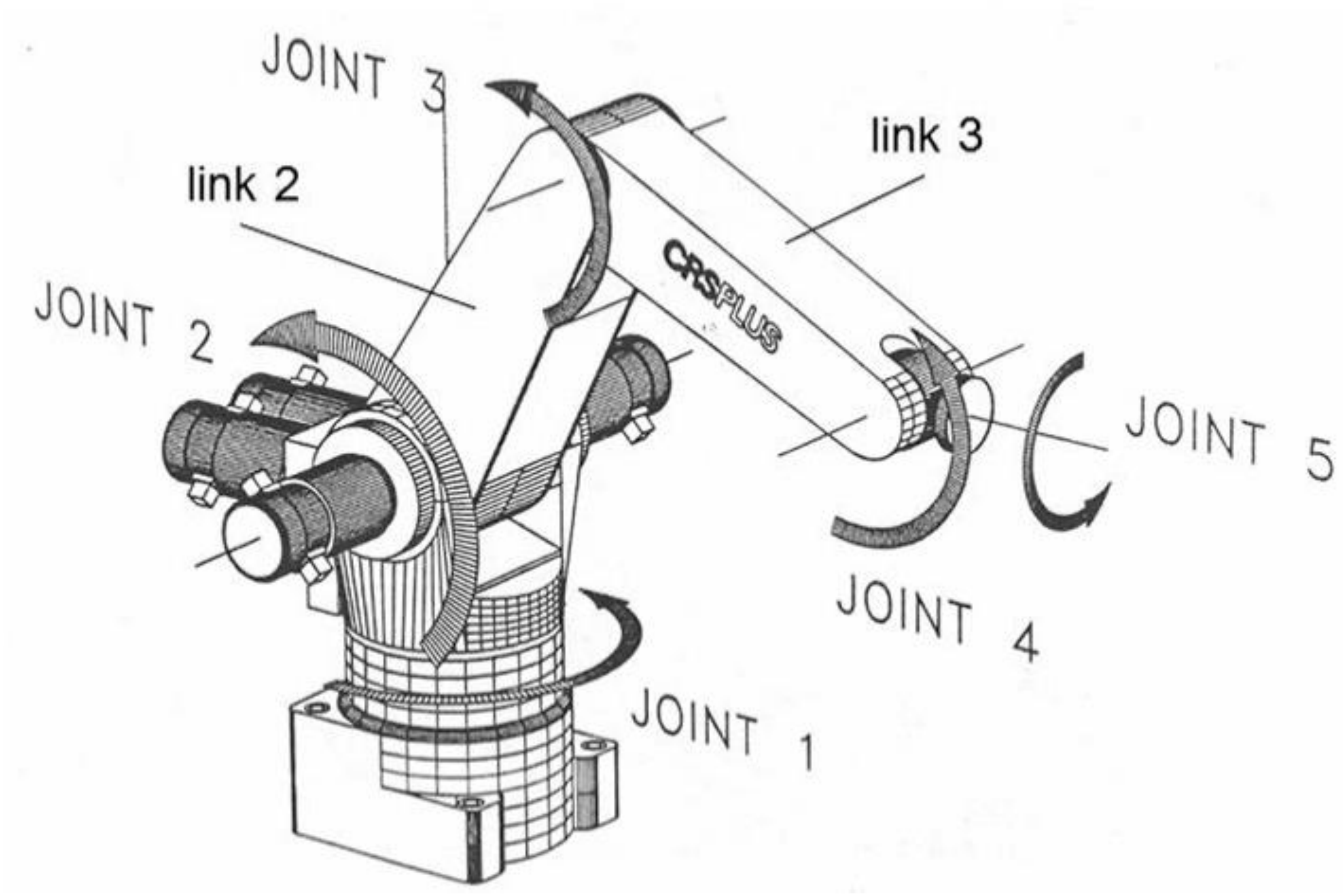
$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{\alpha_2\sin\theta_2}{\alpha_1+\alpha_2\cos\theta_2}\right)$$

Manipulators

- Robot arms, industrial robot
  - Rigid bodies (links) connected by joints
  - Joints: revolute or prismatic
  - Drive: electric or hydraulic
  - End-effector (tool) mounted on a flange or plate secured to the wrist joint of robot

- a robotic manipulator is a kinematic chain
  - i.e. an assembly of pairs of rigid bodies that can move respect to one another via a mechanical constraint
- the rigid bodies are called *links*
- the mechanical constraints are called *joints*

A 150 Robotic Arm



Joints

- most manipulator joints are one of two types

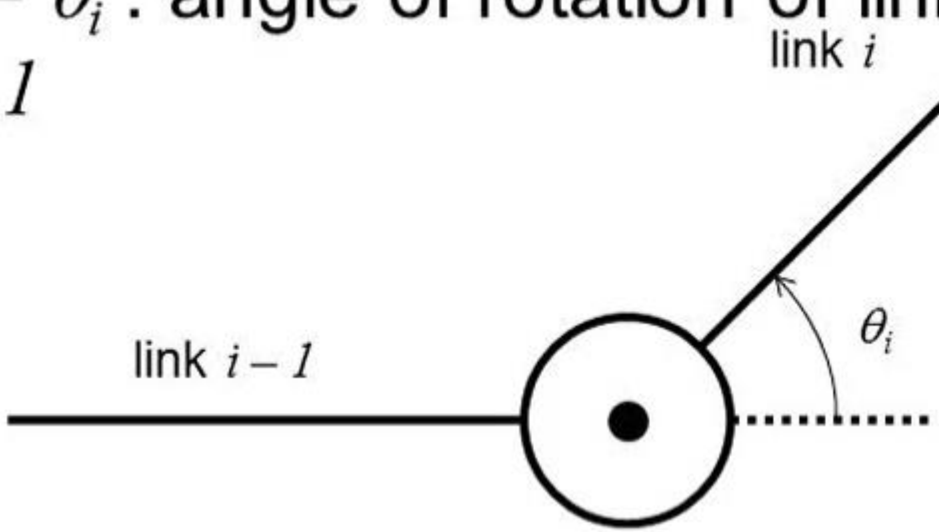
1. revolute (or rotary)
   - like a hinge
   - allows relative rotation about a fixed axis between two links
     - axis of rotation is the z axis by convention
2. prismatic (or linear)
   - like a piston
   - allows relative translation along a fixed axis between two links
     - axis of translation is the z axis by convention
     - our convention: joint i connects link $i-1$ to link i
   - when joint i is actuated, link i moves

Joint Variables

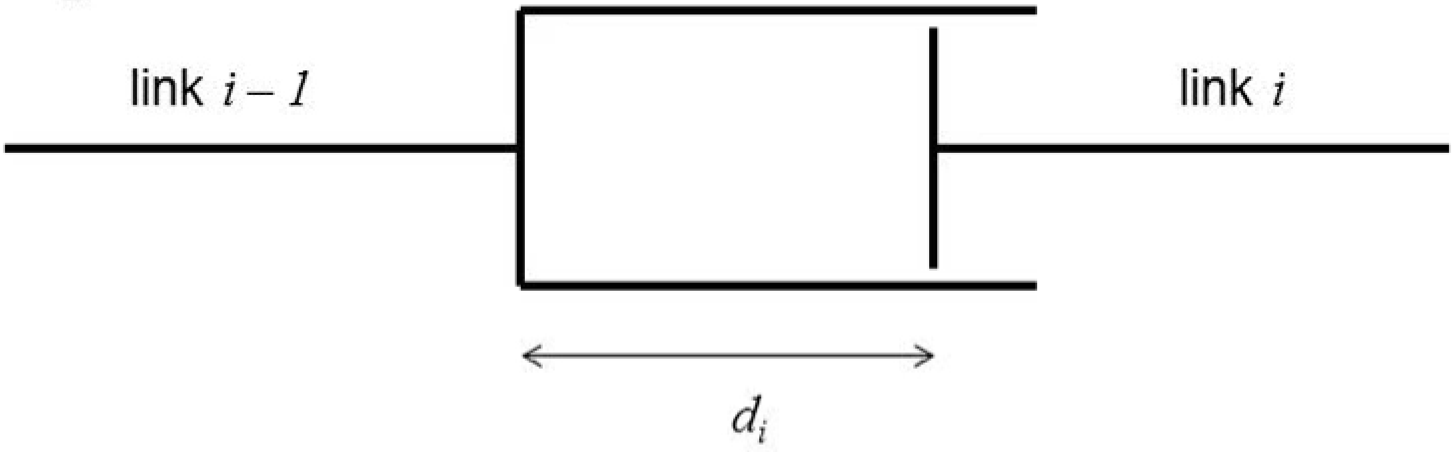
- revolute and prismatic joints are one degree of freedom (DOF) joints; thus, they can be described using a single numeric value called a joint variable
- q_i : joint variable for joint i

1. revolute
   - $q_i = \theta_i$: angle of rotation of link i relative to link $i - 1$
2. prismatic
   - $q_i = d_i$: displacement of link i relative to link $i - 1$

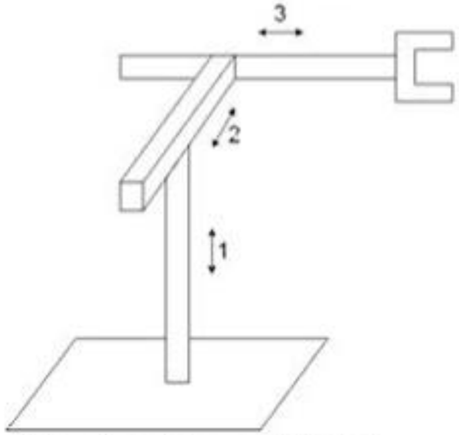
- revolute
  - $q_i = \theta_i$: angle of rotation of link i relative to link $i-1$



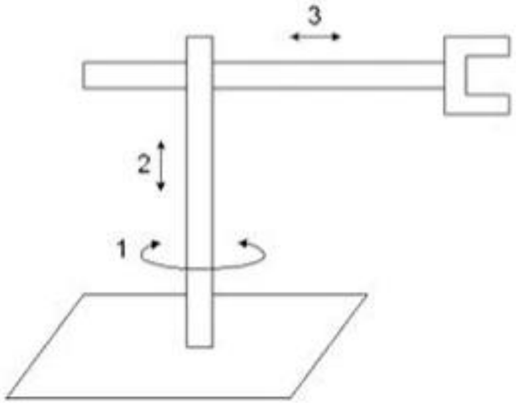
- prismatic
  - $q_i = d_i$: displacement of link i relative to link $i - 1$



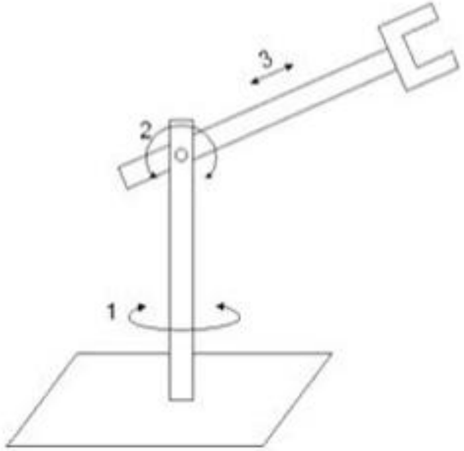
- Robot Configuration:



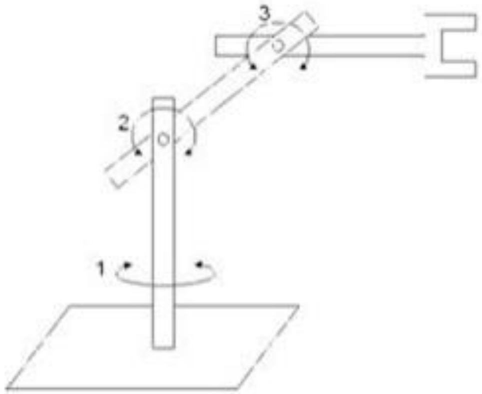
Cartesian: PPP



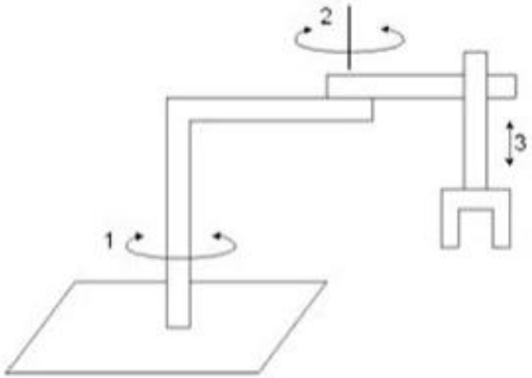
Cylindrical: RPP



Spherical: RRP

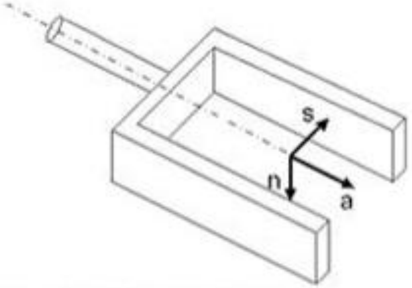


Articulated: RRR



SCARA: RRP

(Selective Compliance Assembly Robot Arm)



Hand coordinate:

n: normal vector; **s**: sliding vector;

a: approach vector

- Robot Specifications
  - Number of Axes
    - Major axes, (1-3) => Position the wrist
    - Minor axes, (4-6) => Orient the tool
    - Redundant, (7-n) => reaching around obstacles, avoiding undesirable configuration
  - Degree of Freedom (DOF)
  - Workspace
  - Payload (load capacity)
  - Precision v.s. Repeatability

how accurately a specified point can be reached

how accurately the same position can be reached

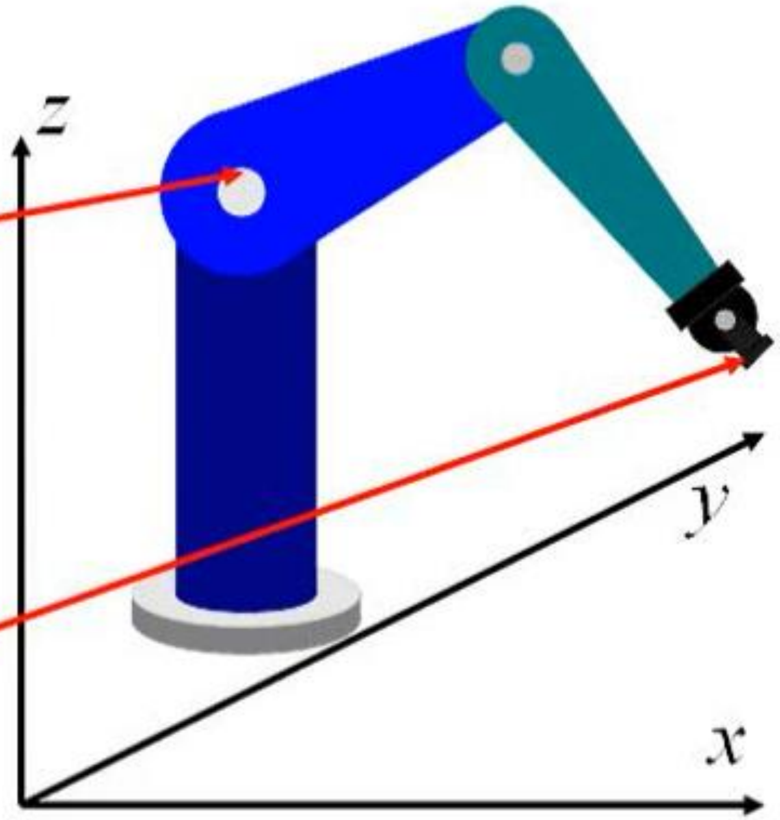
- Forward kinematics

Given joint variables

$$q = (q_1, q_2, q_3, q_4, q_5, q_6, \cdots q_n)$$

$$\Downarrow$$

$$Y = (x, y, z, O, A, T)$$



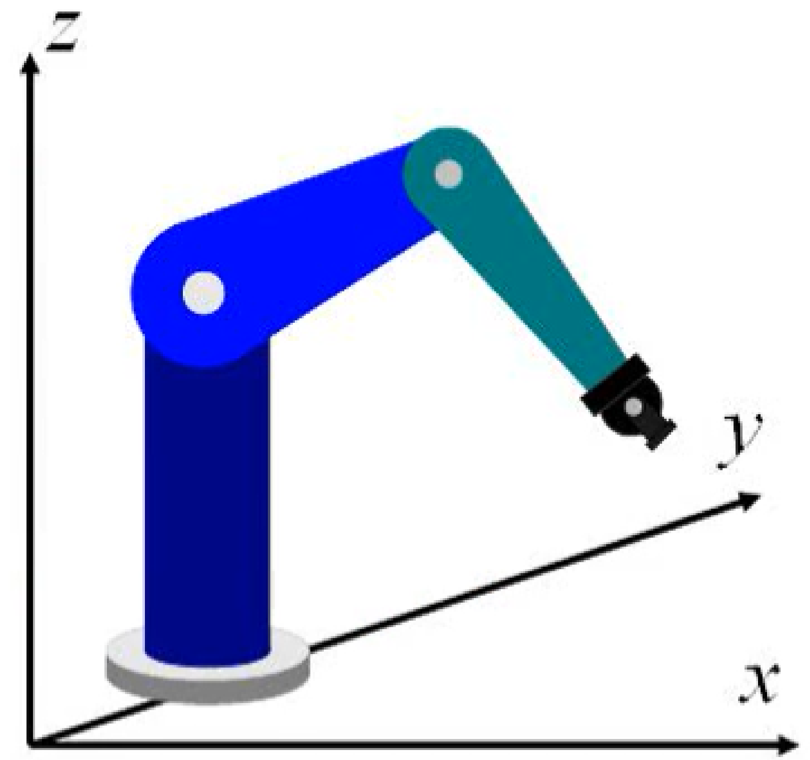
End-effector position and orientation, -Formula?

- Inverse kinematics

End effector position
and orientation

$$(x, y, z, O, A, T)$$

$$q \; = (q_1, q_2, q_3, q_4, q_5, q_6, \cdots q_n)$$



Joint variables -Formula?

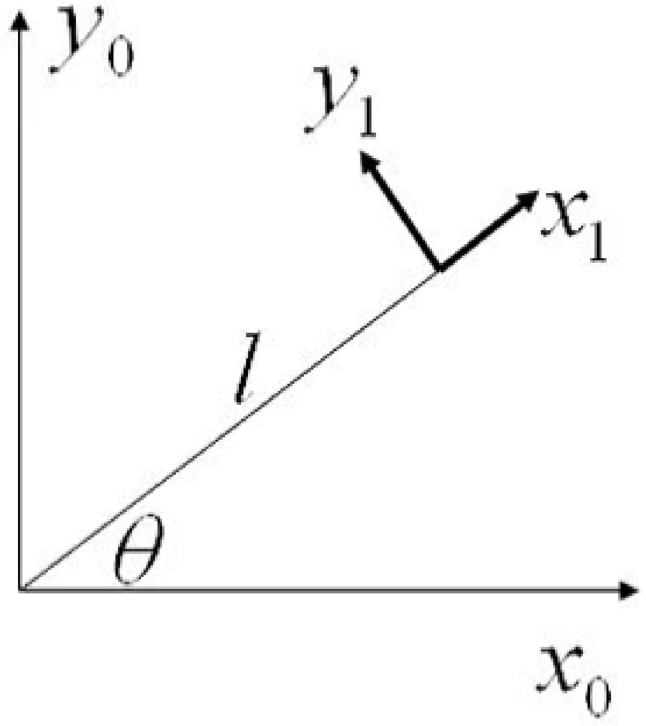
Forward kinematics

$$x_1 = l\cos\theta$$

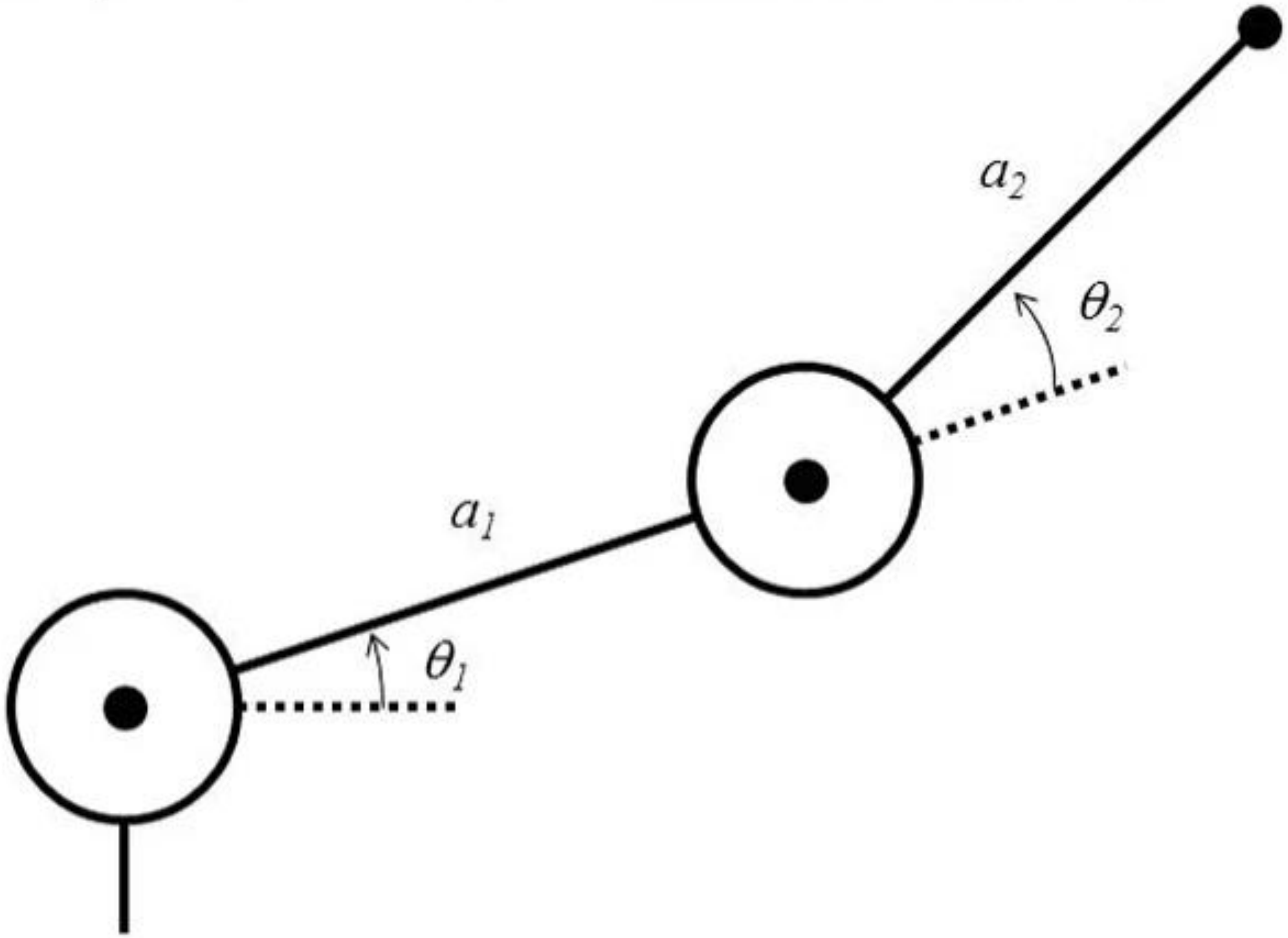
$$y_1 = l\sin\theta$$

Inverse kinematics

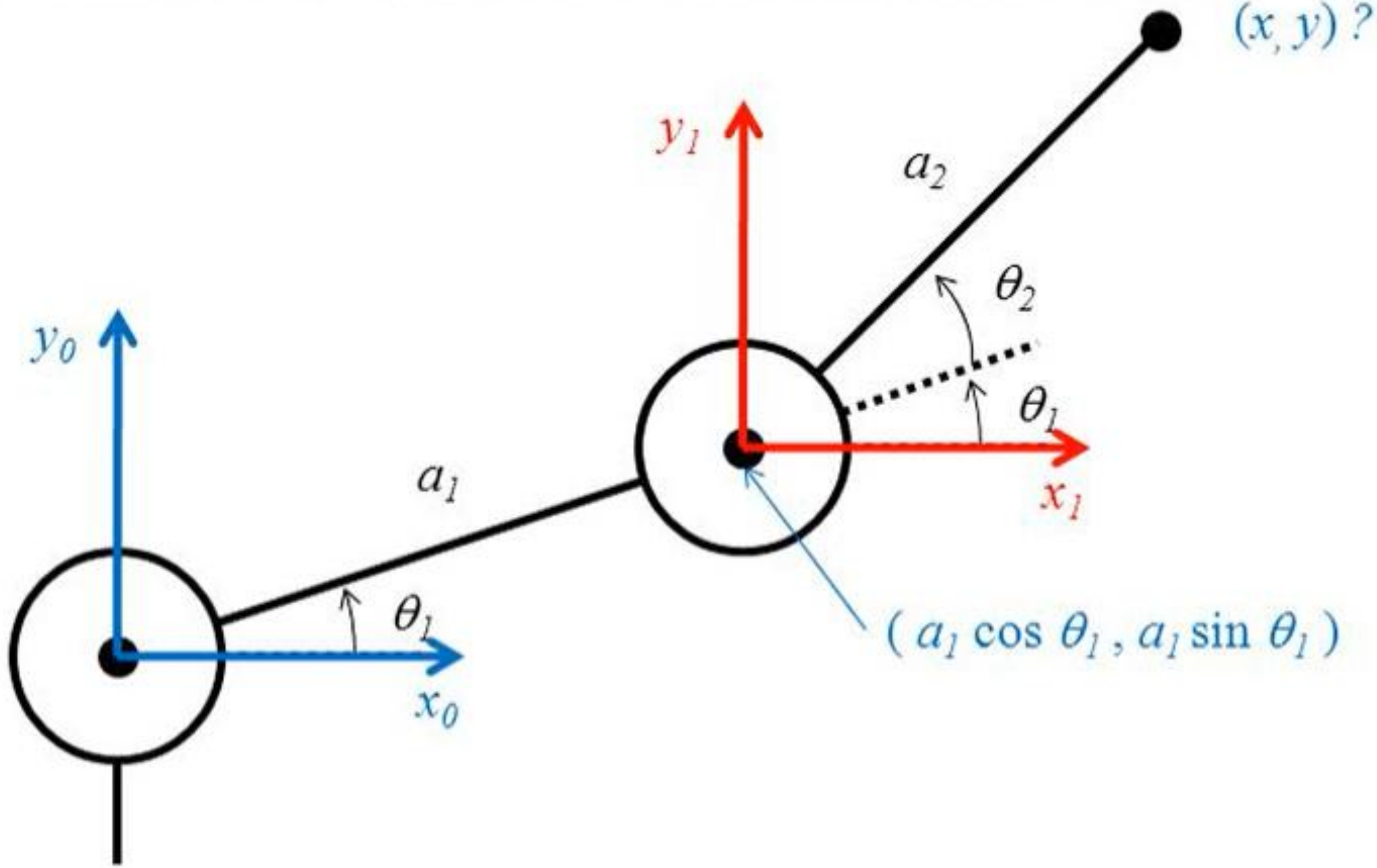
$$\theta = \cos^{-1}(x_1 / l)$$



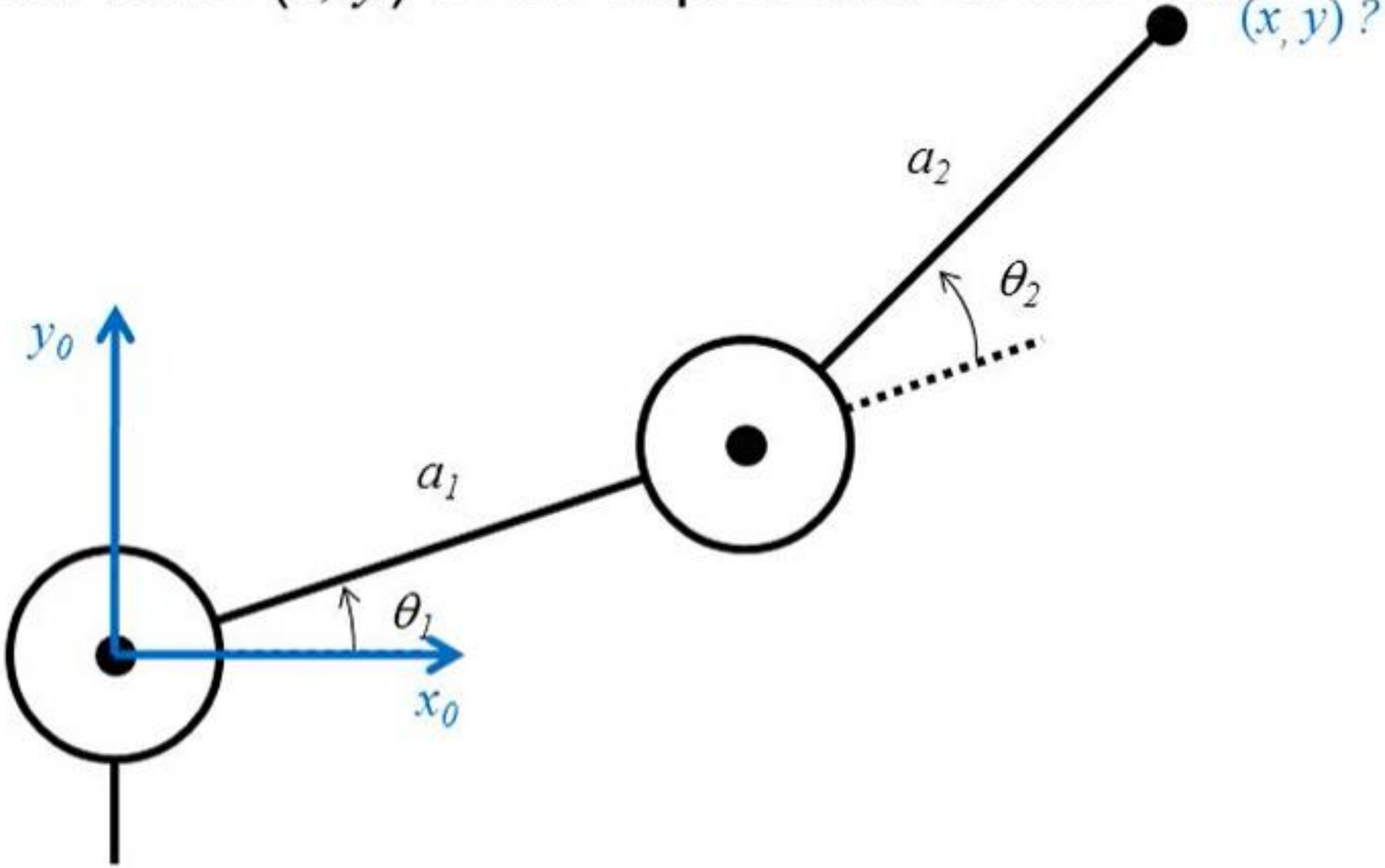
- given the joint variables and dimensions of the links what is the position and orientation of the end effector?



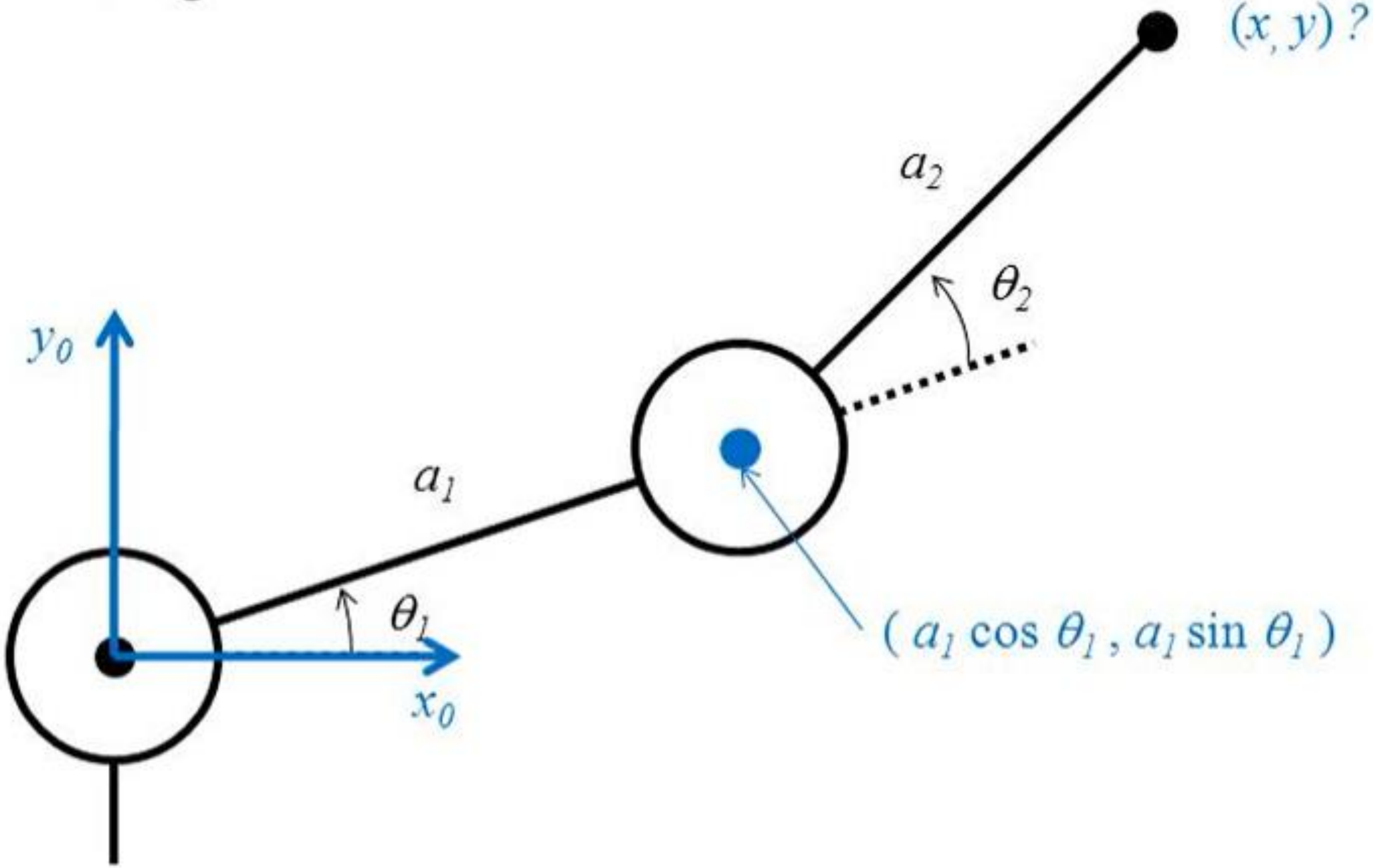
- choose a coordinate frame with origin located on joint *2* with the same orientation as the base frame



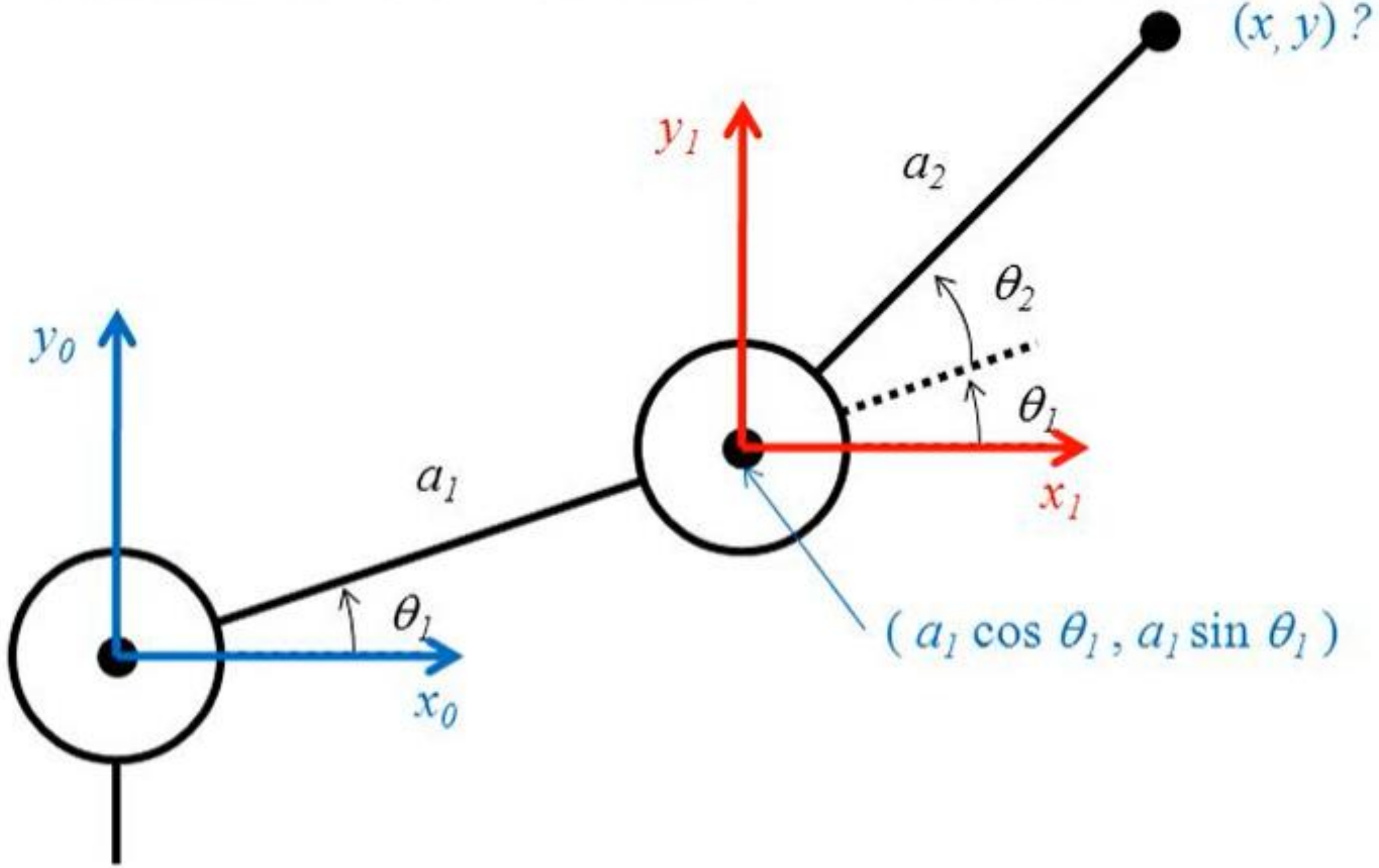
- choose the base coordinate frame of the robot
  - we want (x, y) to be expressed in this frame



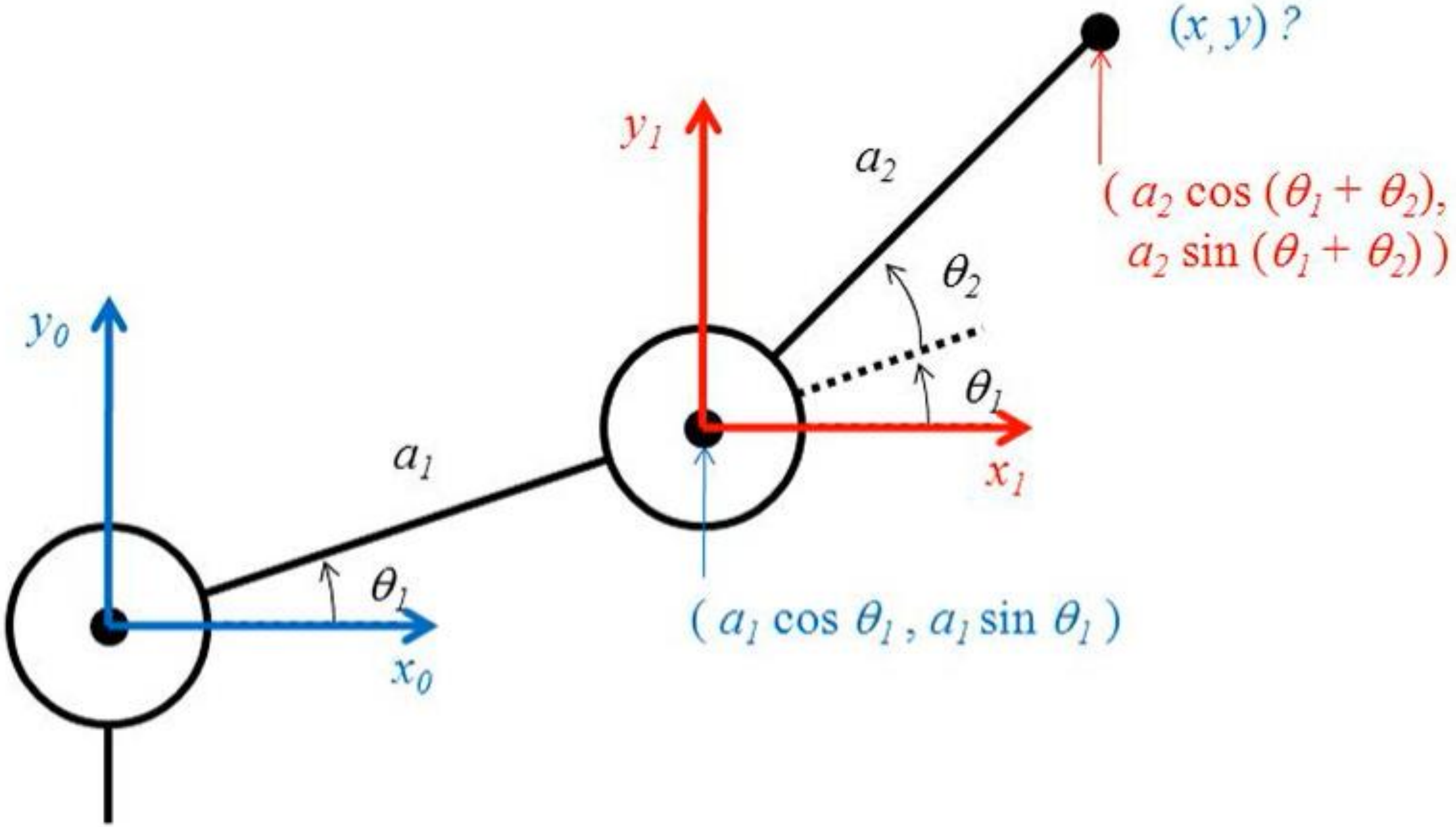
- notice that link *1* moves in a circle centered on the base frame origin



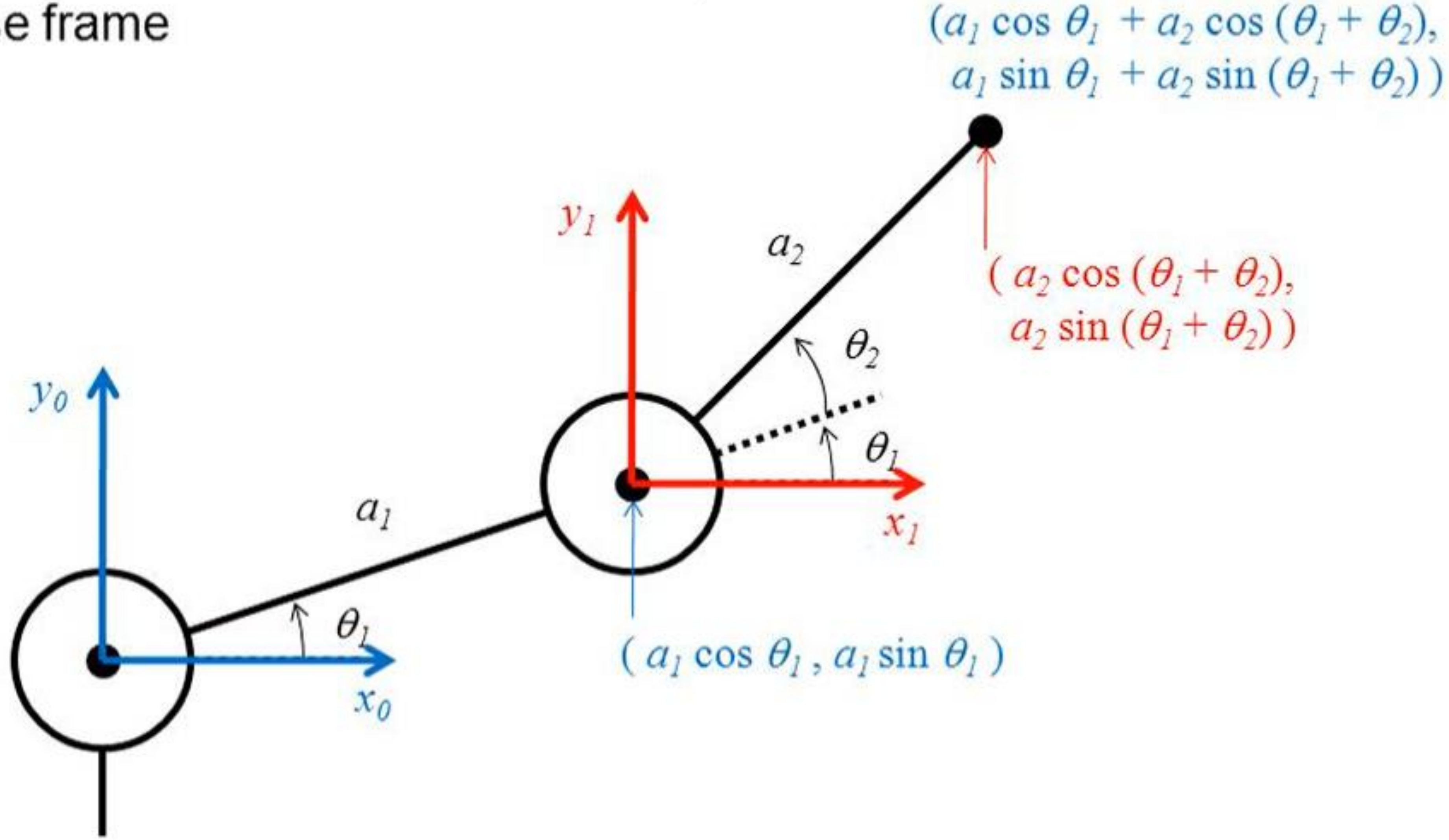
- choose a coordinate frame with origin located on joint *2* with the same orientation as the base frame



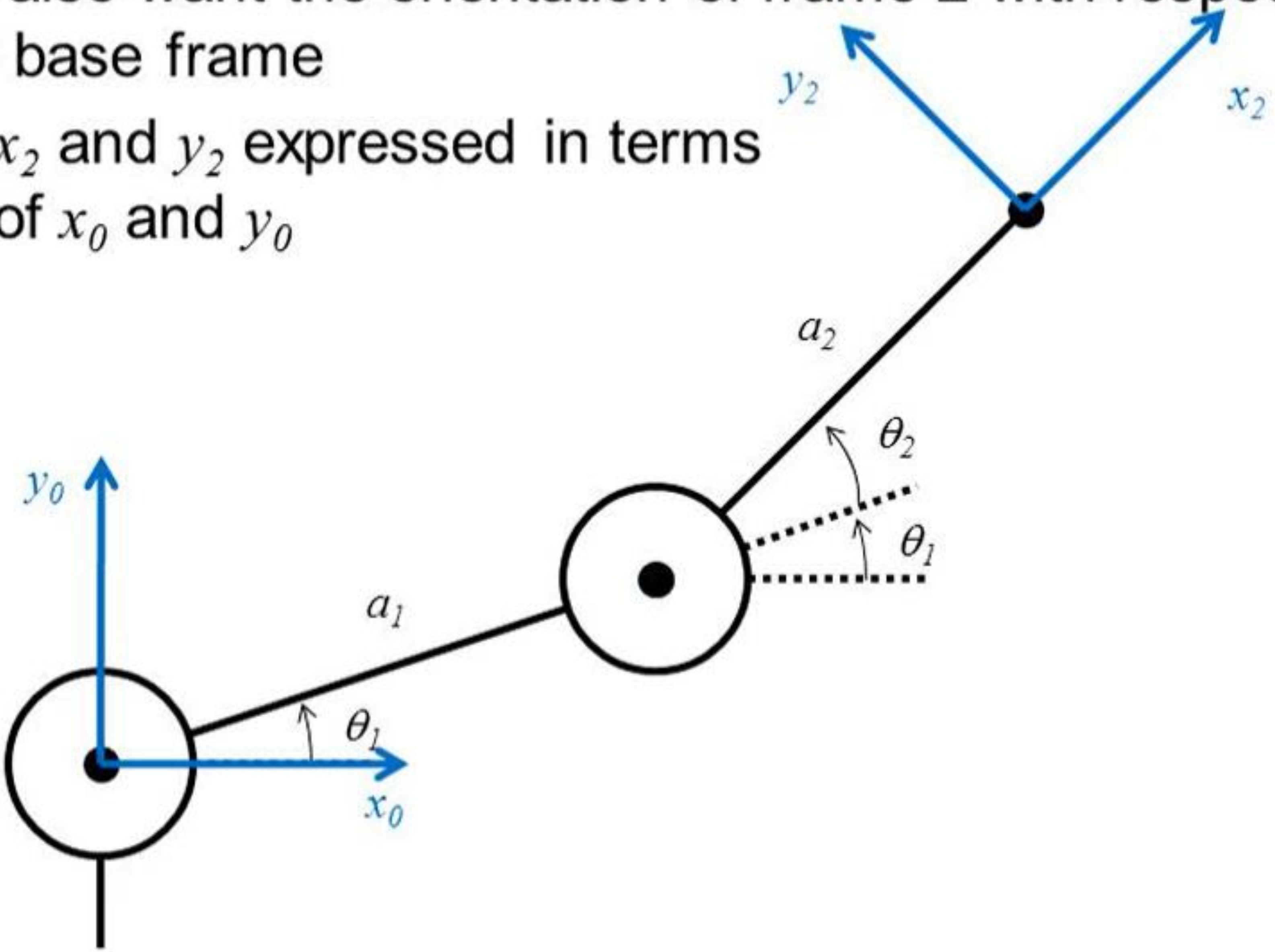
- notice that link *2* moves in a circle centered on frame *1*

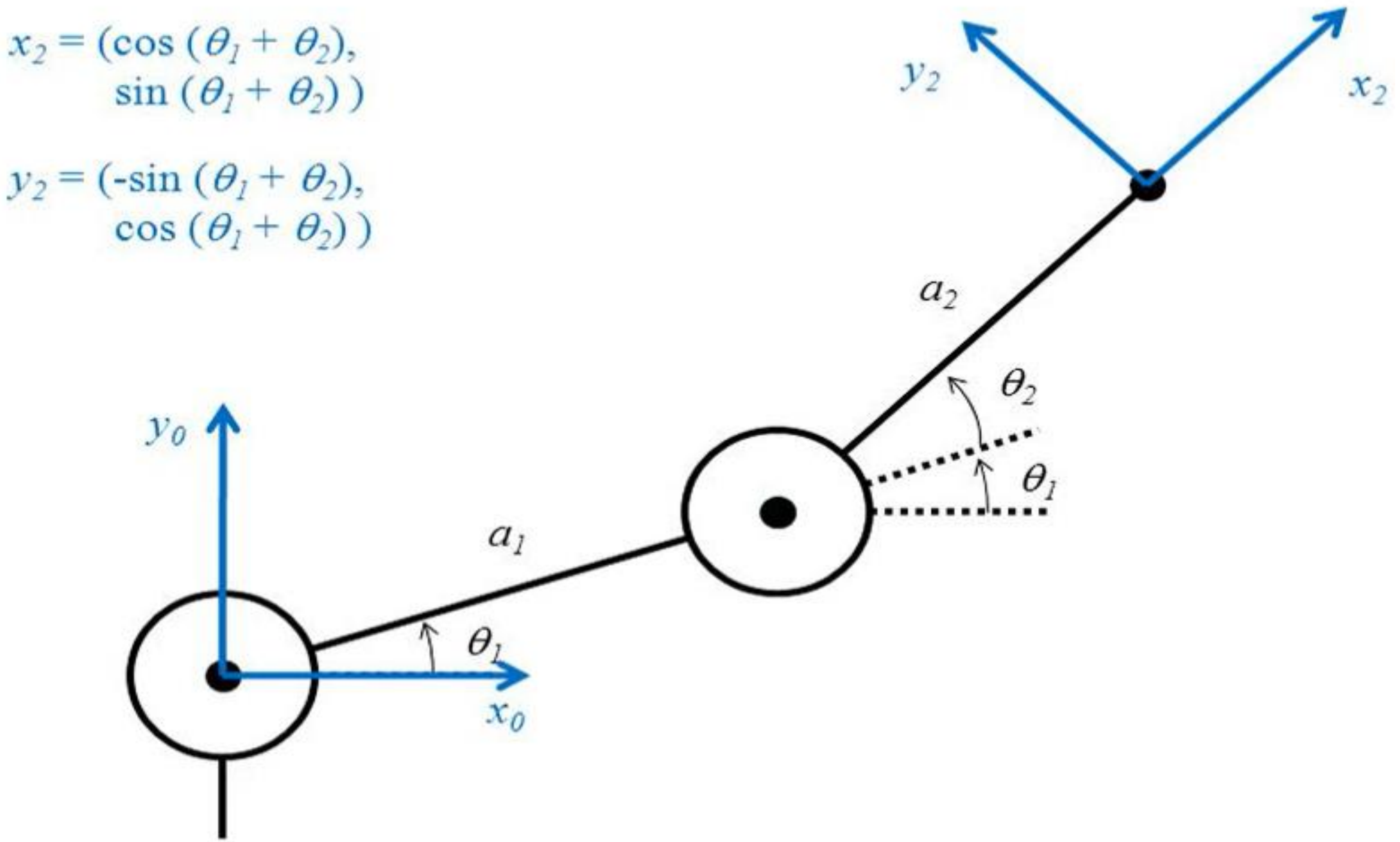


- because the base frame and frame *1* have the same orientation, we can sum the coordinates to find the position of the end effector in the base frame

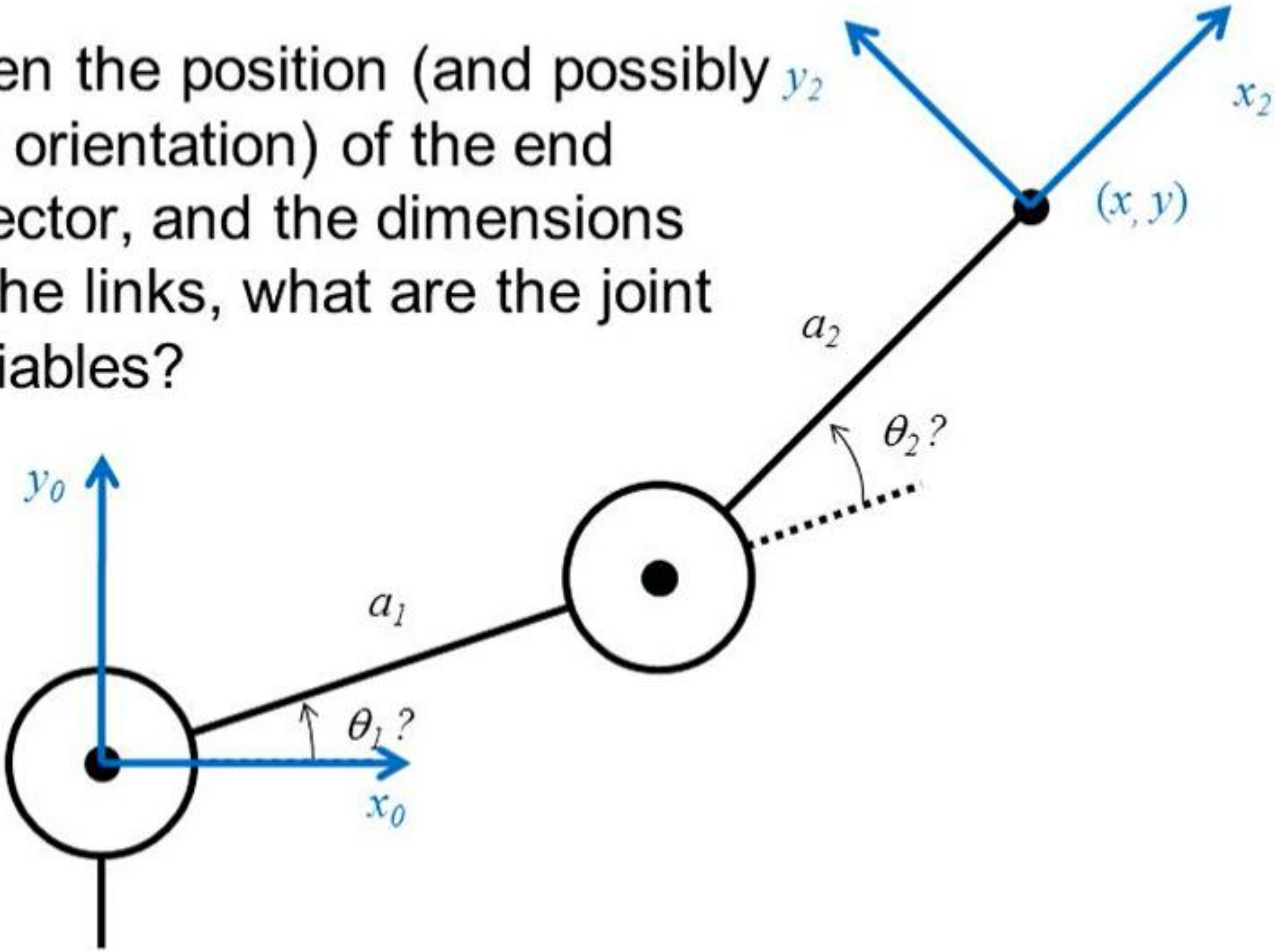


- we also want the orientation of frame 2 with respect to the base frame
  - x_2 and y_2 expressed in terms of x_0 and y_0

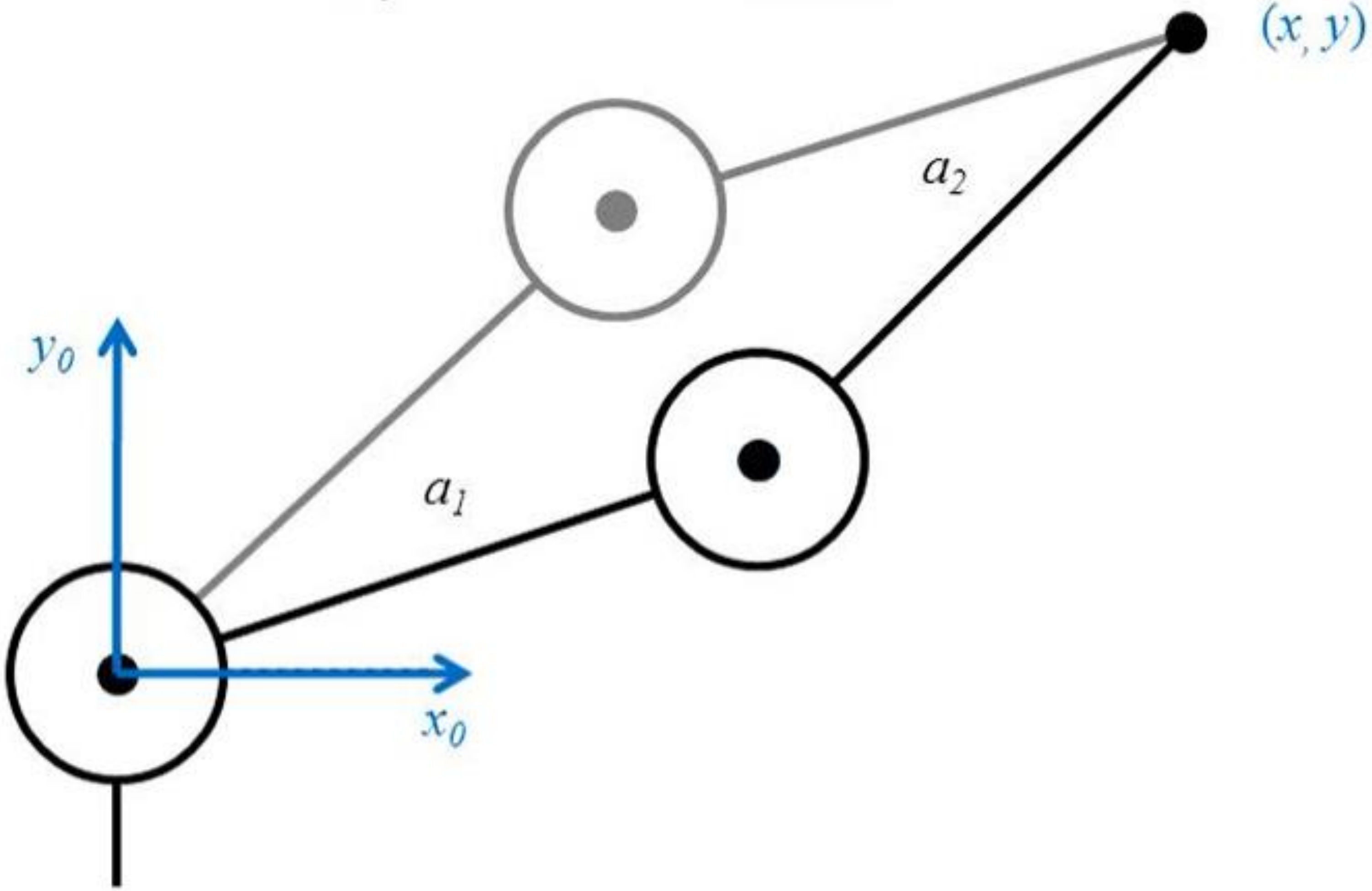



$x_2 = (\cos(\theta_1 + \theta_2),$
$\sin(\theta_1 + \theta_2))$
$y_2 = (-\sin(\theta_1 + \theta_2),$
$\cos(\theta_1 + \theta_2))$
y_2
x_2
a_2
θ_2
θ_1
y_0
a_1
θ_1
x_0

- given the position (and possibly the orientation) of the end effector, and the dimensions of the links, what are the joint variables?

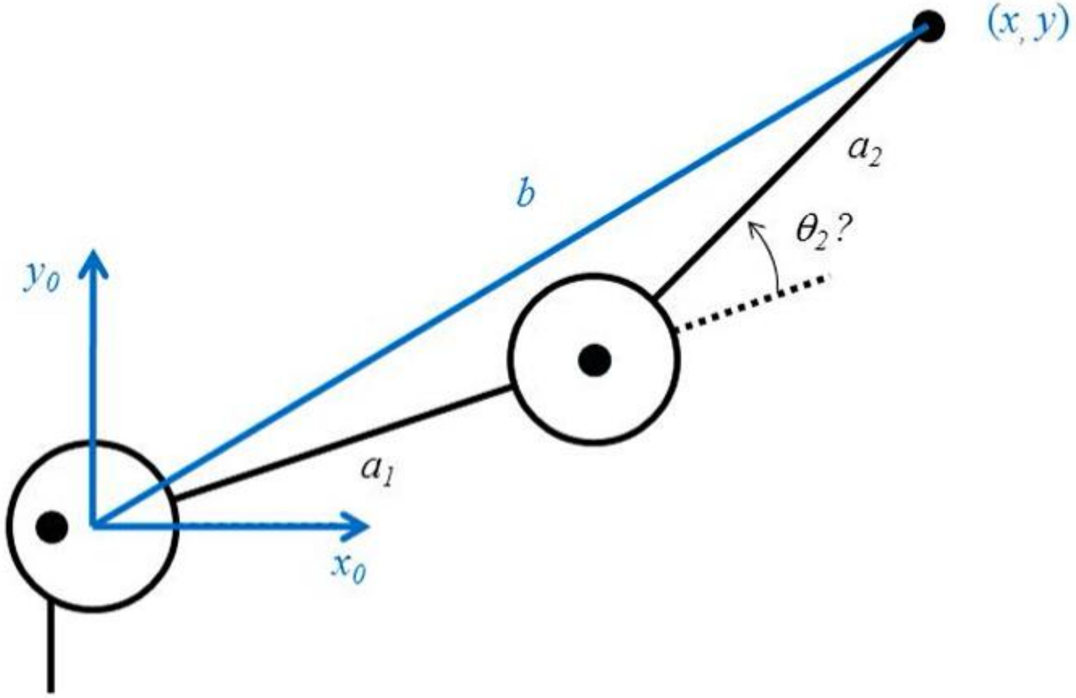


- harder than forward kinematics because there is often more than one possible solution



Law of Cosines

$$b^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2) = x^2 + y^2$$



$$-\cos(\pi-\theta_2)=\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

and we have the trigonometric identity

$$-\cos(\pi-\theta_2)=\cos(\theta_2)$$

therefore,

$$\cos\theta_2=\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}=C_2$$

We could take the inverse cosine, but this gives only one of the two solutions.

Instead, use the two trigonometric identities:

$$\sin^2\theta + \cos^2\theta_2 = 1 \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

to obtain

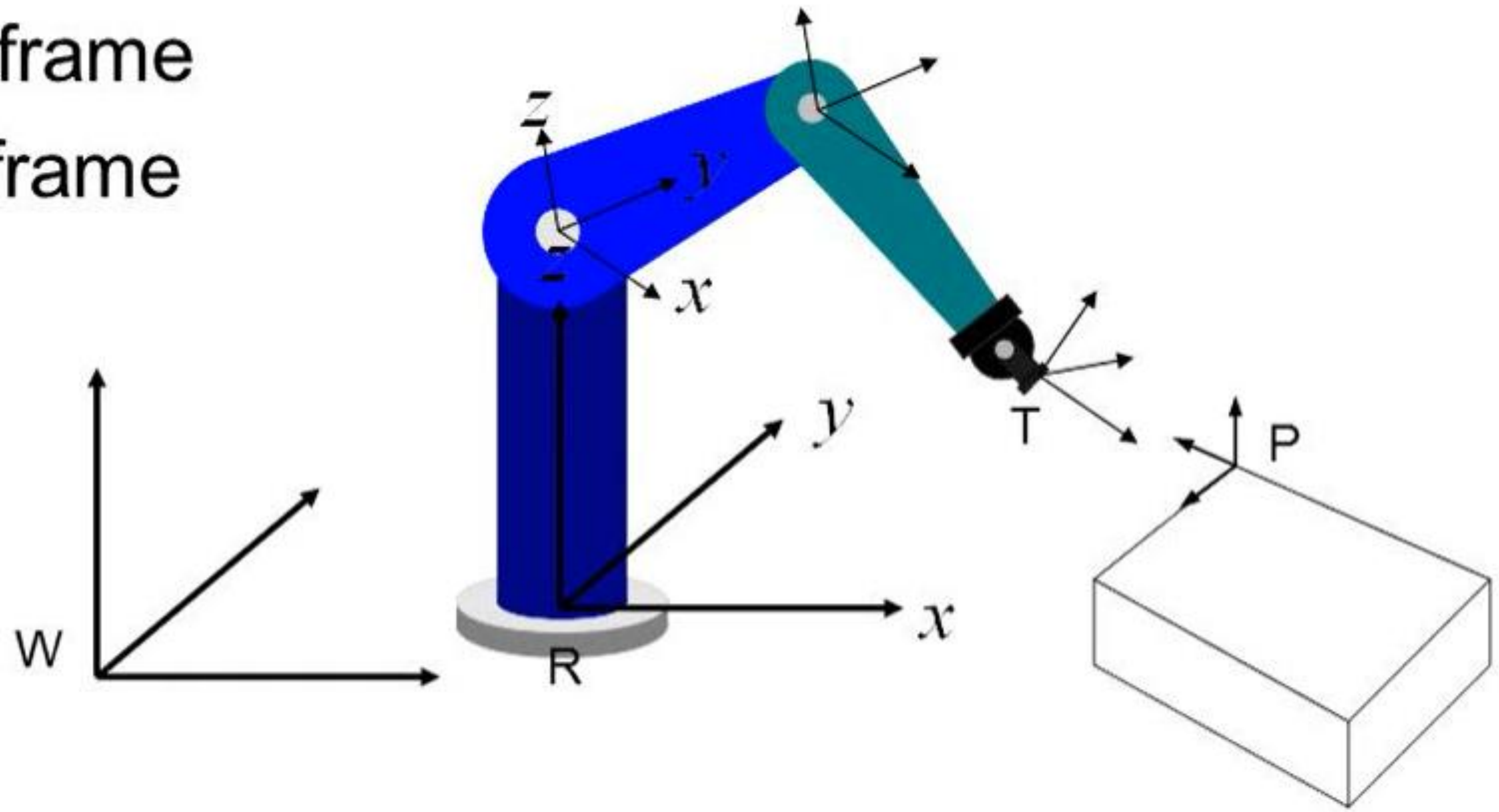
$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1 - C_2^2}}{C_2}$$

which yields both solutions for θ_2. In many programming languages you would use four quadrant inverse tangent function `atan2`

```
c2 = (x*x + y*y - a1*a1 - a2*a2) / (2*a1*a2);
s2 = sqrt(1 - c2*c2);
theta21 = atan2(s2, c2);
theta22 = atan2(-s2, c2);
```

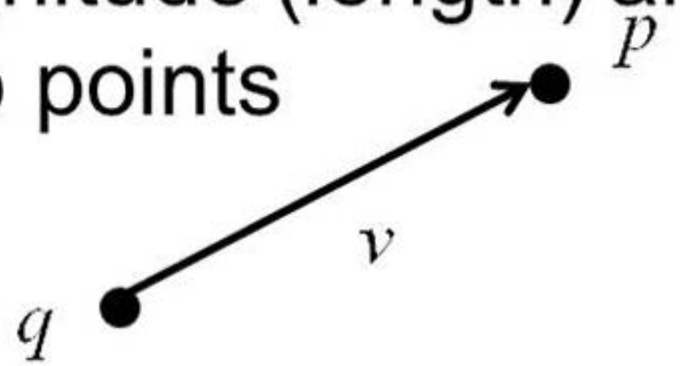
$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{a_2 \sin\theta_2}{a_1 + a_2 \cos\theta_2}\right)$$

- Robot Reference Frames
  - World frame
  - Joint frame
  - Tool frame



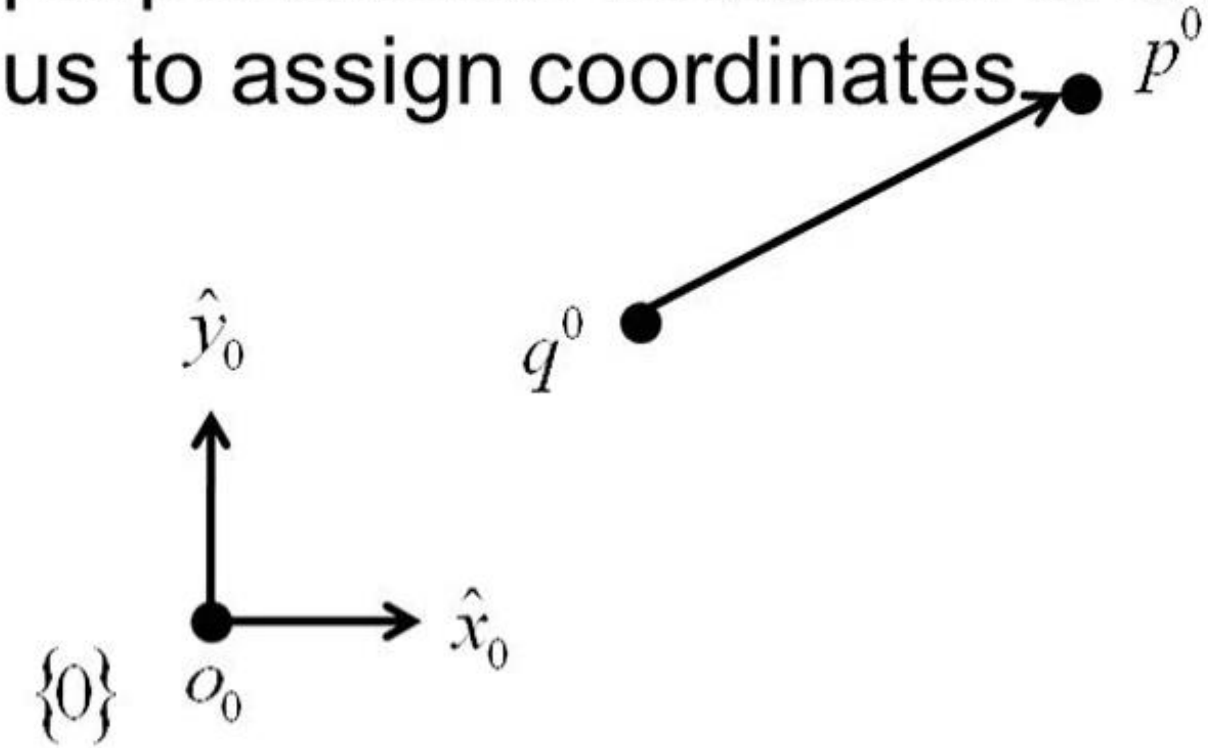
Points and Vectors

- point : a location in space
- vector : magnitude (length) and direction between two points



Coordinate Frames

- choosing a frame (a point and two perpendicular vectors of unit length) allows us to assign coordinates



$$p^0 = \begin{bmatrix} 4 \\ 2.5 \end{bmatrix}$$

$$q^0 = \begin{bmatrix} 2 \\ 1.5 \end{bmatrix}$$

$$v^0 = p^0 - q^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Preliminary

- Coordinate Transformation
  - Reference coordinate frame OXYZ
  - Body-attached frame O'uvw

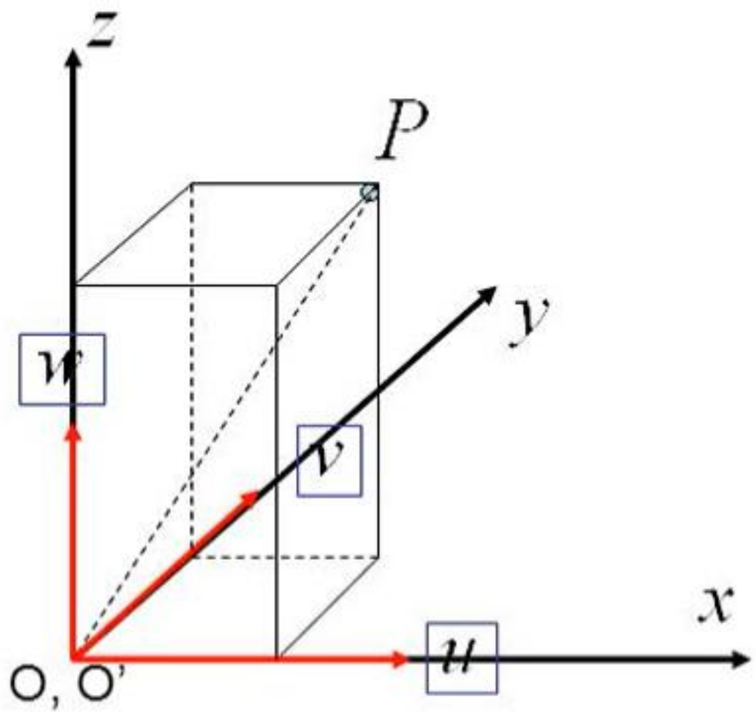
Point represented in OXYZ:

$$P_{xyz} = [p_x, p_y, p_z]^T$$

$$\vec{P}_{xyz} = p_x \mathrm{i}_\mathrm{x} + p_y \mathrm{j}_\mathrm{y} + p_z \mathrm{k}_\mathrm{z}$$

Point represented in O'uvw:

$$\vec{P}_{uvw} = p_u \mathrm{i}_\mathrm{u} + p_v \mathrm{j}_\mathrm{v} + p_w \mathrm{k}_\mathrm{w}$$



Two frames coincide ==> $p_u = p_x \quad p_v = p_y \quad p_w = p_z$

Properties: Dot Product

Let x and y be arbitrary vectors in R^3 and θ be the angle from x to y, then

$$x \cdot y = |x||y| \cos\theta$$

Properties of orthonormal coordinate frame

- Mutually perpendicular

$$\vec{i} \cdot \vec{j} = 0$$

$$\vec{i} \cdot \vec{k} = 0$$

$$\vec{k} \cdot \vec{j} = 0$$

- Unit vectors

$$|\vec{i}| = 1$$

$$|\vec{j}| = 1$$

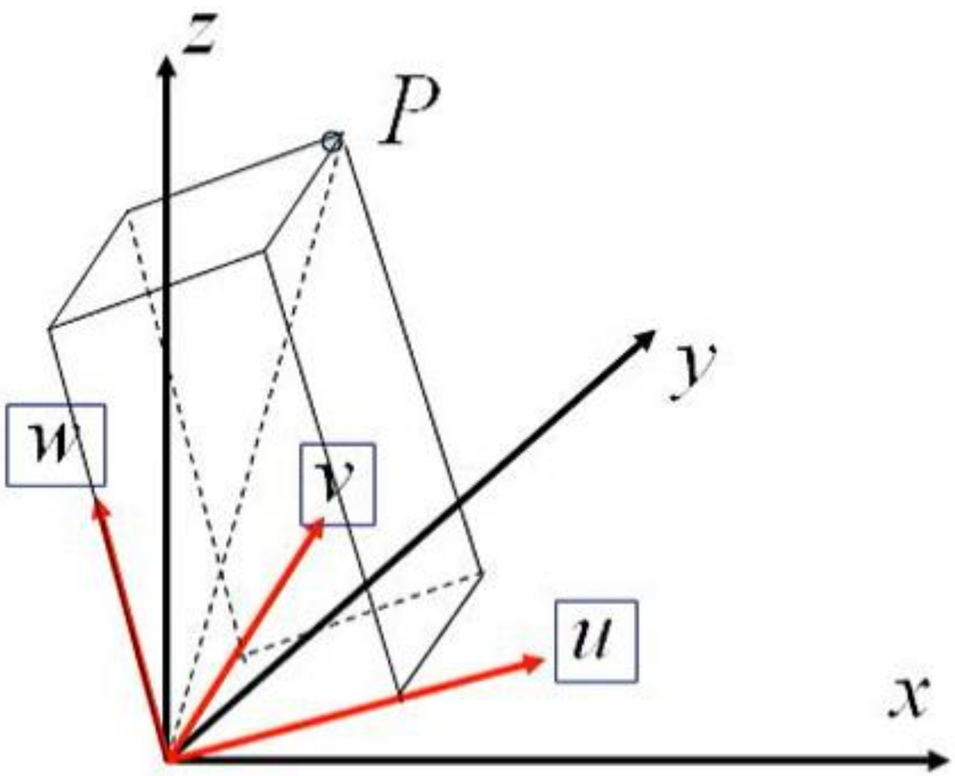
$$|\vec{k}| = 1$$

- Coordinate Transformation
  - Rotation only

$$\vec{P}_{xyz} = p_x \mathrm{i}_\mathrm{x} + p_y \mathrm{j}_\mathrm{y} + p_z \mathrm{k}_\mathrm{z}$$

$$\vec{P}_{uvw} = p_u \mathrm{i}_\mathrm{u} + p_v \mathrm{j}_\mathrm{v} + p_w \mathrm{k}_\mathrm{w}$$

$$P_{xyz} = RP_{uvw}$$



How to relate the coordinate in these two frames?

- Basic Rotation
  - p_x , p_y, and p_z represent the projections of P onto OX, OY, OZ axes, respectively
  - Since $P = p_u \mathrm{i_u} + p_v \mathrm{j_v} + p_w \mathrm{k_w}$

$$p_x = \mathrm{i_x} \cdot P = \mathrm{i_x} \cdot \mathrm{i_u} p_u + \mathrm{i_x} \cdot \mathrm{j_v} p_v + \mathrm{i_x} \cdot \mathrm{k_w} p_w$$

$$p_y = \mathrm{j_y} \cdot P = \mathrm{j_y} \cdot \mathrm{i_u} p_u + \mathrm{j_y} \cdot \mathrm{j_v} p_v + \mathrm{j_y} \cdot \mathrm{k_w} p_w$$

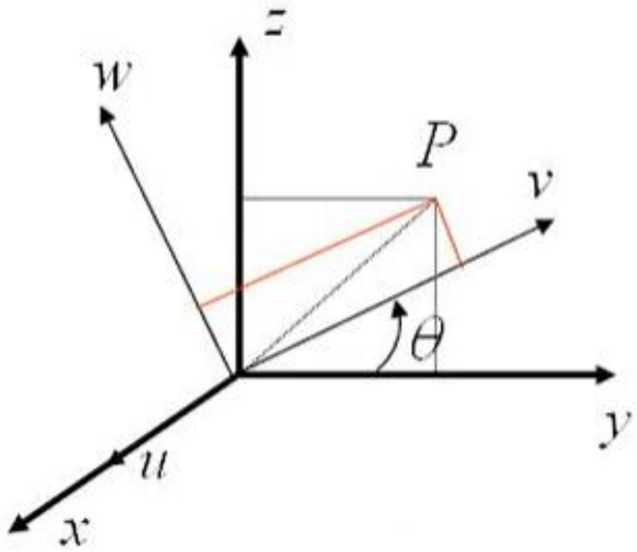
$$p_z = \mathrm{k_z} \cdot P = \mathrm{k_z} \cdot \mathrm{i_u} p_u + \mathrm{k_z} \cdot \mathrm{j_v} p_v + \mathrm{k_z} \cdot \mathrm{k_w} p_w$$

- Basic Rotation Matrix

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

  - Rotation about x-axis with θ

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$



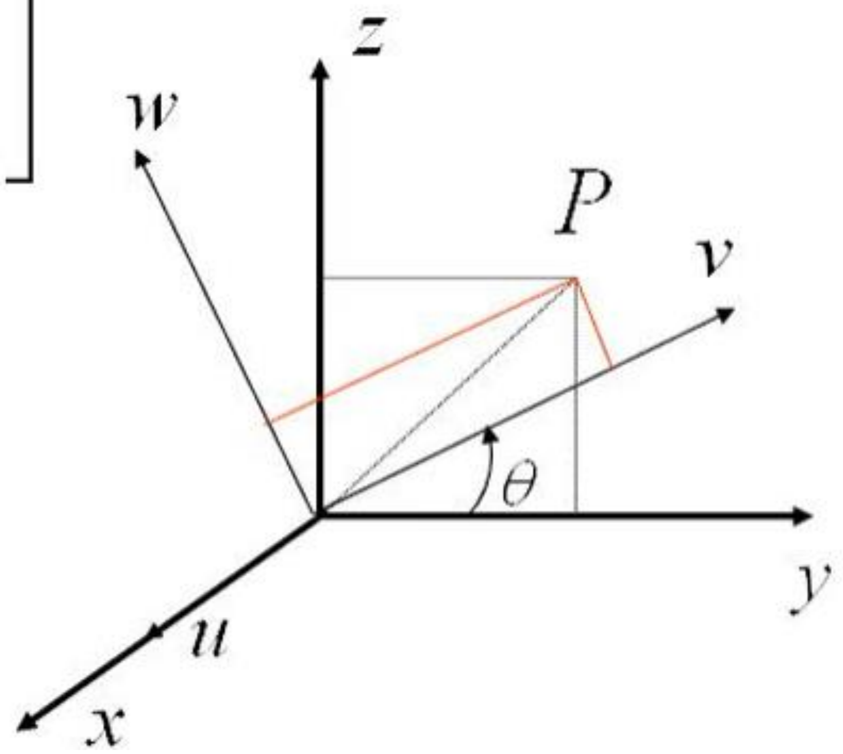
- Is it True?
  - Rotation about x axis with θ

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}$$

$$p_x = p_u$$

$$p_y = p_v \cos\theta - p_w \sin\theta$$

$$p_z = p_v \sin\theta + p_w \cos\theta$$



– Rotation about x-axis with θ

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

– Rotation about y-axis with θ

$$Rot(y,\theta) = \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix}$$

– Rotation about z-axis with θ

$$P_{xyz} = RP_{uvw}$$

$$Rot(z,\theta) = \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Basic Rotation Matrix

$$R = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} \qquad P_{xyz} = RP_{uvw}$$

  - Obtain the coordinate of P_{uvw} from the coordinate of P_{xyz}

Dot products are commutative!

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \mathbf{i}_u \cdot \mathbf{i}_x & \mathbf{i}_u \cdot \mathbf{j}_y & \mathbf{i}_u \cdot \mathbf{k}_z \\ \mathbf{j}_v \cdot \mathbf{i}_x & \mathbf{j}_v \cdot \mathbf{j}_y & \mathbf{j}_v \cdot \mathbf{k}_z \\ \mathbf{k}_w \cdot \mathbf{i}_x & \mathbf{k}_w \cdot \mathbf{j}_y & \mathbf{k}_w \cdot \mathbf{k}_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad P_{uvw} = QP_{xyz}$$

$$Q = R^{-1} = R^T$$

$$QR = R^T R = R^{-1} R = I_3$$

<== 3X3 identity matrix

Properties of Rotation Matrices

- $R^T = R^{-1}$
- the columns of R are mutually orthogonal
- each column of R is a unit vector
- $\det R = 1$ (the determinant is equal to 1)

- A point $a_{uvw} = (4,3,2)$ is attached to a rotating frame, the frame rotates 60 degree about the OZ axis of the reference frame. Find the coordinates of the point relative to the reference frame after the rotation.

$$a_{xyz} = Rot(z,60)a_{uvw}$$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2 \end{bmatrix}$$

- A point $a_{xyz} = (4,3,2)$ is the coordinate w.r.t. the reference coordinate system, find the corresponding point a_{uvw} w.r.t. the rotated OU-V-W coordinate system if it has been rotated 60 degree about OZ axis.

$$a_{uvw} = Rot(z,60)^T a_{xyz}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4.598 \\ -1.964 \\ 2 \end{bmatrix}$$

Composite Rotation Matrix

- A sequence of finite rotations
  - matrix multiplications do not commute
  - rules:
    - if rotating coordinate O-U-V-W is rotating about principal axis of OXYZ frame, then ***Pre-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix
    - if rotating coordinate OUVW is rotating about its own principal axes, then ***post-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix

- Find the rotation matrix for the following operations:

Rotation ϕ about OY axis

Rotation θ about OW axis

Rotation α about OU axis

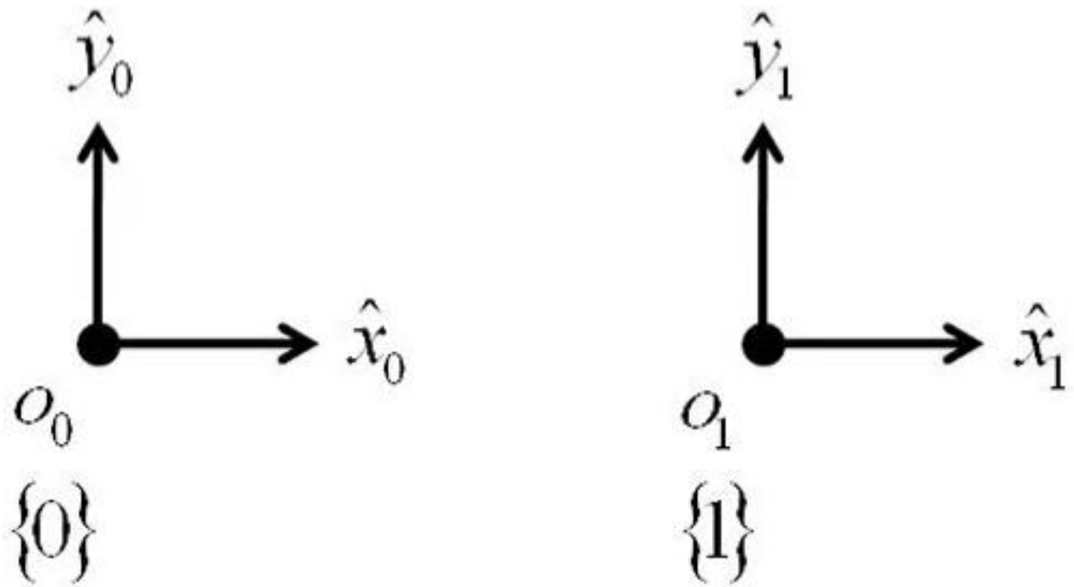
Answer...

$$R = Rot(y,\phi) I_3 Rot(w,\theta) Rot(u,\alpha)$$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

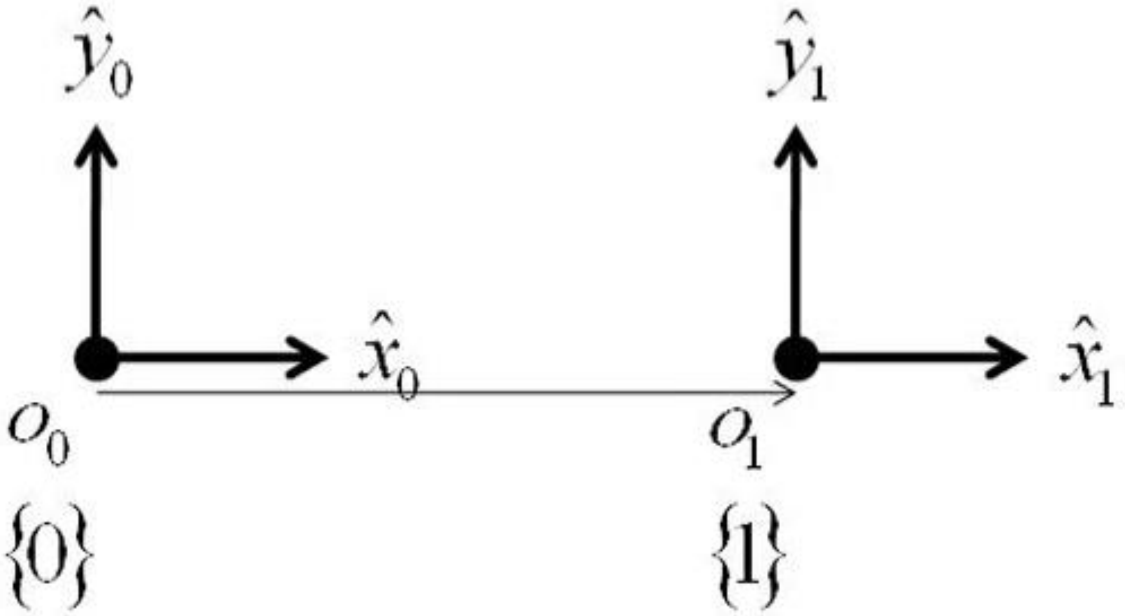
$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

Pre-multiply if rotate about the OXYZ axes
Post-multiply if rotate about the OUVW axes



$$o_1^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

- suppose we are given o_1 expressed in $\{0\}$



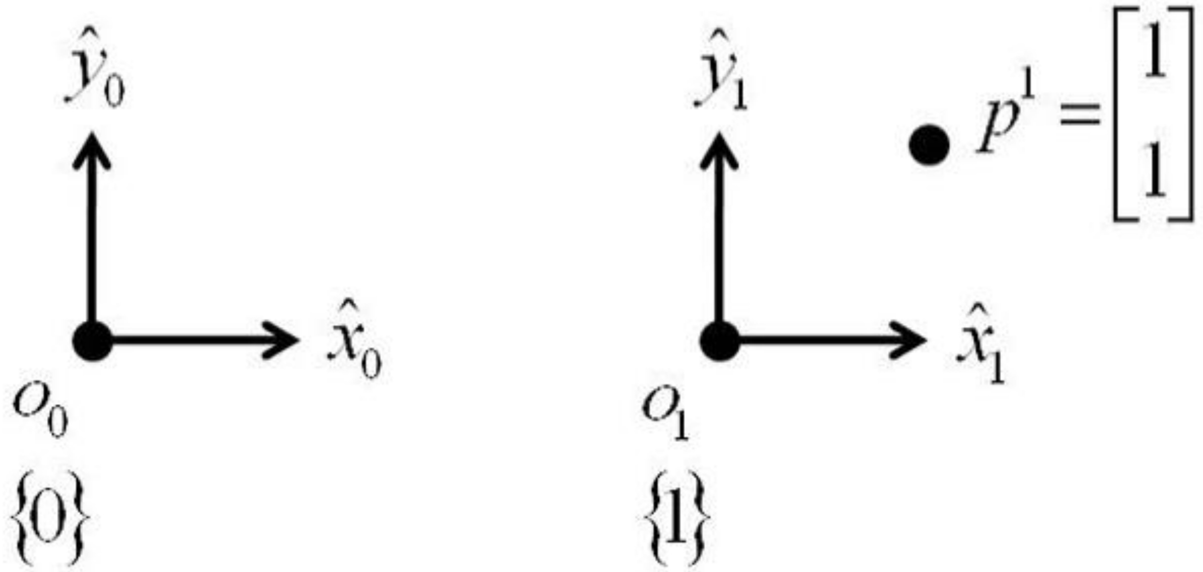
$$d_1^0 = o_1^0 - o_0^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

- the location of $\{1\}$ expressed in $\{0\}$

Translation 1

1. the translation vector d^i_j can be interpreted as the location of frame $\{j\}$ expressed in frame $\{i\}$

Translation 2



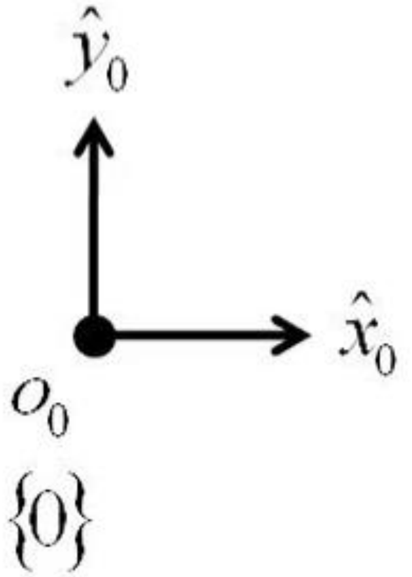
$$p^0 = d_1^0 + p^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

- p^1 expressed in $\{0\}$

2. the translation vector d^i_j can be interpreted as a coordinate transformation of a point from frame $\{j\}$ to frame $\{i\}$

Translation 3

$$p^0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad q^0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$q^0 = d \; + p^0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- q^0 expressed in $\{0\}$

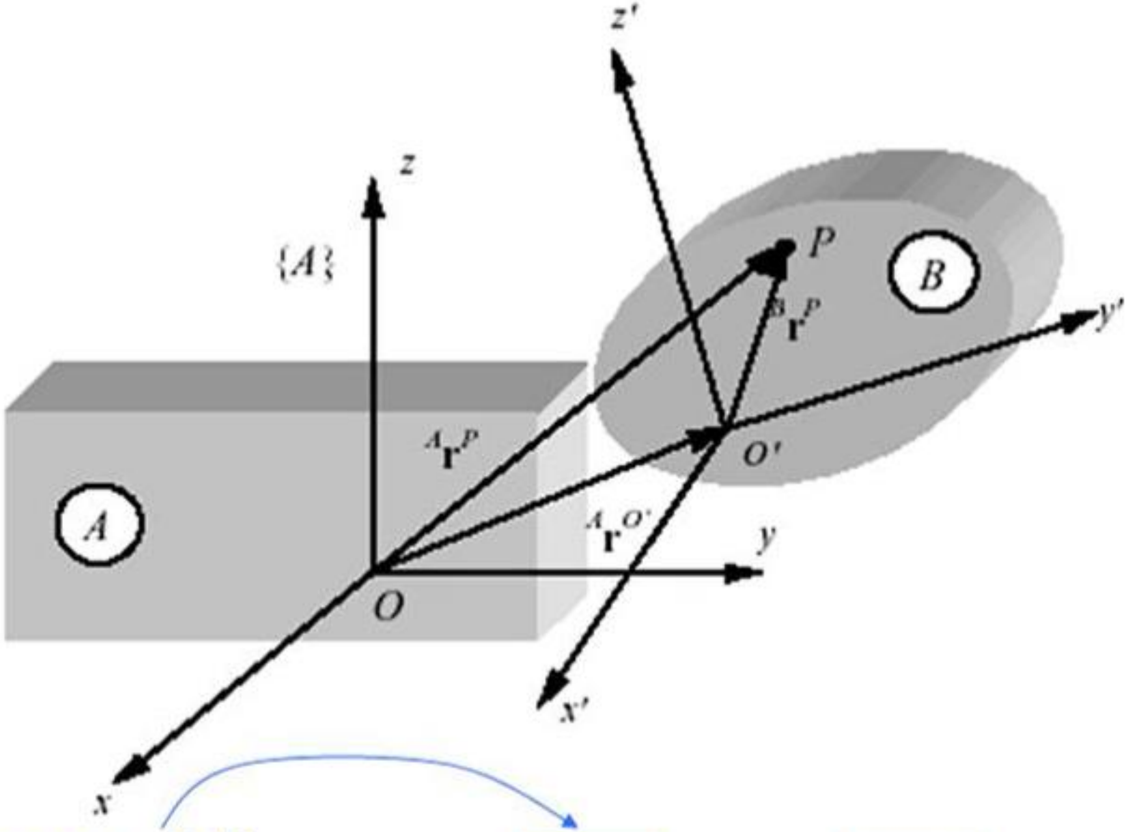
Translation 3

3. the translation vector d can be interpreted as an operator that takes a point and moves it to a new point in the same frame

Coordinate Transformations

- position vector of P in $\{B\}$ is transformed to position vector of P in $\{A\}$

- description of $\{B\}$ as seen from an observer in $\{A\}$



Rotation of $\{B\}$ with respect to $\{A\}$

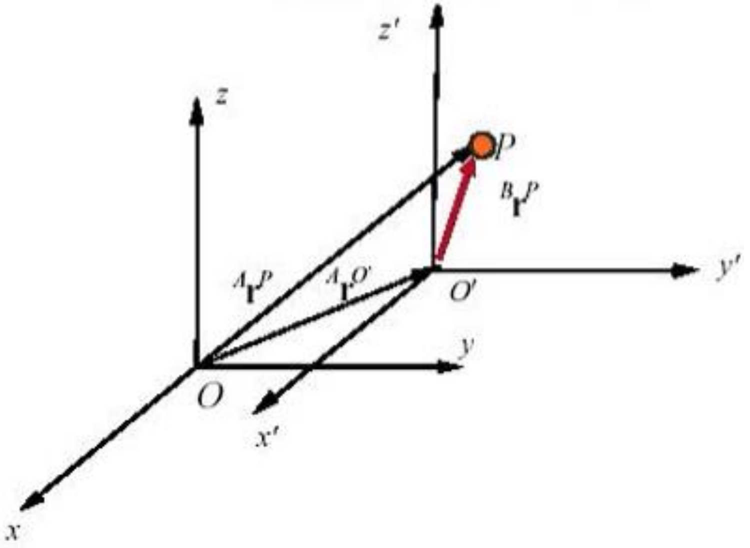
$$\mathbf{r}_P^A = \boxed{{}^A\mathbf{R}_B}\ \mathbf{r}_P^B + \boxed{\mathbf{r}_{O'}^A}$$

Translation of the origin of $\{B\}$ with respect to origin of $\{A\}$

Coordinate Transformations

- Two Special Cases

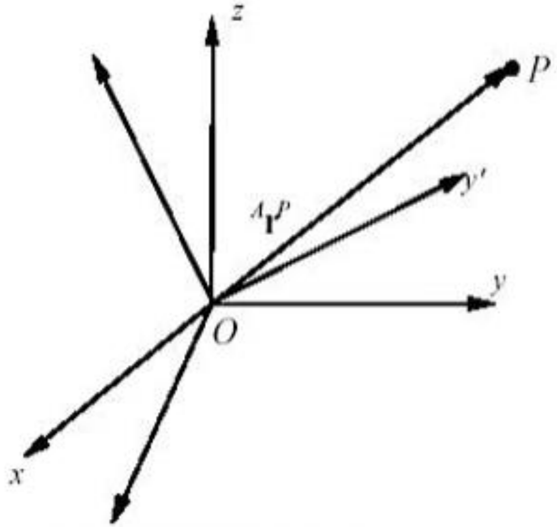
$$\boldsymbol{r}_P^A = {}^A R_B \boldsymbol{r}_P^B + \boldsymbol{r}_{o'}^A$$



1. Translation only

– Axes of {*B*} and {*A*} are parallel

$${}^A R_B = 1$$



2. Rotation only

– Origins of {*B*} and {*A*} are coincident

$$\boldsymbol{r}_{o'}^A = \boldsymbol{0}$$

- Coordinate transformation from $\{B\}$ to $\{A\}$

$$r_P^A = {}^A R_B \, r_P^B + r_{o'}^A$$

$$\begin{bmatrix} r_P^A \\ 1 \end{bmatrix} = \begin{bmatrix} {}^A R_B & r_{o'}^A \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} r_P^B \\ 1 \end{bmatrix}$$

- Homogeneous transformation matrix

$${}^A T_B = \begin{bmatrix} {}^A R_B & r_{o'}^A \\ 0_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} \boxed{R_{3\times 3}} & \boxed{P_{3\times 1}} \\ 0 & \boxed{1} \end{bmatrix}$$

Rotation matrix

Position vector

Scaling

Homogeneous Transformation

- Special cases

1. Translation

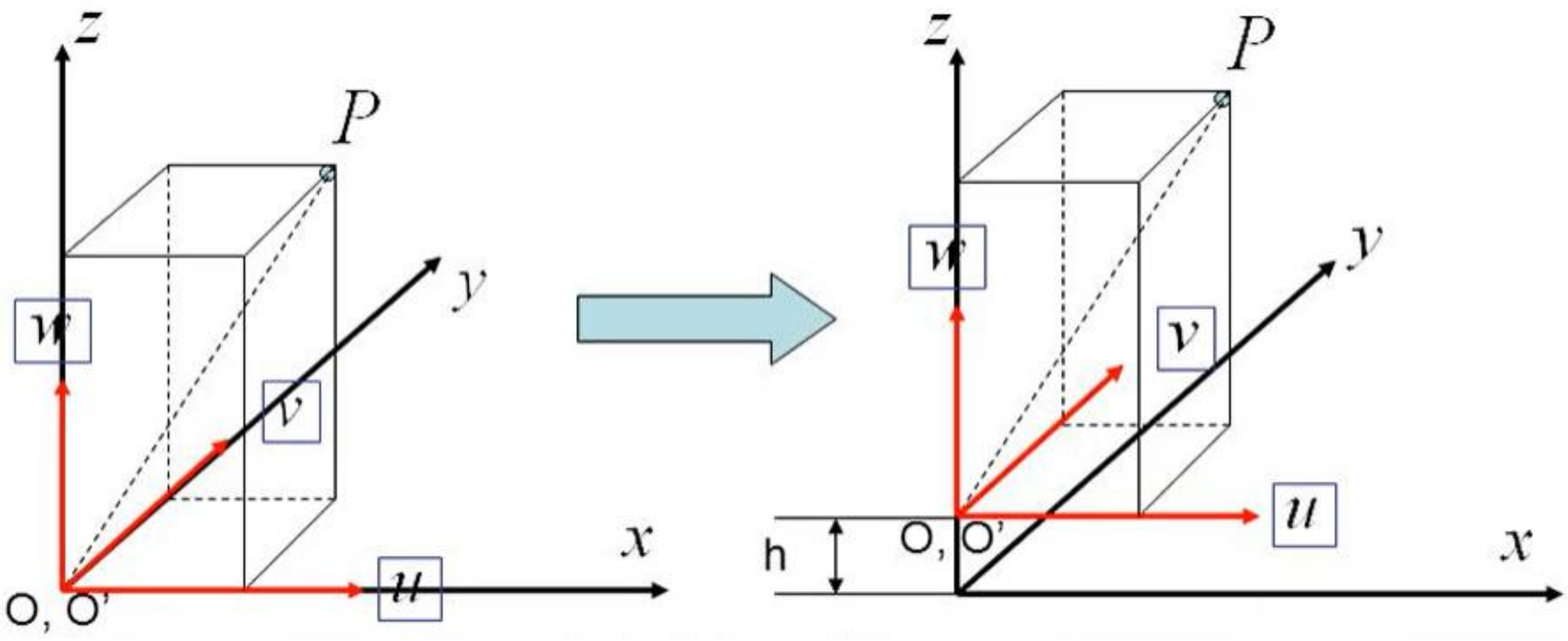
$$ {}^{A}T_{B} = \begin{bmatrix} \boldsymbol{I}_{3\times 3} & \boldsymbol{r}_{o'}^{A} \\ \boldsymbol{0}_{1\times 3} & \boldsymbol{1} \end{bmatrix} $$

2. Rotation

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{bmatrix} $$

- Translation along Z-axis with h:

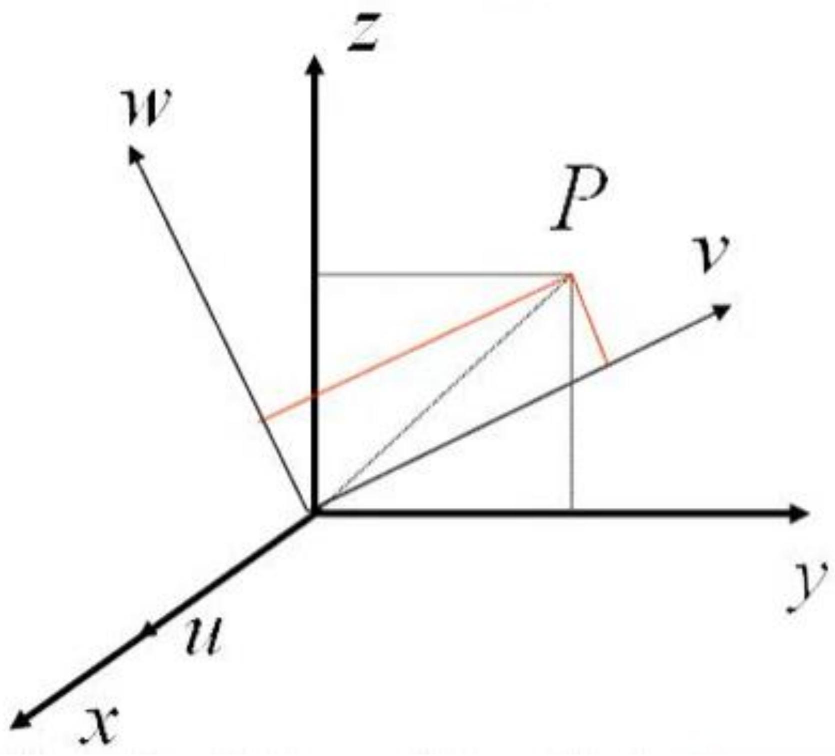
$$Trans(z,h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix} = \begin{bmatrix} p_u \\ p_v \\ p_w + h \\ 1 \end{bmatrix}$$



- Rotation about the X-axis by

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}p_u\\ p_v\\ p_w\\ 1\end{bmatrix}$$



Homogeneous Transformation

- Composite Homogeneous Transformation Matrix
- Rules:
  - Transformation (rotation/translation) w.r.t (X,Y,Z) (OLD FRAME), using pre-multiplication
  - Transformation (rotation/translation) w.r.t (U,V,W) (NEW FRAME), using post-multiplication

Example

- Find the homogeneous transformation matrix (T) for the following operations:

Rotation α about OX axis

Translation of a along OX axis

Translation of d along OZ axis

Rotation of θ about OZ axis

$$T = T_{z,\theta} T_{z,d} T_{x,a} T_{x,\alpha} I_{4\times 4}$$

Answer:

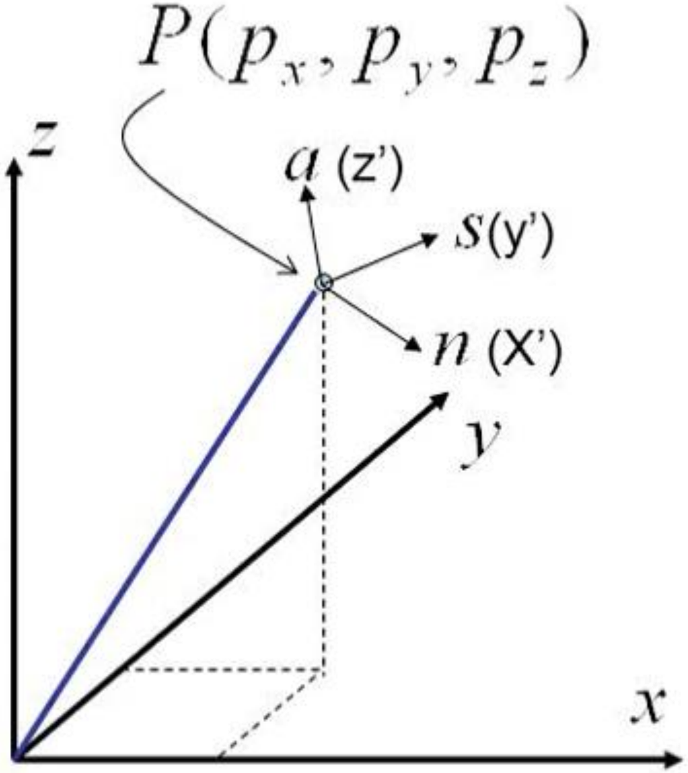
$$= \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Representation

- A frame in space (Geometric Interpretation)

$$F = \begin{bmatrix} R_{3\times3} & P_{3\times1} \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} \boxed{\begin{matrix} n_x \\ n_y \\ n_z \end{matrix}} & \begin{matrix} s_x \\ s_y \\ s_z \end{matrix} & \begin{matrix} a_x \\ a_y \\ a_z \end{matrix} & \begin{matrix} p_x \\ p_y \\ p_z \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

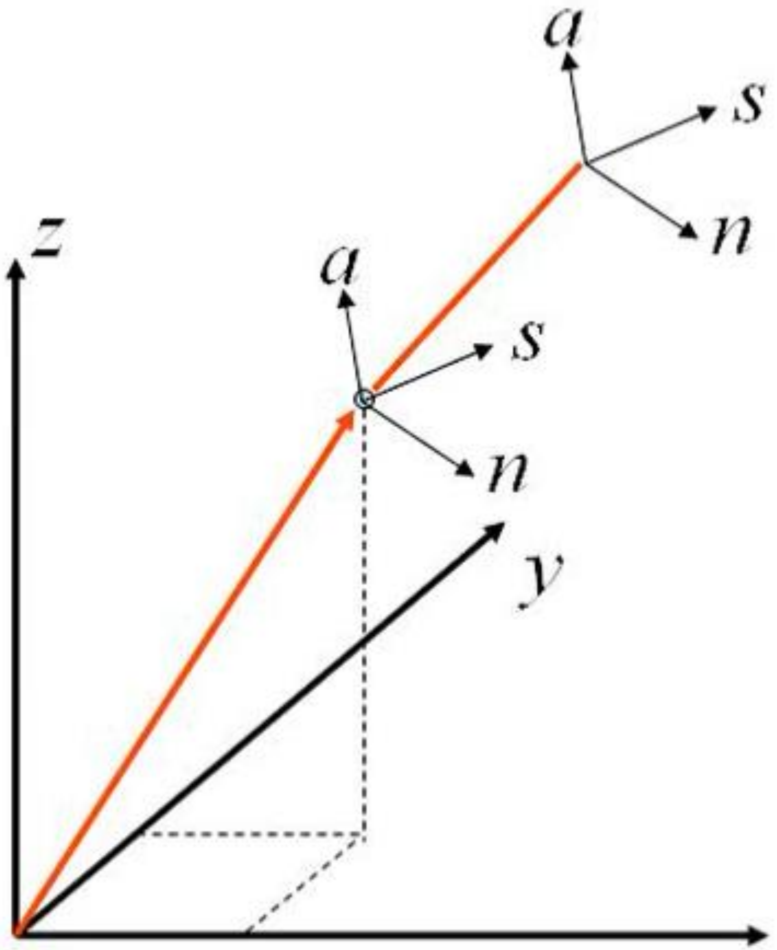


Principal axis *n* w.r.t. the reference coordinate system

- Translation

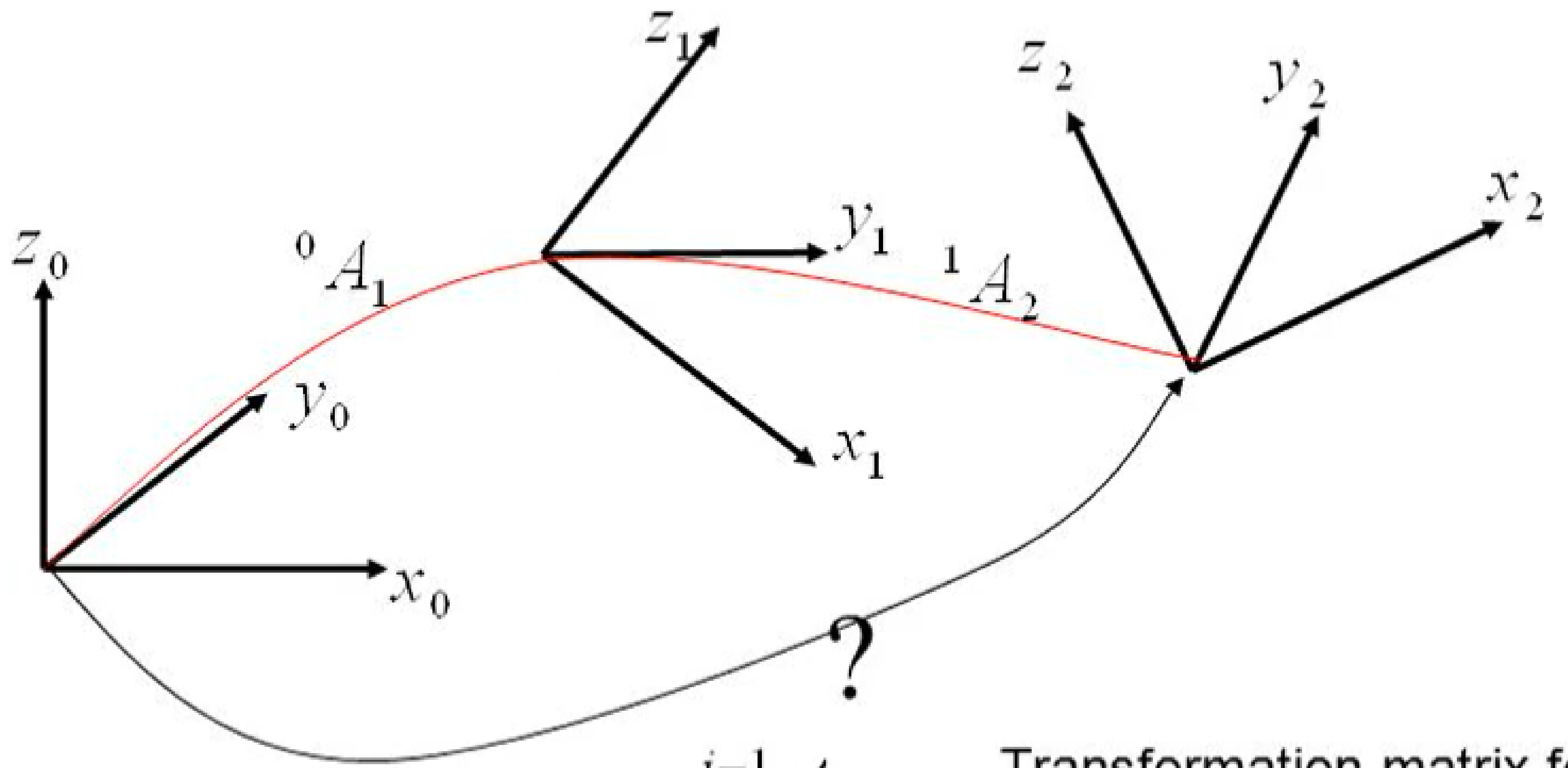
$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & s_x & a_x & p_x + d_x \\ n_y & s_y & a_y & p_y + d_y \\ n_z & s_z & a_z & p_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old}$$

Composite Homogeneous Transformation Matrix

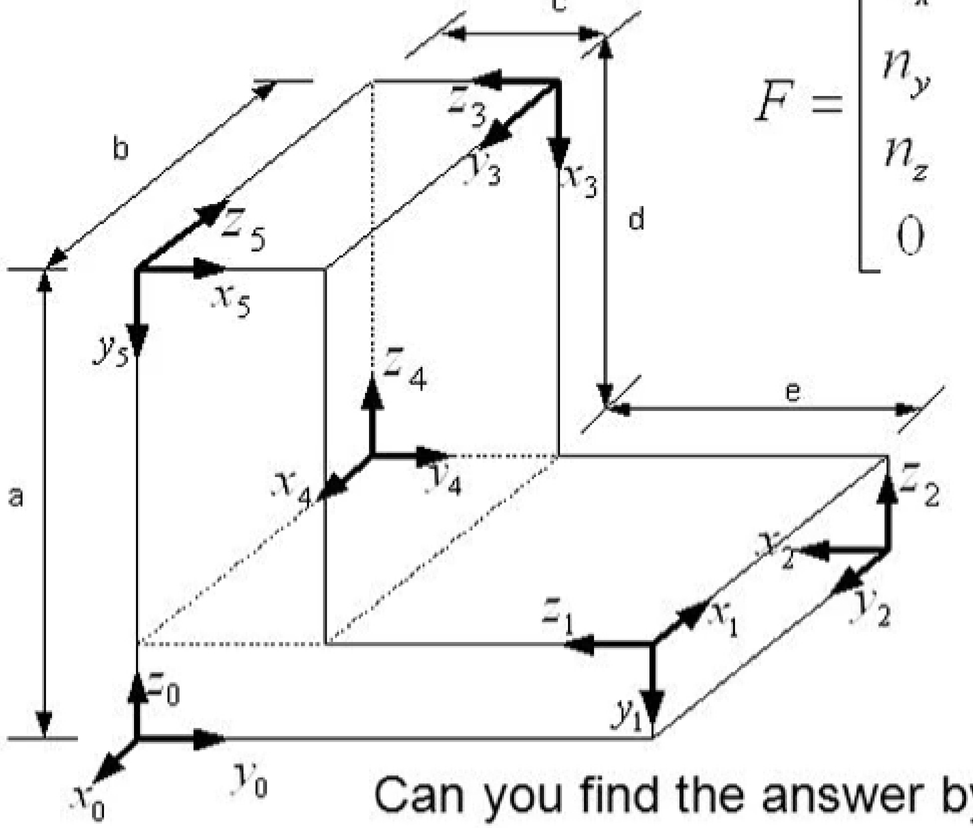


$^{i-1}A_i$ Transformation matrix for adjacent coordinate frames

$${}^{0}A_2 = {}^{0}A_1\,{}^{1}A_2$$

Chain product of successive coordinate transformation matrices

- For the figure shown below, find the 4x4 homogeneous transformation matrices ${}^{i-1}A_i$ and ${}^{0}A_i$ for i=1, 2, 3, 4, 5



$$F = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & e+c \\ 0 & -1 & 0 & a-d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}A_2 = \begin{bmatrix} 0 & -1 & 0 & b \\ 0 & 0 & -1 & a-d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}A_2 = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & e+c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Can you find the answer by observation based on the geometric interpretation of homogeneous transformation matrix?

Orientation Representation

$$F = \begin{bmatrix} R_{3\times 3} & P_{3\times 1} \\ 0 & 1 \end{bmatrix}$$

- Rotation matrix representation needs 9 elements to completely describe the orientation of a rotating rigid body.
- Any easy way?

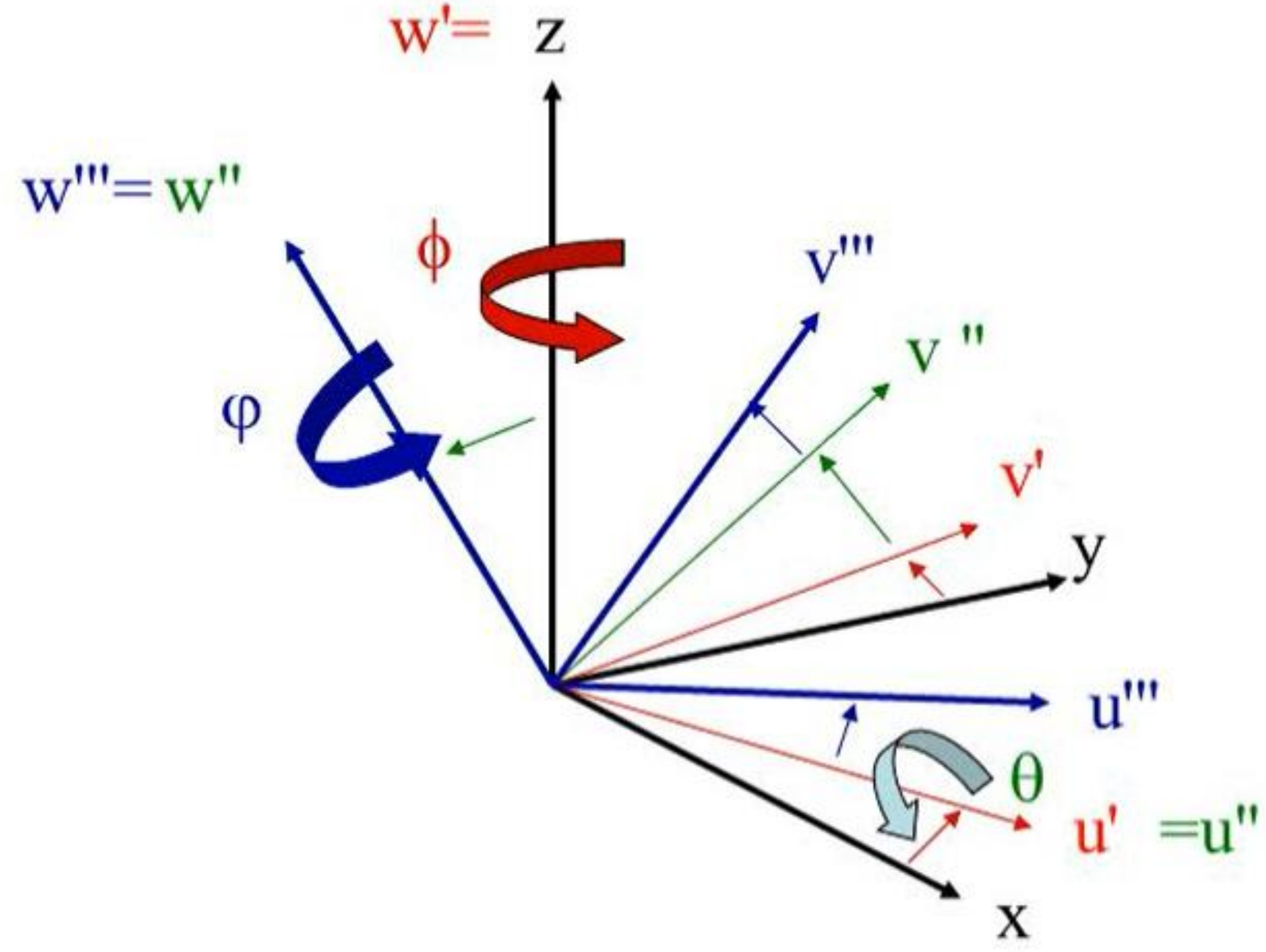
Euler Angles Representation

Orientation Representation

- Euler Angles Representation (ϕ, θ, ψ)
  - Many different types
  - Description of Euler angle representations

	Euler Angle I	Euler Angle II	Roll-Pitch-Yaw
Sequence	ϕ about OZ axis	ϕ about OZ axis	ψ about OX axis
of	θ about OU axis	θ about OV axis	θ about OY axis
Rotations	ψ about OW axis	ψ about OW axis	ϕ about OZ axis

Euler Angle I



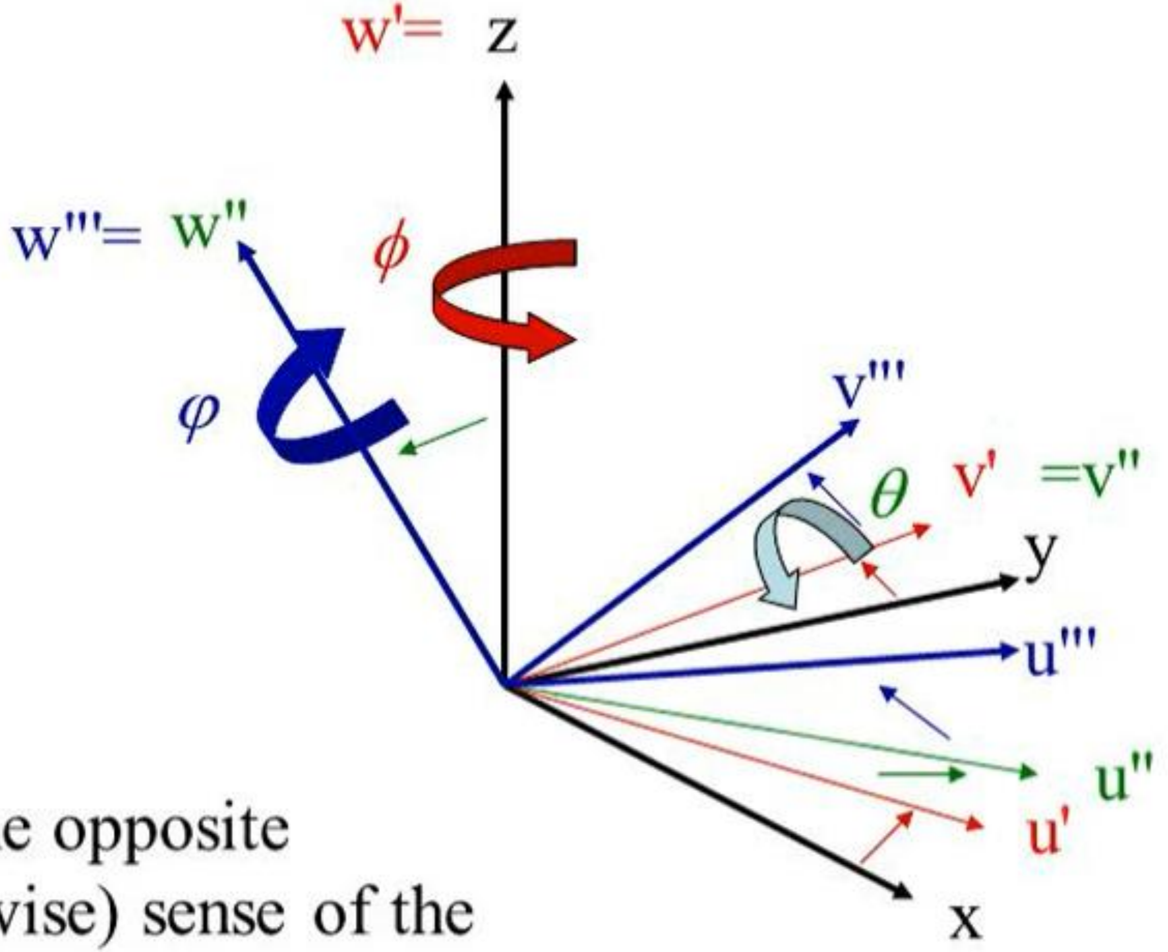
Orientation Representation

- Euler Angle I

$$R_{z\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_{u'\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$R_{w''\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Euler Angle II

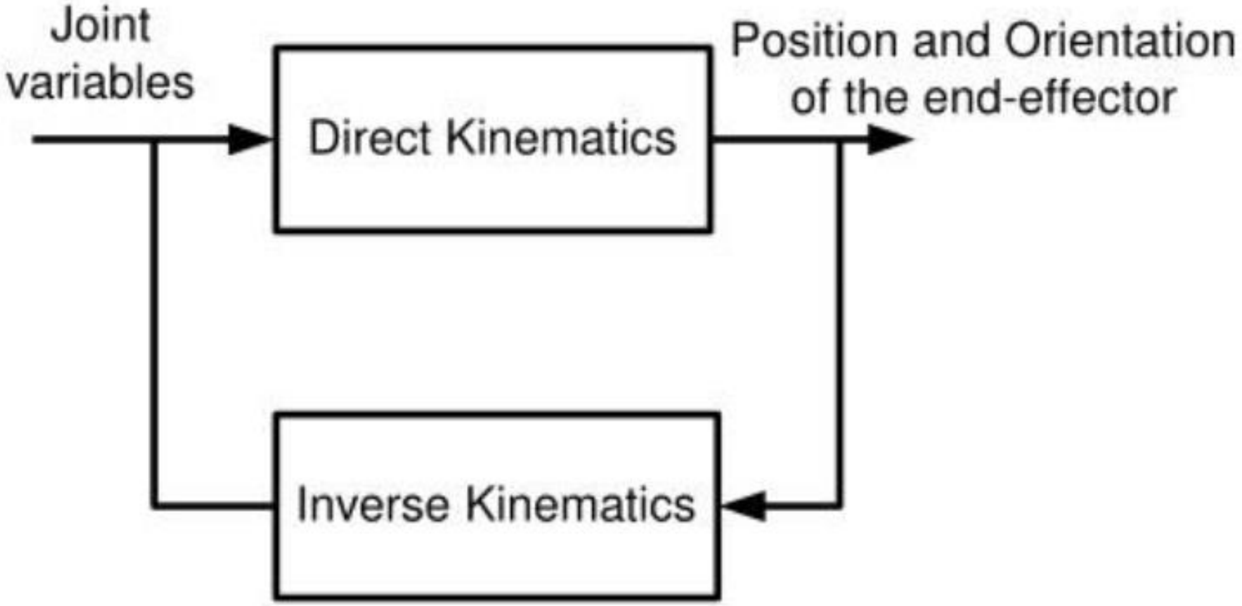


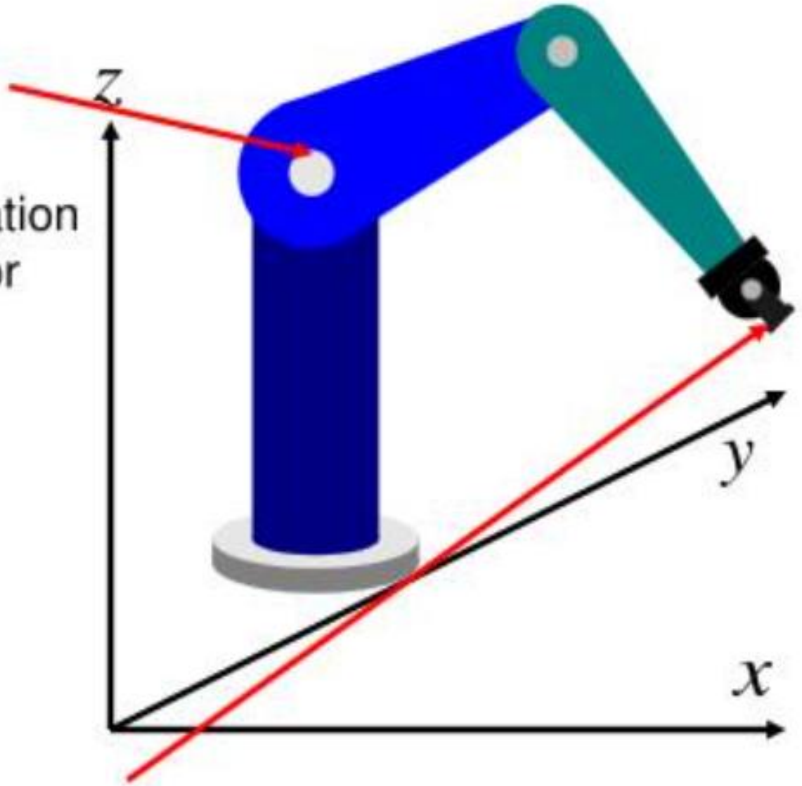
Note the opposite (clockwise) sense of the third rotation, ϕ.

Kinematics Model

- Forward (direct) Kinematics

$$q = (q_1, q_2, \cdots q_n)$$





$$Y = (x, y, z, \phi, \theta, \psi)$$

- Inverse Kinematics

Forward Kinematics

- more generally

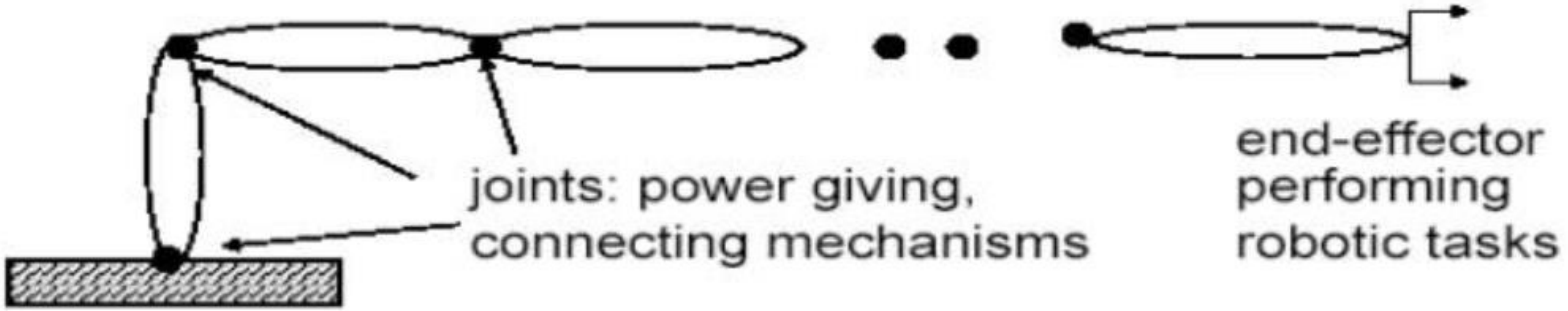
$$T^i_j = \begin{cases} T^i_{i+1}\, T^{i+1}_{j+2} \ldots T^{j-1}_j & \text{if} \quad i < j \\ I & \text{if} \quad i = j \\ \left(T^i_j\right)^{-1} & \text{if} \quad i > j \end{cases}$$

- the forward kinematics problem has been reduced to matrix multiplication

Robot Links and Joints

Robotic Tasks
- positioning/orienting
- force/moment exerted on environment

Chain of rigid bodies connected by joints

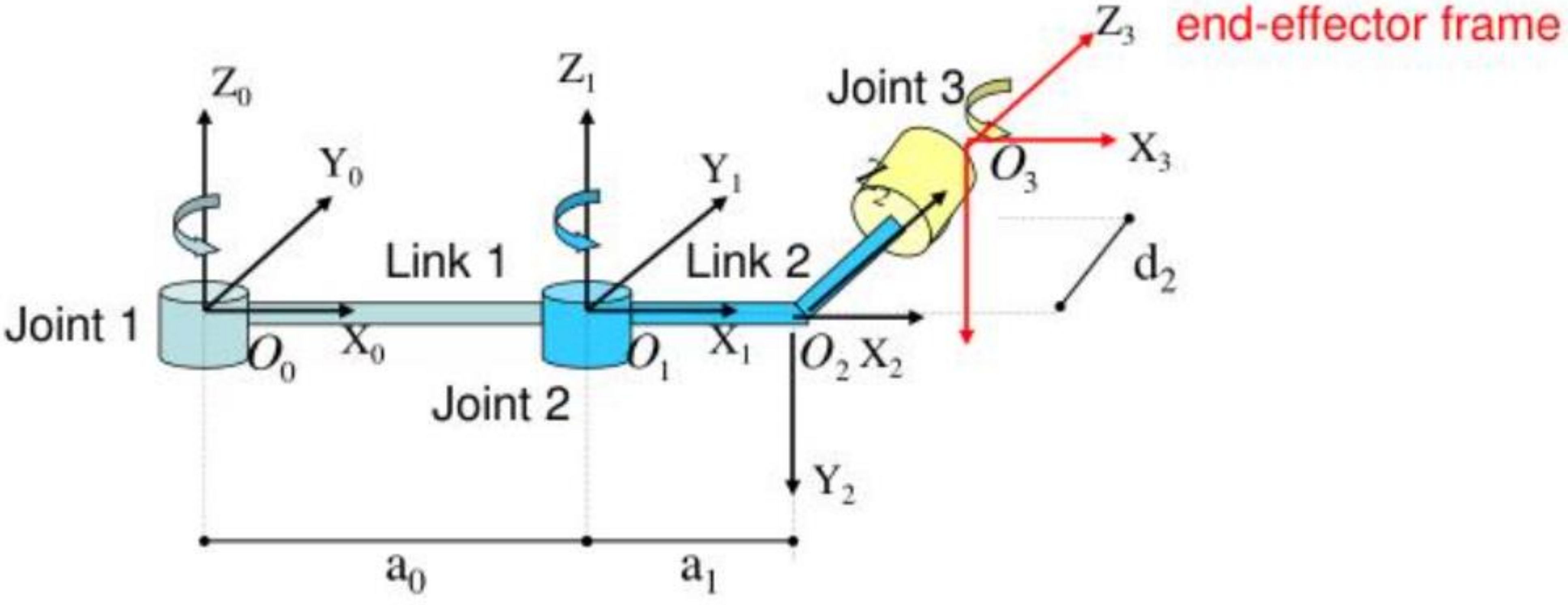


Denavit-Hartenberg Convention

- Number the joints from 1 to n starting with the base and ending with the end-effector.
- *Establish the base coordinate system.* Establish a right-handed orthonormal coordinate system (X_0, Y_0, Z_0) at the supporting base with Z_0 axis lying along the axis of motion of joint 1.
- *Establish joint axis.* Align the Z_i with the axis of motion (rotary or sliding) of joint i+1.
- *Establish the origin of the ith coordinate system.* Locate the origin of the ith coordinate at the intersection of the Z_i & Z_{i-1} or at the intersection of common normal between the Z_i & Z_{i-1} axes and the Z_i axis.
- *Establish X_i axis.* Establish $X_i = \pm(Z_{i-1} \times Z_i) / \|Z_{i-1} \times Z_i\|$ or along the common normal between the Z_{i-1} & Z_i axes when they are parallel.
- *Establish Y_i axis.* Assign $Y_i = +(Z_i \times X_i) / \|Z_i \times X_i\|$ to complete the right-handed coordinate system.
- Find the link and joint parameters

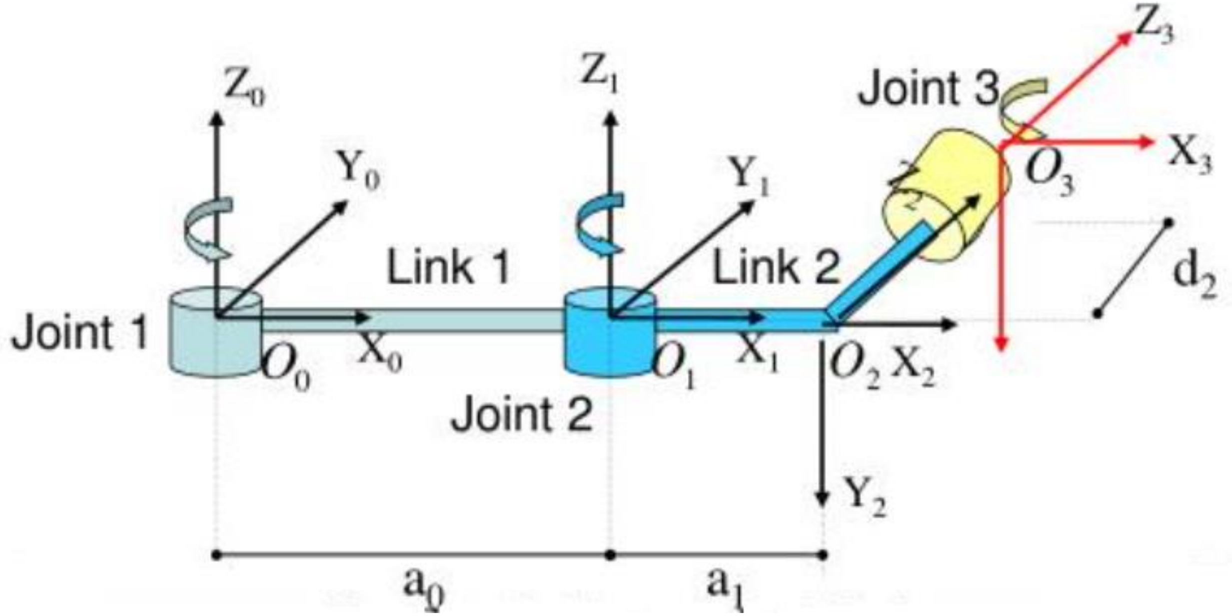
Example I

- 3 Revolute Joints



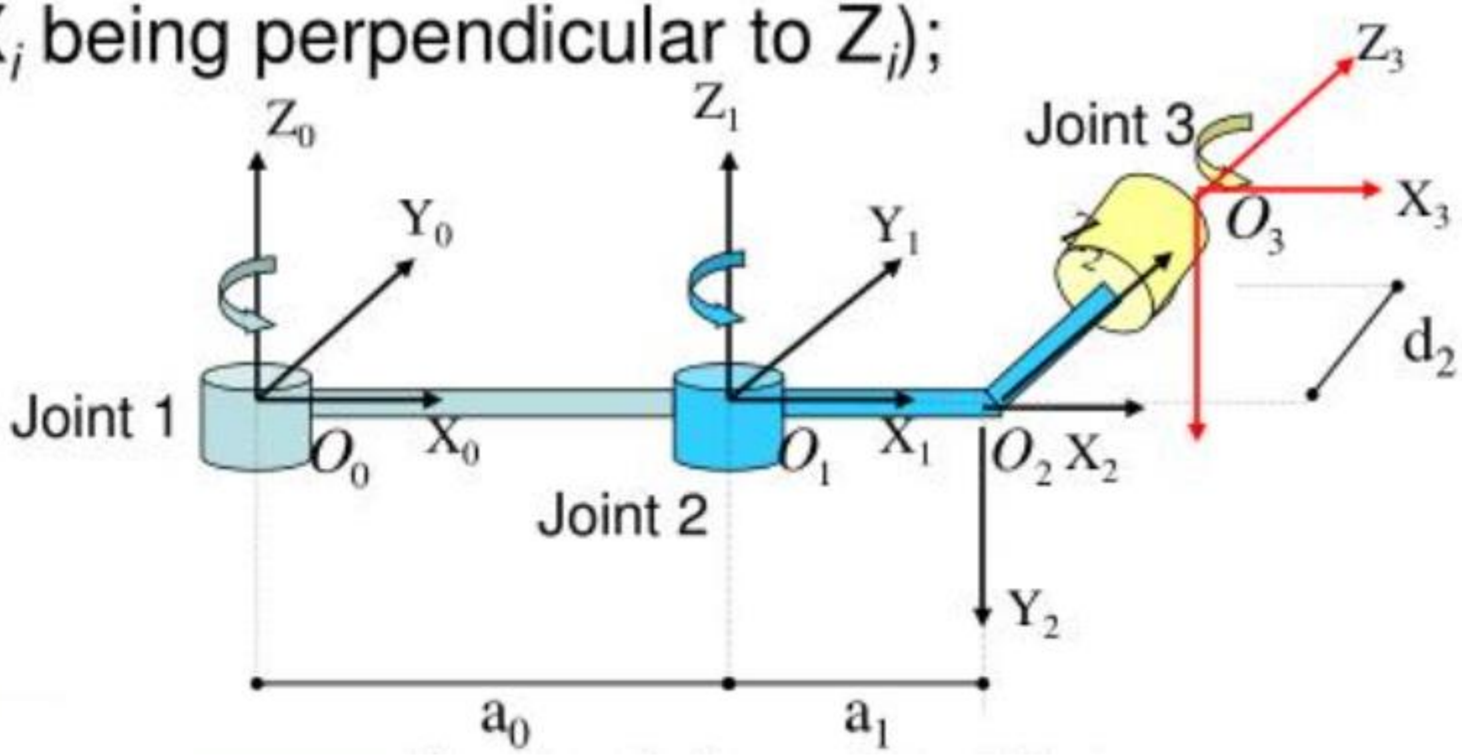
Link Coordinate Frames

- *Assign Link Coordinate Frames:*
  - To describe the geometry of robot motion, we assign a Cartesian coordinate frame (O_i, X_i,Y_i,Z_i) to each link, as follows:
    - establish a right-handed orthonormal coordinate frame O_0 at the supporting base with Z_0 lying along joint 1 motion axis.
    - the Z_i axis is directed along the axis of motion of joint (i + 1), that is, link (i + 1) rotates about or translates along Z_i;



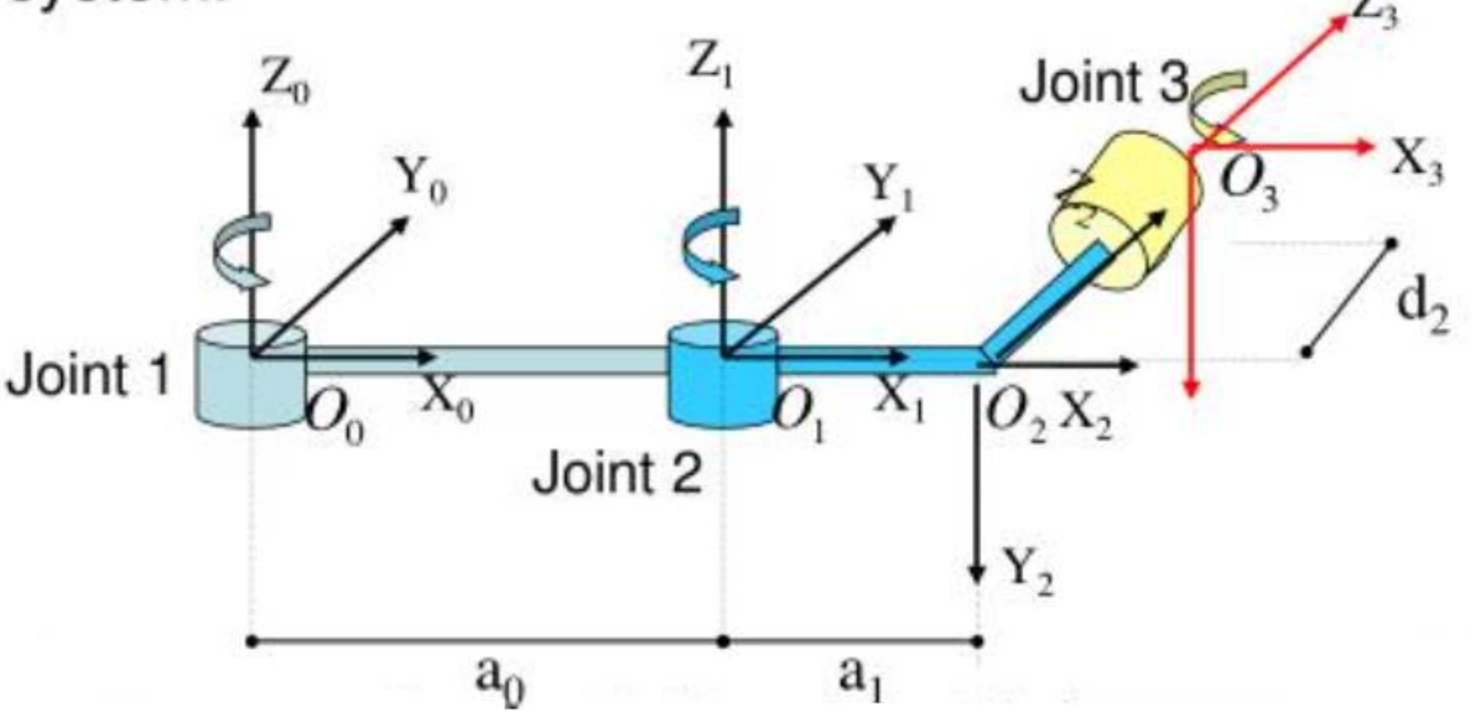
– Locate the origin of the ith coordinate at the intersection of the Z_i & Z_{i-1} or at the intersection of common normal between the Z_i & Z_{i-1} axes and the Z_i axis.

– the X_i axis lies along the common normal from the Z_{i-1} axis to the Z_i axis $X_i = \pm(Z_{i-1} \times Z_i) / \|Z_{i-1} \times Z_i\|$, (if Z_{i-1} is parallel to Z_i, then X_i is specified arbitrarily, subject only to X_i being perpendicular to Z_i);



– Assign $Y_i = +(Z_i \times X_i)/\|Z_i \times X_i\|$ to complete the right-handed coordinate system.

- The hand coordinate frame is specified by the geometry O_n of the end-effector. Normally, establish Z_n along the direction of Z_{n-1} axis and pointing away from the robot; establish X_n such that it is normal to both Z_{n-1} and Z_n axes. Assign Y_n to complete the right-handed coordinate system.



Denavit-Hartenberg

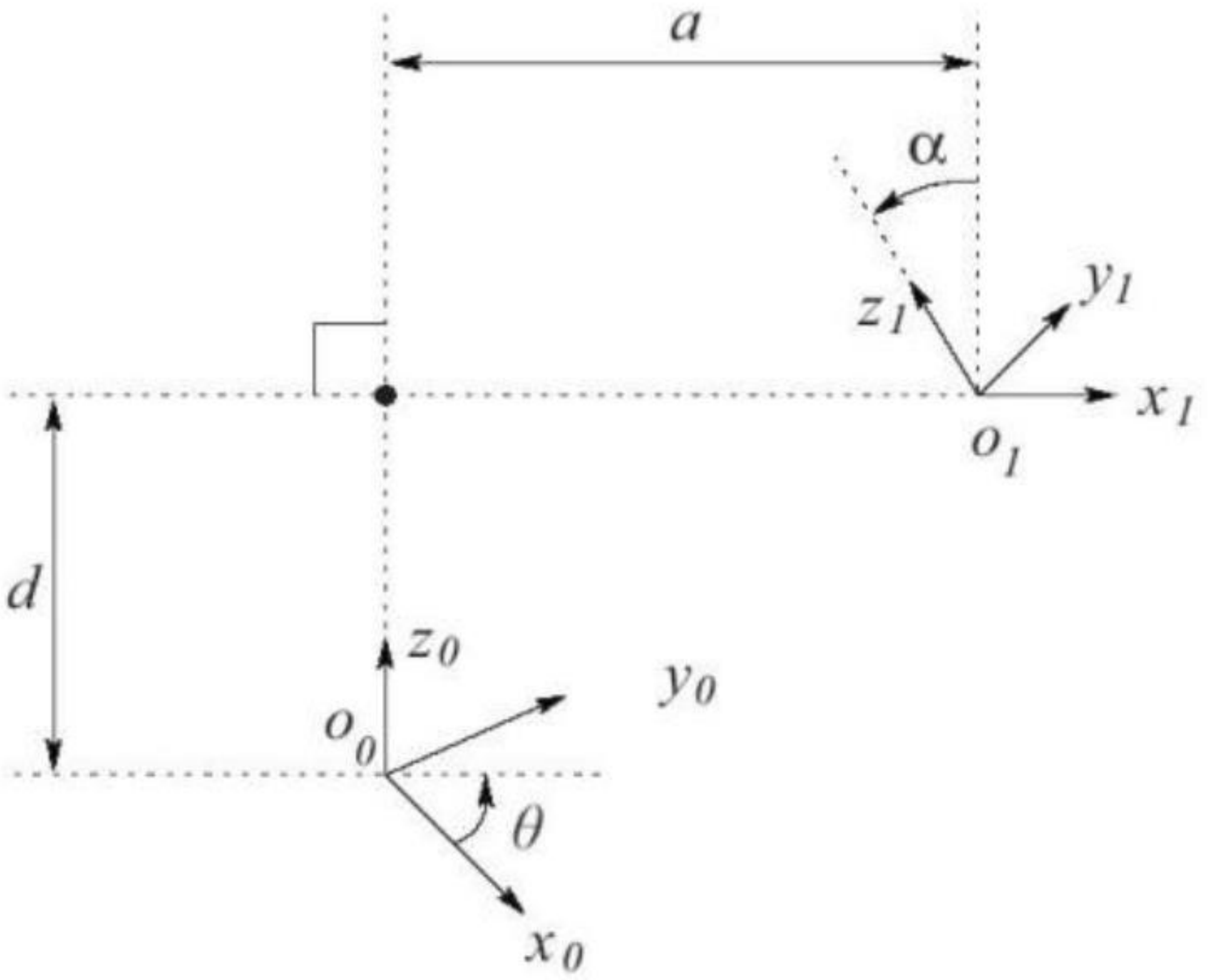


Figure 3.2: Coordinate frames satisfying assumptions DH1 and DH2.

- notice the form of the rotation component

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} \end{bmatrix}$$

  - this does not look like it can represent arbitrary rotations
- can the DH convention actually describe every physically possible link configuration?

- yes, but we must choose the orientation and position of the frames in a certain way
  - (DH1) $\hat{x}_1 \perp \hat{z}_0$
  - (DH2) $\hat{x}_1$ intersects $\hat{z}_0$

$$a, d, \theta, \alpha \text{ such that } T_1^0 = R_{z,\theta} D_{z,d} D_{x,a} R_{x,\alpha}$$

- claim: if DH1 and DH2 are true then there exists unique numbers

$$T_i^{i-1} = R_{z,\theta_i} T_{z,d_i} T_{x,a_i} R_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a_i	link length
α_i	link twist
d_i	link offset
θ_i	joint angle

Denavit-Hartenberg Forward Kinematics

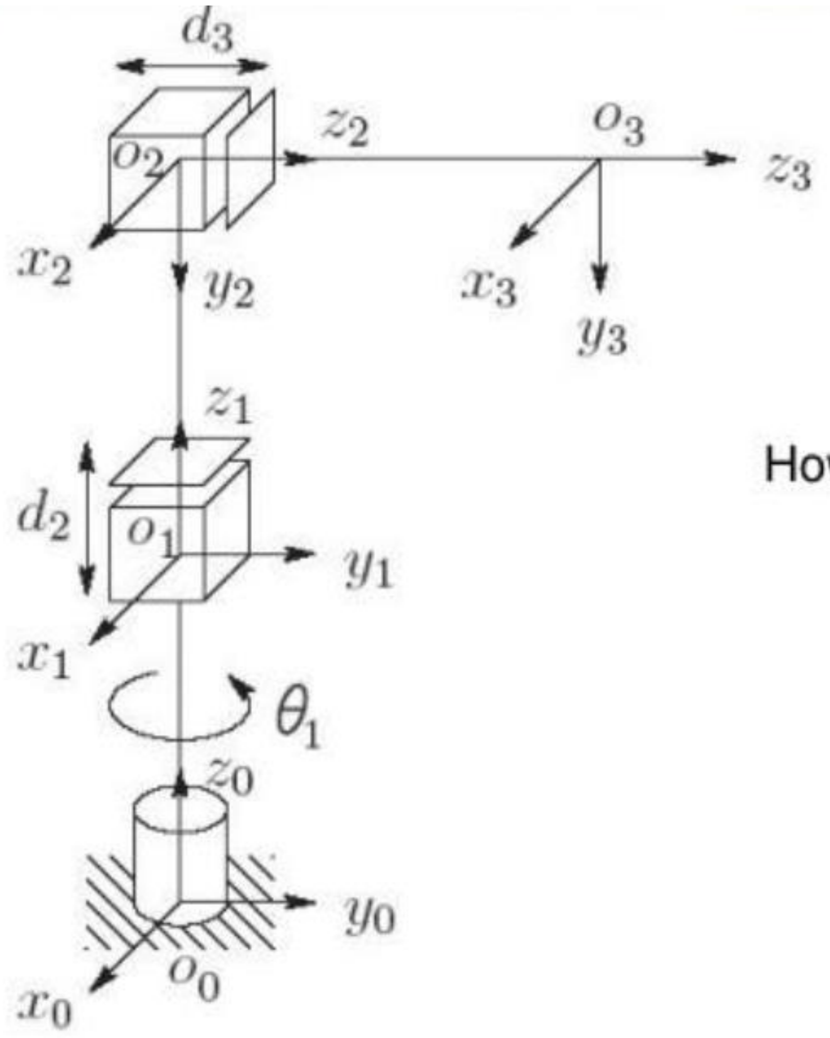


Figure 3.7: Three-link cylindrical manipulator.

How do we place the frames?

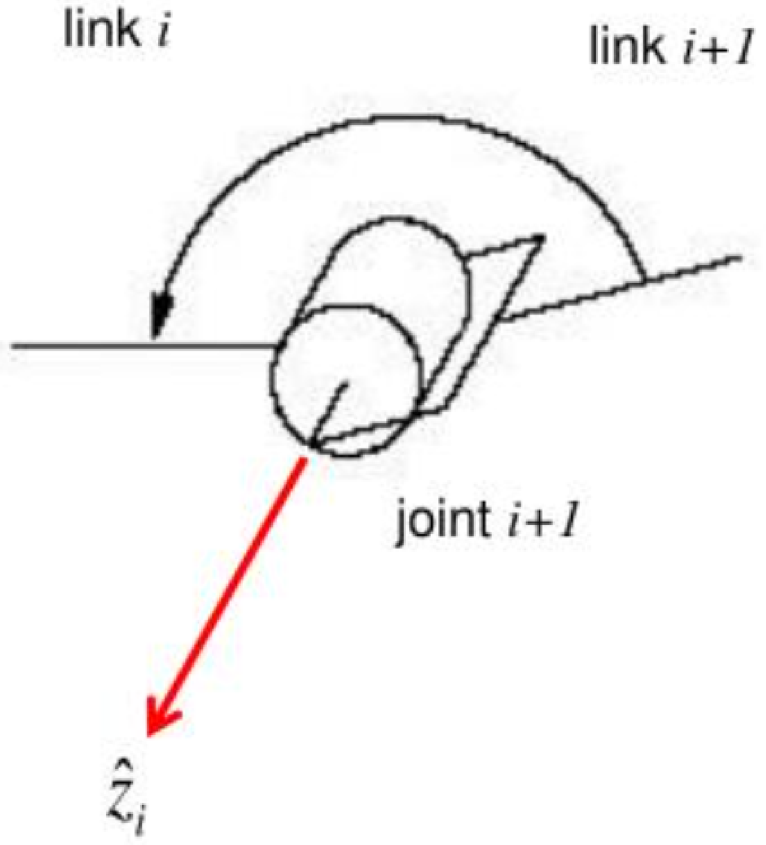
Step 1: Choose the z-axis for each frame

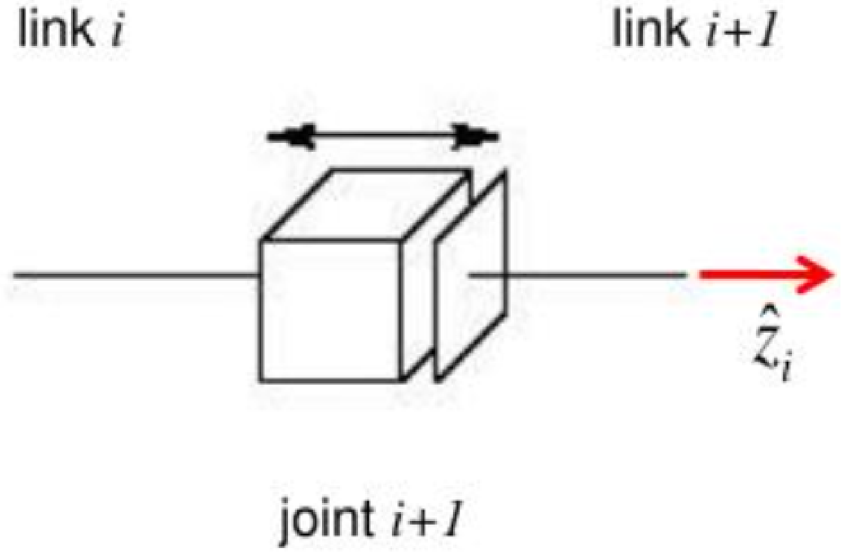
- recall the DH transformation matrix

$$T_i^{i-1} = R_{z,\theta_i} T_{z,d_i} T_{x,a_i} R_{x,\alpha_i}$$

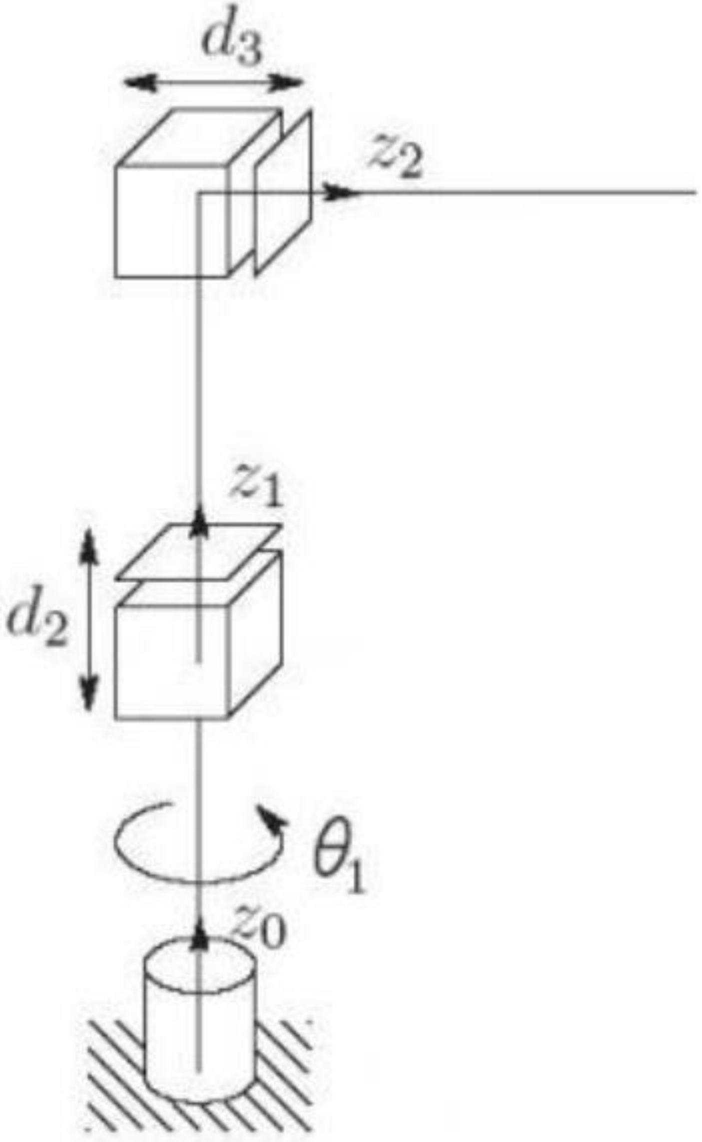
$$= \begin{array}{c} \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} \hat{x}_i^{i-1} & \hat{y}_i^{i-1} & \hat{z}_i^{i-1} & \phantom{a_i c_{\theta_i}} \end{array} \end{array}$$

- $\hat{z}_i \equiv$ axis of actuation for joint $i+1$



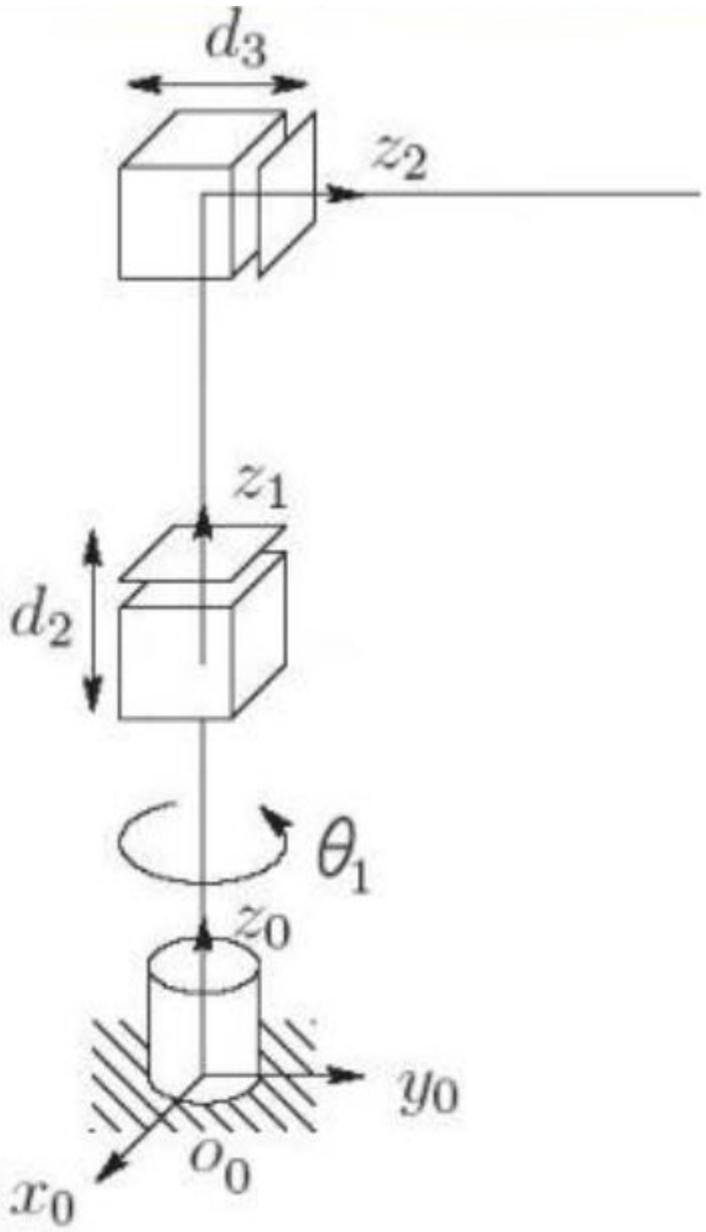


Step 1: Choose the z-axis for each frame



Step 2: Establish frame {0}

- place the origin o_0 anywhere on z_0
  - often the choice of location is obvious
- choose x_0 and y_0 so that $\{0\}$ is right-handed
  - often the choice of directions is obvious

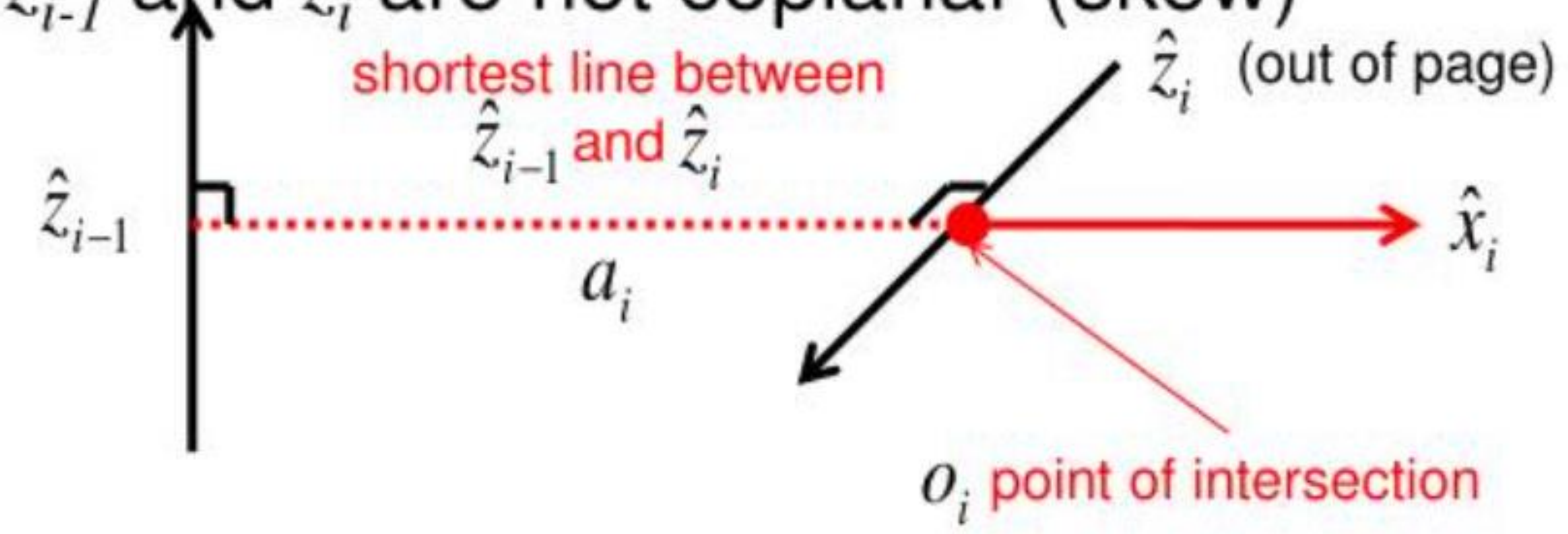


d_3
z_2
z_1
d_2
θ_1
z_0
y_0
o_0
x_0

Step 3: Iteratively construct $\{1\}$, $\{2\}$, ... $\{n\text{-}1\}$

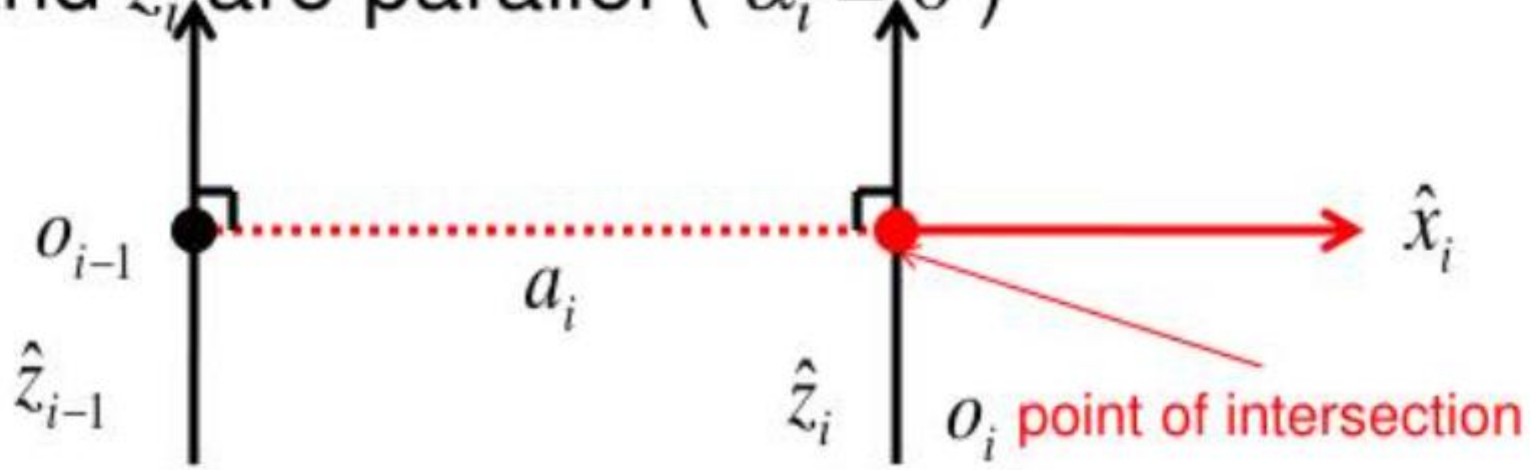
- using frame $\{i\text{-}1\}$ construct frame $\{i\}$
  - DH1: x_i is perpendicular to z_{i-1}
  - DH2: x_i intersects z_{i-1}
- 4 cases to consider depending on the relationship between z_{i-1} and z_i

- Case 1
  - z_{i-1} and z_i are not coplanar (skew)



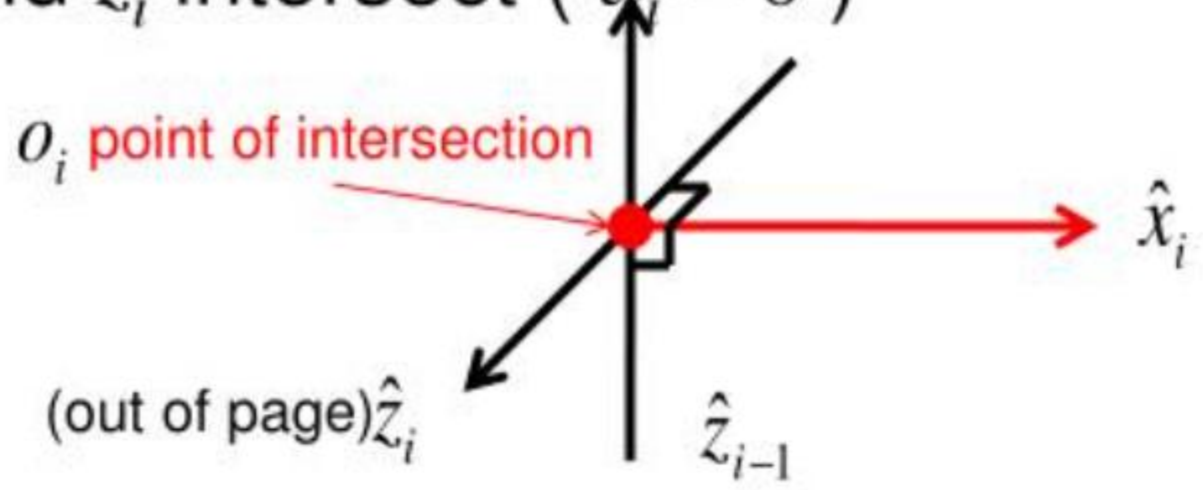
□ α_i angle from z_{i-1} to z_i measured about x_i

- Case 2
  - z_{i-1} and z_i are parallel ($\alpha_i = 0$)

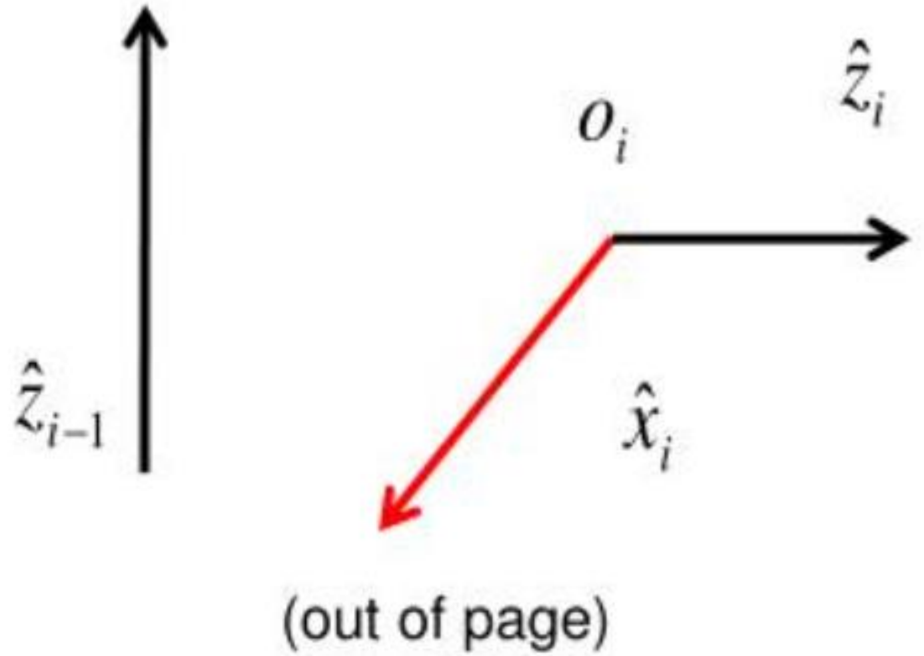


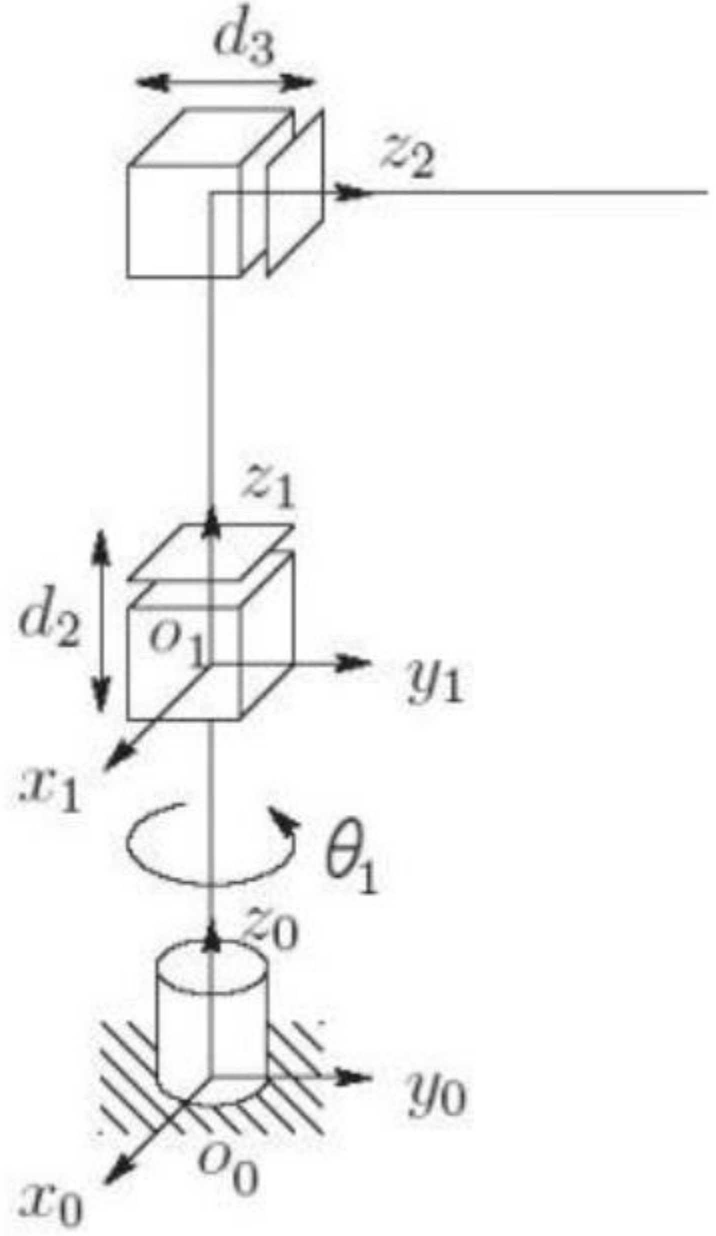
  - notice that this choice results in $d_i = 0$

- Case 3
  - z_{i-1} and z_i intersect ($a_i = 0$)

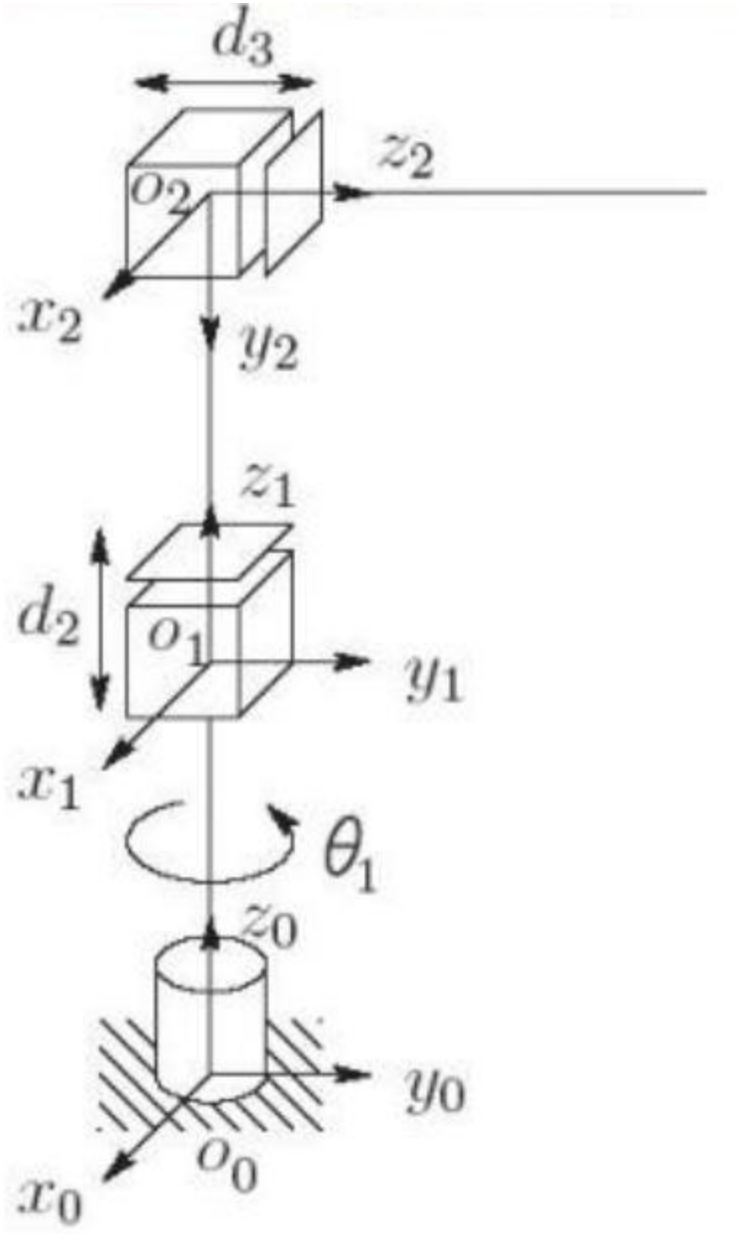


- Case 4
  - *x_i cannot be perpendicular on z_{i-1},* then x_i perpendicular on z_{i-1} and z_i





d_3
z_2
z_1
d_2
o_1
y_1
x_1
θ_1
z_0
y_0
o_0
x_0


d_3
z_2
o_2
x_2
y_2
z_1
d_2
o_1
y_1
x_1
θ_1
z_0
y_0
o_0
x_0

Step 4: Place the end effector

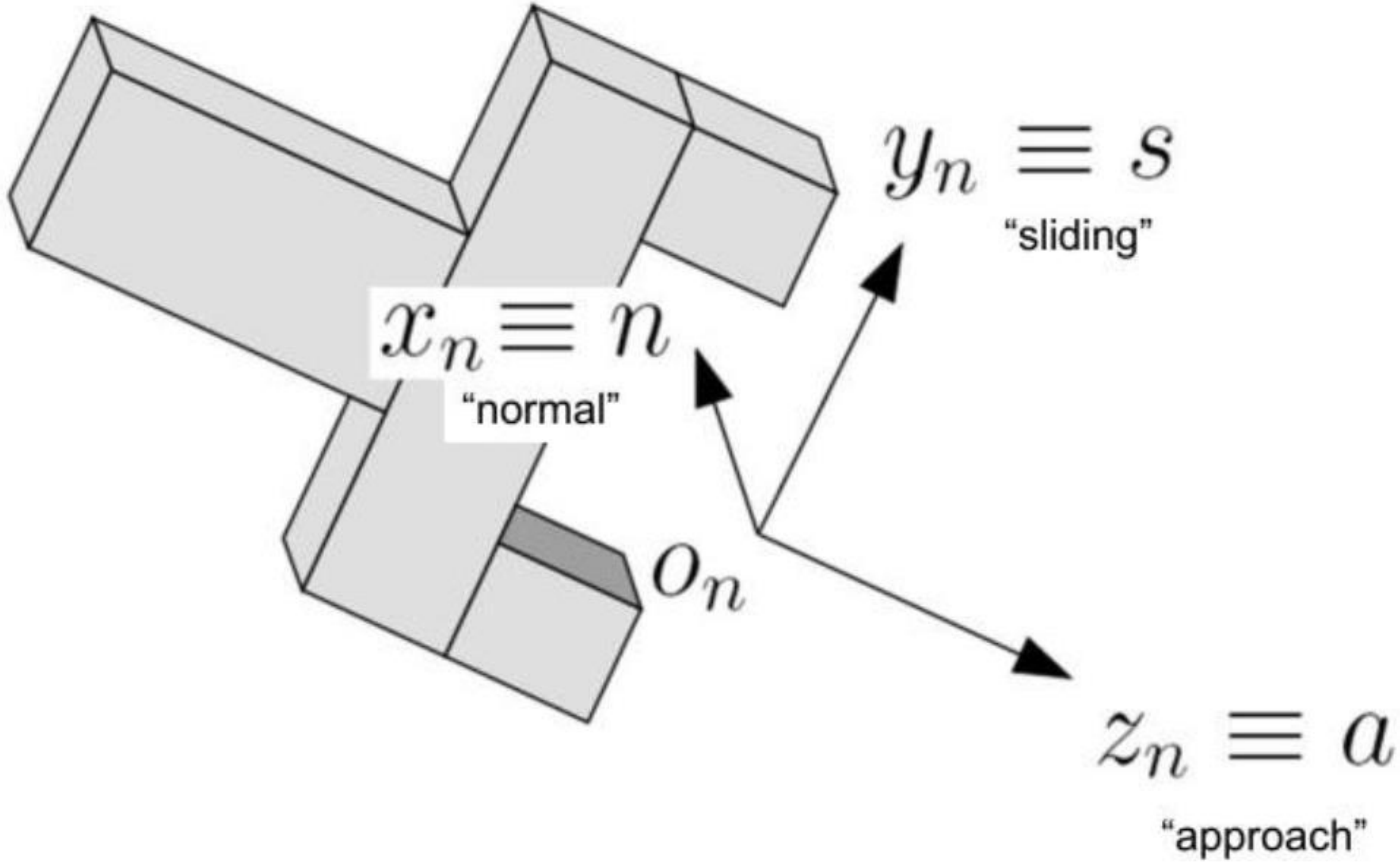


Figure 3.5: Tool frame assignment.

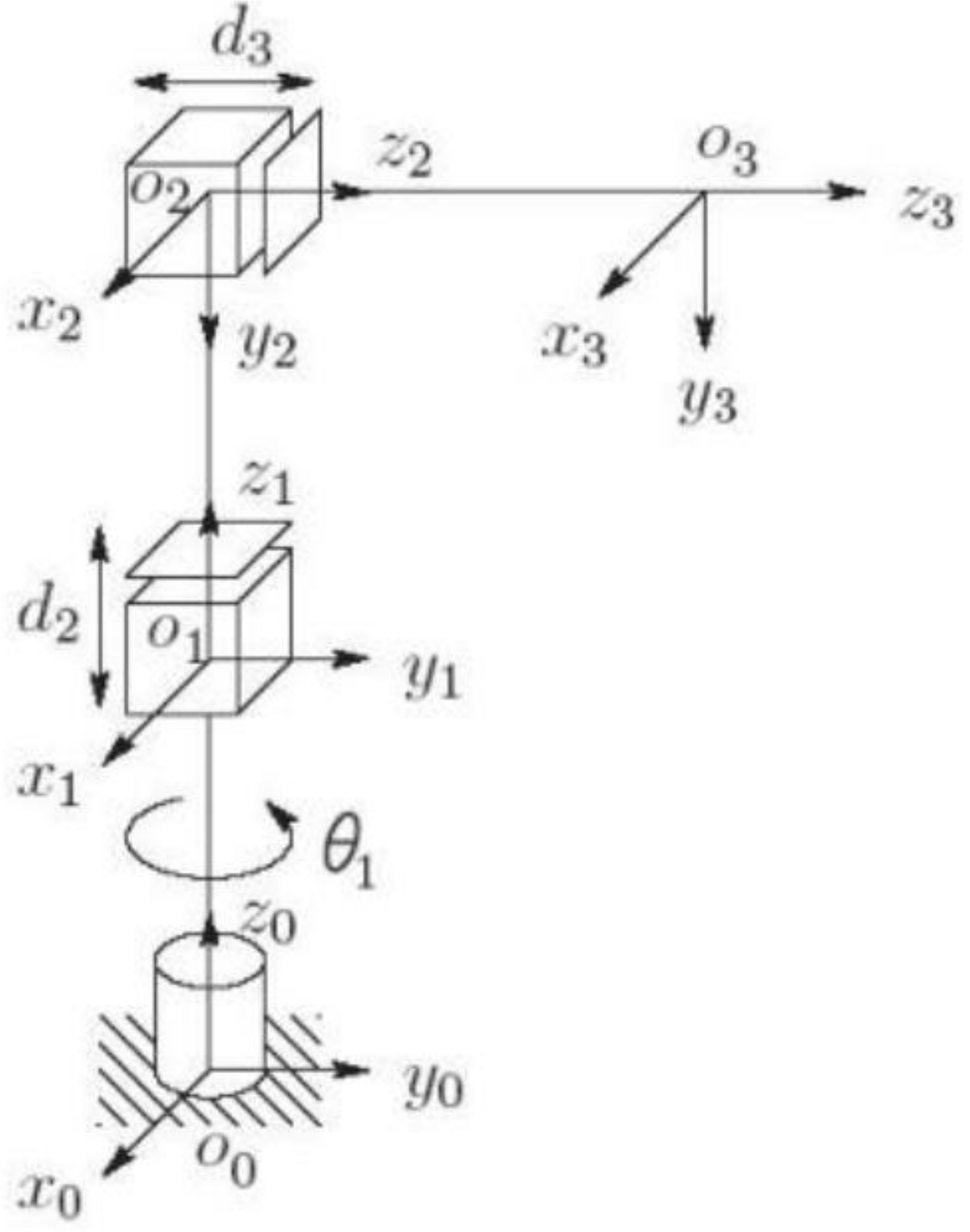


Figure 3.7: Three-link cylindrical manipulator.

Step 5: Find the DH parameters

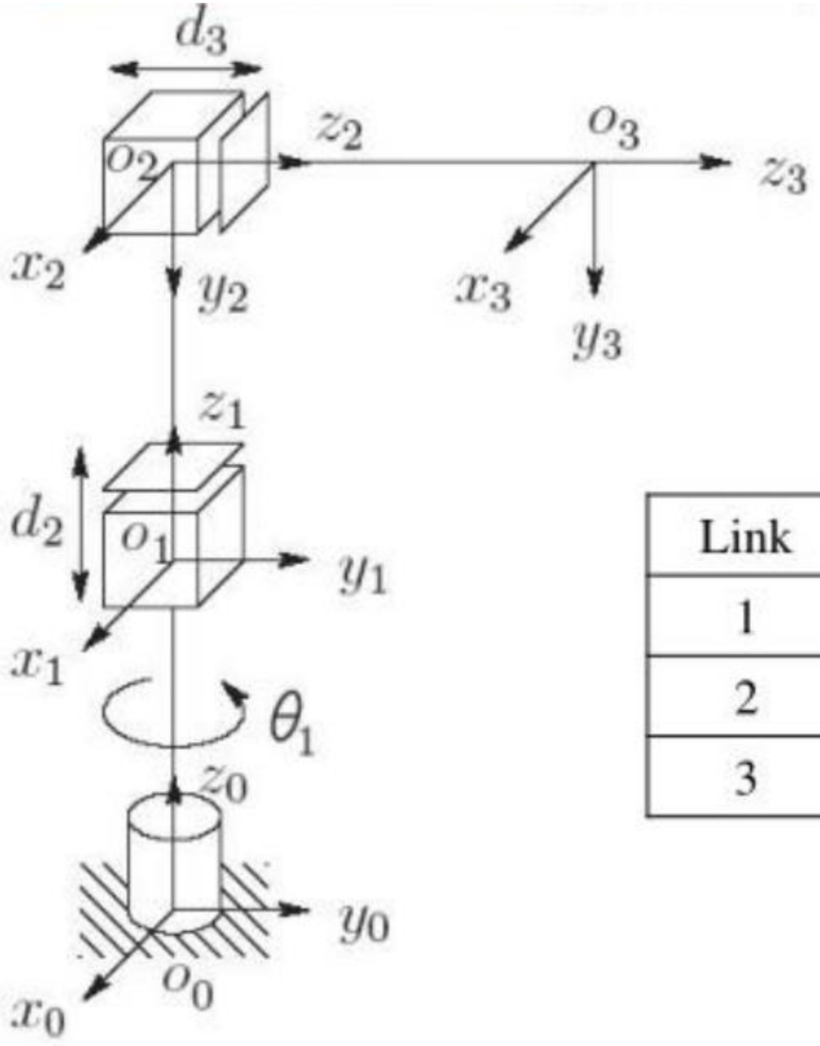
- *Joint angle* θ_i: the angle of rotation from the X_{i-1} axis to the X_i axis about the Z_{i-1} axis. It is the joint variable if joint i is rotary.

- *Joint distance* d_i : the distance from the origin of the (i-1) coordinate system to the intersection of the Z_{i-1} axis and the X_i axis along the Z_{i-1} axis. It is the joint variable if joint i is prismatic.

- *Link length* a_i : the distance from the intersection of the Z_{i-1} axis and the X_i axis to the origin of the ith coordinate system along the X_i axis.

- *Link twist angle* α_i: the angle of rotation from the Z_{i-1} axis to the Z_i axis about the X_i axis.

Step 5: Find the DH parameters



Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

Step 6: Compute the transformation

- once the DH parameters are known, it is easy to construct the overall transformation

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

$$T_1^0 = R_{z,\theta_1} T_{z,d_1} T_{x,a_1} R_{x,\alpha_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

$$T_2^1 = R_{z,\theta_2} T_{z,d_2} T_{x,a_2} R_{x,\alpha_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link	a_i	α_i	d_i	θ_i
1	0	0	d_1	θ_1*
2	0	-90	d_2*	0
3	0	0	d_3*	0

* joint variable

$$T_3^2 = R_{z,\theta_3} T_{z,d_3} T_{x,a_3} R_{x,\alpha_3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = T_1^0 T_2^1 T_3^2 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Spherical Wrist

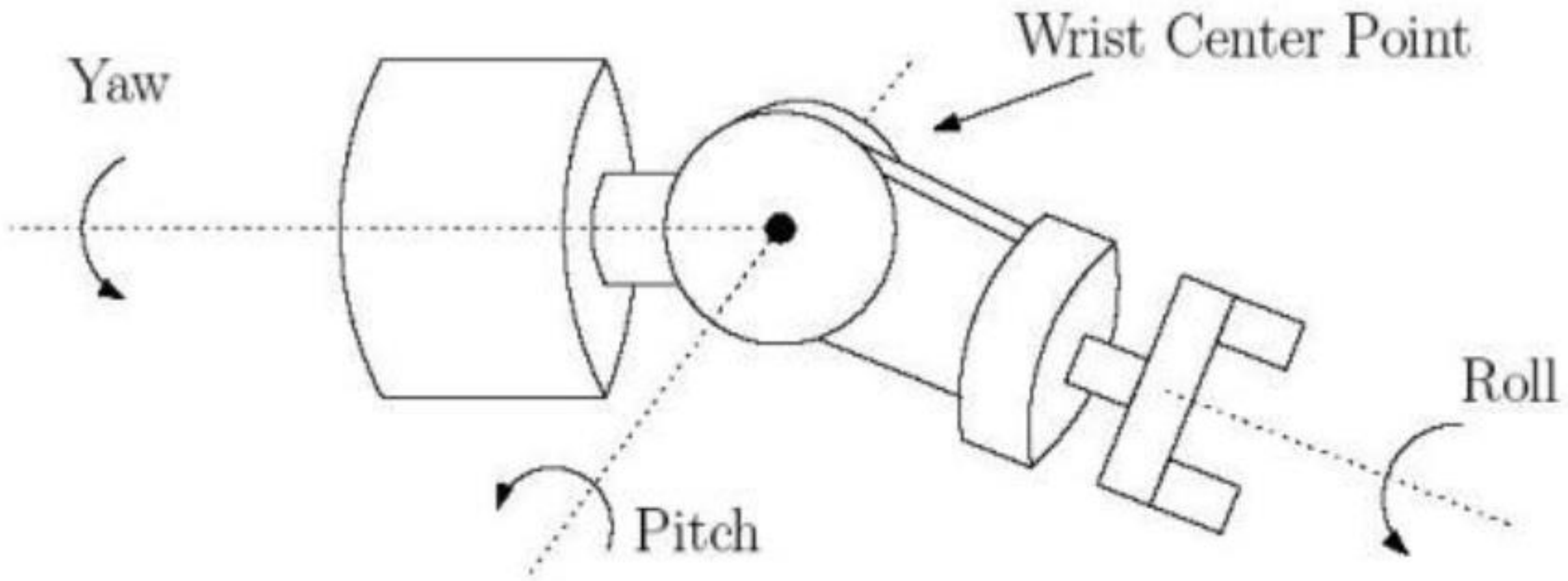
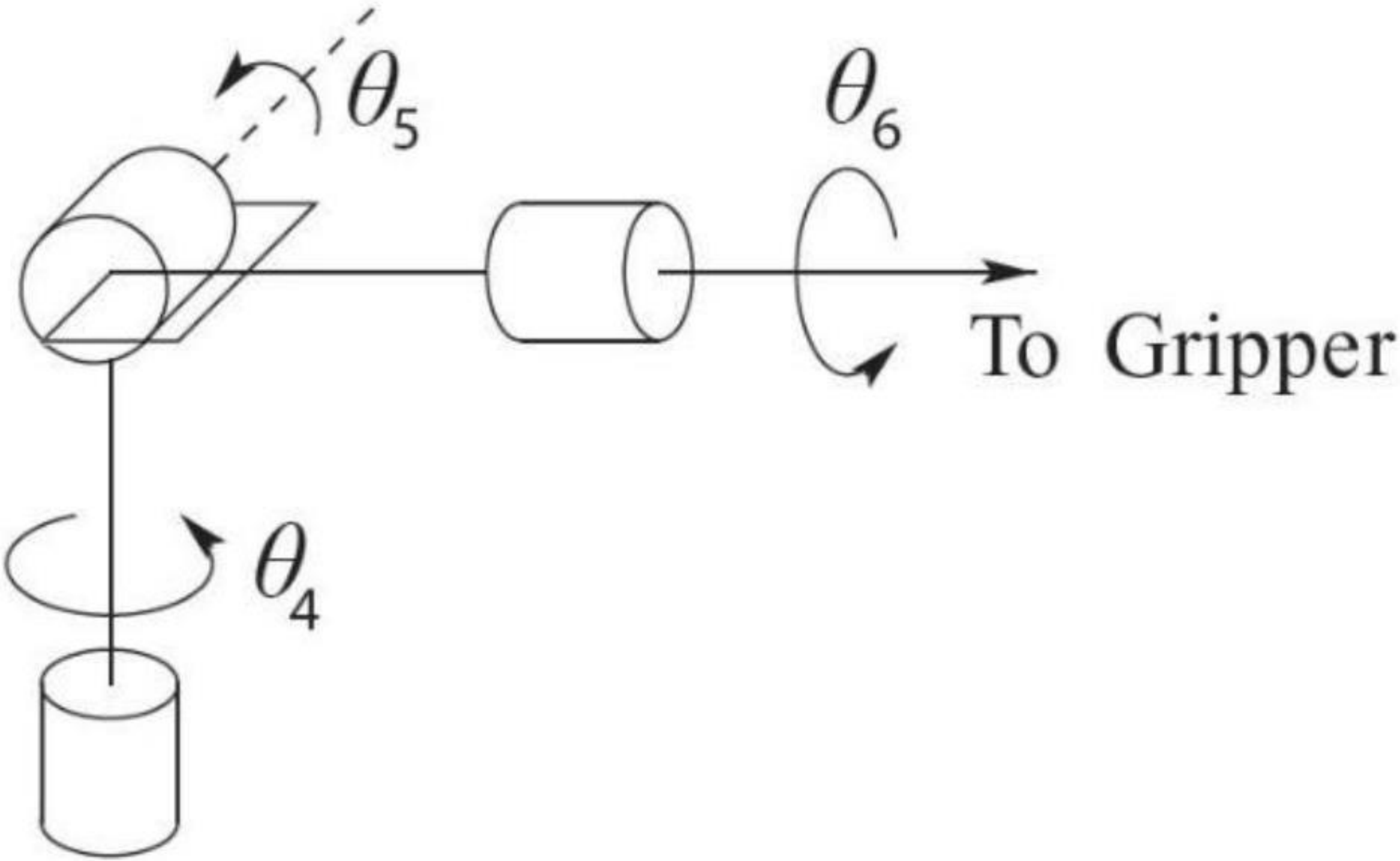
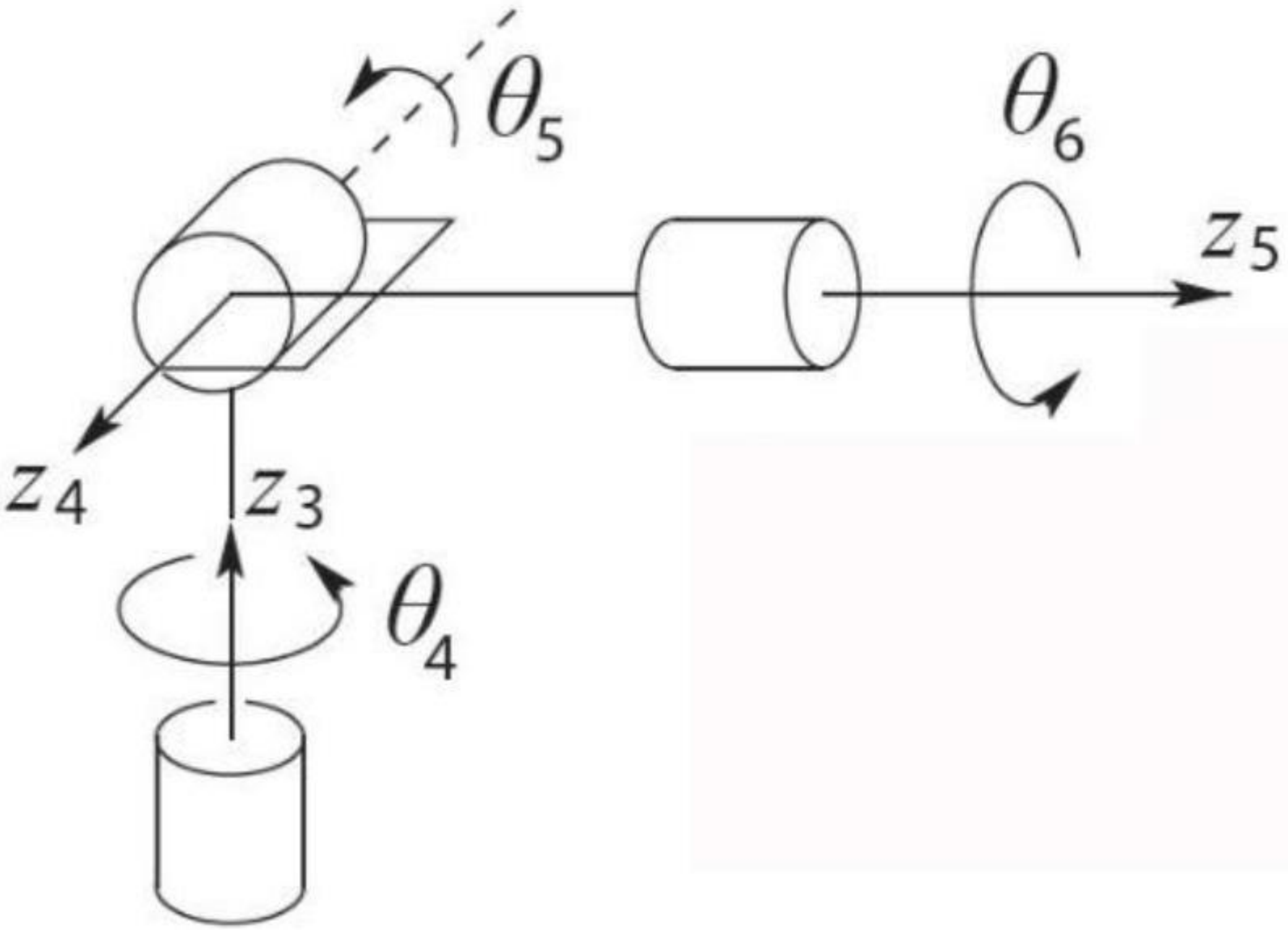


Figure 1.6: The spherical wrist. The axes of rotation of the spherical wrist are typically denoted roll, pitch, and yaw and intersect at a point called the wrist center point.

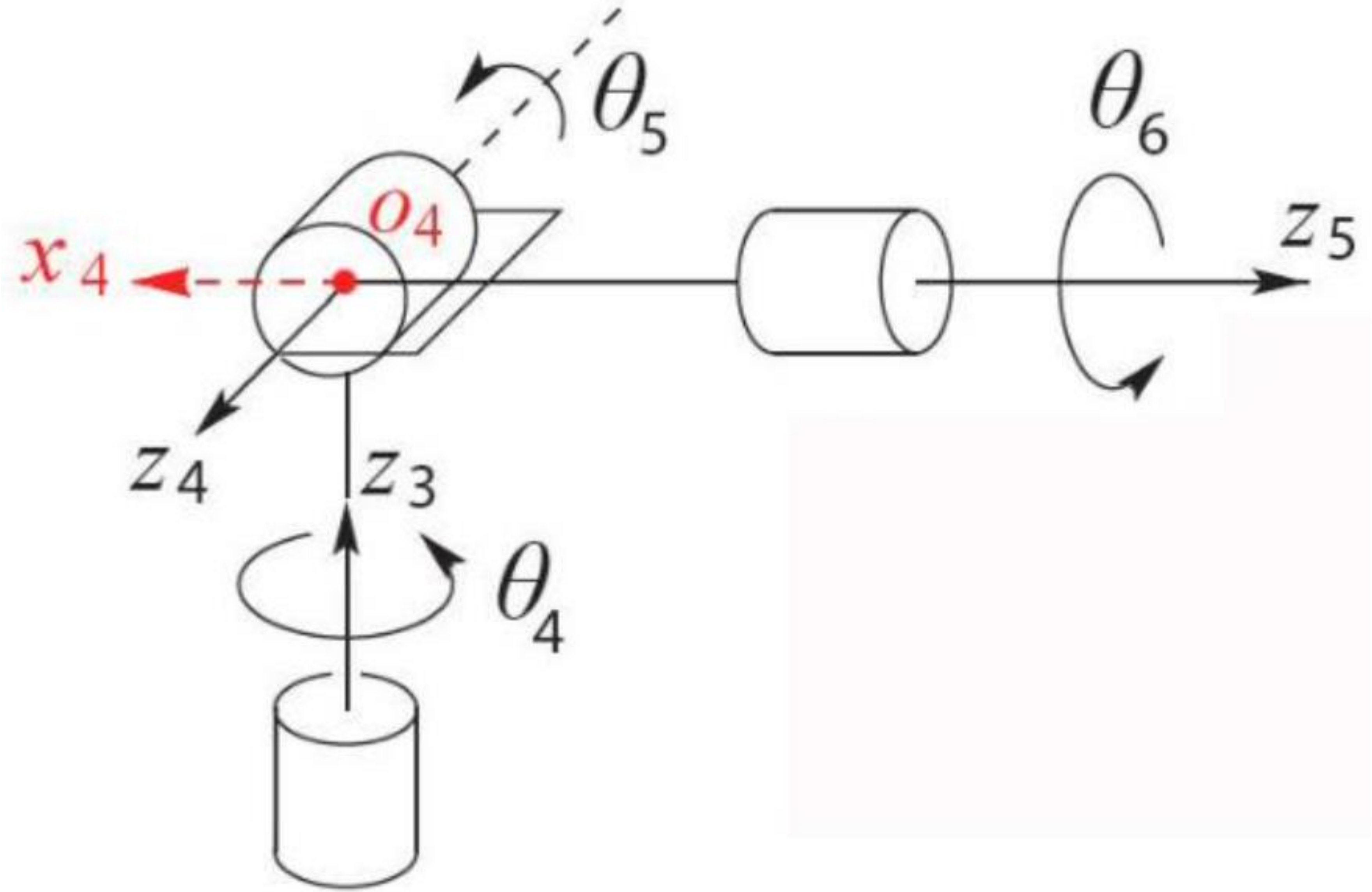


θ_5
θ_6
To Gripper
θ_4

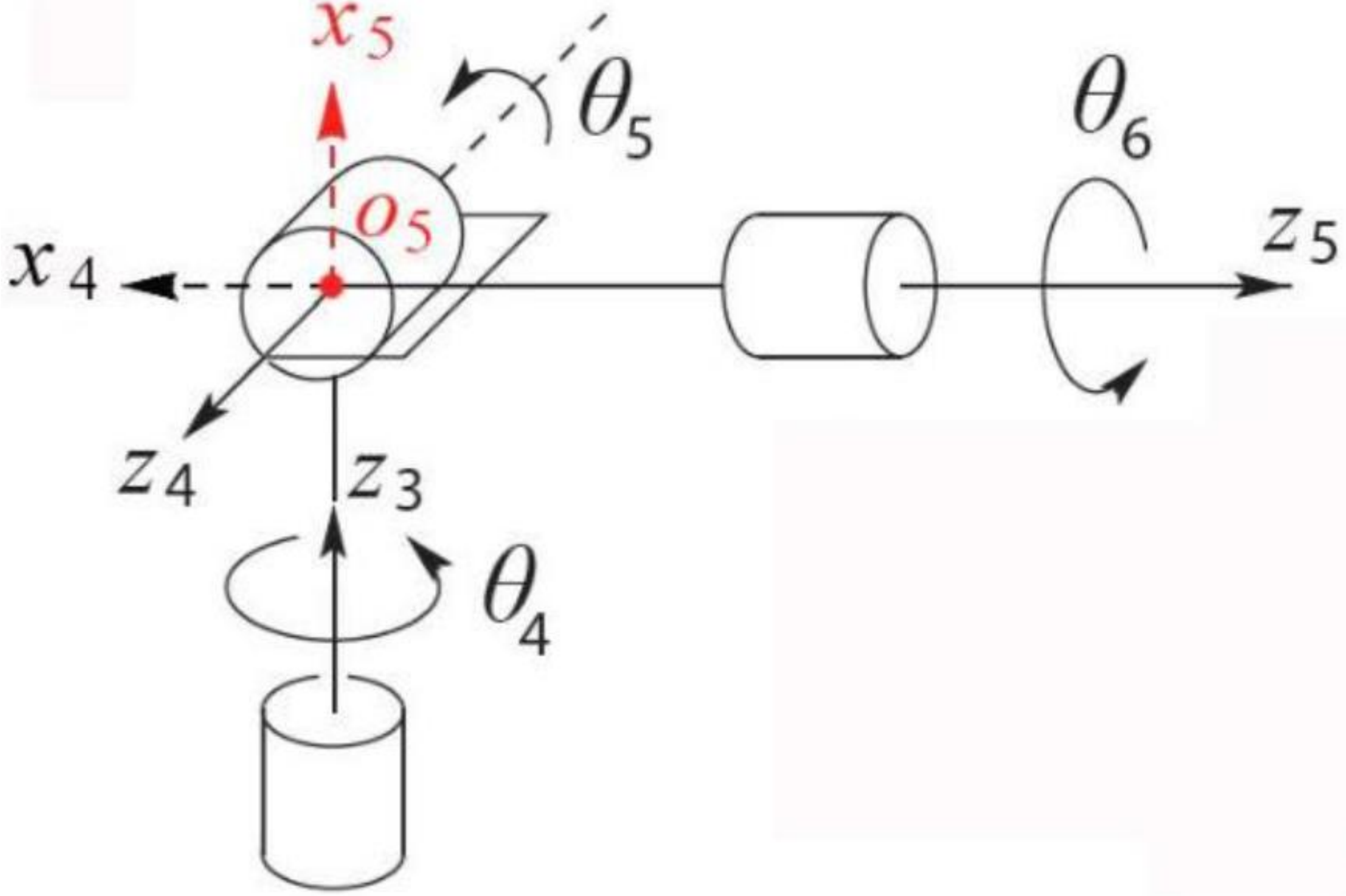
Spherical Wrist: Step 1



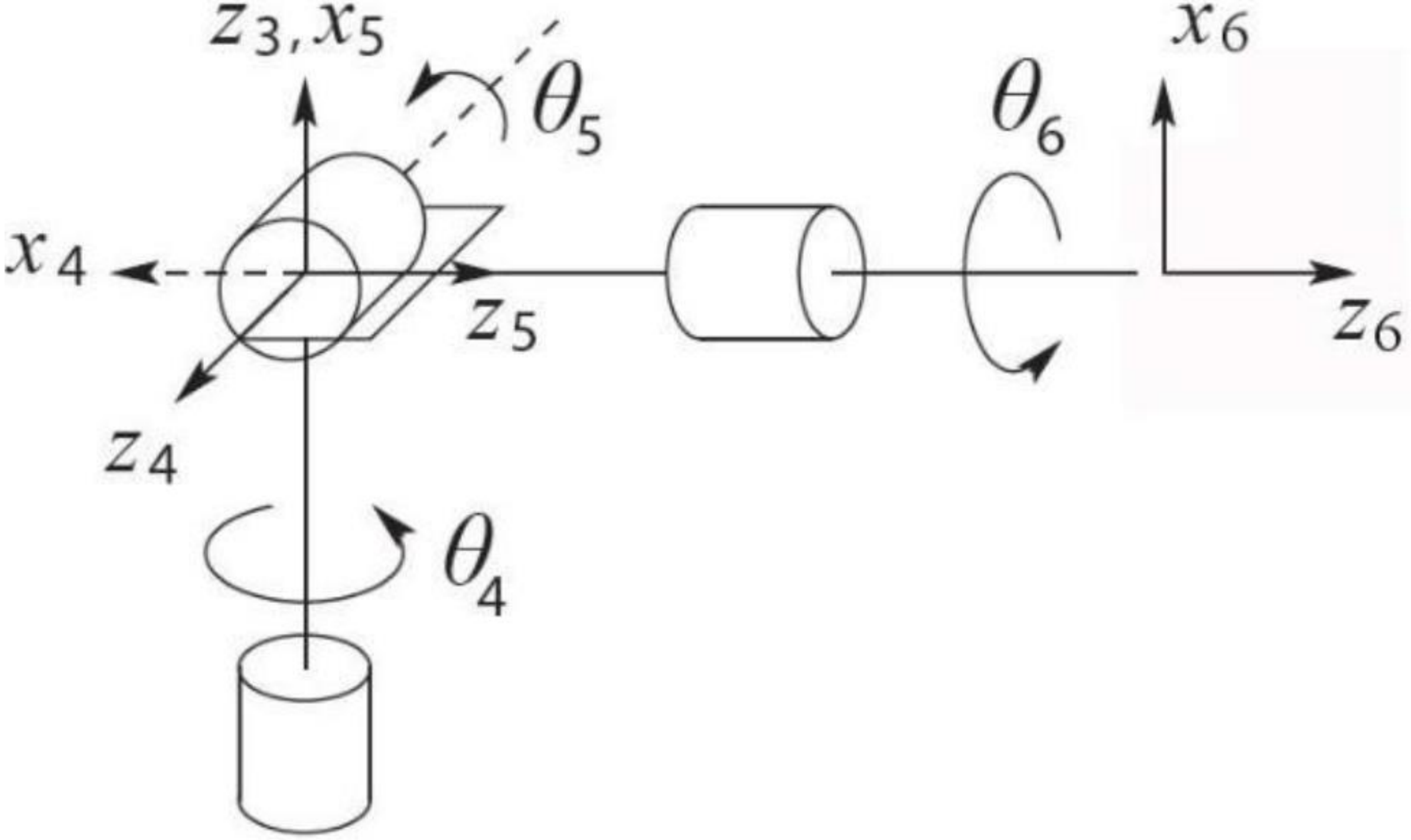
Spherical Wrist: Step 2



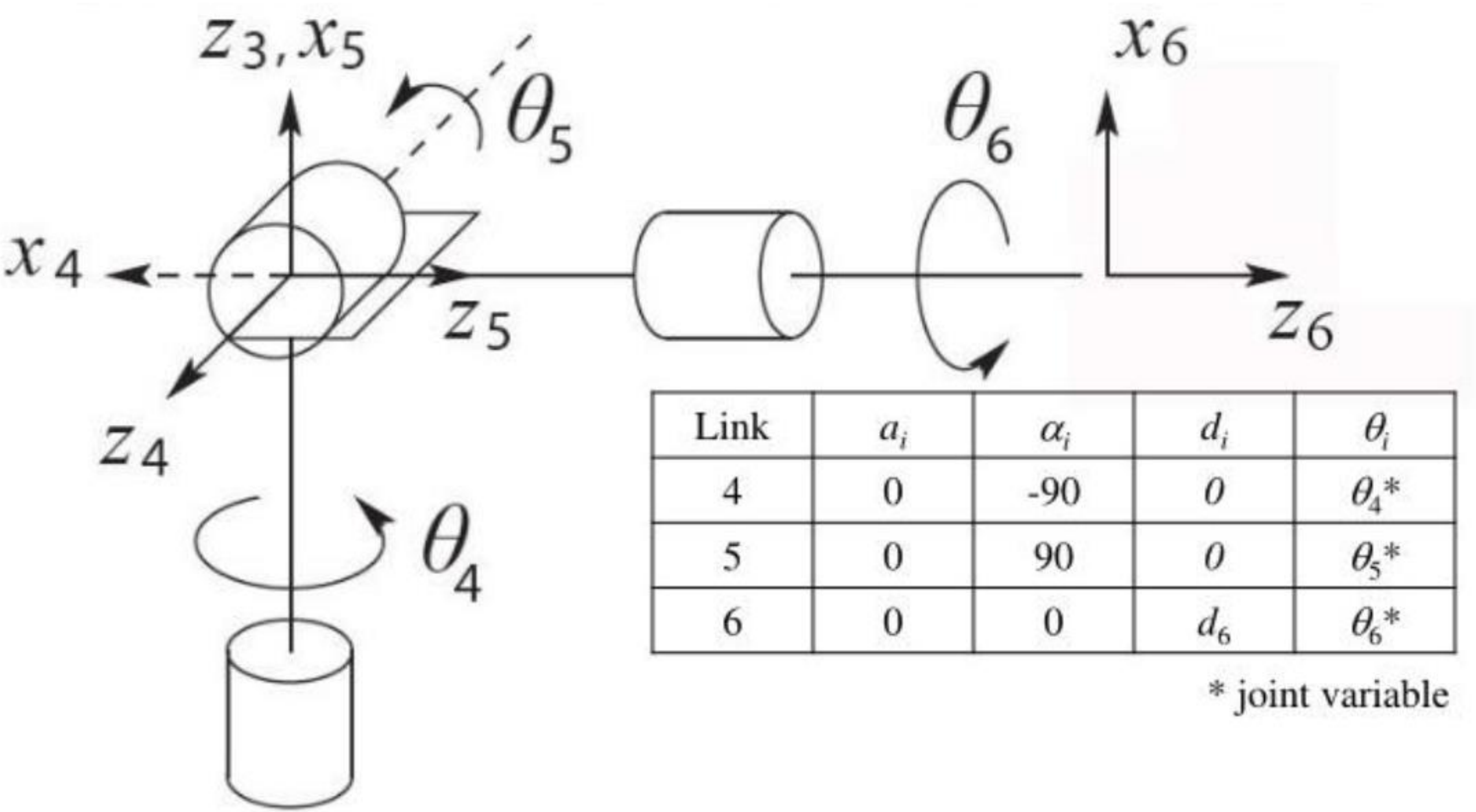
STEP 3



STEP 4



Step 5: DH Parameters



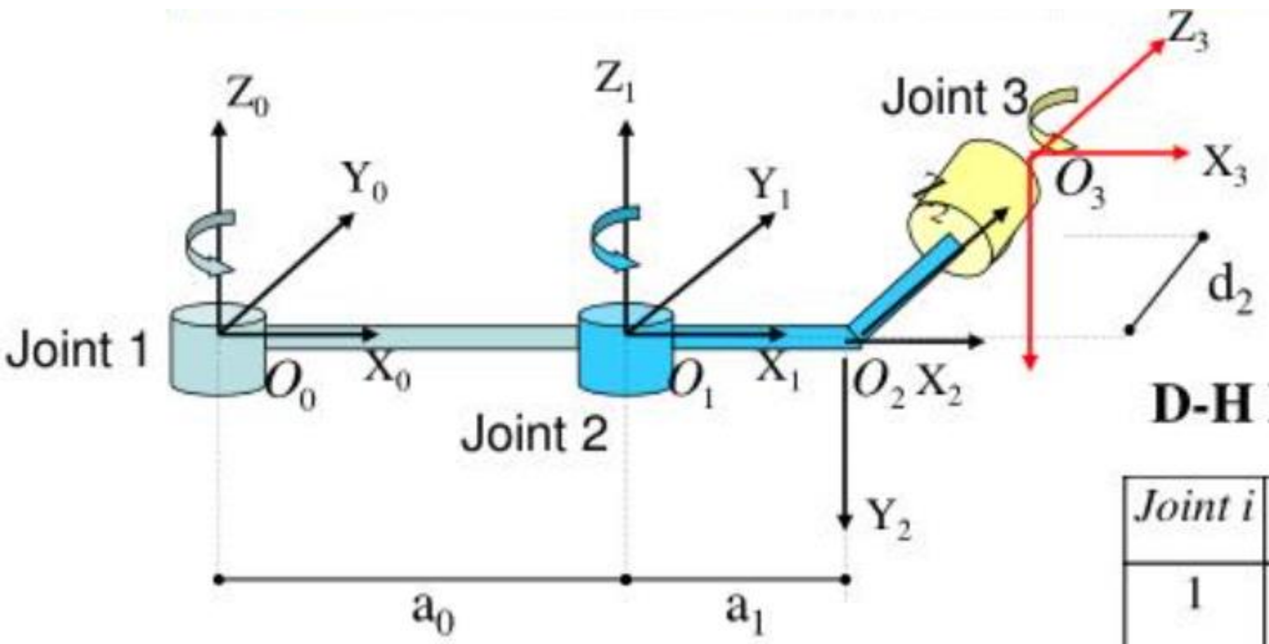
Link	a_i	α_i	d_i	θ_i
4	0	-90	*0*	θ_4*
5	0	90	*0*	θ_5*
6	0	0	d_6	θ_6*

* joint variable

Step 6: Compute the transformation

$$T_6^3 = T_4^3 T_5^4 T_6^5 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example I



D-H Link Parameter Table

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

α_i : rotation angle from Z_{i-1} to Z_i about X_i

a_i : distance from intersection of Z_{i-1} & X_i to origin of i coordinate along X_i

d_i : distance from origin of (i-1) coordinate to intersection of Z_{i-1} & X_i along Z_{i-1}

θ_i : rotation angle from X_{i-1} to X_i about Z_{i-1}

Joint i	α_i	a_i	d_i	θ_i
1	0	a_0	0	θ_1
2	-90	a_1	0	θ_2
3	0	0	d_2	θ_3

$$T_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_0\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_0\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

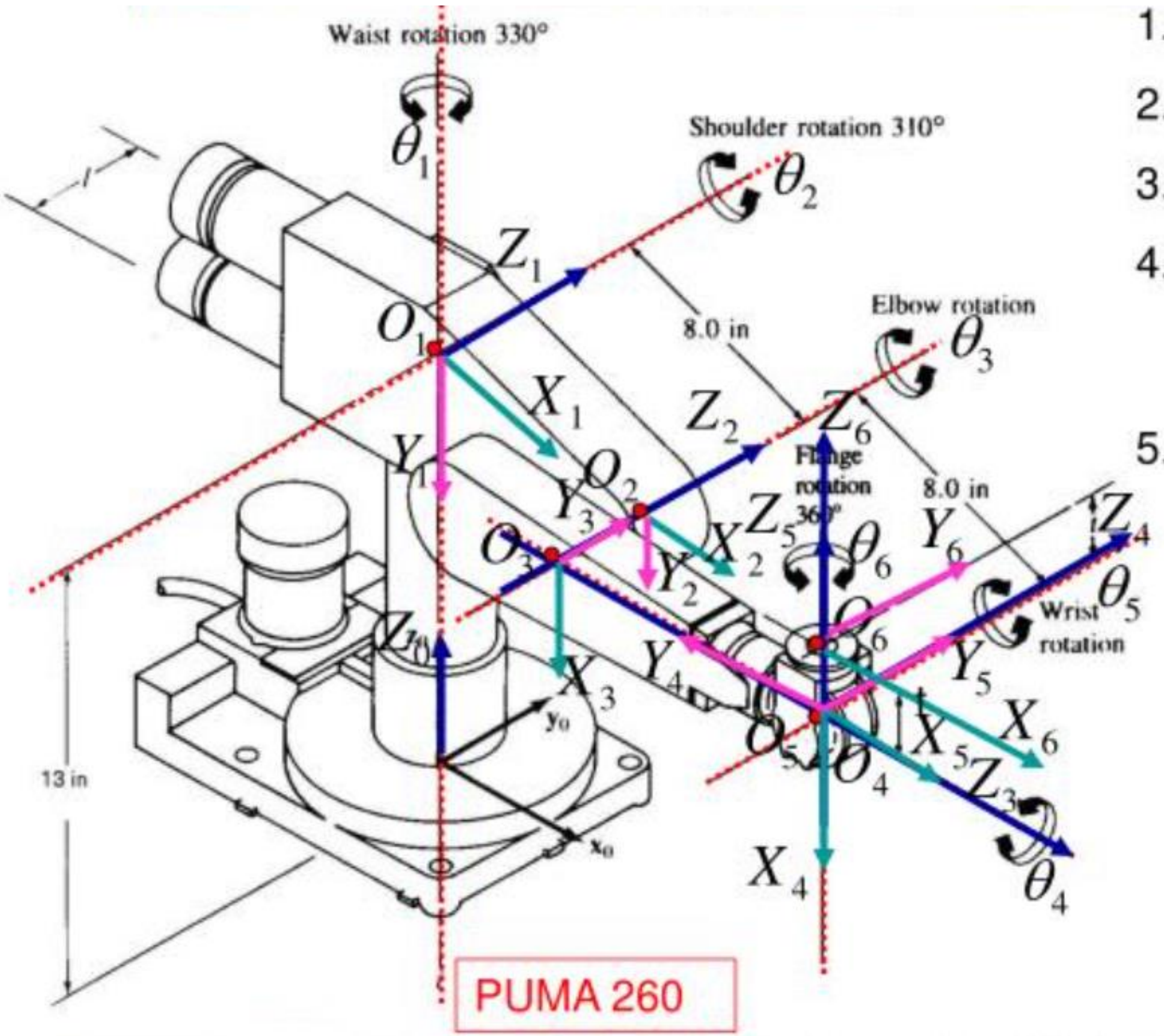
$$T_2^1 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 & a_1\cos\theta_2 \\ \sin\theta_2 & 0 & \cos\theta_2 & a_1\sin\theta_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_i^{i-1} = \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^2 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = (T_1^0)(T_2^1)(T_3^2)$$

Example II: PUMA 260

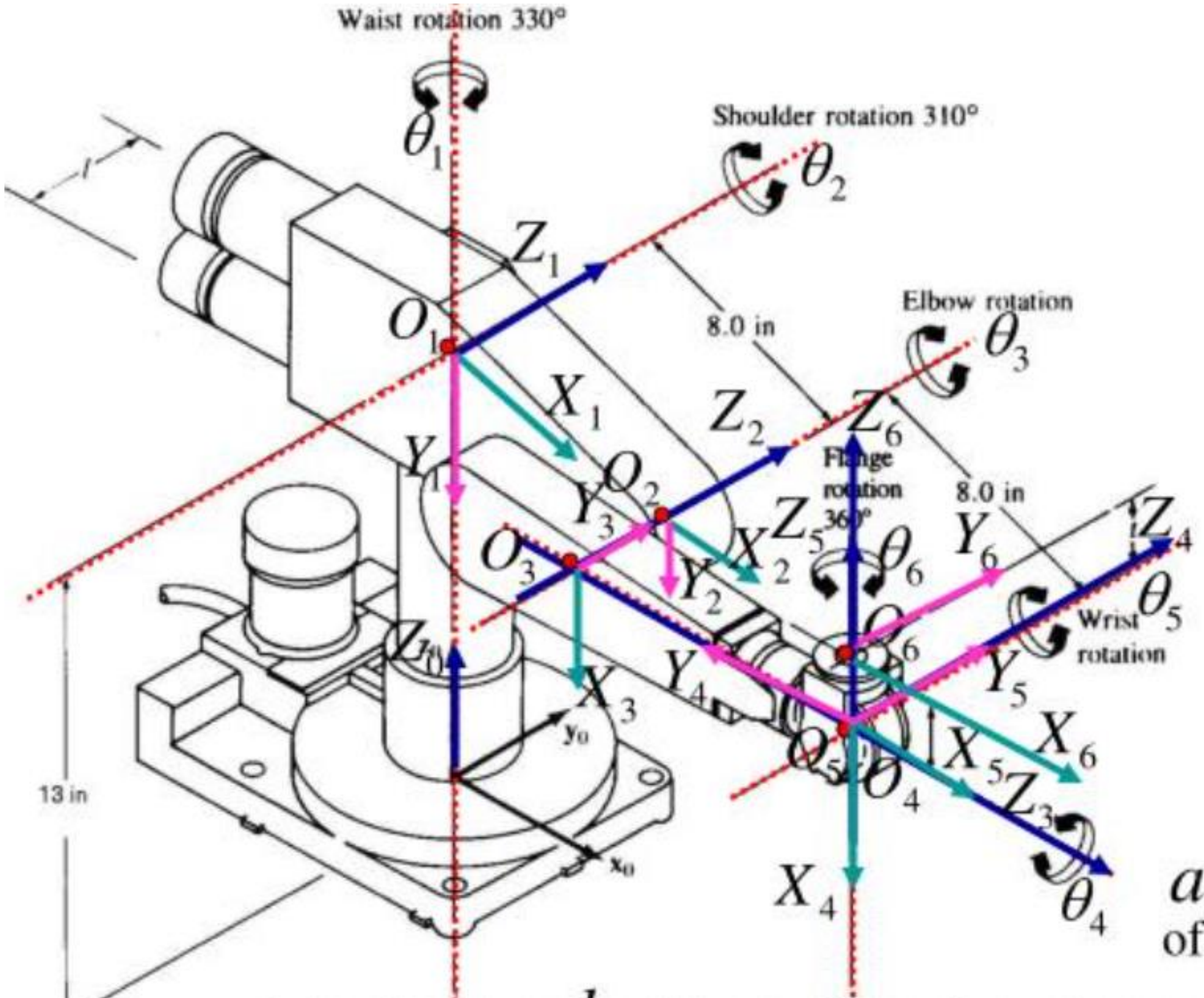


1. Number the joints
2. Establish base frame
3. Establish joint axis Z_i
4. Locate origin, (intersect. of Z_i & Z_{i-1}) OR (intersect of common normal & Z_i)
5. Establish X_i, Y_i

$$X_i = \pm(Z_{i-1} \times Z_i) / \| Z_{i-1} \times Z_i \|$$

$$Y_i = +(Z_i \times X_i) / \| Z_i \times X_i \|$$

Link Parameters



J	θ_i	α_i	a_i	d_i
1	θ_1	-90	0	13
2	θ_2	0	8	0
3	θ_3	90	0	-l
4	θ_4	-90	0	8
5	θ_5	90	0	0
6	θ_6	0	0	t

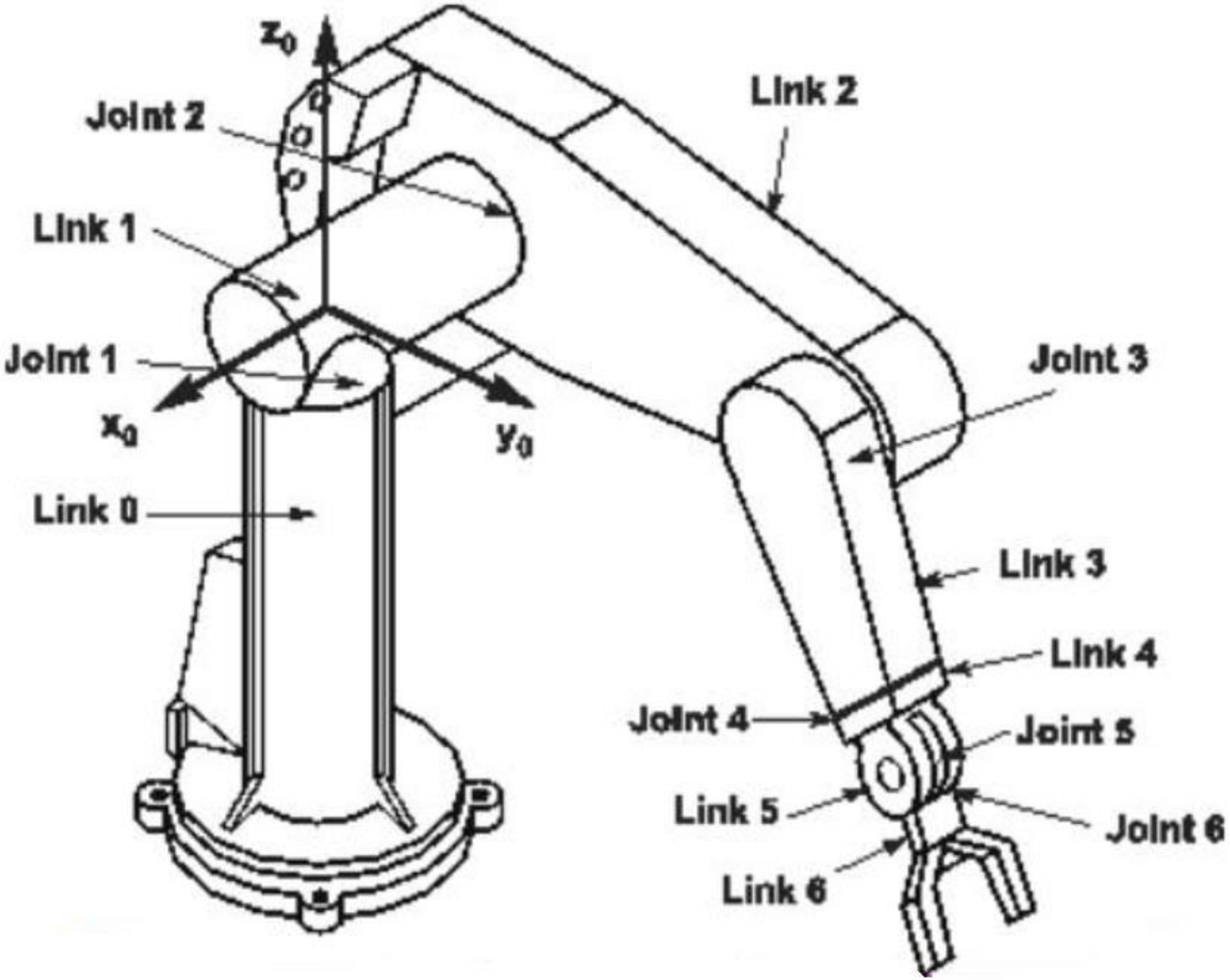
θ_i: angle from X_{i-1} to X_i about Z_{i-1}

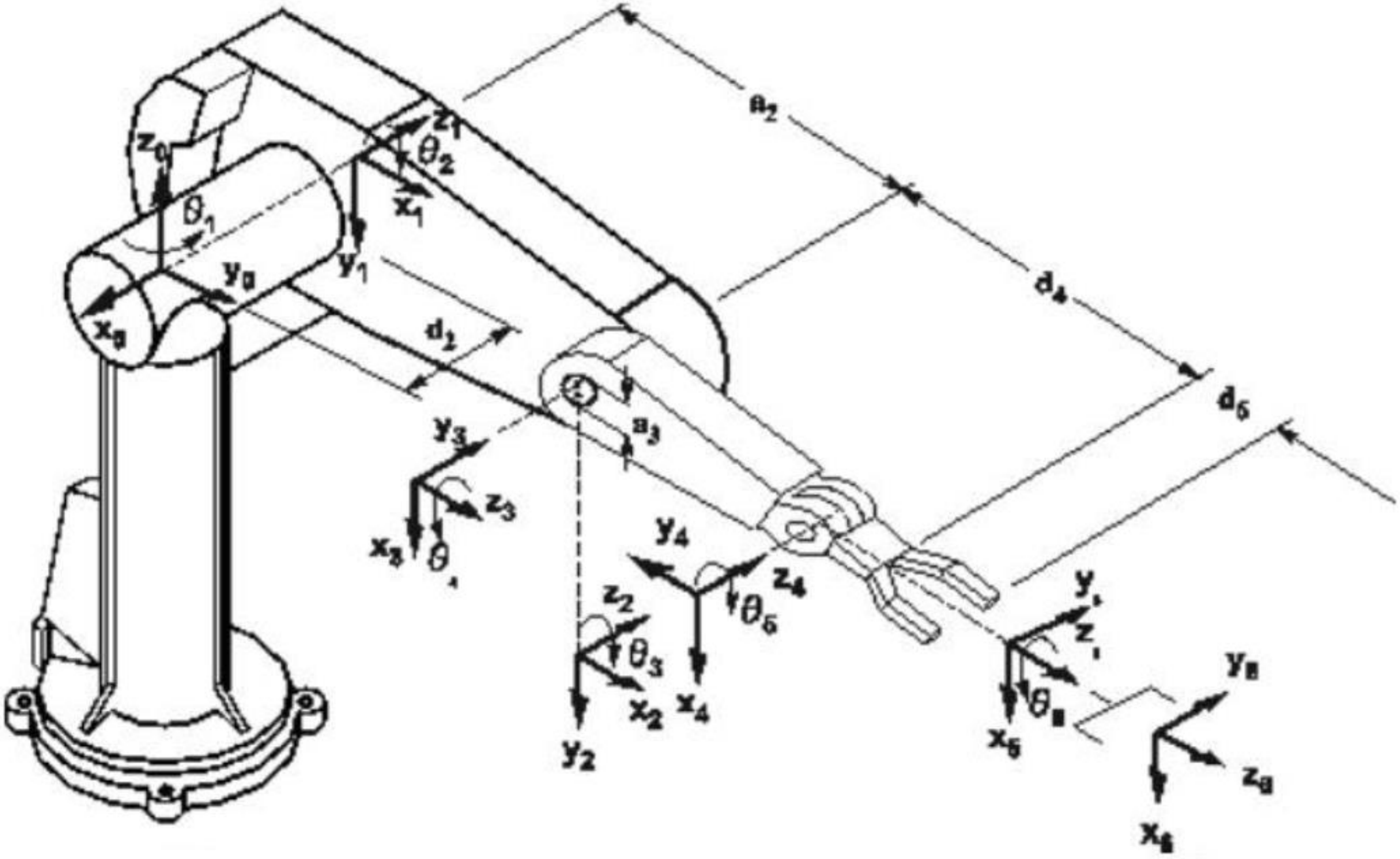
α_i: angle from Z_{i-1} to Z_i about X_i

a_i: distance from intersection of Z_{i-1} & X_i to O_i along X_i

Joint distance d_i: distance from O_{i-1} to intersection of Z_{i-1} & X_i along Z_{i-1}

Example: **Puma 560**





a2
d4
d6
d2
a3
z0
θ1
y0
x0
z1
θ2
x1
y1
y3
z3
x3
z2
θ3
x2
y2
y4
z4
θ5
x4
x6
y6
z6

Link Coordinate Parameters

PUMA 560 robot arm link coordinate parameters

Joint i	θ_i	α_i	$a_i(mm)$	$d_i(mm)$
1	θ_1	-90	0	0
2	θ_2	0	431.8	149.09
3	θ_3	90	-20.32	0
4	θ_4	-90	0	433.07
5	θ_5	90	0	0
6	θ_6	0	0	56.25

$$
{}^{0}\boldsymbol{T}_1 = \begin{pmatrix} \cos\theta_1 & 0 & -\sin\theta_1 & 0 \\ \sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{1}\boldsymbol{T}_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{2}\boldsymbol{T}_3 = \begin{pmatrix} \cos\theta_3 & 0 & \sin\theta_3 & a_3\cos\theta_3 \\ \sin\theta_3 & 0 & -\cos\theta_3 & a_3\sin\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{3}\boldsymbol{T}_4 = \begin{pmatrix} \cos\theta_4 & 0 & -\sin\theta_4 & 0 \\ \sin\theta_4 & 0 & \cos\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
{}^{4}\boldsymbol{T}_5 = \begin{pmatrix} \cos\theta_5 & 0 & \sin\theta_5 & 0 \\ \sin\theta_5 & 0 & -\cos\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad {}^{5}\boldsymbol{T}_6 = \begin{pmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ \sin\theta_6 & \cos\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Example: **Puma 560**

$$^0T_6 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6 = \begin{pmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
n_x &= c_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) - s_1(s_4c_5c_6 + c_4s_6) \\
n_y &= s_1(c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6) + c_1(s_4c_5c_6 + c_4s_6) \\
n_z &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6 \\
s_x &= c_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) - s_1(-s_4c_5s_6 + c_4c_6) \\
s_y &= s_1(-c_{23}(c_4c_5s_6 + s_4c_6) + s_{23}s_5s_6) + c_1(-s_4c_5s_6 + c_4c_6) \\
s_z &= s_{23}(c_4c_5s_6 + s_4c_6) - c_{23}s_5s_6 \\
a_x &= c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5 \\
a_y &= s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5 \\
a_z &= -s_{23}c_4s_5 + c_{23}c_5) \\
p_x &= c_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) - s_1(d_6s_4s_5 + d_2) \\
p_y &= s_1(d_6(c_{23}c_4s_5 + s_{23}c_5) + s_{23}d_4 + a_3c_{23} + a_2c_2) + c_1(d_6s_4s_5 + d_2) \\
p_z &= d_6(c_{23}c_5 - s_{23}c_4s_5) + c_{23}d_4 - a_3s_{23} - a_2s_2
\end{aligned}$$

E III: RPP + Spherical Wrist

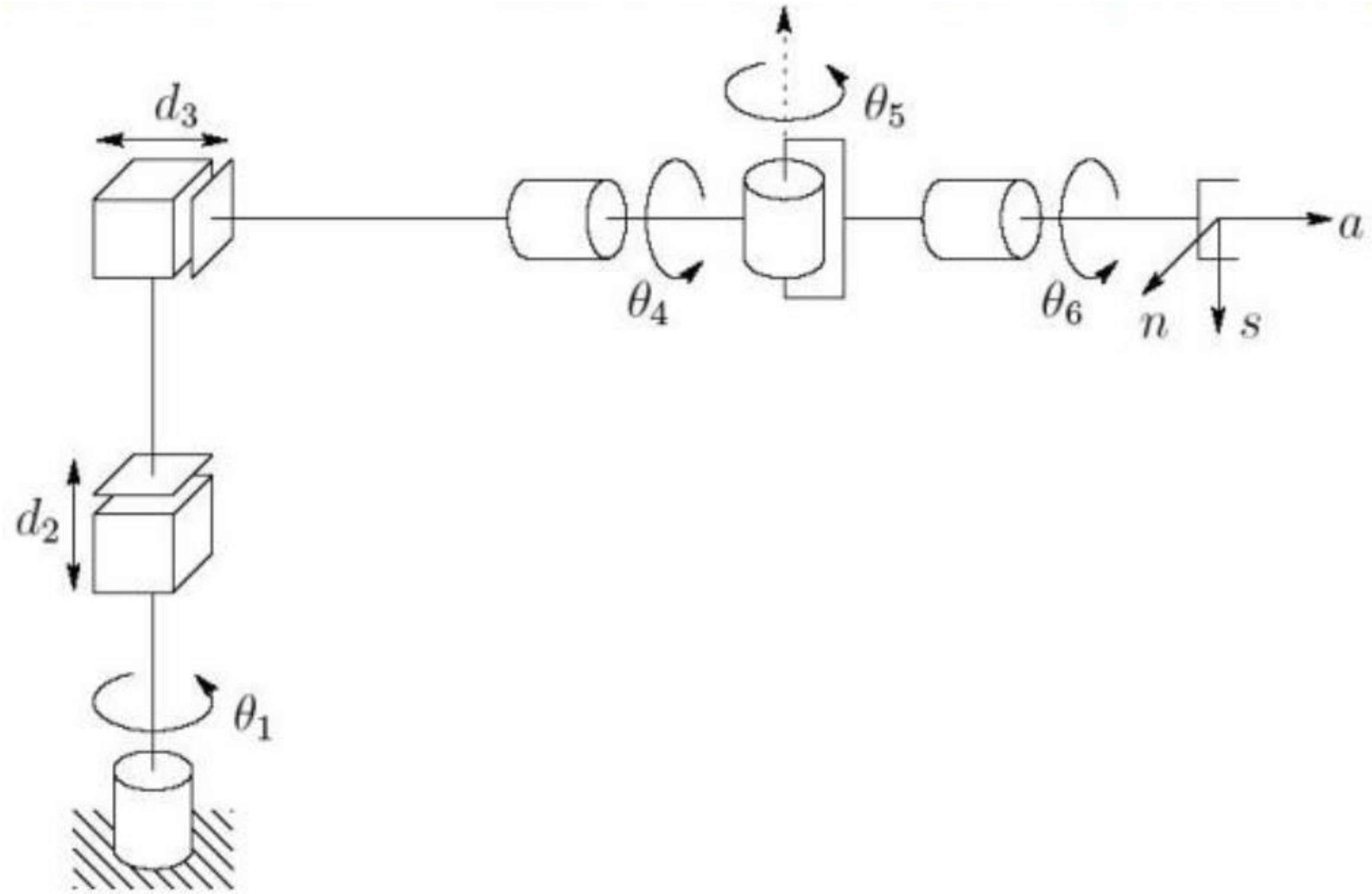


Figure 3.9: Cylindrical robot with spherical wrist.

RPP + Spherical Wrist

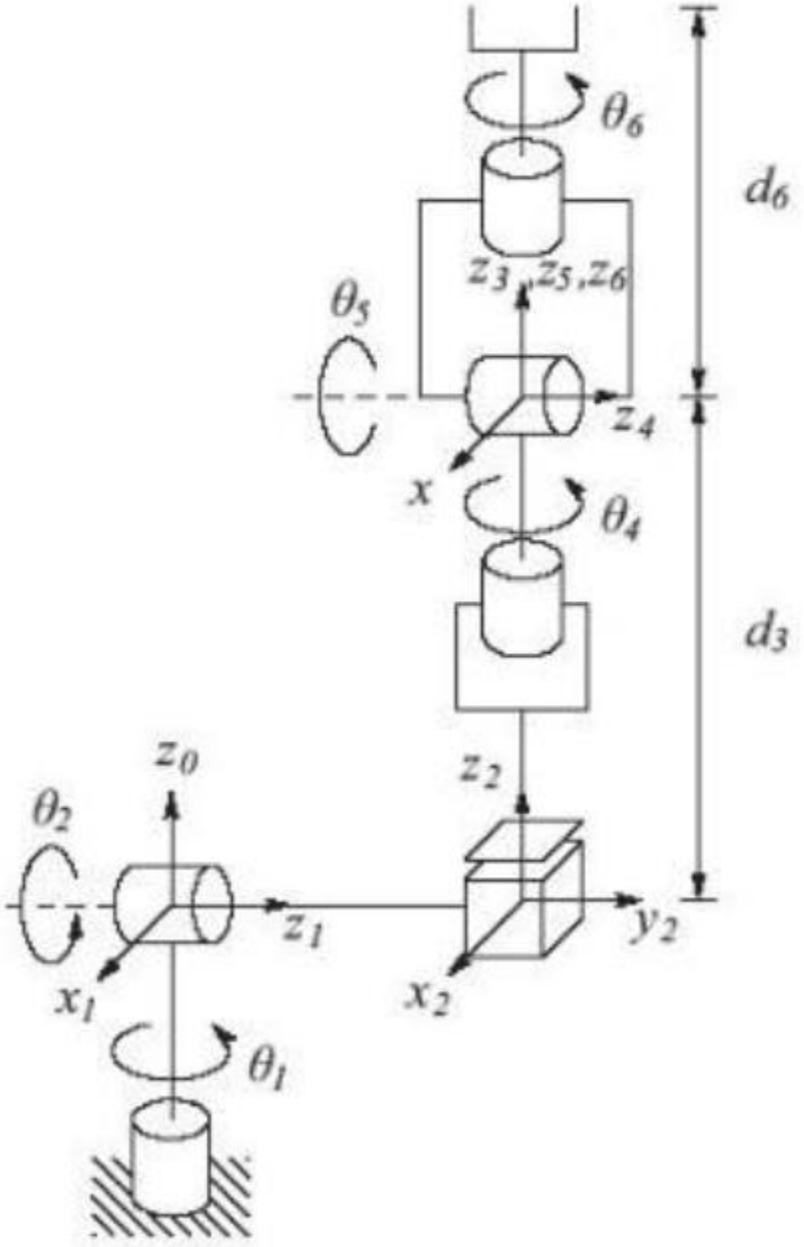
$$T_6^0 = T_3^0 T_6^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1 c_4 c_5 c_6 - c_1 s_4 s_6 + s_1 s_5 c_6$$

$$\vdots$$

$$d_z = -s_4 s_5 d_6 + d_1 + d_2$$

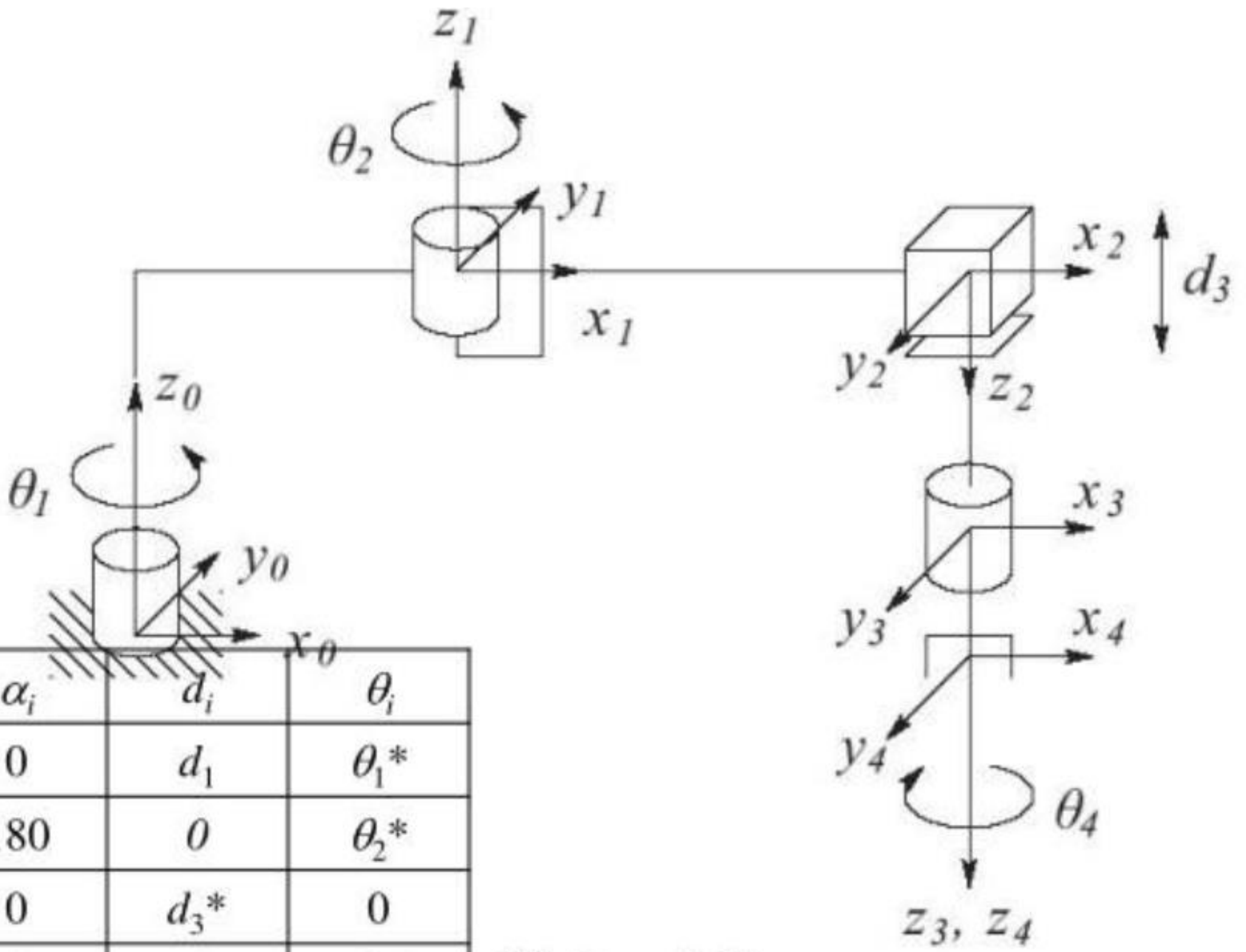
EIV: Stanford Manipulator + Spherical Wrist



Link	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1*
2	0	90	d_2	θ_2*
3	0	0	d_3*	0
4	0	-90	*0*	θ_4*
5	0	90	*0*	θ_5*
6	0	0	d_6	θ_6*

* joint variable

E V: SCARA + 1DOF Wrist



Link	a_i	α_i	d_i	θ_i
1	a_1	0	d_1	θ_1*
2	a_2	180	0	θ_2*
3	0	0	d_3*	0
4	0	0	d_4	θ_4*

* joint variable

Transformation between i-1 and i

- Four successive elementary transformations are required to relate the *i*-th coordinate frame to the (*i*-1)-th coordinate frame:
  - Rotate about the Z_{i-1} axis an angle of θ_i to align the X_{i-1} axis with the X_i axis.
  - Translate along the Z_{i-1} axis a distance of d_i, to bring X_{i-1} and X_i axes into coincidence.
  - Translate along the X_i axis a distance of a_i to bring the two origins O_{i-1} and O_i as well as the X axis into coincidence.
  - Rotate about the X_i axis an angle of α_i (in the right-handed sense), to bring the two coordinates into coincidence.

- D-H transformation matrix for adjacent coordinate frames, *i* and *i-1*.
  - The position and orientation of the *i*-th frame coordinate can be expressed in the (*i-1*)th frame by the following homogeneous transformation matrix:

Reference Coordinate

Source coordinate

$$T_i^{i-1} = T(z_{i-1}, d_i)R(z_{i-1}, \theta_i)T(x_i, a_i)R(x_i, \alpha_i)$$

$$\begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Kinematic Equations

- Forward Kinematics $q = (q_1, q_2, \cdots q_n)$
  - Given joint variables
  - End-effector position & orientation $Y = (x, y, z, \phi, \theta, \psi)$
- Homogeneous matrix T_n^0
  - specifies the location of the ith coordinate frame w.r.t. the base coordinate system
  - chain product of successive coordinate transformation matrices of T_i^{i-1}

$$T_n^0 = T_1^0 T_2^1 \ldots T_n^{n-1}$$

Position vector

Orientation matrix

$$= \begin{bmatrix} \boxed{R_n^0} & \boxed{P_n^0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & P_n^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Other representations
  - reference from, tool frame

$$\boldsymbol{T_{tool}^{ref} = B_0^{ref} T_n^0 H_{tool}^n}$$

  - Yaw-Pitch-Roll representation for orientation

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 & 0 \\ S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Representing forward kinematics

- Forward kinematics

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} \Rightarrow \begin{bmatrix} p_x \\ p_y \\ p_z \\ \phi \\ \theta \\ \varphi \end{bmatrix}$$

- Transformation Matrix

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outline

- Review
  - Kinematics Model
  - Inverse Kinematics
    - Example
- Jacobian Matrix
  - Singularity
- Trajectory Planning

Review

- Steps to derive kinematics model:
  - Assign D-H coordinates frames
  - Find link parameters
  - Transformation matrices of adjacent joints
  - Calculate kinematics matrix
  - When necessary, Euler angle representation

Denavit-Hartenberg Convention

- Number the joints from 1 to n starting with the base and ending with the end-effector.
- *Establish the base coordinate system.* Establish a right-handed orthonormal coordinate system (X_0, Y_0, Z_0) at the supporting base with Z_0 axis lying along the axis of motion of joint 1.
- *Establish joint axis.* Align the Z_i with the axis of motion (rotary or sliding) of joint i+1.
- *Establish the origin of the ith coordinate system.* Locate the origin of the ith coordinate at the intersection of the Z_i & Z_{i-1} or at the intersection of common normal between the Z_i & Z_{i-1} axes and the Z_i axis.
- *Establish X_i axis.* Establish $X_i = \pm(Z_{i-1} \times Z_i)/\|Z_{i-1} \times Z_i\|$ or along the common normal between the Z_{i-1} & Z_i axes when they are parallel.
- *Establish Y_i axis.* Assign $Y_i = +(Z_i \times X_i)/\|Z_i \times X_i\|$ to complete the right-handed coordinate system.
- Find the link and joint parameters

Review

- Link and Joint Parameters
  - *Joint angle* θ_i: the angle of rotation from the X_{i-1} axis to the X_i axis about the Z_{i-1} axis. It is the joint variable if joint i is rotary.
  - *Joint distance* d_i: the distance from the origin of the (i-1) coordinate system to the intersection of the Z_{i-1} axis and the X_i axis along the Z_{i-1} axis. It is the joint variable if joint i is prismatic.
  - *Link length* a_i : the distance from the intersection of the Z_{i-1} axis and the X_i axis to the origin of the ith coordinate system along the X_i axis.
  - *Link twist angle* α_i: the angle of rotation from the Z_{i-1} axis to the Z_i axis about the X_i axis.

Review

- D-H transformation matrix for adjacent coordinate frames, *i* and *i-1*.
  - The position and orientation of the *i*-th frame coordinate can be expressed in the *(i-1)*th frame by the following 4 successive elementary transformations:

Source coordinate

$$T_{i-1}^{i} = T(z_{i-1}, d_i)R(z_{i-1}, \theta_i)T(x_i, a_i)R(x_i, \alpha_i)$$

Reference Coordinate

$$= \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Kinematics Equations
  - chain product of successive coordinate transformation matrices of T_{i-1}^{i}
  - T_0^n specifies the location of the *n*-th coordinate frame w.r.t. the base coordinate system

$$T_0^n = T_0^1 T_1^2 \ldots T_{n-1}^n$$

Orientation matrix

$$= \begin{bmatrix} \boxed{R_0^n} & \boxed{P_0^n} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & P_0^n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Position vector

- Forward Kinematics

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} \Rightarrow \begin{bmatrix} p_x \\ p_y \\ p_z \\ \phi \\ \theta \\ \varphi \end{bmatrix}$$

- Kinematics Transformation Matrix

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Why use Euler angle representation?

What is $a\tan 2(y, x)$?

$$\theta = a\tan 2(y,x) = \begin{cases} 0^\circ \le \theta \le 90^\circ & for \quad +x \quad and \quad +y \\ 90^\circ \le \theta \le 180^\circ & for \quad -x \quad and \quad +y \\ -180^\circ \le \theta \le -90^\circ & for \quad -x \quad and \quad -y \\ -90^\circ \le \theta \le 0^\circ & for \quad +x \quad and \quad -y \end{cases}$$

- Yaw-Pitch-Roll Representation

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi} \Rightarrow R_{z,\phi}^{-1} T = R_{y,\theta} R_{x,\psi}$$

$$\begin{bmatrix} C\phi & S\phi & 0 & 0 \\ -S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x & 0 \\ n_y & s_y & a_y & 0 \\ n_z & s_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} C\phi \cdot n_x + S\phi \cdot n_y & XX & XX & 0 \\ -S\phi \cdot n_x + C\phi \cdot n_y & \boxed{-S\phi \cdot s_x + C\phi \cdot s_y} & \boxed{-S\phi \cdot a_x + C\phi \cdot a_y} & 0 \\ n_z & XX & XX & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & S\theta S\psi & S\theta C\psi & 0 \\ 0 & \boxed{C\psi} & \boxed{-S\psi} & 0 \\ -S\theta & C\theta S\psi & C\theta C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Equation A)

- Compare LHS and RHS of Equation A, we have:

$$-\sin\phi\cdot n_x+\cos\phi\cdot n_y=0 \Longrightarrow \phi=a\tan 2(n_y,n_x)$$

$$\begin{cases}\cos\phi\cdot n_x+\sin\phi\cdot n_y=\cos\theta\\ n_z=-\sin\theta\end{cases} \Longrightarrow \theta=a\tan 2(-n_z,\cos\phi\cdot n_x+\sin\phi\cdot n_y)$$

$$\begin{cases}-\sin\phi\cdot s_x+\cos\phi\cdot s_y=\cos\psi\\ -\sin\phi\cdot a_x+\cos\phi\cdot a_y=-\sin\psi\end{cases}$$

$$\Downarrow$$

$$\psi=a\tan 2(\sin\phi\cdot a_x-\cos\phi\cdot a_y,-\sin\phi\cdot s_x+\cos\phi\cdot s_y)$$

Inverse Kinematics

- Transformation Matrix

$$T_0^6 = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 \Rightarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$$

Robot dependent, Solutions not unique

Systematic closed-form solution in general is not available

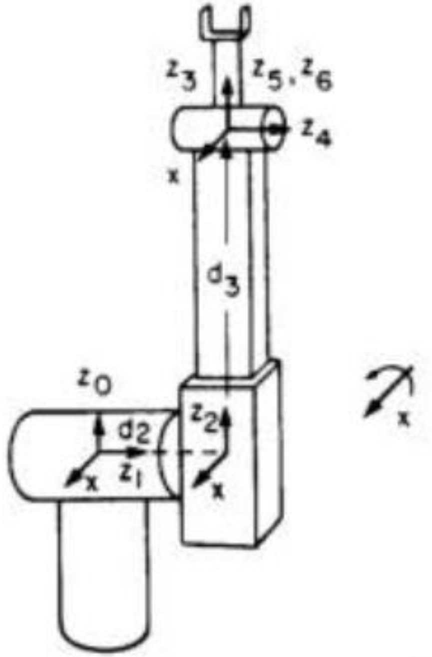
Special cases make the closed-form arm solution possible:

1. Three adjacent joint axes intersecting (PUMA, Stanford)
2. Three adjacent joint axes parallel to one another (MINIMOVER)

Example

- Solving the inverse kinematics of Stanford arm

$$T_0^6 = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6$$



$$(T_0^1)^{-1} T_0^6 = T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 = T_1^6$$

$$T_1^6 = \begin{bmatrix} X & X & X & C\theta_1 p_x + S\theta_1 p_y \\ X & X & X & -p_z \\ X & X & X & -S\theta_1 p_x + C\theta_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X & X & X & S\theta_2 \cdot d_3 \\ X & X & X & -C\theta_2 \cdot d_3 \\ X & X & X & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Solving the inverse kinematics of Stanford arm

$$-Sin\theta_1 \cdot p_x + \cos\theta_1 \cdot p_y = 0.1$$ Equation (1)

$$\cos\theta_1 \cdot p_x + \sin\theta_1 \cdot p_y = \sin\theta_2 \cdot d_3$$ Equation (2)

$$-p_z = -\cos\theta_2 \cdot d_3$$ Equation (3)

In Equ. (1), let

$$p_x = r \cdot \cos\alpha, \qquad p_y = r \cdot \sin\alpha, \qquad r = \sqrt{p_x^2 + p_y^2}, \qquad \alpha = a\tan 2(\frac{p_y}{p_x})$$

$$\sin\alpha \cdot \cos\theta_1 - \sin\theta_1 \cdot \cos\alpha = {}^{0.1}\!/_{r} \Rightarrow \left.\begin{aligned} \sin(\alpha - \theta_1) &= {}^{0.1}\!/_{r} \\ \cos(\alpha - \theta_1) &= \pm\sqrt{1 - (0.1/r)^2} \end{aligned}\right\}$$

$$\theta_1 = a\tan 2(\frac{p_y}{p_x}) - a\tan 2(\frac{0.1}{\pm\sqrt{r^2 - 0.1^2}})$$

$$\theta_2 = a\tan 2(\frac{\cos\theta_1 p_x + \sin\theta_1 p_y}{p_z}) \qquad d_3 = \frac{p_z}{\cos\theta_2}$$

- Solving the inverse kinematics of Stanford arm

$$(T_3^4)^{-1}(T_2^3)^{-1}(T_1^2)^{-1}(T_0^1)^{-1}T_0^6 = T_4^5 T_5^6 = \begin{bmatrix} X & X & S\theta_5 & 0 \\ X & X & -C\theta_5 & 0 \\ X & X & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From term (3,3)

$$-S\theta_4[C\theta_2(C\theta_1 a_x + S\theta_1 a_y) - S\theta_2 a_z] + C\theta_4(-S\theta_1 a_x + C\theta_1 a_y) = 0$$

$$\theta_4 = a\tan 2\left(\frac{-S\theta_1 a_x + C\theta_1 a_y}{C\theta_2(C\theta_1 a_x + S\theta_1 a_y) - S\theta_2 a_z}\right) \qquad \theta_5 = a\tan 2\left(\frac{S\theta_5}{C\theta_5}\right)$$

From term (1,3), (2,3)

$$\begin{cases} S\theta_5 = C\theta_4(C\theta_2(C\theta_1 a_x + S\theta_1 a_y) - S\theta_2 a_z] + S\theta_4(-S\theta_1 a_x + C\theta_1 a_y) \\ C\theta_5 = S\theta_2(C\theta_1 a_x + S\theta_1 a_y) + C\theta_2 a_z) \end{cases}$$

- Solving the inverse kinematics of Stanford arm

$$(T_4^5)^{-1}(T_3^4)^{-1}(T_2^3)^{-1}(T_1^2)^{-1}(T_0^1)^{-1}T_0^6 = T_5^6 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⇩

$$\begin{cases} S\theta_6 = -C\theta_5\{C\theta_4[C\theta_2(C\theta_1 s_x + S\theta_1 s_y) - S\theta_2 s_z] + S\theta_4(-S\theta_1 s_x + C\theta_1 s_y)\} + S\theta_5[S\theta_2(C\theta_1 s_x + S\theta_1 s_y) + C\theta_2 s_z] \\ C\theta_6 = -S\theta_4[C\theta_2(C\theta_1 s_x + S\theta_1 s_y) - S\theta_2 s_z] + C\theta_4(-S\theta_1 s_x + C\theta_1 s_y) \end{cases}$$

⇩

$$\theta_6 = a\tan 2(\frac{S\theta_6}{C\theta_6})$$

Jacobian Matrix

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\end{bmatrix} \underset{\text{Inverse}}{\overset{\text{Forward}}{\rightleftarrows}} \begin{bmatrix}x\\ y\\ z\\ \phi\\ \theta\\ \varphi\end{bmatrix}$$

Kinematics

Joint Space Task Space

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\\ \dot{\theta}_5\\ \dot{\theta}_6\end{bmatrix} \xleftrightarrow{\text{Jacobian Matrix}} \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

Jacobian Matrix: Relationship between joint space velocity with task space velocity

Forward kinematics

$$\begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix} = h\left(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}\right)_{6\times1} = \begin{bmatrix} h_1(q_1, q_2, \cdots, q_6) \\ h_2(q_1, q_2, \cdots, q_6) \\ h_3(q_1, q_2, \cdots, q_6) \\ h_4(q_1, q_2, \cdots, q_6) \\ h_5(q_1, q_2, \cdots, q_6) \\ h_6(q_1, q_2, \cdots, q_6) \end{bmatrix}_{6\times1}$$

$$Y_{6\times1} = h(q_{n\times1})$$

$$\dot{Y}_{6\times1} = \frac{d}{dt} h(q_{n\times1}) = \frac{dh(q)}{dq}\frac{dq}{dt} = \frac{dh(q)}{dq}\dot{q}$$

$$\dot{Y}_{6\times1} = J_{6\times n}\dot{q}_{n\times1}$$

$$\boxed{J = \frac{dh(q)}{dq}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \boxed{\left[\frac{dh(q)}{dq}\right]_{6\times n}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times1}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \left[\frac{dh(q)}{dq} \right]_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

Jacobian is a function of q, it is not a constant!

$$J = \left(\frac{dh(q)}{dq} \right)_{6\times n} = \begin{bmatrix} \frac{\partial h_1}{\partial q_1} & \frac{\partial h_1}{\partial q_2} & \cdots & \frac{\partial h_1}{\partial q_n} \\ \frac{\partial h_2}{\partial q_1} & \frac{\partial h_2}{\partial q_2} & \cdots & \frac{\partial h_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_6}{\partial q_1} & \frac{\partial h_6}{\partial q_2} & \cdots & \frac{\partial h_6}{\partial q_n} \end{bmatrix}_{6\times n}$$

Forward Kinematics

$$T_0^6 = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$\nearrow \quad p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} h_1(q) \\ h_2(q) \\ h_3(q) \end{bmatrix}$$

$$\searrow \quad \{n, s, a\} \rightarrow \begin{bmatrix} \phi(q) \\ \theta(q) \\ \psi(q) \end{bmatrix} = \begin{bmatrix} h_4(q) \\ h_5(q) \\ h_6(q) \end{bmatrix}$$

$$Y_{6\times 1} = h(q) = \begin{bmatrix} h_1(q) \\ h_2(q) \\ \vdots \\ h_6(q) \end{bmatrix}$$

$$\Downarrow$$

$$\dot{Y}_{6\times 1} = J_{6\times n}\dot{q}_{n\times 1}$$

$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} V \\ \Omega \end{bmatrix}$$

Linear velocity

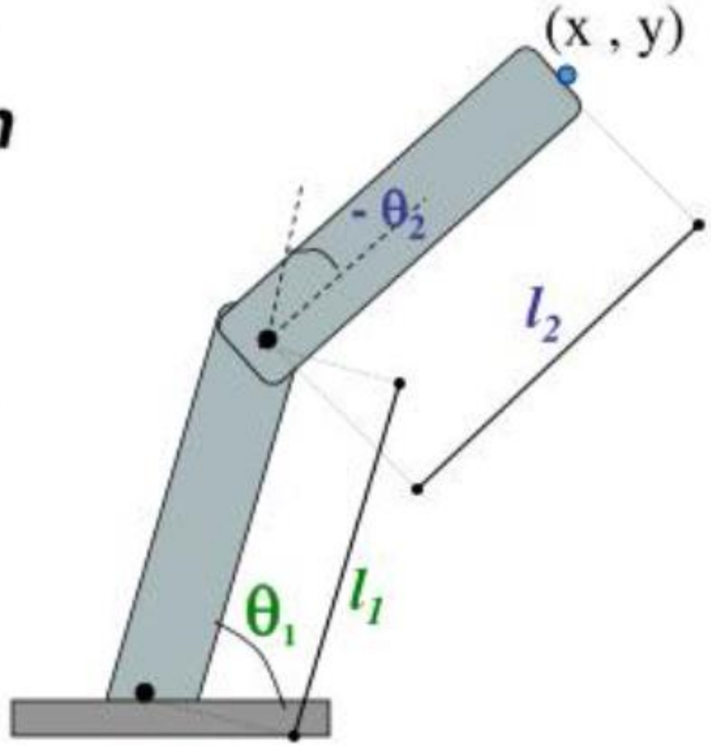
$$V = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

Angular velocity

$$\Omega = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Example

- 2-DOF planar robot arm
  - Given l_1, l_2, ***Find: Jacobian***



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} h_1(\theta_1, \theta_2) \\ h_2(\theta_1, \theta_2) \end{bmatrix}$$

$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial h_1}{\partial \theta_1} & \dfrac{\partial h_1}{\partial \theta_2} \\ \dfrac{\partial h_2}{\partial \theta_1} & \dfrac{\partial h_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Jacobian Matrix

- Physical Interpretation

$$\dot{Y} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

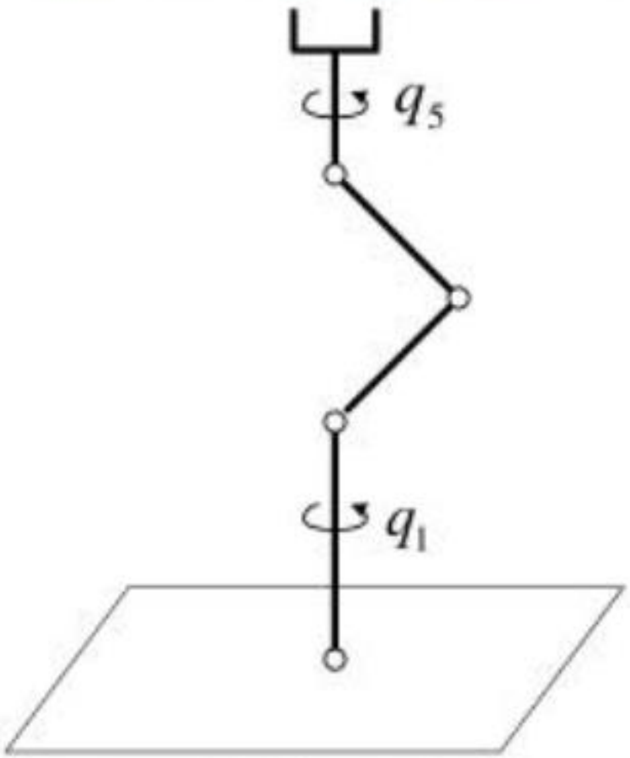
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} J_{11}\dot{q}_1 + J_{12}\dot{q}_2 + \cdots + J_{16}\dot{q}_6 \\ J_{21}\dot{q}_1 + J_{22}\dot{q}_2 + \cdots + J_{26}\dot{q}_6 \\ J_{31}\dot{q}_1 + J_{32}\dot{q}_2 + \cdots + J_{36}\dot{q}_6 \\ J_{41}\dot{q}_1 + J_{42}\dot{q}_2 + \cdots + J_{46}\dot{q}_6 \\ J_{51}\dot{q}_1 + J_{52}\dot{q}_2 + \cdots + J_{56}\dot{q}_6 \\ J_{61}\dot{q}_1 + J_{62}\dot{q}_2 + \cdots + J_{66}\dot{q}_6 \end{bmatrix}$$

How each individual joint space velocity contribute to task space velocity.

- Inverse Jacobian

$$\dot{Y} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\dot{q} = J^{-1}\dot{Y}$$



- Singularity
  - rank(J)<min{6,n}, Jacobian Matrix is less than full rank
  - Jacobian is non-invertable
  - Boundary Singularities: occur when the tool tip is on the surface of the work envelop.
  - Interior Singularities: occur inside the work envelope when two or more of the axes of the robot form a straight line, i.e., collinear

Quiz

- Find the singularity configuration of the 2-DOF planar robot arm

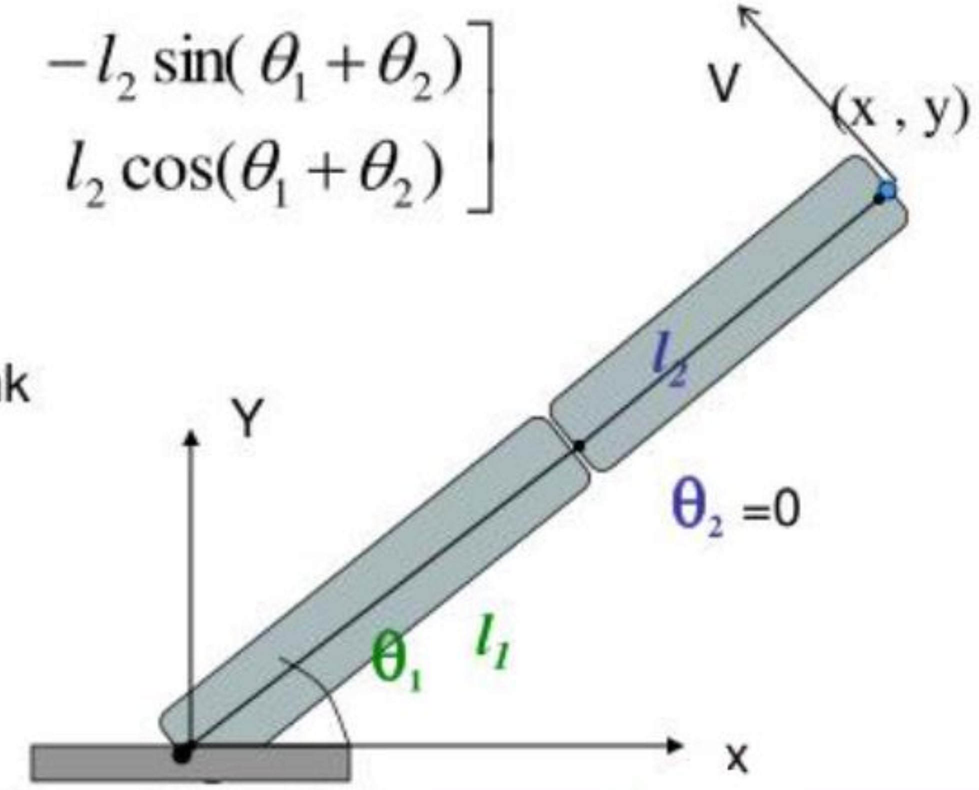
$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

determinant(J)=0 $\Rightarrow$ Not full rank

$$\theta_2 = 0$$

Det(J)=0



Jacobian Matrix

- Pseudoinverse
  - Let A be an mxn matrix, and let A^+ be the pseudoinverse of A. If A is of full rank, then A^+ can be computed as:

$$A^+ = \begin{cases} A^T[AA^T]^{-1} & m \le n \\ A^{-1} & m = n \\ [A^TA]^{-1}A^T & m \ge n \end{cases}$$

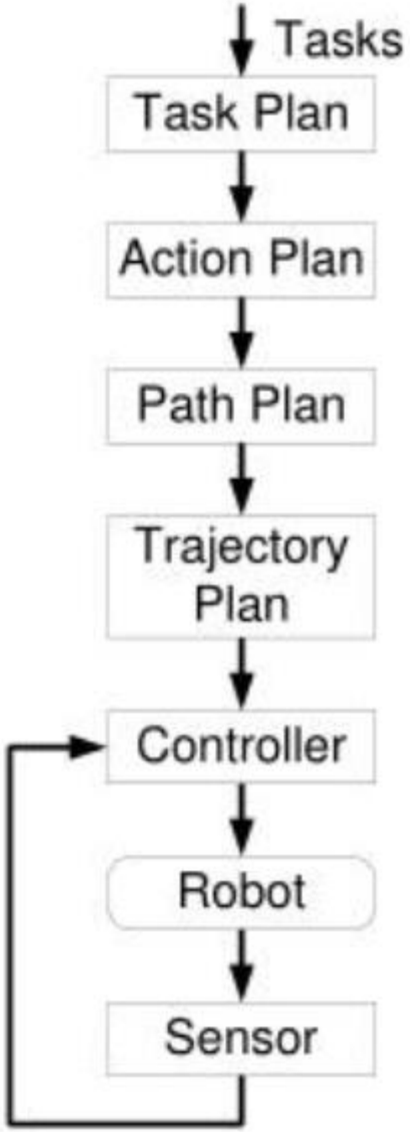
  - Example:

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$x = A^+ b = 1/9[-5,13,16]^T$$

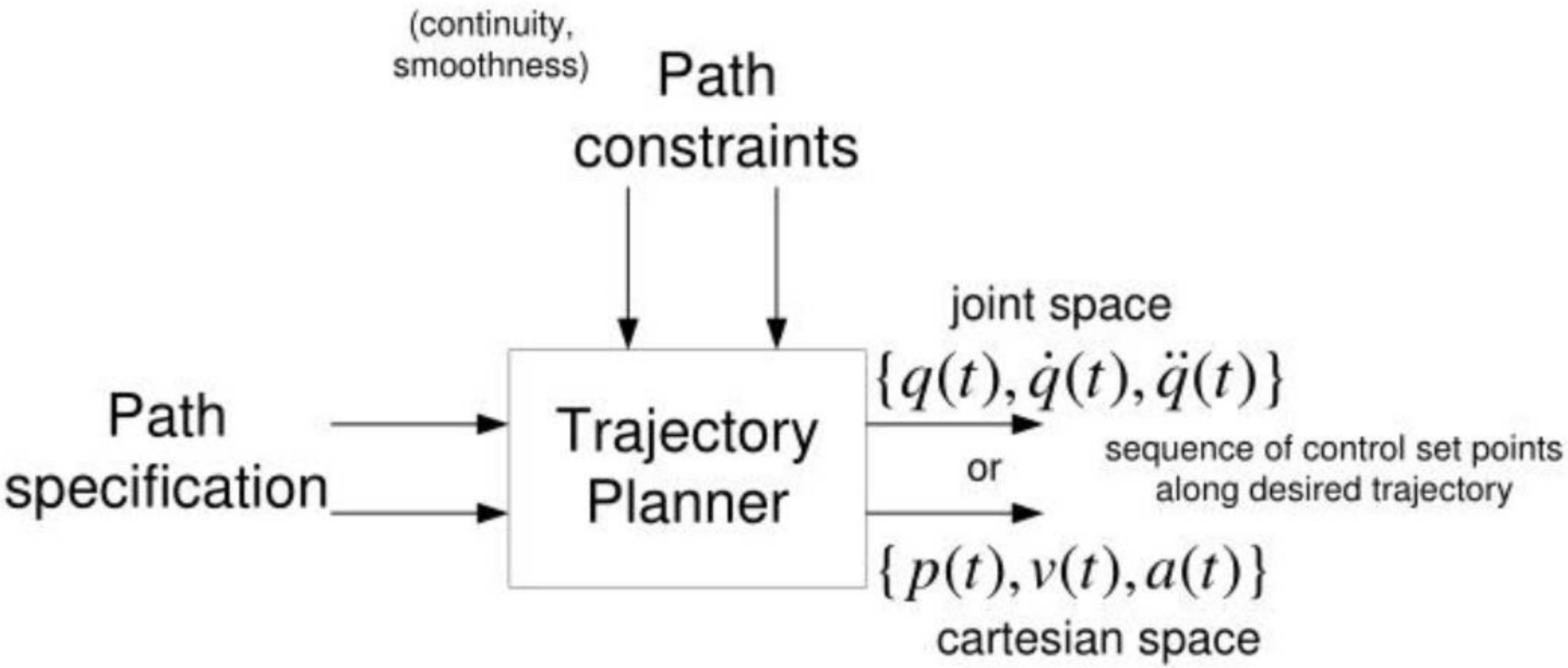
$$A^+ = A^T[AA^T]^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 4 \\ 1 & -5 \\ 4 & -2 \end{bmatrix}$$

Robot Motion Planning



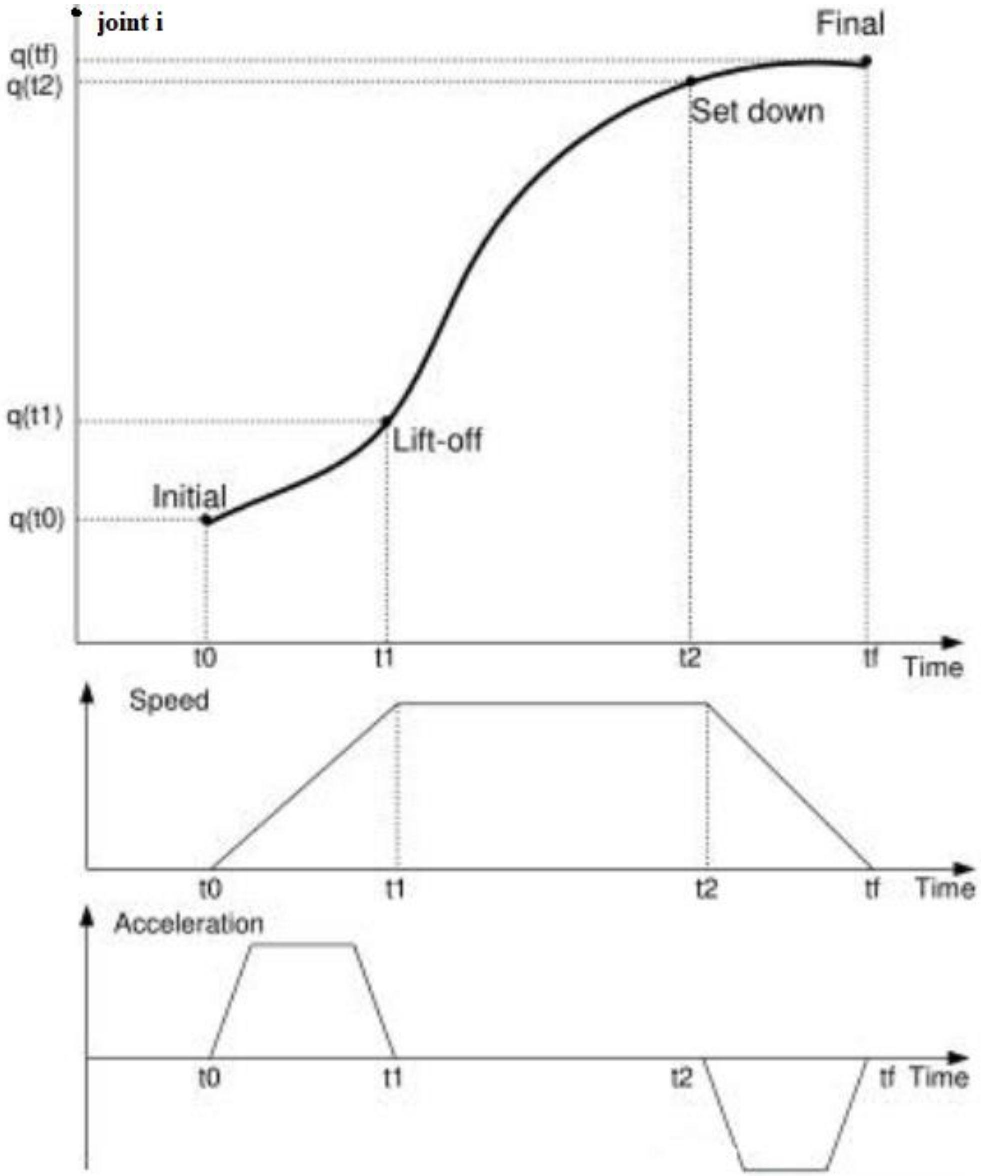
- Path planning
  - Geometric path
  - Issues: obstacle avoidance, shortest path
- Trajectory planning,
  - "interpolate" or "approximate" the desired path by a class of polynomial functions and generates a sequence of time-based "control set points" for the control of manipulator from the initial configuration to its destination.

Trajectory Planning



Trajectory planning

- Path Profile
- Velocity Profile
- Acceleration Profile



The boundary conditions

1) Initial position
2) Initial velocity
3) Initial acceleration
4) Lift-off position
5) Continuity in position at t_1
6) Continuity in velocity at t_1
7) Continuity in acceleration at t_1
8) Set-down position
9) Continuity in position at t_2
10) Continuity in velocity at t_2
11) Continuity in acceleration at t_2
12) Final position
13) Final velocity
14) Final acceleration

Requirements

- Initial Position
  - Position (given)
  - Velocity (given, normally zero)
  - Acceleration (given, normally zero)
- Final Position
  - Position (given)
  - Velocity (given, normally zero)
  - Acceleration (given, normally zero)

- Intermediate positions
  - set-down position (given)
  - set-down position (continuous with previous trajectory segment)
  - Velocity (continuous with previous trajectory segment)
  - Acceleration (continuous with previous trajectory segment)

- Intermediate positions
  - Lift-off position (given)
  - Lift-off position (continuous with previous trajectory segment)
  - Velocity (continuous with previous trajectory segment)
  - Acceleration (continuous with previous trajectory segment)

Trajectory Planning

- n-th order polynomial, must satisfy 14 conditions,
- 13-th order polynomial

$$a_{13}t^{13} + \cdots + a_2 t^2 + a_1 t + a_0 = 0$$

- 4-3-4 trajectory

$$h_1(t) = a_{14}t^4 + a_{13}t^3 + a_{12}t^2 + a_{12}t + a_{10}$$ t0→t1, 5 unknow

$$h_2(t) = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20}$$ t1→t2, 4 unknow

$$h_n(t) = a_{n4}t^4 + a_{n3}t^3 + a_{n2}t^2 + a_{n2}t + a_{n0}$$ t2→tf, 5 unknow

- 3-5-3 trajectory

THANK YOU