University School of Automation and Robotics
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USAR
UNIVERSITY SCHOOL OF AUTOMATION AND ROBOTICS

Engineering Mechanics

By: Dr. Divya Agarwal

- **UNIT- I**

- **Force system:** Introduction, force, principle of transmissibility of force, resultant of a force system, resolution of a force, moment of force about a line, Varigon’s theorem, couple, resolution of a force into force and a couple, properties of couple and their application to engineering problems.
- **Equilibrium:** Force body diagram, equations of equilibrium, and their applications to engineering problems, equilibrium of two force and three force members.
- **Distributed forces:** Determination of centre of gravity, centre of mass and centroid by direct integration and by the method of composite bodies., mass moment of inertia and area moment of inertia by direct integration and composite bodies method, radius of gyration, parallel axis theorem, polar moment of inertia.

- **UNIT- II**

- **Structure:** Plane truss, perfect and imperfect truss, assumption in the truss analysis, analysis of perfect plane trusses by the method of joints, method of section, graphical method.
- **Friction:** Static and Kinetic friction, laws of dry friction, co-efficient of friction, angle of friction, angle of repose, cone of friction, frictional lock, friction in pivot and collar bearing, friction in flat belts.

- **UNIT-III**

- **Kinematics of Particles:** Rectilinear motion, plane curvilinear motion, rectangular coordinates, normal and tangential coordinates

- **Kinetics of Particles:** Equation of motion, rectilinear motion and curvilinear motion, work energy equation, conservation of energy, concept of impulse and momentum, conservation of momentum, impact of bodies, co-efficient of restitution, loss of energy during impact.

- **UNIT-IV**

- **Kinematics of Rigid Bodies:** Concept of rigid body, type of rigid body motion, absolute motion, introduction to relative velocity, relative acceleration (Corioli's component excluded) and instantaneous center of zero velocity, velocity and acceleration.

- **Kinetics of Rigid Bodies:** Equation of motion, translatory motion and fixed axis rotation, application of work energy principles to rigid bodies conservation of energy.

- **Beam:** Introduction, types of loading, methods for the reactions of a beam, space diagram, types of end supports, beams subjected to couple

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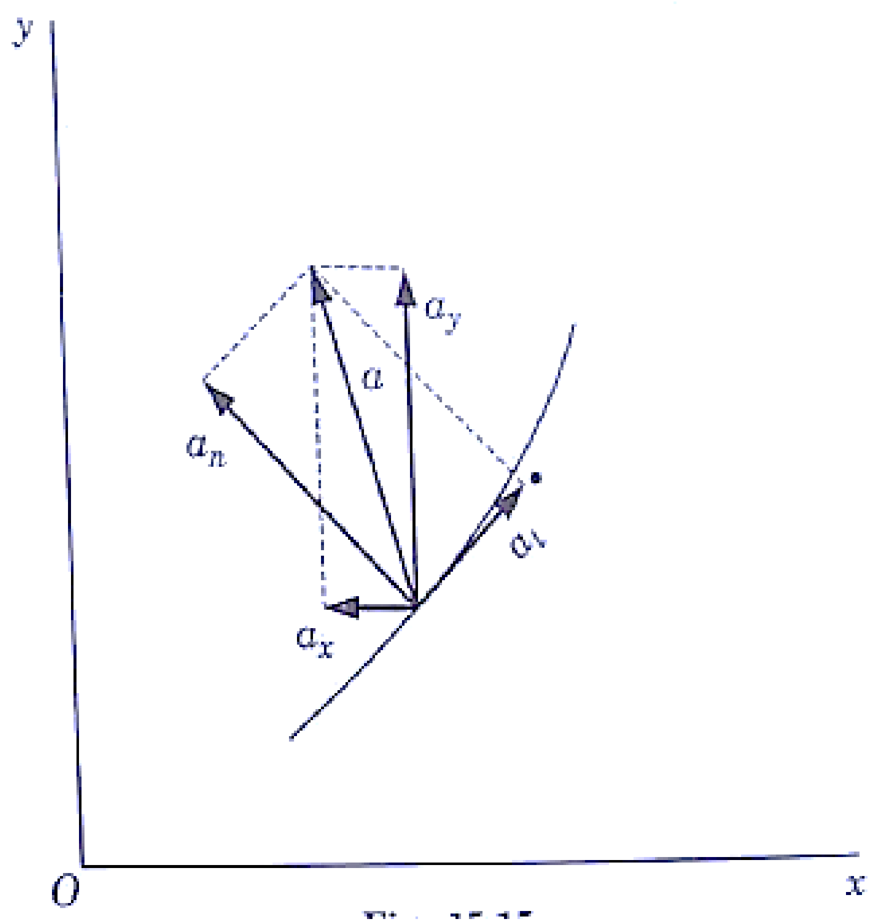
INTRODUCTION

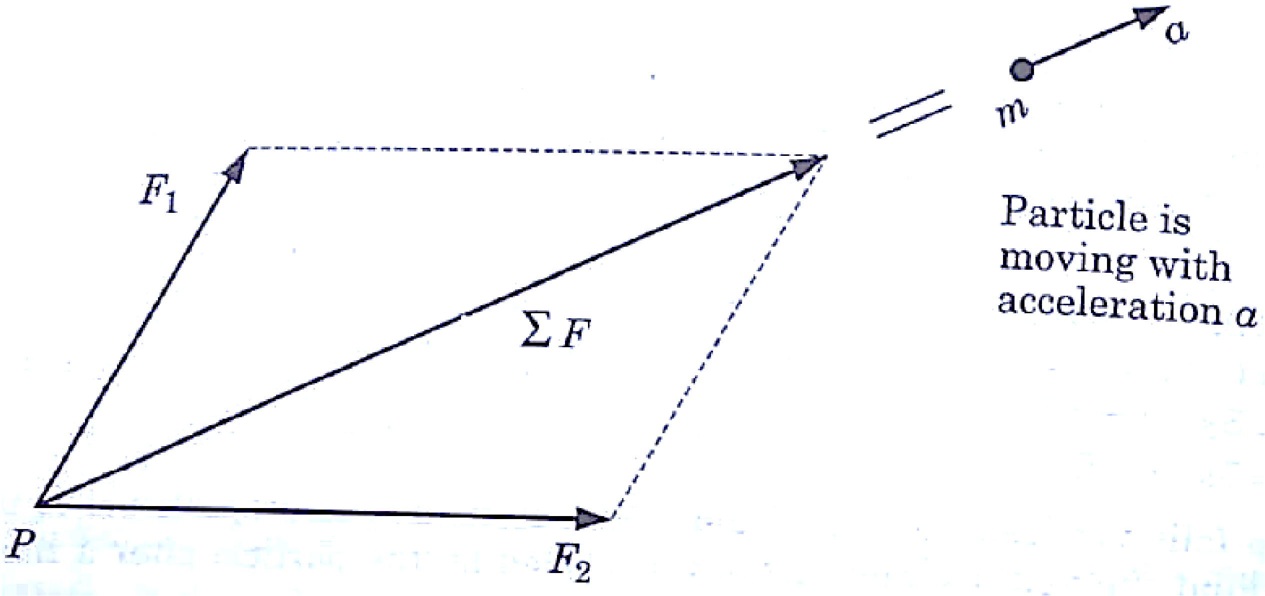
- The equations of motion of a particle relate force, mass and acceleration of the particle.
- The acceleration of a particle moving in a curvilinear path is a vector which can be resolved into two components mutually perpendicular to each other.
- These can be a_x and a_y along the directions of the coordinate axes x and y respectively.
- a_t and a_n along the directions of the tangent and normal to the curve respectively.
- The equations of motion, therefore, can be written choosing either set of the component of acceleration.



EQUATIONS OF RECTILINEAR MOTION

- Consider a particle P of mass m having an acceleration a when acted upon by several forces say F_1 and F_2.
- Let $\sum F$ be the resultant of these forces.
- Apply Newton's second law

$$\sum F = ma$$



- Where acceleration a of particle is in the direction of the resultant force $\sum F$.
- Where vector quantities are involved, their x and y components can be considered separately when applying them to the solution of a given problem.

EQUATIONS OF RECTILINEAR MOTION

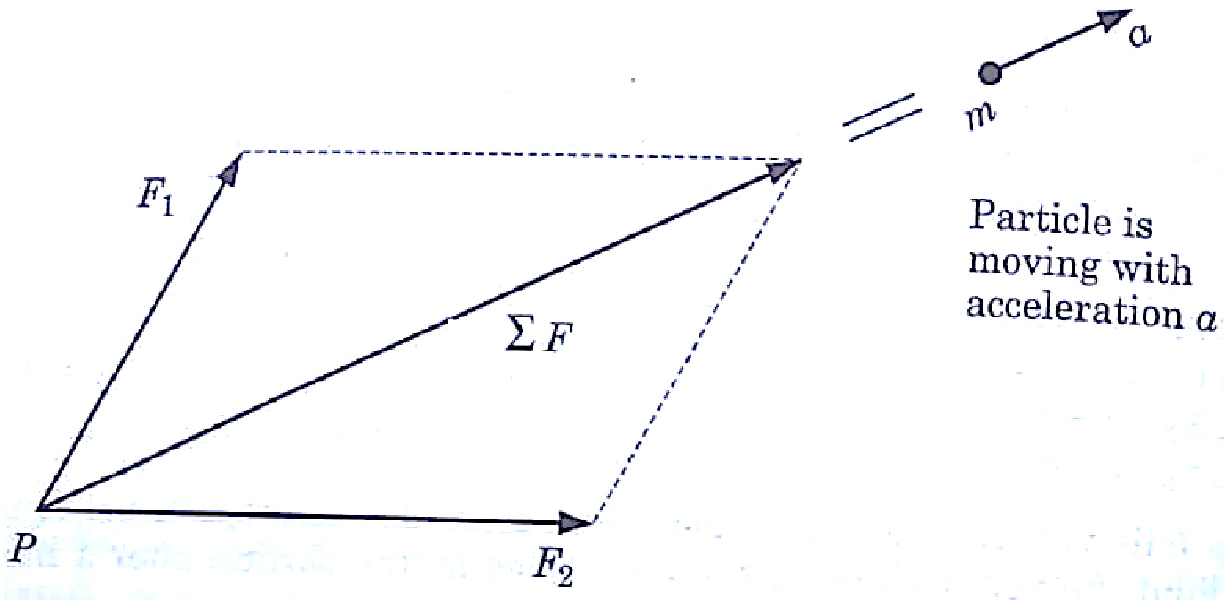
- Therefore, above equation can be written as.

$$\sum F_x = ma_x \quad \text{and} \quad \sum F_y = ma_y$$

- The equations written in the above form are called the equations of motion of the particle.
- When the particle is moving in a straight line along the x-axis we can write,

$$\sum F_x = ma_x \quad \text{or} \quad \sum F_x = m\ddot{x}$$

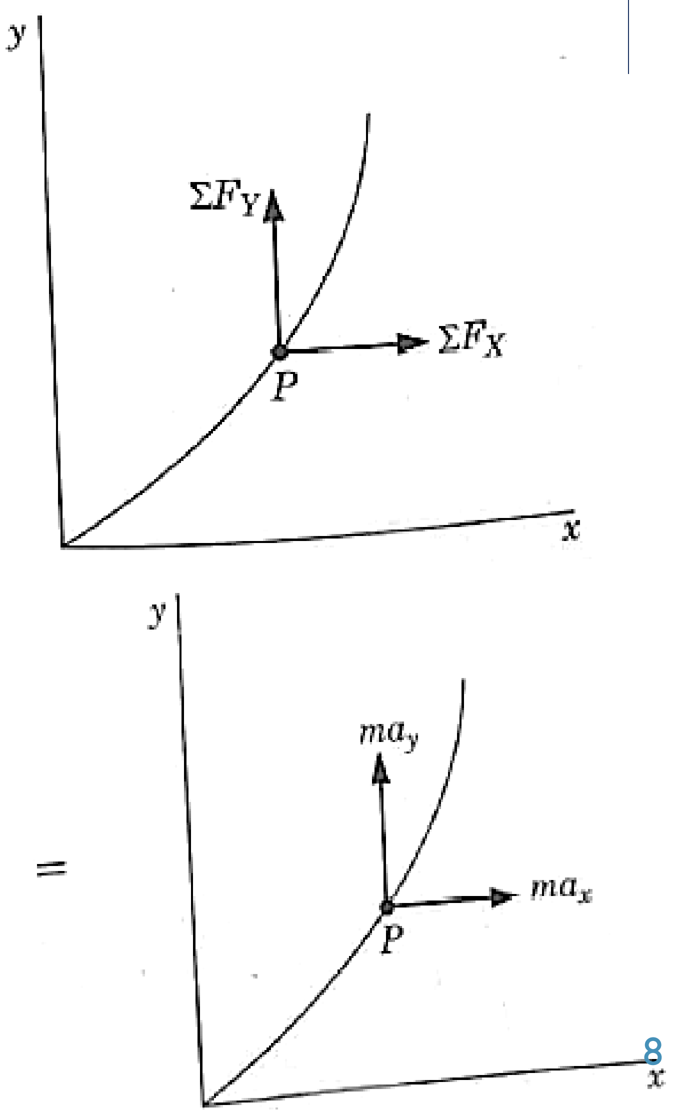
- Where $\ddot{x}$ is the acceleration of the particle along the x axis and $\sum F_x$ is the resultant of all forces acting along the x-axis.



Note: $\ddot{x}$ **means double differentiation i.e.** $\frac{d^2x}{dt^2}$ *while* $\dot{x}$ *means single differentiation* $\frac{dx}{dt}$

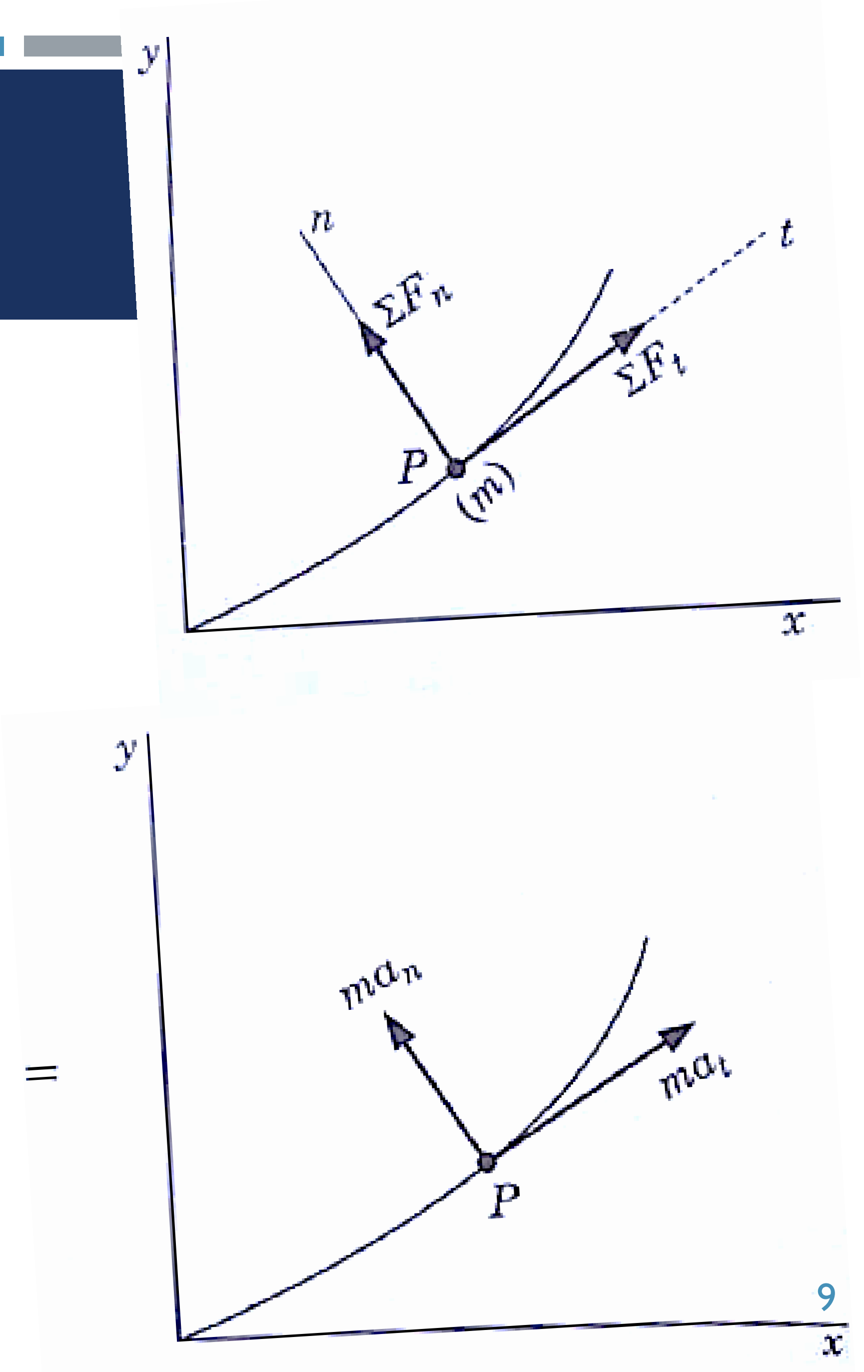
EQUATIONS OF CURVILINEAR MOTION : RECTANGULAR COMPONENTS

- Consider system of forces acting on a particle P as shown.
- Resolve forces acting on the particle in the x and y directions.
- Let $\sum F_x$ and $\sum F_y$ represent the sum of the components of the forces in the x and y directions respectively.
- Let a_x and a_y be the components of acceleration in the direction of the x and y respectively.
- Applying Newton's second law, $\sum F_x = ma_x$
- i.e., sum of components of forces acting in x direction is equal to product of mass of the particle and its acceleration in the x direction.
- Similarly, $\sum F_y = ma_y$



EQUATIONS OF CURVILINEAR MOTION : NORMAL AND TANGENTIAL COORDINATES

- Often it is more convenient to resolve the forces acting on a particle in two components F_t and F_n.
- F_t being the component along the tangent to the path and F_n component along normal to the path and directed towards the centre of curvature of the path.
- Applying Newton's second law $\sum F_t = ma_t$
- Sum of components of forces along tangent to path = Mass x component of acceleration along tangent to path.
- Similarly, $\sum F_n = ma_n$



EQUATIONS OF CURVILINEAR MOTION : NORMAL AND TANGENTIAL COORDINATES

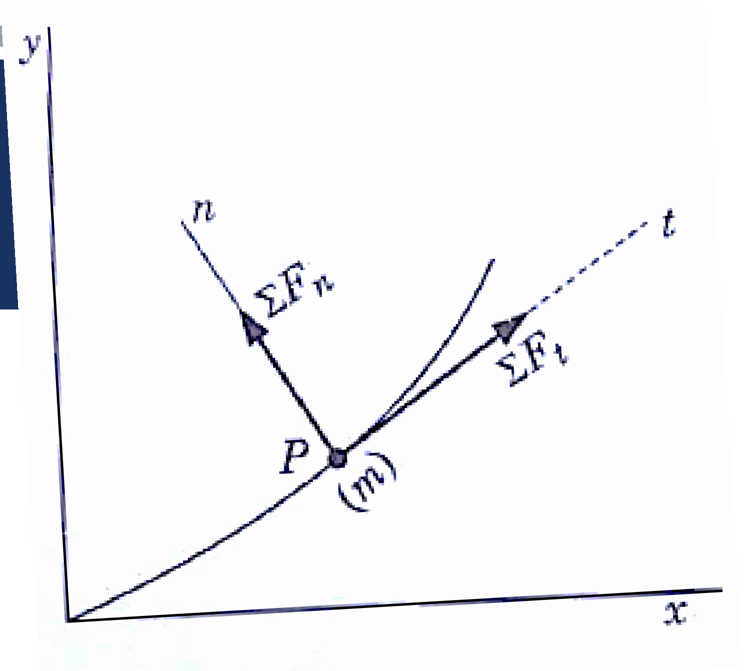
- If a particle is moving in a curved path of radius of curvature ρ and having the velocity v at any instant, we know

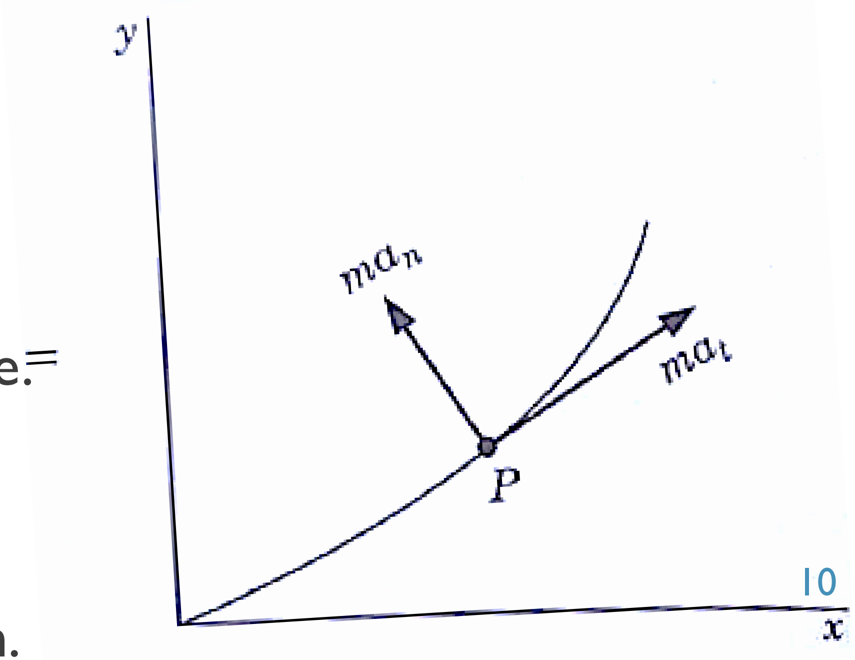
$$a_t = \frac{dv}{dt} \quad and \quad a_n = \frac{v^2}{\rho}$$

- The equation of motion,

$$\sum F_t = ma_t \quad and \quad \sum F_n = ma_n$$

- become, $\sum F_t = m\frac{dv}{dt} \quad and \quad \sum F_n = m\frac{v^2}{\rho}$
- When particle is moving in circular path then, ρ = radius r of circle.=
- If moving with a constant velocity v then, $a_t = 0$
- Whether to resolve forces and acceleration into rectangular or normal and tangential component depend upon nature of problem.





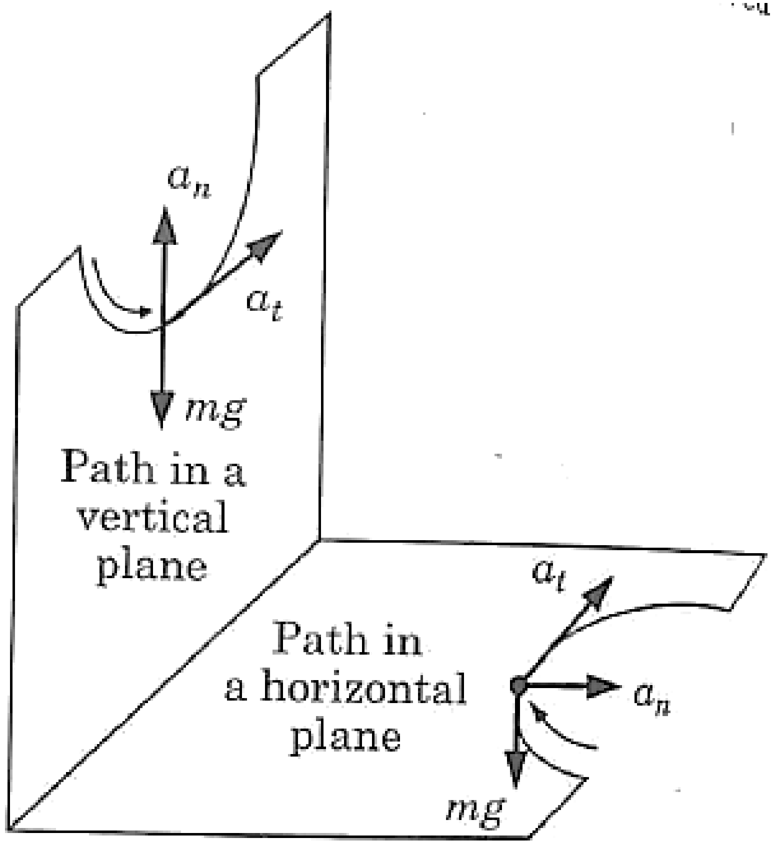
WORKING CONCEPTS CURVILINEAR MOTION

Motion of a particle in a curved frictionless path.

- A particle may move along a curved path which may lie in a vertical plane or in a horizontal plane.
- These planes are called the planes of motion.

1. Weight of particle **mg** acts in plane of motion only when particle moves in a vertical plane. When particle moves in a horizontal plane it acts in a plane normal to the plane of motion.
2. Normal acceleration $\mathbf{a_n}$ of particle is always directed towards centre of curvature of the path and is given by ,

$$a_n = \frac{v^2}{\rho}$$



WORKING CONCEPTS CURVILINEAR MOTION

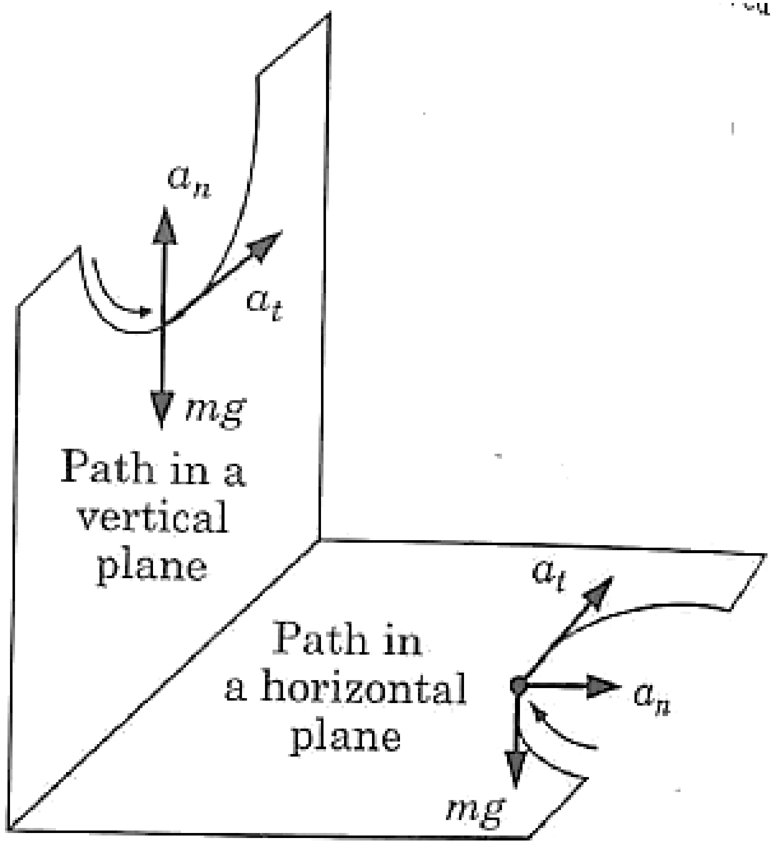
Motion of a particle in a curved frictionless path.

3. Radius of curvature of path may not be the same at all the points of path. In general, radius of curvature at a point is given by,

$$\rho_P = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

- where $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ are to be determined from equations of curve and are to be evaluated at point P on the curve. In case of a circular path ρ = radius r of circular path.

4. The tangential acceleration $\mathbf{a_t = dv/\,dt}$ is considered to be positive in direction of tangent coinciding with the sense of motion of particle.



D' ALEMBERT'S PRINCIPLE

- D' Alembert, a French mathematician, was first to point out that on the lines of *equation of static equilibrium, equation of dynamic equilibrium can also be established by introducing inertia force in direction opposite acceleration in addition to real forces on the plane.*
- Static equilibrium equations are :

$$\Sigma H\ (or\ \Sigma F_x) = 0, \Sigma V\ (or\ \Sigma F_y) = 0, \Sigma M = 0$$

- Similarly when different external forces act on a system in motion, the algebraic sum of all the forces (including the *inertia force*) is zero.
- We know that, ***F = ma*** (Newton's second law of motion) or ***F – ma = 0*** or ***F + (– ma) = 0***
- Expression in the block **(– *ma*)** is the *inertia force* and negative sign signifies that it acts in a direction opposite to that of acceleration/retardation ***a***.
- It is also known as the "*principle of kinostatics*".

KINETICS OF A PARTICLE :WORK AND ENERGY

- If a particle is acted upon by an unknown force F we can determine its acceleration by applying Newton's second law of motion, F = ma and then principles of kinematics to determine its velocity and displacement.
- An alternative method of solving the problems involved in the motion of a particle is to apply the method of work and energy which relates force, mass, velocity and displacement.
- The advantage of this method is that we can directly determine the velocity of the particle without requiring it to determine its acceleration.
- Also as work and energy are scalar, we have to deal with scalar quantities.

CONCEPT OF WORK

- In 'Mechanics' work means accomplishment. A *force is said to have done work if it moves the body, on which it acts, through a certain distance.*
- A force is not able to produce any displacement-translational or rotational no work is said to have been done.
- Work is measured by the *product of force (P) and displacement (S)* both being in the same direction.
- Work is positive or negative, according as the force acts in the same direction or in the direction opposite to the direction of displacement.
- Mathematically, *Work = Force × Displacement in direction of force*

CONCEPT OF WORK

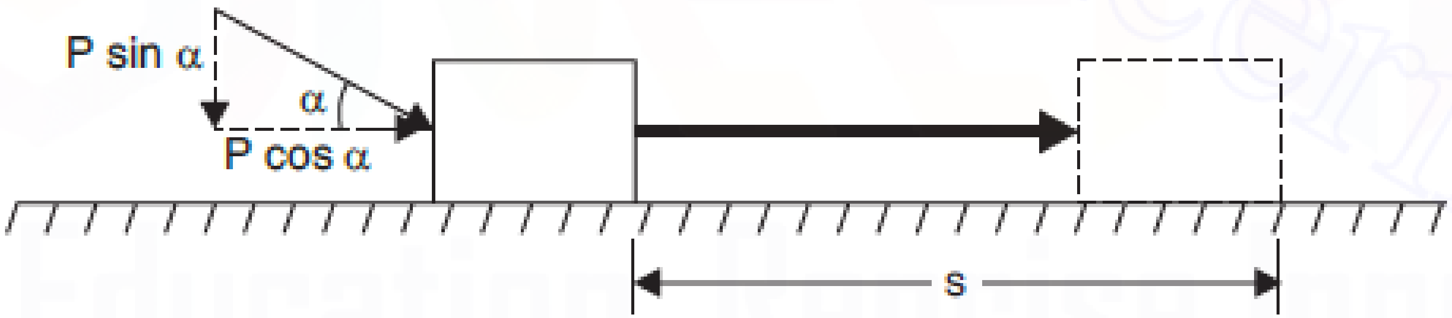


Figure shows force P acting at angle α to direction of displacement.

- Work done by force ‘P’ = Force × displacement in the direction of force

$$\therefore U = P\cos\alpha \times S$$

- Or,

$$U = PS\cos\alpha$$

- If angle α = 0 *i.e.*, force P acts in direction of displacement or motion, then

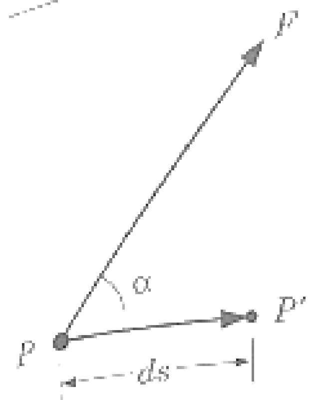
$$U = P \times S$$

- *Only the component of force in the direction of motion does the work.*

WORK OF A FORCE

- If a particle is subjected to force F and displaced by an infinitesimal displacement dS, work done dU by the force is given by,

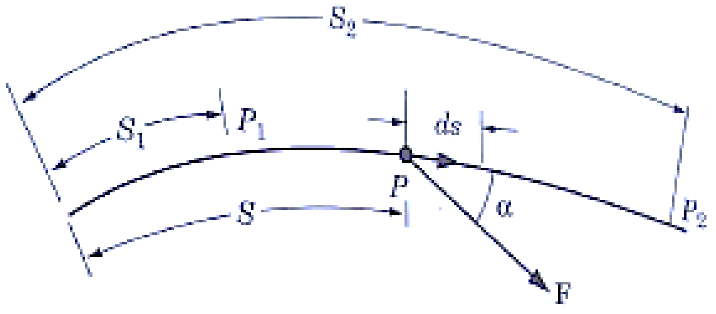
$$dU = FdS\ \cos\alpha$$



- where α is angle between force and displacement vector.

- Thus, work done by a force given during an infinitesimal displacement is equal to product of displacement dS and component of force **F cos α** in the direction of displacement.

- Work done by a force during a finite displacement from position P_1 to P_2 can be obtained by integrating above equation,



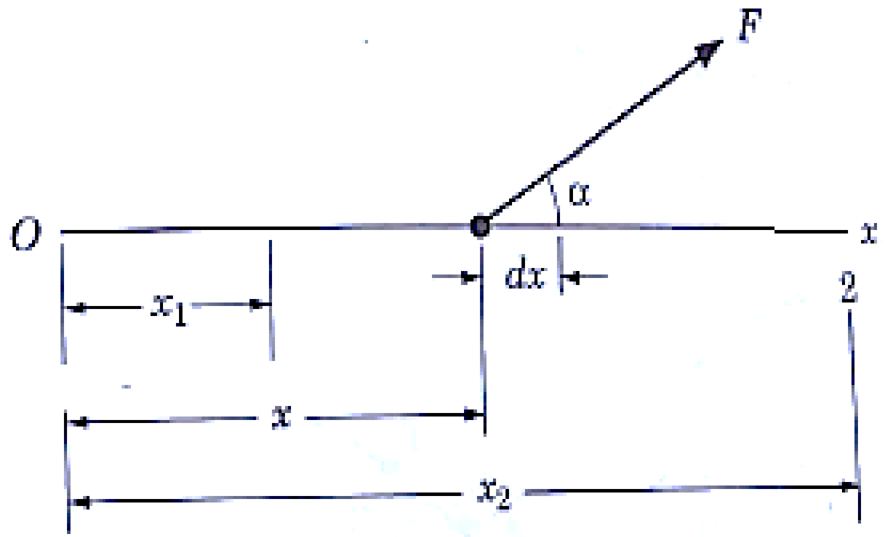
$$U_{1-2} = \int_{S_1}^{S_2} \mathbf{F}\cos\alpha\, dS$$

- where displacements S_1 and S_2 are measured along path

WORK DONE BY A CONSTANT FORCE IN RECTILINEAR MOTION

- If a particle moves along a straight line from position 1 to position 2 under a constant force F then work done is equal to

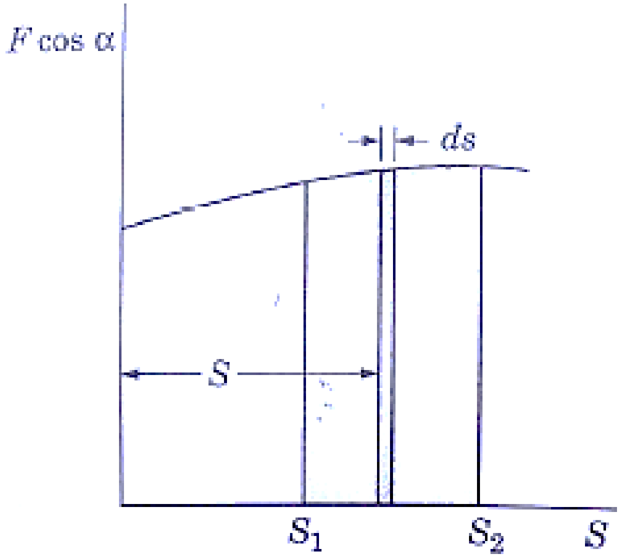
$$U_{1-2} = \int_{x_1}^{x_2} \mathbf{F} \cos \alpha \, dx$$



- Work done for a displacement x from the origin is

$$U_{1-2} = (\mathbf{F} \cos \alpha) x$$

- Graphically work done by a force can be represented by area under the curve shown in figure.
- Unit of work is Newton metre of Joule.



POINTS FOR THE WORK OF A FORCE.

- Work done by a force is zero if either displacement is zero or force acts normal to displacement. **For example**, a gravity force does no work when a body moves horizontally.
- Work done by a force is positive if direction of force and direction of displacement are same. **For example,** work done by force of gravity is positive when a body moves from a higher elevation to a lower elevation. A positive work can be said as work done by a force and negative work as work done against the force.
- Work is scalar quantity and has magnitude and sign but no direction.
- Work done by a force depends on path over which force moves except in the case of conservative forces. Force due to gravity, spring force, elastic force are conservative forces, whereas, friction force is a non conservative force.

ENERGY OF A PARTICLE

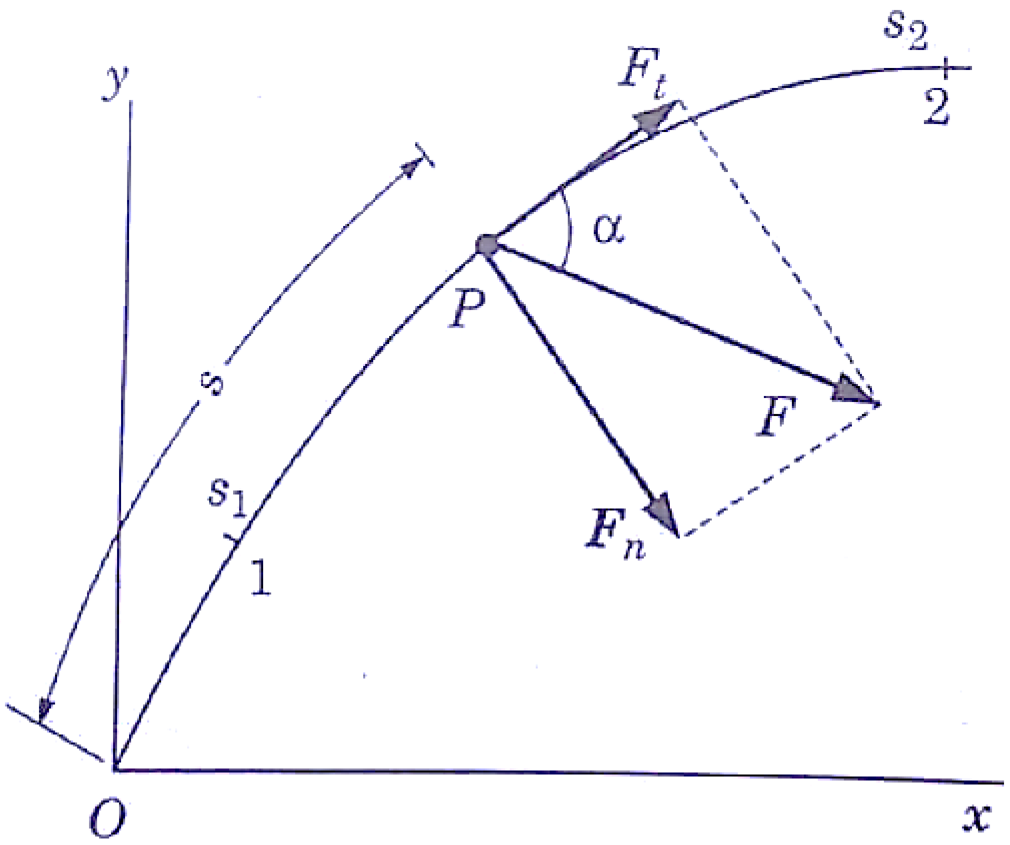
- Energy is defined as capacity to do work.
- Which implies that energy has same unit as work and it is a scalar quantity too.
- Energy can manifest itself in many forms like mechanical, thermal, or electric energy etc.
- Here we are mainly concerned with the mechanical energy of a particle which consists of its potential energy and kinetic energy.
- **Kinetic energy of a particle**. It is the energy possessed by a particle by virtue of its motion. If a particle of mass m is moving with velocity v, its kinetic energy is given by

$$K.E. = \frac{1}{2}mv^2$$

- Unit of Kinetic energy is $kg\left(\frac{m}{s}\right)^2 = Nm \; or \; J$

PRINCIPLE OF WORK AND ENERGY

- Consider a particle **P** of mass **m** acted upon by a force **F** and moving with velocity **v** along a path which can be rectilinear or curved as shown in figure
- At any position P of the particle let its distance from the reference point **O** along the path be **S**.
- Let α be the angle that the force vector makes with the tangent to the path at **P.**
- Resolve the force **f** along the tangent and normal to the path at P.
- Tangential component of force, F_t = **Fcos** α
- Normal component of force, . F_n = **F sin** α



PRINCIPLE OF WORK AND ENERGY

- Equation of motion in tangential direction of the particle is

$$F_t = ma_t$$

- where a_t is acceleration of particle in tangential direction

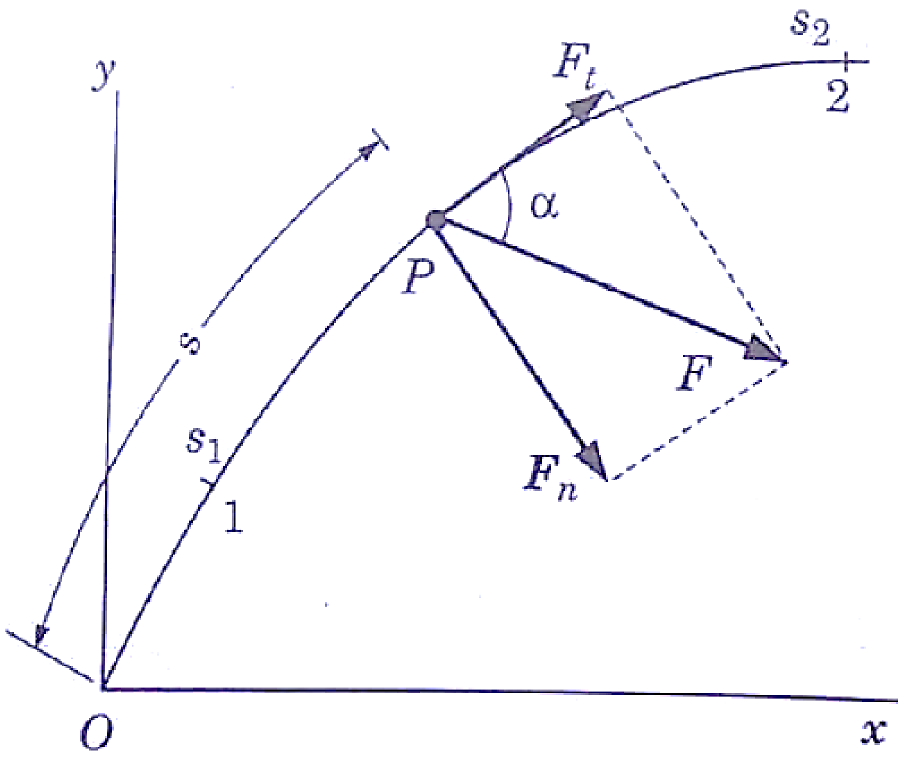
$$F\cos\alpha = ma_t$$

- As, $$a_t = \frac{dv}{dt}$$

- Therefore, $$F\cos\alpha = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt}$$

$$\frac{ds}{dt} = v$$

$$F\cos\alpha = mv\frac{dv}{ds}$$



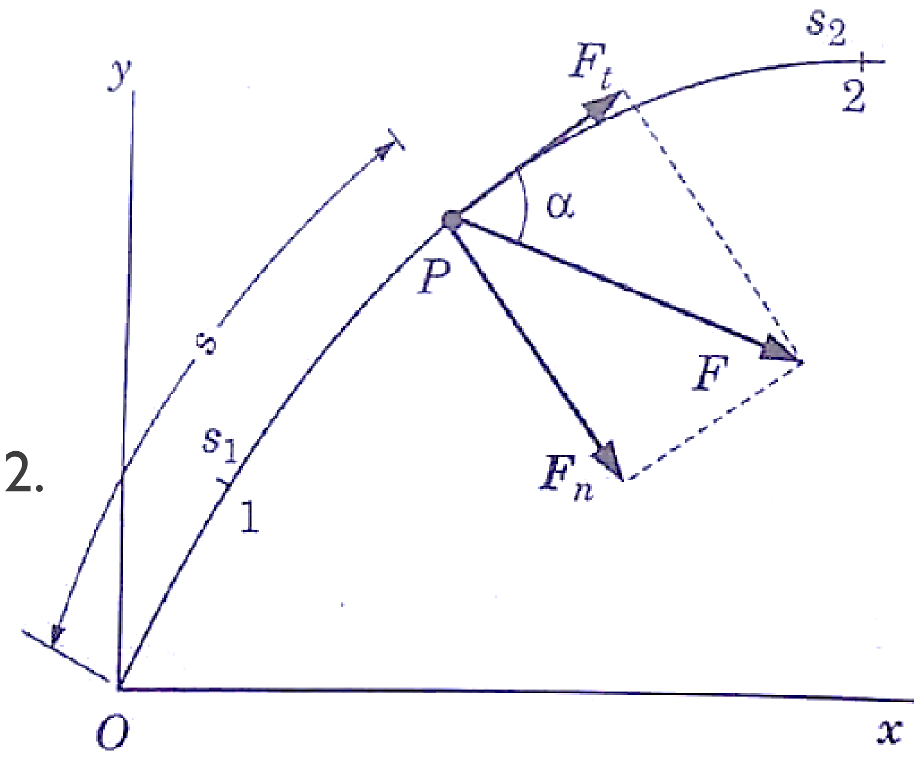
PRINCIPLE OF WORK AND ENERGY

- Let v_1 and v_2 be the velocities of the particle at point 1 and 2 and the corresponding distance be S_1 and S_2.
- Integrating the above equation,

$$\int_{S_1}^{S_2} \mathbf{F} \cos\alpha \, dS = \int_{v_1}^{v_2} mv \, dv$$

$$\int_{S_1}^{S_2} \mathbf{F} \cos\alpha \, dS = \left(\frac{mv_2^2}{2} - \frac{mv_1^2}{2}\right)$$

- Left hand side of equation represents work (U_{1-2}) of force **F** exerted on particle during its displacement from position 1 to 2.
- Right hand side represents change in kinetic energy of particle.



$$U_{1-2} = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}$$

$$U_{1-2} = K.E._2 - K.E._1.$$

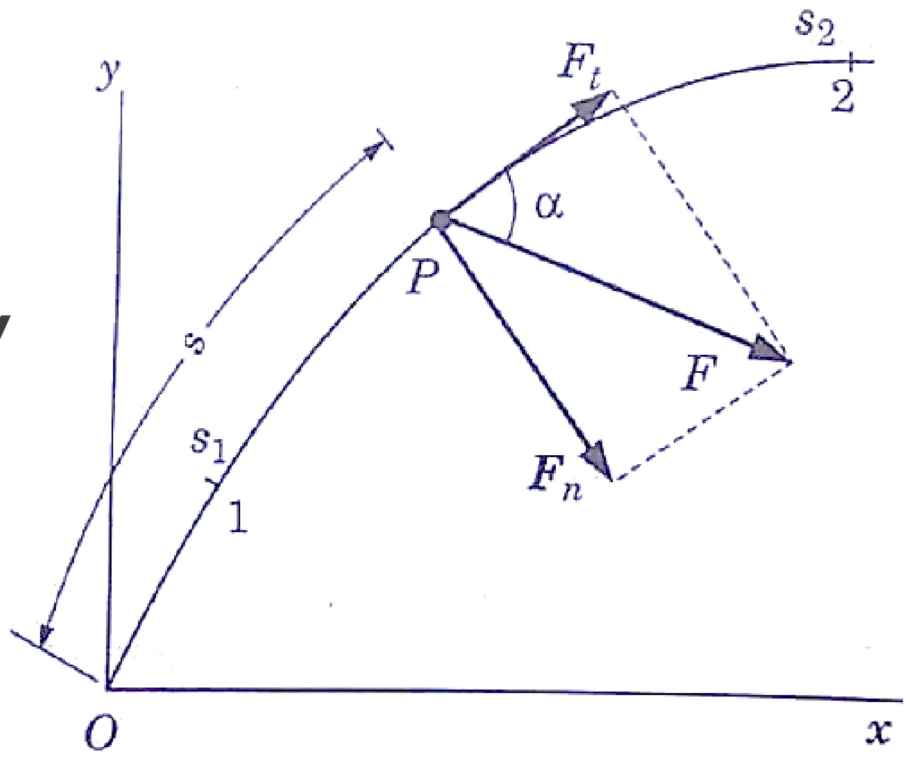
PRINCIPLE OF WORK AND ENERGY

$$\int_{S_1}^{S_2} \mathbf{F} \cos \alpha \, dS = \left(\frac{mv_2^2}{2} - \frac{mv_1^2}{2}\right)$$

- Above expression is a symbolic representation of work energy principle which can be stated as follows:

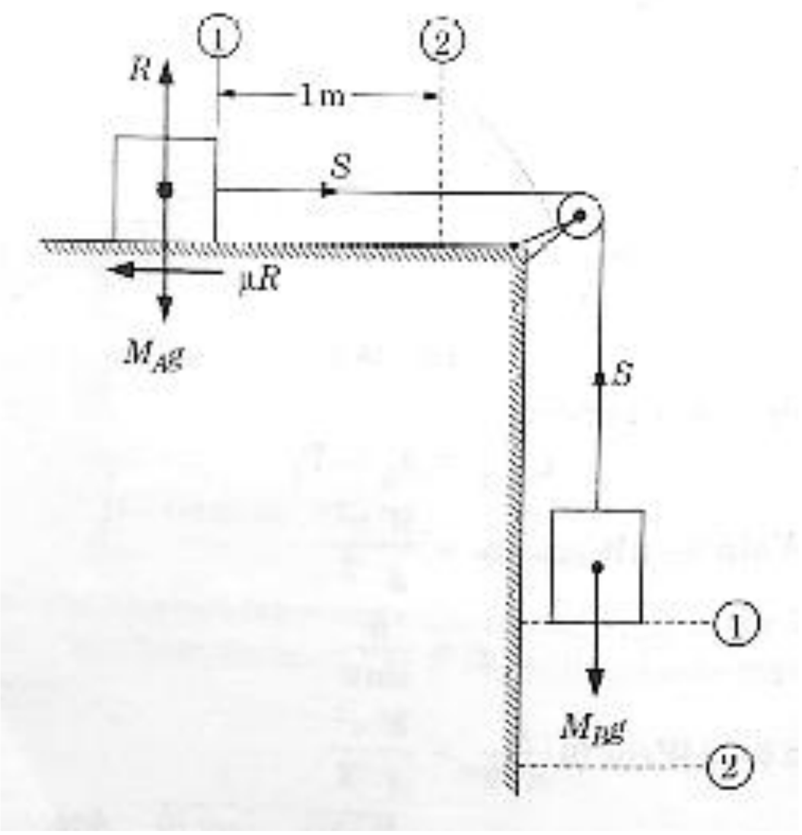
Work done by a force acting on a particle during its displacement is equal to change in its kinetic energy of the particle during the displacement.

- It may be recalled here that work and energy are both scalar quantities.



WORK AND ENERGY PRINCIPLE FOR A SYSTEM OF PARTICLES

- Work and energy principle was stated for a single particle in motion.
- Consider a system of particles in motion, For example, the motion of two bodies connected by a string.
- To apply the principle of work and energy to a system of particles, we add up change of kinetic energy of all particles and equate it to the work of all the forces involved during the displacement of the system of particles.
- i.e., $\sum \boldsymbol{U_{1-2}} = \sum(\boldsymbol{K.E._2} - \boldsymbol{K.E._1})$
- Where, $\sum \boldsymbol{U_{1-2}}$ is the work of all the forces acting on the various particles.



WORK AND ENERGY PRINCIPLE FOR A SYSTEM OF PARTICLES

Work of Internal Forces

- In a problem involving a system of particles, internal forces (like tension in the string) may appear.
- While calculating the work of all the forces, work of internal forces must also be included.
- In some problems, however, the internal forces may appear as a pair of equal and opposite forces (like action and reaction) moving through equal distances.
- In such a problem, therefore, work of internal forces may become zero.

POTENTIAL ENERGY

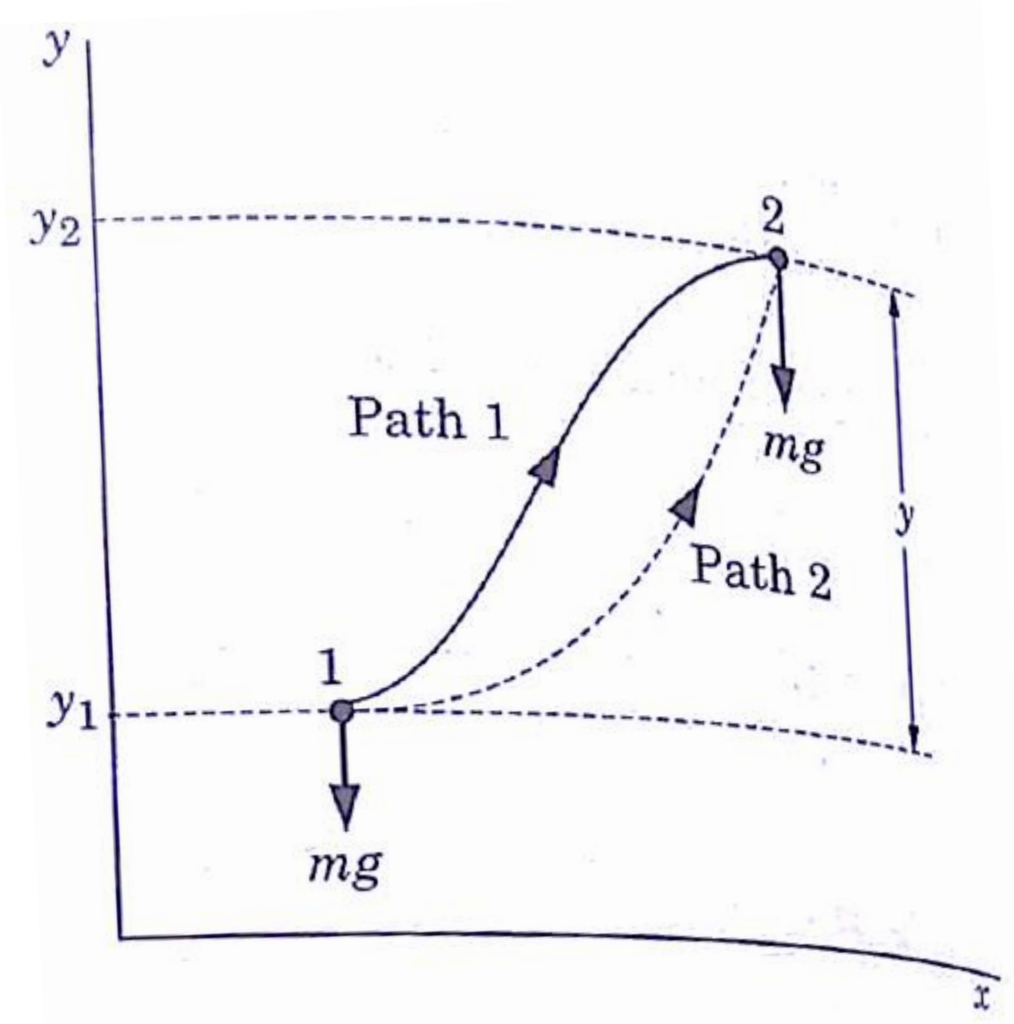
- The potential energy of a particle is the energy possessed by a particle by virtue of its position.
- Consider a particle of mass m which moves from position 1 of elevation y_1 to position 2 of elevation y_2, following the path 1 as shown in figure.
- Work of gravity force,

$$\sum U_{1-2} = \int_{y_1}^{y_2} (-mg)\, dy$$

Or,

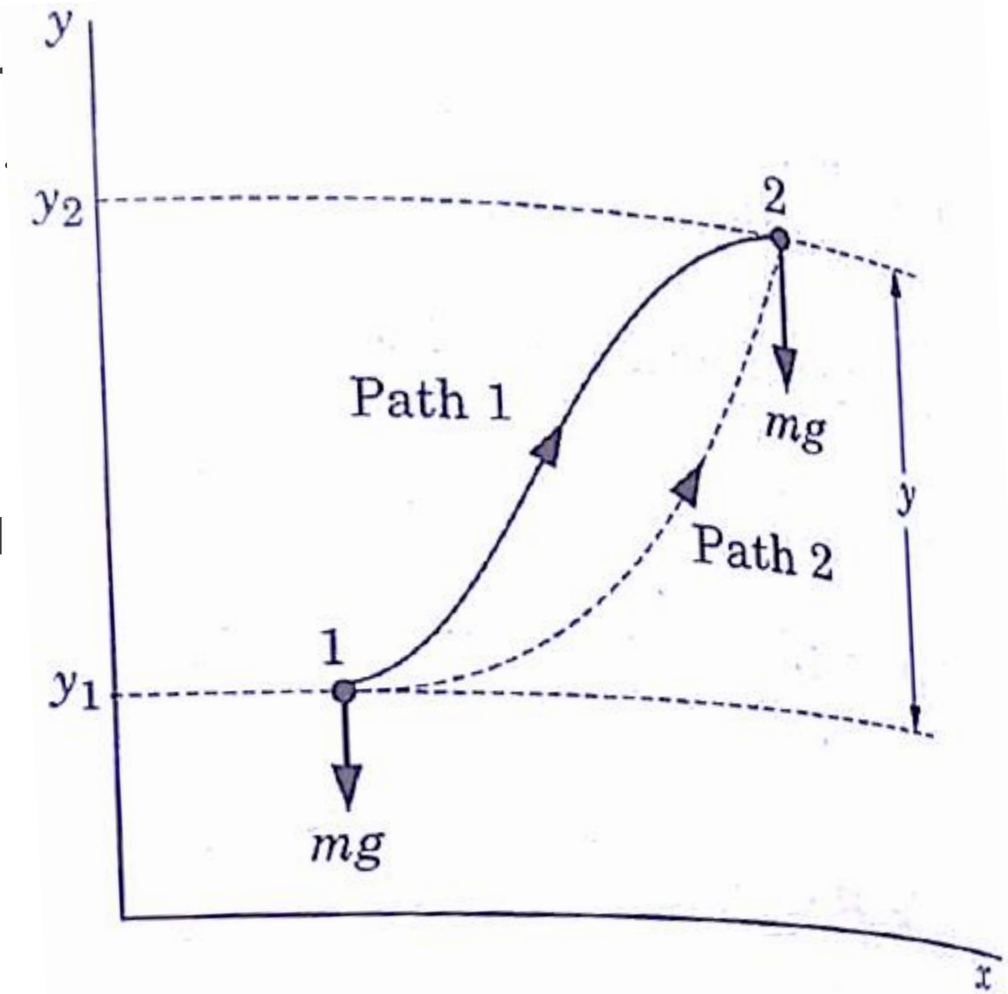
$$\sum U_{1-2} = -\mathrm{mg}(y_2 - y_1)$$

$$\sum U_{1-2} = -mgy$$



POTENTIAL ENERGY

- U_{1-2} is work done against gravity force (hence negative) and represents an increase in potential energy of particle of mass m.
- As, work done against force of gravity is stored as PE of particle.
- Next consider particle to follow a different path 2, between position 1 and 2. The work done against the gravity force would still be the same and equal to mgy.
- So we can conclude that the work of gravity force is independent of the path followed and depends on the initial and final values of the function mgy. This function is called the potential energy of the particle due to gravity.
- Unit of potential energy: $(kg)\left(\frac{m}{s^2}\right)m$ = Nm or Joule (J)
- Potential energy of particle is represented by symbol V.



CONSERVATIVE FORCES.

- If work of a force in moving a particle between two positions is independent of the path followed by the particle and can be expressed as a change in its position energy, then such a force is called a conservative force.
- Gravity force is a conservative force whereas, the frictional force is a non-conservative force.
- This work of friction force depends upon the path followed and this work cannot be expressed as a change in the potential energy so it is a non-conservative force.
- The various conservative forces are, force due to gravity, spring force and elastic force.

PRINCIPLE OF CONSERVATION OF ENERGY

$$Work\ done = \ change\ in\ kinetic\ energy$$

$$U_{1\text{-}2} = K.E._2 - K.E._1$$

- If a particle moves under action of a conservative force, work done is stored as potential energy,

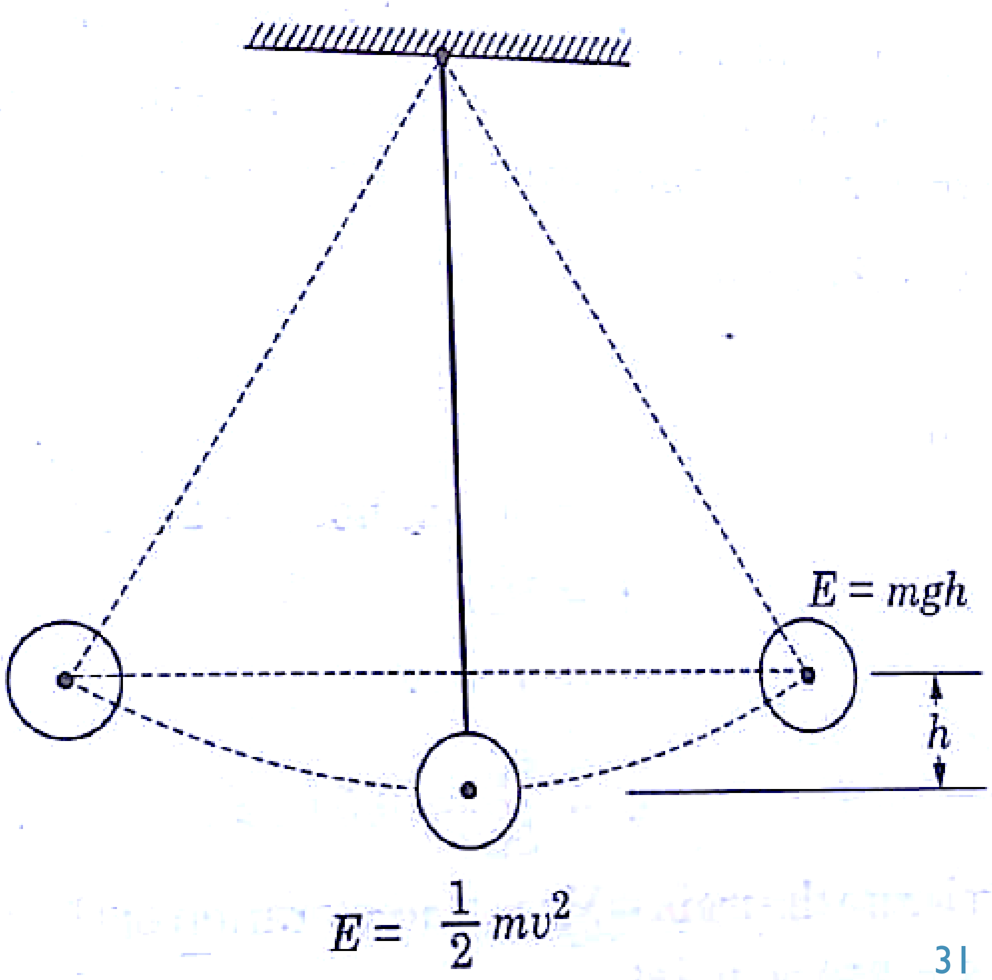
$$U_{1\text{-}2} = -\ (V_2 - V_1)$$

- Work done= -(negative change in potential energy)
- Combining above two equations: $K.E._2 - K.E._1 = -(V_2 - V_1)$
- Or, $K.E._2 + V_2 = K.E._1 + V_1$
- Or, $K.E._1 + P.E._1 = K.E._2 + P.E._2$
- Which means that sum of potential energy and kinetic energy of particle (or of a system of particles) remains constant during the motion under action of conservative forces.
- Denoting sum of potential energy and kinetic energy by E, above equation can be written as:

$$E_1 = E_2$$

PRINCIPLE OF CONSERVATION OF ENERGY

- The principle of conservation of energy can be applied to a particle or to a system of particles only under the action of conservative forces.
- Where frictional force is involved, this principle cannot be applied.
- For example, in case of a simple pendulum, sum of P.E. and K.E. remains constant in any position of the pendulum provided frictional force at support and force due to air resistance are negligibly small; both of which are non-conservative forces.
- i.e., $U_{1-2} = -(V_2 - V_1) = V_1 - V_2$



PRINCIPLE OF CONSERVATION OF ENERGY

- For example, if a particle moves from a lower elevation to a higher elevation, potential energy of particle increases from ($V_2 > V_1$) but work U_{1-2} of gravity force is negative.
- Also, potential energy of a particle has to be determined w.r.t. some reference which can be chosen arbitrarily as per convenience.

KINETICS OF PARTICLES: IMPULSE AND MOMENTUM

- Previously, we discussed two methods for solving problems of motion of the particles.
- These were based on application of Newton's second law and on application of principle of work and energy.
- Here, we shall discuss principle of impulse and momentum and is derived from Newton's second law.
- The principle relates force, mass, velocity and time and is particularly suitable when large forces act for a very small time.

IMPULSE OF FORCE

- Suppose a force *F* acts on a body for a time interval '*t*'.
- Product *Ft* is called the *impulse* of the force, *i.e.*,

$$Impulse = force \times time\ interval$$

- According to Newton's second law,

$$Force = mass \times acceleration \quad \textbf{or} \quad F = ma$$

$$F = m\left(\frac{v-u}{t}\right) \quad \textbf{or} \quad Ft = m\,(v - u)$$

i.e., **Force × time interval = mass × change in velocity**

- or **Impulse = mass × change in velocity**
- Impulse of a force is a useful concept when force acts for a short duration of time.
- It tells how much velocity of a body of mass m will change if impulse is applied to it.
- For example, When we hit a hockey stick we give it an impulse.

IMPULSE OF FORCE

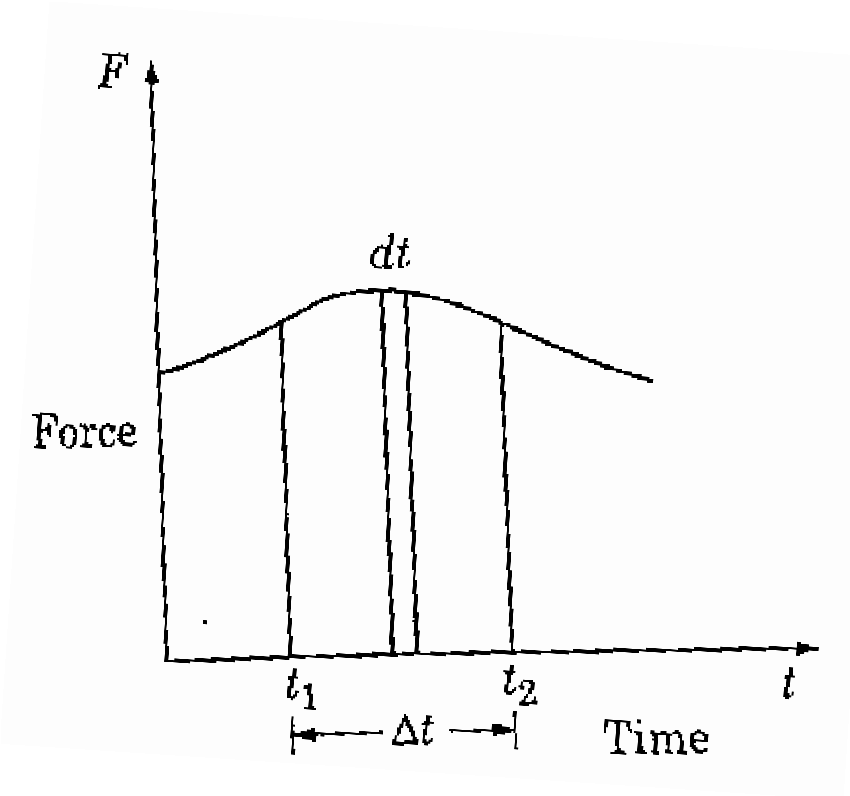
- When large force acts over a short period of time that force is called an impulse force.
- Impulse of force F acting over time interval for t_1 to t_2 is defined by integral

$$I = \int_{t_1}^{t_2} F\,dt$$

- Impulse of force, can be visualised as area under the force vs. time graph as shown in figure.
- When variation of force w.r.t. time is unknown, impulse can also be measured as

$$\mathbf{I = F_{average} \times \Delta t}$$

- Impulse of a force is a vector quantity and has a unit of Newton Second (Ns).

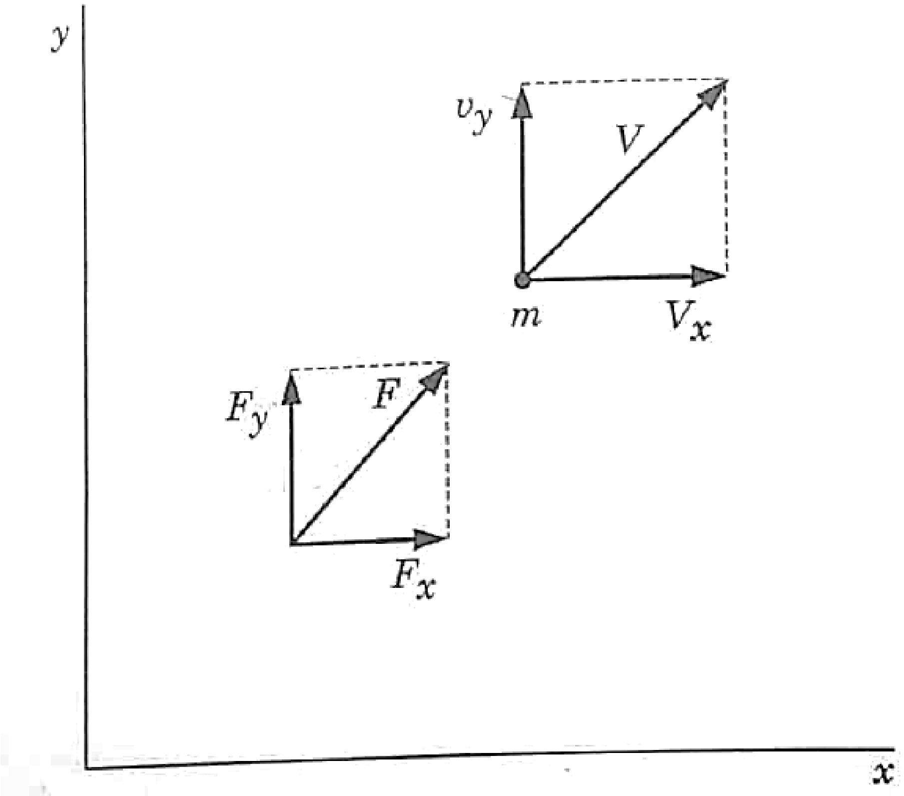


MOMENTUM

- Consider a particle in motion of mass m acted upon by force F.
- Equation of motion of particle in x and y direction are

$$F_x = ma_x \quad \text{and} \quad F_y = ma_y$$

- Or, $F_x = m\frac{dv_x}{dt}$ and $F_y = m\frac{dv_y}{dt}$
- Or, $F_x = \frac{dmv_x}{dt}$ and $F_y = \frac{dmv_y}{dt}$
- A single equation in vector form can be written as, $F = \frac{d}{dt}(mv)$
- Which states that force F acting on particle is equal to rate of change of momentum of particle.
- Unit of momentum is Newton Second (Ns).
- Vector mv is called momentum or linear momentum. It has same direction as velocity of particle.



PRINCIPLE OF IMPULSE AND MOMENTUM

- $F_x = \frac{dmv_x}{dt}$ and $F_y = \frac{dmv_y}{dt}$
- Multiplying both sides of above equation by dt
- $F_x\, dt = d(mv_x)$ and $F_y\, dt = d(mv_y)$
- Where, F_x dt & F_y dt is impulse of force F_x and F_y
- $d(mv_x)$ & $d(mv_y)$ is differential change in momentum of particle in x & y direction in time dt.
- Above equation expresses that differential change in momentum of a particle during time interval dt is equal to impulse of force acting during same interval.
- Integrating above equation from a time t_1 to time t_2 will give:-
- $\int_{t_1}^{t_2} F_x\, dt = (mv_x)_2 - (mv_x)_1$ and $\int_{t_1}^{t_2} F_y\, dt = (mv_y)_2 - (mv_y)_1$

PRINCIPLE OF IMPULSE AND MOMENTUM

- The above equation can be combined into a single vector equation as

$$\int_{t_1}^{t_2} F\,dt = (mv)_2 - (mv)_1$$

- If $t_1 = 0, t_2 = t$. We can write,

$$mv_2 - mv_1 = \int_0^t F dt$$

- Or, **Final Momentum - Initial Momentum = Impulse Of Force.**
- Above equation expresses that total change in momentum of a particle during a time interval is equal to impulse of the force acting during the same interval of time.
- Above equation represents a relation between vector quantities for a single particle.
- In the actual solution of a problem, it should be represented by the two corresponding component equations in the x and y directions.

PRINCIPLE OF IMPULSE AND MOMENTUM: SYSTEM OF PARTICLES

- Principle of impulse and momentum is particularly useful when dealing with system of particles.
- For example, a gun firing a bullet.
- When a problem involves the motion of several particles each particle is to be considered separately.
- We then add vectorially the momentum of all particles and impulses of all the forces involved and above equation can be written as

$$\sum m v_2 - \sum m v_1 = \int_0^t F\,dt$$

- Above equation is also a vector relation which can be replaced by two component equations in the x and y direction.

PRINCIPLE OF IMPULSE AND MOMENTUM

- When impulse forces act on a system, non impulsive forces (like weight of particle) generally can be neglected.
- Internal forces between the particles need not be considered as sum of the impulses of the internal forces is always zero.
- Because the internal forces appear in pairs of equal and opposite forces (as action and reaction) acting for the same interval of time, resulting in equal and opposite impulses.
- But the sum of work done by internal forces may not be zero because they may not move through the same distances.

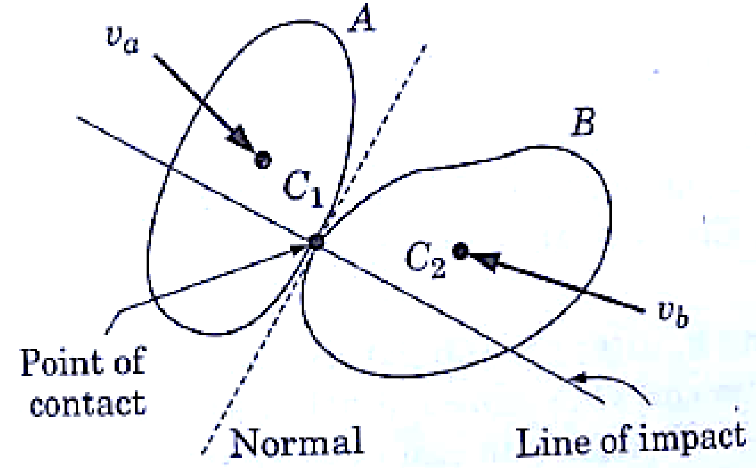
CONSERVATION OF MOMENTUM

- It can be observed from equation ($\sum mv_2 - \sum mv_1 = \int_0^t Fdt$) that when sum of impulses due to external forces is zero, momentum of the system remains constant or is conserved,
- i.e., $$\sum mv_2 = \sum mv_1$$
- Or, **Final momentum of the system = Initial momentum of the system**
- **Examples of conservation of momentum**

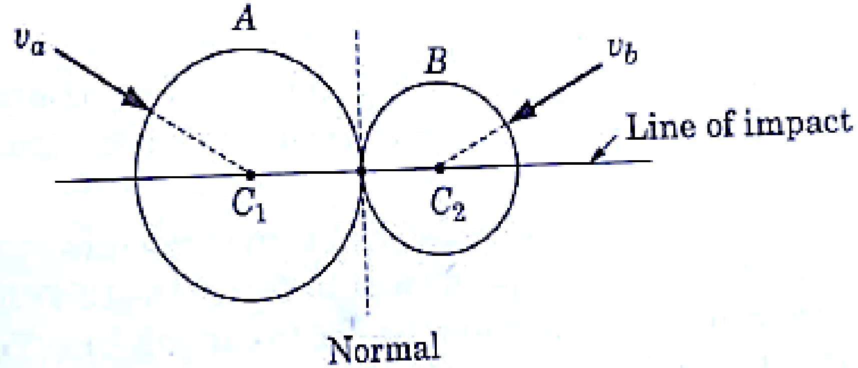
1. **Propulsion of a jet plane.** In jet engine, gases at high speed are ejected through a nozzle at rear of engine. Escaping gases have momentum. Since total momentum must remain constant, engine acquires an equal and opposite momentum. Momentum gained by engine provides force needed to push the plane forward.
2. **Recoil of a gun.** Consider, a gun loaded with bullet. Before firing, gun and bullet are at rest. Therefore, total momentum of system is zero. After firing, bullet moves in one direction but gun recoils *i.e.*, moves in opposite direction. Momentum of bullet is equal in magnitude but opposite in direction to that of gun. Total momentum, being the vector sum of momentum of bullet and that of the gun, is zero. Since mass of the gun is very large compared to that of bullet, velocity of recoil of gun will be much smaller than that of bullet.

IMPACT: COLLISION OF ELASTIC BODIES

- Phenomenon of collision of two bodies which occurs in a very small interval of time and during which two bodies exert very large force on each other is called an impact.
- **Line of impact**. Common normal to surfaces of two bodies in contact during impact, is called line of impact.
- **Central/non-central impact.** When mass centre C_1 and C_2 of colliding bodies lie on line of impact, it is called Central impact, otherwise it is called non-Central or eccentric impact.



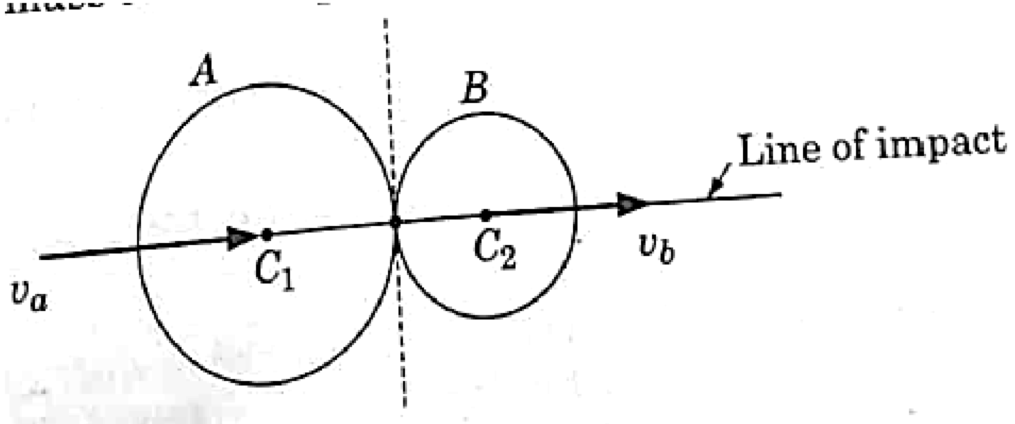
Oblique Non-Central Impact
(a)



Oblique Central Impact
(b)

IMPACT: COLLISION OF ELASTIC BODIES

- **Direct Impact/ Indirect (Oblique) impact**. If velocities of two bodies before collision are collinear with the line of impact it is called direct impact. Otherwise it is called indirect or oblique impact.

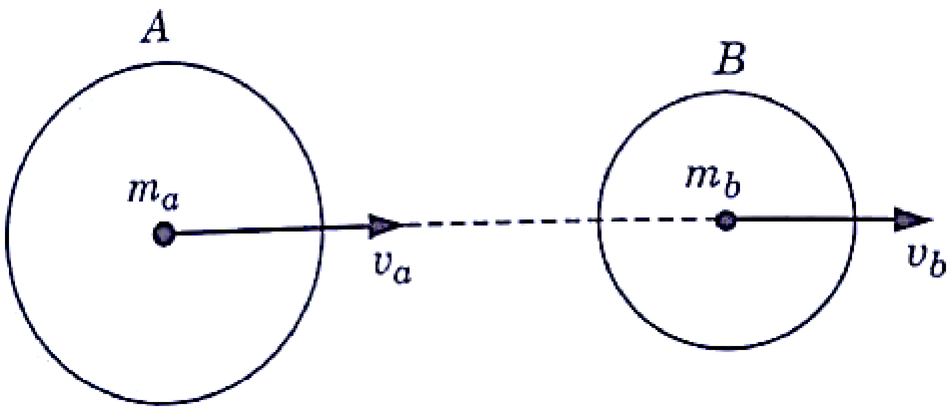


Direct Central Impact
(c)

- In study of phenomenon of impact an important assumption is made that in comparison with the forces of impact, any other finite forces that may act during the impact are negligible.

DIRECT CENTRAL IMPACT

- Consider two spheres A and B of masses m_a and m_b moving in same direction and along the same straight line with unknown velocities v_a and v_b respectively.
- If $\mathbf{v_a > v_b}$, the sphere A will strike sphere B.
- Since no external forces are acting, total momentum of the system of spheres A and B is conserved.



$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$

- Where v_a' and v_b' are velocities after impact.
- As all velocities are directed along line of impact they can be treated as scalars.
- For purpose of fixing sense of velocity and momentum., they are taken as positive when directed to the right.

DIRECT CENTRAL IMPACT

- Now, it is required to determine velocities v_a' and v_b' after impact.
- A single equation obtained above is not sufficient to determine two unknowns.
- For two unknowns - other equation can be derived based on nature of impact and is

$$e = -\frac{(v_b' - v_a')}{(v_b - v_a)}$$

- Where, e is called **coefficient of restitution** and its value depends upon nature of impact.
- In a problem value e is either given for an impact or is to be determined.

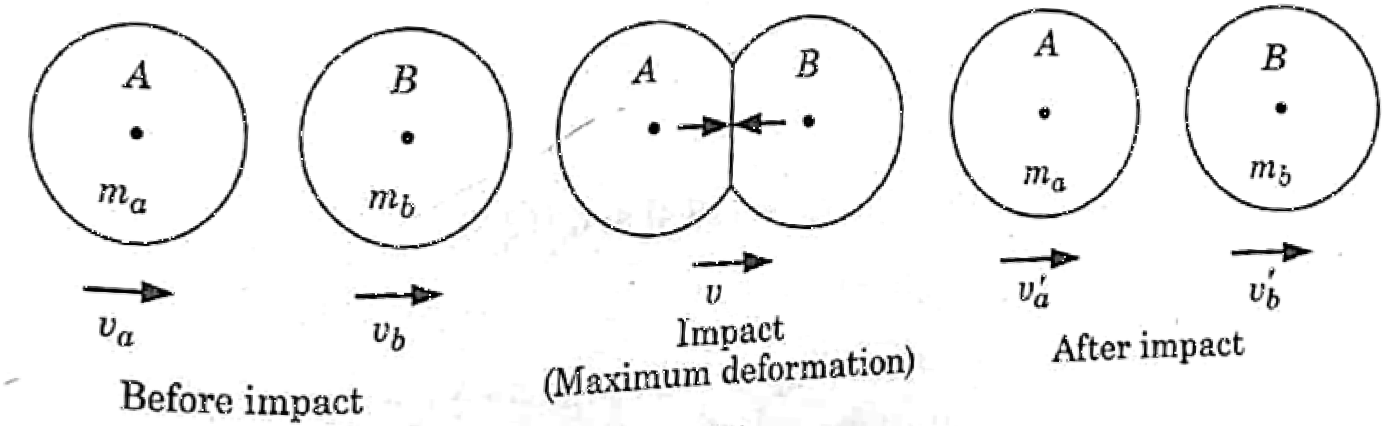
$$e = -\frac{(\text{velocity of separation})}{(\text{velocity of approach})}$$

- With these two equations now, we can solve for unknown velocities v_a' and v_b', if the value of e is known.

PHASES OF IMPACT

1. **Period of deformation**. Just after the impact two bodies deform. Time interval from first contact to the maximum deformation is called period of deformation. At end of this period, both bodies move with same velocity v.

2. **Period of restitution.** Period of deformation is followed by a period of restitution. At its end, two bodies either regain their original shapes fully or partially, or remain permanently deformed. This depends upon magnitude of impact forces and properties of materials involved. Also, at its end, two bodies separate and travel with different velocities except in case of plastic impact.



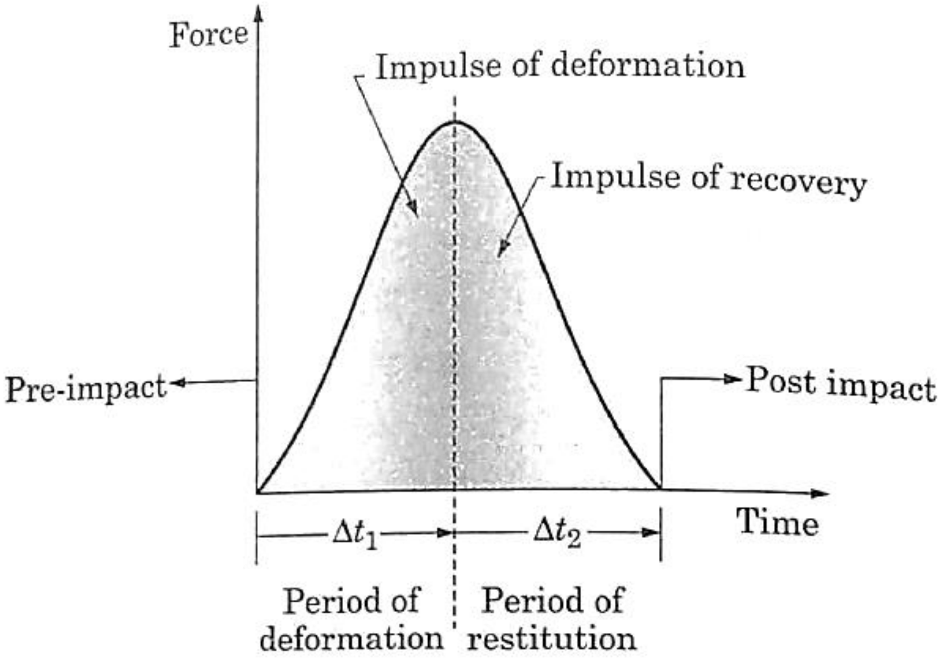
(a)

NATURE OF IMPACT AND COEFFICIENT OF RESTITUTION

- Consider body A.
- It is moving with velocity $\mathbf{v_a}$ and collides with another body B.
- During deformation period an impulse force $\mathbf{F_d}$ is exerted by body B on body A and changes its velocity to $\mathbf{v}$.
- Applying principle of impulse and momentum

$$m_a v_a - m_a v = \int F_d dt$$

- Where, integral extends over period of deformation.
- Next consider its motion during restitution period.



(b)

NATURE OF IMPACT AND COEFFICIENT OF RESTITUTION

- Let $\mathbf{F_r}$ be impulsive force exerted by body B on body A changing its velocity from $\mathbf{v}$ to $\mathbf{v_a}'$. We can write,

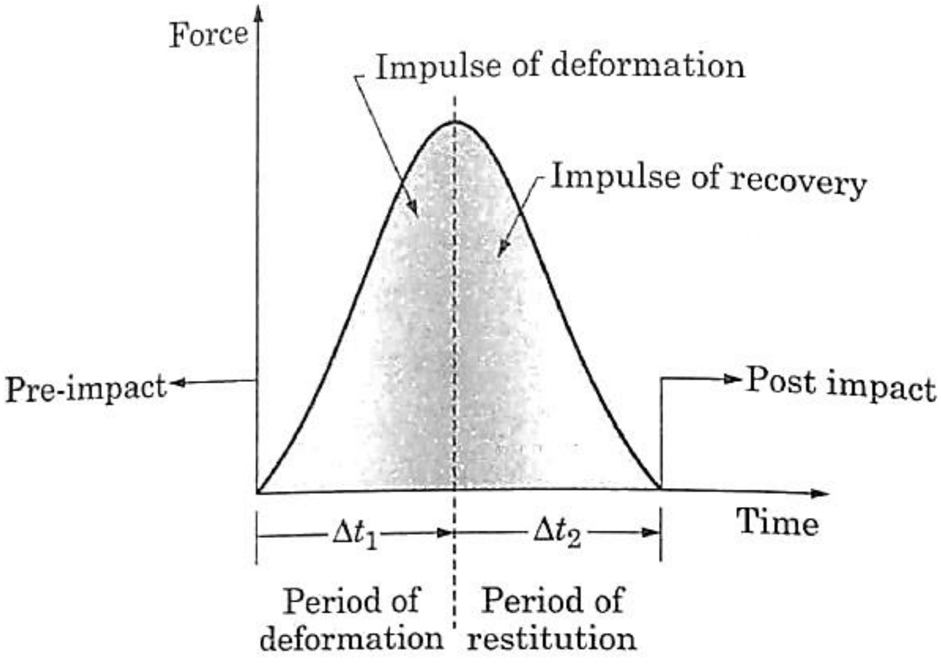
$$m_a v - m_a v'_a = \int F_r dt$$

- Force $\mathbf{F_d}$ exerted on body A during deformation period is different from force exerted during restitution or recovery period.
- The coefficient of restitution is ratio of magnitude of impulses during restitution period and deformation period.

$$e = \frac{Impulse\ during\ restitution\ or\ recovery}{impulse\ during\ deformation} = \frac{\int F_r dt}{\int F_d dt}$$

Using $m_a v_a - m_a v = \int F_d dt$ and $m_a v - m_a v'_a = \int F_r dt$

$$e = \frac{m_a\,(v - v'_a)}{m_a\,(v_a - v)} = \frac{(v - v'_a)}{(v_a - v)}$$



(b)

NATURE OF IMPACT AND COEFFICIENT OF RESTITUTION

- But similar analysis for body B, we can obtain

$$e = \frac{m_b\,(v - v'_b)}{m_b\,(v_b - v)} = \frac{(v'_b - v)}{(v - v_b)} = \frac{(v'_b - v)}{(v - v_b)}$$

- Adding, we get, $$e = \frac{(v - v'_a) + (v'_b - v)}{(v_a - v) + (v - v_b)} = \frac{v'_b - v'_a}{v_a - v_b}$$

- Which is same as defined earlier,

- or $$e = -\frac{(v'_b - v'_a)}{(v_b - v_a)} = -\frac{(\,\text{velocity of separation})}{(\text{velocity of approach})}$$

- Coefficient of restitution is a parameter which indicates energy loss during an impact and can be determined experimentally.

- While using momentum equation and coefficient of restitution relation, velocities should be assigned proper sign and should be added and subtracted algebraically.

IMPORTANT CASES OF IMPACT

1. ***Perfectly elastic impact (when e = 1)***

- For example, an impact between two hardened and polished steel balls.
- The coefficient of restitution relation gives,
- $e = -\frac{(v'_b - v'_a)}{(v_b - v_a)} = 1,$ or $(v'_b - v'_a) = -(v_b - v_a)$ or $(v'_b - v'_a) = (v_a - v_b)$
- Also, in case of a perfectly elastic impact, energy of system is conserved.
- Consider an elastic impact between m_a and m_b.
- Conservation of momentum gives,

$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$

- Which can be written as, $m_a(v_a - v'_a) = m_b(v'_b - v_b)$ I

IMPORTANT CASES OF IMPACT

- The coefficient of restitution relation when e=1 is,

$$(v'_b - v'_a) = (v_a - v_b)$$

- Or, $$(v_b + v'_b) = (v_a + v'_a) \qquad 2$$
- Multiplying equations 1 and 2,

$$m_a(v_a - v'_a)(v_a + v'_a) = m_b(v'_b - v_b)(v_b + v'_b)$$

- Or $$\frac{1}{2}m_a v_a^2 + \frac{1}{2}m_b v_b^2 = \frac{1}{2}m_a {v'}_a^2 + \frac{1}{2}m_b {v'}_b^2$$
- Kinetic energy of two bodies before impact = kinetic energy of two bodies after impact.
- Which shows that in case of a perfectly elastic impact (e=1), energy of system is conserved.
- Thus, in the case of an elastic impact both momentum and energy are conserved.

IMPORTANT CASES OF IMPACT: *PERFECTLY PLASTIC IMPACT (e = 0)*

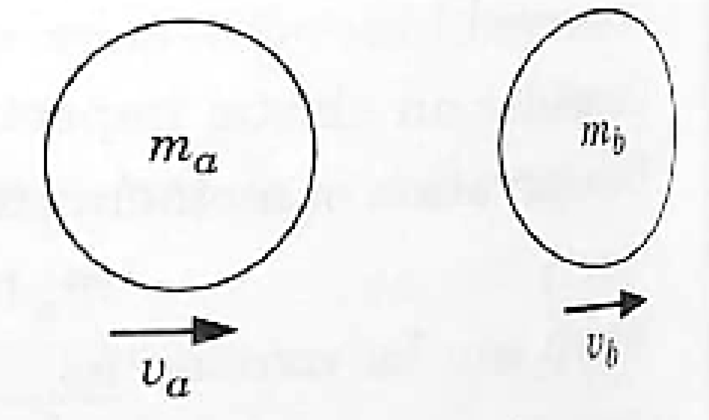
- For example, an impact between two putty balls.
- The coefficient of restitution relation gives, $e = -\frac{(v_b' - v_a')}{(v_b - v_a)} = 0$ or $v_b' = v_a' = v'$
- i.e., After plastic impact, final velocities of both bodies become equal and they move together as one body.
- As two bodies are permanently deformed there is no period of restitution or recovery.
- Note that the kinetic energy of the system is not conserved.
- But total momentum of system of bodies is conserved and we can write,

$$m_a v_a + m_b v_b = (m_a + m_b) v'$$

- Above equation can be solved for v'. Common velocity of the two bodies is, $v' = \frac{(m_a v_a + m_b v_b)}{(m_a + m_b)}$
- In any general case of impact when e is not equal to 1, energy of system is not conserved.
- This can be shown in any given case by comparing kinetic energy before and after impact.

ELASTIC IMPACT: *IMPACT OF TWO EQUAL MASSES*

$$m_a = m_b = m$$



- Conversation of momentum gives,

$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$

- Or,

$$v_a + v_b = v'_a + v'_b$$

- The coefficient of restitution relation gives, $e = -\frac{(v'_b - v'_a)}{(v_b - v_a)} = 1$

$$(v'_b - v'_a) = (v_a - v_b)$$

- Solving the above two equations,

$$v'_a = v_b \qquad \text{and} \qquad v'_b = v_a$$

- After an elastic impact two masses exchange velocities.

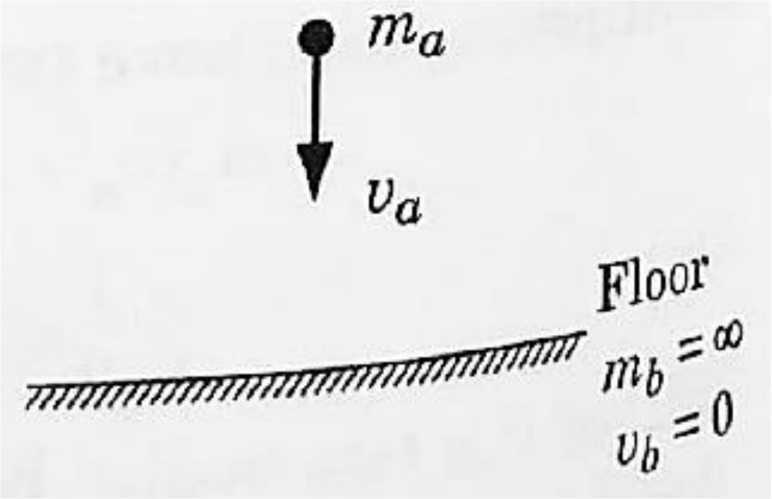
ELASTIC IMPACT: *IMPACT OF TWO BODIES. OF THE TWO BODIES, ONE IS IMMOVABLE AND OF VERY LARGE MASS AS COMPARED TO THE OTHER BODY*

- For example, a ball dropped on the floor.

$$m_b = \infty \quad \text{and} \quad v_b = 0$$

- Conservation of momentum gives,

$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$



- Dividing by $\mathbf{m_b}$, $\frac{m_a}{m_b} v_a + v_b = \frac{m_a}{m_b} v'_a + v'_b$
- When $\mathbf{m_b}$ is very large and $\mathbf{v_b}$ is 0,
- The coefficient of restitution relation gives, $e = -\frac{(v'_b - v'_a)}{(v_b - v_a)} = 1$

$$(v'_b - v'_a) = (v_a - v_b) \qquad (\text{As, } v_b = v'_b = 0)$$

- Therefore, $v_a = v'_a$
- Body m_a would rebound with same velocity with which it strikes the immovable body

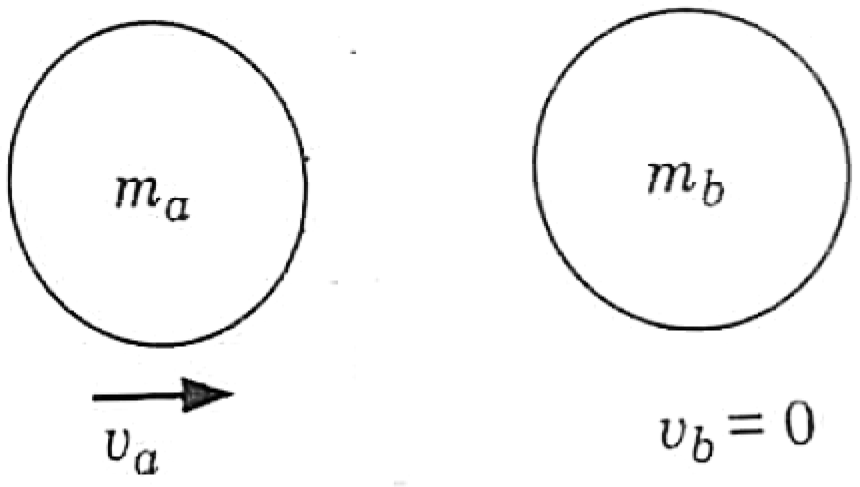
ELASTIC IMPACT: *A BODY STRIKES ANOTHER BODY OF EQUAL MASS AT REST*

$$m_a = m_b = m, \qquad v_b = 0$$

- Conservation of momentum gives

$$m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$$

$$v_a + 0 = v'_a + v'_b$$



- The coefficient of restitution relation gives,

$$e = -\frac{(v'_b - v'_a)}{(v_b - v_a)} = 1$$

$$v_b - v_a = -v'_b + v'_a$$

- Solving the above equations, $v'_a = 0$ and $v'_b = v_a$
- The striking mass m_a stops after imparting its entire velocity to the mass m_b.

LOSS OF KINETIC ENERGY DURING IMPACT

- Let the masses m_a and m_b moving with velocities v_a and v_b collide with central direct impact.
- Velocities after the impact are v_a' and v_b'.
- Kinetic energy of the two masses before impact, $= \frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2$
- Kinetic energy of the two masses after impact, $= \frac{1}{2} m_a v'^2_a + \frac{1}{2} m_b v'^2_b$
- The loss of kinetic energy during impact, $= \left(\frac{1}{2} m_a v_a^2 + \frac{1}{2} m_b v_b^2\right) - \left(\frac{1}{2} m_a v'^2_a + \frac{1}{2} m_b v'^2_b\right)$
- Conservation of momentum gives, $m_a v_a + m_b v_b = m_a v'_a + m_b v'_b$
- The coefficient of restitution relation gives, $e = -\frac{(v_b' - v_a')}{(v_b - v_a)}$
- or $(v_b' - v_a') = -e(v_b - v_a)$

LOSS OF KINETIC ENERGY DURING IMPACT

- To express loss of K.E. in terms of masses of two bodies and their velocities before impact.
- To eliminate $\mathbf{v_a'}$ and $\mathbf{v_b'}$ multiply the numerator and denominator of equation (The loss of kinetic energy during impact,$=\left(\frac{1}{2}m_a v_a^2+\frac{1}{2}m_b v_b^2\right)-\left(\frac{1}{2}m_a v'^2_a+\frac{1}{2}m_b v'^2_b\right)$)by $\mathbf{m_a + m_b}$.
- $Loss\ of\ K.E.=\frac{1}{2(m_a+m_b)}\left[m_a^2v_a^2+m_am_bv_a^2+m_am_bv_b^2+m_b^2v_b^2-m_a^2v'^2_a-m_am_bv'^2_a+m_am_bv'^2_b+m_b^2v'^2_b\right]$
- $Loss\ of\ K.E.=\frac{1}{2(m_a+m_b)}\left[\{(m_av_a+m_bv_b)^2+m_am_b(v_a-v_b)^2\}-\{(m_av'_a+m_bv'_b)^2+m_am_b(v'_a-v'_b)^2\}\right]$
- From conservation of momentum, $(m_av_a+m_bv_b)^2=(m_av'_a+m_bv'_b)^2$
- Hence, $Loss\ of\ K.E.=\frac{1}{2(m_a+m_b)}\left[m_am_b(v_a-v_b)^2-m_am_b(v'_a-v'_b)^2\right]$

LOSS OF KINETIC ENERGY DURING IMPACT

- Hence, $\boldsymbol{Loss\ of\ K.E.} = \frac{m_a m_b}{2(m_a+m_b)}\left[(v_a - v_b)^2 - (v'_a - v'_b)^2\right]$

- Using $(v'_b - v'_a) = -e(v_b - v_a)$

$$Loss\ of\ K.E. = \frac{m_a m_b}{2(m_a + m_b)}\left[(v_a - v_b)^2 - e^2(v_b - v_a)^2\right]$$

$$Loss\ of\ K.E. = \frac{m_a m_b}{2(m_a + m_b)}\left[(v_a - v_b)^2(1 - e^2)\right]$$

Parameters	Work Energy Principle	Principle of Conservation of Energy	Principle of Impulse and Momentum
Equation	$\int_{S_1}^{S_2} \mathbf{F} \cos\alpha \, dS = \left(\frac{mv_2^2}{2} - \frac{mv_1^2}{2}\right)$	$K.E._1 + P.E._1 = K.E._2 + P.E._2$ $V_2 + T_2 = (V_1 + T_1)$ $E_1 = E_2$	$\sum mv_2 - \sum mv_1 = \int_0^t F dt$
Definition	Work done by a force acting on a particle during its displacement is equal to change in its kinetic energy of the particle during the displacement.	Sum of the P.E. and K.E. of a particle remain constant during its motion under the action of conservative force.	The change in the momentum of a particle during a small interval of time (dt) is equal to impulse of the force acting during the same interval. If $\int_0^t F dt = 0$, then momentum is conserved
Application	Applied between two positions of the particle	Applied between two positions of particle	Applied between an interval of time
Applicability	Applicable whether force involved are conservative or non-conservative. Can be applied if friction is present.	Applicable when only conservative forces (gravity, spring or elastic forces) are present. Not applicable when friction is present	Applicable whether forces are conservative or non-conservative but neglects the effect of non-impulsive forces.
Consideration of Internal Forces	When applied to a system of particles, the work of all forces whether external or internal is to be considered. Sum of work of action and reaction between particles may or may not be zero.	When applied to a system of particles, the potential energy corresponding to internal forces must be considered.	When applied to a system of particles, the internal forces between various particles need not be considered as sum of their impulses is always zero.