HW Absolute Minima/Maxima

Find the relative and absolute minimum and maximum values for the following functions in the given region R:-

(1) $x^2+y^2-x-y+1$; R: rectangular region ; $0 \le x \le 2$, $0 \le y \le 2$.

(2) x^2-y^2-2y ; R: $x^2+y^2 \le 1$.

(3) xy ; R: $x^2+y^2 \le 1$.

Que Prove that $f(x,y) = x^2-2xy+y^2+x^4+y^4$ has a minima at the origin.

Que Show that $f(x,y) = x^2-3xy^2+2y^4$ has neither a minimum nor maximum value at the origin.

Sol^n $f(x,y) = x^2+y^2-x-y+1$

$f_x = 2x-1 = 0 \Rightarrow x = 1/2$

$f_y = 2y-1 = 0 \Rightarrow y = 1/2$

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0 = f_{yx}$

$J = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$ Pos. def

$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ is pt of local Min.

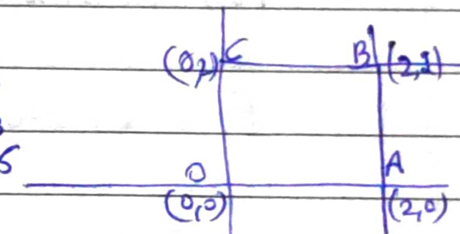
$f\left(\frac{1}{2}, \frac{1}{2}\right) = \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{2} + 1 = \frac{1}{2}$

$f(0,0) = 1$

$f(2,0) = 3$

$f(0,2) = 3$

$f(2,2) = 5$



Along OA, $y=0$

$f(x,y) = g(x) = x^2-x+1$

$g'(x) = 2x-1 = 0 \Rightarrow x = 1/2$

$g''(x) > 0$

$\left(\frac{1}{2}, 0\right) \rightarrow$ Min. $f\left(\frac{1}{2}, 0\right) = \frac{1}{4} - \frac{1}{2} +$

$= \frac{1-2+4}{4} =$

along OC, $x=0$

$f(x,y) = f(0,y) = y^2-y+1 = h(y)$

$h'(y) = 2y-1 = 0 \Rightarrow y = 1/2$

$h''(y) > 0$

$(0, 1/2)$ — Min

$f\left(0, \frac{1}{2}\right) = 3/4$

along AB, $x=2$

$f(2,y) = y^2-y+3$

$f(2, 1/2) = 1/4 - 1/2 + 3 = \frac{1-2+6}{4} = 5/$

Along BC, $y=3$

$f(x,3) = x^2-x+7$

$f(1/2, 3) = \frac{1}{4} - \frac{1}{2} + 7 = \frac{1-2+28}{4} = 27/4$

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Que xy on $x+2y=2$, $x\geq 0, y\geq 0$.

$F = xy + \lambda(x+2y-2)$

$F_x = y+\lambda = 0 \Rightarrow y = -\lambda$

$F_y = x + 2\lambda = 0$

$\Rightarrow x = -2\lambda$

So $-2\lambda - 2\lambda = 2$

$-4\lambda = 2 \Rightarrow \lambda = -1/2$

$x = 1, y = 1/2$

Max value at $(1, \frac{1}{2})$ is $1/2$

Min value $= 0$.

Que $x+2y$ on the circle $x^2+y^2=1$

$F = x+2y + \lambda(x^2+y^2-1)$

$F_x = 1 + 2\lambda x = 0 \Rightarrow x = \frac{-1}{2\lambda}$

$F_y = 2 + 2\lambda y = 0 \Rightarrow y = -1/\lambda$

$x^2+y^2 = 1$

$\Rightarrow \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 1 \Rightarrow \frac{5}{4} = \lambda^2 \Rightarrow \lambda = \pm\frac{\sqrt{5}}{2}$

$\lambda = \frac{\sqrt{5}}{2}$, $x = \frac{-1}{2\times\sqrt{5}/2} = \frac{-1}{\sqrt{5}}$

$y = \frac{-2}{\sqrt{5}}$

So $x+2y = \frac{-1}{\sqrt{5}} - \frac{4}{\sqrt{5}} = \frac{-5}{\sqrt{5}} = -\sqrt{5}$

$\lambda = \frac{-\sqrt{5}}{2}$; $x = \frac{1}{\sqrt{5}}$

$y = 2/\sqrt{5}$

$x+2y = \sqrt{5}$

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Que Find the Smallest and the largest value of $2x-y$ on the Curve $x-\sin y=0;\ 0\le y\le 2\pi$

Sol^n

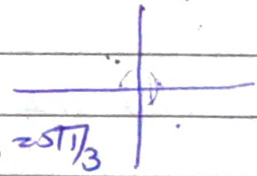
$F=(2x-y)+\lambda(x-\sin y)$

$F_x=2+\lambda=0 \Rightarrow \lambda=-2$

$F_y=-1-\cos y\,\lambda=0 \Rightarrow -1=\cos y(-2)$

$\Rightarrow \frac{1}{2}=\cos y$

$\Rightarrow y=\pi/3,\ 2\pi-\pi/3=5\pi/3$



$x-\sin y=0 \Rightarrow x=\sin\pi/3$

$= x=\sqrt{3}/2$

So $\left(\frac{\sqrt{3}}{2},\frac{\pi}{3}\right)$ $\quad 2x-y=2\frac{\sqrt{3}}{2}-\frac{\pi}{3}=\sqrt{3}-\frac{\pi}{3}$

$=\left(\frac{3\sqrt{3}-\pi}{3}\right)$

$x=\sin\frac{5\pi}{3}=\sin\left(2\pi-\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}$

$\left(\frac{-\sqrt{3}}{2},\frac{5\pi}{3}\right)\ ;\ 2\left(\frac{-\sqrt{3}}{2}\right)-\frac{5\pi}{3}=\frac{-3\sqrt{3}-5\pi}{3}=\frac{-(3\sqrt{3}+5\pi)}{3}$

Que Find the Extreme values of x^2+y^2 when $x^4+y^4=1$

Sol^n

$F=x^2+y^2+\lambda(x^4+y^4-1)$

$F_x=2x+4\lambda x^3=0$

$F_y=2y+4\lambda y^3=0$

$\Rightarrow 1+2\lambda x^2=0$

$\Rightarrow x^2=-1/2\lambda$

$1+2\lambda y^2=0 \Rightarrow y^2=-1/2\lambda$

So $x^4+y^4=1 \Rightarrow \frac{1}{4\lambda^2}+\frac{1}{4\lambda^2}=1$

$\Rightarrow \frac{1}{2\lambda^2}=1 \Rightarrow \boxed{\lambda^2=1/2}$

$\Rightarrow \lambda=\pm\frac{1}{\sqrt{2}}$

$\lambda=\frac{1}{\sqrt{2}};\ x^2=\frac{-1}{2\times\frac{1}{\sqrt{2}}}=\frac{-1}{\sqrt{2}}$

$y^2=\frac{-1}{\sqrt{2}}$

$x^2+y^2=-\sqrt{2}$

$\lambda=\frac{-1}{\sqrt{2}};\ x^2=\frac{1}{\sqrt{2}},\ y^2=\frac{1}{\sqrt{2}}$

$x^2+y^2=\sqrt{2}$

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HW (1) Find the Extreme values of xyz when $x+y+z=a, a>0$

Ans: $\frac{a^3}{27}$ at $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

(2) Find the Extreme values of $x^3+8y^3+64z^3$ when $xyz=1$

Ans:- 24 at $\left(2, 1, \frac{1}{2}\right)$

Qu Find the Extreme values of $f(x,y,z) = 2x+3y+z$ s.t. $x^2+y^2=5$ and $x+z=1$.

Sol^n $F(x,y,z) = 2x+3y+z + \lambda_1(x^2+y^2-5) + \lambda_2(x+z-1)$

$F_x = 2+2\lambda_1 x + \lambda_2 = 0$

$F_y = 3+2\lambda_1 y = 0$

$F_z = 1+\lambda_2 = 0$

$\Rightarrow \lambda_2 = -1$

$\Rightarrow 2+2\lambda_1 x - 1 = 0 \Rightarrow 2\lambda_1 x + 1 = 0$

$\Rightarrow x = \frac{-1}{2\lambda_1}$

and $y = -3/2\lambda_1$

$x^2+y^2=5 \Rightarrow \frac{1}{4\lambda_1^2} + \frac{9}{4\lambda_1^2} = 5$

$\Rightarrow 10 = 20\lambda_1^2 \Rightarrow \lambda_1^2 = 1/2$

$\Rightarrow \lambda_1 = \pm\frac{1}{\sqrt{2}}$

$\lambda_1 = \frac{1}{\sqrt{2}}, \; x = \frac{-1}{\sqrt{2}}, \; y = \frac{-3}{\sqrt{2}}, \; x+z=1 \Rightarrow z = 1+\frac{1}{\sqrt{2}}$

$f(x,y,z) = 2x+3y+z = 2\left(\frac{-1}{\sqrt{2}}\right)+3\left(\frac{-3}{\sqrt{2}}\right)+1+\frac{1}{\sqrt{2}}$

$= \cancel{-\frac{1}{\sqrt{2}} - \frac{3\sqrt{3}}{2} + 1}$

$= \frac{-10}{\sqrt{2}}+1 = -5\sqrt{2}+1$

$\lambda_1 = \frac{-1}{\sqrt{2}}; \; x = \frac{1}{\sqrt{2}}, \; y = \frac{3}{\sqrt{2}}, \; z = 1-\frac{1}{\sqrt{2}}$

$f(x,y,z) = 2/\sqrt{2} + 9/\sqrt{2} + 1 - \frac{1}{\sqrt{2}} = \frac{10}{\sqrt{2}}+1 = 1+5\sqrt{2}$

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Jacobian:→ If f_1, f_2 be 2 functions of two variables x and y possessing partial derivatives of first order at every point of the domain then

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$ is called Jacobian Matrix.

and $$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{vmatrix}$$ is called Jacobian of f_1 & f_2 w.r.t. x_1 & x_2.

and is denoted by $J(f_1, f_2)$ or $\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)}$ or $J_f(x_1, x_2)$
(where $f = (f_1, f_2)$)

Ex If $f = x^2 - x\sin y$; $g = x^2y^2 + x + y$. Find $\frac{\partial(f,g)}{\partial(x,y)}$

$$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

$f_x = 2x - \sin y$; $f_y = -x\cos y$; $g_x = 2xy^2 + 1$; $g_y = 2x^2y + 1$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} 2x - \sin y & -x\cos y \\ 2xy^2 + 1 & 2x^2y + 1 \end{vmatrix}$$

$$= (2x - \sin y)(2x^2y + 1) + x\cos y\,(2xy^2 + 1)$$

(2) $f_1(x_1, x_2, x_3) = x_1^3$; $f_2(x_1, x_2, x_3) = e^{x_2}$; $f_3(x_1, x_2, x_3) = x_1 + \sin x_3$

Then find $\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}$

Sol^n $\frac{\partial f_1}{\partial x_1} = 3x_1^2$ $\frac{\partial f_1}{\partial x_2} = 0 = \frac{\partial f_1}{\partial x_3}$

$\frac{\partial f_2}{\partial x_1} = 0$, $\frac{\partial f_2}{\partial x_2} = e^{x_2}$; $\frac{\partial f_2}{\partial x_3} = 0$

$\frac{\partial f_3}{\partial x_1} = 1$; $\frac{\partial f_3}{\partial x_2} = 0$, $\frac{\partial f_3}{\partial x_3} = \cos x_3$

$$\Rightarrow \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} 3x_1^2 & 0 & 0 \\ 0 & e^{x_2} & 0 \\ 1 & 0 & \cos x_3 \end{vmatrix}$$

$$= 3x_1^2 \cdot e^{x_2} \cdot \cos x_3$$

Que $f(x,y) = (x+y, (x+y)^2)$ find $J_f(x,y)$

Sol^n $f_1(x,y) = x+y, \quad f_2(x,y) = (x+y)^2$

$$\frac{\partial f_1}{\partial x} = 1, \quad \frac{\partial f_1}{\partial y} = 1, \quad \frac{\partial f_2}{\partial x} = 2(x+y); \quad \frac{\partial f_2}{\partial y} = 2(x+y)$$

$$J_f(x,y) = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2(x+y) & 2(x+y) \end{vmatrix} = 0.$$

$\Rightarrow f_1$ & f_2 are functionally dependent.

HW (1) $f(x,y,z) = (x \sin y \cos z, \; x \sin y \sin z, \; x \cos y)$

Find $J_f(x,y,z)$

(2) Let $f_1(x,y) = \dfrac{x+y}{1-xy}; \quad f_2(x,y) = \tan^{-1}x + \tan^{-1}y$

Find $J_f(x,y)$ or $\dfrac{\partial(f_1, f_2)}{\partial(x,y)}$.

Change of variables | Transformation of Coordinates :→

Let $z = f(x,y)$

where $x = \phi(u,v); \; y = \psi(u,v)$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \quad -(1)$$

and
$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} \quad -(2)$$

From (1) and (2)

$$\frac{\partial f/\partial x}{\frac{\partial f}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial y}{\partial u}} = \frac{\partial f/\partial y}{\frac{\partial f}{\partial v}\cdot\frac{\partial x}{\partial u} - \frac{\partial f}{\partial u}\cdot\frac{\partial x}{\partial v}} = \frac{1}{\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}}$$

$\because$

$$\begin{array}{cccc} \frac{\partial y}{\partial u} & \frac{\partial f}{\partial u} & \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial f}{\partial v} & \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{\frac{\partial(f,y)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}}$$

|||ly $$\frac{\partial f}{\partial y} = -\frac{\frac{\partial(f,x)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}}$$

Que $z = f(x,y)$; $x = r\cos\theta$, $y = r\sin\theta$ Then Show that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$

Sol^n

$$\frac{\partial f}{\partial x} = \frac{\frac{\partial(f,y)}{\partial(r,\theta)}}{\frac{\partial(x,y)}{\partial(r,\theta)}}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & +r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(f,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \frac{\partial f}{\partial r}\cdot r\cos\theta - \frac{\partial f}{\partial \theta}\cdot\sin\theta$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{r}\left[r\cos\theta\frac{\partial f}{\partial r} - \sin\theta\frac{\partial f}{\partial \theta}\right] = \cos\theta\frac{\partial f}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial f}{\partial \theta}$$

Similary $$\frac{\partial f}{\partial y} = -\frac{\frac{\partial(f,x)}{\partial(r,\theta)}}{\frac{\partial(x,y)}{\partial(r,\theta)}}$$

$$\frac{\partial(f,x)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \end{vmatrix} = \frac{\partial f}{\partial r}\cdot(-r\sin\theta) - \frac{\partial f}{\partial \theta}(\cos\theta)$$

$$= -r\sin\theta\frac{\partial f}{\partial r} - \cos\theta\cdot\frac{\partial f}{\partial \theta}$$

$$\Rightarrow \frac{\partial f}{\partial y} = \sin\theta\frac{\partial f}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial f}{\partial \theta}$$

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HW (1) If $z = f(x,y)$, $x = r\cosh\theta$, $y = r\sinh\theta$

Then S.T. $\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$

(2) If $z = y + f(u)$; $u = \frac{x}{y}$ then $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1.$

Sol^n $\cancel{\frac{\partial z}{\partial y} = 1}$; $z = y + f\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$$

$$\frac{\partial z}{\partial y} = 1 + f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

So $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{x}{y}\cancel{\left(\frac{1}{y}\right)f'\left(\frac{x}{y}\right)} + 1 - \cancel{\frac{x}{y^2}f'\left(\frac{x}{y}\right)}$

$= 1.$

(3) If $u = f(x-y, y-z, z-x)$ Then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

$w_1 = x-y$, $w_2 = y-z$, $w_3 = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial w_1}\cdot\frac{\partial w_1}{\partial x} + \frac{\partial u}{\partial w_2}\cdot\frac{\partial w_2}{\partial x} + \frac{\partial u}{\partial w_3}\cdot\frac{\partial w_3}{\partial x}$$

$$= \frac{\partial u}{\partial w_1}(1) - \frac{\partial u}{\partial w_3}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial w_1}\cdot\frac{\partial w_1}{\partial y} + \frac{\partial u}{\partial w_2}\cdot\frac{\partial w_2}{\partial y} + \frac{\partial u}{\partial w_3}\cdot\frac{\partial w_3}{\partial y}$$

$$= -\frac{\partial u}{\partial w_1} + \frac{\partial u}{\partial w_2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial w_1}\cdot\frac{\partial w_1}{\partial z} + \frac{\partial u}{\partial w_2}\cdot\frac{\partial w_2}{\partial z} + \frac{\partial u}{\partial w_3}\cdot\frac{\partial w_3}{\partial z}$$

$$= \frac{-\partial u}{\partial w_2} + \frac{\partial u}{\partial w_3}$$

So $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

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Differentiation under Integral Sign :→

Leibnitz Integral Rule

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

and $$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt + f(x,b(x))\cdot\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x)$$

Ex $$\frac{d}{dx}\int_{\sin x}^{\cos x} \cosh t^2\,dt$$

$$= \int_{\sin x}^{\cos x} \frac{\partial}{\partial x}(\cosh t^2)\,dt + \cosh(\cos x)^2\cdot\frac{d}{dx}(\cos x) - \cosh(\sin x)^2\cdot\frac{d}{dx}(\sin x)$$

$$= 0 + \cosh(\cos^2 x)(-\sin x) - \cosh(\sin^2 x)\cdot\cos x$$

$$= -\cosh(\cos^2 x)\sin x - \cosh(\sin^2 x)\cdot\cos x.$$

Ex Using Diff under Integral Sign

P.T. $$\int_0^1 \frac{x^y-1}{\log x}dx = \log(1+y)$$

Sol^n Let $$I(y) = \int_0^1 \frac{x^y-1}{\log x}\,dx$$

Diff w.r.t y

$$\frac{d}{dy}(I(y)) = \frac{d}{dy}\int_0^1 \frac{x^y-1}{\log x}\,dx$$

$$= \int_0^1 \frac{\partial}{\partial y}\left(\frac{x^y-1}{\log x}\right)dx$$

$$= \int_0^1 \frac{\partial}{\partial y}\left(\frac{x^y}{\log x}\right)dx = \int_0^1 \frac{x^y \cancel{\log x}}{\cancel{\log x}}\,dx$$

$$= \int_0^1 x^y\,dx = \left(\frac{x^{y+1}}{y+1}\right)_0^1 = \frac{1}{y+1}$$

$$\Rightarrow I(y) = \int\frac{dy}{y+1} = \log_e(y+1) + \log C = \log_e C(y+1)$$

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Now $I(0) = \int_0^1 \frac{x^0 - 1}{\log x} = 0$

$\Rightarrow I(0) = \log c(1) \Rightarrow 0 = \log c$
$\Rightarrow c = 1$

$\Rightarrow I(y) = \log(y+1)$

Thm Let $\phi(y) = \int_a^b f(x,y)\,dx$ where $f(x,y)$ is a cts function of (x,y) in the Rectangle $R: \{(x,y);\ a \le x \le b,\ c \le y < d\}$ and $f_y(x,y)$ Exist and is cts in R then $\phi'(y)$ Exist and is Equal to $\int_a^b \frac{\partial}{\partial y} f(x,y)\,dx$.

i.e. $\phi'(y) = \frac{d}{dy}\int_a^b f(x,y)\,dy = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx.$

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(3) $\omega = f(x,y)$; $x = \sqrt{u^2+v^2}$; $y = \cot^{-1}\left(\frac{v}{u}\right)$

$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = ?$

Solⁿ

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{u}{\sqrt{u^2+v^2}} + \frac{\partial f}{\partial y}\left(\frac{-1}{1+\frac{v^2}{u^2}}\right)\left(\frac{-v}{u^2}\right)$$

$$= \frac{u}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial x} + \frac{v}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{v}{\sqrt{u^2+v^2}}\cdot\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\left(\frac{-1}{1+\frac{v^2}{u^2}}\right)\left(\frac{1}{u}\right)$$

$$= \frac{v}{\sqrt{u^2+v^2}}\frac{\partial f}{\partial x} - \frac{u}{u^2+v^2}\cdot\frac{\partial f}{\partial y}$$

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \frac{u^2+v^2}{u^2+v^2}\left(\frac{\partial f}{\partial x}\right)^2 + \frac{v^2+u^2}{(u^2+v^2)^2}\left(\frac{\partial f}{\partial y}\right)^2$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \frac{1}{u^2+v^2}\cdot\left(\frac{\partial f}{\partial y}\right)^2$$

$$= \left[(u^2+v^2)\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]\frac{1}{u^2+v^2}$$

Que $f(x,y) = (y-x)^4 + (x-2)^4$ has a Minimum at $(2,2)$.

Solⁿ Let $f(x,y) = (y-x)^4 + (x-2)^4$

$f_x = -4(y-x)^3 + 4(x-2)^3$;

$f_y = 4(y-x)^3$

$f_{xx} = 12(y-x)^3 + 12$

$f_{yy} = 12(y-x)^2$

$f_{yx} = -12(y-x)^2$

$$J_{(2,2)} = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}_{(2,2)} \qquad \underline{|J| = 0} \to \text{Test fails.}$$

$f(2,2) = 0$

$f(x,y) - f(2,2) = (y-x)^4 + (x-2)^4 \geq 0$

$\Rightarrow (2,2)$ is pt of Minima.

Que $f(x,y) = (x-y)^2 + x^3 - y^3 + x^5$ → at origin.

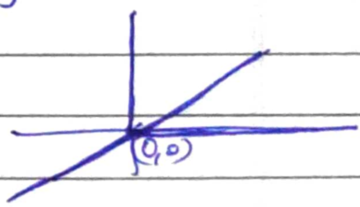
$\hookrightarrow |J| = 0$

Solⁿ $f(0,0) = 0$

$f(x,y) - f(0,0) = (x-y)^2 + (x-y)(x^2+y^2+xy) + x^5$.

along $y = x$ line

$f(x,y) - f(0,0) = x^5$



$\Rightarrow$ In nbd of $(0,0)$, there are points at which $f(x,y)$ is +ive and there are pts at which $f(x,y)$ is -ive

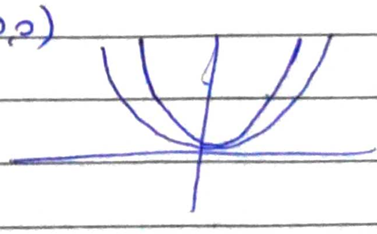
$\Rightarrow f(x,y)$ has neither maxima nor minima at $(0,0)$.

Que $f(x,y) = 2x^4 + y^2 - 3x^2y$ at $(0,0)$

$= 2x^4 - 2x^2y - x^2y + y^2$

$= 2x^2(x^2-y) - y(x^2-y)$

$\Rightarrow (2x^2-y)(x^2-y)$



Neither Max nor Min at $(0,0)$

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Ex $f(x,y) = x^2 - 2xy + y^2 + x^4 + y^4$ at origin.

Sol^n $|J| = 0$ — Check

$f(0,0) = 0$

$f(x,y) = (x-y)^2 + x^4 + y^4$

$f(x,y) > 0 \quad \forall (x,y) \in N_\delta(0,0) - \{0,0\}$.

$\Rightarrow$ $f(x,y)$ has Minimum at $(0,0)$

Ex $f(x,y) = x^2 - 3xy^2 + 2y^4$ at origin.

Sol^n $f(x,y) = x^2 - 2xy^2 - xy^2 + 2y^4$

$= x(x-2y^2) - y^2(x-2y^2)$

$= (x-2y^2)(x-y^2)$

$\underline{<0 \quad >0}$

$\underline{<0}$