Non-Homogeneous linear diff. Equation with Constant Coefficients

$$F(D)\,y = Q(x) \quad -(1)$$

If D is a differential operator, then its inverse D^{-1} is an integral operator s.t. $D^{-1} D(f(x)) = f(x)$

i.e. $\frac{1}{D}$ or $D^{-1}(f) = \int f(x)dx$.

from (1); $y = (F(D))^{-1} Q(x) = \frac{Q(x)}{F(D)}$

Case I If $Q(x) = e^{\alpha x}$.

Then Substitute $D = \alpha$. $\therefore$ Solⁿ is $y(x) = \frac{Q(x)}{F(\alpha)}$ Provided $F(\alpha) \neq 0$.

Ex $y'' - 2y' - 3y = 3e^{2x}$

Solⁿ General solution of non-homogeneous diff. Equation

= General solution of hom. diff Eqⁿ + Particular Integral

= Complementary fun + P.I

i.e. = C.F + P.I

$(D^2 - 2D - 3)y = 0$

Aux Eqⁿ is $m^2 - 2m - 3 = 0$

$\Rightarrow (m-3)(m+1) = 0$

$\Rightarrow m = 3, -1$

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$\therefore y_c(x) = c_1 e^{-x} + c_2 e^{3x}$.

P.I is $y_p(x) = \frac{3e^{2x}}{F(D)} = \frac{3e^{2x}}{D^2-2D-3}$

Put $D=2$

$= \frac{3e^{2x}}{4-4-3} = -e^{2x}$.

$\therefore$ Gen Sol^n is $y(x) = y_c(x) + y_p(x)$

$= c_1 e^{-x} + c_2 e^{3x} - e^{2x}$.

Que $y'' + y' - 6y = 5e^{-3x}$.

Sol^n $(D^2+D-6)y = 5e^{-3x}$

Char. Equ is

$m^2+m-6=0$

$\Rightarrow (m+3)(m-2)=0$

$\Rightarrow m = 2, -3$

$\therefore y_c(x) = c_1 e^{2x} + c_2 e^{-3x}$

$y_p(x) = \frac{5e^{-3x}}{D^2+D-6} = \frac{5e^{-3x}}{(-3)^2-3-6} = \frac{5e^{-3x}}{0}$ — (Case fails)

$= \frac{x \cdot 5e^{-3x}}{2D+1} = \frac{5xe^{-3x}}{-5} = -xe^{-3x}$

$\therefore$ Gen Sol^n is $y(x) = y_c(x) + y_p(x)$

$\Rightarrow y(x) = c_1 e^{2x} + c_2 e^{-3x} - xe^{-3x}$.

Que $4y'' - 4y' + y = e^{x/2}$ $\Rightarrow (4D^2-4D+1)y = e^{x/2}$

Sol^n Char Equ^n is $4m^2 - 4m + 1 = 0$

$\Rightarrow (2m-1)^2 = 0 \Rightarrow m = 1/2, 1/2$

$y_c(x) = c_1 e^{x/2} + c_2 x e^{x/2} = (c_1 + c_2 x) e^{x/2}$.

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$$y_p(x) = \frac{e^{x/2}}{4D^2-4D+1}$$

$$= \frac{e^{x/2}}{4(\frac{1}{2})^2-4(\frac{1}{2})+1} = \frac{e^{x/2}}{1-2+1=0} \text{ — (Test fails)}$$

$$= \frac{xe^{x/2}}{8D-4} = \frac{xe^{x/2}}{4-4=0}$$

$$= \frac{x^2e^{x/2}}{8}$$

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$= (c_1+c_2x)e^{x/2} + \frac{x^2e^{x/2}}{8}.$$

Hw $9y''' + 3y'' - 5y' + y = 42e^x + 64e^{x/3}$.

Solⁿ $(9D^3+3D^2-5D+1)y = 42e^x + 64e^{x/3}$

A.E is $9m^3+3m^2-5m+1=0$

$\Rightarrow m=-1$ Satisfies

-1	9	3	-5	1
		-9	6	-1
	9	-6	1	$\boxed{0}$

$$9m^2-6m+1=0$$

$$(3m-1)^2=0$$

$$\therefore m = 1/3, 1/3, -1$$

$$y_c(x) = (c_1+c_2x)e^{x/3} + c_3e^{-x}.$$

$$y_p(x) = \frac{42e^x}{9D^3+3D^2-5D+1} + \frac{64e^{x/3}}{9D^3+3D^2-5D+1}$$

$$= \frac{42e^x}{9+3-5+1} + \frac{64e^{x/3}}{9\times\frac{1}{27}+3\times\frac{1}{9}-5\times\frac{1}{3}+1}$$

$\frac{1}{3}\frac{4}{3}$

$$= \frac{42e^x}{8} + \frac{64}{0} \rightarrow \text{Test fails}$$

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$$\frac{64e^{x/3}}{9D^3+3D^2-5D+1} \leftarrow \text{Test fails}$$

$$\frac{64\,x\,e^{x/3}}{27D^2+6D-5} = \frac{64xe^{x/3}}{3\cancel{27}\left(\frac{1}{9}\right)+\cancel{6}2\left(\frac{1}{3}\right)-5}$$

$5-5=0 \rightarrow$ again fails

$$\frac{64x^2e^{x/3}}{54D+6} = \frac{64x^2e^{x/3}}{54\left(\frac{1}{3}\right)+6} = \frac{64}{24}x^2e^{x/3}$$

$$= \frac{8}{3}x^2e^{x/3}$$

$$\therefore y(x) = y_c(x)+y_p(x)$$

$$= (c_1+c_2x)e^{x/3}+c_3e^{-x}+\frac{21}{4}e^x+\frac{8}{3}x^2e^{x/3}.$$

Que $16y''+8y'+y=48xe^{-x/4}$.

Ans:- $(c_1x+c_2)e^{-x/4}+\frac{1}{2}x^3e^{-x/4}$.

Case II When $Q(x)=\sin\alpha x$ or $\cos\alpha x$.

Substitute $D^2=-\alpha^2$.

then $y_p(x) = \frac{Q(x)}{F(D^2)} = \frac{Q(x)}{F(-\alpha^2)}$ Provided $F(-\alpha^2)\neq 0$.

Que $y''+4y=6\cos x$

Sol^n $(D^2+4)y=6\cos x$

A.E. is $m^2+4=0$

$\cancel{\Rightarrow m=0,-4}$ $\Rightarrow m=\pm 2i$

$y_c(x) = c_1\cos 2x+c_2\sin 2x$

$$y_p(x) = \frac{6\cos x}{D^2+4} = \frac{6\cos x}{-1+4} = \frac{6}{3}\cos x = 2\cos x$$

$$\therefore y(x) = y_c(x)+y_p(x)$$

$$= c_1\cos 2x+c_2\sin 2x+2\cos x.$$

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Qu $2y'' + y' - y = 16\cos 2x$

Sol^n $(2D^2 + D - 1)y = 16\cos 2x$

A.E is $2m^2 + m - 1 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1+8}}{2} = -1, \tfrac{1}{2}$$

$$y_C(x) = C_1 e^{-x} + C_2 e^{x/2}$$

$$y_p(x) = \frac{16\cos 2x}{2D^2 + D - 1} = \frac{16\cos 2x}{-8 + D - 1} = \frac{16\cos 2x}{D - 9} \times \frac{D+9}{D+9}$$

$$= \frac{(D+9)\, 16\cos 2x}{D^2 - 81}$$

$$= \frac{(D+9)\, 16\cos 2x}{-85}$$

$$= \frac{D(16\cos 2x) + 144\cos 2x}{-85}$$

$$= \frac{-32\sin 2x + 144\cos 2x}{-85}$$

$$= \frac{-16}{85}(9\cos 2x - 2\sin 2x)$$

$$\therefore y(x) = y_c(x) + y_p(x)$$

$$= C_1 e^{-x} + C_2 e^{x/2} - \frac{16}{85}(9\cos 2x - 2\sin 2x)$$

HW (1) $y'' - 5y' + 4y = 65\sin 2x$

Ans $y(x) = Ae^{x} + Be^{4x} + \frac{13}{2}\cos 2x.$

(2) $y''' - y'' + 4y' - 4y = \sin 3x$

Ans:- $Ae^{x} + B\cos 2x + C\sin 2x + (3\cos 3x + \sin 3x)/50.$

Qu $y'' + y = 6\sin x$

Sol^n $(D^2 + 1)y = 6\sin x$

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A.E is $m^2+1=0$

$\Rightarrow m = \pm i$

$y_c(x) = C_1 \cos x + C_2 \sin x$

$y_p(x) = \frac{6 \sin x}{D^2+1} = \frac{6 \sin x}{-1+1=0}$ → Test fails

$\frac{x \cdot 6 \sin x}{2D} = 3x \frac{1}{D}(\sin x)$

$= -3x \cos x$

Que $y'' - 4y' + 13y = 18e^{2x} \sin 3x$

Case III If $Q(x) = e^{\alpha x} h(x)$

Then $y_p(x) = \frac{1}{F(D)} [e^{\alpha x} \cdot h(x)]$

$= e^{\alpha x} \frac{1}{F(D+\alpha)} h(x)$

Ex $y'' - 4y' + 13y = 18e^{2x} \sin 3x$

Sol $(D^2 - 4D + 13)y = 18e^{2x} \sin 3x$

A.E. is

$m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{16-52}}{2}$

$= \frac{4 \pm \sqrt{-36}}{2}$

$= \frac{4 \pm 6i}{2} = 2 \pm 3i$

$y_c(x) = e^{2x}(C_1 \cos 3x + C_2 \sin 3x)$

$y_p(x) = \frac{18 e^{2x} \sin 3x}{(D^2 - 4D + 13)}$

$= 18 e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 13} \sin 3x$

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$$= 18e^{2x}\left[\frac{1}{D^2+9}\sin 3x\right]$$

$$= 18e^{2x}\left[\frac{1}{-9+9=0}\sin 3x\right. \quad \text{Put } D^2=-9$$

$\rightarrow$ Case fails

$$= \frac{18x\,e^{2x}}{2D}\sin 3x$$

$$= 9xe^{2x}\int \sin 3x\,dx$$

$$= 9xe^{2x}\left(\frac{-\cos 3x}{3}\right) = -3xe^{2x}\cos 3x$$

$$y(x) = y_c(x) + y_p(x)$$

Que $(D^2+2D+5)y = e^{-x}\cos 2x.$

Solⁿ:→ AE is $(m^2+2m+5)=0$

$$m = \frac{-2\pm\sqrt{4-20}}{2}$$

$$= \frac{-2\pm 4i}{2} = -1\pm 2i$$

$$y_c(x) = e^{-x}(C_1\cos 2x + C_2\sin 2x)$$

$$y_p(x) = \frac{e^{-x}\cos 2x}{D^2+2D+5}$$

$$= e^{-x}\frac{1}{(D-1)^2+2(D-1)+5}\cos 2x$$

$$= e^{-x}\left[\frac{1}{D^2+4}\cos 2x\right]$$

$$= e^{-x}\left[\frac{1}{-4+4=0}\cos 2x\right] \quad \text{Put } D^2=-4$$

$\rightarrow$ Case fails

$$\text{So } y_p(x) = \frac{e^{-x}\cdot x}{2D}\cos 2x$$

$$= \frac{e^{-x}x}{2}\left[\int \cos 2x\,dx\right] = \frac{xe^{-x}\sin 2x}{4}$$

$\therefore\ y(x) = y_c(x) + y_p(x)$

$\Rightarrow y(x) = e^{-x}(C_1\cos 2x + C_2\sin 2x) + \dfrac{xe^{-x}\sin 2x}{4}$

HW (1) $(2D^2 - 7D + 3)y = \sin 2x.$

(2) $(D^2 - 4D + 5)y = 24e^{2x}\sin x.$

(3) $(D^2 + 4D + 3)y = e^{2x}\cos x.$

Case IV If ~~Ele~~ $Q(x) = x^{\alpha}$, $\alpha > 0$ and Integer.

Then $y_p(x) = \dfrac{1}{F(D)} \cdot x^{\alpha}$.

$\rightarrow$ Then Expand $[F(D)]^{-1}$ in ascending powers of D.

Que $\rightarrow$ $y'' + 16y = 64x^2$

Sol$^n \rightarrow$ $(D^2 + 16)y = 64x^2$

AE is $m^2 + 16 = 0$

$\Rightarrow m = \pm 4i$

$y_c(x) = C_1\cos 4x + C_2\sin 4x.$

$$y_p(x) = \frac{64x^2}{D^2+16}$$

$$= \frac{64}{16}\left[1 + \frac{D^2}{16}\right]^{-1} x^2$$

$$= 4\left[1 - \frac{D^2}{16}\right] x^2$$

$$= 4\left[x^2 - \frac{1}{16}(2)\right] = 4x^2 - \frac{1}{2}.$$

$y(x) = C_1\cos 4x + C_2\sin 4x + 4x^2 - \dfrac{1}{2}.$

Que $(D^2+25)y = 9x^3$

Sol^n ⇒ AE is $m^2+25=0$

$\Rightarrow m = \pm 5i$

$y_c(x) = C_1\cos 5x + C_2 \sin 5x.$

$$y_p(x) = \frac{9x^3}{D^2+25} = \frac{1}{25}\left[1+\frac{D^2}{25}\right]^{-1} 9x^3$$

$$= \frac{1}{25}\left[1 - \frac{D^2}{25} + \frac{(-1)(-2)}{\lfloor 2}\cdot\left(\frac{D^2}{25}\right)^2 + \dots\right] 9x$$

$$= \frac{1}{25}\left[9x^3 - \frac{1}{25}(54x)\right]$$

$$= \frac{225x^3 - 54x}{625}$$

$y(x) = y_c(x) + y_p(x)$

$$= C_1\cos 5x + C_2 \sin 5x + \frac{225x^3-54x}{625}.$$