

16.

## LAPLACE TRANSFORM OF SOME

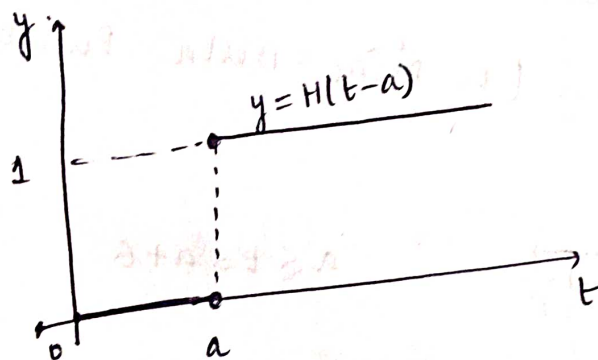
①

SPECIAL FUNCTIONS.(1) Heaviside Unit Step function

An important discontinuous function that finds important application in connection with Laplace transform

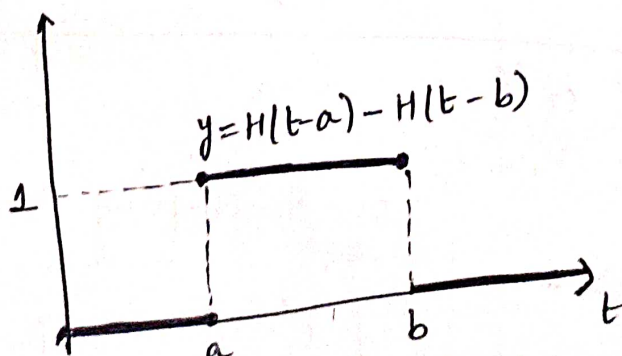
is Unit step function  $H(t-a)$  or  $U(t-a)$  with  $a \geq 0$

It is defined as  $H(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

(2) Unit Pulse Function

It is defined as  $y = H(t-a) - H(t-b)$  is defined as

$$H(t-a) - H(t-b) = \begin{cases} 0 & t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases} \quad (b > a)$$



## Laplace Transform

$$(1) \quad L(H(t-a)) = \int_a^{\infty} e^{-st} dt = \frac{e^{-as}}{s} \quad \text{for } s > a > 0$$

$$(2) \quad L(H(t-a) - H(t-b)) = \left( \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} \right)$$

$$(3) \quad \text{Second Shifting Theorem, } L[f(t-a)u(t-a)] = e^{-as} L[f(t)]$$

If  $L(f(t)) = \bar{f}(s)$  then  $L(f(t-a)H(t-a)) = e^{-as} \bar{f}(s) = e^{-as} L(f(t))$

(3) Unit Impulse Function (or Dirac-Delta Function)

It is defined as

$$\delta(t-a) = \begin{cases} \frac{1}{\epsilon} \\ 0 \end{cases}$$

$$a \leq t \leq a+\epsilon$$

otherwise

$$(4) \quad L(\delta(t-a)) = e^{-as}$$

In particular for  $a=0$   $L(\delta(t)) = 1$ .

$$(5) \quad L(f(t)\delta(t-a)) = e^{-as} f(a)$$

Example To find  $L^{-1}\left(\frac{se^{-3s}}{s^2+4}\right)$ .

We know  $L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$

By II shifting theorem,

$$L^{-1}\left(\frac{e^{-3s}s}{s^2+4}\right) = \cos 2(t-3)H(t-3)$$

Ex 1. Find the Laplace transform of the function (3)

$$f(t) = \begin{cases} t-1 & , 1 < t < 2 \\ 3-t & , 2 < t < 3 \end{cases}$$

Solution

$$f(t) = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

$$\therefore L[f(t)] = L[(t-1)u(t-1)] - 2L[(t-2)u(t-2)] + L[(t-3)u(t-3)]$$

$$= e^{-s}L(t) - 2e^{-2s}L(t) + e^{-3s}L(t)$$

$$= \left( \frac{-s}{e} - 2\frac{-2s}{e} + \frac{-3s}{e} \right) L(t)$$

$$= \left( \frac{-s}{e} - 2\frac{-2s}{e} + \frac{-3s}{e} \right) \cdot \frac{1!}{s^2}$$

$$= \frac{-s}{s^2} (1 - 2e^{-s} + e^{-2s}) = \frac{-e^{-s}}{s^2} (1 - e^{-s})^2$$

---

Ex 2 Express the function  $f(t) = \begin{cases} 2t & 0 < t < 5 \\ 10 & t > 5 \end{cases}$

in terms of unit step function and find its Laplace transform.

Solution

$$f(t) = 2t[u(t) - u(t-5)] + 10[u(t-5)]$$

$$= 2tu(t) - 2(t-5)u(t-5)$$

$$\therefore L(f(t)) = 2L(tu(t)) - 2L((t-5)u(t-5))$$

$$= 2e^{-0s}L(t) - 2e^{-5s}L(t)$$

$$= 2(1 - e^{-5s})L(t) = \frac{2(1 - e^{-5s})}{s^2}$$

Ex 3 Find the Laplace transform of

(1)  $t^2 u(t-3)$

$$t^2 = [t-3+3]^2 = (t-3)^2 + 9 + 6(t-3)$$

(2)  $e^{-3t} u(t-2)$

we have to write  $t^2$  in terms of  $(t-3)$ .

Solution

(1)  $f(t) = t^2 u(t-3)$

$$= ((t-3)^2 + 6t - 9) u(t-3)$$

$$= ((t-3)^2 + 6(t-3) + 9) u(t-3)$$

$$\therefore L(f(t)) = L((t-3)^2 u(t-3)) + 6L((t-3)u(t-3)) + 9L(u(t-3))$$

$$= \frac{e^{-3s}}{s^3} L(t^2) + 6 \frac{e^{-3s}}{s^2} L(t) + \frac{9e^{-3s}}{s}$$

$$= \frac{e^{-3s}}{s^3} \cdot \frac{2!}{s^3} + 6 \frac{e^{-3s}}{s^2} \cdot \frac{1!}{s^2} + \frac{9e^{-3s}}{s}$$



$$= e^{-3s} \left[ \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right] .$$

(5)

(2)

$$f(t) = e^{-3t} V(t-2)$$

(writing  $e^{-3t}$  in term of  $t-2$ )

$$= \left( e^{-3(t-2)} \times e^{-6} \right) V(t-2)$$

$$= e^{-6} \left( e^{-3(t-2)} V(t-2) \right)$$

$$\therefore L[f(t)] = e^{-6} L \left( e^{-3(t-2)} V(t-2) \right)$$

$$= e^{-6} e^{-2s} \cdot L(e^{-3t})$$

$$= \frac{e^{-6-2s}}{s+3} = \frac{e^{-2(s+3)}}{s+3}.$$

Ex 4

Find the inverse Laplace transform of

(1)  $\frac{e^{-\pi s}}{s^2+4}$

(2)  $\frac{(3s+1)e^{-3s}}{s^2(s^2+4)}$

Solution

(1) we know  $L^{-1} \left( \frac{1}{s^2+4} \right) = \frac{1}{2} \sin 2t$

therefore,  $L^{-1} \left( \frac{e^{-\pi s}}{s^2+4} \right) = \frac{1}{2} \sin 2(t-\pi) V(t-\pi)$

(second shifting theorem)

$$= \begin{cases} 0 & 0 < t < \pi \\ \frac{1}{2} \sin 2t & t > \pi \end{cases}$$

$$(2) \quad \text{let } \bar{f}(s) = \frac{3s+1}{s^2(s^2+4)} = \frac{3s+1}{4} \left[ \frac{1}{s^2} - \frac{1}{s^2+4} \right] \quad (6)$$

$$= \frac{3s+1}{4s^2} - \frac{3s+1}{4(s^2+4)}$$

$$\therefore \bar{f}(s) = \frac{3}{4s} + \frac{1}{4s^2} - \frac{3s}{4(s^2+4)} - \frac{1}{4(s^2+4)}$$

Then  $f(t) = \mathcal{L}^{-1}(\bar{f}(s)) = \frac{3}{4} \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{4} \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$

$$- \frac{3}{4} \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \frac{1}{4} \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)$$

$$f(t) = \frac{3}{4}(1) + \frac{1}{4}t - \frac{3}{4} \cos 2t - \frac{\sin 2t}{8}$$

Now,  $\mathcal{L}(f(t-a)U(t-a)) = e^{-as} \bar{f}(s) = e^{-as} \mathcal{L}(f(t))$

$$\therefore \mathcal{L}^{-1}\left(\frac{(3s+1)e^{-3s}}{s^2(s^2+4)}\right) = f(t-a)U(t-a) \quad (a=3)$$

$$= \left( \frac{3+t-3}{4} - \frac{3}{4} \cos 2(t-3) - \frac{\sin 2(t-3)}{8} \right) U(t-3)$$

$$= \frac{1}{8} [2t - \sin 2(t-3) - 6 \cos 2(t-3)] U(t-3).$$


---

Q5 Solve  $\frac{d^2 y}{dx^2} + 4y = U(x-2)$ ,  $y(0)=0$ ,  $y'(0)=1$ . (7)

Solution Taking Laplace transform on both sides

$$s^2 \bar{y}(s) - s y(0) - y'(0) + 4 \bar{y}(s) = \frac{e^{-2s}}{s}$$

$$\therefore \bar{y}(s) (s^2 + 4) = \frac{e^{-2s}}{s} + 1$$

$$\Rightarrow \bar{y}(s) = \frac{e^{-2s}}{s(s^2 + 4)} + \frac{1}{s^2 + 4}$$

Therefore  $L(y(x)) = \frac{e^{-2s}}{s(s^2 + 4)} + \frac{1}{s^2 + 4}$

$$\Rightarrow y(t) = \bar{L}^{-1} \left( \frac{e^{-2s}}{s(s^2 + 4)} \right) + \bar{L}^{-1} \left( \frac{1}{s^2 + 4} \right)$$

$$= \bar{L}^{-1} \left( \frac{e^{-2s}}{s(s^2 + 4)} \right) + \frac{1}{2} \sin 2t$$

Now  $\bar{L}^{-1} \left( \frac{1}{s(s^2 + 4)} \right) = \int_0^t \frac{1}{2} \sin 2t \, dt = \left[ -\frac{\cos 2t}{4} \right]_0^t$

$$= \frac{1 - \cos 2t}{4} = \frac{\sin^2 t}{2}$$

Since  $L(f(t-a)U(t-a)) = \bar{f}(s)e^{-as}$ , we have

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s(s^2+4)}\right) = \frac{1}{2} \sin^2(t-2) U(t-2)$$

Thus,  $y = \frac{1}{2} \sin 2t + \frac{1}{2} \sin^2(t-2) U(t-2).$

Ex Solve the initial value problem

$$y'' + 3y' + 2y = H(t-\pi) \sin 2t, \quad y(0) = 1, \quad y'(0) = 0$$

Solution Taking Laplace transform on both sides, we get

$$s^2 \bar{y}(s) - sy(0) - y'(0) + 3(s\bar{y}(s) - y(0)) + 2\bar{y}(s) = \mathcal{L}(H(t-\pi) \sin 2t)$$

$$= \mathcal{L}(H(t-\pi) \sin 2t)$$

$$\Rightarrow (s^2 + 3s + 2) \bar{y}(s) - s - 3 = \frac{2e^{-\pi s}}{s^2 + 4}$$

$$\Rightarrow \bar{y}(s) = \frac{s+3}{s^2+3s+2} + \frac{2e^{-\pi s}}{(s^2+3s+2)(s^2+4)}$$

$$= \frac{2}{s+1} - \frac{1}{s+2} + e^{-\pi s} \left[ \frac{2}{s(s+1)} - \frac{1}{4(s+2)} - \frac{1}{20} \left( \frac{2}{s^2+4} \right) \right]$$

$$\Rightarrow \bar{y}(s) = 2\mathcal{L}(e^{-t}) - \mathcal{L}(e^{-2t}) + \frac{2}{5} \left[ \frac{e^{-(t-\pi)}}{s} H(t-\pi) - \frac{1}{4} e^{-\frac{3}{4}(t-\pi)} H(t-\pi) \right]$$



$$-\frac{1}{20} \sin 2(t-\pi) H(t-\pi) - \frac{3}{20} \cos 2(t-\pi) H(t-\pi) \quad (9)$$

Ex. Find  $L(tV(t-4) - t^3 \delta(t-2))$

Solution

$$\begin{aligned} L(tV(t-4)) &= L((t-4+4)V(t-4)) \\ &= L((t-4)V(t-4) + 4V(t-4)) \\ &= e^{-4s} L(t) + 4e^{-4s} L(1) \\ &= e^{-4s} \left( \frac{1}{s^2} + \frac{4}{s} \right). \end{aligned}$$

Again,  $L(t^3 \delta(t-2)) = e^{-2s} (2)^3 = 8e^{-2s}.$

Therefore,  $L(tV(t-4) - t^3 \delta(t-2)) = e^{-4s} \left( \frac{1}{s^2} + \frac{4}{s} \right) - 8e^{-2s}.$

Exercise 16B

Ex. Find the Laplace transform of function of

$$e^{-t} (1 - V(t-2)).$$

Solution

$$\begin{aligned} &L(e^{-t} (1 - V(t-2))) \\ &= L(e^{-t}) - L(e^{-t} V(t-2)) \\ &= L(e^{-t}) - L(e^{-(t-2)} * e^{-4} V(t-2)) \\ &= \frac{1}{s+1} - e^{-4} L(e^{-(t-2)} V(t-2)) \end{aligned}$$

$$= \frac{1}{s+1} - e^{-4} \left( \frac{e^{-2s}}{s+1} \right) L(e^{-t})$$

(10)

$$= \frac{1}{s+1} - \frac{e^{-2s-4}}{s+1} = \frac{1-e^{-2(s+2)}}{s+1}$$

Ex 4 find the inverse Laplace transform of

$$(1) \frac{s e^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$$

Solution  $L^{-1} \left( \frac{s e^{-s/2} + \pi e^{-s}}{s^2 + \pi^2} \right)$

$$= L^{-1} \left( \frac{s e^{-s/2}}{s^2 + \pi^2} \right) + \pi L^{-1} \left( \frac{e^{-s}}{s^2 + \pi^2} \right)$$

We know,  $L^{-1} \left( \frac{s}{s^2 + \pi^2} \right) = \cos \pi t$

By second shifting theorem,

$$L^{-1} \left( \frac{e^{-s/2} s}{s^2 + \pi^2} \right) = \cos \left( \pi \left( t - \frac{1}{2} \right) \right) H \left( t - \frac{1}{2} \right)$$

$$= \cos \left( \pi t - \pi/2 \right) H(t - 1/2)$$

$$= \cos \left( \pi/2 - \pi t \right) H(t - 1/2)$$

$$= \sin \pi t H(t - 1/2)$$

(11)

$$\begin{aligned}
 L^{-1}\left(\frac{\bar{e}^s}{s^2 + \pi^2}\right) &= \frac{1}{\pi} \sin \pi(t-1) H(t-1) \\
 &= \frac{1}{\pi} \sin(\pi t - \pi) H(t-1) \\
 &= -\frac{1}{\pi} \sin \pi t H(t-1) .
 \end{aligned}$$

$$\begin{aligned}
 \therefore L^{-1}\left(\frac{s \bar{e}^{-s/2}}{s^2 + \pi^2}\right) + \pi L^{-1}\left(\frac{\bar{e}^s}{s^2 + \pi^2}\right) \\
 = \sin \pi t H(t-1/2) - \frac{\pi}{\pi} \sin \pi t H(t-1) \\
 = \sin \pi t \left( H(t-1/2) - H(t-1) \right) .
 \end{aligned}$$


---

$$(2) \quad L^{-1}\left(\frac{\bar{e}^{-s} - 3\bar{e}^{-3s}}{s^2}\right)$$

Solution  $L^{-1}\left(\frac{\bar{e}^{-s}}{s^2}\right) - 3 L^{-1}\left(\frac{\bar{e}^{-3s}}{s^2}\right)$

Now  $L^{-1}\left(\frac{1}{s^2}\right) = t$  ~~and t~~

Using second shifting theorem,

$$L^{-1}\left(\frac{\bar{e}^s}{s^2}\right) = (t-1) H(t-1)$$

$$L^{-1}\left(\frac{\bar{e}^{-3s}}{s^2}\right) = (t-3) H(t-3) .$$

$$\therefore (t-1) H(t-1) - 3(t-3) H(t-3) .$$

Ex Find the inverse Laplace transform of (12).

$$\left( \frac{1-\sqrt{s}}{s^{3/2}} \right) e^{-s}.$$

Solution

$$\mathcal{L}^{-1} \left( \frac{e^{-s}}{s^{3/2}} \right) = \mathcal{L}^{-1} \left( \frac{e^{-s}}{s} \right)$$

$$\text{Now } \mathcal{L}^{-1} \left( \frac{1}{s^{3/2}} \right) = \frac{t^{1/2}}{\Gamma_{1/2}} = \frac{\sqrt{t}}{\sqrt{\pi/2}}$$

$$\mathcal{L}^{-1} \left( \frac{1}{s} \right) = 1$$

Using second shifting theorem

$$\mathcal{L}^{-1} \left( \frac{e^{-s}}{s^{3/2}} \right) = \sqrt{\frac{2}{\pi}} \cdot \sqrt{t-1} \, H(t-1)$$

$$\text{and } \mathcal{L}^{-1} \left( \frac{e^{-s}}{s} \right) = H(t-1).$$

$$\text{Thus, } \mathcal{L}^{-1} \left( \left( \frac{1-\sqrt{s}}{s^{3/2}} \right) e^{-s} \right) = \left( \sqrt{\frac{2}{\pi}} \cdot \sqrt{t-1} - 1 \right) H(t-1).$$

---