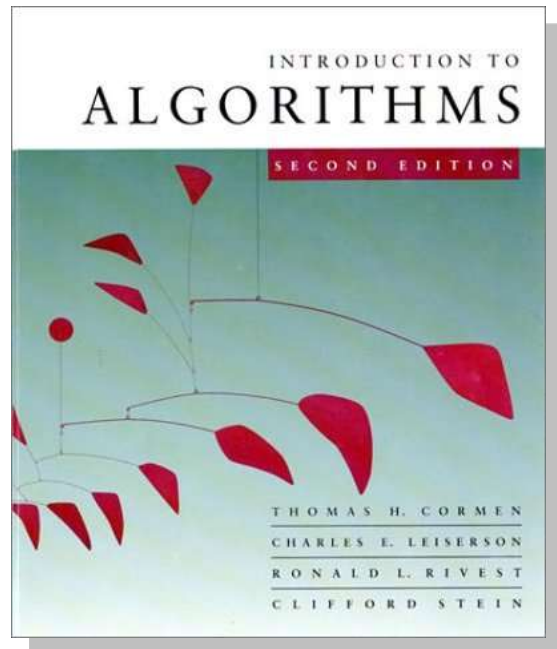


Introduction to Algorithms

6.046J/18.401J

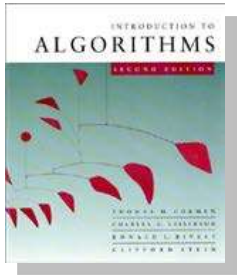


LECTURE 3

Divide and Conquer

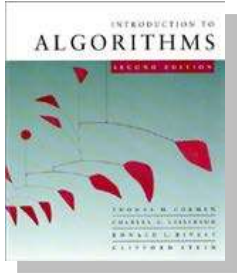
- Binary search
- Powering a number
- Fibonacci numbers
- Matrix multiplication
- Strassen's algorithm
- VLSI tree layout

Prof. Erik D. Demaine



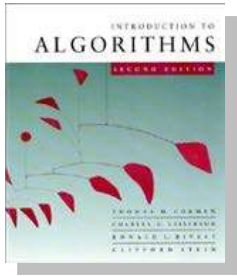
The divide-and-conquer design paradigm

1. *Divide* the problem (instance) into subproblems.
2. *Conquer* the subproblems by solving them recursively.
3. *Combine* subproblem solutions.



Merge sort

1. *Divide*: Trivial.
2. *Conquer*: Recursively sort 2 subarrays.
3. *Combine*: Linear-time merge.



Merge sort

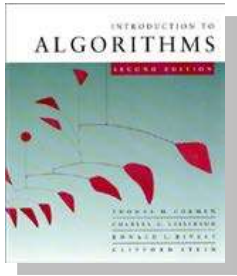
1. **Divide:** Trivial.
2. **Conquer:** Recursively sort 2 subarrays.
3. **Combine:** Linear-time merge.

$$T(n) = 2T(n/2) + \Theta(n)$$

subproblems

subproblem size

work dividing and combining



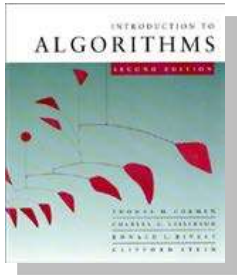
Master theorem (reprise)

$$T(n) = a T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$, constant $\varepsilon > 0$
 $\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, constant $k \geq 0$
 $\Rightarrow T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$, constant $\varepsilon > 0$,
and regularity condition
 $\Rightarrow T(n) = \Theta(f(n))$.



Master theorem (reprise)

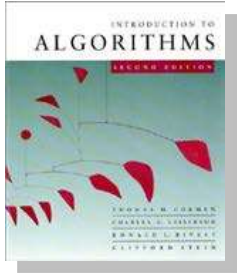
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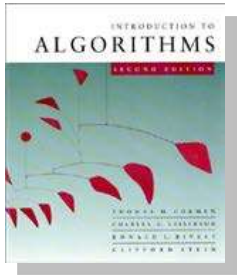
Merge sort: $a = 2, b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 2} = n$
 \Rightarrow **CASE 2** ($k = 0$) $\Rightarrow T(n) = \Theta(n \lg n)$.



Binary search

Find an element in a sorted array:

- 1. *Divide:*** Check middle element.
- 2. *Conquer:*** Recursively search **1** subarray.
- 3. *Combine:*** Trivial.



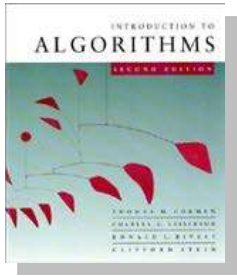
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Example: Find **9**

3 5 7 8 9 12 15



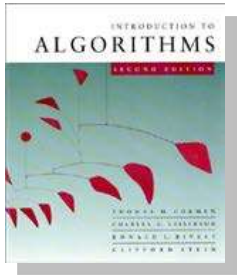
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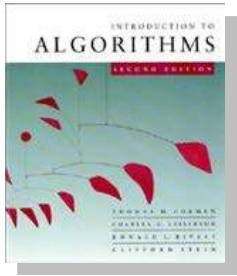
7

8

9

12

15



Binary search

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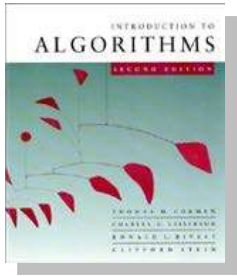
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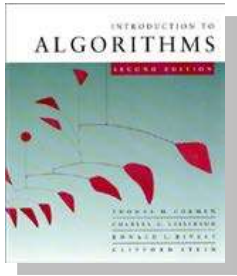
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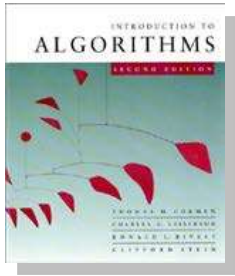
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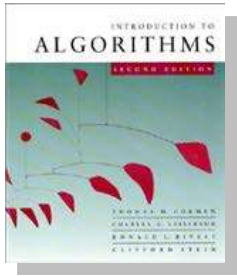
Recurrence for binary search

$$T(n) = 1 T(n/2) + \Theta(1)$$

subproblems

subproblem size

*work dividing
and combining*

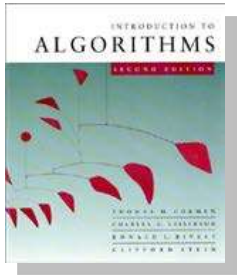


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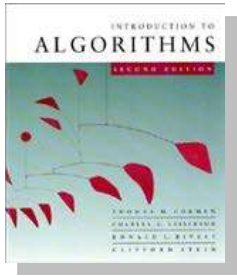
$$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow \text{CASE 2 } (k = 0) \\ \Rightarrow T(n) = \Theta(\lg n) .$$



Powering a number

Problem: Compute a^n , where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.



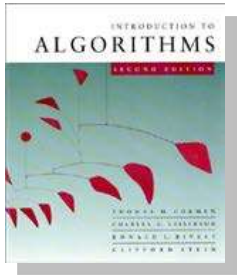
Powering a number

Problem: Compute a^n , where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$



Powering a number

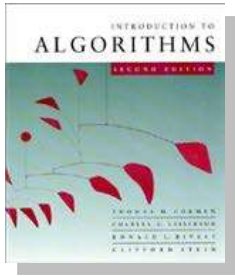
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$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\lg n) .$$

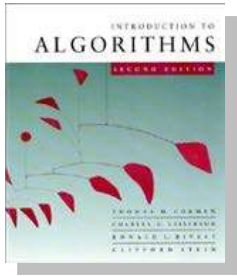


Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

0 1 1 2 3 5 8 13 21 34 ...



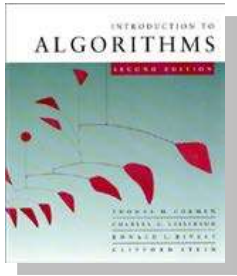
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0 1 1 2 3 5 8 13 21 34 ...

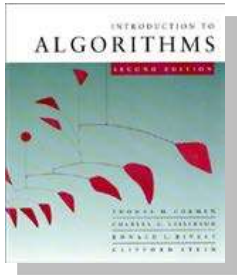
Naive recursive algorithm: $\Omega(\phi^n)$
(exponential time), where $\phi = (1 + \sqrt{5})/2$
is the *golden ratio*.



Computing Fibonacci numbers

Bottom-up:

- Compute $F_0, F_1, F_2, \dots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.



Computing Fibonacci numbers

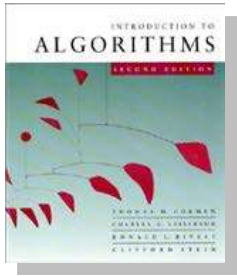
Bottom-up:

- Compute $F_0, F_1, F_2, \dots, F_n$ in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.

Naive recursive squaring:

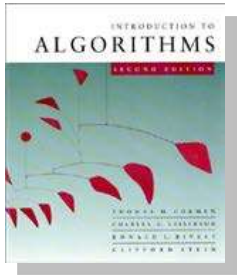
$F_n = \phi^n / \sqrt{5}$ rounded to the nearest integer.

- Recursive squaring: $\Theta(\lg n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.



Recursive squaring

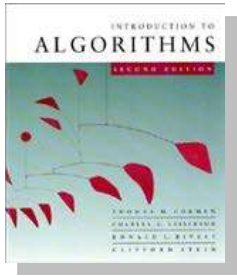
Theorem:
$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n.$$



Recursive squaring

Theorem:
$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n .$$

Algorithm: Recursive squaring.
Time = $\Theta(\lg n)$.



Recursive squaring

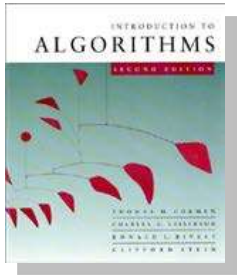
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Algorithm: Recursive squaring.

Time = $\Theta(\lg n)$.

Proof of theorem. (Induction on n .)

Base ($n = 1$):
$$\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1 .$$

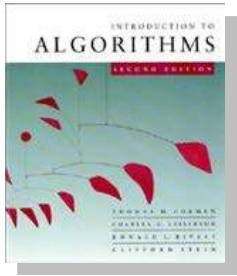


Recursive squaring

Inductive step ($n \geq 2$):

$$\begin{aligned} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} &= \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \end{aligned}$$

■

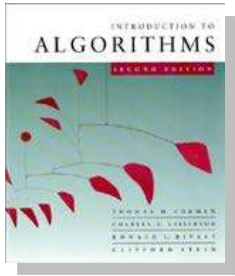


Matrix multiplication

Input: $A = [a_{ij}], B = [b_{ij}].$
Output: $C = [c_{ij}] = A \cdot B.$ $\left. \vphantom{\begin{matrix} A \\ B \\ C \end{matrix}} \right\} i, j = 1, 2, \dots, n.$

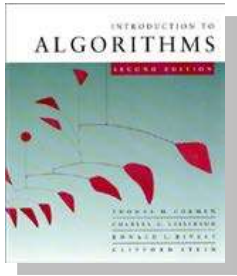
$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$$



Standard algorithm

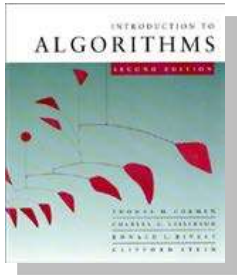
```
for  $i \leftarrow 1$  to  $n$ 
  do for  $j \leftarrow 1$  to  $n$ 
    do  $c_{ij} \leftarrow 0$ 
      for  $k \leftarrow 1$  to  $n$ 
        do  $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$ 
```



Standard algorithm

```
for  $i \leftarrow 1$  to  $n$ 
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```

Running time = $\Theta(n^3)$



Divide-and-conquer algorithm

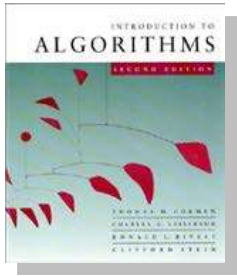
IDEA:

$n \times n$ matrix = 2×2 matrix of $(n/2) \times (n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C = A \cdot B$$

$$\left. \begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dg \\ u = cf + dh \end{array} \right\} \begin{array}{l} 8 \text{ mults of } (n/2) \times (n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} \end{array}$$



Divide-and-conquer algorithm

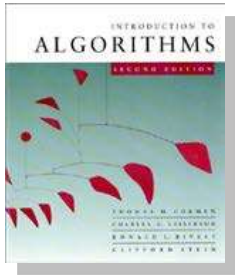
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$$C = A \cdot B$$

$$\left. \begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dh \\ u = cf + dg \end{array} \right\} \begin{array}{l} \text{recursive} \\ 8 \text{ mults of } (n/2) \times (n/2) \text{ submatrices} \\ 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} \end{array}$$



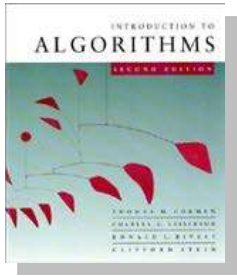
Analysis of D&C algorithm

$$T(n) = 8T(n/2) + \Theta(n^2)$$

submatrices

submatrix size

work adding
submatrices



Analysis of D&C algorithm

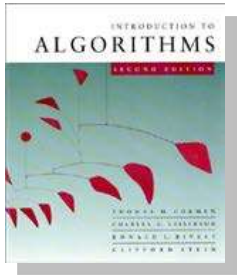
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$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$



Analysis of D&C algorithm

$$T(n) = 8T(n/2) + \Theta(n^2)$$

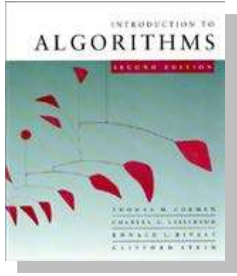
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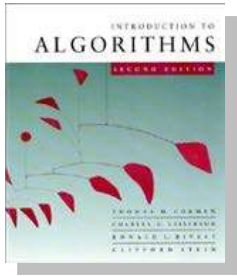
$$n^{\log_b a} = n^{\log_2 8} = n^3 \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$

$$P_2 = (a + b) \cdot h$$

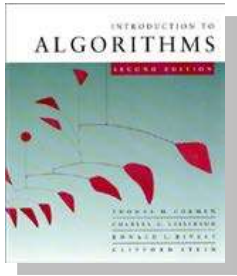
$$P_3 = (c + d) \cdot e$$

$$P_4 = d \cdot (g - e)$$

$$P_5 = (a + d) \cdot (e + h)$$

$$P_6 = (b - d) \cdot (g + h)$$

$$P_7 = (a - c) \cdot (e + f)$$



Strassen's idea

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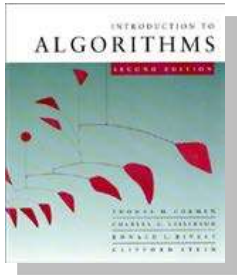
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$

$$s = P_1 + P_2$$

$$t = P_3 + P_4$$

$$u = P_5 + P_1 - P_3 - P_7$$



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$

$$P_2 = (a + b) \cdot h$$

$$P_3 = (c + d) \cdot e$$

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$$P_6 = (b - d) \cdot (g + h)$$

$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$

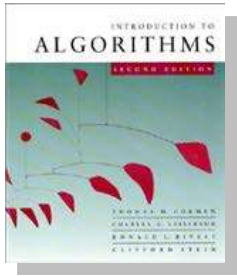
$$s = P_1 + P_2$$

$$t = P_3 + P_4$$

$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.

Note: No reliance on commutativity of mult!



Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$

$$P_2 = (a + b) \cdot h$$

$$P_3 = (c + d) \cdot e$$

$$P_4 = d \cdot (g - e)$$

$$P_5 = (a + d) \cdot (e + h)$$

$$P_6 = (b - d) \cdot (g + h)$$

$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$

$$= (a + d)(e + h)$$

$$+ d(g - e) - (a + b)h$$

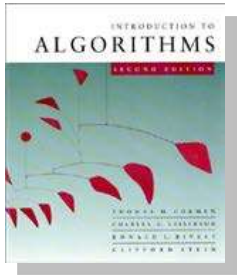
$$+ (b - d)(g + h)$$

$$= ae + ah + de + dh$$

$$+ dg - de - ah - bh$$

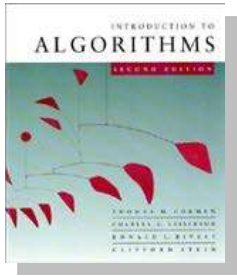
$$+ bg + bh - dg - dh$$

$$= ae + bg$$



Strassen's algorithm

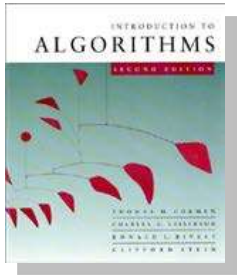
1. **Divide:** Partition A and B into $(n/2) \times (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
2. **Conquer:** Perform 7 multiplications of $(n/2) \times (n/2)$ submatrices recursively.
3. **Combine:** Form C using $+$ and $-$ on $(n/2) \times (n/2)$ submatrices.



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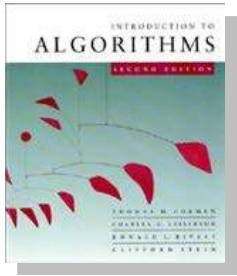
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$$T(n) = 7 T(n/2) + \Theta(n^2)$$



Analysis of Strassen

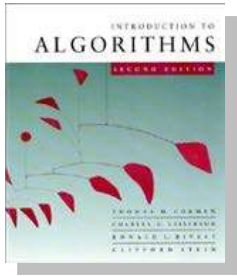
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Analysis of Strassen

$$T(n) = 7 T(n/2) + \Theta(n^2)$$

$$n^{\log_2 7} = n^{\log_2 7} \approx n^{2.81} \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^{\lg 7}).$$

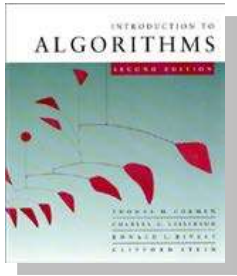


Analysis of Strassen

$$T(n) = 7 T(n/2) + \Theta(n^2)$$

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \Rightarrow \text{CASE 1} \Rightarrow T(n) = \Theta(n^{\lg 7}).$$

The number **2.81** may not seem much smaller than **3**, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 32$ or so.



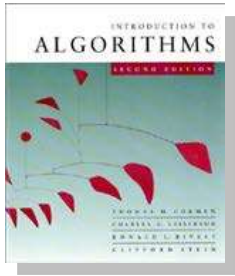
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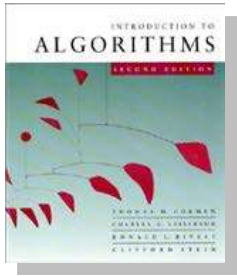
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Best to date (of theoretical interest only): $\Theta(n^{2.376\dots})$.



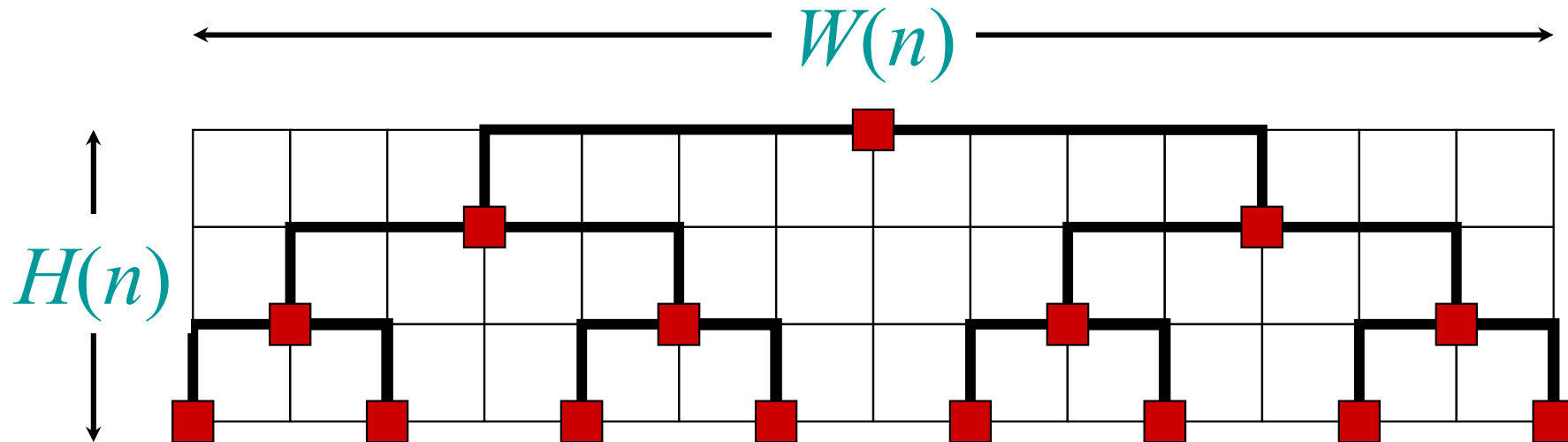
VLSI layout

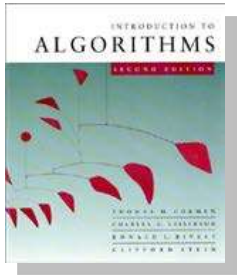
Problem: Embed a complete binary tree with n leaves in a grid using minimal area.



VLSI layout

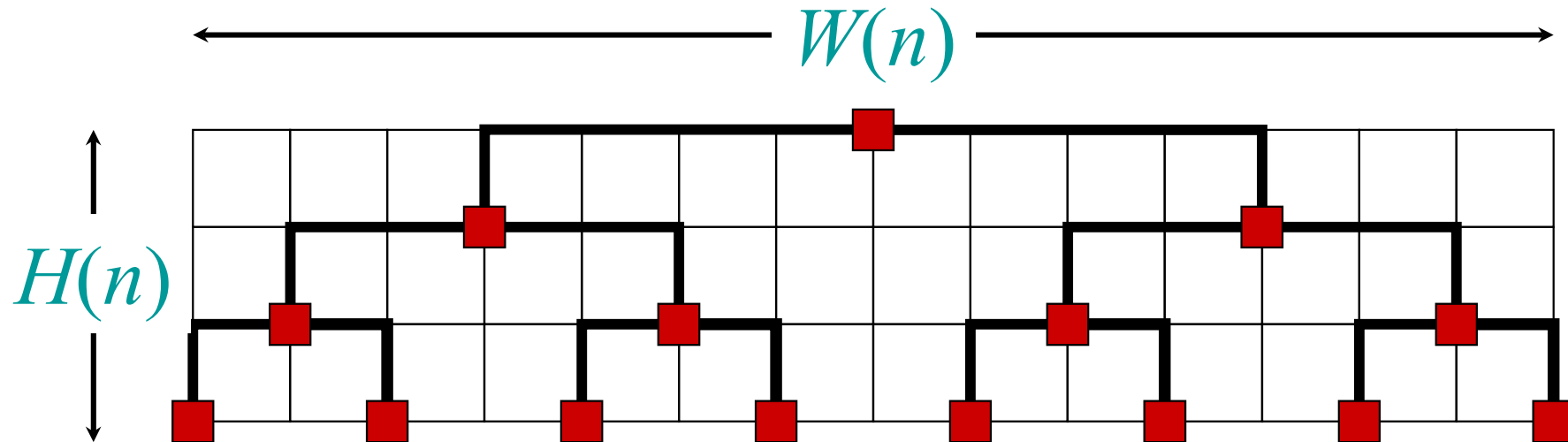
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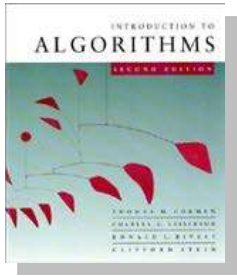


VLSI layout

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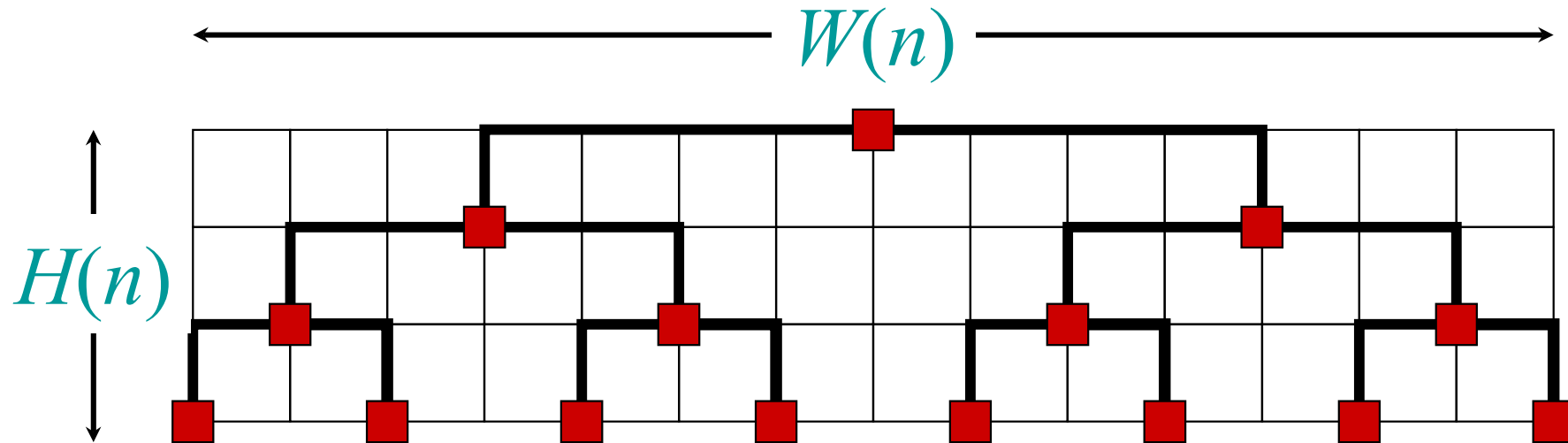


$$\begin{aligned} H(n) &= H(n/2) + \Theta(1) \\ &= \Theta(\lg n) \end{aligned}$$



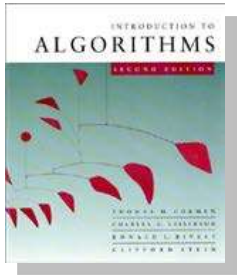
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Problem: Embed a complete binary tree with n leaves in a grid using minimal area.



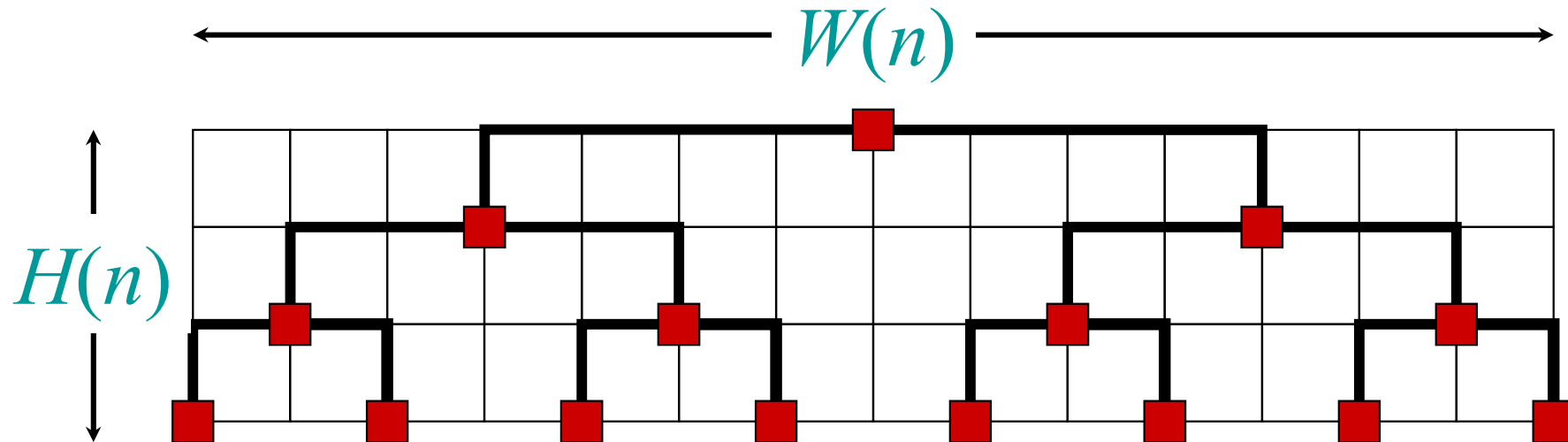
$$\begin{aligned} H(n) &= H(n/2) + \Theta(1) \\ &= \Theta(\lg n) \end{aligned}$$

$$\begin{aligned} W(n) &= 2 W(n/2) + \Theta(1) \\ &= \Theta(n) \end{aligned}$$



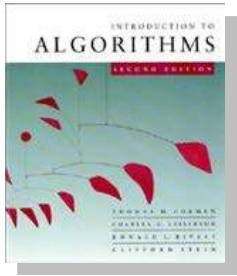
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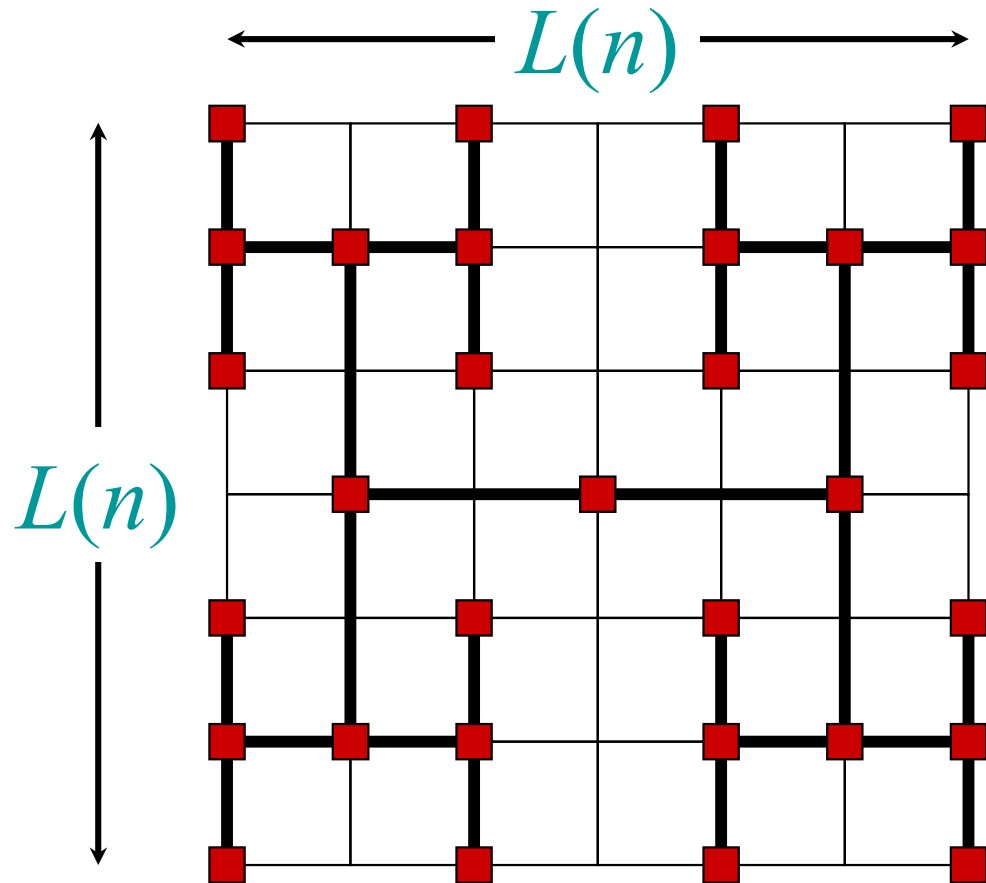


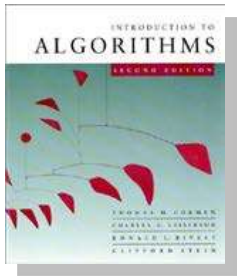
$$\begin{aligned}
 H(n) &= H(n/2) + \Theta(1) & W(n) &= 2W(n/2) + \Theta(1) \\
 &= \Theta(\lg n) & &= \Theta(n)
 \end{aligned}$$

$$\text{Area} = \Theta(n \lg n)$$

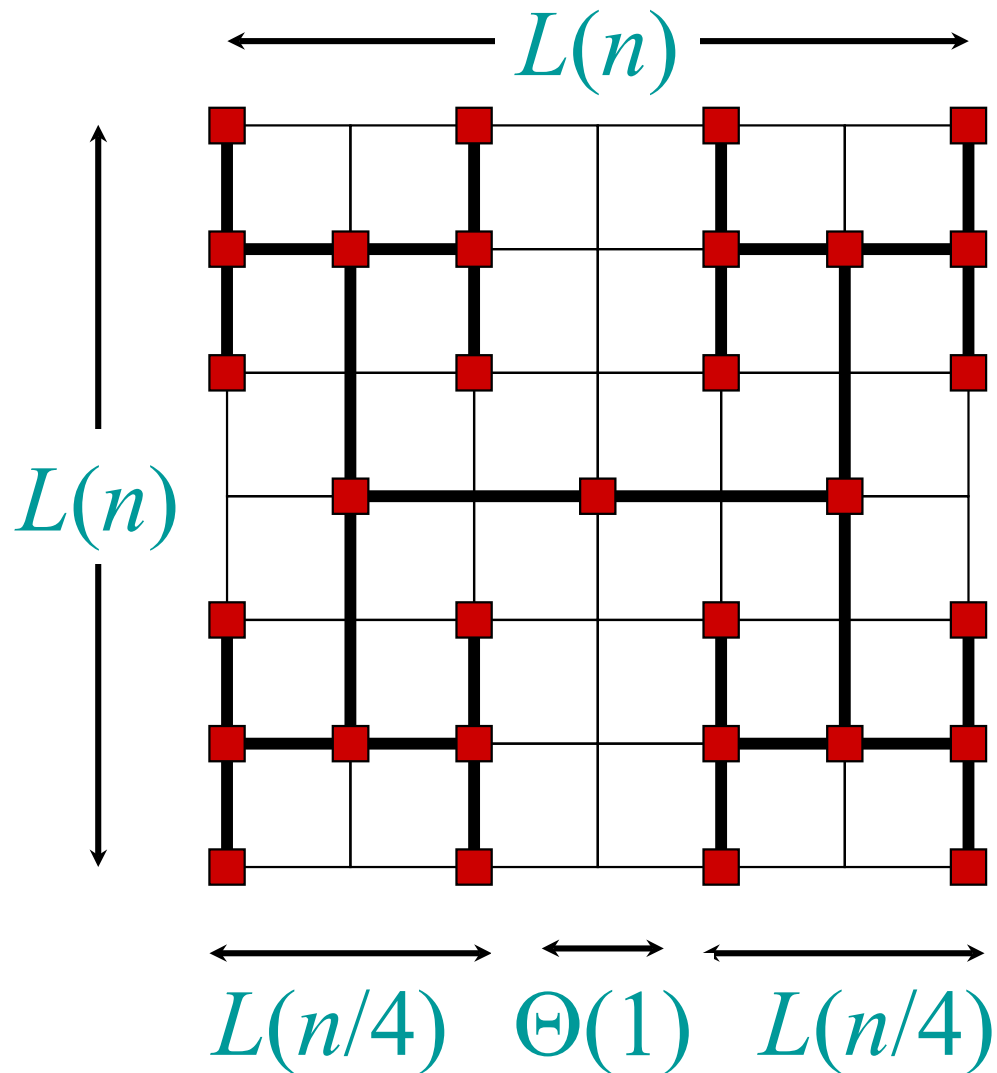


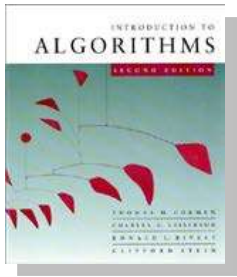
H-tree embedding



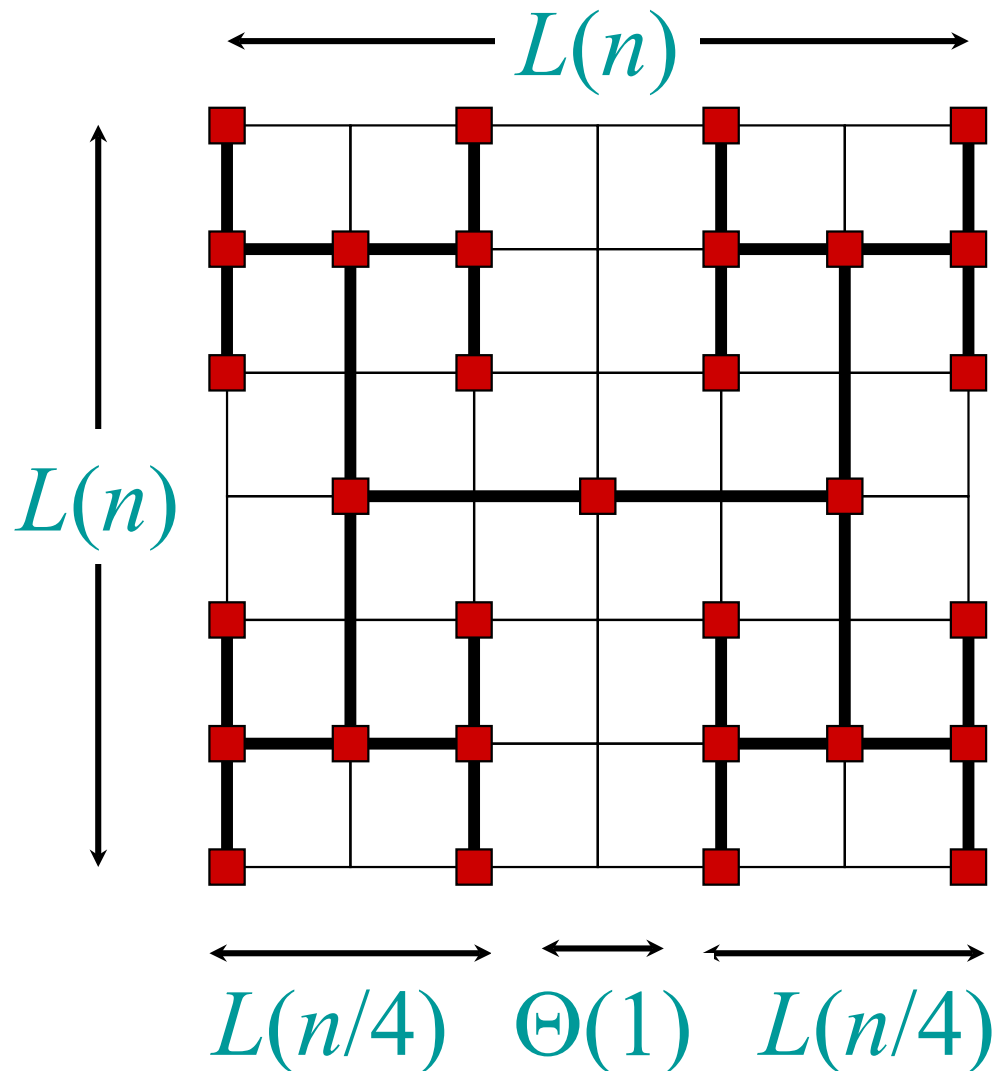


H-tree embedding



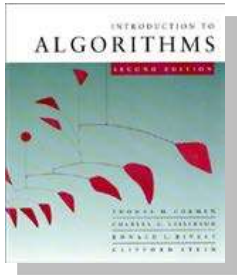


H-tree embedding



$$\begin{aligned} L(n) &= 2L(n/4) + \Theta(1) \\ &= \Theta(\sqrt{n}) \end{aligned}$$

$$\text{Area} = \Theta(n)$$



Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- The divide-and-conquer strategy often leads to efficient algorithms.