(b) $V = \mathbb{R}^2$

$W_1 = \{(a_1, 0) : a_1 \in \mathbb{R}\}$

Give examples of 2 diff subspaces W_2 and W_2' such that
$V = W_1 \oplus W_2$
& $V = W_1 \oplus W_2'$

<u>Solution</u>

(a) Let $W_2 = \{(0, a_2) ; a_2 \in \mathbb{R}\}$
W_2 is a subspace as $(0,0) \in W_2$,
Also, $w_1 + w_2 \in W_2$ $[(0, a_2) + (0 + a_2') = (0, a_2 + a_2')$.

Now claim : $V = W_1 \oplus W_2$ ① $V = W_1 + W_2$
(2) $W_1 \cap W_2 \{0\}$.

(1) $V = W_2 + W1$.
$(0, a_2) + (a_1, 0) = (a_1, a_2) \in V = V$;

(2) $W_1 \cap W_2 = \{0\}$
Let $(x, y) \in W_1 \cap W_2$
$(x,y) \in W_1$ and $(x,y) \in W_2$. $\Rightarrow x = 0, y = 0$; $(0,0) \in W_1 \cap W_2$

(b) $W_2' = \{(t, t) : t \in \mathbb{R}\}$
Clearly, W_2' is subspace $(0,0) \in W_2'$

Now ① $V = W_2' + W_1$

$\alpha(a_1, 0) + \beta(t, t) = (a_1, a_3)$
$(\alpha a_1 + \beta t, \beta t) = (a_1, a_3)$.

$\alpha a_1 + a_3 = a_1$

$\beta t = a_3$

Quotient subspace

$V/W = \{ v + W : v \in V \}$

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$(v+w)+(v'+w) = v+v'+w$ & $\alpha(v+w) = (\alpha v + w)$.

$\alpha = \dfrac{a_1 - a_3}{a_1}$, $\beta = \dfrac{a_3}{t}$

Hence $W_1 + W_2' = V$.

(2) $W_1 \cap W_2 = \{0\}$;

Let $(x,y) \in W_1 \cap W_2)$

$y = 0$ & $x = y$ $\Rightarrow$ $(0,0) \in W_1 \cap W_2$

Ques 35 Let W be subspace of FDVS, V.

Let $\{u_1, u_2, \ldots, u_k\}$ be basis of W.

Let $\gamma = \{u_1, u_2 \ldots u_k, u_{k+1} \ldots u_n\}$ be basis of V

(a) Prove that

$\alpha = \{ u_{k+1} + w, u_{k+2} + w, \ldots u_n + w \}$

is basis of V/W.

$V/W = \{ v + w, v \in V \}$

Proof (a) α is basis for V/W

(1) α is linearly independent

Consider

$a_1(u_{k+1} + W) + a_2(u_{k+2} + w) + \ldots a_n(u_n + w) = W$

$(a_1 u_{k+1} + W) + (a_2 u_{k+2} + w) + \ldots (a_n u_n + W) = W$

(by oper^n of scalar.)

$(a_1 u_{k+1} + a_2 u_{k+2} + \ldots a_n u_n) + w = W$.

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$a_1 u_{k+1} + a_2 u_{k+2} + \dots a_n u_n \in W.$

$a_1 u_{k+1} + a_2 u_{k+2} + \dots a_n u_n = b_1 u_1 + b_2 u_2 + \dots b_k u_k$

$b_i \in F$

$\{\because \beta$ is basis of W

$\Rightarrow a_1 u_{k+1} + \dots a_n u_n - b_1 u_1 - b_2 u_2 \dots - b_k u_k) = 0$

$\in \gamma.$

Hence, the above vector must be L.I.

$\Rightarrow a_i = 0$ & $b_i = 0$

Hence α is LI.

(b) (2) $V/W \in \text{span}\,\alpha.$

Let $v + W \in V/W$

As $v \in V \Rightarrow v = a_1 u_1 + \dots a_k u_k + b_1 u_{k+1} + \dots + b_n u_n$

$\Rightarrow v + W = (a_1 u_1 + a_2 u_2 + \dots a_k u_k + b_1 u_{k+1} + \dots b_n u_n) + W$

$= (b_1 u_{k+1} + \dots b_n u_n) + (a_1 u_1 + a_2 u_2 \dots a_k u_k) + W$

$= (b_1 u_{k+1} + \dots b_n u_n) + W.$

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$= b_1(u_{k+1}+w) + b_2(u_{k+2}+w) + \dots b_n(u_n+w)$

$\Rightarrow v + w \in \text{span}\{u_{k+1}+w, \dots, u_n+w\}$

$\Rightarrow$ Basis for V/W is $\{ \quad ,, \quad \}$.

Now $\dim(V/W) = n-k$.

$\dim(V/W) = \dim V - \dim W$..

Theorem 5.5 (a) *Any set of vectors in R^n having more than n vectors is linearly dependent and hence cannot be basis for R^n.*

(b) *Any set of vectors that span R^n must contain atleast n vectors therefore any set of vectors in R^n having less than n vectors cannot be basis for R^n.*

(c) *Any linearly independent set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ spans R^n and hence is a basis of R^n.*

(d) *Any set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ spanning R^n is linearly independent and hence a basis of R^n.*

(e) *If $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent set of vectors in R^n and $\mathbf{v} \notin L(S)$, then*

$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}\}$ is also linearly independent.

Example 5.46. *Which of the following sets of vectors are bases for R^2?*

(a) $\{(1, 3), (1, -1)\}$ (b) $\{(0, 0), (1, 2), (2, 4)\}$

(c) $\{(1, 2), (2, -3), (3, 2)\}$ (d) $\{(1, 3), (-2, 6)\}$

Solution. (a) Let $\alpha_1 (1,3) + \alpha_2 (1,-1) = (0, 0)$

$$\Rightarrow \quad (\alpha_1 + \alpha_2, 3\alpha_1 - \alpha_2) = (0, 0)$$

or

$$\alpha_1 + \alpha_2 = 0$$

$$3\alpha_1 - \alpha_2 = 0$$

$$4\alpha_1 = 0$$

$$\alpha_1 = 0, \ \alpha_2 = 0$$

$\therefore$ $\{(1,3), (1,-1)\}$ is linearly independent set in R^2

By theorem 5.5(c) $\{(1, 3), (1, -1)\}$ is a basis of R^2.

(b) Let $S = \{(0, 0), (1, 2), (2, 4)\}$

Since S contains 3 vectors therefore by Theorem 5.5(a) S cannot be basis of R^2.

(c) $S = \{(1, 2), (2, -3), (3, 2)\}$

Again as S contains 3 vectors (more than 2)

by theorem 5.5(a) S cannot be basis of R^2.

(d) Let $S = \{(1, 3), (-2, 6)\}$

Let $(a, b) \in R^2$

Let if possible $(a,b) = \alpha_1 (1,3) + \alpha_2 (-2,6)$

or $$(a,b) = (\alpha_1 - 2\alpha_2, 3\alpha_1 + 6\alpha_2)$$

or $$\alpha_1 - 2\alpha_2 = a$$

$$3\alpha_1 + 6\alpha_2 = b$$

In matrix form

$$\begin{pmatrix} 1 & -2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\begin{pmatrix} 1 & -2 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b-3a \end{pmatrix}$$

$\Rightarrow$ $$\alpha_1 - 2\alpha_2 = a$$

$$12\alpha_2 = b - 3a$$

$$\alpha_2 = \frac{b-3a}{12},$$

$$\alpha_1 = a + 2\alpha_2 = a + \frac{2b-6a}{12} = \frac{6a+2b}{12}$$

Hence, $L(S) = R^2$

$\therefore$ By Theorem 5.5(d) S is a basis of R^2.

Example 5.47. *Which of the following sets of vectors are bases of R^3 ?*

(a) $\{(1, 2, 0), (0, 1, -1)\}$

(b) $\{(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)\}$

(c) $\{(3, 2, 2), (-1, 2, 1), (0, 1, 0)\}$

(d) $\{(1, 0, 0), (0, 2, -1), (3, 4, 1), (0, 1, 0)\}$

Solution. (*a*) Let $S = \{(1, 2, 0), (0, 1, -1)\}$

Since S contains two vectors (less than 3) and any basis of R^3 must contain 3 vectors therefore by theorem 5.5(*b*) S cannot be basis of R^3.

(*b*) Let $S = \{(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)\}$

Since S contains four vectors (more than 3) and any basis of R^3 must contain 3 vectors therefore by theorem 5.5(*a*) S cannot be basis of R^3.

(*c*) Let $S = \{(3, 2, 2), (-1, 2, 1), (0, 1, 0)\}$

Let $\alpha_1, \alpha_2, \alpha_3 \in R$ be such that

$$\alpha_1 (3,2,2) + \alpha_2 (-1,2,1) + \alpha_3 (0,1,0) = (0, 0, 0)$$

$$\Rightarrow \quad (3\alpha_1 - \alpha_2, 2\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2) = (0, 0, 0)$$

or

$$3\alpha_1 - \alpha_2 = 0$$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$

$$2\alpha_1 + \alpha_2 = 0$$

In matrix form

$$\begin{pmatrix} 3 & -1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{2}{3} R_1,\ R_2 \to R_3 - \frac{2}{3} R_1$

$$\begin{pmatrix} 3 & -1 & 0 \\ 0 & \frac{8}{3} & 1 \\ 0 & \frac{5}{3} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 3\alpha_1 - \alpha_2 = 0$$

$$\frac{8}{3}\alpha_2 + \alpha_3 = 0$$

$$\frac{5}{3}\alpha_2 = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\therefore$ S is linearly independent. Hence by theorem 5.5(*c*) S is a basis of R^3.

(d) Let $S=\{(1,0,0),(0,2,-1),(3,4,1),(0,1,0)\}$

Since S contains 4 vectors (more than 3) therefore by theorem 5.5(a) S is not a basis of R^3.

Example 5.48. *Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for the vector space R^3. Prove that* $\{\mathbf{u}_1+\mathbf{u}_2, \mathbf{u}_2+\mathbf{u}_3, \mathbf{u}_3+\mathbf{u}_1\}$

and $\{\mathbf{u}_1, \mathbf{u}_1+\mathbf{u}_2, \mathbf{u}_1+\mathbf{u}_2+\mathbf{u}_3\}$ are also basis of R^3.

Solution. Let $\alpha_1, \alpha_2, \alpha_3 \in R$ such that

$$\alpha_1(\mathbf{u}_1+\mathbf{u}_2)+\alpha_2(\mathbf{u}_2+\mathbf{u}_3)+\alpha_3(\mathbf{u}_3+\mathbf{u}_1)=(0,0,0)$$

$$\Rightarrow \quad \alpha_1\mathbf{u}_1+\alpha_3\mathbf{u}_1+\alpha_1\mathbf{u}_2+\alpha_2\mathbf{u}_2+\alpha_2\mathbf{u}_3+\alpha_3\mathbf{u}_3=(0,0,0)$$

$$\Rightarrow \quad (\alpha_1+\alpha_3)\mathbf{u}_1+(\alpha_1+\alpha_2)\mathbf{u}_2+(\alpha_2+\alpha_3)\mathbf{u}_3=(0,0,0)$$

$$\Rightarrow \quad \alpha_1+\alpha_3=0$$

$$\alpha_1+\alpha_2=0$$

$$\alpha_2+\alpha_3=0$$

as $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ being basis of R^3 is linearly independent set.

In matrix form

$$\begin{pmatrix}1 & 0 & 1\\1 & 1 & 0\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\begin{pmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 2\end{pmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

or

$$\alpha_1+\alpha_3=0$$

$$\alpha_2-\alpha_3=0$$

$$2\alpha_3=0$$

$$\Rightarrow \qquad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Hence, $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \mathbf{u}_3 + \mathbf{u}_1\}$ is linearly independent set in R^3. Therefore, by theorem 5.5(*c*) it is a basis of R^3. Now suppose

$$\alpha_1 \mathbf{u}_1 + \alpha_2(\mathbf{u}_1 + \mathbf{u}_2) + \alpha_3(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) = (0, 0, 0)$$

$$\Rightarrow \quad (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{u}_1 + (\alpha_2 + \alpha_3)\mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = 0$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_2 + \alpha_3 = 0$$

$$\alpha_3 = 0$$

(because $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ being basis of R^3 is linearly independent.)

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Hence, $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3\}$ is linearly independent

$\therefore$ By theorem 5.5(*c*) it is a basis of R^3.

Example 5.49. *Extend the set* $\{\mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (1, 0, 0)\}$ *to form a basis of* R^3.

Solution. Let $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 = \mathbf{0}$

or $\qquad (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 0, 0) = (0, 0, 0)$

$$\Rightarrow \qquad (\alpha_1 + \alpha_2, \alpha_1, \alpha_1) = (0, 0, 0)$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 = 0, \ \alpha_1 = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = 0.$$

$\therefore$ $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent set

Now,

$$L(\mathbf{u}_1, \mathbf{u}_2) = \{\alpha_1(1,1,1) + \alpha_2(1,0,0) : \alpha_1, \alpha_2 \in R\}$$

$$= \{(\alpha_1 + \alpha_2, \alpha_1, \alpha_1) : \alpha_1, \alpha_2 \in R\}$$

Clearly,

$$\mathbf{u}_3 = (0, 1, 0) \notin L(\mathbf{u}_1, \mathbf{u}_2)$$

$\therefore$ By theorem 5.5(*e*) $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent and hence a basis of R^3.

Example 5.50. *Extend the set $S=\{(1, 1, 0)\}$ to form two different basis of R^3.*

Solution. Since $(1, 1, 0) \neq (0, 0, 0)$ $\therefore$ S is linearly independent.

$$L(S)=\{\alpha(1,1,0) : \alpha \in R\}$$
$$=\{(\alpha,\alpha,0) : \alpha \in R\}$$

Now, $(1, 0, 0) \notin L(S)$

$\therefore$ $S_1 =\{(1,1,0),(1,0,0)\}$ is linearly independent.

Now,

$$L(S_1)=\{\alpha_1 (1,1,0)+\alpha_2 (1,0,0) : \alpha_1, \alpha_2 \in R\}$$
$$=\{(\alpha_1 +\alpha_2, \alpha_1, 0) : \alpha_1, \alpha_2 \in R\}$$

Now, $(0, 0, 1), (1, 1, 1) \notin L(S_1)$

$\therefore$ $S_2 =\{(1, 1, 0), (1, 0, 0), (0, 0, 1)\}$ and

$S_3 =\{(1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

are linearly independent and hence S_2 and S_3 are two basis of R^3.

Example 5.51. *Let $S=\{\mathbf{u}_1 =(4, 2, 1), \mathbf{u}_2 =(2, 6, -5), \mathbf{u}_3 =(1, -2, 3)\}$. Is S basis of R^3. If $W = L(S)$ what is dimension of W ?*

Solution. Let $\alpha_1 (4,2,1)+\alpha_2 (2,6,-5)+\alpha_3 (1,-2,3)=(0, 0, 0)$

$\Rightarrow (4\alpha_1 +2\alpha_2 +\alpha_3, 2\alpha_1 +6\alpha_2 -2\alpha_3, \alpha_1 -5\alpha_2 +3\alpha_3)=(0, 0, 0)$

$$\Rightarrow \quad \begin{aligned} 4\alpha_1 +2\alpha_2 +\alpha_3 &=0 \\ 2\alpha_1 +6\alpha_2 -2\alpha_3 &=0 \\ \alpha_1 -5\alpha_2 +3\alpha_3 &=0 \end{aligned}$$

Now determinant of coefficient matrix is

$$\begin{vmatrix} 4 & 2 & 1 \\ 2 & 6 & -2 \\ 1 & -5 & 3 \end{vmatrix} =4(18-10)-2(6+2)+1(-10-6)$$

$$=32-16-16=0$$

$\therefore$ The system has infinite solutions.

Hence, S is linearly dependent. $\therefore$ S is not a basis of R^3.

We express one of the vectors as linear combination of other two.

Let $(1,-2,3) = \alpha_1(4,2,1) + \alpha_2(2,6,-5)$

$$= (4\alpha_1 + 2\alpha_2, 2\alpha_1 + 6\alpha_2, \alpha_1 - 5\alpha_2)$$

$$\Rightarrow \quad 4\alpha_1 + 2\alpha_2 = 1$$

$$2\alpha_1 + 6\alpha_2 = -2$$

$$\alpha_1 - 5\alpha_2 = 3$$

Solving last two, we get

$$16\alpha_2 = -8 \Rightarrow \alpha_2 = -\frac{1}{2}$$

$$\alpha_1 = 3 + 5 \times \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2}$$

also satisfies the first equation.

$$\therefore \quad (1,-2,3) = \frac{1}{2}(4,2,1) - \frac{1}{2}(2,6,-5)$$

Also, If $\alpha_1(4,2,1) + \alpha_2(2,6,-5) = (0,0,0)$

Then $\alpha_1 = \alpha_2 = 0$

$$\therefore \quad S_1 = \{(4,2,1), (2,6,-5)\}$$

is linearly independent and $(1,-2,3) \in L(S_1)$

$$\therefore \quad L(S_1) = L(S) = W \qquad \therefore \textbf{dim}\, W = 2$$

Definition 5.7 *Let V be non-zero subspace of R^n. The* ***dimension*** *of V is the number of vectors in a basis of V.* **dim** *V is notation for dimension of V. Since $\{\mathbf{0}\}$ is linearly dependent the dimension of vector space $\{\mathbf{0}\}$ is defined to be 0.*

It is to be understood that if **dim** $V = k$ then V has a basis containing k vectors say $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$

S is linearly independent and $\mathbf{v} \in V$ can be uniquely expressed as

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots + \alpha_k \mathbf{u}_k$$

Examples 5.38. dim $R^2 = 2$ since $\{(1, 0), (0, 1)\}$ is a basis of R^2.

Every basis of R^2 has two vectors.

dim $R^3 = 3$ since $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of R^3.

Every basis of R^3 has three vectors.

dim $R^n = n$ since $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 0, \ldots, 1)\}$ is a basis of R^n.

Every basis of R^n has n–vectors.

Example 5.39. Let $W = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$

We have shown that W is subspace of R^3

Now $(a_1, a_2, 0) = a_1 (1,0,0) + a_2 (0,1,0)$

It can be verified that $S = \{(1, 0, 0), (0, 1, 0)\}$ is linearly independent.

Hence S is basis of W. **dim** $W = 2$

Example 5.40. $W = \{\alpha (1,1,1) : \alpha \in R\}$

W is a subspace of R^3

$\{(1, 1, 1)\}$ is basis of W.

dim $W = 1$

Example 5.41. $W = \{(a_1, a_2) : a_1 + a_2 = 0, a_1, a_2 \in R\}$

We have shown W is subspace of R^2.

$$(a_1, a_2) \in W \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

Any vector in W is of type $(a_1, -a_1) = a_1(1, -1)$

$\therefore$ $\{(1, -1)\}$ is a basis for W

dim $W = 1$

Example 5.42. $W = \{(0, a) : a \in R\}$

We have shown W is a subspace of R^2.

$\{(0, 1)\}$ is a basis for W.

dim $W = 1$

Example 5.43. $W = \{(x, y) : 3x - 4y = 0\}$

We have shown W is a subspace of R^2.

If $(x, y) \in W$ then $3x - 4y = 0$ or $y = \frac{3}{4}x$

Any element of W is of type

$$\left(x, \frac{3}{4}x\right) = x\left(1, \frac{3}{4}\right)$$

$\therefore$ $\left\{\left(1, \frac{3}{4}\right)\right\}$ is a basis of W.

dim $W = 1$

Example 5.44. Let $W = \{(x_1, x_2, x_3) : x_1 - x_2 - x_3 = 0\} \subseteq R^3$

We have shown W is a subspace of R^3.

All elements of W are of type

$$(x + y, x, y), \ x, y \in R$$

Now, $(1, 0, 1)$ and $(1, 1, 0) \in W$

If $\alpha_1(1, 0, 1) + \alpha_2(1, 1, 0) = (0, 0, 0)$

then $(\alpha_1, 0, \alpha_1) + (\alpha_2, \alpha_2, 0) = (0, 0, 0)$

$$(\alpha_1+\alpha_2, \alpha_2, \alpha_1)=(0,0,0)$$

$\Rightarrow$ $\alpha_1=0,\ \alpha_2=0$

$\Rightarrow$

$\therefore$ $\{(1,0,1),(1,1,0)\}$ is linearly independent.

Let $(x+y, x, y)\in W$

Then $(x+y, x, y)=x(1,1,0)+y(1,0,1)$

$\therefore$ $(x+y, x, y)\in L\{(1,1,0),(1,0,1)\}$

So $L((1,1,0),(1,0,1))=W$

$\therefore$ $\{(1,1,0),(1,0,1)\}$ is a basis of W **dim** $W=2$.

Example 5.45. $W=\{(a,b,c): a-2b+3c=0\}\subset R^3$

Find a basis of W

Let $\mathbf{u}=(a_1,b_1,c_1)$, $\mathbf{v}=(a_2,b_2,c_2)\in W$ and $\alpha\in R$.

Then $(a_1,b_1,c_1)+(a_2,b_2,c_2)=(a_1+a_2, b_1+b_2, c_1+c_2)$

Now $(a_1+a_2)-2(b_1+b_2)+3(c_1+c_2)$

$$=(a_1-2b_1+3c_1)+(a_2-2b_2+3c_2)$$

$$=0+0=0$$

Also, $\alpha(a_1,b_1,c_1)=(\alpha a_1,\alpha b_1,\alpha c_1)$

$$\alpha a_1-2\alpha b_1+3\alpha c_1=\alpha(a_1-2b_1+3c_1)=\alpha\cdot 0=0$$

$\therefore$ $\mathbf{u}+\mathbf{v},\ \alpha\mathbf{u}\in W$

Hence, W is a subspace of R^3.

Now every member of W is of type $(2x-3y, x, y)$

Now $(2,1,0),(-3,0,1)\in W$

Let $\alpha_1(2,1,0)+\alpha_2(-3,0,1)=(0,0,0)$

Then $(2\alpha_1-3\alpha_2, \alpha_1, \alpha_2)=(0,0,0)$

$\Rightarrow$ $\alpha_1=0,\ \alpha_2=0$

$\Rightarrow$ $\{(2,1,0),(-3,0,1)\}$ is linearly independent set in R^3.

Also, $(2x-3y, x, y)=x(2,1,0)+y(-3,0,1)$

$\therefore$ $L((2,1,0),(-3,0,1))=W$

$\therefore$ $\{(2,1,0),(-3,0,1)\}$ is a basis of W

dim $W=2$.