Ques In $\mathbb{C}^3$, $x = (2, 1+i, i)$

$y = (2-i, 2, 1+2i)$

i) compute $\langle x, y \rangle$, $\|x\|$, $\|y\|$, $\|x+y\|$

ii) In $C[0,1]$, let $f(t) = t$ and $g(t) = e^t$ compute $\langle f, g \rangle$ $\|f\|$, $\|g\|$ and $\|f+g\|$

$\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$.

Also verify cauchy's schwartz's inequality and the triangle inequality for both the questions

DEFINITIONS

- Let 'V' be an innerproduct space two vectors x and y is V are said to be orthogonal ($\perp$) if $\langle x, y \rangle = 0$
- A subset 'S' of of 'V' is said to be orthogonal if each of the 2 distinct vectors in 'S' are orthogonal
- A vector 'x' in 'V' is s.t.b a unit vector or is a unit vector if and only if $\|x\| = 1$
- So finally, a subset 'S' of 'V' is s.t.b orthonormal if 'S' is orthogonal and consists of entirely unit vectors.

Ques In $\mathbb{R}^3$, for the subset $S = \{\underset{x_1}{(1,1,0)}, \underset{x_2}{(1,-1,1)}, \underset{x_3}{(-1,1,2)}\}$ check if S is orthogonal,

Also check is S is orthonormal.

$\mathbb{R}^3$ over $\mathbb{R}$ is

$\langle x_1, x_2 \rangle = (1)(1) + (1)(-1) = 1 - 1 = 0$

$\langle x_2, x_3 \rangle = -1 - 1 + 2 = 0$

$\langle x_3, x_1 \rangle = -1 + 1 = 0$.

$\therefore$ S is orthogonal.

$\|x_1\| = \sqrt{\langle x_1, x_1 \rangle} = \sqrt{(1+1+0)} = \sqrt{2}$

$\|x_2\| = \sqrt{\langle x_2, x_2 \rangle} = \sqrt{3}$

$\|x_3\| = \sqrt{6}$.

Not orthonormal.

Ques In $\mathbb{R}^3$, for the subset $S=\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\}$ check if S is orthonormal!

S is orthonormal

PROOF OF CAUCHY SCHWARTZ INEQUALITY

Case 1 If $y=0$, then obviously the inequality holds true

Case 2 If $y\neq 0$, then consider $0 \le \|x-cy\|^2 = \langle x-cy, x-cy\rangle$

$\langle x,x\rangle + \langle -cy, x\rangle + \langle x, -cy\rangle + \langle -cy, -cy\rangle$

$= \|x\|^2 - c\langle y,x\rangle - \bar{c}\langle x,y\rangle + c\bar{c}\langle y,y\rangle$ —— (*)

(using $\langle x, cy\rangle = \bar{c}\langle x,y\rangle$)

Let without loss of generality, consider (wlog)

$$c = \frac{\langle x,y\rangle}{\langle y,y\rangle}$$

$$(*) = \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$0 \le \|x\|^2 - \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

$$|\langle x,y\rangle| \le \|x\|\cdot\|y\|$$

$$c = \frac{\langle x,y\rangle}{\langle y,y\rangle}$$

$$\therefore -c\langle y,x\rangle = -\frac{\langle x,y\rangle}{\|y\|^2}\langle y,x\rangle = \frac{-\langle x,y\rangle \overline{\langle x,y\rangle}}{\|y\|^2}$$

$$= \frac{|\langle x,y\rangle|^2}{\|y\|^2}$$

TRIANGLE INEQUALITY

$$\|x+y\| \le \|x\| + \|y\|$$

$$\|x+y\|^2 = \langle x+y, x+y\rangle$$

$$= \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,y\rangle$$

$$= \langle x,x\rangle + \overline{\langle x,y\rangle} + \langle x,y\rangle + \langle y,y\rangle$$

$$= \langle x,x\rangle + 2\mathrm{Re}\langle x,y\rangle + \langle y,y\rangle \le$$

$\|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2$

$$\leq \|x\|^2 + 2\|x\|.\|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

REMARK
If 'V' is an inner product space then equality holds in cauchy schwartz inequality i.e $|\langle x,y\rangle| = \|x\|.\|y\|$ ~~if and only~~ iff one of the vectors x or y is a multiple of one another.

Ques Find out the condn under which Δ inequality gets converted to equality.

If the 2 vectors are
(1) orthogonal to each other then the Δ inequality gets converted to equality.

x — x — x — x — x — x — x — x