Variation of Perameters:→

$$y'' + P(x)\, y' + q(x)\, y = R(x) \quad -(1)$$

Let y_1 and y_2 be two L.I. solutions of $y''+P(x)y'+q(x)y=0$ —(2)

Then Complementary fun. of (1) is $y_c(x) = a y_1 + b y_2$.

Let us assume that $y(x) = A(x)\, y_1 + B(x)\, y_2$ is a general solution of (1). —(3)

$\because$ y_1 and y_2 are sol^n of (2)

So $y_1'' + P(x) y_1' + q(x) y_1 = 0$ — (4)

and $y_2'' + P(x) y_2' + q(x) y_2 = 0$ — (5)

From (3) $y = A y_1 + B y_2$ where A, B are fun of x.

$\Rightarrow y' = A y_1' + A' y_1 + B y_2' + B' y_2$

Let $A' y_1 + B' y_2 = 0$ — (6)

$\Rightarrow y' = A y_1' + B y_2'$

$\Rightarrow y'' = A' y_1' + A y_1'' + B' y_2' + B y_2''$

Put the values of y, y', y'' in (1), we get

$$(A' y_1' + A y_1'' + B' y_2' + B y_2'') + P(x)\,(A y_1' + B y_2') + q(x)\,(A y_1 + B y_2) = R(x)$$

$$\Rightarrow A(y_1'' + P(x) y_1' + q(x) y_1) + B(y_2'' + P(x) y_2' + q(x) y_2) + A' y_1' + B' y_2' = R(x)$$

using (4), (5), we get

$$A' y_1' + B' y_2' = R \quad -(7)$$

From (6) and (7) find A' and B'.

$$A' y_1 + B' y_2 = 0$$

$$A' y_1' + B' y_2' - R = 0$$

$$\begin{array}{cccc} y_2 & 0 & y_1 & y_2 \\ y_2' & -R & y_1' & y_2' \end{array}$$

$$\frac{A'}{-y_2 R} = \frac{B'}{+R y_1} = \frac{1}{y_1 y_2' - y_1' y_2}$$

$$\Rightarrow A' = \frac{-R y_2}{W(y_1, y_2)} \quad \text{and} \quad B' = \frac{+R y_1}{W(y_1, y_2)}$$

$$\Rightarrow A(x) = \int \frac{-Ry_2}{W(y_1,y_2)}dx + c_1 \; ; \; B(x) = \int \frac{Ry_1}{W(y_1,y_2)}dx + c_2$$

So From (3) $y(x) = A(x)y_1 + B(x)y_2$

$$\Rightarrow y(x) = -\int \frac{Ry_2\,dx}{W(y_1,y_2)} \cdot y_1 + c_1 y_1 + \int \frac{Ry_1}{W(y_1,y_2)}dx \cdot y_2 + c_2 y_2$$

$$\Rightarrow y(x) = \underbrace{c_1 y_1 + c_2 y_2}_{C.F.} + \underbrace{y_1 \int \frac{-Ry_2}{W(y_1,y_2)}dx + \int \frac{Ry_1}{W(y_1,y_2)}dx \cdot y_2}_{P.I.}$$

In Short:→ For $y'' + P(x)y' + q(x)y = R(x)$ —(1)

Find its C.F. i.e. $y(x) = c_1 u + c_2 v$.

Let Gen Sol^n of (1) is $y(x) = Au + Bv$

where A, B are fun of x.

where $A = \int \frac{-vR}{W}dx + c_1$; $B = \int \frac{uR}{W}dx + c_2$.

& $W = W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - u'v$.

Que:→ $y'' - y = e^x$ —(1)

Sol^n $(D^2-1)y = e^x$

$m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_c(x) = c_1 e^x + c_2 e^{-x}$

let Gen Sol^n of (1) is

$y(x) = Ae^x + Be^{-x}$

$A = -\int \frac{vR}{W}dx + c_1$ and $B = \int \frac{uR}{W}dx + c_2$

where $u = e^x$, $v = e^{-x}$, $R = e^x$

and $W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x \cdot e^{-x} - e^x e^{-x}$

$= -2$

So $A = +\int \frac{e^{-x} \cdot e^x}{+2}dx = \frac{1}{2}x + c_1$

$B = \int \frac{e^x \cdot e^x}{-2}dx + c_2 = \frac{-1}{4}e^{2x} + c_2$.

Date

Page

So $y(x) = \left(\frac{1}{2}x + c_1\right)e^{x} + \left(-\frac{1}{4}e^{2x} + c_2\right)e^{-x}$

$= \frac{1}{2}xe^{x} + c_1e^{x} - \frac{1}{4}e^{x} + c_2e^{-x}$

$= \cancel{c_1e^{x}}\ \left(c_1 - \frac{1}{4}\right)e^{x} + c_2e^{-x} + \frac{1}{2}xe^{x}$

$= \underbrace{c_3e^{x} + c_2e^{-x}}_{C.F.} + \underbrace{\frac{1}{2}xe^{x}}_{P.I.}$

Que $y'' + y = \sec x$ — (1)

Sol^n $(D^2+1)y = \sec x$

AE $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c(x) = c_1\cos x + c_2\sin x$

let Gen Sol^n of (1) is $y(x) = A\cos x + B\sin x$.

where $A = \int \frac{-VR}{W}dx + c_1$ & $B = \int \frac{UR}{W}dx + c_2$.

$\cancel{A = \int \sin x \sec x\, dx}$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$\therefore A = \int -\sin x \cdot \sec x\, dx + c_1$

$= -\int \tan x\, dx + c_1$

$= \log|\cos x| + c_1$

$B = \int \cos x \cdot \sec x\, dx + c_2 = x + c_2$

$\therefore y(x) = (\log|\cos x| + c_1)\cos x + (x + c_2)\sin x$

$= (c_1\cos x + c_2\sin x) + x\sin x + \log|\cos x| \cdot \cos x$

Qu $y'' + a^2 y = \operatorname{Cosec} ax$ —(1)

Solⁿ $(D^2+a^2)y = \operatorname{Cosec} ax$

A.E $m^2 + a^2 = 0$

$\Rightarrow m = \pm ai$

$y_c(x) = C_1 \cos ax + C_2 \sin ax$

let Gen Solⁿ of (1) is

$y(x) = A \cos ax + B \sin ax$

where $A = -\int \frac{VR}{W} dx + C_1$ & $B = \int \frac{UR}{W} dx + C_2$.

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a$$

$$A = -\int \frac{\sin ax \cdot \operatorname{Cosec} ax}{a} dx + C_1$$

$$\boxed{A = \frac{-1}{a} x + C_1}$$

$$B = \int \frac{\cos ax \cdot \operatorname{Cosec} ax}{a} dx + C_2$$

$$\boxed{B = \frac{1}{a^2} \log|\sin ax| + C_2}$$

$$\therefore \; y(x) = \left(\frac{-x}{a} + C_1\right)\cos ax + \left(\frac{1}{a^2}\log|\sin ax| + C_2\right)\sin ax$$

$$y(x) = C_1 \cos ax + C_2 \sin ax - \frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax \log|\sin ax|.$$

H.W (1) $y'' + 4y = 4\tan 2x$

(2) $y'' + 4y = \cot 2x$

(3) $y'' - 2y' + y = x e^x \log x, \; x > 0.$

(4) $y'' + y = \frac{1}{1+\sin x}$