Solution of Laplace Equation with variable coefficients.

Suppose the given differential equation contain a term of the form $t^m y^n(t)$ i.e. $t^m \frac{d^n}{dt^n} y(t)$, the Laplace transform of which is $(-1)^m \frac{d^m}{dp^m} L[y^n(t)]$

The method is illustracted in the following questions.

Note!.

Ques!. Solve: $\frac{d^2}{dt^2} y(t) + t \frac{d}{dt} y(t) - y(t) = 0$, if $y(0) = 0$, $\left(\frac{dy}{dt}\right)_{t=0} = 1$.

Sol! Taking the Laplace transform of both side of the given equation, we get

$$L[y''] + L[t y'] - L[y(t)] = 0$$

or $$p^2 L[y(t)] - p y(0) - y'(0) - \frac{d}{dp} L[y'] - L[y(t)] = 0$$

or $$p^2 \bar{y} - p y(0) - y'(0) - \frac{d}{dp}[p\bar{y} - y(0)] - \bar{y} = 0$$

where $L[y(t)] = \bar{y}$

Using initial condition, we get

$$p^2 \bar{y} - 1 - \frac{d}{dp}[p\bar{y}] - \bar{y} = 0$$

$$p^2\bar{y} - 1 - \left[\bar{y} + p\frac{d\bar{y}}{dp}\right] - \bar{y} = 0$$

or $$-p\frac{d\bar{y}}{dp} + (p^2-2)\bar{y} = 1$$

or $$\frac{d\bar{y}}{dp} + \left(\frac{2}{p} - p\right)\bar{y} = -\frac{1}{p} \quad -(1)$$

which is linear differential equation.

$$I.F = \exp\int P\,dp$$

$$= \exp\int\left(\frac{2}{p} - p\right)dp = \exp\left[2\log p - \frac{p^2}{2}\right]$$

$$= \exp\left[\log p^2 - p^2/2\right] = e^{\log p^2 - p^2/2}$$

$$= e^{\log p^2}\, e^{-p^2/2} = p^2 e^{-p^2/2}$$

Solution of the equation is

$$\bar{y}\cdot p^2 e^{-p^2/2} = \int\left(-\frac{1}{p}\right)\cdot p^2 e^{-p^2/2}\,dp + C$$

$$= -\int p\, e^{-p^2/2}\,dp + C$$

$$-\frac{p^2}{2} = u \Rightarrow -\frac{2p\,dp}{2} = du$$

$$\Rightarrow -p\,dp = du$$

$$= \int e^u\,du + C = e^u + C$$

$$= e^{-p^2/2} + C$$

$$\bar{y}\, p^2 \times e^{-p^2/2} = C + e^{-p^2/2}$$

Where C is a constant and $\bar{y} = L[y(t)] = y(p)$, say.

C must be vanish if $\bar{y}$ is a Laplace transform.

Since $\bar{y} \to 0$ as $p \to \infty$

$$\therefore \bar{y}\, p^2 \times e^{-p^2/2} = 0 + e^{-p^2/2}$$

$$\text{or} \quad \bar{y} = \frac{1}{p^2}$$

Taking inverse Laplace transform, we have

$$y(t) = L^{-1}\left[\frac{1}{p^2}\right] = t$$

—— Ans.

Note?:

$$\bar{y} = \frac{c}{p^2} \cdot e^{p^2/2} + \frac{1}{p^2}$$

$$= \frac{c}{p^2}\left[1 + \frac{p^2}{2} + \frac{1}{2!}\frac{p^4}{4} + \cdots\right] + \frac{1}{p^2}$$

$$\bar{y} = \left(\frac{c+1}{p^2}\right) + \frac{c}{2} + \frac{1}{8} p^2 c + \cdots$$

$$\therefore y(t) = L^{-1}\left[\frac{c+1}{p^2}\right] + \frac{c}{2} L^{-1}[1] + \frac{1}{8} c L^{-1}[p^2] + \cdots$$

Note $L^{-1}[p^n] = 0, \ \forall n = 0, 1, 2, \ldots$

$y = (c+1)t$

$y'(t) = c+1 \Rightarrow c = 0 \quad \because y'(0) = 1$

$y = t$ which is required solution.

Example. Solve $ty'' + y' + 4ty = 0$ with $y(0)=3, y'(0)=0$.

Sol: We have

$$ty'' + y' + 4t.y = 0 \quad \text{—— (1)}$$

Taking the Laplace transform of both side, of equation (1) of the given differential equation.

$$L[ty''] + L[y'] + 4L[t.y] = 0$$

$$\text{or } -\frac{d}{dp}[L(y'')] + L[y'] + 4(-1)\frac{d}{dp}[L(y)] = 0$$

$$\text{or } -\frac{d}{dp}[p^2\bar{y} - py(0) - y'(0)] + [p\bar{y} - y(0)] - 4\frac{d}{dp}(\bar{y})$$

where $\bar{y} = L[y]$

$$\text{or } -\frac{d}{dp}[p^2\bar{y} - 3p] + (p\bar{y} - 3) - 4\frac{d\bar{y}}{dp} = 0$$

$$\text{or } -[2p\bar{y} + p^2\frac{d\bar{y}}{dp} - 3] + p\bar{y} - 3 - 4\frac{d\bar{y}}{dp} = 0$$

$$\text{or } -(p^2+4)\frac{d\bar{y}}{dp} - p\bar{y} = 0$$

$$\text{or } \frac{d\bar{y}}{dp} + \frac{p}{p^2+4}\bar{y} = 0 \quad \text{—— (2)}$$

$$\text{or } \frac{d\bar{y}}{\bar{y}} + \frac{p}{p^2+4}dp = 0$$

(seperating the variables)

on integrating, we get

$$\log \bar{y} + \frac{1}{2} \log (p^2+4) = \log C$$

$$\text{or } \log \left(\bar{y} \sqrt{p^2+4}\right) = \log C$$

$$\text{or } \quad \bar{y} = \frac{C}{\sqrt{p^2+4}} \qquad -(3)$$

Taking inverse Laplace transform, both side

$$y = L^{-1}\left[\frac{C}{\sqrt{p^2+2^2}}\right] = C \cdot J_0(2t)$$

$$\because L^{-1}\left[\frac{1}{\sqrt{p^2+a^2}}\right] = J_0(at)$$

Since $y(0) = 3 \Rightarrow 3 = C_1 J_0(0)$

$\Rightarrow C_1 = 3 \quad \because J_0(0) = 1$

Hence, $y(t) = 3 J_0(2t)$

which is required solution.

Ques:- Solve : $ty'' + (1-2t)y' - 2y = 0$ with

Condition $y(0) = 1$, $y'(0) = 2$.

Sol:- We have

$$ty'' + (1-2t)y' - 2y = 0 \qquad -(1)$$

Taking the Laplace transform of both sides of the given equation, we have

$$L[ty''] + L[(1-2t)y'] - 2L[y] = 0$$

or $$-\frac{d}{dp}[L(y'')] + L[y'] - 2(-1)\frac{d}{dp}L(y') - 2L(y) = 0$$

or $$-\frac{d}{dp}[p^2\bar{y} - py(0) - y'(0)] + [p\bar{y} - y(0)]$$
$$+2\frac{d}{dp}[p\bar{y} - y(0)] - 2\bar{y} = 0$$

where $L(y) = \bar{y}$

using the initial conditions, we have

$$-\frac{d}{dp}[p^2\bar{y} - p - 2] + (p\bar{y} - 1) + 2\frac{d}{dp}[p\bar{y} - 1]$$
$$-2\bar{y} = 0$$

or $$-[p^2\frac{d\bar{y}}{dp} + 2p\bar{y} - 1] + (p\bar{y} - 1) + 2[p\frac{d\bar{y}}{dp} + \bar{y}]$$
$$-2\bar{y} = 0$$

or $$-[(p^2 - 2p)\frac{d\bar{y}}{dp}] - 2p\bar{y} + p\bar{y} + \cancel{2\bar{y}} - \cancel{2\bar{y}} + \cancel{1} - \cancel{1} = 0$$

or $$-(p^2 - 2p)\frac{d\bar{y}}{dp} - p\bar{y} = 0$$

$p \neq 0$

or $$\frac{d\bar{y}}{\bar{y}} + \frac{dp}{(p-2)} = 0$$

For $p[(p-2)\frac{d\bar{y}}{dp} + \bar{y}] = 0$ $\quad \because p \neq 0$

or $\dfrac{d\bar{y}}{\bar{y}} + \dfrac{dp}{(p-2)} = 0$

on integrating, we get

$$\log(\bar{y}) + \log(p-2) = \log c$$

or $\log[\bar{y}(p-2)] = \log c$

or $\bar{y}(p-2) = c$

$$\bar{y} = \frac{c}{p-2} \quad \text{——— (2)}$$

Taking inverse Laplace transform, both sides.

$$y(t) = L^{-1}\left[\frac{c}{p-2}\right] = c\,e^{2t}$$

$$\therefore \quad y(t) = c\,e^{2t} \quad \text{——— (3)}$$

But $y(0) = 1 \Rightarrow \quad 1 = c e^{0} \quad \Rightarrow c = 1$

Therefore, $y(t) = e^{2t}$ which is required solution of the given equation.