Numerical Linear Algebra:

LU Decomposition or factorization of a matrix →

Lower-upper (LU) decomposition can be defined as the product of a lower and an upper triangular matrices.

Consider the system of eqns in three variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These can be written in the form of $AX = B$ as;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Steps to solve by LU decomposition method :-

Step 1: Generate a matrix $A = LU$ such that L is the lower triangular matrix with principal diagonal elements being equal to 1 and U is the upper triangular matrix. That means

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step 2: Now we can write $AX = B$ as

$$LUX = B \quad \cdots (1) \qquad (\because A = LU)$$

Step 3: Let us assume $UX = Y \quad \cdots (2)$

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Step 4: From eqns (1) and (2), we have; $LY = B$

On solving this eqn, we get y_1, y_2, y_3.

Step 5: Substituting Y in eqn (2), we get $UX = Y$

By solving eqn, we get X i.e, x_1, x_2, x_3.

The above process is also called the process of triangularisation.

Ex. Solve the system of eqns $x_1+x_2+x_3=1$, $3x_1+x_2-3x_3=5$, $x_1-2x_2-5x_3=10$ by LU decomposition method.

Soln: Given System of eqns are:

$$x_1+x_2+x_3=1$$
$$3x_1+x_2-3x_3=5$$
$$x_1-2x_2-5x_3=10$$

These eqns are written in the form $AX=B$ as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

Step 1: Let us write the above matrix as $LU=A$. That means

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

By expanding the left side matrices we get,

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

Thus by equating the corresponding elements, we get;

$u_{11}=1$, $u_{12}=1$, $u_{13}=1$

$l_{21}u_{11}=3$, i.e, $l_{21}=3$

$l_{21}u_{12}+u_{22}=1$ i.e, $3\times1+u_{22}=1 \Rightarrow u_{22}=1-3=-2$

$u_{21}u_{13}+u_{23}=-3 \Rightarrow 3\times1+u_{23}=-3 \Rightarrow u_{23}=-6$

$l_{31}u_{11}=1 \Rightarrow l_{31}=1$

$l_{31}u_{12}+l_{32}u_{22}=-2 \Rightarrow 1\times1+l_{32}\times-2=-2 \Rightarrow 1-2l_{32}=-2$
$\Rightarrow -2l_{32}=-3 \Rightarrow l_{32}=\frac{3}{2}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=-5 \Rightarrow 1\times1+\frac{3}{2}\times-\overset{3}{\cancel{6}}+u_{33}=-5$

Solving these eqns we get $\Rightarrow 1-9+u_{33}=-5 \Rightarrow 3$

$u_{22}=-2$, $u_{23}=-6$, $u_{33}=3$, $l_{21}=3$, $l_{31}=1$, $l_{32}=3/2$

Step 2: $LUX=B$

Step 3: Let $UX=Y$

Step 4: From the previous two steps, we have $LY=B$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

so,

$$y_1 = 1$$

$$3y_1 + y_2 = 5$$

$$y_1 + \frac{3}{2}y_2 + y_3 = 10$$

solving these eqns, we get;

$$y_1 = 1,\ y_2 = 2,\ y_3 = 6$$

Step5: Now consider, $UX = Y$. So,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

By expanding this eqn, we get

$$x_1 + x_2 + x_3 = 1$$

$$-2x_2 - 6x_3 = 2$$

$$3x_3 = 6$$

Solving these eqns, we can get

$$x_3 = 2,\ x_2 = -7,\ x_1 = 6$$

Therefore, the soln of the given system of eqns is $(6, -7, 2)$

Q. Find the soln of the system of eqns. by LU decomposition.

$x+2y+3z=9,\ 4x+5y+6z=24,\ 3x+y-2z=4$.

Cholesky Factorization: Similar to LU factorization method.

It is suitable for symmetric matrix $(A = A^T)$ and positive definite $(x^T A x > 0)$.

$A = L L^T$

Step 1:
$$A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

L (lower triangular) $\qquad L^T$ (upper triangular)

Step 2: $AX = B$

$L \underbrace{L^T X} = B$

$LY = B \rightarrow$ we solve for Y

Step 3: $L^T X = Y \rightarrow$ we solve for X

Ex.
$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} \rightarrow \text{Find } L.$$

Soln. This is symmetric matrix

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

Expanding ~~we get~~ the leftside matrices and equating we get

$l_{11}^2 = 4 \quad \therefore l_{11} = 2$

$l_{11} \cdot l_{21} + 0 = 12 \quad \therefore l_{21} = 6$

$l_{11} l_{31} = -16 \quad \therefore l_{31} = -8$

$l_{21} l_{11} = 12 \quad \therefore l_{21} = 6$

$l_{21} l_{21} + l_{22} l_{22} = 37 \quad \therefore l_{21}^2 + l_{22}^2 = 37 \Rightarrow 36 + l_{22}^2 = 37 \Rightarrow l_{22} = 1$

$l_{31} l_{21} + l_{32} l_{22} = -43 \Rightarrow -8 \times 6 + l_{32} = -43 \Rightarrow l_{32} = -43 + 48 = 5$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 98 \Rightarrow (-8)^2 + (5)^2 + l_{33}^2 = 98 \Rightarrow 64 + 25 + l_{33}^2 = 98$

$\Rightarrow 89 + l_{33}^2 = 98$

$\Rightarrow l_{33}^2 = 98 - 89 = 9$

$\Rightarrow l_{33} = 3$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$$

Ex: Solve the system by cholesky method

$x+2y+3z=5,\ 2x+8y+22z=6,\ 3x+22y+82z=-10$

Soln: Let the given system is

$$Ax = B \quad —(1)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

Let $LL^T = A$ —(2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$\therefore l_{11}^2 = 1 \Rightarrow l_{11} = 1$

$l_{11} l_{21} + 0 = 2 \Rightarrow l_{21} = 2$

$l_{11} l_{31} = 3 \Rightarrow l_{31} = 3$

$l_{21} l_{11} = 2$

$l_{21}^2 + l_{22}^2 = 8 \Rightarrow l_{22}^2 = 8-4 = 4 \Rightarrow l_{22} = 2$

$l_{21} l_{31} + l_{22} l_{32} = 22 \Rightarrow 2 \times 3 + 2 l_{32} = 22$

$\Rightarrow l_{32} = 8$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82 = 9 + 64 + l_{33}^2 = 82$

$\Rightarrow l_{33}^2 = 9 \Rightarrow l_{33} = 3$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

By (1) and (2) $LL^T X = B$ —(3)

Put $L^T X = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ —(4)

Then (3) becomes $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

ie, $y_1 = 5$

$2y_1 + 2y_2 = 6$

$3y_1 + 8y_2 + 3y_3 = -10$

$\therefore y_2 = -2,\ y_3 = -3$

$\therefore \boxed{y_1 = 5,\ y_2 = -2,\ y_3 = -3}$

By eqn (4) ie, $L^T X = Y$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$3z = -3 \Rightarrow z = -1$.

$2y + 8z = -2 \Rightarrow y = 3$

$x + 2y + 3z = 5 \Rightarrow x = 2$

$\therefore \boxed{x = 2,\ y = 3,\ z = -1}$

Singular Value Decomposition

Theorem. Singular Value Theorem for Linear Transformation →

Let V and W be finite-dimensional inner product spaces, and let $T: V \to W$ be a L.T of rank r. Then there exists orthonormal bases $\{v_1, v_2, \cdots, v_n\}$ for V and $\{u_1, u_2, \cdots, u_m\}$ for W and positive scalars $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ such that

$$T(v_i) = \begin{cases} \sigma_i u_i & \text{if } 1 \leq i \leq r \\ 0 & \text{if } i > r \end{cases}$$

Conversely, suppose that the preceding conditions are satisfied. Then for $1 \leq i \leq n$, v_i is an eigen vector of T^*T with corresponding eigen value σ_i^2 if $1 \leq i \leq r$ and 0 if $i > r$. Therefore the scalars $\sigma_1, \sigma_2, \cdots, \sigma_r$ are uniquely determined by T.

Singular Values of Transformation

Def^n → The unique scalars $\sigma_1, \sigma_2, \cdots, \sigma_r$ are called the singular values of T. If r is less than both m and n, then the term singular value is extended to include $\sigma_{r+1} = \cdots = \sigma_k = 0$, where k is the minimum of m and n.

Singular Values of matrix

Def^n → Let A be a $m \times n$ matrix. We define the singular values of A to be the singular values of the linear transformation L_A.

Singular Value Decomposition Theorem for Matrices :

Let A be an $m \times n$ matrix of rank r with the positive singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, and let Σ be the $m \times n$ matrix defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that $A = U \Sigma V^*$.

Singular Value Decomposition of A

Def^n → Let A be an $m \times n$ matrix of rank r with positive singular value $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$. A factorization $A = U \Sigma V^*$ where U and V are unitary matrices and Σ is the $m \times n$ matrix defined

$$\text{by } \Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

is called a singular value decomposition of A.

Let A be an $m \times n$ matrix. Then $A = U \Sigma V^T$ is the singular value decomposition of A.

- U is $m \times m$ orthogonal matrix with columns equal to the unit eigenvectors of AA^T.

$$U = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{u_1} & \vec{u_2} & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

- V is an $n \times n$ orthogonal matrix whose columns are unit eigenvectors $A^T A$:

$$\cancel{A^T A}\, V = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{v_1} & \vec{v_2} & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

- Σ is an $m \times n$ matrix with the singular values of A on the main diagonal and all other entries of zero.

$$\Sigma = \begin{bmatrix} \vec{\sigma_1} & 0 & 0 \\ 0 & \vec{\sigma_2} & \vdots \\ 0 & 0 & \vdots \end{bmatrix}$$

Eigen values → $|A - \lambda I| = 0$

Eigen vectors → $[A - \lambda I]X = 0$

$$A_{2 \times 3} = U_{2 \times 2} \Sigma_{2 \times 3} V^T_{3 \times 3}$$

Singular values of a matrix →

Let A be $m \times n$ matrix

- The singular values of A are the square roots of the positive eigen-values of $A^T A$ or AA^T.
- $A^T A$ and AA^T have the same positive eigenvalues.

Steps →

1. Determine V and then V^T
2. Determine the singular values σ_i and then Σ.
3. Determine U using $A = U \Sigma V^T \rightarrow AV = U\Sigma$ since V is orthogonal to V^T, we know $VV^T = I$.

Q: $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ find the singular values of A.

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$$A^T A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

For eigen values. $|A - \lambda I| = 0$

$$\begin{vmatrix} 10-\lambda & 2 & 6 \\ 2 & 2-\lambda & -2 \\ 6 & -2 & 10-\lambda \end{vmatrix} = 0$$

ch. eqn. is $\lambda(\lambda-16)(\lambda-6) = 0 \Rightarrow \lambda_1 = 16, \lambda_2 = 6, \lambda_3 = 0$
(order from greatest to least)

Corr. eigen vectors for $\lambda_1 = 16$
$[A - \lambda I] X = 0$

$$\begin{bmatrix} 10-16 & 2 & 6 \\ 2 & 2-16 & -2 \\ 6 & -2 & 10-16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 2 & 6 \\ 2 & -14 & -2 \\ 6 & -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - x_3 = 0 \\ x_2 = 0 \\ x_3 = x_3 \end{matrix} \quad \begin{matrix} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{matrix} \quad X_1 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$t \neq 0$

$$\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(normalizing the vector X_1)

For $\lambda_2 = 6$, $[A - \lambda I] X = 0$

$$\begin{bmatrix} 10-6 & 2 & 6 \\ 2 & 2-6 & -2 \\ 6 & -2 & 10-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 6 \\ 2 & -4 & -2 \\ 6 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = x_3 \end{matrix} \quad \begin{matrix} x_1 = -t \\ x_2 = -t \\ x_3 = t \end{matrix} \quad X_2 = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$t \neq 0$

$$\vec{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

For $\lambda_3 = 0$, $[A - \lambda I] X = 0$

$$\begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 - 2x_3 = 0 \\ x_3 = x_3 \end{matrix} \quad \begin{matrix} x_1 = -t \\ x_2 = 2t \\ x_3 = t \end{matrix} \quad X_3 = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

Find SVD for

Q $A = \begin{array}{c} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} \\ \begin{matrix} v_1 & \quad v_2 & \quad v_3 \end{matrix} \end{array}$, $\lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 0, \therefore \sigma_1 = \sqrt{6}$, $\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$, $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$V = [v_1 \; v_2 \; v_3]$

$$\therefore V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \text{ and } V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The singular values of AA^T in order from greatest to least are:

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{6}$

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}$$

[Σ = singular values = $\sqrt{\text{(Eigenvalues)}}$]

Now we need to find U.

$A = U \Sigma V^T$

$AV = U \Sigma V^T V$

$AV = U \Sigma I$

Since V and V^T are orthogonal $V^T V = I$

$AV = U\Sigma$

It follows: $A\vec{v}_1 = \sigma_1 \vec{u}_1$

$A\vec{v}_2 = \sigma_2 \vec{u}_2$

$$Av_1 = \sigma_1 u_1 \rightarrow u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix}$$

$$Av_2 = \sigma_2 u_2 \rightarrow u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ $A = U \Sigma V^T$

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}}_{V^T} \; \underbrace{=}_{=} \; \underbrace{\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}}_{A}$$

Q. Find SVD for the given matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

Soln: $\rightarrow$ $A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$

For eigenvalue $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & 2-\lambda & -2 \\ -2 & -2 & 2-\lambda \end{vmatrix} = 0$$

ch. eqn is

$$(2-\lambda)[(2-\lambda)^2-4] - 2[2(2-\lambda)-4] - 2[-4+2(2-\lambda)] = 0$$

$$\Rightarrow (2-\lambda)[4-4\lambda+\lambda^2-4] - 2[4-2\lambda-4] - 2[-4+4-2\lambda] = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2-4\lambda) + 4\lambda + 4\lambda = 0$$

$$\Rightarrow 2\lambda^2 - \cancel{8\lambda} - \lambda^3 + 4\lambda^2 + \cancel{8\lambda} = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 = 0$$

$$\Rightarrow \lambda^2(\lambda - 6) = 0$$

$$\Rightarrow \lambda = 0, 0, 6$$

$$\therefore \lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 0$$

Corr. eigen vector for $\lambda_1 = 6$ is $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} -4 & 2 & -2 \\ 2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$ normalizing the vector X_1 we get

$$v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Eigen vector corr. to $\lambda_2 = 0 = \lambda_3$

$$\begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$ normalizing the vector X_2

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, normalizing the vector X_3, and $v_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\therefore V = \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt3} & 0 & \frac{2}{\sqrt6} \end{bmatrix}, \quad V^T = \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{2}{\sqrt6} \end{bmatrix}$$

The singular values of A^TA in order from greatest to least are :-
$\sigma_1 = \sqrt6$ is the only non-zero singular value of A.

$$\therefore \Sigma = \begin{bmatrix} \sqrt6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\because$ V and V^T are orthogonal $V^TV = I$.

$$AV = U\Sigma$$

It follows : $Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$

$$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt6}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \\ -\frac{1}{\sqrt3} \end{bmatrix}$$

$$\therefore u_1 = \frac{1}{\sqrt{6\times3}}\begin{bmatrix} 3 \\ 3 \end{bmatrix}_{2\times1} = \frac{3}{3\sqrt2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Next choose $u_2 = \frac{1}{\sigma_2} Av_2 = \frac{1}{\sqrt2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, a unit vector orthogonal to u_1,

to obtain orthonormal basis $U = \{u_1, u_2\}$ for R^2 and set

$$U = \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}$$

Then $A = U\Sigma V^T$ is the desired SVD.

$$\therefore \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}\begin{bmatrix} \sqrt6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} & 0 \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt6} & \frac{2}{\sqrt6} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

Pseudo inverse of A =

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt3} & 0 & \frac{2}{\sqrt6} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

The Pseudoinverse of a Matrix $\rightarrow$ (Generalization of Matrix inverse)

Let A be a $m \times n$ matrix. Then there exists a unique $n \times m$ matrix B such that $(L_A)^{+} : F^m \to F^n$ is equal to the left multiplication transformation L_B. We call B the pseudo inverse of A and denote it by $B = A^{+}$. Thus $(L_A)^{+} = L_{A^+}$.

Th. Let A be an $m \times n$ matrix of rank r with a singular value decomposition $A = U \Sigma V^T$ and non-zero singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$. Let Σ^{+} be the $n \times m$ matrix defined by

$$\Sigma^{+}_{ij} = \begin{cases} \frac{1}{\sigma_i} & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

Then $A^{+} = V \Sigma^{+} U^T$, and this is a SVD of A^{+}.

Note: Σ^{+} is pseudo inverse of Σ.

~~SVD to Pseudo~~ inverse

~~$Ax = b$~~

~~$A = U\Sigma V^T \Rightarrow A^{+} = V\Sigma^{-1}U^T$~~

Q. Find A^{+} for the matrix $\cancel{A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}}$ $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Soln. $\rightarrow$

$$A^T A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

Eigen values $|A - \lambda I| = 0$ are, $\lambda_1 = 16$, $\lambda_2 = 6$, $\lambda_3 = 0$

Corr. Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

$\therefore$ $u_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, $u_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$, $u_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \quad V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Singular values of AA^T are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{6}$

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

we have

$$A^{+} = V\Sigma^{+}U^{T} = \begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt3} & -\frac{1}{\sqrt6} \\ 0 & -\frac{1}{\sqrt3} & \frac{2}{\sqrt6} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{\sqrt6} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}$$

$$= \frac{1}{96}\begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}$$

Least Squares and SVD

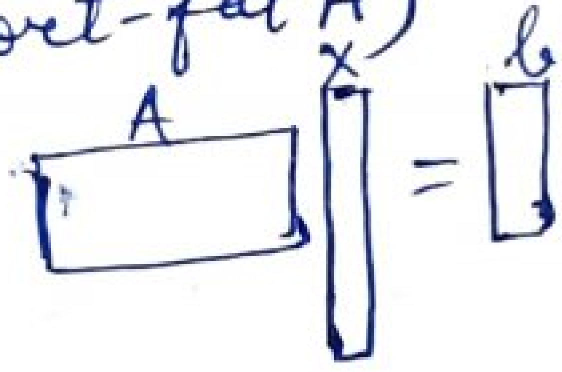
$A\,\underset{\text{solve}}{x} = b$ linear system of eqns.

A known, b known

SVD allows us to generalize to non-square A.

underdetermined, $n < m$ (short-fat A)

- ∞ many solns x given b



$\min \|\tilde{x}\|$ s.t. $A\tilde{x} = b$

(minimum norm soln)

overdetermined, $n > m$ (tall skinny A)

- zero soln x for given b.

(least squares soln)

$\min \|A\tilde{x} - b\|$

$$Ax = b$$
$$U\Sigma V^{T}x = b$$
$$V\Sigma^{-1}U^{T}U\Sigma V^{T}x = \underbrace{V\Sigma^{-1}U^{T}}_{A^{+}}b$$
$$\tilde{x} = V\Sigma^{-1}U^{T}b$$
$$= A^{+}b$$

Q. $A = \begin{bmatrix} 1 & 1 \\ .0001 & 0 \\ 0 & .0001 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ .0001 \\ .0001 \end{bmatrix}$

$U = \begin{pmatrix} -1 & 0 \\ 0 & -0.7071 \\ 0 & 0.7071 \end{pmatrix}$, $\Sigma = \begin{bmatrix} 1.4142 & 0 \\ 0 & 0.0001 \end{bmatrix}$, $V = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$

$$\tilde{x} = V\Sigma^{-1}U^{T}b$$
$$= A^{+}b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Low Rank Approximation $\rightarrow$

SVD provide a very simple solution to low rank approximation problem

Suppose $A \in \mathbb{R}^{m \times n}$, has SVD

$$A = U \Sigma V^{t} = \sum_{i=1}^{N} u_i \sigma_i v_i^{t}$$

then the k- approximation to A is given by

$$A_k = \sum_{i=1}^{k} u_i \sigma_i v_i^{t} \qquad \text{where } k \le \text{rank} A.$$

let A be 5×5 matrix.

$\sigma_1 = 3, \ \sigma_2 = 1, \ \sigma_3 = 0.5, \ \sigma_4 = 0.2, \ \sigma_5 = 0.05$

$$A_{5\times5} = U_{5\times5} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{bmatrix} V^{t}_{5\times5}.$$

$$A_3 = U_{5\times3} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V^{t}_{3\times5}.$$

$$= \begin{bmatrix} u_1 & u_2 & u_3 & \cancel{u_4} & \cancel{u_5} \\ | & | & | & | & | \end{bmatrix} \Sigma_{3\times3} \begin{bmatrix} v_1 & v_2 & v_3 & \cancel{v_4} & \cancel{v_5} \\ | & | & | & | & | \end{bmatrix}^{t}$$

$$A_3 = \underbrace{\begin{bmatrix} u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}}_{5\times 3} \underbrace{\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \\ & & \end{bmatrix}^t}_{3\times 5}.$$

A measure of quality of the approximation is given by

$$\frac{\|A_k\|_F^2}{\|A\|_F^2} = \frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2}{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}.$$

Ex Find rank 2 approximation of

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \Sigma V^t$$

$$A = \begin{bmatrix} 0.91 & 0.42 & 0.02 \\ 0.41 & -0.87 & -0.26 \\ 0.09 & -0.24 & 0.97 \end{bmatrix} \begin{bmatrix} 4.04 & 0 & 0 \\ 0 & 1.20 & 0 \\ 0 & 0 & 0.87 \end{bmatrix} \begin{bmatrix} 0.67 & 0.73 & 0.08 \\ 0.65 & -0.53 & -0.53 \\ 0.35 & -0.41 & 0.84 \end{bmatrix}^t$$

$$A_2 = \begin{bmatrix} 0.91 & 0.42 \\ 0.41 & -0.87 \\ 0.09 & -0.24 \end{bmatrix} \begin{bmatrix} 4.04 & 0 \\ 0 & 1.20 \end{bmatrix} \begin{bmatrix} 0.67 & 0.73 \\ 0.65 & -0.53 \\ 0.35 & -0.41 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2.99 & 2.01 & 0.98 \\ 0.02 & 1.98 & 1.1 \\ -0.07 & 0.45 & 0.4 \end{bmatrix} \rightarrow$$