

CHAPTER 13

Moment generating functions

13.1. Definition and examples

Definition (Moment generating function)

The *moment generating function* (MGF) of a random variable X is a function $m_X(t)$ defined by

$$m_X(t) = \mathbb{E}e^{tX},$$

provided the expectation is finite.

In the discrete case m_X is equal to $\sum_x e^{tx}p(x)$ and in the continuous case $\int_{-\infty}^{\infty} e^{tx}f(x)dx$.

Let us compute the moment generating function for some of the distributions we have been working with.

Example 13.1 (Bernoulli).

$$m_X(t) = e^{0t}(1-p) + e^{1t}p = e^tp + 1 - p.$$

Example 13.2 (Binomial). Using independence,

$$\mathbb{E}e^{t\sum X_i} = \mathbb{E}\prod e^{tX_i} = \prod \mathbb{E}e^{tX_i} = (pe^t + (1-p))^n,$$

where the X_i are independent Bernoulli random variables. Equivalently

$$\sum_{k=0}^n e^{tk} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^n \binom{n}{k} (pe^t)^k (1-p)^{n-k} = (pe^t + (1-p))^n$$

by the Binomial formula.

Example 13.3 (Poisson).

$$\mathbb{E}e^{tX} = \sum_{k=0}^{\infty} \frac{e^{tk} e^{-\lambda} \lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t-1)}.$$

Example 13.4 (Exponential).

$$\mathbb{E}e^{tX} = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t}$$

if $t < \lambda$, and ∞ if $t \geq \lambda$.

Example 13.5 (Standard normal). Suppose $Z \sim \mathcal{N}(0, 1)$, then

$$m_Z(t) = \frac{1}{\sqrt{2\pi}} \int e^{tx} e^{-x^2/2} dx = e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int e^{-(x-t)^2/2} dx = e^{t^2/2}.$$

Example 13.6 (General normal). Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$, then we can write $X = \mu + \sigma Z$, and therefore

$$m_X(t) = \mathbb{E}e^{tX} = \mathbb{E}e^{t\mu} e^{t\sigma Z} = e^{t\mu} m_Z(t\sigma) = e^{t\mu} e^{(t\sigma)^2/2} = e^{t\mu + t^2\sigma^2/2}.$$

Proposition 13.1

Suppose X_1, \dots, X_n are n independent random variables, and the random variable Y is defined by

$$Y = X_1 + \dots + X_n.$$

Then

$$m_Y(t) = m_{X_1}(t) \cdot \dots \cdot m_{X_n}(t).$$

PROOF. By independence of X and Y and Proposition 12.1 we have

$$m_Y(t) = \mathbb{E}(e^{tX_1} \cdot \dots \cdot e^{tX_n}) = \mathbb{E}e^{tX_1} \cdot \dots \cdot \mathbb{E}e^{tX_n} = m_{X_1}(t) \cdot \dots \cdot m_{X_n}(t).$$

□

Proposition 13.2

Suppose for two random variables X and Y we have $m_X(t) = m_Y(t) < \infty$ for all t in an interval, then X and Y have the same distribution.

We will not prove this, but this statement is essentially the uniqueness of the Laplace transform \mathcal{L} . Recall that the *Laplace transform* of a function $f(x)$ defined for all positive real numbers $s \geq 0$

$$(\mathcal{L}f)(s) := \int_0^\infty f(x) e^{-sx} dx$$

Thus if X is a continuous random variable with the PDF such that $f_X(x) = 0$ for $x < 0$, then

$$\int_0^\infty e^{tx} f_X(x) dx = \mathcal{L}f_X(-t).$$

Proposition 13.1 allows to show some of the properties of sums of independent random variables we proved or stated before.

Example 13.7 (Sums of independent normal random variables). If $X \sim \mathcal{N}(a, b^2)$ and $Y \sim \mathcal{N}(c, d^2)$, and X and Y are independent, then by Proposition 13.1

$$m_{X+Y}(t) = e^{at+b^2t^2/2} e^{ct+d^2t^2/2} = e^{(a+c)t+(b^2+d^2)t^2/2},$$

which is the moment generating function for $\mathcal{N}(a + c, b^2 + d^2)$. Therefore Proposition 13.2 implies that $X + Y \sim \mathcal{N}(a + c, b^2 + d^2)$.

Example 13.8 (Sums of independent Poisson random variables). Similarly, if X and Y are independent Poisson random variables with parameters a and b , respectively, then

$$m_{X+Y}(t) = m_X(t)m_Y(t) = e^{a(e^t-1)}e^{b(e^t-1)} = e^{(a+b)(e^t-1)},$$

which is the moment generating function of a Poisson with parameter $a + b$, therefore $X + Y$ is a Poisson random variable with parameter $a + b$.

One problem with the moment generating function is that it might be infinite. One way to get around this, at the cost of considerable work, is to use the *characteristic function* $\varphi_X(t) = \mathbb{E}e^{itX}$, where $i = \sqrt{-1}$. This is always finite, and is the analogue of the *Fourier transform*.

Definition

Joint MGF The *joint moment generating function* of X and Y is

$$m_{X,Y}(s, t) = \mathbb{E}e^{sX+tY}.$$

If X and Y are independent, then

$$m_{X,Y}(s, t) = m_X(s)m_Y(t)$$

by Proposition 13.2. We will not prove this, but the converse is also true: if $m_{X,Y}(s, t) = m_X(s)m_Y(t)$ for *all* s and t , then X and Y are independent.

13.2. Further examples and applications

Example 13.9. Suppose that m.g.f of X is given by $m(t) = e^{3(e^t-1)}$. Find $\mathbb{P}(X = 0)$.

Solution: We can match this MGF to a known MGF of one of the distributions we considered and then apply Proposition 13.2. Observe that $m(t) = e^{3(e^t-1)} = e^{\lambda(e^t-1)}$, where $\lambda = 3$. Thus $X \sim \text{Poisson}(3)$, and therefore

$$\mathbb{P}(X = 0) = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-3}.$$

This example is an illustration to why $m_X(t)$ is called the *moment generating function*. Namely we can use it to find all the moments of X by differentiating $m(t)$ and then evaluating at $t = 0$. Note that

$$m'(t) = \frac{d}{dt} \mathbb{E}[e^{tX}] = \mathbb{E}\left[\frac{d}{dt} e^{tX}\right] = \mathbb{E}[X e^{tX}].$$

Now evaluate at $t = 0$ to get

$$m'(0) = \mathbb{E}[X e^{0 \cdot X}] = \mathbb{E}[X].$$

Similarly

$$m''(t) = \frac{d}{dt} \mathbb{E}[X e^{tX}] = \mathbb{E}[X^2 e^{tX}],$$

so that

$$m''(0) = \mathbb{E}[X^2 e^0] = \mathbb{E}[X^2].$$

Continuing to differentiate the MGF we have the following proposition.

Proposition 13.3 (Moments from MGF)

For all $n \geq 0$ we have

$$\mathbb{E}[X^n] = m^{(n)}(0).$$

Example 13.10. Suppose X is a discrete random variable and has the MGF

$$m_X(t) = \frac{1}{7}e^{2t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{8t}.$$

What is the PMF of X ? Find $\mathbb{E}X$.

Solution: this does not match any of the known MGFs directly. Reading off from the MGF we guess

$$\frac{1}{7}e^{2t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{8t} = \sum_{i=1}^4 e^{tx_i} p(x_i)$$

then we can take $p(2) = \frac{1}{7}, p(3) = \frac{3}{7}, p(5) = \frac{2}{7}$ and $p(8) = \frac{1}{7}$. Note that these add up to 1, so this is indeed a PMF.

To find $\mathbb{E}[X]$ we can use Proposition 13.3 by taking the derivative of the moment generating function as follows.

$$m'(t) = \frac{2}{7}e^{2t} + \frac{9}{7}e^{3t} + \frac{10}{7}e^{5t} + \frac{8}{7}e^{8t},$$

so that

$$\mathbb{E}[X] = m'(0) = \frac{2}{7} + \frac{9}{7} + \frac{10}{7} + \frac{8}{7} = \frac{29}{7}.$$

Example 13.11. Suppose X has the MGF

$$m_X(t) = (1 - 2t)^{-\frac{1}{2}} \text{ for } t < \frac{1}{2}.$$

Find the first and second moments of X .

Solution: We have

$$\begin{aligned} m'_X(t) &= -\frac{1}{2}(1 - 2t)^{-\frac{3}{2}}(-2) = (1 - 2t)^{-\frac{3}{2}}, \\ m''_X(t) &= -\frac{3}{2}(1 - 2t)^{-\frac{5}{2}}(-2) = 3(1 - 2t)^{-\frac{5}{2}}. \end{aligned}$$

So that

$$\begin{aligned} \mathbb{E}X &= m'_X(0) = (1 - 2 \cdot 0)^{-\frac{3}{2}} = 1, \\ \mathbb{E}X^2 &= m''_X(0) = 3(1 - 2 \cdot 0)^{-\frac{5}{2}} = 3. \end{aligned}$$

13.3. Exercises

Exercise 13.1. Suppose that you have a fair 4-sided die, and let X be the random variable representing the value of the number rolled.

- (a) Write down the moment generating function for X .
- (b) Use this moment generating function to compute the first and second moments of X .

Exercise 13.2. Let X be a random variable whose probability density function is given by

$$f_X(x) = \begin{cases} e^{-2x} + \frac{1}{2}e^{-x} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

- (a) Write down the moment generating function for X .
- (b) Use this moment generating function to compute the first and second moments of X .

Exercise 13.3. Suppose that a mathematician determines that the revenue the UConn Dairy Bar makes in a week is a random variable, X , with moment generating function

$$m_X(t) = \frac{1}{(1 - 2500t)^4}$$

Find the standard deviation of the revenue the UConn Dairy bar makes in a week.

Exercise 13.4. Let X and Y be two independent random variables with respective moment generating functions

$$m_X(t) = \frac{1}{1 - 5t}, \quad \text{if } t < \frac{1}{5}, \quad m_Y(t) = \frac{1}{(1 - 5t)^2}, \quad \text{if } t < \frac{1}{5}.$$

Find $\mathbb{E}(X + Y)^2$.

Exercise 13.5. Suppose X and Y are independent Poisson random variables with parameters λ_x, λ_y , respectively. Find the distribution of $X + Y$.

Exercise 13.6. True or False? If $X \sim \text{Exp}(\lambda_x)$ and $Y \sim \text{Exp}(\lambda_y)$ then $X + Y \sim \text{Exp}(\lambda_x + \lambda_y)$. Justify your answer.

13.4. Selected solutions**Solution to Exercise 13.1(A):**

$$\begin{aligned}
m_X(t) &= \mathbb{E} [e^{tX}] = e^{1 \cdot t} \frac{1}{4} + e^{2 \cdot t} \frac{1}{4} + e^{3 \cdot t} \frac{1}{4} + e^{4 \cdot t} \frac{1}{4} \\
&= \frac{1}{4} (e^{1 \cdot t} + e^{2 \cdot t} + e^{3 \cdot t} + e^{4 \cdot t})
\end{aligned}$$

Solution to Exercise 13.1(B): We have

$$\begin{aligned}
m'_X(t) &= \frac{1}{4} (e^{1 \cdot t} + 2e^{2 \cdot t} + 3e^{3 \cdot t} + 4e^{4 \cdot t}), \\
m''_X(t) &= \frac{1}{4} (e^{1 \cdot t} + 4e^{2 \cdot t} + 9e^{3 \cdot t} + 16e^{4 \cdot t}),
\end{aligned}$$

so

$$\mathbb{E}X = m'_X(0) = \frac{1}{4} (1 + 2 + 3 + 4) = \frac{5}{2}$$

and

$$\mathbb{E}X^2 = m''_X(0) = \frac{1}{4} (1 + 4 + 9 + 16) = \frac{15}{2}.$$

Solution to Exercise 13.2(A): for $t < 1$ we have

$$\begin{aligned}
m_X(t) &= \mathbb{E} [e^{tX}] = \int_0^\infty e^{tx} \left(e^{-2x} + \frac{1}{2} e^{-x} \right) dx \\
&= \frac{1}{t-2} e^{tx-2x} + \frac{1}{2(t-1)} e^{tx-x} \Big|_{x=0}^{x=\infty} = \\
&= 0 - \frac{1}{2-t} + 0 - \frac{1}{2(t-1)} \\
&= \frac{1}{t-2} + \frac{1}{2(1-t)} = \frac{t}{2(2-t)(1-t)}
\end{aligned}$$

Solution to Exercise 13.2(B): We have

$$\begin{aligned}
m'_X(t) &= \frac{1}{(2-t)^2} + \frac{1}{2(1-t)^2} \\
m''_X(t) &= \frac{2}{(2-t)^3} + \frac{1}{(1-t)^3}
\end{aligned}$$

and so $\mathbb{E}X = m'_X(0) = \frac{3}{4}$ and $\mathbb{E}X^2 = m''_X(0) = \frac{5}{4}$.

Solution to Exercise 13.3: We have $\text{SD}(X) = \sqrt{\text{Var}(X)}$ and $\text{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$. Therefore we can compute

$$\begin{aligned} m'(t) &= 4(2500)(1 - 2500t)^{-5}, \\ m''(t) &= 20(2500)^2(1 - 2500t)^{-6}, \\ \mathbb{E}X &= m'(0) = 10,000 \\ \mathbb{E}X^2 &= m''(0) = 125,000,000 \\ \text{Var}(X) &= 125,000,000 - 10,000^2 = 25,000,000 \\ \text{SD}(X) &= \sqrt{25,000,000} = \mathbf{5,000}. \end{aligned}$$

Solution to Exercise 13.4: First recall that if we let $W = X + Y$, and using that X, Y are independent, then we see that

$$m_W(t) = m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{1}{(1 - 5t)^3},$$

recall that $\mathbb{E}[W^2] = m_W''(0)$, which we can find from

$$\begin{aligned} m_W'(t) &= \frac{15}{(1 - 5t)^4}, \\ m_W''(t) &= \frac{300}{(1 - 5t)^5}, \end{aligned}$$

thus

$$\mathbb{E}[W^2] = m_W''(0) = \frac{300}{(1 - 0)^5} = 300.$$

Solution to Exercise 13.5: Since $X \sim \text{Pois}(\lambda_x)$ and $Y \sim \text{Pois}(\lambda_y)$ then

$$\begin{aligned} m_X(t) &= e^{\lambda_x(e^t - 1)}, \\ m_Y(t) &= e^{\lambda_y(e^t - 1)}. \end{aligned}$$

Then

$$\begin{aligned} m_{X+Y}(t) &= m_X(t)m_Y(t) \\ &\stackrel{\text{independence}}{=} e^{\lambda_x(e^t - 1)}e^{\lambda_y(e^t - 1)} = e^{(\lambda_x + \lambda_y)(e^t - 1)}. \end{aligned}$$

Thus $X + Y \sim \text{Pois}(\lambda_x + \lambda_y)$.

Solution to Exercise 13.6: We will use Proposition 13.2. Namely, we first find the MGF of $X + Y$ and compare it to the MGF of a random variable $V \sim \text{Exp}(\lambda_x + \lambda_y)$. The MGF of V is

$$m_V(t) = \frac{\lambda_x + \lambda_y}{\lambda_x + \lambda_y - t} \text{ for } t < \lambda_x + \lambda_y.$$

By independence of X and Y

$$m_{X+Y}(t) = m_X(t)m_Y(t) = \frac{\lambda_x}{\lambda_x - t} \cdot \frac{\lambda_y}{\lambda_y - t},$$

but

$$\frac{\lambda_x + \lambda_y}{\lambda_x + \lambda_y - t} \neq \frac{\lambda_x}{\lambda_x - t} \cdot \frac{\lambda_y}{\lambda_y - t}$$

and hence the statement is false.