Solution. We have $x^2 + y^2 + 2ax + c^2 = 0$ ...(*i*)

Differentiating w.r.t. x, $2x + 2y\,dy/dx + 2a = 0$

or $$2a = -2\left(x + y\frac{dy}{dx}\right)$$

Substituting in (*i*), $x^2 + y^2 - 2(x + y\,dy/dx)x + c^2 = 0$

or $$2xy\,dy/dx = y^2 - x^2 + c^2$$

which is the required differential equation.

11.4 (1) SOLUTION OF A DIFFERENTIAL EQUATION

A solution (or integral) of a differential equation is a relation between the variables which satisfies the given differential equation.

For example, $$x = A\cos(nt + \alpha) \quad ...(1)$$

is a solution of $$\frac{d^2x}{dt^2} + n^2x = 0 \text{ [Example 11.1]} \quad ...(2)$$

The **general** (or **complete**) **solution** *of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.* Thus (1) is a general solution (2) as the number of arbitrary constants (A, α) is the same as the order of (2).

A **particular solution** *is that which can be obtained from the general solution by giving particular values to the arbitrary constants.*

For example, $$x = A\cos(nt + \pi/4)$$

is the particular solution of the equation (2) as it can be derived from the general solution (1) by putting $\alpha = \pi/4$.

A differential equation may sometimes have an additional solution which cannot be obtained from the general solution by assigning a particular value to the arbitrary constant. Such a solution is called a **singular solution** and is not of much engineering interest.

Linearly independent solution. *Two solutions $y_1(x)$ and $y_2(x)$ of the differential equation*

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad ...(3)$$

are said to be linearly independent if $c_1y_1 + c_2y_2 = 0$ such that $c_1 = 0$ and $c_2 = 0$

If c_1 and c_2 are not both zero, then the two solutions y_1 and y_2 are said to be *linearly dependent.*

If $y_1(x)$ and $y_2(x)$ any two solutions of (3), then their linear combination $c_1y_1 + c_2y_2$ where c_1 and c_2 are constants, is also a solution of (3).

Example 11.4. *Find the differential equation whose set of independent solutions is $[e^x, xe^x]$.*

Solution. Let the general solution of the required differential equation be $y = c_1e^x + c_2xe^x$ ...(*i*)

Differentiating (*i*) w.r.t. x, we get

$$y_1 = c_1e^x + c_2(e^x + xe^x)$$

$$\therefore \quad y - y_1 = c_2e^x \quad ...(ii)$$

Again differentiating (*ii*) w.r.t. x, we obtain

$$y_1 - y_2 = c_2e^x \quad ...(iii)$$

Subtracting (*iii*) from (*ii*), we get

$$y - y_1 - (y_1 - y_2) = 0 \quad \text{or} \quad y - 2y_1 + y_2 = 0$$

which is the desired differential equation.

(2) Geometrical meaning of a differential equation. Consider any differential equation of the first order and first degree

$$\frac{dy}{dx} = f(x, y) \quad ...(1)$$

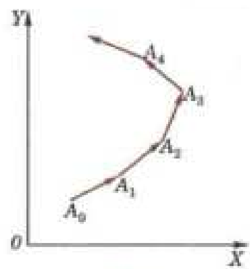


Fig. 11.1

If $P(x, y)$ be any point, then (1) can be regarded as an equation giving the value of dy/dx $(= m)$ when the values of x and y are known (Fig. 11.1). Let the value of m at the point $A_0(x_0, y_0)$ derived from (1) be m_0. Take a neighbouring point $A_1(x_1, y_1)$ such that the slope of A_0A_1 is m_0. Let the corresponding value of m at A_1 be m_1. Similarly take a neighbouring point $A_2(x_2, y_2)$ such that the slope of A_1A_2 is m_1 and so on.

If the successive points A_0, A_1, A_2, A_3 ... are chosen very near one another, the broken curve $A_0A_1A_2A_3$ ... approximates to a smooth curve $C[y = \phi(x)]$ which is a solution of (1) associated with the initial point A_0 (x_0, y_0). Clearly the slope of the tangent to C at any point and the coordinates of that point satisfy (1).

A different choice of the initial point will, in general, give a different curve with the same property. *The equation of each such curve is thus a* **particular solution** *of the differential equation (1). The equation of the whole family of such curves is the general solution of* (1). The slope of the tangent at any point of each member of this family and the co-ordinates of that point satisfy (1).

Such a simple geometric interpretation of the solutions of a second (or higher) order differential equation is not available.

PROBLEMS 11.1

Form the differential equations from the following equations :

1. $y = ax^3 + bx^2$.
2. $y = C_1 \cos 2x + C_2 \sin 2x$ (*Bhopal, 2008*)
3. $xy = Ae^x + Be^{-x} + x^2$. (*U.P.T.U., 2005*)
4. $y = e^x (A \cos x + B \sin x)$. (*P.T.U., 2003*)
5. $y = ae^{2x} + be^{-3x} + ce^x$.

Find the differential equations of :

6. A family of circles passing through the origin and having centres on the x-axis. (*J.N.T.U., 2006*)
7. All circles of radius 5, with their centres on the y-axis.
8. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
9. If $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$ are two solutions of $y'' + 4y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.
10. Determine the differential equation whose set of independent solutions is $[e^x, xe^x, x^2 e^x]$ (*U.P.T.U., 2002*)
11. Obtain the differential equation of the family of parabolas $y = x^2 + c$ and sketch those members of the family which pass through (0, 0), (1, 1), (0, 1) and (1, – 1) respectively.

11.5 EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE

It is not possible to solve such equations in general. We shall, however, discuss some special methods of solution which are applied to the following types of equations :

(*i*) Equations where variables are separable, (*ii*) Homogeneous equations,
(*iii*) Linear equations, (*iv*) Exact equations.

In other cases, the particular solution may be determined numerically (Chapter 31).

11.6 VARIABLES SEPARABLE

If in an equation it is possible to collect all functions of x and dx on one side and all the functions of y and dy on the other side, then the *variables are said to be separable*. Thus the general form of such an equation is $f(y)\, dy = \phi(x)\, dx$

Integrating both sides, we get $\int f(y)\, dy = \int \phi(x)\, dx + c$ as its solution.

Example 11.5. *Solve* $dy/dx = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$. (*V.T.U., 2008*)

Solution. Given equation is $x(2 \log x + 1)\, dx = (\sin y + y \cos y)\, dy$

Integrating both sides, $2 \int (\log x \,.\, x + x)\, dx = \int \sin y\, dy + \int y \cos y\, dy + c$

or $$2\left[\left(\log x \,.\, \frac{x^2}{2} - \int \frac{1}{x} \,.\, \frac{x^2}{2} dx\right) + \frac{x^2}{2}\right] = -\cos y + \left[y \sin y - \int \sin y \,.\, 1\, dy + c\right]$$

or $$2x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} = -\cos y + y \sin y + \cos y + c$$

Hence the solution is $2x^2 \log x - y \sin y = c$.

Example 11.6. *Solve* $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$.

Solution. Given equation is $\frac{dy}{dx} = e^{-2y}(e^{3x} + x^2)$ or $e^{2y}\, dy = (e^{3x} + x^2)\, dx$

Integrating both sides, $\int e^{2y}\, dy = \int (e^{3x} + x^2)\, dx + c$

or $$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + c \quad \text{or} \quad 3e^{2y} = 2(e^{3x} + x^3) + 6c.$$

Example 11.7. *Solve* $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$. *(V.T.U., 2005)*

Solution. Putting $x + y = t$ so that $dy/dx = dt/dx - 1$

The given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

or $$dt/dx = 1 + \sin t + \cos t$$

Integrating both sides, we get $dx = \int \frac{dt}{1 + \sin t + \cos t} + c.$

or $$x = \int \frac{2d\theta}{1 + \sin 2\theta + \cos 2\theta} + c \qquad [\text{Putting } t = 2\theta]$$

$$= \int \frac{2d\theta}{2\cos^2\theta + 2\sin\theta\cos\theta} + c = \int \frac{\sec^2\theta}{1 + \tan\theta}\, d\theta + c$$

$$= \log(1 + \tan\theta) + c$$

Hence the solution is $x = \log\left[1 + \tan \frac{1}{2}(x + y)\right] + c$.

Example 11.8. *Solve* $dy/dx = (4x + y + 1)^2$, *if* $y(0) = 1$.

Solution. Putting $4x + y + 1 = t$, we get $\frac{dy}{dx} = \frac{dt}{dx} - 4$.

$\therefore$ the given equation becomes $\frac{dt}{dx} - 4 = t^2$ or $\frac{dt}{dx} = 4 + t^2$

Integrating both sides, we get $\int \frac{dt}{4 + t^2} = \int dx + c$

or $$\frac{1}{2}\tan^{-1}\frac{t}{2} = x + c \quad \text{or} \quad \frac{1}{2}\tan^{-1}\left[\frac{1}{2}(4x + y + 1)\right] = x + c.$$

or $$4x + y + 1 = 2\tan 2(x + c)$$

When $x = 0, y = 1$ $\therefore$ $\frac{1}{2}\tan^{-1}(1) = c$ *i.e.* $c = \pi/8$.

Hence the solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Example 11.9. *Solve* $\frac{y}{x}\frac{dy}{dx} + \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$. *(V.T.U., 2003)*

Solution. Putting $x^2 + y^2 = t$, we get $2x + 2y\frac{dy}{dx} = \frac{dt}{dx}$ or $\frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dt}{dx} - 1$.

Therefore the given equation becomes $\dfrac{1}{2x}\dfrac{dt}{dx} - 1 + \dfrac{t-1}{2t+1} = 0$

or $$\frac{1}{2x}\frac{dt}{dx} = 1 - \frac{t-1}{2t+1} = \frac{t+2}{2t+1} \quad \text{or} \quad 2x\,dx = \frac{2t+1}{t+2}\,dt$$

or $$2x\,dx = \left(2 - \frac{3}{t+2}\right) dt$$

Integrating, we get $$x^2 = 2t - 3\log(t+2) + c$$

or $$x^2 + 2y^2 - 3\log(x^2 + y^2 + 2) + c = 0 \qquad [\because\ t = x^2 + y^2]$$

which is the required solution.

PROBLEMS 11.2

Solve the following differential equations :

1. $y\sqrt{(1-x^2)}\,dy + x\sqrt{(1-y^2)}\,dx = 0.$
2. $(x^2 - yx^2)\dfrac{dy}{dx} + y^2 + xy^2 = 0.$
3. $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0.$ (P.T.U., 2003)
4. $\dfrac{y}{x}\dfrac{dy}{dx} = \sqrt{(1 + x^2 + y^2 + x^2y^2)}.$ (V.T.U., 2011)
5. $e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0.$ (V.T.U., 2009)
6. $\dfrac{dy}{dx} = xe^{y-x^2}$, if $y = 0$ when $x = 0$. (J.N.T.U., 2006)
7. $x\dfrac{dy}{dx} + \cot y = 0$ if $y = \pi/4$ when $x = \sqrt{2}$.
8. $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0.$
9. $\dfrac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}.$
10. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right).$
11. $(x+1)\dfrac{dy}{dx} + 1 = 2e^{-y}.$ (Madras, 2000 S)
12. $(x-y)^2\dfrac{dy}{dx} = a^2.$
13. $(x+y+1)^2\dfrac{dy}{dx} = 1.$ (Kurukshetra, 2005)
14. $\sin^{-1}(dy/dx) = x + y$ (V.T.U., 2010)
15. $\dfrac{dy}{dx} = \cos(x+y+1)$ (V.T.U., 2003)
16. $\dfrac{dy}{dx} - x\tan(y - x) = 1.$
17. $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$

11.7 HOMOGENEOUS EQUATIONS

are of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{\phi(x,y)}$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y (see page 205).

To solve a homogeneous equation (*i*) Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$,

(*ii*) Separate the variables v and x, and integrate.

Example 11.10. *Solve* $(x^2 - y^2)\,dx - xy\,dy = 0.$

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{xy}$ which is homogeneous in x and y. ...(*i*)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. $\therefore$ (*i*) becomes $v + x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v}$

or $$x\frac{dv}{dx} = \frac{1 - v^2}{v} - v = \frac{1 - 2v^2}{v}.$$

Separating the variables, $\dfrac{v}{1-2v^2}\,dv = \dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v\,dv}{1-2v^2} = \int \frac{dx}{x} + c$

or $$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x} + c \quad \text{or} \quad -\frac{1}{4}\log(1-2v^2) = \log x + c$$

or $$4\log x + \log(1-2v^2) = -4c \quad \text{or} \quad \log x^4(1-2v^2) = -4c \qquad [\text{Put } v = y/x]$$

or $$x^4(1-2y^2/x^2) = e^{-4c} = c'$$

Hence the required solution is $x^2(x^2-2y^2) = c'$.

Example 11.11. *Solve* $(x\tan y/x - y\sec^2 y/x)\,dx - x\sec^2 y/x\,dy = 0.$ *(V.T.U., 2006)*

Solution. The given equation may be rewritten as

$$\frac{dy}{dx} = \left(\frac{y}{x}\sec^2\frac{y}{x} - \tan\frac{y}{x}\right)\cos^2 y/x \qquad ...(i)$$

vy

which is a homogeneous equation. Putting $y = vx$, (*i*) becomes $v + x\dfrac{dv}{dx} = (v\sec^2 v - \tan v)\cos^2 v$

or $$x\frac{dv}{dx} = v - \tan v\cos^2 v - v$$

Separating the variables $\dfrac{\sec^2 v}{\tan v}\,dv = -\dfrac{dx}{x}$

Integrating both sides $\log\tan v = -\log x + \log c$

or $$x\tan v = c \quad \text{or} \quad x\tan y/x = c.$$

Example 11.12. *Solve* $(1+e^{x/y})\,dx + e^{x/y}(1-x/y)\,dy = 0.$
(P.T.U., 2006 ; Rajasthan, 2005 ; V.T.U., 2003)

Solution. The given equation may be rewritten as

$$\frac{dx}{dy} = -\frac{e^{x/y}(1-x/y)}{1+e^{x/y}} \qquad ...(i)$$

which is a homogeneous equation. Putting $x = vy$ so that (*i*) becomes

$$v + y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} \quad \text{or} \quad y\frac{dv}{dy} = -\frac{e^v(1-v)}{1+e^v} - v = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we get

$$-\frac{dy}{y} = \frac{1+e^v}{v+e^v}\,dv = \frac{d(v+e^v)}{v+e^v}$$

Integrating both sides, $-\log y = \log(v+e^v) + c$

or $$y(v+e^v) = e^{-c} \quad \text{or} \quad x + ye^{x/y} = c' \quad \text{(say)}$$

which is the required solution.

PROBLEMS 11.3

Solve the following differential equations :

1. $(x^2-y^2)\,dx = 2xy\,dy$
2. $(x^2y-2xy^2)\,dx-(x^3-3x^2y)\,dy=0.$ *(Bhopal, 2008)*
3. $x^2y\,dx-(x^3+y^3)\,dy=0.$ *(V.T.U., 2010)*
4. $y\,dx-x\,dy=\sqrt{x^2+y^2}\,dx.$ *(Raipur, 2005)*
5. $y^2+x^2\dfrac{dy}{dx}=xy\dfrac{dy}{dx}.$
6. $(3xy-2ay^2)\,dx+(x^2-2axy)\,dy=0.$ *(S.V.T.U., 2009)*

[Equations solvable like homogeneous equations : When a differential equation contains y/x a number of times, solve it like a homogeneous equation by putting $y/x = v$].

7. $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin \dfrac{y}{x}$. (V.T.U., 2000 S)

8. $ye^{x/y}\, dx = (xe^{x/y} + y^2)\, dy$. (V.T.U., 2006)

9. $xy\,(\log x/y)\, dx + [y^2 - x^2 \log (x/y)]\, dy = 0$.

10. $x\, dx + \sin^2 (y/x)\,(ydx - xdy) = 0$.

11. $x \cos \dfrac{y}{x}\,(ydx + xdy) = y \sin \dfrac{y}{x}\,(xdy - ydx)$.

11.8 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

The equations of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$ ...(1)

can be reduced to the homogeneous form as follows :

Case I. When $\dfrac{\mathbf{a}}{\mathbf{a'}} \neq \dfrac{\mathbf{b}}{\mathbf{b'}}$

Putting $x = X + h,\ y = Y + k$, (h, k being constants)

so that $dx = dX,\ dy = dY$, (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad ...(2)$$

Choose h, k so that (2) may become homogeneous.

Put $ah + bk + c = 0$, and $a'h + b'k + c' = 0$

so that $$\frac{h}{bc' - b'c} = \frac{k}{ca' - c'a} = \frac{1}{ab' - ba'}$$

or $$h = \frac{bc' - b'c}{ab' - b'a},\ k = \frac{ca' - c'a}{ab' - ba'} \quad ...(3)$$

Thus when $ab' - ba' \neq 0$, (2) becomes $\dfrac{dY}{dX} = \dfrac{aX + bY}{a'X + b'Y}$ which is homogeneous in X, Y and can be solved by putting $Y = vX$.

Case II. When $\dfrac{\mathbf{a}}{\mathbf{a'}} = \dfrac{\mathbf{b}}{\mathbf{b'}}$.

i.e., $ab' - b'a = 0$, the above method fails as h and k become infinite or indeterminate.

Now $$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m} \text{ (say)}$$

$\therefore$ $a' = am,\ b' = bm$ and (1) becomes

$$\frac{dy}{dx} = \frac{(ax + by) + c}{m(ax + by) + c'} \quad ...(4)$$

Put $ax + by = t$, so that $a + b\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dt}{dx} - a\right) \quad \therefore \quad \text{(4) becomes } \frac{1}{b}\left(\frac{dt}{dx} - a\right) = \frac{t + c}{mt + c'}$$

or $$\frac{dt}{dx} = a + \frac{bt + bc}{mt + c'} = \frac{(am + b)t + ac' + bc}{mt + c'}$$

so that the variables are separable. In this solution, putting $t = ax + by$, we get the required solution of (1).

Example 11.13. *Solve* $\dfrac{dy}{dx} = \dfrac{y + x - 2}{y - x - 4}$. *(Raipur, 2005)*

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{y + x - 2}{y - x - 4}$ $\left[\text{Case } \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$ ...(*i*)

Putting $x = X + h$, $y = Y + k$, (h, k being constants) so that $dx = dX$, $dy = dY$, (*i*) becomes

$$\frac{dY}{dX} = \frac{Y + X + (k + h - 2)}{Y - X + (k - h - 4)} \qquad ...(ii)$$

Put $k + h - 2 = 0$ and $k - h - 4 = 0$ so that $h = -1$, $k = 3$.

$\therefore$ (*ii*) becomes $\dfrac{dY}{dX} = \dfrac{Y + X}{Y - X}$ which is homogeneous in X and Y. ...(*iii*)

$\therefore$ put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ (*iii*) becomes $\quad v + X\dfrac{dv}{dX} = \dfrac{v+1}{v-1}$ or $X\dfrac{dv}{dX} = \dfrac{v+1}{v-1} - v = \dfrac{1 + 2v - v^2}{v - 1}$

or
$$\frac{v - 1}{1 + 2v - v^2}\, dv = \frac{dX}{X}.$$

Integrating both sides, $-\dfrac{1}{2}\displaystyle\int \dfrac{2 - 2v}{1 + 2v - v^2}\, dv = \int \dfrac{dX}{X} + c.$

or
$$-\frac{1}{2}\log(1 + 2v - v^2) = \log X + c$$

or
$$\log\left(1 + \frac{2Y}{X} - \frac{Y^2}{X^2}\right) + \log X^2 = -2c$$

or
$$\log(X^2 + 2XY - Y^2) = -2c \quad \text{or} \quad X^2 + 2XY - Y^2 = e^{-2c} = c' \qquad ...(iv)$$

Putting $X = x - h = x + 1$, $Y = y - k = y - 3$, (*iv*) becomes

$$(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = c'$$

or $\quad x^2 + 2xy - y^2 - 4x + 8y - 14 = c'$ which is the required solution.

Example 11.14. *Solve* $(3y + 2x + 4)\, dx - (4x + 6y + 5)\, dy = 0.$ (*Madras, 2000 S*)

Solution. Given equation is $\dfrac{dy}{dx} = \dfrac{(2x + 3y) + 4}{2(2x + 3y) + 5}$ ...(*i*)

Putting $2x + 3y = t$ so that $2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$ $\quad\therefore\quad$ (*i*) becomes $\dfrac{1}{3}\left(\dfrac{dt}{dx} - 2\right) = \dfrac{t + 4}{2t + 5}$

or
$$\frac{dt}{dx} = 2 + \frac{3t + 12}{2t + 5} = \frac{7t + 22}{2t + 5} \quad \text{or} \quad \frac{2t + 5}{7t + 22}\, dt = dx$$

Integrating both sides, $\displaystyle\int \frac{2t + 5}{7t + 22}\, dt = \int dx + c$

or
$$\int\left(\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7t + 22}\right) dt = x + c \quad \text{or} \quad \frac{2}{7}t - \frac{9}{49}\log(7t + 22) = x + c$$

Putting $t = 2x + 3y$, we have $14(2x + 3y) - 9\log(14x + 21y + 22) = 49x + 49c$

or $\quad 21x - 42y + 9\log(14x + 21y + 22) = c'$ which is the required solution.

PROBLEMS 11.4

Solve the following differential equations :

1. $(x - y - 2)\, dx + (x - 2y - 3)\, dy = 0.$ (*Rajasthan, 2006*)
2. $(2x + y - 3)\, dy = (x + 2y - 3)\, dx.$ (*V.T.U., 2009 S ; Madras, 2000*)
3. $(2x + 5y + 1)\, dx - (5x + 2y - 1)\, dy = 0.$ (*J.N.T.U., 2000*)
4. $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0.$
5. $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}.$
6. $(4x - 6y - 1)\, dx + (3y - 2x - 2)\, dy = 0.$
7. $(x + 2y)(dx - dy) = dx + dy.$ (*Bhopal, 2002 S ; V.T.U., 2001*)

11.9 LINEAR EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur only in the first degree and not multiplied together.

Thus **the standard form of a linear equation of the first order,** *commonly known as* **Leibnitz's linear equation,*** *is*

$$\frac{dy}{dx} + Py = Q \quad \text{where, } P, Q \text{ are the functions of } x. \qquad ...(1)$$

To solve the equation, multiply both sides by $e^{\int P dx}$ so that we get

$$\frac{dy}{dx} \cdot e^{\int P dx} + y(e^{\int P dx} P) = Qe^{\int P dx} \quad i.e., \quad \frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx}$$

Integrating both sides, we get $ye^{\int P dx} = \int Qe^{\int P dx} dx + c$ as the required solution.

Obs. The factor $e^{\int P dx}$ on multiplying by which the left-hand side of (1) becomes the differential coefficient of a single function, is called the **integrating factor (I.F.)** of the linear equation (1).

It is important to remember that **I.F.** $= e^{\int P dx}$

and the solution is y **(I.F.)** $= \int Q(\text{I.F.})\, dx + c.$

Example 11.15. *Solve* $(x + 1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2$.

Solution. Dividing throughout by $(x + 1)$, given equation becomes

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x + 1) \text{ which is Leibnitz's equation.} \qquad ...(i)$$

Here $\quad P = -\frac{1}{x+1}$ and $\int P dx = -\int \frac{dx}{x+1} = -\log(x + 1) = \log(x + 1)^{-1}$

$\therefore \quad$ I.F. $= e^{\int P dx} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$

Thus the solution of (1) is $y(\text{I.F.}) = \int [e^{3x}(x + 1)](\text{I.F.})\, dx + c$

or $\quad \frac{y}{x+1} = \int e^{3x} dx + c = \frac{1}{3}e^{3x} + c \quad$ or $\quad y = \left(\frac{1}{3}e^{3x} + c\right)(x + 1).$

Example 11.16. *Solve* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$

Solution. Given equation can be written as $\frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ $\qquad ...(i)$

$\therefore \quad$ I.F. $= e^{\int x^{-1/2} dx} = e^{2\sqrt{x}}$

Thus solution of (i) is y (I.F.) $= \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$ (I.F.) $dx + c$

or $\quad ye^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}}\, dx + c$

or $\quad ye^{2\sqrt{x}} = \int x^{-1/2} dx + c \quad$ or $\quad ye^{2\sqrt{x}} = 2\sqrt{x} + c.$

---

* See footnote p. 139.

Example 11.17. *Solve* $3x(1-x^2)y^2\dfrac{dy}{dx}+(2x^2-1)y^3=ax^3$. (*Rajasthan, 2006*)

Solution. Putting $y^3=z$ and $3y^2\dfrac{dy}{dx}=\dfrac{dz}{dx}$, the given equation becomes

$$x(1-x^2)\frac{dz}{dx}+(2x^2-1)z=ax^3,\quad\text{or}\quad \frac{dz}{dx}+\frac{2x^2-1}{x-x^3}z=\frac{ax^3}{x-x^3} \quad ...(i)$$

which is Leibnitz's equation in z

$$\therefore \qquad \text{I.F.}=\exp\left(\int\frac{2x^2-1}{x-x^3}dx\right)$$

Now $\displaystyle\int\frac{2x^2-1}{x-x^3}dx=\int\left(-\frac{1}{x}-\frac{1}{2}\frac{1}{1+x}+\frac{1}{2}\cdot\frac{1}{1-x}\right)dx=-\log x-\frac{1}{2}\log(1+x)-\frac{1}{2}\log(1-x)$

$$=-\log[x\sqrt{(1-x^2)}]$$

$$\therefore \qquad \text{I.F.}=e^{-\log[x\sqrt{(1-x^2)}]}=[x\sqrt{(1-x^2)}]^{-1}$$

Thus the solution of (i) is

$$z(\text{I.F.})=\int\frac{ax^3}{x-x^3}(\text{I.F.})\,dx+c$$

or
$$\frac{z}{[x\sqrt{(1-x^2)}]}=a\int\frac{x^3}{x(1-x^2)}\cdot\frac{1}{x\sqrt{(1-x^2)}}dx+c=a\int x(1-x^2)^{-3/2}dx$$

$$=-\frac{a}{2}\int(-2x)(1-x^2)^{-3/2}dx+c=a(1-x^2)^{-1/2}+c$$

Hence the solution of the given equation is

$$y^3=ax+cx\sqrt{(1-x^2)}. \qquad [\because\ z=y^3]$$

Example 11.18. *Solve* $y(\log y)\,dx+(x-\log y)\,dy=0$. (*U.P.T.U., 2000*)

Solution. We have $\dfrac{dx}{dy}+\dfrac{x}{y\log y}=\dfrac{1}{y}$ ... (i)

which is a Leibnitz's equation in x

$$\therefore \qquad \text{I.F.}=e^{\int\frac{1}{y\log y}dy}=e^{\log(\log y)}=\log y$$

Thus the solution of (i) is $x\,(\text{I.F.})=\displaystyle\int\frac{1}{y}(\text{I.F.})\,dy+c$

$$x\log y=\int\frac{1}{y}\log y\,dy+c=\frac{1}{2}(\log y)^2+c$$

i.e.,
$$x=\frac{1}{2}\log y+c(\log y)^{-1}.$$

Example 11.19. *Solve* $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$. (*Bhopal, 2008; V.T.U., 2008; U.P.T.U., 2005*)

Solution. This equation contains y^2 and $\tan^{-1}y$ and is, therefore, not a linear in y; but since only x occurs, it can be written as

$$(1+y^2)\frac{dx}{dy}=\tan^{-1}y-x\quad\text{or}\quad\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}$$

which is a Leibnitz's equation in x.

$$\therefore \qquad \text{I.F.}=e^{\int P\,dy}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

Thus the solution is $x\,(\text{I.F.})=\displaystyle\int\frac{\tan^{-1}y}{1+y^2}(\text{I.F.})\,dy+c$

or $$xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}\, dy + c \qquad \left[\begin{array}{l}\text{Put } \tan^{-1} y = t \\ \therefore \ \dfrac{dy}{1+y^2} = dt\end{array}\right]$$

$$= \int te^t\, dt + c = t \cdot e^t - \int 1 \cdot e^t\, dt + c \qquad \text{(Integrating by parts)}$$

$$= t \cdot e^t - e^t + c = (\tan^{-1} y - 1)\, e^{\tan^{-1} y} + c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Example 11.20. *Solve $r \sin \theta\, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta)\, dr = 0$.*

Solution. Given equation can be rewritten as

$$\sin \theta \frac{d\theta}{dr} + \frac{1}{r}(1 - 2r^2) \cos \theta = -r^2 \qquad ...(i)$$

Put $\cos \theta = y$ so that $-\sin \theta\, d\theta/dr = dy/dr$

Then (i) becomes $-\dfrac{dy}{dr} + \left(\dfrac{1}{r} - 2r\right) y = -r^2$ or $\dfrac{dy}{dr} + \left(2r - \dfrac{1}{r}\right) y = r^2$

which is a Leibnitz's equation $\therefore$ I.F. $= e^{\int (2r - 1/r)\, dr} = e^{r^2 - \log r} = \dfrac{1}{r} e^{r^2}$

Thus its solution is $y \left(\dfrac{1}{r} e^{r^2}\right) = \displaystyle\int r^2 \cdot e^{r^2} \cdot \frac{1}{r}\, dr + c$

or $$ye^{r^2}/r = \frac{1}{2}\int e^{r^2}\, 2r\, dr + c = \frac{1}{2} e^{r^2} + c$$

or $$2e^{r^2} \cos \theta = re^{r^2} + 2cr \quad \text{or} \quad r(1 + 2ce^{-r^2}) = 2 \cos \theta.$$

PROBLEMS 11.5

Solve the following differential equations :

1. $\cos^2 x \dfrac{dy}{dx} + y = \tan x.$
2. $x \log x \dfrac{dy}{dx} + y = \log x^2.$ (*V.T.U., 2011*)
3. $2y' \cos x + 4y \sin x = \sin 2x$, given $y = 0$ when $x = \pi/3$. (*V.T.U., 2003*)
4. $\cosh x \dfrac{dy}{dx} + y \sinh x = 2 \cosh^2 x \sinh x$. (*J.N.T.U., 2003*)
5. $(1 - x^2) \dfrac{dy}{dx} - xy = 1$ (*V.T.U., 2010*)
6. $(1 - x^2) \dfrac{dy}{dx} + 2xy = x\sqrt{(1 - x^2)}$ (*Nagpur, 2009*)
7. $\dfrac{dy}{dx} = -\dfrac{x + y \cos x}{1 + \sin x}.$
8. $dr + (2r \cot \theta + \sin 2\theta)\, d\theta = 0.$ (*J.N.T.U., 2003*)
9. $\dfrac{dy}{dx} + 2xy = 2e^{-x^2}$ (*P.T.U., 2005*)
10. $(x + 2y^3) \dfrac{dy}{dx} = y.$ (*Marathwada, 2008*)
11. $\sqrt{(1 - y^2)}\, dx = (\sin^{-1} y - x)\, dy.$
12. $ye^y\, dx = (y^3 + 2xe^y)\, dy.$
13. $(1 + y^2)\, dx + (x - e^{-\tan^{-1} y})\, dy = 0.$ (*V.T.U., 2006*)
14. $e^{-y} \sec^2 y\, dy = dx + x\, dy.$

11.10 BERNOULLI'S EQUATION

The equation $\dfrac{dy}{dx} + Py = Qy^n \qquad ...(1)$

where P, Q are functions of x, is reducible to the Leibnitz's linear equation and is usually called the Bernoulli's equation.*

*Named after the Swiss mathematician *Jacob Bernoulli* (1654–1705) who is known for his basic work in probability and elasticity theory. He was professor at Basel and had amongst his students his youngest brother *Johann Bernoulli* (1667–1748) and his nephew *Niklaus Bernoulli* (1687–1759). Johann is known for his basic contributions to Calculus while Niklaus had profound influence on the development of Infinite series and probability. His son *Daniel Bernoulli* (1700–1782) is known for his contributions to kinetic theory of gases and fluid flow.

To solve (1), divide both sides by y^n, so that $y^{-n}\dfrac{dy}{dx} + Py^{1-n} = Q$ ...(2)

Put $y^{1-n} = z$ so that $(1-n)y^{-n}\dfrac{dy}{dx} = \dfrac{dz}{dx}$.

$\therefore$ (2) becomes $\dfrac{1}{1-n}\dfrac{dz}{dx} + Pz = Q$ or $\dfrac{dz}{dx} + P(1-n)z = Q(1-n)$,

which is Leibnitz's linear in z and can be solved easily.

Example 11.21. *Solve* $x\dfrac{dy}{dx} + y = x^3y^6$.

Solution. Dividing throughout by xy^6, $y^{-6}\dfrac{dy}{dx} + \dfrac{y^{-5}}{x} = x^2$ ...(*i*)

Put $y^{-5} = z$, so that $-5y^{-6}\dfrac{dy}{dx} = \dfrac{dz}{dx}$ $\therefore$ (*i*) becomes $-\dfrac{1}{5}\dfrac{dz}{dx} + \dfrac{z}{x} = x^2$

or $$\frac{dz}{dx} - \frac{5}{x}z = -5x^2 \text{ which is Leibnitz's linear in } z. \quad ...(ii)$$

$$\text{I.F.} = e^{-\int (5/x)dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$$

$\therefore$ the solution of (*ii*) is $z\,(\text{I.F.}) = \int (-5x^2)(\text{I.F.})\,dx + c$ or $zx^{-5} = \int(-5x^2)\,x^{-5}\,dx + c$

or $$y^{-5}x^{-5} = -5.\frac{x^{-2}}{-2} + c \qquad [\because\ z = y^{-5}]$$

Dividing throughout by $y^{-5}x^{-5}$, $1 = (2.5 + cx^2)\,x^3y^5$ which is the required solution.

Example 11.22. *Solve* $xy\,(1 + xy^2)\dfrac{dy}{dx} = 1$. (*Nagpur, 2009*)

Solution. Rewriting the given equation as

$$\frac{dx}{dy} - yx = y^3x^2$$

and dividing by x^2, we have

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \quad ...(i)$$

Putting $x^{-1} = z$ so that $-x^{-2}\dfrac{dx}{dy} = \dfrac{dz}{dy}$ (*i*) becomes

$$\frac{dz}{dy} + yz = -y^3 \text{ which is Leibnitz's linear in } z.$$

Here $\quad \text{I.F.} = e^{\int y\,dy} = e^{y^2/2}$

$\therefore$ the solution is $z\,(\text{I.F.}) = \int(-y^3)(\text{I.F.})\,dy + c$

or $$ze^{y^2/2} = -\int y^2 . e^{\frac{1}{2}y^2} . y\,dy + c \qquad \left|\begin{array}{l}\text{Put } \frac{1}{2}y^2 = t \\ \text{so that } y\,dy = dt\end{array}\right.$$

$$= -2\int t . e^t\,dt + c \qquad \text{[Integrate by parts]}$$

$$= -2\,[t . e^t - \int 1 . e^t\,dt] + c = -2\,[te^t - e^t] + c = (2 - y^2)\,e^{y^2/2} + c$$

or $$z = (2 - y^2) + ce^{-\frac{1}{2}y^2} \quad \text{or} \quad 1/x = (2 - y^2) + ce^{-\frac{1}{2}y^2}.$$

Note. *General equation reducible to Leibnitz's linear is* $f'(y)\dfrac{dy}{dx} + Pf(y) = Q$ ...(A)

where P, Q are functions of x. To solve it, put $f(y) = z$.

Example 11.23. *Solve* $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$. *(V.T.U., 2011 ; Marathwada, 2008 ; J.N.T.U., 2005)*

Solution. Dividing throughout by $\cos^2 y$, $\sec^2 y \dfrac{dy}{dx} + 2x \dfrac{\sin y \cos y}{\cos^2 y} = x^3$

or $$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \text{ which is of the form } (A) \text{ above.} \qquad ...(i)$$

$\therefore$ put $\tan y = z$ so that $\sec^2 y \dfrac{dy}{dx} = \dfrac{dz}{dx}$ $\quad \therefore$ (*i*) becomes $\dfrac{dz}{dx} + 2xz = x^3$.

This is Leibnitz's linear equation in z. $\quad \therefore$ I.F. $= e^{\int 2x\,dx} = e^{x^2}$

$\therefore$ the solution is $ze^{x^2} = \int e^{x^2} x^3 dx + c = \dfrac{1}{2}(x^2 - 1) e^{x^2} + c$.

Replacing z by $\tan y$, we get $\tan y = \dfrac{1}{2}(x^2 - 1) + ce^{-x^2}$ which is the required solution.

Example 11.24. *Solve* $\dfrac{dz}{dx} + \left(\dfrac{z}{x}\right) \log z = \dfrac{z}{x} (\log z)^2$.

Solution. Dividing by z, the given equation becomes

$$\frac{1}{z}\frac{dz}{dx} + \frac{1}{x} \log z = \frac{1}{x} (\log z)^2 \qquad ...(i)$$

Put $\log z = t$ so that $\quad \dfrac{1}{z}\dfrac{dz}{dx} = \dfrac{dt}{dx}$. $\quad \therefore$ (*i*) becomes

$$\frac{dt}{dx} + \frac{t}{x} = \frac{t^2}{x} \quad \text{or} \quad \frac{1}{t^2}\frac{dt}{dx} + \frac{1}{x}\cdot\frac{1}{t} = \frac{1}{x} \qquad ...(ii)$$

This being Bernoulli's equation, put $1/t = v$ so that (*ii*) reduces to

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x}$$

This is Leibnitz's linear in v. $\quad \therefore$ I.F. $= e^{-\int 1/x\,dx} = 1/x$

$\therefore$ the solution is $$v \cdot \frac{1}{x} = -\int \frac{1}{x}\cdot\frac{1}{x}\,dx + c = \frac{1}{x} + c$$

Replacing v by $1/\log z$, we get $(x \log z)^{-1} = x^{-1} + c \quad$ or $\quad (\log z)^{-1} = 1 + cx$ which is the required solution.

PROBLEMS 11.6

Solve the following equations :

1. $\dfrac{dy}{dx} = y \tan x - y^2 \sec x$. *(P.T.U., 2005)*
2. $r \sin\theta - \cos\theta \dfrac{dr}{d\theta} = r^2$. *(V.T.U., 2005)*
3. $2xy' = 10x^3y^5 + y$.
4. $(x^3y^2 + xy)\,dx = dy$. *(B.P.T.U., 2005)*
5. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$. *(Bhillai, 2005)*
6. $x(x - y)\,dy + y^2\,dx = 0$. *(I.S.M., 2001)*
7. $\dfrac{dy}{dx} - \dfrac{\tan y}{1 + x} = (1 + x) e^x \sec y$. *(Bhopal, 2009)*
8. $e^y\left(\dfrac{dy}{dx} + 1\right) = e^x$. *(V.T.U., 2009)*
9. $\sec^2 y \dfrac{dy}{dx} + x \tan y = x^3$.
10. $\tan y \dfrac{dy}{dx} + \tan x = \cos y \cos^2 x$. *(Sambalpur, 2002)*
11. $\dfrac{dy}{dx} = \dfrac{y}{x - \sqrt{(xy)}}$. *(V.T.U., 2011)*
12. $(y \log x - 2)\,y\,dx - x\,dy = 0$. *(V.T.U., 2006)*

11.11 EXACT DIFFERENTIAL EQUATIONS

(1) Def. *A differential equation of the form* $\mathbf{M}(x, y)\,\mathbf{dx} + \mathbf{N}(x, y)\,\mathbf{dy} = 0$ *is said to be* ***exact*** *if its left hand member is the exact differential of some function* $u(x, y)$ *i.e.,* $du = Mdx + Ndy = 0$. Its solution, therefore, is $u(x, y) = c$.

(2) Theorem. The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial \mathbf{M}}{\partial \mathbf{y}} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}}$$

Condition is necessary :

The equation $Mdx + Ndy = 0$ will be exact, if

$$Mdx + Ndy \equiv du \qquad ...(1)$$

where u is some function of x any y.

But $$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad ...(2)$$

$\therefore$ equating coefficients of dx and dy in (1) and (2), we get $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

But $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \qquad \text{(Assumption)}$$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ which is the necessary condition for exactness.

Condition is sufficient : i.e., if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $Mdx + Ndy = 0$ is exact.

Let $\int Mdx = u$, where y is supposed constant while performing integration.

Then $$\frac{\partial}{\partial x}\left(\int Mdx\right) = \frac{\partial u}{\partial x}, \quad i.e.,\ M = \frac{\partial u}{\partial x} \qquad \left\{\begin{array}{l} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \text{ (given)} \\ \text{and } \dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y} \end{array}\right. \qquad ...(3)$$

$$\therefore \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

Interating both sides w.r.t. x (taking y as constant).

$$N = \frac{\partial u}{\partial y} + f(y), \text{ where } f(y) \text{ is a function of } y \text{ alone.} \qquad ...(4)$$

$$\therefore \qquad Mdx + Ndy = \frac{\partial u}{\partial x}dx + \left\{\frac{\partial u}{\partial y} + f(y)\right\}dy \qquad \text{[By (3) and (4)]}$$

$$= \left\{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right\} + f(y)\,dy = du + f(y)\,dy = d\left[u + \int f(y)dy\right] \qquad ...(5)$$

which shows that $Mdx + Ndy = 0$ is exact.

(3) Method of solution. By (5), the equation $Mdx + Ndy = 0$ becomes $d[u + \int f(y)dy] = 0$

Integrating $\qquad u + \int f(y)dy = 0.$

But $\qquad u = \int_{y\text{ constant}} Mdx$ and $f(y)$ = terms of N not containing x.

$\therefore$ *The solution of $Mdx + Ndy = 0$ is*

$$\int_{(y\text{ cons.})} \mathbf{M\,dx} + \int (\textbf{terms of N not containing x})\,\mathbf{dy} = \mathbf{c}$$

provided $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$ *(V.T.U., 2006)*

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2 e^{xy^2} \,.\, 2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y\text{ const.})} (y^2 e^{y^2 x} + 4x^3) dx + \int (-3y^2)\, dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\, dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

$$\int_{(y\text{ const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\, y + x \cos y = c.$$

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y\text{ const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \,.\, 2x dx - \int 2x\, dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\dfrac{dy}{dx} + \dfrac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0.$$

Here $M = y \cos x + \sin y + y$ and $N = \sin x + x \cos y + x.$

$$\therefore \quad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\text{ const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c$$

i.e.,
$$\int_{(y\text{ const.})} (y \cos x + \sin y + y)\, dx + \int (0)\, dx = c \quad \text{or} \quad y \sin x + (\sin y + y)\, x = c.$$

Example 11.29. *Solve* $(2x^2 + 3y^2 - 7)\,xdx - (3x^2 + 2y^2 - 8)\,ydy = 0.$ *(U.P.T.U., 2005)*

Solution. Given equation can be written as

$$\frac{ydy}{xdx} = \frac{2x^2 + 3y^2 - 7}{3x^2 + 2y^2 - 8}$$

or $$\frac{ydy + xdx}{ydy - xdx} = \frac{5(x^2 + y^2 - 3)}{-x^2 + y^2 + 1} \qquad \text{[By componendo \& dividendo]}$$

or $$\frac{xdx + ydy}{x^2 + y^2 - 3} = 5 \cdot \frac{xdx - ydy}{x^2 - y^2 - 1}$$

Integrating both sides, we get

$$\int \frac{2xdx + 2ydy}{x^2 + y^2 - 3} = 5 \int \frac{2xdx - 2ydy}{x^2 - y^2 - 1} + c$$

or $$\log(x^2 + y^2 - 3) = 5 \log(x^2 - y^2 - 1) + \log c' \qquad \text{[Writing } c = \log c'\text{]}$$

or $$x^2 + y^2 - 3 = c'(x^2 - y^2 - 1)^5$$

which is the required solution.

PROBLEMS 11.7

Solve the following equations :

1. $(x^2 - ay)\,dx = (ax - y^2)\,dy.$
2. $(x^2 + y^2 - a^2)\,xdx + (x^2 - y^2 - b^2)\,ydy = 0$ *(Kurukshetra, 2005)*
3. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$
4. $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$
5. $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$
6. $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy = 0$ *(V.T.U., 2008)*
7. $(3x^2 + 6xy^2)\,dx + (6x^2y + 4y^3)\,dy = 0$
8. $\dfrac{2x}{y^3}dx + \dfrac{y^2 - 3x^2}{y^4}dy = 0$
9. $y \sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$
10. $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$ *(Marathwada, 2008)*
11. $(2xy + y - \tan y)\,dx + x^2 - x\tan^2 y + \sec^2 y)\,dy = 0.$ *(Nagpur, 2009)*

11.12 EQUATIONS REDUCIBLE TO EXACT EQUATIONS

Sometimes a differential equation which is not exact, can be made so on multiplication by a suitable factor called an *integrating factor*. The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows :

(1) I.F. found by inspection. In a number of cases, the integrating factor can be found after regrouping the terms of the equation and recognizing each group as being a part of an exact differential. In this connection the following integrable combinations prove quite useful :

$$xdy + ydx = d(xy)$$

$$\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right); \frac{xdy - ydx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{xdy - ydx}{y^2} = -d\left(\frac{x}{y}\right); \frac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$$

$$\frac{xdy - ydx}{x^2 - y^2} = d\left(\frac{1}{2}\log\frac{x + y}{x - y}\right).$$

Example 11.30. *Solve* $y(2xy + e^x)\,dx = e^x\,dy.$ *(Kurukshetra, 2005)*

Solution. It is easy to note that the terms $ye^x dx$ and $e^x dy$ should be put together.

$\therefore$ $$(ye^x dx - e^x dy) + 2xy^2\,dx = 0$$

Now we observe that the term $2xy^2\,dx$ should not involve y^2. This suggests that $1/y^2$ may be I.F. Multiplying throughout by $1/y^2$, it follows

$$\frac{ye^x\,dx - e^x\,dy}{y^2} + 2xdx = 0 \quad \text{or} \quad d\left(\frac{e^x}{y}\right) + 2xdx = 0$$

Integrating, we get $\dfrac{e^x}{y} + x^2 = c$ which is the required solution.

(2) I.F. of a homogeneous equation. *If $Mdx + Ndy = 0$ be a homogeneous equation in x and y, then $1/(Mx + Ny)$ is an integrating factor ($Mx + Ny \neq 0$).*

Example 11.31. *Solve* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ *(Osmania, 2003 S)*

Solution. This equation is homogeneous in x and y.

$$\therefore \quad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{(x^2y - 2xy^2)\,x - (x^3 - 3x^2y)y} = \frac{1}{x^2y^2}$$

Multiplying throughout by $1/x^2y^2$, the equation becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \text{ which is exact.}$$

$\therefore$ the solution is $\displaystyle\int_{(y\text{ const})} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$ or $\dfrac{x}{y} - 2\log x + 3\log y = c.$

(3) I.F. for an equation of the type $f_1(xy)ydx + f_2(xy)xdy = 0$.
If the equation $Mdx + Ndy = 0$ be of this form, then $1/(Mx - Ny)$ is an integrating factor ($Mx - Ny \neq 0$).

Example 11.32. *Solve* $(1 + xy)\,ydx + (1 - xy)\,xdy = 0.$ *(S.V.T.U., 2008)*

Solution. The given equation is of the form $f_1(xy)\,ydx + f_2(xy)\,xdy = 0$

Here $\qquad M = (1 + xy)\,y, N = (1 - xy)\,x.$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)yx - (1 - xy)xy} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $1/2x^2y^2$, it becomes

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0, \text{ which is an exact equation.}$$

$\therefore$ the solution is $\displaystyle\int_{(y\text{ const})} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or
$$\frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c \quad \text{or} \quad \log\frac{x}{y} - \frac{1}{xy} = c'.$$

(4) In the equation $Mdx + Ndy = 0$,

(a) if $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ *be a function of x only = f(x) say, then* $e^{\int f(x)dx}$ *is an integrating factor.*

(b) if $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ *be a function of y only = F(y) say, then* $e^{\int F(y)dy}$ *is an integrating factor.*

Example 11.33. *Solve* $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0.$ *(S.V.T.U., 2009 ; Mumbai, 2007)*

Solution. Here $M = xy^2 - e^{1/x^3}$ and $N = -x^2y$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x} \text{ which is a function of } x \text{ only.}$$

$\therefore$ $$\text{I.F.} = e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = x^{-4}$$

Multiplying throughout by x^{-4}, we get $\left(\dfrac{y^2}{x^3} - \dfrac{1}{4^4} e^{1/x^2}\right) dx - \dfrac{y}{x^2}\, dy = 0$

which is an exact equation.

$\therefore$ the solution is $\int_{(y\ \text{const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

or $$\int \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^2} \right) dx + 0 = c$$

or $$-\frac{y^2 x^{-2}}{2} + \frac{1}{3} \int e^{x^{-3}} (-3x^{-4})\, dx = c \text{ or } \frac{1}{3} e^{x^{-3}} - \frac{1}{2} \frac{y^2}{x^2} = c.$$

Otherwise it can be solved as a Bernoulli's equation (§ 11.10)

Example 11.34. *Solve* $(xy^3 + y)\, dx + 2\,(x^2y^2 + x + y^4)\, dy = 0.$

Solution. Here $M = xy^3 + y$, $N = 2(x^2y^2 + x + y^4)$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)} (4xy^2 + 2 - 3xy^2 - 1) = \frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$\therefore$ $$\text{I.F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying throughout by y, it becomes $(xy^4 + y^2)\, dx + (2x^2y^3 + 2xy + 2y^5)\, dy = 0$, which is an exact equation.

$\therefore$ its solution is $\int_{(y\ \text{const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = 0$

or $$\int_{(y\ \text{const})} (xy^4 + y^2) dx + \int 2y^5\, dy = c \quad \text{or} \quad \frac{1}{2} x^2y^4 + xy^2 + \frac{1}{3} y^6 = c.$$

Example 11.35. *Solve* $(y \log y)\, dx + (x - \log y)\, dy = 0$ *(U.P.T.U., 2004)*

Solution. Here $M = y \log y$ and $N = x - \log y$

$\therefore$ $$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y \log y} (1 - \log y - 1) = -\frac{1}{y}, \text{ which is a function of } y \text{ alone.}$$

$\therefore$ $$\text{I.F.} = e^{-\int \frac{1}{y} dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying the given equation throughout by $1/y$, it becomes

$$\log y\, dx + \frac{1}{y} (x - \log y)\, dy = 0$$

which is an exact equation $\left[\because \dfrac{\partial}{\partial y} (\log y) = \dfrac{\partial}{\partial x} \left(\dfrac{x - \log y}{y} \right)\right.$

$\therefore$ its solution is $\int_{(y\ \text{const})} (Mdx) + \int (\text{terms of } N \text{ not containing } x)\, dy = c$

or $$\log y \int dx + \int \left(\frac{-\log y}{y} \right) dy = c \qquad \text{or} \quad x \log y - \frac{1}{2} (\log y)^2 = c.$$

(5) For the equation of the type

$$x^a y^b\, (my dx + nx dy) + x^{a'} y^{b'}\, (m' y dx + n' x dy) = 0,$$

an integrating factor is $x^h y^k$

where $$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

Example 11.36. *Solve* $y(xy + 2x^2y^3)\,dx + x\,(xy - x^3y^2)\,dy = 0.$ *(Hissar, 2005 ; Kurukshetra, 2005)*

Solution. Rewriting the equation as $xy\,(ydx + xdy) + x^2y^2\,(2ydx - xdy) = 0$ and comparing with

$$x^ay^b\,(mydx + nxdy) + x^{a'}y^{b'}\,(m'ydx + n'xdy) = 0,$$

we have $a = b = 1$, $m = n = 1$; $a' = b' = 2$, $m' = 2$, $n' = -1$.

$$\therefore \qquad \text{I.F.} = x^hy^k.$$

where
$$\frac{a+h+1}{m} = \frac{b+k+1}{n},\ \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

i.e.
$$\frac{1+h+1}{1} = \frac{1+k+1}{1},\ \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

or
$$h - k = 0,\ h + 2k + 9 = 0$$

Solving these, we get $h = k = -3$. $\therefore$ I.F. $= 1/x^3y^3$.

Multiplying throughout by $1/x^3y^3$, it becomes

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dx = 0, \text{ which is an exact equation.}$$

$\therefore$ The solution is $\displaystyle\int_{(y\ const)} Mdx + \int(\text{terms of } N \text{ not containing } x)\,dy = c$

or
$$\frac{1}{y}\left(-\frac{1}{x}\right) + 2\log x - \log y = c \quad \text{or} \quad 2\log x - \log y - 1/xy = c.$$

PROBLEMS 11.8

Solve the following equations :

1. $xdy - ydx + a(x^2 + y^2)\,dx = 0.$
2. $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2 + y^2}.$ *(U.P.T.U., 2005)*
3. $ydx - xdy + \log x\,dx = 0.$
4. $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}.$
5. $(x^3y^2 + x)\,dy + (x^2y^3 - y)\,dx = 0.$
6. $(x^2y^2 + xy + 1)\,ydx + (x^2y^2 - xy + 1)\,xdy = 0.$
7. $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0.$
8. $(4xy + 3y^2 - x)\,dx + x\,(x + 2y)dy = 0$ *(Mumbai, 2006)*
9. $x^4\dfrac{dy}{dx} + x^3y + \operatorname{cosec}(xy) = 0.$
10. $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ *(Mumbai, 2006)*
11. $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0.$ *(Kurukshetra, 2006)*
12. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)dy = 0.$ *(Rajasthan, 2005)*
13. $2ydx + x(2\log x - y)\,dy = 0.$ *(P.T.U., 2005)*

11.13 EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

As dy/dx will occur in higher degrees, it is convenient to denote dy/dx by p. Such equations are of the form $f(x, y, p) = 0$. Three cases arise for discussion :

Case. I. Equation solvable for p. A differential equation of the first order but of the nth degree is of the form

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ are functions of x and y.

Splitting up the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)]\,\ldots\,[p - f_n(x, y)] = 0.$$

Equating each of the factors to zero,

$$p = f_1(x, y),\ p = f_2(x, y),\ \ldots,\ p = f_n(x, y)$$

Solving each of these equations of the first order and first degree, we get the solutions

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0,\ \ldots,\ F_n\,(x, y, c) = 0.$$

These n solutions constitute the general solution of (1).

Otherwise, the general solution of (1) may be written as

$$F_1(x, y, c) \,.\, F_2(x, y, c) \,\ldots\ldots\, F_n(x, y, c) = 0.$$

Example 11.37. *Solve* $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

Solution. Given equation is $p - \dfrac{1}{p} = \dfrac{x}{y} - \dfrac{y}{x}$ where $p = \dfrac{dy}{dx}$ or $p^2 + p\left(\dfrac{y}{x} - \dfrac{x}{y}\right) - 1 = 0$.

Factorising $(p + y/x)(p - x/y) = 0$.

Thus we have $\quad p + y/x = 0 \quad \ldots(i) \quad$ and $\quad p - x/y = 0 \quad \ldots(ii)$

From (*i*), $\quad \dfrac{dy}{dx} + \dfrac{y}{x} = 0$ or $xdy + ydx = 0$

i.e., $\quad d(xy) = 0$. Integrating, $xy = c$.

From (*ii*), $\quad \dfrac{dy}{dx} - \dfrac{x}{y} = 0$ or $xdx - ydx = 0$

Integrating, $x^2 - y^2 = c$. Thus $xy = c$ or $x^2 - y^2 = c$, constitute the required solution.

Otherwise, combining these into one, the required solution can be written as

$$(xy - c)(x^2 - y^2 - c) = 0.$$

Example 11.38. *Solve* $p^2 + 2py \cot x = y^2$. *(Bhopal, 2008 ; Kerala, 2005)*

Solution. We have $p^2 + 2py \cot x + (y \cot x)^2 = y^2 + y^2 \cot^2 x$

or $\quad p + y \cot x = \pm y \operatorname{cosec} x$

i.e., $\quad p = y(-\cot x + \operatorname{cosec} x) \quad \ldots(i)$

or $\quad p = y(-\cot x - \operatorname{cosec} x) \quad \ldots(ii)$

From (*i*), $\dfrac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

Integrating, $\log y = \log \tan \dfrac{x}{2} - \log \sin x + \log c = \log \dfrac{c \tan x/2}{\sin x}$

or $\quad y = \dfrac{c}{2 \cos x^2/2}$ or $y(1 + \cos x) = c \quad \ldots(iii)$

From (*ii*), $\quad \dfrac{dy}{dx} = -y(\cot x + \operatorname{cosec} x)$ or $\dfrac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx$

Integrating, $\log y = -\log \sin x - \log \tan \dfrac{x}{2} + \log c = \log \dfrac{c}{\sin x \tan \dfrac{x}{2}}$

or $\quad y = \dfrac{c}{2 \sin^2 \dfrac{x}{2}}$ or $y(1 - \cos x) = c \quad \ldots(iv)$

Thus combining (*iii*) and (*iv*), the required general solution is

$$y(1 \pm \cos x) = c.$$

PROBLEMS 11.9

Solve the following equations :

1. $y\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - x = 0$.
2. $p(p + y) = x(x + y)$. *(V.T.U., 2011)*
3. $y = x\{p + \sqrt{(1 + p^2)}\}$.
4. $xy\left(\dfrac{dy}{dx}\right)^2 - (x^2 + y^2)\dfrac{dy}{dx} + xy = 0$.
5. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$. *(Madras, 2003)*

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \qquad ...(1)$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $F(x, p, c) = 0$. ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Obs. This method is especially useful for equations which do not contain x.

Example 11.39. *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$ ...(*i*)

Differentiating both sides with respect to x, $\dfrac{dy}{dx} = p = 2\left(p + x\dfrac{dp}{dx}\right) + \dfrac{p^2 + 2xp\dfrac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right) \cdot \frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\int\dfrac{dx}{x} + 2\int\dfrac{dp}{p}$ = a constant

or $$\log x + 2\log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

whence $$xp^2 = c \quad \text{or} \quad p = \sqrt{(c/x)} \qquad ...(ii)$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$

or $y = 2\sqrt{(cx)} + \tan^{-1} c$ which is the general solution of (*i*).

Obs. The significance of the factor $1 + p/(1 + x^2p^4) = 0$ which we didn't consider, will not be considered here as it concerns 'singular solution' of (*i*) whereas *we are interested only in finding general solution*.

Caution. Sometimes one is tempted to write (*ii*) as

$$\frac{dy}{dx} = \sqrt{\left(\frac{c}{x}\right)}$$

and integrating it to say that the required solution is $y = 2\sqrt{(cx)} + c'$. Such a reasoning is *incorrect*.

Example 11.40. *Solve* $y = 2px + p^n$. (*Bhopal, 2009*)

Solution. Given equation is $y = 2px + p^n$ ...(*i*)

Differentiating it with respect to x, we get

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} \quad \text{or} \quad p\frac{dx}{dp} + 2x = -np^{n-1}$$

or $$\frac{dx}{dp} + \frac{2x}{p} = -np^{n-2} \qquad ...(ii)$$

This is Leibnitz's linear equation in x and p. Here I.F. $= e^{\int \frac{2}{p}dp} = e^{\log p^2} = p^2$

$\therefore$ the solution of (*ii*) is

$$x(\text{I.F.}) = \int(-np^{n-2}).(\text{I.F.})\,dp + c \quad \text{or} \quad xp^2 = -n\int p^n\,dp + c = -\frac{np^{n+1}}{n+1} + c$$

or
$$x = cp^{-2} - \frac{np^{n-1}}{n+1} \qquad ...(iii)$$

Substituting this value of x in (*i*), we get $y = \dfrac{2c}{p} + \dfrac{1-n}{1+n}p^n$ ...(*iv*)

The equations (*iii*) and (*iv*) taken together, with parameter p, constitute the general solution (*i*).

Obs. In general, the equations of the form $y = xf(p) + \phi(p)$, known as *Lagrange's equation*, are solvable for y and lead to Leibnitz's equation in dx/dp.

PROBLEMS 11.10

Solve the following equations :

1. $y = x + a\tan^{-1}p$.
2. $y + px = x^4p^2$. (*S.V.T.U., 2007*)
3. $x^2\left(\dfrac{dy}{dx}\right)^4 + 2x\dfrac{dy}{dx} - y = 0$.
4. $xp^2 + x = 2yp$.
5. $y = xp^2 + p$.
6. $y = p\sin p + \cos p$.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad ...(1)$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

Obs. This method is especially useful for equations which do not contain y.

Example 11.41. *Solve $y = 2px + y^2p^3$.* (*Bhopal, 2008*)

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(=\dfrac{1}{p}\right) = \dfrac{1}{2}\cdot\dfrac{p\left(1 - 2y.p^3 - y^2 3p^2\dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

or
$$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

or
$$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

or
$$\left(p + y\frac{dp}{dy}\right)(1 + 2py^3) = 0 \text{ This gives } p + y\frac{dp}{dy} = 0. \text{ or } \frac{d}{dy}(py) = 0.$$

Integrating $\qquad py = c.$ ...(*i*)

Thus eliminating from the given equation and (*i*), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

which is the required solution.

PROBLEMS 11.11

Solve the following equations :

1. $p^3 - 4xyp + 8y^2 = 0$. (*Kanpur, 1996*)
2. $p^2y + 2px = y$.
3. $x - yp = ap^2$. (*Andhra, 2000*)
4. $p = \tan\left(x - \dfrac{p}{1+p^2}\right)$. (*S.V.T.U., 2008*)

11.14 CLAIRAUT'S EQUATION*

An equation of the form $y = px + f(p)$ is known as Clairaut's equation ...(1)

Differentiating with respect to x, we have $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

or $$[x + f'(p)]\frac{dp}{dx} = 0 \qquad \therefore \frac{dp}{dx} = 0, \text{ or } x + f'(p) = 0$$

$$\frac{dp}{dx} = 0, \text{ gives } p = c \qquad ...(2)$$

Thus eliminating p from (1) and (2), we get $y = cx + f(c)$ ...(3)

as the general solution of (1).

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

Obs. If we eliminate p from $x + f'(p) = 0$ and (1), we get an equation involving no constant. *This is the* **singular solution** of (1) *which gives the envelope of the family of straight lines* (3).

To obtain the singular solution, we proceed as follows :

(*i*) Find the general solution by replacing p by c *i.e.*, (3)

(*ii*) Differentiate this w.r.t. c giving $x + f'(c) = 0$. ...(4)

(*iii*) Eliminate c from (3) and (4) which will be the *singular solution*.

Example 11.42. *Solve $p = \sin(y - xp)$. Also find its singular solutions.*

Solution. Given equation can be written as

$$\sin^{-1} p = y - xp \text{ or } y = px + \sin^{-1} p \text{ which is the } \textit{Clairaut's equation}.$$

$\therefore$ its solution is $y = cx + \sin^{-1} c$.

To find the singular solution, differentiate (*i*) w.r.t. c giving

$$0 = x + \frac{1}{\sqrt{1-c^2}} \qquad ...(ii)$$

To eliminate c from (*i*) and (*ii*), we rewrite (*ii*) as

$$c = \sqrt{(x^2 - 1)}/x$$

Now substituting this value of c in (*i*), we get

$$y = \sqrt{(x^2 - 1)} + \sin^{-1}\{\sqrt{(x^2 - 1)}/x\}$$

which is the desired singular solution.

Obs. Equations reducible to Clairaut's form. Many equations of the first order but of higher degree can be easily reduced to the Clairaut's form by making suitable substitutions.

Example 11.43. *Solve $(px - y)(py + x) = a^2p$.* (*V.T.U., 2011 ; J.N.T.U., 2006*)

Solution. Put $x^2 = u$ and $y^2 = v$ so that $2xdx = du$ and $2ydy = dv$

$$\therefore \qquad p = \frac{dy}{dx} = \frac{dv}{y}\Big/\frac{du}{x} = \frac{x}{y}P, \text{ where } P = \frac{dv}{du}$$

*After the name of a youthful prodigy *Alexis Claude Clairaut* (1713–65) who first solved this equation. A French mathematician who is also known for his work in astronomy and geodesy.

Then the given equation becomes $\left(\frac{xP}{y}\cdot x-y\right)\left(\frac{xP}{y}\cdot y+x\right)=a^2\,\frac{xP}{y}$

or $$(uP-v)(P+1)=a^2P \text{ or } uP-v=\frac{a^2P}{P+1}$$

or $$v=uP-a^2P/(P+1), \text{ which is Clairaut's form.}$$

$\therefore$ its solution is $$v=uc-a^2c/(c+1),\ i.e.,\ y^2=cx^2-a^2c/(c+1).$$

PROBLEMS 11.12

1. Find the general and singular solution of the equations :

   (i) $xp^2-yp+a=0$. (*J.N.T.U., 2006*) (ii) $p=\log(px-y)$.

   (iii) $y=px+\sqrt{(a^2p^2+b^2)}$ (*W.B.T.U., 2005*) (iv) $\sin px\cos y=\cos px\sin y+p$ (*P.T.U., 2006*)

Solve the following equations :

2. $y+2\left(\frac{dy}{dx}\right)^2=(x+1)\frac{dy}{dx}$.
3. $(y-px)(p-1)=p$.
4. $(x-a)\left(\frac{dy}{dx}\right)^2+(x-y)\frac{dy}{dx}-y=0$.
5. $x^2(y-px)=yp^2$.
6. $(px+y)^2=py^2$.
7. $(px-y)(x+py)=2p$.

11.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 11.13

Fill up the blanks or choose the correct answer in the following problems :

1. $y=cx-c^2$, is the general solution of the differential equation

   (i) $(y')^2-xy'+y=0$ (ii) $y''=0$ (iii) $y'=c$ (iv) $(y')^2+xy'+y=0$.
2. The differential equation having a basis for its solution as $\sinh 6x$ and $\cosh 6x$ is

   (i) $y''+36y=0$ (ii) $y''-36y=0$ (iii) $y''+6y=0$ (iv) none of these.
3. The differential equation $(dx/dy)^2+5y^{1/3}=x$ is

   (i) linear of degree 3 (ii) non-linear of order 1 and degree 6

   (iii) non-linear of order 1 and degree 2.
4. The differential equation $y\,dx/dy+1=y$, $y(0)=1$, has

   (i) a unique solution (ii) two solutions

   (iii) infinite number of solutions (iv) no solution
5. Solution of $(x^2+y^2)\,dy=xy\,dx$ is ....
6. Solution of $(3x-2y)\,dx=x\,dy$ is ....
7. Solution of $dy/dx-y=2xy^2e^{-x}$ is ....
8. The differential equation $(y^2e^{xy^2}+6x)\,dx+(2xye^{xy^2}-4y)\,dy=0$ is

   (i) linear, homogeneous and exact (ii) non-linear, homogeneous and exact

   (iii) non-linear, non-homogeneous and exact (iv) non-linear, non-homogeneous and inexact.
9. Solution of $x\,dx+y\,dy+\frac{x\,dy-y\,dx}{x^2+y^2}$ is ....
10. Solution of $dy/dx=\frac{x^3+y^3}{xy^2}$ is ....
11. The differential equation $(x+x^8+ay^2)\,dx+(y^8-y+bxy)\,dy=0$ is exact if

    (i) $b=2a$ (ii) $a=b$ (iii) $a\neq 2b$ (iv) $a=1,\ b=3$.
12. Solution of $xy\,(1+xy^2)\,dy=dx$ is ....
13. Solution of $xp^2-yp+a=0$ is ....
14. The differential equation $p=\log(px-y)$ has the solution ....
15. Solution of $\frac{dy}{dx}-\frac{dx}{dy}=\frac{x}{y}-\frac{y}{x}$ is ....

16. The order of the differential equation $(1+y_1^2)^{3/2}/y_2 = c$ is ......

17. The general solution of $\dfrac{1}{x^2y^2}(xdy + ydx) = 0$ is ......

18. Integrating factor of the differential equation $\dfrac{dx}{dy}+\dfrac{3x}{y}=\dfrac{1}{y^2}$ is

(a) e^{y^3} (b) y^3 (c) x^3 (d) $-y^3$. (V.T.U., 2009)

19. Solution of the equation $\dfrac{dy}{dx}=\dfrac{y}{x}-\operatorname{cosec}\dfrac{y}{x}$ is

(a) $\cos(y/x) - \log x = c$ (b) $\cos(y/x) + \log x = c$
(c) $\cos^2(y/x) + \log x = c$ (d) $\cos^2(y/x) - \log x = c$. (V.T.U., 2010)

20. Solution of $x\sqrt{(1+x^2)} + y\sqrt{(1+y^2)}\, dy/dx = 0$ is ....

21. Solution of $dy/dx + y = 0$ given $y(0) = 5$ is ....

22. The substitution that transforms the equation $\dfrac{dy}{dx}=\dfrac{x+y+1}{2x+2y+3}$ to homogeneous form is ....

23. Integrating factor of $xy' + y = x^3y^6$ is ....

24. Solution of the exact differential equation $Mdx + Ndy = 0$ is ....

25. Solution of $(2x^3y^2 + x^4)\,dx + (x^4y + y^4)\,dy = 0$ is ....

26. The general solution of the differential equation $\dfrac{dy}{dx}+\dfrac{y}{x}=\tan 2x$ is ....

27. Degree of the differential equation $\left(\dfrac{d^2y}{dx^2}\right)^2 + x\left(\dfrac{dy}{dx}\right)^5 x^2y = 0$ is

(a) 2 (b) 0 (c) 3 (d) 5 (Bhopal, 2008)

28. Integrating factor of the differential equation $\dfrac{dy}{dx} + y\cos x = \dfrac{\sin 2x}{2}$ is

(a) $e^{\sin^2 x}$ (b) $e^{\sin^3 x}$ (c) $e^{\sin x}$ (d) $\sin x$

29. The differential equation of the family of circles with centre as origin is ..... (Nagarjuna, 2008)

30. Solution of $xe^{-x^2}\,dx + \sin y\,dy = 0$ is .... (Nagarjuna, 2008)

31. Solution of $p = \sin(y - xp)$ is

(a) $y = \dfrac{c}{x} + \sin^{-1} c$ (b) $y = cx + \sin c$ (c) $y = cx + \sin^{-1} c$ (d) $y = x + \sin^{-1} c$ (V.T.U., 2011)

32. Differential equation obtained by eliminating A and B form $y = A\cos x + B\sin x$ is $d^2y/dx^2 - y = 0$ (True or False)

33. $(x^3 - 3xy^2)\,dx + (y^3 - 2x^2y)\,dy = 0$ is an exact differential equation. (True or False)

CHAPTER 12

Applications of Differential Equations of First Order

1. Introduction. 2. Geometric applications. 3. Orthogonal trajectories. 4. Physical applications. 5. Simple electric circuits. 6. Newton's law of cooling. 7. Heat flow. 8. Rate of decay of radio-active materials. 9. Chemical reactions and solutions. 10. Objective Type of Questions.

12.1 INTRODUCTION

In this chapter, we shall consider only such practical problems which give rise to differential equations of the first order. The fundamental principles required for the formation of such differential equations are given in each case and are followed by illustrative examples.

12.2 GEOMETRIC APPLICATIONS

(*a*) *Cartesian coordinates.* Let $P(x, y)$ be any point on the curve $f(x, y) = 0$ (Fig. 12.1), then [as per 4.6 §(1) & 4.11(1) & (4)], we have

(*i*) slope of the tangent at $P (= \tan \psi) = dy/dx$

(*ii*) equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x)$$

so that its x-intercept $(= OT)$

$$= x - y \, . \, dx/dy$$

and y-intercept $(= OT') = y - x \, . \, dy/dx$

(*iii*) equation of the normal at P is $Y - y = -\dfrac{dx}{dy}(X - x)$

(*iv*) length of the tangent $(= PT) = y\sqrt{[1 + (dx/dy)^2]}$

(*v*) length of the normal $(= PN) = y\sqrt{[1 + (dy/dx)^2]}$

(*vi*) length of the sub-tangent $(= TM) = y \, . \, dx/dy$

(*vii*) length of the sub-normal $(= MN) = y \, . \, dy/dx$

(*viii*) $\dfrac{ds}{dx} = [1 + (dy/dx)^2]; \dfrac{ds}{dy} = \sqrt{[1 + (dx/dy)^2]}$

(*ix*) differential of the area $= ydx$ or xdy

(*x*) ρ, radius of curvature at $P = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}$

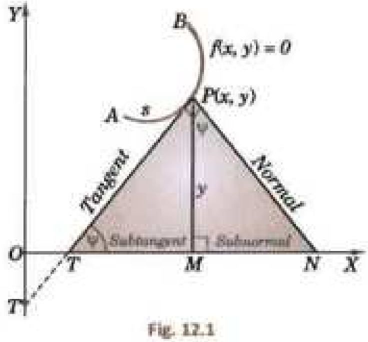


Fig. 12.1

(*b*) *Polar coordinates.* Let $P(r, \theta)$ be any point on the curve $r = f(\theta)$ (Fig. 12.2), then [as per § 4.7, 4.9 (2) & 4.11 (4)], we have

(*i*) $\psi = \theta + \phi$

(*ii*) $\tan\phi = r d\theta/dr$, $p = r\sin\phi$

(*iii*) $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

(*iv*) polar sub-tangent $(= OT) = r^2 d\theta/dr$

(*v*) polar sub-normal $(ON) = dr/d\theta$

(*vi*) $\dfrac{ds}{dr} = \sqrt{\left[1 + \left(r\dfrac{d\theta}{dr}\right)^2\right]};\ \dfrac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]}$

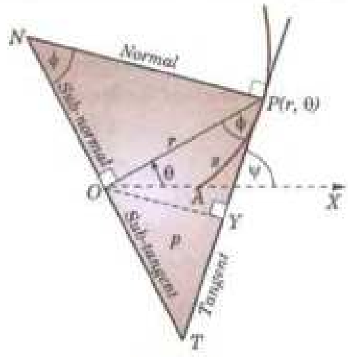


Fig. 12.2

Example 12.1. *Show that the curve in which the portion of the tangent included between the co-ordinates axes is bisected at the point of contact is a rectangular hyperbola.*

Solution. Let the tangent at any point $P(x, y)$ of a curve cut the axes at T and T' (Fig. 12.3).

We know that its x-intercept $(= OT) = x - y \,.\, dx/dy$

and $\qquad y$-intercept $(= OT') = y - x \,.\, dy/dx$

$\therefore$ the co-ordinates of T and T' are

$$(x - y \,.\, dx/dy, 0), (0, y - x\, dy/dx)$$

Since P is the mid-point of TT'

$$\therefore \qquad \frac{[x - y.dx/dy] + 0}{2} = x$$

or $\qquad x - y \,.\, dx/dy = 2x$ or $x\,dy + y\,dx = 0$

or $\qquad d(xy) = 0$ Integrating, $xy = c$

which is the equation of a rectangular hyperbola, having x and y axes as its asymptotes.

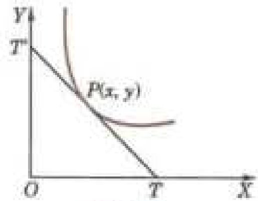


Fig. 12.3

Example 12.2. *Find the curve for which the normal makes equal angles with the radius vector and the initial line.*

Solution. Let PT and PN be the tangent and normal at $P(r, \theta)$ of the curve so that

$$\tan\phi = r\, d\theta/dr$$

By the condition of the problem,

$$\angle OPN = 90° - \phi = \angle ONP \text{ (Fig. 12.4).}$$

$$\therefore \qquad \theta = \angle PON = 180° - (180° - 2\phi) = 2\phi$$

or $\qquad \theta/2 = \phi \quad \therefore \quad \tan\dfrac{\theta}{2} = \tan\phi = r\dfrac{d\theta}{dr}.$

Here the variables are separable.

$$\therefore \qquad \frac{dr}{r} = \frac{\cos\theta/2}{\sin\theta/2}\,d\theta$$

Integrating both sides $\log r = 2\log\sin\theta/2 + \log c$

or $\qquad r = c\sin^2\theta/2 = \dfrac{1}{2}\,c(1 - \cos\theta)$

Thus the curve is the cardioid $r = a(1 - \cos\theta)$.

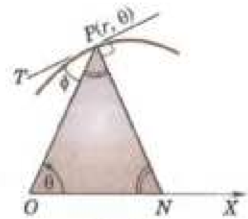


Fig. 12.4

Example 12.3. *Find the shape of a reflector such that light coming from a fixed source is reflected in parallel rays.*

Solution. Taking the fixed source of light as the origin and the X-axis parallel to the reflected rays; the reflector will be a surface generated by the revolution of a curve $f(x, y) = 0$ about X-axis (Fig. 12.5).

In the XY-plane, let PP' be the reflected ray, where P is the point (x, y) on the curve $f(x, y) = 0$.

If TPT' be the tangent at P, then

$\because$ angle of incidence = angle of reflection,

$$\therefore \quad \phi = \angle OPT = \angle P'PT' = \angle OTP = \psi$$

i.e.,
$$p = \frac{dy}{dx} = \tan \angle XOP = \tan 2\phi$$

$$= \frac{2 \tan \phi}{1 - \tan^2 \phi} = \frac{2p}{1 - p^2}$$

or
$$2x = \frac{y}{p} - yp \text{ which is solvable for } x \quad \ldots(i)$$

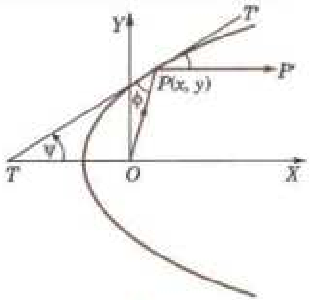


Fig. 12.5

$\therefore$ differentiating (i) w.r.t. y, $\dfrac{2}{p} = \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy} - p - y\dfrac{dp}{dy}$

i.e.,
$$\left(\frac{1}{p} + p\right) + \left(\frac{1}{p^2} + 1\right) y \frac{dp}{dy} = 0 \quad \text{or} \quad \left(\frac{1}{p} + p\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

This gives $\qquad dp/p = - dy/y$

Integrating, $\qquad \log p = \log c - \log y, \quad$ i.e., $p = c/y \quad \ldots(ii)$

Thus eliminating p from (i) and (ii), we have family of curves $y^2 = 2cx + c^2$.

Hence the reflector is a member of the family of paraboloids of revolution $y^2 + z^2 = 2cx + c^2$.

PROBLEMS 12.1

1. Find the equation of the curve which passes through
   (i) the point $(3, -4)$ and has the slope $2y/x$ at the point (x, y) on it.
   (ii) the origin and has the slope $x + 3y - 1$.
2. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through $(0, 1)$. Find the equation of the curve.
3. A curve is such that the length of the perpendicular from origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is
   $$y^2 - 2xy\frac{dy}{dx} - x^2 = 0, \text{ and hence find the curve.}$$
4. A plane curve has the property that the tangents from any point on the y-axis to the curve are of constant length a. Find the differential equation of the family to which the curve belongs and hence obtain the curve.
5. Determine the curve whose sub-tangent is twice the abscissa of the point of contact and passes through the point $(1, 2)$. *(Sambalpur, 1998)*
6. Determine the curve in which the length of the sub-normal is proportional to the square of the ordinate.
7. The tangent at any point of a certain curve forms with the coordinate axes a triangle of constant area A. Find the equation to the curve.
8. Find the curve which passes through the origin and is such that the area included between the curve, the ordinate and the x-axis is twice the cube of that ordinate.
9. Find the curve whose (i) polar sub-tangent is constant.
   (ii) polar sub-normal is proportional to the sine of the vectorial angle.
10. Determine the curve for which the angle between the tangent and the radius vector is twice the vectorial angle. *(Kanpur, 1996)*
11. Find the curve for which the tangent at any point P on it bisects the angle between the ordinate at P and the line joining P to the origin.
12. Find the curve for which the tangent, the radius vector r and the perpendicular from the origin on the tangent form a triangle of area kr^2.

12.3 (1) ORTHOGONAL TRAJECTORIES

Two families of curves such that every member of either family cuts each member of the other family at right angles are called **orthogonal trajectories** *of each other (Fig. 12.6).*

The concept of the orthogonal trajectories is of wide use in applied mathematics especially in field problems. For instance, in an electric field, the paths along which the current flows are the orthogonal trajectories of the equipotential curves and *vice versa*. In fluid flow, the stream lines and the equipotential lines (lines of constant velocity potential) are orthogonal trajectories. Likewise, the lines of heat flow for a body are perpendicular to the isothermal curves. The problem of finding the orthogonal trajectories of a given family of curves depends on the solution of the first order differential equations.

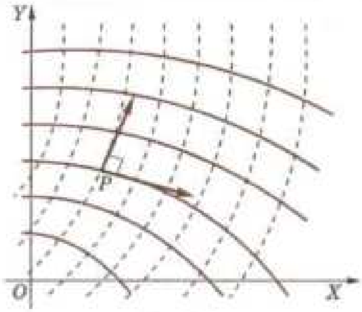


Fig. 12.6

(2) To find the orthogonal trajectories of the family of curves F(x, y, c) = 0.

(*i*) *Form its differential equation in the form* $f(x, y, dy/dx) = 0$ *by eliminating* c.

(*ii*) *Replace, in this differential equation,* dy/dx *by* $-dx/dy$, (so that the product of their slopes at each point of intersection is -1).

(*iii*) *Solve the differential equation of the orthogonal trajectories i.e.,* $f(x, y, -dx/dy) = 0$.

Example 12.4. *If the stream lines (paths of fluid particles) of a flow around a corner are* xy = *constant find their orthogonal trajectories (called equipotential lines-§ 20.6)* (*Marathwada, 2008*)

Solution. Taking the axes as the walls, the stream lines of the flow around the corner of the walls is

$$xy = c \qquad ...(i)$$

Defferentiating, we get, $x\dfrac{dy}{dx} + y = 0$ ...(*ii*)

as the differential equation of the given family (*i*).

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (*ii*), we obtain $x\left(-\dfrac{dx}{dy}\right) + y = 0$

or $$xdx - ydy = 0 \qquad ...(iii)$$

as the differential equation of the orthogonal trajectories.

Integrating (*iii*), we get $x^2 - y^2 = c'$ as the required orthogonal trajectories of (*i*) *i.e.*, the *equipotential lines*, shown dotted in Fig. 12.7.

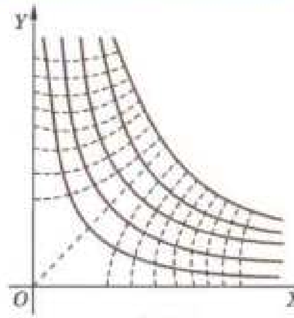


Fig. 12.7

Example 12.5. *Find the orthogonal trajectories of the family of confocal conics* $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2+\lambda} = 1$, *where* λ *is the parameter.* (V.T.U., 2009 S)

Solution. Differentiating the given equation, we get $\dfrac{2x}{a^2} + \dfrac{2y}{a^2+\lambda}\dfrac{dy}{dx} = 0$

or $$\frac{y}{a^2+\lambda} = -\frac{x}{a^2\,(dy/dx)} \quad \text{or} \quad \frac{y^2}{a^2+\lambda} = \frac{-xy}{a^2\,(dy/dx)}$$

Substituting this in the given equation, we get

$$\frac{x^2}{a^2} - \frac{xy}{a^2\,(dy/dx)} = 1 \quad \text{or} \quad (x^2 - a^2)\frac{dy}{dx} = xy \qquad ...(i)$$

which is the differential equation of the given family.

Changing dy/dx to $-dx/dy$ in (*i*), we get $(a^2 - x^2)\, dx/dy = xy$ as the differential equation of the orthogonal trajectories.

Separating the variables and integrating, we obtain

$$\int ydy = \int \frac{a^2 - x^2}{x} dx + c \quad \text{or} \quad \frac{1}{2}y^2 = a^2 \log x - \frac{1}{2}x^2 + c$$

or $$x^2 + y^2 = 2a^2 \log x + c' \qquad [c' = 2c]$$

which is the equation of the required orthogonal trajectories.

Example 12.6. *Find the orthogonal trajectories of a system of confocal and coaxial parabolas.*

Solution. The equation of the family of confocal parabolas having x-axis as their axis, is of the form

$$y^2 = 4a(x + a) \qquad ...(i)$$

Differentiating, $$y\frac{dy}{dx} = 2a \qquad ...(ii)$$

Substituting the value of a from (*ii*) in (*i*), we get $y^2 = 2y\dfrac{dy}{dx}\left(x + \dfrac{1}{2}y\dfrac{dy}{dx}\right)$

i.e., $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ as the differential equation of the family.} \qquad ...(iii)$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in (*iii*), we obtain $y\left(\dfrac{dx}{dy}\right)^2 - 2x\dfrac{dx}{dy} - y = 0$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0 \text{ which is the same as } (iii).$$

Thus we see that a system of confocal and coaxial parabolas is *self-orthogonal, i.e.*, each member of the family (*i*) cuts every other member of the same family orthogonally.

(3) To find the orthogonal trajectories of the curves F(r, θ, c) = 0.

(*i*) *Form its differential equation in the form $f(r, \theta, dr/d\theta) = 0$ by eliminating c.*

(*ii*) *Replace in this differential equation,*

$$\frac{dr}{d\theta} \text{ by } -r^2\frac{d\theta}{dr}$$

[$\because$ for the given curve through $P(r, \theta)$ $\tan\phi = r d\theta/dr$

and for the orthogonal trajectory through P

$$\tan\phi' = \tan(90° + \phi) = -\cot\phi = -\frac{1}{r}\frac{dr}{d\theta}$$

Thus for getting the differential equation of the orthogonal trajectory

$$r\frac{d\theta}{dr} \text{ is to be replaced by } -\frac{1}{r}\frac{dr}{d\theta}$$

or $$\frac{dr}{d\theta} \text{ is to be replaced by } -r^2\frac{d\theta}{dr}.]$$

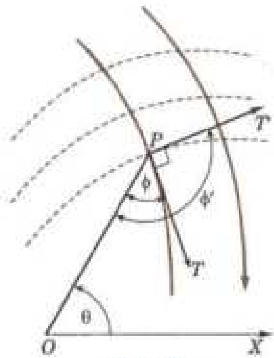


Fig. 12.8

(*iii*) *Solve the differential equation of the orthogonal trajectories*

i.e., $$f(r, \theta, -r^2 d\theta/dr) = 0.$$

Example 12.7. *Find the orthogonal trajectory of the cardioids $r = a(1 - \cos\theta)$.* (*Kurukshetra, 2005*)

Solution. Differentiating $$r = a(1 - \cos\theta). \qquad ...(i)$$

with respect to θ, we get $$\frac{dr}{d\theta} = a\sin\theta \qquad ...(ii)$$

Eliminating a from (*i*) and (*ii*), we obtain

$$\frac{dr}{d\theta}\cdot\frac{1}{r} = \frac{\sin\theta}{1 - \cos\theta} = \cot\frac{\theta}{2} \text{ which is the differential equation of the given family.}$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \cot\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} + \tan\frac{\theta}{2}\,d\theta = 0$$

as the differential equation of orthogonal trajectories. It can be rewritten as

$$\frac{dr}{r} = -\frac{(\sin\theta/2)\,d\theta}{\cos\theta/2}$$

Integrating, $\log r = 2 \log \cos \theta/2 + \log c$

or $$r = c \cos^2 \theta/2 = \frac{1}{2} c(1 + \cos \theta) \quad \text{or} \quad r = a'(1 + \cos \theta)$$

which is the required orthogonal trajectory.

Example 12.8. *Find the orthogonal trajectory of the family of curves $r^n = a \sin n\theta$.* (V.T.U., 2006)

Solution. We have $n \log r = \log a + \log \sin n\theta$.

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \qquad \text{or} \qquad \frac{1}{r}\frac{dr}{d\theta} = \cot n\theta$$

Replacing $dr/d\theta$ by $-r^2\, d\theta/dr$, we obtain

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta \qquad \text{or} \qquad \tan n\theta.\ d\theta - \frac{dr}{r} = 0$$

Integrating, $$\int \frac{dr}{r} + \int \frac{\sin n\theta}{\cos n\theta}\, d\theta = c,$$

i.e., $$\log r - \frac{1}{n} \log \cos n\theta = c \qquad \text{or} \qquad \log (r^n/\cos n\,\theta) = nc = \log b. \text{ (say)}$$

or $r^n = b \cos n\theta$, which is the required orthogonal trajectory.

PROBLEMS 12.2

Find the orthogonal trajectories of the family of :

1. Parabolas $y^2 = 4ax$. (*Marathwada, 2009*)
2. Parabolas $y = ax^2$. (*J.N.T.U., 2006*)
3. Semi-cubical parabolas $ay^2 = x^3$. (*J.N.T.U., 2005*)
4. Coaxial circles $x^2 + y^2 + 2\lambda x + c = 2$, λ being the parameter. (*J.N.T.U., 2006*)
5. Confocal conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$, λ being the parameter. (*Kurukshetra, 2006*)
6. Cardioids $r = a(1 + \cos \theta)$. (*J.N.T.U., 2003*)
7. $r = 2a (\cos \theta + \sin \theta)$ (*V.T.U., 2010 S*)
8. Confocal and coaxial parabolas $r = 2a/(1 + \cos \theta)$. (*Nagpur, 2008*)
9. Curves $r^2 = a^2 \cos 2\theta$. (*V.T.U., 2009 S*)
10. $r^n \cos n\theta = a^n$. (*V.T.U. 2011*)
11. Show that the family of parabolas $x^2 = 4a (y + a)$ is self orthogonal. (*Kerala, 2005*)
12. Show that the family of curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ are orthogonal. (*Mumbai, 2005*)
13. The *electric lines of force* of two opposite charges of the same strength at $(\pm 1, 0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their *equipotential lines* (orthogonal trajectories).

    [**Isogonal trajectories.** *Two families of curves such that every member of either family cuts each member of the other family at a constant angle α (Say), are called* ***isogonal trajectories*** *of each other.* The slopes m, m' of the tangents to the corresponding curves at each point, are connected by the relation $\dfrac{m \,\square\, m'}{1+mm'} = \tan \alpha = \text{const.}$]
14. Find the isogonal trajectories of the family of circles $x^2 + y^2 = a^2$ which intersect at 45°.

12.4 PHYSICAL APPLICATIONS

(1) Let a body of mass m start moving from O along the straight line OX under the action of a force F. After any time t, let it be moving at P where $OP = x$, then

(*i*) its velocity $(v) = \dfrac{dx}{dt}$

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}$ or $\dfrac{v\,dv}{dx}$ or $\dfrac{d^2x}{dt^2}$

If, however, the body be moving along a curve, then

(*i*) its velocity $(v) = ds/dt$ and

(*ii*) its acceleration $(a) = \dfrac{dv}{dt}, v\dfrac{dv}{ds}$ or $\dfrac{d^2s}{dt^2}$.

The quantity mv is called the *momentum*.

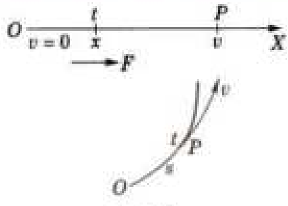


Fig. 12.9

(2) Newton's second law *states that* $F = \dfrac{d}{dt}(mv)$.

If m is constant, then $F = m\dfrac{dv}{dt} = ma$, *i.e., net force = mass × acceleration.*

(3) Hooke's law* *states that tension of an elastic string (or a spring) is proportional to extension of the string (or the spring) beyond its natural length.*

Thus $\mathbf{T = \lambda e/1}$,

where e is the extension beyond the natural length l and λ is the *modulus of elasticity.*

Sometimes for a spring, we write $\mathbf{T = ke}$,

where e is the extension beyond the natural length and k is the *stiffness of the spring.*

(4) Systems of units

I. **F.P.S.** [foot (*ft.*) pound (*lb.*), second (*sec.*)] **system**. If mass m is in *pounds* and acceleration (a) is in ft/sec^2, then the force $F(= ma)$ is in *poundals.*

II. **C.G.S.** [centimetre (*cm.*), gram (*g*), second (*sec*)] **system**. If mass m is in *grams* and acceleration a is in cm/sec^2 then the force $F(= ma)$ is *dynes.*

III. **M.K.S.** [metre (*m*), kilogram (*kg.*), second (*sec*)] **system**. If mass m is in kilograms and acceleration a in m/sec^2, then the force $F(= ma)$ is in *newtons (nt).*

These are called *absolute units.* If g is the acceleration due to gravity and w is the weight of the body, then w/g is the mass of the body in *gravitational units.*

$$g = 32 \text{ ft/sec}^2 = 980 \text{ cm/sec}^2 = 9.8 \text{ m/sec}^2 \text{ approx.}$$

Example 12.9. Motion of a boat across a stream. *A boat is rowed with a velocity u directly across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the path of the boat and the distance down stream to the point where it lands.*

Solution. Taking the origin at the point from where the boat starts, let the axes be chosen as in Fig. 12.10.

At any time t after its start from O, let the boat be at $P(x, y)$, so that

dx/dt = velocity of the current $= ky(a - y)$

dy/dt = velocity with which the boat is being rowed $= u$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{u}{ky(a-y)} \quad \ldots(i)$$

This gives the direction of the resultant velocity of the boat which is also the direction of the tangent to the path of the boat.

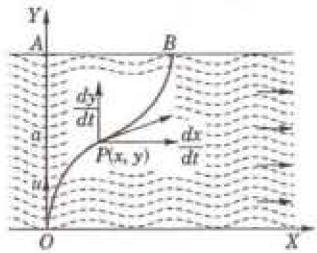


Fig. 12.10

Now (*i*) is of variables separable form and we can write it as

$$y(a - y)dy = \frac{u}{k}\,dx$$

Integrating, we get $\quad \dfrac{ay^2}{2} - \dfrac{y^3}{3} = \dfrac{u}{k}x + c$

Since $y = 0$ when $x = 0$, $\therefore$ $c = 0$.

Hence the equation to the path of the boat is $x = \dfrac{k}{6u}y^2(3a - 2y)$

Putting $y = a$, we get the distance AB, down stream where the boat lands $= ka^3/6u$.

*Named after an English physicist *Robert Hooke* (1635–1703) who had discovered the law of gravitation earlier than Newton.

Example 12.10. Resisted motion. *A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.* (Marathwada, 2008)

Solution. By Newton's second law, the equation of motion of the body is $v\dfrac{dv}{dx} = -cx - bv^2$

or $$v\frac{dv}{dx} + bv^2 = -cx \qquad ...(i)$$

This is Bernoulli's equation. $\therefore$ Put $v^2 = z$ and $2v\, dv/dx = dz/dx$, so that (*i*) becomes

$$\frac{dz}{dx} + 2bz = -2cx \qquad ...(ii)$$

This is Leibnitz's linear equation and I.F. $= e^{2bx}$.

$\therefore$ the solution of (*ii*) is $ze^{2bx} = -\int 2cxe^{2bx}\,dx + c'$ [Integrate by parts]

$$= -2c\left[x\cdot\frac{e^{2bx}}{2b} - \int 1\cdot\frac{e^{2bx}}{2b}\,dx\right] + c' = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

or $$v^2 = \frac{c}{2b^2} + c'e^{-2bx} - \frac{cx}{b} \qquad ...(iii)$$

Initially $v = 0$ when $x = 0$ $\therefore$ $0 = c/2b^2 + c'$.

Thus, substituting $c' = -c/2b^2$ in (*iii*), we get $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$.

Example 12.11. Resisted vertical motion. *A particle falls under gravity in a resisting medium whose resistance varies with velocity. Find the relation between distance and velocity if initially the particle starts from rest.* (U.P.T.U., 2003)

Solution. After falling a distance s in time t from rest, let v be velocity of the particle. The forces acting on the particle are its weight mg downwards and resistance $m\lambda v$ upwards.

$\therefore$ equating of motion is $m\dfrac{dv}{dt} = mg - m\lambda v$

or $$\frac{dv}{dt} = g - \lambda v \quad \text{or} \quad \frac{dv}{g - \lambda v} = dt$$

Integrating, $\displaystyle\int\frac{dv}{g - \lambda v} = \int dt + c$ or $-\dfrac{1}{\lambda}\log(g - \lambda v) = t + c$

Since $v = 0$ when $t = 0$, $\therefore$ $c = -\dfrac{1}{\lambda}\log g$

Thus $$\frac{1}{\lambda}\log\left[\frac{g}{g - \lambda v}\right] = t \quad \text{or} \quad \frac{g - \lambda v}{g} = e^{-\lambda t}$$

or $$\frac{ds}{dt} = v = \frac{g}{\lambda}(1 - e^{-\lambda t}) \qquad ...(i)$$

Integrating, $$s = \frac{g}{\lambda}\int(1 - e^{-\lambda t})dt + c' \quad \text{or} \quad s = \frac{g}{\lambda}\left(t + \frac{1}{\lambda}e^{-\lambda t}\right) + c'$$

Since $s = 0$ when $t = 0$, $\therefore$ $c' = -g/\lambda^2$

Thus $$s = \frac{g}{\lambda}t + \frac{g}{\lambda^2}(e^{-\lambda t} - 1) \qquad ...(ii)$$

Eliminating t from (*i*) and (*ii*), we get

$$s = \frac{g}{\lambda^2}\log\left(\frac{g}{g - \lambda v}\right) - \frac{v}{\lambda}$$

which is the desired relation between s and v.

Example 12.12. *A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that*

$$\frac{2kx}{m} = \log \frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

Solution. If the body be moving with the velocity v after having fallen through a distance x, then its equation of motion is

$$mv\frac{dv}{dx} = mg - kv^2 \quad \text{or} \quad mv\frac{dv}{dx} = k(a^2 - v^2). \qquad [\because mg = ka^2] \quad ...(i)$$

$\therefore$ separating the variables and integrating, we get $\int \frac{vdv}{a^2 - v^2} = \int \frac{k}{m} dx + c$

or
$$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + c \qquad ...(ii)$$

Initially, when $x = 0$, $v = 0$. $\therefore -\frac{1}{2}\log a^2 = c$ ...(iii)

Subtracting (*iii*) from (*ii*), we have $\frac{1}{2}[\log a^2 - \log(a^2 - v^2)] = kx/m$

or
$$\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Obs. *When the resistance becomes equal to the weight, the acceleration becomes zero and particle continues to fall with a constant velocity, called the* **limiting** *or* **terminal** *velocity.* From (*i*), it follows that the acceleration will become zero when $v = a$. *Thus, the limiting velocity, i.e., the maximum velocity which the particle can attain is a.*

Example 12.13. Velocity of escape from the earth. *Find the initial velocity of a particle which is fired in radial direction from the earth's centre and is supposed to escape from the earth. Assume that it is acted upon by the gravitational attraction of the earth only.*

Solution. According to Newton's law of gravitation, the acceleration α of the particle is proportional to $1/r^2$ where r is the variable distance of the particle from the earth's centre. Thus

$$\alpha = v\frac{dv}{dr} = -\frac{\mu}{r^2}$$

where v is the velocity when at a distance r from the earth's centre. The acceleration is negative because v is decreasing. When $r = R$, the earth's radius then $\alpha = -g$, the acceleration of gravity at the surface.

i.e.,
$$-g = -\mu/R^2, \text{ i.e., } \mu = gR^2 \quad \therefore \quad v\frac{dv}{dr} = -\frac{gR^2}{r^2}$$

Separating the variables and integrating, we obtain $\int v dv = -gR^2 \int \frac{dr}{r^2} + c$

i.e.,
$$v^2 = \frac{2gR^2}{r} + 2c \qquad ...(i)$$

On the earth's surface $r = R$ and $v = v_0$ (say), the initial velocity. Then

$$v_0^2 = 2gR + 2c, \quad \text{i.e.,} \quad 2c = v_0^2 - 2gR$$

Inserting this value of c in (*i*), we get $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$

When v vanishes, the particle stops and the velocity will change from positive to negative and the particle will return to the earth. Thus the velocity will remain positive, if and only if $v_0^2 \geq 2gR$ and then the particle projected fron the earth with this velocity will escape from the earth. Hence the minimum such velocity of projection $v_0 = \sqrt{(2gR)}$ *is called the velocity of escape* from the earth [See Problem 9, page 454].

Example 12.14. Rotating cylinder containing liquid. *A cylindrical tank of radius r is filled with water to a depth h. When the tank is rotated with angular velocity ω about its axis, centrifugal force tends to drive the water outwards from the centre of the tank. Under steady conditions of uniform rotation, show that the section of the free surface of the water by a plane through the axis, is the curve*

$$y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h.$$

Solution. Let the figure represent an axial section of the cylindrical tank. Forces acting on a particle of mass m at $P(x, y)$ on the curve, cut out from the free surface of water, are :

(*i*) the weight mg acting vertically downwards,

(*ii*) the centrifugal force $m\omega^2 x$ acting horizontally outwards.

As the motion is steady, P moves just on the surface of the water and, therefore, there is no force along the tangent to the curve. Thus the resultant R of mg and $m\omega^2 x$ is along the outward normal to the curve.

$\therefore \quad R \cos \psi = mg$ and $R \sin \psi = m\omega^2 x$

whence $$\frac{dy}{dx} = \tan \psi = \frac{m\omega^2 x}{mg} = \frac{\omega^2 x}{g} \qquad ...(i)$$

This is the differential equation of the surface of the rotating liquid.

Integrating (*i*), we get

$$\int dy = \frac{\omega^2}{g} \int x\, dx + c$$

i.e., $$y = \frac{\omega^2 x^2}{2g} + c \qquad ...(ii)$$

To find c, we note that the volume of the liquid remains the same in both cases (Fig. 12.11).

When $x = 0$ in (*ii*), OA $(= y) = c$. When $x = r$

in (*ii*), h' $(= y) = \dfrac{\omega^2 r^2}{2g} + c \qquad ...(iii)$

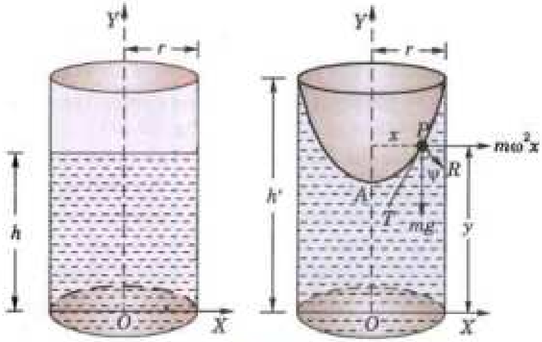


Fig 12.11

Now the volume of the liquid in the non-rotational case $= \pi r^2 h$, and the volume of the liquid in the rotational case

$$= \pi r^2 h' - \int_{OA}^{h'} \pi x^2 dy = \pi r^2 h' - \frac{2\pi g}{\omega^2} \int_c^{h'} (y - c)\, dy \qquad \text{[From (ii)]}$$

$$= \pi r^2 h' - \frac{\pi g}{\omega^2}(h' - c)^2 = \pi r^2 \left(\frac{\omega^2 r^2}{4g} + c\right) \qquad \text{[By (iii)]}$$

Thus $$\pi r^2 h = \pi r^2 \left(\frac{\omega^2 r^2}{4g} + c\right) \text{ whence } c = h - \frac{\omega^2 r^2}{4g}$$

$\therefore$ (*ii*) becomes, $$y = \frac{\omega^2 x^2}{2g} + h - \frac{\omega^2 x^2}{4g} \quad \text{or} \quad y = \frac{\omega^2}{2g}\left(x^2 - \frac{r^2}{2}\right) + h$$

which is the desired equation of the curve.

Example 12.15. Discharge of water through a small hole. *If the velocity of flow of water through a small hole is $0.6\sqrt{2gy}$ where g is the gravitational acceleration and y is the height of water level above the hole, find the time required to empty a tank having the shape of a right circular cone of base radius a and height h filled completely with water and having a hole of area A_0 in the base.*

Solution. At any time t, let the height of the water level be y and radius of its surface be r (Fig. 12.12) so that

$$\frac{h - y}{r} = \frac{h}{a} \quad \text{or} \quad r = a(h - y)/h$$

$\therefore$ surface area of the liquid $= \pi r^2 = \pi a^2 (1 - y/h)^2$

Volume of water drained through the hole per unit time

$$= 0.6\sqrt{(2gy)}\,A_0 = 4.8\sqrt{y}\,A_0 \qquad [\because \ g = 32$$

$\therefore$ rate of fall of liquid level $= 4.8\,A_0\sqrt{y} \div \pi a^2(1 - y/h)^2$

i.e., $$\frac{dy}{dt} = -\frac{4.8\,A_0\sqrt{y}}{\pi a^2(1 - y/h)^2} \qquad \text{(– ve is taken since the water level decreases)}$$

Hence time to empty the tank (= t)

$$= -\int_h^0 \frac{\pi a^2(1 - y/h)^2}{4.8\,A_0\sqrt{y}}\,dy = \frac{\pi a^2}{4.8\,A_0}\int_0^h (y^{-1/2} - 2y^{1/2}/h + y^{3/2}/h^2)\,dy$$

$$= \frac{\pi a^2}{4.8\,A_0}\left|2y^{1/2} - \frac{4}{3h}y^{3/2} + \frac{2}{5h^2}y^{5/2}\right|_0^h = 0.2\,\pi a^2\sqrt{h}/A_0.$$

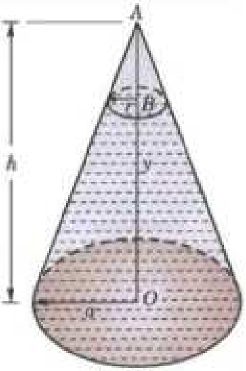


Fig. 12.12

Example 12.16. Atmospheric pressure. *Find the atmospheric pressure p lb. per ft. at a height z ft. above the sea-level, both when the temperature is constant or variable.*

Solution. Take a vertical column of air of unit cross-section.

Let p be the pressure at a height z above the sea-level and $p + \delta p$ at height $z + \delta z$. Let ρ be the density at a height z. (Fig. 12.13)

Now since the thin column δz of air is being pressured upwards with pressure p and downwards with $p + \delta p$, we get by considering its equilibrium;

$$p = p + \delta p + g\rho\delta z.$$

Taking the limit, we get $\qquad dp/dz = -g\rho \qquad$ ...(*i*)

which is the differential equation giving the atmospheric pressure at height z.

(*i*) *When the temperature is constant,* we have by Boyle's law*, $p = k\rho \qquad$ ...(*ii*)

$\therefore$ Substituting the value of ρ from (*ii*) in (*i*), we get

$$\frac{dp}{dz} = -gp/k \quad \text{or} \quad \int\frac{dp}{p} = -\frac{g}{k}\int dz + c \quad \text{or} \quad \log p = -\frac{g}{k}z + c$$

At the sea-level, where $z = 0$, $p = p_0$ (say) then $c = \log p_0$

$$\therefore \qquad \log p - \log p_0 = -\frac{g}{k}z \ \ i.e., \ \log p/p_0 = -gz/k$$

Hence p is given by $p = p_0e^{-gz/k}$.

(*ii*) *When the temperature varies,* we have $p = k\rho^n$.

Proceeding as above, we shall find that p is given by $\dfrac{n}{n-1}(p_0^{1-1/n} - p^{1-1/n}) = gk^{-1/n}z$.

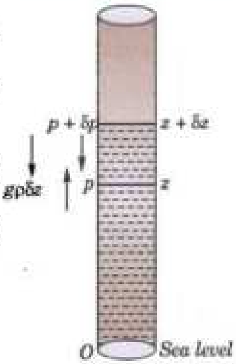


Fig. 12.13

PROBLEMS 12.3

1. A particle of mass m moves under gravity in a medium whose resistance is k times its velocity, where k is a constant. If the particle is projected vertically upwards with a velocity v, show that the time to reach the highest point is $\dfrac{m}{k}\log_e\left(1 + \dfrac{kv}{mg}\right)$
2. A body of mass m falls from rest under gravity and air resistance proportional to square of velocity. Find velocity as function of time. (*Marathwada, 2008*)
3. A body of mass m falls from rest under gravity in a field whose resistance is mk times the velocity of the body. Find the terminal velocity of the body and also the time taken to acquire one half of its limiting speed.
4. A particle is projected with velocity v along a smooth horizontal plane in the medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is $\dfrac{1}{\mu v}(\sqrt{1 + 2\mu v^2 t} - 1)$.

*Named after the English physicist *Robert Boyle* (1627–1691) who was one of the founders of the Royal Society.

5. When a bullet is fired into a sand tank, its retardation is proportional to the square root of its velocity. How long will it take to come to rest if it enters the sand bank with velocity v_0?
6. A particle of mass m is attached to the lower end of a light spring (whose upper end is fixed) and is released. Express the velocity v as a function of the stretch x feet.
7. A chain coiled up near the edge of a smooth table just starts to fall over the edge. The velocity v when a length x has fallen is given by $xv\dfrac{dv}{dx} + v^2 = gx$. Show that $v = 8\sqrt{(x/3)}$ ft/sec.
8. A toboggan weighing 200 lb., descends from rest on a uniform slope of 5 in 13 which is 15 yards long. If the coefficient of friction is 1/10 and the air resistance varies as the square of the velocity and is 3 lb. weight when the velocity is 10 ft/sec.; prove that its velocity at the bottom is 38.6 ft/sec and show that however long, the slope is the velocity cannot exceed 44 ft per sec. [**Hint.** Fig. 12.14. Equation of motion is

$$\frac{W}{g}\cdot v\frac{dv}{dx} = -\mu R - kv^2 + W\sin\alpha]$$

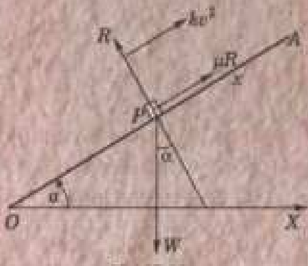


Fig. 12.14

9. Show that a particle projected from the earth's surface with a velocity of 7 miles/sec. will not return to the earth. [Take earth's radius = 3960 miles and $g = 32.17$ ft/sec²].
10. A cylindrical tank 1.5 m. high stands on its circular base of diameter 1 m. and is initially filled with water. At the bottom of the tank there is a hole of diameter 1 cm., which is opened at some instant, so that the water starts draining under gravity. Find the height of water in the tank at any time t sec. Find the times at which the tank is one-half full, one quarter full, and empty.

[**Hint.** Take $g = 980$ cm/sec² in $v = 0.6\sqrt{(2gy)}$]
11. The rate at which water flows from a small hole at the bottom of a tank is proportional to the square root of the depth of the water. If half the water flows from a cylindrical tank (with vertical axis) in 5 minutes, find the time required to empty the tank.
12. A conical cistern of height h and semi-vertical angle α is filled with water and is held in vertical position with vertex downwards. Water leaks out from the bottom at the rate of kx^2 cubic cms per second, k is a constant and x is the height of water level from the vertex. Prove that the cistern will be empty in $(\pi h\tan^2\alpha)/k$ seconds.
13. Upto a certain height in the atmosphere, it is found that the pressure p and the density ρ are connected by the relation $p = k\rho^n$ $(n > 1)$. If this relation continued to hold upto any height, show that the density would vanish at a finite height.

12.5 SIMPLE ELECTRIC CIRCUITS

We shall consider circuits made up of

(*i*) three passive elements—resistance, inductance, capacitance and

(*ii*) an active element—voltage source which may be a battery or a generator.

(1) Symbols

Element	*Symbol*	*Unit**
1. *Quantity of electricity*	q	coulomb
2. *Current* (= time rate flow of electricity)	i	ampere (A)
3. *Resistance, R*		ohm (Ω)
4. *Inductance, L*		henry (H)
5. *Capacitance, C*		farad (F)

*These units are respectively named after the French engineer and physicist *Charles Augustin de Coulomb* (1736–1806); French physicist *Andre Marie Ampere* (1775–1836); German physicist *George Simon Ohm* (1789–1854); Italian physicist *Joseph Henry* (1797–1878); American physicist *Michael Faraday* (1791–1867) and the Italian physicist *Alessandro Volta* (1745–1827).

Element	Symbol	Unit
6. *Electromotive force (e.m.f.) or voltage, E*	 *Battery, E = Constant*  *Generator, E = Variable*	volt (V)

7. *Loop* is any closed path formed by passing through two or more elements in series.
8. *Nodes* are the terminals of any of these elements.

(2) Basic relations

(*i*) $i = \dfrac{dq}{dt}$ or $q = \int i dt$ [$\because$ current is the rate of flow of electricity]

(*ii*) Voltage drop across resistance $R = Ri$ [Ohm's Law]

(*iii*) Voltage drop across inductance $L = L\dfrac{di}{dt}$

(*iv*) Voltage drop across capacitance $C = \dfrac{q}{C}$.

(3) Kirchhoff's laws*. The formulation of differential equations for an electrical circuit depends on the following two Kirchhoff's laws which are of cardinal importance :

I. *The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit.*

II. *The algebraic sum of the currents flowing into (or from) any node is zero.*

(4) Differential equations

(*i*) *R, L series circuit.* Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) E. (Fig. 12.15).

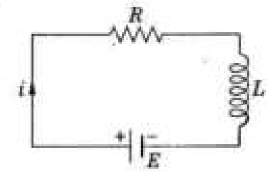


Fig. 12.15

Let i be the current flowing in the circuit at any time t. Then by Kirchhoff's first law, we have sum of voltage drops across R and $L = E$

i.e., $$Ri + L\frac{di}{dt} = E \quad \text{or} \quad \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \qquad ...(1)$$

This is a Leibnitz's linear equation.

$$\text{I.F.} = e^{\int \frac{R}{L}dt} = e^{Rt/L} \text{ and therefore, its solution is } i(\text{I.F.}) = \int \frac{E}{L}(\text{I.F.})dt + c$$

or $$i \cdot e^{Rt/L} = \int \frac{E}{L} e^{Rt/L}\, dt + c = \frac{E}{L} \cdot \frac{L}{R} \cdot e^{Rt/L} + c \text{ whence } i = \frac{E}{R} + ce^{-Rt/L} \qquad ...(2)$$

If initially there is no current in the circuit, *i.e.*, $i = 0$, when $t = 0$, we have $c = -E/R$.

Thus (2) becomes $i = \dfrac{E}{R}\left(1 - e^{-Rt/L}\right)$ which shows that i increases with t and attains the maximum value E/R.

(*ii*) *R, L, C series circuit.* Now consider a circuit containing resistance R, inductance L and capacitance C all in series with a constant e.m.f. E (Fig. 12.16)

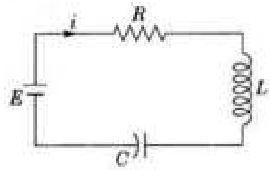


Fig. 12.16

If i be the current in the circuit at time t, then the charge q on the condenser $= \int i\, dt$, *i.e.*, $i = \dfrac{dq}{dt}$.

Applying Kirchhoff's law, we have, sum of the voltage drops across R, L and $C = E$.

i.e., $$Ri + L\frac{di}{dt} + \frac{q}{C} = E$$

*Named after the German physicist *Gustav Robert Kirchhoff* (1824–1887).

or $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E.$$

This is the desired differential equation of the circuit and will be solved in § 14.5.

Example 12.17. *Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation L di/dt + Ri = E sin ωt.*

Find the value of the current at any time t, if initially there is no current in the circuit.

(*Kurukshetra, 2005*)

Solution. By Kirchhoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$

i.e., $$Ri + L\frac{di}{dt} = E \sin \omega t.$$

This is the required differential equation which can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \sin \omega t$

This is a *Leibnitz's equation*. Its I.F. $= e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i(\text{I.F.}) = \int \frac{E}{L} \sin \omega t \cdot (\text{I.F.})\, dt + c$

or $$ie^{Rt/L} = \frac{E}{L}\int e^{Rt/L} \sin \omega t\, dt + c = \frac{E}{L}\frac{e^{Rt/L}}{\sqrt{[(R/L)^2 + \omega^2]}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c$$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin(\omega t - \phi) + ce^{-Rt/L} \text{ where } \tan\phi = L\omega/R \qquad ...(i)$$

Initially when $t = 0$; $i = 0$. $\therefore$ $0 = \frac{E \sin(-\phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + c$, *i.e.*, $c = \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}}$

Thus (*i*) takes the form $i = \frac{E \sin(\omega t - \phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}} \cdot e^{-Rt/L}$

or $$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} [\sin(\omega t - \phi) + \sin\phi \,.\, e^{-Rt/L}]$$ which gives the current at any time t.

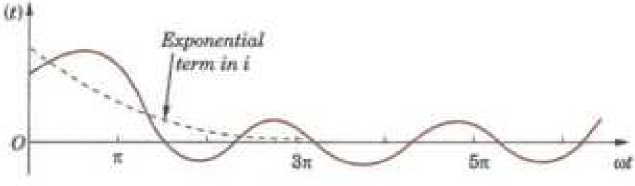


Fig. 12.17

Obs. As t increases indefinitely, the exponential term will approach zero. This implies that after sometime the current $i(t)$ will execute nearly harmonic oscillations only (Fig. 12.17).

PROBLEMS 12.4

1. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current i builds up at a rate given by $L\, di/dt + Ri = E$.
   Find i as a function of t. How long will it be, before the current has reached one-half its final value if $E = 6$ volts, $R = 100$ ohms and $L = 0.1$ henry ?
2. When a resistance R ohms is connected in series with an inductance L henries with an e.m.f. of E volts, the current i amperes at time t is given by $L\, di/dt + Ri = E$.
   If $E = 10 \sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.

3. A resistance of 100 Ω, an inductance of 0.5 henry are connected in series with a battery of 20 volts. Find the current in the circuit at $t = 0.5$ sec, if $i = 0$ at $t = 0$. *(Marathwada, 2008)*
4. The equation of electromotive force in terms of current i for an electrical circuit having resistance R and condenser of capacity C in series, is
$$E = Ri + \int \frac{i\,dt}{C}$$
Find the current i at any time t when $E = E_m \sin \omega t$. *(S.V.T.U., 2008; P.T.U., 2006)*
5. A resistance R in series with inductance L is shunted by an equal resistance R with capacity C. An alternating e.m.f. $E \sin pt$ produces currents i_1 and i_2 in two branches. If initially there is no current, determine i_1 and i_2 from the equations
$$L\frac{di_1}{dt} + Ri_1 = E \sin pt \quad \text{and} \quad \frac{i_2}{C} + R\frac{di_2}{dt} = pE \cos pt.$$
Verify that if $R^2C = L$, the total current $i_1 + i_2$ will be $(E \sin pt)/R$.

12.6 NEWTON'S LAW OF COOLING*

According to this law, *the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.*

If θ_0 is the temperature of the surroundings and θ that of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \text{ where } k \text{ is a constant.}$$

Example 12.18. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?*

Solution. If θ be the temperature of the body at any time t, then
$$\frac{d\theta}{dt} = -k(\theta - 40), \qquad \text{where } k \text{ is a constant.}$$
Integrating,
$$\int \frac{d\theta}{\theta - 40} = -k \int dt + \log c, \qquad \text{where } c \text{ is a constant.}$$
or
$$\log(\theta - 40) = -kt + \log c \quad i.e., \quad \theta - 40 = ce^{-kt} \qquad ...(i)$$
When $t = 0$, $\theta = 80°$ and when $t = 20$, $\theta = 60°$. $\therefore$ $40 = c$, and $20 = ce^{-20k}$; $k = \frac{1}{20}\log 2$.

Thus (*i*) becomes $\quad \theta - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$

When $t = 40$ min., $\quad \theta = 40 + 40e^{-2\log 2} = 40 + 40e^{\log(1/4)} = 40 + 40 \times \frac{1}{4} = 50°\text{C}.$

12.7 HEAT FLOW

The fundamental principles involved in the problems of heat conduction are :

(*i*) *Heat flows from a higher temperature to the lower temperature.*

(*ii*) *The quantity of heat in a body is proportional to its mass and temperature.*

(*iii*) *The rate of heat-flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.*

If q (cal./sec.) be the quantity of heat that flows across a slab of area α (cm^2) and thickness δx in one second, where the difference of temperature at the faces is δT, then by (*iii*) above
$$q = -k\alpha dT/dx \qquad ...(A)$$
where k is a constant depending upon the material of the body and is called the *thermal conductivity*.

*Named after the great English mathematician and physicist *Sir Issac Newton* (1642–1727) whose contributions are of utmost impertance. He discovered many physical laws, invented Calculus alongwith *Leibnitz* (see footnote p. 139) and created analytical methods of investigating physical problems. He became professor at Cambridge in1699, but his '*Mathematical Principles of Natural Philosophy*' containing development of classical mechanics had been completed in 1687.

Example 12.19. *A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick for which $k = 0.0025$. If the temperature of the outer surface of the covering is 40°C, find the temperature half-way through the covering under steady state conditions.*

Solution. Let q cal./sec. be the constant quantity of heat flowing out radially through a surface of the pipe having radius x cm. and length 1 cm (Fig. 12.18). Then the area of the lateral surface (belt) $= 2\pi x$.

$\therefore$ the equation (A) above gives

$$q = -k \cdot 2\pi x \cdot \frac{dT}{dx} \quad \text{or} \quad dT = -\frac{q}{2\pi k} \cdot \frac{dx}{x}$$

Integrating, we have

$$T = -\frac{q}{2\pi k} \log_e x + c$$

Since $T = 150$, when $x = 10$. $\therefore$ $150 = -\frac{q}{2\pi k} \log_e 10 + c$ ...(i)

Again since $T = 40$, when $x = 15$, $40 = -\frac{q}{2\pi k} \log_e 15 + c$ ...(ii)

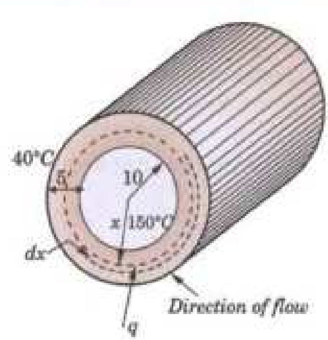


Fig. 12.18

Subtracting (ii) from (i), $110 = \frac{q}{2\pi k} \log_e 1.5$ ...(iii)

Let $T = t$, when $x = 12.5$ $\therefore$ $t = -\frac{q}{2\pi k} \log_e 12.5 + c$ ...(iv)

Subtracting (i) from (iv), $t - 150 = -\frac{q}{2\pi k} \log_e 1.25$ ...(v)

Dividing (v) by (iii), $\frac{t-150}{110} = -\frac{\log_e 1.25}{\log_e 1.5}$, whence $t = 89.5$°C.

PROBLEMS 12.5

1. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will 40°C.
2. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.
3. Two friends A and B order coffee and receive cups of equal temperature at the same time. A adds a small amount of cool cream immediately but does not drink his coffee until 10 minutes later, B waits for 10 minutes and adds the same amount of cool cream and begins to drink. Assuming the Newton's law of cooling, decide who drinks the hotter coffee ?
4. A pipe 20 cm. in diameter contains steam at 200°C. It is covered by a layer of insulations 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C, find the heat loss per hour from two metre length of the pipe.
5. A steam pipe 20 cm. in diameter contains steam at 150°C and is covered with asbestos 5 cm thick. The outside temperature is kept at 60°C. By how much should the thickness of the covering be increased in order that the rate of heat loss should be decreased by 25% ?

12.8 RATE OF DECAY OF RADIO-ACTIVE MATERIALS

This law states that disintegration at any instant is proportional to the amount of material present.

of material at any time t, then $\frac{du}{dt} = -ku$, where k is a constant.

Example 12.20. *Uranium disintegrates at a rate proportional to the amount then present at any instant. If M_1 and M_2 grams of uranium are present at times T_1 and T_2 respectively, find the half-life of uranium.*

Solution. Let the mass of uranium at any time t be m grams.

Then the equation of disintegration of uranium is $\frac{dm}{dt} = -\mu m$, where μ is a constant.

Integrating, we get $\int \frac{dm}{dt} = -\mu \int dt + c$ or $\log m = c - \mu t$ ...(i)

Initially, when $t = 0$, $m = M$ (say) so that $c = \log M$ $\therefore$ (i) becomes, $\mu t = \log M - \log m$ ...(ii)

Also when $t = T_1$, $m = M_1$ and when $t = T_2$, $m = M_2$

$\therefore$ From (ii), we get $\mu T_1 = \log M - \log M_1$ ...(iii)

$\mu T_2 = \log M - \log M_2$ ...(iv)

Subtracting (iii) from (iv), we get

$$\mu(T_2 - T_1) = \log M_1 - \log M_2 = \log (M_1/M_2) \text{ whence } \mu = \frac{\log (M_1/M_2)}{T_2 - T_1}$$

Let the mass reduce to half its initial value in time T. *i.e.*, when $t = T$, $m = \frac{1}{2}M$.

$\therefore$ from (ii), we get $\mu T = \log M - \log (M/2) = \log 2$.

Thus
$$T = \frac{1}{\mu} \log 2 = \frac{(T_2 - T_1)\log 2}{\log (M_1/M_2)}.$$

12.9 CHEMICAL REACTIONS AND SOLUTIONS

A type of problems which are especially important to chemical engineers are those concerning either chemical reactions or chemical solutions. These can be best explained through the following example :

Example 12.21. *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?* (Andhra, 1997)

Solution. Let the salt content at time t be u lb. so that its rate of change is du/dt

$= 2$ gal. $\times$ 2 lb. $= 4$ lb./min.

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow

$= 2$ gal. $\times$ C lb. $= 2C$ lb./min.

$$\therefore \quad \frac{du}{dt} = 4 - 2C \quad \text{...(i)}$$

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

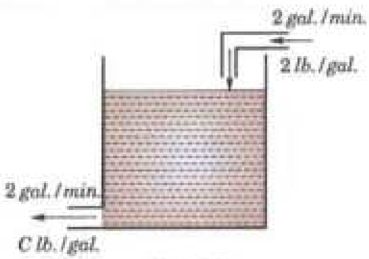


Fig. 12.19

$\therefore$ (i) becomes $\frac{du}{dt} = 4 - 2\frac{u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log_e (100 - u) + k \quad \text{...(ii)}$$

Initially when $t = 0$, $u = 0$ $\therefore$ $0 = -25 \log_e 100 + k$ ...(iii)

Eliminating k from (ii) and (iii), we get $t = 25 \log_e \frac{100}{100 - u}$.

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25 \log_e \frac{100}{60} \text{ and } t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time $(t_2 - t_1) = 25 \log_e 5 - 25 \log_e 5/3$

$= 25 \log_e 3 = 25 \times 1.0986 = 27$ min. 28 sec.

PROBLEMS 12.6

1. The number N of bacteria in a culture grew at a rate proportional to N. The value of N was initially 100 and increased to 332 in one hour. What would be the value of N after $1\frac{1}{2}$ hours ? *(Nagarjuna, 2008 ; J.N.T.U., 2003)*
2. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, in how many hours will it triple ? *(Andhra, 2000)*
3. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will remain at the end of 21 years ?
4. If 30% of radio active substance disappeared in 10 days, low long will it take for 90% of it to disappear ? *(Madras, 2000 S)*
5. Under certain conditions cane-sugar in water is converted into dextrose at a rate which is proportional to the amount unconverted at any time. If of 75 gm. at time $t = 0$, 8 gm. are converted during the first 30 minutes, find the amount converted in $1\frac{1}{2}$ hours.
6. In a chemical reaction in which two substances A and B initially of amounts a and b respectively are concerned, the velocity of transformation dx/dt at any time t is known to be equal to the product $(a - x)(b - x)$ of the amounts of the two substances then remaining untransformed. Find t in terms of x if $a = 0.7$, $b = 0.6$ and $x = 0.3$ when $t = 300$ seconds.
7. A tank contains 1000 gallons of brine in which 500 lt. of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons /minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lt. of salt is left in the tank ?

   [**Hint.** If u be the amount of salt after t minutes, then $du/dt = -10u/1000$.]
8. A tank is initially filled with 100 gallons of salt solution containing 1 lb. of salt per gallon. Fresh brine containing 2 lb. of salt per gallon runs into the tank at the rate of 5 gallons per minute and the mixture assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time, and determine how long it will take for this amount to reach 150 lb.

12.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 12.7

Fill up the blanks or choose the correct answer in the following problems :

1. If a coil having a resistance of 15 ohms and an inductance of 10 henries is connected to 90 volts supply then the current after 2 secs is .....
2. A tennis ball dropped from a height of 6 m, rebounds infinitely often. If it rebounds 80% of the distance that it falls, then the total distance for these bounces is .....
3. Radium decomposes at a rate proportional to the amount present. If 5% of the original amount disappears in 50 years then ......% will remain after 100 years.
4. The curve whose polar subtangent is constant is .....
5. The curve in which the length of the subnormal is proportional to the square of the ordinate, is ......
6. The curve in which the portion of the tangent between the axes is bisected at the point of contact, is ......
7. If the stream lines of a flow around a corner are $xy = c$, then the equipotential lines are ......
8. The orthogonal trajectories of a system of confocal and coaxial parabolas is ......
9. When a bullet is fired into a sand tank, its retardation is proportional to $\sqrt{(\text{velocity})}$. If it enters the sand tank with velocity v_0, it will come to rest after ...... seconds.
10. The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in two hours, then it will triple after ..... hours.
11. Ram and Sunil order coffee and receive cups simultaneously at equal temperature. Ram adds a spoon of cold cream but doesn't drink for 10 minutes; Sunil waits for 10 minutes and adds a spoon of cold cream and begins to drink. Who drinks the hotter coffee ?
12. The equation $y - 2x = c$ represents the orthogonal trajectories of the family

    (*i*) $y = ae^{-2x}$ (*ii*) $x^2 + 2y^2 = a$ (*iii*) $xy = a$ (*iv*) $x + 2y = a$.

13. In order to keep a body in air above the earth for 12 seconds, the body should be thrown vertically up with a velocity of

(a) $\sqrt{6}$ g m/sec (b) $\sqrt{12}$ g m/sec (c) 6 g m/sec (d) 12g m/sec.

14. The orthogonal trajectory of the family $x^2 + y^2 = c^2$ is

(a) $x + y = c$ (b) $xy = c$ (c) $x^2 + y^2 = x + y$ (d) $y = cx$. (V.T.U., 2010)

15. If a thermometer is taken outdoors where the temperature is 0°C, from a room having temperature 21°C and the reading drops to 10°C in 1 minute then its reading will be 5°C after ......minutes.

16. The equation of the curve for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = 2a \sin 2\theta$. This satisfies the differential equation

(a) $r\dfrac{dr}{d\theta} = \tan 2\theta$ (b) $r\dfrac{dr}{d\theta} = \cos 2\theta$ (c) $r\dfrac{d\theta}{dr} = \tan 2\theta$ (d) $r\dfrac{d\theta}{dr} = \cos 2\theta$.

17. Two balls of m_1 and m_2 grams are projected vertically upwards such that the velocity of projection of m_1 is double that of m_2. If the maximum height to which m_1 and m_2 rise be h_1 and h_2 respectively then

(a) $h_1 = 2h_2$ (b) $2h_1 = h_2$ (c) $h_1 = 4h_2$ (d) $4h_1 = h_2$.

18. Two balls are projected simultaneously with same velocity from the top of a tower, one vertically upwards and the other vertically downwards. If they reach the ground in times t_1 and t_2, then the height of the tower is

(a) $\frac{1}{2} g t_1 t_2$ (b) $\frac{1}{2} g(t_1^2 + t_2^2)$ (c) $\frac{1}{2} g(t_1^2 - t_2^2)$ (d) $\frac{1}{2} g(t_1 + t_2)^2$.

19. A particle projected from the earth's surface with a velocity of 7 miles/sec will return to the earth.
(Taking $g = 32.17$ and earth's radius = 3960 miles) (True/False)

20. If a particle falls under gravity with air resistance k times its velocity, then its velocity cannot exceed g/k. (True/False)

CHAPTER 13

Linear Differential Equations

13.1 DEFINITIONS

Linear differential equations *are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.* Thus the general linear differential equation of the nth order is of the form

$$\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + p_n y = X,$$

where $p_1, p_2, \ldots, p_n$ and X are functions of x only.

Linear differential equations with constant co-efficients are of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = X$$

where $k_1, k_2, \ldots, k_n$ are constants. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

13.2 (1) THEOREM

If y_1, y_2 are only two solutions of the equation

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + k_n y = 0 \quad \ldots(1)$$

then $c_1 y_1 + c_2 y_2$ $(= u)$ is also its solution.

Since $y = y_1$ and $y = y_2$ are solutions of (1).

$$\therefore \quad \frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1} y_1}{dx^{n-1}} + k_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \ldots + k_n y_1 = 0 \quad \ldots(2)$$

and
$$\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1} y_2}{dx^{n-1}} + k_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \ldots + k_n y_2 = 0 \quad \ldots(3)$$

If c_1, c_2 be two arbitrary constants, then

$$\frac{d^n (c_1 y_1 + c_2 y_2)}{dx^n} + k_1 \frac{d^{n-1} (c_1 y_1 + c_2 y_2)}{dx^{n-1}} + \ldots + k_n (c_1 y_1 + c_2 y_2)$$

$$= c_1\left(\frac{d^n y_1}{dx^n} + k_1 \frac{d^{n-1}y_1}{dx^{n-1}} + \dots + k_n y_1\right) + c_2\left(\frac{d^n y_2}{dx^n} + k_1 \frac{d^{n-1}y_2}{dx^{n-1}} + \dots + k_n y_2\right)$$

$$= c_1(0) + c_2(0) = 0 \qquad \text{[By (2) and (3)]}$$

i.e.,
$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1}u}{dx^{n-1}} + \dots + k_n y = 0 \qquad \dots(4)$$

This proves the theorem.

(2) Since the general solution of a differential equation of the nth order contains n arbitrary constants, it follows, from above, that if $y_1, y_2, y_3, \dots, y_n$, are n independent solutions of (1), then $c_1y_1 + c_2y_2 + \dots + c_ny_n$ $(= u)$ is its complete solution.

(3) If $y = v$ be any particular solution of

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + \dots + k_n y = X \qquad \dots(5)$$

then
$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1}v}{dx^{n-1}} + \dots + k_n v = X \qquad \dots(6)$$

Adding (4) and (6), we have $\dfrac{d^n(u+v)}{dx^n} + k_1 \dfrac{d^{n-1}(u+v)}{dx^{n-1}} + \dots + k_n(u+v) = X$

This shows that $y = u + v$ is the complete solution of (5).

The part u is called the **complementary function (C.F.)** *and the part v is called the* **particular integral (P.I.)** *of (5).*

$\therefore$ the complete solution (C.S.) of (5) is **y = C.F. + P.I.**

Thus in order to solve the question (5), we have to first find the C.F., i.e., the complete solution of (1), and then the P.I., i.e. a particular solution of (5).

13.3 OPERATOR D

Denoting $\dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}$ etc. by D, D^2, D^3 etc., so that

$\dfrac{dy}{dx} = Dy$, $\dfrac{d^2y}{dx^2} = D^2y$, $\dfrac{d^3x}{dx^3} = D^3y$ etc., the equation (5) above can be written in the *symbolic form* $(D^n + k_1D^{n-1} + \dots + k_n)y = X$, *i.e.,* $f(D)y = X$,
where $f(D) = D^n + k_1D^{n-1} + \dots + k_n$, *i.e.,* a polynomial in D.

Thus *the symbol D stands for the operation of differentiation and can be treated much the same as an algebraic quantity i.e., f (D) can be factorised by ordinary rules of algebra and the factors may be taken in any order.* For instance

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)\,y = (D+3)(D-1)y \text{ or } (D-1)(D+3)\,y.$$

13.4 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

To solve the equation $\dfrac{d^n y}{dx^n} + k_1 \dfrac{d^{n-1}y}{dx^{n-1}} + k_2 \dfrac{d^{n-2}y}{dx^{n-2}} + \dots + k_n y = 0 \qquad \dots(1)$

where k's are constants.

The equation (1) in *symbolic form is*

$$(D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n)y = 0 \qquad \dots(2)$$

Its symbolic co-efficient equated to zero *i.e.*

$$D^n + k_1D^{n-1} + k_2D^{n-2} + \dots + k_n = 0$$

is called the *auxiliary equation* (A.E.). Let $m_1, m_2, \dots, m_n$ be its roots.

Case I. *If all the roots be real and different*, then (2) is equivalent to

$$(D - m_1)(D - m_2)\dots(D - m_n)y = 0 \qquad \dots(3)$$

Now (3) will be satisfied by the solution of $(D - m_n)y = 0$, *i.e.*, by $\frac{dy}{dx} - m_n y = 0$.

This is a Leibnitz's linear and I.F. $= e^{-m_n x}$

$\therefore$ its solution is $y e^{-m_n x} = c_n$, *i.e.*, $y = c_n e^{m_n x}$

Similarly, since the factors in (3) can be taken in any order, it will be satisfied by the solutions of $(D - m_1) y = 0$, $(D - m_2)y = 0$ etc. *i.e.*, by $y = c_1 e^{m_1 x}$, $y = c_2 e^{m_2 x}$ etc.

Thus the complete solution of (1) is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$ ...(4)

Case II. *If two roots are equal (i.e., $m_1 = m_2$),* then (4) becomes

$$y = (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$y = C e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x} \qquad [\because \ c_1 + c_2 = \text{one arbitrary constant } C]$$

It has only $n - 1$ arbitrary constants and is, therefore, not the complete solution of (1). In this case, we proceed as follows :

The part of the complete solution corresponding to the repeated root is the complete solution of $(D - m_1)(D - m_1)y = 0$

Putting $(D - m_1)y = z$, it becomes $(D - m_1) z = 0$ or $\frac{dz}{dx} - m_1 z = 0$

This is a Leibnitz's linear in z and I.F. $= e^{-m_1 x}$. $\therefore$ its solution is $z e^{-m_1 x} = c_1$ or $z = c_1 e^{m_1 x}$

Thus $\qquad (D - m_1)y = z = c_1 e^{m_1 x}$ or $\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ ...(5)

Its I.F. being $e^{-m_1 x}$, the solution of (5) is

$$y e^{-m_1 x} = \int c_1 e^{m_1 x} e^{-m_1 x} dx + c_2 = c_1 x + c_2 \quad \text{or} \quad y = (c_1 x + c_2) e^{m_1 x}$$

Thus the complete solution of (1) is $y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$

If, however, the A.E. has three equal roots (*i.e.*, $m_1 = m_2 = m_3$), then the complete solution is

$$y = (c_1 x^2 + c_2 x + c_3)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$$

Case III. *If one pair of roots be imaginary, i.e.,* $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

$[\because$ by *Euler's Theorem*, $e^{i\theta} = \cos \theta + i \sin \theta]$

$$= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$$

where $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$.

Case IV. *If two points of imaginary roots be equal i.e.,* $m_1 = m_2 = \alpha + i\beta$, $m_3 = m_4 = \alpha - i\beta$, then by case II, the complete solution is

$$y = e^{\alpha x}[(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] + \ldots + c_n e^{m_n x}.$$

Example 13.1. *Solve* $\frac{d^2 x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$, *given* $x(0) = 0$, $\frac{dx}{dt}(0) = 15$. *(V.T.U., 2010)*

Solution. Given equation in *symbolic form is* $(D^2 + 5D + 6)\, x = 0$.

Its A.E. is $D^2 + 5D + 6 = 0$, *i.e.*, $(D + 2)(D + 3) = 0$ whence $D = -2, -3$.

$\therefore$ $\qquad$ C.S. is $x = c_1 e^{-2t} + c_2 e^{-3t}$ and $\frac{dx}{dt} = -2ae^{-2t} - 3c_2 e^{-3t}$

When $t = 0$, $x = 0$. $\quad \therefore \quad 0 = c_1 + c_2$ ...(*i*)

When $t = 0$, $dx/dt = 15$ $\quad \therefore \quad 15 = -2c_1 - 3c_2$ ...(*ii*)

Solving (*i*) and (*ii*), $c_1 = 15, c_2 = -15$.
Hence the required solution is $x = 15\,(e^{-2t} - e^{-3t})$.

Example 13.2. *Solve* $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0.$

Solution. Given equation in *symbolic form is* $(D^2 + 6D + 9) = 0$

$\therefore$ A.E. is $D^2 + 6D + 9 = 0$, *i.e.*, $(D + 3)^2 = 0$ whence $D = -3, -3$.

Hence the C.S. is $x = (c_1 + c_2 t)\, e^{-3t}$.

Example 13.3. *Solve* $(D^3 + D^2 + 4D + 4) = 0.$

Solution. Here the A.E. is $D^3 + D^2 + 4D + 4 = 0$ *i.e.*, $(D^2 + 4)(D + 1) = 0$ $\therefore$ $D = -1, \pm 2i$.

Hence the C.S. is $y = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

i.e., $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$.

Example 13.4. *Solve* (*i*) $(D^4 - 4D + 4)\, y = 0$ *(Bhopal, 2008)*
(*ii*) $(D^2 + 1)^3\, y = 0$ *where* $D = d/dx$.

Solution. (*i*) The A.E. equation is $D^4 - 4D^2 + 4 = 0$ or $(D^2 - 2)^2 = 0$

$\therefore$ $D^2 = 2, 2$ *i.e.*, $D = \pm\sqrt{2}, \pm\sqrt{2}$.

Hence the C.S. is $((c_1 + c_2 x)\, e^{\sqrt{2}x} + (c_3 + c_4 x)\, e^{-\sqrt{2}x})$ [Roots being repeated]

(*ii*) The A.E. equation is $(D^2 + 1)^3 = 0$

$\therefore$ $D = \pm i, \pm i, \pm i$.

Hence the C.S. is $y = e^{0x}\,[(c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x]$

i.e., $y = (c_1 + c_2 + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$.

Example 13.5. *Solve* $\dfrac{d^4x}{dt^4} + 4x = 0.$

Solution. Given equation in *symbolic form is* $(D^4 + 4)\, x = 0$

$\therefore$ A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$ or $(D^2 + 2)^2 - (2D)^2 = 0$

or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$\therefore$ either $D^2 + 2D + 2 = 0$ or $D^2 - 2D + 2 = 0$

whence $D = \dfrac{-2 \pm \sqrt{(-4)}}{2}$ and $\dfrac{2 \pm \sqrt{(-4)}}{2}$ *i.e.*, $D = -1 \pm i$ and $1 \pm i$.

Hence the required solution is $x = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{t}(c_3 \cos t + c_4 \sin t)$.

PROBLEMS 13.1

Solve :

1. $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 13x = 0, x(0), \dfrac{dx(0)}{dt} = 2.$ *(V.T.U., 2008)*
2. $y'' - 2y' + 10y = 0, y(0) = 4, y'(0) = 1.$
3. $4y''' + 4y'' + y' = 0.$
4. $\dfrac{d^3y}{dx^3} + y = 0.$ *(V.T.U., 2000 S)*
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0.$
6. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0.$ *(J.N.T.U., 2005)*
7. $(4D^4 - 8D^3 - 7D^2 + 11D + 6)y = 0.$ *(V.T.U., 2008)*
8. $(D^2 + 1)^2 (D - 1)y = 0.$
9. If $\dfrac{d^4x}{dt^4} = m^4 x$, show that $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt.$

13.5 INVERSE OPERATOR

(1) Definition. $\frac{1}{f(D)}X$ is that function of x, not containing arbitrary constants which when operated upon by $f(D)$ gives X.

i.e.,
$$f(D)\left\{\frac{1}{f(D)}X\right\} = X$$

Thus $\frac{1}{f(D)}X$ *satisfies the equation* $f(D)y = X$ *and is, therefore, its particular integral.*

Obviously, $f(D)$ and $1/f(D)$ are inverse operators.

(2)
$$\mathbf{\frac{1}{D}X = \int X dx}$$

Let
$$\frac{1}{D}X = y \qquad ...(i)$$

Operating by D,
$$D\frac{1}{D}X = Dy \quad i.e., \; X = \frac{dy}{dx}$$

Integrating both sides w.r.t. x, $y = \int X\,dx$, no constant being added as (*i*) does not contain any constant.

Thus
$$\frac{1}{D}X = \int X\,dx.$$

(3)
$$\mathbf{\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx.}$$

Let
$$\frac{1}{D-a}X = y \qquad ...(ii)$$

Operating by $D-a$, $(D-a)\,.\,\frac{1}{D-a}X = (D-a)y.$

or
$$X = \frac{dy}{dx} - ay\,,\; i.e.,\; \frac{dy}{dx} - ay = X$$ which is a Leibnitz's linear equation.

$\therefore$ I.F. being e^{-ax}, its solution is

$$ye^{-ax} = \int Xe^{-ax}dx,$$ no constant being added as (*ii*) doesn't contain any constant.

Thus
$$\frac{1}{D-a}X = y = e^{ax}\int Xe^{-ax}dx.$$

13.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + ... + k_n y = X$

which is *symbolic form of* $(D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n)y = X.$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^n + k_1D^{n-1} + k_2D^{n-2} + ... + k_n}X.$$

Case I. When $\mathbf{X = e^{ax}}$

Since
$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
....
....
$$D^ne^{ax} = a^ne^{ax}$$

$$\therefore \quad (D^n + k_1D^{n-1} + ... + k_n)e^{ax} = (a^n + k_1a^{n-1} + ... + k_n)e^{ax}, \quad i.e., \quad f(D)e^{ax} = f(a)e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$, $\frac{1}{f(D)} f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$ or $e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$

$\therefore$ dividing by $f(a)$,

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \text{ provided } f(a) \neq 0 \qquad ...(1)$$

If f(a) = 0, *the above rule fails and we proceed further.*

Since a is a root of A.E. $f(D) = D^n + k_1 D^{n-1} + ... + k_n = 0$.

$\therefore$ $D - a$ is a factor of $f(D)$. Suppose $f(D) = (D - a)\,\phi(D)$, where $\phi(a) \neq 0$. Then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)} e^{ax} \qquad \text{[By (1)]}$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a} e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax} \int e^{ax} . e^{-ax} dx \qquad \text{[By §13.5 (3)]}$$

$$= \frac{1}{\phi(a)} e^{ax} \int dx = x \frac{1}{\phi(a)} e^{ax} \quad i.e., \quad \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax} \qquad ...(2)$$

$$\left[\begin{array}{l} \because f'(D) = (D-a)\phi'(D) + 1 . \phi(D) \\ \therefore f'(a) = 0 \times \phi'(a) + \phi(a) \end{array}\right.$$

If $f'(a) = 0$, then applying (2) again, we get $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}$, *provided* $f''(a) \neq 0$ ...(3)

and so on.

Example 13.6. *Find the P.I. of* $(D^2 + 5D + 6)y = e^x$.

Solution. P.I. $= \frac{1}{D^2 + 5D + 6} e^x$ [Put $D = 1$] $= \frac{1}{1^2 + 5.1 + 6} e^x = \frac{e^x}{12}$.

Example 13.7. *Find the P.I. of* $(D + 2)(D - 1)^2 y = e^{-2x} + 2 \sinh x$.

Solution. P.I. $= \frac{1}{(D+2)(D-1)^2} [e^{-2x} + 2 \sinh x] = \frac{1}{(D+2)(D-1)^2} [e^{-2x} + e^x - e^{-x}]$

Let us evaluate each of these terms separately.

$$\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2} \cdot \left[\frac{1}{(D-1)^2} e^{-2x}\right]$$

$$= \frac{1}{D+2} \cdot \frac{1}{(-2-1)^2} e^{-2x} = \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} . x . \frac{1}{1} e^{-2x} = \frac{x}{9} e^{-2x} \qquad \left[\because \frac{d}{dD}(D+2) = 1\right]$$

$$\frac{1}{(D+2)(D-1)^2} e^x = \frac{1}{1+2} \cdot \frac{1}{(D-1)^2} e^x = \frac{1}{3} . x^2 . \frac{1}{2} e^x = \frac{x^2}{6} e^x \qquad \left[\because \frac{d^2}{dD^2}(D-1)^2 = 2\right]$$

and $$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{e^{-x}}{4}$$

Hence, P.I. $= \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{1}{4} e^{-x}$.

Case II. When X = sin (ax + b) or cos (ax + b).

Since $D \sin(ax + b) = a \cos(ax + b)$

$D^2 \sin(ax + b) = -a^2 \sin(ax + b)$

$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

i.e.,
$$D^2 \sin(ax+b) = (-a^2)\sin(ax+b)$$
$$(D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

In general
$$(D^2)^r \sin(ax+b) = (-a^2)^r \sin(ax+b)$$
$$\therefore \quad f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$$

Operating on both sides $1/f(D^2)$,

$$\frac{1}{f(D^2)} \cdot f(D^2)\sin(ax+b) = \frac{1}{f(D^2)} f(-a^2)\sin(ax+b)$$

or
$$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b)$$

$\therefore$ Dividing by $f(-a^2)$ $\quad \mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = \dfrac{1}{f(-a^2)}\sin(ax+b)}$ *provided* $f(-a^2) \neq 0$ ...(4)

If $\mathbf{f(-a^2) = 0}$, *the above rule fails and we proceed further.*

Since $\cos(ax+b) + i\sin(ax+b) = e^{i(ax+b)}$ [Euler's theorem]

$$\therefore \quad \frac{1}{f(D^2)}\sin(ax+b) = \text{I.P. of } \frac{1}{f(D^2)} e^{i(ax+b)} \qquad [\text{Since } f(-a^2) = 0 \;\therefore\; \text{by (2)}]$$

$$= \text{I.P. of } x\frac{1}{f'(D^2)} e^{i(ax+b)} \qquad \text{where } D^2 = -a^2$$

$\therefore \quad \mathbf{\dfrac{1}{f(D^2)}\sin(ax+b) = x\dfrac{1}{f'(-a^2)}\sin(ax+b)}$ *provided* $f'(-a^2) \neq 0$ ...(5)

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)} \cdot \sin(ax+b) = x^2\dfrac{1}{f''(-a^2)}\sin(ax+b)}$, *provided* $f''(-a^2) \neq 0$, and so on.

Similarly, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = \dfrac{1}{f(-a^2)}\cos(ax+b)}$, *provided* $f(-a^2) \neq 0$

If $f(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x \cdot \dfrac{1}{f'(-a^2)}\cos(ax+b)}$, *provided* $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, $\mathbf{\dfrac{1}{f(D^2)}\cos(ax+b) = x^2\dfrac{1}{f''(-a^2)}\cos(ax+b)}$, *provided* $f''(-a^2) \neq 0$ and so on.

Example 13.8. *Find the P.I. of $(D^3 + 1)y = \cos(2x-1)$.*

Solution. P.I. $= \dfrac{1}{D^3+1}\cos(2x-1)$ [Put $D^2 = -2^2 = -4$]

$$= \frac{1}{D(-4)+1}\cos(2x-1) \qquad [\text{Multiply and divide by } 1+4D]$$

$$= \frac{(1+4D)}{(1-4D)(1+4D)}\cos(2x-1) = (1+4D)\cdot\frac{1}{1-16D^2}\cos(2x-1) \qquad [\text{Put } D^2 = -2^2 = -4]$$

$$= (1+4D)\frac{1}{1-16(-4)}\cos(2x-1) = \frac{1}{65}[\cos(2x-1) + 4D\cos(2x-1)]$$

$$= \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)].$$

Example 13.9. *Find the P.I. of $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = \sin 2x$.*

Solution. Given equation in *symbolic form* is $(D^3 + 4D)y = \sin 2x$

$\therefore$ $$\text{P.I.} = \frac{1}{D(D^2+4)}\sin 2x \qquad [\because D^2+4=0 \text{ for } D^2=-2^2, \therefore \text{ Apply (5) 477}]$$

$$= x\frac{1}{3D^2+4}\sin 2x \qquad \left[\because \frac{d}{dD}[D^3+4D]=3D^2+4\right]$$

[Put $D^2=-2^2=-4$]

$$= x\frac{1}{3(-4)+4}\sin 2x = -\frac{x}{8}\sin 2x.$$

Case III. When X = x^m.

Here $$\text{P.I.} = \frac{1}{f(D)}x^m = [f(D)]^{-1}x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $(m+1)$th and higher derivatives of x^m are zero, we need not consider terms beyond D^m.

Example 13.10. *Find the P.I. of* $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}=x^2+2x+4.$

Solution. Given equation in *symbolic form* is $(D^2+D)y=x^2+2x+4$.

$\therefore$ $$\text{P.I.} = \frac{1}{D(D+1)}(x^2+2x+4) = \frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$= \frac{1}{D}(1-D+D^2-\ldots)(x^2+2x+4) = \frac{1}{D}[x^2+2x+4-(2x+2)+2]$$

$$= \int(x^2+4)dx = \frac{x^3}{3}+4x.$$

Case IV. When X = e^{ax} V, *V being a function of x.*

If u is a function of x, then

$$D(e^{ax}u) = e^{ax}Du + ae^{ax}u + e^{ax}(D+a)u$$

$$D^2(e^{ax}u) = a^{ax}D^2u + 2ae^{ax}Du + a^2e^{ax}u = e^{ax}(D+a)^2u$$

and in general, $D^n(e^{ax}u) = e^{ax}(D+a)^n u$

$\therefore$ $f(D)(e^{ax}u) = e^{ax}f(D+a)u$

Operating both sides by $1/f(D)$,

$$\frac{1}{f(D)}\cdot f(D)(e^{ax}u) = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

$$e^{ax}u = \frac{1}{f(D)}[e^{ax}f(D+a)u]$$

Now put $f(D+a)u=V$, *i.e.*, $u=\dfrac{1}{f(D+a)}V$, so that $e^{ax}\dfrac{1}{f(D+a)}V = \dfrac{1}{f(D)}(e^{ax}V)$

i.e., $$\mathbf{\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.} \qquad \ldots(6)$$

Example 13.11. *Find P.I. of* $(D^2-2D+4)y=e^x\cos x.$

Solution. $$\text{P.I.} = \frac{1}{D^2-2D+4}e^x\cos x \qquad [\text{Replace } D \text{ by } D+1]$$

$$= e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x = e^x\frac{1}{D^2+3}\cos x \qquad [\text{Put } D^2=-1^2=-1]$$

$$= e^x\frac{1}{-1+3}\cos x = \frac{1}{2}e^x\cos x.$$

Case V. When X is any other function of x.

Here $$\text{P.I.} = \frac{1}{f(D)} X.$$

If $f(D) = (D - m_1)(D - m_2) \ldots (D - m_n)$, resolving into partial fractions,

$$\frac{1}{f(D)} = \frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}$$

$$\therefore \quad \text{P.I.} = \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots + \frac{A_n}{D - m_n}\right] X$$

$$= A_1 \frac{1}{D - m_1} X + A_2 \frac{1}{D - m_2} X + \ldots + A_n \frac{1}{D - m_n} X$$

$$= A_1 . e^{m_1 x} \int X e^{-m_1 x} dx + A_2 . e^{m_2 x} \int X e^{-m_2 x} dx + \ldots + A_n . e^{m_n x} \int X e^{-m_n x} dx \qquad \text{[By §13.5 ...(3)]}$$

Obs. This method is a general one and can, therefore, be employed to obtain a particular integral in any given case.

13.7 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_{n-1} \frac{dy}{dx} + k_n y = X$$

of which the *symbolic form* is

$$(D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n) y = X .$$

Step I. To find the complementary function

(*i*) *Write the A.E.*

i.e., $D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n = 0$ and solve it for D.

(*ii*) Write the *C.F.* as follows :

Roots of A.E.	*C.F.*
1. m_1, m_2, m_3 ... (real and different roots)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2. m_1, m_1, m_3 ... (two real and equal roots)	$(c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots$
3. m_1, m_1, m_1, m_4 ... (three real and equal roots)	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots$
4. $\alpha + i\beta, \alpha - i\beta, m_3$ ... (a pair of imaginary roots)	$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots$
5. $\alpha \pm i\beta, \alpha \pm i\beta, m_5$ ... (2 pairs of equal imaginary roots)	$e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots$

Step II. To find the particular integral

From symbolic form P.I. $= \dfrac{1}{D^n + k_1 D^{n-1} + \ldots + k_{n-1} D + k_n} X = \dfrac{1}{f(D)}$ or $\dfrac{1}{\phi(D^2)} X$

(*i*) *When* $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) \neq 0]$$

$$= x \frac{1}{f'(D)} e^{ax}, \text{ put } D = a, \qquad [f(a) = 0, f'(a) \neq 0]$$

$$= x^2 \frac{1}{f''(D)} e^{ax}, \text{ put } D = a, \qquad [f'(a) = 0, f''(a) \neq 0]$$

and so on.

where $f'(D)$ = diff. coeff. of $f(D)$ w.r.t. D

$f''(D)$ = diff. coeff. of $f'(D)$ w.r.t. D, etc.

(*ii*) *When* $X = \sin(ax + b)$ *or* $\cos(ax + b)$.

$$\text{P.I.} = \frac{1}{\phi(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) \neq 0]$$

$$= x\frac{1}{\phi'(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)\text{], put } D^2 = -a^2 \qquad [\phi(-a^2) = 0, \phi'(-a^2) \neq 0]$$

$$= x^2\frac{1}{\phi''(D^2)} \sin(ax+b) \text{ [or } \cos(ax+b)\text{], put } D^2 = -a^2 \qquad [\phi'(-a^2) \neq 0, \phi''(-a^2) \neq 0]$$

and so on.

where $\phi'(D^2)$ = diff. coeff. of $\phi(D^2)$ w.r.t. D,

$\phi''(D^2)$ = diff. coeff. of $\phi'(D^2)$ w.r.t. D, etc.

(*iii*) *When* $X = x^m$, m being a positive integer.

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m$$

To evaluate it, expand $[f(D)]^{-1}$ in ascending powers of D by Binomial theorem as far as D^m and operate on x^m term by term.

(*iv*) *When* $X = e^{ax}V$, where V is a function of x.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$$

and then evaluate $\frac{1}{f(D+a)}V$ as in (*i*), (*ii*), and (*iii*).

(*v*) *When X is any function of x.*

$$\text{P.I.} = \frac{1}{f(D)} X$$

Resolve $\frac{1}{f(D)}$ into partial fractions and operate each partial fraction on X remembering that

$$\frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx.$$

Step III. To find the complete solution

Then the C.S. is y = C.F. + P.I.

Example 13.12. *Solve* $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = (1 - e^x)^2$.

Solution. Given equation in *symbolic form* is $(D^2 + D + 1)y = (1 - e^x)^2$

(*i*) *To find C.F.*

Its A.E. is $D^2 + D + 1 = 0$, $\therefore\ D = \frac{1}{2}(-1 \pm \sqrt{3}i)$

Thus C.F. $= e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2+D+1}(1 - 2e^x + e^{2x}) = \frac{1}{D^2+D+1}(e^{0x} - 2e^x + e^{2x})$$

$$= \frac{1}{0^2+0+1}e^{0x} - 2\cdot\frac{1}{1^2+1+1}e^x + \frac{1}{2^2+2+1}e^{2x} = 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$$

(*iii*) Hence the C.S. is $y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + 1 - \frac{2}{3}e^x + \frac{e^{2x}}{7}$.

Example 13.13. *Solve* $y'' + 4y' + 4y = 3 \sin x + 4 \cos x$, $y(0) = 1$ *and* $y'(0) = 0$. (J.N.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 + 4D + 4)y = 3 \sin x + 4 \cos x$

(*i*) *To find C.F.*

Its A.E. is $(D + 2)^2 = 0$ where $D = -2, -2$ $\therefore$ C.F. $= (c_1 + c_2x)e^{-2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + 4D + 4}(3 \sin x + 4 \cos x) = \frac{1}{-1 + 4D + 4}(3 \sin x + 4 \cos x)$$

$$= \frac{4D - 3}{16D^2 - 9}(3 \sin x + 4 \cos x) = \frac{(4D - 3)}{-16 - 9}(3 \sin x + 4 \cos x)$$

$$= \frac{-1}{25}\{3(4 \cos x - 3 \sin x) + 4(-4 \sin x - 3 \cos x)\} = \sin x$$

(*iii*) C.S. is $y = (c_1 + c_2)\, e^{-2x} + \sin x$

When $x = 0, y = 1,$ $\therefore$ $1 = c_1$

Also $y' = c_2e^{-2x} + (c_1 + c_2x)(-2)e^{-2x} + \cos x$.

When $x = 0, y' = 0,$ $\therefore$ $0 = c_2 - 2c_1 + 1$, *i.e.*, $c_2 = 1$.

Hence the required solution is $y = (1 + x)\, e^{-2x} + \sin x$.

Example 13.14. *Solve* $(D - 2)^2 = 8(e^{2x} + \sin 2x + x^2)$.

Solution. (*i*) *To find C.F.*

Its A.E. is $(D - 2)^2 = 0$, $\therefore$ $D = 2, 2$.

Thus C.F. $= (c_1 + c_2x)\, e^{2x}$.

(*ii*) *To find P.I.*

$$\text{P.I.} = 8\left[\frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}\sin 2x + \frac{1}{(D-2)^2}x^2\right]$$

Now $\dfrac{1}{(D-2)^2}e^{2x} = x^2\dfrac{1}{2(1)}e^{2x}$ [$\because$ by putting $D = 2$, $(D - 2)^2 = 0$, $2(D - 2) = 0$]

$$= \frac{x^2e^{2x}}{2}$$

$$\frac{1}{(D-2)^2}\sin 2x = \frac{1}{D^2 - 4D + 4}\sin 2x = \frac{1}{(-2^2) - 4D + 4}\sin 2x$$

$$= -\frac{1}{4}\int \sin 2x \, dx = -\frac{1}{4}\left(\frac{-\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

and
$$\frac{1}{(D-2)^2}x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2 = \frac{1}{4}\left[1 + (-2)\left(\frac{D}{2}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{D}{2}\right)^2 + \ldots\right]x^2$$

$$= \frac{1}{4}\left(1 + D + \frac{3D^2}{4} + \ldots\right)x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

Thus P.I. $= 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

(*iii*) Hence the C.S. is $y = (c_1 + c_2x)\, e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2 + 4x + 3$.

Example 13.15. *Find the complete solution of* $y'' - 2y' + 2y = x + e^x \cos x$. (U.P.T.U., 2002)

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 2)y = x + e^x \cos x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 2 = 0$ $\therefore$ $D = \dfrac{2 \pm \sqrt{(4 - 8)}}{2} = 1 \pm i$.

Thus C.F. $= e^x\,(c_1 \cos x + c_2 \sin x)$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2-2D+2}(x) + \frac{1}{D^2-2D+2}(e^x \cos x)$$

$$= \frac{1}{2}\left[1-\left(D-\frac{D^2}{2}\right)\right]^{-1}(x) + e^x \frac{1}{(D+1)^2-2(D+1)+2}(\cos x)$$

$$= \frac{1}{2}\left(1+D-\frac{D^2}{2}\right)x + e^x \frac{1}{D^2+1}\cos x \qquad \text{[Case of failure]}$$

$$= \frac{1}{2}(x+1-0) + e^x \cdot x\frac{1}{2D}\cos x = \frac{1}{2}(x+1) + \frac{xe^x}{2}\int \cos x\, dx = \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$$

(iii) Hence the C.S. is $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x+1) + \frac{xe^x}{2}\sin x$.

Example 13.16. *Solve* $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x$.

(V.T.U., 2008 ; Kottayam, 2005 ; U.P.T.U., 2003)

Solution. Given equation in *symbolic form* is $(D^2 - 3D + 2)y = xe^{3x} + \sin 2x$

(i) To find C.F.

Its A.E. is $D^2 - 3D + 2 = 0$ or $(D-2)(D-1) = 0$ whence $D = 1, 2$.

Thus C.F. $= c_1e^x + c_2e^{2x}$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2-3D+2}(xe^{3x} + \sin 2x) = \frac{1}{D^2-3D+2}(e^{3x}\,.\,x) + \frac{1}{D^2-3D+2}(\sin 2x)$$

$$= e^{3x}\cdot\frac{1}{(D+3)^2-3(D+3)+2}(x) + \frac{1}{-4-3D+2}(\sin 2x)$$

$$= e^{3x}\cdot\frac{1}{D^2+3D+2}(x) - \frac{3D-2}{9D^2-4}(\sin 2x) = \frac{e^{3x}}{2}\cdot\left[1+\left\{\frac{3D+D^2}{2}\right\}\right]^{-1}x - \frac{(3D-2)}{9(-4)-4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(1-\frac{3D}{2}...\right)x + \frac{1}{40}(6\cos 2x - 2\sin 2x) = \frac{e^{3x}}{2}\left(x-\frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

(iii) Hence the C.S. is $y = c_1e^x + c_2e^{2x} + e^{3x}\left(\frac{x}{2}-\frac{3}{4}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$.

Example 13.17. *Solve* $\frac{d^2y}{dx^2} - 4y = x \sinh x$. *(Madras, 2000 S)*

Solution. Given equation in *symbolic form* is $(D^2 - 4)y = x \sinh x$.

(i) To find C.F.

Its A.E. is $D^2 - 4 = 0$, whence $D = \pm 2$.

Thus C.F. $= c_1e^{2x} + c_2e^{-2x}$

(ii) To find P.I.

$$\text{P.I.} = \frac{1}{D^2-4}\, x \sinh x = \frac{1}{D^2-4}\, x\left(\frac{e^x - e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2-4}e^x \cdot x - \frac{1}{D^2-4}e^{-x}\cdot x\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{(D+1)^2-4}x - e^{-x}\frac{1}{(D-1)^2-4}x\right] = \frac{1}{2}\left[e^x\frac{1}{D^2+2D-3}x - e^{-x}\frac{1}{D^2-2D-3}x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{-3}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1}.x - \frac{e^{-x}}{-3}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1}.x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}+\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}+\ldots\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$$

(*iii*) Hence the C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$.

Example 13.18. *Solve* $(D^2 - 1)y = x \sin 3x + \cos x$.

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 1 = 0$, whence $D = \pm 1$. $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2-1}(x\sin 3x + \cos x) = \frac{1}{D^2-1}x\,(\text{I.P. of } e^{3ix}) + \frac{1}{D^2-1}\cos x$$

$$= \text{I.P. of } \frac{1}{D^2-1}e^{3ix}.x + \frac{1}{(-1)^2-1}\cos x = \text{I.P. of }\left[e^{3ix}\frac{1}{(D+3i)^2-1}x\right] - \frac{\cos x}{2}$$

[Replacing D by $D + 3i$]

$$= \text{I.P. of }\left[e^{3ix}\frac{1}{D^2+6iD-10}x\right] - \frac{\cos x}{2}$$

$$= \text{I.P. of }\left[e^{3ix}.\frac{1}{-10}\left(1-\frac{3iD}{5}-\frac{D^2}{10}\right)^{-1}x\right] - \frac{\cos x}{2}$$

[Expand by Binomial theorem]

$$= \text{I.P. of }\left[e^{3ix}.\frac{1}{-10}\left(1+\frac{3iD}{5}+\ldots\ldots\right)x\right] - \frac{\cos x}{2} = \text{I.P. of }\left[-\frac{e^{3ix}}{10}\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$$

$$= \text{I.P. of }\left[\frac{-1}{10}(\cos 3x + i\sin 3x)\left(x+\frac{3i}{5}\right)\right] - \frac{\cos x}{2}$$

$$= -\frac{1}{10}\,\text{I.P. of }\left[\left(x\cos 3x - \frac{3\sin 3x}{5}\right) + i\left(x\sin 3x + \frac{3}{5}\cos 3x\right)\right] - \frac{\cos x}{2}$$

$$= -\frac{1}{10}\left(x\sin 3x + \frac{3}{5}\cos 3x\right) - \frac{\cos x}{2}.$$

(*iii*) Hence the C.S. is $y = c_1e^x + c_2e^{-x} - \frac{1}{50}(5x\sin 3x + 3\cos 3x + 25\cos x)$.

Example 13.19. *Solve* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$. *(S.V.T.U., 2007 ; J.N.T.U., 2006 ; U.P.T.U., 2005)*

Solution. Given equation in *symbolic form* is $(D^2 - 2D + 1)y = xe^x \sin x$

(*i*) *To find C.F.*

Its A.E. is $D^2 - 2D + 1 = 0$, *i.e.*, $(D-1)^2 = 0$

$\therefore$ $D = 1, 1$. Thus C.F. $= (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x \cdot x \sin x = e^x \cdot \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x \, dx \qquad \text{[Integrate by parts]}$$

$$= e^x \frac{1}{D}\left[x(-\cos x) - \int 1 \cdot (-\cos x)\, dx\right] = e^x \int [-x\cos x + \sin x]\, dx$$

$$= e^x\left[-\left\{x \sin x - \int 1 \cdot \sin x \, dx\right\} - \cos x\right] = e^x[-x\sin x - \cos x - \cos x]$$

$$= -e^x(x \sin x + 2\cos x).$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2 \cos x)$.

Example 13.20. *Solve* $(D^4 + 2D^2 + 1)y = x^2 \cos x.$ *(Nagarjuna, 2008 ; Rajasthan, 2005)*

Solution. (*i*) *To find C.F.*

Its A.E. is $(D^2+1)^2 = 0$ whose roots are $D = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2x)\cos x + (c_3 + c_4 x)\sin x$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \frac{1}{(D^2+1)^2} x^2 (\text{Re.P. of } e^{ix})$$

$$= \text{Re.P. of} \left\{\frac{1}{(D^2+1)^2} e^{ix} \cdot x^2\right\} = \text{Re.P. of} \left\{e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2\right\}$$

$$= \text{Re.P. of} \left\{e^{ix} \frac{1}{(D^2+2iD)^2} x^2\right\} = \text{Re.P. of} \left[e^{ix}\left\{-\frac{1}{4D^2}\left(1 - \frac{i}{2}D\right)^{-2} x^2\right\}\right]$$

$$= \text{Re.P. of} \left[-\frac{1}{4} e^{ix} \cdot \frac{1}{D^2}\left\{1 + 2\frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \dots\right\} x^2\right]$$

$$= \text{Re.P. of} \left\{-\frac{1}{4}e^{ix} \cdot \frac{1}{D^2}\left(x^2 + 2ix - \frac{3}{2}\right)\right\} = \text{Re.P. of} \left\{-\frac{1}{4}e^{ix} \cdot \frac{1}{D}\left(\frac{x^3}{3} + ix^2 - \frac{3}{2}x\right)\right\}$$

$$= -\frac{1}{4} \text{Re.P. of} \left\{e^{ix}\left(\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right)\right\} = \frac{-1}{48} \text{ Re.P. of } [(\cos x + i \sin x)(x^4 + 4ix^3 - 9x^2)]$$

$$= -\frac{1}{48}[(x^4 - 9x^2)\cos x - 4x^3 \sin x]$$

(*iii*) Hence the C.S. is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + \dfrac{1}{48}[4x^3 \sin x - x^2(x^2 - 9)\cos x]$.

Example 13.21. *Solve* $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x.$ *(J.N.T.U., 2006 ; U.P.T.U., 2004)*

Solution. (*i*) *To find C.F.*

Its A.E. is $D^2 - 4D + 4 = 0$ *i.e.*, $(D-2)^2 = 0$. $\therefore$ $D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)e^{2x}$

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2 e^{2x} \sin 2x) = 8e^{2x} \frac{1}{(D+2-2)^2}(x^2 \sin 2x)$$

$$= 8e^{2x} \frac{1}{D^2}(x^2 \sin 2x) = 8e^{2x} \cdot \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= 8e^{2x} \cdot \frac{1}{D}\left\{x^2\left(\frac{-\cos 2x}{2}\right) - \int 2x\left(\frac{-\cos 2x}{2}\right)dx\right\}$$

$$= 8e^{2x}\frac{1}{D}\left\{-\frac{x^2}{2}\cos 2x + x\frac{\sin 2x}{2} - \int 1 . \frac{\sin 2x}{2}\,dx\right\}$$

$$= 8e^{2x}\int\left\{-\frac{x^2}{2}\cos 2x + \frac{x}{2}\sin 2x + \frac{\cos 2x}{4}\right\}dx$$

$$= 8e^{2x}\left[\left\{\frac{-x^2}{2}\frac{\sin 2x}{2} - \int(-x)\frac{\sin 2x}{2}dx\right\} + \left\{\int\frac{x}{2}\sin 2x\,dx\right\} + \frac{\sin 2x}{8}\right]$$

$$= 8e^{2x}\left[\left(\frac{-x^2}{4} + \frac{1}{8}\right)\sin 2x + \int x\sin 2x\,dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x\left(\frac{-\cos 2x}{2}\right) - \int 1 . \left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$= 8e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

(*iii*) Hence the C.S. is $y = e^{2x}[c_1 + c_2x + (3 - 2x^2)\sin 2x - 4x\cos 2x]$.

Example 13.22. *Solve* $\dfrac{d^2y}{dx^2} + a^2y = \sec ax.$

Solution. Given equation in *symbolic form* is $(D^2 + a^2)y = \sec ax$.

(*i*) *To find C.F.*

Its A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia.$

Thus C.F. $= c_1 \cos ax + c_2 \sin ax$.

(*ii*) *To find P.I.*

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{(D + ia)(D - ia)}\sec ax \qquad \text{[Resolving into partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right]\sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia}\sec ax - \frac{1}{D + ia}\sec ax\right]$$

Now $\quad \dfrac{1}{D - ia}\sec ax = e^{iax}\displaystyle\int \sec ax . e^{-iax}\,dx \qquad \left[\because \dfrac{1}{D - a}X = e^{ax}\displaystyle\int Xe^{-ax}\,dx\right]$

$$= e^{iax}\int\frac{\cos ax - i\sin ax}{\cos ax}dx = e^{iax}\int(1 - i\tan ax)dx = e^{iax}\left(x + \frac{i}{a}\log\cos ax\right)$$

Changing i to $-i$, we have

$$\frac{1}{D + ia}\sec ax = e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}$$

Thus $\quad \text{P.I.} = \dfrac{1}{2ia}\left[e^{iax}\left\{x + \dfrac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \dfrac{i}{a}\log\cos ax\right\}\right]$

$$= \frac{x}{a}\frac{e^{iax} - e^{-iax}}{2i} + \frac{1}{a^2}\log\cos ax . \frac{e^{iax} + e^{-iax}}{2} = \frac{x}{a}\sin ax + \frac{1}{a^2}\log\cos ax . \cos ax .$$

(*iii*) Hence the C.S. is

$$y = c_1\cos ax + c_2\sin ax + (1/a)x\sin ax + (1/a^2)\cos ax\log\cos ax.$$

PROBLEMS 13.2

Solve :

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$ (V.T.U., 2005)
2. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$. Also find y when $y = 0$, $\frac{dy}{dx} = 1$ at $x = 0$.
3. $\frac{d^2x}{dt^2} + n^2x = k\cos(nt + \alpha)$.
4. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$.
5. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$. (Bhopal, 2002 S)
6. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$. (Madras, 2000)
7. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$. (V.T.U., 2004)
8. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$. (Delhi, 2002)
9. $(D^3 - 5D^2 + 7D - 3)y = e^{2x}\cosh x$. (Nagarjuna, 2008)
10. $\frac{d^2y}{dx^2} - y = e^x + x^2e^x$. (Nagpur, 2009)
11. $(D^3 - D)y = 2x + 1 + 4\cos x + 2e^x$. (Mumbai, 2006)
12. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = e^{2x} + \sin x + x$. (V.T.U., 2006)
13. $(D^2 + 1)^2 y = x^4 + 2\sin x \cos 3x$. (Madras, 2006)
14. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = e^{-2x}\sin 2x$. (Bhopal, 2008)
15. $(D^4 + D^2 + 1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$. (Rajasthan, 2006)
16. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = e^x\cos x$. (V.T.U., 2010)
17. $(D^2 + 4D + 3)y = e^{-x}\sin x + xe^{3x}$. (Raipur, 2005 ; Anna, 2002 S)
18. $\frac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$.
19. $\frac{d^4y}{dx^4} - y = \cos x \cosh x$.
20. $(D^3 + 2D^2 + D)y = x^2e^{2x} + \sin^2 x$. (P.T.U., 2003)
21. $\frac{d^2y}{dx^2} + 16y = x\sin 3x$. (V.T.U., 2010 S)
22. $(D^2 + 2D + 1)y = x\cos x$. (Rajasthan, 2006)
23. $(D^2 - 1)y = x\sin x + (1 + x^2)e^x$.
24. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$. (S.V.T.U., 2009)
25. $(D^2 + a^2)y = \tan ax$. (V.T.U., 2005)

13.8 TWO OTHER METHODS OF FINDING P.I.

I. Method of variation of parameters. *This method is quite general and applies to equations of the form*

$$y'' + py' + qy = X \quad ...(1)$$

where p, q, and X are functions of x. It gives P.I. $= -y_1\int\frac{y_2X}{W}dx + y_2\int\frac{y_1X}{W}dx$ ...(2)

where y_1 and y_2 are the solutions of $y'' + py' + qy = 0$ ...(3)

and $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ *is called the Wronskian* of y_1, y_2.*

Proof. Let the C.F. of (1) be $y = c_1y_1 + c_2y_2$

Replacing c_1, c_2 (regarded as *parameters*) by unknown functions $u(x)$ and $v(x)$, let the P.I. be

$$y = uy_1 + vy_2 \quad ...(4)$$

Differentiating (4) w.r.t. x, we get $y' = uy'_1 + vy'_2 + u'y_1 + v'y_2$

---

*Named after the Polish mathematician and philosopher *Hoene Wronsky* (1778–1853).

$$= uy_1' + vy_2' \qquad ...(5)$$

on assuming that $u'y_1 + v'y_2 = 0$ ...(6)

Differentiate (4) and substitute in (1). Then noting that y_1 and y_2, satisfy (3), we obtain

$$u'y_1' + v'y_2' = X \qquad ...(7)$$

Solving (6) and (7), we get

$$u' = -\frac{y_2X}{W}, v' = \frac{y_1X}{W} \qquad \text{where } W = y_1y_2' - y_2y_1'$$

Integrating $u = -\int \frac{y_2X}{W} dx, v = \int \frac{y_1X}{W} dx$. Substituting these in (4), we get (2).

Example 13.23. *Using the method of variation of parameters, solve*

$$\frac{d^2y}{dx^2} + 4y = \tan 2x. \qquad \text{(V.T.U., 2008 ; Bhopal, 2007 ; S.V.T.U., 2006 S)}$$

Solution. Given equation in *symbolic form* is $(D^2 + 4)y = \tan 2x$.

(i) To find C.F.

Its A.E. is $D^2 + 4 = 0$, $\therefore$ $D = \pm 2i$

Thus C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$.

(ii) To find P.I.

Here $y_1 = \cos 2x$, $y_2 = \sin 2x$ and $X = \tan 2x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Thus,
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W} dx + y_2 \int \frac{y_1X}{W} dx$$

$$= -\cos 2x \int \frac{\sin 2x \tan 2x}{2} dx + \sin 2x \int \frac{\cos 2x \tan 2x}{2} dx$$

$$= -\frac{1}{2}\cos 2x \int (\sec 2x - \cos 2x)dx + \frac{1}{2}\sin 2x \int \sin 2x\, dx$$

$$= -\frac{1}{4} \cos 2x[\log(\sec 2x + \tan 2x) - \sin 2x] - \frac{1}{4} \sin 2x \cos 2x$$

$$= -\frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$$

Hence the C.S. is $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$.

Example 13.24. *Solve, by the method of variation of parameters,* $d^2y/dx^2 - y = 2/(1 + e^x)$.

(V.T.U., 2005 ; Hissar, 2005)

Solution. Given equation is $D^2 - 1 = 2/(1 + e^x)$

A.E. is $D^2 - 1 = 0$, $D = \pm 1$, $\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

Here $y_1 = e^x$, $y_2 = e^{-x}$ and $X = 2/(1 + e^x)$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^xe^{-x} - e^xe^{-x} = -2.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W} dx + y_2 \int \frac{y_1X}{W} dx = -e^x \int \frac{e^{-x}}{-2} \cdot \frac{2}{1+e^x} dx + e^{-x} \int \frac{e^x}{-2} \cdot \frac{2}{1+e^x} dx$$

$$= e^x \int \left(\frac{1}{e^x} - \frac{1}{1+e^x}\right) dx - e^{-x} \log(1 + e^x) = e^x \left[e^{-x} - \int \frac{e^{-x}}{e^{-x}+1} dx\right] - e^{-x} \log(1 + e^x)$$

$$= e^x[-e^{-x} + \log(e^{-x} + 1)] - e^{-x} \log(1 + e^x) = -1 + e^x \log(e^{-x} + 1) - e^{-x} \log(e^x + 1)$$

Hence C.S. is $\quad y = c_1e^x + c_2e^{-x} - 1 + e^x \log(e^{-x} + 1) - e^{-x} \log(e^x + 1)$.

Example 13.25. *Solve by the method of variation of parameters $y'' - 6y' + 9y = e^{3x}/x^2$.*
(Nagpur, 2009 ; S.V.T.U., 2009)

Solution. Given equation is $(D^2 - 6D + 9)y = e^{3x}/x^2$

A.E. is $D^2 - 6D + 9 = 0$ *i.e.* $(D - 3)^2 = 0$ $\therefore$ C.F. $= (c_1 + c_2x)e^{3x}$

Here $y_1 = e^{3x}$, $y_2 = xe^{3x}$ and $X = e^{3x}/x^2$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}.$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx = -e^{3x} \int \frac{xe^{3x}}{e^{6x}} \frac{e^{3x}}{x^2}\,dx + xe^{3x} \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2}\,dx$$

$$= -e^{3x} \int \frac{dx}{x} + xe^{3x} \int x^{-2}\,dx = -e^{3x}(\log x + 1)$$

Hence C.S. is $\qquad y = (c_1 + c_2x)e^{3x} - e^{3x}(\log x + 1).$

Example 13.26. *Solve, by the method of variation of parameters, $y'' - 2y' + y = e^x \log x$.*
(V.T.U., 2006 ; Kurukshetra, 2005 ; Madras, 2003)

Solution. Given equation in symbolic form is $(D^2 - 2D + 1)\,y = e^x \log x$

(*i*) *To find C.F.*

Its A.E. is $(D - 1)^2 = 0$, $\therefore$ $D = 1, 1$

Thus C.F. is $y = (c_1 + c_2x)e^x$

(*ii*) *To find P.I.*

Here $y_1 = e^x$, $y_2 = xe^x$ and $X = e^x \log x$

$$\therefore \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (1 + x)\,e^x \end{vmatrix} = e^{2x}$$

Thus
$$\text{P.I.} = -y_1 \int \frac{y_2X}{W}\,dx + y_2 \int \frac{y_1X}{W}\,dx$$

$$= -e^x \int \frac{xe^x \cdot e^x \log x}{e^{2x}}\,dx + xe^x \int \frac{e^x \cdot e^x \log x}{e^{2x}}\,dx = -e^x \int x \log x\,dx + xe^x \int \log x\,dx$$

$$= -e^x \left(\frac{x^2}{2} \log x - \int \frac{1}{x} \cdot \frac{x^2}{2}\,dx \right) + x \cdot e^x \left(x \log x - \int \frac{1}{x} \cdot x\,dx \right)$$

$$= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + x\;e^x (x \log x - x) = \frac{1}{4}\,x^2e^x\,(2 \log x - 3)$$

Hence C.S. is $y = (c_1 + c_2x)\,e^x + \dfrac{1}{4}\,x^2e^x\,(2 \log x - 3).$

II. Method of undetermined coefficients

To find the P.I. of $f(D)\,y = X$, we assume a trial solution containing unknown constants which are determined by substitution in the given equation. The trial solution to be assumed in each case, depends on the form of X. Thus when (*i*) $X = 2e^{3x}$, trial solution $= ae^{3x}$.

(*ii*) $X = 3 \sin 2x$, trial solution $= a_1 \sin 2x + a_2 \cos 2x$

(*iii*) $X = 2x^3$, trial solution $= a_1x^3 + a_2x^2 + a_3x + a_4$

However when $X = \tan x$ or $\sec x$, this method fails, since the number of terms obtained by differentiating $X = \tan x$ or $\sec x$ is infinite.

The above method holds so long as no term in the trial solution appears in the C.F. If any term of the trial solution appears in the C.F., we multiply this trial solution by the lowest positive integral power of x which is large enough so that none of the terms which are then present, appear in the C.F.

Example 13.27. *Solve* $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 2x^2 + 3e^{-x}$. (V.T.U., 2008)

Solution. Here C.F. $= e^{-x}(c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x)$

Assume P.I. as $y = a_1x^2 + a_2x + a_3 + a_4e^{-x}$

$\therefore$ $Dy = 2a_1x + a_2 - a_4e^{-x}$ and $D^2y = 2a_1 + a_4e^{-x}$

Substituting these in the given equation, we get

$$4a_1x^2 + (4a_1 + 4a_2)x + (2a_1 + 2a_2 + 4a_3) + 3a_4e^{-x} = 2x^2 + 3e^{-x}$$

Equating corresponding coefficients on both sides, we get

$$4a_1 = 2,\ 4a_1 + 4a_2 = 0,\ 2a_1 + 2a_2 + 4a_3 = 0,\ 3a_4 = 3$$

Then $a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_3 = 0, a_4 = 1$. Thus P.I. $= \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$

$\therefore$ C.S. is $y = e^{-x}(c_1 \cos \sqrt{3x} + c_2 \sin \sqrt{3x}) + \frac{1}{2}x^2 - \frac{1}{2}x + e^{-x}$.

Example 13.28. *Solve* $(D^2 + 1)y = \sin x$.

Solution. Here C.F. $= c_1 \cos x + c_2 \sin x$

We would normally assume a trial solution as $a_1 \cos x + a_2 \sin x$.

However, since these terms appear in the C.F., we multiply by x and assume the trial P.I. as

$$y = x(a_1 \cos x + a_2 \sin x)$$

$\therefore$ $Dy = (a_1 + a_2x)\cos x + (a_2 - a_1x)\sin x$ and $D^2y = (2a_2 - a_1x)\cos x - (2a_1 + a_2x)\sin x$

Substituting these in the given equation, we get $2a_1 \cos x - 2a_2 \sin x = \sin x$

Equating corresponding coefficients,

$$2a_1 = 0, \quad -2a_2 = 1 \quad \text{so that } a_1 = 0, a_2 = -\frac{1}{2}. \quad \text{Thus P.I.} = -\frac{1}{2}x \sin x$$

$\therefore$ C.S. is $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x \sin x$.

Example 13.29. *Solve by the method of undetermined coefficients,*

$$\frac{d^2y}{dx^2} - y = e^{3x}\cos 2x - e^{2x}\sin 3x.$$

Solution. Its A.E. is $D^2 - 1 = 0$, $\therefore$ $D = \pm 1$.

Thus C.F. $= c_1e^x + c_2e^{-x}$

Assume P.I. as $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - e^{2x}(c_3 \cos 3x + c_4 \sin 3x)$

$$\therefore \quad \frac{dy}{dx} = e^{3x}\{(3c_1 + 2c_2)\cos 2x + (3c_2 - 2c_1)\sin 2x\} - e^{2x}\{(2c_3 + 3c_4)\cos 3x + (2c_4 - 3c_3)\sin 3x\}$$

and $$\frac{d^2y}{dx^2} = e^{3x}\{(5c_1 + 12c_2)\cos 2x + (5c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 5c_3)\cos 3x - (5c_4 + 12c_3)\sin 3x\}$$

Substituting these in the given equation, we get

$$e^{3x}\{(4c_1 + 12c_2)\cos 2x + (4c_2 - 12c_1)\sin 2x\} - e^{2x}\{(12c_4 - 6c_3)\cos 3x - (6c_4 + 12c_3)\sin 3x\}$$
$$= e^{3x}\cos 2x - e^{2x}\sin 3x$$

Equating corresponding coefficients,

$$4c_1 + 12c_2 = 1,\ 4c_2 - 12c_1 = 0\ ;\ 12c_4 - 6c_3 = 0,\ 6c_4 + 12c_3 = -1$$

whence $c_1 = 1/40, c_2 = 3/40, c_3 = -1/15, c_4 = -1/30$

Thus P.I. $= \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x) + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x)$

Hence C.S. is $y = c_1e^x + c_2e^{-x} + \frac{1}{30}e^{2x}(2\cos 3x + \sin 3x) + \frac{1}{40}e^{3x}(\cos 2x + 3\sin 2x)$.

PROBLEMS 13.3

Solve by the method of variation of parameters :

1. $\frac{d^2y}{dx^2} + a^2y = \text{cosec } ax.$
2. $\frac{d^2y}{dx^2} + y = \sec x.$ (*Bhopal, 2007*)
3. $\frac{d^2y}{dx^2} + y = \tan x.$ (*P.T.U., 2005 ; Raipur, 2004*)
4. $\frac{d^2y}{dx^2} + y = x \sin x.$ (*S.V.T.U., 2007 ; J.N.T.U., 2005*)
5. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x/x.$ (*V.T.U., 2006*)
6. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}.$ (*V.T.U., 2010 S ; U.P.T.U., 2005*)
7. $y'' - 2y' + 2y = e^x \tan x.$ (*V.T.U., 2010*)
8. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x.$ (*U.P.T.U., 2003*)
9. $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}.$ (*V.T.U., 2004*)

Solve by the method of undetermined coefficients :

10. $(D^2 - 3D + 2)\,y = x^2 + e^x.$ (*V.T.U., 2003 S*)
11. $\frac{d^2y}{dx^2} + y = 2 \cos x.$ (*V.T.U., 2000 S*)
12. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x} + \sin x.$ (*V.T.U., 2008*)
13. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x.$ (*V.T.U., 2010*)
14. $(D^2 - 2D + 3)\,y = x^3 + \cos x.$
15. $(D^2 - 2D)\,y = e^x \sin x.$ (*V.T.U., 2006*)

13.9 EQUATIONS REDUCIBLE TO LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Now we shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substitutions.

I. Cauchy's homogeneous linear equation*. An equation of the form

$$x^n \frac{d^ny}{dx^n} + k_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + k_{n-1}x\frac{dy}{dx} + k_ny = X \qquad \ldots(1)$$

where X is a function of x, is called *Cauchy's homogeneous linear equation.*

Such equations can be reduced to linear differential equations with constant coefficients, by putting

$$x = e^t \quad \text{or} \quad t = \log x. \text{ Then if } D = \frac{d}{dt}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x}, \quad i.e., \quad x\frac{dy}{dx} = Dy.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

i.e., $$x^2\frac{d^2y}{dx^2} = D(D-1)\,y. \text{ Similarly, } x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)\,y \text{ and so on.}$$

After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Example 13.30. *Solve* $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x.$ (*V.T.U., 2010*)

Solution. This is a Cauchy's homogeneous linear.

*See footnote p. 144.

Put $x = e^t$, *i.e.*, $t = \log x$, so that $x\dfrac{dy}{dx} = Dy, x^2\dfrac{d^2y}{dx^2} = D(D-1)y$ where $D = \dfrac{d}{dt}$

Then the given equation becomes $[D(D-1) - D + 1]\, y = t$ or $(D-1)^2 y = t$ ...(*i*)

which is a linear equation with constant coefficients.

Its A.E. is $(D-1)^2 = 0$ whence $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 t)e^t$ and P.I. $= \dfrac{1}{(D-1)^2}\, t = (1-D)^{-2} t = (1 + 2D + 3D^2 + \ldots)\, t = t + 2.$

Hence the solution of (*i*) is $y = (c_1 + c_2 t)e^t + t + 2$ or, putting $t = \log x$ and $e^t = x$, we get

$y = (c_1 + c_2 \log x)\, x + \log x + 2$ as the required solution of (*i*).

Example 13.31. *Solve* $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$. (*P.T.U., 2003*)

Solution. Put $x = e^t$ *i.e.*, $t = \log x$ so that $x\, dy/dx = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$[D(D-1) + 3D + 1]\, y = \frac{1}{(1-e^t)^2} \quad \text{or} \quad (D^2 + 2D + 1)\, y = \frac{1}{(1-e^t)^2}$$

Its A.E. is $D^2 + 2D + 1 = 0$ or $(D+1)^2 = 0$ *i.e.*, $D = -1, -1$.

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{-t} = (c_1 + c_2 \log x)\, \frac{1}{x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\frac{1}{(1-e^t)^2} = \frac{1}{D+1}\, u, \text{ where } u = \frac{1}{D+1}\cdot\frac{1}{(1-e^t)^2} \; i.e. \; \frac{du}{dt} + u = (1-e^t)^{-2}$$

which is Leibnitz's linear equation having I.F. $= e^t$

$\therefore$ its solution is $ue^t = \displaystyle\int \frac{e^t}{(1-e^t)^2}\, dt = \frac{1}{1-e^t}$ or $u = \dfrac{e^{-t}}{1-e^t}$

$$\therefore \quad \text{P.I.} = \frac{1}{D+1}\left(\frac{e^{-t}}{1-e^t}\right) = e^{-t}\int \frac{1}{1-e^t}\, dt = \frac{1}{x}\int \frac{dx}{x(1-x)}$$

$$= \frac{1}{x}\int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx = \frac{1}{x}\,[\log x - \log(1-x)] = \frac{1}{x}\log\frac{x}{x-1}$$

Hence the solution is $y = \left\{c_1 + c_2 \log x + \log\dfrac{x}{x-1}\right\}\dfrac{1}{x}$.

Example 13.32. *Solve* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

(*Kurukshetra, 2006 ; Madras, 2006 ; Kerala, 2005*)

Solution. Putting $x = e^t$ *i.e.* $t = \log x$, the given equation becomes

$[D(D-1) + D + 1]\, y = t \sin t$ *i.e.* $(D^2 + 1)\, y = t \sin t$ ...(*i*)

Its A.E. is $D^2 + 1 = 0$ *i.e.* $D = \pm i$.

$\therefore$ C.F. $= c_1 \cos t + c_2 \sin t$

and $\quad P.I. = \dfrac{1}{D^2+1}\, t \sin t = \dfrac{1}{D^2+1}\, t\, (\text{I.P. of } e^{it})$

$$= \text{I.P. of } e^{it}\,\frac{1}{(D+i)^2+1}\, t = \text{I.P. of } e^{it}\cdot\frac{1}{D^2+2iD}\, t$$

$$= \text{I.P. of } e^{it} \frac{1}{2iD(1 + D/2i)} t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 - \frac{iD}{2}\right)^{-1} t$$

$$= \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(1 + \frac{iD}{2} + \ldots\right) t = \text{I.P. of } \frac{1}{2i} e^{it} \frac{1}{D}\left(t + \frac{i}{2}\right)$$

$$= \text{I.P. of } \frac{e^{it}}{2i} \int\left(t + \frac{i}{2}\right) dt = \text{I.P. of } \frac{e^{it}}{2i}\left(\frac{t^2}{2} + \frac{it}{2}\right)$$

$$= \text{I.P. of } e^{it}\left(-\frac{i}{4}t^2 + \frac{t}{4}\right) = \text{I.P. of } (\cos t + i \sin t)\left(-\frac{it^2}{4} + \frac{t}{4}\right) = -\frac{t^2}{4}\cos t + \frac{t}{4}\sin t$$

Hence the C.S. of (*i*) is $y = c_1 \cos t + c_2 \sin t - \dfrac{t^2}{4}\cos t + \dfrac{t}{4}\sin t$

or $\quad y = c_1 \cos(\log x) + c_2 \sin(\log x) - \dfrac{1}{4}(\log x)^2 \cos(\log x) + \dfrac{1}{4}\log(\log x)\sin(\log x)$

which is the required solution.

Example 13.33. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}$ *(I.S.M., 2001)*

Solution. Put $x = e^t$, *i.e.*, $t = \log x$ so that $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

Then the given equation becomes

$$\{D(D-1) - 3D + 1\}y = t\frac{\sin t + 1}{e^t} \qquad \text{or} \qquad (D^2 - 4D + 1)y = e^{-t}t(\sin t + 1)$$

which is a linear equation with constant coefficients.

Its A.E. is $D^2 - 4D + 1 = 0$ whence $D = 2 \pm \sqrt{3}$

$\therefore \qquad \text{C.F.} = c_1 e^{(2+\sqrt{3})t} + c_2 e^{(2-\sqrt{3})t} = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t})$

and $\qquad \text{P.I.} = \dfrac{1}{D^2 - 4D + 1} e^{-t}t(\sin t + 1) = e^{-t}\dfrac{1}{(D-1)^2 - 4(D-1) + 1}t(\sin t + 1)$

$$= e^{-t}\left\{\frac{1}{D^2 - 6D + 6}t + \frac{1}{D^2 - 6D + 6}t\sin t\right\}$$

$$\frac{1}{D^2 - 6D + 6}t = \frac{1}{6}\left(1 - \frac{6D - D^2}{6}\right)^{-1} t = \frac{1}{6}(1 + D)t = \frac{1}{6}(t + 1)$$

$$\frac{1}{D^2 - 6D + 6}t\sin t = \text{I.P. of } \frac{1}{D^2 - 6D + 6}e^{it}.t$$

$$= \text{I.P. of } e^{it}\frac{1}{(D+i)^2 - 6(D+i) + 6}t = \text{I.P. of } e^{it}\frac{1}{D^2 + (2i-6)D + (5-6i)}t$$

$$= \text{I.P. of } \frac{e^{it}}{5-6i}\left\{1 + \frac{(2i-6)D + D^2}{5-6i}\right\}^{-1} t = \text{I.P. } \frac{e^{it}}{5-6i}\left(1 - \frac{2i-6}{5-6i}D\right)t$$

$$= \text{I.P. of } \frac{(5+6i)}{61}(\cos t + i\sin t)\left(t - \frac{2i-6}{5-6i}\right)$$

$$= \text{I.P. of } \frac{1}{61}\{(5\cos t - 6\sin t) + i(5\sin t + 6\cos t)\}\left(t + \frac{42+26i}{61}\right)$$

$$= \frac{26}{3721}(5\cos t - 6\sin t) + \frac{1}{61}(5\sin t + \cos t)\left(t + \frac{42}{61}\right)$$

$$= \frac{t}{61}(5\sin t + \cos t) + \frac{2}{3721}(27\sin t + 191\cos t)$$

$\therefore$ $$\text{P.I.} = e^{-t}\left[\frac{1}{6}(t+1) + \frac{1}{61}(5\sin t + 6\cos t) + \frac{2}{3721}(27\sin t + 191\cos t)\right]$$

Hence $$y = e^{2t}(c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t}) + e^{-t}\left[\frac{1}{6}(t+1) + \frac{t}{61}(5\sin t + 6\cos t) + \frac{2}{3721}(27\sin t + 191\cos t)\right]$$

or $$y = x^2(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}) + \frac{1}{x}\left[\frac{1}{6}(\log x + 1) + \frac{\log x}{61}\{5\sin(\log x) + 6\cos(\log x)\} + \frac{2}{3721}\{27\sin(\log x) + 191\cos(\log x)\}\right].$$

Example 13.34. *Solve* $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$. *(Kurukshetra, 2005 ; U.P.T.U., 2005)*

Solution. Putting $x = e^t$, *i.e.*, $t = \log x$, the given equation becomes

$[D(D-1) + 4D + 2]\,y = e^{e^t}$ *i.e.*, $(D^2 + 3D + 2)\,y = e^{e^t}$

Its A.E. is $D^2 + 3D + 2 = 0$ whence $D = -1, -2$.

$\therefore$ C.F. $= c_1 e^{-t} + c_2 e^{-2t} = c_1 x^{-1} + c_2 x^{-2}$

and $$\text{P.I.} = \frac{1}{(D^2+3D+2)}e^{e^t} = \frac{1}{(D+1)(D+2)}e^{e^t} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{e^t}$$

Now $$\frac{1}{D+1}e^{e^t} = \frac{1}{D+1}e^{-t}\,.\,e^t e^{e^t} = e^{-t}\frac{1}{(D-1)+1}e^t e^{e^t}$$

$$= e^{-t}\frac{1}{D}e^t e^{e^t} = e^{-t}\int e^{e^t}\,d(e^t) = x^{-1}\int e^x dx = x^{-1}e^x$$

$$\frac{1}{D+2}e^{e^t} = \frac{1}{D+2}e^{-2t}\,.\,e^{2t}e^{e^t} = e^{-2t}\frac{1}{(D-2)+2}e^{2t}e^{e^t}$$

$$= e^{-2t}\frac{1}{D}e^{e^t}e^{2t} = e^{-2t}\int e^{e^t}e^t\,d(e^t)$$

$$= x^{-2}\int e^x\,x\,dx \qquad [\because\ e^t = x]$$

$$= x^{-2}(xe^x - e^x) \qquad \text{[Integrating by parts]}$$

$\therefore$ $$\text{P.I.} = x^{-1}e^x - x^{-2}(xe^x - e^x) = x^{-2}e^x$$

Hence the required solution is $y = c_1 x^{-1} + x^{-2}(c_2 + e^x)$.

II. Legendre's linear equation*. An equation of the form

$$(ax+b)^n\frac{d^ny}{dx^n} + k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + k_n y = X \qquad ...(2)$$

where k's are constants and X is a function of x, is called *Legendre's linear equation*.

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax + b = e^t$, *i.e.*, $t = \log(ax+b)$.

Then, if $$D = \frac{d}{dt},\ \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a}{ax+b}\cdot\frac{dy}{dt} \text{ i.e. } (ax+b)\frac{dy}{dx} = a\,Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

* A French mathematician *Adrien Marie Legender* (1752 – 1833) who made important contributions to number theory, special functions, calculus of variations and elliptic integrals.

i.e., $(ax+b)^2 \dfrac{d^2y}{dx^2} = a^2 D(D-1)y$. Similarly, $(ax+b)^3 \dfrac{d^3y}{dx^3}$ $a^3D(D-1)(D-2)y$ and so on.

After making these replacements in (2), there results a linear equation with constant coefficients.

Example 13.35. *Solve* $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$. ...(*i*)

(*V.T.U., 2009 ; J.N.T.U., 2005 ; Kerala, 2005*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $1+x=e^t$, *i.e.*, $t=\log(1+x)$, so that $(1+x)\dfrac{dy}{dx} = Dy$

and $$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)\,y, \text{ where } D = \frac{d}{dt}$$

Then (*i*) becomes $D(D-1)y + Dy + y = 2\sin t$

or $$(D^2+1)\,y = 2\sin t \qquad ...(ii)$$

This is a linear equation with constant co-efficients

Its A.E. is $D^2+1=0$, whence $D=\pm i$ $\therefore$ C.F. $= c_1\cos t + c_2 \sin t$

and $$\text{P.I.} = 2\frac{1}{D^2+1}\sin t = 2t \cdot \frac{1}{2D}\sin t$$

$$= t\int \sin t\,dt = -t\cos t \qquad [\because \text{ on replacing } D^2 \text{ by } -1^2,\ D^2+1=0]$$

Hence the solution of (*ii*) is $y = c_1\cos t + c_2\sin t - t\cos t$ and on replacing t by $\log(1+x)$, we get $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$ as the required solution.

Example 13.36. *Solve* $(2x-1)^2\dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} - 2y = 8x^2 - 2x + 3$. (*V.T.U., 2006*)

Solution. This is a Legendre's linear equation.

$\therefore$ put $2x-1=e^t$ *i.e.*, $t=\log(2x-1)$ so that $(2x-1)\dfrac{dy}{dx} = 2Dy$

and $$(2x-1)^2\frac{d^2y}{dx^2} = 4D(D-1)\,y, \text{ where } D=\frac{d}{dt}.$$

Then the given equation becomes

$$4D(D-1)y + 2Dy - 2y = 8\left(\frac{1+e^t}{2}\right)^2 - 2\left(\frac{1+e^t}{2}\right) + 3$$

or $$2D^2y - Dy - y = e^{2t} + \frac{3}{2}e^t + 2 \qquad ...(i)$$

This is a linear equation with constant coefficients.

Its A.E. is $2D^2 - D - 1 = 0$ whence $D = 1, -1/2$.

$\therefore$ $$\text{C.F.} = c_1e^t + c_2e^{-t/2}$$

and $$\text{P.I.} = \frac{1}{2D^2-D-1}\left(e^{2t} + \frac{3}{2}e^t + 2\right) = \frac{1}{2.4-2-1}e^{2t} + \frac{3}{2}\frac{t}{4D-1}e^t + 2\cdot\frac{1}{2.0^2-0-1}e^{0t}$$

$$[\because \text{ on putting } t=1,\ 2D^2-D-1=0$$

$$= \frac{1}{5}e^{2t} + \frac{3t}{2}\cdot\frac{1}{4-1}e^t - 2 = \frac{1}{5}e^{2t} + \frac{t}{2}e^t - 2$$

Hence the solution of (*i*) is

$$y = c_1e^t + c_2e^{-t/2} + \frac{1}{5}e^{2t} + \frac{1}{2}te^t - 2 \text{ and on replacing } t \text{ by } \log(2x-1),$$

$$y = c_1 (2x-1) + c_2 (2x-1)^{-1/2} + \frac{1}{5}(2x-1)^2 + \frac{1}{2}(2x-1)\log(2x-1) - 2.$$

which is the required solution.

PROBLEMS 13.4

Solve :

1. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2.$

2. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4.$

3. $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = (1+x^2).$ (*S.V.T.U., 2007*)

4. $x \frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}.$ (*V.T.U., 2005 S*)

5. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the conditions $u = 0$ when $r = 0$, $u = 0$ when $r = a$.

Solve :

6. $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = \log x.$ (*Bhopal, 2009*)

7. $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$ (*Bhopal, 2008*)

8. $x^2y'' + xy' + y = 2\cos^2(\log x).$ (*V.T.U., 2011*)

9. $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$ (*S.V.T.U., 2006 ; P.T.U., 2003*)

10. $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$ (*P.T.U., 2003*)

11. $x^2 \frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x \log x.$ (*U.P.T.U., 2004*)

12. $x^2 \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = x^3 \log x.$ (*Bhopal, 2008*)

13. $(2x+3)^2 \frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x.$ (*V.T.U., 2007 ; Kerala, 2005 ; Anna, 2002 S*)

14. $(x-1)^3 \frac{d^3y}{dx^3} + 2(x-1)^2 \frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y = 4\log(x-1).$ (*Nagpur, 2009*)

15. $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[2\log(1+x)]$ (*P.T.U., 2006 ; V.T.U., 2004*)

16. $(3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2 + x + 1.$ (*Mumbai, 2006*)

13.10 (1) LINEAR DEPENDENCE OF SOLUTIONS

Consider the initial value problem consisting of the homogeneous linear equation

$$y'' + py' + qy = 0 \quad \text{...(1)}$$

with variable coefficients $p(x)$ and $q(x)$ and two initial conditions $y(x_0) = k_0, y'(x_0) = k_1$ ...(2)

Let its general solution be $y = c_1 y_1 + c_2 y_2$ ...(3)

which is made up of two linearly dependent solutions y_1 and y_2.*

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is any fixed point on I, then the above initial value problem has a **unique solution** *$y(x)$ on the interval I.*

* As in §2.12, y_1, y_2 are said to be *linearly dependent* in an interval I, if and only if there exist numbers λ_1, λ_2 not both zero such that $\lambda_1 y_1 + \lambda_2 y_2 = 0$ for all x in I.

If no such numbers other than zero exist, then y_1, y_2 are said to be *linearly independent*.

(2) Theorem. *If $p(x)$ and $q(x)$ are continuous on an open interval I, then the solutions y_1 and y_2 of (1) are linearly dependent in I if and only if the Wronskian*† $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ for some x_0 on I. If there is an $x = x_1$ in I at which $W(y_1, y_2) \neq 0$, then y_1, y_2 *are linearly independent on I.*

Proof. If y_1, y_2 are linearly dependent solutions of (1) then there exist two constants c_1, c_2 not both zero, such that

$$c_1y_1 + c_2y_2 = 0 \qquad \text{...(4)}$$

Differentiating w.r.t. x, $c_1y_1' + c_2y_2' = 0$ ...(5)

Eliminating c_1, c_2 from (4) and (5), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Conversely, suppose $W(y_1, y_2) = 0$ for some $x = x_0$ on I and show that y_1, y_2 are linearly dependent.

Consider the equation

$$\left.\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) = 0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) = 0 \end{aligned}\right\} \qquad \text{...(6)}$$

which, on eliminating c_1, c_2, give $W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = 0$

Hence the system has a solution in which c_1, c_2 are not both zero.

Now introduce the function $\bar{y}(x) = c_1y_1(x) + c_2y_2(x)$

Then $y(x)$ is a solution of (1) on I. By (6), this solution satisfies the initial conditions $y(x_0) = 0$ and $y'(x_0) = 0$. Also since $p(x)$ and $q(x)$ are continuous on I, this solution must be unique. But $\mathbf{y = 0}$ is obviously another solution of (1) satisfying the given initial conditions. Hence $\bar{y} = y$ *i.e.*, $c_1y_1 + c_2y_2 = 0$ in I. Now since c_1, c_2 are not both zero, it implies that y_1 and y_2 are linearly dependent on I.

Example 13.37. *Show that the two functions $\sin 2x$, $\cos 2x$ are independent solutions of $y'' + 4y = 0$.*

Solution. Substituting $y_1 = \sin 2x$ (or $y_2 = \cos 2x$) in the given equation we find that y_1, y_2 are its solutions.

Also
$$W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$$

for any value of x. Hence the solutions y_1, y_2 are linearly independent.

PROBLEMS 13.5

Solve :

1. Show that e^{-x}, xe^{-x} are independent solutions of $y'' + 2y' + y = 0$ in any interval.
2. Show that $e^x \cos x$, $e^x \sin x$ are independent solutions of the equation $xy'' - 2y' = 0$.
3. If y_1, y_2 be two solutions of $y'' + p(x)y' + q(x)y = 0$, show that the *Wronskian* can be expressed as $W(y_1, y_2) = ce^{-\int p dx}$

13.11 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

Quite often we come across linear differential equations in which there are two or more dependent variables and a single independent variable. Such equations are known as *simultaneous linear equations*. Here we shall deal with systems of linear equations with constant coefficients only. Such a system of equations is solved by eliminating all but one of the dependent variables and then solving the resulting equations as before. Each of the dependent variables is obtained in a similar manner.

Example 13.38. *Solve the simultaneous equations :*

$$\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$$

being given $x = y = 0$ when $t = 0$. (*S.V.T.U., 2009 ; Kurukshetra, 2005*)

† See footnote on p. 486.

Solution. Taking $d/dt = D$, the given equations become

$$(D+5)x - 2y = t \qquad ...(i)$$

$$2x + (D+1)y = 0 \qquad ...(ii)$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (i) by 2 and operating on (ii) by $D+5$ and then subtracting, we get

$$[-4-(D+5)(D+1)]\,y = 2t \text{ or } (D^2+6D+9)\,y = -2t$$

Its auxiliary equation is $D^2 + 6D + 9 = 0$, *i.e.*, $(D+3)^2 = 0$

whence $\qquad D = -3, -3 \quad \therefore \quad \text{C.F.} = (c_1 + c_2 t)\, e^{-3t}$

and $\qquad \text{P.I.} = \dfrac{1}{(D+3)^2}(-2t) = -\dfrac{2}{9}\left(1+\dfrac{D}{3}\right)^{-2} t = -\dfrac{2}{9}\left(1-\dfrac{2D}{3}+\ldots\right)t = -\dfrac{2t}{9}+\dfrac{4}{27}$

Hence $\qquad y = (c_1 + c_2 t)\, e^{-3t} - \dfrac{2t}{9} + \dfrac{4}{27} \qquad ...(iii)$

Now to find x, either eliminate y from (i) and (ii) and solve the resulting equation or substitute the value of y in (ii). Here, it is more convenient to adopt the latter method.

From (iii), $\qquad Dy = c_2 e^{-3t} + (c_1 + c_2 t)(-3)\, e^{-3t} - \dfrac{2}{9}$

$\therefore$ Substituting for y and Dy in (ii), we get

$$x = -\frac{1}{2}[Dy + y] = \left[\left(c_1 - \frac{1}{2}c_2\right) + c_2 t\right] e^{-3t} + \frac{t}{9} + \frac{1}{27} \qquad ...(iv)$$

Hence (iii) and (iv) constitute the solutions of the given equations.

Since $x = y = 0$ when $t = 0$, the equations (iii) and (iv) give

$$0 = c_1 + \frac{4}{27} \text{ and } c_1 - \frac{1}{2}c_2 + \frac{1}{27} = 0 \text{ whence } c_1 = -\frac{4}{27},\ c_2 = -\frac{2}{9}.$$

Hence the desired solutions are

$$x = -\frac{1}{27}(1+6t)\, e^{-3t} + \frac{1}{27}(1+3t),\ y = -\frac{2}{27}(2+3t)\, e^{-3t} + \frac{2}{27}(2-3t).$$

Example 13.39. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 2y + \sin t = 0,\ \dfrac{dy}{dt} - 2x - \cos t = 0$ *given that* $x = 0$ *and* $y = 1$ *when* $t = 0$.

Solution. Given equations are

$$Dx + 2y = -\sin t \quad ...(i); \qquad -2x + Dy = \cos t \qquad ...(ii)$$

Eliminating x by multiplying (i) by 2 and (ii) by D and then adding, we get

$$4y + D^2 y = -2\sin t - \sin t \text{ or } (D^2+4)y = -3\sin t$$

Its A.E. is $D = \pm 2i \quad \therefore \quad \text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$

$$\text{P.I.} = -3\,\frac{1}{D^2+4}\sin t = -3\,\frac{1}{-1+4}\sin t = -\sin t$$

$\therefore \qquad y = c_1 \cos 2t + c_2 \sin 2t - \sin t \qquad ...(iii)$

and $\qquad dy/dt = -2\sin 2t + 2c_2 \cos 2t - \cos t \qquad ...(iv)$

Substituting (iii) in (ii), we get

$$2x = Dy - \cos t = -2c_1 \sin 2t + 2c_2 \cos 2t - 2\cos t$$

or $\qquad x = -c_1 \sin 2t + c_2 \cos 2t + -\cos t \qquad ...(v)$

When $t = 0, x = 0, y = 1$, (iii) and (v) give $1 = c_1,\ 0 = c_2 - 1$

Hence $x = \cos 2t - \sin 2t - \cos t,\ y = \cos 2t + \sin 2t - \sin t$.

Example 13.40. *Solve the simultaneous equations*

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t,\ \frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t. \qquad \text{(U.P.T.U., 2001)}$$

Solution. Given equations are

$$Dx + (D-2)y = 2\cos t - 7\sin t \qquad \text{...(i)}$$

$$(D+2)x - Dy = 4\cos t - 3\sin t \qquad \text{...(ii)}$$

Eliminate y by operating on (*i*) by D and (*ii*) by $(D-2)$ and then adding, we get

$$D^2x + (D-2)(D+2)x = -2\sin t - 7\cos t + 4(-\sin t - 2\cos t) - 3(\cos t - 2\sin t)$$

or

$$2(D^2-2)x = -18\cos t \text{ or } (D^2-2)x = -9\cos t$$

Its A.E. is $\quad D^2 - 2 = 0$ or $D = \pm\sqrt{2}$, $\quad\therefore$ C.F. $= c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t}$

$$\text{P.I.} = (-9)\frac{1}{D^2-2}\cos t = \frac{-9\cos t}{-1-2} = 3\cos t.$$

Hence $x = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t}$ $3\cos t$.

Now substituting this value of x in (*ii*), we get

$$Dy = (D+2)(c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + 3\cos t) - 4\cos t + 3\sin t$$

$$= c_1\sqrt{2}e^{\sqrt{2}t} + 2c_1e^{\sqrt{2}t} + c_2(-\sqrt{2}e^{-2t}) + 2c_2e^{-\sqrt{2}t} - 3\sin t + 6\cos t - 4\cos t + 3\sin t$$

$$= (2+\sqrt{2})c_1e^{\sqrt{2}t} + (2-\sqrt{2})c_2e^{-\sqrt{2}t} + 2\cos t$$

Hence $\qquad y = (\sqrt{2}+1)c_1e^{\sqrt{2}t} - (\sqrt{2}-1)c_2e^{-\sqrt{2}t} + 2\sin t + c_3.$

Example 13.41. *The small oscillations of a certain system with two degrees of freedom are given by the equations*

$$D^2x + 3x - 2y = 0$$

$$D^2x + D^2y - 3x + 5y = 0$$

where $D = d/dt$. If $x = 0, y = 0, Dx = 3, Dy = 2$ when $t = 0$, find x and y when $t = 1/2$.

Solution. Given equations are $(D^2+3)x - 2y = 0 \qquad$ ...(*i*)

$$(D^2-3)x + (D^2+5)y = 0 \qquad \text{...(ii)}$$

To eliminate x, operate these equations by $D^2 - 3$ and $D^2 + 3$ respectively and subtract (*i*) from (*ii*). Then

$$[(D^2+3)(D^2+5) + 2(D^2-3)]y = 0 \quad\text{or}\quad (D^4 + 10D^2 + 9)y = 0$$

Its auxiliary equation is $D^4 + 10D^2 + 9 = 0$ whence $D = \pm i, \pm 3i$

Thus $\qquad y = c_1\cos t + c_2\sin t + c_3\cos 3t + c_4\sin 3t \qquad$ ...(*iii*)

To find x, we eliminate y from (*i*) and (*ii*).

$\therefore$ operating (*i*) by $D^2 + 5$ and multiplying (*ii*) by 2 and adding, we get

$$(D^4 + 10D^2 + 9)x = 0. \text{ Thus } x = k_1\cos t + k_2\sin t + k_3\cos 3t + k_4\sin 3t \qquad \text{...(iv)}$$

To find the relations between the constants in (*iii*) and (*iv*), substitute these values of x and y either of the given equations, say (*i*). This gives

$$2(k_1 - c_1)\cos t + 2(k_2 - c_2)\sin t - 2(3k_3 + c_3)\cos 3t - 2(3k_4 + c_4)\sin 3t = 0$$

which must hold for all values of t.

$\therefore$ Equating to zero the coefficients of $\cos t$, $\sin t$, $\cos 3t$ and $\sin 3t$, we get

$$k_1 = c_1, k_2 = c_2, k_3 = -c_3/3, k_4 = -c_4/3$$

Thus $\qquad x = c_1\cos t + c_2\sin t - \frac{1}{3}(c_3\cos 3t + c_4\sin 3t) \qquad$ ...(*v*)

Hence (*iii*) and (*iv*) constitute the solutions of (*i*) and (*ii*).

Since $\qquad x = y = 0$, when $t = 0$; $\therefore$ (*iii*) and (*v*) give

$$0 = c_1 + c_3 \text{ and } c_1 - \frac{1}{3}c_3 = 0 \text{ i.e. } c_1 = c_3 = 0$$

Thus (*iii*) and (*v*) reduce to

and
$$\left.\begin{aligned} y &= c_2\sin t + c_4\sin 3t \\ x &= c_2\sin t - \frac{c_4}{3}\sin 3t \end{aligned}\right\} \qquad \text{...(vi)}$$

$\therefore$ $Dx = c_2 \cos t - c_4 \cos 3t$ and $Dy = c_2 \cos t + 3c_4 \cos 3t$.

Since $Dx = 3$ and $Dy = 2$ when $t = 0$

$\therefore$ $3 = c_2 - c_4$ and $2 = c_2 + 3c_4$, whence $c_2 = 11/4$, $c_4 = -\frac{1}{4}$.

Hence equation (*vi*) becomes $x = \frac{1}{4}\left(11 \sin t + \frac{1}{3} \sin 3t\right), y = \frac{1}{4}(11 \sin t - \sin 3t)$ ...(*vii*)

$\therefore$ when $t = 1/2$, $x = \frac{1}{4}\left[11 \sin (0.5) + \frac{1}{3} \sin (1.5)\right] = \frac{1}{4}\left[11 (0.4794 + \frac{1}{3}(0.9975)\right] = 1.4015$

and $y = \frac{1}{4}[11 \sin (0.5) - \sin (1.5)] = 1.069.$

Example 13.42. *Solve the simultaneous equations:* $\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x$.

(S.V.T.U., 2006 S ; U.P.T.U., 2004)

Solution. Differentiating first equation w.r.t. t, $\frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 2(2z)$

Again differentiating w.r.t. t, $\frac{d^3x}{dt^2} = 4\frac{dz}{dt} = 4(2x)$ ...(*i*)

or $(D^3 - 8)x = 0$

Its A.E. is $D^3 - 8 = 0$ or $(D - 2)(D^2 + 2D + 4) = 0$

or $D = 2, -1 \pm i\sqrt{3}$

$\therefore$ the solution of (*i*) is $x = c_1 e^{2t} + e^{-t}(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t})$ ...(*ii*)

From the first equation, we have $y = \frac{1}{2}\frac{dx}{dt}$

$$\therefore \quad y = \frac{1}{2}[2c_1e^{2t} + (-1)e^{-t}(c_2 \cos \sqrt{3t} + c_3 \sin \sqrt{3t}) + e^t(-\sqrt{3}\, c_2 \sin \sqrt{3t} + \sqrt{3}\, c_3 \cos \sqrt{3t})]$$

or $y = c_1e^{2t} + \frac{1}{2}e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin \sqrt{3t}\right\}$ ...(*iii*)

From the second equation, we have $z = \frac{1}{2}\frac{dy}{dt}$

$$\therefore \quad z = \frac{1}{2}2c_1e^{2t} + \frac{1}{4}\Big[(-1)e^{-t}\left\{(\sqrt{3}c_3 - c_2)\cos \sqrt{3t} - (c_3 + \sqrt{3}c_2)\sin \sqrt{3t}\right\}$$
$$+ e^{-t}\left\{\sqrt{3}(c_2 - \sqrt{3}c_3)\sin \sqrt{3t} - \sqrt{3}(c_3 + \sqrt{3}c_2)\cos \sqrt{3t}\right\}\Big]$$
$$= c_1e^{2t} + \frac{1}{4}e^{-t}\left\{(-2c_2 - 2\sqrt{3}c_3)\cos \sqrt{3t} - (2\sqrt{3}\, c_2 - 2c_3)\sin \sqrt{3t}\right\}$$

or $z = c_1e^{2t} - \frac{1}{2}e^{-t}\left\{(\sqrt{3}\, c_2 - c_3)\sin \sqrt{3t} + (c_2 + \sqrt{3}c_3)\cos \sqrt{3t}\right\}$ ...(*iv*)

Hence the equations (*ii*), (*iii*) and (*iv*) taken together give the required solution.

PROBLEMS 13.6

Solve the following simultaneous equations :

1. $\frac{dx}{dt} = 5x + y, \frac{dy}{dt} = y - 4x$.
2. $\frac{dx}{dt} + y = \sin t, \frac{dy}{dt} + x = \cos t$; given that $x = 2$ and $y = 0$ when $t = 0$.

(Bhopal, 2009 ; J.N.T.U., 2006 ; Kerala, 2005)

3. $\frac{dx}{dt} + 2x + 3y = 0,\ 3x + \frac{dy}{dt} + 2y = 2e^{2t}$. (*Delhi, 2002*)
4. $\frac{dx}{dt} - 7x + y = 0,\ \frac{dy}{dt} - 2x - 5y = 0$.
5. $\frac{dx}{dt} + 2y = e^t,\ \frac{dy}{dt} - 2x = e^{-t}$. (*Bhopal, 2002 S*)
6. $\frac{dx}{dt} + 2x - 3y = t;\ \frac{dy}{dt} - 3x + 2y = e^{2t}$. (*Nagpur, 2009*)
7. $(D - 1)x + Dy = 2t + 1,\ (2D + 1)x + 2Dy = t$.
8. $(D + 1)x + (2D + 1)y = e^t,\ (D - 1)x + (D + 1)y = 1$.
9. $Dx + Dy + 3x = \sin t,\ Dx + y - x = \cos t$. (*U.P.T.U., 2003*)
10. $t\frac{dx}{dt} + y = 0,\ t\frac{dy}{dt} + x = 0$ given $x(1) = 1,\ y(-1) = 0$.
11. $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t,\ \frac{dx}{dt} + y - x = \cos t$.
12. $\frac{d^2x}{dt^2} - 3x - 4y = 0,\ \frac{d^2y}{dt^2} + x + y = 0$. (*U.P.T.U., 2005*)
13. $\frac{d^2x}{dt^2} + y = \sin t,\ \frac{d^2y}{dt^2} + x = \cos t$. (*U.P.T.U., 2004*)
14. A mechanical system with two degrees of freedom satisfies the equations

$$2\frac{d^2x}{dt^2} + 3\frac{dy}{dt} = 4;\ 2\frac{d^2y}{dt^2} - 3\frac{dx}{dt} = 0.$$

Obtain expression for x and y in terms of t, given $x, y, dx/dt, dy/dt$ all vanish at $t = 0$.

13.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 13.7

Fill up the blanks or choose the correct answer in the following problems :

1. The complementary function of $(D^4 - a^4)y = 0$ is ....
2. P.I. of the differential equation $(D^2 + D + 1)y = \sin 2x$ is ....
3. P.I. of $y'' - 3y' + 2y = 12$ is ....
4. The Wronskian of x and e^x is ....
5. The C.F. of $y'' - 2y' + y = xe^x \sin x$ is
   (*a*) $C_1e^x + C_2e^{-x}$ (*b*) $(C_1x + C_2)e^x$ (*c*) $(C_1 + C_2x)e^{-x}$ (*d*) None of these. (*V.T.U., 2010*)
6. The general solution of the differential equation $(D^4 - 6D^3 + 12D^2 - 8D)y = 0$ is ....
7. The particular integral of $(D^2 + a^2)y = \sin ax$ is
   (*a*) $-\frac{x}{2a}\cos ax$ (*b*) $\frac{x}{2a}\cos ax$ (*c*) $-\frac{ax}{2}\cos ax$ (*d*) $\frac{ax}{2}\cos ax$.
8. The solution of the differential equation $(D^2 - 2D + 5)^2 y = 0$, is ....
9. The solution of the differential equation $y'' + y = 0$ satisfying the conditions $y(0) = 1$ and $y(\pi/2) = 2$, is ....
10. $e^{-x}(c_1 \cos\sqrt{3x} + c_2 \sin\sqrt{3x}) + c_3e^{2x}$ is the general solution of
    (*a*) $d^3y/dx^3 + 4y = 0$ (*b*) $d^3y/dx^3 - 8y = 0$
    (*c*) $d^3y/dx^3 + 8y = 0$ (*d*) $d^3y/dx^3 - 2d^2y/dx^2 + dy/dx - 2 = 0$.
11. The solution of the differential equation $(D^2 + 1)^2 y = 0$ is ....
12. The particular integral of $d^2y/dx^2 + y = \cosh 3x$ is ....
13. The solution of $x^2y'' + xy' = 0$ is ....
14. The general solution of $(D^2 - 2)^2 y = 0$ is.......
15. P.I. of $(D + 1)^2 y = xe^{-x}$ is ....
16. If $f(D) = D^2 - 2$, $\frac{1}{f(D)}e^{2x} =$ ....
17. If $f(D) = D^2 + 5$, $\frac{1}{f(D)}\sin 2x =$ ....
18. The particular integral of $(D + 1)^2 y = e^{-x}$ is ....
19. The general solution of $(4D^3 + 4D^2 + D)y = 0$ is ....

20. P.I. of $(D^2 + 4)y = \cos 2x$ is

(a) $\frac{1}{2} \sin 2x$ (b) $\frac{1}{2} x \sin 2x$ (c) $\frac{1}{4} x \sin 2x$ (d) $\frac{1}{2} x \cos 2x$. (Bhopal, 2008)

21. By the method of undetermined coefficients y_p of $y'' + 3y' + 2y = 12x^2$ is of the form

(a) $a + bx + cx^2$ (b) $a + bx$ (c) $ax + bx^2 + cx^3$ (d) None of these. (V.T.U., 2010)

22. In the equation $\frac{dx}{dt} + y = \sin t + 1$, $\frac{dy}{dt} + x = \cos t$ if $y = \sin t + 1 + e^{-t}$, then $x =$ ....

23. $(x^2D^2 + xD + 7)y = 2/x$ converted to a linear differential equation with constant coefficients is ......

24. P.I. of $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$ is

(a) $\frac{x^2}{3} + 4x$ (b) $\frac{x^3}{3} + 4$ (c) $\frac{x^3}{3} + 4x$ (d) $\frac{x^3}{3} + 4x^2$.

25. The solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{3x}$ is given by

(a) $y = C_1e^x + C_2e^{2x} + \frac{1}{2}e^{3x}$ (b) $y = C_1e^{-x} + C_2e^{-2x} + \frac{1}{2}e^{3x}$

(c) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{3x}$ (d) $y = C_1e^{-x} + C_2e^{2x} + \frac{1}{2}e^{-3x}$.

26. The particular integral of the differential equation $(D^3 - D)y = e^x + e^{-x}$, $D = \frac{d}{dx}$ is

(a) $\frac{1}{2}(e^x + e^{-x})$ (b) $\frac{1}{2}x(e^x + e^{-x})$ (c) $\frac{1}{2}x^2(e^x + e^{-x})$ (d) $\frac{1}{2}x^2(e^x - e^{-x})$.

27. The complementary function of the differential equation $x^2y'' - xy' + y = \log x$ is ....

28. The homogeneous linear differential equation whose auxiliary equation has roots $1, -1$ is ......

29. The particular integral of $(D^2 - 6D + 9)y = \log 2$ is .... (V.T.U., 2011)

30. To transform $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{1}{x}$ into a linear differential equation with constant coefficients, put $x =$ ....

31. The particular integral of $(D^2 - 4)y = \sin 3x$ is

(a) 1/4 (b) $-1/13$ (c) 1/5 (d) None of these. (V.T.U., 2010)

32. The solution of $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$ is ....

33. The differential equation whose auxiliary equation has the roots $0, -1, -1$ is ......

34. Complementary function of $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 2x \log x$ is

(a) $(C_1 + C_2x)e^x$ (b) $(C_1 + C_2 \log x)x$ (d) $(C_1 + C_2x) \log x$ (d) $(C_1 + C_2 \log x)e^x$. (Bhopal, 2008)

35. The general solution of $(D^2 - D - 2)x = 0$ is $x = c_1e^t + c_2e^{-2t}$ (True or False)

36. $\frac{1}{f(D)}(x^2e^{ax}) = \frac{1}{f(D+a)}(e^{ax}x^2)$. (True or False)

CHAPTER 14

Applications of Linear Differential Equations

14.1 INTRODUCTION

The linear differential equations with constant coefficients find their most important applications in the study of electrical, mechanical and other linear systems. In fact such equations play a dominant role in unifying the theory of electrical and mechanical oscillatory systems.

We shall begin by explaining the types of oscillations of the mechanical systems and the equivalent electrical circuits. Then we shall study at some length the slightly less striking applications such as deflection of beams and whirling of shafts. At the end, we'll take up some of the applications of simultaneous linear differential equations.

14.2 SIMPLE HARMONIC MOTION

When the acceleration of a particle is proportional to its displacement from a fixed point and is always directed towards it, then the motion is said to be *simple harmonic*.

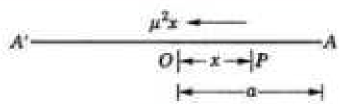


Fig. 14.1

If the displacement of the particle at any time t, from fixed point O is x (Fig. 14.1), then

$$\frac{d^2x}{dt^2} = -\mu^2 x \quad \text{or} \quad (D^2 + \mu^2)x = 0, \qquad \text{...(1)}$$

$\therefore$ its solution is $\quad x = c_1 \cos \mu t + c_2 \sin \mu t \qquad$ ...(2)

$\therefore$ its velocity at $\quad P = \dfrac{dx}{dt} = \mu(-c_1 \sin \mu t + c_2 \cos \mu t) \qquad$ ...(3)

If the particle starts from rest at A, where $OA = a$,

then from (2), $\quad$ (at $t = 0, x = a$) $\quad a = c_1$

and from (3), $\quad$ (at $t = 0, dx/dt = 0$) $\quad 0 = c_2$.

Thus $\quad x = a \cos \mu t \qquad$ ...(4)

and $\quad \dfrac{dx}{dt} = -a\mu \sin \mu t = -\sqrt{(a^2 - x^2)} \qquad$ ...(5)

which give the displacement and the velocity of the particle at any time t.

Nature of motion. The particle starts from A towards O under acceleration directed towards O which gradually decreases until it vanishes at O, when the particle has acquired the maximum velocity. On passing through O, retardation begins and the particle comes to an instantaneous rest at A', where $OA' = OA$. It then retraces its path and goes on oscillating between A and A'.

The **amplitude** *or maximum displacement from the centre is a.*

The **periodic time,** *i.e., the time of complete oscillation is* $2\pi/\mu$, for when t is increased by $2\pi/\mu$, the values of x and dx/dt remain unaltered.

The **frequency** *or the number of oscillations per second is*

$$1/\text{periodic time, i.e., } \mu/2\pi.$$

Example 14.1. *In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to weight mg. Also find the displacement s of particle when initially* $s = 0$, $v = 0$.

Solution. Let $OA(= l)$ be the elastic horizonta string with the end O fixed and having a particle of mass m attached to the end A. (Fig. 14.2)

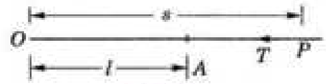


Fig. 14.2

At any time t, let the particle be at P where $OP = s$; so that the elongation $AP = s - l$.

Since for the elongation e, tension $= mg$

$\therefore$ for the elongation $s - l$, tension $T = \dfrac{mg(s-l)}{e}$

Tension being the only horizontal force, the equation of motion is

$$m\frac{d^2s}{dt^2} = -T \quad \text{or} \quad \frac{d^2s}{dt^2} = -\frac{T}{m} = -\frac{g(s-l)}{e} \qquad ...(i)$$

which is the required equation of motion.

Now (i) can be written as $(D^2 + g/e)s = gl/e$, where $D = d/dt$ ...(ii)

$\therefore$ the auxiliary equation is $D^2 + g/e = 0$ or $D = \pm i\sqrt{(g/e)}$

$$\therefore \qquad \text{C.F.} = c_1 \cos\sqrt{(g/e)}\,t + c_2 \sin\sqrt{(g/e)}\,t$$

and
$$\text{P.I.} = \frac{1}{D^2 + g/e}\cdot\frac{gl}{e} = \frac{gl}{e}\cdot\frac{1}{D^2 + g/e}e^{0t} = l$$

Thus the solution of (ii) is

$$s = c_1 \cos\sqrt{(g/e)}\,t + c_2 \sin\sqrt{(g/e)}\,t + l \qquad ...(iii)$$

When $t = 0$, $s = s_0$, $\quad \therefore \quad s_0 = c_1 + 0 + l \quad$ i.e., $c_1 = s_0 - l$

Again from (iii), $\quad \dfrac{ds}{dt} = -c_1\sqrt{(g/e)}\sin\sqrt{(g/e)}\,t + c_2\sqrt{(g/e)}\cos\sqrt{(g/e)}\,t$

When $t = 0$, $ds/dt = 0$; $\quad \therefore \quad 0 = c_2$.

Substituting the values of c_1 and c_2 in (iii), we have

$$s = (s_0 - l)\cos\sqrt{(g/e)}\,t + l \text{ which is the required result.}$$

Example 14.2. *Two particles of masses m_1 and m_2 are tied to the ends of an elastic string of natural length a and modulus λ. They are placed on a smooth table so that the string is just taut and m_2 is projected with any velocity directly away from m_1. Show that the string will become slack after the lapse of time* $\pi\sqrt{[am_1m_2/\lambda(m_1 + m_2)]}$.

Solution. Taking O as fixed point of reference, let particle m_1 be at O and m_2 at a distance a from m_1 at time $t = 0$ Fig. 14.3. At any time t, let m_1 be of a distance x from O and m_2 be at a distance y from O. Then the equation of motion of m_1 is

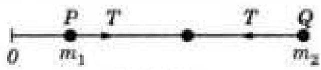


Fig. 14.3

$$m_1 d^2x/dt^2 = T \qquad ...(i)$$

and equation of motion of m_2 is $m_2 d^2y/dt^2 = -T$ ...(ii)

where $T = \lambda(y - x)/a$

From (*i*) and (*ii*) $d^2y/dt^2 - d^2x/dt^2 = -\frac{T}{m_2} - \frac{T}{m_1}$

or $$\frac{d^2(y-x)}{dt^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\lambda(y-x)}{a} \quad \text{or} \quad \frac{d^2u}{dt^2} = -\frac{\lambda(m_1+m_2)u}{m_1 m_2 a} \text{ where } u = y - x$$

This is S.H.M. with periodic time $\tau = 2\pi\sqrt{\left\{\frac{am_1m_2}{\lambda(m_1+m_2)}\right\}}$

The string will acquire its original length (*i.e.* become slack) after time τ_1 of m_2 moving towards m_1 such that

$$\tau_1 = \frac{\tau}{4} + \frac{\tau}{4} = \frac{\tau}{2} = \pi\sqrt{\left\{\frac{am_1m_2}{\lambda(m_1+m_2)}\right\}}.$$

Example 14.3. *A particle of mass m executes S.H.M. in the line joining the points A and B, on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T. If l, l′ be the extensions of the strings beyond their natural lengths, find the time of an oscillation.*

Solution. In the equilibrium position, let the particle be at C so that $AC = a + l$ and $BC = a' + l'$, where a, a' are natural lengths of the strings (Fig. 14.4). Then the tensions (at this time) are given by

$$T = \lambda l/a = \lambda' l'/a' \quad \ldots(i)$$

Fig. 14.4

At any time t, let the particle be at P, so that $CP = x$. Then

$$T_1 = \lambda\frac{l+x}{a} \text{ and } T_2 = \lambda'\frac{l'-x}{a'}$$

$\therefore$ the equation of motion is $m\frac{d^2x}{dt^2} = T_2 - T_1 = \lambda'\frac{l'-x}{a'} - \lambda\frac{l+x}{a}$

$$= \left(\frac{\lambda' l'}{a'} - \frac{\lambda l}{a}\right) - \left(\frac{\lambda'}{a'} + \frac{\lambda}{a}\right)x = -\left(\frac{T}{l'} + \frac{T}{l}\right)x \quad \text{[By (i)]}$$

or $$\frac{d^2x}{dt^2} = -\mu x \quad \text{where } \mu = \frac{l+l'}{ll'}\cdot\frac{T}{m}$$

Hence the periodic time $= \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\left\{\frac{mll'}{(l+l')T}\right\}}$.

14.3 (1) SIMPLE PENDULUM

A heavy particle attached by a light string to a fixed point and oscillating under gravity constitutes a *simple pendulum*.

Let O be the fixed point, l be the length of the string and A be the position of the bob initially (Fig. 14.5). If P be the position of the bob at any time t, such that arc $AP = s$ and $\angle AOP = \theta$, then $s = l\theta$.

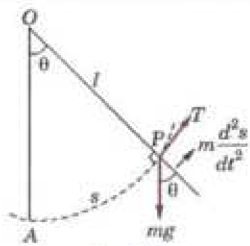


Fig. 14.5

$\therefore$ the equation of motion along PT is $m\frac{d^2s}{dt^2} = -mg\sin\theta$

i.e., $$\frac{d^2(l\theta)}{dt^2} = -g\sin\theta$$

or $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\left(\theta - \frac{\theta^3}{3!} + \ldots\right) = -\frac{g\theta}{l} \text{ to a first approx.}$$

Here the auxiliary equation being $D^2 + g/l = 0$, we have $D = \pm\sqrt{(g/l)}\,i$

$\therefore$ its solution is $\theta = c_1\cos\sqrt{(g/l)}\,t + c_2\sin t$.

Thus the motion of the bob is simple harmonic and the time of an oscillation is $2\pi\sqrt{(l/g)}$.

Obs. The movement of the bob from one end to the other constitutes half an oscillation and is called a *beat or a swing*. Thus the time of one beats $= \pi\sqrt{(l/g)}$.

A seconds pendulum beats 86400 times a day for there are 86,400 seconds in 24 hours.

(2) Gain or loss of oscillations. Let a pendulum of length l make n beats in time T, so that

$$T = \text{time of } n \text{ beats} = n\pi\sqrt{(l/g)} \quad \text{or} \quad n = \frac{T}{\pi}(g/l)^{1/2}$$

Taking logs, $\log n = \log(T/\pi) + \frac{1}{2}(\log g - \log l)$.

Taking differentials of both sides, we get $\dfrac{dn}{n} = \dfrac{1}{2}\left(\dfrac{dg}{g} - \dfrac{dl}{l}\right)$.

If only g changes, l remaining constant, $\dfrac{\mathbf{dn}}{\mathbf{n}} = \dfrac{\mathbf{dg}}{\mathbf{2g}}$ ...(1)

If only l changes, g remaining constant, $\dfrac{\mathbf{dn}}{\mathbf{n}} = -\dfrac{\mathbf{dl}}{\mathbf{2l}}$. ...(2)

Example 14.4. *Find how many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900 : 901.*

Solution. If the original length l of the string be increased to $l + dl$, then

$$\frac{l+dl}{dl} = \frac{901}{900}. \quad \therefore \quad \frac{dl}{l} = \frac{901}{900} - 1 = \frac{1}{900}.$$

$\therefore$ using (2) above, we have $\dfrac{dn}{n} = -\dfrac{dl}{2l} = -\dfrac{1}{1800}$

i.e.,
$$dn = -\frac{n}{1800} = -\frac{86400}{1800} = -48.$$

Since dn is negative, the clock will lose 4 seconds per day.

Example 14.5. *A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to the velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.*

Solution. Let the position of the bob (of mass m), at any time t be P and O be the point of suspension such that $OP = l$, $\angle AOP = \theta$ and therefore, arc $AP = s = l\theta$. (Fig. 14.6)

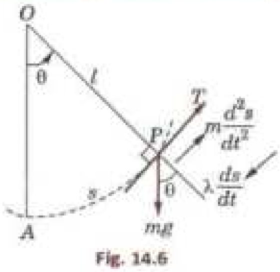


Fig. 14.6

$\therefore$ the equation of motion along the tangent PT is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta - \lambda\frac{ds}{dt} \text{ where } \lambda \text{ is a constant.}$$

or
$$\frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m}\frac{d(l\theta)}{dt} + g\sin\theta = 0$$

Replacing $\sin\theta$ by θ since it is small and writing $\lambda/m = 2k$, we get

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g\theta}{l} = 0 \quad ...(i)$$

which is the required differential equation.

Its auxiliary equation has roots $D = k \pm \sqrt{(k^2 - w^2)}$ where $w = g/l$.

The oscillatory motion of the bob is only possible when $k < w$.

Then the roots of the auxiliary equation are $-k \pm i\sqrt{(w^2 - k^2)}$.

$\therefore$ the solution of (i) is

$$\theta = e^{-kt}$$

which gives a vibratory motion of period $2\pi/\sqrt{(w^2 - k^2)}$.

Example 14.6. *A pendulum of length l has one end of the string fastened to a peg on a smooth plane inclined to the horizon at an angle α. With the string and the weight on the plane, its time of oscillation is t sec. If the pendulum of length l′ oscillates in one sec. when suspended vertically, prove that* $\alpha = \sin^{-1}\left(\dfrac{l}{l't^2}\right)$.

(*Kurukshetra, 2006*)

Solution. At any time t, let the bob of mass m be at P and O be the point of suspension so that $OP = l$ and $\angle AOP = \theta$ (Fig. 14.7).

The component of weight along the plane being $mg \sin \alpha$, the equation of motion of the bob along the tangent at P is

$$m\frac{d^2s}{dt^2} = -\,mg \sin \alpha \sin \theta$$

or $$\frac{d^2(l\theta)}{dt^2} = -g \sin \alpha \sin \theta \qquad [\because \; s = l\theta]$$

or $$\frac{d^2\theta}{dt^2} = -g \sin \alpha \left(\theta - \frac{\theta^3}{3!} + \dots\right)$$

or $$\frac{d^2\theta}{dt^2} = -\mu\theta \quad \text{where } \mu = \frac{g \sin \alpha}{l}, \text{ to a first approximation.}$$

∴ the motion being simple harmonic, the time of oscillation t.

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{l}{g \sin \alpha}} \qquad \dots(i)$$

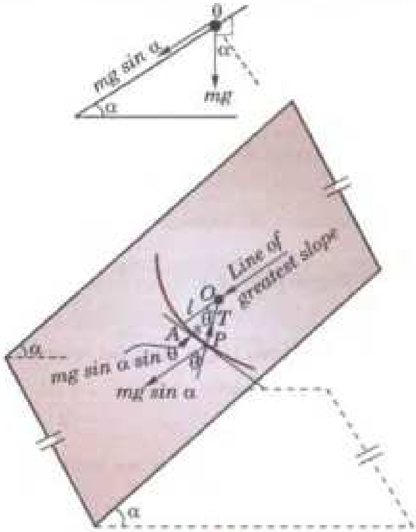


Fig. 14.7

We know that for a pendulum of length l' when suspended vertically, the time of oscillation

$$1 = 2\pi\sqrt{l'/g} \qquad \dots(ii)$$

∴ dividing (*i*) by (*ii*), we have $t = \sqrt{\left(\dfrac{l}{l' \sin \alpha}\right)}$

or $$t^2 = l/l' \sin \alpha \quad \text{or} \quad \alpha = \sin^{-1}(l/l't^2).$$

PROBLEMS 14.1

1. A particle is executive simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.
2. At the ends of three successive seconds, the distances of a point moving with S.H.M. from its mean position are x_1, x_2, x_3. Show that the time of a complete oscillation is
$$2\pi / \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$
3. An elastic string of natural length $2a$ and modulus λ is stretched between two points A and B distant $4a$ apart on a smooth horizontal table. A particle of mass m is attached to the middle of the string. Show that it can vibrate in line AB with period $2\pi/\omega$, where $\omega^2 = 2\lambda/am$.
4. A particle of mass m moves in a straight line under the action of force $mn^2(OP)$, which is always directed towards fixed point O in the line. If the resistance to the motion is $2\lambda\, mnv$, where v is the speed and $0 < \lambda < 1$, find the displacement x in terms of the time t given that when $t = 0$, $x = 0$ and $dx/dt = u$ where $OP = x$.
5. A point moves in a straight line towards the centre of force $\mu/(\text{distance}^3)$ starting from rest at a distance a from the centre of force, show that the time of reaching a point b from the centre of force is $a\sqrt{(a^2 - b^2)}/\sqrt{\mu}$ and that its velocity then is $\dfrac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$. (*U.P.T.U., 2001*)

6. A clock loses five seconds a day, find the alteration required in the length of its pendulum in order that it may keep correct time.
7. A clock with a seconds pendulum loses 10 seconds per day at a place where $g = 32$ ft/sec^2. What change in the gravity is necessary to make it accurate ?
8. A seconds pendulum which gains 10 seconds per day at one place loses 10 seconds per day at another; compare the acceleration due to gravity at the two places. *(Kurukshetra, 2005)*
9. Show that the free oscillations of a galvanometer needle, as affected by the viscosity of the surrounding air which varies directly as the angular velocity of the needle, are determined by the equation $\frac{d^2\theta}{dt^2} + K\frac{d\theta}{dt} + \mu\theta = 0$, where k is the co-efficient of viscosity and θ is the angular deflection of the needle at time t. Obtain θ in terms of t and discuss the different cases that can arise.
10. If $I = \frac{d^2\theta}{dt^2} = -mgl\sin\theta$, where I, m, g, l are constant, given that at $t = 0$, $\theta = 0$ and $d\theta/dt = \omega_0 = m\sqrt{(mgl)}/I$, then show that $t = \frac{2}{\omega_0}\log\frac{\pi+\theta}{4}$. *(Nagpur, 2009)*

14.4 OSCILLATIONS OF A SPRING

(*i*) **Free oscillations.** Suppose a mass m is suspended from the end A of a light spring, the other end of which is fixed at O. (Fig. 14.8)

Let e (= AB) be the elongation produced by the mass m hanging in equilibrium. If k be the restoring force per unit stretch of the spring due to elasticity, then for the equilibrium at B,

$$mg = T = ke \qquad \text{...(1)}$$

At any time t, after the motion ensues, let the mass be at P, where $BP = x$. Then the equation of motion of m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) = -kx \qquad \text{[By (1)]}$$

Or writing $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + \mu^2 x = 0 \qquad \text{...(2)}$$

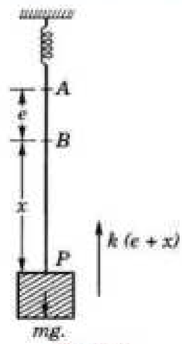


Fig. 14.8

This equation represents the free vibrations of the spring which are of the simple harmonic form having centre of oscillation at B—its equilibrium position and the *period of oscillation*

$$= \frac{2\pi}{\mu} = 2\pi\sqrt{\left(\frac{e}{g}\right)}. \qquad \left[\because \frac{1}{\mu} = \sqrt{\left(\frac{m}{k}\right)} = \sqrt{\left(\frac{e}{g}\right)}, \text{[By (1)]}\right]$$

(*ii*) **Damped oscillations.** If the mass m be subjected to do damping force proportional to velocity (say : $r\,dx/dt$) (Fig. 14.9), then the equation of motion becomes

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - r\frac{dx}{dt}$$

$$= -kx - r\frac{dx}{dt} \qquad \text{[By (1)]}$$

Or writing $r/m = 2\lambda$ and $k/m = \mu^2$, it becomes

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = 0 \qquad \text{...(3)}$$

$\therefore$ its auxiliary equation is

$$D^2 + 2\lambda D + \mu^2 = 0 \quad \text{whence} \quad D = -\lambda \pm .$$

Case I. *When* $\lambda > \mu$, the roots of the auxiliary equation are real and distinct (say γ_1, γ_2).

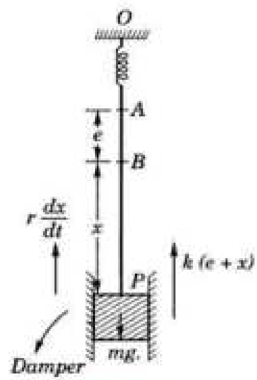


Fig. 14.9

$\therefore$ the solution of (3) is $x = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}$ ...(4)

To determine c_1, c_2 let the spring be stretched to a length $x = l$ and then released so that

$x = l$ and $dx/dt = 0$ at $t = 0$.

$\therefore$ from (4), $l = c_1 + c_2$

Also from $\dfrac{dx}{dt} = c_1\gamma_1 e^{\gamma_1 t} + c_2\gamma_2 e^{\gamma_2 t}$, we get

$0 = c_1\gamma_1 + c_2\gamma_2$

whence $c_1 = \dfrac{-l\gamma_2}{\gamma_1 - \gamma_2}$ and $c_2 = \dfrac{l\gamma_1}{\gamma_1 - \gamma_2}$

Hence the solution of (3) is

$$x = \frac{l}{\gamma_1 - \gamma_2}(\gamma_1 e^{\gamma_2 t} - \gamma_2 e^{\gamma_1 t}) \quad ...(5)$$

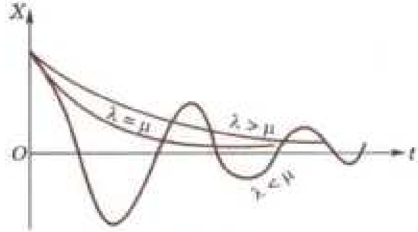


Fig. 14.10

which shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The restoring force, in this case, is so great that the motion is non-oscillatory and is, therefore, referred to as *over-damped* or *dead-beat* motion.

Case II. *When* $\lambda = \mu$, the roots of the auxiliary equation are real and equal, (each being $= -\lambda$).

$\therefore$ The general solution of (3) becomes $x = (c_1 + c_2 t)e^{-\lambda t}$.

As in case I, if $x = l$ and $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l$.

Hence the solution of (3) is $x = l(1 + \lambda t)e^{-\lambda t}$ which also shows that x is always positive and decreases to zero as $t \to \infty$ (Fig. 14.10).

The nature of motion is similar to that of the previous case and is called the *critically damped motion* for it separates the non-oscillatory motion of case I from the most interesting oscillatory motion of case III.

Case III. *When* $\lambda < \mu$, the roots of the auxiliary equation are imaginary, *i.e.* $D = -\lambda \pm i\alpha$, where $\alpha^2 = \mu^2 - \lambda^2$.

$\therefore$ the solution of (3) is $x = e^{-\lambda t}(c_1 \cos \alpha t + c_2 \sin \alpha t)$

As in case I, $x = l$, $dx/dt = 0$ at $t = 0$, then $c_1 = l$ and $c_2 = \lambda l/\alpha$

Thus the solution of (3) becomes $x = le^{-\lambda t}\left(\cos \alpha t + \dfrac{\lambda}{\alpha}\sin \alpha t\right)$.

which can be put in the form $x = l\sqrt{\left\{1 + \left(\dfrac{\lambda}{\alpha}\right)^2\right\}}e^{-\lambda t}\cos\left\{\alpha - \tan^{-1}\dfrac{\lambda}{\alpha}\right\}$ ...(7)

Here the presence of the trigonometric factor in (7) shows that the *motion is oscillatory*, having

(*a*) the variable amplitude $= l\sqrt{[1 + (\lambda/\alpha)^2]}e^{-\lambda t}$ which decreases with time,

(*b*) the periodic time $T = 2\pi/\alpha$.

But the periodic time of free oscillations is $T' = 2\pi/\mu$.

As $\alpha = \sqrt{(\mu^2 - \lambda^2)} < \mu$

$\therefore$ $\dfrac{2\pi}{\alpha} > \dfrac{2\pi}{\mu}$, *i.e.* $T > T'$.

This shows that the *effect of damping is to increase the period of oscillation and the motion ultimately dies away*. Such a motion is termed as *damped oscillatory motion*.

(*iii*) **Forced oscillations** (*without damping*). If the point of the support of the spring is also vibrating with some external periodic force, then the resulting motion is called the *forced oscillatory motion*.

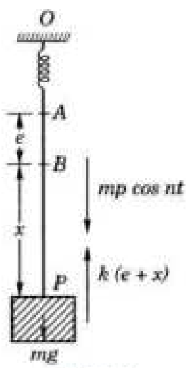


Fig. 14.11

Taking the external periodic force to be $mp \cos nt$, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(e + x) + mp\cos nt$$

$$= -kx + mp\cos nt \quad [\because\ mg = ke] \quad ...(8)$$

Or writing $k/m = \mu^2$, (8) takes the form

$$\frac{d^2x}{dt^2} + \mu^2 x = p \cos nt \quad \text{...(9)}$$

Its C.F. $= c_1 \cos \mu t + c_2 \sin \mu t$ and P.I. $= p \dfrac{1}{D^2 + \mu^2} \cos nt$.

New two cases arise :

Case I. *When* $\mu \neq n$.

$$\text{P.I.} = \frac{p}{\mu^2 - n^2} \cos nt.$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{p}{\mu^2 - n^2} \cos nt$.

On writing $c_1 \cos \mu t + c_2 \sin \mu t$ as $r \cos (\mu t + \phi)$, we have

$$x = r \cos (\mu t + \phi) + \frac{p}{\mu^2 - n^2} \cos nt \quad \text{...(10)}$$

This shows that the motion is compounded of two oscillatory motions : the first (due to the C.F.) *gives free oscillations of period* $2\pi/\mu$, *and the second* (due to the P.I.) *gives forced oscillations of period* $2\pi/n$.

Also we observe that if the frequency of free oscillations is very high (*i.e.*, μ is large), then the amplitude of forced oscillations is small.

Case II. *When* $\mu = n$.

$$\text{P.I.} = pt \cdot \frac{1}{2D} \cos \mu t = \frac{pt}{2} \int \cos \mu t \, dt = \frac{pt}{2\mu} \sin \mu t$$

$\therefore$ the complete solution of (9) is $x = c_1 \cos \mu t + c_2 \sin \mu t + \dfrac{pt}{2\mu} \sin \mu t$

$$= \left(c_2 + \frac{pt}{2\mu}\right) \sin \mu t + c_1 \cos \mu t.$$

Putting $c_2 + pt/2\mu = \rho \cos \psi$ and $c_1 = \rho \sin \psi$, we get

$$x = \rho \sin (\mu t + \psi) \quad \text{...(11)}$$

This shows that the oscillations are of period $2\pi/\mu$ and amplitude $\rho = \sqrt{[(c_2 + pt/2\mu)^2 + c_1^2]}$, which clearly increases with time (Fig. 14.12).

Thus the amplitude of the oscillations may become abnormally large causing over-strain and consequently breakdown of the system. In practice, however, collapse rarely occurs, though the amplitudes may become dangerously large since there is always some resistance present in the system.

This phenomenon of the impressed frequency becoming equal to the natural frequency of the system, is refered to as **resonance.**

Thus, while designing a machine or a structure, the occurrence of resonance should always be avoided to check the rupture of the system at any stage. That is why, the soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to 'resonance'.

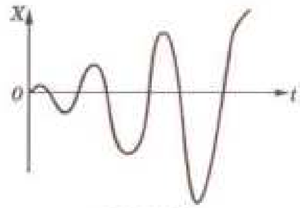


Fig. 14.12

(*iv*) **Forced oscillations** (*with damping*). If, in addition, there is a damping force proportional to velocity (say : $r\, dx/dt$) (Fig. 14.13), then the equation (8) becomes

$$m \frac{d^2x}{dt^2} = mg - k(e + x) + mp \cos nt - r \frac{dx}{dt}$$

$$= - kx + mp \cos nt - r \frac{dx}{dt} \quad | \because \; mg = ke$$

On writing $r/m = 2\lambda$ and $k/m = \mu^2$, it takes the form

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \mu^2 x = p\cos nt \qquad ...(12)$$

Its auxiliary equation is $D^2 + 2\lambda D + \mu^2 = 0$ whence $D = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$.

$$\therefore \qquad \text{C.F.} = e^{-\lambda t}[c_1 e^{t\sqrt{(\lambda^2-\mu^2)}} + c_2 e^{-t\sqrt{(\lambda^2-\mu^2)}}].$$

It represents the free oscillations of the system which die out as $t \to \infty$.

Also the P.I.

$$= p\frac{1}{D^2 + 2\lambda D + \mu^2}\cos nt = p\frac{1}{-n^2 + 2\lambda D + \mu^2}\cos nt$$

$$= p\frac{(\mu^2 - n^2) - 2\lambda D}{(\mu^2 - n^2)^2 - 4\lambda^2 D^2}\cos nt = p\frac{(\mu^2 - n^2)^2 \cos nt + 2\lambda n \sin nt}{(\mu^2 - n^2)^2 + 4\lambda^2 n^2}$$

Putting $\mu^2 - n^2 = R\cos\theta$ and $2\lambda n = R\sin\theta$, we get

$$\text{P.I.} = \frac{p}{\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}}\cos(nt - \theta)$$

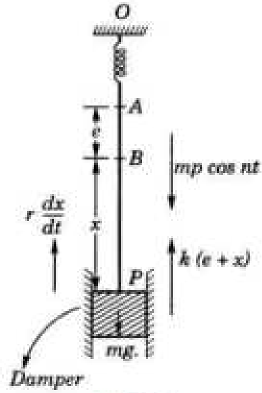


Fig. 14.13

which represents the forced oscillations of the system having

(*a*) a constant amplitude

$$= p/\sqrt{[(\mu^2 - n^2)^2 + 4\lambda^2 n^2]}$$

and (*b*) the period $= 2\pi/n$ which is the same as that of the impressed force.

Thus *with the increase of time, the free oscillations die away while the forced oscillations continue giving the steady state motion.*

Example 14.7. *A body weighing 10 kg is hung from a spring. A pull of 20 kg. wt. will stretch the spring to 10 cm. The body is pulled down to 20 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t sec., the maximum velocity and the period of oscillation.*

Solution. Let O be the fixed end and A, the lower end of the spring (Fig. 14.14).

Since a pull of 20 kg wt. at A stretches the spring by 0.1 m.

$$\therefore \qquad 20 = T_0 = k \times 0.1, \text{ i.e. } k = 200 \text{ kg/m}.$$

Let B be the equilibrium position when a body weighing $W = 10$ kg is hung from A ; then

$$10 = T_B = k \times AB$$

i.e.,
$$AB = \frac{10}{200} = 0.05 \text{ m}$$

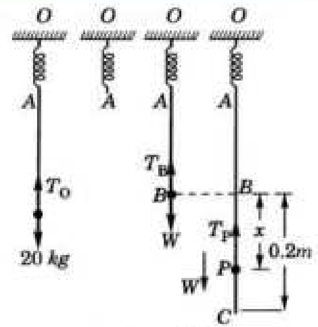


Fig. 14.14

Now the weight is pulled down to C, where $BC = 0.2$ m. After any time t sec. of its release from C, let the weight be at P where $BP = x$.

Then the tension $T_P = k \times AP = 200(0.05 + x) = 10 + 200x$.

$\therefore$ The equation of motion of the body is

$$\frac{W}{g}\frac{d^2x}{dt^2} = W - T_P, \text{ where } g = 9.8 \text{ m/sec}^2.$$

i.e.,
$$\frac{10}{9.8}\frac{d^2x}{dt^2} = 10 - (10 + 200x) \quad \text{or} \quad \frac{d^2x}{dt^2} = -\mu^2 x, \quad \text{where } \mu = 14.$$

This shows that the motion of the body is simple harmonic about B as centre and the period of oscillation $= 2\pi/\mu = 0.45$ sec.

Also the amplitude of motion being $BC = 0.2$ m., the displacement of the body from B at time t is given by $x = 0.2\cos\mu t = 0.2\cos 14t$ m

and the maximum velocity $= \mu$ (amplitude) $= 14 \times 0.2 = 2.8$ m/sec.

Example 14.8. *A spring fixed at the upper end supports a weight of 980 gm at its lower end. The spring stretches $\frac{1}{2}$ cm under a load of 10 gm and the resistance (in gm wt.) to the motion of the weigth is numerically equal to $\frac{1}{10}$ of the speed of the weight in cm/sec. The weight is pulled down $\frac{1}{4}$ cm. below its equilibrium position and then released. Find the expression for the distance of weight from its equilibrium position at time t during its first upward motion.*

Also find the time it takes the damping factor to drop to $\frac{1}{10}$ of its initial value.

Solution. Let O be the fixed end and A the other end of the spring (Fig. 14.15).

Since load of 10 gm attached to A stretches the spring by $\frac{1}{2}$ cm.

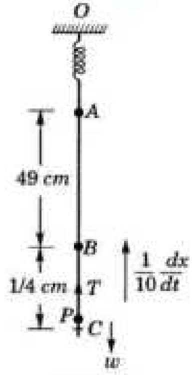


Fig. 14.15

$$\therefore \qquad 10 = T_0 = k \cdot \frac{1}{2} \text{ i.e., } k = 20 \text{ gm/cm.}$$

Let B be the equilibrium position when 980 gm. weight is attached to A, then

$$980 = T_B = k \times AB, \quad i.e., \quad AB = \frac{980}{20} = 49 \text{ cm.}$$

Now the 980 gm weight is pulled down to C, where $BC = \frac{1}{4}$ cm.

After any time t of its release from C, let the weight be at P, where $BP = x$.

Then the tension

$$T = k \times AP = 20\,(49 + x) = 980 + 20x \text{ and the resistance to motion} = \frac{1}{10}\frac{dx}{dt}.$$

$\therefore$ the equation of motion is

$$\frac{980}{g}\frac{d^2x}{dt^2} = w - T - \frac{1}{10}\frac{dx}{dt} \qquad [\because g = 980 \text{ cm/sec}^2 \text{ (p. 449)}$$

$$= 980 - (980 + 20x) - \frac{1}{10}\frac{dx}{dt} \quad i.e. \quad 10\frac{d^2x}{dt^2} + \frac{dx}{dt} + 200x = 0 \qquad ...(i)$$

Its auxiliary equation is $10D^2 + D + 200 = 0$,

whence
$$D = \frac{-1 \pm \sqrt{[1 - 4 \times 10 \times 200]}}{20} = \frac{-1 \pm i(89.4)}{20} = -0.05 \pm i\,(4.5)$$

$\therefore$ the solution of (i) is $x = e^{-0.05t}[c_1 \cos (4.5)t + c_2 \sin (4.5)t]$ ...(ii)

Also
$$\frac{dx}{dt} = e^{-0.05t}(-0.05)\,[c_1 \cos (4.5)t + c_2 \sin (4.5)t]$$
$$+ e^{-0.05t}\,[-c_1 \sin (4.5)t + c_2 \cos (4.5)t](4.5) \qquad ...(iii)$$

Initially when the mass is at C, $t = 0$, $x = \frac{1}{4}$ cm. and $dx/dt = 0$.

From (ii), $c_1 = \frac{1}{4}$, and from (iii) $0 = (-0.05)c_1 + c_2(4.5)$, *i.e.*, $c_2 = -0.003$.

Thus, substituting these value in (ii), we get

$$x = e^{-0.05t}[0.25 \cos (4.5)t + 0.003 \sin (4.5)t]$$

which gives the displacement of the weight from the equilibrium position at any time t.

Here damping factor $= re^{-0.05t}$, where r is a constant of proportionality.

Its initial value $= re^0 = r$.

Suppose after time t, the damping factor $= r/10$. $\therefore$ $r/10 = re^{-0.05t}$ or $e^{t/20} = 10$.

Thus
$$t = 20 \log_e 10 = 20 \times 2.3 = 46 \text{ sec.}$$

Example 14.9. *A spring which stretches by an amount e under a force mk^2e is suspended from a support O and has a mass m at the lower end. Initially the mass is at rest in its equilibrium position at a point A below O. A vertical oscillation is now given to the support O such that at any time $(t > 0)$ its displacement below its initial position is a sin nt. Show that the displacement x of the mass below A is given by*

$$d^2x/dt^2 + \lambda^2 x = \lambda^2 a \sin nt.$$

Hence show that if $n \neq \lambda$, *the displacement is given by* $x = \lambda a\,(\lambda \sin nt - n \sin \lambda t)/(\lambda^2 - n^2)$. *What happens when* $n = \lambda$?

Solution. If k be the stiffness of the spring then $m\lambda^2 e = ke$ *i.e.,* $k = m\lambda^2$.

Also in equilibrium $\qquad mg = ke \qquad$ ...(*i*)

Initially the mass is in equilibrium at A (Fig. 14.7). At time t, the support P is given a downward displacement $a \sin nt$. If the mass is displaced through a further distance x from A, then the equation of motion of the mass is given by

$$m\frac{d^2x}{dt^2} = mg - k(x + e) + ka \sin nt$$

$$= -kx + ka \sin nt \qquad \text{[By (i)]}$$

or
$$\frac{d^2x}{dt^2} + \lambda^2 x = \lambda^2 a \sin nt \qquad [\because\ k = m\lambda^2]$$

or
$$(D^2 + \lambda^2)x = \lambda^2 a \sin nt \qquad \text{...(ii)}$$

Its A.E. $= c_1 \cos \lambda t + c_2 \sin \lambda t$

$$\text{P.I.} = \frac{1}{D^2 + \lambda^2}\lambda^2 a \sin nt.$$

Now two cases arise :

Case I. *When* $n \neq \lambda$

$$\text{P.I.} = \lambda^2 a \frac{1}{n^2 + \lambda^2}\sin nt$$

$\therefore$ the complete solution of (*ii*) is $x = c_1 \cos \lambda t + c_2 \sin \lambda t + \dfrac{\lambda^2 a}{\lambda^2 - n^2}\sin nt \qquad$ ...(*iii*)

$$\therefore \qquad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2\lambda \cos \lambda t + \frac{\lambda^2 an}{\lambda^2 - n^2}\cos nt$$

Initially when $t = 0$, $x = 0$ and $dx/dt = 0$.

$\therefore \qquad c_1 = 0$ and $0 = c_2\lambda + \lambda^2 an/(\lambda^2 - n^2)$ *i.e.,* $c_2 = \lambda an/(\lambda^2 - n^2)$

Thus, substituting the values of c_1 and c_2 in (*iii*), we have

$$x = -\frac{\lambda an}{\lambda^2 - n^2}\sin \lambda t + \frac{\lambda^2 a}{\lambda^2 - n^2}\sin nt = \frac{\lambda a}{\lambda^2 - n^2}(\lambda \sin nt - n \sin \lambda t)$$

Case II. *When* $n = \lambda$

$$\text{P.I.} = \lambda^2 a\frac{1}{D^2 + \lambda^2}\sin nt = \lambda^2 at \cdot \frac{1}{2D}\sin \lambda t = \frac{\lambda^2 at}{2}\int \sin \lambda t\, dt = -\frac{\lambda at}{2}\cos \lambda t$$

$\therefore$ the complete solution is

$$x = c_1 \cos \lambda t + c_2 \sin \lambda t - \frac{\lambda at}{2}\cos \lambda t \qquad \text{...(iv)}$$

$$\therefore \qquad \frac{dx}{dt} = -c_1\lambda \sin \lambda t + c_2\lambda \cos \lambda t + \frac{\lambda^2 at}{2}\sin \lambda t - \frac{\lambda a}{2}\cos \lambda t$$

When $t = 0$, $x = 0$ and $dx/dt = 0$

$\therefore \qquad 0 = c_1$ and $0 = c_2\lambda - \lambda a/2$ *i.e.,* $c_2 = a/2$.

Thus, substituting the values of c_1 and c_2 in (*iv*), we get

$$x = \frac{a}{2}\sin \lambda t - \frac{\lambda at}{2}\cos \lambda t$$

$$= \frac{a}{2}(\sin \lambda t - \lambda t \cos \lambda t) \qquad [\text{Put } 1 = r\cos\phi \text{ and } \lambda t = r \sin\phi]$$

$$= \frac{ar}{2}\sin(\lambda t - \phi)$$

Its amplitude $\left(\frac{ar}{2}\right) = \frac{a}{2}\sqrt{(1+\lambda^2 t^2)}$, which increases with time. Hence the phenomenon of *resonance* occurs.

Example 14.10. *A spring of negligible weight which stretches 1 inch under tension of 2 lb is fixed at one end and is attached to a weight of w lb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2t lb acts on the weight. Show that when the free vibrations have died out, the forced vibrations are given by x = ct sin 2t, and find the values of w and c.*

Solution. As a weight of 2 lb attached to the lower end A of the spring stretched it by $\frac{1}{12}$ ft.

$$\therefore \quad 2 = T = k \cdot \frac{1}{12}, \quad i.e., \ k = 24 \text{ lb/ft.}$$

Let B be the equilibrium position of the weight w attached to A (Fig. 14.16), then

$$w = T_B = k \times AB = 24 \times AB$$

$$\therefore \quad AB = w/24 \text{ ft.}$$

At any time t, let the weight be at P, where $BP = x$.

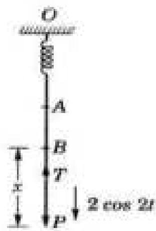


Fig. 14.16

Then the tension T at $P = k \times AP = 24\left(\frac{w}{24}+x\right) = w + 24x$

$\therefore$ its equation of motion is

$$\frac{w}{g}\frac{d^2x}{dt^2} = -T + w + 2\cos 2t = -w - 24x + w + 2\cos 2t$$

or
$$w\frac{d^2x}{dt^2} + 24gx = 2g\cos 2t \qquad ...(i)$$

The phenomenon of **resonance** *occurs when the period of free oscillations is equal to the period of forced oscillations.*

Writing (i) as $\frac{d^2x}{dt^2} + \mu^2 x = \frac{2g}{w}\cos 2t$, where $\mu^2 = 24g/w$, the period of free oscillations is found to be $2\pi/\mu$ and the period of the force $(2g/w)\cos 2t$ is π.

$\therefore \quad 2\pi/\mu = \pi$ or $24g/w = \mu^2 = 4$. Thus the weight, $w = 6g$.

Taking this value of w, (i) takes the form

$$\frac{d^2x}{dt^2} + 4x = \frac{1}{3}\cos 2t \qquad ...(ii)$$

We know that the free oscillations are given by the C.F. and the forced oscillations by the P.I.

Thus, when the free oscillations have died out, the forced oscillations are given by the P.I. of (ii).

Now P.I. of $(ii) = \frac{1}{3}\cdot\frac{1}{D^2+4}\cos 2t = \frac{1}{3}t\cdot\frac{1}{2D}\cos 2t = \frac{1}{12}t\sin 2t.$

Hence $\quad c = \frac{1}{12}.$

PROBLEMS 14.2

1. An elastic string of natural length a is fixed at one end and a particle of mass m hangs freely from the other end. The modulus of elasticity is mg. The particle is pulled down a further distance l below its equilibrium position and released from rest. Show that the motion of the particle is simple harmonic and find the periodicity.
2. A mass of 4 lb suspended from a light elastic string of natural length 3 feet extends it to a distance 2 feet. One end of the string is fixed and a mass of 2 lb is attached to other. The mass is held so that the string is just unstretched and is then let go. Find the amplitude, the period and the maximum velocity of the ensuing simple harmonic motion.

3. A light elastic string of natural length l has one extremity fixed at a point A and the other end attached to a stone, the weight of which in equilibrium would extend the string to a depth l_1. Show that if the stone be dropped from rest at A, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position.
4. A 4 lb weight on a string stretches it 6 in. Assuming that a damping force in lb wt. equal to λ times the instantaneous velocity in ft/sec. acts on the weight, show that the motion is over damped, critically damped or oscillatory according as $\lambda > = < 2$. Find the period of oscillation when $\lambda = 1.5$.
5. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force 196,000 dynes. The spring is pulled 5 cm and released. Find the displacement t seconds after release if there be a damping force of 2000 dynes per cm per second.
6. A body weighing 16 lb is suspended by a spring in a fluid whose resistance in lb wt. is twice the speed of the body in ft/sec. A pull of 25 lb wt. would stretch the spring 3 inches. The body is drawn 3 inches below the equilibrium position in the fluid and then released. Find the period of oscillations and the time required for the damping factor to be reduced to one-tenth of its initial value. *(Sambhalpur, 1998)*
7. A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F \sin \omega t$ in the direction of its length. The force f is measured positive vertically downwards and at zero time M is at rest. If the spring stiffness is S, prove that the displacement of M at time t from the commencement of motion is given by

$$x = \frac{F}{M(p^2 - \omega^2)}\left[\sin \omega t - \frac{\omega}{p}\sin pt\right]$$

where $p^2 = S/M$ and damping effects are neglected. *(U.P.T.U., 2002)*

8. A vertical spring having 4.5 lb/ft. has 16 lb wt. suspended from it. An external force of $24 \sin 9t$ $(t \geq 0)$ lb wt. is applied. A damping force given numerically in lb. wt. by four times its velocity in ft/sec, is assumed to act. Initially the weight is at rest at its equilibrium position. Determine the position of the weight at any time. Also find the amplitude, period and the frequency of the steady-state solution.
9. A body weighing 4 lb hangs at rest on a spring producing in the spring an extension of 1ft. The upper end of the spring in now made to execute a vertical simple harmonic oscillation $x = \sin 4t$, x being measured vertically downwards in feet. If the body is subject to a frictional resistance whose magnitude in lb wt. is one-quarter of its velocity in feet per second, obtain the differential equation for the motion of the body and find the expression for its displacement at time t, when t is large.
10. A body executes damped forced vibrations given by the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2x = e^{-kt}\sin nt.$$

Solve the equation for both the cases, when $n^2 \neq b^2 - k^2$ and $n^2 = b^2 - k^2$. *(U.P.T.U., 2004)*

14.5 OSCILLATORY ELECTRICAL CIRCUIT

(i) L-C circuit

Consider an electrical circuit containing an inductance L and capacitance C (Fig. 14.17).

Let i be the current and q the charge in the condenser plate at any time t, so that the voltage drop across

$$L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$$

and the voltage drop across $C = q/C$.

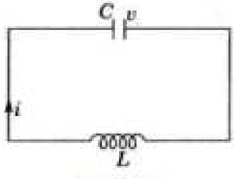


Fig. 14.17

As there is no applied e.m.f. in the circuit, therefore, by Kirchhoff's first law, we have

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$$

Or dividing by L and writing $1/LC = \mu^2$, we get $\dfrac{d^2q}{dt^2} + \mu^2 q = 0$ ...(1)

This equation is precisely same as (2) on page 507 and, therefore, *it represents free electrical oscillations of the current having period $2\pi/\mu = 2\pi\sqrt{LC}$.*

Thus the discharging of a condenser through an inductance L is same as the motion of the mass m at the end of a spring.

(*ii*) **L–C–R circuit**

Now consider the discharge of a condenser C through an inductanc L and the resistance R (Fig. 14.18). Since the voltage drop across L, C and R are respectively

$$L\frac{d^2q}{dt^2}, \frac{q}{C} \text{ and } R\frac{dq}{dt}.$$

$\therefore$ by Kirchhoff's law, we have $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = 0$ ...(2)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + 2\lambda\dfrac{dq}{dt} + \mu^2 q = 0$

This equation is same as (3) on page 507 and, therefore has the same solution as for the mass m on a spring with a damper.

Thus *the charging or discharging of a condenser through the resistance R and an inductance L is an electrical analogue of the damped oscillations of mass m on a spring.*

(*iii*) **L–C circuit with e.m.f.** = $p \cos nt$.

The equation (1) for an L–C circuit (Fig. 14.19), now becomes $L\dfrac{d^2q}{dt^2} + \dfrac{q}{C} = p \cos nt$.

Or writing $1/LC = \mu^2$, we have $\dfrac{d^2q}{dt^2} + \mu^2 q = \dfrac{p}{L}\cos nt$ ...(3)

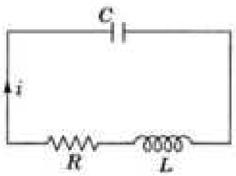


Fig. 14.18

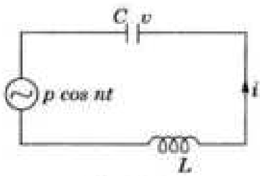


Fig. 14.19

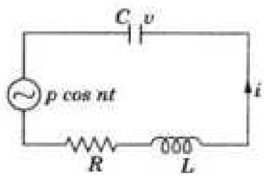


Fig. 14.20

This equation is of the same form as (9) on page 509 and, therefore, has the solution as for the motion of a mass m on a spring with external periodic force $p \cos nt$ acting on it.

Thus the condenser placed in series with source of e.m.f. (= p cos nt) and discharging through a coil containing inductance L is an electrical analogue of the forced oscillations of the mass m on a spring.

An electrical instance of *resonance* phenomena occurs while tuning a radio-station, for the natural frequency of the tuning of L–C circuit is made equal to the frequency of the desired radio-station, giving the maximum output of the receiver at the said receiving station.

(*iv*) **L–C–R circuit with e.m.f.** = $p \cos nt$.

The equation of (2) above, now becomes $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = p \cos nt$. (Fig. 14.20)

Or writing $R/L = 2\lambda$ and $1/LC = \mu^2$ as before, we have

$$\frac{d^2q}{dt^2} + 2\lambda\frac{dq}{dt} + \mu^2 q = \frac{p}{L}\cos nt \quad ...(4)$$

This equation is exactly same as (12) on page 510 and, therefore, its C.F. represents the free oscillations of the circuit whereas the P.I. represents the forced oscillations.

Here also as t increases, the free oscillations die out while the forced oscillations persist giving steady motion.

Thus the L–C–R circuit with a source of alternating e.m.f. is an electrical equivalent of the mechanical phenomena of forced oscillations with resistance.

14.6 ELECTRO-MECHANICAL ANALOGY

We have just seen, how merely by renaming the variables, the differential equation representing the oscillation of a weight on a spring represents an analogous electrical circuit. As electrical circuits are easy to assemble and the currents and voltages are accurately measured with ease, this affords a practical method of studying the oscillations of complicated mechanical systems which are expensive to make and unwieldy to handle by considering an equivalent electrical circuit. While making an electrical equivalent of a mechanical system, the following correspondences between the elements should be kept in mind, noting that the circuit may be in series or in parallels :

Mech. System	*Series circuit*	*Parallel circuit*
Displacement	Current i	Voltage E
Force or couple	Voltage E	Current i
Mass m or $M.I.$	Inductance L	Capacitance C
Damping force	Resistance R	Conductance $1/R$
Spring modulus	Elastance $1/C$	Susceptance $1/L$

Example 14.11. *An uncharged condenser of capacity C is charged by applying an e.m.f. $E \sin t/\sqrt{(LC)}$, through leads of self-inductance L and negligible resistance. Prove that at any time t, the charge on one of the plates is* $\frac{EC}{2}\left\{\sin\frac{t}{\sqrt{(LC)}} - \frac{t}{\sqrt{(LC)}}\cos\frac{t}{\sqrt{(LC)}}\right\}$ *(U.P.T.U., 2003)*

Solution. If q be the change on the condenser, the differential equation of the circuit is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E\sin\frac{t}{\sqrt{(LC)}} \quad ...(i)$$

Its A.E. is $LD^2 + 1/C = 0$ or $D = \pm 1/\sqrt{(LC)}$

$\therefore$ C.F. $= c_1 \cos t/\sqrt{LC} + c_2 \sin t/\sqrt{(LC)}$

and P.I. $= \dfrac{1}{LD^2 + \frac{1}{C}} E\sin\dfrac{t}{\sqrt{(LC)}}$ $\left[\text{Putting } D^2 = -\frac{1}{LC}, \text{denom.} = 0\right]$

$$= Et\frac{1}{2LD}\sin\frac{t}{\sqrt{(LC)}} = \frac{Et}{2L}\int \sin\frac{t}{\sqrt{(LC)}}dt = -\frac{Et}{2L}\sqrt{(LC)}\cos\frac{t}{\sqrt{(LC)}} = -\frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$$

Thus the C.S. of (*i*) is $q = c_1\cos\frac{t}{\sqrt{LC}} + c_2\sin\frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}}$

When $t = 0, q = 0, c_1 = 0$

$$\therefore \quad q = c_2\sin\frac{t}{\sqrt{LC}} - \frac{Et}{2}\sqrt{\frac{C}{L}}\cos\frac{t}{\sqrt{LC}} \quad ...(ii)$$

Differentiating (*ii*) w.r.t. t, we get

$$\frac{dq}{dt} = \frac{c_2}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}}\left\{\cos\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\sin\frac{t}{\sqrt{LC}}\right\}$$

Also when $t = 0$, $dq/dt = i = 0$,

$$\therefore \quad \frac{c_2}{\sqrt{LC}} - \frac{E}{2}\sqrt{\frac{C}{L}} = 0 \quad \text{or} \quad c_2 = \frac{EC}{2}.$$

Substituting the value of c_2 in (*ii*), q at any time t is given by

$$q = \frac{EC}{2}\left\{\sin\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\frac{t}{\sqrt{LC}}\right\}.$$

Example 14.12. *In an L—C—R circuit, the charge q on a plate of a condenser is given by*

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin pt.$$

The circuit is tuned to resonance so that $p^2 = 1/LC$. If initially the current i and the charge q be zero, show that, for small values of R/L, the current in the circuit at time t is given by

$$(Et/2L)\sin pt. \qquad \textit{(U.P.T.U., 2004)}$$

Solution. Given differential equation is $(LD^2 + RD + 1/C)q = E \sin pt$ ...(i)

Its auxiliary equation is $LD^2 + RD + 1/C = 0$,

which gives $$D = \frac{1}{2L}\left[-R \pm \sqrt{\left(R^2 - \frac{4L}{C}\right)}\right] = -\frac{R}{2L} + \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}$$

As R/L is small, therefore, to the first order in R/L,

$$D = -\frac{R}{2L} \pm i\frac{1}{\sqrt{(LC)}} = -\frac{R}{2L} \pm ip \qquad \left[\because\ p^2 = \frac{1}{LC}\right]$$

$\therefore$ $$\text{C.F.} = e^{-(Rt/2L)}(c_1 \cos pt + c_2 \sin pt)$$
$$= (1 - Rt/2L)(c_1 \cos pt + c_2 \sin pt) \text{ rejecting terms in } (R/L)^2 \text{ etc.}$$

and $$\text{P.I.} = \frac{1}{LD^2 + RD + 1/C} E \sin pt = E\frac{1}{-Lp^2 + RD + 1/C} \sin pt$$
$$= \frac{E}{R}\int \sin pt\, dt = -\frac{E}{Rp} \cos pt \qquad \left[\because p^2 = \frac{1}{LC}\right]$$

Thus the complete solution of (i) is $q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \frac{E}{Rp}\cos pt$ ...(ii)

$\therefore$ $$i = \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin pt + c_2 \cos pt)p - \frac{R}{2L}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R}\sin pt \quad ...(iii)$$

Initially, when $t = 0, q = 0, i = 0$ $\therefore$ from (ii), $0 = c_1 - E/Rp$ $\therefore$ $c_1 = E/Rp$ and from (iii),

$0 = c_2 p - Rc_1/2L$ $\therefore$ $c_2 = Rc_1/2Lp = E/2Lp^2$

Thus, substituting these values of c_1 and c_2 in (iii), we get

$$i = \left(1 - \frac{Rt}{2L}\right)\left(-\frac{E}{Rp}\sin pt + \frac{E}{2Lp^2}\cos pt\right)p - \frac{R}{2L}\left(\frac{E}{Rp}\cos pt + \frac{E}{2Lp^2}\sin pt\right) + \frac{E}{R}\sin pt$$
$$= \frac{Et}{2L}\sin pt. \qquad [\because\ R/L \text{ is small}]$$

PROBLEMS 14.3

1. Show that the frequency of free vibrations in a closed electrical circuit with inductance L and capacity C in series is $\frac{30}{\pi\sqrt{(LC)}}$ per minute.
2. The differential equation for a circuit in which self-inductance and capacitance neutralize each other is $L\frac{d^2i}{dt^2} + \frac{i}{C} = 0$. Find the current i as a function of t given that I is the maximum current, and $i = 0$ when $t = 0$.
3. A constant e.m.f. E at $t = 0$ is applied to a circuit consisting of inductance L, resistance R and capacitance C in series. The initial values of the current and the charge being zero, find the current at any time t, if $CR^2 < 4L$. Show that the amplitudes of the successive vibrations are in geometrical progression.
4. The damped LCR circuit is governed by the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$ where, L, R, C are positive constants. Find the conditions under which the circuit is over damped, under damped and critically damped. Find also the critical resistance. (U.P.T.U., 2005)
5. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$. Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2 \times 10^{-6}$ farads, and that when $t = 0$, charge q is 0.002 coulombs and the current $dq/dt = 0$, obtain the value of q in terms of t.
6. An e.m.f. $E \sin pt$ is applied at $t = 0$ to a circuit containing a capacitance C and inductance L. The current i satisfies the equation $L\frac{di}{dt} + \frac{1}{C}\int i\, dt = E \sin pt$. If $p^2 = 1/LC$ and initially the current i and the charge q are zero, show that the current at time t is $(Et/2L) \sin pt$, where $i = dq/dt$.

7. For an L—R—C circuit, the charge q on a plate of the condenser is given by $L\dfrac{d^2q}{dt^2} + R\dfrac{dq}{dt} + \dfrac{q}{C} = E \sin \omega t$, where $i = \dfrac{dq}{dt}$. The circuit is tuned to resonance so that $\omega^2 = 1/LC$.

   If $CR^2 < 4L$ and initially $q = 0$, $i = 0$, show that $q = \dfrac{E}{R\omega}\left[e^{-Rt/2L}\left(\cos pt + \dfrac{R}{2Lp}\sin pt\right) - \cos \omega t\right]$

   where $$p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}.$$ (U.P.T.U., 2003)

8. An alternating E.M.F. $E \sin pt$ is applied to a circuit at $t = 0$. Given the equation for the current i as $L\dfrac{d^2i}{dt^2} + R\dfrac{di}{dt} + \dfrac{i}{C} = pE \cos pt$, find the current i when (i) $CR^2 > 4L$, (ii) $CR^2 < 4L$.

14.7 DEFLECTION OF BEAMS

Consider a uniform beam as made up of fibres running lengthwise. We have to find its deflection under given loadings.

In the bent form, the fibres of the lower half are stretched and those of upper half are compressed. In between these two, there is a layer of unstrained fibres called the *neutral surface*. The fibre which was initially along the x-axis (the central horizontal axis of the beam) now lies in the neutral surface, in the form of a curve called the *deflection curve* or the *elastic curve*. We shall encounter differential equations while finding the equation of this curve.

Consider a cross-section of the beam cutting the elastic curve in P and the neutral surface in the line AA'—called the neutral axis of this section (Fig. 14.21).

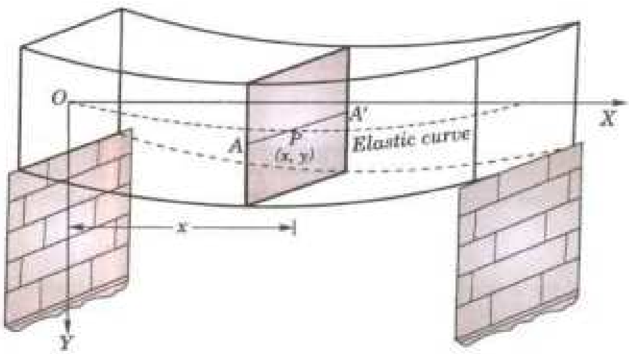


Fig. 14.21

It is well-known from mechanics that the bending moment M about AA', of all forces acting on either side of the two portions of the beam separated by this cross-section, is given by the *Bernoulli-Euler law*

$$M = EI/R$$

where E = modulus of elasticity of the beam,

I = moment of inertia of the cross-section about AA',

and R = radius of curvature of the elastic curve at $P(x, y)$.

If the deflection of the beam is small, the slope of the elastic curve is also small so that we may neglect $(dy/dx)^2$ in the formula,

$$R = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}.$$ Thus for small deflections, $R = 1/(d^2y/dx^2)$.

Hence (1) *Bending moment* $\mathbf{M} = \mathbf{EI}\,\dfrac{\mathbf{d^2y}}{\mathbf{dx^2}}$

(2) *Shear force* $\left(=\frac{dM}{dx}\right) = EI\,\frac{d^3y}{dx^3}$;

(3) *Intensity of loading* $\left(=\frac{d^2M}{dx^2}\right) = EI\,\frac{d^4y}{dx^4}$

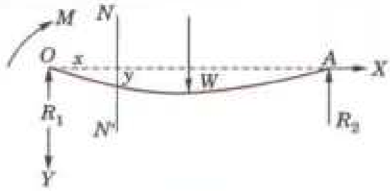


Fig. 14.22

(4) *Convention of signs.* The sum of the moments about a section NN' due to external forces on the left of the section, if anti-clockwise is taken as positive and if clockwise (as in Fig. 14.22) is taken as negative.

The deflection y downwards and length x to the right are taken as positive. The slope dy/dx will be positive if downwards in the direction of x-positive.

(5) *End conditions.* The arbitrary constants appearing in the solution of the differential equation (1) for a given problem are found from the following end conditions :

(*i*) *At a freely supported end* (Fig. 14.23), there being no deflection and no bending moment, we have $y = 0$ and $d^2y/dx^2 = 0$.

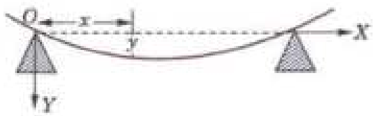


Fig. 14.23

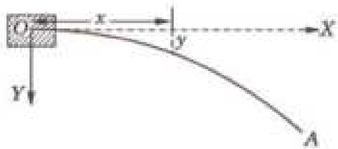


Fig. 14.24

(*ii*) *At a (horizontal) fixed end* (Fig. 14.24), the deflection and the slope of the beam being both zero, we have

$$y = 0 \text{ and } dy/dx = 0.$$

(*ii*) *At a perfectly free end* (*A* in Fig. 14.24), there being no bending moment or shear force, we have

$$\frac{d^2y}{dx^2} = 0 \quad \text{and} \quad \frac{d^3y}{dx^3} = 0$$

(6) *A member of a structure or a machine when subjected to end thrusts only is called a* **strut** *and a vertical strut is called a* **column**.

There are four possible ways of the end fixation of a strut:

(*i*) Both ends fixed, called a *built-in* or *encastre* strut.

(*ii*) One end fixed and the other freely supported, hinged or pin-jointed.

(*iii*) One end fixed and the other end free, called a *cantilever.*

(*iv*) Both ends freely supported or pin-jointed.

Example 14.13. *The deflection of a strut of length l with one end* ($x = 0$) *built-in and the other supported and subjected to end thrust P, satisfies the equation*

$$\frac{d^2y}{dx^2} + a^2y = \frac{a^2R}{P}(l - x).$$

Prove that the deflection curve is $y = \frac{R}{P}\left(\frac{\sin ax}{a} - l\cos ax + l - x\right)$, *where* $al = \tan al$.

(U.P.T.U., 2001)

Solution. Given differential equation is $(D^2 + a^2)y = \frac{a^2R}{P}(l - x)$ ...(*i*)

Its auxiliary equation is $D^2 + a^2 = 0$, whence $D = \pm\, ai$.

$$\therefore \quad C.F. = \frac{1}{D^2 + a^2}\,\frac{a^2R}{P}(l - x) = \frac{R}{P}\left(1 + \frac{D^2}{a^2}\right)^{-1}(l - x)$$

$$= \frac{R}{P}\left(1 - \frac{D^2}{a^2} + \dots\right)(l - x) = \frac{R}{P}(l - x)$$

Thus the complete solution of (*i*) is $y = c_1 \cos ax + c_2 \sin ax + \frac{R}{P}(l - x)$ ...(*ii*)

Also $$\frac{dy}{dx} = -c_1 a \sin ax + c_2 a \cos ax - \frac{R}{P}$$ ...(*iii*)

Now as the end O is built in (Fig. 14.25). $\therefore y = dy/dx = 0$ at $x = 0$.

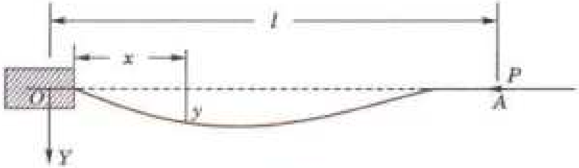


Fig. 14.25

$\therefore$ from (*ii*) and (*iii*), we have

$$0 = c_1 + Rl/P \text{ and } 0 = c_2 a - R/P$$

whence $$c_1 = -Rl/P \quad \text{and} \quad c_2 = R/aP$$

Thus (*ii*) becomes $$y = \frac{R}{P}\left(\frac{\sin ax}{a} - l\cos ax + l - x\right)$$ ...(*iv*)

which is the desired equation of the deflection curve.

The end A being freely supported $y = 0$ when $x = l$ (We don't need the other condition $d^2y/dx^2 = 0$].

$\therefore$ (*iv*) gives $$0 = \frac{R}{P}\left(\frac{\sin al}{a} - l\cos al\right) \text{ whence } al = \tan al.$$

Example 14.14. *A horizontal tie-rod is freely pinned at each end. It carries a uniform load w lb per unit length and has a horizontal pull P. Find the central deflection and the maximum bending moment, taking the origin at one of its ends.*

Solution. Let OA be the given beam of length l (Fig.14.26).

At each end there is a vertical reaction $R = wl/2$.

The external forces acting to the left of the section NN' are : (*i*) the horizontal pull P, (*ii*) the reaction $R = wl/2$ and (*iii*) the weight of the portion $ON = wx$ acting mid-way.

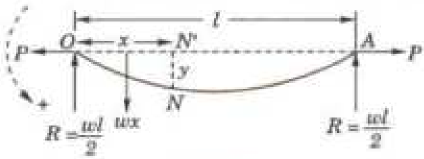


Fig. 14.26

Taking moments about, N, we have

$$EI\frac{d^2y}{dx^2} = Py - \frac{wl}{2}x + wx \cdot \frac{x}{2}$$

or $$EI\frac{d^2y}{dx^2} - Py = \frac{w}{2}(x^2 - lx) \quad \text{or} \quad \frac{d^2y}{dx^2} - a^2y = \frac{w}{2EI}(x^2 - lx), \text{ where } a^2 = \frac{P}{EI}$$ ...(*i*)

This is the differential equation of the elastic curve. Its auxiliary equation is $D^2 - a^2 = 0$, whence $D = \pm a$.

$\therefore$ $$\text{C.F.} = c_1 \cosh ax + c_2 \sinh ax$$

and $$\text{P.I.} = \frac{1}{D^2 - a^2}\frac{w}{2EI}(x^2 - lx) = \frac{-w}{2EIa^2}\left(1 - \frac{D^2}{a^2}\right)^{-1}(x^2 - lx)$$

$$= -\frac{w}{2P}\left(1 + \frac{D^2}{a^2}\dots\right)(x^2 - lx) = -\frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right).$$

Thus the complete solution of (*i*) is $y = c_1 \cosh ax + c_2 \sinh ax - \frac{W}{2P}\left(x^2 - lx + \frac{2}{a^2}\right)$ ...(*ii*)

At the end O, $y = 0$ when $x = 0$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (ii) gives $0 = c_1 - w/Pa^2$, or $c_1 = w/Pa^2$ ...(iii)

At the end A, $y = 0$ when $x = l$, [We don't need the other condition $d^2y/dx^2 = 0$]

$\therefore$ (ii) gives $0 = c_1 \cosh al + c_2 \sinh al - w/Pa^2$ or $c_2 \sinh al = \dfrac{W}{Pa^2}(1 - \cosh al)$

whence $$c_2 = -\frac{w}{Pa^2}\tanh\frac{al}{2} \qquad ...(iv)$$

Substituting these values of c_1 and c_2 in (ii), we get

$$y = \frac{w}{Pa^2}\left(\cosh ax - \tanh\frac{al}{2}\sinh ax\right) - \frac{w}{2P}\left(x^2 - lx + \frac{2}{a^2}\right)$$

which gives the deflection of the beam at N.

Thus the central deflection $= y$ (at $x = l/2$)

$$= \frac{w}{Pa^2}\left(\cosh\frac{al}{2} - \tanh\frac{al}{2}\sinh\frac{al}{2} - 1\right) + \frac{wl^2}{8P} = \frac{w}{Pa^2}\left(\text{sech}\frac{al}{2} - 1\right) + \frac{wl^2}{8P}$$

Also the bending moment is maximum at the point of maximum deflection ($x = l/2$).

$\therefore$ *The maximum bending moment*

$$= EI\frac{d^2y}{dx^2}(\text{at } x = l/2) = Py + \frac{w}{2}(x^2 - lx)\ (\text{at } x = l/2) = \frac{w}{a}\left(\text{sech}\frac{al}{2} - 1\right)$$

Example 14.15. *A cantilever beam of length l and weighing w lb/unit is subjected to a horizontal compressive force P applied at the free end. Taking the origin at the free end and y-axis upwards, establish the differential equation of the beam and hence find the maximum deflection.*

Solution. Let $N(x, y)$ be any point of the beam referred to axes through the free end as shown (Fig. 14.27).

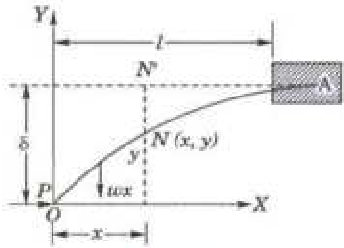


Fig. 14.27

The external forces acting to the left of the section NN', are

(i) the compressive force P,

(ii) the weight of the portion $ON = wx$ acting midway.

$\therefore$ Taking moments about N, we get $EI\dfrac{d^2y}{dx^2} = -Py - wx \cdot \dfrac{x}{2}$

or $$EI\frac{d^2y}{dx^2} + Py = -\frac{wx^2}{2} \qquad ...(i)$$

which is the desired differential equation.

Dividing by EI and taking $P/EI = n^2$, we get

$$\frac{d^2y}{dx^2} + n^2y = -\frac{wn^2}{2P}\cdot x^2$$

Its auxiliary equation is $D^2 + n^2 = 0$, whence $D = \pm ni$.

$$\text{C.F.} = c_1 \cos nx + c_2 \sin nx$$

$\therefore$ $$\text{P.I.} = \frac{1}{D^2 + n^2}\left(-\frac{wn^2}{2P}x^2\right) = -\frac{w}{2P}\left(1 + \frac{D^2}{n^2}\right)^{-1}x^2 = -\frac{w}{2P}\left(1 - \frac{D^2}{n^2} + \ldots\right)x^2 = \frac{w}{2P}\left(\frac{2}{n^2} - x^2\right)$$

Thus the complete solution of (i) is $y = c_1 \cos nx + c_2 \sin nx + \dfrac{w}{2P}\left(\dfrac{2}{n^2} - x^2\right)$ ...(ii)

The boundary conditions at the fixed end are

$x = l$, $y = \delta$, the maximum deflection and $dy/dx = 0$.

Using the first condition (*i.e.* $y = \delta$, when $x = l$), (ii) gives

$$\delta = c_1 \cos nl + c_2 \sin nl + \frac{w}{2P}\left(\frac{2}{n^2} - l^2\right) \qquad ...(iii)$$

Differentiating (*ii*), we get $\frac{dy}{dx} = n(-c_1 \sin nx + c_2 \cos nx) - \frac{wx}{P}$.

Applying the second condition, it gives $0 = n(-c_1 \sin nl + c_2 \cos nl) - wl/P$ ...(*iv*)

Also imposing the boundary condition for the free end (*i.e.* $x = 0$, $d^2y/dx^2 = 0$) on

$$\frac{d^2y}{dx^2} = -n^2(c_1 \cos nx + c_2 \sin nx) - \frac{w}{P},$$

we get $$0 = -n^2c_1 - w/P, \text{ i.e., } c_1 = -w/Pn^2.$$

Substituting this value of c_1 in (*iv*), we get $c_2 = \frac{wl}{Pn} \sec nl - \frac{w}{Pn^2} \tan nl$

Thus, substituting the values of c_1 and c_2 in (*iii*), we get

the *maximum deflection* $\delta = \frac{w}{Pn^2}\left(1 - \frac{l^2n^2}{2} - \sec nl + nl \tan nl\right)$.

14.8 WHIRLING OF SHAFTS

(1) Critical or whirling speeds. A shaft seldom rotates about its geometrical axis for there is always some non-symmetrical crookedness in the shaft. In fact, the dead weight of the shaft causes some deflection which tends to become large at certain speeds. Such speeds at which the deflection of the shaft reaches a stage, where the shaft will fracture unless the speed is lowered are called the *critical or whirling speeds* of the shaft.

(2) Differential equation of the rotating shaft. Consider a shaft of weight W per unit length which is rotating with angular velocity ω.

Take its original horizontal position and the vertical downwards through the end O as the axes of x and y (Fig. 14.28). We know that for a uniformly loaded beam, the intensity of loading at $P(x, y) = EI\, d^4y/dx^4$.

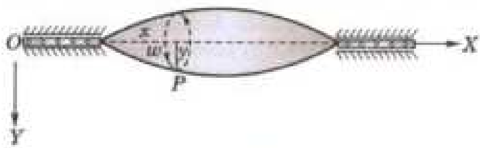


Fig. 14.28

$\therefore$ the restoring force (*i.e.* the internal action to oppose bending at $P(x, y) = EI\, d^4y/dx^4$.

Also the centrifugal force per unit length at $P = mr\omega^2$, *i.e.* $\frac{Wy}{g}\omega^2$.

As the restoring force arising out of the rigidity or stiffness of the shaft balances the centrifugal force which causes further deflection.

$$\therefore \qquad EI\frac{d^4y}{dx^4} = \frac{W}{g}y\omega^2 \quad \text{or} \quad \frac{d^4y}{dx^4} - a^4y = 0, \text{ where } a^4 = \frac{W\omega^2}{gEI}$$

which is the desired differential equation.

Its auxiliary equation being $D^4 - a^4 = 0$, we have

$$D = \pm a, \pm ai.$$

Hence its solution is $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$ which may be put in the form

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax.$$

(3) End conditions. To determine the arbitrary constants A, B, C, D we use the following end conditions :

(*i*) *At an end in a short or flexible bearings* (Fig. 14.29), there being no deflection and also no bending moment, we have

$$y = 0 \text{ and } \frac{d^2y}{dx^2} = 0.$$

Fig. 14.29

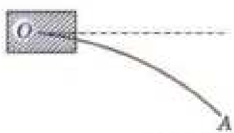


Fig. 14.30

(*ii*) *At an end in long or fixed bearings* (Fig 14.30), the deflection and the slope of the shaft being both zero, we have

$$y = 0 \text{ and } \frac{dy}{dx} = 0.$$

(*ii*) *At a perfectly free end* (such as A in Fig. 14.30), there being no bending moment and no shear force, we have

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = 0.$$

Example 14.16. *The differential equation for the displacement y of a whirling shaft when the weight of the shaft is taken into account is*

$$EI\frac{d^4y}{dx^4} - \frac{W\omega^2}{g}y = W.$$

Taking the shaft of length $2l$ with the origin at the centre and short bearings at both ends, show that the maximum deflection of the shaft is

$$\frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

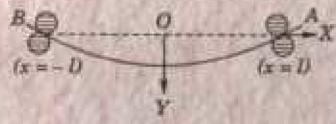


Fig. 14.31

Solution. Given differential equation can be written as

$$\frac{d^4y}{dx^4} - a^4y = \frac{W}{EI}, \text{ where } a^4 = \frac{W\omega^2}{EIg} \quad ...(i)$$

Its C.F. $= A \cosh ax + B \sinh ax + C \cos ax + D \sin ax$

and P.I. $= \dfrac{1}{D^4 - a^4} \cdot \dfrac{W}{EI} = \dfrac{W}{EI} \cdot \dfrac{1}{D^4 - a^4} e^{0 \cdot x} = -\dfrac{W}{EIa^4} = -\dfrac{g}{\omega^2}$

Thus the complete solution of (*i*) is

$$y = A \cosh ax + B \sinh ax + C \cos ax + D \sin ax - \frac{g}{\omega^2} \quad ...(ii)$$

Differentiating it twice, we get

$$\frac{1}{a}\frac{dy}{dx} = A \sinh ax + B \cosh ax - C \sin ax + D \cos ax$$

$$\frac{1}{a^2}\frac{d^2y}{dx^2} = A \cosh ax + B \sinh ax - C \cos ax - D \sin ax \quad ...(iii)$$

As the end A of the shaft is in short bearings (Fig. 14.31)

$\therefore$ when $x = l$; $y = 0$, $d^2y/dx^2 = 0$

$\therefore$ from (*ii*) and (*iii*), we have

$$0 = A \cosh al + B \sinh al + C \cos al + D \sin al - \frac{g}{\omega^2} \quad ...(iv)$$

$$0 = A \cosh al + B \sinh al - C \cos al - D \sin al \quad ...(v)$$

Similarly *at the end B*, $x = -l$, $y = 0$, $d^2y/dx^2 = 0$.

$\therefore$ from (*ii*) and (*iii*), we get

$$0 = A \cosh al - B \sinh al + C \cos al - D \sin al - \frac{g}{\omega^2} \quad ...(vi)$$

$$0 = A \cosh al - B \sinh al - C \cos al + D \sin al \quad ...(vii)$$

Adding (*iv*) and (*vi*), and (*v*) and (*vii*), we get

$$A \cosh al + C \cos al = \frac{g}{\omega^2} \quad \text{and} \quad A \cosh al - C \cos al = 0.$$

whence $\quad A = \dfrac{g}{2\omega^2 \cosh al}$ and $C = \dfrac{g}{2\omega^2 \cos al}$

Again subtracting (*vi*) from (*iv*) and (*vii*) from (*v*), we get

$$D \sinh al + D \sin al = 0 \text{ and } B \sinh al - D \sin al = 0, \text{ whence } B = 0 \text{ and } D = 0.$$

Substituting the values of A, B, C and D in (*ii*), we get

$$y = \frac{g}{2\omega^2}\left[\frac{\cosh ax}{\cosh al} + \frac{\cos ax}{\cos al} - 2\right]$$

Thus the maximum deflection = value of y at the centre (x = 0)

$$= \frac{g}{2\omega^2}(\text{sech } al + \sec al - 2).$$

Example 14.17. *The whirling speed of a shaft of length l is given by*

$$\frac{d^4 y}{dx^4} - m^4 y = 0 \text{ where } m^4 = \frac{W\omega^2}{gEI},$$

and y is the displacement at distance x from one end. If the ends of the shaft are constrained in long bearings, show that the shaft will whirl when cos ml cosh ml = 1.

Solution. The solution of the given differential equation is

$$y = A \cosh mx + B \sinh mx + C \cos mx + D \sin mx \quad \ldots(i)$$

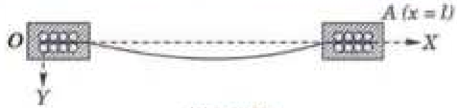


Fig. 14.32

which on differentiation gives,

$$\frac{1}{m}\frac{dy}{dx} = A \sinh mx + B \cosh mx - C \sin mx + D \cos mx \quad \ldots(ii)$$

As the end O of the shaft is fixed in long bearings (Fig. 14.32).

$\therefore$ when $x = 0$, $y = 0$, $dy/dx = 0$,

$\therefore$ from (*i*) and (*ii*), we have

$$0 = A + C \quad \text{or} \quad C = -A \quad \ldots(iii)$$

and

$$0 = B + D \quad \text{or} \quad D = -B \quad \ldots(iv)$$

Similarly, at the end A, $x = l$, $y = 0$, $dy/dx = 0$.

$\therefore$ From (*i*) and (*ii*), we have

$$0 = A \cosh ml + B \sinh ml + C \cos ml + D \sin ml \quad \ldots(v)$$

$$0 = A \sinh ml + B \cosh ml - C \sin ml + D \cos ml \quad \ldots(vi)$$

Substituting the values of C and D in (*v*) and (*vi*), we get

$$A(\cosh ml - \cos ml) + B(\sinh ml - \sin ml) = 0$$

and

$$A(\sinh ml + \sin ml) + B(\cosh ml - \cos ml) = 0$$

Eliminating A and B from these equations, we get

$$\frac{\cosh ml - \cos ml}{\sinh ml - \sin ml} = -\frac{B}{A} = \frac{\sinh ml + \sin ml}{\cosh ml - \cos ml}$$

or

$$\cosh^2 ml - 2\cosh ml \cos ml + \cos^2 ml = \sinh^2 ml - \sin^2 ml$$

or

$$-2\cosh ml \cos ml + 2 = 0 \text{ or } \cos ml \cosh ml = 1$$

which must be satisfied when the shaft whirls.

The solution of this equation gives $ml = 4.73 = 3\pi/2$ radians approximately.

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\, l^2 = m^2 l^2 = \frac{9\pi^2}{4}$$

Thus the *whirling speed of a shaft with ends in long bearings.*

$$= \omega = \frac{9\pi^2}{4l^2}\sqrt{\left(\frac{gEI}{W}\right)} \text{ approximately.}$$

Obs. 1. *When the shaft has one long bearing and the other short bearing,* the condition to be satisfied is $\tan ml = \tanh ml$, of which the solution is $ml = 3.927$

or $$\omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2l^2 = (3.927)^2 = 15.4 \text{ nearly.}$$

Thus *the whirling speed* $= \omega = \dfrac{15.4}{l^2}\sqrt{\left(\dfrac{gEI}{W}\right)}$

Obs. 2. *When the shaft has both short bearings,* the condition to be satisfied is $\sin ml = 0$ *i.e.* $ml = \pi$ (least non-zero value).

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2 l^2 = \pi^2. \text{ Thus the whirling speed } = \omega = \frac{\pi^2}{l^2}\sqrt{\left(\frac{gEI}{W}\right)}.$$

Obs. 3. *When the shaft has one long bearing,* the condition to the satisfied is $\cos ml \cosh ml = -1$.

Its solution gives $ml = 1.865$ [See Example 1.25]

$$\therefore \quad \omega\sqrt{\left(\frac{W}{gEI}\right)}\cdot l^2 = m^2l^2 = (1.865)^2 = 3.5 \text{ nearly. Thus the whirling speed } \omega = \frac{3.5}{l^2}\sqrt{\left(\frac{gEI}{W}\right)}.$$

PROBLEMS 14.4

1. A horizontal tie-rod of length $2l$ with concentrated load W at the centre and ends freely hinged, satisfies the differential equation $EI\dfrac{d^2y}{dx^2} = Py - \dfrac{W}{2}x$. With conditions $x = 0, y = 0$ and $x = l$, $dy/dx = 0$, prove that the deflection δ and the bending moment M at the centre $(x = l)$ are given by $\delta = \dfrac{W}{2Pn}(nl - \tanh nl)$ and $M = -\dfrac{W}{2n}\tanh nl$, where $n^2EI = P$.
2. A light horizontal strut AB is freely pinned at A and B. It is under the action of equal and opposite compressive forces P at its ends and it carries a load W at its centre. Then for $0 < x < l/2$, $EI\dfrac{d^2y}{dx^2} + Py + \dfrac{1}{2}Wx = 0$. Also $y = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.

   Prove that $y = \dfrac{W}{2P}\left(\dfrac{\sin nx}{n\cos nl/2} - x\right)$ where $n^2 = \dfrac{P}{EI}$.
3. A uniform horizontal strut of length l freely supported at both ends, carries a uniformly distributed load W per unit length. If the thrust at each end is, P, prove that the maximum deflection is $\dfrac{W}{Pa^2}\left(\sec\dfrac{al}{2} - 1\right) - \dfrac{Wl^2}{8P}$, where $\dfrac{P}{EI} = a^2$.

   Prove also that the maximum bending moment is of the magnitude $\dfrac{W}{a^2}\left(\sec\dfrac{al}{2} - 1\right)$.
4. The shape of a strut of length l subjected to an end thrust P and lateral load w per unit length, when the ends are built in, is given by $EI\dfrac{d^2y}{dx^2} + Py = \dfrac{wx^2}{2} - \dfrac{wlx}{2} + M$, where M is the moment at a fixed end. Find y in terms of x, given that $y = 0$, $dy/dx = 0$ at $x = 0$ and $dy/dx = 0$ at $x = l/2$.
5. A light horizontal strut of length l is clamped at one end carries a vertical load W at the free end. If the horizontally thrust at the free end is P, show that the strut satisfies the differential equation

   $EI\dfrac{d^2y}{dx^2} = (\delta - y)P + W(l - x)$, where y is the displacement of a point at a distance x from the fixed end and δ, the deflection at the free end.

   Prove that the deflection at the free end is given by $\dfrac{W}{nP}(\tan nl - nl)$, where $n^2EI = P$.
6. A long column fixed at one end $(x = 0)$ and hinged at the other $(x = l)$ is under the action of axial load P. If a force F is applied laterally at the hinge to prevent lateral movement, show that it satisfies the equation $\dfrac{d^2y}{dx^2} + n^2y = \dfrac{En^2}{P}(l - x)$, where $EIn^2 = P$. Hence determine the equation of the deflection curve.
7. A long column of length l is fixed at one end and is completely free at the other end. If y is the lateral deflection at a point distance x from the fixed end, when load P is axially applied, find the differential equation satisfied by x and y. Show that the deflection curve is given by $y = a\,[1 - \cos\sqrt{P/EI}\,x]$ and find the least value of the critical load (a is the lateral deflection of the free end).

8. The differential equation for the displacement y of a heavy whirling shaft is $\frac{d^4y}{dx^4} = a^4\left(y + \frac{g}{\omega^2}\right)$, where $a^4 = \frac{W\omega^2}{gEI}$. If both ends are in short bearings, the ends being $x = 0$ and $x = l$, find the bending moment of the centre of the shaft.

14.9 APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS

So far we have considered engineering systems having only one degree of freedom. The analysis of a system having more than one degree of freedom depends on the solution of simultaneous linear equations. In fact such equations form the basis of the theory of projectiles and the coupled circuits having self and mutual inductance. The details of such applications are best explained through the following examples :

Example 14.18. Projectile with resistance. *Find the path of a particle projected with a velocity v at an angle α to the horizon in a medium whose resistance, apart from gravity, varies as velocity. Also find the greatest height attained.*

Solution. Let the axes of x and y be respectively horizontal and vertical with origin at the point of projection (Fig. 14.33).

Let $P(x, y)$ be the position of the projectile at the time t, where the velocity components parallel to the axes are

$$v_x = \frac{dx}{dt}, v_y = \frac{dy}{dt}$$

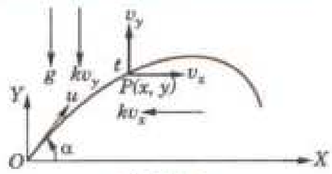


Fig. 14.33

∴ the equations of motion are:

Parallel to x-axis

$$m\frac{dv_x}{dt} = -mkv_x$$

or $$\frac{dv_x}{dt} = -kv_x$$

Parallel to y-axis

$$m\frac{dv_y}{dt} = -mg - mkv_y$$

$$\frac{dv_y}{dt} = -(g + kv_y)$$

Separating the variables and integrating, we have

$$\int\frac{dv_x}{v_x} = -k\int dt + c_1 \qquad \Bigg| \qquad \frac{dv_y}{g + kv_y} = -\int dt + c_2$$

or $$\log v_x = -kt + c_1 \qquad \Bigg| \qquad \frac{1}{k}\log(g + kv_y) = -t + c_2.$$

Initially when $t = 0$, $v_x = u\cos\alpha$, $v_y = u\sin\alpha$.

$$\log u\cos\alpha = c_1 \qquad \Bigg| \qquad \frac{1}{k}\log(g + ku\sin\alpha) = c_2$$

Subtracting,

$$\log\left(\frac{v_x}{u\cos\alpha}\right) = -kt \qquad \Bigg| \qquad \frac{1}{k}\log\left(\frac{g + kv_y}{g + ku\sin\alpha}\right) = -t$$

or $$\frac{dx}{dt} = v_x = u\cos\alpha e^{-kt} \quad ...(i) \qquad \Bigg| \qquad \frac{dy}{dt} = v_y = \frac{1}{k}[(g + ku\sin\alpha)e^{-kt} - g] \quad ...(ii)$$

Again integrating, we get

$$x = \frac{u\cos\alpha}{-k}e^{-kt} + c_3, \; y = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right)e^{-kt} - \frac{g}{k}t + c_4$$

Initially when $t = 0$, $x = 0$, $y = 0$.

∴ $$0 = \frac{u\cos\alpha}{k} + c_3, \; 0 = -\frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right) + c_4$$

Subtracting, we get $$x = \frac{u\cos\alpha}{k}(1 - e^{-kt}) \qquad ...(iii)$$

$$y = \frac{1}{k}\left(\frac{g}{k} + u \sin \alpha\right)\left(1 - e^{-kt}\right) - \frac{gt}{k} \qquad ...(iv)$$

Eliminating t from (*iii*) and (*iv*), we obtain $y = \left(\frac{g}{k} + u \sin \alpha\right)\frac{x}{u \cos \alpha} + \frac{g}{k^2} \log\left(1 - \frac{kx}{u \cos \alpha}\right)$

which is the required equation of the trajectory.

The projectile will attain the greatest height when $dy/dt = 0$.

i.e., when $e^{-kt} = g/(g + ku \sin \alpha)$, *i.e.*, at time $t = \frac{1}{k} \log\left(1 + \frac{ku \sin \alpha}{g}\right)$. [From (*ii*)]

Subtituting the value of t in (iv), we get the greatest height attained

$$(= y) = \frac{u \sin \alpha}{k} - \frac{g}{k^2} \log\left(1 + \frac{ku \sin \alpha}{g}\right).$$

Example 14.19. *Two particles each of mass m gm are suspended from two springs of same stiffness k as in Fig. 14.34. After the system comes to rest, the lower mass is pulled l cm downwards and released. Discuss their motion.*

Solution. Let x and y denote the displacement of the upper and lower masses at time t from their respective positions of equilibrium.

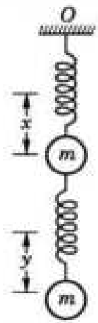


Fig. 14.34

Then the stretch of the upper spring is x and that of the lower spring is $y - x$.

$\therefore$ the restoring force acting on the upper mass

$$= -kx + k(y - x) = k(y - 2x)$$

and that on the lower mass $= -k(y - x)$.

Thus their equations of motion are

$$m\,\frac{d^2x}{dt^2} = k(y - 2x) \text{ and } m\,\frac{d^2y}{dt^2} = -k(y - x)$$

or $\quad (mD^2 + 2k)x - ky = 0 \qquad ...(i)$

and $\quad (mD^2 + k)y - kx = 0 \qquad ...(ii)$

Operating (*i*) by $(mD^2 + k)$ and adding to k times (*ii*), we get

$[(mD^2 + k)(mD^2 + 2k) - k^2]\,x = 0$ or $(D^4 + 3\lambda D^2 + \lambda^2)x = 0$, where $\lambda^2 = k/m$.

Its auxiliary equation is $D^4 + 3\lambda D^2 + \lambda^2 = 0$

which gives $\quad D^2 = \dfrac{-3\lambda \pm \sqrt{(9\lambda^2 - 4\lambda^2)}}{2} = -2.62\lambda$ or $-0.38\lambda = -\alpha^2, -\beta^2$ (say)

so that $D = \pm i\alpha, \pm i\beta$.

Thus $\quad x = c_1 \cos \alpha t + c_2 \sin \alpha t + c_3 \cos \beta t + c_4 \sin \beta t \qquad ...(iii)$

Also from (*i*), $y = \left(\frac{D^2}{\lambda} + 2\right)x = (2 - \alpha^2/\lambda)(c_1 \cos \alpha t + c_2 \sin \alpha t) + (2 - \beta^2/\lambda)(c_3 \cos \beta t + c_4 \sin \beta t) \qquad ...(iv)$

Initially when $\quad t = 0, x = y = l, dx/dt = dy/dt = 0$.

$\therefore$ from (*iii*), $\quad l = c_1 + c_3;\ 0 = \alpha c_2 + \beta c_4$

and from (*iv*) $\quad l = (2 - \alpha^2/\lambda)c_1 + (2 - \beta^2/\lambda)c_3$ and $0 = (2 - \alpha^2/\lambda)\alpha c_2 + (2 - \beta^2/\lambda)\beta c_4$

whence $\quad c_1 = \dfrac{l(\lambda - \beta^2)}{\alpha^2 - \beta^2},\ c_3 = \dfrac{l(\lambda - \alpha^2)}{\beta^2 - \alpha^2},\ c_3 = c_4 = 0.$

Substituting these values of constants in (*iii*) and (*iv*), we get x and y which show that the motion of the spring is a combination of two simple harmonic motions of periods $2\pi/\alpha$ and $2\pi/\beta$.

Example 14.20. *Two coils of a transformer are identical with resistance R, inductance L, mutual inductance M and a voltage E is impressed on the primary. Determine the currents in the coils at any instant, assuming that there is no current in either initially.*

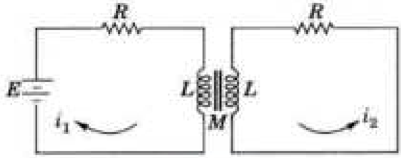


Fig. 14.35

Solution. Let i_1, i_2 ampere be the currents flowing through the primary and secondary coils at time t sec (Fig. 14.35). Then by Kirchhoff's law, we know that sum of the voltage drops across R, L and M = applied voltage.

$\therefore$ for the primary circuit,

$$Ri_1 + L\frac{di_1}{dt} + M\frac{di_2}{dt} = E$$

and for the secondary circuit, $Ri_2 + L\frac{di_2}{dt} + M\frac{di_1}{dt} = 0.$

Replacing d/dt by D and rearranging the terms,

$$(LD + R)i_1 + MDi_2 = E \quad \text{...(i)}$$

$$MDi_1 + (LD + R)i_2 = 0 \quad \text{...(ii)}$$

Eliminating i_2, we get $[(LD + R)^2 - M^2D^2]i_1 = (LD + R)E$

i.e.,
$$[(L^2 - M^2)D^2 + 2LRD + R^2]i_1 = RE \quad \text{...(iii)}$$

Its auxiliary equation is $(L^2 - M^2)D^2 + 2LRD + R^2 = 0$ whence $D = \frac{-R}{L+M}, \frac{-R}{L-M}$.

As L is usually $> M$, therefore, both values of D are negative and real.

$$\therefore \quad \text{C.F.} = c_1 e^{-\frac{Rt}{L+M}} + c_2 e^{-\frac{Rt}{L-M}} \text{ and P.I.} = RE \cdot \frac{1}{(L^2 - M^2)D^2 + 2LRD + R^2} e^{0 \cdot t} = E/R.$$

Thus the complete solution of (*iii*) is $i_1 = c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)} + E/R$ ...(iv)

and from (*ii*), we have $i_2 = -\frac{MD}{LD+R} i_1$

$$= -\frac{MD}{LD+R}\left(c_1 e^{-Rt/(L+M)} + c_2 e^{-Rt/(L-M)}\right) - \frac{MD}{LD+R}\left(\frac{E}{R}\right)$$

$$= -\frac{Mc_1}{L\left(\frac{-R}{L+M}\right)+R} \cdot De^{-Rt/(L+M)} - \frac{Mc_2}{L\left(\frac{-R}{L-M}\right)+R} \cdot De^{-Rt/(L-M)}$$

$$= c_1 e^{-Rt/(L+M)} - c_2 e^{-Rt/(L-M)}$$

Initially, when $t = 0$, $i_1 = i_2 = 0$.

$$\therefore \quad c_1 + c_2 = -E/R, \; c_1 - c_2 = 0 \qquad \therefore c_1 = c_2 = -E/2R.$$

Substituting the values of c_1, c_2 in (*iv*) and (*v*), we get

$$i_1 = \frac{E}{2R}\left[2 - e^{-Rt/(L+M)} - e^{-Rt/(L-M)}\right] \quad \text{...(vi)}$$

and
$$i_2 = \frac{E}{2R}\left[e^{-Rt/(L-M)} - e^{-Rt/(L+M)}\right] \quad \text{...(vii)}$$

Thus (*vi*) and (*vii*) give the currents at any instant.

PROBLEMS 14.5

1. A particle is projected with velocity u, at an elevation α. Neglecting air resistance, show that the equation to its path is the parabola $y = x \tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$. Also find the time of flight and range on the horizontal plane.
2. An inclined plane makes angle α with the horizontal. A projectile is launched from the bottom of the inclined plane with speed V in a direction making angle β with the horizontal. Set up the differential equations and find (*i*) the range on the incline, (*ii*) the maximum range up the incline.
3. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time $\frac{1}{k}\log\left\{1 + \frac{2ku}{g}\sin\alpha\right\}$.

4. A particle moving in a plane is subjected to a force directed towards a fixed point O and proportional to the distance of the particle from O. Show that the differential equations of motion are of the form $\frac{d^2x}{dt^2} = -k^2x$, $\frac{d^2y}{dt^2} = -k^2y$. Find the cartesian equation of the path of the particle if $x = 1$, $y = 0$, $\frac{dx}{dt} = 0$ and $dy/dt = 2$, when $t = 0$.
5. The currents i_1 and i_2 in mesh are given by the differential equations
$$\frac{di_1}{dt} - \omega i_2 = a \cos pt,\ \frac{di_2}{dt} + \omega i_1 = a \sin pt.$$ Find the currents i_1 and i_2 if $i_1 = i_2 = 0$ at $t = 0$.
6. The currents i_1 and i_2 in two coupled circuits are given by
$$L\frac{di_1}{dt} + Ri_1 + R(i_1 - i_2) = E;\ L\frac{di_2}{dt} + Ri_2 - R(i_1 - i_2) = 0,$$
where L, R, E are constants. Find i_1 and i_2 in terms of t given that $i_1 = i_2 = 0$ at $t = 0$.
7. The motion of a particle is governed by the equations
$$\frac{d^2x}{dt^2} - n\frac{dy}{dt} = 0,\ \frac{d^2y}{dt^2} + n\frac{dx}{dt} = n^2a,$$ when $x = y = \frac{dx}{dt} = \frac{dy}{dt} = 0$ at $t = 0$. Find x and y in terms of t.
8. Under certain conditions, the motion of an electron is given by the equations $m\frac{d^2x}{dt^2} + eH\frac{dy}{dt} = eE$ and $m\frac{d^2y}{dt^2} - eH\frac{dx}{dt} = 0$. Find the path of the electron, if it started from rest at the origin.
9. The voltage V and the current i at a distance x from the source satisfy the equations $-dV/dt = Ri$, $-di/dx = GV$, where R, G are constants. If $V = V_0$ at $x = 0$ and $V = 0$ at the receiving end $x = l$, show that $V = V_0 \sinh n(l - x)/\sinh nl$, $i = V_0\sqrt{(G/R)}.\cosh n(l - x)/\sinh nl$, where $n^2 = RG$.

14.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 14.6

Fill up the blanks or choose the correct answer in the following problems:

1. A particle executing simple harmonic motion of amplitude 5 cm has a speed of 8 cm/sec when at a distance of 3 cm from the centre of the path. The period of the motion of the particle will be
(*a*) $\pi/2$ sec (*b*) π sec (*c*) 2π sec (*c*) 4π sec.
2. A ball of mass m is suspended from a fixed point O by a light string of natural length l and modulus of elasticity λ. If the ball is displaced vertically, its motion will be S.H.M. of period
(*a*) $2\pi\sqrt{(m/\lambda l)}$ (*b*) $2\pi\sqrt{(ml/\lambda)}$ (*c*) $2\pi\sqrt{(l/m\lambda)}$ (*d*) $2\pi\sqrt{(\lambda m/l)}$.
3. The periodic time of the motion described by the differential equation $\frac{d^2x}{dt^2} + 4x = 0$ is
(*a*) $\pi/2$ (*b*) π (*c*) 2π.
4. A particle is projected with a velocity u at an angle of 60° to the horizontal. The time of flight of the projectile is equal to
(*a*) $\sqrt{3}u/2g$ (*b*) $\sqrt{3}u/g$ (*c*) u/g (*d*) $u/2g$.
5. A body of 6.5 kg is suspended by two strings of lengths 5 and 12 metres attached to two points in the same horizontal line whose distance apart is 13 meters. The tension of the strings are
(*a*) 2 kg & 6.5 kg (*b*) 2.5 kg & 6 kg (*c*) 2.25 kg & 6.25 kg (*d*) 3 kg & 5.5 kg.
6. A particle is projected at an angle of 30° to the horizontal with a velocity of 1962 cm/sec then the time of flight is
(*a*) 1 sec (*b*) 2 sec (*c*) 2.5 sec (*d*) 3 sec.
7. A point moves with S.H.M. whose period is 4 seconds. If it starts from rest at a distance of 4 meters from the centre of its path, then the time it takes, before it has described 2 metres is
(*a*) $\frac{1}{3}$ second (*b*) $\frac{2}{3}$ second (*c*) $\frac{3}{4}$ second (*d*) $\frac{4}{5}$ second.

8. If the length of the pendulum of a clock be increased in the ratio 720 : 721, it would loose .... seconds per day.
9. The frequency of free vibrations in a closed circuit with inductance L and capacity C in series is .... per minute.
10. If a clock with a seconds pendulum loses 10 seconds per day at a place having $g = 32$ ft/sec^2, g should be increased by .... ft/sec^2, to keep correct time.
11. The soldiers break step while marching over a bridge for the fear that their steps may not be in rhym with the natural frequency of the bridge causing its collapse due to ....
12. A horizontal tie-rod is freely pinned at each end. If it carries a uniform load w lb per unit length and has a horizontal pull P, then the differential equation of the elastic curve is ....
13. The conditions for an end of a whirling shaft to be in fixed bearings are .... and ....

CHAPTER 15

Differential Equations of Other Types

15.1 INTRODUCTION

In this chapter, we propose to study some other important types of ordinary differential equations which require special methods for their solution and have varied applications as illustrated side by side.

15.2 EQUATIONS OF THE FORM $d^2y/dx^2 = f(x)$

Integrating with respect to x, we have $\frac{dy}{dx} = \int f(x)dx + c = F(x)$, (say)

Again integrating, we get $y = \int F(x)dx + c'$ as the required solution.

In general, the solution of the equations of the form $\frac{d^n y}{dx^n} = f(x)$ is obtained by integrating it n times successively.

Example 15.1. *Solve* $\frac{d^2y}{dx^2} = xe^x$.

Solution. Integrating, we get $\frac{dy}{dx} = xe^x - \int e^x\,dx + c_1 = (x-1)\,e^x + c_1$

Again integrating, we get

$$y = (x-1)e^x - \int e^x dx + c_1x + c_2 = (x-2)e^x + c_1x + c_2.$$

PROBLEMS 15.1

Solve :

1. $\frac{d^2y}{dx^2} = x^2 \sin x$.

2. $\frac{d^3y}{dx^3} = x + \log x$.

3. A beam of length $2l$ with uniform load w per unit length is freely supported at both ends. Prove that the maximum deflection of the beam is $\dfrac{5wl^4}{24EI}$.

[**Hint.** Taking the origin at the left end, we have $EI\, d^4y/dx^4 = w$. At each end, $y = 0$ and $d^2y/dx^2 = 0$.]

4. For a cantilever beam of length l with a uniform load of w per unit length, show that the maximum deflection at the free end is wl^4/EI, where the symbols have the usual meaning.

15.3 EQUATIONS OF THE FORM $d^2y/dx^2 = f(y)$

Multiplying both sides by $2dy/dx$, we have $2\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2} = 2f(y)\dfrac{dy}{dx}$

Integrating with respect to x, $\left(\dfrac{dy}{dx}\right)^2 = 2\int f(y)\, dy + c = F(y)$ (say)

or $$\frac{dy}{dx} = \sqrt{[F(y)]}$$

Separating the variables and integrating, we get $\int \dfrac{dy}{\sqrt{[F(y)]}} = x + c$, whence follows the desired solution.

Such equations occur quite frequently in Dynamics.

Example 15.2. *Solve $d^2y/dx^2 = 2(y^3 + y)$ under the conditions $y = 0$, $dy/dx = 1$, when $x = 0$.* *(U.P.T.U., 2003)*

Solution. Multiplying by $2\, dy/dx$, the given equation becomes

$$2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = 4(y^3 + y)\,\frac{dy}{dx}$$

Integrating w.r.t. x, $\left(\dfrac{dy}{dx}\right)^2 = 4\left(\dfrac{y^4}{4} + \dfrac{y^2}{2}\right) + c = y^4 + 2y^2 + c$ ...(*i*)

As $dy/dx = 1$ for $y = 0$, $\therefore\ c = 1$

$\therefore$ (*i*) takes the form $(dy/dx)^2 = y^4 + 2y^2 + 1 = (y^2 + 1)^2$ or $dy/dx = y^2 + 1$

Separating the variables and integrating, we have $\int \dfrac{dy}{1 + y^2} = \int dx + c'$

or $$\tan^{-1} y = x + c' \quad ...(ii)$$

Thus (*ii*) becomes $\tan^{-1} y = x$ or $y = \tan x$ which is the required solution.

Example 15.3. *A point moves in a straight line towards a centre of force $\mu/(\text{distance})^3$, starting from rest at a distance 'a' from the centre of force, show that the time of reaching a point distant 'b' from the centre of force is $\dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$ and that its velocity is $\dfrac{\sqrt{\mu}}{ab}\sqrt{(a^2 - b^2)}$.* *(U.P.T.U., 2001)*

Solution. Let O be the centre of force and A the point of start so that $OA = a$. At any time t, let the point be at P where $OP = x$ so that

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^3} \quad ...(i)$$

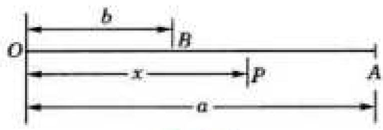


Fig. 15.1

Multiplying both sides by $2\, dx/dt$, we get

$$\frac{2dx}{dt}\cdot\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}\cdot\frac{2dx}{dt}$$

Integrating both sides, we obtain

$$\left(\frac{dx}{dt}\right)^2 = -\mu \int \frac{2}{x^3}\frac{dx}{dt} \cdot dt + c = +\frac{\mu}{x^2} + c$$

When $x = a$, velocity $dx/dt = 0$. $\therefore\ c = -\mu/a^2$.

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{1}{x^2} - \frac{1}{a^2}\right) = \frac{\mu(a^2 - x^2)}{a^2x^2} \qquad ...(ii)$$

At B $(x = b)$, velocity towards $O = \dfrac{\sqrt{\mu(a^2 - b^2)}}{ab}$

Again (ii) can be rewritten as $\dfrac{-ax\,dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}\,dt$ [– ve is taken since point is moving towards O]

Integrating both sides, we get

$$\sqrt{\mu}\int dt = -\int \frac{ax\,dx}{\sqrt{(a^2 - x^2)}} + c' \quad \text{or} \quad \sqrt{\mu}t = a\sqrt{(a^2 - x^2)} + c' \qquad ...(iii)$$

Since $t = 0$ at $x = a$, $\therefore\ c' = 0$

Thus (iii) gives $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - x^2)}$

Hence at B $(x = b)$ $t = \dfrac{a}{\sqrt{\mu}}\sqrt{(a^2 - b^2)}$.

PROBLEMS 15.2

Solve :

1. $d^2y/dx^2 = 3\sqrt{y}$ given that $y = 1$, $dy/dx = 2$ when $x = 0$.
2. $\dfrac{d^2y}{dx^2} = \dfrac{36}{y^2}$, given that when $x = 0$, $\dfrac{dy}{dx} = 0$, $y = 8$.
3. If $d^2r/dt^2 = \omega^2 r$, find the value of r in terms of t, if $r = a$ and $dr/dt = v$, when $t = 0$.
4. The motion of a particle let fall from a point outside the earth is given by $d^2x/dt^2 = -ga^2/x^2$. Given that $x = h$ and $dx/dt = 0$, when $t = 0$, find t in terms of x.
5. A particle is acted upon by a force $\mu\,(x + a^4/x^3)$ per unit mass towards the origin, where x is the distance from the origin at time t. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi/(4\sqrt{\mu})$.

15.4 EQUATIONS WHICH DO NOT CONTAIN y

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, x) = 0$$

On putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, it becomes

$$f(dp/dx, p, x) = 0.$$

This is an equation of the first order in x and p and can, therefore, be solved easily.

If its solution is $(p =)\ dy/dx = \phi(x)$, then $y = \int \phi(x)dx + c$ is the required solution.

Obs. This method may be used to reduce any such equation of the nth order to one of the $(n-1)$th order. If, however, the lowest derivative in such an equation is d^ry/dx^r

(i) put $d^ry/dx^r = p$; (ii) find p and therefrom find y, (*See* Ex. 15.5).

Example 15.4. *Solve* $x\dfrac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}$.

Solution. Putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, the given equation becomes

$$x\,dp/dx = \sqrt{(1+p^2)}.$$

Separating the variables and integrating, we get

$$\int \frac{dp}{\sqrt{(1+p^2)}} = \int \frac{dx}{x} + \text{constant}$$

or $$\log\left[p + \sqrt{(1+p^2)}\right] = \log x + \log c = \log cx.$$

$\therefore$ $$p + \sqrt{(1+p^2)} = cx \quad \text{or} \quad 1 + p^2 = (cx - p)^2$$

or $$(p =)\ \frac{dy}{dx} = \frac{1}{2}\left(cx - \frac{1}{cx}\right).$$

$\therefore$ integrating again, we have $y = \dfrac{1}{2}\left(c\dfrac{x^2}{2} - \dfrac{1}{c}\log x\right) + c'$ as the required solution.

Example 15.5. *Solve* $\dfrac{d^4y}{dx^4} \cdot \dfrac{d^3y}{dx^3} = 1.$

Solution. Putting $d^3y/dx^3 = p$ and $d^4y/dx^4 = dp/dx$, the given equation becomes $\dfrac{dp}{dx}\,p = 1.$

Integrating w.r.t. x, $\int p\,dp = x + c_1$, *i.e.* $p^2/2 = x + c_1$ or $(p =)\ d^3y/dx^3 = \sqrt{2}(x + c_1)^{1/2}$.

Integrating thrice successively, we get

$$\frac{d^2y}{dx^2} = \sqrt{2}\frac{(x+c_1)^{3/2}}{3/2} + c_2,\ \frac{dy}{dx} = \frac{2\sqrt{2}}{3} \cdot \frac{(x+c_1)^{5/2}}{5/2} + c_2x + c_3$$

$$y = \frac{4\sqrt{2}}{15}\,\frac{(x+c_1)^{7/2}}{7/2} + c_2\frac{x^2}{2} + c_3x + c_4$$

Hence $y = \dfrac{8\sqrt{2}}{105}(x + c_1)^{7/2} + \dfrac{1}{2}c_2x^2 + c_3x + c_4$ is the desired solution.

PROBLEMS 15.3

Solve the following equations :

1. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 6x = 0.$

2. $(1+x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0.$

3. $2x\dfrac{d^3y}{dx^3} \cdot \dfrac{d^2y}{dx^2} = \left(\dfrac{d^2y}{dx^2}\right)^2 - a^2.$

4. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2} = a\dfrac{d^2y}{dx^2}.$

5. A particle of mass m grammes is constrained to move in a horizontal circular path of radius a cm and is subjected to a resistance proportional to the square of the speed at any instant. Show that the differential equation of motion is of the form $m\dfrac{d^2\theta}{dt^2} + \mu a\left(\dfrac{d\theta}{dt}\right)^2 = 0$. If the particle starts with an angular velocity ω, find its angular displacement θ at time t sec.

6. When the inner of two concentric spheres of radii r_1 and $r_2 (r_1 < r_2)$ carries an electric charge, the differential equation for the potential v at any point between two spheres at a distance r from their common centre is $\dfrac{d^2v}{dr^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0$. Solve for v given $v = v_1$ when $r = r_1$ and $v = v_2$ when $r = r_2$.

15.5 EQUATIONS WHICH DO NOT CONTAIN x

A second order equation of this form is

$$f(d^2y/dx^2, dy/dx, y) = 0.$$

On putting $\frac{dy}{dx} = p$ and $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = p\frac{dp}{dy}$, it becomes

$$f(p\, dp/dy, p, y) = 0.$$

This is an equation of the first order in y and p and can, therefore, be solved easily.

Example 15.6. *Solve* $y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx} - 2y\right) = 0.$

Solution. On putting $dy/dx = p$ and $d^2y/dx^2 = p\, dp/dy$, the given equation becomes

$$yp\frac{dp}{dy} + p(p - 2y) = 0.$$

This gives *either* $p = 0$, of which the solution is $y = c$;

or $$\left(y\frac{dp}{dy} + p\right) - 2y = 0 \quad i.e., \quad (ydp + pdy) = 2ydy \ i.e., d(py) = 2ydy.$$

Integrating, $\quad py = 2\int ydy + c_1 = y^2 + c_1.$

Separating the variables and integrating, we get

$$\int \frac{ydy}{y^2 + c_1} = \int dx + c_2 \quad \text{or} \quad \frac{1}{2}\log(y^2 + c_1) = x + c_2 \text{ whence } y^2 + c_1 = c_3e^{2x}$$

Hence the required solutions are $y = c$ and $y^2 + c_1 = c_3 e^{2x}$.

Example 15.7. *Find the curve in which the radius of curvature is twice the normal and in the opposite direction.*

Solution. At any point $P(x, y)$ of a curve, the radius of curvature

$$\rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}$$

and the length of the normal (PN)

$$= y\sqrt{[1 + (dy/dx)^2]}.$$

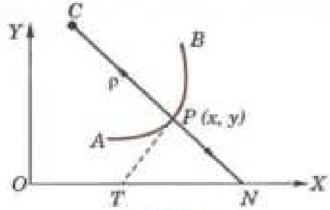


Fig. 15.2

Also we know that ρ is measured inwards and the normal is measured outwards, *i.e.*, both of them are positive when measured in opposite directions. So the sign will be positive (or negative) according as ρ and the normal run in the opposite (or same) directions.

Thus for the given curve $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \div \frac{d^2y}{dx^2} = 2y\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}$

or $$1 + \left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}.$$

On putting $dy/dx = p$ and $d^2y/dx^2 = p\, dp/dy$, the given equation becomes

$$1 + p^2 = 2y \,.\, p\, dp/dy.$$

∴ separating variables and integrating, we have

$$\int \frac{2pdp}{1 + p^2} = \int \frac{dy}{y} + \text{constant}$$

or $$\log(1 + p^2) = \log y + \log a = \log ay$$

∴ $$1 + p^2 = ay \text{ or } (p =)\ dy/dx = \sqrt{(ay - 1)}$$

$\therefore$ separating the variables and integrating, we get

$$\int dx + b = \int (ay-1)^{-1/2} dy$$

or
$$x + b = \frac{2}{a}(ay-1)^{1/2} \quad \text{or} \quad a^2(x+b)^2 = 4(ay-1)$$

which is required equation of the curve and represents a system of parabolas having axes parallel to y-axis.

PROBLEMS 15.4

Solve the following equations :

1. $2\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + 4 = 0.$
2. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1.$
3. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log y.$
4. $y(1-\log y)\dfrac{d^2y}{dx^2} + (1+\log y)\left(\dfrac{dy}{dx}\right)^2 = 0.$
5. Find the curve in which the radius of curvature is equal to the normal and is in the same direction.

15.6 EQUATIONS WHOSE ONE SOLUTION IS KNOWN

Consider the equation $d^2y/dx^2 + P\,dy/dx + Q = R$, where P, Q and R are functions of x only. If $y = u(x)$ is a known solution of this equation, then put $y = uv$ in it. It *reduces the differential equation to one of first order in dv/dx which can be completely solved.*

One integral belonging to the C.F. can be found by inspection as follows ;

(*i*) If $1 + P + Q = 0$, then $y = e^x$ is a solution,

(*ii*) If $I - P + Q = 0$, then $y = e^{-x}$ is a solution,

(*iii*) If $P + Qx = 0$, then $y = x$ is a solution.

Example 15.8. *Solve* $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0.$ *(Bhopal, 2008 S)*

Solution. The given equation is $\dfrac{d^2y}{dx^2} - \left(2 - \dfrac{1}{x}\right)\dfrac{dy}{dx} + \left(1 - \dfrac{1}{x}\right)y = 0$ ...(*i*)

Here $\quad 1 + P + Q = 1 - (2 - 1/x) + (1 - 1/x) = 0$

$\therefore \quad y = e^x$ is a part of C.F. of (*i*)

Now let $y = e^x v$ ...(*ii*)

so that
$$\frac{dy}{dx} = e^x v + e^x \frac{dv}{dx} \quad ...(iii) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = e^x v + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2} \quad ...(iv)$$

Substituting (*iv*), (*iii*) and (*ii*) in (*i*), we get

$$x\left(e^x v + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}\right) - (2x-1)\left(e^x v + e^x \frac{dv}{dx}\right) + (x-1)e^x v = 0$$

or cancelling e^x, it becomes $x\dfrac{d^2v}{dx^2} + \dfrac{dv}{dx} = 0$ or $x\dfrac{dp}{dx} + p = 0$, where $p = \dfrac{dv}{dx}$.

Integrating, we get $\displaystyle\int \frac{dp}{p} = -\int \frac{dx}{x} + c$ or $\log p = -\log x + \log c_1$

i.e.,
$$p = \frac{c_1}{x} \quad \text{or} \quad \frac{dv}{dx} = \frac{c_1}{x}.$$

Again integrating, we obtain $v = c_1 \log x + c_2$

Hence the complete solution of (*i*) is $y = e^x(c_1 \log x + c_2)$.

Example 15.9. *Solve* $(1 - x^2)\,y'' - 2xy' + 2y = 0$ *given that* $y = x$ *is a solution.* (B.P.T.U., 2005 S)

Solution. Let $y = xv$ so that $y' = v + x\dfrac{dv}{dx}$

and $$y'' = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Substituting these in the given equation, we get

$$(1 - x^2)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - 2x\left(v + x\frac{dv}{dx}\right) + 2xv = 0$$

or $$(x - x^3)\,\frac{d^2v}{dx^2} + (2 - 4x^2)\frac{dv}{dx} = 0$$

or $$(x - x^3)\,\frac{dp}{dx} + (2 - 4x^2)\,p = 0 \text{ where } p = \frac{dv}{dx}$$

Integrating, we get $\displaystyle\int \frac{dp}{p} + \int \frac{2 - 4x^2}{x - x^3}\,dx = c$

or $$\log p + \int \frac{2}{x}dx - \int \frac{dx}{1 - x} - \int \frac{dx}{1 + x} = c$$

or $$\log p + 2\log x + \log(1 - x) - \log(1 + x) = \log c_1$$

$$px^2\,(1 - x)/(1 + x) = c_1 \text{ or } \frac{dv}{dx} = \frac{c_1(1 + x)}{x^2\,(1 - x)}$$

Again integrating, $v = c_1 \displaystyle\int \left(\frac{2}{x} + \frac{1}{x^2} + \frac{2}{1 - x}\right)dx + c_2$

or $$v = c_1\,\{2\log(x/1 - x) - 1/x\} + c_2$$

Hence the required complete solution is $y = x\,[c_1\,\{\log(x/1 - x)^2 - 1/x\} + c_2]$

Obs. Here $P + Qx = 0$. That is why $y = x$ is a solution of the given equation.

PROBLEMS 15.5

1. If $y = e^{x^2}$ is a solution of $y'' - 4xy' + (4x^2 - 2)\,y = 0$, find a second independent solution. (U.P.T.U., 2004)
2. Solve $x^2y'' - (x^2 + 2x)\,y' + (x + 2)y = x^3e^x$.
3. Solve $x\dfrac{d^2y}{dx^2} - (2x - 1)\dfrac{dy}{dx} + (x - 1)\,y = e^x$ given that $y = e^x$ is one integral. (Bhopal, 2007 S)
4. Solve $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is a solution. (Bhopal, 2007)
5. Solve $\dfrac{d^2y}{dx^2} - \cot x\dfrac{dy}{dx} - (1 - \cot x)\,y = e^x \sin x$.

15.7 EQUATIONS WHICH CAN BE SOLVED BY CHANGING THE INDEPENDENT VARIABLE

Consider the equation $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

To change the independent variable x to z, let $z = f(x)$

Then $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \qquad ...(2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right) = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} \qquad ...(3)$$

Substituting (2) and (3) in (1), we get $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$ ...(4)

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right) \Big/ \left(\frac{dz}{dx}\right)^2, Q_1 = Q \Big/ \left(\frac{dz}{dx}\right)^2, R_1 = R \Big/ \left(\frac{dz}{dx}\right)^2$

Now equation (4) can be solved by taking Q_1 = a constant.

Example 15.10. *Solve, by changing the independent variable,* $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$ *(U.P.T.U., 2003)*

Solution. Given equation is $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$ ...(*i*)

Here $P = -1/x$, $Q = 4x^2$ and $R = x^4$.

Choose z so that $Q/(dz/dx)^2$ = const. or $(dz/dx)^2 = 4x^2$ (say)

or $$\frac{dz}{dx} = 2x \quad \text{or} \quad z = x^2$$

Changing the independent variable x to z by $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P \cdot \frac{dy}{dz} + Q_1 y = R_1 \qquad ...(ii)$$

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right) \Big/ \left(\frac{dz}{dx}\right)^2 = [2 + (-x^{-1})\, 2x]/4x^2 = 0$

$$Q_1 = \frac{Q}{(dz/dx)^2} = \frac{4x^2}{4x^2} = 1, R_1 = \frac{R}{(dz/dx)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

$\therefore$ (*ii*) takes the form $\frac{d^2y}{dz^2} + y = \frac{z}{4}$ or $(D^2 + 1)\,y = \frac{z}{4}$

Its A.E. is $D^2 + 1 = 0$, *i.e.*, $D = \pm i$

C.F. $= c_1 \cos z + c_2 \sin z$

$$\text{P.I.} = \frac{1}{D^2+1}\frac{z}{4} = \frac{1}{4}(1 + D^2)^{-1} z = \frac{1}{4}(1 - D^2 \,.........)\, z = \frac{z}{4}.$$

Hence the complete solution of (*i*) is

$$y = c_1 \cos z + c_2 \sin z + \frac{z}{4} \quad \text{or} \quad y = c_1 \cos x^2 + c_2 \sin x^2 + \frac{x^2}{4}.$$

Example 15.11. *Solve* $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0.$ ...(*i*)

Solution. Here $P = \cot x$, $Q = 4 \operatorname{cosec}^2 x$

Choosing z so that $Q \Big/ \left(\frac{dz}{dx}\right)^2$ = const. or $\left(\frac{dz}{dx}\right)^2 = \operatorname{cosec}^2 x$ (say)

$$dz/dx = \operatorname{cosec} x \text{ or } z = \int \operatorname{cosec} x\, dx = \log \tan x/2$$

Changing the independent variable x to z, we get

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad ...(ii)$$

where $P_1 = \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right) \Big/ \left(\frac{dz}{dx}\right)^2 = (-\operatorname{cosec} x \cot x + \cot x \operatorname{cosec} x)/\operatorname{cosec}^2 x = 0$

$$Q_1 = Q \Big/ \left(\frac{dz}{dx}\right)^2 = \frac{4 \operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 4, R_1 = 0$$

$\therefore$ (*ii*) takes the form $\dfrac{d^2y}{dz^2} + 4y = 0$

Its solution is $y = c_1 \cos(2z) + c_2 \sin(2z)$

i.e., $$y = c_1 \cos(2 \log \tan x/2) + c_2 \sin(2 \log \tan x/2)$$

This is the required complete solution of (*i*).

PROBLEMS 15.6

Solve the following equations (by changing the independent variable) :

1. $\dfrac{d^2y}{dx^2} + \tan x \dfrac{dy}{dx} + y \cos^2 x = 0.$ (*Bhopal, 2005*)
2. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{y}{x^4} = 0.$
3. $\dfrac{d^2y}{dx^2} - \cot x \dfrac{dy}{dx} - \sin^2 x y = 0.$
4. $x\dfrac{d^2y}{dx^2} + (4x^2 - 1)\dfrac{dy}{dx} + 4x^3y = 2x^3.$ (*U.P.T.U., 2006*)
5. $\cos x \dfrac{d^2y}{dx^2} + \sin x \dfrac{dy}{dx} - 2y \cos^3 x = 2 \cos^5 x.$ (*Bhopal, 2006 S*)

15.8 TOTAL DIFFERENTIAL EQUATIONS

(1) An ordinary differential equation of the first order and first degree involving three variables is of the form

$$P + Q\frac{dy}{dx} + R\frac{dz}{dx} = 0 \qquad ...(1)$$

where P, Q, R are functions of x, y, z and x is the independent variable.

In terms of differentials, (1) can be written as

$$Pdx + Qdy + Rdz = 0 \qquad ...(2)$$

which is integrable only if

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad ...(3)$$

(2) Rule *to solve* $Pdx + Qdy + Rdz = 0$

If the condition of integrability is satisfied, consider one of the variables say : z, as constant so that dz = 0. Then integrate the equation Pdx + Qdy = 0. Replace the arbitrary constant appearing in its integral by $\phi(z)$. *Now differentiate the integral just obtained with respect to x, y, z. Finally, compare this result with the given differential equation to determine* $\phi(z)$.

Example 15.12. *Solve* $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$

Solution. Here $P = y^2 + yz$, $Q = z^2 + zx$, $R = y^2 - xy$.

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (y^2 + yz)[2z + x - (2y - x)] + (z^2 + zx)[-y - y] + (y^2 - xy)[(2y + z) - z] = 0$$

Hence the condition of integrability is satisfied.

Considering z as constant, the given equation becomes

$$(y^2 + yz)dx + (z^2 + zx)dy = 0, \quad \text{or} \quad \frac{dx}{z(z + x)} + \frac{dy}{y(y + z)} = 0$$

Integrating and noting that z is a constant, we get

$$\frac{1}{z}\int \frac{dx}{z + x} + \frac{1}{z}\int \left(\frac{1}{y} - \frac{1}{y + z}\right) dy = \text{constant}$$

i.e., $$\log(z + x) + \log y - \log(y + z) = \text{constant.}$$

i.e., $$\frac{y(z + x)}{y + z} = \text{constant} = \phi(z), \text{ say} \qquad ...(i)$$

or $$y(z + x) - (y + z)\,\phi(z) = 0$$

Differentiating w.r.t. x, y, z, we obtain

$$y(dz + dx) + (z + x)dy - [(y + z)\phi'(z)dz + (dy + dz)\phi(z)] = 0$$

or $$ydx + [z + x - \phi(z)]dy + [y - (y + z)\,\phi'(z) - \phi(z)]\,dz = 0 \qquad ...(ii)$$

Comparing (*ii*) with the given differential equation, we get

$$\frac{y^2 + yz}{y} = \frac{z^2 + zx}{z + x - \phi(z)} = \frac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}.$$

The relation $\dfrac{y^2 + yz}{y} = \dfrac{z^2 + zx}{z + x - \phi(z)}$ reduces to (*i*). $\therefore$ it gives no information about $\phi(z)$.

Taking $\dfrac{y^2 + yz}{y} = \dfrac{y^2 - xy}{y - (y + z)\phi'(z) - \phi(z)}$, we get

$$y^2 - xy = (y + z)\,[y - (y + z)\phi'(z) - \phi(z)] = y^2 + yz - (y + z)^2\,\phi'(z) - (y + z)\,\phi(z)$$

$$= y^2 + yz - (y + z)^2\,\phi'(z) - y(z + x) \qquad \text{[From (i)]}$$

$$= y^2 - xy - (y + z)^2\,\phi'(z)$$

i.e., $(y + z)^2\,\phi'(z) = 0$, *i.e.*, $\phi'(z) = 0$ so that $\phi(z) = c$

Hence the required solution is $y(z + x) = (y + z)\,c$. [From (*i*)]

Obs. *Sometimes the integral is readily obtained by simply regrouping the terms in the given equation as is illustrated below.*

Example 15.13. *Solve* $xdx + zdy + (y + 2z)\,dz = 0$.

Solution. Regrouping the terms, we can write the given equation as

$$xdx + (ydz + zdy) + 2z\,dz = 0$$

of which the integral is $\dfrac{x^2}{2} + yz + z^2 = c$.

PROBLEMS 15.7

Solve :

1. $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0$.
2. $(y^2 + z^2 - x^2)dx - 2xydy - 2xzdz = 0$.
3. $yzdx - 2zxdy - 3xydz = 0$.
4. $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$.
5. $(x + z)^2dy + y^2(dx + dz) = 0$.
6. $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)\,dz = 0$.

15.9 SIMULTANEOUS TOTAL DIFFERENTIAL EQUATIONS

These equations in three variables are given by

$$\left.\begin{aligned} Pdx + Qdy + Rdz = 0 \\ P'dx + Q'dy + R'dz = 0 \end{aligned}\right\} \qquad ...(1)$$

where P, Q, R and P', Q', R' are any functions of x, y, z.

(*a*) If each of these equations is integrable and have solutions $f(x, y, z) = c$ and $Y(x, y, z) = c'$ respectively, then these taken together constitute the solution of the simultaneous equations (1).

(*b*) If one or both the equations (1) is not integrable, then we write these as follows :

$$\frac{dx}{QR' - Q'R} = \frac{dy}{RP' - R'P} = \frac{dz}{PQ' - P'Q}$$

and solve these by the methods explained below.

15.10 EQUATIONS OF THE FORM dx/P = dy/Q = dz/R

(1) Method of grouping

See if it is possible to take two fractions $dx/P = dz/R$ from which y can be cancelled or is absent, leaving equations in x and z only.

If so, integrate it by giving $\phi(x, z) = c$. ...(1)

Again see if one variable say : x is absent or can be removed may be with the help of (1), from the equation $dy/Q = dz/R$.

Then integrate it by giving $\psi(y, z) = c'$ ...(2)

These two independent solutions (1) and (2) taken together constitute the complete solution required.

Example 15.14. *Solve* $\dfrac{dx}{z^2y} = \dfrac{dy}{z^2x} = \dfrac{dz}{y^2x}$

Solution. Taking the first two fractions and cancelling z^2, we get

$$\frac{dx}{y} = \frac{dy}{x} \quad \text{or} \quad xdx - ydy = 0$$

which on integration gives $x^2 - y^2 = c$. ...(*i*)

Again taking the second and third fractions and cancelling x, we have

$$\frac{dy}{z^2} = \frac{dz}{y^2}, \text{ i.e., } y^2dy - z^2dz = 0.$$

Its integral is $y^3 - z^3 = c'$. ...(*ii*)

Thus (*i*) and (*ii*) taken together constitute the required solution of the given equations.

(2) Method of multipliers

By a proper choice of the multipliers l, m, n which are not necessarily constants, we write

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR} \text{ such that } lP + mQ + nR = 0.$$

Then $ldx + mdy + ndz = 0$ can be solved giving the integral $\phi(x, y, z) = c$ ...(1)

Again search for another set of multipliers λ, μ, γ

so that $\lambda P + \mu Q + \gamma R = 0$

giving $\lambda dx + \mu dy + \gamma dz = 0$,

which on integration gives the solution $\psi(x, y, z) = c'$ ...(2)

These two solutions (1) and (2) taken together constitute the required solution.

Example 15.15. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$

Solution. Using the multipliers x, y, z

each fraction $= \dfrac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \dfrac{xdx + ydy + zdz}{0}$

$\therefore \quad xdx + ydy + zdz = 0$, which on integration gives the solution $x^2 + y^2 + z^2 = c$ ...(*i*)

Again using the multipliers $1/x, -1/y, -1/z$

each fraction $= \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \dfrac{\frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz}{0}$ so that $\dfrac{dx}{x} - \dfrac{dy}{y} - \dfrac{dz}{z} = 0$

which on integration gives $\log x - \log y - \log z = \text{constant}$ or $yz = c'x$. ...(*ii*)

Hence the solution of the given equation is $x^2 + y^2 + z^2 = c$; $yz = c'x$.

PROBLEMS 15.8

Solve :

1. $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$
2. $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dx}{ly - mx}$
3. $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$
4. $\dfrac{dx}{y - zx} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$
5. $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$
6. $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$

CHAPTER 16

Series Solution of Differential Equations and Special Functions

16.1 INTRODUCTION

Many differential equations arising from physical problems are linear but have variable coefficients and do not permit a general solution in terms of known functions. Such equations can be solved by numerical methods (Chapter 28), but in many cases it is easier to find a solution in the form of an infinite convergent series.

The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's polynomial, Lagurre's polynomial, Hermite's polynomial, Chebyshev polynomials. Strum-Lioville problem based on the orthogonality of functions is also included which shows that Bessel's, Legendre's and other equations can be considered from a common point of view. These special functions have many applications in engineering.

16.2 VALIDITY OF SERIES SOLUTION OF THE EQUATION

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad ...(i)$$

can be determined with the help of the following theorems :

Def. 1. If $P_0(a) \neq 0$, then $x = a$ is called and **ordinary point** of (i), otherwise a **singular point.**

2. *A singular point $x = a$ of (1) is called* **regular** *if, when (i) is put in the form*

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0,$$

$Q_1(x)$ and $Q_2(x)$ possess derivatives of all orders in the neighbourhood of a.

3. *A singular point which is not regular is called an* **irregular singular point.**

Theorem I. *When $x = a$ is an ordinary point of (i), its every solution can be expressed in the form*

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + ... \qquad ...(ii)$$

Theorem II. *When $x = a$ is a regular singularity of (i), at least one of the solutions can be expressed as*

$$y = (x-a)^m [a_0 + a_1(x-a) + a_2(x-a)^2 + ...] \qquad ...(iii)$$

Theorem III. *The series (ii) and (iii) are convergent at every point within the circle of convergence at a.* A solution in series will be valid only if the series is convergent.

16.3 SERIES SOLUTION WHEN X = 0 IS AN ORDINARY POINT OF THE EQUATION

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0 \quad ...(1)$$

where P's are polynomials in x and $P_0 \neq 0$ at $x = 0$.

(*i*) Assume its solution to be of the form $y = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ ...(2)

(*ii*) Calculate dy/dx, d^2y/dx^2 from (2) and substitute the values of y, dy/dx, d^2y/dx^2 in (1).

(*iii*) Equate to zero the coefficients of the various powers of x and determine a_2, a_3, a_4 ... in terms of a_0, a_1. (The result obtained by equating to zero is the coefficient of x^n that is called the *recurrence relation*).

(*iv*) Substituting the values of a_2, a_3, a_4 ... in (2), we get the desired series solution having a_0, a_1 as its arbitrary constants.

Example 16.1. *Solve in series the equation* $\frac{d^2y}{dx^2} + xy = 0$. *(V.T.U., 2010)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume its solution is $y = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ ...(*i*)

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1} + ...$$

and $$\frac{d^2y}{dx^2} = 2 . 1a_2 + 3 . 2a_3x + 4 . 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ...$$

Substituting in the given differential equation

$1 . 1a_2 + 3 . 2a_3x + 4 . 3a_4x^2 + ... + n(n-1)a_nx^{n-2} + ... + x(a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...) = 0$

or $2 . 1a_2 + (3 . 2a_3 + a_0)x + (4 . 3a_4 + a_1)x^2 + (5 . 4a_5 + a_2)x^3 + ... + [(n+2)(n+1)a_{n+2} + a_{n-1}]x^n + ... = 0.$

Equating to zero the co-efficients of the various powers of x,

$a_2 = 0,$ [Coeff. of $x^0 = 0$]

$3 . 2a_3 + a_0 = 0$, *i.e.*, $a_3 = -\frac{a_0}{3!}$ [Coeff. of $x = 0$]

$4 . 3a_4 + a_1 = 0$, *i.e.*, $a_4 = -\frac{2a_1}{4!}$ [Coeff. of $x^2 = 0$]

$5 . 4a_5 + a_2 = 0$, *i.e.*, $a_5 = -\frac{a_2}{5 . 4} = 0$ and so on. [Coeff. of $x^3 = 0$]

In general, $(n+2)(n+1)a_{n+2} + a_{n-1} = 0$ [Coeff. of $x^n = 0$]

i.e., $$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)} \quad ..(ii)$$

which is the *recurrence relation.*

Putting $n = 4, 5, 6, ...$ in (*ii*) successively, $a_6 = -\frac{a_3}{6 . 5} = \frac{4a_0}{6!}$; $a_7 = -\frac{a_4}{7 . 6} = \frac{5 . 2a_1}{7!}$

$$a_8 = -\frac{a_5}{8 . 7} = 0; \; a_9 = -\frac{a_6}{9 . 8} = -\frac{7 . 4a_0}{9!} \text{ and so on.}$$

Substituting these values in (*i*), we get

$$y = a_0\left(1 - \frac{x^3}{3!} + \frac{1 . 4x^6}{6!} - \frac{1 . 4 . 7x^9}{9!} + ...\right) + a_1\left(x - \frac{2x^4}{4!} + \frac{2 . 5x^7}{7!} - ...\right)$$

which is the required solution.

Example 16.2. *Solve in series* $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+4y=0.$ *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Here $x = 0$ is an ordinary point since coefficient of $y'' \neq 0$ at $x = 0$.

Assume the solution of the given equation to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots \qquad \dots(i)$$

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$

and $$\frac{d^2y}{dx^2} = 2a_2 + 3.2a_3x + 4.3a_4x^2 + \dots + n(n-1)a_nx^{n-2} + \dots$$

Substituting in the given equation, we get

$$(1-x^2)[2a_2 + 3.2a_3x + 4.3a_4x^2 + \dots + n(n-1)a_nx^{n-2} + \dots]$$
$$- x[a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots] + 4[a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots] = 0$$

Equating to zero the coefficients of the various powers of x,

$$2a_2 + 4a_0 = 0 \quad i.e., \quad a_2 = -2a_0 \qquad [\text{coeff. of } x^0 = 0]$$

$$3.2a_3 - a_1 + 4a_1 = 0 \quad i.e., \quad a_3 = -\frac{1}{2}a_1 \qquad [\text{coeff. of } x^1 = 0]$$

$$4.3a_4 - 2a_2 - 2a_2 + 4a_2 = 0 \quad i.e., \quad a_4 = 0 \qquad [\text{coeff. of } x^2 = 0]$$

$$5.4a_5 - 3.2a_3 - 3a_3 + 4a_3 = 0 \qquad [\text{coeff. of } x^3 = 0]$$

i.e., $$20a_5 - 5a_3 = 0 \quad i.e., \quad a_5 = -\frac{a_1}{8} \text{ and so on.}$$

In general, $(n+2)(n+1)a_{n+2} - n(n-1)a_n - na_n + 4a_n = 0$

or $$a_{n+2} = \frac{n-2}{n+1}a_n \qquad \dots(ii)$$

which is the *recurrence relation*

Putting $n = 4, 5, 6, 7\dots$ in (ii) successively,

$$a_6 = 0; \quad a_7 = \frac{3}{6}a_5 = -\frac{3}{6}\frac{a_1}{8}; \; a_8 = 0; \; a_9 = -\frac{5.3}{8.6}\cdot\frac{a_1}{8} \dots$$

Substituting these values in (i), we get

$$y = a_0(1-2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{3}{6}\cdot\frac{x^6}{8} - \frac{5.3}{8.6}\cdot\frac{x^8}{8} - \dots\right).$$

PROBLEMS 16.1

Solve the following equations in series :

1. $\dfrac{d^2y}{dx^2} + y = 0$, given $y(0) = 0$. (B.P.T.U., 2005 S)
2. $\dfrac{d^2y}{dx^2} + x^2y = 0$.
3. $y'' + xy' + y = 0$. (V.T.U., 2008)
4. $(1-x^2)y'' + 2y = 0$, given $y(0) = 4$, $y'(0) = 5$. (P.T.U., 2006)
5. $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$. (S.V.T.U., 2008)
6. $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = 0$. (U.P.T.U., 2004)

16.4 FROBENIUS* METHOD : Series solution when x = 0 is a regular sigularity of the equation

$$\mathbf{P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0} \qquad \dots(1)$$

*A German mathematician *F.G. Frobenius* (1849–1917) who is known for his contributions to the theory of matrices and groups.

(*i*) Assume the solution to be $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots)$ ...(2)

(*ii*) Substitute from (2) for y, dy/dx, d^2y/dx^2 in (1) as before.

(*iii*) Equate to zero the coefficient of the lowest degree term in x. It gives a quadratic equation known as the *indicial equation*.

(*iv*) Equating to zero the coefficients of the other powers of x, find the values of $a_1, a_2, a_3, \dots$ in terms of a_0.

The complete solution depends on the nature of roots of the indicial equation.

Case I. When roots of the indicial equation are distinct and do not differ by an integer, *the complete solution is*

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

where m_1, m_2 are the roots.

Example 16.3. *Solve in series the equation* $9x(1-x)\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0.$

(Madras, 2006 ; Roorkee, 2000)

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots$

$$\therefore \quad \frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots$$

in the given equation, we obtain

$$9x(1-x)\,[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots]$$
$$- 12[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots] + 4[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives

$$a_0(9m(m-1) - 12m] = 0, \text{ i.e., } m(3m-7) = 0 \quad \text{as} \quad a_0 \neq 0.$$

Thus the roots of the *indicial equation* are $m = 0, 7/3$. *i.e., Roots are distinct and do not differ by an integer.*

The coefficient of x^m equated to zero gives $a_1\{9(m+1)m - 12(m+1)\} + a_0\{4 - 9m(m-1)\} = 0$

i.e., $$3a_1(3m-4)(m+1) - a_0(3m-4)(3m+1) = 0$$

i.e., $$3a_1(m+1) = a_0(3m+1).$$

Similarly $3a_2(m+2) = a_1(3m+4)$, $3a_3(m+3) = a_2(5m+7)$ and so on.

$$\therefore \quad a_1 = \frac{3m+1}{3(m+1)}a_0,\ a_2 = \frac{(3m+4)a_1}{3(m+2)} = \frac{(3m+4)(3m+1)}{3^2(m+2)(m+1)}a_0,\ a_3 = \frac{(3m+7)(3m+4)(3m+1)}{3^3(m+3)(m+2)(m+1)}a_0 \text{ etc.}$$

When $m = 0$, $a_1 = \dfrac{1}{3}a_0$, $a_2 = \dfrac{1.4}{3.6}a_0$, $a_3 = \dfrac{1.4.7}{3.6.9}a_0$ etc. giving the particular solution

$$y_1 = a_0\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \dots\right]$$

When $m = 7/3$, the particular solution is

$$y_2 = a_0x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \dots\right]$$

Thus the complete solution is $y = c_1y_1 + c_2y_2$

i.e.,
$$y = C_1\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \dots\right]$$
$$+ C_2x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \dots\right], \qquad \text{where } C_1 = c_1a_0,\ C_2 = c_2a_0.$$

Case II. When roots of the indicial equation are equal *the complete solution is*

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

where m_1, m_1 are the roots.

Example 16.4. *Solve in series the equation* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0.$ *(V.T.U., 2010 ; S.V.T.U., 2007)*

Solution. Substituting $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots$ ...(*i*)

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and $$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$x[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$+ [ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots]$$
$$+ x[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0.$$

The lowest power of x is x^{m-1}. Its coefficient equated to zero gives $a_0[m(m-1) + m] = 0.$

i.e., $m^2 = 0$ as $a_0 \neq 0.$ $\therefore$ $m = 0, 0.$

The coefficients of $x^m, x^{m+1}, \ldots$ equated to zero give

$$a_1[(m+1)m + m + 1] = 0, \quad i.e., \quad a_1 = 0$$
$$a_2(m+2)^2 + a_0 = 0, a_3(m+3)^2 + a_1 = 0, a_4(m+4)^2 + a_2 = 0 \text{ and so on.}$$

Clearly $a_3 = a_5 = a_7 \ldots\ldots = 0.$

Also $$a_2 = -\frac{a_0}{(m+2)^2}, a_4 = -\frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2} \text{ etc.}$$

$$\therefore \quad y = a_0x^m\left[1 - \frac{x^2}{(m+2)^2} + \frac{x^4}{(m+2)^2(m+4)^2} - \frac{x^6}{(m+2)^2(m+4)^2(m+6)^2} + \ldots\ldots\right] \quad \ldots(ii)$$

Putting $m = 0$, the first solution is

$$y_1 = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \ldots\ldots\right] \quad \ldots(iii)$$

This gives only one solution instead of two. To get the second solution, differentiate (*ii*) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = y\log x + a_0x^m\left\{\frac{x^2}{(m+2)^2}\frac{2}{m+2} - \frac{x^4}{(m+2)^2(m+4)^2}\left[\frac{2}{m+2} + \frac{2}{m+4}\right] + \ldots\right\}$$

$\therefore$ the second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$= y_1\log x + a_0\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\} \quad \ldots(iv)$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$. [From (*iii*) & (*iv*)]

i.e., $$y = (C_1 + C_2\log x)\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}x^4 - \frac{1}{2^2.4^2.6^2}x^6 + \ldots\right]$$
$$+ C_2\left\{\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1 + \frac{1}{2}\right)x^4 + \frac{1}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\}$$

where $C_1 = a_0c_1, C_2 = a_0c_2.$

Obs. The above differential equation is called *Bessel's equation of order zero*, y_1 is called *Bessel function of the first kind of order zero* and is denoted by $J_0(x)$. It is absolutely convergent for all values of x whether real or complex.

y_2 is called the *Bessel function of the second kind of order zero or the Neumann function* and is denoted by $Y_0(x)$.

Thus the complete solution of the *Bessel's equation of order zero* is $y = AJ_0(x) + BY_0(x)$.

Case III. When roots of indicial equation are distinct and differ by an integer, making a coefficient of y infinite.

Let m_1 and m_2 be the roots such that $m_1 < m_2$. If some of the coefficients of y series become infinite when $m = m_1$, we modify the form of y by replacing a_0 by $b_0(m - m_1)$. Then the complete solution is

$$y = C_1(y)_{m_2} + C_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Obs. 1. Two independent solution can also be obtained by putting $m = m_1$ (lesser of the two roots) in the modified form of y and $\partial y/\partial m$.

Obs. 2. *If one of the coefficients (say : a_1) becomes* **indeterminate** *when $m = m_2$, the complete solution is given by putting $m = m_2$ in y which contains two arbitrary constants.*

Example 16.5. *Obtain the series solution of the equation*

$$x(1 - x)\frac{d^2y}{dx^2} - (1 + 3x)\frac{dy}{dx} - y = 0.$$

Solution. Here $x = 0$ is a singular point, since coefficient of y'' is zero at $x = 0$.

$\therefore$ substituting $\quad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \quad \ldots(i)$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots$$

and
$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots$$

in the given equation, we obtain

$$x(1-x)[m(m-1)a_0x^{m-2} + (m+1)ma_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots]$$
$$-(1+3x)[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \ldots] - [a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x, we get $a_0[m(m-1) - m] = 0$, $(a_0 \neq 0)$,

i.e., $\quad m(m-2) = 0$, *i.e.* $m = 0, 2$ *i.e.*, the two roots are distinct and differ by an integer.

Equating to zero the coefficients of successive powers of x, we get

$$(m-1)a_1 = (m+1)a_0,\ ma_2 = (m+2)a_1,\ (m+1)a_3 = (m+3)a_2 \text{ and so on.}$$

i.e.,
$$a_1 = \frac{m+1}{m-1}a_0,\ a_2 = \frac{(m+1)(m+2)}{(m-1)m}a_0,\ a_3 = \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}a_0 \text{ etc.}$$

Thus (i) becomes

$$y = a_0x^m\left[1 + \frac{m+1}{m-1}x + \frac{(m+1)(m+2)}{(m-1)m}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)m(m+1)}x^3 + \ldots\right] \quad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is

$$y_1 = a_0x^2\left[1 + 3x + \frac{3.4}{2}x^2 + \frac{4.5}{2}x^3 + \ldots\right]$$

If we put $m = 0$ in (ii), the coefficients become infinite.

To obviate this difficulty, put $a_0 = b_0(m - 0)$ so that

$$y = b_0x^m\left[m + \frac{m(m+1)}{m-1}x + \frac{(m+1)(m+2)}{m-1}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3 + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0x^m\log x\left[m + \frac{m(m+1)x}{m-1} + \frac{(m+1)(m+2)}{m-1}x^2 + \frac{(m+1)(m+2)(m+3)}{(m-1)(m+1)}x^3 + \ldots\right]$$
$$+ b_0x^m\left[1 + \frac{m^2-2m-1}{(m-1)^2}x + \frac{m^2-m-5}{(m-1)^2}x^2 + \frac{m^2-2m-11}{(m-1)^2}x^3 + \ldots\right]$$

$\therefore$ the second solution is $y_2 = \left(\dfrac{\partial y}{\partial m}\right)_{m=0}$

$$= b_0 \log x\,[-1.2x^2 - 2.3x^3 - 3.4x^4 - \ldots] + b_0[1 - x - 5x^2 - 11x^3 - \ldots]$$

Hence the complete solution is $y = c_1y_1 + c_2y_2$

i.e., $$y = \frac{1}{2}\,c_1a_0[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots] - b_0c_2 \log x\,[1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots]$$
$$- b_0c_2\,[-1 + x + 5x^2 + 11x^3 + \ldots]$$

i.e., $$y = (C_1 + C_2 \log x)\,(1.2x^2 + 2.3x^3 + 3.4x^4 + \ldots) + C_2\,(-1 + x + 5x^2 + 11x^3 + \ldots)$$

where $C_1 = \dfrac{1}{2}\,c_1a_0,\ C_2 = -\,b_0c_2$

Example 16.6. *Solve in series* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - 4)\,y = 0.$ *(Bhopal, 2008 S; Rajasthan, 2003)*

Solution. $x = 0$ is a singular point, since coeff. of y'' is zero at $x = 0$.

Substituting $$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \qquad \ldots(i)$$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots$$

and $$\frac{d^2y}{dx^2} = m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots$$

in the given equation, we get

$$x^2\,[m\,(m-1)\,a_0x^{m-2} + (m+1)\,ma_1x^{m-1} + (m+2)\,(m+1)\,a_2x^m + \ldots]$$
$$+ x\,[ma_0x^{m-1} + (m+1)\,a_1x^m + (m+2)\,a_2x^{m+1} + \ldots]$$
$$+ (x^2 - 4)\,[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots] = 0$$

Equating to zero the coefficients of the lowest power of x.

$$a_0\,[m\,(m-1) + m - 4] = 0 \text{ so that } m = \pm\,2.$$

i.e., the two roots are distinct and differ by an integer.

Now equating to zero the coefficiens of successive powers of x, we get

$$m(m+4)\,a_2 = -\,a_0,\ i.e.,\ a_2 = \frac{-1}{m\,(m+4)}\,a_0,\ a_3 = 0$$

$$a_4 = \frac{1}{(m+2)\,(m+6)}\cdot\frac{1}{m\,(m+4)}\,a_0,\ a_5 = a_7 = \ldots = 0.$$

$$a_6 = \frac{-\,a_0}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} \text{ etc.}$$

Substituting these values in (i), we get

$$y = a_0x^m\left[1 - \frac{x^2}{m\,(m+4)} + \frac{x^4}{m\,(m+2)\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+2)\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right] \qquad \ldots(ii)$$

Putting $m = 2$ (greater of the two roots) in (ii), the first solution is

$$y_1 = a_0x^2\left\{1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 6\cdot 8} - \frac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10} + \ldots\right\}$$

If we put $m = -\,2$ in (ii), the coefficients become infinite. To obviate this difficulty, let $a_0 = b_0\,(m+2)$, so that

$$y = b_0x^m\left[(m+2)\left\{1 - \frac{x^2}{m\,(m+4)}\right\} + \frac{x^4}{m\,(m+4)\,(m+6)} - \frac{x^6}{m\,(m+4)^2\,(m+6)\,(m+8)} + \ldots\right]$$

$$\therefore \quad \frac{\partial y}{\partial m} = b_0 x^m \log x \left[(m+2)\left\{1 - \frac{x^2}{m(m+4)}\right\} + \frac{x^4}{m(m+4)(m+6)} - \dots \right]$$
$$+ b_0 x^m \left[1 - \frac{(m+2)}{m(m+4)} \left\{ \frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4} \right\} x^2 \right.$$
$$\left. + \frac{1}{m(m+4)(m+6)} \left\{ -\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6} \right\} x^4 + \dots \right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-2}$

$$= b_0 x^{-2} \log x \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 \cdot 4 \cdot 6} \cdots \right] + b_0 x^{-2} \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \dots \right]$$

Hence the complete solution $y = c_1 y_1 + c_2 y_2$

i.e.,
$$y = C_1 x^2 \left\{ 1 - \frac{x^2}{2 \cdot 6} + \frac{x^4}{2 \cdot 4 \cdot 6 \cdot 8} - \frac{x^6}{2 \cdot 4 \cdot 6^2 \cdot 8 \cdot 10} + \dots \right\}$$
$$+ C_2 \left[x^2 \log x \left\{ -\frac{1}{2^2 \cdot 4} + \frac{x^4}{2^3 \cdot 4 \cdot 6} \cdots \right\} + x^{-2} \left\{ 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \cdots \right\} \right]$$

where $C_1 = c_1 a_0$, $C_2 = c_2 b_0$.

Example 16.7. *Solve in series* $xy'' + 2y' + xy = 0$. *(U.P.T.U., 2003)*

Solution. Here $x = 0$ is a singular point since coefficient of $y'' = 0$ at $x = 0$.

$\therefore$ Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \dots$ ...(*i*)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \dots$$

and
$$\frac{d^2 y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \dots$$

in the given equation, we get

$$x\,[m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \dots]$$
$$+ 2\,[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \dots]$$
$$+ x\,[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \dots]$$

Equating to zero, the coefficients of the lowest power of x,

$$m(m-1) a_0 + 2 m a_0 = 0 \text{ so that } m = 0, -1.$$

i.e., the roots are distinct of and differ by an integer.

Equating to zero, the coefficient of x^m, we get

$$(m+1) m a_1 + 2(m+1) a_1 = 0 \text{ i.e. } (m+1)(m+2) a_1 = 0$$

or $(m+1) a_1 = 0$ $[\because \ m + 2 \neq 0]$

When $m = -1$, $a_1 = 0/0$ *i.e., indeterminate.*

Hence the complete solution will be given by putting $m = -1$ *in y itself* (containing two arbitrary constants a_0 and a_1).

Now equating to zero, the coefficients of successive powers of x, we get

$(m+2)(m+3) a_2 + a_0 = 0$ [Coeff. of $x^{m+1} = 0$]

$(m+3)(m+4) a_3 + a_1 = 0$ [Coeff. of $x^{m+2} = 0$]

$(m+4)(m+5) a_4 + a_2 = 0$ [Coeff. of $x^{m+3} = 0$]

$(m+5)(m+6) a_5 + a_3 = 0$ etc. [Coeff. of $x^{m+4} = 0$]

i.e.,
$$a_2 = -\frac{a_0}{(m+2)(m+3)},\ a_3 = \frac{-a_1}{(m+3)(m+4)},\ a_4 = \frac{a_0}{(m+2)(m+3)(m+4)(m+5)},$$

$$a_5 = \frac{a_1}{(m+3)(m+4)(m+5)(m+6)} \text{ and so on.}$$

Substituting the values in (*i*), we get

$$y = x^m \left[a_0 + a_1 x - \frac{a_0}{(m+2)(m+3)} x^2 - \frac{a_1}{(m+3)(m+4)} x^3 \right.$$

$$\left. + \frac{a_0}{(m+2)(m+3)(m+4)(m+5)} x^4 + \frac{a_1}{(m+3)(m+4)(m+5)(m+6)} x^5 - \cdots \right]$$

Putting $m = -1$, the complete solution is

$$y = x^{-1} \left\{ a_0 \left(1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \cdots \right) + a_1 \left(x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \cdots \right) \right\}$$

$$= x^{-1} (a_0 \cos x + a_1 \sin x).$$

PROBLEMS 16.2

Solve the following equations in power series :

1. $4x \dfrac{d^2y}{dx^2} + 2 \dfrac{dy}{dx} + y = 0.$ (*P.T.U., 2005*)
2. $y'' + xy' + (x^2 + 2)y = 0.$ (*P.T.U., 2007*)
3. $x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0.$
4. $3x \dfrac{d^2y}{dx^2} + (1 - x) \dfrac{dy}{dx} - y = 0.$ (*S.V.T.U., 2008*)
5. $x \dfrac{d^2y}{dx^2} + (1 + x) \dfrac{dy}{dx} + 2y = 0.$ (*J.N.T.U., 2006*)
6. $2x^2y'' + xy' - (x + 1)y = 0.$ (*U.P.T.U., 2005*)
7. $8x^2 \dfrac{d^2y}{dx^2} + 10x \dfrac{dy}{dx} - (1 + x)y = 0.$ (*P.T.U., 2009*)
8. $2x(1 - x) \dfrac{d^2y}{dx^2} + (1 - x) \dfrac{dy}{dx} + 3y = 0.$ (*U.P.T.U., 2004*)
9. $x(1 - x) \dfrac{d^2y}{dx^2} - 3 \dfrac{dy}{dx} + 2y = 0.$
10. $(2x + x^3) \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0.$ (*Bhopal, 2008*)

16.5 BESSEL'S EQUATION*

One of the most important differential equations in applied mathematics is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \ldots(1)$$

which is known as *Bessel's equation of order n*. Its particular solutions are called *Bessel functions of order n*. Many physical problems involving vibrations or heat conduction in cylindrical regions give rise to this equation.

Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$

(1) takes the form

$$a_0(m^2 - n^2) x^m + a_1[(m+1)^2 - n^2] x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\} x^{m+2} + \ldots = 0.$$

Equating to zero the coefficient of x^m, we obtain the indicial equation $m^2 - n^2 = 0$ (as $a_0 \neq 0$) where $m = n$ or $-n$.

$$a_1 = a_3 = a_5 = a_7 = \ldots = 0$$

and $$a_2 = -\frac{a_0}{(m+2)^2 - n^2}, \quad a_4 = -\frac{a_2}{(m+4)^2 - n^2} \text{ etc.}$$

These give $$y = a_0 x^m \left(1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^4 - \cdots \right)$$

---

* Named after the German mathematician and astronomer *Friederich Wilhelm Bessel* (1784 – 1846) whose paper on Bessel functions appeared in 1826. He studied Astronomy of his own and became director of Königsberg observatory.

For $m = n$, we get

$$y_1 = a_0 x^n \left\{1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(n+1)(n+2)} x^4 - \frac{1}{4^3 \cdot 3!\,(n+1)(n+2)(n+3)} x^6 + \cdots \right\} \qquad ...(2)$$

and for $m = -n$, we have

$$y_2 = a_0 x^{-n} \left\{1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(-n+1)(-n+2)} x^4 - \frac{1}{4^3 \cdot 3!\,(-n+1)(-n+2)(-n+3)} x^6 + \cdots \right\} \qquad ...(3)$$

Case I. *When n is not integral or zero,* the complete solution of (1) is $y = c_1 y_1 + c_2 y_2$.

If we take $a_0 = 1/2^n \,\Gamma(n+1)$, then the solution given by (2) is called the *Bessel function of the first kind of order n* and is denoted by $J_n(x)$. Thus

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{\frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)}\left(\frac{x}{2}\right)^6 + \cdots \right\} \quad (n > 0)$$

i.e.

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)} \qquad ...(4)$$

and corresponding to (3), we have $J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \dfrac{1}{r!\,\Gamma(-n+r+1)}$ ...(5)

which is called the *Bessel function of the first kind of order* $-n$.

Hence complete solution of the Bessel's equation (1) may be expressed in the form.

$$y = AJ_n(x) + BJ_{-n}(x). \qquad ...(6)$$

Case II. *When n is zero,* $y_1 = y_2$ and the complete solution of (1), which reduces to the *Bessel's equation of order zero,* is obtained as in Example 16.4.

Case III. *When n is integral,* y_2 fails to give a solution for positive values of n and y_1 fails to give a solution for negative values. Thus another independent integral of the Bessel's equation (1) is needed to form its general solution. We now proceed to find an independent solution of (1), when n is an integer.

Let $y = u(x)J_n(x)$ be a solution of (1). Substituting the values of y, $y' = u'J_n + u\,J_n'$ and $y'' = u''J_n + 2u'J_n' + uJ_n''$ in (1), we obtain

$$x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + u\,J_n') + (x^2 - n^2)uJ_n = 0$$

or

$$u\{x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n\} + x^2u''J_n + 2x^2u'J_n' + xu'J_n = 0. \qquad ...(7)$$

Now since J_n is a solution of (1), therefore, $x^2 J_n'' + x\,J_n' + (x^2 - n^2)J_n = 0$

$\therefore$ (7) reduces to $x^2u''J_n + 2x^2u'\,J_n' + xu'\,J_n = 0$.

Dividing throughout by $x^2u'J_n$, it becomes $\dfrac{u''}{u'} + 2\dfrac{J_n'}{J_n} + \dfrac{1}{x} = 0$

i.e.,

$$\frac{d}{dx}(\log u') + 2\frac{d}{dx}(\log J_n) + \frac{d}{dx}(\log x) = 0 \text{ or } \frac{d}{dx}\{\log(u'J_n^2 x)\} = 0.$$

Integrating, $\log(u'J_n^2 x) = \log B$, whence $xu'J_n^2 = B$.

$$\therefore \qquad u' = \frac{B}{xJ_n^2} \text{ or } u = B\int \frac{dx}{xJ_n^2} + A.$$

Thus

$$y = AJ_n(x) + BJ_n(x)\int \frac{dx}{x[J_n(x)]^2}.$$

Hence the complete solution of the Bessel's equation (1) is

$$y = AJ_n(x) + BY_n(x) \qquad ...(8) \quad (V.T.U., 2006)$$

where

$$Y_n(x) = J_n(x)\int \frac{dx}{x[J_n(x)]^2} \qquad ...(9)$$

$Y_n(x)$ is called the *Bessel function of the second kind of order n or Neumann function**.

---

* Named after the German mathematician and physicist *Carl Neumann* (1832–1925) whose work on potential theory gave impetus for development of integral equations by *Volterra* of Rome, *Fredholm* of Stockholm and *Hilbert* of Gottingen.

Obs. Putting $k = -n + r$, *i.e.* $r = k + n$, and noting that $\Gamma(k+1) = k!$ where k is an integer, (5) may be written as

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n} (x/2)^{2k+n}}{(k+n)!\, k!} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+2k}}{k!\, \Gamma(k+n+1)}$$

Hence $J_{-n}(x) = (-1)^n J_n(x)$. ...(10) (*Bhopal, 2008 ; S.V.T.U., 2008 ; V.T.U., 2006*)

16.6 RECURRENCE FORMULAE FOR $J_n(x)$

The following recurrence formulae can easily be derived from the series expression for $J_n(x)$:

(1) $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$.

(2) $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$.

(3) $J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$.

(4) $J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$.

(5) $J_n'(x) = \dfrac{n}{x} J_n(x) - J_{n+1}(x)$.

(6) $J_{n+1}(x) = \dfrac{2n}{x} J_n(x) - J_{n-1}(x)$.

These formulae are very useful in the solution of boundary value problems and in establishing the various properties of Bessel functions.

Proofs. (1) Multiplying (4) of page 551 by x^n, we have

$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\, \Gamma(n+r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2(n+r)}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r) x^{2(n+r)-1}}{2^{n+2r}\, r!\, \Gamma(n+r+1)} = x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n-1+2r}}{r!\, \Gamma(n-1+r+1)} = x^n J_{n-1}(x).$$

(*Bhopal, 2008 ; V.T.U., 2005 ; U.P.T.U., 2005*)

(2) Multiplying (4) of page 551 by x^{-n}, we have

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r} r!\, \Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}[x^{-n} J_n(x)] = \sum_{r=0}^{\infty} \frac{(-1)^r 2r\, x^{2r-1}}{2^{n+2r} r!\, \Gamma(n+r+1)} = -x^{-n} \sum_{r=1}^{\infty} \frac{(-1)^{r-1} x^{n+1+2(r-1)}}{2^{n+1+2(r-1)} (r-1)!\, \Gamma(n+r+1)}$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+1+2k}}{k!\, \Gamma(n+1+k+1)} = -x^{-n} J_{n+1}(x), \text{ where } k = r-1.$$

(*P.T.U., 2006 ; B.P.T.U., 2005*)

(3) From (1), we have $x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$

or dividing by x^n, $\quad J_n'(x) + (n/x) J_n(x) = J_{n-1}(x)$ ...(*i*)

Similarly from (2), we get $x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$

or $\quad -J_n'(x) + \dfrac{n}{x} J_n(x) = J_{n+1}(x)$ ...(*ii*)

Adding (*i*) and (*ii*), we obtain $\dfrac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$

i.e., $\quad J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$ (*S.V.T.U., 2008 ; Anna, 2005 S*)

(4) Subtracting (*ii*) from (*i*), we get $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

i.e., $\quad J_n'(x) = \dfrac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$. (*S.V.T.U., 2007 ; P.T.U., 2005*)

(5) is another way of writing (*ii*). (*J.N.T.U., 2006 ; Anna, 2005*)

(6) is another way of writing (3). (*Madras, 2006 ; V.T.U., 2005*)

16.7 (1) EXPANSIONS FOR J_0 AND J_1

We have from (4) of page 551,

$$J_0(x) = 1 - \frac{1}{(1!)}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \quad ...(1)$$

and

$$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right] \quad \textit{(B.P.T.U., 2005)} ...(2)$$

Because of their special importance, the values of $J_0(x)$ and $J_1(x)$ are given in Appendix 2 : Table II to four decimal places at intervals of 0.1. With the help of these values, the graphs of $J_0(x)$ and $J_1(x)$ can be drawn as shown in Fig. 16.1, for $x > 0$. Their close resemblance to graphs of $\cos x$ and $\sin x$ is interesting.

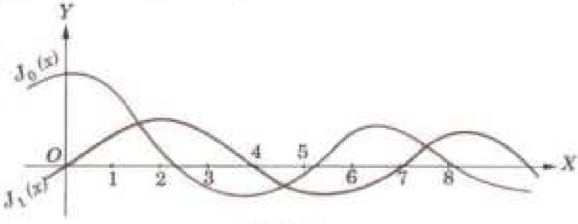


Fig. 16.1

Obs. The roots of the equation $J_0(x) = 0$ are useful in some physical problems. This equation has no complex roots but an infinite number of real roots. Its first four roots are $x = 2.4, 5.52, 8.65, 11.79$ approximately.

16.8 VALUE OF $J_{1/2}$

We may think that $J_0(x)$ is the simplest of the J's but actually $J_{1/2}(x)$ is simpler, for it can be expressed in a finite form. Taking $n = \frac{1}{2}$ in (4) of page 551, we have

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\,\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \frac{\sqrt{x}}{\sqrt{2}\,\Gamma\left(\frac{1}{2}\right)}\left\{\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \ldots\right\}$$

Now multiplying the series by $x/2$ and outside by $2/x$, we get

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right\} = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x. \quad ...(3) \quad \textit{(V.T.U., 2009 ; J.N.T.U., 2003)}$$

Similarly taking $n = \frac{1}{2}$ in (5) of page 551, it can be shown that

$$J_{-\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x. \quad ...(4) \quad \textit{(Anna, 2005 ; W.B.T.U., 2005 ; V.T.U., 2003)}$$

Example 16.8. *Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.* *(Bhopal, 2008 S ; V.T.U., 2001)*

Solution. We know that

$$J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)] \text{ i.e. } J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

Putting $n = 1, 2, 3, 4$ successively, $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$ ...(*i*) $\qquad J_3(x) = \frac{4}{x}J_2(x) - J_1(x)$ ...(*ii*)

$J_4(x) = \frac{6}{x}J_3(x) - J_2(x)$ ...(*iii*) $\qquad J_5(x) = \frac{8}{x}J_4(x) - J_3(x)$ ...(*iv*)

Substituting the value of $J_2(x)$ in (*ii*), we have

$$J_3(x) = \frac{4}{x}\left\{\frac{2}{x}J_1(x) - J_0(x)\right\} - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \quad ...(v)$$

(*W.B.T.U., 2005 ; Madras, 2003*)

Now substituting the values of $J_3(x)$ from (v) and $J_2(x)$ from (i) in (iii), we get

$$J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x) \quad ...(vi) \quad (V.T.U., 2003\ S)$$

Finally putting the values of $J_4(x)$ from (vi) and $J_3(x)$ from (v) in (iv), we obtain

$$J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x).$$

Example 16.9. *Prove that* $J_{5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$. (*J.N.T.U., 2006*)

Solution. We know that $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ ...(i)

Putting $n = \frac{1}{2}$, we get $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$

(*Bhopal, 2007 ; V.T.U., 2006*)

Again putting $n = \frac{3}{2}$ in (i), we get $J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$

$$= \frac{3}{x}\left[\sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)\right] - \sqrt{\left(\frac{2}{\pi x}\right)}\sin x = \sqrt{\left(\frac{2}{\pi x}\right)}\left[\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right]$$

which is the required result.

Example 16.10. *Prove that*

(a) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$. (*J.N.T.U., 2006*)

(b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)]$. (*V.T.U., 2006*)

Solution. (a) We know that $J_n'(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$ ...(i)

Differentiating both sides, we get $J_n''(x) = \frac{1}{2}\{J_{n-1}'(x) - J_{n+1}'(x)\}$ ...(ii)

Changing n to $n-1$ in (i), we obtain $J_{n-1}'(x) = \frac{1}{2}\{J_{n-2}(x) - J_n(x)\}$ ...(iii)

Changing n to $n+1$ in (i), we have $J_{n+1}'(x) = \frac{1}{2}\{J_n(x) - J_{n+2}(x)\}$ ...(iv)

Substituting the values of $J_{n-1}'(x)$ and $J_{n+1}'(x)$ from (iii) and (iv) in (ii), we get

$$J_n'' = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

(b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x[J_n(x)J'_{n+1}(x) + J_n'(x)J_{n+1}(x)]$ ...(i)

From (5) of § 16.6, we have $J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$ ...(ii)

Changing n to $n+1$ in (i) of page 499, we get $J'_{n+1}(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x)$ ...(iii)

Now substituting from (*iii*) and (*ii*) in (*i*), we get

$$\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x\left[J_n(x)\left\{J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right\} + \left\{\frac{n}{x}J_n(x) - J_{n+1}(x)\right\}J_{n+1}(x)\right]$$

$$= x\{J_n^2(x) - J_{n+1}^2(x)\}.$$

Example 16.11. *Prove that :*

(*a*) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x).$

(*b*) $\int xJ_0^2(x)dx = \frac{1}{2}x^2\{J_0^2(x) + J_1^2(x)\}.$ (*U.P.T.U., 2004 ; Osmania, 2002*)

Solution. (*a*) We know that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ [§ 16.6 (2)] ...(*i*)

or $\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ ...(*ii*)

$\therefore$ $\int J_3(x)dx = \int x^2 \cdot x^{-2}J_3(x)dx + c$ [Integrate by parts]

$$= x^2 \cdot \int x^{-2}J_3(x)dx - \int 2x\left[\int x^{-2}J_3(x)dx\right]dx + c$$

$$= x^2[-x^{-2}J_2(x)] - \int 2x[-x^{-2}J_2(x)]\,dx + c \quad \text{[By (ii) when } n = 2]$$

$$= c - J_2(x) + \int \frac{2}{x}J_2(x)\,dx = c - J_2(x) - \frac{2}{x}J_1(x) \quad \text{[By (ii) when } n = 1]$$

(*b*) $\int xJ_0^2(x)dx = \int J_0^2(x)\cdot x\,dx$ [Integrate by parts]

$$= J_0^2(x)\cdot\frac{1}{2}x^2 - \int 2J_0(x)J_0'(x)\cdot\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}x^2J_0^2(x) + \int x^2J_0(x)J_1(x)\,dx \quad \text{[By (i) when } n = 0]$$

$$= \frac{1}{2}x^2J_0^2(x) + \int xJ_1(x)\cdot\frac{d}{dx}[xJ_1(x)]\,dx \quad \left[\because \frac{d}{dx}[xJ_1(x)] = xJ_0(x) \text{ by § 16.6 (1)}\right]$$

$$= \frac{1}{2}x^2J_0^2(x) + \frac{1}{2}[xJ_1(x)]^2 = \frac{1}{2}x^2\{J_0^2(x) + J_1^2(x)\}.$$

16.9 GENERATING FUNCTION FOR $J_n(x)$

To prove that $e^{\frac{1}{2}x(t-1/t)} = \sum_{n=-\infty}^{\infty} t^nJ_n(x).$

We have $e^{\frac{1}{2}x(t-t^{-1})} = e^{xt/2} \times e^{-x/2t}$

$$= \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3!}\left(\frac{xt}{2}\right)^3 + \cdots\right] \times \left[1 - \left(\frac{x}{2t}\right) + \frac{1}{2!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3!}\left(\frac{x}{2t}\right)^3 + \cdots\right]$$

The coefficient of t^n in this product

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} - \ldots = J_n(x).$$

As all the integral powers of t, both positive and negative occur, we have

$$e^{\frac{1}{2}x(t-t^{-1})} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

$$= \sum_{n=-\infty}^{\infty} t^nJ_n(x)$$ (*V.T.U., 2007*)

This shows that Bessel functions of various orders can be derived as coefficients of different powers of t in the expansion of $e^{\frac{1}{2}x(t-1/t)}$. For this reason, it is known as the *generating function of Bessel functions.*

Example 16.12. *Show that*

(a) $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$, *n being an integer.* (*V.T.U., 2006*)

(b) $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi$. (*Madras, 2006*)

(c) $J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \ldots = 1.$ (*Kerala M. Tech, 2005 ; U.P.T.U., 2003; V.T.U., 2003 S*)

Solution. (*a*) We know that

$$e^{\frac{1}{2}x(t-1/t)} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \ldots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$

$$\therefore \quad e^{\frac{1}{2}x(t-1/t)} = J_0 + J_1(t - 1/t) + J_2(t^2 + 1/t^2) + J_3(t^3 - 1/t^3) + \ldots \qquad \ldots(i)$$

Now put $t = \cos\theta + i\sin\theta$

so that $t^p = \cos p\theta + i\sin p\theta$ and $1/t^p = \cos p\theta - i\sin p\theta$

giving $t^p + 1/t^p = 2\cos p\theta$ and $t^p - 1/t^p = 2i\sin p\theta.$

Substituting these in (*i*), we get

$$e^{ix\sin\theta} = J_0 + 2[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] + 2i\,[J_1\sin\theta + J_3\sin 3\theta + \ldots] \qquad \ldots(ii)$$

Since $e^{ix\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta).$

$\therefore$ equating the real and imaginary parts in (*ii*), we get

$$\cos(x\sin\theta) = J_0 + 2\,[J_2\cos 2\theta + J_4\cos 4\theta + \ldots] \qquad \ldots(iii)$$

$$\sin(x\sin\theta) = 2[J_1\sin\theta + J_3\sin 3\theta + \ldots] \qquad \ldots(iv)$$

which are known as *Jacobi series**. (*V.T.U., 2006*)

Now multiplying both sides of (*iii*) by $\cos n\theta$ and both sides of (*iv*) by $\sin n\theta$ and integrating each of the resulting expressions between 0 and π, we obtain

$$\frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\cos n\theta\,d\theta = \begin{cases} J_n(x), & n \text{ even or zero} \\ 0, & n \text{ odd} \end{cases}$$

and

$$\frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin n\theta\,d\theta = \begin{cases} 0 & n \text{ even} \\ J_n(x), & n \text{ odd} \end{cases}$$

Hence generally, if n is a positive integer,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\sin n\theta]\,d\theta = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

[This is Bessel's original definition of $J_n(x)$ given in 1824 while investigating Planetary motion.]

(*b*) Changing θ to $\frac{1}{2}\pi - \phi$ in (*iii*), we get

$$\cos(x\cos\phi) = J_0 + 2J_2\cos(\pi - 2\phi) + 2J_4\cos(2\pi - 4\phi) + \ldots$$
$$= J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \ldots$$

Integrating both sides w.r.t. ϕ from 0 to π, we get

$$\int_0^\pi \cos(x\cos\phi)\,d\phi = \int_0^\pi [J_0(x) - 2J_2(x)\cos 2\phi + 2J_4(x)\cos 4\phi - \cdots]\,d\phi$$

$$= \left| J_0(x)\cdot\phi - 2J_2(x)\cdot\frac{1}{2}\sin 2\phi + 2J_4(x)\cdot\frac{1}{4}\sin 4\phi - \cdots \right|_0^\pi = J_0(x)\,.\,\pi \text{ whence follows the result.}$$

* See footnot p. 215.

(c) Squaring (*iii*) and (*iv*) and integrating w.r.t. ϕ from 0 to π and noting that (m, n being integers),

$$\int_0^{\pi} \cos m\theta \cos n\theta \, d\theta = \int_0^{\pi} \sin m\theta \sin n\theta \, d\theta = 0, \qquad (m \neq n)$$

and
$$\int_0^{\pi} \cos^2 n\theta \, d\theta = \int_0^{\pi} \sin^2 n\theta \, d\theta = \pi/2, \text{ we obtain}$$

$$[J_0(x)]^2 \pi + 4\,[J_2(x)]^2 \frac{\pi}{2} + 4\,[J_4(x)]^2 \frac{\pi}{2} + \ldots = \int_0^{\pi} \cos^2 (x \sin \theta)\, d\theta$$

$$4\,[J_1(x)]^2 \frac{\pi}{2} + 4\,[J_3(x)]^2 \frac{\pi}{2} + \ldots = \int_0^{\pi} \sin^2 (x \sin \theta)\, d\theta$$

Adding, $\quad \pi\,[J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots] = \int_0^{\pi} d\theta = \pi$

Hence $\quad J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \cdots = 1.$

PROBLEMS 16.3

1. Compute $J_0(2)$, $J_1(1)$ correct to three decimal places.
2. Show that (*i*) $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$. (*ii*) $J_1(x) + J_3(x) = \frac{4}{x} J_2(x)$ (P.T.U., 2003)
3. Show that
   (*i*) $J_{-1/2}(x) = J_{1/2}(x) \cot x$. (S.V.T.U., 2008) (*ii*) $J'_{1/2}(x) J_{-1/2}(x) - J'_{-1/2}(x) J_{1/2}(x) = 2/\pi x$ (Delhi, 2002)
   (*iii*) $J_{-3/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\left(\sin x + \frac{\cos x}{x}\right)$ (*iv*) $J_{-5/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)\left(\frac{3}{x}\sin x + \frac{3 - x^2}{x^2}\cos x\right)}$ (V.T.U., 2000)
4. Prove that (*i*) $\frac{d}{dx} J_0(x) = -J_1(x)$. (*ii*) $\frac{d}{dx}\,[xJ_1(x)] = xJ_0(x)$.
   (*iii*) $\frac{d}{dx}\,[x^n J_n(ax)] = ax^n J_{n-1}(ax)$. (Madras, 2000 S) (*iv*) $J'_n(x) = -\frac{n}{2} J_n(x) + J_{n-1}(x)$ (P.T.U., 2009 S)
5. Show by the use of recurrence formula, that
   (*i*) $J_0''(x) = \frac{1}{2}\,[J_2(x) - J_0(x)]$ (*ii*) $J_1''(x) = J_1(x) - \frac{1}{x} J_2(x)$.
   (*iii*) $4J_0'''(x) + 3J_0'(x) + J_3(x) = 0$. (Osmania, 2003)
6. Prove that
   (*i*) $\frac{d}{dx}\,[J_n^2(x)] = \frac{x}{2n}\,[J_{n-1}^2(x) - J_{n+1}^2(x)]$ (S.V.T.U., 2008 ; Kerala M.E., 2005)
   (*ii*) $\frac{d}{dx}\,[J_n^2(x) + J_{n+1}^2(x)] = 2\left\{\frac{n}{2} J_n^2(x) - \frac{n+1}{x} J_{n+1}^2(x)\right\}$ (U.P.T.U., 2005 ; V.T.U., 2000 S)
7. Prove that (*i*) $\int_0^{\pi/2} \sqrt{(\pi x)}\, J_{1/2}(2x)\, dx = 1$. (P.T.U., 2005) (*ii*) $\int_0^{r} xJ_0(ax)dx = \frac{r}{a} J_1(ar)$.
   (*iii*) $\int x^2 J_1(x)dx = x^2 J_2(x)$. (P.T.U., 2007)
8. Prove that (*i*) $\int J_0(x) J_1(x)\, dx = -\frac{1}{2}\,[J_0(x)]^2$. (*ii*) $\int_0^{\infty} e^{-ax} J_0(bx)dx = \frac{1}{\sqrt{a^2 + b^2}}$
9. Starting with the series of § 16.9, prove that
   $$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \text{ and } xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x).$$
10. Establish the *Jacobi series*
   $$\cos (x \cos \theta) = J_0 - 2J_2 \cos 2\theta + 2J_4 \cos 4\theta - \ldots$$
   $$\sin (x \cos \theta) = 2[J_1 \cos \theta - J_3 \cos 3\theta + J_5 \cos 5\theta - \ldots]$$ (Madras, 2003 S)
11. Prove that (*i*) $\sin x = 2[J_1 - J_3 + J_5 - \ldots]$ (Anna, 2005 S)
   (*ii*) $\cos x = J_0 - 2J_2 + 2J_4 - 2J_6 + \ldots$ (Kerala M. Tech., 2005)
   (*iii*) $1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \ldots$

16.10 EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

In many problems, we come across such differential equations which can easily be reduced to Bessel's equation and, therefore, can be solved by means of Bessel functions.

(1) *To reduce the differential equation* $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+(k^2x^2-n^2)\,y=0$ *to Bessel form.*

Put $t = kx$, so that $\dfrac{dy}{dx}=k\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dx^2}=k^2\dfrac{d^2y}{dt^2}$.

Then (1) becomes $t^2\dfrac{d^2y}{dt^2}+t\dfrac{dy}{dt}+(t^2-n^2)y=0$

$\therefore$ its solution is $y=c_1J_n(t)+c_2J_{-n}(t)$, n is non-integral,

or $\quad y=c_1J_n(t)+c_2Y_n(t)$, n is integral.

Hence the solution of (1) is

$$y=c_1J_n(kx)+c_2J_{-n}(kx),\ n \text{ is non-integral}$$

or

$$y=c_1J_n(kx)+c_2Y_n(kx),\ n \text{ is integral.}$$

(2) *To reduce the differential equation* $x\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}+k^2xy=0$ *to Bessel's equation,* (*Madras, 2006*)

put $\quad y=x^nz,$

so that $\quad \dfrac{dy}{dx}=x^n\dfrac{dz}{dx}+nx^{n-1}z$ and $\dfrac{d^2y}{dx^2}=x^n\dfrac{d^2z}{dx^2}+2nx^{n-1}\dfrac{dz}{dx}+n(n-1)x^{n-2}z$

Then (2) takes the form $x^{n+1}\dfrac{d^2z}{dx^2}+(2n+a)x^n\dfrac{dz}{dx}+[k^2x^2+n^2+(a-1)n]x^{n-1}z=0.$

Dividing throughout by x^{n-1} and putting $2n+a=1$,

$$x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}+(k^2x^2-n^2)z=0.$$

Its solution by (1) is $z=c_1J_n(kx)+c_2J_{-n}(kx)$, n is non-integral

or $\quad z=c_1J_n(kx)+c_2Y_n(kx)$, n is integral

Hence the solution of (2) is $y=x^n\,[c_1J_n(kx)+c_2J_{-n}(kx)]$, n is non-integral

or $\quad y=x^n\,[c_1J_n(kx)+c_2Y_n(kx)]$, n is integral, where $n=(1-a)/2$.

(3) *To reduce the differential equation* $x\dfrac{d^2y}{dx^2}+c\dfrac{dy}{dx}+k^2x^ry=0$ *to Bessel form,* put $x=t^m$, *i.e.* $t=x^{1/m}$,

so that $\quad \dfrac{dy}{dx}=\dfrac{dy}{dt}\cdot\dfrac{dt}{dx}=\dfrac{1}{m}t^{1-m}\dfrac{dy}{dt}$

and $\quad \dfrac{d^2y}{dx^2}=\dfrac{d}{dt}\left(\dfrac{1}{m}t^{1-m}\dfrac{dy}{dt}\right)\cdot\dfrac{1}{m}t^{1-m}\ \dfrac{1}{m^2}t^{2-2m}\dfrac{d^2y}{dt^2}+\dfrac{1-m}{m^2}t^{1-2m}\dfrac{dy}{dt}$

Then (3) takes the form $\dfrac{1}{m^2}t^{2-m}\dfrac{d^2y}{dt^2}+\dfrac{1-m+cm}{m^2}t^{1-m}\dfrac{dy}{dt}+k^2t^{mr}y=0$

or multiplying throughout by m^2/t^{1-m}, $t\dfrac{d^2y}{dt^2}+(1-m+cm)\dfrac{dy}{dt}+(km)^2t^{mr+m-1}y=0.$

In order to reduce it to (2), we set $mr+m-1=1$, *i.e.* $m=2/(r+1)$

and $\quad a=1-m+cm=(r+2c-1)/(r+1).$

Thus it reduces to $t\dfrac{d^2y}{dt^2}+a\dfrac{dy}{dt}+(km)^2ty=0$ which is similar to (2).

Hence the solution of (3) is $y=x^{n/m}[c_1J_n(k_mx^{1/m})+c_2J_{-n}(kmx^{1/m})]$, n is a fraction

or $\quad y=x^{n/m}\,[c_1J_n(kmx^{1/m})+c_2Y_n(kmx^{1/m})]$, n is an integer

where $\quad n=\dfrac{1-a}{2}=\dfrac{1-c}{1+r}$ and $m=\dfrac{2}{1+r}$.

Example 16.13. *Solve the differential equations :*

(i) $y'' + \frac{y'}{x} + \left(8 - \frac{1}{x^2}\right) y = 0.$ (ii) $4y'' + 9xy = 0.$ (iii) $xy'' + y' + \frac{1}{4}y = 0.$ (Anna, 2005)

Solution. (i) Rewriting the given equation as $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (8x^2 - 1)y = 0,$

and comparing with (1) above, we see that $n = 1$ and $k = 2\sqrt{2}$.

$\therefore$ The solution of the given equation is $y = c_1 J_n(kx) + c_2 Y_n(kx)$

i.e., $$y = c_1 J_1(2\sqrt{2x}) + c_2 Y_1(2\sqrt{2x}).$$

(ii) Rewriting the given equation as $x\frac{d^2y}{dx^2} + \frac{9}{4}x^2 y = 0$ ...(α)

and comparing with (3) above, we find that $c = 0$, $k = 3/2$ and $r = 2$.

$$\therefore \quad n = \frac{1-c}{1+r} = \frac{1}{3}, \quad m = \frac{2}{1+r} = \frac{2}{3} \quad \text{and} \quad \frac{n}{m} = \frac{1}{2}.$$

Hence the solution of (α) is $y = x^{n/m} [c_1 J_n(kmx^{1/m}) + c_2 Y_{-n}(kmx^{1/m})]$

$$y = \sqrt{x}\,[c_1 J_{1/3}(x^{3/2}) + c_2 J_{-1/3}(x^{3/2})].$$

(iii) Multiplying by x, the given equation becomes

$$x^2y'' + xy' + \frac{1}{4}xy = 0 \qquad ...(\alpha)$$

Comparing with (3) above, we get $c = 1$, $k = 1/2$ & $r = 0$. $\therefore\ m = \frac{2}{1+r} = 2, n = \frac{1-c}{1+r} = 0\ \&\ \frac{n}{m} = 0$

Hence the solution of (α)

$$y = x^{n/m} [c_1 J_n(kmx^{1/m}) + c_2 Y_n(kmx^{1/m})] = x^0 \left\{c_1 J_0\left(\frac{1}{2}2x^{1/2}\right) + c_2 Y_0\left(\frac{1}{2}.2x^{1/2}\right)\right\}$$

i.e., $$y = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x})$$

16.11 (1) ORTHOGONALITY OF BESSEL FUNCTIONS

We shall prove that

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}, \text{ where } \alpha, \beta \text{ are the roots of } J_n(x) = 0.$$

We know that the solution of the equation

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad ...(1)$$

and $$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad ...(2)$$

are $$u = J_n(\alpha x) \text{ and } v = J_n(\beta x) \text{ respectively.}$$

Multiplying (1) by v/x and (2) by u/x and subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

or $$\frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2)\,xuv.$$

Now integrating both sides from 0 to 1,

$$(\beta^2 - \alpha^2)\int_0^1 xuv\,dx = [x(u'v - uv')]_0^1 = (u'v - uv')_{x=1} \qquad ...(3)$$

Since $$u = J_n(\alpha x),$$

$$\therefore \quad u' = \frac{d}{dx}[J_n(\alpha x)] = \frac{d}{d(\alpha x)}[J_n(\alpha x)].\frac{d(\alpha x)}{dx} = \alpha J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x)$ and $v' = \beta J_n'(\beta x)$. Substituting these values in (3), we get

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \frac{\alpha J_n'(\alpha)J_n(\beta) - \beta J_n(\alpha)J_n'(\beta)}{\beta^2 - \alpha^2} \quad ...(4)$$

If α and β are distinct roots of $J_n(x) = 0$, then $J_n(\alpha) = J_n(\beta) = 0$, and (4) reduces to

$$\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = 0 \quad ...(5)$$

This is known as the *orthogonality relation of Bessel functions.*

When $\beta = \alpha$, the right side of (4) is of 0/0 form. Its value can be found by considering α as a root of $J_n(x) = 0$ and β as a variable approaching α. Then (4) gives

$$\underset{\beta\to\alpha}{\text{Lt}} \int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n(\beta)}{\beta^2 - \alpha^2}$$

or by L'Hospital's rule, $\displaystyle\int_0^1 xJ_n^2(\alpha x)dx = \underset{\beta\to\alpha}{\text{Lt}} \frac{\alpha J_n'(\alpha)J_n'(\beta)}{2\beta} = \frac{1}{2}[J_n'(\alpha)]^2$

$$= \frac{1}{2}[J_{n+1}(\alpha)]^2 \quad ...(6) \quad \text{[By (5) of p. 552]}$$

Obs. If however, the interval be from 0 to 1, it can be shown that

$$\int_0^1 xJ_n^2(\alpha x)dx = \frac{1}{2}[J_n'(\alpha)]^2 \quad \text{where } \alpha \text{ is the root of } J_n(x) = 0. \quad ...(7) \quad (V.T.U., 2006)$$

(2) Fourier-Bessel expansion. If $f(x)$ is a continuous function having finite number of oscillations in the interval $(0, a)$, then we can write

$$f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + ... + c_nJ_n(\alpha_n x) + ... \quad ...(8)$$

where α_1, α_2 ... are the positive roots of $J_n(x) = 0$.

To determine the coefficients c_n, multiply both sides of (8) by $xJ_n(\alpha_n x)$ and integrate from 0 to a. Then all integrals on the right of (1) vanish by (5), except the term in c_n. This gives

$$\int_0^a xf(x)J_n(\alpha_n x)dx = c_n\int_0^a xJ_n^2(\alpha_n x)dx = c_n\frac{a^2}{2}J_{n+1}^2(a\alpha_n) \quad \text{[By (7)]}$$

$$\therefore \quad c_n = \frac{2}{a^2J_{n+1}^2(a\alpha_n)}\int_0^a xf(x)J_n(\alpha_n x)\,dx$$

Equation (8) is known as the *Fourier-Bessel expansion of $f(x)$.*

Example 16.14. *If $\alpha_1, \alpha_2, ... \alpha_n$ are the positive roots of $J_0(x) = 0$, show that*

$$\frac{1}{2} = \sum_{n=1}^{\infty} [J_0(\alpha_n x)/\alpha_n J_1(\alpha_n)].$$

Solution. If $\quad f(x) = c_1J_n(\alpha_1 x) + c_2J_n(\alpha_2 x) + ... + c_rJ_n(\alpha_n x) + ... \quad ...(i)$

then $\quad c_r = \dfrac{2}{a^2J_{n+1}^2(a\alpha_r)}\displaystyle\int_0^a x\,f(x)J_n(\alpha_n x)dx$

Taking $f(x) = 1$, $a = 1$ and $n = 0$, we get

$$c_r = \frac{2}{J_1^2(\alpha_r)}\int_0^1 xJ_0(\alpha_r x)dx = \frac{2}{J_1^2(\alpha_r)}\left|\frac{xJ_1(\alpha_r x)}{\alpha_r}\right|_0^1 = \frac{2}{\alpha_rJ_1(\alpha_r)}$$

From (i), $\quad 1 = \displaystyle\sum_{r=1}^{\infty}\frac{2}{\alpha_rJ_1(\alpha_r)}J_0(\alpha_r x) \quad$ or $\quad \frac{1}{2} = \sum_{n=1}^{\infty}\frac{J_0(\alpha_n x)}{\alpha_nJ_1(\alpha_n)}.$

Example 16.15. *Expand $f(x) = x^2$ in the interval $0 < x < 2$ in terms of $J_2(\alpha_n x)$, where α_n are determined by $J_2(2\alpha_n) = 0$.*

Solution. Let the Fourier-Bessel expansion of $f(x)$ be $x^2 = \displaystyle\sum_{n=1}^{\infty} c_nJ_2(\alpha_n x).$

Multiplying both sides by $xJ_2(\alpha_n x)$ and integrating w.r.t. x from 0 to 2, we get

$$\int_0^2 x^3 J_2(\alpha_n x)dx = c_n \int_0^2 xJ_2^2(\alpha_n x)dx = c_n \frac{(2)^2}{2} J_3^2(2\alpha_n) \quad \text{[By (7)]}$$

or
$$\left| \frac{x^3 J_3(\alpha_n x)}{\alpha_n} \right|_0^2 = 2c_n J_3^2(2\alpha_n)$$

$\therefore$
$$c_n = \frac{4}{\alpha_n J_3(2\alpha_n)}$$

Hence
$$x^2 = 4\sum_{n=1}^{\infty} \frac{J_2(\alpha_n x)}{\alpha_n J_3(2\alpha_n)}.$$

16.12 BER AND BEI FUNCTIONS

Consider the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - ixy = 0 \quad ...(1)$$

which occurs in certain problems of electrical engineering. This is equation (1) of §16.10 with $n = 0$ and $k^2 = -i$, so that its particular solution is

$$y = J_0(kx) = J_0[(-i)^{1/2} x] = J_0(i^{3/2} x)$$

Replacing $i^{3/2} x$ in the series for $J_0(x)$ [§16.8], we get

$$y = 1 - \frac{i^3 x^2}{2^2} + \frac{i^6 x^4}{(2\,!)^2\, 2^4} - \frac{i^9 x^6}{(3\,!)^2 2^6} + \frac{i^{12} x^8}{(4!)^2\, 2^8} - ...$$

$$= \left[1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - ...\right] + i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - ...\right] \quad ...(2)$$

which is complex for x real. The series in the above brackets are taken to define *Bessel-real (or ber)* and *Bessel-imaginary (or bei) functions.*

Thus
$$ber\, x = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 ... (4m)^2} \quad ...(3)$$

and
$$bei\, x = -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 ... (4m-2)^2} \quad ...(4)$$

so that $y = ber\, x + i\, bei\, x$ is a solution of (1).

Tables giving numerical values of $ber\, x$ and $bei\, x$ are also available.

Example 16.16. *Prove that* (i) $\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$ (ii) $\frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$

Solution. We have $x\, ber'\, x = x\sum_{m=1}^{\infty} (-1)^m \frac{4mx^{4m-1}}{2^2 \cdot 4^2 \cdot 6^2 ... (4m)^2}$

$$= \sum_{m=1}^{\infty} (-1)^m \cdot \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 ... (4m-2)^2\, 4m} = -\int_0^x x\, bei\, x\, dx$$

or
$$\frac{d}{dx}(x\, ber'\, x) = -x\, bei\, x$$

Again
$$\int_0^x x\, ber\, x\, dx = \frac{x^2}{2} + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m+2}}{2^2 \cdot 4^2 \cdot 6^2 ... (4m)^2 (4m+2)}$$

$$= -\sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 ... (4m-4)^2 (4m-2)} = x\, bei'\, x \quad \text{or} \quad \frac{d}{dx}(x\, bei'\, x) = x\, ber\, x.$$

PROBLEMS 16.4

Obtain the solutions of the following differential equations in terms of Bessel functions :

1. $y'' + \dfrac{y'}{x} + \left(1 - \dfrac{1}{9x^2}\right) y = 0.$
2. $y'' + \dfrac{y'}{2} + \left(1 - \dfrac{1}{6.25x^2}\right) y = 0.$
3. $xy'' + ay' + k^2xy = 0.$ *(V.T.U., 2010)*
4. $x^2y'' - xy' + 4x^2y = 0.$
5. $xy'' + y = 0.$
6. Show that (*i*) $x^nJ_n(x)$ is a solution of the equation $xy'' + (1 - 2n)y' + xy = 0.$ *(V.T.U., 2001)*
   (*ii*) $x^{-n}J_n(x)$ is a solution of the equation $xy'' + (1 + 2n)y' + xy = 0.$
7. Show that under the transformation $y = u/\sqrt{x}$, Bessel equation becomes
   $$u'' + \left(1 + \frac{1 - 4n^2}{4x^2}\right)u = 0.$$ Hence find the solution of this equation.
8. By the use of substitution $y = u/\sqrt{x}$, show that the solution of the equation $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + \left(x^2 - \dfrac{1}{4}\right)y = 0$ can be written in the form $y = c_1\dfrac{\sin x}{\sqrt{x}} + c_2\dfrac{\cos x}{\sqrt{x}}.$
9. Show that $\int_0^p x(ber^2 x + bei^2 x)\,dx = p(ber\,p\,bei'\,p - bei\,p\,ber'\,p).$
10. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the positive roots of $J_0(x) = 0$, prove that
    $$x^2 = 2\sum_{n=1}^{\infty} \frac{\alpha_n^2 - 4}{\alpha_n^3 J_1(\alpha_n)} J_0(\alpha_n x).$$
11. Expand $f(x) = x^2$ in the interval $0 < x < 3$ in terms of functions $J_1(\alpha_n x)$ where α_n are determined by $J_1(3\alpha) = 0.$

16.13 LEGENDRE'S EQUATION*

Another differential equation of importance in Applied Mathematics, particularly in boundary value problems for spheres, is *Legendre's equation,*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \ldots(1)$$

Here n is a real number. But in most applications only integral values of n are required.

Substituting $\quad y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots \; (a_0 \neq 0),$

(1) takes the form

$$a_0(m)(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + \ldots + [a_{r+2}(m+r+2)(m+r+1)$$
$$-\{(m+r)(m+r+1) - n(n+1)\}a_r]x^{m+r} + \ldots = 0$$

Equating to zero the coefficient of the lowest power of x, *i.e.*, of x^{m-2}, we get

$$a_0m(m-1) = 0, \; m = 0, 1 \qquad [\because a_0 \neq 0]$$

Equating to zero the coefficients of x^{m-1} and x^{m+r}, we get $a_1(m+1)m = 0 \quad \ldots(2)$

$$a_{r+2}(m+r+2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\}a_r = 0 \quad \ldots(3)$$

When $m = 0$, (2) is satisfied and therefore, $a_1 \neq 0$. Then (3) gives, taking $r = 0, 1, 2, \ldots$ in turn,

$$a_2 = -\frac{n(n+1)}{2!}a_0, \quad a_3 = -\frac{(n-1)(n+2)}{3!}a_1$$

$$a_4 = \frac{-(n-2)(n+3)}{4.3}a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5.4}a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1, \text{ etc.}$$

Hence for $m = 0$, there are two independent solutions of (1) :

$$y_1 = a_0\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \ldots\ldots\right\} \quad \ldots(4)$$

---

*See footnote p. 493.

$$y_2 = a_1\left\{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \ldots\ldots\right\} \qquad \ldots(5)$$

When $m = 1$, (2) shows that $a_1 = 0$. Therefore, (3) gives

$$a_3 = a_5 = a_7 = \ldots = 0$$

and
$$a_2 = -\frac{(n-1)(n+2)}{3!}a_0$$

$$a_4 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_0, \text{ etc.}$$

Thus for $m = 1$, we get the solution (5) again. Hence $y = y_1 + y_2$ is the general solution of (1).

If n is a positive even integer, the series (4) terminates at the term in x^n and y_1 becomes a polynomial. Similarly if n is an odd integer, (5) becomes a polynomial of degree n. Thus, whenever n is a positive integer, the general solution of (1) consists of a polynomial solution and an infinite series solution.

These polynomial solutions, with a_0 or a_1 so chosen that the value of the polynomial is 1 for $x = 1$, are called *Legendre polynomials* of order n and are denoted by $P_n(x)$. The infinite series solution with (a_0 or a_1 properly chosen) is called *Legendre function of the second kind* and is denoted by $Q_n(x)$. *(V.T.U., 2006)*

16.14 (1) RODRIGUE'S FORMULA*

We shall prove that $P_n(x) = \dfrac{1}{n!\,2^n}\dfrac{d^n}{dx^n}(x^2-1)^n$ $\qquad \ldots(1)$

Let $\qquad v = (x^2-1)^n$. Then $v_1 = \dfrac{dv}{dx} = 2nx(x^2-1)^{n-1}$

i.e., $\qquad (1-x^2)v_1 + 2nxv = 0 \qquad \ldots(2)$

Differentiating (2), $(n+1)$ times by Leibnitz's theorem

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{1}{2!}(n+1)n(-2)v_n + 2n[xv_{n+1} + (n+1)v_n] = 0$$

or
$$(1-x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)v_n = 0$$

which is Legendre's equation and cv_n is its solution. Also its finite series solution is $P_n(x)$.

$\therefore \qquad P_n(x) = cv_n = c\dfrac{d^n}{dx^n}(x^2-1)^n \qquad \ldots(3)$

To determine the constant c, put $x = 1$ in (3). Then

$$1 = c\left[\frac{d^n}{dx^n}\{(x-1)^n(x+1)^n\}\right]_{x=1}$$

$= c[n!\,(x+1)^n]$

+ terms containing $(x-1)$ and its powers$]_{x=1}$

$= c\,.\,n!\,2^n$, *i.e.* $c = 1/n!\,2^n$.

Substituting this value of c in (3), we get (1), which is known as the *Rodrigue's formula*.

(V.T.U., 2008 ; Bhopal, 2007 ; U.P.T.U., 2004)

Obs. All roots of $P_n(x) = 0$ are real and lie between -1 and $+1$. *(Madras, 2003 S)*

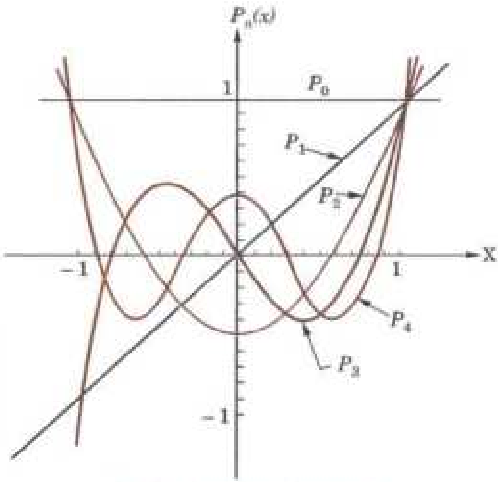


Fig. 16.2. Legendre polynomials.

(2) Legendre polynomials. Using (1), we get

$P_0(x) = 1, \qquad P_1(x) = x,$

$P_2(x) = \dfrac{1}{2}(3x^2-1), \qquad P_3(x) = \dfrac{1}{2}(5x^3-3x),$

* Named after the French mathematician and economist *Olinde Rodrigue* (1794–1851).

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),\ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \text{ etc.} \qquad (V.T.U., 2009)$$

In general, we have $$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r} \qquad ...(4)$$

where $N = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Let us derive (4) from (1).

By Binomial theorem, $$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n}(-1)^r \frac{n!}{r!(n-r)!} x^{2n-2r}$$

$\therefore$ by (1), $$P_n = \frac{1}{n!2^n} \sum_{n=0}^{n} \frac{(-1)^r n!}{r!(n-r)!} \frac{d^n(x^{2n-2r})}{dx^n} = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!(n-2r)!} x^{n-2r}$$

This is same as (4), and the last term ($r = N$) is such that the power of x (*i.e.*, $n - 2r$) for this term is either 0 or 1.

Example 16.17. *Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ in terms of Legendre polynomials.*
(V.T.U., 2010 ; S.V.T.U., 2007)

Solution. Since $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$. $\therefore x^4 = \frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}$

$$\therefore \quad f(x) = \left[\frac{8}{35}P_4(x) + \frac{6}{7}x^2 - \frac{3}{35}\right] + 3x^3 - x^2 + 5x - 2$$

$$= \frac{8}{35}P_4(x) + 3x^3 - \frac{1}{7}x^2 + 5x - \frac{73}{35} \qquad \left[\because x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x\ ;\ x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}\right]$$

$$= \frac{8}{35}P_4(x) + 3\left[\frac{2}{5}P_3(x) + \frac{3}{5}x\right] - \frac{1}{7}\left[\frac{2}{3}P_2(x) + \frac{1}{3}\right] + 5x - \frac{73}{35}$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}x - \frac{224}{105} \qquad [\because x = P_1(x),\ 1 = P_0(x)]$$

$$= \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1x - \frac{224}{105}P_0(x).$$

Example 16.18. *Show that for any function $f(x)$, for which the nth derivative is continuous,*

$$\int_{-1}^{1} f(x) P_n(x) dx = \frac{1}{2^n n!}\int_{-1}^{1}(1-x^2)^n f^n(x)\,dx.$$

Solution. Using Rodrigue's formula : $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$,

$$\int_{-1}^{1} f(x) P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\frac{d^n(x^2-1)^n}{dx^n}dx \qquad \text{[Integrate by parts]}$$

$$= \frac{1}{2^n n!}\left[\left| f(x)\cdot\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\right|_{-1}^{1} - \int_{-1}^{1} f'(x)\cdot\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}dx\right]$$

$$= \frac{(-1)}{2^n n!}\int_{-1}^{1} f'(x)\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

$$= \frac{(-1)^2}{2^n n!}\int_{-1}^{1} f''(x)\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n dx \qquad \text{[Again integrating by parts]}$$

$$= \frac{(-1)^n}{2^n\, n!} \int_{-1}^{1} f^n(x)\, (x^2-1)^n\, dx \qquad \text{[Integrating by parts } (n-2) \text{ times]}$$

$$= \frac{(-1)^{2n}}{2^n\, n!} \int_{-1}^{1} f^n(x)(1-x^2)^n\, dx = \frac{1}{2^n\, n!} \int_{-1}^{1} f^n(x)(1-x^2)^n\, dx$$

16.15 GENERATING FUNCTION FOR $P_n(x)$

To show that $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$.

Since
$$(1-z)^{-\frac{1}{2}} = 1 + \frac{1}{2}z + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}z^3 + \ldots$$

$$= 1 + \frac{2!}{(1!)^2\, 2^2} z + \frac{4!}{(2!)^2\, 2^4} z^2 + \frac{6!}{(3!)^2\, 2^6} z^3 + \ldots$$

$\therefore$
$$[1-t(2x-t)]^{-\frac{1}{2}} = 1 + \frac{2!}{(1!)^2\, 2^2} t(2x-t) + \frac{4!}{(2!)^2\, 2^4} t^2(2x-t)^2 + \ldots$$

$$+ \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} t^{n-r}(2x-t)^{n-r} + \ldots + \frac{(2n)!}{(n!)^2\, 2^{2n}} t^n (2x-t)^n + \ldots \qquad \ldots(1)$$

The term in t^n from the term containing $t^{n-r}(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}}\, t^{n-r} \cdot {}^{n-r}C_r (-t)^r (2x)^{n-2r}$$

$$= \frac{(2n-2r)!}{[(n-r)!]^2\, 2^{2n-2r}} \times \frac{(n-r)!}{r!\,(n-2r)!} (-1)^r t^n \,.\, (2x)^{n-2r} = \frac{(-1)^r\,(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r} \,.\, t^n.$$

Collecting all terms in t^n which will occur in the term containing $t^n (2x-t)^n$ and the preceding terms, we see that terms in t^n.

$$= \sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r} \,.\, t^n = P_n(x) t^n$$

where $N = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

Hence (1) may be written as $[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n$ $\qquad \ldots(2)$

(Kerala M.E., 2005 ; U.P.T.U., 2005)

This shows that $P_n(x)$ is the coefficient of t^n in the expansion of $(1-2xt+t^2)^{-1/2}$. That is why, it is known as the *generating function of Legendre polynomials.*

Cor. 1. $P_n(1) = 1$. *(V.T.U., 2003 S ; Delhi, 2002)*

Taking $x = 1$ in (2), we have $(1-2t+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)\, t^n$

i.e.,
$$\sum_{n=0}^{\infty} P_n(1)\, t^n = (1-t)^{-1} = 1 + t + t^2 + \ldots + t^n + \ldots$$

Equating coefficients of t^n, we get $P_n(1) = 1$.

Cor. 2. $P_n(-1) = (-1)^n$. *(B.P.T.U., 2005 S ; V.T.U., 2003)*

Taking $x = -1$ in (2), we have

$$\sum_{n=0}^{\infty} P_n(-1)\, t^n = (1+t)^{-1} = 1 - t + t^2 - \ldots + (-1)^n t^n + \ldots$$

Equating coefficients of t^n, we get the desired result.

Cor. 3. $P_n(0) = \begin{cases} (-1)^{n/2} \dfrac{1 \times 3 \times 5 \ldots (n-1)}{2 \times 4 \times 6 \times \ldots n}, & \text{when } n \text{ is even} \\ 0, & \text{when } n \text{ is odd} \end{cases}$ (V.T.U., 2005)

Putting $x = 0$ in (2), we get $\sum_{n=0}^{\infty} P_n(0)\, t^n = (1 + t^2)^{-1/2}$

$$= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}\, t^4 \ldots + (-1)^r \frac{1 \cdot 3 \cdot 5 \cdots (2r+1)}{2 \cdot 4 \cdot 6 \cdots 2r} t^{2r} + \ldots$$

Equating coefficient of t^{2m}, we get $P_{2m}(0) = (-1)^m \dfrac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdot 6 \cdots 2m}$

Similarly equating coefficients of t^{2m+1}, we have $P_{2m+1}(0) = 0$.

Cor. 4. $P'_n(1) = \dfrac{1}{2} n(n+1)$ (U.P.T.U. 2003)

Since $P_n(x)$ is a solution of Legendre's eduation, $(1 - x^2) P''_n(x) - 2xP'_n(x) + n(n+1) P_n(x) = 0$

Putting $x = 1$, $-2P'_n(1) + n(n+1) P_n(1) = 0$ or $P'_n(1) = \dfrac{1}{2} n(n+1)$ $[\because P_n(1) = 1]$

16.16 RECURRENCE FORMULAE FOR $P_n(x)$

The following recurrence formulae can be easily derived from the generating function for $P_n(x)$:

(1) $(n+1) P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

(2) $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$

(3) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

(4) $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$.

(5) $(1 - x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$.

Proofs. **(1)** We know that $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n$ ...(i)

Differentiating partially w.r.t. t, we get

$$-\frac{1}{2}(1 - 2xt + t^2)^{-3/2}(-2x + 2t) = \Sigma nP_n(x)\, t^{n-1}$$

or $$(x - t)(1 - 2xt + t^2)^{-1/2} = (1 - 2xt + t^2)\, \Sigma nP_n(x)\, t^{n-1}$$

or $$(x - t)\, \Sigma P_n(x)\, t^n = (1 - 2xt + t^2)\, \Sigma nP_n(x)\, t^{n-1}$$

Equating coefficients of t^n from both sides, we get

$$xP_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nxP_n(x) + (n-1) P_{n-1}(x)$$

whence follows the required result. (S.V.T.U., 2007 ; V.T.U., 2003)

(2) Differentiating (*i*) partially w.r.t. x,

$$-\frac{1}{2}(1 - 2xt + t^2)^{-3/2} \cdot (-2t) = \Sigma P_n'(x)t^n$$

i.e., $$t(1 - 2tx + t^2)^{-3/2} = \Sigma P_n'(x)\, t^n \quad \ldots(ii)$$

Again differentiating (*i*) partially w.r.t. t, we have

$$(x - t)(1 - 2tx + t^2)^{-3/2} = \Sigma nP_n(x)t^{n-1} \quad \ldots(iii)$$

Dividing (*iii*) by (*ii*), we get $\dfrac{x - t}{t} = \dfrac{\Sigma nP_n(x)t^{n-1}}{\Sigma P_n'(x)t^n}$

i.e., $$\Sigma nP_n(x)\, t^n = (x - t)\, \Sigma P_n'(x)\, t^n$$

Equating coefficients of t^n from both sides, we get (2). (J.N.T.U., 2006 ; U.P.T.U., 2006)

(3) Differentiating (**1**) w.r.t. x, we get

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)xP'_n(x) - nP'_{n-1}(x) \quad \ldots(iv)$$

Substituting for $xP'_n(x)$ from (2) in (*iv*), we obtain

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1) [nP_n(x) + P'_{n-1}(x)] - nP'_{n-1}(x)$$

or $$(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$ (Madras, 2006)

(4) Rewriting (*iv*) as

$$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (n+1)xP_n'(x) + n[xP_n'(x) - P'_{n-1}(x)]$$

$$= (2n+1)P_n(x) + (n+1)xP_n'(x) + n^2P_n(x) \quad \text{[by (2)]}$$

$$= (n+1)xP'_n(x) + (n^2+2n+1)P_n(x)$$

or $$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x)$$

Replacing n by $(n-1)$, we get (4).

(5) Rewriting (2) and (4) as

$$xP_n'(x) - P'_{n-1}(x) = nP_n(x) \quad ...(v)$$

and $$P_n'(x) - xP'_{n-1}(x) = nP_{n-1}(x) \quad ...(vi)$$

Multiplying (*v*) by x and substracting from (*vi*), we get

$$(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)].$$

Example 16.19. *Prove that* $(2n+1)(1-x^2)P_n'(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$.

Solution. We have the recurrence formula

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

or $$(\overline{n+1}+n)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

or $$(n+1)[xP_n(x) - P_{n+1}(x)] = n[P_{n-1}(x) - xP_n(x)]$$

$$= (1-x^2)P_n'(x) \quad [\because (1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)] \quad ...(i)$$

or $$xP_n(x) = P_{n+1}(x) + \frac{(1-x^2)P_n'(x)}{n+1} \quad ...(ii)$$

Also from (*i*) $$xP_n(x) = P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n} \quad ...(iii)$$

From (*ii*) and (*iii*), $$P_{n-1}(x) - \frac{(1-x^2)P_n'(x)}{n} = P_{n+1}(x) + \frac{(1-x^2)P_n'(x)}{n+1}$$

or $$n(n+1)P_{n-1}(x) - (n+1)(1-x^2)P'_n(x) = n(n+1)P_{n+1}(x) + n(1-x^2)P'_n(x)$$

or $$(2n+1)(1-x^2)P'_n(x) = n(n+1)[P_{n-1}(x) - P_{n+1}(x)]$$

16.17 (1) ORTHOGONALITY OF LEGENDRE POLYNOMIALS

We shall prove that, $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

We know that the solutions of

$$(1-x)^2u'' - 2xu' + m(m+1)u = 0 \quad ...(1)$$

and $$(1-x^2)v'' - 2xv' + n(n+1)v = 0 \quad ...(2)$$

are $P_m(x)$ and $P_n(x)$ respectively.

Multiplying (1) by v and (2) by u and subtracting, we get

$$(1-x^2)(u''v - uv'') - 2x(u'v - uv') + [m(m+1) - n(n+1)]uv = 0$$

or $$\frac{d}{dx}\{(1-x^2)(u'v - uv')\} + (m-n)(m+n+1)uv = 0.$$

Now integrating from -1 to 1, we get

$$(m-n)(m+n+1)\int_{-1}^{1} uv\,dx = \Big|(1-x^2)(uv' - u'v)\Big|_{-1}^{1} = 0.$$

Hence $$\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0. \quad (m \neq n) \quad ...(3)$$

This is known as the *orthogonality property of Legendre polynomials.*

(S.V.T.U., 2008 ; Madras, 2006 ; V.T.U., 2006)

When $m = n$, we have from Rodrigue's formula,

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = \int_{-1}^{1} D^n(x^2-1)^n \cdot D^n(x^2-1)^n\,dx \qquad \text{[Integrate by parts]}$$

$$= \Big| D^n(x^2-1)^n \cdot D^{n-1}(x^2-1)^n \Big|_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

Since $D^{n-1}(x^2-1)^n$ has x^2-1 as a factor, the first term on the right vanishes for $x = \pm 1$. Thus

$$(n\,!\,2^n)^2 \int_{-1}^{1} P_n^2(x)\,dx = -\int_{-1}^{1} D^{n+1}(x^2-1)^n \,.\, D^{n-1}(x^2-1)^n\,dx$$

[Integrate by parts $(n-1)$ times]

$$= (-1)^n \int_{-1}^{1} D^{2n}(x^2-1)^n \,.\, (x^2-1)^n\,dx = (-1)^n \int_{-1}^{1} (2n)\,!\,(x^2-1)^n\,dx$$

$$= 2(2n)\,! \int_0^1 (1-x^2)^n\,dx \qquad \text{[Put } x = \sin\theta\text{]}$$

$$= 2(2n)\,! \int_0^{\pi/2} \cos^{2n+1}\theta\,d\theta = 2(2n)\,!\,\frac{2n(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 2\,.\,1}$$

$$= 2(2n)\,!\,[2n(2n-2)\ldots 4\,.\,2]^2/(2n+1)\,! = \frac{2}{2n+1}\,(2^n n\,!)^2$$

Hence $$\int_{-1}^{1} P_n^2(x)dx = 2/(2n+1). \qquad \ldots(4)$$ (*Bhopal, 2008 ; V.T.U., 2007 ; J.N.T.U., 2006*)

(2) Fourier-Legendre expansion of f(x). If $f(x)$ be a function defined from $x = -1$ to $x = 1$, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \qquad \ldots(5)$$

To determine the coefficient c_n, multiply both sides by $P_n(x)$ and integrate from -1 to 1. Then (3) and (4) give

$$\int_{-1}^{1} f(x)P_n(x)dx = c_n \int_{-1}^{1} P_n^2(x)\,dx = \frac{2c_n}{2n+1} \quad \text{or} \quad c_n = \left(n+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)\,dx$$

Equation (5) is known as *Fourier-Legendre expansion of* $f(x)$.

Example 16.20. *Show that* $\int_{-1}^{1} xP_n(x)\,P_{n-1}(x)\,dx = \dfrac{2n}{4n^2-1}$.

Solution. The recurrence formula (1) can be written as

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

or $$(2n-1)\,xP_{n-1} = nP_n + (n-1)\,P_{n-2} \qquad \text{[Changing } n \text{ to } n-1\text{]}$$

Multiplying by P_n, we get $xP_nP_{n-1} = \dfrac{1}{2n-1}\,[nP_n^2 + (n-1)\,P_nP_{n-2}]$

Integrating both sides w.r.t. x from $x = -1$ to $x = 1$, we get

$$\int_{-1}^{1} xP_nP_{n-1}dx = \frac{n}{2n-1}\int_{-1}^{1} P_n^2 dx + \frac{n-1}{2n-1}\int_{-1}^{1} P_nP_{n-2}\,dx$$

$$= \frac{n}{2n-1}\left(\frac{2}{2n+1}\right) + \frac{n-1}{2n-1}(0), \text{ by Orthogonal property}$$

Hence $$\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}.$$

Example 16.21. *Show that* $\int_{-1}^{1} (1-x^2)\,P_m'(x)\,dx, = \begin{cases} 0, & \text{when } m \neq n \\ \dfrac{2n(n+1)}{2n+1}, & \text{when } m = n \end{cases}$

(*S.V.T.U., 2008 ; U.P.T.U., 2006*)

Solution. Integrating by parts,

$$\int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)dx = \left|(1-x^2)P'_m(x)\cdot P_n(x)\right|_{-1}^{1} - \int_{-1}^{1}\cdot\frac{d}{dx}\left\{(1-x^2)\,P'_m(x)\right\}P_n\,dx$$

$$= -\int_{-1}^{1}P_n\left\{(1-x^2)\,P''_m(x) - 2x\,P'_m(x)\right\}dx \qquad ...(i)$$

Now $P_m(x)$ being a solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + m\,(m+1)\,y = 0, \text{ we have}$$

$$(1-x^2)\,P''_m(x) - 2x\,P'_m(x) = -m\,(m+1)\,P_m(x)$$

Substituting this in (*i*), we get

$$\int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = -\int_{-1}^{1}P_n\ \{-m\,(m+1)\,P_m(x)\}\,dx$$

$$= m\,(m+1)\int_{-1}^{1}P_m(x)\,P_n(x)dx \qquad ...(ii)$$

When $m \neq n$, $\int_{-1}^{1}P_m(x)\,P_n(x)\,dx = 0$, by orthogonality property.

$$\therefore \qquad \int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = m(m+1)\,.\,0 = 0 \qquad \text{[from } (ii)\text{]}$$

When $m = n$, $\int_{-1}^{1}P_m(x)\,P_n(x)\,dx = \dfrac{2}{2n+1}$, by orthogonality property.

$$\therefore \qquad \int_{-1}^{1}(1-x^2)\ P'_m(x)\,P'_n(x)\,dx = n(n+1)\,.\,\frac{2}{2n+1} = \frac{2n\,(n+1)}{(2n+1)}.$$

Example 16.22. *Show that* $\int_{-1}^{1}x^2\ P_{n-1}\,P_{n+1}\,dx = \dfrac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$

(J.N.T.U., 2006 ; Kerala M. Tech., 2005)

Solution. We have from the recurrence relation (1),

$$(2n+1)\,x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$$

$$\therefore \qquad xP_{n-1} = \frac{1}{2n-1}\ \{nP_n + (n-1)\,P_{n-2}\}$$

and

$$xP_{n+1} = \frac{1}{2n+3}\ \{(n+2)\,P_{n+2} + (n+1)\,P_n\}P$$

$$\therefore \qquad x^2P_{n-1}P_{n+1} = \frac{1}{(2n-1)\,(2n+3)}\{n(n+2)\,P_nP_{n+2} + n(n+1)P_n^{\,2}$$
$$+ (n-1)\,(n+2)P_{n-2}P_{n+2} + (n^2-1)P_n\,P_{n-2}\}$$

Integrating both sides from -1 to 1 and using orthogonality of Legendre polynomials, we get

$$\int_{-1}^{1}x^2P_{n-1}P_{n+1}\,dx = \frac{n(n+1)}{(2n-1)\,(2n+3)}\int_{-1}^{1}P_n^{\,2}\,dx = \frac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}.$$

Example 16.23. *If* $f(x) = 0, \quad -1 < x \leq 0$
$\qquad\qquad\qquad = x, \quad 0 < x < 1,$

show that $f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}\ P_2(x) - \frac{3}{32}P_4(x) + ...$ *(U.P.T.U., 2003)*

Solution. Let $\qquad f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$

Then c_n is given by $c_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_n(x)\, dx$

$$= \left(n + \frac{1}{2}\right)\left[\int_{-1}^{0} 0 \,.\, P_n(x)\, dx + \int_{0}^{1} xP_n(x)\, dx\right] = \left(n + \frac{1}{2}\right) \int_{0}^{1} x\, P_n(x)\, dx$$

$\therefore$
$$c_0 = \frac{1}{2}\int_0^1 xP_0(x)\, dx = \frac{1}{2}\int_0^1 x\, dx = \frac{1}{4}$$

$$c_1 = \frac{3}{2}\int_0^1 xP_1(x)\, dx = \frac{3}{2}\int_0^1 x^2\, dx = \frac{1}{2}$$

$$c_2 = \frac{5}{2}\int_0^1 xP_2(x)\, dx = \frac{5}{2}\int_0^1 x \,.\, \frac{3x^2 - 1}{2}\, dx = \frac{5}{4}\left|\frac{3x^4}{4} - \frac{x^2}{2}\right|_0^1 = \frac{5}{16}$$

$$c_3 = \frac{7}{2}\int_0^1 xP_3(x)\, dx = \frac{7}{2}\int_0^1 x \,.\, \frac{5x^3 - 3x}{2}\, dx = \frac{7}{4}\left|5\frac{x^5}{5} - 3\,\frac{x^3}{3}\right|_0^1 = 0$$

$$c_4 = \frac{9}{2}\int_0^1 xP_4(x)\, dx = \frac{9}{2}\int_0^1 x \,.\, \frac{35x^4 - 30x^2 + 3}{8}\, dx$$

$$= \frac{9}{16}\left|35\frac{x^6}{6} - 35\,\frac{x^4}{4} + 3\frac{x^2}{2}\right|_0^1 = -\frac{3}{32} \text{ and so on.}$$

Hence
$$f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) - \frac{3}{32}P_4(x) + \dots .$$

PROBLEMS 16.5

1. Show that $P_n(-x) = (-1)^n P_n(x)$. *(Bhopal, 2008 ; V.T.U., 2003 S)*
2. Prove that (*i*) $P_{2n}'(0) = 0$ (*ii*) $P_{2n+1}'(0) = \dfrac{(-1)^n\,(2n+1)!}{2^{2n}(n!)^2}$. (*iii*) $P_n'(-1) = (-1)^n\,\dfrac{n(n+1)}{2}$ *(S.V.T.U., 2008)*
3. Express the following in terms of Legendre polynomials : (*i*) $5x^3 + x$
   (*ii*) $x^3 + 2x^2 - x - 3$. *(Osmania, 2003)* (*iii*) $4x^3 + 6x^2 + 7x + 2$. *(S.V.T.U., 2008)*
   (*iv*) $x^4 + 3x^3 - x^2 + 5x - 2$ *(Bhopal, 2008 ; Madras, 2006)*
4. Prove that (*i*) $(1 - x^2)\, P_n'(x) = (n + 1)\, [xP_n(x) - P_{n+1}(x)]$.
   (*ii*) $P_n(x) = P'_{n+1}(x) - 2xP_n'(x) + P'_{n-1}(x)$. (*iii*) $P_n(x)\, P_{n+1/2}(x) = \dfrac{\sqrt{\pi}}{2^{2n+1}}\, P_{2n}(x)$ *(Anna, 2005 S)*
5. Prove that (*i*) $\int_{-1}^{1} [P_2(x)]^2\, dx = \frac{2}{5}$. *(P.T.U., 2002)* (*ii*) $\int_0^1 P_{2n}(x)\, dx = 0$.
6. Prove that $\int_{-1}^{1} P_n(x)(1 - 2hx + h^2)^{-1/2}\, dx = \dfrac{2h^n}{2n+1}$.
7. Show that $\int_{-1}^{1} (1 - x^2)[P_n'(x)]^2\, dx = \dfrac{2n(n+1)}{2n+1}$. *(U.P.T.U., 2006 ; Kerala M.E., 2005)*
8. Using Rodrigue's formula, show that $P_n(x)$ satisfies the differential equation
$$\frac{d}{dx}\left[(1 + x^2)\frac{d}{dx}[P_n(x)]\right] + n\,(n + 1)\, P_n(x) = 0.$$
9. Expand the following functions in terms of Legendre polynomials in the interval $-1 < x < 1$:
   (*i*) $f(x) = x^3 + 2x^2 - x - 3$ *(V.T.U., 2008)* (*ii*) $f(x) = x^4 + x^3 + 2x^2 - x - 3$.
10. If $f(x) = 0, \quad -1 < x < 0$
    $= 1, \quad 0 < x < 1$, show that $f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \dots\dots$

16.18 OTHER SPECIAL FUNCTIONS

The following special functions occur in numerous engineering problems. We state below their important properties which can be verified by similar methods :

(1) Laguerre's polynomials*. These are the solutions of *Laguerre's differential equation*

$$xy'' + (1-x)y' + ny = 0 \qquad ...(1)$$

These polynomials $L_n(x)$, are given by the corresponding Rodrigue's formula

$$L_n(x) = e^x \frac{d^n}{dx^n}(x^n e^{-x}) \qquad ...(2)$$

In particular, $L_0(x) = 1$; $L_1(x) = 1 - x$, $L_2(x) = 2 - 4x + x^2$; $L_3(x) = 6 - 18x + 9x^2 - x^3$. *(Madras, 2006)*

Their *generating function* is given by

$$\frac{e^{-xt/(1-t)}}{1-t} = \sum_{n=0}^{\infty} \frac{L_n(x)}{n!} t^n \qquad ...(3)$$

The *orthogonal property* for these polynomials is

$$\int_0^{\infty} e^{-x} L_m(x) L_n(x)\, dx = \begin{cases} 0, & m \neq n \\ (n!)^2, & m = n \end{cases} \qquad ...(4)$$

(2) Hermite's polynomials†. These are the solutions of Hermite's differential equation

$$y'' - 2xy' + 2ny = 0 \qquad ...(5)$$

These polynomials $H_n(x)$, are given by the Rodrigue's formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^{2n}}{dx^n}(e^{-x^2}) \qquad ...(6)$$

In particular, $H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 - 2$; $H_3(x) = 8x^3 - 12x$. *(Madras, 2006)*

Their *generating function is* given by

$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \qquad ...(7) \quad \textit{(Madras, 2002 S)}$$

The *orthogonal property* of these polynomials is

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = \begin{cases} 0, & m \neq n \\ 2^n\, n!\sqrt{\pi}, & m = n \end{cases} \qquad ...(8)$$

(3) Chebyshev polynomials**. These polynomials denoted by $T_n(x)$, are the solutions of the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0 \qquad ...(9)$$

Their *generating function* is

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n \qquad ...(10)$$

and

$$T_n(x) = \frac{n}{2} \sum_{r=0}^{N} (-1)^r \frac{(n-r-1)!}{r!(n-2r)!} (2x)^{n-2r} \qquad ...(11)$$

(J.N.T.U., 2006)

where $N = \frac{n}{2}$, if n is even and $N = \frac{1}{2}(n-1)$, if n is odd.

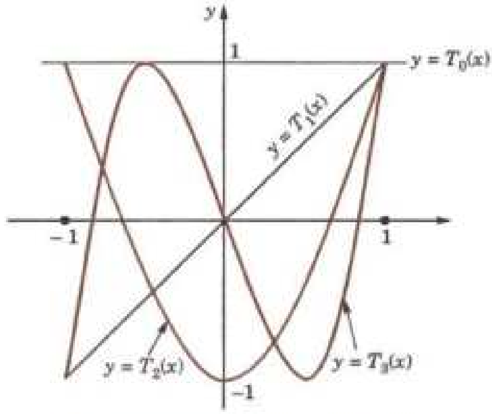


Fig. 16.3. Graphs of $T_0(x)$, $T_1(x)$, $T_2(x)$, $T_3(x)$.

* Named after the French mathematician *Edmond Laguerre* (1834–86) who is known for his work in infinite series and geometry.

† See footnote p. 68.

** Named after the Russian mathematician *Pafnuti Chebyshev* (1821–1894) who is known for his work in the theory of numbers and approximation theory.

In particular, $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$. Also, we have the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x) \qquad ...(12) \quad (Bhopal,\ 2002)$$

which defines T_{n+1} in terms of T_n and T_{n-1}.

Their *orthogonal property* is

$$\int_{-1}^{1} (1-x^2)^{-1/2}\, T_m(x) T_n(x)\, dx = \begin{cases} 0, & m \neq n \\ \dfrac{\pi}{2}, & m = n \\ \pi, & m = n = 0 \end{cases} \qquad ...(13)$$

Example 16.24. *Prove that* $\int_0^\infty e^{-x} L_m(x) L_n(x)\, dx = 0,\ m \neq n.$ *(Anna, 2006)*

Solution. Since $L_m(x)$ and $L_n(x)$ are the solutions of the Laguerre's differential equation (1).

$$\therefore \qquad xL_m'' + (1-x)L_m' + mL_m = 0 \qquad ...(i)$$

$$xL_n'' + (1-x)L_n' + nL_n = 0 \qquad ...(ii)$$

Multiplying (*i*) by L_n and (*ii*) by L_m and subtracting, we get

$$x(L_n L_m'' - L_m L_n'') + (1-x)(L_n L_m' - L_m L_n') = (n-m) L_m L_n$$

or
$$\frac{d}{dx}(L_n L_m' - L_m L_n') + \frac{1-x}{x}(L_n L_m' - L_m L_n') = \frac{(n-m) L_m L_n}{x}$$

This is Leibnitz's linear equation and its

$$\text{I.F.} = e^{\int \left(\frac{1}{x} - 1\right) dx} = e^{\log x - x} = xe^{-x}.$$

$$\therefore \quad \text{Its solution is } \Big| (L_n L_m' - L_m L_n')\, xe^{-x} \Big|_0^\infty = \int_0^\infty \frac{(n-m) L_m L_n}{x}\, xe^{-x}\, dx$$

or
$$\int_0^\infty e^{-x} L_m L_n\, dx = \left| \frac{(L_n L_m' - L_m L_n') xe^{-x}}{n-m} \right|_0^\infty = 0 \text{ which proves the result.}$$

Example 16.25. *Prove that* $H_n(x) = (-1)^n e^{x^2} \dfrac{d^{2n}}{dx^n}(e^{-x^2})$.

Solution. The generating function for $H_n(x)$ is $e^{2tx - t^2} = e^{x^2} \cdot e^{-(t-x)^2} = \sum_{n=0}^{\infty} H_n(x) \cdot \dfrac{t^n}{n!}$

Then
$$\left[\frac{\partial^n}{\partial t^n}(e^{2tx - t^2})\right]_{t=0} = H_n(x) \qquad ...(i)$$

Also
$$\left[\frac{\partial^n}{\partial t^n}(e^{2tx - x^2})\right]_{t=0} = e^{x^2}\left[\frac{\partial^n}{\partial t^n}\{e^{-(t-x)^2}\}\right]_{t=0}$$

$$= e^{x^2}\left[\frac{\partial^n}{\partial (-x)^n}\{e^{-(t-x)^2}\}\right]_{t=0} = (-1)^n \frac{d^n}{dx^n}(e^{-x^2}) \qquad ...(ii)$$

Equating (*i*) and (*ii*), we get the desired result.

PROBLEMS 16.6

1. Using the generating function (3) page 571, obtain the recurrence formula
$$L_{n+1}(x) = (2n+1-x)\, L_n(x) - n^2 L_{n-1}(x).$$
2. Show that (*i*) $nL_{n-1}(x) = nL_{n-1}'(x) - L_n'(x)$, (*ii*) $L_n'(x) = L_{n-1}'(x) - L_{n-1}(x)$. *(Anna, 2005)*
3. Show that (*i*) $H_{2n}(0) = (-1)^n \dfrac{2n!}{n!}$ (*ii*) $H_{2n+1}(0) = 0$ *(Anna, 2005)*

4. Prove that (i) $H_n'(x) = 2n\,H_{n-1}(x)$ (ii) $\dfrac{d^m}{dx^m}[H_n(x)] = \dfrac{2^m \cdot n!}{(n-m)!} H_{n-m}(x), m < n.$
5. Using the generating function (7) page 515, obtain the recurrence formula $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$.
6. Prove that (i) $\int_{-\infty}^{\infty} e^{-x^2} H_2(x)\,H_3(x) = 0,$ (ii) $\int_{-\infty}^{\infty} e^{-x^2} [H_2(x)]^2 = 8\sqrt{\pi}.$ (*Madras, 2003*)
7. Express x^3 in terms of Chebyshev polynomials T_1 and T_3. (*U.P.T.U., 2009*)
8. Show that (i) $T_5 = 16x^5 - 20x^3 + 5x$. (*Bhopal, 2002*)
   (ii) $(1-x^2)T_n' = nT_{n-1}(x) - nxT_n(x)$. (*Osmania, 2003*)
9. Prove that $\dfrac{1-t^2}{1-2xt+t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\,t^n$. (*J.N.T.U., 2006*)

16.19 (1) STRUM*-LIOUVILLE† PROBLEM

Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ...(i)

can be written as, $[(1-x^2)y']' + \lambda y = 0$ $[\lambda = n(n+1)]$

Bessel's equation $X^2 \dfrac{d^2y}{dx^2} + X\dfrac{dy}{dx} + (X^2 - n^2)y = 0$ can be transformed by putting $X = kx$ (so that $\dfrac{dy}{dx} = \dfrac{dy}{dx}\cdot\dfrac{dx}{dX} = \dfrac{y'}{k}, \dfrac{d^2y}{dX^2} = \dfrac{y''}{k^2}$ to the form

$$x^2y'' + xy' + (k^2x^2 - n^2)\,y = 0$$

or
$$(xy'' + y') + (\lambda x - n^2/x)\,y = 0 \qquad [\lambda = k^2]$$

or
$$(xy')' + (\lambda x - n^2/x)\,y = 0 \qquad \text{...(ii)}$$

Both the equations (i) and (ii) are of the form

$$[r(x)\,y']' + [\lambda p(x) + q(x)]y = 0 \qquad \text{...(1)}$$

which is known as the *Strum-Liouville equation*. Similarly Laguerre's, Hermite's equations etc. can also be reduced to (1). Thus all the above equations of engineering utility can be considered with a common approach by means of Strum-Liouville's equation.

Eqution (1) considered on some interval $a \le x \le b$, satisfying the conditions

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0,\ \beta_1 y(b) + \beta_2 y'(b) = 0 \qquad \text{...(2)}$$

with the real constants : α_1, α_2 not both zero and β_1, β_2 not both zero. The conditions (2) at the end points are called *boundary condtions.*

A differential equation together with the boundary conditions, is called a **boundary value problem.** *Equation (1) together with boundary conditions (2) is called a* **Strum-Liouville problem.**

Obviously $y = 0$ is a solution of the problem for any value of the parameter λ which is a trivial solution and as such is of no practical utility. Any other solution of (1) satisfying (2) is called an *eigen function* of the problem and the corresponding value of λ is called an eign value of the problem.

A special case. Taking $r = p = 1$ and $q = 0$ in (1), we get

$$y'' + \lambda y = 0 \qquad \text{...(3)}$$

Also if $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$, then the boundary conditions (2) become

$$y(a) = 0, \quad y(b) = 0 \qquad \text{...(4)}$$

Thus (3) and (4) constitute the *simplest form of Strum-Liouville problem.*

(2) **Orthogonality.** Of the various properties of eigen functions of Strum-Liouville problem the orthogonality is of special importance.

---

* Named after the Swiss mathematician *J.C.F. Strum* (1803–1855) who later became Poisson's successor at Sorbonne university, Paris.

† Named after the French professor *Joseph Liouville* (1809–1882) who is known for his important contributions to complex analysis, special functions, number theory and differential geometry.

Def. *Two functions $y_m(x)$ and $y_n(x)$ defined on some interval $a \le x \le b$, are said to be orthogonal on this interval w.r.t. the weight function $p(x) > 0$, if*

$$\int_a^b p(x)\, y_m(x) y_n(x)\, dx = 0 \text{ for } m \ne n.$$

The **norm** *of y_m, denoted by $\| y_m \|$, is defined to be the non-negative square root of $\int_a^b p(x) [y_m(x)]^2\, dx$. Thus*

$$\| y_m \| = \sqrt{\left\{ \int_a^b p(x) [y_m(x)]^2\, dx \right\}}$$

The functions which are orthogonal on $a \le x \le b$ and have norm equal to 1, are called **orthonormal** on this interval.

(3) Orthogonality of eigen functions.

Theorem. *If (i) the functions p, q, r and r' in the Strum-Liouville equation (1) be continuous in $a \le x \le b$;*

(ii) $y_m(x), y_n(x)$ be two eigen functions of the Strum-Liouville problem corresponding to eigen values λ_m and λ_n respectively ;

then $y_m(x)$ and $y_n(x)$ $(m \ne n)$ are orthogonal on that interval w.r.t. the weight function $p(x)$.

Proof. Since y_m and y_n satisfy (1) above

$$(ry'_m)' + (\lambda_m p + q)\, y_m = 0$$
$$(ry'_n)' + (\lambda_n p + q)\, y_n = 0$$

Multiplying the first equation by y_n and the second by $-y_m$ and adding, we get

$$(\lambda_m - \lambda_n)\, p y_m y_n = y_m (ry'_n) - y_n (ry'_m)$$
$$= \frac{d}{dx} [(ry'_n)\, y_m - (ry'_m) y_n], \text{ after differentiation.}$$

Now integrating both sides w.r.t. x from a to b, we obtain

$$(\lambda_m - \lambda_n) \int_a^b p y_m y_n\, dx = \left[(ry'_n) y_m - (ry'_m) y_n \right]_a^b$$
$$= r(b)\, [y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)] - r(a)\, [y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)] \quad \text{...(A)}$$

The R.H.S. will vanish if the boundary conditions are of one of the following forms :

I. $y(a) = y(b) = 0$; II. $y'(a) = y'(b) = 0$; III. $\alpha_1 y(a) + \alpha_2 y'(a) = 0$, $\beta_1 y(b) + \beta_2 y'(b) = 0$ where either α_1 and α_2 is not zero and either β_1 or β_2 is not zero.

Thus in each case (A) reduces to $\int_a^b p y_m y_n\, dx = 0 \quad (m \ne n)$

which shows that the eigen functions y_m and y_n are orthogonal on $a \le x \le b$ w.r.t. the weight function $p(x) = 0$.

Obs. The third form of the boundary conditions in fact contains the first two forms as special cases.

Cor. 1. Orthogonality of Legendre polynomials has already been established directly in § 16.17. But if follows at once from the above theorem.

We have already seen in para (1) that Legendre's equation is Strum-Liouville equation

$$[(1 - x^2)\, y']' + \lambda y = 0 \qquad [\lambda = n(n+1)]$$

with $r(x) = 1 - x^2$, $p(x) = 1$ and $q(x) = 0$.

Since $y(-1) = y(1) = 0$ and for $n = 0, 1, 2, \ldots, \lambda = 0, 1.2, 2.3, \ldots$, the Legendre polynomials are the solutions of the problem *i.e.*, these are the eigen functions. Thus it follows by the above theorem, that they are orthogonal on $-1 \le x \le 1$.

Cor. 2. Orthogonality of Bessel functions has also been established directly in § 16.11. But it can easily be seen to follow from the above theorem.

In para (1), we transformed the Bessel's equation

$$X^2 \frac{d^2 J_n}{dx^2} + X \frac{dJ_n}{dx} + (X^2 - n^2)\, J_n(x) = 0$$

into $[xJ'_n(kx)]' + (k^2 x - n^2/x)\, J_n(kx) = 0$ which is Strum-Liouville equation with $r(x) = x$, $p(x) = x$, $q(x) = -n^2/x$ and $\lambda = k^2$. Since $r(0) = 0$, it follows from the above theorem that those solutions of $J_n(kx)$ which are zero at $x = 0$ form an orthogonal set on $0 \le x \le R$ with weight function $p(x) = x$.

Example 16.26. *For the Strum-Liouville problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(l) = 0$, find the eigen functions and show that they are orthogonal.*

Solution. For $\lambda = -\gamma^2$, the general solution of the equation is $y(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}$

The above boundary conditions give $c_1 = c_2 = 0$ and $y = 0$ which is not an eigen function.

For $\lambda = \gamma^2$, the general solution is $y(x) = A \cos \gamma x + B \sin \gamma x$

The first boundary condition gives $y(0) = A = 0$ and the second boundary condition gives $y(l) = B \sin \gamma l = 0$, $\gamma = 0, \pm \pi/l, \pm 2\pi/l, \ldots$ Thus the eigen values are $\lambda = 0, \pi^2/l^2, 4\pi^2/l^2, \ldots$ and taking $B = 1$, the corresponding eigen functions are

$$y_n(x) = \sin (n\pi x/l) \qquad n = 0, 1, 2, \ldots$$

From the above theorem, it follows that the said eigen functions are orthogonal on the interval $0 \le x \le l$.

Obs. This problem concerns an elastic string stretched between fixed points $x = 0$ and $x = l$ and allowed to vibrate. Here $y(x)$ is the space function of the deflection $u(x, t)$ of the string where t is the time. (See § 18.4).

PROBLEMS 16.7

Find the eigen functions of each of the following *Strum-Liouville problems* and verify their orthogonality :

1. $y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0$.
2. $y'' + \lambda y = 0, y(0) = 0, y'(l) = 0$.
3. $y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0$.
4. $y'' + \lambda y = 0, y(\pi) = y(-\pi), y'(\pi) = y'(-\pi)$.
5. $(xy')' + \lambda x^{-1} y = 0, y(1) = 0, y'(e) = 0$.

Transform each of the following equations to the *Strum-Liouville equations* indicating the weight function :

6. *Laguerre's equation* : $xy'' + (1 - x)y' + ny = 0$.
7. Hermite's equation : $y'' - 2xy' + 2ny = 0$.

16.20 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 16.8

Fill up the blanks or choose the correct answer in the following problems :

1. In terms of Legendre polynomials $2 - 3x + 4x^2$ is ......
2. $J_{-1/2} =$ ....
3. $\int_{-1}^{1} P_n^2(x)\, dx =$ ....
4. $P_{2n+1}(0) =$ ....
5. $\int_{-1}^{1} x^m P_n(x)\, dx =$ ......(*m* being an integer $< n$)
6. The recurrence relation connecting $J_n(x)$ to $J_{n-1}(x)$ and $J_{n+1}(x)$ is ....
7. Orthogonality relation for Bessel functions is ....
8. Bessel's equation of order zero is ....
9. $J_{1/2} =$ ....
10. $\frac{d}{dx}[x^n J_n(x)] =$ ....
11. Value of $P_2(x)$ is ....
12. $\int_{-1}^{1} P_3(x) P_4(x)\, dx =$ ....
13. $P_n(-1) = (-1)^n$ (True or False)
14. Rodrigue's formula for $P_n(x)$ is ....
15. $\int_0^1 xJ_n(\alpha x) J_n(\beta x)\, dx = 0$, if ....
16. Expansion of $5x^3 + x$ interms of Legendre polynomials is ....
17. Generating function of $P_n(x)$ is ....
18. $\frac{d}{dx}[J_0(x)] =$ ....
19. Bessel equation of order 4 is $x^2y'' + xy' + (x^2 - 4)y = 0$. (True or False)
20. $\frac{d}{dx}[x^2 J_2(x)] = x^2 J_1(x)$. (True or False)
21. Legendre's polynomial of first degree $= x$. (True or False)

22. If α is a root of $P_n(x) = 0$, then $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ are of opposite signs. (True or False)
23. $x = 0$ is a regular singular point of $2x^2y'' + 3xy' + (x^2 - 4)y = 0$. (True or False)
24. $\cos x = 2J_1 - 2J_3 + 2J_5 - \ldots\ldots$ (True or False)
25. If J_0 and J_1 are Bessel functions, then $J_1'(x)$ is given by

    (*a*) $-J_0$ (*b*) $J_0(x) - 1/x\, J_1(x)$ (*c*) $J_0(x) + \frac{1}{x} J_1(x)$.

26. If $J_n(x)$ is the Bessel function of first kind, then $\int_0^{\pi} [J_{-2}(x) - J_2(x)]\, dx =$

    (*a*) 2 (*b*) -2 (*c*) 0 (*d*) 1.

27. If $J_{n+1}(x) = \frac{2}{x} J_n(x) - J_0(x)$, then n is

    (*a*) 0 (*b*) 2 (*c*) -1 (*d*) none of these.

28. The series $x - \frac{x^3}{2^2(1!)^2} + \frac{x^5}{2^4(2!)^2} - \frac{x^7}{2^6(3!)^2} + \ldots\ldots \infty$ equals

    (*a*) $J_{1/2}(x)$ (*b*) $J_0(x)$ (*c*) $xJ_0(x)$ (*d*) $xJ_{1/2}(x)$.

29. If $\int_{-1}^{1} P_n(x)\, dx = 2$, then n is

    (*a*) 0 (*b*) 1 (*c*) -1 (*d*) none of these.

30. The value of $\int_{-1}^{1} (2x + 1)P_3(x)\, dx$ where $P_3(x)$ is the third degree Legendre polynomial, is

    (*a*) 1 (*b*) -1 (*c*) 2 (*d*) 0.

31. The value of the integral $\int_{-1}^{1} x^3 P_3(x)\, dx$, where $P_3(x)$ is a Legendre polynomial of degree 3, is

    (*a*) 0 (*b*) $\frac{2}{35}$ (*c*) $\frac{4}{35}$ (*d*) $\frac{11}{35}$.

32. The polynomial $2x^2 + x + 3$ in terms of Legendre polynomials is

    (*a*) $\frac{1}{3}(4P_2 - 3P_1 + 11P_0)$ (*b*) $\frac{1}{3}(4P_2 + 3P_1 - 11P_0)$

    (*c*) $\frac{1}{3}(4P_2 + 3P_1 + 11P_0)$ (*d*) $\frac{1}{3}(4P_2 - 3P_1 - 11P_0)$.

33. If $P_n(x)$ be the Legendre polynomial, then $P_n'(-x)$ is equal to

    (*a*) $(-1)^n P_n(x)$ (*b*) $(-1)^n P_n'(x)$ (*c*) $(-1)^{n+1} P_n'(x)$ (*d*) $P_n''(x)$.

34. Legendre polynomial $P_5(x) = \lambda(63x^5 - 70x^3 + 15x)$ where λ is equal to

    (*a*) 1/2 (*b*) 1/5 (*c*) 1/8 (*d*) 1/10.

35. $\int_{-1}^{1} (1 + x) P_n(x)\, dx$, $(n > 1)$, is equal to

    (*a*) $\frac{1}{2n+1}$ (*b*) $\frac{2}{2n+1}$ (*c*) $\frac{n}{2n+1}$ (*d*) 0.

36. The singular points of the differential equation $x^3(x - 1)y'' + 2(x - 1)y' + y = 0$ are .... (P.T.U., 2009)

CHAPTER 17

Partial Differential Equations

17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z, the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; *the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.* The method is best illustrated by the following examples :

Example 17.1. *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \qquad \ldots(i)$$

Solution. Differentiating (*i*) partially with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (*i*), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. *Form the partial differential equations (by eliminating the arbitrary functions) from*
(*a*) $z = (x + y)\,\phi\,(x^2 - y^2)$ (P.T.U., 2009)
(*b*) $z = f(x + at) + g(x - at)$ (V.T.U., 2009) (*c*) $f(x^2 + y^2, z - xy) = 0$ (S.V.T.U., 2007)

Solution. (*a*) We have $z = (x + y)\,\phi\,(x^2 - y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,2x + \phi\,(x^2 - y^2), \qquad ...(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,(-2y) + \phi\,(x^2 - y^2) \qquad ...(ii)$$

From (*i*), $\quad p - \dfrac{z}{x + y} = 2x\,(x + y)\,\phi'\,(x^2 - y^2)$

From (*ii*), $\quad q - \dfrac{z}{x + y} = -2y\,(x + y)\,\phi'\,(x^2 - y^2)$

Division gives $\dfrac{p - z/(x + y)}{q - z/(x + y)} = -\dfrac{x}{y}$

i.e., $\quad [p(x + y) - z]y + [q\,(x + y) - z]x$

i.e., $\quad (x + y)\,(py + qx) - z(x + y) = 0$

Hence $\quad py + qz = z$ is required equation.

(*b*) We have $\quad z = f\,(x + at) + g(x - at) \qquad ...(i)$

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'\,(x + at) + g'(x - at),\ \frac{\partial^2 z}{\partial x^2} = f''\,(x + at) + g''(x - at) \qquad ...(ii)$$

$$\frac{\partial z}{\partial t} = af'\,(x + at) - ag'(x - at),\ \frac{\partial^2 z}{\partial t^2} = a^2 f''\,(x + at) + a^2 g''(x - at) = a^2\,\frac{\partial^2 z}{\partial x^2} \qquad [\text{By } (ii)]$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

(*c*) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f\,(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or
$$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + p) = 0 \qquad ...(i)$$

and
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x + q) = 0 \qquad ...(ii)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (*i*) and (*ii*), we get

$$\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0 \quad \text{or} \quad xq - yp = x^2 - y^2.$$

Example 17.3. *Find the differential equation of all planes which are at a constant distance a from the origin.* (V.T.U., 2009 S ; Kurukshetra, 2006)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \qquad ...(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $\qquad l^2 + m^2 + n^2 = 1$ or $n = \sqrt{(1 - l^2 - m^2)}$

$\therefore$ (*i*) becomes $\qquad lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a \qquad$ ...(*ii*)

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ .\ p = 0 \qquad ...(iii)$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ .\ q = 0 \qquad ...(iv)$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ .\ p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ .\ q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $\qquad (l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2$ or $1 - l^2 - m^2 = 1 - \dfrac{p^2 + q^2}{1 + p^2 + q^2} = \dfrac{1}{1 + p^2 + q^2}$

Also $\qquad l = -\dfrac{p}{\sqrt{(1 + p^2 + q^2)}}$ and $m = -\dfrac{q}{\sqrt{(1 + p^2 + q^2)}}$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $\qquad z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

1. $z = ax + by + a^2 + b^2$.
2. $(x - a)^2 + (y - b)^2 + z^2 = c^2$. (*Kottayam, 2005*)
3. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$ (*Anna, 2009*)
4. $z = a \log \left\{\dfrac{b(y-1)}{1-x}\right\}$ (*J.N.T.U., 2002 S*)
5. Find the differential equation of all spheres of fixed radius having their centres in the *xy*-plane. (*Madras 2000 S*)
6. Find the differential equation of all spheres whose centres lie on the *z*-axis. (*Kerala, 2005*)

Form the partial differential equations (by eliminating the arbitrary functions) from :

7. $z = f(x^2 - y^2)$ (*S.V.T.U., 2008*)
8. $z = f(x^2 + y^2) + x + y$ (*Anna, 2009*)
9. $z = yf(x) + xg(y)$. (*V.T.U., 2004*)
10. $z = x^2 f(y) + y^2 g(x)$. (*Anna, 2003*)
11. $z = f(x) + e^y g(x)$.
12. $xyz = \phi(x + y + z)$.
13. $z = f_1(x) f_2(y)$.
14. $z = e^{my} \phi(x - y)$. (*P.T.U., 2002*)
15. $z = y^2 + 2f\left(\dfrac{1}{x} + \log y\right)$. (*V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000*)
16. $z = f_1(y + 2x) + f_2(y - 3x)$. (*Kurukshetra, 2005*)
17. $v = \dfrac{1}{r}[f(r - at) + F(r + at)]$.
18. $z = xf_1(x + t) + f_2(x + t)$.
19. $F(xy + z^2, x + y + z) = 0$. (*V.T.U., 2006*)
20. $F(x + y + z, x^2 + y^2 + z^2) = 0$. (*S.V.T.U., 2007*)
21. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\dfrac{\partial u}{\partial x} + (x^2 - yz)\dfrac{\partial u}{\partial y} + (z^2 - xy)\dfrac{\partial u}{\partial z} = 0$.

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $\qquad f(x, y, z, a, b) = 0 \qquad$ ...(1)

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. *Solve* $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ *(V.T.U., 2010)*

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \dfrac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1$.

Solution. If z were function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$

$$\therefore \qquad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0$; $z = e^y$, $\quad \therefore e^y = \phi(y)$. $\qquad$ When $x = 0$, $\dfrac{\partial z}{\partial x} = 1$, $\quad \therefore \; 1 = f(y)$.

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 17.6. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *for which* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$ *when* y *is an odd multiple of* $\pi/2$. *(V.T.U., 2010 S)*

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \ldots(i)$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2\sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \qquad \text{...}(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$$\therefore \qquad 0 = 0 + 0 + g(x) \quad \text{or} \quad g(x) = 0 \qquad [\because \cos(2n+1)\pi/2 = 0]$$

Hence from (*ii*), the complete solution is $z = (1 + \cos x)\cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a.$
2. $\frac{\partial^2 z}{\partial x^2} = xy.$
3. $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x.$
4. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y).$
5. $\frac{\partial^2 z}{\partial y^2} = z$, given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.
6. $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$.

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as *Lagrange's Linear equation**, is of the form

$$Pp + Qq = R \qquad \text{...(1)}$$

where P, Q and R are functions of x, y, z. This equation is called a *quasi-linear equation*. When P, Q and R are independent of z it is known as *linear equation*.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$ ...(2)
where u, v are some functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0 \text{ and } \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0.$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get $\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q \end{vmatrix} = 0$

which simplifies to $\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$ ...(3)

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = du = 0$$

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = dv = 0.$$

*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

or
$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \text{...(4) [By virtue of (1) and (3)]}$$

The solutions of these equations are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(*i*) *form the subsidiary equations* $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(*ii*) *solve these simultaneous equations by the method of § 16.10 giving $u = a$ and $v = b$ as its solutions.*

(*iii*) *write the complete solution as $\phi(u, v) = 0$ or $u = f(v)$.*

Example 17.7. *Solve* $\frac{y^2z}{x}p + xzq = y^2$. (*Kottayam, 2005*)

Solution. Rewriting the given equation as

$$y^2zp + x^2zq = y^2x,$$

The subsidiary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

The first two fractions give $x^2dx = y^2dy$.

Integrating, we get $x^3 - y^3 = a$ ...(*i*)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $x^2 - z^2 = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. *Solve* $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx.$ (*V.T.U., 2010 ; S.V.T.U., 2009*)

Solution. Here the subsidiary equations are $\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$

Using multipliers x, y, and z, we get each fraction $= \frac{xdx + ydy + zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$ ...(*i*)

Again using multipliers l, m and n, we get each fraction $= \frac{ldx + mdy + ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz.$ (*V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008*)

Solution. Here the subsidiary equations are $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$

From the last two fractions, we have $\frac{dy}{y}=\frac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$ ...(*i*)

Using multipliers x, y and z, we have

$$\text{each fraction} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \quad \therefore \quad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

which on integration gives $\log(x^2+y^2+z^2) = \log z + \log b$

or $$\frac{x^2+y^2+z^2}{z} = b \qquad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the required solution is $x^2+y^2+z^2 = zf(y/z)$.

Example 17.10. *Solve* $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$. (*P.T.U., 2009 ; Bhopal, 2008 ; S.V.T.U. 2007*)

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \qquad \text{...(i)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\therefore \quad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \qquad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Example 17.11. *Solve* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$. (*Bhopal, 2008 ; V.T.U., 2006 ; Madras, 2000*)

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \text{...(i)}$$

$$\text{Each of these equations} = \frac{dx - dy}{x^2 - y^2 - (y-x)z} = \frac{dy - dz}{y^2 - z^2 - x(z-y)}$$

i.e., $$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(x+y+z)} \quad \text{or} \quad \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

Integrating, $$\log(x-y) = \log(y-z) + \log c \quad \text{or} \quad \frac{x-y}{y-z} = c \qquad \text{...(ii)}$$

$$\text{Each of the subsidiary equations (i)} = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x+y+z)(x^2+y^2+z^2-yz-zx-xy)} \qquad \text{...(iii)}$$

$$\text{Also each of the subsidiary equations} = \frac{dx + dy + dz}{x^2+y^2+z^2-yz-zx-xy} \qquad \text{...(iv)}$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z) d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad ...(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p.\cos(x + y) + q \sin(x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log(x + y)$.
8. $xp - yq = y^2 - x^2$ (*J.N.T.U., 2002 S*)
9. $(y + z)p - (z + x)q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$. (*Bhopal, 2007*)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$. (*V.T.U., 2010 ; Anna, 2008*)
12. $y^2p - xyq = x(z - 2y)$. (*S.V.T.U., 2008*)
13. $(y^2 + z^2)p - xyq + zx = 0$. (*P.T.U., 2009 ; V.T.U., 2009*)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (*Kerala, 2005*)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. f(p, q) = 0, *i.e., equations containing p and q only.*

Its complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation $f(a, b) = 0$ ...(2)

[Since from (1), $p = \dfrac{\partial z}{\partial x} = a$ and $q = \dfrac{\partial z}{\partial y} = b$, which when substituted in (2) give $f(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. *Solve $p - q = 1$.* (*Anna, 2009*)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + (a - 1)y + c$ is the desired solution.

Example 17.13. *Solve $x^2p^2 + y^2q^2 = z^2$.* (*Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005*)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z} \cdot \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \cdot \frac{\partial z}{\partial y}\right)^2 = 1 \qquad ...(i)$$

and setting $\dfrac{dx}{x} = du, \dfrac{dy}{y} = dv, \dfrac{dz}{z} = dw$ so that $u = \log x, v = \log y, w = \log z$.

Then (*i*) becomes $$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$$

i.e., $$P^2 + Q^2 = 1 \quad \text{where} \quad P = \frac{\partial w}{\partial u} \quad \text{and} \quad Q = \frac{\partial w}{\partial v}.$$

Its complete solution is $w = au + bv + c$ ...(*ii*)

where $a^2 + b^2 = 1$ or $b = \sqrt{(1-a^2)}$.

$\therefore$ (*ii*) becomes $w = au + \sqrt{(1-a^2)}\, v + c$

or $$\log z = a \log x + \sqrt{(1-a^2)} \log y + c$$ which is the required solution.

Form II. f(z, p, q) = 0, *i.e., equations not containing x and y.*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{dz}{du}\right) = 0$$ which is an ordinary differential equation of the first order.

Rewiting it as $\dfrac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$F(z, a) = u + b$, or $x + ay + b = F(z, a)$ which is the desired complete solution.

Thus to solve $f(z, p, q) = 0$,

(*i*) *assume $u = x + ay$ and substitute $p = dz/du$, $q = a\, dz/du$ in the given equation;*

(*ii*) *solve the resulting ordinary differential equation in z and u;*

(*iii*) *replace u by $x + ay$.*

Example 17.14. *Solve $p(1 + q) = qz$.* (*Madras, 2000 S*)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a\, dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \quad \text{or} \quad a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a\,dz}{az-1} = \int du + b$$

or $$\log(az - 1) = u + b \text{ or } \log(az - 1) = x + ay + b$$

which is the required complete solution.

Example 17.15. *Solve $q^2 = z^2p^2(1 - p^2)$.* (*J.N.T.U., 2005 ; Kerala, 2005*)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a\frac{dz}{du} \quad \text{and} \quad q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

$\therefore$ The given equation becomes $$\left(\frac{dz}{du}\right)^2 = a^2z^2\left(\frac{dz}{du}\right)^2\left\{1 - a^2\left(\frac{dz}{du}\right)^2\right\} \qquad ...(i)$$

or $$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2-1)}}{a^2z}$$

Integrating, $$\int \frac{a^2z}{\sqrt{(a^2z^2-1)}}\,dz = \int du + c \quad \text{or} \quad (a^2z^2 - 1)^{1/2} = u + c$$

i.e., $$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (*i*) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. *Solve $z^2(p^2x^2 + q^2) = 1$.* (*Bhopal, 2008 S*)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(i)$$

Putting $X = \log x$, so that $x\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}$, (i) takes the standard form

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \quad ...(ii)$$

Let $u = X + ay$ and put $\dfrac{\partial z}{\partial X}=\dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y}=a\dfrac{dz}{du}$ in (ii), so that

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1 \quad \text{or} \quad \sqrt{(1+a^2)}\,zdz=\pm du$$

Integrating, $\sqrt{(1+a^2)}\,z^2=\pm 2u+b=\pm 2(X+ay)+b$

or $z^2\sqrt{(1+a^2)}=\pm 2(\log x+ay)+b$

which is the complete solution required.

Form III. f(x, p) = F(y, q), *i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q.*

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p, we get $p = \phi(x)$

and solving for q, we get $q = \psi(y)$

Since $dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy=pdx+qdy$

$\therefore$ $dz=\phi(x)dx+\psi(y)dy$

Integrating, $z=\int\phi(x)dx+\int\psi(y)dy+b$

which is the desired complete solution containing two constants a and b.

Example 17.17. *Solve* $p^2+q^2=x+y$. *(Bhopal, 2006; Madras, 2003)*

Solution. Given equation is $p^2-x=y-q^2=a$, say

$\therefore$ $p^2-x=a$ gives $p=\sqrt{(a+x)}$

and $y-q^2=a$ gives $q=\sqrt{(y-a)}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz=\sqrt{(a+x)}\,dx+\sqrt{(y-a)}\,dy$$

$\therefore$ integrating gives, $z=\dfrac{2}{3}(a+x)^{3/2}+\dfrac{2}{3}(y-a)^{3/2}+b$

which is the required complete solution.

Example 17.18. *Solve* $z^2(p^2+q^2)=x^2+y^2$. *(Bhopal, 2008)*

Solution. The equation can be reduced to the above form by writing it as

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2=x^2+y^2 \quad ...(i)$$

and putting $zdz=dZ$, *i.e.*, $Z=\dfrac{1}{2}z^2$

$\therefore$ $\dfrac{\partial Z}{\partial x}=\dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial x}=z\dfrac{\partial z}{\partial x}=P$

and $\dfrac{\partial Z}{\partial y}=\dfrac{\partial Z}{\partial z}\cdot\dfrac{\partial z}{\partial y}=z\dfrac{\partial z}{\partial y}=Q$

$\therefore$ (i) becomes $$P^2 + Q^2 = x^2 + y^2$$

or $$P^2 - x^2 = y^2 - Q^2 = a, \text{ say.}$$

$\therefore$ $$P = \sqrt{(x^2 + a)} \text{ and } Q = \sqrt{(y^2 - a)}.$$

$\therefore$ $$dZ = Pdx + Qdy \text{ gives}$$

$$dZ = \sqrt{(x^2 + a)}\, dx + \sqrt{(y^2 - a)}\, dy$$

Integrating, we have $$Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}]$$

$$+ \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$$

or $$z^2 = x\sqrt{(x^2 + a)} + y\sqrt{(y^2 - a} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$$

which is the required complete solution.

Example 17.19. *Solve* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

(*Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003*)

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = P + Q$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = P - Q, \text{ where } P = \frac{\partial z}{\partial u}, Q = \frac{\partial z}{\partial v}$$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2u = 1 - 4Q^2v = a \text{ (say)}$$

$$P = \pm \frac{1}{2}\sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2}\sqrt{\frac{1-a}{v}}$$

$\therefore$ $$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$$

$$= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1-a}}{2} \frac{dv}{\sqrt{v}}$$

Integrating, we have $$z = \pm \sqrt{a}\sqrt{u} \pm \sqrt{1-a}\sqrt{v} + b$$

or $$z = \pm \sqrt{a(x + y)} \pm \sqrt{(1 - a)(x - y)} + b$$

which is the required complete solution.

Form IV. z = px + qy + f(p, q): *an equation analogous to the Clairaut's equation (§ 11.14).*

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. *Solve* $z = px + qy + \sqrt{(1 + p^2 + q^2)}$. (*Anna, 2009*)

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

$\therefore$ Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.
2. $p^2 + q^2 = 1$. (*Osmania, 2000*)
3. $z = p^2 + q^2$. (*Anna, 2005 S ; J.N.T.U., 2002 S*)
4. $p(1 - q^2) = q(1 - z)$ (*Anna, 2006*)
5. $yp + xq + pq = 0$.
6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.
8. $\sqrt{p} + \sqrt{q} = x + y$.
9. $p^2 + q^2 = x^2 + y^2$. *(Osmania, 2003)*
10. $z = px + qy + \sin(x + y)$.
11. $\sqrt{p} + \sqrt{q} = 2x$. *(J.N.T.U., 2006)*
12. $z = px + qy - 2\sqrt{(pq)}$.
13. $(x - y)(px - qy) = (p - q)^2$. **[Hint.** Use $x + y = u, xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y, z, p, q) = 0 \qquad \text{...(1)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy \qquad \text{...(2)}$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)

then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(4)}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad \text{...(5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(6)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{...(7)}$$

Eliminating $\dfrac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad \text{...(8)}$$

Also eliminating $\dfrac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \qquad \text{...(9)}$$

Adding (8) and (9) and using $\dfrac{\partial q}{\partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0 \qquad \text{...(10)}$$

i.e.,

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0 \qquad \text{...(11)}$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

Example 17.21. *Solve* $(p^2 + q^2)y = qz$. *(V.T.U., 2007 ; Hissar, 2005)*

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $\quad p^2 + q^2 = c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2 + q^2 = c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2 - c^2y^2)}/z$

Hence $$dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2 - c^2y^2)}\,dx + \frac{c^2y}{z}dy$$

or $$zdz - c^2y\,dy = c\sqrt{(z^2 - c^2y^2)}\,dx \quad \text{or} \quad \frac{\frac{1}{2}d(z^2 - c^2y^2)}{\sqrt{(z^2 - c^2y^2)}} = c\,dx$$

Integrating, we get $\sqrt{(z^2 - c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Example 17.22. *Solve* $2xz - px^2 - 2qxy + pq = 0$. *(Rajasthan, 2006)*

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore \quad dq = 0 \quad \text{or} \quad q = a.$

Putting $q = a$ in (*i*), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady \quad \text{or} \quad \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a) \quad \text{or} \quad z = ay + b(x^2 - a)$

which is the required complete solution.

Example 17.23. *Solve* $2z + p^2 + qy + 2y^2 = 0$. *(J.N.T.U., 2005 ; Kurukshetra, 2005)*

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a - x)^2]$$

$\therefore \qquad dz = pdx + qdy = (a - x)dx - \dfrac{1}{y}[2z + 2y^2 + (a - x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\,dy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

Integrating $\qquad 2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $\qquad y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2y$.
2. $z^2 = pq\,xy$. (Anna, 2009 ; V.T.U., 2004)
3. $1 + p^2 = qz$.
4. $pxy + pq + qy = yz$. (J.N.T.U., 2006 ; Kurukshetra, 2006)
5. $p(p^2 + 1) + (b - z)q = 0$.
6. $q + xp = p^2$. (Osmania, 2003)

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \dots(1)$$

in which k's are constants, is called a *homogeneous linear partial differential equation of the nth order with constant coefficients*. It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\dfrac{\partial^r}{\partial x^r} = D^r$ and $\dfrac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1D^{n-1}D'^r + D' + \dots + k_nD'^n)\,z = F(x, y)$

or briefly $\qquad f(D, D')z = F(x, y) \qquad \dots(2)$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and *the particular integral*.

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + k_1 \dfrac{\partial^2 z}{\partial x \partial y} + k_2 \dfrac{\partial^2 z}{\partial y^2} = 0 \qquad \dots(1)$

which in *symbolic form is* $(D^2 + k_1DD' + k_2D'^2)z = 0 \qquad \dots(2)$

Its symbolic operator equated to zero, *i.e.*, $D^2 + k_1DD' + k_2D'^2 = 0$ is called the *auxiliary equation* (A.E.)

Let its root be $\quad D/D' = m_1, m_2$.

Case I. *If the roots be real and distinct* then (2) is equivalent to

$$(D - m_1D')(D - m_2D')z = 0 \qquad \dots(3)$$

It will be satisfied by the solution of

$$(D - m_2D')z = 0, \text{ i.e., } p - m_2q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2x = a \text{ and } z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0, \quad i.e., \quad \text{by } z = f(y + m_1x)$$

Hence the complete solution of (1) is $z = f(y + m_1x) + \phi(y + m_2x)$.

Case II. *If the roots be equal* (*i.e.*, $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1D')^2z = 0 \quad \text{...(4)}$$

Putting $(D - m_1D')z = u$, it becomes $(D - m_1D')u = 0$ which gives

$$u = \phi(y + m_1x)$$

$\therefore$ (4) takes the form $(D - m_1'D)z = \phi(y + m_1x)$ or $p - m_1q) = \phi(y + m_1x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1x)}$$

giving $y + m_1x = a$ and $dz = \phi(a)\,dx$, *i.e.*, $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1x) = f(y + m_1x). \quad i.e., \quad z = f(y + m_1x) + x\phi(y + m_1x).$$

Example 17.24. *Solve* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. Given equation in *symbolic form* is $(2D^2 + 5DD' + 2D'^2)z = 0$.

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives $m = -2, -1/2$.

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example 17.25. *Solve* $4r + 12s + 9t = 0$. *(P.T.U., 2010)*

Solution. Given equation in *symbolic form* is $(4D^2 + 12DD' + 9D'^2)z = 0$

for $r = \frac{\partial^2 z}{\partial x^2} = D^2 z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2 z.$

$\therefore$ Its *auxiliary equation* is $4m^2 + 12m + 9 = 0$, whence $m = -3/2, -3/2$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

17.10 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1DD' + k_2D'^2)z = F(x, y)$ *i.e.*, $f(D, D')z = F(x, y)$.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

Case I. *When* $F(x, y) = e^{ax + by}$

Since $De^{ax+by} = ae^{ax+by}\ ;\ D'e^{ax+by} = be^{ax+by}$

$\therefore$ $D^2e^{ax+by} = a^2e^{ax+by}\ ;\ DD'e^{ax+by} = abe^{ax+by}$

and $D'^2e^{ax+by} = b^2e^{ax+by}$

$\therefore$ $(D^2 + k_1DD' + k_2D'^2)e^{ax+by} = (a^2 + k_1ab + k_2b^2)\,e^{ax+by}$

i.e., $f(D, D')e^{ax+by} = f(a, b)\,e^{ax+by}$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$$

Case II. *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny)$.

$\therefore$ $f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(mx + ny) = \frac{1}{f(-m^2 - mn, -n^2)} \sin(mx + ny)$$

Similarly about the P.I. for $\cos(mx + ny)$.

Case III. *When* $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. *When $F(x, y)$ is any function of x and y.*

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

17.11 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $\quad (D^n + k_1 D^{n-1} D' + \dots + k_n D'^n) z = F(x, y)$

or briefly $\quad f(D, D') z = F(x, y)$

Step I. To find the C.F.

(*i*) *Write the A.E.*

i.e., $\quad m^n + k_1 m^{n-1} + \dots + k_n = 0$ *and solve it for* m.

(*ii*) *Write the C.F. as follows*

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \dots$ (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots$
2. $m_1, m_1, m_3 \dots$ (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \dots$
3. $m_1, m_1, m_1 \dots$ (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \dots$

Step II. To find the P.I.

From the *symbolic form*, $\text{P.I.} = \dfrac{1}{f(D, D')} F(x, y)$.

(*i*) *When* $F(x, y) = e^{ax + by}$ $\text{P.I.} = \dfrac{1}{f(D, D')} e^{ax + by}$ [Put $D = a$ and $D' = b$]

(*ii*) *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny) \qquad [\text{Put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(*iii*) *When* $F(x, y) = x^m y^n$, $\text{P.I.} = \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$.

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

(*iv*) *When $F(x, y)$ is any function of x and y* $\text{P.I.} = \dfrac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration.}$$

Example 17.26. *Solve* $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$. *(Madras, 2006)*

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ *i.e.*, $m = 1, -5$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - 5x)$

$$\text{P.I.} = \frac{1}{D^2 + 4DD' - 5D'^2} \sin(2x + 3y) \quad [\text{Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2$$

$$= \frac{1}{-4 + 4(-6) - 5(-9)} \sin(2x + 3y) = \frac{1}{17} \sin(2x + 3y).$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17} \sin(2x + 3y)$.

Example 17.27. *Solve* $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$. *(Bhopal, 2008 S)*

Solution. Given equation in *symbolic form* is $(D^2 - DD')z = \cos x \cos 2y$.

Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \cos x \cos 2y = \frac{1}{2} \frac{1}{D^2 - DD'} [\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'} \cos(x + 2y) \right. \qquad [\text{Put } D^2 = -1, DD' = -2]$$

$$\left. + \frac{1}{D^2 - DD'} \cos(x - 2y)\right] \qquad [\text{Put } D^2 = -1, DD' = 2]$$

$$= \frac{1}{2}\left[\frac{1}{-1 + 2} \cos(x + 2y) + \frac{1}{-1 - 2} \cos(x - 2y)\right] = \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$$

Hence the C.S. is $z = f_1(y) + f_2(y + x) + \frac{1}{2} \cos(x + 2y) - \frac{1}{6} \cos(x - 2y)$.

Example 17.28. *Solve* $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$. *(S.V.T.U., 2007)*

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. $= f_1(y) + x f_2(y) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 D'} (2e^{2x} + 3x^2 y) = 2\frac{1}{D^3 - 2D^2 D'} e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)} x^2 y$$

$$= 2\frac{1}{2^3 - 2 \cdot 2^2(0)} e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1} x^2 y = \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots\right) x^2 y$$

$$= \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(x^2 y + \frac{2}{D} x^2 \cdot 1\right) = \frac{1}{4} e^{2x} + \frac{3}{D^3}\left(x^2 y + \frac{2}{3} x^3\right) \qquad \left[\because \frac{1}{D} f(x) = \int f(x)\, dx\right]$$

$$= \frac{1}{4} e^{2x} + 3y \frac{x^5}{3 \cdot 4 \cdot 5} + 2 \cdot \frac{x^6}{4 \cdot 5 \cdot 6} \qquad \left[\because \frac{1}{D^3} f(x) = \int\left[\int\left(\int f(x)\, dx\right) dx\right] dx\right]$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is $z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$.

Example 17.29. *Solve* $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in *symbolic form* $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m-2)^2 = 0$, whence $m = 2, 2$.

$\therefore$ C.F. $= f_1(y + 2x) + xf_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\,dx = \int e^{2x + (c-2x)}\,dx = xe^c = xe^{2x+y} \qquad [\because y = c - mx = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x+(c-2x)}dy = \frac{1}{2}x^2e^c = \frac{1}{2}x^2e^{2x+y} \qquad [\because y = c - mx = c - 2x]$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{1}{2}x^2e^{2x+y}$.

Example 17.30. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = \cos(2x + y)$. (*P.T.U., 2010 ; S.V.T.U., 2009*)

Solution. Given equation in *symbolic form* is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$\therefore$ C.F. $= f_1(y - 3x) + f_2(y + 2x)$.

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')}\cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})dx\right]_{c \to y+2x} = \frac{1}{D + 3D'}\left[\int \cos c\,dx\right]_{c \to y+2x}$$

$[\because y = c - mx = c - 2x$

$$= \frac{1}{D + 3D'} x\cos(y + 2x) = \left[\int x\cos(\overline{c + 3x} + 2x)dx\right]_{c \to y-3x} = \left[\int x\cos(5x + c)dx\right]_{c \to y-3x}$$

$$= \left[\frac{x\sin(5x + c)}{5} + \frac{\cos(5x + c)}{25}\right]_{c \to y-3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x}) = \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y).$$

Example 17.31. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \partial y} - 6\dfrac{\partial^2 z}{\partial y^2} = y \cos x.$ *(Anna, 2005 S ; U.P.T.U., 2003)*

or $r + s - 6t = y \cos x.$ *(Bhopal, 2008 ; S.V.T.U., 2008)*

Solution. Its *symbolic form* is $(D^2 + DD' - 6D'^2)z = y \cos x$
and the A.E. is $m^2 + m - 6 = 0$, whence $m = -3, 2$.

$$\therefore \quad \text{C.F.} = f_1(y - 3x) + f_2(y + 2x)$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} y \cos x = \frac{1}{D - 2D'}\left[\int (c + 3x) \cos x \, dx\right]_{c \to y - 3x}$$

$$[\because \ y = c - mx = c + 3x]$$

$$= \frac{1}{D - 2D'}[(c + 3x) \sin x + 3 \cos x]_{c \to y - 3x} \quad \text{[Integrating by parts]}$$

$$= \frac{1}{D - 2D'}(y \sin x + 3 \cos x) = \left[\int \{(c - 2x) \sin x \; 3 \cos x\} \, dx\right]_{c \to y - 2x}$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x]_{c \to y + 2x}$$

$$= -y \cos x + \sin x$$

Hence the C.S. is $z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x.$

Example 17.32. *Solve* $4\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 16 \log (x + 2y).$

Solution. Its *symbolic form* is $4D^2 - 4DD' + D'^2 = 16 \log (x + 2y)$
and the A.E. is $4m^2 - 4m + 1 = 0$, $m = 1/2, 1/2$.

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y) = 4 \frac{1}{\left(D - \frac{1}{2}D'\right)} \left\{\frac{1}{D - \frac{1}{2}D'} \log (x + 2y)\right\}$$

$$= 4 \frac{1}{D - \frac{1}{2}D'}\left[\int \log \left\{x + 2\left(c - \frac{x}{2}\right)\right\} dx\right]_{c \to y + x/2} \quad [\because \ y = c - mx = c - x/2]$$

$$= 4 \frac{1}{D - \frac{1}{2}D'}\left[\int \log (2c) \, dx\right]_{c \to y + x/2} = 4 \frac{1}{D - \frac{1}{2}D'}[x \log (x + 2y)]$$

$$= 4\left[\int \left\{x \log \left[x + 2\left(c - \frac{x}{2}\right)\right]\right\} dx\right]_{c \to y + x/2} = 4\left[\log 2c \int x \, dx\right]_{c \to y + x/2} = 2x^2 \log (x + 2y)$$

Hence the C.S. is $z = f_1\left(y + \dfrac{x}{2}\right) + x f_2\left(y + \dfrac{x}{2}\right) + 2x^2 \log (x + 2y).$

PROBLEMS 17.6

Solve the following equations :

1. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 x}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 0.$

2. $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial y^3} = e^{x + 2y}.$ *(Burdwan, 2003)*

3. $(D^2 - 2DD' + D'^2)z = e^{x + y}.$ *(Bhopal, 2007)*

4. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 5\dfrac{\partial^3 z}{\partial x \partial y^2} - 2\dfrac{\partial^3 z}{\partial y^3} = e^{2x + y}.$ *(Bhopal, 2008)*

5. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x.$ *(P.T.U., 2009 S)*

6. $\dfrac{\partial^2 y}{\partial t^2} - a^2 \dfrac{\partial^2 y}{\partial x^2} = E \sin pt.$

7. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{4\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 2 \sin (3x + 2y).$ *(S.V.T.U., 2007)*

8. $(D^3 - 7DD'^2 - 6D'^3)z = \cos (x + 2y) + 4.$ *(Anna, 2008)*

9. $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos (x + 2y).$ *(U.P.T.U., 2006)*

10. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$ *(U.P.T.U., 2003)*

11. $(D^2 - DD')z = \cos 2y (\sin x + \cos x).$

12. $(D^2 - D'^2)z = e^{x-y} \sin (x + 2y).$ *(Anna, 2009)*

13. $(D^2 + 3DD' + 2D'^2)z = 24xy.$

14. $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2.$

15. $(D^2 - DD' - 2D'^2)z = (y - 1) e^x.$ *(Bhopal, 2006)*

16. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y.$

17. $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y.$ *(P.T.U., 2005)*

17.12 NON-HOMOGENEOUS LINEAR EQUATIONS

If in the equation $\quad f(D, D')z = F(x, y) \quad$ ...(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F. + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations.

To find the C.F., we factorize $f(D, D')$ into factors of the form $D - mD' - c$. To find the solution of $(D - mD' - c)z = 0$, we write it as $p - mq = cz$ ...(2)

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Its integrals are $y + mx = a$ and $z = be^{cx}$.

Taking $b = \phi(a)$, we get $z = e^{cx} \phi(y + mx)$

as the solution of (2). The solution corresponding to various factors added up, give the C.F. of (1).

Example 17.32. *Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin (x + 2y)$.* *(U.P.T.U., 2004)*

Solution. Here $\quad f(D, D') = (D + D')(D + D' - 2)$

Since the solution corresponding to the factor $D - mD' - c$ is known to be

$$z = e^{cx} \phi(y + mx)$$

$$\therefore \quad \text{C.F.} = \phi_1(y - x) + e^{2x} f_2(y - x)$$

$$\therefore \quad \text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin (x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'} \sin (x + 2y)$$

$$= -\frac{1}{2(D + D') + 9} \sin (x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81} \sin (x + 2y)$$

$$= \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)]$$

Hence the complete solution is

$$z = \phi_1(y - x) + e^{2x} \phi_2(y - x) + \frac{1}{39} [2 \cos (x + 2y) - 3 \sin (x + 2y)].$$

PROBLEMS 17.7

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = e^{-x}$.
2. $(D - D' - 1)(D - D' - 2)z = e^{2x - y}$.
3. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$.
4. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = x^2 + y^2$. (*Madras, 2000 S*)
5. $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$. (*S.V.T.U., 2009*)
6. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$.

17.13 NON-LINEAR EQUATIONS OF THE SECOND ORDER

We now give a method due to *Monge**, for integrating the equation $Rr + Ss + Tt = V$ ...(1)
in which R, S, T, V are functions of x, y, z, p and q.

Since $$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + tdy, \text{ and } dq = sdx + tdy,$$

we have $$r = (dp - tdy)/dx \text{ and } t = (dq - sdx)/dy.$$

Substituting these values of r and t in (1), and rearranging the terms, we get

$$(Rdpdy + Tdqdx - Vdxdy) - s(Rdy^2 - Sdydx + Tdx^2) = 0 \quad ...(2)$$

Let us consider the equations

$$Rdy^2 - Sdydx + Tdx^2 = 0 \quad ...(3)$$

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad ...(4)$$

which are known as *Monge's equations.*

Since (3) can be factorised, we obtain its integral first. In case the factors are different, we may get two distinct integrals of (3). Either of these together with (4) will give an integral of (4). If need be, we may also use the relation $dz = pdx + qdy$ while solving (3) and (4).

Let $u(x, y, z, p, q) = a$ and $v(x, y, z, p, q) = b$ be the integrals of (3) and (4) respectively. Then $u = a, v = b$ evidently constitute a solution of (2) and therefore, of (1) also. Taking $b = \phi(a)$, we find a general solution of (1) to be $v = \phi(u)$, which should be further integrated by methods of first order equations.

Example 17.34. *Solve* $(x - y)(xr - xs - ys + yt) = (x + y)(p - q)$. (*S.V.T.U., 2007*)

Solution. Monge's equations are

$$xdy^2 + (x + y)dy\,dx + ydx^2 = 0 \quad ...(i)$$

$$xdpdy + ydqdx - \frac{x + y}{x - y}(p - q)\,dydx = 0 \quad ...(ii)$$

(*i*) may be factorised as $(xdy + ydx)(dx + dy) = 0$ whose integrals are $xy = c$ and $x + y = c$.

Taking $xy = c$ and dividing each term of (*ii*) by xdy or its equivalent $-ydx$, we get

$$dp - dq - \frac{dx - dy}{x - y}(p - q) = 0 \quad \text{or} \quad \frac{d(p - q)}{p - q} - \frac{d(x - y)}{x - y} = 0$$

This gives on integration $(p - q)/(x - y) = c$.

Hence a first integral of the given equation is $p - q = (x - y)\,\phi(xy)$ which is a Lagrange's linear equation. Its subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x - y)\phi(xy)}$$

From the first two equations, we have $x + y = a$

Using this, we have

$$dz = -\,\phi(ax - x^2)\,.\,(a - 2x)\,dx \quad \text{which gives } z = \phi_1(ax - x^2) + b$$

Writing $b = \phi_2(a)$ and $a = x + y$, we get

$$z = \phi_1(xy) + \phi_2(x + y).$$

---

* Named after *Gaspard Monge* (1746–1818), Professor at Paris.

Obs. Had we started with the integral $x + y = c$ and divided each term of (*ii*) by dx or $-dy$, we would have arrived at the same solution.

Example 17.35. *Solve* $y^2r - 2ys + t = p + 6y$. (*Osmania, 2002*)

Solution. Monge's equations are $y^2dy^2 + 2ydydx + dx^2 = 0$ ...(*i*)

and $$y^2dpdy + dqdx - (p + 6y)dydx = 0 \quad ...(ii)$$

(*i*) gives $$(ydy + dx)^2 = 0 \text{ i.e. } y^2 + 2x = c \quad ...(iii)$$

Putting $ydy = -dx$ in (*ii*), we get

$$ydp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (ydp + pdy) - dq + 6ydy = 0$$

whose integral is $$py - q + 3y^2 = a$$

Combining this with (*iii*), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2 + 2x) - 3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\, dy = 0$

whose solution is $$z + y\phi(c) - y^3 = b.$$

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

PROBLEMS 17.8

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p\tan x = 0$.
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$. (*J.N.T.U., 2006*)
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

17.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 17.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. The equation $\dfrac{\partial^2 z}{\partial x^2} + 2xy\left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{\partial z}{\partial y} = 5$ is of order .... and degree ....
2. The complementary function of $(D^2 - 4DD' + 4D'^2)z = x + y$ is ......
3. The solution of $\dfrac{\partial^2 z}{\partial y^2} = \sin(xy)$ is ......
4. A solution of $(y - z)p + (z - x)q = x - y$ is ......
5. The particular integral of $(D^2 + DD')z = \sin(x + y)$ is ....
6. The partial differential equation obtained from $z = ax + by + ab$ by eliminating a and b is ......
7. Solution of $\sqrt{p} + \sqrt{q} = 1$ is ......
8. Solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ is ......
9. Solution of $p - q = \log(x + y)$.
10. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$, is ......
11. The solution of $x\, \partial z/\partial x = 2x + y$ is
12. By eliminating a and b from $z = a(x + y) + b$, the p.d.e. formed is ......
13. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is ......
14. By eliminating the arbitrary constants from $z = a^2x + ay^2 + b$, the partial differential equation formed is ...... (*Anna, 2008*)
15. A solution of $u_{xy} = 0$ is of the form ....
16. If $u = x^2 + t^2$ is a solution of $c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$, then $c =$ ......

17. The general solution of $u_{xy} = xy$ is ......

18. The complementary function of $r - 7s + 6t = e^{x+y}$ is ......

19. The solution of $xp + yq = z$ is

(i) $f(x^2, y^2) = 0$ (ii) $f(xy, yz)$ (iii) $f(x, y) = 0$ (iv) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

20. The solution of $(y - z)p + (z - x)q = x - y$, is

(i) $f(x^2 + y^2 + z^2) = xyz$ (ii) $f(x + y + z) = xyz$

(iii) $f(x + y + z) = x^2 + y^2 + z^2$ (iv) $f(x^2 + y^2 + z^2, xyz) = 0$.

21. The partial differential equation from $z = (c + x)^2 + y$ is

(i) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (ii) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$ (iii) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (iv) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$.

22. The solution of $p + q = z$ is

(i) $f(xy, y \log z) = 0$ (ii) $f(x + y, y + \log z) = 0$

(iii) $f(x - y, y - \log z) = 0$ (iv) None of these.

23. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(i) $\frac{1}{2}e^{x+2y}$ (ii) $-\frac{x}{2}e^{x+2y}$ (iii) xe^{x+2y} (iv) x^2e^{x+2y}.

24. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(i) $z = (1 + x + x^2)f(y)$ (ii) $z = (1 + y + y^2)f(x)$

(iii) $z = f_1(x) + y f_2(x) + y^2 f_3(x)$ (iv) $z = f_1(y) + x f_2(y) + x^2 f_3(y)$.

25. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(i) $x \cos(x + y)$ (ii) $\frac{x}{2}\cos(x + y)$ (iii) $x \sin(x + y)$ (iv) $\frac{x}{2}\sin(x + y)$

26. The solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ is

(i) $z = f_1(y + x) + f_2(y - x)$ (ii) $z = f_1(y + x) + f_1(y - x)$

(iii) $z = f(x^2 - y^2)$ (iv) $z = f(x^2 + y^2)$.

27. $xu_x + yu_y = u^2$ is a non-linear partial differential equation. (True or False)

28. $xu_x + u_{xx} = 0$ is a non-linear partial differential equation. (True or False)

29. $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$. (True or False)

30. $u = e^{-t}\sin x$ is a solution of $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$. (True or False)

31. $x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 2u$ is an ordinary differential equation. (True or False)

CHAPTER 18

Applications of Partial Differential Equations

18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem*.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$ *(P.T.U., 2009 S ; Bhopal 2008 ; U.P.T.U., 2005)*

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \quad \text{where } X' = \frac{dX}{dx}, Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

∴ $$\frac{X'' - 2X'}{X} = a, \text{ i.e. } X'' - 2X' - aX = 0 \quad ...(iii)$$

and $$-Y'/Y = a, \quad \text{i.e.,} \quad Y' + aY = 0 \quad ...(iv)$$

To solve the ordinary linear equation (*iii*), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

∴ the solution of (*iii*) is $X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (*iv*) is $Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (*i*), we get

$$z = \{c_1 e^{[1+\sqrt{(1+a)}]x} + c_2 e^{[1-\sqrt{(1+a)}]x}\} \cdot c_3 e^{-ay}$$

i.e., $$z = \{k_1 e^{[1+\sqrt{(1+a)}]x} + k_2 e^{[1-\sqrt{(1+a)}]x}\} \, e^{-ay}$$

which is the required complete solution.

Obs. In practical problems, the unknown constants a, k_1, k_2 are determined from the given boundary conditions.

Example 18.2. *Using the method of separation of variables, solve* $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$.

(V.T.U., 2009 ; Kurukshetra, 2006 ; Kerala, 2005)

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\,T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

∴ $$X' - X - 2kX = 0 \quad \text{or} \quad \frac{X'}{X} = 1 + 2k \quad ...(i) \quad \text{and} \quad \frac{T'}{T} = k \quad ...(ii)$$

Solving (*i*), $\log X = (1 + 2k)x + \log c \quad \text{or} \quad X = ce^{(1 + 2k)x}$

From (*ii*), $\log T = kt + \log c' \quad \text{or} \quad T = c'e^{kt}$

Thus $u(x, t) = XT = cc'\, e^{(1 + 2k)x}\, e^{kt}$ ...(*iii*)

Now $6e^{-3x} = u(x, 0) = cc'\, e^{(1 + 2k)x}$

∴ $cc' = 6$ and $1 + 2k = -3 \quad \text{or} \quad k = -2$

Substituting these values in (*iii*), we get

$$u = 6e^{-3x}\, e^{-2t} \quad \text{i.e.,} \quad u = 6e^{-(3x + 2t)}$$ which is the required solution.

PROBLEMS 18.1

Solve the following equations by the method of *separation of variables* :

1. $py^3 + qx^2 = 0$. *(V.T.U., 2011 ; S.V.T.U., 2008)*
2. $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$. *(V.T.U., 2008)*
3. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$. *(J.N.T.U., 2006)*
4. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$. *(S.V.T.U., 2008)*
5. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$. *(V.T.U., 2008 S)*
6. Find a solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\,g(y)$. Solve the equation subject to the conditions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$, when $x = 0$ for all values of y. *(Andhra, 2000)*

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(i) *Wave equation* : $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

(ii) *One dimensional heat flow equation* : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

(iii) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation* : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(iv) *Transmission line equations.*

(v) *Vibrating membrane.* Two dimensional wave equation.

(vi) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

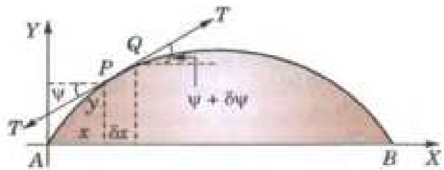


Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right] \quad i.e., \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, where $c^2 = \frac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $\qquad \dfrac{\partial^2 y}{\partial t^2} = X \, . \, T''$ and $\dfrac{\partial^2 y}{\partial x^2} = X'' \, . \, T$

Substituting these in (1), we get $XT'' = c^2 X''T$ i.e., $\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(i) *When k is positive* and $= p^2$, say $X = c_1 e^{px} + c_2 e^{-px}$; $T = c_3 e^{cpt} + c_4 e^{-cpt}$.

(ii) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(iii) *When k is zero.* $X = c_9 x + c_{10}$; $T = c_{11} t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad ...(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(8)$$

is the only suitable solution of the *wave equation*. *(Bhopal, 2008)*

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l)$. (V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004)

Solution. The vibration of the string is given by $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii) \qquad \text{and} \qquad y(l, t) = 0 \quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $\qquad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(iv)

Also $\qquad y(x, 0) = a \sin (\pi x/l)$ ...(v)

Now we have to solve (i) subject to the *boundary conditions* (ii) and (iii) and *initial conditions* (iv) and (v). Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vi)$$

By (ii), $\qquad y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $\qquad y(x, t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt)$ ...(vii)

and $\qquad \dfrac{\partial y}{\partial t} = C_2 \sin px \, \{C_3(-cp \, . \, \sin cpt) + C_4(cp \, . \, \cos cpt)\}$

$\therefore$ By (iv), $\qquad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \, . \, (C_4 \, cp) = 0$, whence $C_2 C_4 cp = 0$.

If $C_2 = 0$, (vii) will lead to the trivial solution $y(x, t) = 0$.

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (vii) becomes $y(x, t) = C_2 C_3 \sin px \cos cpt$ ...(viii)

$\therefore$ By (*iii*), $\quad y(l, t) = C_2C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore$ $pl = n\pi$, *i.e.*, $p = n\pi/l$, where n is an integer.

Hence (*i*) reduces to $\quad y(x, t) = C_2C_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

[These are the solutions of (*i*) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum.**]

Finally, imposing the last condition (*v*), we have $y(x, 0) = C_2C_3 \sin \dfrac{n\pi x}{l} = a \sin \dfrac{\pi x}{l}$

which will be satisfied by taking $C_2C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l}$ ...(*ix*)

Obs. We have from (*ix*) $\dfrac{\partial^2 y}{\partial t^2} = -a\left(\dfrac{\pi c}{l}\right)^2 \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l} = -\left(\dfrac{\pi c}{l}\right)^2 y.$

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, *i.e.*, $2l/c$.

Thus we can look upon (*ix*) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l, wave-velocity c and amplitude $y_0 = a \cos (\pi ct/l)$ which varies harmonically with time t. Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called *nodes*, remain undisturbed during wave motion. Thus (*ix*) represents a *stationary sine wave* of varying amplitudes whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $\quad y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = y_0 \sin^3 \left(\dfrac{\pi x}{l}\right)$ ...(*iii*)

and
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \text{...(iv)}$$

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $\quad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore \quad y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*), $\quad y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus
$$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{cn\pi t}{l} + c_4 \sin \frac{cn\pi t}{l}\right) \qquad \text{...(v)}$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{cn\pi}{l}\left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l}\right)$$

By (*iv*), $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l}\right)\dfrac{cn\pi}{l} \cdot c_4 = 0$, *i.e.* $c_4 = 0$.

Thus (*v*) becomes $y(x, t) = c_2c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \text{...(vi)}$$

$\therefore$ from (*iii*), $$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing both sides, we have

$$b_1 = 3\, y_0/4,\ b_2 = 0,\ b_3 = -\, y_0/4,\ b_4 = b_5 = \ldots = 0.$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.*

(Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0,\ y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \ldots(iv)$$

The solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*) $y(l, t) = c_2 \sin pl(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \qquad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right) \frac{n\pi c}{l} \left(-\, c_3 \sin \frac{n\pi ct}{l} + c_4 \cos \frac{n\pi ct}{l}\right)$$

$\therefore$ by (*iv*) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l}\right) \frac{n\pi c}{l} \cdot c_4 = 0$$

Thus (*v*) becomes $$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \ldots(vi)$$

From (*iii*), $$\mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l} \int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l} \left\{ \left| (lx - x^2)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) \right|_0^l - \int_0^l (l - 2x)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) dx \right\}$$

$$= \frac{2\mu}{l} \cdot \frac{1}{n\pi} \left\{ \int_0^l (l-2x) \frac{\cos n\pi x}{l} dx \right\} = \frac{2\mu}{n\pi} \left\{ \left| (l-2x) \frac{\sin n\pi x/l}{n\pi/l} \right|_0^l - \int_0^l (-2) \frac{\sin n\pi x/l}{n\pi/l} dx \right\}$$

$$= \frac{2\mu}{n\pi} \cdot \frac{2l}{n\pi} \int_0^l \sin \frac{n\pi x}{l} dx = \frac{4\mu l}{n^2\pi^2} \left| \frac{-\cos n\pi x/l}{n\pi/l} \right|_0^l = \frac{4\mu l^2}{n^3\pi^3} \{1 - (-1)^n\}$$

Hence from (vi), the desired solution is

$$y(x, t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi}{l} x \cos \frac{(2m-1)\pi ct}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement $y(x, t)$.*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l} \qquad ...(iv)$$

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $\quad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore \quad y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also $\quad y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $\quad pl = n\pi$ or $p = \dfrac{n\pi}{l}$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t \right)$$

By (iii), $\quad 0 = c_2 c_3 \sin \dfrac{n\pi x}{l}$ $\quad$ for all x i.e., $c_2 c_3 = 0$

$\therefore \quad y(x, t) = b_n \sin \dfrac{n\pi x}{l} \sin \dfrac{cn\pi t}{l}$ $\quad$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l} \qquad ...(v)$$

Now $$\frac{\partial y}{\partial t} = \sum b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$$

By (iv), $$v_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$$

or $$\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l} \qquad [\because \sin 3\theta = 3 \sin\theta - 4\sin^3\theta]$$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \ldots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$\therefore \qquad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_5 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $\quad y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $\quad y(x, 0) = 0$ ...(iii)

and
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l - x) \qquad \text{...(iv)}$$

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \sin \frac{n\pi ct}{l} \qquad \text{...(v)}$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} . \cos \frac{n\pi ct}{l} . \left(\frac{n\pi c}{l}\right)$$

By (iv),
$$\lambda x(l - x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\therefore \qquad \frac{\pi c}{l} nb_n = \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2\lambda}{l} \left| (lx - x^2)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right) \right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3} (1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3} [1 - (-1)^n]$$

or
$$b_n = \frac{4\lambda l^3}{c\pi^4 n^4} [1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4 (2m - 1)^4} \text{ taking } n = 2m - 1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^4} \sin \frac{(2m - 1)\pi x}{l} \sin \frac{(2m - 1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.* (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA (= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(i)$$

and the boundary conditions are

$$y(0, t) = 0 \quad ...(ii)$$

$$y(l, t) = 0 \quad ...(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

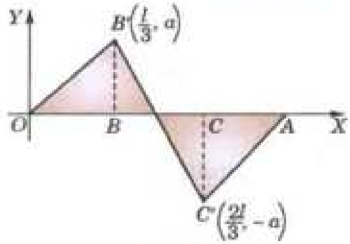


Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\,x$;

the equation of $B'C'$ is $\quad y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, i.e., $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\quad y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}x, \ 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x), \ \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l), \ \frac{2l}{3} \le x \le l \end{aligned}\right\} \quad ...(v)$$

As in example 18.6, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad [\text{Where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad ...(vi)$$

Putting $t = 0$, we have $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(vii)

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x) \sin \frac{n\pi x}{l}\,dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l) \sin \frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left| x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l - 2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left. + \left|(x - l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1).\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}[1 + (-1)^n] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (*vi*) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad ...(vii)$$

Putting $x = l/2$ in (*vii*), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\dfrac{\partial y}{\partial x} = \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}$

and
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\dfrac{\partial^2 y}{\partial u \partial v} = 0$ ...(2)

Integrating (2) w.r.t. v, we get $\dfrac{\partial y}{\partial u} = f(u)$ ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.
$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad ...(4)$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\dfrac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $\quad \phi'(x) = \psi'(x)$ ...(5)

and $\quad y(x, 0) = \phi(x) + \psi(x) = f(x)$ ...(6)

(5) gives, $\quad \phi(x) = \psi(x) + k$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or
$$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] = f(x + ct) + f(x - ct) \quad ...(7)$$

which is the *d'Alembert's solution** *of the wave equation* (1) *(V.T.U., 2011 S)*

Obs. The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* *(V.T.U., 2011)*

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

$$= \frac{1}{2}[k\{\sin(x + ct) - \sin 2(x + ct)\} + k\{\sin(x - ct) - \sin 2(x - ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $\quad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\quad \partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$, $0 < x < l$. *(Bhopal, 2007 S ; U.P.T.U., 2005)*

2. Solve the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0$, $u(l, t) = 0$ for all t ; $u(x, 0) = f(x)$ and $\left(\dfrac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.

3. Find the solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection

$$f(x) = \frac{2k}{l}x \text{ when } 0 < x < \frac{l}{2}, = \frac{2k}{l}(l - x) \text{ when } \frac{l}{2} < x < l,$$

and initial velocity zero. *(Bhopal, 2006 ; Kerala, M.E., 2005)*

4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release. *(Anna, 2005)*

5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by

$$f(x) = \begin{cases} x, & 0 \le x \le \dfrac{1}{2} \\ 1 - x, & \dfrac{1}{2} \le x \le 1 \end{cases}$$

If it is released from this position with velocity a, perpendicular to the x-axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by

$$u(x, t) = \frac{4\sqrt{2}}{\pi^2}\left[\sum_{n=1}^{\infty}\left\{\frac{\sin[(4pi - 3)\pi x]\cos[(4pi - 3)\pi at - \pi/4]}{(4n - 3)^2} - \frac{\sin[(4pi - 1)\pi x]\cos[(4pi - 1)\pi at - \pi/4]}{(4n - 1)^2}\right\}\right]$$

6. If a string of length l is initially at rest in equilibrium position and each of its points is given a velocity v such that $v = cx$ for $< x < l/2$
$c(l \,.\, x)$ for $l/2 < x < l$, determine the displacement $y(x, t)$ at anytime t. *(Anna, 2008)*

7. Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection :
(*i*) $f(x) = a(x - x^2)$ *(Kerala, M. Tech., 2005)* (*ii*) $f(x) = a \sin^2 \pi x$.

---

*See footnote of p. 373.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section α(cm^2). Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, -t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (*ii*) p. 466, the quantity of heat in this slab $= s\rho\alpha\,\delta x\delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x\dfrac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

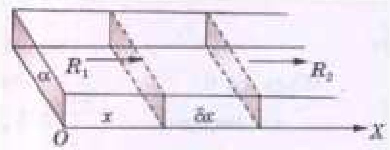


Fig. 18.3

Now by (*A*) of p. 466, $R_1 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (*i*) on p. 466.

Hence $s\rho\alpha\delta x\,\dfrac{\partial u}{\partial t} = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x + k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$ *i.e.*, $\dfrac{\partial u}{\partial t} = \dfrac{k}{s\rho}\left\{\dfrac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \quad ...(1)$$

This is the *one-dimensional heat-flow equation.* (*V.T.U., 2011*)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x)\,.\,T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2X''T, \text{ i.e., } X''/X = T'/c^2T \quad ...(2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3\,e^{c^2p^2t}\,;$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4\cos px + c_5\sin px,\ T = c_6\,e^{-c^2p^2t}\,;$$

(*iii*) *When k is zero :*

$$X = c_7x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1e^{px} + c_2e^{-px})c_3e^{c^2p^2t} \quad ...(5)$$

$$u = (c_4\cos px + c_5\sin px)c_6e^{-c^2p^2t} \quad ...(6)$$

$$u = (c_7x + c_8)c_9 \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1\cos px + C_2\sin px)e^{-c^2p^2t} \quad ...(8)$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3\sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $$u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-p^2 t} \quad ...(ii)$$

When $x = 0$, $u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $u(x, t) = c_2 \sin px e^{-p^2 t}$ ...(*iii*)

When $x = 1$, $u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $p = n\pi$.

$\therefore$ (*iii*) reduces to $u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2\pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3\sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(*i*) *u is not infinite for* $t \to \infty$, (*ii*) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(*ii*) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. (P.T.U., 2007)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \text{ and } \frac{dT}{dt} + k^2\alpha^2 T = 0 \quad ...(1)$$

Their solutions are $X = c_1 \cos kx + c_2 \sin kx$, $T = c_3 e^{-k^2\alpha^2 t}$ ...(2)

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx},\ T = c_6 e^{k^2\alpha^2 t} \quad ...(3)$$

If $k^2 = 0$, the solutions are $X = c_7 x + c_8$, $T = c_9$ ...(4)

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.*, the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \quad ...(5)$$

From (2), $\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, kc_3 e^{-k^2\alpha^2 t}$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \quad u = c_1 \cos kx \,.\, c_3 e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right)\frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \quad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l)\, e^{-n^2\pi^2 a^2 t/l^2} \qquad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and
$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2 a^2 t/l^2}.$$

Example 18.12. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* (U.P.T.U., 2005)

(*b*) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* (Madras, 2000 S)

Solution. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(*ii*)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (*ii*) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \qquad ...(iv)$$

and
$$u(l, t) = 0 \text{ for all values of } t \qquad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1 \cos px + C_2 \sin px)e^{-c^2p^2t} \qquad ...(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2 \sin pl \,.\, e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\therefore\ p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n \sin\dfrac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$, where $b_n = C_2$.

[These are the solutions of (*i*) satisfying the boundary conditions (*iv*) and (*v*). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (*i*) satisfying the boundary conditions (*iv*) and (*v*) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2} \qquad ...(viii)$$

Putting $t = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad ...(ix)$$

In order that the condition (*iii*) may be satisfied, (*iii*) and (*ix*) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence (*viii*) gives $u(x, t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(*b*) Here the initial condition remains the same as (*iii*) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad ...(x)$$

$$u(l, t) = 80 \text{ for all values of } t \qquad ...(xi)$$

In part (*a*), the boundary values (*i.e.*, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad ...(xii)$$

where $u_s(x)$ is a solution of (*i*) involving x only and satisfying the boundary conditions (*x*) and (*xi*); $u_t(x, t)$ is then a function defined by (*xii*). Thus $u_s(x)$ is a steady state solution of the form (*ii*) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (*ii*) we get

$$u_s(x) = 20 + (60/l)x \qquad ...(xiii)$$

Putting $x = 0$ in (*xii*), we have by (*x*),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad ...(xiv)$$

Putting $x = l$ in (*xii*), we have by (*xi*),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad ...(xv)$$

Also $$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \qquad \text{[by } (iii) \text{ and } (xiii)]$$

$$= \frac{40x}{l} - 20 \qquad ...(xvi)$$

Hence (*xiv*) and (*xv*) give the boundary conditions and (*xvi*) gives the initial condition relative to the transient solution. Since the boundary values given by (*xiv*) and (*xv*) are both zero, therefore, as in part (*a*), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2 p^2 t}$

By (*xiv*), $u_t(0, t) = C_1 e^{-c^2 p^2 t} = 0$, for all values of t.

Hence $C_1 = 0$ and $$u_t(x, t) = C_2 \sin px \,.\, e^{-c^2 p^2 t} \qquad ...(xvii)$$

Applying (*xv*), it gives $u_t(l, t) = C_2 \sin pl e^{-c^2 p^2 t} = 0$ for all values of t.

This requires $\sin pl = 0$, *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence (*xvii*) reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \quad \text{...(xviii)}$$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(*xix*)

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\, x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{where} \quad b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin \frac{n\pi x}{l}\, dx = -\frac{40}{nx}(1 + \cos n\pi)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4,...}^{\infty} \left(\frac{-80}{n\pi}\right) \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2}$ [Take $n = 2m$]

$$= -\frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t/l^2} \quad \text{...(xx)}$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2 m^2 \pi^2 t/l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

In steady state condition, u is independent of time and depends on x only, (*i*) reduces to

$$\partial^2 u / \partial x^2 = 0. \quad \text{...(ii)}$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u\,(x, 0) = 30 + \frac{5}{2}x \quad \text{...(iii)}$$

The boundary conditions are $u\,(0, t) = 40$, $u\,(20, t) = 60$

Using (*ii*), the steady state temperature is

$$u\,(x, 0) = 40 + \frac{60 - 40}{20}x = 40 + x \quad \text{...(iv)}$$

To find the temperature u in the intermediate period,

$$u\,(x, t) = u_s\,(x) + u_t\,(x, t)$$

where $u_s\,(x)$ is the steady state temperature distribution of the form (*iv*) and $u_t\,(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t\,(x, t)$ satisfies one dimensional heat equation

$$\therefore \quad u(x, t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2 t} \quad \text{...(v)}$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin px e^{-p^2 t} \qquad ...(vi)$$

Also $$u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20\, p e^{-p^2 t}$$

or $$\sum_{n=1}^{\infty} b_n \sin 20\, p e^{-p^2 t} = 0 \text{ i.e., } \sin 20p = 0 \text{ i.e., } p = n\pi/20$$

Thus (vi) becomes $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \qquad ...(vii)$$

Using (iii), $$30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

or $$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $b_n = \frac{2}{20}\int_0^{20} \left(\frac{3x}{2} - 10\right) \sin \frac{n\pi x}{20}\, dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{1 + 2\cos n\pi}{n} \sin \frac{n\pi x}{20} e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \qquad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l$. ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px) e^{-c^2 p^2 t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px) e^{-c^2 p^2 t} \qquad ...(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 p e^{-c^2 p^2 t} = 0$ for all t. [By (ii)]

$\therefore$ $c_2 = 0$

Putting $x = l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl e^{-c^2 p^2 t}$ for all t. [By (ii)]

$\therefore$ $c_1 p \sin pl = 0$ i.e., p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $pl = n\pi$ or $p = n\pi/l$, $n = 0, 1, 2, ...$

Hence (*iv*) becomes $u(x, t) = c_1 \cos \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$.

$\therefore$ the most general solution of (*i*) satisfying the boundary conditions (*ii*) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \qquad (\text{where } A_n = c_1) \quad ...(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{l} = x$ [by (*iii*)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad \text{where } a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

and $$a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l}\,dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$

$= 0$, where n is even ; $= -4l/n^2\pi^2$, when n is odd.

$\therefore$ $$A_0 = \frac{a_0}{2} = l/2, \text{ and } A_n = a_n = 0 \text{ for } n \text{ even} ; = -4l/n^2\pi^2 \text{ for } n \text{ odd.}$$

Hence (*vi*) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1,3,...}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2 (2n-1)^2 \pi^2 t/l^2} \quad ...(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

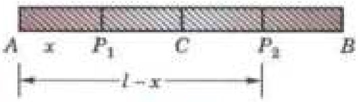


Fig. 18.4

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (*vii*), we get the temperature at P_2 as

$$u(l - x, t) = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi(l-x)}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}} \quad ...(viii)$$

$$\left\{\because \cos \frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos \frac{(2n-1)\pi x}{l}\right.$$

Adding (*vii*) and (*viii*), we get $u(x, t) + u(l - x, t) = l = 100°C$.

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x, 0) = x, \qquad 0 \le x \le 50$$
$$= 100 - x, \qquad 50 \le x \le 100.$$

Find the temperature $u(x, t)$ at any time. (*Bhopal, 2007 ; S.V.T.U., 2007 ; Kurukshetra, 2006*)

2. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length l, whose ends are kept at temperature 0°C and whose initial temperature in (°C) is given by $ax(l-x)/l^2$. (P.T.U., 2009)
3. A rod 30 cm. long, has its ends A and B kept at 20° and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x = 0$ at A. (Anna, 2008)
4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t. (P.T.U., 2010)
   Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.
5. Solve the following boundary value problem :
   $$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \qquad \frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(l,t)}{\partial x} = 0, \ u(x, 0) = x.$$ (S.V.T.U., 2008)
6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.
7. Find the solution of $\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$, such that

   (i) θ is not infinite when $t \to +\infty$; (ii) $\left.\begin{array}{ll}\frac{\partial \theta}{\partial x} = 0 & \text{when } x = 0 \\ \theta = 0, & \text{when } x = l\end{array}\right\}$ for all values of t ;

   (iii) $\theta = \theta_0$, when $t = 0$, for all values of x between 0 and l. (S.V.T.U., 2008)
8. Find the solution of $\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm³), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z-coordinate and depends only on x, y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

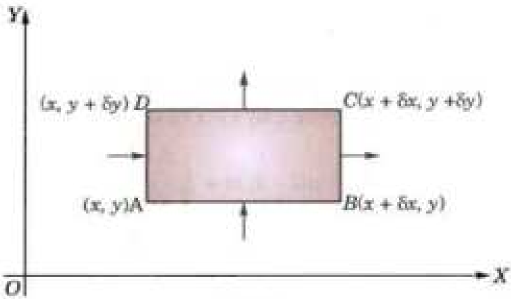


Fig. 18.5

Consider a rectangular element $ABCD$ of the plane with sides δx and δy. By (A) on p. 466, the amount of heat entering the element in 1 sec. from the side AB

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y$$

and the amount of heat entering the element in 1 second from the side $AD = -k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x$

The quantity of heat flowing out through the side CD per sec. $= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second $= -k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

Hence the total gain of heat by the rectangular element $ABCD$ per second

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + k\alpha\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\partial x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\partial y}\right] \qquad ...(1)$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t} \qquad ...(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t}.$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\frac{\partial u}{\partial t}$$

i.e.,
$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2\left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right) \text{ where } c^2 = k/\rho s \text{ is the } \textit{diffusivity}. \qquad ...(3)$$

Hence the equation (3) gives the temperature distribution of the plane in the *transient state*.

Cor. In the *steady state*, u is independent of t, so that $\partial u/\partial t = 0$ and the above equation reduces to,

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0}$$

which is the well known **Laplace's equation** *in two dimensions.*

Obs. When the stream lines are curves in space, *i.e.*, the heat flow is three dimensional, we shall similarly arrive at the equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

In a *steady state*, it reduces to $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z}^2} = \mathbf{0}$

which is the *three dimensional* **Laplace's equation.**

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0} \qquad ...(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\dfrac{d^2X}{dx^2}Y + X\dfrac{\partial^2 Y}{\partial y^2} = 0$

or separating the variables, $\dfrac{1}{X}\dfrac{d^2X}{dx^2} = -\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}$ ...(2)

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1 e^{px} + c_2 e^{-px}, \; Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px, \; Y = c_7 e^{py} + c_8 e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9 x + c_{10}$, $Y = c_{11} y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{...(3)}$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \text{...(4)}$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \text{...(5)}$$

Of these we take that solution which is consistent with the given boundary conditions.

(*V.T.U., 2011 S ; Kerala, 2005*)

Temperature distribution in long plates

Example 18.15. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.*

(*P.T.U., 2005 ; J.N.T.U., 2002 S*)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{...(i)}$$

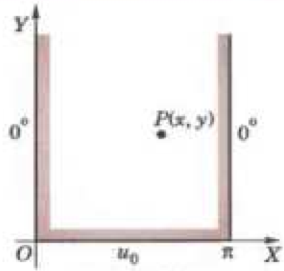


Fig. 18.6

The boundary conditions are $u(0, y) = 0$ for all values of y ...(*ii*)

$u(\pi, y) = 0$ for all values of y ...(*iii*)

$u(x, \infty) = 0$ in $0 < x < \pi$ ...(*iv*)

$u(x, 0) = u_0$ in $0 < x < \pi$ ...(*v*)

Now the three possible solutions of (*i*) are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \text{...(vi)}$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \text{...(vii)}$$

$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \text{...(viii)}$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (*vi*) cannot satisfy the condition (*ii*) for $u \neq 0$ for $x = 0$, for all values of y. The solution (*viii*) cannot satisfy the condition (*iv*). Thus the only possible solution is (*vii*), *i.e.* of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \quad \text{...(ix)}$$

By (*ii*), $\quad u(0, y) = C_1(C_3 e^{py} + C_4 e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (*ix*) reduces to

$$u(x, y) = C_2 \sin px \,(C_3 e^{py} + C_4 e^{-py}) \quad \text{...(x)}$$

By (*iii*), $\quad u(\pi, y) = C_2 \sin p\pi (C_3 e^{py} + C_4 e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.* $p\pi = n\pi$ as $C_2 \neq 0$. $\quad\therefore\quad p = n$, an integer.

Also to satisfy the condition (*iv*), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (*x*) takes the form $u(x, y) = b_n \sin nx \,.\, e^{-ny}$, where $b_n = C_2 C_4$.

$\therefore$ the most general solution satisfying (*ii*), (*iii*) and (*iv*) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \,.\, e^{-ny} \quad \text{...(xi)}$$

Putting $y = 0$, $$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{...(xii)}$$

In order that the condition (*v*) may be satisfied, (*v*) and (*xii*) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}[1-(-1)^n]$$

i.e., $b_n = 0$, if n is even ; $= 4u_0/n\pi$, if n is odd.

Hence (*xi*) becomes $u(x, y) = \frac{4u_0}{\pi}\left[e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \ldots\right]$.

Temperature distribution in finite plates

Example 18.16. *Solve the Laplace equation* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin n\pi x/l$. *(V.T.U., 2011 ; J.N.T.U., 2006 ; Kerala M. Tech., 2005, U.P.T.U., 2004)*

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(i)$$

are
$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \quad \ldots(ii)$$
$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \quad \ldots(iii)$$
$$u = (c_9 x + c_{10})(c_{11} y + c_{12}) \quad \ldots(iv)$$

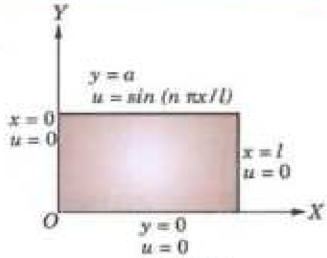


Fig. 18.7

We have to solve (*i*) satisfying the following boundary conditions

$$u(0, y) = 0 \quad \ldots(v) \qquad u(l, y) = 0 \quad \ldots(vi)$$
$$u(x, 0) = 0 \quad \ldots(vii) \qquad u(x, a) = \sin n\pi x/l \quad \ldots(viii)$$

Using (*v*) and (*vi*) in (*ii*), we get

$$c_1 + c_2 = 0, \quad \text{and} \quad c_1 e^{pl} + c_2 e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (*v*) and (*vi*) in (*iv*). Hence the suitable solution for the present problem is solution (*iii*). Using (*v*) in (*iii*), we have $c_5(c_7 e^{py} + c_8 e^{-py}) = 0$ *i.e.*, $c_5 = 0$

$\therefore$ (*iii*) becomes $u = c_6 \sin px(c_7 e^{py} + c_8 e^{-py})$ ...(*ix*)

Using (*vi*), we have $c_6 \sin pl\,(c_7 e^{py} + c_8 e^{-py}) = 0$

$\therefore$ either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, \ldots\ldots$

$\therefore$ (*ix*) becomes $u = c_6 \sin(n\pi x/l)(c_7 e^{n\pi y/l} + c_8 e^{-n\pi y/l})$ ...(*x*)

Using (*vii*), we have $0 = c_6 \sin n\pi x/l\,.\,(c_7 + c_8)$ *i.e.*, $c_8 = -c_7$.

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin\frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \quad \text{where} \quad b_n = c_6 c_7$$

Now using the condition (*viii*), we have

$$u(x, a) = \sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get
$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}}\sin\frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)}\sin\frac{n\pi x}{l}.$$

Example 18.17. *The function $v(x, y)$ satisfies the Laplace's equation in rectangular coordinates (x, y) and for points within the rectangle $x = 0, x = a, y = 0, y = b$, it satisfies the conditions $v(0, y) = v(a, y) = v(x, b) = 0$ and $v(x, 0) = x(a - x)$, $0 < x < a$. Show that $v(x, y)$ is given by*

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x/a}{(2n+1)^3}\frac{\sinh(2n+1)\pi(b-y)/a}{\sinh(2n+1)\pi b/a} \qquad \textit{(Madras, 2003)}$$

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad ...(i)$$

is of the form $\qquad v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad ...(ii)$

The boundary conditions are

$$v(0, y) = 0\ ; \quad v(a, y) = 0 \qquad ...(iii)$$

$$v(x, b) = 0 \qquad ...(iv)$$

$$v(x, 0) = x(a-x),\ 0 < x < a. \qquad ...(v)$$

Using (iii) $\qquad v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0 \quad i.e., \quad c_1 = 0.$

$\therefore$ (ii) becomes $\qquad v(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py}) \qquad ...(vi)$

Again using (iii), $\qquad v(a, y) = c_2 \sin pa\,(c_3 e^{py} + c_4 e^{-py}) = 0.$

i.e., $\qquad \sin pa = 0$, *i.e.* $pa = n\pi$ or $p = n\pi/a$

$\therefore$ (vi) becomes $\qquad v(x, y) = c_2 \sin \frac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

or $\qquad v(x, y) = \sin \frac{n\pi x}{a}\,(Ae^{n\pi y/a} + Be^{-n\pi y/a})$ where $A = c_2c_3, B = c_2c_4 \qquad ...(vii)$

Now using (iv), $\qquad v(x, b) = \sin \frac{n\pi x}{a}\left(Ae^{\frac{n\pi b}{a}} + Be^{-\frac{n\pi b}{a}}\right) = 0$

i.e., $\qquad Ae^{n\pi b/a} + Be^{-n\pi b/a} = 0$ or $Ae^{n\pi b/a} - Be^{-n\pi b/a} = -\frac{1}{2}b_n$ (say)

Thus (vii) becomes $\qquad v(x, y) = \sin \frac{n\pi x}{a}\cdot\frac{1}{2}b_n\left\{e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\right\}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$$

$\therefore$ the most general solution of (i) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \qquad ...(viii)$$

Using the condition (v), we have

$$x(a-x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $\qquad b_n \sinh \frac{n\pi b}{a} = \frac{2}{a}\int_0^a x(a-x) \sin \frac{n\pi x}{a}\,dx$

$$= \frac{2}{a}\left| (ax - x^2)\left(\frac{-\cos n\pi x/a}{n\pi/a}\right) - (a-2x)\left(-\frac{\sin n\pi x/a}{(n\pi/a)^2}\right) + (-2)\left\{\frac{\cos n\pi x/a}{(n\pi/a)^3}\right\}\right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3\pi^3} \text{ when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from $(viii)$, the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or $$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty} \frac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \frac{(2n+1)\pi x}{a}.$$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \qquad \text{(J.N.T.U., 2005)}$$

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin (\pi x/8), \quad 0 < x < 8;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100 e^{-\pi y/8} \sin (\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20\,x \qquad \text{for } 0 \le x \le 5$$

and

$$u = 20(10 - x) \qquad \text{for } 5 \le x \le 10$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{40}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-(2n-1)\pi y/10} \qquad \text{(Anna, 2009)}$$

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$

while other three edges are kept at 0°C. Find the steady state temperature in the plate. *(Madras, 2003)*

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm. and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi} \left[\frac{1}{\cosh \pi/2} - \frac{1}{3 \cosh 3\pi/2} + \frac{1}{5 \cosh 5\pi/2} - \dots \right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \dfrac{(2n-1)\pi x}{2a} \cos \dfrac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \dfrac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$ (See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(1)$$

Assume that a solution of (1) is of the form $u = R(r) . \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\dfrac{r^2R'' + rR'}{R} = -\dfrac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - kR = 0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2\phi}{d\theta^2} + k\phi = 0 \quad ...(4)$$

Putting $r = e^z$, (3) reduces to $\dfrac{d^2R}{dz^2} - kR = 0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},\ \phi = c_3 \cos p\theta + c_4 \sin p\theta$$

(*ii*) *When k is negative* and $= -p^2$, say

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos (p \log r) + c_6 \sin (p \log r),\ \phi = c_7e^{p\theta} + c_8e^{-p\theta}$$

(*iii*) *When k is zero* :

$$R = c_9z + c_{10} = c_9 \log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \qquad ...(6)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7e^{p\theta} + c_8e^{-p\theta}) \qquad ...(7)$$

$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \qquad ...(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (*S.V.T.U., 2008*)

Example 18.18. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r, \theta) = \frac{4T}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1} \sin (2n-1)\theta. \qquad \text{(Kerala M. Tech., 2005)}$$

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(i)$$

Fig. 18.8

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \qquad ...(ii)$$

$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \qquad ...(iii)$$

and

$$u(a, \theta) = T \qquad ...(iv)$$

The three possible solutions of (*i*) are

$$u = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \qquad ...(v)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7e^{p\theta} + c_8e^{-p\theta}) \qquad ...(vi)$$

$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \qquad ...(vii)$$

From (*ii*) and (*iii*), $u = 0$ when $r = 0$ *i.e.*, u must be finite at the origin. Thus the solutions (*vi*) and (*vii*) are to be rejected. Hence the only suitable solution is (*v*).

By (*ii*), $u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_3 = 0$

Hence $c_3 = 0$ and (*v*) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta$ ...(*viii*)

By (*iii*), $u(r, \pi) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi = 0.$

As $c_4 \neq 0$, $\sin p\pi = 0$, *i.e.*, $p = n$, where n is any integer.

Hence (*viii*) reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta$ ...(*ix*)

Since $u = 0$, when $r = 0$, $\therefore$ $c_2 = 0$ and (*ix*) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

$\therefore$ the most general solution of (*i*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(x)$$

Putting $r = a$,
$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \qquad ...(xi)$$

In order that (*iv*) may be satisfied, (*iv*) and (*xi*) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta \, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$$\therefore \qquad b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e., $b_n = 0$, if n is even

$= \dfrac{4T}{n\pi a^n}$, if n is odd.

Hence (*x*) gives $u(r, \theta) = \dfrac{4T}{\pi}\left\{\dfrac{(r/a)}{1}\sin\theta + \dfrac{(r/a)^3}{3}\sin 3\theta + \dfrac{(r/a)^5}{5}\sin 5\theta + ...\right\}$

Example 18.19. *The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by*

$$u(a, \theta) = \begin{cases} 50\theta, \text{ when } 0 < \theta \le \pi/2 \\ 50(\pi - \theta) \text{ when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function u (r, θ). (*Madras, 2003*)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$ ...(*ii*)

and $u(a, \theta) = 50\theta$ for $0 \le \theta \le \pi/2$; $u(a, \theta) = 50(\pi - \theta)$ for $\pi/2 \le \theta \le \pi$ ...(*iii*)

As in example 18.18, the most general solution of (*i*) satisfying the boundary conditions (*ii*) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(iv)$$

Putting $r = a$,
$$r(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (*iii*) is satisfied, we have $r(a, \theta) = \displaystyle\sum_{n=1}^{\infty} B_n \sin n\theta$

where $b_n a^n = B_n = \dfrac{2}{\pi}\left\{\displaystyle\int_0^{\pi/2} 50\theta \sin n\theta \, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta \, d\theta\right\}$ ...(*v*)

$$= \frac{100}{\pi}\left\{\left|\theta\left(\frac{-\cos n\theta}{\theta}\right)-(1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_0^{\pi/2}+\left|(\pi-\theta)\left(\frac{-\cos n\theta}{n}\right)-(-1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{100}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{\sin n\pi/2}{n^2}+\frac{\pi}{2n}\cos\frac{n\pi}{2}+\frac{\sin n\pi/2}{n^2}\right\} = \frac{200}{\pi n^2}\sin n\pi/2.$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (*iv*) gives

$$u(r,\theta) = \sum_{n=1,3,5,...}^{\infty}\left(\frac{200}{\pi n^2}\sin\frac{n\pi}{2}\right)\frac{1}{a^n}\cdot r^n\sin n\theta$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1}\sin(2m-1)\theta.$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \ldots$; gives $m = 1, 2, 3$, $\sin n\pi/2 = \sin(2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.

PROBLEMS 18.5

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta) = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
3. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
4. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
5. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.
6. If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and
$$u(10, \theta) = 15\cos\theta,\ u(20, \theta) = 30\sin\theta,$$
find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus]
7. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are
$$u(2, \theta) = 10\sin\theta + 6\cos\theta,\ u(4, \theta) = 17\sin\theta + 15\cos\theta$$
Find the steady-state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

Consider the forces acting on an element $\delta x\delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

$= T\delta y\sin\beta - T\delta y\sin\alpha$

$= T\delta y(\tan\beta - \tan\alpha)$ approximately, since α and β are small

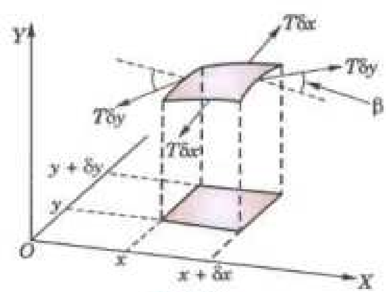


Fig. 18.9

$$= T\delta y\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} = T\delta y\,\frac{\partial^2 u}{\partial x^2}\,\delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x\,\frac{\partial^2 u}{\partial y^2}\,\delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y\,\frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x\delta y \quad \text{or} \quad \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \mathbf{c}^2\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] \quad \text{where} \quad c^2 = T/m \qquad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = \frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2 X}{dx^2} + k^2 X = 0,\ \frac{d^2 Y}{dy^2} + l^2 Y = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} + (k^2 + l^2)\,c^2 T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)\,(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos\sqrt{(k^2 + l^2)}ct + c_6 \sin\sqrt{(k^2 + l^2)}ct\,] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0$, $x = a$, $y = 0$, $y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos\sqrt{(k^2 + l^2)ct} + c_6 \sin\sqrt{(k^2 + l^2)ct}\,] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$.

(m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2 c_4 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$ ...(3)

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2 c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,(A_{mn}\cos pt + B_{mn}\sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, *i.e.*, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

This is *double Fourier series*. Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx = \frac{ab}{4} A_{mn} \qquad \ldots(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula.**

Rectangular Membranes

Example 18.20. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$A_{mn} = 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y\, dydx$$

$$= 4A \int_0^1 \sin \pi x \sin m\pi x\, dx \left(\int_0^1 \sin 2\pi y \sin n\pi y\, dy \right) = 0, \quad \text{for } m \neq 1$$

$$= 4A \left(\frac{1}{2}\right) \int_0^1 \sin 2\pi y \sin n\pi y\, dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x\, dx = \tfrac{1}{2}\right]$$

i.e.,
$$A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y\, dy = 0, \text{ for } n \neq 2$$

$$= 2A\left(\frac{1}{2}\right), \text{ for } n = 2.$$

$\therefore$ $A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $[\because a = b = 1 = c]$

$\therefore$ $p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\,\pi$.

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos(\sqrt{5}\,\pi t)$.

Example 18.21. *Find the vibration $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.*

Solution. Proceeding as in § 18.9 (**2**), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c\sqrt{[(m/a)^2 + (n/b)^2}$$

Since the membrane starts from rest $\partial u/\partial t = 0$ when $t = 0$,

$$\therefore \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-A_{mn}\, p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \qquad \ldots(i)$$

Then
$$hxy(a - x)(b - y) = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where
$$A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b hxy(a - x)(b - y) \,.\, \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx$$

$$= \frac{4h}{ab} \left\{ \int_0^a x(a - x) \sin \frac{m\pi x}{a} dx \right\} \left\{ \int_0^b y(b - y) \sin \frac{n\pi y}{b} dy \right\}$$

$$= \frac{4h}{ab} \left| (ax - x^2)\left(\frac{-\cos m\pi x/a}{m\pi/a}\right) - (a - 2x)\left\{\frac{-\sin \frac{m\pi x}{a}}{(m\pi/a)^2}\right\} + (-2)\frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| (by - y^2)\left(\frac{-\cos n\pi y/b}{n\pi/b}\right) - (b - 2y)\left\{\frac{-\sin n\pi/b}{(n\pi/b)^2}\right\} + (-2)\frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (*i*), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \dfrac{16\, ha^2 b^2}{m^3 n^3 \pi^6} (1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos (c\alpha_m t) \,.\, J_0(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr,$$

and $\alpha_m (m = 1, 2, ...)$ are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \qquad ...(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \qquad ...(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ ...(*iii*)

and initial velocity $\left(\dfrac{\partial u}{\partial t} \right)_{t=0} = 0$ ...(*iv*)

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ ...(*v*)

satisfying the boundary condition (*ii*).

Differentiating and substituting (*v*) in (*i*), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) = -k^2 \text{ (say)}$$

This leads to $\dfrac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(*vi*)

and

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + k^2 R = 0 \qquad ...(vii)$$

Now putting $s = kr$, (*vii*) transforms to $\dfrac{d^2 R}{ds^2} + \dfrac{1}{s} \dfrac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m,\ m = 1, 2, ...$$

---

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (*vii*) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (*vi*) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (*i*) satisfying (*ii*) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (*iii*) and (*iv*), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

Putting $t = 0$ and using (*iii*), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, *i.e.*,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r)\, J_0(\alpha_m r)\, dr$$

Using (*iv*), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\, \pi c t)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25\,(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km ; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

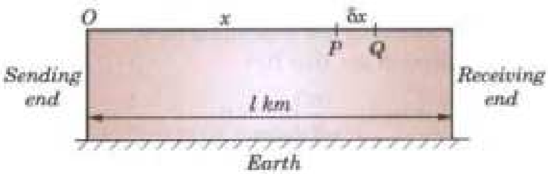


Fig. 18.10

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \qquad -\delta v = iR\delta x + L\delta x \cdot \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \ldots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$$\therefore \qquad -\delta i = C\frac{\partial v}{\partial t}\delta x + Gv\delta x \text{ from which as before, we get}$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right) i + \frac{\partial v}{\partial x} = 0 \qquad \ldots(3)$$

and $$\frac{\partial i}{\partial x}+\left(C\frac{\partial}{\partial t}+G\right)v=0, \quad ...(4)$$

we shall eliminate i and v in turn.

$\therefore$ operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R+L\frac{\partial}{\partial t}\right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2}-\left(R+L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t}+G\right)v=0$$

or $$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2}+(LG+RC)\frac{\partial v}{\partial t}+RGv \quad ...(5)$$

Again operating (3) by $\left(C\frac{\partial}{\partial t}+G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t}+G\right)\left(R+L\frac{\partial}{\partial t}\right)i-\frac{\partial^2 i}{\partial x^2}=0$$

or $$\frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2}+(LG+RC)\frac{\partial i}{\partial t}+RGi \quad ...(6)$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations.*

Cor. 1. *If* $L = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=RC\frac{\partial v}{\partial t} \quad ...(7) \qquad \frac{\partial^2 i}{\partial x^2}=RC\frac{\partial i}{\partial t} \quad ...(8)$$

which are known as the *telegraph equations.*

Rewriting (7) as $\frac{\partial v}{\partial t}=\frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. *If* $R = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2} \quad ...(9) \qquad \frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2} \quad ...(10)$$

which are called the *radio equations.*

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2}=k^2\frac{\partial^2 v}{\partial x^2}$ where $k^2=\frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + kt) + f_2(x - kt)$. [See (4) p. 609]

Similarly from (10), $i(x, t) = F_1(x + kt) + F_2(x - kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If* $L = C = 0$, *e.g.*, in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2}=GRv, \text{ i.e. } (D^2-GR)v=0$$

$$\therefore \quad v(x)=A\cosh(\sqrt{GR}\,.\,x)+B\sinh(\sqrt{GR}\,.\,x) \quad ...(11)$$

Since by (1), $$Ri=-\frac{\partial v}{\partial x}=-\sqrt{GR}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)]$$

$$\therefore \quad i(x)=-\sqrt{G/R}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)] \quad ...(12)$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}B$.

Hence writing $\sqrt{(GR)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x)=v_0\cosh\gamma x-i_0 z_0\sinh\gamma x \quad ...(13)$$

and $$i(x)=i_0\cosh\gamma x-\frac{v_0}{z_0}\sinh\gamma x. \quad ...(14)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. *Neglecting R and G, find the e.m.f. $v(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$.* (S.V.T.U., 2008)

Solution. Since R and G are negligible, we use the *Radio equation* $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$ ...(*i*)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0, \ v(l, t) = 0 \quad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and
$$v(x, 0) = e_1 \sin \pi x/l + e_5 \sin 5\pi x/l \quad ...(iii)$$

$$\therefore \quad \frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \quad \text{gives} \quad \frac{\partial v}{\partial t}(x, 0) = 0 \quad ...(iv)$$

Let $v = X(x)T(t)$ be the solution of (*i*).

$\therefore$ (*i*) gives
$$X''T = LCXT''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -k^2 \text{ (say)}$$

$$\therefore \quad X'' + k^2X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{k}{\sqrt{LC}}.t + c_4 \sin \frac{k}{\sqrt{LC}}t\right)$$

Using the boundary conditions (*ii*), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$$\therefore \quad v = \sin \frac{n\pi x}{l}\left(a_n \cos \frac{n\pi}{l\sqrt{LC}}t + b_n \sin \frac{n\pi}{l\sqrt{LC}}t\right)$$

Using the initial condition (*iv*), we get $b_n = 0$

$$\therefore \quad v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}}t$$

Thus the most general solution of (*i*) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (*iii*), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$$\therefore \quad a_1 = e_1 \quad \text{and} \quad a_5 = e_5$$ while all other a's are zero.

Hence
$$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. *A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time $t = 0$, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.*

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$\therefore$ $$X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3e^{-k^2t/RC}$$

giving $$v = (c_1 \cos kx + c_2 \sin kx)c_3e^{-k^2t/RC} \quad \text{...(i)}$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$\therefore$ $$0 = c_1c_3 ;\ 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e., $$c_1c_3 = 0 \text{ and } kl = n\pi \ (n \text{ an integer})$$

Putting these in (*i*) and adding a linear term, we have

$$v = a_0x + b_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}e^{-n^2\pi^2t/RCl^2} \quad \text{...(ii)}$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (*ii*) gives $b_0 = 0$ and $a_0 = E/l$

Then (*ii*) becomes $$v = \frac{E}{l}x + \sum b_n \sin\frac{n\pi x}{l}e^{-n^2\pi^2t/RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \Sigma b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$\therefore$ $$b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l = \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus $$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}e^{-n^2\pi^2t/RCl^2} \quad \text{...(iii)}$$

Also when $L = 0$, $\dfrac{-\partial v}{\partial x} = Ri$

i.e., $$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l}e^{-n^2\pi^2t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2t/RCl^2}$$

When $t = 1$ sec, $$i = -\frac{E}{lR}\left(1 - 2e^{\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - \ldots\right) \quad \text{...(iv)}$$

Since $$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5\times10^{-7})(3000)^2} = 0.548$$

$\therefore$ $$e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty$, $i \to -E/lR$

Hence from (*iv*), we have

$$i = -\frac{E}{lR}[1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + \ldots]$$
$$= -\frac{E}{lR}[1 - 1.156 + 0.223 - 0.014 + \ldots]$$
$$= i_{\infty}(0.053) = 5.3\% \text{ of } i_{\infty}.$$

Example 18.25. *A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* (Andhra, 2000)

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad \ldots(i)$$

v_s = initial steady voltage satisfying $\frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0)$ ...(*ii*)

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached = $1300 - 1.3x$

$\therefore$ $v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \qquad \text{[By (viii), p. 614]} \qquad \ldots(iii)$$

where l = 1000 kilometers.

Putting $t = 0$, we have from (*ii*) and (*iii*)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e.

$$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l 1.2\sin\frac{n\pi x}{l}\,dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence

$$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}.$$

PROBLEMS 18.7

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$.
   Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $1/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the *radio equation*

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$

appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad ...(1)$$

which is the *Laplace's equation in three dimensions.*

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{[(x-\zeta)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r} \, d\xi \, d\eta \, d\zeta \quad ...(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z, we get

$$\nabla^2 u = \iiint_R \rho \, \nabla^2 (1/r) \, d\xi \, d\eta \, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the* **potential theory** *and its solutions are called the* **harmonic functions.**

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) Cartesian form of $\nabla^2 u = 0$ is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = X(x)Y(y)Z(z)$ ...(2)

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0 \quad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0, \quad \frac{d^2Y}{dy^2} - l^2Y = 0, \quad \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $X = c_1e^{kx} + c_2e^{-kx}, \; Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z$$

Thus a possible solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly}) \{c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z\}.$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\{c_5e^{\sqrt{(k^2+l^2)}z} + c_6e^{-\sqrt{(k^2+l^2)}z}\}.$$

(2) Cylindrical form of $\nabla^2 u = 0$ is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $$u = R(\rho)\, H(\phi)\, Z(z)$$ [(iv) p. 359]

be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)+\frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad ...(2)$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z,$ ...(3)

(2) reduces to $$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)-\frac{n^2}{\rho^2}+k^2=0$$

or $$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+(k^2\rho^2-n^2)R=0.$$

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1J_n(k\rho) + c_2Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi,\ Z = c_5e^{kz} + c_6e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1J_n(k\rho) + c_2Y_n(k\rho)]\,[c_3 \cos n\phi + c_4 \sin n\phi]\,[c_5\, e^{kz} + c_6e^{-kz}]$$

which is known as a *cylindrical harmonic.* (*Assam, 1999*)

(3) Spherical form of $\nabla^2 u = 0$ is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0 \qquad ...(1) \quad [(iv)\text{ p. }361]$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1).

Then $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)+\frac{1}{G}\left(\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}\right)+\frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2}=0$$

Putting $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \qquad ...(2) \qquad \text{and } \frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2, \qquad ...(3)$$

the above equation takes the form

$$\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\operatorname{cosec}^2\theta]G=0 \qquad ...(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y''-2xy'+n(n+1)y=0,$$

m times with respect to x and writing $u = d^my/dx^m$, we get

$$(1-x^2)u''-2(m+1)xu'+(n-m)(n+m+1)u=0 \qquad ...(5)$$

Now putting $G = (1 - x^2)^{m/2}\, u$ in (5), we get

$$(1-x^2)\frac{d^2G}{dx^2}-2x\frac{dG}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]G=0 \qquad ...(6)$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $$R = c_5r^n + c_6r^{-n-1}$$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)](c_3\cos m\phi+c_4\sin m\phi)\times(c_5r^n+c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic.*

Example 18.26. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is* $f(\theta) = \cos^2 \theta$.

Solution. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r)\, G(\theta)$ in (*i*) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{d\theta^2} + \cot\theta \frac{dG}{d\theta} + n(n+1)G = 0 \qquad ...(ii)$$

and
$$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cot\theta = v$, (*ii*) takes the form

$$(1 - v^2)\frac{d^2 G}{dv^2} - 2v\frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (*iii*) are $R_n(r) = r^n$, $\overline{R_n}(r) = 1/r^{n+1}$.

Hence the equation (*i*) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n(\cos\theta) \text{ and } \bar{u}_n(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}, \text{ where } n = 0, 1, 2, ...$$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)$ ...(*iv*)

On the boundary of S, $u(1, \theta) = f(\theta) \quad \therefore \quad f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta)\, P_n(x)\, dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x) dx \qquad [\because \ f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\, dx \qquad [\because \ P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$c_n = 0$, except for $n = 0, 2$. Hence

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\, dx = \frac{1}{3},\ c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{0} P_2^2(x)\, dx = \frac{2}{3}.$$

Substituting in (*iv*), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3}r^2 P_2(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2(\cos^2\theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)\,[e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)].$$

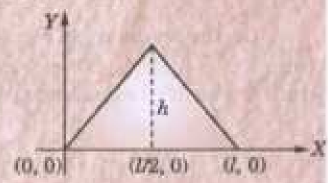
2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by
$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$
and
$$u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3 \text{ for } r > 1.$$
3. Show that in spherical polar coordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form
$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$
where $\mu = \cos\theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation
$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are ....
2. The partial differential equation representing variable heat flow in three dimensions, is ....
3. Temperature gradient is defined as ....
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as ....
5. The partial differential equation of the transverse vibrations of a string is ....
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is ....
7. The equation $u_t = c^2u_{xx}$ is classified as ....
8. The two dimensional steady state heat flow equation in polar coordinates is ....
9. The mathematical function of the initial form of the string given by the following graph is ....



10. When a vibrating string fastened to two points l apart, has an initial velocity u_0, its initial conditions are ....
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is ....
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u, then the boundary conditions for this problem are ....
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are ....
14. D'Alembert's solution of the wave equation is ....
15. The partial differential equation of 2-dimensional heat flow in ....
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is .... *(Anna, 2008)*
17. The three possible solutions of Laplace equation in polar coordinates are ....
18. Solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is ....
19. Solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is ....
20. In the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, α^2 represents ....
21. The telegraph equations for potential and current are ......
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form ....
23. The three possible solutions of Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ are ....

CHAPTER 19

Complex Numbers and Functions

1. Complex Numbers. 2. Argand's diagram. 3. Geometric representation of $z_1 \pm z_2$; $z_1 z_2$ and z_1/z_2. 4. De Moivre's theorem. 5. Roots of a complex number. 6. To expand sin $n\theta$, cos $n\theta$ and tan $n\theta$ in powers of sin θ, cos θ and tan θ respectively; Addition formulae for any number of angles; To expand $\sin^m \theta$, $\cos^n \theta$ and $\sin^m \theta \cos^n \theta$ in a series of sines or cosines of multiples of θ. 7. Complex function: Definition. 8. Exponential function of a complex variable. 9. Circular functions of a complex variable. 10. Hyperbolic functions. 11. Inverse hyperbolic functions. 12. Real and imaginary parts of circular and hyperbolic functions. 13. Logarithmic functions of a complex variable. 14. Summation of series – 'C + iS' method. 15. Approximations and Limits. 16. Objective Type of Questions.

19.1 COMPLEX NUMBERS

Definition. *A number of the form $x +\ iy$, where x and y are real numbers and $i = \sqrt{(-1)}$, is called a* **complex number.**

x is called the *real part* of $x + iy$ and is written as $R(x + iy)$ and y is called the *imaginary part* and is written as $I(x + iy)$.

A pair of complex numbers $x + iy$ and $x - iy$ are said to be conjugate of each other.

Properties : (1) *If $x_1 + iy_1 = x_2 + iy_2$, then $x_1 - iy_1 = x_2 - iy_2$.*

(2) *Two complex numbers $x_1 + iy_1 =$ and $x_2 + iy_2$ are said to be equal when*

$$R(x_1 + iy_1) = R(x_2 + iy_2), \text{ i.e., } x_1 = x_2$$

and
$$I(x_1 + iy_1) = I(x_2 + iy_2), \text{ i.e., } y_1 = y_2.$$

(3) *Sum, difference, product and quotient of any two complex numbers is itself a complex number.*

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

(*i*) their sum $= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

(*ii*) their difference $= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(*i*) their product $= (x_1 + iy_1) + (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

and (*iv*) their quotient
$$= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$$

(4) *Every complex number $x + iy$ can always be expressed in the form r (cos θ + i sin θ).*

Put
$$R(x + iy), \text{ i.e., } x = r\cos\theta \quad ...(i)$$

and
$$I(x + iy), \text{ i.e., } y = r\sin\theta \quad ...(ii)$$

Squaring and adding, we get $x^2 + y^2 = r^2$ *i.e.* $r = \sqrt{(x^2 + y^2)}$ (taking positive square root only)

Dividing (*ii*) by (*i*), we get $y/x = \tan\theta$ *i.e.* $\theta = \tan^{-1}(y/x)$.

Thus $x + iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{(x^2 + y^2)}$ and $\theta = \tan^{-1}(y/x)$.

Definitions. *The number* $r = +\sqrt{(x^2 + y^2)}$ *is called the* **modulus** *of* $x + iy$ *and is written as mod* $(x + iy)$ *or* $|x + iy|$.

The angle θ *is called the amplitude or argument of* $x + iy$ *and is written as amp* $(x + iy)$ *or arg* $(x + iy)$.

Evidently the amplitude θ has an infinite number of values. The value of θ which lies between $-\pi$ and π is called the *principal value of the amplitude.* Unless otherwise specified, we shall take amp (z) to mean the principal value.

Note. *$\cos\theta + i\sin\theta$ is briefly written as cis θ (pronounced as 'sis θ')*

(5) *If the conjugate of* $z = x + iy$ be $\bar{z}$, *then*

(*i*) $R(z) = \frac{1}{2}(z + \bar{z}),\ I(z) = \frac{1}{2i}(z - \bar{z})$ (*ii*) $|z| = \sqrt{R^2(z) + I^2(z)} = |\bar{z}|$

(*iii*) $z\bar{z} = |z|^2$ (*iv*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(*v*) $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ (*vi*) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$, where $\bar{z}_2 \neq 0$.

Example 19.1. *Reduce* $1 - \cos\alpha + i\sin\alpha$ *to the modulus amplitude form.*

Solution. Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

$$\therefore \quad r = (1 - \cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\alpha/2$$

i.e.,
$$r = 2\sin\alpha/2$$

and
$$\tan\theta = \frac{\sin\alpha}{1 - \cos\alpha} = \frac{2\sin\alpha/2\cos\alpha/2}{2\sin^2\alpha/2} = \cot\alpha/2$$

$$= \tan\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \quad \therefore \theta = \frac{\pi - \alpha}{2}.$$

Thus $1 - \cos\alpha + i\sin\alpha = 2\sin\frac{\alpha}{2}\left[\cos\frac{\pi - \alpha}{2} + i\sin\frac{\pi - \alpha}{2}\right]$.

Example 19.2. *Find the complex number* z *if* $\arg(z + 1) = \pi/6$ *and* $\arg(z - 1) = 2\pi/3$.
(*Mumbai, 2009*)

Solution. Let $z = x + iy$ so that $z + 1 = (x + 1) + iy$ and $(z - 1) = (x - 1) + iy$

Since $\arg(z + 1) = \pi/6$, $\therefore \tan^{-1}\left(\frac{y}{x+1}\right) = 30°$

i.e.,
$$\frac{y}{x + 1} = \tan 30° = 1/\sqrt{3}, \text{ or } \sqrt{3}y = x + 1 \qquad ...(i)$$

Also since $\arg(z - 1) = 2\pi/3$, $\therefore \tan^{-1}\left(\frac{y}{x+1}\right) = 120°$

i.e.,
$$\frac{y}{x - 1} = \tan 120° = -\sqrt{3}, \text{ or } y = -\sqrt{3}x + \sqrt{3} \text{ or } \sqrt{3}y = -3x + 3 \qquad ...(ii)$$

Subtracting (*ii*) from (*i*), we get $4x - 2 = 0$ *i.e.,* $x = 1/2$

From (*i*), $\sqrt{3}\,y = 1/2 + 1$, *i.e.,* $y = \sqrt{3}/2$

Hence $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Example 19.3. *Find the real values of* x, y *so that* $-3 + ix^2y$ *and* $x^2 + y + 4i$ *may represent complex conjugate numbers.*

Solution. If $z = -3 + ix^2y$, then $\bar{z} = x^2 + y + 4i$

so that
$$z = (x^2 + y) - 4i$$

$$\therefore \quad -3 + ix^2y = x^2 + y - 4i$$

Equating real and imaginary parts from both sides, we get

$$-3 = x^2 + y, x^2y = -4$$

Eliminating $\quad x, (y + 3)y = -4$

or $\quad y^2 + 3y - 4 = 0 \quad i.e., \quad y = 1 \text{ or } -4$

When $y = 1$, $\quad x^2 = -3 - 1 \quad \text{or} \quad x = +2i$ which is not feasible

When $y = -4$, $\quad x^2 = 1 \quad \text{or} \quad x = \pm 1$

Hence $x = 1$, $\quad y - 4 \quad \text{or} \quad x = -1, y = -4$.

19.2 (1) GEOMETRIC REPRESENTATION OF IMAGINARY NUMBERS

Let all the real numbers be represented along $X'OX$, the positive real numbers being along OX and negative ones along OX'. Let OA be equal to one unit of measurement (Fig. 19.1).

Take a point L on OX such that $OL = x\,(OA)$.

Then L on OX represents the positive real number x and $i.ix = i^2x = -x$ is represented by a point L' on OX' distant OL from O.

From this we infer that the multiplication of the real number x by i twice amounts to the rotation of OL through two right angles to the position OL'.

Thus it naturally follows that the multiplication of a real number by i is equivalent to the rotation of OL through one right angle to the position OL''.

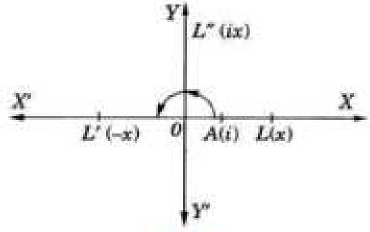


Fig. 19.1

Hence, if $Y'OY$ be a line perpendicular to the real axis $X'OX$, then all imaginary numbers are represented by points on $Y'OY$, called the **imaginary axis**, *the positive ones along OY and negative ones along OY'.**

Obs. Geometric interpretation of i*. From the above, it is clear that i is an operation which when multiplied to any real number makes it imaginary and rotates its direction through a right angle on the complex plane.

(2) Geometric representation of complex numbers†

Consider two lines $X'OX$, $Y'OY$ at right angles to each other.

Let all the real numbers be represented by points on the line $X'OX$ (called the *real axis*), positive real numbers being along OX and negative ones along OX'. Let the point L on OX represent the real number x (Fig. 19.2).

Since the multiplication of a real number by i is equivalent to the rotation of its direction through a right angle. Therefore, let all the imaginary numbers be represented by points on the line $Y'OY$ (called the *imaginary axis*), the positive ones along OY and negative ones along OY'. Let the point M on OY represent the imaginary number iy.

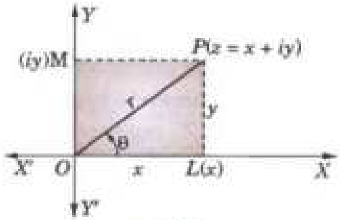


Fig. 19.2

Complete the rectangle $OLPM$. *Then the point whose cartesian coordinates are (x, y) uniquely represents the complex number $z = x + iy$ on the complex plane z.* The diagram in which this representation is carried out is called the **Argand's diagram**.

If (r, θ) be the polar coordinates of P, then r is the modulus of z and θ is its amplitude.

Obs. Since a complex number has magnitude and direction, therefore, it can be represented like a vector. *Hereafter we shall often refer to the complex number $z = x + iy$ as*

(i) the point z whose co-ordinates are (x, y) or (ii) the vector z from O to $P(x, y)$.

Example 19.4. *The centre of a regular hexagon is at the origin and one vertex is given by $\sqrt{3} + i$ on the Argand diagram. Determine the other vertices.*

---

* The first mathematician to propose a geometric representation of imaginary number i was *Kuhn of Denzig* (1750–51).

† The geometric representation of complex numbers came into mathematics through the memoire of *Jean Robert Argand*, Paris 1806.

Solution. Let $\vec{OA} = \sqrt{3} + i$ so that

$$OA = 2 \text{ and } \angle XOA = \tan^{-1} 1/\sqrt{3} = 30°. \text{ (Fig. 19.3)}$$

Being a regular hexagon, $OB = OC = 2$

$$\angle XOB = 30° + 60° = 90°$$

and $$\angle XOC = 30° + 120° = 150°$$

$\therefore$ $$\vec{OB} = 2(\cos 90° + i \sin 90°) = 2i$$

$$\vec{OC} = 2(\cos 150° + i \sin 150°) = -\sqrt{3} + i$$

Since $\vec{AD}, \vec{BE}, \vec{CF}$ are bisected at O,

$\therefore$ $$\vec{OD} = -\vec{OA} = -\sqrt{3} - i$$

$$\vec{OE} = -\vec{OB} = -2i \text{ and } \vec{OF} = -\vec{OC} = \sqrt{3} - i.$$

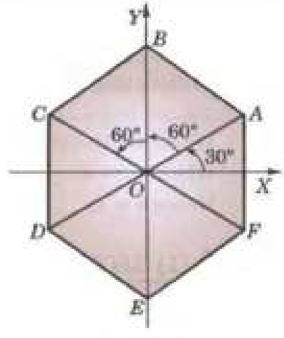


Fig. 19.3

19.3 (1) GEOMETRIC REPRESENTATION OF $z_1 + z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. (Fig. 19.4)

Complete the parallelogram OP_1PP_2. Draw P_1L, P_2M and $PN \perp$ s to OX. Also draw $P_1K \perp PN$.

Since $$ON = OL + LN = OL + OM = x_1 + x_2 \quad [\because LN = P_1K = OM]$$

and $$NP = NK + KP = LP_1 + MP_2 = y_1 + y_2.$$

The coordinates of P are $(x_1 + x_2, y_1 + y_2)$ and it represents the complex number

$$z = x_1 + x_2 + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) = z_1 + z_2.$$

Thus the point P which is the extremity of the diagonal of the parallelogram having OP_1 and OP_2 as adjacent sides, represents the sum of the complex numbers $P_1(z_1)$ and $P_2(z_2)$ such that

$$|z_1 + z_2| = OP \text{ and } \text{amp}(z_1 + z_2) = \angle XOP.$$

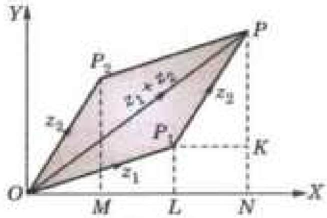


Fig. 19.4

Obs. Vectorially, we have $\vec{OP_1} + \vec{P_1P} = \vec{OP}$.

(2) Geometric representation of $z_1 - z_2$

Let P_1, P_2 represent the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ (Fig. 19.5). Then the subtraction of z_2 from z_1 may be taken as addition of z_1 to $-z_2$.

Produce P_2O backwards to R such that $OR = OP_2$. Then the coordinates of R are evidently $(-x_2, -y_2)$ and so it corresponds to the complex number $-x_2 - iy_2 = -z_2$.

Complete the parallelogram $ORQP_1$, then the sum of z_1 and $(-z_2)$ is represented by OQ *i.e.*, $z_1 - z_2 = \vec{OQ} = \vec{P_2P_1}$.

Hence the complex number $z_1 - z_2$ is represented by the vector P_2P_1.

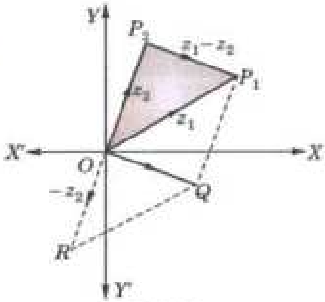


Fig. 19.5

Obs. By means of the relation $\vec{P_2P_1} = \vec{OP_1} - \vec{OP_2}$, any vector $\vec{P_2P_1}$ may be referred to the origin.

Example 19.5. *Find the locus of $P(z)$ when*

(*i*) $|z - a| = k$;

(*ii*) amp $(z - a) = \alpha$, where k and α are constants. *(Gorakhpur, 1999)*

Solution. Let a, z be represented by A and P in the complex plane, O being the origin (Fig. 19.6).

Then $z - a = \vec{OP} - \vec{OA} = \vec{AP}$

(*i*) $|z - a| = k$ means that $AP = k$.

Thus the locus of P(z) is a circle whose centre is A(a) and radius k.

(ii) amp $(z - a)$, *i.e.*, amp $(\vec{AP}) = \alpha$, means that AP always makes a constant angle with the X-axis.

Thus the locus of P(z) is a straight line through A(a) making an ∠α with OX.

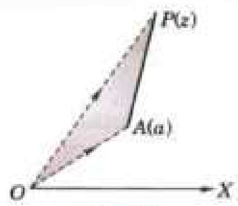


Fig. 19.6

Example 19.6. *Determine the region in the z-plane represented by*
(i) $1 < |z + 2i| \le 3$ (ii) $R(z) > 3$ (iii) $\pi/6 \le amp\ (z) \le \pi/3$.

Solution. (i) $|z + 2i| = 1$ is a circle with centre $(-2i)$ and radius 1 and $|z + 2i| = 3$ is a circle with the same centre and radius 3.

Hence $1 < |z + 2i| \le 3$ represents the region outside the circle $|z + 2i| = 1$ and inside (including circumference of) the circle $|z + 2i| = 3$ [Fig. 19.7].

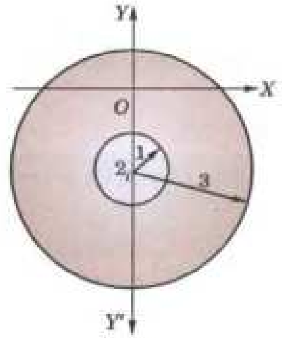


Fig. 19.7

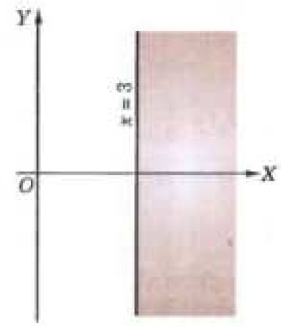


Fig. 19.8

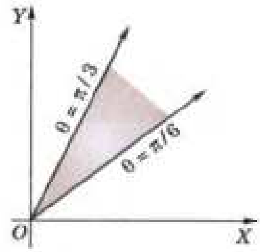


Fig. 19.9

(ii) $R(z) > 3$, defines all points (z) whose real part is greater than 3. Hence it represents the region of the complex plane to the right of the line $x = 3$ [Fig. 19.8].

(iii) If $z = r(\cos\theta + i\sin\theta)$, then amp $(z) = \theta$.

$\therefore \pi/6 \le$ amp $(z) \le \pi/3$ defines the region bounded by and including the lines $\theta = \pi/6$ and $\theta = \pi/3$. [Fig. 19.9].

Example 19.7. *If z_1, z_2 be any two complex numbers, prove that*
(i) $|z_1 + z_2| \le |z_1| + |z_2|$ [*i.e., the modulus of the sum of two complex numbers is less than or at the most equal to the sum of their moduli*].
(ii) $|z_1 - z_2| \ge |z_1| - |z_2|$ [*i.e., the modulus of the difference of two complex numbers is greater than or at the most equal to the difference of their moduli*].

Solution. Let P_1, P_2 represent the complex numbers z_1, z_2 (Fig. 19.10). Complete the parallelogram OP_1PP_2, so that

$$|z_1| = OP_1,\ |z_2| = OP_2 = P_1P,$$

and
$$|z_1 + z_2| = OP.$$

Fig. 19.10

Now from ΔOP_1P, $OP \le OP_1 + P_1P$, the sign of equality corresponding to the case when O, P_1, P are collinear.

Hence $|z_1 + z_2| \le |z_1| + |z_2|$ ...(i)

Again $|z_1| = |(z_1 - z_2) + z_2| \le |z_1 - z_2| + |z_2|$ [By (i)]

Thus $|z_1 - z_2| \ge |z_1| - |z_2|$ ...(ii)

Obs. $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$.
In general, $|z_1 + z_2 + ... + z_n| \le |z_1| + |z_2| + ... + |z_n|$.

Example 19.8. *If $|z_1 + z_2| = |z_1 - z_2|$, prove that the difference of amplitudes of z_1 and z_2 is $\pi/2$.* (Mumbai, 2007)

Solution. Let $z_1 + z_2 = r(\cos\theta + i\sin\theta)$ and $z_1 - z_2 = r(\cos\phi + i\sin\phi)$

Then
$$2z_1 = r[(\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)]$$
$$= r\left\{2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} + 2i\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right\}$$

or
$$z_1 = r\cos\frac{\theta-\phi}{2}\left(\cos\frac{\theta+\phi}{2} + i\sin\frac{\theta+\phi}{2}\right) \text{ i.e., amp }(z_1) = \frac{\theta+\phi}{2} \quad ...(i)$$

Also
$$2z_2 = r(\cos\theta - \cos\phi) + i(\sin\theta - \sin\phi)$$
$$= 2r\sin\frac{\theta-\phi}{2}\left(-\sin\frac{\theta+\phi}{2} + i\cos\frac{\theta+\phi}{2}\right)$$

or
$$z_2 = r\sin\frac{\theta-\phi}{2}\left\{\cos\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\pi}{2}+\frac{\theta+\phi}{2}\right)\right\}$$

i.e.,
$$\text{amp }(z_2) = \frac{\pi}{2} + \frac{\theta+\phi}{2} \quad ...(ii)$$

Hence [(ii) – (i)], gives amp (z_2) – amp $(z_1) = \dfrac{\pi}{2}$.

Example 19.9. *Show that the equation of the ellipse having foci at z_1, z_2 and major axis $2a$, is $|z - z_1| + |z - z_2| = 2a$.*
Also find its eccentricity.

Solution. Let $P(z)$ be any point on the given ellipse (Fig. 19.11) having foci at $S(z_1)$ and $S'(z_2)$ so that $SP = |z - z_1|$ and $S'P = |z - z_2|$.

We know that $SP + S'P = AA' (= 2a)$

i.e.,
$$|z - z_1| + |z - z_2| = 2a$$

which is the desired equation of the ellipse.

Also we know that $SS' = 2ae$, e being the eccentricity.

or
$$\left|\vec{OS'} - \vec{OS}\right| = 2ae \quad \text{or} \quad |z_2 - z_1| = 2ae$$

or
$$|z_1 - z_2| = 2ae \text{ whence } e = |z_1 - z_2|/2a.$$

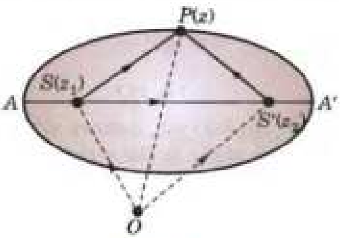


Fig. 19.11

(3) Geometric Representation of z_1z_2. Let P_1, P_2 represent the complex numbers
$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
and
$$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Measure off $OA = 1$ along OX (Fig. 19.12). Construct ΔOP_2P on OP_2 directly similar to ΔOAP_1,

so that
$$OP/OP_1 = OP_2/OA \text{ i.e., } OP = OP_1 \cdot OP_2 = r_1r_2$$
and
$$\angle AOP = \angle AOP_2 + \angle P_2OP = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$$

$\therefore$ P represents the number
$$r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

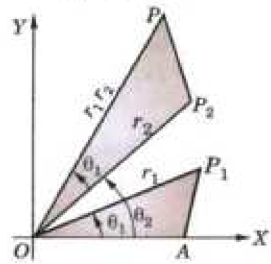


Fig. 19.12

Hence *the product of two complex numbers z_1, z_2 is represented by the point P, such that (i)* $|\mathbf{z_1z_2}| = |\mathbf{z_1}| \cdot |\mathbf{z_2}|$.

(ii) **amp $(\mathbf{z_1z_2})$ = amp $(\mathbf{z_1})$ + amp $(\mathbf{z_2})$.**

Cor. The effect of multiplication of any complex number z by $\cos\theta + i\sin\theta$ is to rotate its direction through an angle θ, for the modulus of $\cos\theta + i\sin\theta$ is unity.

(4) Geometric representation of z_1/z_2.

Let P_1, P_2 represent the complex numbers

$$z_1 = x_1 + iy_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

and
$$z_2 = x_2 + iy_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

Measure off $OA = 1$, construct triangle OAP on OA directly similar to the triangle OP_2P_1 (Fig. 19.13), so that

$$\frac{OP}{OA} = \frac{OP_1}{OP_2} \quad i.e., \quad OP = \frac{OP_1}{OP_2} = \frac{r_1}{r_2}$$

and
$$\angle XOP = \angle P_2OP_1 = \angle AOP_1 - \angle AOP_2 = \theta_1 - \theta_2.$$

$\therefore$ P represents the number

$$(r_1/r_2)\,[\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)].$$

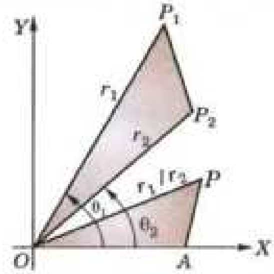


Fig. 19.13

Hence the complex number z_1/z_2 is represented by the point P, such that

(*i*) $|\mathbf{z_1/z_2}| = |\mathbf{z_1}|/|\mathbf{z_2}|$

(*ii*) **amp $(\mathbf{z_1/z_2})$ = amp $(\mathbf{z_1})$ – amp $(\mathbf{z_2})$.**

Note. *If $P_1(z_1)$, $P_2(z_2)$ and $P_3(z_3)$ be any three points, then*

$$\text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \angle P_1P_2P_3.$$

Join O, the origin, to P_1, P_2, and P_3. Then from the figure 19.14, we have

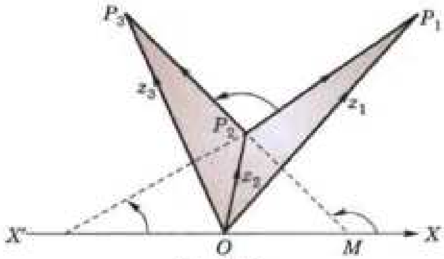


Fig. 19.14

$$\vec{P_2P_1} = z_1 - z_2 \quad \text{and} \quad \vec{P_2P_3} = z_3 - z_2$$

$$\therefore \quad \text{amp}\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \text{amp}\left[\frac{\vec{P_2P_3}}{\vec{P_2P_1}}\right]$$

$$= \text{amp}\,(\vec{P_2P_3}) - \text{amp}\,(\vec{P_2P_1}) = \beta - \alpha = \angle P_1P_2P_3.$$

Example 19.10. *Find the locus of the point z, when*

(*i*) $\left|\dfrac{z-a}{z-b}\right| = k$ (*ii*) *amp* $\left(\dfrac{z-a}{z-b}\right) = \alpha$ *where k and α are constants.*

Solution. Let $A(a)$ and $B(b)$ be any two fixed points on the complex plane and let $P(z)$ be any variable point (Fig. 19.15).

(*i*) Since $|z - a| = AP$ and $|z - b| = BP$.

$\therefore$ The point P moves so that $\left|\dfrac{z-a}{z-b}\right| = \left|\dfrac{z-a}{z-b}\right| = \dfrac{AP}{BP} = k$

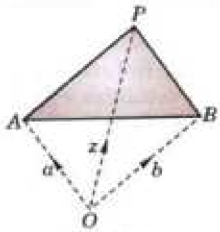


Fig. 19.15

i.e., P moves so that its distances from two fixed points are in a constant ratio, which is obviously the *Appollonius* circle.

When $k = 1$, $BP = AP$ *i.e.*, P moves so that its distance from two fixed points are always equal and thus the locus of P is the right bisector of AB.

Hence the locus of P(z) is a circle (unless k = 1, when the locus is the right bisector of AB).

Obs. For different values of k, the equation represents family of non-intersecting coaxial circles having A and B as its limiting points.

(*ii*) From the figure 19.16, we have amp $\left(\dfrac{z-a}{z-b}\right) = \angle APB = \alpha$.

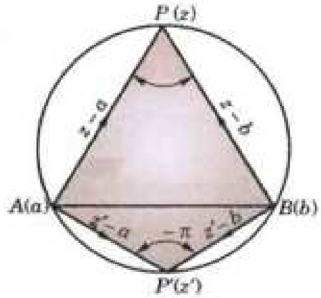


Fig. 19.16

Hence the locus of P(z) is the arc APB of the circle which passes through the fixed points A and B.

If, however, $P'(z')$ be a point on the lower arc AB of this circle, then

amp $\left(\dfrac{z'-a}{z'-b}\right) = \angle BP'A = \alpha - \pi$, which shows that the locus of P' is the arc $AP'B$ of the same circle.

Obs. For different values of α from $-\pi$ to π, the equation represents a family of intersecting coaxial circles having AB as their common radical axis.

Example 19.11. *If z_1, z_2 be two complex numbers, show that*

$$(z_1 + z_2)^2 + (z_1 - z_2)^2 = 2[|z_1|^2 + |z_2|^2].$$

Solution. Let $z_1 = r_1 \text{ cis } \theta_1$ and $z_2 = r_2 \text{ cis } \theta_2$ so that

$$|z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

and
$$|z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$$
$$= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

$$\therefore \quad |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(r_1^2 + r_2^2) = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Example 19.12. *If z_1, z_2, z_3 be the vertices of an isosceles triangle, right angled at z_2, prove that*

$$z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

Solution. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ be the vertices of ΔABC such that

$$AB = BC \text{ and } \angle ABC = \pi/2. \text{ (Fig. 19.17)}$$

Then $\quad |z_1 - z_2| = |z_3 - z_2| = r$ (say).

If amp $(z_1 - z_2) = \theta$ then amp $(z_3 - z_2) = \pi/2 + \theta$

$$\therefore \quad z_1 - z_2 = r(\cos\theta + i \sin\theta),$$

and
$$z_3 - z_2 = r\left[\cos\left(\frac{\pi}{2} + \theta\right) + i \sin\left(\frac{\pi}{2} + \theta\right)\right] = r(-\sin\theta + i\cos\theta)$$

i.e.,
$$z_3 - z_2 = ir(\cos\theta + i\sin\theta) = i(z_1 - z_2)$$

or
$$(z_3 - z_2)^2 = -(z_1 - z_2)^2 \text{ or } z_1^2 + z_3^2 + 2z_2^2 = 2z_2(z_1 + z_3).$$

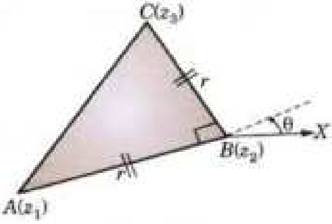


Fig. 19.17

Example 19.13. *If z_1, z_2, z_3 be the vertices of an equilateral triangle, prove that*

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

Solution. Since ΔABC is equilateral, therefore, BC when rotated through 60° coincides with BA (Fig. 19.18). But to turn the direction of a complex number through an $\angle\theta$, we multiply it by $\cos\theta + i\sin\theta$.

$$\therefore \quad \vec{BC}\,(\cos\pi/3 + i\sin\pi/3) = \vec{BA}$$

i.e.,
$$(z_3 - z_2)\left(\frac{1 + i\sqrt{3}}{2}\right) = z_1 - z_2$$

or
$$i\sqrt{3}\,(z_3 - z_2) = 2z_1 - z_2 - z_3$$

Squaring,
$$-3(z_3 - z_2)^2 = (2z_1 - z_2 - z_3)^2$$

or
$$4(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1) = 0$$

whence follows the required condition.

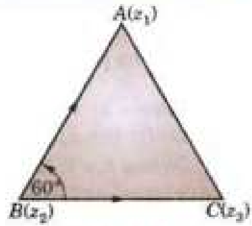


Fig. 19.18

PROBLEMS 19.1

1. Express the following in the modulus-amplitude form:

   (*i*) $1 + \sin\alpha + i\cos\alpha$ $\qquad$ (*ii*) $\dfrac{1}{(2+i)^2} - \dfrac{1}{(2-i)^2}$ $\qquad$ (*V.T.U., 2011 S*)

2. If $\dfrac{1}{x+iy} + \dfrac{1}{u+iv} = 1$; x, y, u, v being real quantities, express v in terms of x and y.

3. If x and y are real, solve the equation $\dfrac{iy}{ix+1} - \dfrac{3y+4i}{3x+y} = 0$.
4. If $\alpha - i\beta = \dfrac{1}{a - ib}$, prove that $(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. (*Mumbai, 2008 S*)
5. Find what curve $z\bar{z} + (1 + i)z + (1 - i)\bar{z} = 0$ represents?
6. In an Argand diagram, show that $9 + i$, $4 + 13i$, $-8 + 8i$ and $-3 - 4i$ form a square.
7. If $|z_1| = |z_2|$ and amp (z_1) + amp $(z_2) = 0$, then show that z_1 and z_2 are conjugate complex numbers.
8. A rectangle is constructed in the complex plane and its sides parallel to the axes and its centre is situated at the origin. If one of the vertices of the rectangle is $1 + i\sqrt{3}$, find the complex numbers representing the other three vertices of the rectangle. Find also the area of the rectangle.
9. An equilateral triangle constructed in the complex plane has its one vertex at the point $1 + i\sqrt{3}$. Find the complex numbers representing the other two vertices, O the origin being its circumcentre.
10. The centre of a regular hexagon is at the origin and one vertex is given by $1 + i$ on the Argand diagram. Find the remaining vertices.
11. What domain of the z-plane is represented by
    (*i*) $2 \le |z + 3| < 4$ (*ii*) $I(z) > 2$
    (*iii*) $\pi/3 <$ amp $(z) < \pi/2$ (*iv*) $|z + 2| + |z - 2| < 4$.
12. If $|z^2 - 1| = |z|^2 + 1$, prove that z lies on the imaginary axis. (*Mumbai, 2007*)
13. What are the loci given by (*i*) $|z - 1| + |z + 1| = 3$ (*ii*) $|z - 3| = k\ |z + 1|$ for $k = 1$ and 2.
14. Find the locus of z given by :
    (*i*) $|z| = |z - 2|$. (*ii*) $|3z - 1| = |z - 3|$.
15. Find the locus of z :
    (*i*) when $\dfrac{z+i}{z+2}$ is real, (*ii*) when $\dfrac{z-i}{z-2}$ is purely imaginary. (*Osmania, 2003 S*)

19.4 DE MOIVRE'S THEOREM*

Statement : *If n be (i) an integer, positive or negative* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$;
(ii) a fraction, positive or negative, one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.

Proof. Case I. *When n is a positive integer.*

By actual multiplication

$$\text{cis }\theta_1 \text{ cis }\theta_2 = (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$
$$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2), \textit{ i.e., } \text{cis }(\theta_1 + \theta_2)$$

Similarly $\text{cis }\theta_1 \text{ cis }\theta_2 \text{ cis }\theta_3 = \text{cis }(\theta_1 + \theta_2)\text{ cis }\theta_3 = \text{cis }(\theta_1 + \theta_2 + \theta_3)$

Proceeding in this way,

$$\text{cis }\theta_1 \text{ cis }\theta_2 \text{ cis }\theta_3 \ldots \text{cis }\theta_n = \text{cis }(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$

Now putting $\theta_1 = \theta_2 = \theta_3 = \ldots = \theta_n = \theta$, we obtain $(\text{cis }\theta)^n = \text{cis } n\theta$.

Case II. *When n is a negative integer.*

Let $n = -m$, where m is a + ve integer.

$$\therefore \quad (\text{cis }\theta)^n = (\text{cis }\theta)^{-m} = \frac{1}{(\text{cis }\theta)^m} = \frac{1}{\text{cis } m\theta} \qquad \text{(By case I)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

[Multiplying the num. and denom. by $(\cos m\theta - i\sin m\theta)$]

---

*One of the remarkable theorems in mathematics; called after the name of its discoverer *Abraham De Moivre* (1667–1754), a French Mathematician.

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i \sin m\theta$$

$$= \cos(-m\theta) + i \sin(-m\theta) = \text{cis}(-m\theta) = \text{cis}\, n\theta \qquad [\because \ -m = n]$$

Case III. *When n is a fraction, positive or negative.*

Let $n = p/q$, where q is a + ve integer and p is any integer + ve or – ve

Now $(\text{cis}\, \theta/q)^q = \text{cis}\,(q\,.\,\theta/q) = \text{cis}\, \theta$

$\therefore$ Taking qth root of both sides cis (θ/q) is one of the q values of $(\text{cis}\, \theta)^{1/q}$,

i.e., one of the values of $(\text{cis}\, \theta)^{1/q} = \text{cis}\, \theta/p$

Raise both sides to power p, then one of the values of $(\text{cis}\, \theta)^{p/q} = (\text{cis}\, \theta/q)^p = \text{cis}\,(p/q)\,\theta$ *i.e.*, one of the values of $(\text{cis}\, \theta)^n = \text{cis}\, n\theta$. (By case I and II)

Thus *the theorem is completely established for all rational values of n.*

Cor. 1. $\text{cis}\, \theta_1 \,.\, \text{cis}\, \theta_2 \ldots \text{cis}\, \theta_n = \text{cis}\,(\theta_1 + \theta_2 + \ldots + \theta_n)$

2. $(\cos\theta - i \sin\theta)^n = \cos n\theta - i \sin n\theta = (\cos\theta + i \sin\theta)^{-n}$

3. $(\text{cis}\, m\theta)^n = \text{cis}\, mn\theta = (\text{cis}\, n\theta)^m$.

Example 19.14. *Simplify* $\dfrac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta - i \sin 4\theta)^5}{(\cos 4\theta + i \sin 4\theta)^3 (\cos 5\theta + i \sin 5\theta)^{-4}}$.

Solution. We have, $(\cos 3\theta + i \sin 3\theta)^4 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i \sin\theta)^{12}$

$(\cos 4\theta - i \sin 4\theta)^5 = \cos 20\theta - i \sin 20\theta = (\cos\theta + i \sin\theta)^{-20}$

$(\cos 4\theta + i \sin 4\theta)^3 = \cos 12\theta + i \sin 12\theta = (\cos\theta + i \sin\theta)^{12}$

$(\cos 5\theta + i \sin 5\theta)^{-4} = \cos 20\theta - i \sin 20\theta = (\cos\theta + i \sin\theta)^{-20}$

$$\therefore \quad \text{The given expression} = \frac{(\cos\theta + i \sin\theta)^{12} (\cos\theta + i \sin\theta)^{-20}}{(\cos\theta + i \sin\theta)^{12} (\cos\theta + i \sin\theta)^{-20}} = 1.$$

Example 19.15. *Prove that*

$$(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cos^n(\theta/2)\,.(\cos n\theta/2).$$

Solution. Put $1 + \cos\theta = r\cos\alpha$, $\sin\theta = r \sin\alpha$.

$$\therefore \quad r^2 = (1 + \cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\theta/2 \quad i.e., \quad r = 2\cos\theta/2$$

and
$$\tan\alpha = \frac{\sin\theta}{1 + \cos\theta} = \frac{2\sin\theta/2\,.\,\cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2 \quad i.e., \quad \alpha = \theta/2.$$

$\therefore$ L.H.S. $= [r(\cos\alpha + i \sin\alpha)]^n + [r(\cos\alpha - i \sin\alpha)]^n$

$= r^n[(\cos\alpha + i \sin\alpha)^n + (\cos\alpha - i \sin\alpha)^n] = r^n(\cos n\alpha + i \sin n\alpha + \cos n\alpha - i \sin n\alpha)$

$= r^n\,.\,2\cos n\alpha$ [Substituting the values of r and α]

$= 2^{n+1}\cos^n(\theta/2)\cos(n\theta/2)$.

Example 19.16. *If* $2\cos\theta = x + \dfrac{1}{x}$, *prove that*

(*i*) $2\cos r\theta = x^r + 1/x^r$, (*ii*) $\dfrac{x^{2n}+1}{x^{2n-1}+x} = \dfrac{\cos n\theta}{\cos(n-1)\theta}$ (*Madras, 2000 S*)

Solution. Since $x + 1/x = 2\cos\theta$ $\therefore$ $x^2 - 2x\cos\theta + 1 = 0$

whence
$$x = \frac{2\cos\theta \pm \sqrt{(4\cos^2\theta - 4)}}{2} = \cos\theta \pm i \sin\theta.$$

(*i*) Taking the + ve sign, $x^r = (\cos\theta + i \sin\theta)^r = \cos r\theta + i \sin r\theta$ (*S.V.T.U., 2009*)

and $x^{-r} = (\cos\theta + i \sin\theta)^{-r} = \cos r\theta - i \sin r\theta$

Adding $x^r + 1/x^r = 2\cos r\theta$. Similarly with the – ve sign, the same result follows.

(ii)
$$\frac{x^{2n}+1}{x^{2n-1}+x}=\frac{(\cos\theta+i\sin\theta)^{2n}+1}{(\cos\theta+i\sin\theta)^{2n-1}+\cos\theta+i\sin\theta}$$
$$=\frac{\cos 2n\theta+i\sin 2n\theta+1}{\cos(2n-1)\theta+i\sin(2n-1)\theta+\cos\theta+i\sin\theta}$$
$$=\frac{(1+\cos 2n\theta)+i\sin 2n\theta}{(\cos\overline{2n-1}\,\theta+\cos\theta)+i(\sin\overline{2n-1}\,\theta+\sin\theta)}$$
$$=\frac{2\cos^2 n\theta+2i\sin n\theta\cos\theta}{2\cos n\theta\cos\overline{n-1}\,\theta+2i\sin n\theta\cos\overline{n-1}\,\theta}$$
$$=\frac{\cos n\theta\,(2\cos n\theta+2i\sin n\theta)}{\cos\overline{n-1}\,\theta\,(2\cos n\theta+2i\sin n\theta)}=\frac{\cos n\theta}{\cos\overline{n-1}\,\theta}.$$

Example 19.17. *If* $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, *prove that* (i) $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\Sigma\sin 2(\alpha+\beta)$

(iv) $\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$ *(Mumbai, 2009)*

Solution. Let $a=\text{cis}\,\alpha,\ b=\text{cis}\,\beta$ and $c=\text{cis}\,\gamma$.

Then $a+b+c=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$ ...(1)

(i)
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$$
$$=\sum\frac{\cos\alpha-i\sin\alpha}{\cos\alpha-i\sin\alpha}\cdot\frac{1}{\cos\alpha+i\sin\alpha}=\sum(\cos\alpha-i\sin\alpha)$$
$$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)=0 \quad \text{(Given)}$$

or $bc+ca+ab=0$ ...(2)

$\therefore\quad a^2+b^2+c^2=(a+b+c)^2-2(bc+ca+ab)=0$ [By (1) & (2) ...(3)]

or $(cis\,\alpha)^2+(cis\,\beta)^2+(cis\,\gamma)^2=cis\,2\alpha+cis\,2\beta+cis\,2\gamma=0$

Equating imaginary parts from both sides, we get

$$\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$$

(ii) Since $a+b+c=0,\quad \therefore\ a^3+b^3+c^3=3abc$

or $(cis\,\alpha)^3+(cis\,\beta)^3+(cis\,\gamma)^3=3\,cis\,\alpha\,cis\,\beta\,cis\,\gamma$

or $cis\,3\alpha+cis\,3\beta+cis\,3\gamma=3\,cis\,(\alpha+\beta+\gamma)$

Equating imaginary parts from both sides, we get

$$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$$

(iii) From (1), $a+b=-c$ or $(a+b)^2=c^2$ or $a^2+b^2-c^2=-2ab$

Again squaring, $a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2=4a^2b^2$

i.e., $a^4+b^4+c^4=2(a^2b^2+b^2c^2+c^2a^2)$

or $(cis\,\alpha)^4+(cis\,\beta)^4+(cis\,\gamma)^4=2\sum(cos\,\alpha)^2\,(cis\,\beta)^2$

or $cis\,4\alpha+cis\,4\beta+cis\,4\gamma=2\sum cis\,2\alpha\,cis\,2\beta=2\sum cis\,2(\alpha+\beta)$

Equating imaginary parts from both sides, we get

$$\sin 4\alpha+\sin 4\beta+\sin 4\gamma=2\sum\sin 2(\alpha+\beta)$$

(iv) From (2), $ab+bc+ca=0$

or $cis\,\alpha\,cis\,\beta+cis\,\beta\,cis\,\gamma+cis\,\gamma\,cis\,\alpha=0$

or $cis\,(\alpha+\beta)+cis\,(\beta+\gamma)+cis\,(\gamma+\alpha)=0$

Equating imaginary parts from both sides, we get

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0$$

PROBLEMS 19.2

1. Prove that (*i*) $\dfrac{(\cos 5\theta - i\sin 5\theta)^2 (\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta - i\sin 4\theta)^9 (\cos\theta + i\sin\theta)^5} = 1$

(*ii*) $\dfrac{(\cos\alpha + i\sin\alpha)^4}{(\sin\beta + i\cos\beta)^5} = \sin(4\alpha + 5\beta) - i\cos(4\alpha + 5\beta)$. (*iii*) $\left(\dfrac{\cos\theta + i\sin\theta)^4}{\sin\theta + i\cos\theta}\right) = \cos 8\theta + i\sin 8\theta$.

2. If $p = \text{cis}\,\theta$ and $q = \text{cis}\,\phi$, show that

(*i*) $\dfrac{p-q}{p+q} = i\tan\dfrac{\theta-\phi}{2}$ (*Mumbai, 2008*) (*ii*) $\dfrac{(p+q)(pq-1)}{(p-q)(pq+1)} = \dfrac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi}$. (*Kurukshetra, 2005*)

3. If $a = \text{cis}\,2\alpha$, $b = \text{cis}\,2\beta$, $c = \text{cis}\,2\gamma$ and $d = \text{cis}\,2\delta$, prove that

(*i*) $\sqrt{\dfrac{ab}{c}} + \sqrt{\dfrac{c}{ab}} = 2\cos(\alpha + \beta - \gamma)$ (*ii*) $\sqrt{\dfrac{ab}{cd}} + \sqrt{\dfrac{cd}{ab}} = 2\cos(\alpha + \beta - \gamma - \delta)$.

4. If $x_r = \text{cis}\,(\pi/2^r)$, show that $\underset{n\to\infty}{\text{Lt}}\; x_1 x_2 x_3 \cdots x_n = -1$. (*S.V.T.U., 2009 ; Mumbai, 2007*)

5. Find the general value of θ which satisfies the equation

$$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1.$$

6. Prove that (*i*) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2(a^2 + b^2)^{m/2n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$.

(*ii*) $(1+i)^n + (1-i)^n = 2^{n/2+1}\cos n\pi/4$.

7. Simplify $[\cos\alpha - \cos\beta + i(\sin\alpha - \sin\beta)]^n + [\cos\alpha - \cos\beta - i(\sin\alpha - \sin\beta)]^n$

8. Prove that (*i*) $(1 + \sin\theta + i\cos\theta)^n + (1 + \sin\theta - i\cos\theta)^n = 2^{n+1}\cos^n\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)\cos\left(\dfrac{n\pi}{4} - \dfrac{n\theta}{2}\right)$.

(*ii*) $\left[\dfrac{1 + \sin\alpha + i\cos\alpha}{1 + \sin\alpha - i\cos\alpha}\right]^n = \cos\left(\dfrac{n\pi}{2} - n\alpha\right) + i\sin\left(\dfrac{n\pi}{2} - n\alpha\right)$. (*S.V.T.U., 2006*)

9. If $2\cos\theta = x + 1/x$ and $2\cos\phi = y + 1/y$, show that one of the values of

(*i*) $x^m y^n + \dfrac{1}{x^m y^n}$ is $2\cos(m\theta + n\phi)$. (*S.V.T.U., 2007*)

(*ii*) $\dfrac{x^m}{y^n} + \dfrac{y^n}{x^m}$ is $2\cos(m\theta - n\phi)$. (*Nagpur, 2009*)

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n + \beta^n = 2^{n+1}\cos n\pi/3$. (*Delhi, 2002*)

11. If α, β are the roots of the equation $z^2\sin^2\theta - z\sin\theta + 1 = 0$, then prove that

(*i*) $\alpha^n + \beta^n = 2\cos n\theta\,\text{cosec}^n\,\theta$ (*ii*) $\alpha^n\beta^n = \text{cosec}^{2n}\,\theta$. (*Mumbai, 2009*)

12. If $x^2 - 2x\cos\theta + 1 = 0$, show that $x^{2n} - 2x^n\cos n\theta + 1 = 0$.

13. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x + y + z = 0$, then prove that

$$x^{-1} + y^{-1} + z^{-1} = 0.$$

14. If $\sin\theta + \sin\phi + \sin\psi = 0 = \cos\theta + \cos\phi + \cos\psi$, prove that

(*i*) $\cos 2\theta + \cos 2\phi + \cos 2\psi = 0$ (*Mumbai, 2009*)

(*ii*) $\cos 3\theta + \cos 3\phi + \cos 3\psi = 3\cos(\theta + \phi + \psi)$

(*iii*) $\cos 4\theta + \cos 4\phi + \cos 4\psi = 2\sum\cos 2(\phi + \psi)$.

15. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

(*i*) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 3/2$

(*ii*) $\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0$ (*Mumbai, 2009 ; S.V.T.U., 2008*)

16. If $\sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, $\cos\alpha + 2\cos\beta + 3\cos\gamma = 0$, prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$ and $\cos 3\alpha + 8\cos 3\beta + 27\sin 3\gamma = 18\cos(\alpha + \beta + \gamma)$.

19.5 ROOTS OF A COMPLEX NUMBER

There are q and only q distinct values of $(\cos\theta + i\sin\theta)^{1/q}$, *q being an integer.*

Since $\cos\theta = \cos(2n\pi + \theta)$ and $\sin\theta = \sin(2n\pi + \theta)$, where n is any integer.

$$\therefore \qquad \text{cis}\,\theta = \text{cis}\,(2n\pi + \theta).$$

By De Moivre's theorem one of the values of

$$(\text{cis}\ \theta)^{1/q} = [\text{cis}\ (2n\pi + \theta)]^{1/q} = \text{cis}\ (2n\pi + \theta)/q \qquad ...(1)$$

Giving n the values 0, 1, 2, 3, ..., $(q-1)$ successively, we get the following q values of $(\text{cis}\ \theta)^{1/q}$;

$$\left.\begin{array}{ll} \text{cis}\ \theta/q & (\text{for } n = 0) \\ \text{cis}\ (2\pi + \theta)/q & (\text{for } n = 1) \\ \text{cis}\ (4\pi + \theta)/q & (\text{for } n = 2) \\ \cdots\cdots & \cdots\cdots \\ \text{cis}\ [2(q-1)\pi + \theta]/q & (\text{for } n = q-1) \end{array}\right\} \qquad ...(2)$$

Putting $n = q$ in (1), we get a value of $(\text{cis}\ \theta)^{1/q} = \text{cis}\ (2\pi + \theta/q) = \text{cis}\ \theta/q$, which is the same as the value of $n = 0$.

Similarly for $n = q + 1$, we get a value of $(\text{cis}\ \theta)^{1/q}$ to be $\text{cis}\ (2\pi + \theta)/q$, which is the same as the value for $n = 1$ and so on.

Thus, the values of $(\text{cis}\ \theta)^{1/q}$ for $n = q, q+1, q+2$ etc. are the mere repetition of the q values obtained in (2).

Moreover, the q values given by (2) are clearly distinct from each other, for no two of the angles involved therein are equal or differ by a multiple of 2π.

Hence $(\text{cis}\ \theta)^{1/q}$ has q and only q distinct values given by (2).

Obs. *$(\text{cis}\ \theta)^{p/q}$ where p/q is a rational fraction in its lowest terms, has also q and only q distinct values; which are* obtained by putting $n = 0, 1, 2, ..., q-1$ successively in $\text{cis}\ p\ (2n\pi + \theta)/q$.

Note that $(\text{cis}\ \theta)^{6/15}$ has only 5 distinct values and not 15; because 6/15 in its lowest terms = 2/5

$\therefore$ *In order to find the distinct values of $(\text{cis}\ \theta)^{p/q}$ always see that p/q is in its lowest terms.*

Note. The above discussion can usefully be employed for extracting any assigned root of a given quantity. We have only to express it in the form $r\ (\cos\theta + i \sin\theta)$ and proceed as above.

Example 19.18. *Find the cube roots of unity and show that they form an equilateral triangle in the Argand diagram.*

Solution. If x be a cube root of unity, then

$$x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3} = (\text{cis}\ 0)^{1/3} = (\text{cis}\ 2n\pi)^{1/3} = \text{cis}\ 2n\pi/3$$

where $n = 0, 1, 2$.

$\therefore$ the three vlaues of x are $\text{cis}\ 0 = 1$,

$$\text{cis}\ 2\pi/3 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

and
$$\text{cis}\ 4\pi/3 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

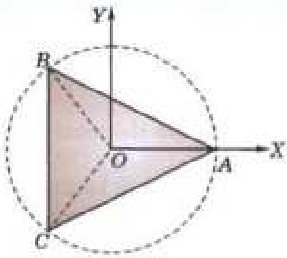


Fig. 19.19

These three cube roots are represented by the points A, B, C on the Argand diagram such that $OA = OB = OC$ and $\angle AOB = 120°$, $\angle AOC = 240°$ (Fig. 19.19).

$\therefore$ these points lie on a circle with centre O and unit radius such that $\angle AOB = \angle BOC = \angle COA = 120°$ *i.e.*, $AB = BC = CA$.

Hence A, B, C form an equilateral triangle.

Example. 19.19. *Find all the values of* $\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^{3/4}$.

Also show that the continued product of these values is 1. (*Nagpur, 2009*)

Solution. Put $1/2 = r \cos\theta$ and $\sqrt{3}/2 = r \sin\theta$ so that $r = 1$ and $\theta = \pi/3$

$$\therefore \quad (1/2 + \sqrt{3}i/2)^{3/4} = [(\cos \pi/3 + i \sin \pi/3)^3]^{1/4} = (\text{cis}\ \pi)^{1/4}$$

$$= [\text{cis}\ (2n+1)\pi]^{1/4} = \text{cis}\ (2n+1)\ \pi/4 \text{ where } n = 0, 1, 2, 3.$$

Hence the required values are $\text{cis}\ \pi/4$, $\text{cis}\ 3\pi/4$, $\text{cis}\ 5\pi/4$ and $\text{cis}\ 7\pi/4$.

$$\therefore \quad \text{their continued product} = \text{cis}\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) = \text{cis}\ 4\pi = 1.$$

Example 19.20. *Use De Moivre's theorem to solve the equation.* (P.T.U., 2005)

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

Solution. '$x^4 - x^3 + x^2 - x + 1$' is a G.P. with common ratio $(-x)$, therefore

$$\frac{1-(-x)^5}{1-(-x)} = 0, \quad x \neq -1 \quad \text{or} \quad x^5 + 1 = 0$$

i.e., $$x^5 = -1 = \text{cis}\ \pi = \text{cis}\ (2n+1)\pi$$

$\therefore$ $$x = [\text{cis}\ (2n+1)\pi]^{1/5} = \text{cis}\ (2n+1)\pi/5, \text{ where } n = 0, 1, 2, 3, 4$$

Hence the values are cis $\pi/5$, cis $3\pi/5$, cis π, cis $7\pi/5$, cis $9\pi/5$

or $$\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}, -1, \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}, \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

Rejecting the value -1 which corresponds to the factor $x + 1$, the required roots are :

$$\cos\pi/5 \pm i\sin\pi/5, \cos 3\pi/5 \pm i\sin 3\pi/5.$$

Example 19.21. *Show that the roots of the equation $(x-1)^n = x^n$, n being a positive integer are $\frac{1}{2}(1 + i\cot r\pi/n)$, where r has the values 1, 2, 3, ..., n − 1.*

Solution. Given equation is $\left(\frac{x-1}{x}\right)^n = 1$ or $1 - \frac{1}{x} = (1)^{1/n}$

or $$\frac{1}{x} = 1 - (1)^{1/n} = 1 - \text{cis}\ \frac{2r\pi}{n}, r = 0, 1, 2, \ldots (n-1). \qquad [\because\ 1 = \text{cis}\ 2\pi r]$$

or $$= \left(1 - \cos\frac{2r\pi}{n}\right) - i\sin\frac{2r\pi}{n} = 2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}$$

$\therefore$ $$x = \frac{1}{2\sin\frac{r\pi}{n}} \cdot \frac{1}{\left(\sin\frac{r\pi}{n} - i\cos\frac{r\pi}{n}\right)} = \frac{\sin\frac{r\pi}{n} + i\cos\frac{r\pi}{n}}{2\sin\frac{r\pi}{n}}$$

$$= \frac{1}{2}\left(1 + i\cot\frac{r\pi}{n}\right), r = 1, 2, \ldots (n-1). \qquad [\because\ \cot 0 \to \infty]$$

Hence the roots of the given equation are $\frac{1}{2}(1 + i\cot r\pi/n)$ where $r = 1, 2, 3, \ldots (n-1)$.

Example 19.22. *Find the 7th roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple number of 7, n being an integer, and then the sum is 7.*

(Mumbai, 2008 ; Kurukshetra, 2005)

Solution. We have $\quad (1)^{1/7} = (\cos 2r\pi + i\sin 2r\pi)^{1/7} = \text{cis}\ \frac{2r\pi}{7} = \left(\text{cis}\ \frac{2\pi}{7}\right)^r$

Putting $r = 0, 1, 2, 3, 4, 5, 6$, we find that 7th roots of unity are $1, \rho, \rho^2, \rho^3, \rho^4, \rho^5, \rho^6$ where $\rho = \cos 2\pi/7$.

$\therefore$ sum S of the nth powers of these roots $= 1 + \rho^n + \rho^{2n} + \ldots + \rho^{6n}$ ...(*i*)

$$= \frac{1-\rho^{7n}}{1-\rho^n}, \text{ being a G.P. with common ratio } \rho$$

When n is not a multiple of 7, $\rho^{7n} = (\rho^7)^n = (\text{cis}\ 2\pi)^n = 1.$

i.e., $$1 - \rho^{7n} = 0 \text{ and } 1 - \rho^n \neq 0, \text{ as } n \text{ is not a multiple of 7.}$$

Thus $S = 0$.

When n is a multiple of 7 = 7p (say)

From (*i*), $\quad S = 1 + (\rho^7)^p + (\rho^7)^{2p} + \ldots + (\rho^7)^{6p} = 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$

Example 19.23. *Find the equation whose roots are $2\cos\pi/7$, $2\cos 3\pi/7$, $2\cos 5\pi/7$.*

Solution. Let $y = \cos\theta + i\sin\theta$, where $\theta = \pi/7, 3\pi/7, \ldots, 13\pi/7$.

Then $\quad y^7 = (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta = -1 \quad$ or $\quad y^7 + 1 = 0$

or $\quad (y+1)(y^6 - y^5 + y^4 - y^3 + y^2 - y + 1) = 0$

Leaving the factor $y + 1$ which corresponds to $\theta = \pi$,

We get $\quad y^6 - y^5 + y^4 - y^3 + y^2 - y + 1 = 0 \quad$ ...(*i*)

Its roots are $\quad y = \text{cis}\,\theta$ where $\theta = \pi/7, 3\pi/7, 5\pi/7, 9\pi/7, 11\pi/7, 13\pi/7$.

Dividing (*i*) by y^3, $(y^3 + 1/y^3) - (y^2 + 1/y^2) + (y + 1/y) - 1 = 0$

or $\quad [(y + 1/y)^3 - 3(y + 1/y)] - [(y + 1/y)^2 - 2] - (y + 1/y) - 1 = 0$

or $\quad x^3 - x^2 - 2x + 1 = 0 \quad$ ...(*ii*)

where $\quad x = y + 1/y = 2\cos\theta$.

Now since $\quad \cos 13\pi/7 = \cos\pi/7, \cos 11\pi/7 = \cos 3\pi/7, \cos 9\pi/7 = \cos 5\pi/7$

Hence the roots of (*ii*) are $2\cos\dfrac{\pi}{7}, 2\cos\dfrac{3\pi}{7}, 2\cos\dfrac{5\pi}{7}$.

PROBLEMS 19.3

1. Find all the values of
   (*i*) $(1+i)^{1/4}$ (*ii*) $(-1+i)^{2/5}$
   (*iii*) $(-1+i\sqrt{3})^{3/2}$ (*iv*) $(1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3}$.
2. If w is a complex cube root of unity, prove that $1 + w + w^2 = 0$.
3. Find all the values of $(-1)^{1/6}$.
4. Mark by points on the Argand diagram, all the values of $(1+i\sqrt{3})^{1/5}$ and verify that they form a pentagon.
5. Use De Moivre's theorem to solve the following equations :
   (*i*) $x^5 + 1 = 0$ (*ii*) $x^7 + x^4 + x^3 + 1 = 0$
   (*iii*) $x^9 + x^5 - x^4 - 1 = 0$ (*Madras, 2000*) (*iv*) $(x-1)^5 + x^5 = 0$.
6. Find the roots common to the equations $x^4 + 1 = 0$ and $x^6 - i = 0$.
7. Solve the equation $x^{12} - 1 = 0$ and find which of its roots satisfy the equation $x^4 + x^2 + 1 = 0$.
8. Show that the roots of $(x+1)^7 = (x-1)^7$ are given by $\pm i\cot r\pi/7$, $r = 1, 2, 3$. (*Mumbai, 2008*)
9. Prove that the nth roots of unity form a geometric progression. (*Mumbai, 2007*)
   Also show that the sum of these n roots is zero and their product is $(-1)^{n-1}$.
10. Find the equation whose roots are $2\cos 2\pi/7, 2\cos 4\pi/7, 2\cos 6\pi/7$.

19.6 (1) TO EXPAND sin nθ, cos nθ AND tan nθ IN POWERS OF sin θ, cos θ AND tan θ RESPECTIVELY (n BEING A POSITIVE INTEGER)

We have $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$ (By De Moivre's theorem)

$$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots$$

(By Binomial theorem)

$$= (\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots) + i\,({}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots)$$

Equating real and imaginary parts from both sides, we get

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots \quad \ldots(1)$$

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots \quad \ldots(2)$$

Replacing every $\sin^2\theta$ by $1 - \cos^2\theta$ in (1) and every $\cos^2\theta$ by $1 - \sin^2\theta$ in (2), we get the desired expansions of $\cos n\theta$ and $\sin n\theta$.

Dividing (2) by (1),

$$\tan n\theta = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

and dividing numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + {}^nC_5\tan^5\theta - \ldots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \ldots}.$$

Example 19.24. *Express* $\cos 6\theta$ *in terms of* $\cos\theta$. (Madras, 2002)

Solution. We know that $\cos n\theta = \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + {}^nC_4 \cos^{n-4}\theta \sin^4\theta - \ldots$

Put $n = 6$, then $\cos 6\theta = \cos^6\theta - {}^6C_2\cos^4\theta\sin^2\theta + {}^6C_4\cos^2\theta\sin^4\theta - {}^6C_6\sin^6\theta$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$$

(2) Addition formulae for any number of angles

We have, $\cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n)$
$$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\ldots(\cos\theta_n+i\sin\theta_n)$$

Now $\cos\theta_1 + i\sin\theta_1 = \cos\theta_1(1+i\tan\theta_1)$, $\cos\theta_2+i\sin\theta_2 = \cos\theta_2(1+i\tan\theta_2)$ and so on.

$\therefore$ $\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n)$
$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)\ldots(1+i\tan\theta_n)$$
$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n[1+i(\tan\theta_1+\tan\theta_2+\ldots+\tan\theta_n)$$
$$+ i^2(\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+\ldots)+i^3(\tan\theta_1\tan\theta_2\tan\theta_3+\ldots)+\ldots+\ldots]$$
$$= \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1+is_1-s_2-is_3+s_4+\ldots)$$

where $s_1 = \tan\theta_1+\tan\theta_2+\ldots+\tan\theta_n$, $s_2 = \Sigma\tan\theta_1\tan\theta_2$, $s_3 = \Sigma\tan\theta_1\tan\theta_2\tan\theta_3$ etc.

Equating real and imaginary parts, we have
$$\cos(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(1-s_2+s_4-\ldots)$$
$$\sin(\theta_1+\theta_2+\ldots+\theta_n) = \cos\theta_1\cos\theta_2\ldots\cos\theta_n(s_1-s_3+s_5-\ldots)$$

and by division, we get $\mathbf{\tan(\theta_1+\theta_2+\ldots+\theta_n) = \dfrac{s_1-s_3+s_5-\ldots}{1-s_2+s_4-s_6+\ldots}}$.

Example 19.25. *If* $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \pi/2$, *show that* $xy+yz+zx = 1$. (P.T.U., 2003)

Solution. Let $\tan^{-1}x = \alpha$, $\tan^{-1}y = \beta$, $\tan^{-1}z = \gamma$ so that $x = \tan\alpha$, $y \tan\beta$, $z = \tan\gamma$

We know that $$\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}$$

$\therefore$ $$\tan\pi/2 = \frac{x+y+z-xyz}{1-xy-yz-zx} \quad\text{or}\quad 1-xy-yz-zx = 0$$

Hence $xy+yz+zx = 1$.

Example 19.26. *If* $\theta_1, \theta_2, \theta_3$ *be three values of* θ *which satisfy the equation* $\tan 2\theta = \lambda\tan(\theta+\alpha)$ *and such that no two of them differ by a multiple of* π, *show that* $\theta_1+\theta_2+\theta_3+\alpha$ *is a multiple of* π.

Solution. Given equation can be written as $\dfrac{2t}{1-t^2} = \lambda\dfrac{t+\tan\alpha}{1-t\,.\tan\alpha}$ where $t = \tan\theta$

or $$\lambda t^3 + (\lambda-2)\tan\alpha\,.\,t^2 + (2-\lambda)t - \lambda\tan\alpha = 0$$

$\therefore$ $\tan\theta_1, \tan\theta_2, \tan\theta_3$, being its roots, we have

$$s_1 = \Sigma\tan\theta_1 = -\frac{\lambda-2}{\lambda}\tan\alpha \qquad \text{[By § 1.3]}$$
$$s_2 = \Sigma\tan\theta_1\tan\theta_2 = \frac{2-\lambda}{\lambda} \quad\text{and}\quad s_3 = \tan\alpha$$

$\therefore$ $$\tan(\theta_1+\theta_2+\theta_3) = \frac{s_1-s_3}{1-s_2} = \frac{(-1+2/\lambda)\tan\alpha-\tan\alpha}{1-(2/\lambda-1)}$$
$$= -\tan\alpha = \tan(n\pi-\alpha)$$

Thus $\theta_1+\theta_2+\theta_3 = n\pi-\alpha$, whence follows the result.

(3) To expand $\sin^m\theta$, $\cos^n\theta$ or $\sin^m\theta\cos^n\theta$ in a series of sines or cosines of multiples of θ

If $z = \cos\theta + i\sin\theta$ then $1/z = \cos\theta - i\sin\theta$.

By De Moivre's theorem, $z^p = \cos p\theta + i\sin p\theta$ and $1/z^p = \cos p\theta - i\sin p\theta$

$\therefore$ $z + 1/z = 2\cos\theta$, $z - 1/z = 2i\sin\theta$; $z^p + 1/z^p = 2\cos p\theta$, $z^p - 1/z^p = 2i\sin p\theta$

These results are used to expand the powers of sin θ or cos θ or their products in a series of sines or cosines of multiples of θ.

Example 19.27. *Expand $\cos^8 \theta$ in a series of cosines of multiples of θ.*

Solution. Let $\quad z = \cos\theta + i \sin\theta$, so that $z + 1/z = 2\cos\theta$ and $z^p + 1/z^p = 2\cos p\theta$.

$$\therefore \quad (2\cos\theta)^8 = (z + 1/z)^8$$

$$= z^8 + {}^8C_1 z^7 \cdot \frac{1}{z} + {}^8C_2 z^6 \cdot \frac{1}{z^2} + {}^8C_3 z^5 \cdot \frac{1}{z^3} + {}^8C_4 z^4 \cdot \frac{1}{z^4} + {}^8C_5 z^3 \cdot \frac{1}{z^5} + {}^8C_6 z^2 \cdot \frac{1}{z^6} + {}^8C_7 z \cdot \frac{1}{z^7} + \frac{1}{z^8}$$

$$= (z^8 + 1/z^8) + {}^8C_1(z^6 + 1/z^6) + {}^8C_2(z^4 + 1/z^4) + {}^8C_3(z^2 + 1/z^2) + {}^8C_4$$

$$= (2\cos 8\theta) + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70.$$

Hence $\quad \cos^8\theta = \dfrac{1}{128}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35].$

Example 19.28. *Expand $\sin^7\theta \cos^3\theta$ in a series of sines of multiples of θ.*

Solution. Let $\quad z = \cos\theta + i\sin\theta$

so that $\quad z + 1/z = 2\cos\theta,\ z - 1/z = 2i\sin\theta$ and $z^p - 1/z^p = 2i\sin p\theta$.

$$\therefore \quad (2i\sin\theta)^7 (2\cos\theta)^3 = (z - 1/z)^7 (z + 1/z)^3$$

$$= (z - 1/z)^4 [(z - 1/z)(z + 1/z)]^3 = (z - 1/z)^4 (z^2 - 1/z^2)^3$$

$$= \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)\left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)$$

$$= \left(z^{10} - \frac{1}{z^{10}}\right) - 4\left(z^8 - \frac{1}{z^8}\right) + 3\left(z^6 - \frac{1}{z^6}\right) + 8\left(z^4 - \frac{1}{z^4}\right) - 14\left(z^2 - \frac{1}{z^2}\right)$$

$$= 2i\sin 10\theta - 4(2i\sin 8\theta) + 3(2i\sin 6\theta) + 8(2i\sin 4\theta) - 14(2i\sin 2\theta)$$

Since $\quad i^7 = -i,$

$$\therefore \quad \sin^7\theta\cos^3\theta = -\frac{1}{2^9}[\sin 10\theta - 4\sin 8\theta + 3\sin 6\theta + 8\sin 4\theta - 14\sin 2\theta].$$

Obs. The expansion of $\sin^m\theta \cos^n\theta$ is a series of sines or cosines of multiples of θ according as m is odd or even.

PROBLEMS 19.4

1. Express sin 6θ/sin θ as a polynomial in cos θ?

Prove that (2-5):

2. $\sin 7\theta/\sin\theta = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$
3. $\dfrac{1 + \cos 7\theta}{1 + \cos\theta} = (x^3 - x^2 - 2x + 1)^2$, where $x = 2\cos\theta$. *(Madras, 2002)*
4. $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta$.
5. $\tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ where $t = \tan\theta$.
6. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, show that $x + y + z = xyz$.
7. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + p = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except in one particular case.

Prove that (8-12):

8. $\cos^7\theta = \dfrac{1}{16}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta).$ *(Madras, 2003 S)*
9. $\cos^6\theta - \sin^6\theta = \dfrac{1}{16}(\cos 6\theta + 15\cos 2\theta).$ *(Mumbai, 2007)*
10. $\sin^8\theta = 2^{-7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35).$
11. $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2.$
12. $\sin^5\theta\cos^2\theta = \dfrac{1}{64}(\sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta).$ *(Madras, 2003)*

13. Expand $\cos^5 \theta \sin^7 \theta$ in a series of sines of multiples of θ ? *(Madras, 2002)*
14. If $\cos^5 \theta = A \cos \theta + B \cos 3\theta + C \cos 5\theta$, find $\sin^5 \theta$ in terms of A, B, C.
15. If $\sin^4 \theta \cos^3 \theta = A_1 \cos \theta + A_3 \cos 3\theta + A_5 \cos 5\theta + A_7 \cos 7\theta$, prove that
$$A_1 + 9A_3 + 25A_5 + 49A_7 = 0.$$

19.7 COMPLEX FUNCTION

Definition. *If for each value of the complex variable z $(= x + iy)$ in a given region R, we have one or more values of w $(= u + iv)$, then w is said to be a* **complex function** *of z* and we write $w = u(x, y) + iv(x, y) = f(z)$ where u, v are real functions of x and y.

If to each value of z, there corresponds one and only one value of w, then w is said to be a *single-valued function of z* otherwise a *multi-valued function*. For example, $w = 1/z$ is a single-valued function and $w = \sqrt{z}$ is a multi-valued function of z. The former is defined at all points of the z-plane except at $z = 0$ and the latter assumes two values for each value of z except at $z = 0$.

19.8 EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

(1) Definition. When x is real, we are already familiar with the exponential function
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \infty.$$
Similarly, *we define the exponential function of the complex variable $z = x + iy$, as*
$$\mathbf{e^z \text{ or } \exp(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots \infty} \qquad \ldots(i)$$

(2) Properties :

I. Exponential form of $z = re^{i\theta}$

Putting $x = 0$ in (i), we get
$$e^{iy} = 1 + \frac{iy}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \ldots \infty$$
$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots\right) = \cos y + i \sin y$$

Thus $\quad e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$

Also $\quad x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$. Thus, $z = re^{i\theta}$

II. e^z is periodic function having imaginary period $2\pi i$, $[\because \; e^{z + 2n\pi i} = e^z \cdot e^{2n\pi i} = e^z]$.

III. e^z is not zero for any value of z.

Since $\quad e^z = e^{x + iy} = re^{i\theta}$ or $e^x \cdot e^{iy} = re^{i\theta}$

$\therefore \quad r = e^x > 0, y = \theta, |e^{iy}| = 1.$

Thus $\quad |e^z| = |e^x| \cdot |e^{iy}| = e^x \neq 0.$

IV. $\quad e^{\bar{z}} = \overline{e^z}$

Since $\quad e^{\bar{z}} = e^{x - iy} = e^x \cdot e^{-iy} = e^x (\cos y - i \sin y)$
$$= \overline{e^x (\cos y + i \sin y)} = \overline{e^z}$$

19.9 CIRCULAR FUNCTIONS OF A COMPLEX VARIABLE

(1) Definitions:

Since $\quad e^{iy} = \cos y + i \sin y$ and $e^{-iy} = \cos y - i \sin y$.

$\therefore$ the circular functions of real angles can be written as

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \cos y = \frac{e^{iy} + e^{-iy}}{2} \text{ and so on.}$$

It is, therefore, natural to *define the circular functions of the complex variable z by the equations :*

$$\mathbf{\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \tan z = \frac{\sin z}{\cos z}}$$

with cosec z, sec z and cot z as their respective reciprocals.

(2) Properties :

I. Circular functions are periodic : $\sin z$, $\cos z$ are periodic functions having real period 2π while $\tan z$, $\cot z$ have period π. [$a \sin (z + 2n\pi) = \sin z$, $\tan (z + n\pi) = \tan z$ etc.]

II. Even and odd functions : cos z, sec z are even functions while sin z, cosec z are odd functions. [$\because \cos z = \dfrac{e^{-iz} + e^{iz}}{2} = \cos z$, and $\sin (-z) = \dfrac{e^{-iz} - e^{iz}}{2i} = \dfrac{e^{iz} - e^{-iz}}{2i} = -\sin z$]

III. Zeros of sin z are given by $z = \pm 2n\pi$ *and zeros of cos z are given by* $z = \pm \frac{1}{2}(2n + 1)\pi$, $n = 0, 1, 2, ...$

IV. All the formulae for real circular functions are valid for complex circular functions
e.g., $\sin^2 z + \cos^2 z = 1$, $\sin (z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$.

(3) Euler's theorem $e^{iz} = \cos z + i \sin z$.

By definition $\cos z + i \sin z = \dfrac{e^{iz} + e^{-iz}}{2} + i\dfrac{e^{iz} - e^{-iz}}{2i} = e^{iz}$ where $z = x + iy$.

Also we have shown that $e^{iy} = \cos y + i \sin y$, where y is real.

Thus $e^{i\theta} = \cos \theta + i \sin \theta$, *where θ is real or complex. This is called the Euler's theorem.**

Cor. *De Moivre's theorem for complex numbers*

Whether θ is real or complex, we have

$$(\cos \theta + i \sin \theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta$$

Thus De Moivre's theorem is true for all θ (real or complex).

Example 19.29. *Prove that* (*i*) $[\sin (\alpha + \theta) - e^{i\alpha} \sin \theta]^n = \sin^n \alpha\, e^{-in\theta}$

(*ii*) $\sin (\alpha - n\theta) + e^{-i\alpha} \sin n\theta = e^{-in\theta} \sin \alpha$.

Solution. (*i*) L.H.S. $= [\sin \alpha \cos \theta + \cos \alpha \sin \theta - (\cos \alpha + i \sin \alpha) \sin \theta]^n$

$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n$

$= \sin^n \alpha (\cos \theta - i \sin \theta)^n = \sin^n \alpha (e^{-i\theta})^n = \sin^n \alpha e^{-in\theta}$

(*ii*) L.H.S. $= \sin \alpha \cos n\theta - \cos \alpha \sin n\theta + (\cos \alpha - i \sin \alpha) \sin n\theta$

$= \sin \alpha \cos n\theta - i \sin \alpha \sin n\theta$

$= \sin \alpha (\cos n\theta - i \sin n\theta) = \sin \alpha \,.\, e^{-in\theta}$.

Example 19.30. *Given* $\dfrac{1}{\rho} = \dfrac{1}{L\rho i} + C\rho i + \dfrac{1}{R}$, *where L, ρ, R are real, express ρ in the form* $Ae^{i\theta}$ *giving the values of A and θ.*

Solution.
$$\frac{1}{\rho} = \frac{R + L\rho^2 CR(-1) + L\rho i}{L\rho Ri} = \frac{(R - L\rho^2 CR) + iLR}{L\rho Ri}$$

or
$$\rho = L\rho \frac{Ri}{(R - L\rho^2 CR) + iL\rho} \times \frac{(R - L\rho^2 CR) - iL\rho}{(R - L\rho^2 CR) - iL\rho}$$

$$= \frac{L^2\rho^2 R + iL\rho R (R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} = A(\cos \theta + i \sin \theta), \text{ say}$$

*See footnote p. 205.

Equating real and imaginary parts, we have

$$A \cos \theta = \frac{L^2\rho^2 R}{(R - L\rho^2 CR)^2 + (L\rho)^2} \qquad ...(i)$$

$$A \sin \theta = \frac{L\rho R(R - L\rho^2 CR)}{(R - L\rho^2 CR)^2 + (L\rho)^2} \qquad ...(ii)$$

Squaring and adding (*i*) and (*ii*),

$$A^2 = \frac{(L^2\rho^2 R)^2 + (L\rho R)^2 (R - L\rho^2 CR)^2}{[(R - L\rho^2 CR)^2 + (L\rho)^2]^2} \quad \text{or} \quad A = \frac{L\rho R}{\sqrt{[(R - L\rho^2 CR)^2 + (L\rho)]^2}} \qquad ...(iii)$$

Dividing (*ii*) by (*i*),

$$\tan \theta = \frac{R - L\rho^2 CR}{L\rho} \quad \text{or} \quad \theta = \tan^{-1}\left\{\frac{R(1 - LC\rho^2)}{L\rho}\right\} \qquad ...(iv)$$

Hence $\qquad P = A(\cos \theta + i \sin \theta) = Ae^{i\theta}$

where A and θ are given by (*iii*) and (*iv*).

19.10 HYPERBOLIC FUNCTIONS

(1) Definitions: *If x be real or complex,*

(*i*) $\dfrac{e^x - e^{-x}}{2}$ *is defined as* **hyperbolic sine of x** and is written as **sinh x.**

(*ii*) $\dfrac{e^x + e^{-x}}{2}$ *is defined as* **hyperbolic cosine of x** *and is written as* **cosh x.**

Thus $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$

Also we define,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}; \ \coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}; \ \text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

(2) Properties

I. Periodic functions : sinh z and cosh z are periodic functions having imaginary period 2πi.

[$\because$ $\sinh(z + 2\pi i) = \sinh z$; $\cosh(z + 2\pi i) = \cosh z$]

II. Even and odd functions : cosh z is an even function while sinh z is an odd function

III. $\sinh 0 = 0$, $\cosh 0 = 1$, $\tanh 0 = 0$.

IV. **Relations between hyperbolic and circular functions.**

Since for all values of θ, $\qquad \sin \theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos \theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\therefore$ Putting $\theta = ix$, we have $\qquad \sin ix = \dfrac{e^{-x} - e^x}{2i} = -\dfrac{e^x - e^{-x}}{2i}$ $\qquad [\because \ e^{i\theta} = e^{i \cdot ix} = e^{-x}]$

$$= i^2 \frac{e^x - e^{-x}}{2i} = i \cdot \frac{e^x - e^{-x}}{2} = i \sinh x$$

and, therefore, $\qquad \cos ix = \dfrac{e^{-x} + e^x}{2} = \cosh x$

Thus $\qquad$ **sin ix = i sinh x** $\qquad ...(i)$

cos ix = cosh x $\qquad ...(ii)$

and $\therefore$ $\qquad$ **tan ix = i tanh x** $\qquad ...(iii)$

Cor. $\qquad$ **sinh ix = i sin x** $\qquad ...(iv)$

cosh ix = cos x $\qquad ...(v)$

tanh ix = i tan x $\qquad ...(vi)$

V. Formulae of hyperbolic functions

(a) Fundamental formulae

(1) $\cosh^2 x - \sinh^2 x = 1$ (2) $\text{sech}^2 x + \tanh^2 x = 1$ (3) $\coth^2 x - \text{cosech}^2 x = 1$.

(b) Addition formulae

(4) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$ (5) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(6) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(c) Functions of 2x.

(7) $\sinh 2x = 2 \sinh x \cosh x$

(8) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

(9) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

(d) Functions of 3x

(10) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(11) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (12) $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

(e) (13) $\sinh x + \sinh y = 2 \sinh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (14) $\sinh x - \sinh y = 2\cosh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$

(15) $\cosh x + \cosh y = 2\cosh \dfrac{x+y}{2} \cosh \dfrac{x-y}{2}$ (16) $\cosh x - \cosh y = 2 \sinh \dfrac{x+y}{2} \sinh \dfrac{x-y}{2}$.

Proofs. (1) Since, for all values of θ, we have $\cos^2\theta + \sin^2\theta = 1$.

$\therefore$ putting $\theta = ix$, we get $\cos^2 ix + \sin^2 ix = 1$ or $\cosh^2 x - \sinh^2 x = 1$

Otherwise : $\cosh^2 x - \sinh^2 x = \left(\dfrac{e^x + e^{-x}}{2}\right)^2 - \left(\dfrac{e^x - e^{-x}}{2}\right)^2 = \dfrac{1}{4}[e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2] = 1.$

Similarly we can establish the formulae (2) and (3).

(4) $\sinh(x+y) = (1/i)\sin i(x+y) = -i[\sin ix \cos iy + \cos ix \sin iy]$

$$= -i[i \sinh x \,.\, \cosh y + \cosh x \,.\, i \sinh y] = \sinh x \cosh y + \cosh x \sinh y.$$

Otherwise : $\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$$

Similarly we can establish the formulae (5) and (6).

(12) $$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

Putting $A = ix$, $\tan 3ix = \dfrac{3 \tan ix - \tan^3 ix}{1 - 3\tan^2 ix}$ or $i \tanh 3x = \dfrac{3(i \tanh x) - (i \tanh x)^3}{1 - 3(i \tanh x)^2}$

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

Similarly, we can establish the formulae (7) to (11).

(16) $\cos C - \cos D = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

Putting $C = ix$, and $D = iy$, $\cos ix - \cos iy = -2 \sin i \dfrac{x+y}{2} \sin i \dfrac{x-y}{2}$

or $$\cosh x - \cosh y = -2\left(i \sinh \frac{x+y}{2}\right)\left(i \sinh \frac{x-y}{2}\right) = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2}$$

Similarly, we can establish the formulae (13) to (15).

19.11 INVERSE HYPERBOLIC FUNCTIONS

(1) Definitions: If *sinh* $u = z$, then u is called the hyperbolic sine inverse of z and is written as $u = \sinh^{-1} z$. Similarly we define $\cosh^{-1} z$, $\tanh^{-1} z$, etc.

The inverse hyperbolic functions like other inverse functions are many-valued, but we shall consider only their principal values.

(2) *To show that (i)* $\sinh^{-1} z = \log [z + \sqrt{(z^2 + 1)}]$ *(Mumbai, 2009)*

(ii) $\cosh^{-1} z = \log [z + \sqrt{(z^2 - 1)}]$, *(iii)* $\tanh^{-1} z = \frac{1}{2} \log \frac{1+z}{1-z}$.

(*i*) Let $\sinh^{-1} z = u$, then $z = \sinh u = \frac{1}{2}(e^u - e^{-u})$

or $$2z = e^u - 1/e^u \quad \text{or} \quad e^{2u} - 2ze^u - 1 = 0$$

This being a quadratic in e^u, we have

$$e^u = \frac{2z \pm \sqrt{(4z^2 + 4)}}{2} = z \pm \sqrt{(z^2 + 1)}$$

$\therefore$ Taking the positive sign only, we have

$$e^u = z + \sqrt{(z^2 + 1)} \quad \text{or} \quad u = \log [z + \sqrt{(z^2 + 1)}]$$

Similarly we can establish (*ii*)

(*iii*) Let $\tanh^{-1} z = u$, then $z = \tanh u$

i.e., $$z = \frac{e^u - e^{-u}}{e^u + e^{-u}}.$$

Applying componendo and dividendo, we get $\frac{1+z}{1-z} = e^u/e^{-u} = e^{2u}$

or $$2u = \log \left(\frac{1+z}{1-z}\right) \text{ whence follows the result.}$$ *(P.T.U., 2005)*

Example 19.31. *If $u = \log \tan (\pi/4 + \theta/2)$, prove that*

(i) $\tanh u/2 = \tan \theta/2$ *(Mumbai, 2008 ; P.T.U., 2006 ; Madras, 2003)*

(ii) $\theta = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)$. *(Kurukshetra, 2006)*

Solution. We have $$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) \quad \text{or} \quad \frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan \theta/2}{1 - \tan \theta/2}$$

By componendo and dividendo, we get

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \tan \theta/2 \quad i.e., \quad \tanh \frac{u}{2} = \tan \frac{\theta}{2} \qquad ...(i)$$

or $$\frac{1}{i} \tan \frac{iu}{2} = \frac{1}{i} \tanh \frac{i\theta}{2} \quad \text{or} \quad \frac{i\theta}{2} = \tanh^{-1}\left(\tan \frac{iu}{2}\right) = \frac{1}{2} \log \frac{1 + \tan iu/2}{1 - \tan iu/2}$$

or $$\theta = \frac{1}{i} \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right). \qquad ...(ii)$$

Example 19.32. *Show that $\tanh^{-1} (\cos \theta) = \cosh^{-1} (\text{cosec } \theta)$.* *(Kurukshetra, 2005)*

Solution. Let $\tanh^{-1} (\cos \theta) = \phi$ so that $\cos \theta = \tanh \phi$

or $\tanh^2 \phi = \cos^2 \theta$ or $1 - \text{sech}^2 \phi = \cos^2 \theta$

or $\text{sech}^2 \phi = 1 - \cos^2 \theta = \sin^2 \theta$ or $\text{sech } \phi = \sin \theta$

or $\cosh \phi = \text{cosec } \theta$ or $\phi = \cosh^{-1} (\text{cosec } \theta)$.

Example 19.33. *Find tanh x, if 5 sinh x – cosh x = 5.* *(Mumbai, 2004)*

Solution. We have $5(\sinh x - 1) = \cosh x$

or $25(\sinh x - 1)^2 = \cosh^2 x = 1 + \sinh^2 x$

or $24 \sinh^2 x - 50 \sinh x + 24 = 0$ or $12 \sinh^2 x - 25 \sinh x + 12 = 0$

or $(3 \sinh x - 4)(4 \sinh x - 3) = 0$ whence $\sinh x = 4/3$ or $3/4$.

$\therefore$ $\cosh x = \sqrt{(1 + \sinh^2 x)} = 5/3$ or $-5/4$ [$\because$ $\cosh x = 5/4$ doesn't satisfy (*i*)]

Hence $\tanh x = \dfrac{4}{5}$ or $-\dfrac{3}{5}$.

PROBLEMS 19.5

1. Separate into real and imaginary parts
   (*i*) $\exp(z^2)$ where $z = x + iy$ (*ii*) $\exp(5 + i\pi/2)$ (*iii*) $\exp(5 + 3i)^2$.
2. From the definitions of $\sin z$ and $\cos z$, prove that
   (*i*) $\cos 2z = 2\cos^2 z - 1$ (*ii*) $\dfrac{\sin 2z}{1 - \cos 2z} = \cot z$ (*iii*) $\sin 3z = 3 \sin z - 4 \sin^3 z$.
3. Prove that $[\sin(\alpha - \theta) + e^{-i\alpha} \sin \theta]^n = \sin^{n-1} \alpha \, [\sin(\alpha - n\theta) + e^{-i\alpha} \sin n\theta]$
4. If $z = e^{i\theta}$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i \tan \theta$.
5. Eliminate z from $p \operatorname{cosech} z + q \operatorname{sech} z + r = 0$, $p' \operatorname{cosech} z + q' \operatorname{sech} z + r' = 0$.
6. If $y = \log \tan x$, show that $\sinh ny = \dfrac{1}{2}(\tan^n x - \cot^n x)$.
7. If $\tan y = \tan \alpha \tanh \beta$ and $\tan z = \cot \alpha \tanh \beta$, prove that $\tan(y + z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.
8. Prove that
   (*i*) $\cosh(\alpha + \beta) - \cosh(\alpha - \beta) = 2 \sinh \alpha \sinh \beta$
   (*ii*) $\sinh(\alpha + \beta) \cosh(\alpha - \beta) = \dfrac{1}{2}(\sinh 2\alpha + \sinh 2\beta)$.
9. Prove that (*i*) $(\cosh \theta \pm \sinh \theta)^n = \cosh n\theta \pm \sinh n\theta$; (*ii*) $\left(\dfrac{1 + \tanh \theta}{1 - \tanh \theta}\right)^3 = \cosh 6\theta + \sinh 6\theta$.
10. Express $\cosh^7 \theta$ in terms of hyperbolic cosines of multiples of θ.
11. If $\sin \theta = \tanh x$, prove that $\tan \theta = \sinh x$.
12. If $\tan x/2 = \tanh u/2$, prove that
    (*i*) $\tan x = \sinh u$ and $\cos x \cosh u = 1$; (*ii*) $u = \log_e \tan(\pi/4 + x/2)$.
13. If $\cosh x = \sec \theta$, prove that
    (*i*) $\tanh^2 x/2 = \tan^2 \theta/2$ (*ii*) $x = \log_e \tan(\pi/4 + \theta/2)$.
14. Show that $\tan^{-1} z = \dfrac{i}{2} \log \dfrac{i + z}{i - z}$.
15. Prove that
    (*i*) $\sinh^{-1} x = \cosh^{-1} \sqrt{(1 + x^2)} = \tanh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}} = \dfrac{1}{2} \operatorname{cosech}^{-1} \dfrac{1}{2x\sqrt{(1 + x^2)}}$
    (*ii*) $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{(1 - x^2)}}$.
16. Show that
    (*i*) $\sinh^{-1}(\tan \theta) = \log \tan(\pi/4 + \theta/2)$ (*ii*) $\operatorname{sech}^{-1}(\sin \theta) = \log \cot \theta/2$.
17. Solve the equation $7 \cosh x + 8 \sinh x = 1$ for real values of x. *(Mumbai, 2008)*
18. Find $\tanh x$ if $\sinh x - \cosh x = 5$.

19.12 REAL AND IMAGINARY PARTS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

(1) *To separate the real and imaginary parts of*

(i) sin (x + iy); (ii) cos (x + iy); (iii) tan (x + iy); (iv) cot (x + iy); (v) sec (x + iy); (vi) cosec (x + iy).

Proofs. (*i*) $\sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$.

Similarly, $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$

(*iii*) Let $\alpha + i\beta = \tan(x+iy)$ then $\alpha - i\beta = \tan(x-iy)$

Adding, $2\alpha = \tan(x+iy) + \tan(x-iy)$

i.e., $$\alpha = \frac{\sin(x+iy+x-iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{\sin 2x}{\cos 2x + \cos 2iy} = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \tan(x+iy) - \tan(x-iy)$

i.e., $$i\beta = \frac{\sin 2iy}{2\cos(x+iy)\cos(x-iy)} = \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

$\therefore$ $$\beta = \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

Similarly, $\cot(x+iy) = \dfrac{\sin 2x - i \sinh 2y}{\cosh 2y - \cos 2x}$.

(*v*) Let $\alpha + i\beta = \sec(x+iy)$ then $\alpha - i\beta = \sec(x-iy)$

Adding, $2\alpha = \sec(x+iy) + \sec(x-iy)$

i.e., $$\alpha = \frac{\cos(x-iy)+\cos(x+iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{2\cos x \cos iy}{\cos 2x + \cos 2iy} = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$

Subtracting, $2i\beta = \sec(x+iy) - \sec(x-iy)$

i.e., $$i\beta = \frac{\cos(x-iy)-\cos(x+iy)}{2\cos(x+iy)\cos(x-iy)} = \frac{2\sin x \sin iy}{\cos 2x + \cos 2iy} = \frac{2i \sin x \sinh y}{\cos 2x + \cosh 2y}$$

$\therefore$ $$\beta = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}$$

Similarly, $\operatorname{cosec}(x+iy) = 2\,\dfrac{\sin x \cosh y - i \cos x \sinh y}{\cosh 2y - \cos 2x}$.

(2) *To separate the real and imaginary parts of*

(i) sinh (x + iy); *(ii) cosh (x + iy);* *(iii) tanh (x + iy).*

Proofs. (*i*) $\sinh(x+iy) = (1/i)\sin i(x+iy) = (1/i)\sin(ix-y)$
$= (1/i)[\sin ix \cos y - \cos ix \sin y] = (1/i)[i \sinh x \cos y - \cosh x \sin y]$
$= \sinh x \cos y + i \cosh x \sin y$

Similarly, $\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y$.

(*iii*) If $\alpha + i\beta = \tanh(x+iy) = (1/i)\tan(ix-y)$

then $\alpha - i\beta = \tanh(x-iy) = (1/i)\tan(ix+y)$

Adding, $2\alpha = (1/i)[\tan(ix-y) + \tan(ix+y)]$

$$\alpha = \frac{\sin(ix-y+ix+y)}{i\,.\,2\cos(ix-y)\cos(ix+y)} = \frac{(1/i)\sin 2ix}{\cos 2ix + \cos 2y} = \frac{\sinh 2x}{\cosh 2x + \cos 2y}.$$

Subtracting, $2i\beta = (1/i)[\tan(ix-y) - \tan(ix+y)]$

i.e., $$i\beta = -\frac{\sin[(ix+y)-(ix-y)]}{i\,.\,2\cos(ix+y)\cos(ix-y)}$$

$\therefore$ $$\beta = \frac{\sin 2y}{\cos 2ix + \cos 2y} = \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$

Example 19.34. *If cosh (u + iv) = x + iy, prove that*

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1 \quad \text{(P.T.U., 2009 S)} \qquad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1. \quad \text{(Madras, 2000)}$$

Solution. Since $x + iy = \cosh(u + iv) = \cos(iu - v)$

$$= \cos iu \cos v + \sin iu \sin v = \cosh u \cos v + i \sinh u \sin v.$$

$\therefore$ equating real and imaginary parts, we get $x = \cosh u \cos v$; $y = \sinh u \sin v$

i.e.,
$$\frac{x}{\cosh u} = \cos v \quad \text{and} \quad \frac{y}{\sinh u} = \sin v$$

Squaring and adding, we get the first result.

Again
$$\frac{x}{\cos v} = \cosh u \quad \text{and} \quad \frac{v}{\sin v} = \sinh u.$$

$\therefore$ squaring and subtracting, we get the second result.

Example 19.35. *If $\tan(\theta + i\phi) = e^{i\alpha}$, show that*

$$\theta = (n + 1/2)\pi/2 \quad \textit{and} \quad \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$ *(S.V.T.U., 2007 ; Rohtak, 2005)*

Solution. Since $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha \quad \therefore \quad \tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$\therefore$
$$\tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$$

$$= \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta + i\phi)\tan(\theta - i\phi)} = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)} = \frac{2\cos\alpha}{0} \to \infty$$

i.e.,
$$2\theta = n\pi + \pi/2 \quad \text{or} \quad \theta = (n + 1/2)\pi/2$$

Also
$$\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)] = \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

or
$$i\tanh 2\phi = \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)} = i\sin\alpha \quad \text{or} \quad \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} = \frac{\sin\alpha}{1}$$

By componendo and dividendo, we get

$$\frac{e^{2\phi}}{e^{-2\phi}} = \frac{1 + \sin\alpha}{1 - \sin\alpha} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2 + 2\sin\alpha/2\cos\alpha/2}{\cos^2\alpha/2 + \sin^2\alpha/2 - 2\sin\alpha/2\cos\alpha/2}$$

or
$$e^{4\phi} = \frac{(\cos\alpha/2 + \sin\alpha/2)^2}{(\cos\alpha/2 - \sin\alpha/2)^2} = \left(\frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}\right)^2$$

or
$$e^{2\phi} = \frac{1 + \tan\alpha/2}{1 - \tan\alpha/2} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right). \quad \text{Hence } \phi = \frac{1}{2}\log\tan(\pi/4 + \alpha/2).$$

Example 19.36. *Separate $\tan^{-1}(x + iy)$ into real and imaginary parts.* *(S.V.T.U., 2009)*

Solution. Let $\alpha + i\beta = \tan^{-1}(x + iy)$. Then $\alpha - i\beta = \tan^{-1}(x - iy)$

Adding,
$$2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)^{*} = \tan^{-1}\frac{(x + iy) + (x - iy)}{1 - (x + iy)(x - iy)}$$

$\therefore$
$$\alpha = \frac{1}{2}\tan^{-1}\frac{2x}{1 - x^2 - y^2}$$

Subtracting,
$$2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \tan^{-1} i\frac{2y}{1 + x^2 + y^2} = i\tanh^{-1}\frac{2y}{1 + x^2 + y^2} \qquad [\because \tan^{-1} iz = i\tanh^{-1} z]$$

$\therefore$
$$\beta = \frac{1}{2}\tanh^{-1}\frac{2y}{1 + x^2 + y^2}.$$

Example 19.37. *Separate $\sin^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.*

---

* $\tan^{-1}A \pm \tan^{-1}B = \tan^{-1}\dfrac{A \pm B}{1 \mp AB}$

Solution. Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

Then $\cos\theta + i\sin\theta = \sin(x+iy) = \sin x\cosh y + i\cos x\sinh y$

$\therefore \quad \cos\theta = \sin x\cosh y \quad ...(i) \qquad \text{and} \qquad \sin\theta = \cos x\sinh y \quad ...(ii)$

Squaring and adding, we have

$$1 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = \sin^2 x(1+\sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$

or $$1 - \sin^2 x = \sinh^2 y, \quad i.e. \quad \cos^2 x = \sinh^2 y.$$

Hence from (*ii*), we have $\sin^2\theta = \cos^4 x$, *i.e.*, $\cos^2 x = \sin\theta$ because θ being a positive acute angle, $\sin\theta$ is positive.

As x is to be between $-\pi/2$ and $\pi/2$, therefore, we have

$$\cos x = +\sqrt{(\sin\theta)} \quad \text{or} \quad x = \cos^{-1}\sqrt{(\sin\theta)}$$

The relation (*ii*), then, gives $\sinh y = \sqrt{(\sin\theta)}$ so that $y = \log[\sqrt{(\sin\theta)} + \sqrt{(1+\sin\theta)}]$.

PROBLEMS 19.6

1. If $\sin(A+iB) = x+iy$, prove that
   (*i*) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$ (*ii*) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1$. (*P.T.U., 2010*)
2. If $\cos(\alpha + i\beta) = r(\cos\theta + i\sin\theta)$, prove that (*i*) $e^{2\beta} = \dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$ (*Kurukshetra, 2005 ; Madras, 2003*)
   (*ii*) $\beta = \dfrac{1}{2}\log\dfrac{\sin(\alpha-\theta)}{\sin(\alpha+\theta)}$. (*V.T.U., 2006*)
3. If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$, prove that
   (*i*) $\sin^2\theta = \pm\sin\alpha$ (*Madras, 2003*) (*ii*) $\cos 2\theta + \cosh 2\phi = 2$.
4. If $\tan(A+iB) = x+iy$, prove that
   (*i*) $x^2+y^2+2x\cot 2A = 1$. (*ii*) $x^2+y^2-2y\coth 2B + 1 = 0$. (*iii*) $x\sinh 2B = y\sin 2A$.
5. If $\tan(\theta+i\phi) = \tan\alpha + i\sec\alpha$, prove that $e^{2\phi} = \pm\cot\alpha/2$ and $2\theta = \left(n+\dfrac{1}{2}\right)\pi + \alpha$. (*Nagpur, 2009 ; S.V.T.U., 2008*)
6. If $\tan(x+iy) = \sin(u+iv)$, prove that $\dfrac{\sin 2x}{\sinh 2y} = \dfrac{\tan u}{\tan v}$. (*S.V.T.U., 2006*)
7. If $\text{cosec}(\pi/4 + ix) = u + iv$, prove that $(u^2+v^2)^2 = 2(u^2-v^2)$. (*Mumbai, 2009*)
8. If $x = 2\cos\alpha\cosh\beta$, $y = 2\sin\alpha\sinh\beta$, prove that $\sec(\alpha+i\beta) + \sec(\alpha-i\beta) = \dfrac{4x}{x^2+y^2}$.
9. If $a+ib = \tanh(v + i\pi/4)$, prove that $a^2+b^2 = 1$.
10. Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $a+ib$. (*Mumbai, 2009*)
    Hence show that $\tan^{-1}(e^{i\theta}) = \dfrac{n\pi}{2} + \dfrac{\pi}{4} - \dfrac{i}{2}\log\tan\left(\dfrac{\pi}{4} - \dfrac{\theta}{2}\right)$.
11. Separate $\cos^{-1}(\cos\theta + i\sin\theta)$ into real and imaginary parts, where θ is a positive acute angle.
12. If $\sin^{-1}(u+iv) = \alpha + i\beta$, prove that $\sin^2\alpha$ and $\cosh^2\beta$ are the roots of the equation
    $$x^2 - x(1+u^2+v^2) + u^2 = 0.$$
13. If $\cos^{-1}(x+iy) = \alpha + i\beta$, show that
    (*i*) $x^2\sec^2\alpha - y^2\,\text{cosec}^2\alpha = 1$, (*ii*) $x^2\,\text{sech}^2\beta + y^2\,\text{cosech}^2\beta = 1$.
14. Prove that (*i*) $\sin^{-1}(ix) = 2n\pi + i\log(\sqrt{1+x^2}+x)$ (*ii*) $\sin^{-1}(\text{cosec}\,\theta) = \pi/2 + i\log\cot\theta/2$.

19.13 LOGARITHMIC FUNCTION OF A COMPLEX VARIABLE

(1) *Definition. If $z(=x+iy)$ and $w(=u+iv)$ be so related that $e^w = z$, then w is said to be a logarithm of z to the base e and is written as $w = \log_e z$.* ...(*i*)

Also $$e^{w+2in\pi} = e^w \cdot e^{2in\pi} = z \qquad [\because \; e^{2in\pi} = 1]$$

$\therefore$ $$\log z = w + 2in\pi \qquad ...(ii)$$

i.e., the logarithm of a complex number has an infinite number of values and is, therefore, a multi-valued function.

The general value of the logarithm of z is written as $Log\, z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log z$. This principal value is obtained by taking $n = 0$ in $Log\, z$.

Thus from (*i*) and (*ii*), $Log\,(x + iy) = 2in\pi + \log (x + iy)$.

Obs. 1. If $y = 0$, then $Log\, x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are imaginary.

2. *We know that the logarithm of a negative quantity has no real value. But we can now evaluate this.*

e.g. $\log_e (-2) = \log_e 2(-1) = \log_e 2 + \log_e (-1) = \log_e 2 + i\pi$ $\quad [\because \ -1 = \cos \pi + i \sin \pi = e^{i\pi}]$

$= 0.6931 + i\,(3.1416)$.

(2) *Real and imaginary parts of Log* $(x + iy)$.

$$Log\,(x + iy) = 2in\pi + \log (x + iy)$$

$$= 2in\pi + \log [r (\cos \theta + i \sin \theta)] \qquad \left\{\begin{array}{l}\text{Put } x = r\cos\theta,\ y = r\sin\theta \text{ so that} \\ r = \sqrt{(x^2 + y^2)} \text{ and } \theta = \tan^{-1}(y/x)\end{array}\right.$$

$$= 2in\pi + \log (re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = \log \sqrt{(x^2 + y^2)} + i\,\{2n\pi + \tan^{-1}(y/x)\}$$

(3) *Real and imaginary parts of* $(\alpha + i\beta)^{x+iy}$

$$(\alpha + i\beta)^{x+iy} = e^{(x+iy)\,Log\,(\alpha + i\beta)} = e^{(x+iy)\,[2in\pi + \log (\alpha + i\beta)]} \qquad \left\{\begin{array}{l}\text{Put } \alpha = r\cos\theta, \beta = r\sin\theta \text{ so that} \\ r = \sqrt{(\alpha^2 + \beta^2)} \text{ and } \theta = \tan^{-1}\beta/\alpha\end{array}\right.$$

$$= e^{(x+iy)[2in\pi + \log re^{i\theta}]} = e^{(x+iy)\,[\log r + i\,(2n\pi + \theta)]}$$

$$= e^{A + iB} = e^A (\cos B + i \sin B).$$

where $A = x \log r - y\,(2n\pi + \theta)$ and $B = y \log r + x\,(2n\pi + \theta)$.

$\therefore$ the required real part $= e^A \cos B$ and the imaginary part $= e^A \sin B$.

Example 19.38. *Find the general value of log* $(-i)$.

Solution. $Log\,(-i) = 2in\pi + \log [0 + i(-1)]$ $\qquad \left\{\begin{array}{l}\text{Put } 0 = r\cos\theta, -1 = r\sin\theta 2 \\ \text{so that } r = 1 \text{ and } \theta = -\pi/2\end{array}\right.$

$$= 2in\pi + \log [r (\cos \theta + i \sin \theta)] = 2in\pi + \log (re^{i\theta})$$

$$= 2in\pi + \log r + i\theta = 2in\pi + \log 1 + i\,(-\pi/2) = i\left(2n - \frac{1}{2}\right)\pi.$$

Example 19.39. *Prove that* (i) $i^i = e^{-(4n+1)\pi/2}$ *and* $Log\, i^i = -\left(2n + \frac{1}{2}\right)\pi$.

(*ii*) $(\sqrt{i})^{\sqrt{i}} = e^{-\alpha}$ *cis* α *where* $\alpha = \pi/4\sqrt{2}$. (*Mumbai, 2008*)

Solution. (*i*) By definition, we have

$$i^i = e^{i\,Log\, i} = e^{i\,(2in\pi + \log i)} = e^{-2n\pi + i \log [\exp (i\pi/2)]} \qquad [\because \ i = \text{cis}\,\pi/2 = \exp (i\pi/2)]$$

$$= e^{-2n\pi + i(i\pi/2)} = e^{-(2n + 1/2)\pi}$$

Taking logarithms, we get (*ii*)

(*ii*) $\qquad (\sqrt{i})^{\sqrt{i}} = e^{\sqrt{i}\log\sqrt{i}}$

Now $\qquad \sqrt{i}\ \log\sqrt{i} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin \pi/2\right)^{1/2} \log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$= \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\log (e^{i\pi/2}) = \frac{1}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{i\pi}{2}$$

$$= \frac{i\pi}{4}\left(\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = -\frac{\pi}{4\sqrt{2}} + i\,\frac{\pi}{4\sqrt{2}}$$

Hence $\quad (\sqrt{i})^{\sqrt{i}} = e^{-\alpha + i\alpha}$ where $\alpha = \pi/4\sqrt{2}$

$$= e^{-\alpha} \cdot e^{i\alpha} = e^{-\alpha}(\cos \alpha + i \sin \alpha).$$

Example 19.40. *If $(a + ib)^p = m^{x+iy}$, prove that one of the values of y/x is*

$$2 \tan^{-1}(b/a) \div \log(a^2 + b^2).$$

Solution. Taking logarithms, $(a + ib)^p = m^{x+iy}$ gives $p \log(a + ib) = (x + iy) \log m$

or $$p\left(\frac{1}{2}\log(a^2 + b^2) + i \tan^{-1}\frac{b}{a}\right) = x \log m + iy \log m$$

Equating real and imaginary parts from both sides, we get

$$\frac{p}{2}\log(a^2 + b^2) = x \log m \quad ...(i), \qquad p \tan^{-1}\frac{b}{a} = y \log m \quad ...(ii)$$

Division of (*ii*) by (*i*) gives

$$y/x = 2\tan^{-1}(b/a)/\log(a^2 + b^2).$$

Example 19.41. *If $i^{i^{\cdot^{\cdot^{\infty}}}} = A + iB$, prove that $\tan \pi A/2 = B/A$ and $A^2 + B^2 = e^{-\pi B}$.* (*S.V.T.U., 2006 S*)

Solution. $\quad i^{i^{\cdot^{\cdot^{\infty}}}} = A + iB \quad$ *i.e.* $\quad i^{A+iB} = A + iB$

or $$A + iB = e^{(A+iB)\log i} = e^{(A+iB)\log(\cos \pi/2 + i \sin \pi/2)}$$

$$= \exp[(A + iB)\log(e^{i\pi/2})] = e^{(A+iB)(i\pi/2)}$$

$$= e^{-B\pi/2} \cdot e^{i\pi A/2} = e^{-B\pi/2}\left(\cos\frac{\pi A}{2} + i \sin\frac{\pi A}{2}\right)$$

Equating real and imaginary parts, we get

$$A = e^{-B\pi/2}\cos\frac{\pi A}{2} \quad ...(i) \qquad B = e^{-B\pi/2}\sin\frac{\pi A}{2} \quad ...(ii)$$

Division of (*ii*) by (*i*) gives $B/A = \tan \pi A/2$

Squaring and adding (*i*) and (*ii*), $A^2 + B^2 = e^{-B\pi}$.

Example 19.42. *Prove that $\log\left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1}\left(\frac{b}{a}\right)$. Hence evaluate $\cos\left[i \log\left(\frac{a+ib}{a-ib}\right)\right]$.* (*P.T.U., 2006*)

Solution. Putting $a = r\cos\theta$, $b = r \sin \theta$ so that $\theta = \tan^{-1} b/a$, we have

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i \sin\theta)}{r(\cos\theta - i\sin\theta)} = \log(e^{i\theta} \div e^{-i\theta})$$

$$= \log e^{2i\theta} = 2i\theta = 2i\tan^{-1} b/a.$$

Thus $$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \cos[i(2i\theta)] = \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-(b/a)^2}{1+(b/a)^2} = \frac{a^2-b^2}{a^2+b^2}.$$

Example 19.43. *Separate into real and imaginary parts $\log \sin(x + iy)$.*

Solution. $$\log\sin(x + iy) = \log(\sin x \cos iy + \cos x \sin iy)$$

$$= \log(\sin x \cosh y + i \cos x \sinh y) = \log r(\cos\theta + i \sin\theta),$$

where $\quad r\cos\theta = \sin x \cosh y$ and $r \sin\theta = \cos x \sinh y$,

so that $\quad r = \sqrt{(\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y)}$

$$= \sqrt{\frac{1-\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1+\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2}} = \sqrt{\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right]}$$

and $\quad \theta = \tan^{-1}(\cot x \tanh y)$.

Thus $\log\sin(x + iy) = \log(re^{i\theta}) = \log r + i\theta$

$$= \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right] + i\tan^{-1}(\cot x \tanh y).$$

Example 19.44. *Find all the roots of the equation*

(i) $\sin z = \cosh 4$ *(ii)* $\sinh z = i$.

Solution. *(i)* $\sin z = \cosh 4 = \cos 4i = \sin (\pi/2 - 4i)$

$\therefore$ $z = n\pi + (-1)^n (\pi/2 - 4i)$ $\left\{\because \text{ If } \sin\theta = \sin\alpha \text{ then } \theta = n\pi + (-1)^n \alpha\right.$

(ii) $$i = \sinh z = \frac{e^z - e^{-z}}{2}$$

or $e^{2z} - 2ie^z - 1 = 0$, *i.e.* $(e^z - i)^2 = 0$ *i.e.*, $e^z = i$

or $$z = Log\, i = 2in\pi + \log i = 2in\pi + \log e^{i\pi/2} = 2in\pi + i\pi/2 = i\left(2n + \frac{1}{2}\right)\pi.$$

PROBLEMS 19.7

1. Find the general value of
(i) $\log (6 + 8i)$ *(Rohtak, 2006)* *(ii)* $\log (-1)$. *(J.N.T.U., 2003)*
2. Show that *(i)* $\log (1 + i \tan \alpha) = \log (\sec \alpha) + i\alpha$, where α is an acute angle.
(ii) $\text{Log}_e \dfrac{3-i}{3+i} = 2i\left(n\pi - \tan^{-1}\dfrac{1}{3}\right)$.
3. If $(a_1 + ib_1)(a_2 + ib_2)\ldots\ldots(a_n + ib_n) = A + iB$, prove that
(i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2)\ldots\ldots(a_n^2 + b_n^2) = A^2 + B^2$
(ii) $\tan^{-1}\dfrac{b_1}{a_1} + \tan^{-1}\dfrac{b_2}{a_2} + \ldots\ldots + \tan^{-1}\dfrac{b_n}{a_n} = \tan^{-1}\dfrac{B}{A}$.
4. Find the modulus and argument of *(i)* $(1 - i)^{1+i}$. *(P.T.U., 2010)* *(ii)* $i^{\log(1+i)}$
5. If $i^{\alpha + i\beta} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$. *(Kurukshetra, 2005)*
6. Prove that $\log \left\{\dfrac{\sin (x + iy)}{\sin (x - iy)}\right\} = 2i \tan^{-1} (\cot x \tanh y)$. *(Mumbai, 2007)*
7. Prove that $\tan \left[i \log \left(\dfrac{a - ib}{a + ib}\right)\right] = \dfrac{2ab}{a^2 - b^2}$.
8. If $\tan \log (x + iy) = a + ib$ where $a^2 + b^2 \neq 1$, show that $\tan \log (x^2 + y^2) = \dfrac{2a}{1 - a^2 - b^2}$.
9. If $\sin^{-1} (x + iy) = \log (A + iB)$, show that $\dfrac{x^2}{\sin^2 u} - \dfrac{y^2}{\cos^2 u} = 1$, where $A^2 + B^2 = e^{2u}$.
10. Separate into real and imaginary parts $\log \cos (x + iy)$.
11. Find all the roots of the equation, *(i)* $\cos z = 2$, *(ii)* $\tanh z + 2 = 0$.

19.14 SUMMATION OF SERIES – 'C + iS' METHOD

This is the most general method and is applied to find the sum of a series of the form

$$a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + \ldots$$

or $$a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta) + \ldots$$

Procedure. *(i)* Put the given series $= S$ (or C) according as it is a series of sines (or cosines).

Then write C (or S) = a similar series of cosines (or sines).

e.g., If $S = a_0 \sin \alpha + a_1 \sin (\alpha + \beta) + a_2 \sin (\alpha + 2\beta) + \ldots$

then $C = a_0 \cos \alpha + a_1 \cos (\alpha + \beta) + a_2 \cos (\alpha + 2\beta)\ldots$

(ii) Multiply the series of sines by i and add to the series of cosines, so that

$$C + iS = a_0 [\cos \alpha + i \sin \alpha] + a_1 [\cos (\alpha + \beta) + i \sin (\alpha + \beta)] + \ldots$$
$$= a_0 e^{i\alpha} + a_1 e^{i(\alpha+\beta)} + a_2 e^{i(\alpha+2\beta)} + \ldots$$

(*iii*) Sum up this last series using any of the following standard series :

(1) Exponential series *i.e.*, $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty = e^x$

(2) Sine, cosine, sinh or cosh series

i.e., $$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \infty = \sin x, \qquad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \infty = \cos x$$
$$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \infty = \sinh x, \qquad 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \infty = \cosh x$$

(3) Logarithmic series

i.e., $$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \infty = \log(1 + x), \qquad -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right) = \log(1 - x)$$

(4) Gregory's series

i.e., $$x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \infty = \tan^{-1} x, \qquad x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \infty = \tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x}$$

(5) Binomial series

i.e., $$1 + nx + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \dots \infty = (1 + x)^n$$
$$1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty = (1 + x)^{-n}$$
$$1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty = (1 - x)^{-n}$$

(6) Geometric series

i.e., $$a + ar + ar^2 + \dots \text{ to } n \text{ terms} = a\,\frac{1 - r^n}{1 - r}, \quad a + ar + ar^2 + \dots \infty = \frac{a}{1 - r}, |r| < 1.$$

(*iv*) Finally express the sum thus obtained in the form $A + iB$ so that by equating the real and imaginary parts, we get $C = A$ and $S = B$.

Series depending on exponential series

Example 19.45. *Sum the series* $\sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \dots \infty$.

Solution. Let $S = \sin \alpha + x \sin(\alpha + \beta) + \frac{x^2}{2!} \sin(\alpha + 2\beta) + \dots \infty$

and $$C = \cos \alpha + x \cos(\alpha + \beta) + \frac{x^2}{2!} \cos(\alpha + 2\beta) + \dots \infty$$

$\therefore$ $$C + iS = [\cos \alpha + i \sin \alpha] + x[\cos(\alpha + \beta) + i \sin(\alpha + \beta)] + \frac{x^2}{2!}[\cos(\alpha + 2\beta) + i \sin(\alpha + 2\beta)] + \dots\dots \infty$$
$$= e^{i\alpha} + xe^{i(\alpha+\beta)} + \frac{x^2}{2!} \cdot e^{i(\alpha+2\beta)} + \dots \infty = e^{i\alpha}\left[1 + \frac{xe^{i\beta}}{1!} + \frac{x^2 e^{2i\beta}}{2!} + \dots \infty\right]$$
$$= e^{i\alpha} \cdot e^{xe^{i\beta}} = e^{i\alpha} e^{x(\cos\beta + i \sin\beta)} = e^{x\cos\beta + i(\alpha + x\sin\beta)} = e^{x\cos\beta} e^{i(\alpha + x\sin\beta)}$$
$$= e^{x\cos\beta}[\cos(\alpha + x\sin\beta) + i \sin(\alpha + x\sin\beta)]$$

Equating imaginary parts from both sides, we have $S = e^{x\cos\beta} \sin(\alpha + x \sin\beta)$.

Series depending on logarithmic series

Example 19.46. *Sum the series*

$$\sin^2\theta - \frac{1}{2}\sin 2\theta \sin^2\theta + \frac{1}{3}\sin 3\theta \sin^3\theta - \frac{1}{4}\sin 4\theta \sin^4\theta + \dots \infty.$$

(P.T.U., 2010 ; V.T.U., 2006 S)

Solution. Let $S = \sin\theta \,.\, \sin\theta - \dfrac{1}{2}\sin 2\theta \,.\, \sin^2\theta + \dfrac{1}{3}\sin 3\theta \,.\, \sin^3\theta - \ldots \infty$

and $C = \cos\theta \,.\, \sin\theta - \dfrac{1}{2}\cos 2\theta \,.\, \sin^2\theta + \dfrac{1}{3}\cos 3\theta \,.\, \sin^3\theta - \ldots \infty$

$$\therefore \quad C + iS = e^{i\theta}\sin\theta - \frac{e^{2i\theta}\sin^2\theta}{2} + \frac{e^{3i\theta}\sin^3\theta}{3} - \ldots \infty$$

$$= \log(1 + e^{i\theta}\sin\theta) = \log[1 + (\cos\theta + i\sin\theta)\sin\theta]$$

$$= \log[1 + \cos\theta\sin\theta + i\sin^2\theta] \; [\text{Put } 1 + \cos\theta\sin\theta = r\cos\alpha \,;\, \sin^2\theta = r\sin\alpha] \quad \ldots(i)$$

$$= \log r(\cos\alpha + i\sin\alpha) = \log re^{i\alpha} = \log r + i\alpha$$

Equating imaginary parts, we have $S = \alpha = \tan^{-1}\left(\dfrac{\sin^2\theta}{1 + \cos\theta\sin\theta}\right)$. [from (*i*)]

Series depending on binomial series

Example 19.47. *Find the sum to infinity of the series*

$$1 - \frac{1}{2}\cos\theta + \frac{1.3}{2.4}\cos 2\theta - \frac{1.3.5}{2.4.6}\cos 3\theta + \ldots \; (-\pi < \theta < \pi). \qquad (S.V.T.U., 2009)$$

Solution. Let $C = 1 - \dfrac{1}{2}\cos\theta + \dfrac{1.3}{2.4}\cos 2\theta - \dfrac{1.3.5}{2.4.6}\cos 3\theta + \ldots \infty$

and $S = 0 - \dfrac{1}{2}\sin\theta + \dfrac{1.3}{2.4}\sin 2\theta - \dfrac{1.3.5}{2.4.6}\sin 3\theta + \ldots \infty$

$$\therefore \quad C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1.3}{2.4}e^{2i\theta} - \frac{1.3.5}{2.4.6}e^{3i\theta} - \ldots$$

$$= 1 + \left(-\frac{1}{2}\right)e^{i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{1.2}e^{2i\theta} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{1.2.3}e^{3i\theta} + \ldots$$

$$= (1 + e^{i\theta})^{-1/2} = (1 + \cos\theta + i\sin\theta)^{-1/2} = \left(2\cos^2\frac{\theta}{2} + i\,.\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2}$$

$$= \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right).$$

Equating real parts, we have $C = (2\cos\theta/2)^{-1/2}\cos\theta/4$.

PROBLEMS 19.8

Sum the following series :

1. $\cos\theta + \sin\theta\cos 2\theta + \dfrac{\sin^2\theta}{1.2}\cos 3\theta + \ldots \infty$. (*P.T.U., 2005*)
2. $\sin\alpha - \dfrac{\sin(\alpha + 2\beta)}{2!} + \dfrac{\sin(\alpha + 4\beta)}{4!} - \ldots \infty$.
3. $x\sin\theta - \dfrac{1}{2}x^2\sin 2\theta + \dfrac{1}{3}x^3\sin 3\theta - \ldots \infty$. (*Kurukshetra, 2005*)
4. $\cos\theta - \dfrac{1}{2}\cos 2\theta + \dfrac{1}{3}\cos 3\theta \ldots \infty$. (*S.V.T.U., 2006*)
5. $e^{\alpha}\cos\beta - \dfrac{e^{3\alpha}}{3}\cos 3\beta + \dfrac{e^{5\alpha}}{5}\cos 5\beta - \ldots \infty$.
6. $c\sin\alpha + \dfrac{c^3}{3}\sin 3\alpha + \dfrac{c^5}{5}\sin 5\alpha + \ldots \infty$.
7. $1 - \dfrac{1}{2}\cos 2\theta + \dfrac{1.3}{2.4}\cos 4\theta - \dfrac{1.3.5}{2.4.6}\cos 6\theta + \ldots \infty$. (*Kurukshetra, 2006*)
8. $n\sin\alpha + \dfrac{n(n+1)}{1.2}\sin 2\alpha + \dfrac{n(n+1)(n+2)}{1.2.3}\sin 3\alpha + \ldots \infty$.
9. $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta)$ (*P.T.U., 2009 S*)
10. $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots$ to n terms. (*Kurukshetra, 2006*)
11. $\sin\alpha\cos\alpha + \sin^2\alpha\cos 2\alpha + \sin^3\alpha\cos 3\alpha + \ldots \infty$.
12. $1 + x\cos\theta + x^2\cos 2\theta + \ldots + x^{n-1}\cos(n-1)\theta$.

19.15 APPROXIMATIONS AND LIMITS

Example 19.48. *If* $\frac{\sin\theta}{\theta} = \frac{599}{600}$, *find an approximate value of θ in radians.*

Solution. Since $\frac{\sin\theta}{\theta} = 1 - \frac{1}{600}$ which is nearly equal to 1. ∴ θ must be very small.

We know that $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$

$$\therefore \quad \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{5!}$$

Omitting θ^4 and higher powers, we have

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = 1 - \frac{1}{600} \quad \text{or} \quad \theta^2 = \frac{1}{100}. \text{ Hence } \theta = 0.1 \text{ radians.}$$

Example 19.49. *Solve approximately* $\sin\left(\frac{\pi}{6} + \theta\right) = 0.51$.

Solution. Since 0.51 is nearly equal to 1/2, which is the value of sin π/6, so θ must be very small.

$$\therefore \quad \sin\left(\frac{\pi}{6} + \theta\right) = \sin\frac{\pi}{6}\cos\theta + \cos\frac{\pi}{6}\sin\theta = \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \ldots\right) + \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\theta, \text{ omitting } \theta^2 \text{ and higher powers of } \theta.$$

Hence the given equation becomes,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}\theta = 0.51 \quad \text{or} \quad \theta = \frac{1}{50\sqrt{3}}$$

or
$$\theta = \frac{1}{50\sqrt{3}} \text{ radian} = \frac{\sqrt{3}}{150} \times 57.29 \text{ degrees nearly} = 39.7'.$$

PROBLEMS 19.9

1. Given $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that θ is 1°58′ nearly.
2. If $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, find an approximate value of θ in radians. *(Madras, 2003)*
3. If $\cos\theta = \frac{1681}{1682}$, find θ approximately.
4. Solve approximately the equation $\cos\left(\frac{\pi}{3} + \theta\right) = 0.49$.

19.16 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 19.10

Choose the correct answer or fill up the blanks in each of the following problems :

1. If $x + iy = \sqrt{2} + 3i$, then $x^2 + y$ is

   (*a*) 7 (*b*) 5 (*c*) 13 (*d*) $\sqrt{2} + 3$

2. The real part of $(\sin x + i\cos x)^5$ is

   (*a*) − cos 5*x* (*b*) − sin 5*x* (*c*) sin 5*x* (*d*) cos 5*x*.

3. The number $(i)^i$ is
   (a) a purely imaginary number (b) an irrational number
   (c) a rational number (d) an integer.
4. The relation $|3 - z| + |3 + z| = 5$ represents
   (a) a circle (b) a parabola (c) an ellipse (d) a hyperbola.
5. z is a complex number with $|z| = 1$ and $\arg(z) = 3\pi/4$. The value of z is
   (a) $(1 + i)/\sqrt{2}$ (b) $(-1 + i)/\sqrt{2}$ (c) $(1 - i)/\sqrt{2}$ (d) $(-1 - i)/\sqrt{2}$.
6. If $f(z) = e^{2z}$, then the imaginary part of $f(z)$ is
   (a) $e^y \sin x$ (b) $e^x \cos y$ (c) $e^{2x} \cos 2y$ (d) $e^{2x} \sin 2y$.
7. Expansion of $\sin^m \theta \cos^n \theta$ is a series of sines of multiples of θ when m is ......
8. Expansion of $\cos 6\theta$ in terms of $\cos \theta$ is ......
9. If $f(z) = 3\bar{z}$, then the value of $f(z)$ at $z = 2 + 4i$ is ......
10. If $x = \cos \theta + i \sin \theta$, then $x^n - 1/x^n =$ ......
11. Imaginary part of $(2 + i3)/(3 - i4)$ is ......
12. Real part of $\cosh(x + iy)$ is ......
13. If $\dfrac{\sin \theta}{\theta} = \dfrac{2165}{2166}$, then $\theta =$ ...... approximately.
14. If $\tan x/2 = \tanh y/2$, then $\cos x \cosh y =$ ......
15. Imaginary part of $\sin \bar{z}$ is ......
16. Modulus of $(\sqrt{i})^{\sqrt{i}} =$ ....
17. If $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$, then $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos$ (..............)
18. $\log(-1) =$ ....
19. $(i)^i$ is purely real or imaginary
20. If $\sin \theta = \tanh \phi$, then $\tan \theta =$ ....
21. Imaginary part of $\tan(\theta + i\phi) =$ ....
22. $\cos 5\alpha =$ (.........) $\cos^5 \alpha$ + (......) $\cos^3 \alpha$ + (......) $\cos \alpha$.
23. Cube roots of unity form .... triangle.
24. If $|z_1 + z_2| = |z_1 - z_2|$ then $\operatorname{amp}(z_1) - \operatorname{amp}(z_2)$ is ....
25. If $-3 + ix^2y$ and $x^2 + y + 4i$ represent conjugate complex number then $x =$ .... and $y =$ ....
26. If $\left|\dfrac{z - a}{z - b}\right| = k \ (\neq 1)$, then the locus of z is ....
27. $(-i)^{-i}$ is purely real. (True or False)
28. The statements $\operatorname{Re} z > 0$ and $|z - 1| < |z + 1|$ are equivalent. (*Mumbai, 2007*) (True or False)
29. Hyperbolic functions are periodic. (True or False)
30. nth roots of unity form a G.P. (True or False)
31. $\sin ix = -i \sinh x$. (*Mumbai, 2008*) (True or False)
32. If the sum and product of two complex numbers are real, then the two numbers must be either real or conjugate. (*Mumbai, 2008*) (True or False)
33. The modulus of the sum of two complex numbers $\geq$ to the sum of their moduli. (True or False)

CHAPTER 20

Calculus of Complex Functions

20.1 INTRODUCTION

In the previous chapter, we have dealt with some elementary complex functions—the exponential, logarithmic, circular and hyperbolic functions, evaluated at specific complex values. These functions are useful in the study of fluid mechanics, thermodynamics and electric fields. It, therefore, seems desirable to study the calculus of such functions.

20.2 (1) LIMIT OF A COMPLEX FUNCTION

A *function* $w = f(z)$ *is said to tend to* **limit** l *as* z *approaches a point* z_0, *if for every real* ϵ, *we can find a positive real* δ *such that*

$$|f(z) - l| < \epsilon \quad \text{for} \quad |z - z_0| < \delta$$

i.e., for every $z \neq z_0$ in the δ–disc (dotted) of z-plane, $f(z)$ has a value lying in the ϵ-disc of w-plane (Fig. 20.1). In symbols, we write $\underset{z \to z_0}{\text{Lt}} f(z) = l$.

This definition of limit though similar to that in ordinary calculus, is quite different for in real calculus x approaches x_0 only along the line whereas here *z approaches z_0 from any direction in the z-plane.*

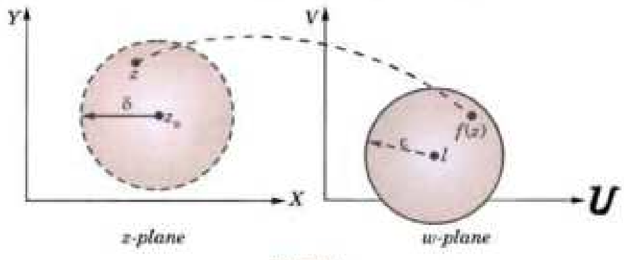


Fig. 20.1

(2) Continuity of f(z). *A function $w = f(z)$ is said to be* **continuous** *at $z = z_0$, if*

$$\underset{z \to z_0}{Lt}\ f(z) = f(z_0).$$

Further $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Also if $w = f(z) = u(x, y) + iv(x, y)$ is continuous at $z = z_0$, then $u(x, y)$ and $v(x, y)$ are also continuous at $z = z_0$, *i.e.*, at $x = x_0$ and $y = y_0$. Conversely if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0), then $f(z)$ will be continuous at $z = z_0$. [cf. § 5.1 (3)].

20.3 (1) DERIVATIVE OF f(z)

Let $w = f(z)$ be a single-valued function of the variable $z = x + iy$. Then *the derivative of $w = f(z)$ is defined to be*

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z + \delta z) - f(z)}{\delta z},$$

provided the limit exists and has the same value for all the different ways in which δz approaches zero.

Suppose $P(z)$ is fixed and $Q(z + \delta z)$ is a neighbouring point (Fig. 20.2). The point Q may approach P along any straight or curved path in the given region, *i.e.*, δz may tend to zero in any manner and dw/dz may not exist. It, therefore, becomes a fundamental problem to determine the necessary and sufficient conditions for dw/dz to exist. The fact is settled by the following theorem.

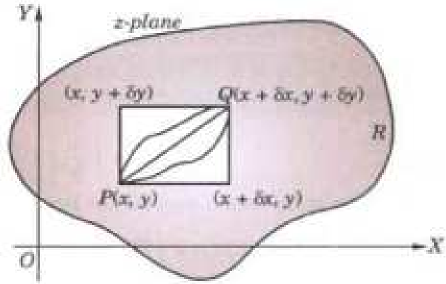


Fig. 20.2

(2) Theorem. *The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R, are*

(*i*) $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ *are continuous functions of x and y in R;*

(*ii*) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The relations (ii) are known as **Cauchy-Riemann* equations** or briefly *C-R* equations.

(*a*) *Condition is necessary.*

If $f(z)$ possesses a unique derivative at $P(z)$, then

$$f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

$$= \underset{\substack{\delta x \to 0 \\ \delta y \to 0}}{\text{Lt}} \frac{[u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)] - [u(x, y) + iv(x, y)]}{\delta x + i\delta y}$$

Since δz can approach zero in any manner, we can first assume δz to be wholly real and then wholly imaginary. When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore \quad f'(z) = \underset{\delta x \to 0}{\text{Lt}} \left(\frac{u(x + \delta x, y) - u(x, y)}{\delta x} + i \frac{v(x + \delta x, y) - v(x, y)}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{...(1)}$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore \quad f'(z) = \underset{\delta y \to 0}{\text{Lt}} \left(\frac{u(x, y + \delta y) - u(x, y)}{i\delta y} + i \frac{v(x, y + \delta y) - v(x, y)}{i\delta y} \right) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad \text{...(2)}$$

Now the existence of $f'(z)$ requires the equality of (1) and (2).

* Named after *Cauchy* (p. 144) and the German mathematician *Bernhard Riemann* (1826—1866) who along with *Weierstrass* (p. 390) laid the foundations of complex analysis. Riemann introduced the concept of integration and made basic contributions to number theory and mathematical analysis. He developed the Riemannian geometry which formed the mathematical base for Einstein's relativity theory.

$$\therefore \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad ...(3)$$

Thus the necessary conditions for the existence of the derivative of f (z) is that the C-R equations should be satisfied. (V.T.U., 2011 S)

(b) *Condition is sufficient.* Suppose $f(z)$ is a single-valued function possessing partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region and the C-R equations (3) are satisfied.

Then by Taylor's theorem for functions of two variables (p. 220)

$$f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y)$$

$$= u(x, y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + ... + i\left[v(x, y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + ...\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

or $$f(z + \delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y.$$

Now using the C-R equation (3), replace $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $-\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x}$ respectively.

Then $$f(z + \delta z) - f(z) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\delta y = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta x + \left[i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right]i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y) = \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\therefore \qquad f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

which by (1) or (2) proves the sufficiency of conditions.

20.4 ANALYTIC FUNCTIONS

A function f (z) which is single-valued and possesses a unique derivative with respect to z at all points of a region R, is called an **analytic function** *of z in that region. An analytic function is also called a regular function* or an *holomorphic function.*

A function which is analytic everywhere in the complex plane, is known as an **entire function.** As derivative of a polynomial exists at every point, a polynomial of any degree is an *entire function.*

A point at which an analytic function ceases to possess a derivative is called a **singular point** *of the function.*

Thus if u and v are real single-valued functions of x and y such that $\delta u/\delta x$, $\delta u/\delta y$, $\delta v/\delta x$, $\delta v/\delta y$ are continuous throughout a region R, then the Cauchy-Riemann equations

$$\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = \frac{\partial \mathbf{v}}{\partial \mathbf{y}} \text{ and } \frac{\partial \mathbf{u}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{v}}{\partial \mathbf{x}} \qquad ...(1)$$

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in R. The derivative of $f(z)$ is then, given by (1) of p. 664 or (2) of p. 665.

The real and imaginary parts of an analytic function are called *conjugate functions.* The relation between two conjugate functions is given by C-R equation (1).

Example 20.1. *If w = log z, find dw / dz and determine where w is non-analytic.*

(U.P.T.U., 2005 ; J.N.T.U., 2005)

Solution. We have $w = u + iv = \log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1} y/x$ [By (2), p. 665]

so that $u = \frac{1}{2}\log(x^2 + y^2),\ v = \tan^{-1} y/x.$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}.$$

Since the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous except at (0, 0). Hence w is analytic everywhere except at $z = 0$.

$$\therefore \quad \frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{x + iy} = \frac{1}{z}\ (z \neq 0).$$

Obs. The definition of the derivative of a function of complex variable is identical in form to that of the derivative of a function of real variable. Hence the rules of differentiation for complex functions are the same as those of real calculus. *Thus* **if, a complex function is once known to be analytic, it can be differentiated just in the ordinary way.**

Example 20.2. *If $f(z)$ is an analytic function with constant modulus, show that $f(z)$ is constant.*
(U.P.T.U., 2008 ; Mumbai, 2005 S ; Madras 2003 ; Bhopal, 2002 S)

Solution. If $f(z) = u + iv$ is an analytic function, then

$|f(z)| = \sqrt{(u^2 + v^2)}$ is constant $= c$ (say) or $u^2 + v^2 = c^2$ ...(i)

Differentiating (i) partially w.r.t. x and y, we get

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0\ ; \qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or $$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad ...(ii) \qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad ...(iii)$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ by C-R equations,

$\therefore$ (iii) becomes $-u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0$ ...(iv)

Squaring and adding (ii) and (iv), we obtain

$$u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 = 0$$

or $$(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] = 0 \text{ or } \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \quad [\because\ u^2 + v^2 = c^2 \neq 0] \quad ...(v)$$

Now $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore \quad |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = 0 \qquad \text{[By (v)]}$$

or $f'(z) = 0.$ or $f(z) =$ constant.

Example 20.3. *Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C.R equations are satisfied thereof.* *(A.M.I.E.T.E., 2005 S ; Osmania, 2003)*

Solution. If $f(z) = \sqrt{|xy|} = u(x, y) + iv(x, y)$, then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin, we have

$$\frac{\partial u}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{0 - 0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{0 - 0}{y} = 0$$

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C.R. equations are satisfied at the origin.

However $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z - 0} = \underset{z \to 0}{\text{Lt}} \frac{\sqrt{|xy|} - 0}{x + iy}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{\sqrt{|mx^2|} - 0}{x(1 + im)}, \text{ when } z \to 0 \text{ along the line } y = mx$$

$$= \frac{\sqrt{|m|}}{1 + im} \text{ which is not unique.}$$

$\therefore$ $f'(0)$ does not exist. Hence $f(z)$ is not analytic at the origin.

Example 20.4. *Prove that the function $f(z)$ defined by*

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \ (z \neq 0), f(0) = 0$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

(S.V.T.U., 2009 ; V.T.U., 2001)

Solution. $$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{x \to 0 \\ y \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{y \to 0}{\text{Lt}} \frac{-y^3(1-i)}{y^2} = \underset{y \to 0}{\text{Lt}} [-y(1-i)] = 0$$

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to 0 \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i)}{x^2} = \underset{x \to 0}{\text{Lt}} [x(1+i)] = 0$$

Also $f(0) = 0$ (given).

Thus $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z \to 0}{\text{Lt}} f(z) = \underset{\substack{y \to mx \\ x \to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}$$

$$= \underset{x \to 0}{\text{Lt}} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2} = \underset{x \to 0}{\text{Lt}} \frac{x[1 + i - m^3(1-i)]}{1 + m^2} = 0$$

Hence $\underset{z \to 0}{\text{Lt}} f(z) = f(0)$, in whatever manner $z \to 0$. $\therefore$ $f(z)$ is continuous at the origin.

Now $$f(z) = \frac{x^3 - y^3}{x^2 + y^2} + i\frac{x^3 + y^3}{x^2 + y^2} = u(x, y) + iv(x, y).$$

Also $u(0, 0) = 0$, and $v(0, 0) = 0$ $\qquad [\because f(0) = 0]$

$$\therefore \qquad \left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x \to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x \to 0}{\text{Lt}} \frac{x}{x} = 1$$

and $$\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y \to 0}{\text{Lt}} \frac{v(0, y) - v(0, 0)}{y} = \underset{y \to 0}{\text{Lt}} \frac{y}{y} = 1.$$

Hence at (0, 0), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus the *C-R* equations are satisfied at the origin.

But $$f'(0) = \underset{z \to 0}{\text{Lt}} \frac{f(z) - f(0)}{z} = \underset{z \to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z \to 0}{\text{Lt}} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}.$$

If $z \to 0$ along the path $y = mx$, then $f'(0) = \frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)(1 + im)}$

which assumes different values as m varies. So $f'(z)$ is not unique at (0, 0) *i.e.*, $f'(0)$ does not exist. Thus $f(z)$ is not analytic at the origin even though it is continuous and satisfies the *C-R* equations thereat.

Example 20.5. *Show that polar form of Cauchy-Riemann equations are*

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$ (U.P.T.U., 2008 ; V.T.U., 2006)

Deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$ (Bhopal, 2009 ; Kurukshetra, 2005)

Solution. If (r, θ) be the coordinates of a point whose cartesian coordinates are (x, y), then $z = x + iy = re^{i\theta}$.

$$\therefore \quad u + iv = f(z) = f(re^{i\theta})$$

where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we have

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

and $$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ...(i) \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad ...(ii)$$

Differentiating (i) partially w.r.t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad ...(iii)$$

Differentiating (ii) partially w.r.t. θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta} \quad ...(iv)$$

Thus using (i), (ii) and (iv)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right) = 0 \qquad \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right]$$

20.5 (1) HARMONIC FUNCTIONS

If $f(z) = u + iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i.e., $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ...(1) \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad ...(2)$$

Differentiating (1) with respect to x and (2) with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ...(3) \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \quad ...(4)$$

Adding (3) and (4) and assuming that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0. \qquad ...(5)$$

Similarly, by differentiating (1) with respect to y and (2) with respect to x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \qquad ...(6)$$

Thus both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as **harmonic functions** and their theory is called **potential theory.** *(Rohtak, 2005)*

(2) Orthogonal system. Consider the two families of curves

$$u(x, y) = c_1 \quad ...(7) \qquad \text{and} \qquad v(x, y) = c_2 \qquad ...(8)$$

Differentiating (7), we get $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}=0$

or
$$\frac{dy}{dx}=-\frac{\partial u/\partial x}{\partial u/\partial y}=\frac{\partial v/\partial y}{\partial v/\partial x}=m_1 \text{ (say)} \qquad \text{[By (1) and (2)]}$$

Similarly (8) gives $\qquad \dfrac{dy}{dx}=-\dfrac{\partial v/\partial x}{\partial v/\partial y}=m_2$ (say)

$\therefore \qquad m_1m_2=-1$, *i.e.*, (7) and (8) form an orthogonal system.

Hence *every analytic function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.* *(U.P.T.U., 2009)*

20.6 APPLICATIONS TO FLOW PROBLEMS

As the real and imaginary parts of an analytic function are the solutions of the Laplace's equation in two variables, the conjugate functions provide solutions to a number of field and flow problems.

As an illustration, consider the irrotational motion of an incompressible fluid in two dimensions. Assuming the flow to be in planes parallel to the xy-plane, the velocity $\mathbf{V}$ of a fluid particle can be expressed as

$$\mathbf{V}=v_x\mathbf{I}+v_y\mathbf{J} \qquad ...(1)$$

Since the motion is irrotational, therefore, by § 6.18 (1), there exist a scalar function $\phi(x, y)$ such that

$$\mathbf{V}=\nabla\phi(x,y)=\frac{\partial\phi}{\partial x}\mathbf{I}+\frac{\partial\phi}{\partial y}\mathbf{J} \qquad ...(2)$$

[The function $\phi(x, y)$ is called the *velocity potential* and the curves $\phi(x, y) = c$ are known as *equipotential lines.*]

Thus from (1) and (2), $v_x=\dfrac{\partial\phi}{\partial x}$ and $v_y=\dfrac{\partial\phi}{\partial y}$ $\qquad ...(3)$

Also the fluid being incompressible div $\mathbf{V} = 0$ [by § 8.7 (1)] *i.e.*, $\dfrac{\partial v_x}{\partial x}+\dfrac{\partial v_y}{\partial y}=0.$

Substituting the values of v_x and v_y from (3), we get $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}=0$

which shows that the velocity potential ϕ is *harmonic*. It follows that there must exist a conjugate harmonic function $\psi(x, y)$ such that $w(z)=\phi(x,y)+i\psi(x,y)$ $\qquad ...(4)$
is analytic.

Also the slope at any point of the curve $\psi(x, y) = c'$ is given by

$$\frac{dy}{dx}=-\frac{\partial\psi/\partial x}{\partial\psi/\partial y}=\frac{\partial\phi/\partial y}{\partial\phi/\partial x} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$=v_y/v_x \qquad \text{[By (3)]}$$

This shows that the velocity of the fluid particle is along the tangent to the curve $\psi(x, y) = c'$, *i.e.* the particle moves along this curve. Such curves are known as *stream lines* and $\psi(x, y)$ is called the *stream function*. Also the equipotential lines $\phi(x, y) = c$ and the stream lines $\psi(x, y) = c'$ cut orthogonally.

From (4), $$\frac{\partial w}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$ [By *C-R* equations]

$$= v_x - iv_y$$ [By (3)]

$\therefore$ The magnitude of the fluid velocity $= \sqrt{(v_x^2 + v_y^2)} = |\, dw/dz \,|$.

Thus the flow pattern is fully represented by the function $w(z)$ which is known as the **complex potential.**

Similarly the complex potential $w(z)$ can be taken to represent any other type of 2-dimensional steady flow. In electrostatics and gravitational fields, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are *equipotential lines* and *lines of force*. In heat flow problems, the curves $\phi(x, y) = c$ and $\psi(x, y) = c'$ are known as *isothermals* and *heat flow lines* respectively.

Given $\phi(x, y)$, we can find $\psi(x, y)$ and *vice-versa*.

Example 20.6. *If $w = \phi + i\psi$ respresents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2 + y}$, determine the function ϕ.* (V.T.U., 2011 ; Mumbai, 2008 ; Bhopal, 2002 S)

Solution. It is readily verified that ψ satisfies the Laplace's equation.

$\therefore$ ϕ and ψ must satisfy the Cauchy-Riemann equations :

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad ...(i) \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad ...(ii)$$

$\therefore$ by (*i*), $$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left[x^2 - y^2 + \frac{x}{x^2 + y^2}\right] = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

Integrating w.r.t. x, we get $\phi = -2xy + \dfrac{y}{x^2 + y^2} + \eta(y)$ where $\eta(y)$ is an arbitrary function of y.

$\therefore$ (*ii*) gives $-2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2} + \eta'(y) = -2x + \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

whence $\eta'(y) = 0$, *i.e.*, $\eta(y) = c$, an arbitrary constant.

Thus $$\phi = -2xy + \frac{y}{x^2 + y^2} + c$$

Otherwise (Milne-Thomson's method*) :

We have $$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x} = \left[-2y - \frac{2xy}{(x^2 + y^2)^2}\right] + i\left[2x + \frac{y^2 - x^2}{(x^2 + y^2)^2}\right]$$

By Milne-Thomson's method, we express dw/dz in terms of z, on replacing x by z and y by 0.

$\therefore$ $$\frac{dw}{dz} = i\left(2z - \frac{1}{z^2}\right)$$

Integrating w.r.t. z, we get $w = i\,(z^2 + 1/z) + A$ where A is a complex constant.

---

* Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z + \bar{z})$$

$\therefore$ $$f(z) = \phi(x, y) + i\psi(x, y) \quad ...(1)$$

$$= \phi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right] + i\psi\left[\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right]$$

Now considering this as a formal identity in the two independent variables z, $\bar{z}$ and putting $\bar{z} = z$, we get

$$f(z) = \phi(z, 0) + i\psi(z, 0) \quad ...(2)$$

$\therefore$ (2) is the same as (1), if we replace x by z and y by 0.

Thus **to express any function in terms of z, replace x by z and y by 0.** *This provides an elegant method of finding $f(z)$ when its real part or the imaginary part is given. It is due to Milne-Thomson.*

Hence $$\phi = R\left[i\left(z^2 + \frac{1}{z}\right) + A\right] = -2xy + \frac{y}{x^2 + y^2} + c.$$

Example 20.7. *Find the analytic function, whose real part is $\sin 2x/(\cosh 2y - \cos 2x)$.*
(J.N.T.U., 2005 ; Anna, 2003)

Solution. Let $f(z) = u + iv$, where $u = \sin 2x/(\cosh 2y - \cos 2x)$

$$\therefore \quad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{[By } C\text{-}R \text{ equations]}$$

$$= \frac{(\cosh 2y - \cos 2x)\, 2\cos 2x - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} - i\,\frac{\sin 2x(-2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i\,\frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By Milne-Thomson's method, we express $f'(z)$ in terms of z by putting $x = z$ and $y = 0$.

$$\therefore \quad f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + i(0) = \frac{-2}{1 - \cos 2z} = \frac{-2}{2\sin^2 z} = -\operatorname{cosec}^2 z$$

Integrating w.r.t. z, we get $f(z) = \cot z + ic$, taking the constant of integration as imaginary since u does not contain any constant.

Example 20.8. *Determine the analytic function $f(z) = u + iv$, if $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$ and $f(\pi/2) = 0$.*
(A.M.I.E.T.E., 2005 ; Osmania, 2003)

Solution. We have $$u - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

$$\therefore \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \cosh y)^2} \qquad \ldots(i)$$

and $$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{(\cos x - \cosh y)\, e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$$

or $$-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = \frac{(\sin x + \cos x)\sinh y + e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \qquad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$2\frac{\partial u}{\partial x} = \frac{\begin{array}{l}(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 \\ \qquad - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)\end{array}}{2(\cos x - \cosh y)^2}$$

Adding (i) and (ii), we have

$$-2\frac{\partial v}{\partial x} = \frac{\begin{array}{l}(\sin x - \cos x)\cosh y + (\sin x + \cos x)\sinh y + 1 \\ \qquad + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)\end{array}}{2(\cos x - \cosh y)^2}$$

Thus $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1 - \cos z}{2(1 - \cos z)^2} \qquad \text{[Putting } x = z \text{ and } y = 0]$$

$$= \frac{1}{2(1 - \cos z)} = \frac{1}{4\sin^2 z/2} = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2} \quad \text{or} \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f(\pi/2) = 0$, $\quad 0 = -\dfrac{1}{2}\cot \pi/4 + c$, whence $c = \dfrac{1}{2}$

Hence $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 20.9. *Find the conjugate harmonic of $v(r, \theta) = r^2 \cos 2\theta - r \cos \theta + 2$. Show that v is harmonic.* *(Marathwada, 2008)*

Solution. Let $f(z) = u + v$. Using C-R equations in polar coordinates (Ex. 20.5),

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta} = -2r^2 \sin 2\theta + r \sin \theta \quad ...(i)$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r} = 2r \cos 2\theta - \cos \theta \quad ...(ii)$$

$\therefore$ (*i*) gives, $\quad \dfrac{\partial u}{\partial r} = -2r \sin 2\theta + \sin \theta$

Integrating w.r.t., r

$$u = -r^2 \sin 2\theta + r \sin \theta + \phi(\theta) \qquad \text{where } \phi(\theta) \text{ is an arbitrary function.}$$

$$\therefore \quad \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta) \quad ...(iii)$$

From (*ii*) and (*iii*), we get

$$-2r^2 \cos 2\theta + r \cos \theta = \frac{\partial u}{\partial \theta} = -2r^2 \cos 2\theta + r \cos \theta + \phi'(\theta)$$

$$\therefore \quad \phi'(\theta) = 0 \quad \text{or} \quad \phi(\theta) = c$$

Thus $\quad u = -r^2 \sin 2\theta + r \sin \theta + c$ is the conjugate harmonic of v.

Now v will be harmonic if it satisfies the Laplace equation $\dfrac{\partial^2 v}{\partial r^2} + \dfrac{1}{r} + \dfrac{\partial v}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 v}{\partial \theta^2} = 0$

From (*i*), $\dfrac{\partial^2 v}{\partial \theta^2} = -4r^2 \cos 2\theta + r \cos \theta$. From (*ii*), $\dfrac{\partial^2 v}{\partial r^2} = 2 \cos 2\theta$

$$\therefore \quad \frac{\partial t^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 2\cos 2\theta + \frac{1}{r}(2r \cos 2\theta - \cos \theta) + \frac{1}{r^2}(-4r^2 \cos 2\theta + r \cos \theta)$$

$$= 4 \cos 2\theta - \frac{1}{r}\cos \theta - 4 \cos 2\theta + \frac{1}{r}\cos \theta = 0$$

Hence v is harmonic.

Example 20.10. (*a*) *Find the orthogonal trajectories of the family of curves*

$$x^4 + y^4 - 6x^2y^2 = \text{constant}$$

(*b*) *Show that the curves $r^n = \alpha \sec n\theta$ and $r^n = \beta \operatorname{cosec} n\theta$ cut orthogonally.*

(Mumbai, 2005 ; J.N.T.U., 2003)

Solution. (*a*) Take $u(x, y) = x^4 + y^4 - 6x^2y^2$. Then the family of curves $v(x, y)$ = constant will be the required trajectories if $f(z) = u + iv$ is analytic.

Now $\quad \dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2, \; \dfrac{\partial u}{\partial y} = 4y^3 - 12x^2y$

$$\therefore \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

Integrating, $\quad v = 4x^3y - 4xy^3 + c(x)$

Differentiating partially w.r.t. x

$$12x^2y - 4y^3 + \frac{dc(x)}{dx} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -4y^3 + 12x^2y$$

$$\therefore \quad \frac{dc(x)}{dx} = 0 \quad \text{or} \quad c = \text{constant}$$

Thus the required orthogonal trajectories are v = constant or $x^3y - xy^3$ = constant.

(*b*) Writing $u(r, \theta) = r^n \cos n\theta = \alpha$ and $v(r, \theta) = r^n \sin n\theta = \beta$,

we have $u(r, \theta) + iv(r, \theta) = \alpha + i\beta = r^n(\cos n\theta + i \sin n\theta) = r^n \cdot e^{in\theta} = (re^{i\theta})^n = z^n$

This is an analytic function.

Thus $f(z) = u + iv$, gives the curves $u = \alpha$ and $v = \beta$ which cut orthogonally.

Example 20.11. *Two concentric circular cylinders of radii r_1, r_2 ($r_1 < r_2$) are kept at potentials ϕ_1 and ϕ_2 respectively. Using complex function $w = a \log z + c$, prove that the capacitance per unit length of the capacitor formed by them is $2\pi\lambda/\log(r_2/r_1)$ where λ is the dielectric constant of the medium.*

Solution. We have $\phi + i\psi = a \log(re^{i\theta}) + c$ where $z = x + iy = re^{i\theta}$

$\therefore$ $\phi = a \log r + c$, and $\psi = a\theta$

so that $\phi_1 = a \log r_1 + c$, $\phi_2 = a \log r_2 + c$

Thus the potential difference $= \phi_2 - \phi_1 = a(\log r_2 - \log r_1)$

Also the total charge (or flux) $= \int_0^{2\pi} d\psi = \int_0^{2\pi} a\, d\theta = 2\pi a$.

The capacitance being the charge required to maintain a unit potential difference ; the capacitance without dielectric

$$= \frac{\text{charge}}{\text{potential difference}} = \frac{2\pi a}{a(\log r_2 - \log r_1)} = \frac{2\pi}{\log(r_2/r_1)}.$$

A medium of dielectric constant λ increases the potential difference to λ times that in vacuum for the same charge. Thus the capacitance with dielectric $= 2\pi\lambda/\log(r_2/r_1)$.

Example 20.12. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2. \qquad \text{(J.N.T.U., 2006 ; Kottayam, 2005)}$$

or

$$\nabla^2\,|f(z)|^2 = 4\,|f'(z)|^2 \qquad \text{(Madras, 2006)}$$

Solution. Let $f(z) = u(x, y) + iv(x, y)$ so that $|f(z)|^2 = u^2 + v^2 = \phi(x, y)$, (say).

$$\therefore \quad \frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \text{ and } \frac{\partial^2 \phi}{\partial x^2} = 2\left\{u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

Similarly,
$$\frac{\partial^2 \phi}{\partial y^2} = 2\left\{u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

Adding, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left\{u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right\} + 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} \quad ...(i)$$

Since u, v have to satisfy Cauchy-Riemann equations and the Laplace's equation.

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 ; \left(\frac{\partial u}{\partial y}\right)^2 = \left(-\frac{\partial v}{\partial x}\right)^2 \text{ and } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

Thus (*i*) takes the form $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 4\left\{\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2\right\}$

Hence
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2 \quad \text{or} \quad \nabla^2\,|f(z)|^2 = 4\,|f'(z)|^2.$$

PROBLEMS 20.1

1. If $f(z) = \begin{cases} x^3y(y - ix)/(x^6 + y^2), & z \neq 0 \\ 0 & , z = 0 \end{cases}$ prove that $|f(z) - f(0)|/z \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ along the curve $y = ax^3$.

2. Show that (*a*) $f(z) = xy + iy$ is everywhere continuous but is not analytic. *(Osmania, 2003 S)*
   (*b*) $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane. *(J.N.T.U., 2003)*
3. If $f(z) = u + iv$ is analytic, then show that $|f'(z)|^2 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$. *(Mumbai, 2007)*
4. Find the constants a, b, c, d and e if $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic. *(Mumbai, 2008)*
5. Show that z^3 is analytic. Hence find its derivative. *(V.T.U., 2010 S)*
6. Determine which of the following functions are analytic :
   (*i*) $2xy + i(x^2 - y^2)$ (*ii*) $(x - iy)/(x^2 + y^2)$ (*iii*) $\cosh z$.
7. (*a*) Determine p such that the function $f(z) = \dfrac{1}{2}\log_e(x^2 + y^2) + i\tan^{-1}(px/y)$ be an analytic function. *(Mumbai, 2007 ; J.N.T.U., 2003)*
   (*b*) Show that $u = \dfrac{1}{2}\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate function. *(U.P.T.U., 2010)*
8. Show that each of the following functions is not analytic at any point :
   (*i*) $\bar{z}$ *(J.N.T.U., 2003)* (*ii*) $|z|^2$.
9. Show that $u + iv = (x - iy)/(x - iy + a)$ where $a \neq 0$, is not an analytic function of $z = x + iy$ whereas $u - iv$ is such a function.
10. Show that $f(z) = \begin{cases} xy^2(x + iy)/(x^2 + y^4), & z \neq 0 \\ 0, & z = 0 \end{cases}$
    is not analytic at $z = 0$, although *C-R* equations are satisfied at the origin. *(J.N.T.U., 2003)*
11. Verify if $f(z) = \dfrac{xy^2(x + iy)}{x^2 + y^4}, z \neq 0; f(0) = 0$ is analytic or not. *(U.P.T.U., 2008)*
12. Examine the nature of the function $f(z) = \dfrac{x^2y^5(x + iy)}{x^4 + y^{10}}, z \neq 0 ; f(0) = 0$. *(Rohtak, 2004)*
13. For the function $f(z)$ defined by $f(z)^2 = (\bar{z})^2/z, z \neq 0; f(0) = 0$, show that the C-R equations are satisfied at (0, 0), but $f(z)$ is not differentiable at (0, 0). *(P.T.U., 2010)*
14. Determine the analytic function whose real part is
    (*i*) $x^3 - 3xy^2 + 3x^2 - 3y^2$ *(Bhopal, 2009)* (*ii*) $\cos x \cosh y$ *(Rohtak, 2004)*
    (*iii*) $y/(x^2 + y^2)$ (*iv*) $y + e^x \cos y$ *(S.V.T.U., 2008 ; V.T.U., 2006)*
    (*v*) $e^{-x}(x \sin y - y \cos y)$ *(U.P.T.U., 2008)*
    (*vi*) $e^{2x}(x \cos 2y - y \sin 2y)$ *(V.T.U., 2008 S : Mumbai, 2005 ; Kottayam, 2005)*
    (*vii*) $x \sin x \cosh y - y \cos x \sinh y$ *(V.T.U., 2006)*
    (*viii*) $e^x[(x^2 - y^2)\cos y - 2xy \sin y]$. *(V.T.U., 2010 S ; Rohtak, 2005)*
15. Find the regular function whose imaginary part is
    (*i*) $(x - y)/(x^2 + y^2)$ (*ii*) $-\sin x \sinh y$ (*iii*) $e^x \sin y$
    (*iv*) $e^{-x}(x \sin y - y \cos y)$ (*v*) $e^{-x}(x \cos y + y \sin y)$ *(U.P.T.U., 2009)* (*vi*) $\dfrac{2 \sin x \sin y}{\cos 2x + \cosh 2y}$. *(Mumbai, 2006)*
16. Find the analytic function $z = u + iv$, if
    (*i*) $u - v = (x - y)(x^2 + 4xy + y^2)$ *(Mumbai, 2008 ; V.T.U., 2007 ; W.B.T.U., 2005)*
    (*ii*) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}}$ when $f\left(\dfrac{\pi}{2}\right) = 0$ *(Mumbai, 2007)*
    (*iii*) $u + v = \dfrac{2 \sin 2x}{e^{2y} - e^{-2y} - 2\cos 2x}$. *(P.T.U., 2002)*
17. An electrostatic field in the xy-plane is given by the potential function $\phi = 3x^2y - y^3$, find the stream function.
18. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function.
19. Prove that $u = x^2 - y^2$ and $v = \dfrac{y}{x^2 + y^2}$ are harmonic functions of (x, y) but are not harmonic conjugates. *(U.P.T.U., 2004 S)*

20. Show that the function $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the conjugate function v and express $u + iv$ as an analytic function of z. *(Bhopal, 2007)*
21. For $w = \exp(z^2)$, find u and v, and prove that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ where c_1 and c_2 are constants, cut orthogonally. *(J.N.T.U., 2003)*
22. Find the orthogonal trajectories of the family of curves
    (*i*) $x^3y - xy^3 = c$ *(Mumbai, 2007)* (*ii*) $e^x \cos y - xy = c$ *(Mumbai, 2008)* (*iii*) $r^2 \cos 2\theta = c$.
23. In a two dimensional fluid flow, the stream function ψ is given, find the velocity potential ϕ :
    (*i*) $\psi = -y/(x^2 + y^2)$ (*ii*) $\psi = \tan^{-1}(y/x)$.
24. Find the analytic function $f(z) = u + iv$, given
    (*i*) $u = a(1 + \cos\theta)$ (*ii*) $v = (r - 1/r)\sin\theta, r \neq 0$.
25. If $f(z)$ is an analytic function of z, show that

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2.$$ *(U.P.T.U., 2009 ; V.T.U., 2008 S ; P.T.U., 2005)*

26. If $f(z)$ is an analytic function of z, prove that
    (*i*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0$ *(Madras, 2000 S)* (*ii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|R\, f(z)|^2 = 2\,|f'(z)|^2$
    (*iii*) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2\,|f'(z)|^2\,|f(z)|^{p-2}$. *(Kerala, 2005)*
27. Prove that $\psi = \log[(x - 1)^2 + (y - 2)^2]$ is harmonic in every region which does not include the point (1, 2). Find a function ϕ such that $\phi + i\psi$ is an analytic function of the complex variable $z = x + iy$. Express $\phi + i\psi$ as a function of z.

20.7 GEOMETRICAL REPRESENTATION OF w = f(z)

To find the geometrical representation of a function of a complex variable, it requires a departure from the usual practice of cartesian plotting, where we associate a curve to a real function $y = f(x)$.

In the complex domain, the function $w = f(z)$

i.e., $$u + iv = f(x + iy) \qquad ...(1)$$

involves four real variables x, y, u, v. Hence a four dimensional region is required to plot (1) in the cartesian fashion. As it is not possible to have 4-dimensional graph papers, we make use of two complex planes, one for the variable $z = x + iy$, and the other for the variable $w = u + iv$. If the point z describes some curve C in the z-plane, the point w will move along a corresponding curve C' in the w-plane, since to each point (x, y), there corresponds a point (u, v) (Fig. 20.3). We then, say that *a curve C in the z-plane is mapped into the corresponding curve C′ in the w-plane by the function w = f(z) which defines a* **mapping** *or* **transformation** *of the z-plane into the w-plane.*

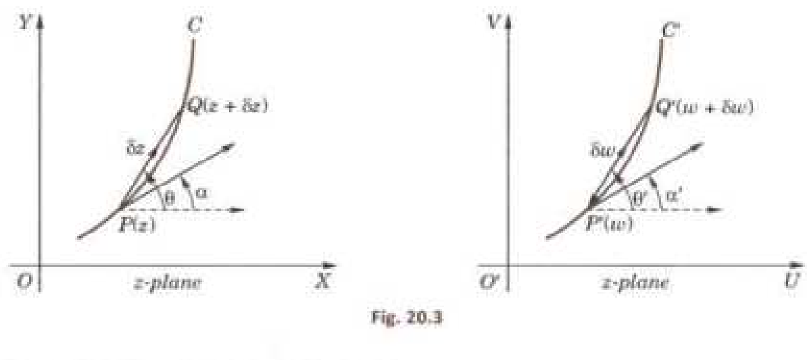


Fig. 20.3

20.8 SOME STANDARD TRANSFORMATIONS

(1) Translation. $w = z + c$, where c is a complex constant.

If $z = x + iy$, $c = c_1 + ic_2$ and $w = u + iv$, then the transformation becomes $u + iv = x + iy + c_1 + ic_2$ whence $u = x + c_1$ and $v = y + c_2$, *i.e.* the point $P(x, y)$ in the z-plane is mapped onto the point $P'(x + c_1, y + c_2)$ in the

w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector c. Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size.

In particular, this transformation changes circles into circles.

(2) Magnification and rotation. w = cz, where c is a complex constant.

If $$c = \rho e^{i\alpha},\ z = re^{i\theta} \text{ and } w = Re^{i\phi}, \text{ then}$$

$$Re^{i\phi} = \rho e^{i\alpha} \cdot re^{i\theta} = \rho r e^{i(\theta+\alpha)}$$

whence $R = \rho r$ and $\phi = \theta + \alpha$, *i.e.* the point $P(r, \theta)$ in the z-plane is mapped onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence the transformation consists of magnification (or contraction) of the radius vector of P by $\rho = |c|$ and its rotation through an $\angle\alpha = \text{amp}(c)$. Accordingly any figure in the z-plane is transformed into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

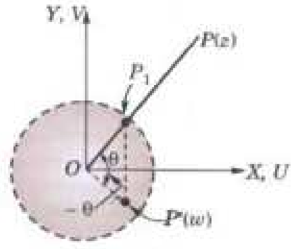


Fig. 20.4

(3) Inversion and reflection. w = 1/z.

Here it is convenient to think the w-plane as superposed on z-plane (Fig. 20.4).

If $z = re^{i\theta}$ and $w = Re^{i\phi}$, then $Re^{i\phi} = \dfrac{1}{r}e^{-i\theta}$

whence $R = 1/r$ and $\phi = -\theta$. Thus, if P be (r, θ) and P_1 be $(1/r, \theta)$, *i.e.* P_1 is the inverse* of P w.r.t. the unit circle with centre O, then the reflection P' of P_1 in the real axis represents $w = 1/z$.

Hence this transformation is an inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Obs. 1. Clearly the function $w = 1/z$ maps the interior of the unit circle $|z| = 1$ onto the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ onto the interior of $|w| = 1$. In particular, the origin $z = 0$ corresponds to the improper point $w = \infty$, called the *point at infinity* and the image of the improper point $z = \infty$ is the origin $w = 0$.

2. *This transformation maps a circle onto a circle or to a straight line if the former goes through the origin.*

To prove this, we write $z = 1/w$ as $x + iy = \dfrac{1}{u + iv} = \dfrac{u - iv}{u^2 + v^2}$

so that $$x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2}. \quad ...(1)$$

Now the general equation of any circle in the z-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad ...(2)$$

which on substituting from (1), becomes $\dfrac{u^2}{(u^2+v^2)^2} + \dfrac{v^2}{(u^2+v^2)^2} + 2g\dfrac{u}{u^2+v^2} + 2f\dfrac{-v}{u^2+v^2} + c = 0$

or $$c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \quad ...(3)$$

This is the equation of a circle in the w-plane. If $c = 0$, the circle (2) passes through the origin and its image, *i.e.*, (3) reduces to a straight line. Hence the result.

Regarding a straight line as the limiting form of a circle with infinite radius, we conclude that the *transformation $w = 1/z$ always maps a circle into a circle.*

(4) Bilinear transformation. The transformation

$$w = \frac{az + b}{cz + d} \quad ...(1)$$

where a, b, c and d are complex constants and $ad - bc \neq 0$ is known as the **bilinear transformation.**** The condition $ad - bc \neq 0$ ensures that $dw/dz \neq 0$, *i.e.*, the transformation is conformal. If $ad - bc = 0$ every point of the z-plane is a *critical point*.

The inverse mapping of (1) is

$$z = \frac{-dw + b}{cw - a} \quad ...(2)$$

which is also a bilinear transformation.

* The inverse of a point A w.r.t. a circle with centre O and radius k is defined as the point B on the line OA such that $OA.OB = k^2$.

** First studied by *Möbius* (p. 337). Hence, sometimes called *Möbius transformation*

Obs. 1. From (1), we see that each point in the z-plane except $z = -d/c$, corresponds a unique point in the w-plane. Similarly, (2) shows that each point in the w-plane except $w = a/c$, maps into a unique point in the z-plane. Including the images of the two exceptional points as the infinite points in the two planes, it follows that *there is one to one correspondence between all points in the two planes.*

Obs. 2. Invariant points of bilinear transformation. If z maps into itself in the w-plane (*i.e.*, $w = z$), then (1) gives

$$z = \frac{az+b}{cz+d} \quad \text{or} \quad cz^2 + (d-a)z - b = 0$$

The roots of this equation (say : z_1, z_2) are defined as the invariant or fixed points of the bilinear transformation (1). If however, the two roots are equal, the bilinear transformation is said to be *parabolic.*

Obs. 3. Dividing the numerator and denominator of the right side of (1) by one of the four constants, it is clear that (1) has only three essential arbitrary constants. Hence *three conditions are required to determine a bilinear transformation.* For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Two important properties :

I. A bilinear transformation maps circles into circles.

By actual division, (1) can be written as $w = \dfrac{a}{c} + \dfrac{bc-ad}{c^2} \cdot \dfrac{1}{z + d/c}$

which is a combination of the transformations

$$w_1 = z + d/c,\ w_2 = 1/w_1,\ w_3 = \frac{bc-ad}{c^2} w_2,\ w = \frac{a}{c} + w_3 .$$

By these transformations, we successively pass from z-plane to w_1-plane, from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Now each of these transformations is one or other of the standard transformations $w = z + c$, $w = cz$, $w = 1/z$ and under each of these a circle always maps onto a circle. Hence the bilinear transformation maps circles into circles.

II. A bilinear transformation preserves cross-ratio† *of four points.*

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation (1). If these points are finite, then from (1), we have

$$w_j - w_k = \frac{az_j+b}{cz_j+d} - \frac{az_k+b}{cz_k+d} = \frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j - z_k).$$

Using this relation for $j, k = 1, 2, 3, 4$, we get $\dfrac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation. If one of the points, say : $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 *i.e.*,

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{z_3-z_4}{z_3-z_2}.$$

Example 20.13. *Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$.*

Hence find (a) the image of $|z| < 1$, (Mumbai, 2006 ; Delhi, 2002)

(b) the invariant points of this transformation. (U.P.T.U., 2008 ; V.T.U., 2000)

Solution. Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$.

Since the cross-ratio remains unchanged under a bilinear transformation.

$$\therefore \quad \frac{(1-i)(-1-z)}{(1-z)(-1-i)} = \frac{(i-0)(-i-w)}{(i-w)(-i-0)} \quad \text{or} \quad \frac{w+i}{w-i} = \frac{(z+1)(1-i)}{(z-1)(1+i)}$$

By componendo dividendo, we get $\dfrac{2w}{2i} = \dfrac{(z+1)(1-i)+(z-1)(1+i)}{(z+1)(1-i)-(z-1)(1+i)}$

---

† **Def.** *If t_1, t_2, t_3, t_4 be any four numbers, then* $\dfrac{(t_1-t_2)(t_3-t_4)}{(t_1-t_4)(t_3-t_2)}$ *is said to be their cross-ratio and is denoted (t_1, t_2, t_3, t_4).*

$$\therefore \qquad w = \frac{1+iz}{1-iz} \qquad ...(i)$$

which is the required bilinear transformation.

(*a*) Rewriting (*i*) as $\quad z = i\dfrac{1-w}{1+w}$

$$\therefore \qquad \left|\frac{i(1-w)}{1+w}\right| = |z| < 1 \quad \text{or} \quad |i|\,|1-w| < |1+w|$$

or $\qquad |1-u-iv| < |1+u+iv| \qquad [\because \ |i| = 1]$

or $\qquad (1-u)^2 + v^2 < (1+u)^2 + v^2$ which reduces to $u > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

(*b*) To find the invariant points of the transformation, we put $w = z$ in (*i*).

$$\therefore \qquad z = \frac{1+iz}{1-iz} \quad \text{or} \quad iz^2 + (i-1)z + 1 = 0$$

or
$$z = \frac{1-i \pm \sqrt{[(i-1)^2 - 4i]}}{2i} = -\tfrac{1}{2}[1 + i \pm \sqrt{(6i)}]$$

which are the required invariant points.

Example 20.14. *Show that* $w = \dfrac{i-z}{i+z}$ *maps the real axis of z-plane into the circle* $|w| = 1$ *and the half plane* $y > 0$ *into the interior of the unit circle* $|w| = 1$ *in the w-plane.* (*Mumbai, 2007*)

Solution. Since $\qquad w = (i-z)/(i+z)$,

$\therefore \qquad |w| = 1$ becomes $|(i-z)/(i+z)| = 1 \quad$ or $\quad |i-z| = |i+z|$

i.e., $\qquad |i-x-iy| = |i+x+iy| \quad$ or $\quad |-x+i(1-y)| = |x+i(1+y)|$

$\therefore \qquad \sqrt{[x^2 + (1-y)^2} = \sqrt{(x^2 + (1+y)^2]}$ or $(1-y)^2 = (1+y)^2$

$\therefore \qquad 4y = 0 \quad$ or $\quad y = 0$ which is the real axis.

Hence the real axis of the z-plane is mapped to the circle $|w| = 1$

Now for the interior of the circle $|w| = 1$

$\qquad |w| < 1 \qquad$ *i.e.*, $\qquad |i-z| < |i+z| \quad$ or $\quad (1-y)^2 < (1+y)^2$

$\therefore \qquad -4y < 0 \qquad$ *i.e.*, $\qquad y > 0$

Hence the half plane $y > 0$ is mapped into the interior of the circle $|w| = 1$.

PROBLEMS 20.2

1. Find the invariant points of the transformation $w = (z-1)/(z+1)$. (*Madras, 2003*)
2. Find the transformation which maps the points $-1, i, 1$ of the z-plane onto $1, i, -1$ of the w-plane respectively. Also find its *invariant points*. (*V.T.U., 2011*)
3. Find the bilinear transformation which maps $1, i, -1$ to $2, i, -2$ respectively. Find the fixed and critical points of the transformation. (*S.V.T.U., 2008 ; Mumbai, 2007 ; V.T.U., 2006*)
4. Determine the bilinear transformation that maps the points $1-2i, 2+i, 2+3i$ respectively into $2+2i, 1+3i, 4$. (*J.N.T.U., 2003 ; Coimbatore, 1999*)
5. Find the bilinear transformation which maps
   (*i*) the points $z = 1, i, -1$ into the points $w = 0, 1, \infty$ (*V.T.U., 2008 ; Mumbai, 2007*)
   (*ii*) the points $z = 0, 1, i$ into the points $w = 1+i, -i, 2-i$ (*V.T.U., 2010 S*)
   (*iii*) $R(z) > 0$ into interior of unit circle so that $z = \infty, i, 0$ map into $w = -1, -i, 1$.
6. Under the transformation $w = \dfrac{z-1}{z+1}$, show that the map of the straight line $x = y$ is a circle and find its centre and radius. (*Marathwada, 2008*)

7. Show that the bilinear transformation $w = (2z + 3)/(z - 4)$ maps the circle $x^2 + y^2 - 4x = 0$ into the line $4u + 3 = 0$.
(Mumbai, 2007 ; J.N.T.U., 2003 ; Bhopal, 2002)
8. Show that the condition for transformation $w = (az + b)/(cz + d)$ to make the circle $| w | = 1$ correspond to a straight line in the z-plane is $| a | = | c |$.
9. Show that the transformation $w = i(1 - z)/(1 + z)$ maps the circle $| z | = 1$ into the real axis of the w-plane and the interior of the circle $| z | < 1$ into the upper half of the w-plane. *(Osmania, 2003 S ; V.T.U., 2001)*
10. If z_0 is the upper half of the z-plane, show that the bilinear transformation $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \bar{z}_0}\right)$ maps the upper half of the z-plane into the interior of the unit circle at the origin in the w-plane.

20.9 (1) CONFORMAL TRANSFORMATION

Suppose two curves C, C_1 in the z-plane intersect at the point P and the corresponding curves C' and C_1' in the w-plane intersect at P' (Fig. 20.5). *If the angle of intersection of the curves at P is the same as the angle of intersection of the curves at P′ in magnitude and sense, then the transformation is said to be* **conformal.**

(2) Theorem. *The transformation effected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) \neq 0$.*

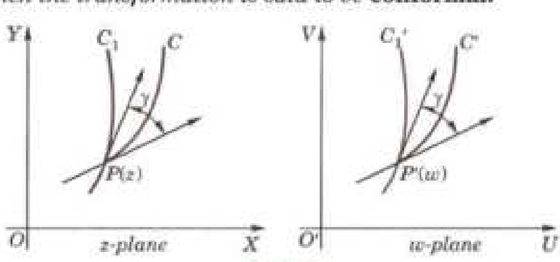


Fig. 20.5

Let $P(z)$ be a point in the region R of the z-plane and $P'(w)$ the corresponding point in the region R' of the w-plane (Fig. 20.3). Suppose z moves on a curve C and w moves on the corresponding curve C'. Let $Q(z + \delta z)$ be a neighbouring point on C and $Q'(w + \delta w)$ be the corresponding point on C' so that $\vec{PQ} = \delta z$ and $\vec{P'Q'} = \delta w$. Then δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x-axis.

$\therefore$
$$\delta z = re^{i\theta}$$

Similarly, if the modulus and amplitude of δw be r' and θ', then $\delta w = r'e^{i\theta'}$.

Hence
$$\frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$

Now if the tangent at P to the curve C makes an $\angle\alpha$ with the x-axis and the tangent at P' to C' makes an $\angle\alpha'$ with the u-axis, then as $\delta z \to 0$, $\theta \to \alpha$ and $\theta' \to \alpha'$.

$\therefore$
$$f'(z) = \frac{dw}{dz} = \left(\text{Lt}\ \frac{r'}{r}\right) . e^{i(\alpha' - \alpha)} \qquad ...(1)$$

If ρ is the modulus and ϕ the amplitude of the function $f(z)$ which is supposed to be non-zero, then
$$f'(z) = \rho e^{i\phi} \qquad ...(2)$$

$\therefore$ from (1) and (2), we have
$$\rho = \text{Lt}\ \frac{r'}{r} \qquad ...(3)$$

and
$$\phi = \alpha' - \alpha. \qquad ...(4)$$

Now let C_1 be another curve through P in the z-plane and C_1' the corresponding curve through P' in the w-plane. If the tangent at P to C_1 makes an $\angle\beta$ with the x-axis and tangent at P' to C_1' makes an $\angle\beta'$ with the u-axis, then as in (4),
$$\psi = \beta' - \beta \qquad ...(5)$$

Equating (4) and (5), $\qquad \alpha' - \alpha = \beta' - \beta \quad \text{or} \quad \beta - \alpha = \beta' - \alpha' = \gamma$ (Fig. 20.5)

Thus the angle between the curves before and after the mapping is preserved in magnitude and direction. Hence the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Obs. 1. *A point at which $f'(z) = 0$ is called a* **critical point** *of the transformation.*

Obs. 2. The relation (4), *i.e.*, $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an $\angle\phi$ = amp $[f'(z)]$ under the given transformation.

Obs. 3. The relation (3) shows that in the transformation, elements of arc passing through P in any direction are changed in the ratio $\rho : 1$, where $\rho = | f'(z) |$, *i.e., an infinitesimal length in the z-plane is magnified by the factor* $|f'(z)|$. Consequently *the infinitesimal areas are magnified by the factor* $| f'(z) |^2$ *in a conformal transformation.*

If $w = f(z)$ is analytic then u and v must satisfy C-R equations.

$$\therefore \quad J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = |f'(z)|^2$$

Hence *in a conformal transformation, infinitesimal areas are magnified by the factor* $J\left(\frac{u,v}{x,y}\right)$.

Also the condition of a conformal mapping is $J\left(\frac{u,v}{x,y}\right) \neq 0$.

Obs. 4. *The angle preserving property of the conformal transformation has many important physical applications.* For instance, consider the flow of an incompressible fluid in a plane with velocity potential $\phi(x, y)$ and stream function $\psi(x, y)$. We know that ϕ and ψ are real and imaginary parts of some analytic function $w = f(z)$. As ϕ = constant and ψ = constant represent a system of orthogonal curves; these are transformed by the function $w = f(z)$ into a set of orthogonal lines in the w-plane and *vice-versa*.

Thus, *the conjugate functions ϕ and ψ when subjected to conformal transformation remain conjugate functions, i.e., the solutions of Laplace's equation remain solutions of the Laplace's equation after the transformation.* This is the main reason for the great importance of the conformal transformation in applications.

20.10 SPECIAL CONFORMAL TRANSFORMATIONS

(1) *Transformation* $w = z^2$.

We have $\quad u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy.$

$$\therefore \quad u = x^2 - y^2 \text{ and } v = 2xy \quad \ldots(1)$$

If u is constant (say, a), then $x^2 - y^2 = a$ which is a rectangular hyperbola. Similarly, if v is constant (say, b), then $xy = b/2$ which also represents a rectangular hyperbola.

Hence a pair of lines $u = a$, $v = b$ parallel to the axes in the w-plane, map into pair of orthogonal rectangular hyperbolae in the z-plane as shown in Fig. 2.7 (p. 455).

Again, if x is constant (say, c), then $y = v/2c$ and $y^2 = c^2 - u$. Elimination of y from these equations gives $v^2 = 4c^2(c^2 - u)$, which represents a parabola. Similarly, if y is a constant (say, d), then elimination of x from the equations (1) gives $v^2 = 4d^2(d^2 + u)$ which is also a parabola.

Hence the pair of lines $x = c$ and $y = d$ parallel to the axes in the z-plane map into orthogonal parabolas in the w-plane as shown in Fig. 20.6.

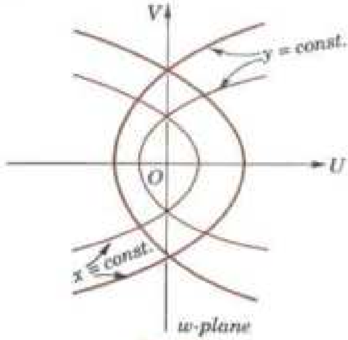


Fig. 20.6

Also since $\dfrac{dw}{dz} = 2z = 0$ for $z = 0$, therefore, it is a *critical point* of the mapping.

Taking $z = re^{i\theta}$ and $w = Re^{i\phi}$ then in polar form $w = z^2$ becomes $Re^{i\phi} = r^2e^{2i\theta}$.

This shows that upper half of the z-plane $0 < \theta < \pi$ transforms into the entire w-plane $0 \le \phi < 2\pi$. The same is true of the lower half. (*P.T.U., 2003*)

Obs. 1. Taking the axes to represent two walls, a single quadrant could be used to represent fluid flow at a corner wall. This transformation can also represent the electrostatic field in the vicinity of a corner conductor.

Obs. 2. *For the tansformation $w = z^n$*, n being a positive integer, we have $dw/dz = 0$ at $z = 0$.

Also $\quad Re^{i\phi} = (re^{i\theta})^n = r^ne^{in\theta}$

$\therefore \quad R = r^n$ and $\phi = n\theta$, when $0 < \theta < \pi/n$, correspondingly $0 < \phi < \pi$.

Hence $w = z^n$ gives a conformal mapping of the z-plane everywhere except at the origin and that is fans out a sector of z-plane of central angle π/n to cover the upper half of the w-plane.

(2) *Joukowski's transformation** $w = z + 1/z$.

Since $\dfrac{dw}{dz} = \dfrac{(z+1)(z-1)}{z^2}$, the mapping is conformal except at the points $z = 1$ and $z = -1$ which correspond to the points $w = 2$ and $w = -2$ of the w-plane.

* Named after the Russian mathematician *Nikolai Jegorovich Joukowski* (1847-1921).

Changing to polar coordinates,

$$w = u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$\therefore$ $$u = (r + 1/r)\cos\theta \text{ and } v = (r - 1/r)\sin\theta$$

Elimination of θ gives $$\frac{u^2}{(r+1/r)^2} + \frac{v^2}{(r-1/r)^2} = 1 \qquad ...(1)$$

while the elimination of r gives $$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad ...(2)$$

From (1), it follows that the circles r = constant of z-plane transform into a family of ellipses of the w-plane (Fig. 20.7). These ellipses are confocal for $(r + 1/r)^2 - (r - 1/r)^2 = 4$, *i.e.*, a constant.

In particular, the unit circle ($r = 1$) in the z-plane flattens out to become the segment $u = -2$ to $u = 2$ of the real axis in w-plane. Thus the exterior of the unit circle in the z-plane maps into the entire w-plane.

(A.M.I.E.T.E., 2005 S)

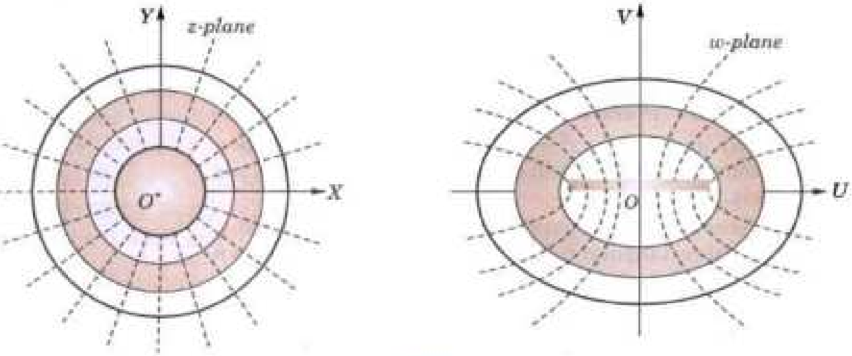


Fig. 20.7

From (2), it is clear that the radial lines θ = constant of the z-plane transform into a family of hyperbolae which are also confocal (Fig. 20.7).

Obs. 1. $v = \left(r - \frac{1}{r}\right)\sin\theta = 0$ gives $r = \pm 1$ or $\theta = 0, \pi$, *i.e.*, this streamline consists of the unit circle $r = 1$ and the x-axis ($\theta = 0$ to $\theta = \pi$). For large z, the flow is nearly uniform and parallel to the x-axis. This can be interpreted as a flow around a circular cylinder of unit radius having two **stagnation points*** at $A(z = 1)$ and $B(z = -1)$. (Fig. 20.8)

[$\because$ $dw/dz = 0$ at $z = \pm 1$]

Obs. 2. This transformation is also used to map the exterior of the profile of an aeroplane wing on the exterior of a nearly circular region. These airfoils are known as *Joukowski airfoils*.

(3) *Transformation* $w = e^z$.

Writing $z = x + iy$ and $w = \rho e^{i\phi}$, we have $\rho e^{i\phi} = e^{x+iy} = e^x \cdot e^{iy}$

whence $\rho = e^x$ ...(1) and $\phi = y$ ...(2)

From (1), it is clear that the lines parallel to y-axis (x = const.) map into circles (ρ = const.) in the w-plane, their radii being less than or greater than 1 according as x is less than or greater than 0 (Fig. 20.9). *(V.T.U., 2011)*

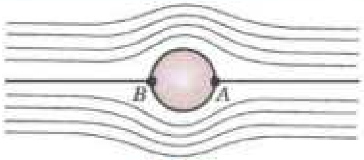


Fig. 20.8

Similarly, it follows from (2) that the lines parallel to the x-axis (y = const.) map into the radial lines (ϕ = const.) of the w-plane. Thus any horizontal strip of height 2π in the z-plane will cover once the entire w-plane.

---

* *Stagnation points* are those at which the fluid velocity is zero.

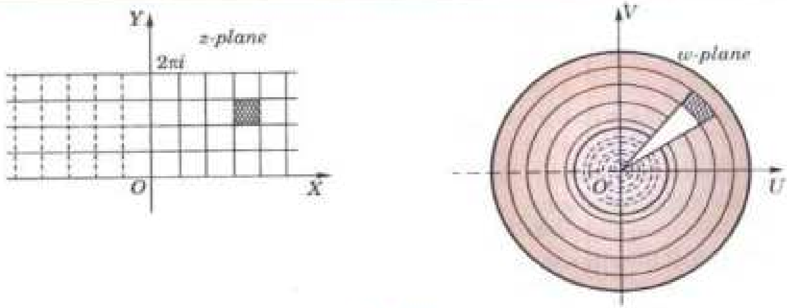


Fig. 20.9

The rectangular region $a_1 \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane (shown shaded) transforms into the region $e^{a_1} \le \rho \le e^{a_2}$, $b_1 \le \phi \le b_2$ in the w-plane bounded by circles and rays (shown shaded).

(*P.T.U., 2005 ; Kerala, 2005*)

Obs. This transformation can be used to obtain the circulation of a liquid around a cylindrical obstacle, the electrostatic field due to a charged circular cylinder etc.

(4) *Transformation* $w = \cosh z$.

We have $\quad u + iv = \cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y \quad$ [By (2) (*ii*), p. 662]

so that $\quad u = \cosh x \cos y$ and $v = \sinh x \sin y$.

Elimination of x from these equations gives

$$\frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad ...(1)$$

while elimination of y gives

$$\frac{u^2}{\cosh^2 x} + \frac{v^2}{\sinh^2 z} = 1 \qquad ...(2)$$

(1) shows that the lines parallel to x-axis (*i.e.*, y = const.) in the z-plane map into hyperbolae in the w-plane.

(2) shows that the lines parallel to the y-axis (*i.e.*, x = const.) in the z-plane map into ellipse in the w-plane (Fig. 20.10). The rectangular region $a_1, \le x \le a_2$, $b_1 \le y \le b_2$ in the z-plane (shown shaded) transforms into the shaded region in the w-plane bounded by the corresponding hyperbolae and ellipses. (*Kerala M. Tech., 2005*)

Obs. This transformation can be used.

(*i*) to obtain the circulation of liquid around an elliptic cylinder,

(*ii*) to determine the electrostatic field due to a charged cylinder;

(*iii*) to determine the potential between two confocal elliptic (or hyperbolic) cylinders.

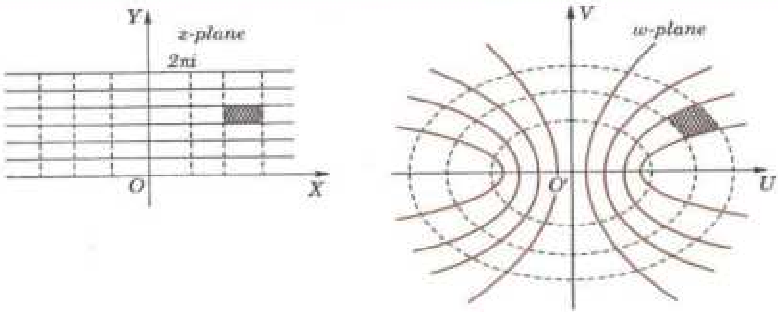


Fig. 20.10

Example 20.15. *Show that under the transformation $w = (z - i)/(z + i)$, real axis in the z-plane is mapped into the circle $|w| = 1$. Which portion of the z-plane corresponds to the interior of the circle?* (*J.N.T.U., 2003*)

Solution. We have $$|w| = \left|\frac{z-i}{z+i}\right| = \frac{|z-i|}{|z+i|} = \frac{|x+i(y-1|}{|x+i(y+1)|}$$
$$= \sqrt{\{x^2+(y-1)^2\}}/\sqrt{\{x^2+(y+1)^2\}}$$

Now the real axis in z-plane *i.e.*, $y = 0$, transforms into
$$|w| = \sqrt{(x^2+1)}/\sqrt{(x^2+1)} = 1.$$

Hence the real axis in the z-plane is mapped into the circle $|w| = 1$.

The interior of the circle, *i.e.*, $|w| < 1$, gives
$$\{x^2+(y-1)^2\}/\{x^2+(y+1)^2\} < 1 \text{ i.e., } -4y < 0 \text{ or } y > 0.$$

Thus the upper half of the z-plane corresponds to the interior of the circle $|w| = 1$.

PROBLEMS 20.3

1. Determine the region of the w-plane into which the following regions are mapped by the transformation $w = z^2$:
   (*i*) first quadrant of z-plane (*J.N.T.U., 2000*)
   (*ii*) region bounded by $x = 1$, $y = 1$, $x + y = 1$ (*Kottayam, 2005 ; V.T.U., 2000 S*)
   (*iii*) the region $1 \le x \le 2$ and $1 \le y \le 2$ (*Osmania, 2003 ; V.T.U., 2000*)
   (*iv*) circle $|z-1| = 2$.
2. Find the transformation which maps the triangular region $0 \le \arg z \le \pi/3$ into the unit circle $w \le 1$.
3. Discuss the transformation $w = \sqrt{z}$. Is it conformal at the origin? (*Delhi, 2002*)
4. Under the transformation $w = 1/z$, find the image of
   (*a*) the circle $|z - 2i| = 2$ (*Bhopal, 2009 ; Kerala M.Tech., 2005*)
   (*b*) the straight line $y - x + 1 = 0$ (*P.T.U., 2007*)
   (*c*) the hyperbola $x^2 - y^2 = 1$. (*Mumbai, 2005 ; J.N.T.U., 2005*)
5. Show that under the transformation $w = 1/z$, (*a*) circle $x^2 + y^2 - 6x = 0$ is transformed into a straight line in the w-plane.
   (*b*) the circle $(x-3)^2 + y^2 = 2$ is transformed into a circle with centre (3/7, 0) and radius $\sqrt{2}/17$. (*Mumbai, 2007*)
6. Show that the transformation $w = 1/z$ transforms all circles and straight lines into the circles and straight lines in the w-plane. Which circles in the z-plane become straight lines in the w-plane, and which straight lines are transformed into other straight lines? (*Anna, 2003*)
7. Show that the transformation $w = z + 1/z$ converts the straight line $\arg z = \alpha$ ($|\alpha| < \alpha/2$) into a branch of hyperbola of eccentricity $\sec \alpha$. (*Mumbai, 2005 S*)
8. Show that the transformation $w = z + (a^2 - b^2)/4z$ transforms the circle of radius $\frac{1}{2}(a+b)$, centre at the origin, in the z-plane into ellipse of semi-axes a, b in the w-plane.
9. Show that the transformation $w = z + a^2/z$ transforms circles with origin at the cent're in the z-plane into co-axial concentric, confocal ellipses in the w-plane. (*Kurukshetra, 2005 ; J.N.T.U., 2005*)
10. Show that the function $w = A(z + a^2/z)$ may be used to represent the flow of a perfect incompressible fluid past a circular cylinder. Also find the stagnation points.
11. Show that by the relation $u + iv = \cos(x + iy)$, the infinite strip bounded by $x = c$, $x = d$, where c and d lie between 0 and $\pi/2$, is mapped into the region between the two branches of the hyperbola lying in $u > 0$. (*Osmania, 2002*)
12. Prove that the transformation $w = \sin z$, maps the families of lines x = constant and y = constant into two families of confocal central conics. (*J.N.T.U., 2003*)
13. Discuss the transformation $w = e^z$, and show that it transforms the region between the real axis and a line parallel to real axis at $y = \pi$, into the upper half of the w-plane.
14. Discuss fully the transformation $w = c \cosh z$, where c is a real number. What physical problem can we study with the help of this transformation?

20.11 SCHWARZ-CHRISTOFFEL TRANSFORMATION

This transformation maps the interior of a polygon of the w-plane into the upper half of the z-plane and the boundary of the polygon into the real axis. The formula of this transformation is

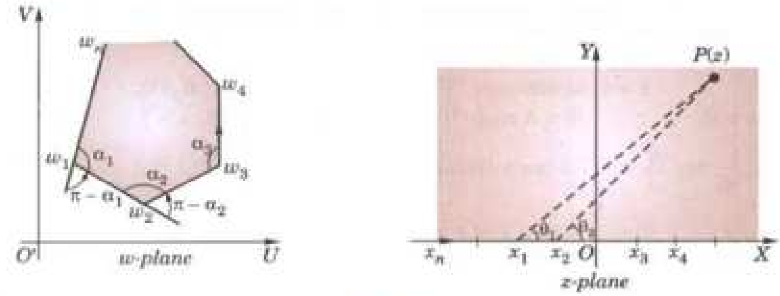


Fig. 20.11

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} \qquad \dots(1)$$

or
$$w = A \int (z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1} dz + B \qquad \dots(2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the interior angles of the polygon having vertices $w_1, w_2, \dots, w_n$ which map into the points $x_1, x_2, \dots, x_n$ on the real-axis of the z-plane (Fig. 20.11). Also A and B are complex constants which determines the size and position of the polygon.

Proof. We have from (1),

$$\text{amp}\left(\frac{dw}{dz}\right) = \text{amp}\,(A) + \left(\frac{\alpha_1}{\pi} - 1\right)\text{amp}\,(z - x_1) + \left(\frac{\alpha_2}{\pi} - 1\right)\text{amp}\,(z - x_2)$$

$$\dots + \left(\frac{\alpha_n}{\pi} - 1\right)\text{amp}\,(z - x_n) \qquad \dots(3)$$

As z moves along the real axis from the left towards x_1, suppose that w moves along the side $w_n w_1$ of the polygon towards w_1. As z crosses x_1 from left to right, $\theta_1 = \text{amp}\,(z - x_1)$ changes from π to 0 while all other terms of (3) remain unaffected. Hence only $\left(\frac{\alpha_1}{\pi} - 1\right)$ amp $(z - x_1)$ decreases by $\left(\frac{\alpha_1}{\pi} - 1\right)\pi = \alpha_1 - \pi$, *i.e.* increases by $\pi - \alpha_1$ in the anti-clockwise direction. In other words, amp (dw/dz) increases by $\pi - \alpha_1$. Thus the direction of w_1 turns through the angle $\pi - \alpha_1$ and w now moves along the side $w_1 w_2$ of the polygon.

Similarly when z passes through x_2, $\theta_1 = \text{amp}\,(z - x_1)$ and $\theta_2 = \text{amp}\,(z - x_2)$ change from π to 0 while all other terms remain unchanged. Hence the side $w_1 w_2$ turns through the angle $\pi - \alpha_2$. Proceeding in this way, we see that as z moves along x-axis, w traces the polygon $w_1\, w_2\, w_3 \dots w_n$ and conversely.

Example 20.16. *Find the transformation which maps the semi-infinite strip in the w-plane (Fig. 20.12) into the upper half of the z-plane* (V.T.U., M.E. 2006 ; Osmania, 2003)

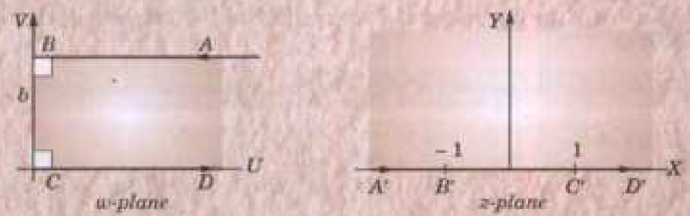


Fig. 20.12

Solution. Consider $ABCD$ as the limiting case of a triangle with two vertices B and C and the third vertex A or D at infinity. Let the vertices B and C map into the points B' (-1) and C' (1) of the z-plane. Since the interior angles at B and C are $\pi/2$, we have by the Schwarz-Christoffel transformation,

$$\frac{dw}{dz} = A(z+1)^{\frac{\pi/2}{\pi} - 1}(z-1)^{\frac{\pi/2}{\pi} - 1} = A/\sqrt{(z^2 - 1)}$$

$$\therefore \qquad w = A\int \frac{dz}{\sqrt{(z^2-1)}} + B = A\cosh^{-1} z + B$$

When $\qquad z = 1, w = 0. \quad \therefore \quad 0 = A\cosh^{-1}(1) + B$, *i.e.*, $B = 0$.

When $z = -1, w = ib. \quad \therefore \quad ib = A\cosh^{-1}(-1) + 0$, *i.e.*, $\cosh(ib/A) = -1$

or $\qquad \cos\dfrac{b}{A} = -1 = \cos\pi$. Thus $A = \dfrac{b}{\pi}$.

Hence $\qquad w = \dfrac{b}{\pi}\cosh^{-1} z$ or $z = \cosh\dfrac{\pi w}{b}$.

PROBLEMS 20.4

1. Find the transformation which maps the semi-infinite strip of width π bounded by the lines $v = 0$, $v = \pi$ and $u = 0$ into the upper half of the z plane.
2. Show how you will use Schwarz-Christoffel transformation to map the semi-infinite strip enclosed by the real axis and the lines $u = \pm 1$ of the w-plane into the upper half of the z-plane.
3. Find the mapping function which maps semi-infinite strip in the z-plane $-\pi/2 \le x \le \pi/2$, $y \ge 0$ into half w-plane for which $v \ge 0$, such that the points $(-\pi/2, 0)$, $(\pi/2, 0)$ in the z-plane are mapped into the points $(-1, 0)$, $(1, 0)$ respectively in w-plane.
4. Find the transformation which will map the interior of the infinite strip bounded by the lines $v = 0$, $v = \pi$ onto the upper half of the z-plane.

20.12 COMPLEX INTEGRATION

We have already discussed the concept of the line integral as applied to vector fields in § 8.11. Now we shall consider the line integral of a complex function.

Consider a continuous function $f(z)$ of the complex variable $z = x + iy$ defined at all points of a curve C having end points A and B. Divide C into n parts at the points

$$A = P_0(z_0), P_1(z_1), \ldots, P_i(z_i), \ldots P_n(z_n) = B.$$

Fig. 20.13

Let $\delta z_i = z_i - z_{i-1}$ and ζ_i be any point on the arc $P_{i-1}P_i$. The limit of the sum $\sum_{i=1}^{n} f(\zeta_i)\,\delta z_i$ as $n \to \infty$ *in such a way that the length of the chord* δz_i *approaches zero, is called the* **line integral of f (z)** *taken along the path C, i.e.,*

$$\int_C f(z)\,dz.$$

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$,

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Obs. *The value of the integral is independent of the path of integration when the integrand is analytic.*

Example 20.17. *Prove that*

(*i*) $\displaystyle\int_C \frac{dz}{z-a} = 2\pi i.$ $\qquad$ (*ii*) $\displaystyle\int_C (z-a)^n\,dz = 0$ [*n, any integer* $\neq -1$]

where C is the circle $|z - a| = r$. *(U.P.T.U., 2003)*

Solution. The parametric equation of C is $z - a = re^{i\theta}$, where θ varies from 0 to 2π as z describes C once in the positive (anti-clockwise) sense. (Fig. 20.14)

(*i*)
$$\int_C \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{re^{i\theta}}\cdot ire^{i\theta}\,d\theta \qquad [\because dz = ire^{i\theta}\,d\theta]$$

$$= i\int_0^{2\pi} d\theta = 2\pi i$$

(ii) $\int_C (z-a)^n\, dz = \int_0^{2\pi} r^n e^{n\theta i} \,.\, ire^{i\theta}\, d\theta$

$$= ir^{n+1} \int_0^{2\pi} e^{(n+1)\theta i}\, d\theta = \frac{r^{n+1}}{n+1}\left| e^{(n+1)\theta i} \right|_0^{2\pi}, \text{ provided } n \neq -1$$

$$= \frac{r^{n+1}}{n+1}\,[e^{2(n+1)\pi i} - 1] = 0 \qquad [\because \; e^{2(n+1)\pi i} = 1]$$

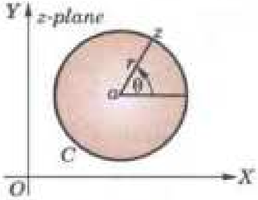


Fig. 20.14

Example 20.18. *Evaluate* $\int_0^{2+i} (\bar{z})^2\, dz$, *along (i) the line* $y = x/2$, *(Bhopal, 2007 ; U.P.T.U., 2002)*
(ii) the real axis to 2 and then vertically to $2 + i$. *(S.V.T.U., 2009 ; P.T.U., 2008 S ; Mumbai, 2006)*

Solution. (i) Along the line OA, $x = 2y$, $z = (2+i)y$, $\bar{z} = (2-i)y$ and $dz = (2+i)\, dy$ (Fig. 20.15)

$$\therefore \qquad I = \int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2 y^2 \,.\, (2+i)\, dy$$

$$= 5(2-i)\left| \frac{y^3}{3} \right|_0^1 = \frac{5}{3}(2-i)$$

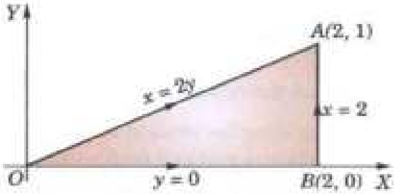


Fig. 20.15

(ii) $$I = \int_{OB} (\bar{z})^2\, dz + \int_{BA} (\bar{z})^2\, dz.$$

Now along OB, $z = x$, $\bar{z} = x$, $dz = dx$;
and along BA, $z = 2 + iy$, $\bar{z} = 2 - iy$, $dz = idy$

$$\therefore \qquad I = \int_0^2 x^2 dx + \int_0^1 (2-iy)^2 \,.\, idy = \left| \frac{x^3}{3} \right|_0^2 + \int_0^1 [\,4y + (4 - y^2)i\,]\, dy$$

$$= \frac{8}{3} + 4 \,.\, \frac{1}{2} + \left(4.1 - \frac{1}{3}\right) i = \frac{1}{3}(14 + 11i).$$

Example 20.19. *Evaluate* $\int_C (z^2 + 3z + 2)\, dz$ *where C is the arc of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *between the points* $(0, 0)$ *and* $(\pi a, 2a)$. *(Rohtak, 2004)*

Solution. $f(z) = z^2 + 3z + 2$ is analytic in the z-plane being a polynomial. As such, the line integral of $f(z)$ between O and A is independent of the path (Fig. 20.16). We therefore, take the path from O to L and L to A so that

$$\int_C f(z)\, dz = \int_{OL} f(z)\, dz + \int_{LA} f(z)\, dz \qquad ...(i)$$

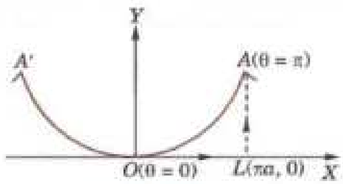


Fig. 20.16

$$\therefore \qquad \int_{OL} f(z)\, dz = \int_0^{\pi a} (x^2 + 3x + 2)\, dx$$

[$\because$ along OL, $y = 0$, $x = 0$ at O, $x = \pi a$ at L

$$= \left| \frac{x^3}{3} + \frac{3x^2}{2} + 2x \right|_0^{\pi a} = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) \qquad ...(ii)$$

and $$\int_{LA} f(z)\, dz = \int_0^{2a} [(\pi a + iy)^2 + 3(\pi a + iy) + 2]\, idy$$

[$\because$ along LA, $x = \pi a$, $z = \pi a + iy$, $dz = idy$ and y varies from 0 (at L) to $2a$ (at A)

$$= L\left| \frac{(\pi a + iy)^3}{3i} + 3\,\frac{(\pi a + iy)^2}{2i} + 2y \right|_0^{2\pi} = \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia \qquad ...(iii)$$

$\therefore$ substituting from (ii) and (iii) in (i), we get

$$\int_C f(z)\, dz = \frac{\pi a}{6}(2\pi^2 a^2 + 9\pi a + 12) + \frac{a^3}{3}(\pi + 2i)^3 + \frac{3a^2}{2}(\pi + 2i)^2 + 4ia$$

PROBLEMS 20.5

1. Evaluate $\int_0^{1+i}(x^2 - iy)\,dz$ along the paths (a) $y = x$ and (b) $y = x^2$. *(U.P.T.U., 2010)*

2. Evaluate $\int_{1-i}^{2+i}(2x + iy + 1)\,dz$, along the two paths: *(U.P.T.U., 2010)*
   (i) $x = t + 1, y = 2t^2 - 1$ (ii) the straight line joining $1 - i$ and $2 + i$. *(U.P.T.U., 2006)*

3. Evaluate $\int_{1-i}^{2+3i}(z^2 + z)\,dz$ along the line joining the points $(1, -1)$ and $(2, 3)$. *(V.T.U., 2004)*

4. Show that for every path between the limits, $\int_{-2}^{-2+i}(2 + z)^2\,dz = -i/3$. *(Delhi, 2002)*

5. Show that $\oint_C (z + 1)\,dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0, z = 1, z = 1 + i$ and $z = i$. *(Rohtak, 2006)*

6. Evaluate $\int_C |z|\,dz$, where C is the contour
   (i) straight line from $z = -i$ to $z = i$.
   (ii) left half of the unit circle $|z| = 1$ from $z = -i$ to $z = i$.
   (iii) circle given by $|z + 1| = 1$ described in the clockwise sense.

7. Find the value of $\int_0^{1+i}(x - y + ix^2)\,dz$
   (i) along the straight line from $z = 0$ to $z = 1 + i$
   (ii) along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. *(U.P.T.U., 2003)*

8. Prove that $\int dz/z = -\pi i$ or πi, according as C is the semi-circular arc $|z| = 1$ above or below the real axis. *(Rohtak, 2005)*

9. Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$.
   What is the value of this integral if C is the lower half of the above circle ?

20.13 CAUCHY'S THEOREM

If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\oint_C f(z)\,dz = 0$.

Writing $f(z) = u(x, y) + iv(x, y)$ and noting that $dz = dx + idy$

$$\oint_C f(z)dz = \oint_C (udx - vdy) + i \oint_C (vdx + udy) \qquad \ldots(1)$$

Since $f'(z)$ is continuous, therefore, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence the Green's theorem (p. 376) can be applied to (1), giving

$$\oint_C f(z)dz = -\iint_D \left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]dxdy + i \iint_D \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right]dxdy \qquad \ldots(2)$$

Now $f(z)$ being analytic, u and v necessarily satisfy the Cauchy-Riemann equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $\qquad \oint_C f(z)dz = 0.$

Obs. 1. The Cauchy-Riemann equations are precisely the conditions for the two real integrals in (1) to be independent of the path. *Hence the line integral of a function $f(z)$ which is analytic in the region D, is independent of the path joining any two points of D.*

Obs. 2. Extension of Cauchy's theorem. *If $f(z)$ is analytic in the region D between two simple closed curves C and C_1*, then $\oint_C f(z)dz = \oint_{C_1} f(z)dz$.

To prove this, we need to introduce the cross-cut AB. Then $\oint f(z)dz = 0$ where the path is as indicated by arrows in Fig. 20.17, *i.e.*, along AB—along C_1 in clockwise sense and along BA—along C in anti-clockwise sense.

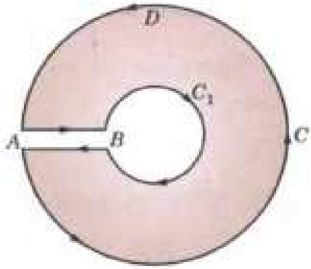


Fig. 20.17(a)

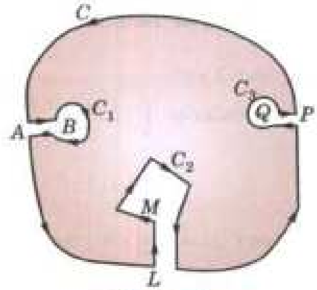


Fig. 20.17(b)

i.e., $$\int_{AB} f(z)dz + \int_{C_1} f(z)dz + \int_{AB} f(z)dz + \int_C f(z)dz = 0$$

But, since the integrals along AB and along BA cancel, it follows that

$$\int_C f(z)dz + \int_{C_1} f(z)dz = 0$$

Reversing the direction of the integral around C_1 and transposing, we get

$\int_C f(z)dz + \int_{C_1} f(z)dz$ each integration being taken in the anti-clockwise sense.

If $C_1, C_2, C_3, \ldots$ *be any number of closed curves within C (Fig. 20.17(b)), then*

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots$$

20.14 CAUCHY'S INTEGRAL FORMULA

If f (z) is analytic within and on a closed curve and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i} = \oint_C \frac{f(z)dz}{z-a}$$

Consider the function $f(z)/(z-a)$ which is analytic at all points within C except at $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.

Now $f(z)/(z-a)$ being analytic in the region enclosed by C and C_1, we have by Cauchy's theorem,

$$\oint_C \frac{f(z)}{z-a}dz = \oint_{C_1} \frac{f(z)}{z-a}dz \qquad \begin{cases}\text{For any point on } C_1, \\ z-a = re^{i\theta} \text{ and } dz = ire^{i\theta}\, d\theta\end{cases}$$

$$= \oint_C \frac{f(a+re^{i\theta})}{re^{i\theta}} . ire^{i\theta}\, d\theta = i\oint_{C_1} f(a+re^{i\theta})d\theta \qquad \ldots(1)$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, as $r \to 0$, the integral (1) will approach to

$$\oint_C f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a) \oint f(z)dz. \text{ Thus } \oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

i.e., $$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \qquad \ldots(2)$$

which is the desired *Cauchy's integral formula*. (*V.T.U., 2011 S*)

Cor. Differentiating both sides of (2) w.r.t. a,

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right]dz = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz \qquad \ldots(3)$$

Similarly, $$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\oint_C dz \qquad \ldots(4)$$

and in general, $$f^n(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz. \quad ...(5)$$

Thus it follows from the results (2) to (5) that *if a function $f(z)$ is known to be analytic on the simple closed curve C then the values of the function and all its derivatives can be found at any point of C.* Incidently, we have established a remarkable fact that **an analytic function possesses derivatives of all orders and these are themselves all analytic.**

Example 20.20. *Evaluate* $\int_C \frac{z^2-z+1}{z-1}\,dz$, *where C is the circle*

(i) $|z| = 1$, *(ii)* $|z| = \frac{1}{2}$. *(S.V.T.U., 2007)*

Solution. *(i)* Here $f(z) = z^2 - z + 1$ and $a = 1$.

Since $f(z)$ is analytic within and on circle $C : |z| = 1$ and $a = 1$ lies on C.

$\therefore$ by Cauchy's integral formula $\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1$ *i.e.*, $\int_C \frac{z^2-z+1}{z-1}\,dz = 2\pi i$.

(ii) In this case, $a = 1$ lies outside the circle $C : |z| = 1/2$. So $(z^2 - z + 1)/(z-1)$ is analytic everywhere within C.

$\therefore$ by Cauchy's theorem $\int_C \frac{z^2-z+1}{z-1}\,dz = 0$.

Example 20.21. *Evaluate, using Cauchy's integral formula:*

(i) $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ *where C is the circle* $|z| = 3$ *(U.P.T.U., 2010)*

(ii) $\oint_C \frac{\cos \pi z}{z^2-1}\,dz$ *around a rectangle with vertices* $2 \pm i, -2 \pm i$

(iii) $\oint_C \frac{e^{iz}}{z^2+1}\,dz$ *where C is the circle* $|z| = 3$. *(U.P.T.U., 2009)*

Solution. *(i)* $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the two singular points $z = 1$ and $z = 2$ lie inside this circle.

$$\therefore \quad \oint_C \frac{f(z)dz}{(z-1)(z-2)} = \oint_C (\sin \pi z^2 + \cos \pi z^2) = \left(\frac{1}{z-2} - \frac{1}{z-1}\right)$$

$$= \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-2}\,dz - \oint_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1}\,dz$$

$$= 2\pi i\,[\sin \pi(2)^2 + \cos \pi(2)^2] - 2\pi i[\sin \pi(1)^2 + \cos \pi(1)^2]$$

[By Cauchy's integral formula]

$$= 2\pi i\,(0 + 1) - 2\pi i\,(0 - 1) = 4\pi i$$

(ii) $f(z) = \cos \pi z$ is analytic in the region bounded by the given rectangle and the two singular points $z = 1$ and $z = -1$ lie inside this rectangle. (Fig. 20.18)

$$\therefore \quad \oint_C \frac{\cos \pi z}{z^2-1}\,dz = \frac{1}{2}\oint_C \left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\,dz$$

$$= \frac{1}{2}\oint_C \frac{\cos \pi z}{z-1}\,dz - \oint_C \frac{\cos \pi z}{z+1}\,dz$$

$$= \frac{1}{2}\{2\pi i \cos \pi(1)\} - \frac{1}{2}\{2\pi i \cos \pi(-1)\} = 0.$$

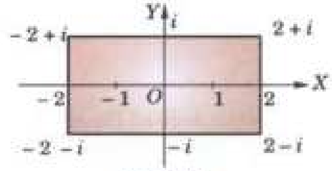


Fig. 20.18

[By Cauchy's integral formula]

(*iii*) $f(z) = e^{tz}$ is analytic within the circle $|z| = 3$.

The singular points are given by $z^2 + 1 = 0$ *i.e.*, $z = i$ and $z = -i$ which lie within this circle.

$$\therefore \quad \oint_C \frac{e^{tz}}{z^2+1}\,dz = \oint_C \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)e^{tz}\,dz = \frac{1}{2i}\left\{\oint_C \frac{e^{tz}}{z-i}\,dz - \oint_C \frac{e^{tz}}{z+i}\,dz\right\}$$

$$= \frac{1}{2i}\{2\pi i e^{t(i)} - 2\pi i e^{t(-i)}\} \qquad \text{[By Cauchy's integral formula]}$$

$$= 2\pi i\left(\frac{e^{it} - e^{-it}}{2i}\right) = 2\pi i \sin t.$$

Example 20.22. *Evaluate*

(*i*) $\oint_C \frac{\sin^2 z}{(z - \pi/6)^3}\,dz$, *where C is the circle* $|z| = 1$ *(Rohtak, 2005)*

(*ii*) $\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$, *where C is the circle* $|z| = 3$ *(U.P.T.U., 2008)*

(*iii*) $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$, *where C is* $|z| = 4$. *(U.P.T.U., 2008; J.N.T.U., 2000)*

Solution. (*i*) $f(z) = \sin^2 z$ is analytic inside the circle C: $|z| = 1$ and the point $a = \pi/6$ (= 0.5 approx.) lies within C.

$\therefore$ by Cauchy's integral formula $f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}\,dz$,

we get
$$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3}\,dz = \pi i\left[\frac{d^2}{dz^2}(\sin^2 z)\right]_{z=\pi/6}$$

$$= \pi i(2\cos 2z)_{z=\pi/6} = 2\pi i \cos \pi/3 = \pi i.$$

(*ii*) $f(z) = e^{2z}$ is analytic within the circle C: $|z| = 3$. Also $z = -1$ lies inside C.

$\therefore$ By Cauchy's integral formula: $f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^4}$

we get
$$\int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{6}\left|\frac{d^3(e^{2z})}{dz^3}\right|_{z=-1} = \frac{\pi i}{3}[8e^{2z}]_{z=-1} = \frac{8\pi i}{3}e^{-2}$$

(*iii*) $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$ is not analytic at $z = \pm\pi i$.

However both $z = \pm \pi i$ lie within the circle $|z| = 4$.

Now
$$\frac{1}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

where $A = 7/2\pi^3 i$, $C = -7/2\pi^3 i$, $B = D = -1/4\pi^2$

$$\therefore \quad \int_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{7}{2\pi^3 i}\left\{\int_C \frac{e^z}{z+\pi i}\,dz - \int_C \frac{e^z}{z-\pi i}\,dz\right\} - \frac{1}{4\pi^2}\left\{\int_C \frac{e^z}{(z+\pi i)^2}\,dz + \int \frac{e^z}{(z-\pi i)^2}\,dz\right\}$$

$$= \frac{7}{2\pi^3 i}[2\pi i\, f(-\pi i) - 2\pi i\, f(\pi i)] - \frac{1}{4\pi^2}[2\pi i\, f'(-\pi i) + 2\pi i\, f'(\pi i)] \qquad \text{where } f(z) = e^z.$$

$$= \frac{7}{\pi^2}(e^{-\pi i} - e^{\pi i}) - \frac{i}{2\pi}(e^{-\pi i} + e^{\pi i}) = -\frac{14i}{\pi^2}\left(\frac{e^{\pi i} - e^{-\pi i}}{2i}\right) - \frac{i}{\pi}\left(\frac{e^{\pi i} + e^{-\pi i}}{2}\right) \qquad [\S\ 19.9]$$

$$= -\frac{14i}{\pi^2}\sin \pi - \frac{i}{\pi}\cos \pi = \frac{i}{\pi}.$$

Example 20.23. *Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the verticles at the points $1 + i$, $-1 + i$ and $-1 - i$.* (U.P.T.U., 2006)

Solution. The boundary of the given triangle consists of three lines AB, BC, CA. (Fig. 29.19).

$$\oint_{ABC} e^{iz} dz = \int_{AB} e^{iz} dz + \int_{BC} e^{iz} dz + \int_{CA} e^{iz} dz$$

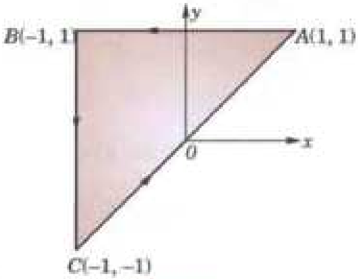


Fig. 20.19

Now $$\int_{AB} e^{iz} dz = \int_1^{-1} e^{i(x+i)} dx \quad \left| \begin{array}{l} \because \text{ Along } AB : y = 1 \\ \therefore z = x + i \text{ and } dz = dx \end{array} \right.$$

$$= \int_1^{-1} e^{(ix-1)} dx = \left| \frac{e^{(ix-1)}}{i} \right|_1^{-1} = \frac{e^{-i-1} - e^{i-1}}{i}$$

$$\int_{BC} e^{iz} dz = \int_1^{-1} e^{i(-1+1iy)} idy$$

$$\left| \begin{array}{l} \because \text{ Along } BC : x = -1 \\ \therefore z = -1 + iy,\ dz = idy \end{array} \right.$$

$$= i \int_1^{-1} e^{-i-y} dy = i \left| \frac{e^{-i-y}}{-1} \right|_1^{-1} = \frac{e^{-i+1} - e^{-i-1}}{i}$$

$$\int_{CA} e^{iz} dz = \int_{-1}^{1} e^{i(1+i)x} (1+i) dx \quad \left| \begin{array}{l} \because \text{ Along } CA : y = x \\ \therefore z = (1+i)x,\ dz = (1+i)dx \end{array} \right.$$

$$= (1+i) \frac{e^{(i-1)} - e^{-(i-1)}}{i(1+i)} = \frac{e^{i-1} - e^{-i+1}}{i}$$

Thus from (*i*) $$\oint_{ABC} e^{iz} dz = \frac{e^{-i-1} - e^{i-1}}{i} + \frac{e^{-i+1} - e^{-i-1}}{i} + \frac{e^{i-1} - e^{-i+1}}{i} = 0 \qquad ...(ii)$$

Also since $f(x) = e^{iz}$ is analytic everywhere,

$\therefore$ by Cauchy's theorem $\oint f(z) = 0$ ...(*iii*)

Hence from (*ii*) and (*iii*), $\oint_{ABC}$ Cauchy's theorem is verified.

Example 20.24. *If* $F(\zeta) = \oint_C \frac{4z^2 + z + 5}{z - \zeta} dz$, *where C is the ellipse* $(x/2)^2 + (y/3)^2 = 1$, *find the value of* (*a*) $F(3.5)$; (*b*) $F(i)$, $F'(-1)$ *and* $F''(-i)$. (Bhopal, 2009 ; Marathwada, 2008 ; Mumbai, 2006)

Solution. (*a*) $$F(3.5) = \oint_C \frac{z^3 + z + 1}{z^2 - 7z + 2} dz$$

Since $\zeta = 3.5$ is the only singular point of $(4z^2 + z + 5)/(z - 3.5)$ and it lies outside the ellipse C, therefore, $(4z^2 + z + 5)/(z - 3.5)$ is analytic everywhere within C.

Hence by Cauchy's theorem,

$$\oint_C \frac{4z^2 + z + 5}{z - 3.5} dz = 0, \text{ i.e., } F(3.5) = 0.$$

(*b*) Since $f(z) = 4z^2 + z + 5$ is analytic within C and $\zeta = i, -1$ and $-i$ all lie within C, therefore, by Cauchy's integral formula

$$f(\zeta) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - \zeta} dz$$

i.e., $$\oint_C \frac{4z^2+z+5}{z-\zeta}dz = 2\pi i(4\zeta^2+\zeta+5)$$

i.e., $$F(\zeta) = 2\pi i(4\zeta^2+\zeta+5)$$

$\therefore$ $$F'(\zeta) = 2\pi i\,(8\zeta+1) \text{ and } F''(\zeta) = 16\pi i$$

Thus $$F(i) = 2\pi i(-4+i+5) = 2\pi(i-1)$$

$$F'(-1) = 2\pi i[8(-1)+1] = -14\pi i \text{ and } F''(-i) = 16\pi i.$$

20.15 (1) CONVERSE OF CAUCHY'S THEOREM: MORERA'S THEOREM*

If $f(z)$ is continuous in a region D and $\oint_C f(z)dz = 0$ around every simple closed curve C in D, then $f(z)$ is analytic in D.

Since $\oint f(z)dz = 0$, then the line integral of $f(z)$ from a fixed point z_0 to a variable point z must be independent of the path and hence must be a function of z only. Thus

$$\int_{z_0}^{z} f(z)dz = \phi(z), \text{ (say)},$$

Let $$\phi(z) = U + iV \text{ and } f(z) = u + iv$$

Then $$U + iV = \int_{(x_0, y_0)}^{(x,y)} (u+iv)(dx+idy) = \int_{(x_0, y_0)}^{(x,y)} (udx - vdy) + i\int_{(x_0, y_0)}^{(x,y)} (vdx + udy)$$

$\therefore$ $$U = \int_{(x_0, y_0)}^{(x,y)} (udx - vdy),\ V = \int_{(x_0, y_0)}^{(x,y)} (vdx + udy)$$

Differentiating under the integral sign,

$$\frac{\partial U}{\partial x} = u,\ \frac{\partial U}{\partial y} = -v,\ \frac{\partial V}{\partial x} = v,\ \frac{\partial V}{\partial y} = u \quad \therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y},\ \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Thus U and V satisfy C-R equations.

Also, since $f(z)$ is given to be continuous, u and v and therefore, $\partial U/\partial x$, $\partial U/\partial y$, $\partial V/\partial x$, $\partial V/\partial y$, are also continuous.

$\therefore$ $\phi(z)$ is an analytic function and

$$\phi'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z).$$

Thus, $f(z)$ is the derivative of an analytic function $\phi(z)$. Hence $f(z)$ is analytic by § 20.14 Cor.

(2) Cauchy's inequality†. *If $f(z)$ is analytic within and on the circle C: $|z-a| = r$, then*

$$|f^n(a)| \le \frac{Mn!}{r^n} \quad \text{...(I)}$$

where M is the maximum value of $|f(z)|$ on C.

From (5) of § 20.14, we get

$$|f^n(a)| = \frac{n!}{2\pi}\left|\oint_C \frac{f(z)dz}{(z-a)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}\oint_C |z| \qquad [\because\ |f(z)| < M]$$

$$= \frac{n!\,M}{2\pi r^{n+1}}\oint_C ds = \frac{Mn!}{2\pi r^{n+1}}2\pi r = \frac{Mn!}{r^n} \qquad \textit{(U.P.T.U., 2005)}$$

(3) Liouville's theorem§. *If $f(z)$ is analytic and bounded for all z in the entire complex plane, then $f(z)$ is a constant.* *(U.P.T.U., 2008)*

---

* Named after the Italian mathematician, *Giacinto Morera* (1856–1909) who worked in Turin.

† See footnote p. 144

§ See footnote p. 573.

Taking $n = 1$ and replacing a by z, (I) gives

$$| f'(z) | \leq M/r$$

As $r \to \infty$, it gives $f'(z) = 0$ *i.e.*, $f(z)$ is constant for all z.

(4) Poisson's integral formulae. *If $f(z)$ is analytic within and on the circle C: $|z| = \rho$ and $z = re^{i\theta}$ is any point within C, then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} f(re^{i\phi})d\phi$$

By Cauchy's integral formula, $f(z) = \dfrac{1}{2\pi i}\oint_C \dfrac{f(w)}{w - z}\, dw$ ...(1)

As the inverse of the point x w.r.t. C lies outside C and is given by $\rho^2/\bar{z}$. [See footnote p. 685]

$\therefore$ by Cauchy's theorem,

$$0 = \frac{1}{2\pi i}\int \frac{f(w)}{w - \rho^2/\bar{z}}\, dw \quad ...(2)$$

Subtracting (2) from (1), $f(z) = \dfrac{1}{2\pi i}\displaystyle\int \left(\frac{1}{w - z} - \frac{1}{w - \rho^2/\bar{z}}\right) f(w)\, dw$

$$= \frac{1}{2\pi i}\oint_C \frac{z\bar{z} - \rho^2}{\bar{z}w^2 - (z\bar{z} + \rho^2)w + z\rho^2)} f(w)\, dw \quad ...(3)$$

Taking $w = \rho e^{i\phi}$ and noting that $\bar{z} = re^{-i\theta}$, (3) gives

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(r^2 - \rho^2) f(\rho e^{i\phi}) \,.\, \rho i e^{i\phi}\, d\phi}{re^{-i\theta} \,.\, \rho^2 e^{2i\phi} - (r^2 + \rho^2)\rho e^{i\phi} + re^{i\theta}\rho^2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2) f(\rho e^{i\phi})d\phi}{\rho^2 + r^2 - r\rho[e^{i(\theta - \phi)} + e^{-i(\theta - \phi)}]} = \frac{1}{2\pi}\int_0^{2\pi} \frac{(\rho^2 - r^2) f(\rho e^{i\phi})d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(4)$$

This is called *Poisson's integral formula** for a circle. It expresses the values of a harmonic function within a circle in terms of its values on the boundary.

Writing $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ and $f(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$ in (4) and equating real and imaginary parts from both sides, we get the formulae:

$$u(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)u(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(5)$$

and

$$v(r, \theta) = \int_0^{2\pi} \frac{(\rho^2 - r^2)v(\rho, \phi)d\phi}{\rho^2 - 2r\rho\cos(\theta - \phi) + r^2} \quad ...(6)$$

PROBLEMS 20.6

1. Evaluate $\oint_C (z - a)^{-1} dz$, where C is a simple closed curve and the point $z = a$ is (*i*) outside C, (*ii*) inside C.
2. Evaluate $\oint_C \dfrac{dz}{(z - a)^n}$, $n = 2, 3, 4, \ldots$, where C is a closed curve containing the point $z = a$.
3. Evaluate (*i*) $\oint_C \dfrac{e^z}{z^2 + 1} dz$, where C is the circle $|z| = 1/2$. (*P.T.U., 2010*)

   (*ii*) $\oint_C \dfrac{e^{3iz}}{(z + \pi)^3} dz$, where C is the circle $|z - \pi| = 3$. (*U.P.T.U., 2007*)

---

* Named after the French mathematician *Simeon Denis Poisson* (1781–1840) who was a professor in Paris and made contributions to partial differential equations, potential theory and probability.

4. Use Cauchy's integral formula to calculate:

(i) $\oint_C \frac{3z-5}{z^2+2z}dz$, where C is $|z| = 1$. *(P.T.U., 2005 S)* (ii) $\oint_C \frac{z^2+1}{z(2z+1)}dz$, where C is $|z| = 1$.

(iii) $\oint_C \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz$ where C is $|z| = 4$. *(U.P.T.U., 2008)*

5. Evaluate (a) $\oint_C \frac{z^2-2z+1}{(z-i)^2}$ where C is $|z| = 2$.

(b) $\oint_C \frac{e^{-z}}{(z-1)(z-2)^2} dz$ where C is $|z| = 3$. *(Rohtak, 2003)*

6. Evaluate, using Cauchy's integral formulae:

(i) $\oint_C \frac{z}{z^2-3z+2}dz$, where C is $|z-2| = \frac{1}{2}$. *(U.P.T.U., 2009 ; Hissar, 2007 ; Madras, 2000)*

(ii) $\oint_C \frac{e^z dz}{(z+1)^2}$, where C is $|z-1| = 3$. *(Bhopal, 2009)*

(iii) $\oint_C \frac{\log z}{(z-1)^3} dz$ where C is $|z-1| = \frac{1}{2}$. *(J.N.T.U., 2003)*

7. Evaluate $f(2)$ and $f(3)$ where $f(a) = \oint_C \frac{2z^2-z-2}{z-a}dz$ and C is the circle $|z| = 2.5$.

8. If $\phi(\zeta) = \oint_C \frac{3z^2+7z+1}{z-\zeta} dz$, where C is the circle $|z| = 2$ find the values of

(i) $\phi(3)$, (ii) $\phi'(1-i)$ (iii) $\phi''(1-i)$. *(Mumbai, 2006)*

9. Evaluate $\oint_C \frac{z^3+z+1}{z^2-7z+2} dz$, where C is the ellipse $4x^2+9y^2=1$. *(Rohtak, 2006)*

10. Verify Cauchy's theorem for the integral of z^2 taken over the boundary of the (i) rectangle with vertices $-1, 1, 1+i, -1+i$; (ii) triangle with vertices $(1, 2), (1, 4), (3, 2)$. *(V.T.U., 2003)*

20.16 (1) SERIES OF COMPLEX TERMS

Let $$(a_1+ib_1)+(a_2+ib_2)+\ldots+(a_n+ib_n)+\ldots \quad \ldots(1)$$
be an infinite series of complex terms ; a's and b's being real numbers. *If the series Σa_n and Σb_n converge to the sums A and B, then series (1) is said to* **converge** *to the sum A + iB.* Also if (1) is a convergent series, then $\underset{n\to\infty}{\text{Lt}} (a_n+ib_n)=0$.

The series (1) is said to be **absolutely convergent** *if the series*
$$|a_1+ib_1|+|a_2+ib_2|+\ldots+|a_n+ib_n|+\ldots$$
is convergent. Since $|a_n|$ and $|b_n|$ are both $\le |a_n+ib_n|$, it follows *that an absolutely convergent series is convergent.*

Next let the series of functions $u_1(z)+u_2(z)+\ldots+u_n(z)+\ldots\ldots$ $\quad\ldots(2)$
converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms. Then *the series (2) is said to be* **uniformly convergent** *in a region R, if corresponding to any positive number ε, there exists a positive number N, depending on ε, but not on z, such that for every z in R.*
$$|S(z)-S_n(z)| < \varepsilon \text{ for } n > N.$$ [*cf. Def. p. 389*]

As in the case of real series (p. 390) **Weirstrass's M-test** holds for series of complex terms. So *the series (2) is uniformly convergent in a region R if there is a convergent series of positive constants ΣM_n such that $|u_n(z)| \le M_n$ for all z in R.*

Also a *uniformly convergent series of continuous complex functions is itself continuous and can be integrated term by term.*

Obs. *If a power series $\Sigma a_n z^n$ converges for $z = z_1$, then it converges absolutely for $|z| < |z_1|$.*

Since $\Sigma a_n z_1^n$ converges, therefore, $\underset{n\to\infty}{\text{Lt}}\ a_n z_1^n = 0$ and so we can find a number k such that $|a_n z_1^n| < k$ for all n. Then $\Sigma a_n z^n = \Sigma |a_n z_1^n| \cdot |z/z_1|^n < \Sigma k t^n$ where $t = |z/z_1|$.

But the series Σt^n converges for $t < 1$. Hence the series $\Sigma a_n z^n$ converges absolutely for $|z| < |z_1|$, *i.e.*, if a circle with centre at the origin and radius $|z|$ be drawn, then the given series converges absolutely at all points inside the circle.

Such a circle $|z| = R$ within which series $\Sigma a_n z^n$ converges, is called the *circle of convergence and R is called the radius of convergence.*

A power series is uniformly convergent in any region which lies entirely within its circle of convergence.

(2) Taylor's series*. *If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,*

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots \qquad \dots(i)$$

Proof. Let z be any point inside C. Draw a circle C_1 with centre at a enclosing z (Fig. 20.20). Let t be a point on C_1. We have

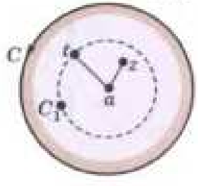


Fig. 20.20

$$\frac{1}{t-z} = \frac{1}{t-a-(z-a)} = \frac{1}{t-a}\left(1 - \frac{z-a}{t-a}\right)^{-1}$$

$$= \frac{1}{t-a}\left[1 + \frac{z-a}{t-a} + \left(\frac{z-a}{t-a}\right) + \dots + \left(\frac{z-a}{t-a}\right)^n + \dots\right] \qquad \dots(ii)$$

As $|z-a| < |t-a|$, *i.e.* $|(z-a)/(t-a)| < 1$, this series converges uniformly. So, multiplying both sides of (*ii*) by $f(t)$, we can integrate over C_1.

$$\therefore \quad \oint_{C_1} \frac{f(t)}{t-z}dz = \oint_{C_1} \frac{f(t)}{t-a}dz + (z-a)\oint_{C_1} \frac{f(t)}{(t-a)^2}dt + \dots + (z-a)^n \cdot \oint_{C_1} \frac{f(t)}{(t-a)^{n+1}}dt + \dots \qquad \dots(iii)$$

Since $f(t)$ is analytic on and inside C_1, therefore, applying the formulae (2) to (5) of p. 697-698 (*iii*), we get (*i*) which is known as *Taylor's series.*

Obs. Another remarkable fact is that *complex analytic functions can always ... represented by power series of the form* (*i*).

(3) Laurent's series†. *If $f(z)$ is analytic in the ring-shaped region R bounded by two concentric circles C and C_1 of radii r and r_1 ($r > r_1$) and with centre at a, then for all z in R*

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots$$

where $a_n = \dfrac{1}{2\pi i}\oint \dfrac{f(t)}{(t-a)^{n+1}}\,dt,$

Γ being any curve in R, encircling C_1 (as in Fig. 20.21).

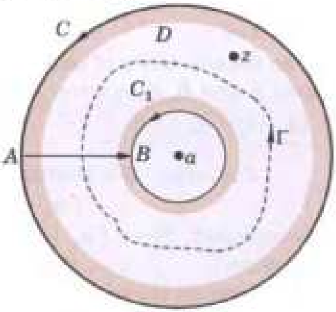


Fig. 20.21

Proof. Introduce cross-out AB, then $f(z)$ is analytic in the region D bounded by AB, C_1 described clockwise, BA and C described anti-clockwise (see Fig. 20.17). Then if z be any point in D, we have

$$f(z) = \frac{1}{2\pi i}\left[\int_{AB}\frac{f(t)}{t-z}dt + \oint_{C_1}\frac{f(t)}{t-z}dt + \int_{BA}\frac{f(t)}{t-z}dt + \oint_{C_1}\frac{f(t)}{t-z}dt\right]$$

$$= \frac{1}{2\pi i}\left[\oint_{C}\frac{f(t)}{t-z}dt - \oint_{C_1}\frac{f(t)}{t-z}dt\right] \qquad \dots(i)$$

where both C and C_1 are described anti-clockwise in (*i*) and integrals along AB and BA cancel (Fig. 20.21).

For the first integral in (*i*), expanding $1/(t-z)$ as in § 20.16 (2), we get

$$\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z}dt = \sum_{n=1}^{\infty}\frac{(z-a)^n}{2\pi i}\oint_C \frac{f(t)}{(t-a)^{n+1}}dt$$

* See footnote p. 145.

† Named after the French engineer and mathematician *Pierre Alphonse Laurent* (1813–1854) who published this theorem in 1843.

$$= \Sigma a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(t)}{(t-a)^{n+1}} dt \qquad ...(ii)$$

For the second integral in (i), let t lie on C_1. Then we write

$$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)} = -\frac{1}{z-a}\left(1-\frac{t-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{z-a}\left[1+\frac{t-a}{z-a}+\left(\frac{t-a}{z-a}\right)^2+\ldots+\left(\frac{t-a}{z-a}\right)^{n-1}+\ldots\right]$$

As $|t-a| < |z-a|$, *i.e.*, $|(t-a)/(z-a)| < 1$, this series converges uniformly. So multiplying both sides by $f(t)$ and integrating over C_1, we get

$$-\frac{1}{2\pi i}\oint_C \frac{f(t)}{t-z} dt = \sum_{n=1}^{\infty} \frac{1}{(z-a)^n} \cdot \frac{1}{2\pi i} \oint_C (t-a)^{n-1} f(t) dt = \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n} \qquad ...(iii)$$

where $$a_{-n} = \frac{1}{2\pi i}\oint_{C_1} \frac{f(t)}{(t-a)^{-n+1}} dt$$

Substituting from (ii) and (iii) in (i), we get

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}. \qquad ...(iv)$$

Now $f(t)/(t-a)^{n+1}$ being analytic in the region between C and Γ, we can take the integral giving a_n over Γ. Similarly we can take the integral giving a_{-n} over Γ. Hence (iv) can be written as

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n \text{ where } a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt$$

which is known as *Laurent's series*.

Obs. 1. *As $f(z)$ is not given to be analytic inside Γ*, $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt \neq \frac{f^n(a)}{n!}$

However, *if $f(z)$ is analytic inside Γ*, then $a_{-n} = 0$; $a_n = \frac{1}{2\pi i}\oint_\Gamma \frac{f(t)}{(t-a)^{n+1}} dt = \frac{f^n(a)}{n!}$

and *Laurent's series reduces to Taylor's series.*

Obs. 2. *To obtain Taylor's or Laurent's series, simply expand f(z) by binomial theorem* instead of finding a_n by complex integration which is quite complicated.

Obs. 3. *Laurent series of a given analytic function f(z) in its annulus of convergence is unique.* There may be different Laurent series of $f(z)$ in two annuli with the same centre.

Example 20.25. *Show that the series $z(1-z) + z^2(1-z) + z^3(1-z) + \ldots \infty$ converges for $|z| < 1$. Determine whether it converges absolutely or not.*

Solution. Let the sum of the first n terms of the series be s_n, so that

$$s_n = z - z^2 + z^2 - z^3 + z^3 - z^4 + \ldots + z^n - z^{n+1} = z - z^{n+1}$$

For $|z| < 1$, $z^{n+1} \to 0$ as $n \to \infty$.

$\therefore$ $\qquad \underset{n\to\infty}{\text{Lt}}(s_n) = z$, *i.e.*, the given series converges for $|z| < 1$.

$$|s_n(z)| = |z(1-z)| + |z^2(1-z)| + \ldots + |z^n(1-z)|$$
$$= |1-z|(|z| + |z|^2 + |z|^3 + \ldots + |z|^n)$$

For $|z| < 1$, $\qquad \underset{n\to\infty}{\text{Lt}} |s_n(z)| = |1-z| \frac{|z|}{1-|z|}$ [G.P.]

Hence the given series converges absolutely.

Example 20.26. *Expand sin z in a Taylor's series about z = 0 and determine the region of convergence.* *(P.T.U., 2009 S)*

Solution. Given $f(z) = \sin z, f'(z) = \cos \theta, f''(z) = -\sin z, f'''(z) = -\cos z, \ldots$

$\therefore \quad f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 1$

By Taylor's series about $z = 0$, we have

$$f(z) = f(0) + \frac{(z-0)}{1!} f'(0) + \frac{(z-0)^2}{2!} f''(0) + \frac{(z-0)^3}{3!} f'''(0) + \ldots$$

i.e.,
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots + (-1)^{n-1} \frac{z^{2n}-1}{(2n-1)!} + \ldots$$

Hence
$$\sin z = \sum_{n=1}^{\infty} a_n (z-0)^{2n-1} \text{ where } a_n = \frac{(-1)^{n-1}}{(2n-1)!}$$

Since
$$\rho = \underset{n\to\infty}{\text{Lt}} \left| \frac{a_{n+1}}{a_n} \right| = \underset{n\to\infty}{\text{Lt}} \left| \frac{(2n-1)!}{(2n+1)!} \right| = 0$$

Thus the radius of convergence of $f(z) = 1/\rho = \infty$

i.e., the region of convergence of $f(z)$ is all reals.

Example 20.27. *Find Taylor's expansion of*

(*i*) $f(z) = \dfrac{1}{(z+1)^2}$ *about the point* $z = -i$. *(V.T.U., 2009 S)*

(*ii*) $f(z) = \dfrac{2z^3+1}{z^2+z}$ *about the point* $z = i$. *(P.T.U., 2003)*

Solution. (*i*) To expand $f(z)$ about $z = -i$, *i.e.*, in powers of $z + i$, put $z + i = t$. Then

$$f(z) = \frac{1}{(t-i+1)^2} = (1-i)^{-2} [1 + t/(1-i)]^{-2} = \frac{i}{2}\left[1 - \frac{2t}{1-i} + \frac{3t^2}{(1-i)^2} - \frac{4t^3}{(1-i)^3} + \ldots\right]$$

[Expanding by Binomial theorem]

$$= \frac{i}{2}\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(n+1)(z+i)^n}{(1-i)^n}\right]$$

(*ii*)
$$f(z) = \frac{2z^3+1}{z(z+1)} = 2z - 2 + \frac{2z+1}{z(z+1)} = (2i-2) + 2(z-i) + \frac{1}{z} + \frac{1}{z+1} \quad \ldots(i)$$

[By partial fractions]

To expand $1/z$ and $1/(z+1)$ about $z = i$, put $z - i = t$, so that

$$\frac{1}{z} = \frac{1}{(t+i)} = \frac{1}{i}\left(1 + \frac{t}{i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{i}\left[1 - \frac{t}{i} + \frac{t^2}{i^2} - \frac{t^3}{i^3} + \frac{t^4}{i^4} - \ldots \infty\right] = \frac{1}{i} + \frac{t}{1} + \frac{t^2}{i^3} - \frac{t^3}{i^4} + \frac{t^4}{i^5} - \ldots \infty$$

$$= -i + (z-i) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \quad \ldots(ii)$$

and
$$\frac{1}{z+1} = \frac{1}{t+i+1} = \frac{1}{1+i}\left(1 + \frac{t}{1+i}\right)^{-1}$$ [Expanding by Binomial theorem]

$$= \frac{1}{1+i}\left[1 - \frac{t}{1+i} + \frac{t^2}{(1+t)^2} - \frac{t^3}{(1+i)^3} + \frac{t^4}{(1+i)^4} - \ldots \infty\right]$$

$$= \frac{1-i}{2} - \frac{t}{2i} + \left[\frac{t^2}{(1+i)^3} - \frac{t^3}{(1+i)^4} + \frac{t^4}{(1+i)^5} - \ldots \infty\right] = \frac{1}{2} - \frac{i}{2} - \frac{z-i}{2i} + \sum_{n=2}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}} \quad \ldots(iii)$$

Substituting from (*ii*) and (*iii*) in (*i*), we get

$$f(z) = \left(2i - 2 - i + \frac{1}{2} - \frac{i}{2}\right) + \left(2 + 1 - \frac{1}{2i}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$$

$$= \left(\frac{i}{2} - \frac{3}{2}\right) + \left(3 + \frac{i}{2}\right)(z-i) + \sum_{n=2}^{\infty} (-1)^n \left\{\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right\}(z-i)^n.$$

Example 20.28. *Expand $f(z) = 1/[(z-1)(z-2)]$ in the region:*
(*a*) $|z| < 1$, (*b*) $1 < |z| < 2$, (*c*) $|z| > 2$, (*d*) $0 < |z-1| < 1$.
(*U.P.T.U., 2010 ; V.T.U., 2010 ; Bhopal, 2009*)

Solution. (*a*) By partial fractions $\dfrac{1}{(z-1)(z-2)} = \dfrac{1}{z-2} - \dfrac{1}{z-1}$ ...(*i*)

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1} \quad ...(ii)$$

For $|z| < 1$, both $|z/2|$ and $|z|$ are less than 1. Hence (*ii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) + (1 + z + z^2 + z^3 + \ldots)$$

$$= \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots$$ which is a *Taylor's series.*

(*b*) *For* $1 < |z| < 2$, we write (*i*) as

$$f(z) = -\frac{1}{2}\frac{1}{(1-z/2)} - \frac{1}{z(1-z^{-1})} \quad ...(iii)$$

and notice that both $|z/2|$ and $|z^{-1}|$ are less than 1. Hence (*iii*) gives on expansion

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right) - \frac{1}{z}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots - z^{-4} - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \ldots$$

which is a *Laurent's series.*

(*c*) *For* $|z| > 2$, we write (*i*) as

$$f(z) = \frac{1}{z(1-2z^{-1})} - \frac{1}{z(1-z^{-1})}$$

$$= z^{-1}(1 + 2z^{-1} + 4z^{-2} + 8z^{-3} + \ldots) - z^{-1}(1 + z^{-1} + z^{-2} + z^{-3} + \ldots)$$

$$= \ldots + 7z^{-4} + 3z^{-3} + z^{-2} + \ldots$$

(*d*) *For* $0 < |z-1| < 1$, we write (*i*) as

$$f(z) = \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -(z-1)^{-1} - [1-(z-1)]^{-1}$$

$$= -(z-1)^{-1} - [1 + (z-1) + (z-1)^2 + (z-1)^3 + \ldots].$$

Example 20.29. *Find the Laurents' expansion of* $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ *in the region* $1 < z + 1 < 3$.
(*S.V.T.U., 2009 ; Anna, 2003 ; V.T.U., 2003*)

Solution. Writing $z + 1 = u$, we have

$$f(z) = \frac{7(u-1)-2}{u(u-1)(u-1-2)} = \frac{7u-9}{u(u-1)(u-3)}$$

$$= -\frac{3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$ (splitting into partial fraction)

$$= -\frac{3}{u} + \frac{1}{u(1-1/u)} - \frac{2}{3(1-u/3)} = -\frac{3}{u} + \frac{1}{u}\left(1 - \frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1 - \frac{u}{3}\right)^{-1}$$

Since $1 < u < 3$ or $1/u < 1$ and $u/3 < 1$, expanding by Binomial theorem,

$$f(z) = \frac{-3}{u} + \frac{1}{u}\left(1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty\right) - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

$$= -\frac{2}{u} + \frac{1}{u^2} + \frac{1}{u^3} + \dots \infty - \frac{2}{3}\left(1 + \frac{u}{3} + \frac{u^2}{3^2} + \frac{u^3}{3^3} + \dots \infty\right)$$

Hence $f(z) = -\dfrac{2}{z+1} + \dfrac{1}{(z+1)^2} + \dfrac{1}{(z+1)^3} + \dots \infty - \dfrac{2}{3}\left[1 + \dfrac{z+1}{3} + \dfrac{(z+1)^2}{3^2} + \dfrac{(z+1)^3}{3^3} + \dots \infty\right]$

which is valid in the region $1 < z + 1 < 3$.

20.17 (1) ZEROS OF AN ANALYTIC FUNCTION

Def. *A zero of an analytic function f(z) is that value of z for which f(z) = 0.*

If $f(z)$ is analytic in the neighbourhood of a point $z = a$, then by Taylor's theorem

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_n(z-a)^n + \dots \qquad \text{where } a_n = \frac{f^n(a)}{n!}.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then $f(z)$ is said to have a *zero of order m* at $z = a$.

When $m = 1$, the zero is said to be *simple*. In the neighbourhood of zero ($z = a$) of order m,

$$f(z) = a_m(z-a)^m + a_{m+1}(z-a)^{m+1} + \dots \infty$$
$$= (z-a)^m \phi(z) \text{ where } \phi(z) = a_m + a_{m+1}(z-a) + \dots$$

Then $\phi(z)$ is analytic and non-zero in the neighbourhood of $z = a$.

(2) Singularities of an analytic function

We have already defined a *singular point of a function as the point at which the function ceases to be analytic.*

(*i*) **Isolated singularity.** *If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood* (*i.e.*, there exists a circle with centre a which has no other singularity), *then $z = a$ is called an* **isolated singularity.**

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \dots \qquad \dots(1)$$

For example, $f(z) = \cot(\pi/z)$ is not analytic where $\tan(\pi/z) = 0$ *i.e.* at the points $\pi/z = 4\pi$ or $z = 1/n$ ($n = 1, 2, 3, \dots$).

Thus $z = 1, 1/2, 1/3, \dots$ are all *isolated singularities* as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood. Thus $z = 0$ is the *non-isolated singularity* of $f(z)$.

(*ii*) **Removable singularity.** If all the negative powers of $(z - a)$ in (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$.

Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called a *removable singularity*.

Thus *if* $\underset{z \to a}{\text{Lt}} f(z)$ *exists finitely, then $z = a$ is a removable singularity.*

(*iii*) **Poles.** *If all the negative powers of $(z - a)$ in (i) after the nth are missing, then the singularity at $z = a$ is called a* **pole of order n.**

A pole of first order is called a **simple pole.**

(*iv*) **Essential singularity.** *If the number of negative powers of $(z - a)$ in (1) is infinite, then $z = a$ is called an essential singularity.* In this case, $\underset{z \to a}{\text{Lt}} f(z)$ does not exist.

Example 20.30. *Find the nature and location of singularities of the following functions:*

(*i*) $\dfrac{z - \sin z}{z^2}$ (*ii*) $(z+1)\sin\dfrac{1}{z-2}$ (*iii*) $\dfrac{1}{\cos z - \sin z}$.

Solution. (*i*) Here $z = 0$ is a singularity.

Also $$\frac{z - \sin z}{z^2} = \frac{1}{z^2}\left\{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right)\right\} = \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!} - \dots$$

Since there are no negative powers of z in the expansion, $z = 0$ *is a removable singularity*.

(*ii*) $(z + 1) \sin \dfrac{1}{z-2} = (t + 2 + 1) \sin \dfrac{1}{t}$ where $t = z - 2$

$$= (t + 3)\left\{\frac{1}{t} - \frac{1}{3!t^3} + \frac{1}{5!t^5} - \dots\right\} = \left(1 - \frac{1}{3!t^2} + \frac{1}{5!t^4} - \dots\right) + \left(\frac{3}{t} - \frac{1}{2t^3} + \frac{3}{5!t^5} - \dots\right)$$

$$= 1 + \frac{3}{t} - \frac{1}{6t^2} - \frac{1}{2t^3} + \frac{1}{120t^4} - \dots = 1 + \frac{3}{z-2} - \frac{1}{6(z-2)^2} - \frac{1}{2(z-2)^3} + \dots$$

Since there are infinite number of terms in the negative powers of $(z - 2)$, $z = 2$ *is an essential singularity*.

(*iii*) Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, *i.e.*, by $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \pi/4$ is a *simple pole of* $f(z)$.

Example 20.31. *What type of singularity have the following functions :*

(*i*) $\dfrac{1}{1-e^z}$ (*ii*) $\dfrac{e^{2z}}{(z-1)^4}$ (*iii*) $\dfrac{e^{1/z}}{z^2}$. (*U.P.T.U., 2009*)

Solution. (*i*) Poles of $f(z) = 1/(1 - e^z)$ are found by equating to zero $1 - e^z = 0$ or $e^z = 1 = e^{2n\pi i}$

$\therefore$ $z = 2n\pi i$ $(n = 0, \pm 1, \pm 2, \dots)$

Clearly $f(z)$ *has a simple pole at* $z = 2\pi i$.

(*ii*) $$\frac{e^{2z}}{(z-1)^4} = \frac{e^{2(t+1)}}{t^4} = \frac{e^2}{t^4} \cdot e^{2t} \qquad \text{where } t = z - 1$$

$$= \frac{e^2}{t^4}\left\{1 + \frac{2t}{1!} + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + \dots\right\} = e^2\left\{\frac{1}{t^4} + \frac{2}{t^3} + \frac{2}{t^2} + \frac{4}{3t} + \frac{2}{3} + \frac{4t}{15} + \dots\right\}$$

$$= e^2\left\{\frac{1}{(z-1)^4} + \frac{2}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{4}{3(z-1)} + \frac{2}{3} + \frac{4}{15}(z-1) + \dots\right\}$$

Since there are finite (4) number of terms containing negative powers of $(z - 1)$,

$\therefore$ $z = 1$ *is a pole of 4th order*.

(*iii*) $$f(z) = \frac{e^{1/z}}{z^2} = \frac{1}{z^2}\left\{1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right\} = z^{-2} + z^{-3} + \frac{z^{-4}}{2} + \dots \infty$$

Since there are infinite number of terms in the negative powers of z, therefore $f(z)$ has *an essential singularity* at $z = 0$.

PROBLEMS 20.7

1. Obtain the expansion of $(z - 1)/z^2$ in a Taylor's series in powers of $(z - 1)$ and determine the region of convergence.
2. Find the first three times of the Taylor's series expansion of $f(z) = 1/(z^2 + 4)$ about $z = -i$. Also find the region of convergence. (*U.P.T.U., 2006*)
3. Expand in Taylor's series (*i*) $(z - 1)/(z + 1)$ about the point $z = 1$. (*Andhra, 2000*)

   (*ii*) $\cos z$ about the point $z = \pi/2$. (*Marathwada, 2008*) (*iii*) $\dfrac{1}{z^2 - z - 6}$ about (*a*) $z = -1$ (*b*) $z = 1$ (*P.T.U., 2009*)
4. Expand the following functions in Laurent's series :

   (*i*) $f(z) = \dfrac{1}{z - z^2}$ for $1 < |z + 1| < 2$. (*Madras, 2006*) (*ii*) $f(z) = \dfrac{1}{(z-1)(z+3)}$ for $1 < |z| < 3$. (*J.N.T.U., 2006*)

(*iii*) $f(z) = z/[(z-1)(z-3)]$ for $|z-1| < 2$. (*V.T.U., 2007*)

5. Find the Laurent's expansion of (*i*) $\dfrac{e^z}{(z-1)^2}$, about $z = 1$. (*Rohtak, 2006*)

   (*ii*) $e^{2z}/(z-1)^3$ about the singularity $z = 1$.

6. Expand the following functions in Laurent series.

   (*i*) $(z-1)/z^2$ for $|z-1| > 1$ (*ii*) $\dfrac{1-\cos z}{z^3}$, about $z = 0$. (*Rohtak, 2004*)

7. Find the Laurent's series expansion of

   (*i*) $\dfrac{z^2-1}{z^2+5z+6}$ about $z = 0$ in the region $2 < |z| < 3$ (*V.T.U., 2011 S ; Osmania, 2003*)

   (*ii*) $\dfrac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$

   (*iii*) $\dfrac{7z^2-9z-18}{z^3-9z}$ in the region (*a*) $|z| > 3$ (*b*) $0 < |z-3| < 3$. (*V.T.U., 2010 S*)

8. Find the Laurent's expansion of $1/[(z^2+1)(z^2+2)]$ for (*a*) $0 < |z| < 1$; (*b*) $1 < |z| < \sqrt{2}$; (*c*) $|z| > 2$.

Find the nature and location of the singularities of the following functions : (*P.T.U., 2005*)

9. $\dfrac{1}{z(2-z)}$. 10. $\sin(1/z)$. (*U.P.T.U., 2009*) 11. $\tan\left(\dfrac{1}{z}\right)$. (*P.T.U., 2006*)

12. $\dfrac{z^2-1}{(z-1)^3}$. (*Osmania, 2003*) 13. $\dfrac{e^z}{(z-1)^4}$. 14. $\dfrac{\cot \pi z}{(z-a)^2}$. (*U.P.T.U., 2008*)

20.18 (1) RESIDUES

The coefficient of $(z-a)^{-1}$ *in the expansion of* $f(z)$ *around an isolated singularity is called the* **residue** *of* $f(z)$ *at that point.* Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$$\therefore \qquad Res\, f(a) = \frac{1}{2\pi i}\oint_C f(z)$$

i.e.,
$$\oint_C f(z)dz = 2\pi i \text{ Res } f(a). \qquad \ldots(1)$$

(2) Residue Theorem

If $f(z)$ *is analytic in a closed curve C except at a finite number of singular points within C, then* $\oint_C f(z)dz = 2\pi i \times$ *(sum of the residues at the singular points within C).*

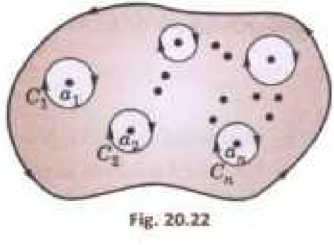


Fig. 20.22

Let us surround each of the singular points $a_1, a_2, \ldots, a_n$ by a small circle such that it encloses no other singular point (Fig. 20.22). Then these circles $C_1, C_2, \ldots, C_n$ together with C, form a multiply connected region in which $f(z)$ is analytic.

$\therefore$ applying Cauchy's theorem, we have

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \ldots + \oint_{C_n} f(z)dz \qquad \text{[by (1)]}$$

$$= 2\pi i\,[\text{Res } f(a_1) + \text{Res } f(a_2) + \ldots + \text{Res } f(a_n)] \text{ which is the desired result.}$$

20.19 CALCULATION OF RESIDUES

(1) *If $f(z)$ has a simple pole at $z = a$, then*

$$\textbf{Res f(a)} = \underset{z \to a}{\text{Lt}} [(z-a)f(z)]. \quad \ldots(1)$$

Laurent's series in this case is

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 \ldots + c_{-1}(z-a)^{-1}$$

Multiplying throughout by $z - a$, we have

$$(z-a)f(z) = c_0(z-a) + c_1(z-a)^2 + \ldots + c_{-1}.$$

Taking limits as $z \to a$, we get

$$\underset{z \to a}{\text{Lt}} [(z-a)f(z)] = c_{-1} = \text{Res } f(a).$$

(2) *Another formula for Res $f(a)$:*

Let $f(z) = \phi(z)/\psi(z)$, where $\psi(z) = (z-a)F(z)$, $F(a) \neq 0$.

Then $\underset{z \to a}{\text{Lt}} [(z-a)\phi(z)/\psi(z)]$

$$= \underset{z \to a}{\text{Lt}} \frac{(z-a)[\phi(a) + (z-a)\phi'(a) + \ldots]}{\psi(a) + (z-a)\psi'(a) + \ldots}$$

$$= \underset{z \to a}{\text{Lt}} \frac{\phi(a) + (z-a)\phi'(a) + \ldots}{\psi'(a) + (z-a)\psi''(a) + \ldots}, \quad \text{since } \psi(a) = 0$$

Thus $$\textbf{Res f(a)} = \frac{\phi(\mathbf{a})}{\psi'(\mathbf{a})}.$$

(3) *If $f(z)$ has a pole of order n at $z = a$, then*

$$\textbf{Res f(a)} = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a}$$

Here $$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_{-1}(z-a)^{-1} + \ldots + c_{-n}(z-a)^{-n}.$$

Multiplying throughout by $(z-a)^n$, we get

$$(z-a)^n f(z) = c_0(z-a)^n + c_1(z-a)^{n+1} + c_2(z-a)^{n+2} + \ldots + c_{-1}(z-a)^{n-1} + c_{-2}(z-a)^{n-2} + \ldots + c_{-n}.$$

Differentiating both sides w.r.t. z, $n-1$ times and putting $z = a$, we get

$$\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]\right\}_{z=a} = (n-1)!\, c_{-1} \text{ whence follows the result.}$$

Obs. In many cases, the residue of a pole $(z = a)$ can be found, by putting $z = a + t$ in $f(z)$ and expanding it in powers of t where $|t|$ is quite small.

Example 20.32. *Find the sum of the residues of $f(z) = \dfrac{\sin z}{z \cos z}$ at its poles inside the circle $|z| = 2$.* (Rohtak, 2004)

Solution. $f(z)$ has simple poles at $z = 0, \pm \pi/2, \pm 3\pi/2, \ldots$

Only the poles $z = 0$ and $z = \pm \pi/2$ lies inside $|z| = 2$.

$$\therefore \quad \text{Res } f(0) = \underset{z \to 0}{\text{Lt}} [z \cdot f(z)] = \underset{z \to 0}{\text{Lt}} \left(\frac{\sin z}{\cos z}\right) = 0.$$

$$\text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \left[\left(z - \frac{\pi}{2}\right) f(z)\right] = \underset{z \to \pi/2}{\text{Lt}} \left\{\frac{(z - \pi/2)\sin z}{z \cos z}\right\}$$

$$= \underset{z \to \pi/2}{\text{Lt}} \frac{(z - \pi/2)\cos z + \sin z}{\cos z - z \sin z} \quad \left[\text{Being } \frac{0}{0} \text{ form}\right]$$

$$= \frac{1}{-\pi/2} = -\frac{2}{\pi}$$

and $\quad \text{Res}\, f(-\pi/2) = \underset{z\to-\pi/2}{\text{Lt}} \left\{\dfrac{(z+\pi/2)\sin z}{z\cos z}\right\} = \underset{z\to-\pi/2}{\text{Lt}} \dfrac{(z+\pi/2)\cos z + \sin z}{\cos z - z\sin z} = \dfrac{-1}{-\pi/2} = \dfrac{2}{\pi}$

Hence sum of residues $= 0 - \dfrac{2}{\pi} + \dfrac{2}{\pi} = 0.$

Example 20.33. *Determine the poles of the function*
$f(z) = z^2/(z-1)^2(z+2)$ and the residue at each pole. (S.V.T.U., 2008 ; J.N.T.U., 2005)
Hence evaluate $\oint_C f(z)dz$, where C is the circle $|z| = 2.5$.

Solution. Since $\underset{z\to-2}{\text{Lt}} [(z+2)f(z)] = \underset{z\to-2}{\text{Lt}} \dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}$,
which is finite and non-zero, the function has a simple pole at $z = -2$ and Res $f(-2) = 4/9$.

Also since $\underset{z\to1}{\text{Lt}} [(z-1)^2 f(z)]$ is finite and non-zero, $f(z)$ has a pole of order two at $z = 1$.

$$\therefore \quad \text{Res}\, f(1) = \frac{1}{1!}\left[\frac{d}{dz}[(z-1)^2 f(z)]\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{z^2}{z+2}\right)\right]_{z=1} = \left[\frac{z^2+4z}{(z+2)^2}\right]_{z=1} = \frac{5}{9}.$$

[*Otherwise writing $z = 1 + t$,*

$$f(z) = \frac{(1+t)^2}{t^2(3+t)} = \frac{1}{3t^2}(1+t)^2(1+t/3)^{-1} = \frac{1}{3t^2}(1+t)^2\left(1 - \frac{t}{3} + \frac{t^2}{9} - \ldots\right)$$

$$= \frac{1}{3t^2}\left(1 + \frac{5}{3}t + \frac{4}{9}t^2 - \ldots\right) = \frac{1}{3t^2} + \frac{5}{9t} + \frac{4}{27} - \ldots \qquad \ldots(i)$$

$$\therefore \quad \text{Res}\, f(1) = \text{coefficient of } \frac{1}{t} \text{ in } (i) = \frac{5}{9}.]$$

Clearly $f(z)$ is analytic on $|z| = 2.5$ and at all points inside except the poles $z = -2$ and $z = 1$. Hence by residue theorem

$$\oint_C f(z)dz = 2\pi i\, [\text{Res}\, f(-2) + \text{Res}\, f(1)] = 2\pi i\left[\frac{4}{9} + \frac{5}{9}\right] = 2\pi i.$$

Example 20.34. *Find the residue of $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$ at its poles and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z| = 2.5$.* (U.P.T.U., 2003)

Solution. The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$.

$\therefore \quad z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res}\, f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$\therefore \quad z = 1$ is a pole of order 4, while $z = 2$ and $z = 3$ are simple poles.

$$\text{Res}\, f(1) = \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 \cdot \frac{z^3}{(z-1)^4(z-2)(z-3)}\right\}_{z=1} = \frac{1}{6}\frac{d^3}{dz^3}\left\{\frac{z^3}{(z-2)(z-3)}\right\}_{z=1}$$

$$= \frac{1}{6}\frac{d^3}{dz^3}\left[z + 5 - \frac{8}{z-2} + \frac{27}{z-3}\right] = \frac{1}{6}\left[-8 \cdot \frac{(-1)^3\, 3!}{(z-2)^4} + \frac{27 \cdot (-1)^3\, 3!}{(z-2)^4}\right]_{z=1}$$

$$= -\left[-8 + \frac{27}{16}\right] = \frac{101}{16}.$$

$$\text{Res}\, f(2) = \underset{z \to 2}{\text{Lt}} \left\{ (z-2) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \underset{z \to 2}{\text{Lt}} \left\{ \frac{z^3}{(z-1)^4 (z-3)} \right\} = \frac{8}{(1)^4(-1)} = -8$$

$$\text{Res}\, f(3) = \underset{z \to 3}{\text{Lt}} \left\{ (z-3) \cdot \frac{z^3}{(z-1)^4 (z-2)(z-3)} \right\} = \frac{27}{(2)^4 \cdot 1} = \frac{27}{16}$$

Now $$\oint_C f(z)dz = 2\pi i\, [\text{Res}\, f(1) + \text{Res}\, f(2)]$$ [$\because$ Pole $z = 3$ is outside C]

$$= 2\pi i \left(\frac{101}{16} - 8 \right) = \frac{-27\pi i}{8}.$$

Example 20.35. *Evaluate*

$$\oint_C \frac{z-3}{z^2+2z+5} dz, \text{ where } C \text{ is the circle}$$

(*i*) $|z| = 1$, (*ii*) $|z + 1 - i| = 2$, (*iii*) $|z + 1 + i| = 2$. (*J.N.T.U., 2003*)

Solution. The poles of $f(z) = \dfrac{z-3}{z^2+2z+5}$ are given by $z^2 + 2z + 5 = 0$

i.e., by $$z = \frac{-2 \pm \sqrt{(4-20)}}{2} = -1 \pm 2i.$$

(*i*) Both the poles $z = -1 + 2i$ and $z = -1 - 2i$ lie outside the circle $|z| = 1$. Therefore, $f(z)$ is analytic everywhere within C.

Hence by Cauchy's theorem, $$\oint_C \frac{z-3}{z^2+2z+5} dz = 0.$$

(*ii*) Here only one pole $z = -1 + 2i$ lies inside the circle $C : |z + 1 - i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res}\, f(-1+2i) = \underset{z \to -1+2i}{\text{Lt}} [\{z - (-1+2i)\} f(z)] = \underset{z \to -1+2i}{\text{Lt}} \frac{(z+1-2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1+2i}{\text{Lt}} \frac{z-3}{z+1+2i} = \frac{-4+2i}{4i} = i + 1/2.$$

Hence by residue theorem $$\oint_C f(z)dz = 2\pi i\, \text{Res}\, f(-1+2i) = 2\pi i (i + 1/2) = \pi(i - 2).$$

(*iii*) Here only the pole $z = -1 - 2i$ lies inside the circle $C : |z + 1 + i| = 2$. Therefore, $f(z)$ is analytic within C except at this pole.

$$\therefore \quad \text{Res}\, f(-1-2i) = \underset{z \to -1-2i}{\text{Lt}} \frac{(z+1+2i)(z-3)}{z^2+2z+5}$$

$$= \underset{z \to -1-2i}{\text{Lt}} \frac{z-3}{z+1-2i} = \frac{-4-2i}{-4i} = \tfrac{1}{2} - i$$

Hence by residue theorem, $$\oint_C f(z)dz = 2\pi i\, \text{Res}\, f(-1-2i) = 2\pi i \left(\tfrac{1}{2} - i\right) = \pi(2 + i).$$

Example 20.36. *Evaluate* $\displaystyle\oint_C \frac{e^z}{\cos \pi z} dz$, *where C is the unit circle* $|z| = 1$. (*Rohtak, 2006*)

Solution. $f(z) = e^z/\cos \pi z$ has simple poles at $z = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$

Out of these only the poles at $z = 1/2$ and $z = -1/2$ lie inside the given circle $|z| = 1$.

$$\therefore \quad \text{Res}\, f(1/2) = \underset{z \to 1/2}{\text{Lt}} \left[\left(z - \frac{1}{2}\right) f(z) \right] = \underset{z \to 1/2}{\text{Lt}} \left\{ \frac{\left(z - \frac{1}{2}\right) e^z}{\cos \pi z} \right\}$$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \underset{z \to 1/2}{\text{Lt}} \frac{e^z + \left(z - \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{1/2}}{-\pi}$$

and $$\text{Res } f(-1/2) = \underset{z \to -1/2}{\text{Lt}} \left\{ \frac{\left(z + \frac{1}{2}\right)e^z}{\cos \pi z} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{z \to -1/2}{\text{Lt}} \frac{e^z + \left(z + \frac{1}{2}\right)e^z}{-\pi \sin \pi z} = \frac{e^{-1/2}}{\pi}$$

Hence $$\oint_C \frac{e^z}{\cos \pi z} dz = 2\pi i \left\{ \text{Res } f\left(\frac{1}{2}\right) + \text{Res } f\left(-\frac{1}{2}\right) \right\}$$

$$= 2\pi i \left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi} \right) = -4i \left(\frac{e^{1/2} - e^{-1/2}}{2} \right) = -4i \sinh \frac{1}{2}.$$

Example 20.37. *Evaluate* $\oint_C \tan z \, dz$ *where C is the circle* $|z| = 2$. *(V.T.U., 2010 S)*

Solution. The poles of $f(z) = \sin z/\cos z$ are given by $\cos z = 0$ *i.e.* $z = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$
Of these poles, $z = \pi/2$, and $-\pi/2$ only are within the given circle.

$$\therefore \quad \text{Res } f(\pi/2) = \underset{z \to \pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = \underset{z \to \pi/2}{\text{Lt}} \left(\frac{\sin z}{-\sin z} \right) = -1 \qquad \text{[By § 20.19 (2)]}$$

Similarly $$\text{Res } f(-\pi/2) = \underset{z \to -\pi/2}{\text{Lt}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1.$$

Hence by residue theorem,

$$\oint_C f(z) dz = 2\pi i \{\text{Res } f(\pi/2) + \text{Res } f(-\pi/2)\} = 2\pi i(-1 - 1) = -4\pi i.$$

Example 20.38. *Evaluate* $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)} dz$, *where C is the circle* $|z| = 3$.
(V.T.U., 2010 ; Anna, 2003 S ; U.P.T.U., 2002)

Solution. $$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$$

is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$.
Since $z = 1$ is a pole of order 2.

$$\therefore \quad \text{Res } f(1) = \frac{1}{1!}\left[\frac{d}{dz}\{(z-1)^2 f(z)\}\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

Also $$\text{Res } f(2) = \underset{z \to 2}{\text{Lt}} [(z-2) f(z)] = \underset{z \to 2}{\text{Lt}} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

Hence by residue theorem,

$$\oint_C f(z) \, dz = 2\pi i \, [\text{Res } f(1) + \text{Res } f(2)] = 2\pi i \, (2\pi + 1 + 1) = 4\pi(\pi + 1)i.$$

PROBLEMS 20.8

1. Expand $f(z) = 1/[z^2 (z - i)]$ as a Laurent's series about i and hence find the residue thereat.
2. Find the residue of (i) $ze^z/(z-1)^3$ at its pole. (J.N.T.U., 2003)
   (ii) $z^2/(z^2 + a^2)$ at $z = ai$. (P.T.U., 2009 S)
3. Determine the poles of the following functions and the residue at each pole :

   (i) $\dfrac{z^2+1}{z^2-2z}$ (ii) $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$ (J.N.T.U., 2005) (iii) $\dfrac{2z+4}{(z+1)(z^2+1)}$ (J.N.T.U., 2006)
4. Find the residues of the following functions at each pole.
   (i) $(1-e^{2z})/z^4$ (ii) $ze^{iz}/(z^2+1)$ (P.T.U., 2010) (iii) $\cot z$.
5. $\oint_C \dfrac{z^2+4}{(z-2)(z+3)}\, dz$, where C is (i) $|z+1| = 2$ (ii) $|z-2| = 2$. (Mumbai, 2006)
6. Evaluate the following integrals :

   (i) $\oint_C \dfrac{e^{2z}\, dz}{(z+2)(z+4)(z+7)}$ for C as circle $|z| = 3$. (V.T.U., 2009)

   (ii) $\oint_C \dfrac{4z^2-4z+1}{(z-2)(4+z^2)}\, dz$, $C : |z| = 1$

   (iii) $\oint_C \dfrac{3z^2+z+1}{(z^2-1)(z+3)}\, dz$, $C : |z| = 2$. (U.P.T.U., 2004)
7. Evaluate

   (i) $\int_C \dfrac{2z+1}{(2z-1)^2}\, dz$, where C is $|z| = 1$ (ii) $\oint_C \dfrac{z+4}{z^2+2z+5}\, dz$, where C is $|z+1-i| = 2$

   (iii) $\int_C \dfrac{z^2-2z}{(z+1)^2(z^2+4)}\, dz$, where C is the circle $|z| = 10$. (U.P.T.U., 2009)
8. Evaluate :

   (i) $\oint_C \dfrac{z\, dz}{(z-1)(z-2)^2}$, $C : |z-2| = \frac{1}{2}$. (Madras, 2006)

   (ii) $\oint_C \dfrac{3z^2+2}{(z-1)(z^2+9)}\, dz$, $C : |z-2| = 2$. (Rohtak, 2005)

   (iii) $\oint_C \dfrac{dz}{(z^2+4)^2}$, $C : |z-i| = 2$. (Hissar, 2007 ; Anna, 2003 S ; Osmania, 2003)
9. Evaluate :

   (i) $\oint_C \dfrac{e^{-z}}{z^2}\, dz$, $C : |z| = 1$. (ii) $\oint_C z^2 e^{1/z}$, $C : |z| = 1$.

   (iii) $\oint_C \dfrac{e^z\, dz}{z^2+4}$, $C : |z-i| = 2$. (V.T.U., 2006) (iv) $\oint_C \dfrac{e^{2z}\, dz}{(z+1)^4}$, $C : |z| = 2$.
10. Evaluate the following integrals : (i) $\oint_C \dfrac{\sin^6 z}{(z-\pi/6)^3}\, dz$, $C : |z| = 1$

    (ii) $\oint_C \dfrac{z \sec z}{(1-z)^2}\, dz$, $C : |z| = 3$ (iii) $\oint_C \dfrac{z \cos z}{(z-\pi/2)^3}\, dz$, $C : |z-1| = 1$. (V.T.U., 2007)
11. Evaluate $\oint_C \dfrac{dz}{\sinh 2z}$ where C is the circle $|z| = 2$. (Marathwada, 2008)
12. Obtain Laurent's expansion for the function $f(z) = 1/z^2 \sinh z$ and evaluate

    $\oint_C \dfrac{z}{z^2 \sinh z}$, where C is the circle $|z-1| = 2$. (J.N.T.U., 2005)

20.20 EVALUATION OF REAL DEFINITE INTEGRALS

Many important definite integrals can be evaluated by applying the Residue theorem to properly chosen integrals. The contours chosen will consist of straight lines and circular arcs.

(*a*) **Integration around the unit circle.** An integral of the type $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$ can be evaluated by writing $e^{i\theta} = z$.

Since $\sin\theta = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$ and $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, then integral takes the form $\int_C f(z)\, dz$, where $f(z)$ is a rational function of z and C is a unit circle $|z| = 1$.

Hence *the integral is equal to $2\pi i$ times the sum of the residues at those poles of $f(z)$ which are within C.*

Example 20.39. *Show that*

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1 - a^2}, (a^2 < 1).$$ *(Bhopal, 2009 ; Rohtak, 2003)*

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \frac{1}{2}(z + 1/z)$ and $\cos 2\theta = \frac{1}{2}(e^{2i\theta} + e^{-2i\theta}) = \frac{1}{2}(z^2 + 1/z^2)$

$\therefore$ the given integral

$$I = \int_C \frac{\frac{1}{2}(z^2 + 1/z^2)}{1 - a(z + 1/z) + a^2} \cdot \frac{dz}{iz} = \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - az^2 - a + a^2 z)}$$

$$= \frac{1}{2i}\int_C \frac{(z^4 + 1)\, dz}{z^2(z - a)(1 - az)} = \int_C f(z)\, dz \qquad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has simple poles at $z = a$, $1/a$ and the second order pole at $z = 0$, of which the poles at $z = 0$ and $z = a$ lie within the unit circle.

$$\therefore \qquad \text{Res}\, f(a) = \underset{z\to a}{\text{Lt}}\, [(z - a) f(z)] = \frac{1}{2i}\underset{z\to a}{\text{Lt}}\left[\frac{z^4 + 1}{z^2(1 - az)}\right] = \frac{a^4 + 1}{2ia^2(1 - a^2)}$$

$$\text{and} \qquad \text{Res}\, f(0) = \underset{z\to 0}{\text{Lt}}\, \frac{d}{dz}[z^2 f(z)] = \frac{1}{2i}\underset{z\to 0}{\text{Lt}}\, \frac{d}{dz}\left[\frac{z^4 + 1}{z - az^2 - a + a^2 z}\right]$$

$$= \frac{1}{2i}\underset{z\to 0}{\text{Lt}}\, \frac{(z - az^2 - a + a^2 z)(4z^3) - (z^4 + 1)(1 - 2az + a^2)}{(z - az^2 - a + a^2 z)^2} = -\frac{1 + a^2}{2ia^2}$$

$$\text{Hence} \qquad I = 2\pi i\, [\text{Res}\, f(a) + \text{Res}\, f(0)] = 2\pi i\left[\frac{a^4 + 1}{2ia^2(1 - a^2)} - \frac{1 + a^2}{2ia^2}\right] = \frac{2\pi a^2}{1 - a^2}.$$

Example 20.40. *By integrating around a unit circle, evaluate* $\int_0^{2\pi} \dfrac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta$.

(S.V.T.U, 2009 ; U.P.T.U., 2009 ; Madras, 2003)

Solution. Putting $z = e^{i\theta}$, $d\theta = dz/iz$, $\cos\theta = \dfrac{1}{2}(z + 1/z)$

$$\text{and} \qquad \cos 3\theta = \frac{1}{2}(e^{3i\theta} + e^{-3i\theta}) = \frac{1}{2}(z^3 + 1/z^3).$$

$$\therefore \text{ the given integral} \quad I = \int_C \frac{\frac{1}{2}(z^3 + 1/z^3)}{5 - 2(z + 1/z)} \cdot \frac{dz}{iz}$$

$$= -\frac{1}{2i}\int_C \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)}\, dz = -\frac{1}{2i}\int_C \frac{(z^6 + 1)\, dz}{z^3(2z - 1)(z - 2)}$$

$$= -\frac{1}{2i}\int_C f(z)\,dz, \quad \text{where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 3 at $z = 0$ and simple poles at $z = \frac{1}{2}$ and $z = 2$. Of these only $z = 0$ and $z = 1/2$ lie within the unit circle.

$$\therefore \quad \text{Res}\, f(1/2) = \underset{z\to 1/2}{\text{Lt}} \frac{(z-1/2)(z^6+1)}{(2z-1)(z-2)} = \underset{z\to 1/2}{\text{Lt}} \left\{\frac{z^6+1}{2z^3(z-2)}\right\} = -\frac{65}{24}$$

$$\text{Res}\, f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0} \quad \text{where } n = 3$$

$$= \frac{1}{2}\left\{\frac{d^2}{dz^2}\left(\frac{z^6+1}{2z^2-5z+2}\right)\right\}_{z=0} = \frac{d}{dz}\left[\frac{(2z^2-5z+2)\,6z^5-(z^6+1)(4z-5)}{2(2z^2-5z+2)^2}\right] \text{ at } z = 0$$

$$= \left\{\frac{d}{dz}\left[\frac{8z^7-25z^6+12z^5-4z+5}{2(2z^2-5z+2)^2}\right]\right\}_{z=0}$$

$$= \left[\frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4)-(8z^7-25z^6+12z^5-4z+5)2(2z^2-5z+2)(4z-5)}{2(2z^2-5z+2)^4}\right]_{z=0}$$

$$= \frac{4(-4)-5(-20)}{2\times 16} = \frac{84}{32} = \frac{21}{8}$$

Hence $$I = \frac{-1}{2i}\{2\pi i\,[\text{Res}\, f(1/2) + \text{Res}\, f(0)]\} = -\pi\left[-\frac{65}{24}+\frac{21}{8}\right] = -\pi\left(-\frac{1}{12}\right) = \frac{\pi}{12}.$$

(*b*) **Integration around a small semi-circle.** To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)$, where C is the contour consisting of the semi-circle $C_R : |z| = R$, together with the diameter that closes it.

Supposing that $f(z)$ has no singular point on the real axis, we have, by the Residue theorem,

$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res}\, f(a).$$

Finally making R tend to ∞, we find the value of $\int_{-\infty}^{\infty} f(x)\,dx$, provided $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$.

Example 20.41. *Evaluate* $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. (*U.P.T.U., 2008*)

Solution. Consider $\int_C \frac{z^2 dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz$ ...(*i*)

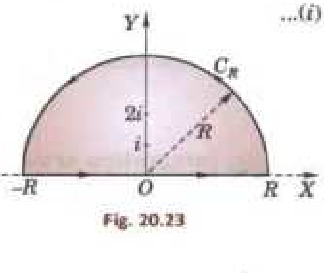


Fig. 20.23

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = \pm i$, $z = \pm 2i$ of which $z = i$, $2i$ only lie inside C.

$\therefore$ by the Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Res}\, f(i) + \text{Res}\, f(2i)]$$

$$= 2\pi i\,[\underset{z\to i}{\text{Lt}}(z-i)f(z) + \underset{z\to 2i}{\text{Lt}}(z-2i)f(z)]$$

$$= 2\pi i\left[\frac{i^2}{2i(i^2+4)} + \frac{4i^2}{(4i^2+1)(4i)}\right] = 2\pi i\left(\frac{i}{6}-\frac{i}{3}\right) = \frac{\pi}{3} \qquad ...(ii)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(iii)$$

Now let $R \to \infty$, so as to show that the second integral in (*iii*) vanishes. For any point on C_R as $|z| \to \infty$

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{(1+z^{-2})(1+4z^{-2})}$$

decreases as $1/z^2$ and tends to zero whereas the length of C_R increases with z.

Consequently, $\underset{|z|\to\infty}{\text{Lt}} \int_{C_R} f(z)\,dz = 0.$

Hence from (*i*), (*ii*) and (*iii*), we get $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$

Example 20.42. *Evaluate* $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx.$ (*U.P.T.U., 2006 ; Delhi, 2002*)

Solution. Consider $\displaystyle\int_C \frac{e^{iaz}}{z^2+1}\,dz = \int_C f(z)\,dz$

where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R as shown in Fig. 20.23.

The integrand has simple poles at $z = i$ amd $z = -i$, of which $z = i$ only lies inside C.

$\therefore$ by Residue theorem, $\displaystyle\int_C f(z)\,dz = 2\pi i \text{ Res } f(i) = 2\pi i \underset{z\to i}{\text{Lt}} [(z-i)\,f(z)]$

$$= 2\pi i \underset{z\to i}{\text{Lt}} \frac{(z-i)\,e^{iaz}}{z^2+1} = 2\pi i \underset{z\to i}{\text{Lt}} \frac{e^{iaz}}{z+i} = 2\pi i\,\frac{e^{-a}}{2i} = \pi e^{-a} \qquad ...(i)$$

Also $$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz \qquad ...(ii)$$

Now $\quad |z| = R$ on C_R and $|z^2+1| \geq R^2 - 1.$

Also $\quad |e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax} \cdot e^{-ay}| = e^{-ay} < 1 \qquad [\because \; y > 0]$

$$\therefore \qquad \left|\frac{e^{iaz}}{z^2+1}\right| = |e^{iaz}| \cdot \frac{1}{|z^2+1|} < 1 \cdot \frac{1}{R^2-1}$$

Thus $$\int_{C_R} f(z)\,dz = \left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| < \int_{C_R} \frac{1}{R^2-1}\,|dz| < \frac{\pi R}{R^2-1} \text{ which} \to \text{to } 0 \text{ as } R \to \infty. \qquad ...(iii)$$

Hence from (*i*), (*ii*) and (*iii*), we get

$$\pi e^{-a} = \int_{-\infty}^{\infty} f(x)\,dx + 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$$

Equating real parts from both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}$$

Since $\cos ax/(x^2+1)$ is an even function of x, we have

$$2\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a} \quad \text{or} \quad \int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}.$$

(*c*) **Integration around rectangular contours**

Example 20.43. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}\,dx.$

Solution. Consider $\displaystyle\int_C \frac{e^{az}}{e^z+1}\,dz = \int_C f(z)\,dz$ where C is the rectangle $ABCD$ with vertices at $(R, 0)$, $(R, 2\pi)$, $(-R, 2\pi)$ and $(-R, 0)$, R being positive (Fig. 20.24).

$f(z)$ has finite poles given by

$$e^z = -1 = e^{(2n+1)\pi i}$$

or $$z = (2n+1)\pi i, \text{ where } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The only pole inside the rectangle is $z = \pi i$.

$\therefore$ by Residue theorem,

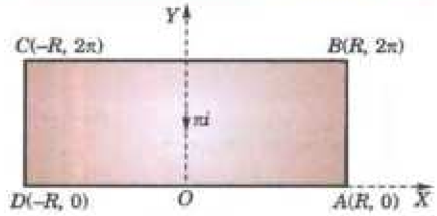


Fig. 20.24

$$\int_C f(z)dz = 2\pi i \text{ Res } f(\pi i)$$

$$= 2\pi i \left[e^{az} \Big/ \frac{d}{dz}(e^z + 1) \right]_{z=\pi i}$$

$$= 2\pi i e^{a\pi i}/e^{\pi i} = -2\pi i e^{a\pi i} \qquad [\because \; e^{\pi i} = -1] \qquad \ldots(i)$$

Also $$\int_C f(z)dz = \int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz$$

$$= \int_0^{2\pi} f(R+iy)idy + \int_R^{-R} f(x+2\pi i)dx + \int_{2\pi}^0 f(-R+iy)idy + \int_{-R}^R f(x)dx$$

[$\because$ $z = R + iy$ along AB, $z = x + 2\pi i$ along BC, $z = -R + iy$ along CD and $z = x$ along DA.]

or $$\int_C f(z)dz = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{e^{R+iy}+1}dy - \int_{-R}^R \frac{e^{a(x+2\pi i)}}{e^{x+2\pi i}+1}dx - i\int_0^{2\pi} \frac{e^{a(-R+iy)}}{e^{-R+iy}+1}dy + \int_{-R}^R \frac{e^{ax}}{e^x+1}dx \qquad \ldots(ii)$$

Now for any two complex numbers z_1, z_2

$$|z_1| \geq |z_2|, \text{ we have } |z_1 + z_2| \geq |z_1| - |z_2|$$

so that $|e^{R+iy}+1| \geq e^R - 1$. Also $|e^{a(R+iy)}| = e^{aR}$

$\therefore$ for the integrand of first integral in (*ii*), we have

$$\left| \frac{e^{a(R+iy)}}{e^{R+iy}+1} \right| \leq \frac{e^{aR}}{e^R - 1} \text{ which } \to 0 \text{ as } R \to \infty. \qquad [\because \; a > 1]$$

Similarly, for the integrand of the third integral in (*ii*), we get

$$\left| \frac{e^{a(-R+iy)}}{e^{-R+iy}+1} \right| \leq \frac{e^{-aR}}{1 - e^{-R}} \text{ which also } \to 0 \text{ as } R \to \infty. \qquad [\because \; a < 0]$$

Hence as $R \to \infty$, since the first and third integrals in (*ii*) approach zero, we get

$$\int_C f(z)dz = -e^{2a\pi i}\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \left(1 - e^{2a\pi i}\right)\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx \qquad \ldots(iii)$$

Thus from (*i*) and (*iii*), we obtain $(1 - e^{2a\pi i}) \int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = -2\pi i e^{a\pi i}$

$\therefore$ equating real parts, we get $\int_{-\infty}^{\infty} \frac{e^{ax}}{e^x+1}dx = \frac{\pi}{\sin a\pi}$.

Example 20.44. *Show that* $\int_0^{\infty} e^{-x^2} \cos 2mx \, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$.

Solution. Integrate $f(z) = e^{-z^2}$ along the rectangle ABCDA having vertices $A(-l), B(l), C(l+im), D(-l+im)$ (Fig. 20.25). $f(z)$ has no poles inside this contour. As such

$$\int_{AB} f(z)dz + \int_{BC} f(z)dz + \int_{CD} f(z)dz + \int_{DA} f(z)dz = 0 \qquad \ldots(i)$$

On $AB: z = x$, on $BC: z = l + iy$, on $CD: z = x + im$ and on $DA: z = -l + iy$.

Therefore, (*i*) becomes

$$\int_{-l}^{l} e^{-x^2}dx + \int_0^m e^{-(l+iy)^2} idy + \int_l^{-l} e^{-(x+im)^2}dx + \int_m^0 e^{-(-l+iy)^2}dy = 0$$

or $$\int_{-l}^{l} e^{-x^2}dx - \int_{-l}^{l} e^{-x^2-2imx+m^2}dx + \int_0^m e^{-l^2-2ily+y^2}.\,idy$$

$$-\int_0^m e^{-l^2+2ily+y^2}.\,idy = 0 \qquad \ldots(ii)$$

Now let $l \to \infty$. Then the last two integrals

$$= ie^{-l^2} \int_0^m e^{y^2} (e^{-2ily} - e^{2ily})\, dy = 2e^{-l^2} \int_0^m e^{y^2} \sin 2ly\, dy \to 0$$

[$\because$ As $l \to \infty$, $e^{-l^2} \to 0$ and $\sin 2ly$ is finite]

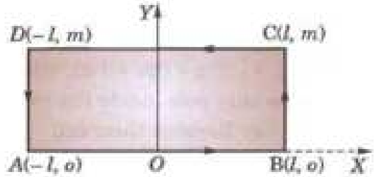


Fig. 20.25

Hence (*ii*) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} dx - e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} (\cos 2mx - i \sin 2mx) dx = 0$$

Equating real parts, we get

$$e^{m^2} \int_{-\infty}^{\infty} e^{-x^2} \cos 2mx\, dx = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$ [See p. 289]

or $$\int_0^{\infty} e^{-x^2} \cos 2mx\, dx = \frac{1}{2}\sqrt{\pi e^{-m^2}}$$

(*d*) **Indenting the contours having poles on the real axis.** So far we have considered such cases in which there is no pole on the real axis. When the integrand has a simple pole on the real axis, we delete it from the region by indenting the contour (*i.e.*, by drawing a small semi-circle having the pole for the centre). The method will be clear from the following example.

Example 20.45. *Evaluate* $\int_0^{\infty} \frac{\sin mx}{x} dx$, *when* $m > 0$. *(U.P.T.U., 2007)*

Solution. Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z) dz$ where C consists of

(*i*) the real axis from r to R,

(*ii*) the upper half of the circle $C_R : |z| = R$,

(*iii*) the real axis $-R$ to $-r$,

(*iv*) the upper half of the circle $C_r : |z| = r$ (Fig. 20.26).

Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

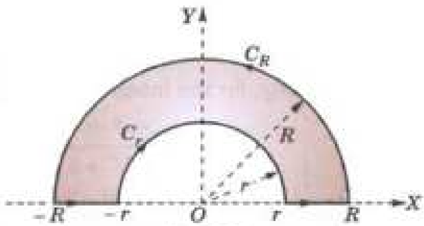


Fig. 20.26

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz = 0 \quad ...(i)$$

Now $$\int_{C_R} f(z)dz = \int_0^{\pi} \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} . Rie^{i\theta} d\theta$$ [$\because$ $z = Re^{i\theta}$]

$$= i \int_0^{\pi} e^{imR(\cos\theta + i \sin\theta)} d\theta$$

Since $|e^{imR(\cos\theta + i\sin\theta)}| = |e^{-mR\sin\theta + imR\cos\theta}| = e^{-mR\sin\theta}$

$\therefore$ $$\left|\int_{C_R} f(z)dz\right| \le \int_0^{\pi} e^{-mR\sin\theta} d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta} d\theta$$

$$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi} d\theta$$ [$\because$ for $0 \le \theta \le \pi/2$, $\sin\theta/\theta \ge 2/\pi$]

$$= \frac{\pi}{mR}\left(1 - e^{-mR}\right) \text{ which } \to 0 \text{ as } R \to \infty,$$

Also $$\int_{C_r} f(z)dz = i\int_{\pi}^{0} e^{imr(\cos\theta + i\sin\theta)} d\theta \to i\int_{\pi}^{0} d\theta \text{ i.e., } -i\pi \text{ as } r \to 0.$$

Hence as $r \to 0$ and $R \to \infty$, we get from (*i*) $\int_0^{\infty} f(x)dx + 0 + \int_{-\infty}^{0} f(x)dx - i\pi = 0$

or $$\int_{-\infty}^{\infty} f(x)dx = i\pi \text{ i.e., } \int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi \quad ...(ii)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi. \text{ Hence } \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

Obs. Equating real parts from both sides of (*ii*), we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x} dx = 0.$$

Example 20.46. *Show that* $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}, 0 < p < 1.$

Solution. Integrate $f(z) = \frac{z^{p-1}}{1+z}$ along the contour consisting of the circles α and γ of radii a and R and the lines AB and FG along x-axis (Fig. 20.27). There is a simple pole at $z = -1$ which is within the contour.

$$\therefore \quad \text{Res} f(-1) = \underset{z \to -1}{\text{Lt}} (1+z) \cdot \frac{z^{p-1}}{1+z} = \underset{z \to -1}{\text{Lt}} z^{p-1} = (-1)^{p-1} = e^{i\pi(p-1)}$$

Thus $$\int_{AB} f(z)dz + \int_{\gamma} f(z)dz + \int_{FG} f(z)dz + \int_{\alpha} f(z)dz = 2\pi i\, e^{i\pi(p-1)} \quad \text{...(i)}$$

On $AB : z = x$ and on $FG : z = xe^{2\pi i}$

$$\therefore \quad \int_{AB} f(z)dz + \int_{FG} f(z)dz = \int_a^R \frac{x^{p-1}}{1+x} dx + \int_R^a \frac{(xe^{2\pi i})^{p-1}}{1+xe^{2\pi i}} dx\, e^{2\pi i}$$

$$= \int_a^R \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}]\, dx$$

On the circle $\gamma : z = Re^{i\theta}$. So

$$\int_{\gamma} f(z)dz = \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1}}{1+Re^{i\theta}} Re^{i\theta}\, i\, d\theta$$

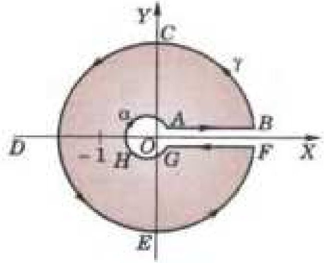


Fig. 20.27

For large R, the integrand is of the order $\frac{R^{p-1} \cdot R}{1+R}$ *i.e.* R^{p-1} which tends to zero as $R \to \infty$. $(\because \; p < 1)$

Hence $$\int_{\gamma} f(z)dz \to 0 \text{ as } R \to \infty$$

On the circle $\alpha : z = ae^{i\theta}$. So

$$\int_{\alpha} f(z)dz = \int_{2\pi}^{0} \frac{(ae^{i\theta})^{p-1}}{1+ae^{i\theta}} ae^{i\theta}\, id\theta$$

For small α, the integrand is of the order a^p which tends to zero as $a \to 0$. $(\because \; p > 0)$

Thus on taking limits as $a \to 0$ and $R \to \infty$, (*i*) gives

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} [1 - e^{2\pi i(p-1)}] dx = 2\pi i\, e^{i\pi(p-1)}$$

or $$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{i\pi(p-1)}}{1 - e^{2\pi i(p-1)}} = \frac{2\pi i\, e^{ip\pi}(-1)}{1 - e^{2ip\pi}\,(1)} = \frac{2i \cdot \pi}{e^{ip\pi} - e^{-ip\pi}} = \frac{\pi}{\sin p\pi}.$$

Example 20.47. *Prove that* $\int_0^{\infty} \sin x^2\, dx = \int_0^{\infty} \cos x^2\, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}.$ *(Osmania, 2003)*

Solution. Consider $\int_C e^{-z^2} dz$ where C consists of the real axis from O to A, part of circle AB of radius R and the line $\theta = \frac{\pi}{4}$. (Fig. 20.28).

e^{-z^2} has no singularity within C.

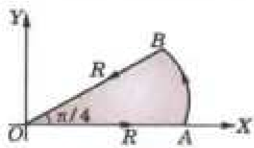


Fig. 20.28

$$\therefore \quad \int_{OA} e^{-z^2} dz + \int_{AB} e^{-z^2} dz + \int_{BO} e^{-z^2} dz = 0 \quad \text{...(i)}$$

On $OA : z = x$, $\therefore$ $\int_{OA} e^{-z^2} dz = \int_0^R e^{-x^2} dx \to \sqrt{\pi/2}$ as $R \to \infty$ [See p. 289]

On $AB : z = Re^{i\theta}$,

$$\therefore \quad \int_{AB} e^{-z^2} dz = \int_0^{\pi/4} e^{-R^2(\cos 2\theta + i \sin 2\theta)} \cdot Re^{i\theta} \cdot id\theta \to 0 \quad \text{as} \quad R \to \infty.$$

[$\because$ integrand $\to 0$ as $R \to \infty$]

On $BO : z = re^{i\pi/4}$ and $z^2 = r^2 e^{i\pi/2} = ir^2$

$$\therefore \quad \int_{BO} e^{-z^2} dz = \int_R^0 e^{-ir^2} \cdot e^{i\pi/4} dr = -\int_0^R e^{-ix^2} \frac{1+i}{\sqrt{2}} dx$$

$$\to -\int_0^\infty (\cos x^2 - i \sin x^2) \frac{1+i}{\sqrt{2}} dx \qquad \text{when } R \to \infty$$

Substituting these in (*i*), we get

$$\frac{1}{2}\sqrt{\pi} + 0 - \int_0^\infty (\cos x^2 - i \sin x^2)\left(\frac{1+i}{\sqrt{2}}\right) dx = 0$$

Equating real and imaginary parts, we obtain

$$\int_0^\infty (\cos x^2 + \sin x^2) dx = \frac{1}{2}\sqrt{(2\pi)} \quad \text{and} \quad \int_0^\infty (\cos x^2 - \sin x^2) dx = 0$$

Hence $$\int_0^\infty \sin x^2 \, dx = \int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\left(\frac{\pi}{2}\right)}.$$

PROBLEMS 20.9

Apply the calculus of residues, to prove that

1. $\int_0^{2\pi} \frac{d\theta}{1 - 2p \sin\theta + p^2} = \frac{2\pi}{1-p^2} (0 < p < 1)$. (*Hissar, 2007 ; Mumbai, 2006 ; Kerala, 2005*)
2. $\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2} = \frac{\pi}{1 - r^2}$. (*J.N.T.U., 2006 ; Madras, 2006 ; Anna, 2003*)
3. $\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{(a^2-1)}} (a > 1)$. (*P.T.U., 2010*)
4. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$. (*U.P.T.U., 2010*)
5. $\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}[a - \sqrt{(a^2 - b^2)}], (0 < b < a)$. (*J.N.T.U., 2003*)
6. $\int_0^{2\pi} \frac{a d\theta}{a^2 + \sin^2\theta} = \frac{2\pi}{\sqrt{(1+a^2)}}, (a > 0)$. (*S.V.T.U., 2009*)
7. $\int_0^{2\pi} \frac{d\theta}{(5 - 3\cos\theta)^2} = \frac{5\pi}{32}$.
8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a+b} (a, b > 0)$. (*P.T.U., 2007 ; Mumbai, 2006 ; Anna, 2003*)
9. $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$. (*A.M.I.E.T.E., 2003 ; Delhi, 2002*)
10. $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{\sqrt{2}}$. (*J.N.T.U., 2006*)
11. $\int_0^{\infty} \frac{dx}{(1 + x^2)^2} = \frac{\pi}{4}$. (*Madras, 2006 ; Kerala, 2005*)
12. $\int_0^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}$. (*Kerala, 2005*)
13. $\int_0^{\infty} \frac{x^2}{x^6 + 1} dx = \frac{\pi}{6}$. (*Rohtak, 2006*)
14. $\int_{-\infty}^{\infty} \frac{\cos mx}{e^x + e^{-x}} dx = \frac{\pi}{2} \operatorname{sech} \frac{m\pi}{2}$.
15. $\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \pi/2$. (*P.T.U., 2005*)
16. $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$. (*Kerala, 2005*)
17. $\int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 4x + 5} dx = \frac{-\pi \sin 2}{e}$.
18. $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) (a > b > 0)$.
19. $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$.

20.12. OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 20.10

Select the correct answer or fill up the blanks in each of the following questions :

1. The only function that is analytic from the following is
   (i) $f(z) = \sin z$ (ii) $f(z) = \bar{z}$ (iii) $f(z) = \text{Im}(z)$ (iv) $R(iz)$.
2. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then $f'(z) =$
   (i) $\dfrac{\partial u}{\partial x} - i\dfrac{\partial v}{\partial x}$ (ii) $\dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ (iii) $\dfrac{\partial v}{\partial y} - i\dfrac{\partial v}{\partial x}$.
3. If $2x - x^2 + ay^2$ is to be harmonic, then a should be
   (a) 1 (b) 2 (c) 3 (d) 0.
4. The analytic function which maps the angular region $0 \le \theta \le \pi/4$ onto the upper half plane is
   (i) z^2 (ii) $4z$ (iii) z^4 (iv) 2θ.
5. An angular domain in the complex plane is defined by $0 < \text{amp}(z) < \pi/4$. The mapping which maps this region onto the left half plane is
   (i) $w = z^4$ (ii) $w = iz^4$ (iii) $w = -z^4$ (iv) $w = -iz^4$.
6. The mapping $w = z^2 - 2z - 3$ is
   (i) conformal within $|z| = 1$ (ii) not conformal at $z = 1$
   (iii) not conformal at $z = -1$ and $z = 3$ (iv) conformal everywhere.
7. If $z = re^{i\theta}$, then the image of θ = constant under the mapping $w(z) = Re^{i\phi} = iz^3$ is
   (i) $\phi = 3\theta$ (ii) $\phi = 3\theta + \pi/2$ (iii) $\phi = 3\theta - \pi/2$ (iv) $\phi = \theta^3$.
8. The fixed points of the mapping $w = (5z + 4)/(z + 5)$ are
   (i) 2, 2 (ii) 2, – 2 (iii) – 2, – 2 (iv) – 4/5, 5.
9. The value of $\int_C (4x^3\,dx + 3y^2z^2\,dy + 2y^3z\,dz)$ where C is any path joining $A(-1, 1, 0)$ to $B(1, 2, 1)$ is
   (i) 0 (ii) 1 (iii) 8 (iv) – 8.
10. The value of $\int_C \dfrac{3z^2 + 7z + 1}{z + 1}\,dz$ where C is $|z| = 1/2$ is
    (i) $2\pi i$ (ii) 0 (iii) πi (iv) $\pi i/2$.
11. The value of $\int_C \dfrac{3z + 4}{z(2z + 1)}\,dz$ where C is the circle $|z| = 1$ is
    (i) $2\pi i$ (ii) $3\pi i$ (iii) 4 (iv) – 4.
12. The residue of a function can be found if the pole is an isolated singularity :
    (i) True (ii) False (iii) Partially false (iv) none of these.
13. The value of $\int_C \dfrac{z\,dz}{\sin z}$ where $C : |z| = 4$ is
    (i) $2\pi i$ (ii) 0 (iii) $-2\pi i$ (iv) $4\pi i$.
14. The value of $\int_C \tanh z\,dz$, where $C : |z| = 3$, is
    (i) 0 (ii) πi (iii) $2\pi i$ (iv) $4\pi i$.
15. The harmonic conjugate of the function $u(x, y) = 2x(1 - y)$ is ...... (U.P.T.U., 2009)
16. Harmonic conjugate of $x^3 - 3xy^2$ is ....
17. The curves $u(x, y) = c$ and $v(x, y) = c'$ are orthogonal if ......
18. The value of $\int_0^{1+i} z^2\,dz$ along the line $x = y$ is ...... 19. Residue of $\dfrac{\cos z}{z}$ at $z = 0$ is ......
20. The critical point of the transformation $w^2 = (z - a)(z - b)$ is ......
21. Image of $|z + 1| = 1$ under the mapping $w = 1/z$ is......
22. The poles of $f(z) = (z^3 - 1)/(z^3 + 1)$ are z = ... 23. $w = \log z$ is analytic everywhere except at z = ...
24. If $f(z) = -\dfrac{1}{z - 1} - 2[1 + (z - 1) + (z - 1)^2 + ...]$, then the residue of $f(z)$ at $z = 1$ is ......
25. If $|z| < 1$ then Taylor's series expansion of $\log(1 + z)$ about $z = 0$ is ......

26. The value of $\int_C \frac{4z^2+z+5}{z-4}\,dz$ where C is $9x^2+4y^2=36$, is ......
27. The value of $\int z^4 e^{1/z}\,dz$, where C is $|\ z\ |=1$, is
(i) πi (ii) $\pi i/12$ (iii) $\pi i/60$ (iv) $-\pi i/60$.
28. If $f(z)$ has a pole of order three at $z=a$ Res $[f(a)]=$ ......
29. The value of $\int_C \frac{e^z\,dz}{(z-3)^2}$, C being $|\ z\ |=2$, is ......
30. The CR equations for $f(z)=u(x,y)+iv(x,y)$ to be analytic are ......
31. If $f(z)$ is analytic in a simply connected domain D and C is any simple closed path then $\int_C f(z)\,dz=$ ......
32. The harmonic conjugate of $e^x \cos y$ is ......
33. The value of $\oint_C \cos z\,dz$ where C is the circle $|\ z\ |=1$, is ......
34. The singularity of $f(z)=z/(z-2)^3$ is ......
35. The function $f(z)=\bar{z}$ is analytic at ......
36. C-R equations for a function to be analytic, in polar form, are ......
37. If C is the circle $|z-a|=r$, $\int_C (z-a)^n\,dz$ [n, any integer $\neq -1$] $=$ ......
38. A simply connected region is that ......
39. A holomorphic function is that ......
40. The poles of the function $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ are at $z=$ ......
41. The cross-ratio of four points z_1, z_2, z_3, z_4 is ......
42. The value of $\int_C |z|\,dz$, where C is the contour represented by the straight line from $z=-i$ to $z=i$, is ......
43. Taylor's series expansion of $\left(\frac{1}{z-2}-\frac{1}{z-1}\right)$ in the region $|z|<1$, is ......
44. The invariant points of the transformation $w=(1+z)/(1-z)$ are $z=$ ......
45. The residue at $z=0$ of $\frac{1+e^z}{z\cos z+\sin z}$ is ......
46. The transformation $w=Cz$ consists of ......
47. The residue of $f(z)$ at a pole is ......
48. The value of $\int_C \frac{1}{z-1}\,dz$, C being $|z|=2$, is ......
49. If C is $|z|=1/2$, $\int_C \frac{z^2-z+1}{z-1}\,dz=$ ......
50. Singular points of $\frac{\cos \pi z}{(z-1)(z-2)}$ are ......
51. Taylor series expansion of $\frac{1}{z-2}$ in $|z|<1$ is ......
52. $\underset{z\to\infty}{\text{Lt}} \frac{iz^3+iz-1}{(2z+3)(z-1)^2}=$ .... (P.T.U., 2007)
53. The poles of $\frac{(z-1)^2}{z(z-2)^2}$ are at $z=$ ......
54. Cauchy's integral theorem states that ......
55. The critical points of the transformation $w=z+1/z$ are ......
56. $\int_c \frac{dz}{2z-3}$, where $|z|=1$, is ......
57. The zeroes and singularities of $\frac{z^2+1}{1-z^2}$ are ......
58. Residue of $\tan z$ at $z=\pi/2$ is ......
59. Singularity of $e^{z^{-1}}$ at $z=0$ is of the type ......
60. Res $(e^{1/z})_{z=0}=$ ......
61. Taylor's series expansion of $\sin z$ about $z=\pi/4$ is ......
62. Image of $|z|=2$ under $w=z+3+2i$ is ....
63. The poles of $\cot z$ are ....
64. If a is simple pole, then Res $[\phi(z)/\psi(z)]_{z=a}=$ ....
65. Bilinear transformation always transforms circles into ....
66. If $f(z)$ and $\overline{f(z)}$ are analytic functions, then $f(z)$ is constant. (True or False) (*Mumbai, 2006*)
67. The function $u(x,y)=2xy+3xy^2-2y^3$ is a harmonic functions. (True or False) (*P.T.U., 2009 S*)
68. The function $e^x \cos y$ is harmonic. (True or False)

69. $\int_C \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$, if $z = a$ is a point within C. (True or False)
70. The transformation affected by an analytic function $w = f(z)$ is conformal at every point of the z-plane where $f'(z) = 0$. (True or False)
71. The function $\bar{z}$ is not analytic at any point. (True or False)
72. Under the transformation $w = 1/z$, circle $x^2 + y^2 - 6x = 0$ transforms into a straight line in the w-plane. (True or False)
73. If $w = f(z)$ is analytic, then $\frac{dw}{dz} = -i\frac{\partial w}{\partial y}$. (True or False)
74. An analytic function with constant imaginary part is constant. (True or False)
75. If $u + iv$ is analytic, then $v - iu$ is also analytic. (True or False)
76. $f(z) = I_m(z)$ is not analytic. (True or False)
77. The cross-ratio of four points is not invariant under bilinear transformation. (True or False)
78. $z = 0$ is not a critical point of the mapping $w = z^2$. (True or False)
79. $f(z) = \text{Re}(z^2)$ is analytic. (True or False)
80. An analytic function with constant modulus is constant. (True or False)
81. The function $|\bar{z}|^2$ is not analytic at any point. (True or False)
82. If $f(z) = z^2$, then the family of curves $x^2 - y^2 = C_1$, and $xy = C_2$ are orthogonal. (True or False)

CHAPTER 21

Laplace Transforms

21.1 INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering.

This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850–1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916–17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms.*

The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

21.2 (1) DEFINITION

Let $f(t)$ be a function of t defined for all positive values of t. Then the **Laplace transforms** *of $f(t)$, denoted by $L\{f(t)\}$ is defined by*

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt \qquad \text{...(1)}$$

provided that the integral exists. s is a parameter which may be a real or complex number.

$L\{f(t)\}$ being clearly a function of s is briefly written as $\bar{f}(s)$ *i.e.*, $L\{f(t)\} = \bar{f}(s)$,

which can also be written as $f(t) = L^{-1}\{\bar{f}(s)\}$.

Then $f(t)$ is called the **inverse Laplace transform** of $\bar{f}(s)$. The symbol L, which transforms $f(t)$ into $\bar{f}(s)$, is called the *Laplace transformation operator*.

**Pierre de Laplace* (1749–1827) (See footnote p. 18) used such transforms, much earlier in 1799, while developing the theory of probability.

(2) Conditions for the existence

The Laplace transform of $f(t)$ *i.e.,* $\int_0^\infty e^{-st} f(t)\,dt$ *exists for* $s > a$, *if*

(*i*) $f(t)$ *is continuous* (*iii*) $\underset{t\to\infty}{\text{Lt}} \{e^{-at} f(t)\}$ *is finite.*

It should however, be noted that the above conditions are sufficient and not necessary.

For example, $L(1/\sqrt{t})$ exists, though $1/\sqrt{t}$ is infinite at $t = 0$.

21.3 TRANSFORMS OF ELEMENTARY FUNCTIONS

The direct application of the definition gives the following formulae :

(1) $\mathbf{L(1) = \dfrac{1}{s}}$ $(s > 0)$

(2) $\mathbf{L(t^n) = \dfrac{n!}{s^{n+1}}}$, when $n = 0, 1, 2, 3, \ldots$ $\left[\text{Otherwise } \dfrac{\Gamma(n+1)}{s^{n+1}}\right]$

(3) $\mathbf{L(e^{at}) = \dfrac{1}{s-a}}$ $(s > a)$

(4) $\mathbf{L(\sin at) = \dfrac{a}{s^2+a^2}}$ $(s > 0)$

(5) $\mathbf{L(\cos at) = \dfrac{s}{s^2+a^2}}$ $(s > 0)$

(6) $\mathbf{L(\sinh at) = \dfrac{a}{s^2-a^2}}$ $(s > |a|)$

(7) $\mathbf{L(\cosh at) = \dfrac{s}{s^2-a^2}}$ $(s > |a|)$

Proofs. **(1)** $$L(1) = \int_0^\infty e^{-st} \cdot 1\,dt = \left| -\frac{e^{-st}}{s} \right|_0^\infty = \frac{1}{s} \text{ if } s > 0.$$

(2) $$L(t^n) = \int_0^\infty e^{-st} \cdot t^n\,dt = \int_0^\infty e^{-p} \cdot \left(\frac{p}{s}\right)^n \frac{dp}{s}, \text{ on putting } st = p$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p} \cdot p^n\,dp = \frac{\Gamma(n+1)}{s^{n+1}}, \text{ if } n > -1 \text{ and } s > 0. \text{ [Page 302]}$$

In particular $L(t^{-1/2}) = \dfrac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\dfrac{\pi}{s}}$; $L(t^{1/2}) = \dfrac{\Gamma(3/2)}{s^{3/2}} = \dfrac{\sqrt{\pi}}{2s^{3/2}}$

In n be a positive integer, $\Gamma(n+1) = n!$ [(*v*) p. 302],

therefore, $L(t^n) = n!/s^{n+1}$.

(3) $$L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left| \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty = \frac{1}{s-a}, \text{ if } s > a.$$

(4) $$L(\sin at) = \int_0^\infty e^{-st} \sin at\,dt = \left| \frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at) \right|_0^\infty = \frac{a}{s^2+a^2}$$

Similarly, the reader should prove **(5)** himself.

(6) $$L(\sinh at) = \int_0^\infty e^{-st} \sinh at\,dt = \int_0^\infty e^{-st} \left(\frac{e^{at} - e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2} \text{ for } s > |a|.$$

Similarly, the reader should prove **(7)** himself.

21.4 PROPERTIES OF LAPLACE TRANSFORMS

I. Linearity property. *If a, b, c be any constants and f, g, h any functions of t, then*

$$\mathbf{L\,[af(t) + bg(t) - ch(t)] = aL\,\{f(t)\} + bL\,\{g(t)\} - cL\,\{h(t)\}}$$

For by definition,

$$\text{L.H.S.} = \int_0^\infty e^{-st}\,[af(t) + bg(t) - ch(t)]\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt - c\int_0^\infty e^{-st} h(t)\,dt = aL\{f(t)\} + bL\{g(t)\} - cL\{h(t)\}$$

This result can easily be generalised.

Because of the above property of L, it is called a *linear operator*.

II. First shifting property. *If* $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\,\{e^{at} f(t)\} = \bar{f}(s - a)}.$$

By definition, $L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt$

$$= \int_0^\infty e^{-rt} f(t)\,dt, \text{ where } r = s - a = \bar{f}(r) = \bar{f}(s-a).$$

Thus, if we know the transform $\bar{f}(s)$ of $f(t)$, we can write the transform of $e^{at} f(t)$ simply replacing s by $s - a$ to get $\bar{f}(s - a)$.

Application of this property leads us to the following useful results :

(1) $\mathbf{L\,(e^{at}) = \dfrac{1}{s-a}}$ $\left[\because\ L(1) = \dfrac{1}{s}\right.$

(2) $\mathbf{L\,(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}}$ $\left[\because\ L(t^n) = \dfrac{n!}{s^{n+1}}\right.$

(3) $\mathbf{L\,(e^{at}\sin bt) = \dfrac{b}{(s-a)^2 + b^2}}$ $\left[\because\ L(\sin bt) = \dfrac{b}{s^2 + b^2}\right.$

(4) $\mathbf{L\,(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2 + b^2}}$ $\left[\because\ L(\cos bt) = \dfrac{s}{s^2 + b^2}\right.$

(5) $\mathbf{L\,(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2 - b^2}}$ $\left[\because\ L(\sinh bt) = \dfrac{b}{s^2 - b^2}\right.$

(6) $\mathbf{L\,(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}}$ $\left[\because\ L(\cosh bt) = \dfrac{s}{s^2 - b^2}\right.$

where in each case $s > a$.

Example 21.1. *Find the Laplace transforms of*
(i) sin 2t sin 3t *(ii) cos² 2t* *(iii) sin³ 2t.*

Solution. (*i*) Since $\sin 2t \sin 3t = \frac{1}{2}[\cos t - \cos 5t]$

$$\therefore\quad L(\sin 2t \sin 3t) = \tfrac{1}{2}[L(\cos t) - L(\cos 5t)] = \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

(*ii*) Since $\cos^2 2t = \dfrac{1}{2}(1 + \cos 4t)$

$$\therefore\quad L(\cos^2 2t) = \frac{1}{2}[L(1) + L\cos 4t] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

(*iii*) Since $\sin 6t = 3 \sin 2t - 4 \sin^3 2t$

or $\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t$

$\therefore$ $L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$

$$= \frac{3}{4} \cdot \frac{2}{s^2 + 2^2} - \frac{1}{4} \cdot \frac{2}{s^2 + 6^2} = \frac{48}{(s^2 + 4)(s^2 + 36)}.$$

Example 21.2. *Find the Laplace transform of*
(*i*) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$. (*ii*) $e^{2t} \cos^2 t$ (V.T.U., 2006) (*iii*) $\sqrt{t}e^{3t}$. (P.T.U., 2009)

Solution. (*i*) $L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} = 2L(e^{-3t} \cos 5t) - 3L(e^{-3t} \sin 5t)$

$$= 2 \cdot \frac{s + 3}{(s + 3)^2 + 5^2} - 3 \cdot \frac{5}{(s + 3)^2 + 5^2} = \frac{2s - 9}{s^2 + 6s + 34}.$$

(*ii*) Since $L(\cos^2 t) = \frac{1}{2} L(1 + \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2 + 4}\right\}$

$\therefore$ by shifting property, we get

$$L(e^{2t} \cos^2 t) = \frac{1}{2}\left\{\frac{1}{s - 2} + \frac{s - 2}{(s - 2)^2 + 4}\right\}.$$

(*iii*) Since $L(\sqrt{t}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{(1/2) \cdot \Gamma\pi}{s^{3/2}}$

$\therefore$ by shifting property, we obtain $L(e^{3t}\sqrt{t}) = \frac{\sqrt{\pi}}{2} \frac{1}{(s - 3)^{3/2}}$.

Example 21.3. *If* $L f(t) = \bar{f}(s)$, *show that*

$$L[(\sinh at) f(t)] = \frac{1}{2}[\bar{f}(s - a) - \bar{f}(s + a)]$$

$$L[(\cosh at) f(t)] = \frac{1}{2}[\bar{f}(s - a) + \bar{f}(s + a)]$$

Hence evaluate (*i*) $\sinh 2t \sin 3t$ (*ii*) $\cosh 3t \cos 2t$.

Solution. We have $L[(\sinh at) f(t)] = L\left[\frac{1}{2}(e^{at} - e^{-at}) f(t)\right] = \frac{1}{2}[L\{e^{at} f(t)\} - L\{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s - a) - \bar{f}(s + a)], \text{ by shifting property.}$$

Similarly, $L[(\cosh at) f(t)] = \frac{1}{2}[L\{e^{at} f(t)\} + \{e^{-at} f(t)\}]$

$$= \frac{1}{2}[\bar{f}(s - a) + \bar{f}(s + a)], \text{ by shifting property.}$$

(*i*) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, the first result gives

$$L(\sinh 2t \sin 3t) = \frac{1}{2}\left\{\frac{3}{(s - 2)^2 + 3^2} - \frac{3}{(s + 2)^2 + 3^2}\right\} = \frac{12s}{s^4 + 10s^2 + 169}$$

(*ii*) Since $L(\cos 2t) = \frac{s}{s^2 + 2^2}$, the second result gives

$$L(\cosh 3t \cos 2t) = \frac{1}{2}\left\{\frac{s - 3}{(s - 3)^2 + 2^2} + \frac{s + 3}{(s + 3)^2 + 2^2}\right\} = \frac{2s(s^2 - 5)}{s^4 - 10s^2 + 169}.$$

Example 21.4. *Show that*

(i) $L(t \sin at) = \dfrac{2as}{(s^2+a^2)^2}$ *(Bhopal, 2001)* (ii) $L(t \cos at) = \dfrac{s^2-a^2}{(s^2+a^2)^2}$.

Solution. Since $L(t) = 1/s^2$. $\therefore$ $L(te^{iat}) = \dfrac{1}{(s-ia)^2} = \dfrac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$

or
$$L[t(\cos at + i \sin at)] = \frac{(s^2-a)^2 + i(2as)}{(s^2+a^2)^2}$$

Equating the real and imaginary parts from both sides, we get the desired results.

Example 21.5. *Find the Laplace transform of $f(t)$ defined as*

(i) $f(t) = t/\tau$, *when* $0 < t < \tau$
$= 1$, *when* $t > \tau$. *(Kerala, 2005)*

(ii) $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$ *(J.N.T.U., 2006 ; W.B.T.U., 2005)*

Solution. (i)
$$Lf(t) = \int_0^\tau e^{-st} \cdot \frac{t}{\tau} dt + \int_\tau^\infty e^{-st} \cdot 1\, dt = \frac{1}{\tau}\left[\left| t \cdot \frac{e^{-st}}{-s}\right|_0^\tau - \int_0^\tau 1 \cdot \frac{e^{-st}}{-s} dt\right] + \left|\frac{e^{-st}}{-s}\right|_\tau^\infty$$
$$= \frac{1}{\tau}\left[\frac{\tau e^{-s\tau} - 0}{-s} - \left|\frac{e^{-st}}{s^2}\right|_0^\tau\right] + \frac{0 - e^{-s\tau}}{-s} = \frac{-e^{-s\tau}}{s} - \frac{e^{-s\tau}-1}{\tau s^2} + \frac{e^{-s\tau}}{s} = \frac{1-e^{-s\tau}}{\tau s^2}.$$

(ii)
$$L\{f(t)\} = \int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot (0)\, dt$$
$$= \left|\frac{e^{-st}}{-s}\right|_0^1 + \left| t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right|_1^2 = \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\}$$
$$= \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.$$

Example 21.6. *Find the Laplace transform of* (i) $\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)^3$. *(Kurukshetra, 2005)*

(ii) $\dfrac{\cos \sqrt{t}}{\sqrt{t}}$ *(Mumbai, 2009)*

Solution. (i) Since $(\sqrt{t} - 1/\sqrt{t})^3 = t^{3/2} - 3t^{1/2} + 3t^{-1/2} - t^{-3/2}$

$$\therefore \quad L(\sqrt{t} - 1/\sqrt{t}) = L(t^{3/2}) - 3L(t^{1/2}) + 3L(t^{-1/2}) - L(t^{-3/2})$$
$$= \frac{\Gamma(3/2+1)}{s^{3/2+1}} - 3\frac{\Gamma(1/2+1)}{s^{1/2+1}} + 3\frac{\Gamma(-1/2+1)}{s^{-1/2+1}} - \frac{\Gamma(-3/2+1)}{s^{-3/2+1}}$$
$$= \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)}{s^{5/2}} - 3\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} + 3\frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} - \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}}$$
$$= \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} + \frac{2\sqrt{\pi}}{s^{-1/2}} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}\right]$$
$$= \frac{\sqrt{\pi}}{4}\left(\frac{3}{s^{5/2}} - \frac{6}{s^{3/2}} + \frac{12}{s^{1/2}} + \frac{8}{s^{-1/2}}\right).$$

(*ii*) We know that $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \infty$

$$\therefore \qquad \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \dots$$

and
$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \dots$$

and
$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \dots$$

$$= \frac{\Gamma 1/2}{\sqrt{s}} - \frac{1}{2}\cdot\frac{1/2\,\Gamma 1/2}{s^{3/2}} + \frac{1}{4!}\frac{3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{5/2}} - \frac{1}{6!}\frac{5/2\,.\,3/2\,.\,1/2\,.\,\Gamma(1/2)}{s^{7/2}} + \dots$$

$$= \sqrt{\left(\frac{\pi}{2}\right)}\left[1 - \frac{1}{(4s)} + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3}\dots\right] = \sqrt{\left(\frac{\pi}{s}\right)}e^{-1/4s}.$$

Example 21.7. *Find the Laplace transform of the function*

(*i*) $f(t) \mid t-1 \mid + \mid t+1 \mid, t \geq 0$ (*S.V.T.U., 2009*)

(*ii*) $f(t) = [t]$, *where* [ ] *stands for the greatest integer function.* (*P.T.U., 2010*)

Solution. (*i*) Given function is equivalent to

$$f(t) = \begin{cases} 2, & 0 \le t < 1 \\ 2t, & t \ge 1 \end{cases}$$

$$\therefore \qquad L\,f(t) = \int_0^1 e^{-st}(2)\,dt + \int_1^\infty e^{-st}\,(2t)dt = 2\left[\left|\frac{e^{-st}}{-s}\right|_0^1 + 2\left|\frac{t.e^{-st}}{-s}\right|_1^\infty - \left|\frac{e^{-st}}{(-s)^2}\right|_1^\infty\right]$$

$$= 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{0-e^{-s}}{-s} - \frac{0-e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right)$$

(*ii*) Given function is equivalent to

$$[t] = 0 \text{ in } (0,1) + 1 \text{ in } (1,2) + 2 \text{ in } (2,3) + 3 \text{ in } (3,4) + \dots$$

$$\therefore \qquad L[f(t)] = \int_0^\infty e^{-st}[f(t)]dt = \int_0^\infty e^{-st}\,[t]dt$$

$$= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(1)dt + \int_2^3 e^{-st}(2)dt + \int_3^4 e^{-st}(3)dt + \dots \infty$$

$$= 0 + \left|\frac{e^{-st}}{-s}\right|_1^2 + 2\left|\frac{e^{-st}}{-s}\right|_2^3 + 3\left|\frac{e^{-st}}{-s}\right|_3^4 + \dots \infty$$

$$= -\frac{1}{s}[(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \dots \infty]$$

$$= \frac{1}{s}(e^{-s} + e^{-2s} + e^{-3s} + \dots \infty) = \frac{1}{s}\left(\frac{e^{-s}}{1-e^{-s}}\right) = \frac{1}{s(e^s - 1)}.$$

III. Change of scale property. *If* $L\{f(t)\} = \bar{f}(s)$, *then* $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\,dt = \int_0^\infty e^{-su/a} f(u)\,.\,du/a \qquad \left|\begin{array}{l}\text{Put } at = u \\ \quad dt = du/a\end{array}\right.$$

$$= \frac{1}{a}\int_0^\infty e^{-su/a} f(u)du = \frac{1}{a}\bar{f}(s/a).$$

Example 21.8. *Find* $L\left\{\frac{\sin at}{t}\right\}$, *given that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left\{\frac{1}{s}\right\}$.

Solution. By the above property,

$$L\left\{\frac{\sin at}{at}\right\} = \frac{1}{a}\tan^{-1}\left\{\frac{1}{(s/a)}\right\} = \frac{1}{a}\tan^{-1}\left(\frac{a}{s}\right) \text{ i.e., } L\left\{\frac{\sin at}{t}\right\} = \tan^{-1}\left\{\frac{a}{s}\right\}.$$

PROBLEMS 21.1

Find the Laplace transforms of

1. $e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$. *(J.N.T.U., 2003)*
2. $1 + 2\sqrt{t} + 3/\sqrt{t}$.
3. $3\cosh 5t - 4\sinh 5t$. *(Nagarjuna, 2006)*
4. $\cos(at + b)$.
5. $(\sin t - \cos t)^2$.
6. $\sin 2t \cos 3t$. *(Kottayam, 2005)*
7. $\sin\sqrt{t}$.
8. $\sin^5 t$. *(Mumbai, 2007)*
9. $\cos^3 2t$.
10. $e^{-at}\sinh bt$
11. $e^{2t}(3t^5 - \cos 4t)$. *(P.T.U., 2007)*
12. $e^{-3t}\sin 5t \sin 3t$. *(V.T.U., 2006)*
13. $e^{-t}\sin^2 t$. *(Mumbai, 2009)*
14. $e^{2t}\sin^4 t$ *(Mumbai, 2007)*
15. $\cosh at \sin at$. *(Delhi, 2002)*
16. $\sinh 3t \cos^2 t$ *(Madras, 2000)*
17. $t^2 e^{2t}$. *(V.T.U., 2008 S)*
18. $(1 + te^{-t})^3$
19. $t\sqrt{(1 + \sin t)}$ *(Mumbai, 2007)*
20. $f(t) = \begin{cases} 4, & 0 \le t <= 1 \\ 3, & t > 1 \end{cases}$ *(U.P.T.U., 2009)*
21. $f(t) = \begin{cases} \sin t, 0 < t < \pi \\ 0, t > \pi. \end{cases}$ *(Madras, 2000 S)*
22. $f(x) = \begin{cases} \sin(x - \pi/3), & x > \pi/3 \\ 0 & , x < \pi/3 \end{cases}$ *(Rajasthan, 2006)*
23. $f(t) = \begin{cases} \cos(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < 2\pi/3 \end{cases}$
24. $f(t) = \begin{cases} t^2, 0 < t < 2 \\ t - 1, 2 < t < 3 \\ 7, t > 3. \end{cases}$ *(Mumbai, 2007)*
25. If $L\{f(t)\} = \dfrac{1}{s(s^2+1)}$, find $L\{e^{-t} f(2t)\}$.

21.5 TRANSFORMS OF PERIODIC FUNCTIONS

If $f(t)$ is a periodic function with period T, i.e., $f(t + T) = f(t)$, then

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)dt}{1 - e^{-sT}}$$

We have $\quad L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^T e^{-st} f(t)dt + \int_T^{2T} e^{-st} f(t)dt + \int_{2T}^{3T} e^{-st} f(t)dt + \ldots$

In the second integral put $t = u + T$, in the third integral put $t = u + 2T$, and so on. Then

$$L\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_0^T e^{-s(u+T)} f(u+T)du + \int_0^T e^{-s(u+2T)} f(u+2T)du + \ldots$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-su} f(u)du + e^{-2sT}\int_0^T e^{-su} f(u)du + \ldots$$

$$[\because \; f(u) = f(u+T) = f(u+2T) \text{ etc.}]$$

$$= \int_0^T e^{-st} f(t)dt + e^{-sT}\int_0^T e^{-st} f(t)dt + e^{-2sT}\int_0^T e^{-st} f(t)dt + \ldots$$

$$= (1 + e^{-sT} + e^{-2sT} + \ldots \infty)\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)dt. \qquad \textit{(V.T.U., 2008 ; Mumbai, 2006)}$$

Example 21.9. *Find the Laplace transform of the function*

$$f(t) = \sin \omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < 2\pi/\omega.$$

(*Kurukshetra, 2005 ; Madras, 2003*)

Solution. Since $f(t)$ is a periodic function with period $2\pi/\omega$.

$$\therefore \quad L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} . 0 \, dt \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left| \frac{e^{-st}(-s \sin \omega t - \omega \cos \omega t)}{s^2+\omega^2} \right|_0^{\pi/\omega} = \frac{\omega e^{-\pi s/\omega} + \omega}{(1-e^{-2\pi s/\omega})(s^2+\omega^2)} = \frac{\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}.$$

Example 21.10. *Draw the graph of the periodic function*

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi. \end{cases}$$

and find its Laplace transform. (*U.P.T.U., 2003*)

Solution. Here the period of $f(t) = 2\pi$ and its graph is as in Fig. 21.1.

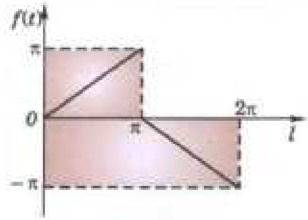


Fig. 21.1

$$\therefore \quad L f(t) = \frac{1}{1-e^{-2\pi s}} \left\{ \int_0^{\pi} e^{-st} t \, dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t) dt \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \left| t\left(\frac{e^{-st}}{-s}\right) - 1 . \left(\frac{e^{-st}}{s^2}\right) \right|_0^{\pi} + \left| (\pi - t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right) \right|_{\pi}^{2\pi} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{-\pi e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2} + \frac{1}{s^2} + \frac{\pi e^{-2\pi s}}{s} + \frac{e^{-2\pi s}}{s^2} - \frac{e^{-\pi s}}{s^2} \right\}$$

$$= \frac{1}{1-e^{-2\pi s}} \left\{ \frac{\pi}{s}\left(e^{-2\pi s} - e^{-\pi s}\right) + \frac{1}{s^2}\left(1 + e^{-2\pi s} - 2e^{-\pi s}\right) \right\}.$$

21.6 TRANSFORMS OF SPECIAL FUNCTIONS

(1) Transform of Bessel functions $J_0(x)$ and $J_1(x)$.

We know that $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots$$ [§ 16.7 (1), p. 553]

$$\therefore \quad L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 . 4^2} . \frac{4!}{s^5} - \frac{1}{2^2 . 4^2 . 6^2} . \frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1 . 3}{2 . 4}\left(\frac{1}{s^4}\right) - \frac{1 . 3 . 5}{2 . 4 . 6}\left(\frac{1}{s^6}\right) + \ldots\right\}$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{(s^2+1)}} \quad \ldots(1)$$

Also since $$J_0'(x) = -J_1(x).$$ [Problem 4(*i*), p. 557]

$$\therefore \quad L\{J_1(x)\} = -L\{J_0'(x)\} = -\{sL\{J_0(x)\} - 1\} = 1 - \frac{s}{\sqrt{(s^2+1)}} \quad \ldots(2)$$

(2) Transform of Error function

We know that $erf(\sqrt{x}) = \dfrac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} e^{-t^2} dt$ (§ 7.18, p. 312)

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{x}} \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots\right) dt = \frac{2}{\sqrt{\pi}} \left(x^{1/2} - \frac{x^{3/2}}{3} + \frac{x^{5/2}}{5.2!} - \frac{x^{7/2}}{7.3!} + \ldots\right)$$

$$\therefore \quad L\{erf(\sqrt{x})\} = \frac{2}{\sqrt{\pi}} \left\{\frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} + \frac{\Gamma(7/2)}{5.2!\,s^{7/2}} - \frac{\Gamma(9/2)}{7.3!\,s^{9/2}} + \ldots\right\}$$

$$= \frac{1}{s^{3/2}} - \frac{1}{2}\frac{1}{s^{5/2}} + \frac{1.3}{2.4} \cdot \frac{1}{s^{7/2}} - \frac{1.3.5}{2.4.6} \cdot \frac{1}{s^{9/2}} + \ldots$$

$$= \frac{1}{s^{3/2}} \left\{1 - \frac{1}{2} \cdot \frac{1}{s} + \frac{1.3}{2.4} \cdot \frac{1}{s^2} - \frac{1.3.5}{2.4.6} \cdot \frac{1}{s^3} + \ldots\right\}$$

$$= \frac{1}{s^{3/2}} \left[1 + \frac{1}{s}\right]^{-1/2} = \frac{1}{s\sqrt{(s+1)}}. \qquad \textit{(Mumbai, 2009)} \quad \ldots(3)$$

(3) Transform of Laguerre's polynomials $L_n(x)$

We know that $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$ (§ 16.18, p. 571)

$$L[L_n(t)] = \int_0^\infty e^{-st} e^t \frac{d^n}{dt^n}(t^n e^{-t}) dt = \int_0^\infty e^{-(s-1)t} \frac{d^n}{dt^n}(e^{-t} t^n) dt$$

$$= \left| e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) \right|_0^\infty + \int_0^\infty e^{-(s-1)t\,(s-1)}\,(s-1) \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt$$

$$= (s-1) \int_0^\infty e^{-(s-1)t} \frac{d^{n-1}}{dt^{n-1}}(e^{-t} t^n) dt. \qquad \text{(Integrating by parts)}$$

$$= (s-1)^n \int_0^\infty e^{-(s-1)t} . e^{-t} . t^n\, dt = (s-1)^n \int_0^\infty e^{-st} . t^n\, dt$$

$$= (s-1)^n L(t^n) = (s-1)^n . \frac{n!}{s^{n+1}}$$

Hence $\quad L[L_n(x)] = \dfrac{n!(s-1)^n}{s^n + 1} \; (s > 1).$

Example 21.11. *Evaluate (i)* $L\{e^{-at} J_0(at)\}$ *(ii)* $L(erf\, 2\sqrt{t})$. *(Mumbai, 2006)*

Solution. (*i*) We know that $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2 + a^2)}}$

By shifting property, we get

$$L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2 + a^2]}} = \frac{1}{\sqrt{(s^2 + 2sa + 2a^2)}}$$

(*ii*) We know that $\quad L(erf\sqrt{t}) = \dfrac{1}{s(s+1)}$

$$\therefore \quad L(erf\; 2\sqrt{t}) = L[erf\sqrt{(4t)}] = \frac{1}{4} . \frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}.$$

PROBLEMS 21.2

1. Find the Laplace transform of the *saw-toothed wave* of period T, given $f(t) = t/T$ for $0 < t < T$. *(V.T.U., 2007)*
2. Find the Laplace transform of the *full-wave rectifier*

   $f(t) = E \sin \omega t,\ 0 < t < \pi/\omega$, having period π/ω.

3. Find the Laplace transform of the *square-wave* (or *meander*) function of period a defined as

$$f(t) = k, \quad \text{when} \quad 0 < t < a$$
$$= -k, \quad \text{when} \quad a < t < 2a.$$
(V.T.U., 2011)

4. Find the Laplace transform of the *triangular wave* of period $2a$ given by

$$f(t) = t, \quad 0 < t < a$$
$$= 2a - t, \quad a < t < 2a.$$
(Nagarjuna, 2008 ; V.T.U., 2008 S ; U.P.T.U., 2002)

Find the Laplace transform of the following functions :

5. $J_0(ax)$. 6. $e^{-at} J_0(bt)$. 7. $e^{2t} \operatorname{erf}(\sqrt{t})$.

21.7 TRANSFORMS OF DERIVATIVES

(1) *If $f'(t)$ be continuous and $L\{f(t)\} = \bar{f}(s)$, then $L\{f'(t)\} = s\bar{f}(s) - f(0)$.*

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \qquad \text{[Integrate by parts]}$$

$$= \left| e^{-st} f(t) \right|_0^\infty - \int_0^\infty (-s) e^{-st} \,.\, f(t)\,dt.$$

Now assuming $f(t)$ to be such that $\underset{t\to\infty}{\mathrm{Lt}}\ e^{-st} f(t) = 0$. When this condition is satisfied, $f(t)$ is said to be *exponential order s.*

Thus, $$L\{f'(t)\} = f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

whence follows the desired result.

(2) *If $f'(t)$ and its first $(n-1)$ derivatives be continuous, then*

$$\mathbf{L\{f^n(t)\} = s^n\,\bar{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0).}$$

Using the general rule of integration by parts (Footnote p. 398).

$$L\{f^n(t)\} = \int_0^\infty e^{-st} f^n(t)\,dt$$

$$= |e^{-st} f^{n-1}(t) - (-s)e^{-st} f^{n-2}(t) + (-s)^2 e^{-st} f^{n-3}(t) - \dots$$

$$+ (-1)^{n-1}(-s)^{n-1} e^{-st} \,.\, f(t)\,|_0^\infty + (-1)^n(-s)^n \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f^{n-1}(0) - sf^{n-2}(0) - s^2 f^{n-3}(0) - \dots - s^{n-1} f(0) + s^n \int_0^\infty e^{-st} f(t)\,dt$$

Assuming that $\underset{t\to\infty}{\mathrm{Lt}}\ e^{-st} f^m(t) = 0$ for $m = 0, 1, 2, \dots, n-1$.

This proves the required result.

21.8 TRANSFORMS OF INTEGRALS

If $L\{f(t)\} = \bar{f}(s)$, then $\mathbf{L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\,\bar{f}(s).}$

Let $$\phi(t) = \int_0^t f(u)\,du, \text{ then } \phi'(t) = f(t) \text{ and } \phi(0) = 0$$

$\therefore$ $$L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0) \qquad \text{[By § 21.7 (1)]}$$

or $$\bar{\phi}(s) = \frac{1}{s} L\{\phi'(t)\} \quad i.e., \quad L\left(\int_0^t f(u)\,du\right) = \frac{1}{s}\bar{f}(s).$$

21.9 MULTIPLICATION BY t^n

If $L\{f(t)\} = \bar{f}(s)$, then

$$\mathbf{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]}, \textit{ where } n = 1, 2, 3 \dots$$

We have $\int_0^\infty e^{-st} f(t)\,dt = \bar{f}(s)$.

Differentiating both sides with respect to s, $\frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\,dt\right\} = \frac{d}{ds}\{\bar{f}(s)\}$

or By Leibnitz's rule for differentiation under the integral sign (p. 233).

$$\int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt = \frac{d}{ds}\{\bar{f}(s)\}$$

or $$\int_0^\infty [-te^{-st} f(t)]\,dt = \frac{d}{ds}[\bar{f}(s)] \quad \text{or} \quad \int_0^\infty e^{-st}[tf(t)]\,dt = -\frac{d}{ds}[\bar{f}(s)]$$

which proves the theorem for $n = 1$.

Now assume the theorem to be true for $n = m$ (say), so that

$$\int_0^\infty e^{-st}[t^m f(t)]\,dt = (-1)^m \frac{d^m}{ds^m}[\bar{f}(s)]$$

Then $$\frac{d}{ds}\left[\int_0^\infty e^{-st} t^m f(t)\,dt\right] = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$$

or By Leibnitz's rule, $\int_0^\infty (-te^{-st}) . t^m f(t)\,dt = (-1)^m \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)]$

or $$\int_0^\infty e^{-st}[t^{m+1} f(t)]\,dt = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}[\bar{f}(s)].$$

This shows that, if the theorem, is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 1$. Hence it is true for $n = 1 + 1 = 2$, and $n = 2 + 1 = 3$ and so on.

Thus the theorem is true for all positive integral values of n. *(U.P.T.U., 2005)*

Example 21.12. *Find the Laplace transforms of*

(i) $t \cos at$ *(Raipur, 2005)* *(ii)* $t^2 \sin at$ *(S.V.T.U., 2007)*

(iii) $t^3 e^{-3t}$ *(Kottayam, 2005)* *(iv)* $te^{-t} \sin 3t$. *(Kurukshetra, 2005)*

Solution. *(i)* Since $L(\cos at) = s/(s^2 + a^2)$

$$\therefore \quad L(t \cos at) = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = -\frac{s^2 + a^2 - s . 2s}{(s^2 + a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2 + a^2)^2}$$ [*cf.* Example 21.4]

(ii) Since $\sin at = \frac{a}{s^2 + a^2}$,

$$\therefore \quad L(t^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2 + a^2}\right) = \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\} = \frac{2a(3s^2 - a^2)}{(s^2 + a^2)^3}.$$

(iii) Since $L(e^{-3t}) = 1/(s + 3)$,

$$\therefore \quad L(t^3 e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s + 3}\right) = -\frac{(-1)^3 . 3!}{(s + 3)^{3+1}} = 6/(s + 3)^4.$$

(iv) Since $L(\sin 3t) = \frac{3}{s^2 + 3^2}$, therefore $L(t \sin 3t) = -\frac{d}{ds}\left(\frac{3}{s^2 + 3^2}\right) = \frac{6s}{(s^2 + 9)^2}$

Now using the shifting property (§ 21.4 II), we get

$$L(e^{-t} t \sin 3t) = \frac{6(s + 1)}{[(s + 1)^2 + 9]^2} = \frac{6(s + 1)}{(s^2 + 2s + 10)^2}.$$

Example 21.13. *Evaluate (i) $L\{t\, J_0(at)\}$ (ii) $L\{t\, J_1(t)\}$ (iii) $L\{t \text{ erf } 2\sqrt{t}\}$.*

Solution. (*i*) Since $L\{J_0(at)\} = \dfrac{1}{\sqrt{(s^2+a^2)}}$

$$\therefore \quad L\{tJ_0(at)\} = -\frac{d}{ds}\{L\, J_0(at)\} = -\frac{d}{ds}\frac{1}{\sqrt{(s^2+a^2)}} = \frac{s}{(s^2+a^2)^{3/2}}$$

(*ii*) Since $\quad L\{J_1(t)\} = 1 - \dfrac{s}{\sqrt{(s^2+1)}}$

$$\therefore \quad L\{t\, J_1(t)\} = -\frac{d}{ds}\{L\, J_0(t)\} = -\frac{d}{ds}\left\{1 - \frac{s}{(\sqrt{s^2+1})}\right\} = \frac{1}{(s^2+1)^{3/2}}$$

(*iii*) Since $\quad L(\text{erf } \sqrt{t}) = \dfrac{1}{s\sqrt{(s+1)}}$

$$\therefore \quad L(\text{erf } 2\sqrt{t}) = L \text{ erf } \sqrt{4(t)} = \frac{1}{4} \cdot \frac{1}{\frac{s}{4}\sqrt{\left(\frac{s}{4}+1\right)}} = \frac{2}{s\sqrt{(s+4)}}$$

Thus $$L(t \text{ erf } 2\sqrt{t}) = -\frac{d}{ds}\left\{\frac{2}{s\sqrt{(s+4)}}\right\} = -\frac{d}{ds}\left\{\frac{2}{\sqrt{(s^3+4s^2)}}\right\} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

21.10 DIVISION BY t

If $L\{f(t)\} = \bar{f}(s)$, *then* $\mathbf{L}\left\{\dfrac{\mathbf{1}}{\mathbf{t}}\,\mathbf{f(t)}\right\} = \displaystyle\int_s^\infty \bar{\mathbf{f}}(\mathbf{s})\,\mathbf{ds}$ *provided the integral exists.*

We have $\bar{f}(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides with respect to s from s to ∞.

$$\int_s^\infty \bar{f}(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds = \int_0^\infty \int_s^\infty f(t)\, e^{-st}\, ds\, dt$$

[Changing the order of integration]

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\, ds\right] dt \qquad [\because \; t \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left|\frac{e^{-st}}{-t}\right|_s^\infty dt = \int_0^\infty e^{-st} \cdot \frac{f(t)}{t}\, dt = L\left\{\frac{1}{t} f(t)\right\}.$$

Example 21.14. *Find the Laplace transform of (i) $(1-e^t)/t$* (*Madras, 2000*)
(*ii*) $\dfrac{\cos at - \cos bt}{t} + t \sin at.$ (*V.T.U., 2010*)

Solution. (*i*) Since $\quad L(1-e^t) = L(1) - L(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$\therefore \quad L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = |\log s - \log(s-1)|_0^\infty$$

$$= \left|\log\left(\frac{s}{s-1}\right)\right|_s^\infty = -\log\left[\frac{1}{1-1/s}\right] = \log\left(\frac{s-1}{s}\right)$$

(*ii*) Since $\quad L(\cos at - \cos bt) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$

$$\therefore \quad L\left(\frac{\cos at - \cos bt}{t}\right) + L(t \sin at) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds - \frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \left| \frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2) \right|_s^\infty - a\,\frac{-2s}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\operatorname*{Lt}_{s\to\infty} \log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} + \frac{2as}{(s^2+a^2)^2}$$

$$= \frac{1}{2}\log\left(\frac{1+0}{1+0}\right) - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) + \frac{2as}{(s^2+a^2)^2} = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2} + \frac{2as}{(s^2+a^2)^2}$$

$[\because \quad \log 1 = 0]$

Example 21.15. *Evaluate (i)* $L\left\{e^{-t}\int_0^t \frac{\sin t}{t}\,dt\right\}$ *(Madras, 2006)*

(ii) $L\left\{t\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$ *(P.T.U., 2005)* *(iii)* $L\left\{\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,dt\,dt\right\}$. *(Mumbai, 2006)*

Solution. (*i*) We know that $L(\sin t) = \dfrac{1}{s^2+1}$

$$L\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{1}{s^2+1}\,ds = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore \quad L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$$

Thus by shifting property, $L\left\{e^{-t}\left(\int_0^t \frac{\sin t}{t}\,dt\right)\right\} = \dfrac{1}{s+1}\cot^{-1}(s+1)$.

(*ii*) Since $\quad L\left(\dfrac{\sin t}{t}\right) = \cot^{-1}s$

$$\therefore \quad L\left(e^{-t}\cdot\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$$

and $\quad L\left\{\displaystyle\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = \dfrac{1}{s}\cot^{-1}(s+1)$

Hence $\quad L\left\{t\cdot\displaystyle\int_0^t e^{-t}\frac{\sin t}{t}\,dt\right\} = -\dfrac{d}{ds}\left\{\dfrac{\cot^{-1}(s+1)}{s}\right\}$

$$= -\frac{s\cdot\left[\dfrac{-1}{1+(s+1)^2}\right] - \cot^{-1}(s+1)}{s^2} = \frac{s + (s^2+2s+2)\cot^{-1}(s+1)}{s^2(s^2+2s+2)}$$

(*iii*) Since $\quad L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore \quad L(t\sin t) = -\frac{d}{ds}\frac{1}{(s^2+1)} = \frac{2s}{(s^2+1)^2}$$

Thus $\quad L\left\{\displaystyle\int_0^t\int_0^t\int_0^t (t\sin t)\,dt\,.\,dt\,.\,dt\right\} = \dfrac{1}{s^3}L(t\sin t) = \dfrac{1}{s^3}\cdot\dfrac{2s}{(s^2+1)^2} = \dfrac{2}{s^2(s^2+1)^2}$

21.11 EVALUATION OF INTEGRALS BY LAPLACE TRANSFORMS

Example 21.16. *Evaluate* (*i*) $\int_0^\infty te^{-3t}\sin t\,dt$ (*V.T.U., 2007*)

(*ii*) $\int_0^\infty \frac{\sin mt}{t}\,dt$ (*iii*) $\int_0^\infty e^{-t}\left(\frac{\cos at-\cos bt}{t}\right)dt$ (*Mumbai, 2009*)

(*iv*) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t}\,dt\right\}$.

Solution. (*i*) $\int_0^\infty te^{-3t}\sin t\,dt = \int_0^\infty e^{-st}(t\sin t)\,dt$ where $s = 3$

$= L(t\sin t)$, by definition.

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2} = \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50}.$$

(*ii*) Since $L(\sin mt) = m/(s^2+m^2) = f(s)$, say.

$\therefore$ Using § 21.10, $L\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)\,ds = \int_0^\infty \frac{m\,ds}{s^2+m^2} = \left|\tan^{-1}\frac{s}{m}\right|_s^\infty$

or by Def., $\int_0^\infty e^{-st}\frac{\sin mt}{t}\,dt = \frac{\pi}{2} - \tan^{-1}\frac{s}{m}$

Now $\underset{s\to 0}{\text{Lt}}\tan^{-1}(s/m) = 0$ if $m > 0$ or π if $m < 0$.

Thus taking limits as $s \to 0$, we get

$$\int_0^\infty \frac{\sin mt}{t}\,dt = \frac{\pi}{2} \text{ if } m > 0 \quad\text{or}\quad -\pi/2 \text{ if } m < 0$$

(*iii*) We know that $L(\cos at) = \frac{s}{s^2+a^2}$ and $L(\cos bt) = \frac{s}{s^2+b^2}$

$$\therefore \quad L\,\frac{\cos at - \cos bt}{t} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}_s^\infty = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

This implies that $\int_0^\infty e^{-st}\left(\frac{\cos at-\cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Taking $s = 1$, we get $\int_0^\infty \left(e^{-t}\,\frac{\cos at - \cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{1+b^2}{1+a^2}\right)$

(*iv*) Since $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \tan^{-1} s = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s.$

$\therefore$ $L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1)$, by shifting property (§ 21.4 *II*).

Thus $L\left[\int_0^t \left\{e^t\left(\frac{\sin t}{t}\right)\right\}dt\right] = \frac{1}{s}\cot^{-1}(s-1)$, by § 21.8.

PROBLEMS 21.3

1. Find $L\left(\int_0^t e^{-t}\cos t\,dt\right)$.

2. Given $L\{2\sqrt{(t/\pi)}\} = 1/s^{3/2}$, show that $L\{1/\sqrt{(\pi t)}\} = 1/\sqrt{s}$. *(U.P.T.U., 2005 ; Madras, 2003)*

3. Given $L\{\sin(\sqrt{t})\} = \dfrac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$, prove that $L\left[\dfrac{\cos(\sqrt{t})}{\sqrt{t}}\right] = \sqrt{\left(\dfrac{\pi}{s}\right)} e^{-1/4s}$. *(Mumbai, 2009)*

Find the Laplace transforms of the following functions :

4. $t\sin^2 t$ *(Nagarjuna, 2008)*
5. $\sin 2t - 2t\cos 2t$. *(Anna, 2003)*
6. $t^2\cos at$.
7. $t\sinh at$.
8. $te^{2t}\sin 3t$. *(Madras, 2003)*
9. $te^{-2t}\sin 4t$. *(V.T.U., 2008)*
10. $t^2 e^{-3t}\sin 2t$. *(Madras, 2000 S)*
11. $(e^{-at} - e^{-bt})/t$. *(Anna, 2005 S)*
12. $(\sin t)/t$. *(P.T.U., 2010)*
13. $\dfrac{(\sin t\sin 5t)}{t}$. *(Mumbai, 2008)*
14. $(e^{at} - \cos bt)/t$. *(U.P.T.U., 2003)*
15. $(e^{-t}\sin t)/t$. *(V.T.U., 2009 S)*
16. $(1 - \cos 3t)/t$. *(V.T.U., 2006)*
17. $(1 - \cos t)/t^2$. *(Hazaribag, 2008)*
18. $2^t + \dfrac{\cos 2t - \cos 3t}{t} + t\sin t$. *(V.T.U., 2004)*

19. Evaluate (*i*) $\displaystyle\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dT$ *(Mumbai, 2008 ; P.T.U., 2006)*

(*ii*) $\displaystyle\int_0^\infty \frac{e^{-\sqrt{2}t}\sinh t\sin t}{t}\,dt$ *(Mumbai, 2005)* (*iii*) $\displaystyle\int_0^\infty te^{-2t}\sin 3t\,dt$ *(V.T.U., 2008)*

(*iv*) $\displaystyle\int_0^\infty te^{-t}\sin^4 t\,dt$.

20. Prove that (*i*) $\displaystyle\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\,dt = \log\frac{b}{a}$, *(S.V.T.U., 2009 ; Mumbai, 2007 ; J.N.T.U., 2006)*

(*ii*) $\displaystyle\int_0^\infty \frac{e^{-2t}\sinh t}{t}\,dt = \frac{1}{2}\log 3$ *(Mumbai, 2008)* (*iii*) $\displaystyle\int_0^\infty \frac{e^{-t}\sin t}{t}\,dt = \frac{\pi}{4}$. *(V.T.U., 2009 S)*

(*iv*) $\displaystyle\int_0^\infty \frac{e^{-t}\sin^2 t}{t}\,dt = \frac{1}{4}\log 5$. *(Kurukshetra, 2006)*

21. Evaluate (*i*) $L\left(\displaystyle\int_0^t \frac{\sin t}{t}\,dt\right)$ *(J.N.T.U., 2005)*

(*ii*) $L\left(\displaystyle\int_0^t e^{-t}\cos t\,dt\right)$ (*iii*) $L\displaystyle\int_0^t \frac{e^t\sin t}{t}\,dt$. *(P.T.U., 2009 S ; S.V.T.U., 2009 ; Bhopal, 2008)*

22. Show that (*i*) $L\{t\,J_0(at)\} = \dfrac{s}{(s^2+a^2)^{3/2}}$ (*ii*) $\displaystyle\int_0^\infty te^{-3t}J_0(4t)\,dt = 3/125$.

21.12 INVERSE TRANSFORMS — METHOD OF PARTIAL FRACTIONS

Having found the Laplace transforms of a few functions, let us now determine the inverse transforms of given functions of s. We have seen that $L\{f(t)\}$ in each case, is a rational algebraic function. Hence *to find the inverse transforms, we first express the given function of s into partial fractions which will, then, be recognizable as one of the following standard forms* :

(1) $\mathbf{L^{-1}\left[\dfrac{1}{s}\right] = 1.}$ **(2)** $\mathbf{L^{-1}\left[\dfrac{1}{s-a}\right] = e^{at}.}$

(3) $L^{-1}\left[\dfrac{1}{s^n}\right]=\dfrac{t^{n-1}}{(n-1)!}$, n = 1, 2, 3, ... (4) $L^{-1}\left[\dfrac{1}{(s-a)^n}\right]=\dfrac{e^{at}\,t^{n-1}}{(n-1)!}$.

(5) $L^{-1}\left(\dfrac{1}{s^2+a^2}\right)=\dfrac{1}{a}\sin at.$ (6) $L^{-1}\left(\dfrac{s}{s^2+a^2}\right)=\cos at.$

(7) $L^{-1}\left(\dfrac{1}{s^2-a^2}\right)=\dfrac{1}{a}\sinh at.$ (8) $L^{-1}\left(\dfrac{s}{s^2-a^2}\right)=\cosh at.$

(9) $L^{-1}\left[\dfrac{1}{(s-a)^2+b^2}\right]=\dfrac{1}{b}e^{at}\sin bt.$ (10) $L^{-1}\left[\dfrac{s-a}{(s-a)^2+b^2}\right]=e^{at}\cos bt.$

(11) $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right)=\dfrac{1}{2a}t\sin at.$ (12) $L^{-1}\left[\dfrac{1}{(s^2+a^2)^2}\right]=\dfrac{1}{2a^3}(\sin at-at\cos at).$

The reader is strongly advised to commit these results to memory. The results (1) to (10) follow at once from their corresponding results in § 21.3 and 21.4. As illustrations, we shall prove (11) and (12). Example 21.4 gives

$$L(t\sin at)=\frac{2as}{(s^2+a^2)^2} \text{ and } L(t\cos at)=\frac{s^2-a^2}{(s^2+a^2)^2}$$

$\therefore$ $$t\sin at=2a\,L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right], \text{ whence follows (11).}$$

Also $$t\cos at=L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right]$$

$$=L^{-1}\left[\frac{1}{s^2+a^2}\right]-2a^2\,L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$=\frac{1}{a}\sin at-2a^2\,L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] \text{ whence follows (12).}$$

Obs. *Go through the note on the 'partial fractions' given in para 10 of 'useful information' in Appendix I.*

Example 21.17. *Find the inverse transforms of*

(i) $\dfrac{s^2-3s+4}{s^3}$ (ii) $\dfrac{s+2}{s^2-4s+13}$ *(V.T.U., 2008)*

Solution. (i) $L^{-1}\left(\dfrac{s^2-3s+4}{s^3}\right)=L^{-1}\left(\dfrac{1}{s}\right)-3L^{-1}\left(\dfrac{1}{s^2}\right)+4L^{-1}\left(\dfrac{1}{s^3}\right)=1-3t+4\,.\,t^2/2\,!=1-3t+2t^2.$

(ii) $$L^{-1}\left(\frac{s+2}{s^2-4s+13}\right)=L^{-1}\left[\frac{s+2}{(s-2)^2+9}\right]=L^{-1}\left[\frac{s-2+4}{(s-2)^2+3^2}\right]$$

$$=L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]=e^{2t}\cos 3t+\tfrac{4}{3}e^{2t}\sin 3t.$$

Example 21.18. *Find the inverse transforms of*

(i) $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ *(V.T.U., 2007 ; U.P.T.U., 2004)*

(ii) $\dfrac{4s+5}{(s-1)^2(s+2)}$ *(Kurukshetra, 2005)*

Solution. (*i*) Here the denominator = $(s-1)(s-2)(s-3)$.

So let $$\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

Then $A = [2 \,.\, 1^2 - 6 \,.\, 1 + 5]/(1-2)(1-3) = \frac{1}{2}$

$B = [2 \,.\, 2^2 - 6 \,.\, 2 + 5]/(2-1)(2-3) = -1$

and $C = [2 \,.\, 3^2 - 6 \,.\, 3 + 5]/(3-1)(3-2) = \frac{5}{2}$.

$$\therefore \quad L^{-1}\left(\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$= \tfrac{1}{2}e^t - e^{2t} + \tfrac{5}{2}e^{3t}.$$

(*ii*) Let $$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$$

Multiplying both sides by $(s-1)^2(s+2)$, $4s+5 = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$

Putting $s = 1$, $9 = 3B$, $\therefore \; B = 3$.

Equating the coefficients of s^2 from both sides,

$$0 = A - \tfrac{1}{3}, \quad \therefore \quad A = \tfrac{1}{3}.$$

$$\therefore \quad L^{-1}\left[\frac{4s+5}{(s-1)^2(s+2)}\right] = \frac{1}{3}L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \tfrac{1}{3}e^t + 3te^t - \tfrac{1}{3}e^{-2t}.$$

Example 21.19. *Find the inverse transforms of*

(*i*) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$ *(Rohtak, 2009 ; U.P.T.U., 2005)*

(*ii*) $\dfrac{s}{s^4+4a^4}$. *(Mumbai, 2008)*

Solution. (*i*) Let $$\frac{5s+3}{(s-1)(s^2+2s+5)} = \frac{5(1)+3}{(s-1)(1^2+2\,.\,1+5)} + \frac{As+B}{s^2+2s+5}$$

Multiplying both sides by $(s-1)(s^2+2s+5)$,

$$5s+3 = 1\,.\,(s^2+2s+5) + (As+B)(s-1).$$

Equating the coefficients of s^2 from both sides,

$$0 = 1 + A, \quad \therefore \quad A = -1.$$

Putting $s = 0$, $3 = 5 - B$, $\therefore \; B = 2$.

$$\therefore \quad L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right] = L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left(\frac{-s+2}{s^2+2s+5}\right)$$

$$= L^{-1}\left(\frac{1}{s-1}\right) + L^{-1}\left[\frac{-(s+1)+3}{(s+1)^2+4}\right] = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^t - e^{-t}\cos 2t + \tfrac{3}{2}e^{-t}\sin 2t.$$

(*ii*) Since $s^4 + 4a^4 = (s^2+2a^2)^2 - (2as)^2 = (s^2+2as+2a^2)(s^2-2as+2a^2)$

$\therefore$ Let $$\frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

Multiplying both sides by s^4+4a^4,

$$s = (As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2)$$

Equating coefficients of s^3, $0 = A + C$ ...(i)

Equating coefficients of s^2, $0 = -2aA + B + 2aC + D$ ...(ii)

Equating coefficients of s, $1 = 2a^2A - 2aB + 2a^2C + 2aD$ ...(iii)

Putting $s = 0$, $\quad 0 = 2a^2B + 2a^2D$ ...(iv)

From (iv), $B + D = 0$ ...(v)

$\therefore$ (ii) becomes $-A + C = 0$, and by (i), we get $A = C = 0$.

Then (iii) reduces to $D - B = 1/2a$ and by (v), $B = -1/4a$, $D = 1/4a$.

$$\therefore \quad L^{-1}\left(\frac{s}{s^4 + 4a^4}\right) = -\frac{1}{4a} L^{-1}\left(\frac{1}{s^2 + 2as + 2a^2}\right) + \frac{1}{4a} L^{-1}\left(\frac{1}{s^2 - 2as + 2a^2}\right)$$

$$= -\frac{1}{4a} L^{-1}\left[\frac{1}{(s+a)^2 + a^2}\right] + \frac{1}{4a} L^{-1}\left[\frac{1}{(s-a)^2 + a^2}\right]$$

$$= -\frac{1}{4a}\cdot\frac{1}{a} e^{-at} \sin at + \frac{1}{4a}\cdot\frac{1}{a} e^{at} \sin at = \frac{1}{2a^2} \sin at \left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2a^2} \sin at \sinh at.$$

PROBLEMS 21.4

Find the inverse Laplace transforms of :

1. $\dfrac{2s-5}{4s^2+25} + \dfrac{4s-18}{9-s^2}$.
2. $\dfrac{1}{s^2-5s+6}$. (S.V.T.U., 2008)
3. $\dfrac{s}{(2s-1)(3s-1)}$. (V.T.U., 2010)
4. $\dfrac{3s}{s^2+2s-8}$.
5. $\dfrac{3s+2}{s^2-s-2}$. (V.T.U., 2010 S)
6. $\dfrac{1}{s(s^2-1)}$. (Nagarjuna, 2008)
7. $\dfrac{1-7s}{(s-3)(s-1)(s+2)}$. (B.P.T.U., 2005 S)
8. $\dfrac{s^2-10s+13}{(s-7)(s^2-5s+6)}$.
9. $\dfrac{2p^2-6p+5}{p^3-6p^2+11p-6}$. (U.P.T.U., 2004)
10. $\dfrac{s}{(s^2-1)^2}$. (Kurukshetra, 2005)
11. $\dfrac{1+2s}{(s+2)^2(s-1)^2}$.
12. $\dfrac{s}{(s-3)(s^2+4)}$.
13. $\dfrac{s}{(s+1)^2(s^2+1)}$.
14. $\dfrac{s^3}{s^4-a^4}$. (Kurukshetra, 2005)
15. $\dfrac{1}{s^3-a^3}$.
16. $\dfrac{s^2+6}{(s^2+1)(s^2+4)}$.
17. $\dfrac{2s-3}{s^2+4s+13}$.
18. $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$.
19. $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$. (Mumbai, 2008)
20. $\dfrac{s}{s^4+s^2+1}$. (Raipur, 2005)
21. $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$. (Mumbai, 2009)

21.13 OTHER METHODS OF FINDING INVERSE TRANSFORMS

We have seen that the most effective method of finding the inverse transforms is by means of partial fractions. However, various other methods are available which depend on the following *important inversion formulae.*

I. Shifting property for inverse Laplace transforms.

If $L^{-1}[\bar{f}(s)] = f(t)$, *then*

$$L^{-1}[\bar{f}'(s-a)] = e^{at} f(t) = e^{at} L^{-1}[\bar{f}(s)].$$

II. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$ *and* $f(0) = 0$, *then*

$$L^{-1}[s\,\bar{f}(s)] = \frac{d}{dt}\{f(t)\}$$

In general, $L^{-1}\{s^n \bar{f}(s)\} = \dfrac{d^n}{dt^n}\{f(t)\}$ provided $f(0) = f'(0) = \ldots = f^{n-1}(0) = 0$.

The above formulae at once follow from the results of § 21.7 (Transforms of derivatives).

III. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)\,dt$$

This result follows from § 21.8 (Transforms of integrals)

Also $$L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^t f(t)\,dt\right\} dt$$

$$L^{-1}\left\{\frac{\bar{f}(s)}{s^3}\right\} = \int_0^t \left\{\int_0^t \left(\int_0^t f(t)\,dt\right) dt\right\} dt \text{ and so on.}$$

IV. *If* $L^{-1}\{\bar{f}(s)\} = f(t)$, *then*

$$t\,f(t) = L^{-1}\left\{-\frac{d}{ds}[\bar{f}(s)]\right\}$$

This result follows from $L\{t\,f(t)\} = -\dfrac{d}{ds}\{\bar{f}(s)\}$ (§ 21.9)

V. The formula of § 21.10, *i.e.*,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)\,ds$$

is useful in finding $f(t)$ when $f(s)$ is given, provided the inverse transform of $\int_s^\infty \bar{f}(s)\,ds$ can be conveniently calculated.

Example 21.20. *Find the inverse Laplace transforms of the following :*

(*i*) $\dfrac{s^2}{(s-2)^3}$ (*ii*) $\dfrac{s+3}{s^2-4s+13}$ (*iii*) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ (*Mumbai, 2005*)

Solution. (*i*) Since $s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \quad \frac{s^2}{(s-2)^3} = \frac{1}{s-2} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore \quad L^{-1}\left\{\frac{s^2}{(s-2)^3}\right\} = L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}\,t + 2e^{2t}\,t^2.$$ [using shifting property]

(*ii*) $$\frac{s+3}{s^2-4s+13} = \frac{s-2}{(s-2)^2+3^2} + \frac{5}{(s-2)^2+3^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s+3}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + \frac{5}{3}L^{-1}\left\{\frac{3}{(s-2^2)+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{5}{3}e^{2t}\sin 3t. \qquad \text{[Using shifting property]}$$

(iii) $$L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{(s^2+4s+4+4)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+4]^2}$$

$$= e^{-2t}L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}L^{-1}\left\{\frac{1}{s^2+4} - \frac{4}{(s^2+4)^2}\right\} = \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}L^{-1}\left\{\frac{1}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{2} - 4e^{-2t}\left\{\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\} = e^{-2t}\left\{\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}.$$

Example 21.21. *Find the inverse transform of (i) $1/s(s^2+a^2)$ (ii) $1/s(s+a)^3$.* (P.T.U., 2003)

Solution. (i) Since $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at.$

therefore, by formula III above,

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\sin at\, dt = \frac{1}{a^2}\left|-\cos at\right|_0^t = (1-\cos at)/a^2$$

(ii) $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = L^{-1}\left\{\frac{1}{[(s+a)-a](s+a)^3}\right\} = e^{-at}L^{-1}\left\{\frac{1}{(s-a)s^3}\right\}$$

Now $$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \quad \therefore \quad L^{-1}\left\{\frac{1}{(s-a)s}\right\} = \int_0^t e^{at}\,dt = \frac{e^{at}}{a} - \frac{1}{a}, \text{ by III above}$$

$\therefore$ $$L^{-1}\left\{\frac{1}{(s-a)s^2}\right\} = \frac{1}{a}\int_0^t (e^{at}-1)\,dt = \frac{1}{a^2}(e^{at}-at-1)$$

$$L^{-1}\left\{\frac{1}{(s-a)s^3}\right\} = \frac{1}{a^2}\int_0^t (e^{at}-at-1)\,dt = \frac{1}{a^3}\left(e^{at} - \frac{a^2}{2}t^2 - at - 1\right)$$

Hence $$L^{-1}\left\{\frac{1}{s(s+a)^3}\right\} = e^{-at}\cdot\frac{1}{a^3}\left(e^{at} - \frac{a^2t^2}{2} - at - 1\right) = \frac{1}{a^3}\left(1 - e^{-at} - ate^{-at} - \frac{a^2}{2}t^2e^{-at}\right).$$

Example 21.22. *Find the inverse Laplace transforms of :*

(i) $\frac{s}{(s^2+a^2)^2}$ (S.V.T.U., 2009) *(ii)* $\frac{s^2}{(s^2+a^2)^2}$ (Hazaribag, 2009) *(iii)* $\frac{1}{(s^2+a^2)^2}$.

Solution. (i) If $f(t) = L^{-1}\frac{s}{(s^2+a^2)^2}$, then by formula V above,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2+a^2)^2}ds = \frac{1}{2}\int_s^\infty \frac{2s}{(s^2+a^2)^2}ds = -\frac{1}{2}\left(\frac{1}{s^2+a^2}\right)_s^\infty = \frac{1}{2}\cdot\frac{1}{s^2+a^2}$$

$\therefore$ $$\frac{f(t)}{t} = \frac{1}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{2a}$$

Hence, $f(t) = \dfrac{1}{2a}\, t \sin at.$

Otherwise : Let $\quad f(t) = L^{-1}\left(\dfrac{1}{s^2 + a^2}\right) = \dfrac{\sin at}{a}$ so that $\bar{f}(s) = \dfrac{1}{s^2 + a^2}$

Then by (IV) above, $\quad t\, f(t) = L^{-1}\left\{-\dfrac{d}{ds}[\bar{f}(s)]\right\} = L^{-1}\left\{-\dfrac{d}{ds}\left(\dfrac{1}{s^2 + a^2}\right)\right\}$

or $$\frac{t \sin at}{a} = L^{-1}\left\{\frac{2s}{(s^2 + a^2)^2}\right\}. \text{ Hence } L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}\, t \sin at.$$

(*ii*) In (*i*), we have proved that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}\, t \sin at = f(t), \text{ say}$$

Since $f(0) = 0$, we get from formula II above, that

$$L^{-1}\left\{\frac{s^2}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{s \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \frac{d}{dt}[f(t)]$$

$$= \frac{d}{dt}\left(\frac{1}{2a}\, t \sin at\right) = \frac{1}{2a}(\sin at + at \cos at)$$

(*iii*) In (*i*), we have shown that

$$L^{-1}\left\{\frac{s}{(s^2 + a^2)^2}\right\} = \frac{1}{2a}(t \sin at) = f(t), \text{ say}$$

By formula III above, we have

$$L^{-1}\left\{\frac{1}{(s^2 + a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s} \cdot \frac{s}{(s^2 + a^2)^2}\right\} = \int_0^t f(t)\, dt = \int_0^t \frac{t \sin at}{2a}\, dt$$

$$= \frac{1}{2a}\left\{\left| t \cdot \frac{-\cos at}{a}\right|_0^t - \int_0^t 1 \cdot \left(\frac{-\cos at}{a}\right) dt\right\}$$

$$= \frac{1}{2a}\left\{\frac{-t \cos at}{a} + \frac{\sin at}{a^2}\right\} = \frac{1}{2a^3}(\sin at - at \cos at).$$

Example 21.23. *Find the inverse Laplace transforms of*

(*i*) $\dfrac{s + 2}{s^2(s + 1)(s - 2)}$ (*V.T.U., 2003*) (*ii*) $\dfrac{s + 2}{(s^2 + 4s + 5)^2}$. (*S.V.T.U., 2009 ; P.T.U., 2005*)

Solution. (*i*) $L^{-1}\left\{\dfrac{s + 2}{(s + 1)(s - 2)}\right\} = \dfrac{4}{3} L^{-1}\left(\dfrac{1}{s - 2}\right) - \dfrac{1}{3} L^{-1}\left(\dfrac{1}{s + 1}\right) = \dfrac{4}{3} e^{2t} - \dfrac{1}{3} e^{-t}$

By III above, $L^{-1}\left\{\dfrac{s + 2}{s(s + 1)(s - 2)}\right\} = \displaystyle\int_0^t L^{-1}\left(\dfrac{s + 2}{(s + 1)(s - 2)}\right) dt$

$$= \int_0^t \left(\frac{4}{3}e^{2t} - \frac{1}{3}e^{-t}\right) dt = \frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1$$

Again by III above, $L^{-1}\dfrac{s + 2}{s^2(s + 1)(s - 2)} = \displaystyle\int_0^t L^{-1}\left\{\dfrac{s + 2}{s(s + 1)(s - 2)}\right\} dt$

$$= \int_0^t \left(\frac{2}{3}e^{2t} + \frac{1}{3}e^{-t} - 1\right) dt = \frac{1}{3}(e^{2t} - e^{-t} - t).$$

(ii) $$L^{-1}\left(\frac{1}{s^2+4s+5}\right)=L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}=e^{-2t}\sin t$$

By II above, $L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\}=(-1)^1 t\,.\,e^{-2t}\sin t$

i.e., $$L^{-1}\left\{\frac{-(2s+4)}{(s^2+4s+5)^2}\right\}=-t\,.\,e^{-2t}\sin t$$

or $$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}=\frac{1}{2}t\,.\,e^{-2t}\sin t.$$

Example 21.24. *Find the inverse Laplace transforms of the following :*

(i) $\log\frac{s+1}{s-1}$ *(S.V.T.U., 2009 ; Bhopal, 2008)* *(ii)* $\log\frac{s^2+1}{s(s+1)}$ *(S.V.T.U., 2009 ; V.T.U., 2008)*

(iii) $\cot^{-1}\left(\frac{s}{2}\right)$ *(iv)* $\tan^{-1}\left(\frac{2}{s^2}\right)$. *(V.T.U., 2011 ; Mumbai, 2005 S)*

Solution. (i) If $f(t)=L^{-1}\log\frac{s+1}{s-1}$, then by IV above,

$$t\,f(t)=L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\}=-L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log(s-1)\right\}$$

$$=-L^{-1}\left(\frac{1}{s+1}\right)+L^{-1}\left(\frac{1}{s-1}\right)=-e^{-t}+e^{t}=2\sinh t$$

Thus $f(t)=(2\sinh t)/t$.

(ii) If $f(t)=L^{-1}\log\frac{s^2+1}{s(s+1)}$, then by IV above,

$$t\,f(t)=L^{-1}\left\{-\frac{d}{ds}\log\left(\frac{s^2+1}{s(s+1)}\right)\right\}=-L^{-1}\left\{\frac{d}{ds}\log(s^2+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log s\right\}$$

$$+L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}$$

$$=-L^{-1}\left(\frac{2s}{s^2+1}\right)+L^{-1}\left(\frac{1}{s}\right)+L^{-1}\left(\frac{1}{s+1}\right)=-2\cos t+1+e^{-t}$$

Thus $f(t)=\frac{1}{t}(1+e^{-t}-2\cos t)$.

(iii) If $f(t)=L^{-1}\cot^{-1}\left(\frac{s}{2}\right)$, then by IV above,

$$t\,f(t)=L^{-1}\left\{-\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\}=L^{-1}\left(\frac{2}{s^2+2^2}\right)=\sin 2t$$

Thus $f(t)=(\sin 2t)/t$.

(iv) If $f(t)=L^{-1}\left(\tan^{-1}\frac{2}{s^2}\right)$, then by IV above,

$$t\,f(t)=L^{-1}\left\{-\frac{d}{ds}\tan^{-1}\left(\frac{2}{s^2}\right)\right\}=L^{-1}\left\{\frac{4s}{s^4+4}\right\}$$

$$= L^{-1}\left\{\frac{4s}{(s^2+2)^2-(2s)^2}\right\} = L^{-1}\left\{\frac{4s}{(s^2+2+2s)(s^2+2-2s)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right\}$$

$$= e^t \sin t - e^{-t}\sin t = 2\sinh t \sin t.$$

21.14 CONVOLUTION THEOREM

If $\quad L^{-1}\{\bar{f}(s)\} = f(t)$, *and* $L^{-1}\{\bar{g}(s)\} = g(t)$,

then $\quad L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du = F * G$

*[F * G is called the* **convolution** *or* **falting** *of F and G.]*

Let $\quad \phi(t) = \int_0^t f(u)\,g(t-u)\,du$

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t f(u)\,g(t-u)\,du\right\}dt = \int_0^\infty \int_0^t e^{-st} f(u)\,g(t-u)\,du\,dt \quad ...(1)$$

The domain of integration for this double integral is the entire area lying between the lines $u = 0$ and $u = t$ (Fig. 21.2).

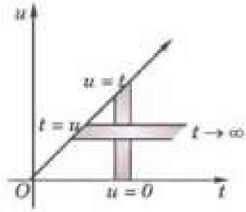


Fig. 21.2

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u)\,g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-s(t-u)}\,g(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv}\,g(v)\,dv\right\}du \text{ on putting } t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\,g(s)\,du = \int_0^\infty e^{-su} f(u)\,du\,.\,\bar{g}(s)$$

$$= \bar{f}(s)\,.\,\bar{g}(s) \quad \text{whence follows the desired result.}$$

Example 21.25. *Apply Convolution theorem to evaluate*

(i) $L^{-1}\dfrac{s}{(s^2+a^2)^2}$. *(V.T.U., 2010)*

(ii) $L^{-1}\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$. *(V.T.U., 2011 S ; Bhopal, 2008 ; Mumbai, 2007)*

Solution. (i) Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \dfrac{1}{a}\sin at$

∴ by Convolution theorem, we get

$$L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\,\frac{\sin a(t-u)}{a}\,du \qquad \left[\because \; f(u) = \cos au,\; g(t-u) = \frac{1}{a}\sin a(t-u)\right.$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]dt = \frac{1}{2a}\left|u\sin at + \frac{1}{2a}\cos(2au-at)\right|_0^t = \frac{1}{2a}\,t\sin at$$

Hence $\quad L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\} = \dfrac{1}{2a}\,t\sin at.$

(*ii*) Since $f(t) = L^{-1}\left(\dfrac{s}{s^2+a^2}\right) = \cos at$ and $g(t) = L^{-1}\left(\dfrac{s}{s^2+b^2}\right) = \cos bt$,

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right) = \int_0^t \cos au \cos b(t-u)\,du \qquad [\because f(u) = \cos au, g(t-u) = \cos b(t-u)]$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\}\,du$$

$$= \frac{1}{2}\left|\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right|_0^t$$

$$= \frac{1}{2}\left\{\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right\} = \frac{a\sin at - b\sin bt}{a^2-b^2}.$$

Example 21.26. *Evaluate* (*i*) $L^{-1}\dfrac{1}{(s^2+1)(s^2+9)}$ *(Mumbai, 2005 S)*

(*ii*) $L^{-1}\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$. *(Madras, 2006)*

Solution. (*i*) Since $L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$, $L^{-1}\left(\dfrac{1}{s^2+9}\right) = \dfrac{\sin 3t}{3}$

$\therefore$ by Convolution theorem, we get

$$L^{-1}\left(\frac{1}{s^2+1}\cdot\frac{1}{s^2+9}\right) = \int_0^t \sin u\,.\,\frac{\sin 3(t-u)}{3}du$$

$$= \frac{1}{6}\int_0^t [\cos(4u-3t) - \cos(3t-2u)]\,du = \frac{1}{6}\left|\frac{\sin(4u-3t)}{4} - \frac{\sin(3t-2u)}{-2}\right|_0^t$$

$$= \frac{1}{6}\left\{\frac{1}{4}(\sin t + \sin 3t) + \frac{1}{2}(\sin t - \sin 3t)\right\} = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right)$$

(*ii*) Since $L^{-1}\left(\dfrac{s}{s^2+4}\right) = \cos 2t$ and $L^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+9)}\right\} = \dfrac{1}{8}\left[\sin t - \dfrac{1}{3}\sin 3t\right]$ [By (*i*)]

$\therefore$ by Convolution theorem, we get

$$L^{-1}\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = L^{-1}\left\{\frac{1}{(s^2+1)(s^2+9)}\cdot\frac{s}{s^2+4}\right\}$$

$$= \int_0^t \frac{1}{8}\left(\sin u - \frac{1}{3}\sin 3u\right).\cos 2(t-u)\,du$$

$$= \frac{1}{8}\int_0^t \left[\sin u\cos 2(t-u) - \frac{1}{3}\sin 3u\cos 2(t-u)\right]du$$

$$= \frac{1}{8}\int_0^t \left[\frac{1}{2}\{\sin(2t-u) - \sin(3u-2t)\} - \frac{1}{6}\{\sin(u+2t) - \sin(5u-2t)\}\right]du$$

$$= \frac{1}{16}\left[\left|\frac{-\cos(2t-u)}{-1} + \frac{\cos(3u-2t)}{3}\right|_0^t\right] - \frac{1}{48}\left[\left|-\cos(u+2t) + \frac{\cos(5u-2t)}{5}\right|_0^t\right]$$

$$= \frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t.$$

PROBLEMS 21.5

Find the inverse transforms of :

1. $\dfrac{1}{s^2(s+5)}$. (*Madras, 2003 S*)
2. $\dfrac{1}{s(s+2)^3}$.
3. $\dfrac{s}{a^2s^2+b^2}$. (*Madras, 2000 S*)
4. $\dfrac{1}{s^2(s^2+a^2)}$.
5. $\dfrac{1}{s^3(s^2+1)}$.
6. $\dfrac{s+2}{(s^2+4s+8)^2}$. (*Mumbai, 2006*)
7. $\dfrac{2as}{(s^2+a^2)^2}$.
8. $\dfrac{s^2}{(s+a)^3}$.
9. $\log\left(\dfrac{1+s}{s}\right)$.
10. $\log\left(\dfrac{s+a}{s+b}\right)$. (*Anna, 2003 ; U.P.T.U., 2003*)
11. $\log\left\{\dfrac{s+1}{(s+2)(s+3)}\right\}$.
12. $\dfrac{1}{2}\log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$. (*Mumbai, 2008 ; V.T.U., 2008*)
13. $\log\left(1-\dfrac{a^2}{s^2}\right)$.
14. $\log\dfrac{s^2+1}{(s-1)^2}$. (*Madras, 2000 S*)
15. $\tan^{-1}\left(\dfrac{2}{s}\right)$. (*Mumbai, 2007 ; P.T.U., 2005*)
16. $\cot^{-1}(s)$. (*V.T.U., 2005*)
17. $s\log\dfrac{s-1}{s+1}$. (*Madras, 1999*)

Using Convolution theorem, evaluate :

18. $L^{-1}\left\{\dfrac{1}{(s+a)(s+b)}\right\}$.
19. $L^{-1}\dfrac{1}{(s^2+a^2)^2}$.
20. $L^{-1}\dfrac{1}{s^2(s^2+a^2)}$.
21. $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$.
22. $L^{-1}\left\{\dfrac{1}{(s-2)(s+2)^2}\right\}$. (*Mumbai, 2009*)
23. $L^{-1}\left\{\dfrac{s}{(s+2)(s^2+9)}\right\}$. (*V.T.U., 2008 S*)
24. $\dfrac{1}{s^3(s^2+1)}$. (*V.T.U., 2007 ; U.P.T.U., 2005*)
25. $\dfrac{1}{(s^2+4s+13)^2}$. (*Mumbai, 2008*)
26. Show that (*i*) $L^{-1}\left(\dfrac{1}{s}\sin\dfrac{1}{s}\right)=t-\dfrac{t^3}{(3!)^2}+\dfrac{t^5}{(5!)^2}-\dfrac{t^7}{(7!)^2}+\ldots$

    (*ii*) $L^{-1}\left(\dfrac{1}{s}\cos\dfrac{1}{s}\right)=1-\dfrac{t^2}{(2!)^2}+\dfrac{t^4}{(4!)^2}-\dfrac{t^6}{(6!)^2}+\ldots$

21.15 (1) APPLICATION TO DIFFERENTIAL EQUATIONS

The Laplace transform method of solving differential equations yields particular solutions without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is, in general, shorter than our earlier methods and is specially useful for solving linear differential equations with constant coefficients.

(2) Working procedure *to solve a linear differential equation with constant coefficients by transform method :*

1. Take the Laplace transform of both sides of the differential equation using the formula of § 21.7, and the given initial conditions.

2. Transpose the terms with minus signs to the right.

3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

4. Resolve this function of s into partial fractions and take the inverse transform of both sides. This gives y as a function of t which is the desired solution satisfying the given conditions.

Example 21.27. *Solve by the method of transforms, the equation*
$$y''' + 2y'' - y' - 2y = 0 \text{ given } y(0) = y'(0) = 0 \text{ and } y''(0) = 6.$$ (V.T.U., 2011 S)

Solution. Taking the Laplace transform of both sides, we get
$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$
Using the given conditions, it reduces to
$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$
$$\therefore \quad \bar{y} = \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$
On inversion, we get $\quad y = L^{-1}\frac{1}{(s-1)} - 3L^{-1}\frac{1}{(s+2)} + 2L^{-1}\left(\frac{1}{s+2}\right)$

or $\quad y = e^t - 3e^{-t} + 2e^{-2t}$ which is the desired result.

Example 21.28. *Use transform method to solve*
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \text{ with } x = 2,\ \frac{dx}{dt} = -1 \text{ at } t = 0.$$ (Anna, 2005 S)

Solution. Taking the Laplace transforms of both sides, we get
$$[s^2\bar{x} - sx(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$
Using the given conditions, it reduces to
$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$
$$\therefore \quad \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$ on breaking into partial fractions.

On inversion, we obtain $x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\frac{1}{(s-1)^2} + L^{-1}\frac{1}{(s-1)^3}$
$$= 2e^t - \frac{3e^t \cdot t}{1!} + \frac{e^t \cdot t^2}{2!} = 2e^t - 3te^t + \tfrac{1}{2}t^2e^t.$$

Example 21.29. *Solve* $(D^2 + n^2)x = a\sin(nt + \alpha)$, $x = Dx = 0$ *at* $t = 0$.

Solution. Taking the Laplace transforms of both sides, we get
$$[s^2\bar{x} - sx(0) - x'(0)] + n^2\bar{x} = aL\{\sin nt \cdot \cos\alpha + \cos nt \cdot \sin\alpha\}$$
On using the given conditions,
$$(s^2 + n^2)\bar{x} = a\cos\alpha \cdot \frac{n}{s^2 + n^2} + a\sin\alpha \cdot \frac{s}{s^2 + n^2}$$
$$\therefore \quad \bar{x} = an\cos\alpha \cdot \frac{1}{(s^2 + n^2)^2} + a\sin\alpha \cdot \frac{s}{(s^2 + n^2)^2}$$
On inversion, we obtain
$$x = an\cos\alpha \cdot \frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha \cdot \frac{t}{2n}\sin nt$$ [By (11) and (12) p. 741]
$$= a\{\sin nt\cos\alpha - nt\cos(nt + \alpha)\}/2n^2.$$

Example 21.30. *Solve* $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ *given that* $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2.$ (S.V.T.U., 2009)

Solution. Taking the Laplace transforms of both sides, we get

$$[s^3 \bar{y} - s^2 y(0) - sy'(0) - y''(0)] - 3[s^2 \bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3}$$

Using the given conditions, it reduces to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain $y = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\frac{1}{(s-1)^2} - L^{-1}\frac{1}{(s-1)^3} + 2L^{-1}\frac{1}{(s-1)^6}$

$$= e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right).$$

Example 21.31. *Solve* $\frac{d^2x}{dt^2} + 9x = \cos 2t$, *if* $x(0) = 1, x(\pi/2) = -1.$ *(Bhopal, 2008 ; U.P.T.U., 2006)*

Solution. Since $x'(0)$ is not given, we assume $x'(0) = a$.

Taking the Laplace transforms of both sides of the equation, we have

$$L(x'') + 9L(x) = L(\cos 2t) \text{ i.e., } [s^2\bar{x} - s\,x(0) - x'(0)] + 9\bar{x} = \frac{s}{s^2+4}$$

$$(s^2+9)\,\bar{x} = s + a + \frac{s}{s^2+4} \quad \text{or} \quad \bar{x} = \frac{s+a}{s^2+9} + \frac{s}{(s^2+4)(s^2+9)}$$

or $$\bar{x} = \frac{a}{s^2+9} + \frac{1}{5}\cdot\frac{s}{s^2+4} + \frac{4}{5}\cdot\frac{s}{s^2+9}.$$

On inversion, we get $\quad x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$

When $t = \pi/2$, $-1 = -\frac{a}{3} - \frac{1}{5}$ or $\frac{a}{3} = \frac{4}{5}$ $\quad\left[\because\ x\left(\frac{\pi}{2}\right) = -1\right]$

Hence the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$.

Obs. *Laplace transform method can also be used for solving ordinary differential equations with variable coefficients of the form* $t^m y^{(n)}(t)$ *because* $L[t^m y^{(n)}(t)] = (-1)^m \frac{d^m}{ds^m}[L\,y^{(n)}(t)]$.

Example 21.32. *Solve* $ty'' + 2y' + ty = \cos t$ *given that* $y(0) = 1.$ *(S.V.T.U., 2009)*

Solution. Taking Laplace transform of both sides of the equation and noting that

$$L[t\,\{f(t)\}] = -\frac{d}{ds}[L\,\{f(t)\}], \text{ we get}$$

$$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or $$-\left(s^2\frac{d\bar{y}}{ds} + 2s\bar{y}\right) + y(0) + 0 + 2s\bar{y} - 2y(0) - \frac{d}{ds}(\bar{y}) = \frac{s}{s^2+1}$$

or $$(s^2+1)\frac{d\bar{y}}{ds} + 1 = -\frac{s}{s^2+1} \quad \text{or} \quad \frac{d\bar{y}}{ds} = -\frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

On inversion and noting that $L^{-1}\{\bar{f}'(s)\} = -t f(t)$, we get

$$-ty = -\sin t - \tfrac{1}{2} t \sin t \qquad \text{[See § 21.12 (11)]}$$

or $$y = \frac{1}{2}\left(1 + \frac{2}{t}\right)\sin t$$ which is the desired solution.

Example 21.33. *Solve* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0,\ y(0) = 2,\ y'(0) = 0.$

Solution. Taking Laplace transform of both sides of the equation, we get

$$L(xy'') + L(y') + L(xy) = 0$$

or $$-\frac{d}{ds}[s^2\bar{y} - sy(0) - y'(0)] + [s\bar{y} - y(0)] - \frac{d\bar{y}}{ds} = 0 \quad \text{or} \quad (s^2+1)\frac{d\bar{y}}{ds} + s\bar{y} = 0$$

Separating the variables, $\displaystyle\int \frac{d\bar{y}}{\bar{y}} + \int \frac{s\,ds}{s^2+1} = c$

or $$\log \bar{y} + \tfrac{1}{2}\log(s^2+1) = \log c' \quad \text{or} \quad \bar{y} = \frac{c'}{\sqrt{(s^2+1)}}$$

Inversion gives $\qquad y = c' J_0(x)$

To find c', we have $\qquad y(0) = c' J_0(0)$, *i.e.*, $c' = 2$

Hence $\qquad y = 2J_0(x).$

Example 21.34. *An alternating e.m.f. E sin ωt is applied to an inductance L and a capacitance C in series. Show by transform method, that the current in the circuit is* $\dfrac{E\omega}{(p^2-\omega^2)L}(\cos \omega t - \cos pt)$, *where* $p^2 = 1/LC$.

Solution. If i be a current and q the charge at time t in the circuit, then its differential equation is

$$L\frac{di}{dt} + \frac{q}{c} = E\sin\omega t \qquad [\because\ R = 0]$$

Taking Laplace transform of both sides, we get

$$L[s\bar{i}(s) - i(0)] + \frac{1}{C}L(q) = E\cdot\frac{\omega}{s^2+\omega^2}$$

Since $i = 0$ and $q = 0$ at $t = 0$

$$\therefore \quad Ls\bar{i}(s) + \frac{1}{C}L(q) = \frac{E\omega}{s^2+\omega^2} \qquad ...(i)$$

Also taking Laplace transform of $i = dq/dt$, we get

$$\bar{i}(s) = L(dq/dt) = s\,L(q) - q(0)$$

i.e. $$L(q) = \bar{i}(s)/s \qquad [\because\ q(0) = 0]$$

$\therefore$ (*i*) becomes $Ls\bar{i}(s) + \dfrac{1}{C}[\bar{i}(s)/s] = \dfrac{E\omega}{s^2+\omega^2}$

or $$\left(Ls + \frac{1}{Cs}\right)\bar{i}(s) = \frac{E\omega}{s+\omega^2} \quad \text{or} \quad \bar{i}(s) = \frac{E\omega s}{L(s^2+1/LC)(s^2+\omega^2)}$$

or $$\bar{i}(s) = \frac{E\omega}{L}\cdot\frac{s}{(s^2+p^2)(s^2+\omega^2)} \quad \text{where } p^2 = 1/LC$$

$$\bar{i}(s) = \frac{E\omega}{L(p^2-\omega^2)}\left\{\frac{s}{s^2+\omega^2} - \frac{s}{s^2+p^2}\right\}$$

Now taking inverse Laplace transform of both sides, we get

$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} L^{-1}\left\{\frac{s}{s^2 + \omega^2} - \frac{s}{(s^2 + p^2)}\right\}$$

or
$$i(t) = \frac{E\omega}{L(p^2 - \omega^2)} (\cos \omega t - \cos pt).$$

PROBLEMS 21.6

Solve the following equations by the transform method :

1. $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1.$ (V.T.U., 2008 S ; Kurukshetra, 2005)
2. $(D^2 - 1)x = a \cosh t, x(0) = x'(0) = 0.$
3. $y'' + y = t, y(0) = 1, y'(0) = 0.$ (Mumbai, 2009)
4. $y'' - 3y' + 2y = e^{3t}$, when $y(0) = 1$ and $y'(0) = 0$. (V.T.U., 2010)
5. $(D^2 - 3D + 2)y = 4e^{2t}$ with $y(0) = -3, y'(0) = 5$. (Mumbai, 2008)
6. $y'' + 25y = 10 \cos 5t$ given that $y(0) = 2, y'(0) = 0$. (S.V.T.U., 2008)
7. $(D^2 + \omega^2)y = \cos \omega t, t > 0$, given that $y = 0$ and $Dy = 0$ at $t = 0$.
8. $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} - 3y = \sin t, y = \dfrac{dy}{dt} = 0$ when $t = 0$. (Kurukshetra, 2005 ; Madras, 2003)
9. $\dfrac{d^4y}{dt^4} - k^4y = 0$, where $y(0) = 1, y'(0) = y''(0) = y'''(0) = 0$.
10. $y'''(t) + 2y''(t) + y(t) = \sin t$, when $y(0) = y'(0) = y''(0) = y'''(0) = 0$.
11. $\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 5y = e^{-t} \sin t$, where $y(0) = 0$ and $y'(0) = 1$. (P.T.U., 2010)
12. $y'' + 2y' + 5y = 5y = 5(t - 2), y(0) = 0, y'(0) = 0.$ (P.T.U., 2005 S)
13. $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2e^{2t}$, where $y = 1, \dfrac{dy}{dt} = 0, \dfrac{d^2y}{dt^2} = -2$ at $t = 0$.
14. $(D^2 + 1)x = t \cos 2t, x = Dx = 0$ at $t = 0$. (Raipur, 2005 ; U.P.T.U., 2005)
15. $ty'' + 2y' + ty = \sin t$, when $y(0) = 1$.
16. $ty'' + (1 - 2t)y' - 2y = 0$, when $y(0) = 1, y'(0) = 2$. (P.T.U., 2002)
17. $y'' + 2ty' - y = t$, when $y(0) = 0, y'(0) = 1$. (U.P.T.U., 2003)
18. $ty'' + y' + 4ty = 0$ when $y(0) = 3, y'(0) = 0$.
19. A voltage Ee^{-at} is applied at $t = 0$ to a circuit of inductance L and resistance R. Show (by the transform method) that the current at time t is $\dfrac{E}{R - aL}(e^{-at} - e^{-Rt/L})$. (V.T.U., 2000)
20. Workout example 12.17, p. 465 by the transform method.
21. Obtain the equation for the forced oscillation of a mass m attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is k, when the driving force is $F_0 \sin at$. Solve this equation (using the Laplace transforms) when $a^2 \neq k/m$, given that initial velocity and displacement (from equilibrium position) are zero.

   $\left[\textbf{Hint :} \text{ The equation of motion is } \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = \dfrac{F_0}{m}\sin at \text{ and } x = \dfrac{dx}{dt} = 0 \text{ when } t = 0.\right]$

21.16 SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

The Laplace transform method can also be applied with advantage to the solution of simultaneous linear differential equations.

Example 21.35. *Solve the simultaneous equations* $\dfrac{dx}{dt} + 5x - 2y = t$, $\dfrac{dy}{dt} + 2x + y = 0$ *being given* $x = y = 0$ *when* $t = 0$. [Ex. 13.38]

Solution. Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = 1/s^2 \quad i.e., \quad (s+5)\bar{x} - 2\bar{y} = 1/s^2 \quad \ldots(i)\ [\because\ x(0)=0]$$

and
$$s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0 \quad i.e., \quad 2\bar{x} + (s+1)\bar{y} = 0 \quad \ldots(ii)\ [\because\ y(0)=0]$$

Solving (*i*) and (*ii*) for $\bar{x}$, we get

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} \div \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Substituting the value of $\bar{x}$ in (*ii*), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On inversion, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t},\ y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}.$$

Example 21.36. *The coordinates (x, y) of a particle moving along a plane curve at any time t, are given by dy/dt + 2x = sin 2t, dx/dt − 2y = cos 2t, (t > 0). If at t = 0, x = 1 and y = 0, show by transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.* (U.P.T.U., 2003)

Solution. Taking the Laplace transforms of the given equations and noting that $y(0) = 0$, $x(0) = 1$,

we get
$$[s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2+2^2} \quad \text{or} \quad 2\bar{x} + s\bar{y} = \frac{2}{s^2+4} \quad \ldots(i)$$

and
$$[s\bar{x} - x(0)] - 2\bar{y} = \frac{s}{s^2+2^2} \quad \text{or} \quad s\bar{x} - 2\bar{y} = \frac{s}{s^2+4} + 1 \quad \ldots(ii)$$

Multiplying (*i*) by *s* and (*ii*) by 2 and subtracting, we get

$$(s^2+4)\bar{y} = -2 \quad \text{or} \quad \bar{y} = -2/(s^2+4)$$

On inversion,
$$y = -2L^{-1}\left[\frac{1}{s^2+4}\right] = -\sin 2t$$

From the given first equation,

$$2x = \sin 2t - dy/dt = \sin 2t - \frac{d}{dt}(-\sin 2t)$$

or
$$2x = \sin 2t + 2\cos 2t \quad \text{or} \quad 4x^2 = (\sin 2t + 2\cos 2t)^2 \quad \ldots(iii)$$

Also
$$4xy = (\sin 2t + 2\cos 2t)(-2\sin 2t) = -2(\sin^2 2t + 2\sin 2t\cos 2t) \quad \ldots(iv)$$

and
$$5y^2 = 5\sin^2 2t. \quad \ldots(v)$$

Adding (*iii*), (*iv*), and (*v*), we obtain

$$4x^2 + 4xy + 5y^2 = \sin^2 2t + 4\sin 2t\cos 2t + 4\cos^2 2t - 2\sin^2 2t$$
$$- 4\sin 2t\cos 2t + 5\sin^2 2t = 4\sin^2 2t + 4\cos^2 2t = 4.$$

Example 21.37. *The small oscillations of a certain system with two degrees of freedom are given by the equations : $D^2x + 3x - 2y = 0$, $D^2x + D^2y - 3x + 5y = 0$ where D = d/dt. If x = 0, y = 0, ẋ = 3, ẏ = 2 when t = 0, find x and y when t = 1/2.* [Example 13.41]

Solution. Taking the Laplace transform of both the equations, we get

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0 \quad i.e., \quad (s^2+3)\bar{x} - 2\bar{y} = 3 \quad \ldots(i)$$

and
$$[s^2\bar{x} - sx(0) - x'(0)] + [s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0 \ i.e., (s^2-3)\bar{x} + (s^2+5)\bar{y} = 5 \quad \ldots(ii)$$

Solving (*i*) and (*ii*) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \begin{vmatrix} 3 & -2 \\ 5 & s^2+5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{3s^2+25}{(s^2+1)(s^2+9)}$$
$$= \frac{11}{4}\cdot\frac{1}{s^2+1} + \frac{1}{4}\cdot\frac{1}{s^2+9}$$

and $$\bar{y} = \begin{vmatrix} s^2+3 & 3 \\ s^2-3 & 5 \end{vmatrix} \div \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{2s^2+24}{(s^2+1)(s^2+9)} = \frac{11}{4}\cdot\frac{1}{s^2+1} + \frac{3}{4}\cdot\frac{1}{s^2+9}.$$

On inversion, we get $x = \frac{11}{4}\sin t + \frac{1}{12}\sin 3t$; $y = \frac{11}{4}\sin t - \frac{1}{4}\sin 3t$

which are the same as the solution in (*vii*) on p. 499.

Obs. The student should compare the earlier solutions of the above examples with those given now and appreciate the superiority of the transform method over others.

PROBLEMS 21.7

Solve the following simultaneous equations (by using Laplace transforms):

1. $\frac{dx}{dt} - y = e^t$, $\frac{dy}{dt} + x = \sin t$, given $x(0) = 1, y(0) = 0$. (*U.P.T.U., 2006 ; Delhi, 2002*)
2. $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$, given that $x = 2$ and $y = 0$ when $t = 0$. (*Kerala, 2005 ; U.P.T.U., 2004*)
3. $\frac{d^2x}{dt^2} - x = y$, $\frac{d^2y}{dt^2} + y = -x$, given that at $t = 0$; $x = 2, y = -1$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. (*P.T.U., 2009 S*)
4. $3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$, $\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$; given $x = 0, y = 0$ when $t = 0$. (*Madras, 2003 S*)
5. $(D-2)x - (D+1)y = 6e^{3t}$; $(2D-3)x + (D-3)y = 6e^{3t}$ given $x = 3, y = 0$ when $t = 0$.
6. The currents i_1 and i_2 in mesh are given by the differential equations ; $di_1/dt - \omega i_2 = a\cos pt$, $di_2/dt + \omega i_1 = a\sin pt$. Find the currents i_1 and i_2 by Laplace transform, if $i_1 = i_2 = 0$ at $t = 0$.

21.17 (1) UNIT STEP FUNCTION

At times, we come across such fractions of which the inverse transform cannot be determined from the formulae so far derived. In order to cover such cases, we introduce the *unit step function* (or *Heaviside's unit function**).

Def. *The unit step function $u(t-a)$ is defined as follows :*

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \ge a \end{cases}$$

where, a is always positive (Fig. 21.3). It is also denoted as $H(t-a)$.

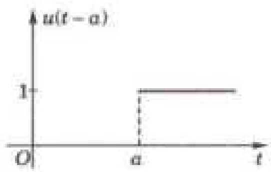


Fig. 21.3

(2) Transform of unit function.

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\,u(t-a)\,dt = \int_0^a e^{-st}\,.\,0\,dt + \int_a^\infty e^{-st}\,.\,1\,dt = 0 + \left|\frac{e^{-st}}{-s}\right|_a^\infty$$

Thus $\quad L\{u(t-a)\} = e^{-as}/s.$

The product $f(t)\,u(t-a) = \begin{cases} 0 & \text{for } t < a \\ f(t) & \text{for } t \ge a. \end{cases}$

The function $f(t-a)\,.\,u(t-a)$ represents the graph of $f(t)$ shifted through a distance a to the right and is of special importance.

Second shifting property. If $L\{f(t)\} = \bar{f}(s)$, *then*

$$\mathbf{L\{f(t-a).\,u(t-a)\} = e^{-as}\,\bar{f}(s)}$$

$$L\{f(t-a)\,.\,u(t-a)\} = \int_0^\infty e^{-st}\,f(t-a)\,u(t-a)\,dt$$

*Named after the British Electrical Engineer Oliver Heaviside (1850–1925).

$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)\,dt \qquad \text{[Put } t-a=u]$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\bar{f}(s).$$

Example 21.38. *Express the following function (Fig. 21.4) in terms of unit step function and find its Laplace transform.* (U.P.T.U., 2002)

Solution. We have $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & t>2 \end{cases}$

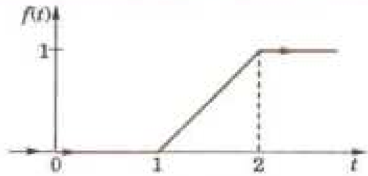


Fig. 21.4

or

$$f(t) = (t-1)[u(t-1) - u(t-2)] + u(t-2)$$
$$= (t-1)\,u(t-1) - (t-2)\,u(t-2)$$

By second shifting property,

$$L[f(t-a)\,u(t-a)] = e^{-as} L[f(t)].$$

Also $\quad L[f(t)] = L(t) = 1/s^2.$

$\therefore \quad L[(t-1)\,u(t-1)]$

$$= e^{-s}\cdot\frac{1}{s^2} \text{ and } L[(t-2)\,u(t-2)] = e^{-2s}\cdot\frac{1}{s^2}$$

Hence $\quad L[f(t)] = L[(t-1)\,u(t-1) - (t-2)\,u(t-2)] = \dfrac{e^{-s} - e^{-2s}}{s^2}.$

Example 21.39. *Using unit step function, find the Laplace transform of*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi. \\ \sin 3t, & t \ge 2\pi \end{cases}$$ (V.T.U., 2004)

Solution. $\quad f(t) = \sin t\,[u(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)] + \sin 3t\,.\,u(t-2\pi)$

$$= \sin t + (\sin 2t - \sin t)\,u(t-\pi) + (\sin 3t - \sin 2t)\,u(t-2\pi)$$

Since $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$,

$$L[f(t)] = L(\sin t) + L[(\sin 2t - \sin t)\,.\,u(t-\pi)] + L[(\sin 3t - \sin 2t)\,.\,u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right).$$

Example 21.40. (*i*) *Express the function (Fig. 21.5) in terms of unit step function and find its Laplace transform.* (P.T.U., 2005 S)

(*ii*) *Obtain the Laplace transform of* $e^{-t}[1 - u(t-2)]$.

Solution. (*i*) We have $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3. \end{cases}$

or

$$f(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$$
$$= (t-1)\,u(t-1) - 2(t-2)\,u(t-2) + (t-3)\,u(t-3)$$

Since $L\{f(t-a)\,u(t-a)\} = e^{-as}\bar{f}(s)$ ...(λ)

$$\therefore \quad L\{f(t)\} = e^{-s}\,.\,\frac{1}{s^2} - 2e^{-2s}\,.\,\frac{1}{s^2} + e^{-3s}\,.\,\frac{1}{s^2} = \frac{e^{-s}(1-e^{-s})^2}{s^2} \qquad [\because\ f(t) = t]$$

(*ii*) $L\{e^{-t}[1 - u(t-2)]\} = L(e^{-t}) - L\{e^{-t}u(t-2)\} = \dfrac{1}{s+1} - e^{-2}L\{e^{-(t-2)}u(t-2)\}$

Taking $f(t) = e^{-t}$, $\bar{f}(s) = \dfrac{1}{s+1}$ and using (λ) above,

$$L\{e^{-(t-2)} u(t-2)\} = e^{-2s} \cdot \frac{1}{s+1}$$

Hence $Le^{-t}\{1 - u(t-2)\} = \{1 - e^{-2(s+1)}\}/(s+1)$.

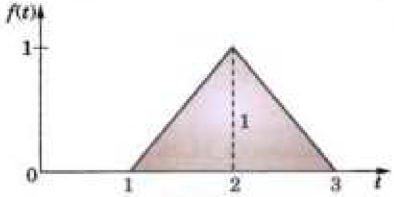


Fig. 21.4

Example 21.41. *Using Laplace transform, evaluate* $\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3) H(t-1)\,dt$. *(Mumbai, 2007)*

Solution. We have $L\{(1 + 2t - t^2 + t^3) H(t-1)\}$

$$= e^{-s} L[1 + 2(t+1) - (t+1)^2 + (t+1)^3] = e^{-s} L(3 + 3t + 2t^2 + t^3)$$

$$= e^{-s}\left(3 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s^2} + 2 \cdot \frac{2!}{s^3} + \frac{3!}{s^4}\right) = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

By definition, this implies that

$$\int_0^\infty e^{-st}(1 + 2t - t^2 + t^3) H(t-1)\,dt = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

Taking $s = 1$, we obtain

$$\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3) H(t-1)\,dt = e^{-1}(3 + 3 + 4 + 6) = 16/e.$$

Example 21.42. *Evaluate* (i) $L^{-1}\left\{\dfrac{e^{-s} - 3e^{-3s}}{s^2}\right\}$ *(U.P.T.U., 2002)*

(ii) $L^{-1}\left\{\dfrac{se^{-as}}{s^2 - w^2}\right\}$, $a > 0$.

Solution.

$$L^{-1}\left\{e^{-s} \cdot \frac{1}{s^2}\right\} = \begin{cases} t-1, & t > 1 \\ 0, & t < 1 \end{cases} = (t-1)\,u(t-1)$$

$$L^{-1}\left\{e^{-3s} \cdot \frac{1}{s^2}\right\} = \begin{cases} t-3, & t > 3 \\ 0, & t < 3 \end{cases} = (t-3)\,u(t-3)$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-s} - 3e^{-3s}}{s^2}\right\} = L^{-1}\left(\frac{e^{-s}}{s^2}\right) - 3L^{-1}\left(\frac{e^{-3s}}{s^2}\right) = (t-1)\,u(t-1) - 3(t-3)\,u(t-3)$$

(*ii*) We know that $L^{-1}\left(\dfrac{s}{s^2 - w^2}\right) = \cosh wt$

$$\therefore \quad L^{-1}\left(\frac{se^{-as}}{s^2 - w^2}\right) = \begin{cases} \cosh w(t-a), & t > a \\ 0, & t < a \end{cases}$$

$= \cosh w(t-a)\,u(t-a)$, by second shifting property.

Example 21.43. *Find the inverse Laplace transform of :*

(*i*) $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2}$ *(V.T.U., 2000)* (*ii*) $\dfrac{e^{-cs}}{s^2(s+a)}$ $(c > 0)$. *(Kurukshetra, 2005)*

Solution. (*i*) Since $L^{-1}\dfrac{s}{s^2 + \pi^2} = \cos \pi t$, $L^{-1}\left(\dfrac{\pi}{s^2 + \pi^2}\right) = \sin \pi t$

and $$L^{-1}[e^{-as}\,\bar{f}(s)] = f(t-a)\,.\,u(t-a) \qquad ...(\lambda)$$

$\therefore$ $$L^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\} = L^{-1}\left\{e^{-s/2}\,.\,\frac{s}{s^2+\pi^2}\right\} + L^{-1}\left\{e^{-s}\,.\,\frac{\pi}{s^2+\pi^2}\right\}$$
$$= \cos\pi\,(t-1/2)\,.\,u\,(t-1/2) + \sin\pi\,(t-1)\,.\,u\,(t-1)$$
$$= \sin\pi t\,.\,u\,(t-1/2) - \sin\pi t\,.\,u\,(t-1) = \{u\,(t-1/2) - u\,(t-1)\}\sin\pi t$$

(*ii*) $$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = L^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2}\,.\,\frac{1}{s}+\frac{1}{a}\,.\,\frac{1}{s^2}+\frac{1}{a^2}\,.\,\frac{1}{s+a}\right)\right\}$$

Using (λ) above, we have

$$L^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = -\frac{1}{a^2}\{1\,.\,u\,(t-c)\} + \frac{1}{a}\{(t-c)\,.\,u(t-c)\} + \frac{1}{a^2}\,\{e^{-a\,(t-c)}\,.\,u(t-c)\}$$
$$= \frac{1}{a^2}\{a\,(t-c) - 1 + e^{-a\,(t-c)}\}\,u(t-c).$$

Example 21.44. *A particle of mass m can oscillate about the position of equilibrium under the effect of a restoring force mk^2 times the displacement. It started from rest by a constant force F which acts for time T and then ceases. Find the amplitude of the subsequent oscillation.*

Solution. The constant force F acting from $t = 0$ to $t = T$ can be expressed as

$$F\,[1 - u(t-T)],\ 0 < t < T$$

$\therefore$ equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = F\,[1 - u\,(t-T)] - mk^2x \quad \text{or} \quad \frac{d^2x}{dt^2} + k^2x = \frac{F}{m}\,[1 - u(t-T)]$$

Taking Laplace transform of both sides, we get

$$(s^2+k^2)\,\bar{x} = \frac{F}{ms}\,(1-e^{-sT}) \qquad [\because\ x = 0,\ \dot{x} = 0 \text{ at } t = 0]$$

or $$\bar{x} = \frac{F}{m}\,.\,\frac{1-e^{-sT}}{s(s^2+k^2)} = \frac{F}{m}(1-e^{-sT})\,.\,\frac{1}{k^2}\left(\frac{1}{s} - \frac{s}{s^2+k^2}\right)$$
$$= \frac{F}{mk^2}\left\{(1-e^{-sT})\frac{1}{s} - (1-e^{-sT})\,.\,\frac{s}{s^2+k^2}\right\}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{F}{mk^2}\,[(1-\cos kt) - \{1-\cos k\,(t-T)\}]\,u\,(t-T)$$

i.e., $$x = \frac{F}{mk^2}(1-\cos kt) \text{ for } 0 < t < T$$

and $$x = \frac{F}{mk^2}(1-\cos kt) - \{1-\cos k\,(t-T)\} \text{ for } t > T$$
$$= \frac{F}{mk^2}\,\{\cos k\,(t-T) - \cos kT\} \text{ for } t > T$$

or $$x = \frac{2F}{mk^2}\sin\frac{kT}{2}\,.\,\sin k\,(t-T/2) \text{ for } t > T$$

Hence the amplitude of subsequent oscillation (*i.e.*, for $t > T$) = $\dfrac{2F}{mk^2}\sin\dfrac{kT}{2}$.

Example 21.45. *In an electrical circuit with e.m.f. E(t), resistance R and inductance L, the current i builds up at the rate given by*

$$L\,di/dt + Ri = E(t). \qquad ...(i)$$

If the switch is connected at t = 0 and disconnected at t = a, find the current i at any instant.

Solution. We have $i = 0$ at $t = 0$ and $E(t) = \begin{cases} E \text{ for } 0 < t < a \\ 0 \text{ for } t > a \end{cases}$

$\therefore$ taking the Laplace transform of both sides, (i) becomes

$$(Ls + R)\bar{i} = \int_0^\infty e^{-st}\, E(t)\, dt = \int_0^a e^{-st}\, E dt = \frac{E}{s}(1 - e^{-as})$$

or
$$\bar{i} = \frac{E}{s(Ls+R)} - \frac{Ee^{-as}}{s(Ls+R)}$$

On inversion, we get
$$i = L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} - L^{-1}\left\{\frac{Ee^{-as}}{s(Ls+R)}\right\} \qquad ...(ii)$$

Now
$$L^{-1}\left\{\frac{E}{s(Ls+R)}\right\} = \frac{E}{R}\left\{L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+R/L}\right)\right\} = \frac{E}{R}(1 - e^{-Rt/L})$$

and
$$L^{-1}\left\{\frac{Ee^{-as}}{s(Ls+R)}\right\} = \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t-a) \qquad \text{[By the second shifting property]}$$

Thus (ii) becomes
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} - \frac{E}{R}\{1 - e^{-R(t-a)/L}\}\, u(t-a)$$

Hence
$$i = \frac{E}{R}\{1 - e^{-Rt/L}\} \text{ for } 0 < t < a$$

and
$$i = \frac{E}{R}[\{1 - e^{-Rt/L}\} - \{1 - e^{-R(t-a)/L}\}] = \frac{E}{R}e^{-Rt/L}(e^{-Ra/L} - 1) \text{ for } t > a.$$

Example 21.46. *Calculate the maximum deflection of an encastre beam 1 ft. long carrying a uniformly distributed load w lb./ft. on its central half length.*

Solution. Taking the origin at the end A, we have

$$EI\frac{d^4y}{dx^4} = w(x)$$

where $w(x) = w\{u(x - l/4) - u(x - 3l/4)\}$

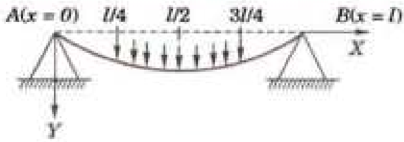


Fig. 21.6

Taking the Laplace transform of both sides, (Fig. 21.6), we get

$$EI[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)]$$
$$= w\left(\frac{e^{-ls/4}}{s} - \frac{e^{-3ls/4}}{s}\right)$$

Using the conditions $y(0) = y'(0) = 0$ and taking $y''(0) = c_1$ and $y'''(0) = c_2$, we have

$$EI\bar{y} = w\left(\frac{e^{-ls/4}}{s^5} - \frac{e^{-3ls/4}}{s^5}\right) + \frac{c_1}{s^3} + \frac{c_2}{s^4}$$

On inversion, we get $EIy = \frac{w}{24}[(x - l/4)^4 u(x - l/4) - (x - 3l/4)^4 u(x - 3l/4)] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$ $\qquad ...(i)$

For $x > 3l/4$,
$$EIy = \frac{w}{24}[(x - l/4)^2 - (x - 3l/4)^2] + \frac{1}{2}c_1x^2 + \frac{1}{6}c_2x^3$$

and
$$EIy' = \frac{w}{6}[(x - l/4)^3 - (x - 3l/4)^3] + c_1x + \frac{1}{2}c_2x^2$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get $0 = \frac{w}{24}\left\{\left(\frac{3l}{4}\right)^4 - \left(\frac{l}{4}\right)^4\right\} + \frac{1}{2}c_1l^2 + \frac{1}{6}c_2l^3$

and
$$0 = \frac{w}{6}\left\{\left(\frac{3l}{4}\right)^3 - \left(\frac{l}{4}\right)^3\right\} + c_1l + \frac{1}{2}c_2l^2$$

whence $c_1 = 11\, wl^2/192\,;\ c_2 = -\,wl/4.$

Thus for $l/4 < x < 3l/4$, (*i*) gives $EIy = \frac{w}{24}\left(x + \frac{1}{4}\right)^4 + \frac{11wl^2}{384}x^2 - \frac{wl}{24}x^3$

Hence the maximum deflection $= y(l/2) = \frac{13wl^4}{6144EI}$.

21.18 (1) UNIT IMPULSE FUNCTION

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. To deal with such and similar ideas, we introduce the *unit impulse function* (also called *Dirac delta function**).

Thus unit impulse function is considered as the limiting form of the function (Fig. 21.7) :

$$\delta_\varepsilon(t-a) = 1/\varepsilon, \quad a \le t \le a+\varepsilon$$
$$= 0, \quad \text{otherwise}$$

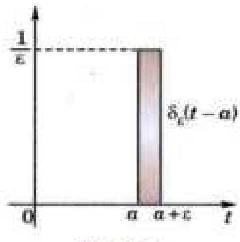


Fig. 21.7

as $\varepsilon \to 0$. It is clear from Fig. 21.7 that as $\varepsilon \to 0$, the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows :

$$\delta(t-a) = \infty \text{ for } t = a \; ; = 0 \text{ for } t \ne a,$$

such that $\int_0^\infty \delta(t-a)\,dt = 1. \quad (a \ge 0)$

As an illustration, a load w_0 acting at the point $x = a$ of a beam may be considered as the limiting case of uniform loading w_0/ε per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) = w_0/\varepsilon \quad a < x < a+\varepsilon,$$
$$= 0, \quad \text{otherwise}$$

i.e., $w(x) = w_0\delta(x + a)$.

(2) Transform of unit impulse function. If $f(t)$ be a function of t continuous at $t = a$, then

$$\int_0^\infty f(t)\,\delta_\varepsilon(t-a)\,.\,dt = \int_0^{a+\varepsilon} f(t)\,.\,\frac{1}{\varepsilon}\,dt$$

$$= (a+\varepsilon-a)f(\eta)\,.\,\frac{1}{\varepsilon} = f(\eta), \qquad \text{where } a < \eta < a+\varepsilon.$$

by Mean value theorem for integrals.

As $\varepsilon \to 0$, we get $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $\mathbf{L\{\delta(t-a)\} = e^{-as}}$.

Example 21.47. *Evaluate* (*i*) $\int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt$ (*ii*) $L\frac{1}{t}\,\delta(t-a)$.

Solution. (*i*) We know that $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\therefore \quad \int_0^\infty \sin 2t\,\delta(t-\pi/4)\,dt = \sin(2\,.\,\pi/4) = 1$$

(*ii*) We know that $L\{\delta(t-a)\} = e^{-as}$

$$\therefore \quad L\left[\frac{1}{t}\,\delta(t-a)\right] = \int_s^\infty L\{\delta(t-a)\}\,ds = \int_s^\infty e^{-as}\,ds$$

$$= \left|\frac{e^{-as}}{-a}\right|_s^\infty = \frac{1}{a}\,e^{-as}.$$

* After the English physicist *Paul Dirac* (1902-84) who was awarded the Nobel prize in 1933 for his work in Quantum mechanics.

Example 21.48. *An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of L, R, C in series with zero initial conditions. If i be the current at any subsequent time t, find the limit of i as $t \to 0$?*

Solution. The equation of the circuit governing the current i is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E\delta(t) \qquad \text{where } i = 0, \text{ when } t = 0.$$

Taking Laplace transform of both sides, we get

$$L\,[s\bar{i} - i(0)] + R\bar{i} + \frac{1}{C}\frac{1}{s}\bar{i} = E \qquad \text{[Using § 21.7 and 21.8]}$$

or $$\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{i} = \frac{E}{L}s \quad \text{or} \quad (s^2 + 2as + a^2 + b^2)\bar{i} = (E/L)\,s$$

where $R/L = 2a$ and $1/CL = a^2 + b^2$

or $$\bar{i} = \frac{E}{L}\frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L}\left\{\frac{s+a}{(s+a)^2+b^2} - a\frac{1}{(s+a)^2+b^2}\right\}$$

On inversion, we get

$$i = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$$

Taking limits as $t \to 0$, $i \to E/L$

Although the current $i = 0$ initially, yet a large current will develop instantaneously due to impulsive voltage applied at $t = 0$. In fact, we have determined the limit of this current which is E/L.

Example 21.49. *A beam is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. It carries a load w at $x = l/4$. Find the resulting deflection at any point.*

Solution. The differential equation for deflection is

$$\frac{d^4y}{dx^4} = \frac{w}{EI}\,\delta(x - l/4)$$

Taking the Laplace transform, we have $s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{w}{EI}e^{-ls/4}$

Using the conditions $y(0) = 0$, $y''(0) = 0$ and taking $y'(0) = c_1$ and $y'''(0) = c_2$, we get

$$\bar{y} = \frac{c_1}{s^2} + \frac{c_2}{s^4} + \frac{w}{EI}\frac{e^{-ls/4}}{s^4}.$$

On inversion, it gives $$y = c_1x + c_2\frac{x^3}{3!} + \frac{w}{EI}\frac{(x - l/4)^3}{3!}\,u(x - l/4)$$

$$\left.\begin{aligned} &\text{i.e.,} \quad y = c_1x + \tfrac{1}{6}c_2x^3, && 0 < x < l/4 \\ &\text{and} \quad y = c_1x + \tfrac{1}{6}c_2x^3 + \frac{\omega}{6EI}(x - l/4)^3, && l/4 < x < l \end{aligned}\right\} \qquad \ldots(i)$$

Using the conditions $y(l) = 0$ and $y'(l) = 0$, we get

$$0 = c_1l + \tfrac{1}{6}c_2l^3 + 9wl^3/128EI \quad \text{and} \quad 0 = c_1 + \tfrac{1}{2}c_2l^2 + 9wl^2/32EI$$

whence $$c_1 = 9wl^2/256\,EI, \qquad c_2 = -81w/128EI.$$

Substituting the values of c_1 and c_2 in (i), we get the deflection at any point.

PROBLEMS 21.8

1. Represent $f(t) = \sin 2t$, $2\pi < t < 4\pi$ and 0 otherwise, in terms of the unit step function and hence find its Laplace transform. *(Mumbai, 2005)*
2. Sketch the graph of the following functions and express them in terms of unit step function. Hence find their Laplace transforms :

(i) $f(t) = 2t$ for $0 < t < \pi$, $f(t) = 1$ for $t > \pi$ *(Assam, 1999)*

(ii) $f(t) = t^2$ for $0 < t \le 2$, $f(t) = 0$ for $t > 2$

(iii) $f(t) = \cos(\omega t + \phi)$ for $0 < t < T$, $f(t) = 0$ for $t > T$.

3. Express the following functions in terms of unit step function and hence find its Laplace transform.

(i) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ 1, & \pi < t < 2\pi \\ \sin t, & t > 2\pi \end{cases}$ *(V.T.U., 2007)* (ii) $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > \pi \end{cases}$

(Mumbai, 2008 ; V.T.U., 2003 S)

(iii) $f(t) = \begin{cases} t^2, & 0 < t < 2 \\ 4t, & 2 < t < 4 \\ 8, & t > 4 \end{cases}$ *(V.T.U., 2011)*

4. Evaluate (i) $L\{e^{t-1} u(t-1)\}$ (ii) $L\{(t-1)^2 u(t-1)\}$

(iii) $L(1 + 2t - 3t^2 + 4t^3) H(t-2)$ *(Mumbai, 2007)* (iv) $L\{t^2 u(t-1) + \delta(t-1)\}$.

5. Evaluate $\int_0^\infty e^{-t}(1 + 3t + t^2) u(t-2)\, dt$.

6. Find the inverse Laplace transforms of :

(i) $\dfrac{e^{-\pi s}}{s^2 + 1}$ (ii) $\dfrac{e^{-2s}}{s^2 + 8s + 25}$ *(Mumbai, 2006)*

(iii) $\dfrac{e^{-s}}{(s+1)^3}$ *(P.T.U., 2010)* (iv) $\dfrac{3}{s} - 4\dfrac{e^{-s}}{s^2} + 4\dfrac{e^{-3s}}{s^2}$. *(P.T.U., 2002 S)*

7. Solve using Laplace transforms $\dfrac{d^2y}{dt^2} + 4y = f(t)$ with conditions

$$y(0) = 0, y'(0) = 1 \text{ and } f(t) = \begin{cases} 1 \text{ when } 0 < t < 1 \\ 0 \text{ when } t > 1 \end{cases}$$ *(Mumbai, 2007)*

8. Using Laplace transforms, solve $x''(t) + x(t) = u(t)$, $x(0) = 1$, $x'(0) = 0$

where $$u(t) = \begin{cases} 3, & 0 \le t \le 4 \\ 2t - 5, & t > 4. \end{cases}$$

9. A beam has its ends clamped at $x = 0$ and $x = l$. A concentrated load W acts vertically downwards at the point $x = l/3$. Find the resulting deflection.

[**Hint.** The differential equation and the boundary conditions are $\dfrac{d^4y}{dx^4} = \dfrac{W}{EI}\,\delta(x - l/3)$ and

$y(0) = y'(0) = 0$, $y(l) = y'(l) = 0$.]

10. A cantilever beam is clamped at the end $x = 0$ and is free at the end $x = l$. It carries a uniform load w per unit length from $x = 0$ to $x = l/2$. Calculate the deflection y at any point. *(Kurukshetra, 2006)*

[**Hint.** The differential equation and boundary conditions are

$$\frac{d^4y}{dx^4} = \frac{W(x)}{EI}\ (0 < x < l) \text{ where } \quad W(x) = \begin{cases} W_0, & 0 < x < l/2 \\ 0, & x > l/2 \end{cases}$$

and $$y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 0]$$

11. An impulse I (kg-sec) is applied to a mass m attached to a spring having a spring constant k. The system is damped with damping constant μ. Derive expressions for displacement and velocity of the mass, assuming initial conditions $x(0) = x'(0) = 0$.

[**Hint.** The equation of motion is $m\dfrac{d^2x}{dt^2} = I\,\delta(x) - kx - \mu\,\dfrac{dx}{dt}$.]

21.19 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 21.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. Laplace transform of $(t \sin t) =$ ....
2. $L\{\delta(t)\} =$ ....
   (a) 0 (b) e^{-as} (c) ∞ (d) 1.
3. If $L\{f(t)\} = f(s)$, then $L\{e^{-at} f(t)\}$ is
   (a) $f(s-a)$ (b) $f(s+a)$ (c) $f(s)$ (d) none of these.
4. $L(e^{2t} \sin t) =$ ....
5. Inverse Laplace transform of $(s+2)^{-2}$ is ....
6. Inverse Laplace transform of $1/(s^2+4s+13) =$ ....
7. Laplace transform of $f'(t) =$ ....
8. $L^{-1}\left[\dfrac{s}{(2s+3)^2}\right] =$ ....
9. $L(\cosh^2 2t) =$ ....
10. $L(e^t) =$ ....
11. $L(e^{-t} t^4) =$ ....
12. $\int_0^\infty e^{-2t} \cos 3t \, dt =$ ....
13. $L\{u(t-a)\} =$ ....
14. $L^{-1}(\sqrt{t}) =$ ....
15. If $L\{F(t)\} = f(s)$, then $L\left\{\dfrac{d^2F(t)}{dt^2}\right\} =$ ....
16. $L(\cos^3 4t) =$ ....
17. $L\left(\dfrac{\sin at}{t}\right) =$ ....
18. $L^{-1}\left\{\dfrac{1}{(s+3)^2}\right\} =$ ....
19. $L \cos(2t+3) =$ ....
20. $L^{-1}(1/s^n)$ is possible only when n is
    (a) zero (b) –ve integer (c) +ve integer (d) negative rational.
21. If $L^{-1}\{\phi(s)\} = f(t)$, the $L^{-1}\{e^{-as} \phi(s)\} =$ ....
22. $L\{u(t+2)\} =$ ....
    (a) e^{-2s}/s^2 (b) e^{2s} (c) $\dfrac{e^{2s}}{s}$ (d) $\dfrac{e^{-2s}}{s}$ (V.T.U., 2011 S)
23. $L^{-1}\left\{\dfrac{s^2-3s+4}{s^3}\right\} =$ .... (V.T.U., 2010 S)
24. If $L\{f(t)\} = \bar{f}(s)$, then $L^{-1}\left\{\dfrac{\bar{f}(s)}{s}\right\} =$ ....
25. If $f(t)$ is a periodic function with period T, then $L[f(t)] =$ ....
26. If y satisfies $y'' + 3y' + 2y = e^{-t}$ with $y(0) = y'(0) = 0$, then $L[y(t)] =$ ....
27. $L[e^{3t}(2 \cos 5t + 3 \sin 4t)] =$ ....
28. $L(4^t) =$ ....
29. $L^{-1}\left\{\dfrac{1}{\sqrt{(s+3)}}\right\} =$ ....
30. Laplace transform of $\sin 2t \, \delta(t-2)$ is
    (a) $e^{2s} \sin 4$ (b) $e^{-2s} \sin 2$ (c) $e^{-4s} \sin 2$ (d) $e^{-2s} \sin 4$. (V.T.U., 2009 S)
31. If $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\} = \dfrac{t \sin t}{2}$ then $L^{-1}\left\{\dfrac{8s}{(4s^2+1)^2}\right\}$ (P.T.U., 2009)
32. $L^{-1}\{e^{-as} F(s)\} =$
    (a) $f(t)\, u(t)$ (b) $f(t-a)\, u(t)$ (c) $f(t-a)\, u(t-a)$ (d) None of these. (V.T.U., 2009 S)
33. $L^{-1}\left\{\dfrac{1}{(s+a)^2}\right\} =$
    (a) e^{at} (b) e^{-at} (c) te^{-at}
    (d) te^{at} (e) $-t$.

34. Laplace transform of $t^4 e^{-at}$ is

(i) $\dfrac{4!}{(s+a)^4}$ (ii) $\dfrac{4!}{(s+a)^5}$ (iii) $\dfrac{4!}{(s-a)^4}$ (iv) $\dfrac{5!}{(s-a)^5}$

35. Laplace transform of $te^{at} \sin (at)$, $t > 0$, is

(i) $\dfrac{s-a}{(s-a)^2+a^2}$ (ii) $\dfrac{a(s-a)}{(s-a)^2+a^2}$ (iii) $\dfrac{2a(s-a)}{[(s-a)^2+a^2]^2}$ (iv) $\dfrac{(s-a)^2}{(s-a)^2+a^2}$

36. $L^{-1} \dfrac{s^2}{(s^2+4)^2}$ is

(i) $\dfrac{1}{4} \sin 2t + t \cos 2t$ (ii) $\dfrac{1}{4} \sin 2t + \dfrac{1}{2} \cos 2t$ (iii) $\sin 2t + \dfrac{t}{2} \cos 2t$ (iv) $\dfrac{1}{2} \sin 2t + \dfrac{t}{4} \cos 2t$

37. $L^{-1} \dfrac{1}{s(s^2+1)}$ is

(i) $1 + \sin t$ (ii) $1 - \sin t$ (iii) $1 + \cos t$ (iv) $1 - \cos t$.

38. $L\,[u(t-a)]$ where $u(t-a)$ is a unit step function, is

(i) $\dfrac{e^{-as}}{s}$ (ii) $\dfrac{e^{as}}{s}$ (iii) $\dfrac{e^{-as}}{s^2}$ (iv) $\dfrac{e^{as}}{s^2}$ *(V.T.U., 2011)*

39. For a periodic function of period 2π, $\displaystyle\int_{a+2\pi}^{b+2\pi} f(x)\, dx =$ .... *(P.T.U., 2009)*

40. $L\,[\delta(t-a)]$ where $\delta(t-a)$ is a unit impulse function, is

(i) e^{as} (ii) e^{-as} (iii) e^{a} (iv) e^{-as}/s *(V.T.U., 2010 S)*

41. Laplace transform of $\sin^2 3t$ is

(i) $\dfrac{3}{s^2+36}$ (ii) $\dfrac{6}{(s^2+36)}$ (iii) $\dfrac{18}{s(s^2+36)}$ (iv) $\dfrac{18}{s^2+36}$ *(V.T.U., 2010)*

42. $[L\,(t^2 e^{-3t})] =$

(i) $\dfrac{1}{(s+3)^3}$ (ii) $\dfrac{2}{(s+3)^2}$ (iii) $\dfrac{3}{(s+3)^3}$ (iv) $\dfrac{2}{(s+3)^3}$ *(V.T.U., 2011)*

43. $\dfrac{d^2}{ds^2}[L\,f(t)] - L[t^2 f(t)] = 0.$ (True or False)

44. Laplace transform of $f(t)$ is defined for +ve and –ve values of t. (True or False)

45. If $L[f(t)] = \phi(s)$, then $L\,[t\,f(t)] = \dfrac{d}{ds}[\phi(s)]$. (True or False)

CHAPTER 22

Fourier Transforms

22.1 INTRODUCTION

In the previous chapter, the reader has already been acquainted with the use of Laplace transforms in the solution of ordinary differential equations. In this chapter, the well-known Fourier transforms will be introduced and their properties will be studied which will be used in the solution of partial differential equations. The choice of a particular transform to be employed for the solution of an equation depends on the boundary conditions of the problem and the ease with which the transform can be inverted. A Fourier transform when applied to a partial differential equation reduces the number of its independent variables by one.

The theory of integral transforms afford mathematical devices through which solutions of numerous boundary value problems of engineering can be obtained *e.g.*, conduction of heat, transverse vibrations of a string, transverse oscillations of an elastic beam, free and forced vibrations of a membrane, transmission lines etc. Some of these applications will be illustrated in the last section.

22.2 DEFINITION

The integral transform of a function f(x) denoted by I[f(x)], is defined by

$$\bar{f}(s) = \int_{x_1}^{x_2} f(x) K(s, x)\, dx$$

where $K(s, x)$ is called the *kernel* of the transform and is a known function of s and x. The function $f(x)$ is called the *inverse transform* of $\bar{f}(s)$.

Three simple examples of a kernel are as follows :

(*i*) *When* $K(s, x) = e^{-sx}$, *it leads to the* **Laplace transform** *of* $f(x)$, *i.e.*,

$$\bar{f}(s) = \int_0^\infty f(x) e^{-sx}\, dx. \qquad \text{[Chap. 21]}$$

(*ii*) *When* $K(s, x) = e^{isx}$, *we have the* **Fourier transform** *of* $f(x)$, *i.e.*,

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx}\, dx.$$

(*iii*) When $K(s, x) = x^{s-1}$, it gives the *Mellin transform* of $f(x)$ *i.e.*,

$$M(s) = \int_0^\infty f(x) x^{s-1}\, dx.$$

Other special transforms arise when the kernel is a sine or a cosine function or a Bessel's function. These lead to *Fourier sine* or *cosine transforms* and the *Hankel transform* respectively.

In order to introduce the *Fourier transforms*, we shall first derive the Fourier integral theorem.

22.3 (1) FOURIER INTEGRAL THEOREM

Consider a function $f(x)$ which satisfies the Dirichlet's conditions (Art. 10.3) in every interval $(-c, c)$ so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c} \right) \quad ...(1)$$

where $a_0 = \frac{1}{c}\int_{-c}^{c} f(t)\, dt,\ a_n = \frac{1}{c}\int_{-c}^{c} f(t) \cos \frac{n\pi t}{c}\, dt$, and $b_n = \frac{1}{c}\int_{-c}^{c} f(t) \sin \frac{n\pi t}{c}\, dt.$

Substituting the values of a_0, a_n and b_n in (1), it takes the form

$$f(x) = \frac{1}{2c}\int_{-c}^{c} f(t)\, dt + \frac{1}{c}\sum_{n=1}^{\infty} \int_{-c}^{c} f(t) \cos \frac{n\pi(t-x)}{c}\, dt \quad ...(2)$$

If we assume that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, the first term on the right side of (2) approaches 0 as $c \to \infty$, since

$$\left| \frac{1}{2c}\int_{-c}^{c} f(t)\, dt \right| \le \frac{1}{2c}\int_{-\infty}^{\infty} |f(t)|\, dt$$

The second term on the right side of (2) tends to

$$\underset{c\to\infty}{\text{Lt}}\ \frac{1}{c}\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi(t-x)}{c}\, dt$$

$$= \underset{\delta\lambda\to 0}{\text{Lt}}\ \frac{1}{\pi}\sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda\,(t-x)\, dt, \text{ on writing } \pi/c = \delta\lambda$$

This is of the form $\underset{\delta\lambda\to 0}{\text{Lt}} \sum_{n=1}^{\infty} F(n\delta\lambda)$, *i.e.*, $\int_0^\infty F(\lambda)\, d\lambda$

Thus as $c \to \infty$, (2) becomes $f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x)\, dt d\lambda$ ...(3)

which is known as the **Fourier integral** of $f(x)$.

Obs. We have given a heuristic demonstration of the Fourier integral theorem which simply helps in deriving the result (3). It cannot however, be taken as a rigorous proof for that would, involve a proof of the convergence of the Fourier integral which is beyond the scope of this book. When $f(x)$ satisfies the above-mentioned conditions, equation (3) holds good at a point of continuity. If however, x is point of discontinuity, we replace $f(x)$ by $\frac{1}{2}[f(x+0) + f(x-0)]$ as in the case of Fourier series.

(2) Fourier sine and cosine integrals. Expanding $\cos \lambda(t-x)$, (3) may be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda + \frac{1}{\pi}\int_0^\infty \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t\, dt\, d\lambda \quad ...(4)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \lambda x \int_0^\infty f(t) \sin \lambda t\, dt\, d\lambda \quad ...(5)$$

which is known as the *Fourier sine integral*.

Similarly, if $f(x)$ is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^\infty f(t) \cos \lambda t \, dt \, d\lambda \qquad ...(6)$$

which is known as the *Fourier cosine integral*.

Obs. A function $f(x)$ defined in the interval $(0, \infty)$ is expressed either as a Fourier sine integral or as a Fourier cosine integral, merely looking upon it as an odd or even function in $(-\infty, \infty)$ on the lines of half-range Fourier series.

(3) **Complex form of Fourier integrals.** Equation (3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t - x) dt d\lambda \qquad ..(7)$$

because $\cos \lambda(t - x)$ is an even function of λ. Also since $\sin \lambda(t - x)$ is an odd function of λ, we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t) \sin \lambda(t - x) dt d\lambda \qquad ...(8)$$

Now multiply (8) by i and add it to (7), so that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\, e^{i\lambda(t-x)} dt d\lambda \qquad ...(9)$$

which is the *complex* form of the *Fourier integral*.

(4) **Fourier integral representation of a function**

Using (4), *a function F(x) may be represented by a Fourier integral as*

$$F(x) = \frac{1}{\pi} \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] \, d\lambda$$

where $$A(\lambda) = \int_{-\infty}^\infty f(t) \cos \lambda t \, dt \, ; \, B(\lambda) = \int_{-\infty}^\infty f(t) \sin \lambda t \, dt \qquad ...(10)$$

If $f(x)$ is an odd function, then [By (5)]

$$f(x) = \frac{1}{\pi} \int_0^\infty B(\lambda) \sin \lambda x \, d\lambda \text{ where } B(\lambda) = 2 \int_0^\infty f(t) \sin \lambda t \, dt \qquad ...(11)$$

If $f(x)$ is an even function, then [By (6)]

$$f(x) = \frac{1}{\pi} \int_0^\infty A(\lambda) \cos \lambda x \, d\lambda \text{ where } A(\lambda) = 2 \int_0^\infty f(t) \cos \lambda t \, dt \qquad ...(12)$$

Example 22.1. *Express* $f(x) = 1$ *for* $0 \le x \le \pi$,
$= 0$ *for* $x > \pi$,
as a Fourier sine integral and hence evaluate

$$\int_0^\infty \frac{1 - \cos(\pi\lambda)}{\lambda} \sin(x\lambda) d\lambda.$$ *(Kottayam, 2005 ; J.N.T.U., 2004 S)*

Solution. The Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\infty f(t) \sin(\lambda t) \, dt$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \int_0^\pi \sin(\lambda t) \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \, d\lambda \left| \frac{-\cos(\lambda t)}{\lambda} \right|_0^\pi = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda$$

$$\therefore \quad \int_0^\infty \frac{1 - \cos(\lambda\pi)}{\lambda} \sin(\lambda x) d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \pi/2 & \text{for } 0 \le x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

At $x = \pi$, which is a point of discontinuity of $f(x)$, the value of the above integral

$$= \frac{\pi}{2}\left[\frac{f(\pi - 0) + f(\pi + 0)}{2}\right] = \frac{\pi}{2} \cdot \frac{1 + 0}{2} = \frac{\pi}{4}.$$

22.4 (1) FOURIER TRANSFORMS

Rewriting (9) of § 22.3 as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\,ds \int_{-\infty}^{\infty} f(t)e^{ist}\,dt,$$

it follows that if
$$F(s) = \int_{-\infty}^{\infty} f(t)\,e^{ist}\,dt \qquad ...(1)$$

then
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds \qquad ...(2)$$

The function $F(s)$, defined by (1), is called the **Fourier transform** of $f(x)$. *Also the function $f(x)$, as given by* (2), *is called the* **inverse Fourier transform** *of $F(s)$*. Sometimes, we call (2) as an *inversion formula* corresponding to (1).

(2) Fourier sine and cosine transforms. From (5) of § 22.3, it follows that if

$$F_s(s) = \int_0^{\infty} f(x)\sin sx\,dx \qquad ...(3)$$

then
$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(s)\sin sx\,ds \qquad ...(4)$$

The function $F_s(s)$, as defined by (3), *is known as the* **Fourier sine transform** *of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by* (4) *is called the* **inverse Fourier sine transform** *of $F_s(s)$*.

Similarly, if follows from (6) of § 22.3 that if

$$F_c(s) = \int_0^{\infty} f(x)\cos sx\,dx \qquad ...(5)$$

then
$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s)\cos sx\,ds \qquad ...(6)$$

The function $F_c(s)$ as defined by (5) *is known as the* **Fourier cosine transform** *of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$, as given by* (6), *is called the* **inverse Fourier cosine transform** *of $F_c(s)$*.

(3) **Finite Fourier sine and cosine transforms.** These transforms are useful for such a boundary-value problem in which at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** *of $f(x)$, in $0 < x < c$, is defined as*

$$F_s(n) = \int_0^{c} f(x)\sin\frac{n\pi x}{c}\,dx \qquad ...(7)$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier sine transform** *of $F_s(n)$ which is given by*

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c} \qquad ...(8)$$

The **finite Fourier cosine transform** *of $f(x)$, in $0 < x < c$, is defined as*

$$F_c(n) = \int_0^{c} f(x)\cos\frac{n\pi x}{c}\,dx \qquad ...(9)$$

where n is an integer.

The function $f(x)$ is then called the **inverse finite Fourier cosine transform** *of $F_c(n)$ which is given by*

$$f(x) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{c} \qquad ...(10)$$

Obs. The finite Fourier sine transform is useful for problems involving boundary conditions of heat distribution on two parallel boundaries, while the finite cosine transform is useful for problems in which the velocities normal to two parallel boundaries are among the boundary conditions.

22.5 PROPERTIES OF FOURIER TRANSFORMS

(1) **Linear property.** *If $F(s)$ and $G(s)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then*

$$F[a\,f(x) + bg(x)] = a\,F(s) + bG(s)$$

where a and b are constants.

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$ and $G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$

$\therefore$ $F[af(x) + bg(x)] = \int_{-\infty}^{\infty} e^{isx} [af(x) + bg(x)]\, dx = a \int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b \int_{-\infty}^{\infty} e^{isx} g(x) dx$

$= aF(s) + bG(s)$

(2) Change of scale property. *If F(s) is the complex Fourier transform of f(x), then*

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right), a \neq 0$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$ ...(*i*)

$\therefore$ $F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx} f(ax) dx$ $\left|\begin{array}{l}\text{Put } ax = t \\ \text{so that } dx = dt/a\end{array}\right.$

$= \int_{-\infty}^{\infty} e^{ist/a} f(t) dt / a = \frac{1}{a} \int_{-\infty}^{\infty} e^{i(s/a)t} f(t) dt = \frac{1}{a} F\left(\frac{s}{a}\right)$ [By (*i*)]

Cor. *If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of $f(x)$ respectively, then*

$$F_s\{f(ax)\} = \frac{1}{a} F_s\left(\frac{s}{a}\right) \quad \text{and} \quad F_c\{f(ax)\} = \frac{1}{a} F_c\left(\frac{s}{a}\right).$$

(3) Shifting property. *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x-a)\} = e^{isa} F(s)$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$ ...(*i*)

$\therefore$ $F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a) dx$ $\left|\begin{array}{l}\text{Put } x - a = t \\ \text{so that } dx = dt\end{array}\right.$

$= \int_{-\infty}^{\infty} e^{is(t+a)} f(t) dt = e^{isa} \int_{-\infty}^{\infty} e^{ist} f(t) dt = e^{isa} F(s)$ [By (*i*)]

(4) Modulation theorem. *If F(s) is the complex Fourier transform of f (x), then*

$$F\{f(x) \cos ax\} = \frac{1}{2} [F(s+a) + F(s-a)]$$

We have $F(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx$ ...(*i*)

$\therefore$ $F\{f(x) \cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x) \cos ax\, dx = \int_{-\infty}^{\infty} e^{isx} . f(x) . \frac{e^{iax} + e^{-iax}}{2}\, dx$

$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(s+a)x} f(x) dx + \int_{-\infty}^{\infty} e^{i(s-a)x} f(x) dx\right] = \frac{1}{2} [F(s+a) + F(s-a)].$

Cor. *If $F_s(s)$ and $F_c(s)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then*

(*i*) $F_s\{f(x) \cos ax\} = \frac{1}{2} [F_s(s+a) + F_s(s-a)]$ (*Anna, 2008*)

(*ii*) $F_c\{f(x) \sin ax\} = \frac{1}{2} [F_s(s+a) - F_s(s-a)]$

(*iii*) $F_s\{f(x) \sin ax\} = \frac{1}{2} [F_c(s-a) - F_c(s+a)]$

Obs. This theorem is of great importance in radio and television where the harmonic carrier wave is modulated by an envelope.

Example 22.2. *Find the Fourier transform of*

$$f(x) = \begin{cases} 1 \text{ for } |x| < 1 \\ 0 \text{ for } |x| > 1 \end{cases}.$$

Hence evaluate $\int_0^{\infty} \frac{\sin x}{x}\, dx.$ (*V.T.U., 2010 ; S.V.T.U., 2009 ; U.P.T.U., 2008*)

Solution. The Fourier transform of $f(x)$, *i.e.*,

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \int_{-1}^{1}(1)\,e^{isx}\,dx = \left|\frac{e^{isx}}{is}\right|_{-1}^{1} = \frac{e^{is}-e^{-is}}{is}$$

Thus $\quad F[f(x)] = F(s) = 2\,\dfrac{\sin s}{s}$, $s \neq 0$. For $s = 0$, we have $F(s) = 2$.

Now by the inversion formula, we get

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds, \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\sin s}{s}\,e^{-isx}ds = \begin{cases}1 \text{ for } |x|<1\\ 0 \text{ for } |x|>1\end{cases}$$

Putting $x = 0$, we get

$$\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds = \pi \;\therefore\; \int_0^{\infty}\frac{\sin s}{s}\,ds = \frac{\pi}{2}, \text{ since the integrand is even.}$$

Example 22.3. *Find the Fourier transform of :*

$$f(x) = \begin{cases}1-x^2, & |x| \le 1\\ 0 & ,|x|>1\end{cases}$$

Hence evaluate $\displaystyle\int_0^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\,dx.$ (V.T.U., 2011 S ; Anna, 2005 S ; Mumbai, 2005 S)

Solution. $F[f(x)] = \displaystyle\int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = F(s)$, say

$$= \int_{-\infty}^{-1}(0)\,e^{isx}\,dx + \int_{-1}^{1}(1-x^2)\,e^{isx}\,dx + \int_{1}^{\infty}(0)\,e^{isx}\,dx = \left|(1-x^2)\frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right|_{-1}^{1}$$

$$= 2\left(\frac{e^{is}+e^{-is}}{-s^2}\right) - 2\left(\frac{e^{is}-e^{-is}}{-is^3}\right) = -\frac{4}{s^3}(s\cos s - \sin s)$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

or $$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)\,e^{-isx}\,ds = \begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

Putting $x = 1/2$, we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{s^3}(s\cos s - \sin s)e^{-is/2}ds = \frac{3}{4}$$

or $$\int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}$$

or $$\int_{-\infty}^{\infty}\frac{s\cos s-\sin s}{s^3}\,.\cos\frac{s}{2}\,ds = -\frac{3\pi}{8}$$

or $$\int_0^{\infty}\frac{x\cos x-\sin x}{x^3}\,.\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}, \text{ since the integral is even.}$$

Example 22.4. (*a*) *Find the Fourier transform of* $e^{-a^2x^2}$, $a<0$. *Hence deduce that* $e^{-x^2/2}$ *is self reciprocal in respect of Fourier transform.* (Madras, 2006 ; Kottayam, 2005)

(*b*) *Find Fourier transform of* (*i*) $e^{-2(x-3)^2}$ (*ii*) $e^{-x^2}\cos 3x$.

Solution. (*a*) $\quad F(e^{-a^2x^2}) = \displaystyle\int_{-\infty}^{\infty} e^{-a^2x^2}\,.\,e^{isx}\,dx = \int_{-\infty}^{\infty} e^{-a^2(x^2-isx/a^2)}dx$

$$= \int_{-\infty}^{\infty} e^{-a^2(x-is/2a^2)^2}\,.\,e^{-s^2/4a^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-t^2} . e^{-s^2/4a^2} \, dt/a \qquad \text{[Putting } a(x - is/2a^2) = t, dx = dt/a$$

$$= \frac{e^{-s^2/4a^2}}{a} \int_{-\infty}^{\infty} e^{-t^2} \, dt = \frac{e^{-s^2/4a^2}}{a} \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}\right]$$

Hence $\quad F(e^{-a^2x^2}) = \dfrac{\sqrt{\pi}}{a} e^{-s^2/4a^2}$

Taking $a^2 = 1/2$, we have

$$F(e^{-x^2/2}) = \frac{\sqrt{\pi}}{(1/\sqrt{2})} e^{-s^2/2} = \sqrt{2\pi} e^{-s^2/2}$$

i.e., Fourier transform of $e^{-x^2/2}$ is a constant times $e^{-s^2/2}$. Also the functions $e^{-x^2/2}$ and $e^{-s^2/2}$ are the same. Hence it follows that $e^{-x^2/2}$ *is self-reciprocal under the Fourier transform.*

(*b*) Since $e^{-2x^2} = e^{-(2x)^2/2} = f(2x)$ where $f(x) = e^{-x^2/2}$

$\therefore$ by change of scale property, $F[f(2x)] = \dfrac{1}{2} F(s/2)$

i.e., $$F(e^{-2x^2}) = F[e^{-(2x)^2/2}] = \sqrt{2\pi} e^{-(s/2)^2/2} = \sqrt{2\pi} e^{-s^2/8}$$

By shifting property $F f(x-3) = e^{i3s} F(3)$

$\therefore$ $$F[e^{-2(x-3)^2}] = e^{3is} \sqrt{2\pi} e^{-s^2/8} = \sqrt{2\pi} e^{(3is - s^2/8)} \qquad ...(i)$$

Also by modulation theorem,

$$F[f(x) \cos 2x] = \frac{1}{2} [F(s+a) + F(s-a)]$$

$$F(e^{-x^2} \cos 3x) = \frac{1}{2} \sqrt{2\pi} [e^{-(s+3)^2/2} + e^{-(s-3)^2/2}]. \qquad ...(ii)$$

Example 22.5. *Find the Fourier cosine transform of* e^{-x^2}. (V.T.U., 2010; Rajasthan, 2006)

Solution. We have $F_c(e^{-x^2}) = \int_0^{\infty} e^{-x^2} \cos sx \, dx = I$ (say)

Differentiating under the integral sign w.r.t. s,

$$\frac{dI}{ds} = -\int_0^{\infty} xe^{-x^2} \sin sx \, dx = \frac{1}{2} \int_0^{\infty} (\sin sx)(-2xe^{-x^2}) dx$$

$$= \frac{1}{2} \left\{ \left| \sin sx . e^{-x^2} \right|_0^{\infty} - s \int_0^{\infty} \cos sx . e^{-x^2} \, dx \right\}$$

$$= -\frac{s}{2} \int_0^{\infty} e^{-x^2} \cos sx \, dx = -\frac{s}{2} I \quad \text{or} \quad \frac{dI}{I} = -\int \frac{s}{2} ds + \log c$$

or $$\log I = -\frac{s^2}{4} + \log c = \log (ce^{-s^2/4})$$

$\therefore$ $$I = ce^{-s^2/4} \quad \text{or} \quad \int_0^{\infty} e^{-x^2} \cos sx \, dx = ce^{-s^2/4}$$

Putting $s = 0$, $$c = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \quad i.e. \; I = \frac{\sqrt{\pi}}{2} e^{-s^2/4}.$$

Hence $\quad F_c(e^{-x^2}) = \dfrac{\sqrt{\pi}}{2} e^{-s^2/4}$.

Example 22.6. *Find the Fourier sine transform of* $e^{-|x|}$.

Hence show that $\int_0^{\infty} \dfrac{x \sin mx}{1+x^2} dx = \dfrac{\pi e^{-m}}{2}$, $m > 0$. (V.T.U., 2010 ; S.V.T.U., 2008 ; Kottayam, 2005)

Solution. x being positive in the interval $(0, \infty)$, $e^{-|x|} = e^{-x}$

$\therefore$ Fourier sine transform of $f(x) = e^{-|x|}$ is given by

$$F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty e^{-x}\sin sx\,dx$$

$$= \left|\frac{e^{-x}}{1+s^2}(-\sin sx - s\cos sx)\right|_0^\infty = \frac{s}{1+s^2}$$

Using Inversion formula for Fourier sine transforms, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin sx\,dx \quad \text{or} \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s}{1+s^2}\sin sx\,ds$$

or changing x to m, $$e^{-m} = \frac{2}{\pi}\int_0^\infty \frac{s\sin ms}{1+s^2}\,ds = \frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+m^2}\,dx$$

Hence $$\int_0^\infty \frac{x\sin mx}{1+m^2}\,dx = \frac{\pi e^{-m}}{2}.$$

Example 22.7. *Find the Fourier cosine transform of* $f(x) = \begin{cases} x, \text{ for } 0 < x < 1 \\ 2-x, \text{ for } 1 < x < 2 \\ 0, \text{ for } x > 2 \end{cases}$ (J.N.T.U., 2006)

Solution. Fourier cosine transform of $f(x)$ *i.e.*, $F_c\{f(x)\}$

$$= \int_0^\infty f_c(x)\cos sx\,dx = \int_0^1 x\cos sx\,dx + \int_1^2 (2-x)\cos sx\,dx + \int_2^\infty 0\,.\,dx$$

$$= \left|x\frac{\sin sx}{s} - \left(\frac{-\cos sx}{s^2}\right)\right|_0^1 + \left|(2-x)\frac{\sin sx}{s} - (-1)\frac{-\cos sx}{s^2}\right|_1^2$$

$$= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right)$$

$$= \frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}.$$

Example 22.8. *Find the Fourier sine transform of* e^{-ax}/x. (V.T.U., 2010 S ; P.T.U., 2006 ; Rohtak, 2005)

Solution. Let $f(x) = e^{-ax}/x$, then its Fourier sine transform

i.e. $$F_s\{f(x)\} = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx = F(s), \text{ say}$$

Differentiating both sides w.r.t. s, we get

$$\frac{d}{ds}\{F(s)\} = \int_0^\infty \frac{xe^{-ax}\cos sx}{x}\,dx = \int_0^\infty e^{-ax}\cos sx\,dx = \frac{a}{s^2+a^2}$$

Integrating w.r.t. s, we obtain $F(s) = \int_0 \frac{a}{s^2+a^2}\,ds = \tan^{-1}\frac{s}{a} + c$

But $F(s) = 0$, when $s = 0$; $\therefore$ $c = 0$. Hence $F(s) = \tan^{-1}(s/a)$.

Example 22.9. *Find the Fourier cosine transform of* $f(x) = 1/(1+x^2)$. (V.T.U., 2011 S ; Anna, 2009)
Hence derive Fourier sine transform of $\phi(x) = x/(1+x^2)$. (V.T.U., 2009 S)

Solution. $$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2}\,dx = I, \text{ say} \quad \text{...(i)}$$

$\therefore$ $$\frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}\,dx = -\int_0^\infty \frac{x^2\sin sx}{x(1+x^2)}\,dx \quad \text{...(ii)}$$

$$= -\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)}dx = -\int_0^\infty \frac{\sin sx}{x}dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx$$

or $$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx \quad ...(iii)$$

$\therefore$ $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx = I$$

or $$\frac{d^2I}{ds^2} - I = 0 \quad \text{or} \quad (D^2-1)I = 0, \text{ where } D = \frac{dI}{ds}$$

Its solution is $$I = c_1e^s + c_2e^{-s} \quad ...(iv)$$

$\therefore$ $$dI/ds = c_1e^s - c_2e^{-s} \quad ...(v)$$

When $s = 0$, (i) and (iv) give $c_1 + c_2 = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

Also when $s = 0$, (iii) and (v) give $c_1 - c_2 = -\pi/2$.

Solving these, $c_1 = 0$, $c_2 = \pi/2$.

Thus from (i) and (iv), we have $F_c[f(x)] = I = (\pi/2)e^{-s}$

Now $$F_s[\phi(x)] = \int_0^\infty \frac{x\sin sx}{1+x^2}dx = -\frac{dI}{ds}, \text{ from } (ii)$$

$$= (\pi/2)e^{-s}, \text{ from } (v), \text{ with } c_1 = 0, c_2 = \pi/2.$$

Example 22.10. *Find the Fourier sine and cosine transform of x^{n-1}, $n > 0$.* (Madras, 2006)

Solution. We know that $F_s(x^{n-1}) = \int_0^\infty x^{n-1}\sin sx\,dx \quad ...(i)$

and $$F_c(x^{n-1}) = \int_0^\infty x^{n-1}\cos sx\,dx \quad ...(ii)$$

$\therefore$ $$F_c(x^{n-1}) + i\,F_s(x^{n-1}) = \int_0^\infty (\cos sx + i\sin sx)x^{n-1}\,dx$$

$$= \int_0^\infty e^{isx}x^{n-1}\,dx = \int_0^\infty e^{-t}\left(-\frac{t}{is}\right)^{n-1}\left(-\frac{dt}{is}\right) \quad [\text{Where } isx = -t]$$

$$= \left(-\frac{1}{(i)}\right)^n \int_0^\infty e^{-t}t^{n-1}\,dt = \frac{(i)^{2n}}{(i)^n s^n}\Gamma(n) = \frac{(i)^n}{s^n}\Gamma(n)$$

$$= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^n \Gamma(n)/s^n = \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)\Gamma(n)/s^n$$

Equating real and imaginary parts, we get

$$F_c(x^{n-1}) = \frac{\Gamma(n)}{s^n}\cos\frac{n\pi}{2} \quad \text{and} \quad F_s(x^{n-1}) = \frac{\Gamma(n)}{s^n}\sin\frac{n\pi}{2}.$$

Example 22.11. *(a) Show that $F_s[x f(x)] = -\frac{d}{ds}\{F_c(s)\}$; $F_c[x f(x)] = \frac{d}{ds}\{F_s(s)\}$.*

(b) Find the Fourier sine and cosine transform of xe^{-ax} (Madras, 2006)

Solution. (a) $$\frac{d}{ds}\{F_c(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\cos sx\,dx\right\} = \int_0^\infty f(x)(-x\sin sx)dx$$

$$= -\int_0^\infty \{x f(x)\}\sin sx dx = -F_s[x f(x)] \quad ...(i)$$

$$\frac{d}{ds}\{F_s(s)\} = \frac{d}{ds}\left\{\int_0^\infty f(x)\sin sx\,dx\right\} = \int_0^\infty f(x)(x\cos sx)dx$$

$$= \int_0^\infty \{x f(x)\}\cos sx dx = F_c[x f(x)] \quad ...(ii)$$

(b) We have $$F_s(e^{-ax}) = \int_0^\infty e^{-ax} \sin sx \, dx = \frac{e^{-ax}}{a^2+s^2} \Big| -a \sin sx - s \cos sx \Big|_0^\infty$$

$$= \frac{s}{a^2+s^2} \qquad \text{...(iii)}$$

and $$F_c(e^{-ax}) = \int_0^\infty e^{-ax} \cos sx \, dx = \frac{e^{-ax}}{a^2+s^2} \Big| -a \cos sx + s \sin sx \Big|_0^\infty$$

$$= \frac{a}{a^2+s^2} \qquad \text{...(iv)}$$

Now $$F_c(xe^{-ax}) = -\frac{d}{ds}\{F_c(e^{-ax})\} \qquad \text{[by (i)}$$

$$= -\frac{d}{ds}\left(\frac{a}{a^2+s^2}\right) = \frac{2as}{(a^2+s^2)^2} \qquad \text{[by (iv)}$$

$$F_c(xe^{-ax}) = \frac{d}{ds}\{F_s(e^{-ax})\} \qquad \text{[by (ii)}$$

$$= \frac{d}{ds}\left(\frac{s}{a^2+s^2}\right) = \frac{(a^2+s^2) - s(2s)}{(a^2+s^2)^2} = \frac{a^2-s^2}{(a^2+s^2)^2}. \qquad \text{[by (iii)}$$

Example 22.12. *If the Fourier sine transform of $f(x) = \dfrac{1-\cos n\pi}{n^2\pi^2}$ $(0 \le x \le \pi)$, find $f(x)$.* (Delhi, 2002)

Solution. We have $f(x)$ = inverse finite Fourier sine transform of $F_s(n)$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin \frac{n\pi x}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2\pi^2}\right\} \sin nx$$

$$= \frac{2}{\pi^3}\sum_{n=1}^{\infty}\left\{\frac{1-\cos n\pi}{n^2}\right\} \sin nx.$$

Example 22.13. *Solve the integral equation**

$$\int_0^\infty f(\theta) \cos \alpha\theta \, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

Hence evaluate $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$. (V.T.U., 2011 S ; Kurukshetra, 2005)

Solution. We have $$\int_0^\infty f(\theta) \cos \alpha\theta \, d\theta = F_c(\alpha)$$

$$\therefore \qquad F_c(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases} \qquad \text{...(i)}$$

By the inversion formula, we have

$$f(\theta) = \frac{2}{\pi}\int_0^\infty F_c(\alpha) \cos \alpha\theta \, d\alpha = \frac{2}{\pi}\int_0^1 (1-\alpha) \cos \alpha\theta \, d\alpha \qquad \text{[Integrating by parts]}$$

$$= \frac{2}{\pi}\left[\left|(1-\alpha)\frac{\sin \alpha\theta}{\theta}\right|_0^1 - \int_0^1 (-1)\frac{\sin \alpha\theta}{\theta}\, d\alpha\right] = \frac{2}{\pi\theta}\left|-\frac{\cos \alpha\theta}{\theta}\right|_0^1 = \frac{2(1-\cos\theta)}{\pi\theta^2}$$

Now $$F_c(\alpha) = \int_0^\infty f(\theta) \cos \alpha\theta \, d\theta = \int_0^\infty \frac{2(1-\cos\theta)}{\pi\theta^2} \cos \alpha\theta \, d\theta \qquad \text{...(ii)}$$

* Refer to Chapter 26.

$\therefore$ From (*i*) and (*ii*), we have

$$\frac{2}{\pi}\int_0^{\infty} \frac{1-\cos\theta}{\theta^2}\cos\alpha\theta \, d\theta = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0 & , \alpha > 1 \end{cases}$$

Now letting $\alpha \to 0$, we get $\dfrac{2}{\pi}\displaystyle\int_0^{\infty} \frac{1-\cos\theta}{\theta^2}\, d\theta = 1$ (*V.T.U., 2008*)

or $$\int_0^{\infty} \frac{2\sin^2\theta/2}{\theta^2}\, d\theta = \pi/2$$ [Put $\theta/2 = t$, so that $d\theta = 2dt$]

$\therefore$ $$\int_0^{\infty} \frac{\sin^2 t}{t^2} = \pi/2.$$

PROBLEMS 22.1

1. Express the function $f(x) = \begin{cases} 1 \text{ for } |x| \le 1 \\ 0 \text{ for } |x| > 1 \end{cases}$ as a Fourier integral.

   Hence evaluate $\displaystyle\int_0^{\infty} \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda$. (*Kottayam, 2005*)

2. Find the Fourier integral representation for

   (*i*) $f(x) = \begin{cases} 1-x^2, \text{ for } |x| \le 1 \\ 0, \text{ for } |x| > 1 \end{cases}$ (*Mumbai, 2008*) (*ii*) $f(x) = \begin{cases} e^{ax}, \text{ for } x \le 0, a > 0 \\ e^{-ax}, \text{ for } x \ge 0\ a < 0 \end{cases}$

3. Using the Fourier integral representation, show that

   (*i*) $\displaystyle\int_0^{\infty} \frac{\omega\sin x\omega}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ (x > 0)$ (*ii*) $\displaystyle\int_0^{\infty} \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}\ \ (x \ge 0)$ (*U.P.T.U., 2008*)

   (*iii*) $\displaystyle\int_0^{\infty} \frac{\sin\omega\cos x\omega}{\omega}\, d\omega = \frac{\pi}{2}$ when $0 \le x < 1$. (*iv*) $\displaystyle\int_0^{\infty} \frac{\sin\pi\alpha\sin\alpha\theta}{1-\alpha^2}\, d\alpha = \begin{cases} \frac{1}{2}\pi\sin\theta, & 0 \le \theta \le \pi \\ 0, & \theta > \pi \end{cases}$

4. Find the Fourier transforms of

   (*i*) $f(x) = \begin{cases} 1, |x| < a \\ 0, |x| > a \end{cases}$ (*W.B.T.U., 2005 ; Madras, 2003 ; P.T.U., 2003*)

   Hence evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{x}\, dx$ (*Mumbai, 2009*)

   (*ii*) $f(x) = \begin{cases} x^2, |x| < a \\ 0, |x| > a \end{cases}$ (*S.V.T.U., 2008*)

5. Find the Fourier transform of $f(x) = \begin{cases} a^2 - x^2 \text{ for } |x| \le a \\ 0 \text{ for } |x| > a \end{cases}$ (*V.T.U., 2007*)

   Hence deduce that $\displaystyle\int_0^{\infty} \frac{\sin t - t\cos t}{t^3}\, dt = \frac{\pi}{4}$. (*Anna, 2009*)

6. Given $F(e^{-x^2}) = \sqrt{\pi}e^{-s^2/4}$, find the Fourier transform of

   (*i*) $e^{-x^2/3}$ (*ii*) $e^{-4(x-3)^2}$

7. Find the Fourier sine and cosine transforms of $f(x) = \begin{cases} 1, 0 \le x < 2 \\ 0, x \ge 2 \end{cases}$ (*V.T.U., 2008*)

8. Using the Fourier sine transform of e^{-ax} $(a > 0)$, show that $\displaystyle\int_0^{\infty} \frac{x\sin kx}{a^2+x^2}\, dx = \frac{\pi}{2}e^{-ak}\ (k > 0)$.

   Hence obtain the Fourier sine transform of $x/(a^2 + x^2)$. (*Rohtak, 2006 ; Madras, 2003 S*)

9. Find the Fourier cosine transform of e^{-ax}. (*Anna, 2009*)

   Hence evaluate $\displaystyle\int_0^{\infty} \frac{\cos\lambda x}{x^2+a^2}\, dx$. (*V.T.U., 2003 S*)

10. If the Fourier sine transform of $f(x)$ is e^{-as}/s, find $f(x)$. Hence obtain the inverse Fourier sine transform of $1/s$. (*Mumbai, 2009*)

11. Find the Fourier cosine transform of e^{-x^2} and hence evaluate Fourier sine transform of xe^{-x^2}.
12. Find the Fourier cosine transform of $e^{-a^2x^2}$ for any $a > 0$ and hence prove that $e^{-x^2/2}$ is self-reciprocal under Fourier cosine transform. (Anna, 2009)
13. Find the Fourier sine transform of (i) $\dfrac{1}{x(x^2+a^2)}$. (Rohtak, 2006)

    (ii) $[e^{-ax}/x]$, $a > 0$ (U.P.T.U., 2008)
14. Obtain Fourier sine transform of

    (i) $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$ (Madras, 2000) (ii) $f(x) = \begin{cases} 4x, & \text{for } 0 < x < 1 \\ 4 - x, & \text{for } 1 < x < 4 \\ 0, & \text{for } x > 4 \end{cases}$ (V.T.U., 2006)
15. Find the Fourier cosine transform of $(1 - x/\pi)^2$. (P.T.U., 2006)
16. Find the finite Fourier sine and cosine transforms of $f(x) = 2x$, $0 < x < 4$. (V.T.U., 2011)
17. Find the finite sine transform of $f(x) = \begin{cases} -x, & x < c \\ \pi - x, & x > c \end{cases}$ where $0 \le c \le \pi$.
18. Show that the inverse finite Fourier sine transform of $F_s(n) = \dfrac{1}{\pi}\left\{1 + \cos n\pi - 2\cos\dfrac{n\pi}{2}\right\}$ is

$$f(x) = \begin{cases} 1, & 0 < x < \pi/2 \\ -1, & \pi/2 < x < \pi \end{cases}$$ (V.T.U., 2008)

19. Solve the integral equation $\displaystyle\int_0^\infty f(x)\sin tx\,dx = \begin{cases} 1, & 0 \le t < 1, \\ 2, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases}$ (Kottayam, 2005)
20. Solve the integral equation $\displaystyle\int_0^\infty f(x)\cos \alpha x\,dx = e^{-\alpha}$. (S.V.T.U., 2009 ; Rohtak, 2004)

22.6 (1) CONVOLUTION

The convolution of two functions $f(x)$ and $g(x)$ over the interval $(-\infty, \infty)$ is defined as

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\,du = h(x).$$

(2) **Convolution theorem for Fourier transforms.** *The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms, i.e.,*

$$F[f(x) * g(x)] = F[f(x)] \,.\, F[g(x)]$$

We have
$$F[f(x) * g(x)] = F\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)\,du\right\} e^{isx}\,dx = \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)\,.\,e^{isx}\,dx\right\} du$$

[Changing the order of integration]

$$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} e^{is(x-u)}\,.\,g(x-u)\,d(x-u)\right\} e^{isu}\,du$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\left\{\int_{-\infty}^{\infty} e^{ist}\, g(t)\,dt\right\} du \text{ where } x - u = t$$

$$= \int_{-\infty}^{\infty} e^{isu} f(u)\,du\,.\,F[g(t)] = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx\,.\,F[g(x)] = F\,[f(x)]\,.\,F[g(x)]$$

22.7 PARSEVAL'S IDENTITY FOR FOURIER TRANSFORMS

If the Fourier transforms of $f(x)$ and $g(x)$ are $F(s)$ and $G(s)$ respectively, then

(i) $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\bar{G}(s)\,ds = \int_{-\infty}^{\infty} f(x)\bar{g}(x)\,dx$ (ii) $\displaystyle\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\,ds = \int_{-\infty}^{\infty} |f(x)|^2\,dx$

where bar implies the complex conjugate.

(*i*) $\int_{-\infty}^{\infty} f(x)\bar{g}\,(dx) \int_{-\infty}^{\infty} f(x)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\, e^{isx}\, ds\right\} dx$ [Using the inversion formula for Fourier transform]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\left\{\int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx\right\} ds$$ [Changing the order of integration]

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)F(s)ds, \text{ by definition of F-transform.}$$

(*ii*) Taking $g(x) = f(x)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,\overline{F}\,(s)ds = \int_{-\infty}^{\infty} f(x)\,\bar{f}(x)dx \text{ or } \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(s)|^2\, ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

Obs. The following Parseval's identities for Fourier cosine and sine transforms can be proved as above :

(*i*) $\frac{2}{\pi}\int_0^{\infty} F_c(s)\,G_c(s)ds = \int_0^{\infty} f(x)\,g(x)dx$ (*ii*) $\frac{2}{\pi}\int_0^{\infty} F_s(s)\,G_s(s)ds = \int_0^{\infty} f(x)\,g(x)dx$

(*iii*) $\frac{2}{\pi}\int_0^{\infty} |F_c(s)|^2\, ds = \int_0^{\infty} |f(x)|^2\, dx$ (*iv*) $\frac{2}{\pi}\int_0^{\infty} |F_s(s)|^2\, ds = \int_0^{\infty} |f(x)|^2\, dx.$

Example 22.14. *Using Parseval's identities, prove that*

(*i*) $\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$ (*S.V.T.U., 2009 ; ... 2008*)

(*ii*) $\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\, dt = \frac{\pi}{4}$ (*iii*) $\int_0^{\infty} \frac{\sin at}{t(a^2+t^2)}\, dt = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}.$

Solution. (*i*) Let $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{b}{b^2+s^2}$

Now using Parseval's identity for Fourier cosine transforms, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} F_c(s)G_c(s)ds = \int_0^{\infty} f(x)\, g(x)dx \qquad ...(1)$$

We have $$\frac{2}{\pi}\int_0^{\infty} \frac{ab}{(a^2+s^2)(b^2+s^2)}\, ds = \int_0^{\infty} e^{-(a+b)x}dx$$

or $$\frac{2ab}{\pi}\int_0^{\infty} \frac{ds}{(a^2+s^2)(b^2+s^2)} = \left|\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^{\infty} = \frac{1}{a+b}$$

Thus $$\int_0^{\infty} \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$$

(*ii*) Let $f(x) = \frac{x}{x^2+1}$ so that $F_s[f(x)] = \frac{\pi}{2}e^{-s}$

Now using Parseval's identity for sine transform, *i.e.*,

$$\frac{2}{\pi}\int_0^{\infty} [F_s\{f(x)\}]^2\, ds = \int_0^{\infty} [f(x)]^2\, dx$$

or $$\int_0^{\infty}\left(\frac{x}{x^2+1}\right)^2 dx = \frac{2}{\pi}\int_0^{\infty}\left(\frac{\pi}{2}e^{-s}\right)^2 ds = \frac{\pi}{2}\left| e^{-2s}/-2\right|_0^{\infty} = \frac{\pi}{-4}(0-1) = \frac{\pi}{4}$$

Hence $$\int_0^{\infty} \frac{t^2}{(t^2+1)^2}\, dt = \frac{\pi}{4}$$

(*iii*) Let $f(x) = e^{-ax}$ and $g(x) = \begin{cases} 1, & 0 < x < a \\ 0, & x > a \end{cases}$. Then $F_c(s) = \frac{a}{a^2+s^2}, G_c(s) = \frac{\sin as}{s}$

Now using (1) above, we have $\frac{2}{\pi}\int_0^{\infty}\frac{a \sin as}{s(a^2+s^2)}ds = \int_0^a e^{-as}.1\,dx = \frac{1-e^{-a^2}}{a}$

Thus $$\int_0^{\infty}\frac{\sin at}{t(a^2+t^2)}dt = \frac{\pi}{2a^2}(1-e^{-a^2}).$$

Example 22.15. *Find the Fourier transform of $f(x)$ given by* $f(x) = \begin{cases} a-|x|, & for\ |x|<a \\ 0 & ,for\ |x|>a \end{cases}$.

Hence show that $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$ *and* $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \pi/3.$ (Anna, 2008)

Solution. Fourier transform of $f(x)$ *i.e.* $F[f(x)] = \int_{-\infty}^{\infty} f(x)\,e^{isx}\,dx = \int_{-a}^{a}[a-|x|]e^{isx}\,dx$

$$= \int_{-a}^{a}[a-|x|](\cos x + i\sin sx)dx$$

$$= 2\int_0^a (a-x)\cos sx\,dx + 0 \qquad \left\{\because \begin{array}{l}[a-|x|]\cos x \text{ in an even function} \\ [a-|x|]\sin x \text{ is an odd function}\end{array}\right.$$

$$= 2\left|(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2}\right|_0^a = 2\,\frac{1-\cos as}{s^2} = 4\,\frac{\sin^2 as/2}{s^2}$$

(*i*) By inversion formula,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\,e^{-isx}\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4\sin^2 as/2}{s^2}e^{-isx}\,ds$$

To evaluate $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt$, put $x = 0$ and $a = 2$ so that

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 s}{s^2}ds = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds \qquad \left[\because \frac{\sin s}{s} \text{ is an even function}\right]$$

$$\therefore \quad \int_0^{\infty}\left(\frac{\sin s}{s}\right)^2 ds = \frac{\pi}{4}f(0) = \frac{\pi}{2}. \qquad [\because f(0) = a = 2$$

(*ii*) Using Parseval's identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}[F(s)]^2 ds = \int_{-\infty}^{\infty}|f(x)|^2\,dx$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{4\sin^2 as/2}{s^2}\right)^2 dx = \int_{-a}^{a}|[a-|x|]|^2\,dx$$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin as/2}{s}\right)^4 ds = 2\int_0^a (a-x)^2\,dx = 2\left|\frac{(a-x)^3}{-3}\right|_0^a = \frac{2}{3}a^3$$

Putting $t = as/2$ and $dt = ads/2$

$$\frac{16}{\pi}\int_0^{\infty}\left(\frac{\sin t}{2t/a}\right)\frac{2}{a}dt = \frac{2}{3}a^3 \quad \text{or} \quad \frac{2a^3}{\pi}\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{2}{3}a^3$$

Hence $$\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}.$$

PROBLEMS 22.2

1. Verify Convolution theorem for $f(x) = g(x) = e^{-x^2}$. (*V.T.U., 2000 S*)
2. Use Convolution theorem to find the inverse Fourier transform of $\frac{i}{(1+s^2)^2}$, given that $\frac{2}{(1+s^2)}$ is the Fourier transform of $e^{-|x|}$. (*V.T.U., 2010 S*)
3. Using Parseval's identity, show that

   (*i*) $\int_0^\infty \frac{dx}{(t^2+1)^2} = \frac{\pi}{4}$, (*Hissar, 2007*) (*ii*) $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)} dt = \frac{\pi}{10}$. (*Rohtak, 2003*)
4. Find the Fourier transform of $f(x) = \begin{cases} 1 \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1 \end{cases}$.

   Hence deduce that $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. (*Anna, 2009*)
5. Evaluate $\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx$.

22.8 RELATION BETWEEN FOURIER AND LAPLACE TRANSFORMS

If
$$f(t) = \begin{cases} e^{-xt} g(t), & t > 0 \\ 0, & t < 0 \end{cases} \quad ...(i)$$
then
$$F\{f(t)\} = L\{g(t)\}.$$

We have
$$F\{f(t)\} = \int_{-\infty}^{\infty} e^{ist} f(t) dt = \int_{-\infty}^{0} e^{ist} . 0 . dt + \int_0^\infty e^{ist} . e^{-xt} g(t) dt$$
$$= \int_0^\infty e^{(is-x)t} g(t)\, dt = \int_0^\infty e^{-pt} g(t) dt \quad \text{where } p = x - is$$

Hence *the Fourier transform of* $f(t)$ [defined by (*i*)] is the *Laplace transform of* $g(t)$.

22.9 FOURIER TRANSFORMS OF THE DERIVATIVES OF A FUNCTION

The Fourier transform of the function $u(x, t)$ is given by
$$F[u(x, t)] = \int_{-\infty}^{\infty} u e^{isx}\, dx$$

Then the Fourier transform of $\partial^2 u/\partial x^2$, *i.e.*
$$F\left[\frac{\partial^2 u}{\partial x^2}\right] = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{isx}\, dx = \left| e^{isx} \frac{\partial u}{\partial x} - is\, e^{isx} . u \right|_{-\infty}^{\infty} + (is)^2 \int_{-\infty}^{\infty} u e^{isx}\, dx,$$

on applying the general rule of integration by parts (p. 398). If u and $\frac{\partial u}{\partial x}$ tend to zero as x tends to $\pm\infty$, then
$$\mathbf{F}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\mathbf{s}^2\, \mathbf{F}[\mathbf{u}] \quad ...(1)$$

Similarly in the case of Fourier sine and cosine transforms, we have
$$\mathbf{F_s}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = \mathbf{s}(\mathbf{u})_{\mathbf{x}=0} - \mathbf{s}^2 \mathbf{F_s}[\mathbf{u}] \quad ...(2)$$
and
$$\mathbf{F_c}\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}\right] = -\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)_{\mathbf{x}=0} - \mathbf{s}^2\, \mathbf{F_c}[\mathbf{u}] \quad ...(3)$$

In general, the *Fourier transform of the nth derivative of* $f(x)$ is given by

$$\mathbf{F}\left[\frac{\mathbf{d^n f}}{\mathbf{dx^n}}\right] = (-\mathbf{is})^n\, \mathbf{F[f(x)]} \qquad ...(4)$$

provided the first $n-1$ derivatives vanish as $x \to \pm\infty$.

For $\quad F[f^n(x)] = \int_{-\infty}^{\infty} f^n(x)\, e^{isx}\, dx$

$$= \left| e^{isx} f^{n-1} - ise^{isx} f^{n-2} + (is)^2 e^{isx} f^{n-3} - \ldots \right|_{-\infty}^{\infty} + (-is)^n \int_{-\infty}^{\infty} f \cdot e^{isx}\, dx$$

by the general rule of integration by parts, whence follows (4).

22.10 INVERSE LAPLACE TRANSFORMS BY METHOD OF RESIDUES

Let the Laplace transform of $f(x)$ be $\bar{f}(s)$ so that

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st}\, dt \qquad ...(1)$$

Multiply both sides by e^{xs} and integrate w.r.t. s within the limits $a - ir$ and $a + ir$. Then

$$\int_{a-ir}^{a+ir} e^{xs}\, \bar{f}(s) ds = \int_{a-ir}^{a+ir} e^{xs} \int_0^{\infty} f(t) e^{-st}\, dt\, ds \qquad [\text{Put } s = a - iu]$$

$$= \int_r^{-r} e^{x(a-iu)} \int_0^{\infty} f(t) e^{-(a-iu)t}\, dt\, (-idu) = ie^{ax} \int_{-r}^{r} e^{-ixu} \int_0^{\infty} [e^{-at} f(t)] e^{iut}\, dt du$$

$$= ie^{ax} \int_{-r}^{r} e^{-ixu} \int_{-\infty}^{\infty} \phi(t)\, e^{iut}\, dt\, du$$

where $\phi(t) = \begin{cases} e^{-at} f(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$.

Proceeding to limits as $r \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s)\, ds = ie^{ax} \cdot 2\pi\phi(x), \text{ by (2) of § 22.4} = 2\pi i e^{ax} . e^{-ax} f(x) \text{ for } x > 0.$$

Hence $\qquad f(x) = \int_{a-i\infty}^{a+i\infty} e^{xs}\, \bar{f}(s) ds \; (x > 0) \qquad ...(2)$

which is called the *complex inversion formula*. It provides a direct means for obtaining the inverse Laplace transform of a given function.

The integration in (2) is performed along a line LM parallel to the imaginary axis in the complex plane $z = x + iy$ such that all the singularities of $\bar{f}(s)$ lie to its left* (Fig. 22.1). Let us take a contour C which is composed of the line LM and the semi-circle C' (*i.e.*, MNL). Then from (2)

$$\frac{1}{2\pi i} \int_{LM} e^{xs}\, \bar{f}(s)\, ds = \frac{1}{2\pi i} \int_C e^{xs}\, \bar{f}(s)\, ds - \frac{1}{2\pi i} \int_{C'} e^{xs}\, \bar{f}(s)\, dx$$

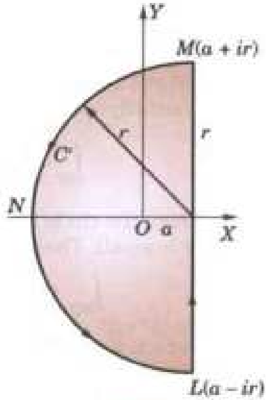


Fig. 22.1

The integral over C' tends to zero as $r \to \infty$ (under certain conditions†). Therefore,

$$\mathbf{f(x)} = \lim_{r \to \infty} \frac{1}{2\pi i} \int_C e^{xs}\, \bar{f}(s)\, dx$$

$$= \textbf{sum of the residues of } e^{xs}\bar{f}(s) \textbf{ at the poles of } f(s) \qquad ...(3)$$

[By §20.18]

* This has been so assumed simply to ensure the convergence of the integral (1).

† If positive constants A and k can be so found that $|\bar{f}(s)| < Ar^{-k}$ for every point on C', then

$$\lim_{r \to \infty} \frac{1}{2\pi i} \int_{C'} e^{xs} \bar{f}(s)\, ds = 0. \qquad \textit{(Jordan's Lemma)}$$

Example 22.16. *Evaluate* $L^{-1}\left\{\dfrac{1}{(s-1)(s^2+1)}\right\}$ *by the method of residues.*

Solution. Since $\left|\dfrac{1}{(s-1)(s^2+1)}\right| \sim \left|\dfrac{1}{s^3}\right|$ for $|s| \to \infty$, therefore,

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \text{sum of Res}\left[\frac{e^{xs}}{(s-1)(s^2+1)}\right] \text{at the poles } s = 1, \pm i$$

Now
$$(\text{Res})_{s=1} = \underset{s\to 1}{\text{Lt}}\left[\frac{(s-1)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} \qquad \text{[By § 20.19 (1)]}$$

$$(\text{Res})_{s=i} = \underset{s\to i}{\text{Lt}}\left[\frac{(s-i)\,.\,e^{xs}}{(s-1)(s^2+1)}\right] = \frac{e^{ix}}{(i-1)(i-1)} = -\frac{1}{2}\cdot\frac{e^{ix}}{1+i}$$

Changing i to $-i$, we get $(\text{Res})_{s=-i} = -\dfrac{1}{2}\cdot\dfrac{e^{-ix}}{1-i}$

$$\therefore \quad L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{e^x}{2} - \frac{1}{2}\left(\frac{e^{ix}}{1+i} + \frac{e^{-ix}}{1-i}\right) = \frac{1}{2}(e^x - \sin x - \cos x).$$

Example 22.17. *Prove that* $L^{-1}\left(\dfrac{e^{-c\sqrt{s}}}{s}\right) = 1 - \text{erf}\left(\dfrac{c}{\sqrt{2x}}\right)$.

Solution. By the complex inversion formula,

$$L^{-1}\left(\frac{e^{-c\sqrt{s}}}{s}\right) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{xs}\,.\,\frac{e^{-c\sqrt{s}}}{s}\,ds.$$

Since $s = 0$ is a branch point of the integrand, we take a contour $LMNPQST$ as shown in Fig. 22.2, so that it doesn't include any singularity. Therefore, by Cauchy's theorem (§ 20.13), we have

$$\left\{\int_{LM} + \int_{MN} + \int_{NP} + \int_{PQS} + \int_{ST} + \int_{TL}\right\} \times e^{xs}\,\frac{e^{-c\sqrt{s}}}{s}\,ds = 0 \qquad \text{...(i)}$$

If $ON = \rho$ and $OP = \varepsilon$, then along NP, $s = Re^{i\pi}$, therefore,

$$\int_{NP} = \int_{\rho}^{\varepsilon} e^{-xR}\,\frac{e^{-ic\sqrt{R}}}{R}\,dR$$

Similarly along ST, $s = Re^{-i\pi}$, therefore,

$$\int_{ST} = \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{ic\sqrt{R}}}{R}\,dR$$

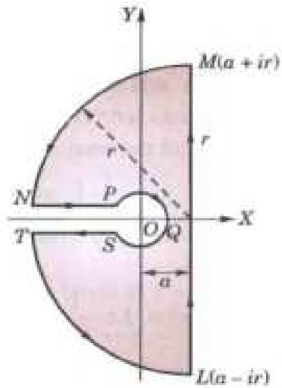


Fig. 22.2

Along the circle PQS, $s = \varepsilon\, e^{i\theta}$. Also e^{sx} and $e^{-c\sqrt{s}}$ are both approximately 1 since ε is small. Therefore,

$$\int_{PQS} = \int_{\pi}^{-\pi} \frac{1}{\varepsilon e^{i\theta}}\,.\,\varepsilon e^{i\theta}\,i d\theta = -2\pi i \text{ approximately.}$$

For $c > 0$, $|e^{-c\sqrt{s}}/s| < |s|^{-1}$.

But $\int_{MN}$ and $\int_{TL}$ both tend to zero as $r \to \infty$

Thus (i) takes the form

$$\int_{a-ir}^{a+ir} \frac{e^{xs-c\sqrt{s}}}{s}\,ds + \int_{\varepsilon}^{\rho} e^{-xR}\,\frac{e^{ic\sqrt{R}} - e^{-ic\sqrt{R}}}{R}\,dR - 2\pi i = 0$$

Taking limits as $\varepsilon \to 0$ and $\rho \to \infty$, we get

$$\int_{a-i\infty}^{a+i\infty} \frac{e^{xs-c\sqrt{s}}}{s}\,ds = 2\pi i - 2i\int_0^{\infty} e^{-xR}\,\frac{\sin c\sqrt{R}}{R}\,dR$$

or
$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{xs-c\sqrt{s}}}{s}\,ds = 1-\frac{2}{\pi}\int_0^{\infty}e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt^*, \text{ where } R = t^2/x$$

$$= 1-\frac{2}{\pi}\cdot\frac{\pi}{2}\,erf\left(\frac{C}{2\sqrt{x}}\right) \text{ whence follows the result.}$$

PROBLEMS 22.3

Using the method of residues, evaluate the inverse Laplace transform of each of the following :

1. $\dfrac{1}{(s+1)(s-2)^2}$.
2. $\dfrac{1}{(s-2)(s^2+1)}$.
3. $\dfrac{1}{s^2(s^2-a^2)}$.
4. $\dfrac{1}{(s-1)^2(s^2+1)}$.
5. $\dfrac{1}{(s^2+1)^2}$. (V.T.U., 2008 S)

22.11 APPLICATION OF TRANSFORMS TO BOUNDARY VALUE PROBLEMS

In one dimensional boundary value problems, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of the complex inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

(*i*) If in a problem $u(x, t)_{x=0}$ is given then we use infinite sine transform to remove $\partial u^2/\partial x^2$ from the differential equation.

In case $[\partial u(x, t)/\partial x]_{x=0}$ is given then we employ infinite cosine transform to remove $\partial^2 u/\partial x^2$.

(*ii*) If in a problem $u(0, t)$ and $u(l, t)$ are given, then we use finite sine transform to remove $\partial^2 u/\partial x^2$ from the differential equation.

In case $(\partial u/\partial x)_{x=0}$ and $(\partial u/\partial x)_{x=l}$ are given, then we employ finite cosine transform to remove $\partial^2 u/\partial x^2$.

The method of solution is best explained through the following examples.

Heat conduction

Example 22.18. *Determine the distribution of temperature in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial distribution of temperature is $f(x)$.* (Osmania, 2003)

Solution. Let $u(x, t)$ be the temperature at any point x and at any time t. We have to solve the heat-flow equation (§ 18.5)

$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}\ (x > 0, t > 0) \qquad ...(i)$$

subject to the initial condition $u(x, 0) = f(x)$ ...(*ii*)

and the boundary condition $u(0, t) = 0$ ...(*iii*)

Taking Fourier sine transform of (1) and denoting $F_s[u(x, t)]$ by $\bar{u}_s$, we have

$$\frac{d\bar{u}_s}{dt} = c^2\,[su(0, t) - s^2\,\bar{u}_s] \qquad \text{[By (2) of § 22.9]}$$

---

* We know that $\int_0^{\infty} e^{-t^2}\cos 2mt\,dt = \frac{1}{2}\sqrt{\pi}e^{-m^2}$ [Example 20.44

Integrating both sides w.r.t. m from 0 to $c/2\sqrt{x}$.

$$\int_0^{\infty} e^{-t^2}\left|\frac{\sin 2mt}{2t}\right|_0^{c/2\sqrt{x}} dt = \frac{1}{2}\sqrt{\pi}\int_0^{c/2\sqrt{x}} e^{-m^2}\,dm$$

or
$$\int_0^{\infty} e^{-t^2}\,\frac{\sin(ct/\sqrt{x})}{t}\,dt = \frac{\pi}{2}\,erf\left(\frac{c}{2\sqrt{x}}\right). \qquad \text{[By § 7.18(1)]}$$

or $$\frac{d\bar{u}_s}{dt} + c^2s^2\bar{u}_s = 0 \qquad \text{[By (iii)] ...(iv)}$$

Also the Fourier sine transform of (*ii*) is $\bar{u}_s = \bar{f}(s)$ at $t = 0$. ...(*v*)

Solving (*iv*) and using (*v*), we get $\bar{u}_s = \bar{f}_s(s)e^{-c^2s^2t}$

Hence taking its inverse Fourier sine transform, we obtain

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(s)e^{-c^2s^2t}\ \sin xs\, ds.$$

Example 22.19. *Solve $\partial u/\partial t = 2\partial^2 u/\partial x^2$, if $u(0, t) = 0$, $u(x, 0) = e^{-x}$ $(x > 0)$, $u(x, t)$ is bounded where $x > 0$, $t > 0$.* (*Rohtak, 2006*)

Solution. Given $\partial u/\partial t = 2\partial^2 u/\partial x^2, x > 0, t > 0$ ...(*i*)

with boundary conditions : $u(0, t) = 0$, $u(x, t)$ is bounded ...(*ii*)

and initial condition $u(x, 0) = e^{-x}, x > 0$ ...(*iii*)

Since $u(0, t)$ is given, we take Fourier sine transform of both sides of (*i*) so that

$$\int_0^\infty \frac{\partial u}{\partial t}\sin px\, dx = 2\int_0^\infty \frac{\partial^2 u}{\partial x^2}\sin px\, dx$$

or $$\frac{d}{dt}\int_0^\infty u(x, t)\sin px\, dx = 2\left[\left|\frac{\partial u}{\partial x}\sin px\right|_0^\infty - \int_0^\infty \frac{\partial u}{\partial x}\cdot p\cos px\, dx\right] \qquad \text{(Integrating by parts)}$$

or $$\frac{d\bar{u}_s}{dt} = -2p\int_0^\infty \frac{\partial u}{\partial x}\cos px\, dx, \text{ if } \frac{\partial u}{\partial x} \to \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty \text{ where } \bar{u}_s(p, t) = \int_0^\infty u(x, t)\sin px\, dx$$

$$= -2p\left[|u(x, t)\cos px|_0^\infty - \int_0^\infty u(x, t) - (-p\sin px)\, dx\right] \qquad \text{[Again integrating by parts]}$$

$$= -2p\left[0 - u(0, t) + p\int_0^\infty u(x, t)\sin px\, dx\right] \qquad [\because\ u(x, t) \to 0 \text{ as } x \to \infty \text{ by (ii)}]$$

$$= 2pu(0, t) - 2p^2\bar{u}_s$$

or $$\frac{d\bar{u}_s}{dt} = -2p^2\bar{u}_s \qquad \text{[By (ii)]}$$

Integrating $\int \frac{\partial\bar{u}_s}{\bar{u}_s} - \log c = -2p^2\int dt$ or $\log \bar{u}_s - \log c = -2p^2t$

$\therefore$ $\bar{u}_s(p, t) = ce^{-2p^2t}$ ...(*iv*)

Taking Fourier sine transform of both sides of (*iii*), we get

$$\int_0^\infty u(x, 0)\sin px\, dx = \int_0^\infty e^{-x}\sin px\, dx$$

or $$\bar{u}_s(p, 0) = \left|\frac{e^{-x}}{1+p^2}(-\sin px - p\cos px)\right|_0^\infty = \frac{p}{1+p^2} \qquad ...(v)$$

Putting $t = 0$ in (*iv*) and using (*v*), we obtain $p/(1 + p^2) = c$

Thus (*iv*) becomes $\bar{u}_s(p, t) = \frac{p}{1+p^2}e^{-2p^2t}$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \frac{pe^{-2p^2t}}{1+p^2}\sin px\, dp.$$

Example 22.20. *Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $(x > 0, t > 0)$ subject to the conditions*

(i) $u = 0$, when $x = 0$, $t > 0$ (ii) $u = \begin{cases} 1, & 0 < x < 1 \\ 0, & x \le 1, \text{ when } t = 0 \end{cases}$ (iii) $u(x, t)$ is bounded. (*U.P.T.U., 2003 S*)

Solution. Since $u(0, t) = 0$, we take Fourier sine transform of both sides of the given equation, we get

$$\int_0^\infty \frac{\partial u}{\partial t} \sin sx \, dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin sx \, dx$$

$$\frac{\partial}{\partial t} \int_0^\infty u \sin sx \, dx = -s^2 \, \bar{u}(s) + s \, u(0) \qquad [\because u = 0, \text{ when } x = 0]$$

or $$\frac{\partial \bar{u}}{\partial t} = -s^2 \bar{u} \quad \text{or} \quad \frac{\partial \bar{u}}{\partial t} + s^2 \bar{u} = 0 \quad \text{or} \quad (D^2 + s^2)\bar{u} = 0 \text{ i.e., } D = \pm is$$

$\therefore$ Its solution is $\bar{u}(s, t) = e^{-s^2 t}$ ...(1)

Since $$\bar{u}(s, t) = \int_0^\infty u(x, t) \sin sx \, dx$$

$\therefore$ $$\bar{u}(s, 0) = \int_0^\infty u(x, 0) \sin sx \, dx = \int_0^1 1 \,.\, \sin sx \, dx \qquad [\text{By } (ii)]$$

$$= \frac{1 - \cos s}{s} \qquad ...(2)$$

From (1) and (2), $$c = \bar{u}(s, 0) = \frac{1 - \cos s}{s}$$

Thus (1) gives $$\bar{u}(s, t) = \frac{1 - \cos s}{s} e^{-s^2 t}$$

Now taking inverse Fourier sine transform, we get

$$u(x, t) = \int_0^\infty \frac{1 - \cos s}{s} e^{-s^2 t} \, ds$$

which is the desired solution.

Example 22.21. *Using finite Fourier transform, solve*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

given $u(0, t) = 0$, $u(4, t) = 0$ *and* $u(x, 0) = 2x$ *where* $0 < x < 4$, $t > 0$. (*Rajasthan, 2006*)

Solution. Since $u(0, t) = 0$, we take finite Fourier sine transform of both sides of the given equation

$$\int_0^4 \frac{\partial u}{\partial t} \sin \frac{n\pi}{4} x \, dx = \int_0^4 \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi}{4} x \, dx$$

or $$\frac{d}{dt}(\bar{u}s) = F_s\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s + \frac{n\pi}{4} [u(0, t) - (-1)^n u(4, t)]$$

$$= -\frac{n^2\pi^2}{16} \bar{u}_s \qquad [\because u(0, t) = 0, u(4, t) = 0.]$$

or $$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{16} \, dt$$

Integrating both sides, $$\log \bar{u}_s = -\frac{n^2\pi^2}{16} t + c$$

or $$\bar{u}_s(x, 0) = \alpha e^{\frac{-n^2\pi^2 t}{16}} \qquad ...(i)$$

Putting $t = 0$, $$a = \bar{u}_s(x, 0) = \int_0^4 u(x, 0) \sin \frac{n\pi x}{4} dx \qquad [\because u(x, 0) = 2x]$$

$$= \int_0^4 2x \sin \frac{n\pi x}{4} dx = -\frac{32}{n\pi} \cos n\pi$$

Thus (*i*) gives, $$\bar{u}_s(x, 0) = -\frac{32}{n\pi}\cos n\pi e^{-n^2\pi^2 t/16} = -\frac{32}{n\pi}(-1)^n\, e^{-n^2\pi^2 t/16}$$

Now taking inverse Fourier sine transform, we get

$$u(x, 0) = \frac{2}{4}\sum_{n=1}^{\infty}\frac{32}{n\pi}(-1)^{n+1}\, e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right)$$

$$= 16\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\, e^{-n^2\pi^2 t/16}\sin\left(\frac{n\pi x}{4}\right).$$

Example 22.22. *If the initial temperature of an infinite bar is given by*

$$\theta(x) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a,\end{cases}$$

determine the temperature at any point x and at any instant t. (S.V.T.U., 2008 ; Rohtak, 2004)

Solution. To determine the temperature $\theta(x, t)$ at any point at any time, we have to solve the equation

$$\frac{\partial\theta}{\partial t} = c^2\frac{\partial^2\theta}{\partial x^2}\quad (t > 0) \qquad ...(i)$$

subject to the initial condition $\theta(x, 0) = \begin{cases}\theta_0 \text{ for } |x| < a\\ 0 \text{ for } |x| > a\end{cases}$ ...(*ii*)

Taking Fourier transform of (*i*) and denoting $F[\theta(x, t)]$ by $\bar{\theta}$, we find

$$\frac{d\bar{\theta}}{dt} = -c^2 s^2\bar{\theta} \qquad \text{[by (1) of § 22.9] ...(iii)}$$

Also the Fourier transform of (2) is

$$\bar{\theta}(s, 0) = \int_{-\infty}^{\infty}\theta(x, 0)\, e^{isx}\, dx = \int_{-a}^{a}\theta_0\, e^{isx}\, dx = \theta_0\frac{e^{isa} - e^{-isa}}{is} = 2\theta_0\frac{\sin as}{s} \qquad ...(iv)$$

Solving (*iii*) and using (*iv*), we get $\bar{\theta} = \dfrac{2\theta_0 \sin as}{s}e^{-c^2s^2t}$

Hence taking its inverse Fourier transform, we get

$$\theta(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\theta_0\sin as}{s}e^{-c^2s^2t}\, e^{-isx}\, ds = \frac{\theta_0}{\pi}\int_{-\infty}^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}\,(\cos xs - i\sin xs)\, ds$$

$$= \frac{2\theta_0}{\pi}\int_0^{\infty}\frac{\sin as}{s}e^{-c^2s^2t}\cos xs\, ds$$

$\left\{\begin{array}{l}\text{The second integral vanishes as}\\ \text{its integrand is an odd function}\end{array}\right.$

$$= \frac{\theta_0}{\pi}\int_0^{\infty}e^{-c^2s^2t}\,\frac{\sin(a + x)s + \sin(a - x)s}{s}\, ds$$

$$= \frac{\theta_0}{\pi}\int_0^{\infty}e^{-v^2}\left\{\sin\frac{(a + x)v}{c\sqrt{t}} + \sin\frac{(a - x)v}{c\sqrt{t}}\right\}\frac{dv}{v} \text{ where } v^2 = c^2s^2t$$

$$= \frac{\theta_0}{\pi}\left\{erf\,\frac{(a + x)}{2c\sqrt{t}} + erf\,\frac{(a - x)}{2c\sqrt{t}}\right\}.$$

[See footnote on p. 783]

Example 22.23. *A bar of length a is at zero temperature. At t = 0, the end x = a is suddenly raised to temperature u_0 and the end x = 0 is insulated. Find the temperature at any point x of the bar at any time t > 0, assuming that the surface of the bar is insulated.*

Solution. Here we have to solve the differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}\quad (0 < x < a, t > 0) \qquad ...(i)$$

subject to the conditions

$u(x, 0) = 0$ ...(*ii*) ; $u_x(0, t) = 0$ ...(*iii*) and $u(a, t) = u_0$ (*Rohtak, 2005*) ...(*iv*)

The Laplace transform of (*i*), if $L[u(x, t)] = \bar{u}(x, s)$, is

$$s\bar{u} - u(x, 0) = c^2 \frac{d^2\bar{u}}{dx^2}$$

Using (*ii*), we get $$\frac{d^2\bar{u}}{dx^2} - \frac{s}{c^2}\bar{u} = 0 \quad ...(v)$$

Similarly the Laplace transform of (*iii*) and (*iv*) are

$$\bar{u}_x(0, s) = 0 \quad ...(vi) \; ; \qquad \bar{u}(a, s) = \frac{u_0}{s} \quad ...(vii)$$

Solving (*v*), we have $$\bar{u} = C_1 e^{x\sqrt{s}/c} + C_2 e^{-x\sqrt{s}/c}$$

Using (*vi*), we find $C_1 = C_2$ so that

$$\bar{u} = C_1 (e^{\sqrt{s}x/c} + e^{-\sqrt{s}x/c}) = 2C_1 \cosh(\sqrt{s}x/c)$$

Now using (*vii*), we have $$\bar{u} = \frac{u_0 \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)}$$

By the inversion formula (3) § 22.10, we get

$$u(x, t) = \text{sum of the residues of } \left(\frac{e^{st} \cdot u_0 \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)}\right) \text{ at all the poles which occur at } s = 0$$

and $$\cosh(\sqrt{s}a/c) = 0 \text{ i.e., at } s = 0,\ \sqrt{s}a/c = \left(n - \frac{1}{2}\right)\pi i,\ n = 0, \pm 1, \pm 2, ...$$

or at $$s = 0,\ s\ (= s_n) = -\frac{(2n-1)^2 c^2 \pi^2}{4a^2} = 0, 1, 2, ...$$

Now $$(\text{Res})_{s=0} = \underset{s \to 0}{\text{Lt}} \left\{ s \cdot \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)} \right\} = u_0$$

$$(\text{Res})_{s=s_n} = u_0 \underset{s \to s_n}{\text{Lt}} \left\{ (s - s_n) \cdot \frac{u_0 e^{st} \cosh(\sqrt{s}x/c)}{s \cosh(\sqrt{s}a/c)} \right\}$$

$$= u_0 \underset{s \to s_n}{\text{Lt}} \left\{ \frac{s - s_n}{\cosh(\sqrt{s}a/c)} \right\} \cdot \underset{s \to s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{s}x/c)}{s} \right\} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= u_0 \underset{s \to s_n}{\text{Lt}} \frac{1}{\sinh(\sqrt{s}a/c) \cdot (a/2\sqrt{s}/c)} \cdot \underset{s \to s_n}{\text{Lt}} \left\{ \frac{e^{st} \cosh(\sqrt{s}x/c)}{s} \right\}$$

$$= \frac{4u_0(-1)^n}{(2n-1)\pi} e^{-(2n-1)^2 \pi^2 c^2 t/4a^2} \cos\frac{(2n-1)\pi x}{2a}$$

Thus we get $$u(x, t) = u_0 + \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-(2n-1)^2 \pi^2 c^2 t/4a^2} \cos\frac{(2n-1)\pi x}{2a}.$$

Vibrations of a string

Example 22.24. *An infinite string is initially at rest and that the initial displacement is $f(x)$, $(-\infty < x < \infty)$. Determine the displacement $y(x, t)$ of the string.* (Rohtak, 2000)

Solution. The equation for the vibration of the string is

$$\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2 \quad ...(i)$$

and the initial conditions are

$$(\partial y/\partial t)_{t=0} = 0 \; ; \; y(x, 0) = f(x) \quad ...(ii)$$

Multiplying (*i*) by e^{isx} and integrating w.r.t. x from $-\infty$ to ∞, we get

$$\partial^2 Y/\partial t^2 = c^2(-s^2 Y) \qquad \text{provided } y \text{ and } \partial y/\partial x \to 0 \text{ as } x \to \infty$$

$\therefore$ a solution of $d^2Y/dt^2 + c^2 s^2 Y = 0$ is $Y = A_1 \cos cst + A_2 \sin cst$ ...(*iii*)

Also Fourier transforms of (*ii*) are

$$\partial y/\partial t = 0 \quad \text{and} \quad Y = F(s) \text{ when } t = 0$$

Applying these to (*iii*), we get

$$A_2 = 0 \quad \text{and} \quad A_1 = F(s)$$

Thus $\qquad Y = F(s) \cos cst$

Now taking inverse Fourier transforms, we get

$$y(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) \cos cst \,.\, e^{-isx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s) \frac{e^{icst} + e^{-icst}}{2} \,.\, e^{-isx}\, dx$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} [F(s)e^{-is(x-ct)} + F(s)e^{-is(x+ct)}]\, ds$$

$$= \frac{1}{2}[f(x - ct) + f(x + ct)] \qquad \left[\because f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds\right.$$

Example 22.25. *An infinitely long string having one end at x = 0, is initially at rest along the x-axis. The end x = 0 is given a transverse displacement f(t), t > 0. Find the displacement of any point of the string at any time.*

Solution. Let $y(x, t)$ be the transverse displacement of any point x of the string at any time t. Then we have to solve the wave equation (§ 18.4)

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0) \qquad ...(i)$$

subject to the conditions $y(x, 0) = 0, y_t(x, 0) = 0, y(0, t) = f(t)$ and the displacement $y(x, t)$ is bounded.

The Laplace transform of (*i*), writing $L[y(x, t)] = \bar{y}(x, s)$ is

$$s^2 \bar{y} - sy(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2}$$

Using the first two conditions, we have

$$\frac{\partial^2 \bar{y}}{\partial x^2} = \left[\frac{s}{c}\right]^2 \bar{y} \qquad ...(ii)$$

Similarly the Laplace transforms of the third and fourth conditions are

$\bar{y}(0, s) = \bar{f}(s)$ at $x = 0$ ...(*iii*) and $\bar{y}(x, s)$ is bounded. ...(*iv*)

Solving (*ii*), we get

$$\bar{y}(x, s) = C_1 e^{sx/c} + C_2 e^{-sx/c}$$

To satisfy condition (*iv*), we must have $C_1 = 0$

Using the condition (*iii*), we get $C_2 = \bar{f}(s)$.

$$\therefore \qquad \bar{y}(x, s) = \bar{f}(s) e^{-sx/c}$$

Using the complex inversion formula, we obtain

$$y = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{(t-x/c)s}\, \bar{f}(s)\, ds = f(t - x/c).$$

Example 22.26. *A tightly stretched flexible string has its ends fixed at x = 0 and x = l. At time t = 0, the string is given a shape defined by F(x) = μx(l − x), where μ is a constant and then released. Find the displacement of any point x of the string at any time t > 0.* (V.T.U., M.E., 2006)

Solution. We have to solve the wave equation $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2} \qquad (x > 0, t > 0)$

subject to the conditions $y(0, t) = 0, y(l, t) = 0$
and $y(x, 0) = \mu x(l - x), y_t(x, 0) = 0$

Now taking Laplace transform, writing $L[y(x, t)] = \bar{y}(x, s)$, we get

$$s^2\bar{y} - s\bar{y}(x, 0) - \frac{\partial y(x, 0)}{\partial t} = c^2 \frac{\partial^2 \bar{y}}{\partial x^2} \quad ...(i)$$

where $\bar{y}(0, s) = 0,\ \bar{y}(l, s) = 0$ ...(ii)

$\therefore$ (i) reduces to $\dfrac{\partial^2 \bar{y}}{\partial x^2} - \left(\dfrac{s}{c}\right)^2 \bar{y} = -\dfrac{\mu s x(l - x)}{c^2}$

Its solution is $\bar{y}(x, s) = c_1 \cosh(sx/c) + c_2 \sinh(sx/c) + \dfrac{\mu x(l - x)}{s} - \dfrac{2c^2\mu}{s^3}$

Applying the conditions (ii), we get

$$c_1 = 2c^2\mu/s^2 \quad \text{and} \quad c_2 = \frac{2c^2\mu}{s^3}\left[\frac{1 - \cosh(sl/c)}{\sinh(sl/c)}\right] - \frac{2c^2\mu}{s^3} \tanh(s/2c)$$

Thus $$\bar{y}(x, s) = \frac{2c^2\mu}{s^3}\left[\frac{\cosh\{s(2x - l)/2c\}}{\cosh(sl/2c)}\right] + \frac{\mu x(l - x)}{s} - \frac{2c^2\mu}{s^3}$$

Now using the inversion formula (3) § 22.10, we get

$y(x, t) =$ sum of the residues of

$$2c^2\mu\left[e^{st}\frac{\cosh\{s(2x - l)/2c\}}{s^3\cosh(sl/2c)}\right] \text{ at all the poles } + \mu x(l - x) - c^2\mu t^2$$

Proceeding exactly as in Example 22.23, we have,

sum of the residues of $2c^2\mu\left[\dfrac{e^{st}\cosh\{s(2x - l)/2c\}}{s^3\cosh sl/2c}\right]$ at all the poles

$$= c^2\mu\left[t^2 + \left(\frac{2x - l}{2c}\right)^2 - \left(\frac{l}{2c}\right)^2\right]$$

$$- \frac{32c^2\mu}{\pi^3}\left(\frac{l}{2c}\right)^2 \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{(2n - 1)^3}\cos\left\{\frac{(2n - 1)\pi(2x - l)}{2l}\right\}\cos\left\{\frac{(2n - 1)\pi ct}{l}\right\}\right]$$

$$= c^2\mu t^2 - \mu x(l - x) + \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right]$$

Hence $$y(x, t) = \frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\left[\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{l}\cos\frac{(2n - 1)\pi ct}{l}\right].$$

Transmission lines

Example 22.27. *A semi-infinite transmission line of negligible inductance and leakance per unit length has its voltage and current equal to zero. A constant voltage v_0 is applied at the sending end ($x = 0$) at $t = 0$. Find the voltage and current at any point ($x > 0$) and at any instant.*

Solution. Let $v(x, t)$ and $i(x, t)$ be the voltage and current at any point x and at any time t. If $L = 0$ and $G = 0$, then the transmission line equations [(1) and (2) of § 18.10] become

$$\frac{\partial v}{\partial x} = -Ri, \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \quad i.e., \quad \frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(i)$$

The boundary conditions are $v(0, t) = v_0$ and $i(x, t)$ is finite for all x and t.

The initial conditions are $v(x, 0) = 0, i(x, 0) = 0$. ...(ii)

Laplace transforms of (*i*), are

$$\frac{d^2\bar{v}}{dx^2} = RC(s\bar{v} - 0) \quad \text{or} \quad \frac{d^2\bar{v}}{dx^2} - RCs\bar{v} = 0 \qquad ...(iii)$$

Laplace transforms of the conditions in (*ii*), are

$$\bar{v}(0, s) = \frac{v_0}{s} \text{ at } x = 0 \qquad ...(iv)$$

and $\bar{v}(x, s)$ remains finite as $x \to \infty$ ...(*v*)

$\therefore$ the solution of (*iii*) is

$$\bar{v}(x, s) = C_1 e^{\sqrt{RCs}\,x} + C_2 e^{-\sqrt{RCs}\,x}$$

To satisfy condition (*v*), we must have $C_1 = 0$.

Using the condition (*iv*), we get $C_2 = v_0/s$

Thus $$\bar{v}(x, s) = \frac{v_0}{s} e^{-\sqrt{RCs}\,x}$$

Using the inversion formula, we obtain

$$v(x, t) = v_0 L^{-1}\left\{\frac{e^{-\sqrt{RC}\,x \,.\, \sqrt{s}}}{s}\right\} = v_0\, erfc\left(x\frac{\sqrt{RC}}{2\sqrt{t}}\right) \qquad \text{[By Ex. 22.17]}$$

$$= v_0 \frac{x\sqrt{RC}}{2\sqrt{\pi}} \int_0^t u^{-3/2} e^{-(RCx^2/4u)}\, du$$

$\therefore$ since $i = -\dfrac{1}{R}\dfrac{\partial v}{\partial x}$, we obtain by differentiation,

$$i(x, t) = \frac{v_0 x}{2\sqrt{x}} \sqrt{\frac{C}{R}}\, t^{-3/2} e^{(-RCx^2/4t)}.$$

Example 22.28. *A transmission line of length l has negligible inductance and leakance. A constant voltage v_0 is applied at the sending end (x = 0) and is open circuited at the far end. Assuming the initial voltage and current to be zero, determine the voltage and current.*

Solution. For a transmission line with $L = G = 0$, the voltage v and current i are given by the equations

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \text{ and } \frac{\partial v}{\partial x} + Ri = 0 \qquad ...(i)$$

The boundary conditions are (for $t > 0$)

$$v = v_0 \text{ at } x = 0 \text{ and } i = \frac{\partial v}{\partial x} = 0 \quad \text{at } x = l \qquad ...(ii)$$

The initial condition is $v = 0$ at $t = 0$ $(x > 0)$

Laplace transforms of (*i*) and (*ii*) are

$$\frac{\partial^2 \bar{v}}{\partial x^2} = RC(s\bar{v} - 0) \qquad ...(iii)$$

and $$\bar{v} = v_0/s \text{ at } x = 0,\ \frac{\partial \bar{v}}{\partial x} = 0 \text{ at } x = l \qquad ...(iv)$$

$\therefore$ the solution of (*iii*) is

$$\bar{v} = c_1 \cosh \sqrt{(RCs)}\, x + c_2 \sinh \sqrt{RCs}\, x$$

Applying conditions (*iv*), it gives

$$v_0/s = c_1,\ 0 = c_1 \sinh \sqrt{(RCs)}\, l + c_2 \cosh \sqrt{(RCs)}\, l$$

$$\therefore \quad \bar{v} = \frac{v_0}{s}\left[\cosh \sqrt{(RCs)}x - \frac{\sinh \sqrt{(RCs)}l}{\cosh \sqrt{(RCs)}l} \sinh \sqrt{(RCs)}x\right]$$

$$= \frac{v_0}{s} \frac{\cosh pq\sqrt{s}}{\cosh p\sqrt{s}} \qquad \text{where } p = \sqrt{(RC)}\,l \text{ and } q = (l - x)/l$$

By the inversion formula (3) § 22.10, we get

$$v(x, t) = \text{sum of the residues of } (e^{st}\,\bar{v}) \text{ at all poles of } e^{st}\,\bar{v}. \qquad ...(iv)$$

These poles are at $s = 0$ and $p\sqrt{s} = \pm i(2n-1)\pi/2 = \pm ipk$ (say)

Now $$Res\,(e^{st}\,\bar{v})_{s=0} = \underset{s \to 0}{\text{Lt}} \frac{se^{st}\, v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} = v_0$$

and $$Res\,(e^{st}\,\bar{v})_{s=-k^2} = \underset{s \to -k^2}{\text{Lt}} \frac{(s+k^2)e^{st}\, v_0 \cosh pq\sqrt{s}}{s \cosh p\sqrt{s}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \underset{s \to -k^2}{\text{Lt}} \frac{v_0 . e^{st} \cosh pq\sqrt{s} + (s+k^2)(...)}{\cosh p\sqrt{s} + s \sinh p\sqrt{s} \,.\, \frac{1}{2} ps^{-1/2}}$$

$$= \frac{v_0\, e^{-k^2 t} \cosh (ipqk) + 0}{0 + 1/2\,(ipk) \sinh (ipk)} = \frac{2v_0 e^{-k^2 t} \cos (pqk)}{-pk \sin pk}$$

Adding up all the residues, (*iv*) gives

$$v(x, t) = v_0 + \frac{4v_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} e^{-\{(2n-1)^2 \pi^2 t/4RCl^2\}} \cos [(2n-1)\,\pi(l-x)/2l]$$

$$\left[\because \quad pk = (2n-1)\,\pi/2, -\sin pk = (-1)^n, pqk = \tfrac{1}{2}(2n-1)\,\pi(l-x)/l, \right.$$
$$\left. k^2 = (2n-1)^2\pi^2/4RCl^2 \right]$$

Also $$i = -\frac{1}{R} \frac{\partial v}{\partial x}. \qquad \text{[By } (i)\text{]}$$

PROBLEMS 22.4

1. Solve the differential equation using Laplace transform method, $\dfrac{\partial y}{\partial t} = 3\dfrac{\partial^2 y}{\partial x^2}$
   where $y(\pi/2, t) = 0$, $(\partial y/\partial x)_{x=0} = 0$ and $y(x, 0) = 30 \cos 5x$. (*U.P.T.U., 2005*)
2. Using suitable transforms, solve the differential equation $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{\partial V}{\partial t}$, $0 \le x \le \pi$, $t \ge 0$.
   where $V(0, t) = 0 = V(\pi, t)$ and $V(x, 0) = V_0$ constant.
3. The initial temperature along the length of an infinite bar is given by $u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$. If the temperature $u(x, t)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$, $-\infty < x < \infty$, $t > 0$, find the temperature at any point of the bar at any point t. (*Rohtak, 2006*)
4. Use the complex form of the Fourier transform to show that
   $$V = \frac{1}{2\sqrt{(\pi t)}} \int_{-\infty}^{\infty} \bar{f}(u) e^{\{-(x-u)^2/4t\}}\, du$$
   is the solution of the boundary value problem
   $$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, -\infty < x < \infty, t > 0\,;\, V = f(x) \text{ when } t = 0. \qquad (\textit{U.P.T.U., 2008})$$
5. A semi-infinite solid $(x > 0)$ is initially at temperature zero. At time $t = 0$, a constant temperature $\theta_0 > 0$ is applied and maintained at the face $x = 0$. Show that the temperature at any point x and at any time t, is given by $\theta(x, t) = \theta_0\, erfc(x/2c\sqrt{t})$.

6. A solid is initially at constant temperature θ_0, while the ends $x = 0$ and $x = a$ are maintained at temperature zero. Determine the temperature at any point of the solid at any later time $t > 0$.
7. An infinite string is initially at rest along the x-axis. Its one end which is at $x = 0$, is given a periodic transverse displacement $a_0 \sin \omega t$, $t > 0$. Show that the displacement of any point of the string at any time is given by

$$y(x, t) = \begin{cases} a_0 \sin \omega(t - x/c), & t > x/c \\ 0, & t < x/c, \end{cases}$$

where c is the wave velocity.
8. An infinite string has an initial transverse displacement $y(x, 0) = f(x)$, $-\infty < x < \infty$, and is initially at rest. Show that

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)].$$

9. A semi-infinite transmission line has negligible inductance and leakance per unit length. A voltage v is applied at the sending end $(x = 0)$ which is given by

$$v(0, t) = \begin{cases} v_0, & 0 < t < \tau \\ 0, & t > \tau \end{cases}$$

Show that the voltage at any point $x > 0$ at any time $t > 0$ is given by

$$v(x, t) = v_0\, erfc\left[\frac{x}{2}\sqrt{(RC/t)}\right].$$

22.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 22.5

Fill in the blanks or choose the correct answer in each of the following problems :

1. Fourier cosine transform of $f(t)$ is ....
2. Fourier sine transform of $1/x$ is ....
3. Convolution theorem for Fourier transforms states that ....
4. If Fourier transform of $f(x)$ is $F(s)$, then the inversion formula is ....
5. $F[x^n f(x)] =$ ....
6. If $F\{f(x)\} = F(s)$, then $F\{f(x - a)\} =$ ....
7. Fourier sine integral representation of a function $f(x)$ is given by ....
8. If $F_c\{f(ax)\} = k F_c(s/a)$, then $k =$ ....
9. Fourier transform of second derivative of $u(x, t)$ is ....
10. If $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$, then Fourier sine integral of $f(x)$ is ....
11. Fourier sine transform of $f''(x)$ in the interval $(0, l)$ is ....
12. If $F(\lambda)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is ....
13. Inverse finite Fourier sine transform of $F_s(p) = \dfrac{1 - \cos p\pi}{(p\pi)^2}$ for $p = 1, 2, 3, \ldots$ and $0 < x < \pi$ is ....
14. If Fourier transform of $f(x) = F(s)$, then Fourier Transform of $f(2x)$ is ....
15. Fourier cosine transform of e^{-x} is ....
16. $f(x) = 1$, $0 < x < \infty$ cannot be represented by a Fourier integral. (True or False)
17. $\int_0^\infty |f(x)|^2\, dx = \int_0^\infty |F_c(s)|^2\, dx.$ (True or False)
18. Fourier transform is a linear operation. (True or False)
19. $F_s[x f(x)] = -\dfrac{d}{ds} F_c(s).$ (True or False)
20. Kernel of Fourier transform is e^{isx}. (True or False)
21. Finite Fourier cosine transform of $f(x) = 1$ in $(0, \pi)$ is zero. (True or False)

CHAPTER 23

Z-Transforms

23.1 INTRODUCTION

The development of communication branch is based on discrete analysis. Z-transform plays the same role in discrete analysis as Laplace transform in continuous systems. As such, Z-transform has many properties similar to those of the Laplace transform (§ 21.2). The main difference is that the Z-transform operates not on functions of continuous arguments but on sequences of the discrete integer-valued arguments, *i.e.* $n = 0, \pm 1, \pm 2,$ ...... The analogy of Laplace transform to Z-transform can be carried further. For every operational rule of Laplace transforms, there is a corresponding operational rule of Z-transforms and for every application of the Laplace transform, there is a corresponding application of Z-transform. A discrete system is expressible as a difference equation (§ 30.2) and its solutions are found using Z-transforms.

23.2 DEFINITION

If the function u_n is defined for discrete values ($n = 0, 1, 2, \ldots$) and $u_n = 0$ for $n < 0$, then its Z-transform is defined to be

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} \text{ whenever the infinite series converges.} \qquad \ldots(i)$$

The inverse Z-transform is written as $Z^{-1}[U(z)] = u_n$.

If we insert a particular complex number z into the power series (i), the resulting value of $Z(u_n)$ will be a complex number. Thus the Z-transform $U(z)$ is a complex valued function of a complex variable z.

23.3 SOME STANDARD Z-TRANSFORMS

The direct application of the definition gives the following results :

(1) $Z(a^n) = \dfrac{z}{z-a}$ **(2)** $Z(n^p) = -z \dfrac{d}{dz} Z(n^{p-1})$, *p being a +ve integer.*

Proof. **(1)** By definition, $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$

$$= 1 + (a/z) + (a/z)^2 + (a/z)^3 + \dots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$ (*Kottayam, 2005*)

(2) $$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n} = z \sum_{n=0}^{\infty} n^{p-1} . n . z^{-(n+1)} \quad \dots(i)$$

Changing p to $p-1$, we get $Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} . z^{-n}$

Differentiating it w.r.t. z,

$$\frac{d}{dz}[Z(n^{p-1})] = \sum_{n=0}^{\infty} n^{p-1} . (-n) z^{-(n+1)} \quad \dots(ii)$$

Substituting (*ii*) in (*i*), we obtain $Z(n^p) = -z\frac{d}{dz}[Z(n^{p-1})]$

which is the desired *recurrence formula.*

In particular, we have the following formulae :

(**3**) $Z(1) = \frac{z}{z-1}$ [Taking $a = 1$ in (1)]

(**4**) $Z(n) = \frac{z}{(z-1)^2}$ [Taking $p = 1$ in (2)]

(**5**) $Z(n^2) = \frac{z^2+z}{(z-1)^3}$ (*V.T.U., 2006*)

(**6**) $Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$

(**7**) $Z(n^4) = \frac{z^4+11z^3+11z^2+z}{(z-1)^5}$.

23.4 LINEARITY PROPERTY

If a, b, c be any constants and u_n, v_n, w_n be any discrete functions, then

$$Z(au_n + bv_n - cw_n) = aZ(u_n) + bZ(v_n) - cZ(w_n)$$

Proof. By definition, $Z(au_n + bv_n - cw_n) = \sum_{n=0}^{\infty} (au_n + bv_n - cw_n)z^{-n}$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} - c\sum_{n=0}^{\infty} w_n z^{-n}$$

$$= aZ(u_n) + bZ(v_n) - cZ(w_n).$$

23.5 DAMPING RULE

If $Z(u_n) = U(z)$, then $Z(a^{-n} u_n) = U(az)$

Proof. By definition, $Z(a^{-n} u_n) = \sum_{n=0}^{\infty} a^{-n} u_n . z^{-n} = \sum_{n=0}^{\infty} u_n . (az)^{-n} = U(az)$. (*Madras, 2006*)

Cor. $\mathbf{Z(a^n u_n) = U(z/a)}$

Obs. The geometric factor a^{-n} when $|a| < 1$, damps the function u_n, hence the name *damping rule.*

23.6 SOME STANDARD RESULTS

The application of the damping rule leads to the following standard results :

(**1**) $Z(na^n) = \frac{az}{(z-a)^2}$

(**2**) $Z(n^2a^n) = \frac{az^2+a^2z}{(z-a)^3}$

(3) $Z(\cos n\theta) = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$ **(4)** $Z(\sin n\theta) = \dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

(5) $Z(a^n\cos n\theta) = \dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$ **(6)** $Z(a^n\sin n\theta) = \dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$.

Proofs. **(1)** We know that $Z(n) = \dfrac{z}{(z-1)^2}$. Applying damping rule, we have

$$Z(na^n) = U(a^{-1}z) = \frac{a^{-1}z}{(a^{-1}z - 1)^2} = \frac{az}{(z-a)^2}.$$ *(Madras, 2000 S)*

(2) We know that $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$. Applying damping rule, we have

$$Z(n^2a^n) = U(a^{-1}z) = \frac{(a^{-1}z)^2 + a^{-1}z}{(a^{-1}z - 1)^3} = \frac{a(z^2 + az)}{(z-a)^3}.$$

(3) and **(4)** We know that $Z(1) = \dfrac{z}{z-1}$. Applying damping rule, we have

$$Z(e^{-in\theta}) = Z(e^{-i\theta})^n \,.\, 1 = \frac{ze^{i\theta}}{ze^{i\theta} - 1} = \frac{z}{z - e^{-i\theta}} = \frac{z(z - e^{i\theta})}{(z - e^{-i\theta})(z - e^{i\theta})}$$

$$= \frac{z(z - \cos\theta) - iz\sin\theta}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} = \frac{z(z - \cos\theta) - iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary parts, we get (3) and (4). *(V.T.U., 2010 S; Anna, 2009)*

(5) We know that $Z(\cos n\theta) = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$. By damping rule, we have

$$Z(a^n\cos n\theta) = \frac{a^{-1}z\,(a^{-1}z - \cos\theta)}{(a^{-1}z)^2 - 2(a^{-1}z)\cos\theta + 1} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$$ *(V.T.U., 2006)*

Similarly using (4) above, we get (6).

Example 23.1. *Find the Z-transform of the following :*
(i) $3n - 4\sin n\pi/4 + 5a$ *(ii)* $(n+1)^2$ *(V.T.U., 2010)*
(iii) $\sin(3n+5)$. *(V.T.U., 2009 S ; Kottayam, 2005)*

Solution. *(i)* $Z\left(3n - 4\sin\dfrac{n\pi}{4} + 5a\right) = 3Z(n) - 4Z\left(\sin\dfrac{n\pi}{4}\right) + 5a\,Z(1)$ [By Linearty property[

$$= 3\,.\,\frac{z}{(z-1)^2} - 4\,.\,\frac{z\sin n\pi/4}{z^2 - 2z\cos\pi/4 + 1} + 5a\,.\,\frac{z}{z-1}$$ [Using formulae for $Z(1)$, $Z(n)$, $Z(\sin n\theta)$]

$$= \frac{(3 - 5a)z + 5az^2}{(z-1)^2} - \frac{2\sqrt{2}z}{z^2 - \sqrt{2}z + 1}$$

(ii) $$Z(n+1)^2 = Z(n^2 + 2n + 1) = Z(n^2) + 2Z(n) + Z(1)$$

$$= \frac{z^2 + z}{(z-1)^3} + 2\frac{z}{(z-1)^2} + \frac{z}{z-1} = \frac{z^2(2z+1)}{(z-1)^3}$$

(iii) $$Z[\sin(3n+5)] = Z(\sin 3n\cos 5 + \cos 3n\sin 5)$$

$= \cos 5.Z(\sin 3n) + \sin 5.Z(\cos 3n)$ (using formulae for $Z(\sin n\theta)$, $Z(\cos n\theta)$)

$$= \cos 5\,.\,\frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \sin 5\,.\,\frac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1} = z\,.\,\frac{(z\sin 5 - \sin 2)}{z^2 - 2z\cos 3 + 1}.$$

Example 23.2. *Find the Z-transforms of the following*
(*i*) e^{an} (*ii*) ne^{an} (*iii*) n^2e^{an}.

Solution. (*i*) Let $u_n = 1$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$. By damping rule $Z(k^{-n} u_n) = U(kz)$.

$$\therefore \quad Z(e^{an}) = Z(k^{-n} . 1) = U(kz) = \frac{kz}{kz-1} \qquad \left[\because \ U(z) = Z(1) = \frac{z}{z-1}\right]$$

$$= \frac{z}{z-1/k} = \frac{z}{z-e^a}$$

(*ii*) Let $u_n = n$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule, $Z(e^{an} . n) = Z(k^{-n} . n) = U(k z)$ where $U(z) = Z(n) = \dfrac{z}{(z-1)^2}$

$$\frac{kz}{(kz-1)^2} = \frac{z}{k(z-1/k)^2} = \frac{e^a z}{(z-e^a)^2}$$

(*iii*) Let $u_n = n^2$, $e^{an} = (e^{-a})^{-n} = k^{-n}$ where $k = e^{-a}$

By damping rule,

$$Z(e^{an} . n^2) = Z(k^{-n} . n^2) = U(kz) \quad \text{where} \quad U(z) = Z(n^2) = \frac{z^2+z}{(z-1)^3}$$

$$= \frac{(kz)^2 + kz}{(kz-1)^3} = \frac{z(z+1/k)}{(z-1/(k)^3} = \frac{ze^a(z+e^a)}{(z-e^a)^3}.$$

Example 23.3. *Find the Z-transform of* (*i*) $\cosh n\theta$ (*V.T.U., 2011*) (*ii*) $a^n \cosh n\theta$

Solution. (*i*)

$$Z(\cosh n\theta) = Z\left(\frac{e^{n\theta}+e^{-n\theta}}{2}\right)$$

$$= \frac{1}{2}\left[Z\left\{(e^{-\theta})^{-n} . 1\right\} + Z\left\{(e^{\theta})^{-n} . 1\right\}\right]$$

Apply damping rule to both terms, taking $u_n = 1$.

$$Z(\cosh n\theta) = \frac{1}{2}\left[\frac{ze^{-\theta}}{ze^{-\theta}-1} + \frac{ze^{\theta}}{ze^{\theta}-1}\right] \qquad \left[\because \ z(1) = \frac{z}{z-1}\right]$$

$$= \frac{1}{2}\left\{\frac{2z^2 - z(e^{\theta}+e^{-\theta})}{z^2 - z(e^{\theta}+e^{-\theta})+1}\right\} = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(*ii*)

$$Z(a^n \cosh n\theta) = Z[(a^{-1})^{-n} . \cosh n\theta] \qquad \text{[Apply damping rule using (i)]}$$

$$= \frac{(a^{-1}z)^2 - (a^{-1}z)\cosh\theta}{(a^{-1}z)^2 - 2(a^{-1}z)\cosh\theta + 1} = \frac{z(z-a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}.$$

Example 23.4. *Find the Z-transforms of*
(*i*) $e^t \sin 2t$ (*Madras, 2003*) (*ii*) $c^k \cos k\alpha$, $(k \ge 0)$ (*U.P.T.U., 2004 S*)

Solution. (*i*) We know that $Z(\sin 2t) = \dfrac{z \sin 2}{z^2 - 2z\cos 2 + 1}$ ...(A)

$$\therefore \quad Z(e^t \sin 2t) = Z[(e^{-1})^{-t} . \sin 2t] \qquad \text{[Apply damping rule, using (A)]}$$

$$= \frac{(e^{-1}z)\sin 2}{(e^{-1}z)^2 - 2(e^{-1}z)\cos 2 + 1} = \frac{ez\sin 2}{z^2 - 2ez\cos 2 + e^2}.$$

(*ii*) We know that $Z(\cos k\alpha) = \dfrac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$ ...(B)

$$\therefore \quad Z(c^k \cos k\alpha) = Z[(c^{-1})^{-k} . \cos k\alpha] \qquad \text{[Apply damping rule, using (B)]}$$

$$= \frac{(c^{-1}z)[c^{-1}z - \cos\alpha]}{(c^{-1}z)^2 - 2(c^{-1}z)\cos\alpha + 1} = \frac{z(z-c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

Example 23.5. *Find the Z-transforms of*

(*i*) $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$ (*V.T.U., 2011 S*) (*ii*) $\cosh\left(\frac{n\pi}{2}+\theta\right)$. (*U.P.T.U., 2008*)

Solution. (*i*) $Z\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right]=Z\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4}-\sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)$

$$=\cos\frac{\pi}{4}\cdot Z\left(\cos\frac{n\pi}{2}\right)-\sin\frac{\pi}{2}\cdot Z\left(\sin\frac{n\pi}{2}\right)$$ [Using formulae for $Z(\sin n\alpha)$ and $Z(\cos n\alpha)$]

$$=\frac{1}{\sqrt{2}}\left\{\frac{z(z-\cos\pi/2)}{z^2-2z\cos\pi/2+1}-\frac{z\sin\pi/2}{z^2-2z\cos\pi/2+1}\right\}=\frac{1}{\sqrt{2}}\left(\frac{z^2}{z^2+1}-\frac{z}{z^2+1}\right)=\frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

(*ii*) $$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]=Z\left[\frac{e^{n\pi/2+\theta}+e^{-(n\pi/2+\theta)}}{2}\right]=\frac{1}{2}\left[e^{\theta}\,Z(e^{n\pi/2})+e^{-\theta}\,Z(e^{-n\pi/2})\right]$$

Since, $Z(a^n)=\frac{z}{z-a}$, $\therefore$ $Z(e^{n\pi/2})=Z(e^{\pi/2})^n=\frac{z}{z-e^{\pi/2}}$, $Z(e^{-n\pi/2})=\frac{z}{z-e^{-\pi/2}}$

Thus $$Z\left[\cosh\left(\frac{n\pi}{2}+\theta\right)\right]=\frac{1}{2}\left\{e^{\theta}\cdot\frac{z}{z-e^{\pi/2}}+e^{-\theta}\cdot\frac{z}{z-e^{-\pi/2}}\right\}$$

$$=\frac{z}{2}\left\{\frac{z(e^{\theta}+e^{-\theta})-[e^{(\pi/2-\theta)}+e^{-(\pi/2-\theta)}]}{z^2-z(e^{\pi/2}+e^{-\pi/2})+1}\right\}=\frac{z^2\cosh\theta-z\cosh\left(\frac{\pi}{2}-\theta\right)}{z^2-2z\cosh\left(\frac{\pi}{2}\right)+1}.$$

Example 23.6. *Find the Z-transform of*

(*i*) ${}^nC_p$ $(0\le p\le n)$ (*ii*) ${}^{n+p}C_p$.

Solution. (*i*) $$Z({}^nC_p)=\sum_{p=0}^{n}\left({}^nC_p z^{-p}\right)=1+{}^nC_1z^{-1}+{}^nC_2z^{-2}+\dots+{}^nC_nz^{-n}=(1+z^{-1})^n$$

(*ii*) $$Z({}^{n+p}C_n)=\sum_{p=0}^{n}{}^{n+p}C_pz^{-p}$$

$$=1+{}^{n+1}C_1z^{-1}+{}^{n+2}C_2z^{-2}+{}^{n+3}C_3z^{-3}+\dots\infty$$

$$=1+(n+1)z^{-1}+\frac{(n+2)(n+1)}{2!}z^{-2}+\frac{(n+3)(n+2)(n+1)}{3!}z^{-3}+\dots\infty$$

$$=1+(-n-1)(-z^{-1})+\frac{(-n-1)(-n-2)}{2!}(-z^{-1})^2+\frac{(-n-1)(-n-2)(-n-3)}{3!}(-z^{-1})^3+\dots\infty$$

$$=(1-z^{-1})^{-n-1}.$$

Example 23.7. *Find the Z-transform of*

(*i*) *unit impulse sequence* $\delta(n)=\begin{cases}1 & n=0\\ 0 & n\ne 0\end{cases}$ (*ii*) *unit step sequence* $u(n)=\begin{cases}0 & n<0\\ 1 & n\ge 0\end{cases}$.

Solution. (*i*) $Z[\delta(n)]=\sum_{n=0}^{\infty}\delta(n)z^{-n}=1+0+0+\dots=1$

(*ii*) $Z[u(n)]=\sum_{n=0}^{\infty}u(n)z^{-n}=1+z^{-1}+z^{-2}+z^{-3}+\dots=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$.

23.7 (1) SHIFTING U_N TO THE RIGHT

If $Z(u_n) = U(z)$, *then* $Z(u_{n-k}) = z^{-k} U(z)$ $\quad (k > 0)$

Proof. By definition,

$$Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} z^{-n} = u_0 z^{-k} + u_1 z^{-(k+1)} + \ldots = z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k} U(z)$$

Obs. This rule will be very useful in applications to difference equations.

(2) **Shifting u_n to the left.** If $Z(u_n) = U(z)$, then

$$Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$$

Proof. $\quad Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$

$$= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n} \right]$$

Hence $\quad Z(u_{n+k}) = z^k [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \ldots - u_{k-1} z^{-(k-1)}]$ (*J.N.T.U., 2002*)

In particular, we have the following standard results :

(1) $Z(u_{n+1}) = z[U(z) - u_0]$; (2) $Z(u_{n+2}) = z^2 [U(z) - u_0 - u_1 z^{-1}]$

(3) $Z(u_{n+3}) = z^3 [U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}]$.

Example 23.8. *Show that* $Z\left(\frac{1}{n!}\right) = e^{1/z}$.

Hence evaluate $Z[1/(n+1)!]$ *and* $Z[1/(n+2)!]$. (*Madras, 2006*)

Solution. We have $\quad Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \ldots = e^{1/z}$.

Shifting $(1/n!)$ one unit to the left gives

$$Z\left[\frac{1}{(n+1)!}\right] = z\left[Z\left(\frac{1}{n!}\right) - 1\right] = z(e^{-1/z} - 1)$$

Similarly shifting $(1/n!)$ two units to the left gives

$$Z\left[\frac{1}{(n+2)!}\right] = z^2 (e^{1/z} - 1 - z^{-1}).$$

23.8 MULTIPLICATION BY n

If $Z(u_n) = u(z)$, then $Z(nu_n) = -z \frac{dU(z)}{dz}$

Proof. $\quad Z(nu_n) = \sum_{n=0}^{\infty} n \cdot u_n z^{-n} = -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} = -z \sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n})$.

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz}(u_n z^{-n}) = -z \frac{d}{dz}\left(\sum_{n=0}^{\infty} u_n z^{-n}\right) = -z \frac{d}{dz} U(z).$$

Obs. We have, $Z(n^2 u_n) = \left(-z \frac{d}{dz}\right)^2 u(z)$ (*Madras, 2006*)

In general, $Z(n^m u_n) = \left(-z \frac{d}{dz}\right)^m u(z)$.

Example 23.9. *Find the Z-transform of (i) $n \sin n\theta$ (ii) $n^2 e^{n\theta}$.*

Solution. (*i*) We know that $Z(nu_n) = -z\dfrac{dU(z)}{dz}$ and $Z(\sin n\theta) = \dfrac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

$$\therefore \quad Z(n \sin n\theta) = -z\frac{d}{dz}[Z(\sin n\theta)] = -z\frac{d}{dz}\left(\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right)$$

$$= -z\frac{\sin\theta - z^2\sin\theta}{(z^2 - 2z\cos\theta + 1)^2} = \frac{z(z^2 - 1)\sin\theta}{(z^2 - 2z\cos\theta + 1)^2}$$

(*ii*) We know that $\quad Z(e^{n\theta}) = \dfrac{z}{z - e^{\theta}}$

$$\therefore \quad Z(n^2 e^{n\theta}) = \left(-z\frac{d}{dz}\right)^2 (Ze^{n\theta}) = \left(-z\frac{d}{dz}\right)\left[-z\frac{d}{dz}\left(\frac{z}{z - e^{\theta}}\right)\right]$$

$$= \left(-z\frac{d}{dz}\right)\left\{-z\frac{(z - e^{\theta})(1) - z(1)}{(z - e^{\theta})^2}\right\} = -z\frac{d}{dz}\left\{\frac{ze^{\theta}}{(z - e^{\theta})^2}\right\}$$

$$= -ze^{\theta}\left\{\frac{(z - e^{\theta})^2(1) - z[2(z - e^{\theta})]}{(z - e^{\theta})^4}\right\} = -ze^{\theta}\frac{z - e^{\theta} - 2z}{(z - e^{\theta})^3} = \frac{z(z + e^{\theta})e^{\theta}}{(z - e^{\theta})^3}.$$

23.9 TWO BASIC THEOREMS

In applications, we often need the values of u_n for $n = 0$ or as $n \to \infty$ without requiring complete knowledge of u_n. We can find this as the behaviour of u_n for small values of n is related to the behaviour of $U(z)$ as $z \to \infty$ and *vice-versa*. The precise relationship is given by the following *initial and final value theorems :*

(1) Initial value theorem. *If $Z(u_n) = U(z)$, then $u_0 = \underset{z\to\infty}{Lt}\, U(z)$*

Proof. We know that $U(z) = Z(u_n) = u_0 + u_1 z^{-1} + u_2 z^{-2} + \ldots$

Taking limits as $z \to \infty$, we get $\underset{z\to\infty}{\text{Lt}}\, [U(z)] = u_0$, as required.

Similarly additional initial values can be found successively, giving :

$$u_1 = \underset{z\to\infty}{\text{Lt}}\left\{z[U(z) - u_0]\right\};\ u_2 = \underset{z\to\infty}{\text{Lt}}\left\{z^2[U(z) - u_0 - u_1 z^{-1}]\right\} \text{ and so on.}$$

(2) Final value theorem. *If $Z(u_n) = U(z)$, then*

$$\underset{n\to\infty}{Lt}(u_n) = \underset{z\to 1}{Lt}(z - 1)U(z)$$

Proof. By definition, $Z(u_{n+1} - u_n) = \displaystyle\sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

or $$Z(u_{n+1}) - Z(u_n) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$z[U(z) - u_0] - U(z) = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $$U(z)(z - 1) - u_0 z = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

Taking limits of both sides as $z \to 1$, we get

$$\underset{z\to 1}{\text{Lt}}\,[(z - 1)U(z)] - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n) = \underset{n\to\infty}{\text{Lt}}\,[(u_1 - u_0) + (u_2 - u_1) + \ldots + (u_{n+1} - u_n)]$$

$$= \underset{n \to \infty}{\text{Lt}} [u_{n+1}] - u_0 = u_\infty - u_0$$

Hence $$u_\infty = \underset{z \to 1}{\text{Lt}} [(z-1)\, U(z)].$$ *(Anna, 2005 S)*

Example 23.10. *If* $U(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$, *evaluate* u_2 *and* u_3.

Solution. Writing $$U(z) = \frac{1}{z^2} \cdot \frac{2 + 5z^{-1} + 14z^{-2}}{(1 - z^{-1})^4}$$

By initial value theorem, $$u_0 = \underset{z \to \infty}{\text{Lt}}\, U(z) = 0$$

Similarly, $$u_1 = \underset{z \to \infty}{\text{Lt}} \{z\, [U(z) - u_0]\} = 0$$

Now $$u_2 = \underset{z \to \infty}{\text{Lt}} \{z^2\, [U(z) - u_0 - u_1 z^{-1}]\} = 2 - 0 - 0 = 2$$

and $$u_3 = \underset{z \to \infty}{\text{Lt}}\, z^3[U(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}] = \underset{z \to \infty}{\text{Lt}}\, z^3[U(z) - 0 - 0 - 2z^{-2}]$$

$$= \underset{z \to \infty}{\text{Lt}}\, z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - \frac{2}{z^2}\right] = \underset{z \to \infty}{\text{Lt}}\, z^3 \left\{\frac{13z^3 + 2z^2 + 8z - 2}{z^2 (z-1)^4}\right\} = 13.$$

PROBLEMS 23.1

1. Find the Z-transforms of the following sequences :

 (*i*) $\dfrac{a^n}{n!}$ $(n \geq 0)$ *(S.V.T.U., 2009)* (*ii*) $\dfrac{1}{(n+1)!}$ (*iii*) $(\cos\theta + i \sin\theta)^n$.

2. Using the linearity property, find the Z- transforms of the following functions :

 (*i*) $2n + 5 \sin n\pi/4 - 3a^4$ (*ii*) $\dfrac{1}{2}(n-1)(n+2)$ *(S.V.T.U., 2007)*

 (*iii*) $(n+1)(n+2)$ *(Anna, 2008)* (*iv*) $(2n-1)^2$ *(V.T.U., 2011 S)*

3. Show that (*i*) $Z(\sinh n\theta) = \dfrac{z \sinh\theta}{z^2 - 2z\cosh\theta + 1}$ *(V.T.U., 2011)* (*ii*) $Z(a^n \sinh n\theta) = \dfrac{az \sinh\theta}{z^2 - 2az\cosh\theta + a^2}$.

4. Show that (*i*) $Z(e^{-an} \cos n\theta) = \dfrac{ze^a(ze^a - \cos\theta)}{z^2e^{2a} - 2ze^a\cos\theta + 1}$; (*ii*) $Z(e^{-an} \sin n\theta) = \dfrac{ze^a \sin\theta}{z^2e^{2a} - 2ze^a\cos\theta + 1}$

 Also evaluate $Z(e^{2n} \sin 2n)$. *(S.V.T.U., 2007)*

5. Using $Z(n^2) = \dfrac{z^2 + z}{(z-1)^3}$, show that $Z(n+1)^2 = \dfrac{z^3 + z^2}{(z-1)^3}$.

6. Find the Z-transforms of (*i*) $\sin(n+1)\theta$, (*ii*) $\cos\left(\dfrac{k\pi}{8} + \alpha\right)$. *(Marathwada, 2008)*

7. Find the Z-transform of $\cos n\theta$ and hence find $Z(n \cos n\theta)$. *(Anna, 2009)*

8. Find the Z-transform of $\cos(n\pi/2)$ and $a^n \cos(n\pi/2)$. *(Anna, 2008 S)*

9. Find the Z-transforms of the following

 (*i*) e^{-an} (*ii*) e^{-2n} *(V.T.U., 2010 S)* (*iii*) $e^{-an} n^2$.

10. Show that (*i*) $Z[\delta(n+1)] = 1/z$ (*ii*) $(1/2)^n u(n) = \dfrac{2z}{2z-1}$.

11. Show that $Z({}^{n+p}C_p) = (1 - 1/z)^{-(p+1)}$. Using the damping rule, deduce that

 $$Z({}^{n+p}C_p\, a^n) = (1 + a/z)^{-(p+1)}.$$

12. If $Z(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$, find the Z-transform of u_{n+2}. *(S.V.T.U., 2009)*

13. If $U(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of u_2 and u_3.

14. Given that $Z(u_n) = \dfrac{2z^2 + 3z + 4}{(z-3)^3}$, $|z| > 3$, show that $u_1 = 2$, $u_2 = 21$, $u_3 = 139$.

15. Show that (i) $Z\left(\dfrac{1}{n}\right) = z \log \dfrac{z}{z-1}$. (*Madras, 2003 S*) (ii) $Z\left\{\dfrac{1}{n(n+1)}\right\}$. (*Anna, 2005 S*)

16. Using $Z(n) = \dfrac{z}{(z-1)^2}$, show that $Z(n \cos n\theta) = \dfrac{(z^3 + z)\cos\theta - 2z^2}{(z^2 - 2z\cos\theta + 1)^2}$.

23.10 SOME USEFUL Z-TRANSFORMS

Sr. No.	*Sequence* u_n $(n \geq 0)$	*Z-transform* $U(z) = Z(u_n)$
1.	k	$kz/(z-1)$
2.	$-k$	$kz/(z+1)$
3.	n	$z/(z-1)^2$
4.	n^2	$(z^2+z)/(z-1)^3$
5.	n^p	$-z\, d/dz\, [Z(n^{p-1})]$, p + ve integer.
6.	$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$	1
7.	$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$	$z/(z-1)$
8.	a^n	$z/(z-a)$
9.	na^n	$az/(z-a)^2$
10.	n^2a^n	$(az^2 + a^2z)/(z-a)^3$
11.	$\sin n\theta$	$\dfrac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$
12.	$\cos n\theta$	$\dfrac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$
13.	$a^n \sin n\theta$	$\dfrac{az\sin\theta}{z^2 - 2az\cos\theta + a^2}$
14.	$a^n \cos n\theta$	$\dfrac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}$
15.	$\sinh n\theta$	$\dfrac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}$
16.	$\cosh n\theta$	$\dfrac{z(z-\cosh\theta)}{z^2 - 2z\cosh\theta + 1}$
17.	$a^n \sinh n\theta$	$\dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$
18.	$a^n \cosh n\theta$	$\dfrac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}$
19.	$a^n u_n$	$U(z/a)$
20.	u_{n+1}	$z[U(z) - u_0]$
	u_{n+2}	$z^2[U(z) - u_0 - u_1z^{-1}]$
	u_{n+3}	$z^3[U(z) - u_0 - u_1z^{-1} - u_2z^{-2}]$
21.	u_{n-k}	$z^{-k}U(z)$
22.	nu_n	$-z\,d/dz[U(z)]$
23.	u_0	$\underset{z \to \infty}{\text{Lt}}\ U(z)$
24.	$\underset{n \to \infty}{\text{Lt}}\ (u_n)$	$\underset{z \to 1}{\text{Lt}}\ [(z-1)\,U(z)]$

23.11 SOME USEFUL INVERSE Z-TRANSFORMS

Sr. No.	$U(z)$	*Inverse Z-transform* $u_n = z^{-1}[U(z)]$
1.	$\frac{1}{z-a}$	a^{n-1}
2.	$\frac{1}{z+a}$	$(-a)^{n-1}$
3.	$\frac{1}{(z-a)^2}$	$(n-1)a^{n-2}$
4.	$\frac{1}{(z-a)^3}$	$\frac{1}{2}(n-1)(n-2)a^{n-3}$
5.	$\frac{z}{z-a}$	a^n
6.	$\frac{z}{z+a}$	$(-a)^n$
7.	$\frac{z^2}{(z-a)^2}$	$(n+1)a^n$
8.	$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(n+1)(n+2)a^n u(n)$

23.12 CONVOLUTION THEOREM

If $Z^{-1}[U(z)] = u_n$ *and* $Z^{-1}[V(z)] = v_n$, *then*

$$Z^{-1}[U(z) \,.\, V(z)] = \sum_{m=0}^{n} u_m \,.\, v_{n-m} = u_n * v_n$$

where the symbol $*$ *denotes the convolution operation.*

Proof. We have $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, $V(z) = \sum_{n=0}^{\infty} v_n z^{-n}$

$$\therefore \quad U(z)\,V(z) = (u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_n z^{-n} + \dots \infty) \times (v_0 + v_1 z^{-1} + v_2 z^{-2} + \dots + v_n z^{-n} + \dots \infty)$$

$$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_n v_0) z^{-n} = Z(u_0 v_n + u_1 v_{n-1} + \dots + u_n v_0)$$

whence follows the desired result.

Obs. The convolution theorem plays an important role in the solution of difference equations and in probability problems involving sums of two independent random variables.

Example 23.11. *Use convolution theorem to evaluate* $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$.

Solution. We know that $Z^{-1}\left\{\frac{z}{z-a}\right\} = a^n$ and $Z^{-1}\left\{\frac{z}{z-b}\right\} = b^n$

$$\therefore \quad Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\} = Z^{-1}\left\{\frac{z}{z-a} \,.\, \frac{z}{z-b}\right\} = a^n * b^n$$

$$= \sum_{m=0}^{n} a^m \,.\, b^{n-m} = b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m \text{ which is a G.P.}$$

$$= b^n \,.\, \frac{(a/b)^{n+1} - 1}{a/b - 1} = \frac{a^{n+1} - b^{n+1}}{a - b}.$$

23.13 CONVERGENCE OF Z-TRANSFORMS

Z-transform operation is performed on a sequence u_n which may exist in the range of integers $-\infty < n < \infty$, and we write

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad \text{...(1)}$$

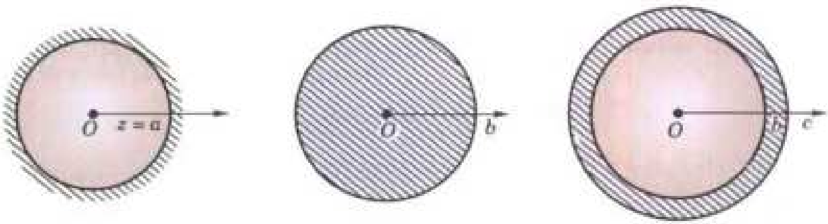
where u_n represents a number in the sequence for n = an integer. *The region of the z-plane in which (1) converges absolutely is known as the region of convergence (ROC) of U(z).*

We have so far discussed *one-sided Z-transform only for which* $n \geq 0$. Here the sequence is always *right-sided* and the *ROC* is always outside a prescribed circle say $|z| > |a|$ Fig. 23.2 (*i*)]. For a left-handed sequence, the ROC is always inside any prescribed contour $|z| < |b|$. [Fig. 23.2 (*ii*)].

23.14 TWO-SIDED Z-TRANSFORM OF u_n IS DEFINED BY

$$U(z) = \sum_{n=-\infty}^{\infty} u_n z^{-n} \qquad \text{...(2)}$$

In this case, the sequence is two-sided and the region of convergence for (2) is the *annular region* $|b| < |z| < |c|$ [Fig. 23.2 (*iii*)]. The inner circle bounds the terms in negative powers of z and the outer circle bounds the terms in positive powers of z. *The shaded annulus of convergence is necessary for the two sided sequence and its Z-transform to exist.*



$|z| > |a|$ (*i*) $\quad |z| < |b|$ (*ii*) $\quad |b| < |z| < |c|$ (*iii*)

Fig. 23.1

Example 23.12. *Find the Z-transfrom and region of convergence of*

(*a*) $u(n) = \begin{cases} 4^n & \text{for } n < 0 \\ 2^n & \text{for } n \geq 0 \end{cases}$ $\qquad$ (*b*) $u(n) = {}^n c_k,\ n \geq k.$

Solution. By definition $\quad Z[u(n)] = \sum_{-\infty}^{\infty} u(n) Z^{-n} = \sum_{-\infty}^{-1} 4^n z^{-n} + \sum_{n=0}^{\infty} 2^n z^{-n}$

Putting $-n = m$ in the first series, we get

$$Z[u(n)] = \sum_{1}^{\infty} 4^{-m} z^m + \sum_{n=0}^{\infty} 2^n z^{-n}$$

$$\left\{\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + \ldots\right\} + \left\{1 + \frac{2}{z} + \frac{2^2}{z^2} + \ldots\right\}$$

$$= \frac{z}{4}\left\{1 + \left(\frac{z}{4}\right) + \left(\frac{z}{4}\right)^2 + \ldots\right\} + \left\{1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \ldots\right\} \qquad \text{...}(i)$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(2/z)} = \frac{z}{4-z} + \frac{z}{z-2} = \frac{2z}{(4-z)(z-2)}$$

Now the two series in (*i*) being G.P. will be convergent if $|z/4| < 1$ and $|2/z| < 1$ *i.e.*, if $|z| < 4$ and $2 < |z|$. *i.e.* $2 < z < 4$.

Hence $Z[u(n)]$ is convergent if z lies between the annulus as shown shaded in Fig. 23.3. Hence ROC is $2 < z < 4$.

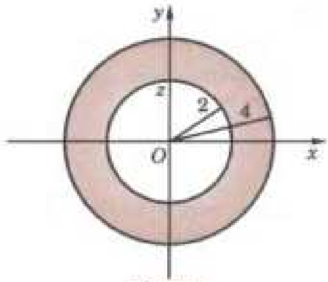


Fig. 23.2

(*b*) By definition, $Z[u(n)] = \sum_{-\infty}^{\infty} {}^nC_k z^{-n} = \sum_{n=k}^{\infty} {}^nC_k 2^n z^{-n}$

To find the sum of this series, we replace n by $k + r$

$$\therefore \quad Z[u(n)] = \sum_{r=0}^{\infty} {}^{k+r}C_k z^{-(k+r)} = z^{-k} \sum_{r=0}^{\infty} {}^{k+r}C_r z^{-r} \qquad [\because \ {}^kC_r = {}^kC_{k-r}]$$

$$= z^{-k}[1 + {}^{k+1}C_1 z^{-1} + {}^{k+1}C_2 z^{-2} + ...]$$

$$= z^{-k}(1 - 1/z)^{-(k+1)}$$

This series is convergence for $|1/z| < 1$ *i.e.*, for $|z| > 1$.

Hence ROC is $|z| > 1$.

Example 23.13. *Find the Z-transform and the radius of convergence of*

(*a*) $f(n) = 2^n, n < 0$ (*b*) $f(n) = 5^n/n!, n \geq 0$. (*Mumbai, 2009*)

Solution. (*a*) Assuming that $f(n) = 0$ for $n \geq 0$ we have

$$Z[f(n)] = \sum_{-\infty}^{\infty} f(n)z^{-n} = \sum_{n=-\infty}^{-1} 2^n z^{-n} = \sum_{m=1}^{\infty} 2^{-m} z^m \quad \text{where} \quad m = -n$$

$$= \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\infty = \frac{z}{2}\{1 + (z/2) + (z/2)^2 + ... \infty\}$$

$$= \frac{z}{2} \cdot \frac{1}{1-(z/2)} = \frac{z}{2-z}$$

This series being a G.P. is convergent if $|z/2| < 1$ *i.e.*, $|z| < 2$.

Hence ROC is $|z| < 2$.

(*b*) By definition, $Z[u(n)] = \sum_{n=0}^{\infty} \frac{5^n}{n!} \cdot z^{-n} = \sum_{0}^{\infty} \frac{(5/z)^n}{n!} = 1 + \left(\frac{5}{z}\right) + \frac{1}{2!}\left(\frac{5}{z}\right)^2 + \frac{1}{3!}\left(\frac{5}{z}\right)^3 + ... \infty$

$$= e^{5/z}$$

The above series is convergent for all values of z.

Hence ROC is the entire z-plane.

PROBLEMS 23.2

Find the Z-transform and its ROC in each of the following sequences :

1. $u(n) = 4^n, n \geq 0$.
2. $u(n) = 2^n, n < 0$.
3. $u(n) = 4^n$, for $n < 0$ and $= 3^n$ for $n \geq 0$.
4. $u(n) = n5^n, n \geq 0$.
5. $u(n) = 2^n/n, n > 1$.
6. $u(n) = 3^n/n!, n \geq 0$.
7. $u(n) = e^{an}, n \geq 0$.

23.15 EVALUATION OF INVERSE Z-TRANSFORMS

We can obtain the inverse Z-transforms using any of the following three methods :

I. **Power series method.** This is the simplest of all the methods of finding the inverse Z-transform. If $U(z)$ is expressed as the ratio of two polynomials which cannot be factorized, we simply divide the numerator by the denominator and take the inverse Z-transform of each term in the quotient.

Example 23.14. *Find the inverse Z-transform of $\log (z/z+1)$ by power series method.*

Solution. Putting $z = 1/t$, $U(z) = \log\left(\dfrac{1/y}{1/y+1}\right) = -\log(1+y) = -y + \dfrac{1}{2}y^2 - \dfrac{1}{3}y^3 + \ldots$

$$= -z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{3}z^{-3} + \ldots$$

Thus
$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ (-1)^n/n & \text{otherwise} \end{cases}.$$

Example 23.15. *Find the inverse Z-transform of $z/(z+1)^2$ by division method.*

Solution.
$$U(z) = \frac{z}{z^2+2z+1} = z^{-1} - \frac{2+z^{-1}}{z^2+2z+1}, \text{ by actual division}$$

$$= z^{-1} - 2z^{-2} + \frac{3z^{-1}+2z^{-2}}{z^2+2z+1} = z^{-1} - 2z^{-2} + 3z^{-3} - \frac{4z^{-2}+3z^{-3}}{z^2+2z+1}$$

Continuing this process of division, we get an infinite series *i.e.*,

$$U(z) = \sum_{n=0}^{\infty} (-1)^{n-1}\, nz^{-n}$$

Thus $\quad u_n = (-1)^{n-1}n.$

II. **Partial fractions method.** This method is similar to that of finding the inverse Laplace transforms using partial fractions. The method consists of decomposing $U(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

Example 23.16. *Find the inverse Z-transforms of*

(i) $\dfrac{2z^2+3z}{(z+2)(z-4)}$ (V.T.U., 2008 S ; S.V.T.U., 2007) *(ii)* $\dfrac{z^3-20z}{(z-2)^3(z-4)}$. (V.T.U., 2011)

Solution. *(i)* We write $U(z) = \dfrac{2z^2+3z}{(z+2)(z-4)}$ as $\dfrac{U(z)}{z} = \dfrac{2z+3}{(z+2)(z-4)} = \dfrac{A}{z+2} + \dfrac{B}{z-4}$

where $A = 1/6$ and $B = 11/6$

$$\therefore \quad U(z) = \frac{1}{6}\frac{z}{z+2} + \frac{11}{6}\frac{z}{z-4}$$

On inversion, we have

$$u_n = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n \qquad \text{[Using § 23.10 (9)]}$$

(ii) We write
$$U(z) = \frac{z^3-20z}{(z-2)^3(z-4)}$$

as
$$\frac{U(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)} = \frac{A+Bz+Cz^2}{(z-2)^3} + \frac{D}{z-4}$$

Readily we get $D = 1/2$. Multiplying throughout by $(z-2)^3(z-4)$, we get

$$z^2 - 20 = (A + Bz + Cz^2)(z-4) + D(z-2)^3.$$

Putting $z = 0, 1, -1$ successively and solving the resulting simultaneous equations, we get $A = 6, B = 0, C = 1/2$.

Thus
$$U(z) = \frac{1}{2}\cdot\frac{12z+z^3}{(z-2)^3} - \frac{z}{z-4} = \frac{1}{2}\frac{z(z-2)^2+4z^2+8z}{(z-2)^3} - \frac{z}{z-4}$$

$$= \frac{1}{2}\left\{\frac{z}{z-2} + 2\frac{2z^2+4z}{(z-2)^3}\right\} - \frac{z}{z-4}$$

On inversion, we get $u_n = \frac{1}{2}(2^n + 2 \cdot n^2 2^n) - 4^n$ [Using § 23.10 (9) & (11)]

$$= 2^{n-1} + n^2 2^n - 4^n.$$

Example 23.17. *Find the inverse Z-transform of*

$2(z^2 - 5z + 6.5)/[(z-2)(z-3)^2]$, for $2 < |z| < 3$.

Solution. Splitting into partial fractions, we obtain

$$U(z) = \frac{2(z^2 - 5z + 6.5)}{(z-2)(z-3)^2} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{(z-3)^2} \quad \text{where } A = B = C = 1$$

$$\therefore \quad U(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

$$= \frac{1}{2}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2} \quad \text{so that } 2/z < 1 \text{ and } z/3 < 1$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \dots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \dots\right) + \frac{1}{9}\left(1 + \frac{2z}{3} + \frac{3z^2}{9} + \frac{4z^3}{27} + \dots\right)$$

where $2 < |z| < 3$.

$$= \left(\frac{1}{2} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \dots\right) - \left(\frac{1}{3} + \frac{z}{3^2} + \frac{z^2}{3^3} + \frac{z^3}{3^4} + \dots\right) + \left(\frac{1}{3^2} + \frac{2z}{3^3} + \frac{3z^2}{3^4} + \frac{4z^3}{3^5} + \dots\right)$$

$$= \sum_{n=1}^{\infty} 2^{n-1} z^{-n} - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+1} z^n + \sum_{n=0}^{\infty} (n+1)\left(\frac{1}{3}\right)^{n+2} z^n$$

On inversion, we get $u_n = 2^{n-1}, n \geq 1$ and $u_n = -(n+2)3^{n-2}, n \leq 0$.

III. **Inversion integral method.** *The inverse Z-transform of $U(z)$ is given by the formula*

$$u_n = \frac{1}{2\pi i}\int_c U(z) z^{n-1}\, dz$$

= *sum of residues of $U(z)\, z^{n-1}$ at the poles of $U(z)$ which are inside the contour C drawn according to the ROC given.*

The following examples will illustrate the application of this formula :

Example 23.18. *Using the inversion integral method, find the inverse Z-transform of*

$$\frac{z}{(z-1)(z-2)}$$

(V.T.U., 2010 S)

Solution. Let $U(z) = \dfrac{z}{(z-1)(z-2)}$. Its poles are at $z = 1$ and $z = 2$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z) z^{n-1}\, dz,$$

where C is a circle large enough to enclose both the poles of $U(z)$.

= sum of residues of $U(z)\, z^{n-1}$ at $z = 1$ and $z = 2$.

Now $\quad Res\, [U(z)\, z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}} \left\{(z-1) \cdot \frac{z^n}{(z-1)(z-2)}\right\} = -1$

and $\quad Res\, [U(z)\, z^{n-1}]_{z=2} = \underset{z \to 2}{\text{Lt}} \left\{(z-2) \cdot \frac{z^n}{(z-1)(z-2)}\right\} = 2^n$

Thus the required inverse Z-transform $u_n = 2^n - 1, n = 0, 1, 2, \dots$

Example 23.19. *Find the inverse Z-transform of $2z/[(z-1)(z^2+1)]$.* *(Madras, 2000 S)*

Solution. Let $U(z) = \dfrac{2z}{(z-1)(z+i)(z-i)}$. It has three poles at $z = 1, z = \pm i$.

Using $U(z)$ in the inversion integral, we have

$$u_n = \frac{1}{2\pi i}\int_C U(z)\,.\,z^{n-1}\,dz\text{, where } C \text{ is a circle large enough to enclose the poles of } U(z).$$

$$= \text{sum of residues of } U(z)\,.\,z^{n-1} \text{ at } z = 1, z = \pm i.$$

Now
$$Res\,[U(z)\,z^{n-1}]_{z=1} = \underset{z \to 1}{\text{Lt}}\left\{(z-1)\frac{2z^n}{(z-1)(z^2+1)}\right\} = 1$$

$$Res\,[U(z)\,z^{n-1}]_{z=i} = \underset{z \to i}{\text{Lt}}\left\{(z-i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{-(i)^n}{1+i}$$

$$Res\,[U(z)\,z^{n-1}]_{z=-i} = \underset{z \to -i}{\text{Lt}}\left\{(z+i)\frac{2z^n}{(z-1)(z+i)(z-i)}\right\} = \frac{(-i)^n}{i-1}$$

Hence
$$u_n = 1 - \frac{(i)^n}{1+i} - \frac{(-i)^n}{1-i}.$$

PROBLEMS 23.3

Using convolution theorem, evaluate the inverse Z-transforms of the following :

1. $\dfrac{z^2}{(z-1)(z-3)}$.
2. $\left(\dfrac{z}{z-a}\right)^2$ *(Madras, 2003)*
3. $\left(\dfrac{z}{z-1}\right)^3$.
4. Show that (*a*) $\dfrac{1}{n!} * \dfrac{1}{n!} = \dfrac{2^n}{n!}$ (*b*) $Z^{-1}\left(\dfrac{z^2}{(z+a)(z+b)}\right) = \dfrac{(-1)}{b-a}(b^{n+1} - a^{n+1})$. *(Anna, 2009)*

Find the inverse Z-transforms of the following :

5. $\dfrac{4z}{z-a}$, $|z| > |a|$. *(Kottayam, 2005)*
6. $\dfrac{5z}{(2-z)(3z-1)}$. *(Madras, 1999)*
7. $\dfrac{z}{(z-1)^2}$.
8. $\dfrac{18z^2}{(2z-1)(4z+1)}$. *(S.V.T.U., 2009)*
9. $\dfrac{8z-z^3}{(4-z)^3}$.
10. $\dfrac{3z^2-18z+26}{(z-2)(z-3)(z-4)}$. *(Anna, 2005 S)*
11. $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$. *(V.T.U., 2011 S)*
12. $\dfrac{z^3+3z}{(z-1)^2(z^2+1)}$. *(Anna, 2009)*
13. $\dfrac{(1-e^{at})z}{(z-1)(z-e^{-at})}$.
14. Obtain $Z^{-1}[1/[(z-2)(z-3)]]$ for (*i*) $|z| < 2$; (*ii*) $2 < |z| < 3$; (*iii*) $|z| > 3$. *(Marathwada, 2008)*
15. Evaluate $Z^{-1}\{(z-5)^{-3}\}$ for $|z| > 5$. *(Mumbai, 2009)*

Using inversion integral, find the inverse Z-transform of the following functions :

16. $\dfrac{z+3}{(z+1)(z-2)}$.
17. $\dfrac{(2z-1)z}{2(z-1)(z+0.5)}$.
18. $\dfrac{1}{z(z-1)(z+0.5)}$. *(S.V.T.U., 2008)*
19. $\dfrac{z^2+z}{(z-1)(z^2+1)}$. *(Madras, 2003)*
20. $\dfrac{2z(z^2-1)}{(z^2+1)^2}$.

23.16 (1) APPLICATION TO DIFFERENCE EQUATIONS

Just as the Laplace transforms method is quite effective for solving linear differential equations (§ 21.15), the Z-transforms are quite useful for solving linear difference equations.

The performance of discrete systems is expressed by suitable difference equations. Also Z-transform plays an important role in the analysis and representation of discrete-time systems. To determine the frequency response of such systems, the solution of difference equations is required for which Z-transform method proves useful.

(2) **Working procedure** *to solve a linear difference equation with constant coefficients by Z-transforms :*

1. *Take the Z-transform of both sides of the difference equations using the formulae of § 26.16 and the given conditions.*

2. *Transpose all terms without $U(z)$ to the right.*

3. *Divide by the coefficient of $U(z)$, getting $U(z)$ as a function of z.*

4. *Express this function in terms of the Z-transforms of known functions and take the inverse Z-transform of both sides. This gives u_n as a function of n which is the desired solution.*

Example 23.20. *Using the Z-transform, solve*

$$u_{n+2} + 4u_{n+1} + 3u_n = 3^n \text{ with } u_0 = 0, u_1 = 1. \qquad \text{(U.P.T.U., 2003)}$$

Solution. If $\quad Z(u_n) = U(z)$, then $Z(u_{n+1}) = z[U(z) - u_0]$,

$$Z(u_{n+2}) = z^2[U(z) - u_0 - u_1 z^{-1}]$$

Also $\quad Z(2^n) = z/(z-2)$

$\therefore$ taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] + 4z\,[U(z) - u_0] + 3U(z) = z/(z-3)$$

Using the given conditions, it reduces to

$$U(z)\,(z^2 + 4z + 3) = z + z/(z-3)$$

$$\therefore \quad \frac{U(z)}{z} = \frac{1}{(z+1)(z+3)} + \frac{1}{(z-3)(z+1)(z+3)} = \frac{3}{8}\frac{1}{z+1} + \frac{1}{24}\frac{1}{z-3} - \frac{5}{12}\frac{1}{z+3},$$

on breaking into partial fractions.

$$U(z) = \frac{3}{8}\frac{z}{z+1} + \frac{1}{24}\frac{z}{z-3} - \frac{5}{12}\frac{z}{z+3}$$

On inversion, we obtain

$$u_n = \frac{3}{8}Z^{-1}\left(\frac{z}{z+1}\right) + \frac{1}{24}Z^{-1}\left(\frac{z}{z-3}\right) - \frac{5}{12}Z^{-1}\left(\frac{z}{z+3}\right) = \frac{3}{8}(-1)^n + \frac{1}{24}3^n - \frac{5}{12}(-3)^n.$$

Example 23.21. *Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$, using Z-transforms.*

(V.T.U., 2011 ; Anna, 2009 ; S.V.T.U., 2009)

Solution. If $Z(y_n) = Y(z)$, then $Z(y_{n+1}) = z\,[(Y(z) - y_0]$, $Z(y_{n+2}) = z^2[Y(z) - y_0 - y_1 z^{-1}]$

Also $\quad Z(2^n) = z/(z-2)$.

Taking Z-transforms of both sides, we get

$$z^2[Y(z) - y_0 - y_1 z^{-1}] + 6z[Y(z) - y_0] + 9Y(z) = z/(z-2)$$

Since $\;y_0 = 0$, and $y_1 = 0$, we have $Y(z)\,(z^2 + 6z + 9) = z/(z-2)$

or $$\frac{Y(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{1}{25}\left[\frac{1}{z-2} - \frac{1}{z+3} - \frac{5}{(z+3)^2}\right],$$ on splitting into partial fractions.

or $$Y(z) = \frac{1}{25}\left[\frac{z}{z-2} - \frac{z}{z+3} - 5\frac{z}{(z+3)^2}\right]$$

On taking inverse Z-transform of both sides, we obtain

$$y_n = \frac{1}{25}\left[Z^{-1}\left(\frac{z}{z-2}\right) - Z^{-1}\left(\frac{z}{z+3}\right) + \frac{5}{3}Z^{-1}\left(-\frac{3z}{(z+3)^2}\right)\right]$$

$$= \frac{1}{25}\{2^n - (-3)^n + \frac{5}{3}n(-3)^n\} \qquad \left[\because\ Z^{-1}\left\{\frac{az}{(z-a)^2}\right\} = na^n\right]$$

Example 23.22. *Find the response of the system $y_{n+2} - 5y_{n+1} + 6y_n = u_n$, with $y_0 = 0, y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \ldots$ by Z-transform method.* (V.T.U., 2010)

Solution. Taking Z-transform of both sides of the given equation, we get

$$z^2(Y(z) - y_0 - y_1 z^{-1}) - 5z(Y(z) - y_0) + 6Y(z) = \frac{z}{z-1}$$

Substituting the values $y_0 = 0, y_1 = 1$, it reduces to

$$(z^2 - 5z + 6)\, Y(z) = \frac{z}{z-1} + z = \frac{z^2}{z-1}$$

or
$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-2)(z-3)}$$

$$= \frac{A}{z-1} + \frac{B}{z-2} + \frac{C}{z-3} \qquad \text{where } A = \frac{1}{2}, B = -2, C = \frac{3}{2}$$

so that
$$Y(z) = \frac{1}{2}\frac{z}{z-1} - 2\frac{z}{z-2} + \frac{3}{2}\frac{z}{z-3}$$

On inversion, we obtain
$$y_n = \frac{1}{2} - 2(2)^n + \frac{3}{2}(3)^n$$

Obs. The initial values given in the problem automatically appear in the generated sequence.

Example 23.23. *Solve the difference equation $y_n + \frac{1}{4}y_{n-1} = u_n + \frac{1}{3}u_{n-1}$ where u_n is a unit step sequence.*

Solution. Taking Z-transform of both sides of the given equation, we get

$$Y(z) + \frac{1}{4}z^{-1}\, Y(z) = 1 + \frac{1}{3}z^{-1}$$

or
$$Y(z) = \left(1 + \frac{1}{3}z^{-1}\right) \Big/ \left(1 + \frac{1}{4}z^{-1}\right) = \left(z + \frac{1}{3}\right) \Big/ \left(z + \frac{1}{4}\right)$$

There being only one simple pole at $z = -1/4$, consider the contour $|z| > 1/4$.

$$\therefore \quad Res\, [Y(z)\, z^{n-1}]_{z=-1/4} = \underset{z \to -1/4}{\text{Lt}} \left\{\left(z + \frac{1}{4}\right) \cdot \left(z + \frac{1}{3}\right) z^{n-1} \Big/ \left(z + \frac{1}{4}\right)\right\}$$

$$= \underset{z \to -1/4}{\text{Lt}} \left(z + \frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4} + \frac{1}{3}\right)\left(-\frac{1}{4}\right)^{n-1} = \frac{1}{12} \cdot \left(-\frac{1}{4}\right)^{n-1}$$

Hence by inversion integral method, we have

$$y_n = \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}.$$

Example 23.24. *Using the Z-transform, solve $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.* (S.V.T.U., 2007)

Solution. Given equation is $u_{n+2} - 2u_{n+1} + u_n = 3n + 5$.

Taking the Z-transforms of both sides, we get

$$z^2[U(z) - u_0 - u_1 z^{-1}] - 2z[U(z) - u_0] + U(z) = 3 \cdot \frac{z}{(z-1)^2} + 5 \cdot \frac{z}{z-1}$$

or
$$U(z)\,(z^2 - 2z + 1) = \frac{5z^2 - 2z}{(z-1)^2} + u_0(z^2 - 2z) + u_1 z$$

or
$$U(z) = \frac{5z^2 - 2z}{(z-1)^4} + u_0\frac{z^2 - 2z}{(z-1)^2} + u_1\frac{z}{(z-1)^2}$$

On inversion, we obtain

$$u_n = Z^{-1}\left\{\frac{5z^2 - 2z}{(z-1)^4}\right\} + u_0 Z^{-1}\left\{\frac{z^2 - 2z}{(z-1)^2}\right\} + u_1 Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} \quad ...(i)$$

Noting that $Z(1) = \dfrac{z}{z-1}, Z(n) = \dfrac{z}{(z-1)^2}$

$$Z(n^2) = \frac{z^2+z}{(z-1)^3}, Z(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}$$

We write $\dfrac{5z^2-2z}{(z-1)^4} \equiv A\dfrac{z^3+4z^2+z}{(z-1)^4} + B\dfrac{z^2+z}{(z-1)^3} + C\dfrac{z}{(z-1)^2} + D\dfrac{z}{z-1}$

Equating coefficients of like powers of z, we find

$$A = \frac{1}{2}, B = 1, C = -\frac{3}{2}, D = 0$$

$$\therefore \quad Z^{-1}\left\{\frac{5z^2-2z}{(z-1)^4}\right\} = \frac{1}{2}n^3 + n^2 - \frac{3}{2}n = \frac{1}{2}n(n-1)(n+3)$$

Also $$Z^{-1}\left\{\frac{z^2-2z}{(z-1)^2}\right\} = Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = 1 - n$$

and $$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = n.$$

Substituting these values in (i) above, we get

$$u_n = \frac{1}{2}n(n-1)(n+3) + u_0(1-n) + u_1 n$$

$$= \frac{1}{2}n(n-1)(n+3) + c_0 + c_1 n. \qquad \text{where } c_0 = u_0, c_1 = u_1 - u_0$$

Example 23.25. *Using residue method, solve* $y_k + \frac{1}{9}y_{k-2} = \frac{1}{3^k}\cos\frac{k\pi}{2}, k \geq 0.$

Solution. Taking Z-transform of both sides of the given equation, we get

$$Z\left\{y_k + \frac{1}{9}y_{k-2}\right\} = Z\left\{\frac{1}{3^k}\cos\frac{k\pi}{2}\right\}$$

or $$Y(z) + \frac{1}{9}z^{-2}Y(z) = \frac{z^2}{z^2+1/9} \quad \text{or} \quad \left(1+\frac{1}{9}z^{-2}\right)Y(z) = \frac{z^2}{z^2+\frac{1}{9}}$$

or $$Y(z) = \frac{z^2}{\left(1+\frac{1}{9}z^{-2}\right)\left(z^2+\frac{1}{9}\right)} = \frac{z^4}{\left(z^2+\frac{1}{9}\right)^2}$$

There are two poles of second order at $z = i/3$ and $z = -i/3$.

$$\therefore \quad \text{Residue at } (z = i/3) = \left[\frac{d}{dz}\left\{\left(\frac{z-i}{3}\right)^2 \frac{z^{k-1}z^4}{(z^2+1/9)^2}\right\}\right]$$

$$= \left[\frac{d}{dz}\left\{\frac{z^{k+3}}{(z+i/3)^2}\right\}\right]_{z=i/3} = \left[\frac{(z+i/3)^2(k+3)z^{k+2} - z^{(k+3)}\cdot 2(z+i/3)}{(z+i/3)^4}\right]_{z=i/3}$$

$$= \left[\frac{(z+i/3)(k+3)z^{k+2} - 2z^{k+3}}{(z+i/3)^3}\right]_{z=i/3} = \left(\frac{3}{2i}\right)^3\left[(2k+6)\left(\frac{i}{3}\right)^{k+3} - 2\left(\frac{i}{3}\right)^{k+3}\right]$$

$$= \frac{1}{8}(2k+4)\left(\frac{i}{3}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^k = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2}+i\sin\frac{k\pi}{2}\right) \quad ...(i)$$

Changing i to $-i$ in (i), we have

$$\text{Residue at } (z = -i/3) = \frac{1}{4}(k+2)\left(\frac{1}{3}\right)^k\left(\cos\frac{k\pi}{2} - i\sin\frac{k\pi}{2}\right) \quad ...(ii)$$

Adding (i) and (ii), we obtain $y_k = \frac{1}{2}(k+2)\left(\frac{1}{3}\right)^k \cos\frac{k\pi}{2}$.

PROBLEMS 23.4

Solve the following difference equations using Z-transforms (1 – 8):

1. $6y_{k+2} - y_{k+1} - y_k = 0$, given that $y(0) = y(1) = 1$. *(Kottayam, 2005)*
2. $y(n+2) + 2y(n+1) + y(n) = 0$, given that $y(0) = y(1) = 0$. *(V.T.U., 2008 S)*
3. $y_{n+2} - 4y_n = 0$ given that $y_0 = 0$, $y_1 = 2$. *(U.P.T.U., 2008)*
4. $f(n) + 3f(n-1) - 4f(n-2) = 0$, $n \geq 2$, given that $f(0) = 3$, $f(1) = -2$. *(Madras, 2003 S)*
5. $y_{(n+2)} - 3y(n+1) + 2y(n) = 0$, given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$. *(Anna, 2005 S)*
6. $y_{n+2} - 5y_{n+1} + 6y_n = 36$, given that $y(0) = y(1) = 0$. *(Anna, 2009)*
7. $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$.
8. $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.
9. $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n$ $(n \geq 0)$, $y_0 = 0$.
10. $u_{x+2} + u_x = 5(2^x)$ given that $u_0 = 1$, $u_1 = 0$. *(Marathwada, 2008)*
11. $y_{n+2} + 4y_{n+1} + 3y_n = 2^n$ with $y_0 = 0$, $y_1 = 1$. *(Madras, 2006)*
12. $u_{k+2} - 2u_{k+1} + u_k = 2^k$ with $y_0 = 2$, $y_1 = 1$.
13. $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$.
14. $y_k + \frac{1}{25}y_{k-2} = \left(\frac{1}{5}\right)^k \cos\frac{k\pi}{2}$, $(k \geq 0)$.
15. Find the response of the system given by $y_n + 3y_{(n-1)} = u_n$ where u_n is a unit step sequence and $y_{(-1)} = 1$.
16. Find the impulse response of a system described by $y_{(n+1)} + 2y_{(n)} = \delta_n$; $y_0 = 0$.

23.1 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 23.5

Choose the correct answer or fill up the blanks in each of the following problems :

1. $Z(1) =$ ......
2. If u_n is defined for $n = 0, 1, 2,$ ...... only, then $Z(u_n) =$ ....
3. Z-transform of $n =$ .... *(Anna, 2009)*
4. $Z(na^n) =$ ......
5. $Z(\sin n\theta) =$ ......
6. Z-transform of $(1/n!)$ is
7. $Z(n^2) =$ ......
8. Linear property of Z-transform states that...
9. $Z^{-1}\left(\frac{1}{z-2}\right) =$ ....
10. $Z^{-1}\left\{\frac{z}{(z+1)^2}\right\} =$ ......
11. Initial value theorem on Z-transform states that ....
12. Z-transform is linear. (True or False)
13. If $Z(u_n) = u(z)$, then $\underset{n\to\infty}{\text{Lt}}(u_n) = \underset{z\to\infty}{\text{Lt}}(z-1)u(z)$. (True or False)
14. Z-transform of the sequence $\{2^k\}$, $k \geq 0$ is $z/(z-2)$. (True or False)
15. Z-transform of $\{a^k/k!\}$, $k \geq 0 = e^{a/z}$. (True or False)
16. Z-transform of $\{^nC_r\}$, $(0 \leq r \leq n)$ is $(1+z)^n$. (True or False)
17. Z-transform of unit impulse sequence $\delta(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$, is $z/z-1$. (True or False)
18. Z-transform of unit step sequence $u(n) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0, \end{cases}$ is 1. (True or False)

CHAPTER 24

Empirical Laws and Curve-fitting

24.1 INTRODUCTION

In many branches of applied mathematics, it is required to express a given data, obtained from observations, in the form of a *Law* connecting the two variables involved. Such a *Law* inferred by some scheme is known as *Empirical Law*. For example, it may be desired to obtain the law connecting the length and the temperature of a metal bar. At various temperatures, the length of the bar is measured. Then, by one of the methods explained below, a law is obtained that represents the relationship existing between temperature and length for the observed values. This relation can then be used to predict the length at an arbitrary temperature.

(2) Scatter diagram. To find a relationship between the set of paired observations x and y (say), we plot their corresponding values on the graph taking one of the variables along the x-axis and other along the y-axis *i.e.* (x_1, y_1), (x_2, y_2), (x_n, y_n). The resulting diagram showing a collection of dots is called a *scatter diagram*. A smooth curve that approximates the above set of points is known as the *approximating curve*.

(3) Curve fitting. Several equations of different types can be obtained to express the given data approximately. But the problem is to find the equation of the curve of *'best fit'* which may be most suitable for predicting the unknown values. The process of finding such an equation of 'best fit' is known as *curve-fitting*.

If there are n pairs of observed values then it is possible to fit the given data to an equation that contains n arbitrary constants for we can solve n simultaneous equations for n unknowns. If it were desired to obtain an equation representing these data but having less than n arbitrary constants, then we can have recourse to any of the four methods : *Graphical method, Method of Least squares, Method of Group averages* and *Method of Moments*. The graphical method fails to give the values of the unknowns uniquely and accurately while the other methods do. *The method of Least squares is, probably, the best to fit a unique curve to a given data.* It is widely used in applications and can be easily implemented on a computer.

24.2 GRAPHICAL METHOD

When the curve representing the given data is a **linear law** $y = mx + c$ *; we proceed as follows :*

(*i*) *Plot the given points on the graph paper* taking a suitable *scale*.

(*ii*) *Draw the straight line of best fit* such that the points are evenly distributed about the line.

(*iii*) *Taking two suitable points* (x_1, y_1) *and* (x_2, y_2) *on the line, calculate* m, the slope of the line and c, its intercept on y-axis.

When the points representing the observed values do not approximate to a straight line, a smooth curve is drawn through them. From the shape of the graph, we try to infer the law of the curve and then reduce it to the form $y = mx + c$.

24.3 LAWS REDUCIBLE TO THE LINEAR LAW

We give below some of the laws in common use, indicating the way these can be reduced to the linear form by suitable substitutions :

(1) *When the law is* $y = mx^n + c$.

Taking $x^n = X$ and $y = Y$ the above law becomes $Y = mX + c$

(2) *When the law is* $y = ax^n$.

Taking logarithms of both sides, it becomes $\log_{10} y = \log_{10} a + n \log_{10} x$

Putting $\log_{10} x = X$ and $\log_{10} y = Y$, it reduces to the form $Y = nX + c$, where $c = \log_{10} a$.

(3) *When the law is* $y = ax^n + b \log x$.

Writing it as $\dfrac{y}{\log x} = a \dfrac{x^n}{\log x} + b$ and taking $x^n/\log x = X$ and $y/\log x = Y$,

the given law becomes, $Y = aX + b$.

(4) *When the law is* $y = ae^{bx}$

Taking logarithms, it becomes $\log_{10} y = (b \log_{10} e) x + \log_{10} a$

Putting $x = X$ and $\log_{10} y = Y$, it takes the form $Y = mX + c$ where $m = b \log_{10} e$ and $c = \log_{10} a$.

(5) *When the law is* $xy = ax + by$.

Dividing by x, we have $y = b \dfrac{y}{x} + a$.

Putting $y/x = X$ and $y = Y$, it reduces to the form $Y = bX + a$.

Example 24.1. *R is the resistance to maintain a train at speed V; find a law of the type* $R = a + bV^2$ *connecting R and V, using the following data :*

V (miles/hour) :	*10*	*20*	*30*	*40*	*50*
R (lb/ton) :	*8*	*10*	*15*	*21*	*30*

Solution. Given law is $R = a + bV^2$ ...(*i*)

Taking $V^2 = x$ and $R = y$, (*i*) becomes

$$y = a + bx \qquad ...(ii)$$

which is a *linear law*.

Table for the values of x and y is as follows :

x	100	400	900	1600	2500
y	8	10	15	21	30

Plot these points. Draw the straight line of best fit through these points (Fig. 24.1)

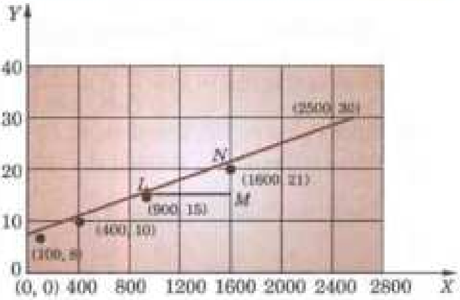


Fig. 24.1

Slope of this line (= b)

$$= \frac{MN}{LM} = \frac{21 - 15}{1600 - 900} = \frac{6}{700} = 0.0085 \text{ nearly.}$$

Since L (900, 15) lies on (*ii*),

$\therefore$ $$15 = a + 0.0085 \times 900,$$

whence $$a = 15 - 7.65 = 7.35 \text{ nearly.}$$

Example 24.2. *The following values of x and y are supposed to follow the law* $y = ax^2 + b \log_{10} x$. *Find graphically the most probable values of the constants a and b.*

x	*2.85*	*3.88*	*4.66*	*5.69*	*6.65*	*7.77*	*8.67*
y	*16.7*	*26.4*	*35.1*	*47.5*	*60.6*	*77.5*	*93.4*

Solution. Given law is $y = ax^2 + b \log_{10} x$

i.e. $$\frac{y}{\log_{10} x} = a\frac{x^2}{\log_{10} x} + b \quad ...(i)$$

Taking $x^2/\log_{10} x = X$ and $y/\log_{10} x = Y$

(*i*) becomes $Y = aX + b$ ...(*ii*)

This is a *linear law*. Table for the values of X and Y is as follows :

$X = x^2/\log_{10} x$	17.93	25.56	32.49	42.87	53.75	67.80	80.83
$Y = y/\log_{10} x$	35.59	44.83	52.50	62.90	73.65	87.04	99.56
Points	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Plot these points and draw the straight line of best fit through these points (Fig. 24.2).

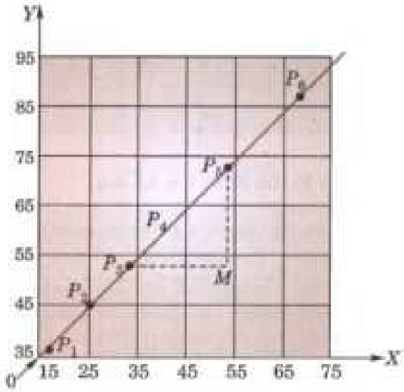


Fig. 24.2

Slope of this line (= a) = $\frac{MP_5}{P_3M} = \frac{73.65 - 52.50}{53.75 - 32.49} = \frac{21.15}{21.26} = 0.99$

Since P_3 lies on (*ii*), therefore, $52.50 = 0.99 \times 32.49 + b$ whence $b = 20.2$

Hence (*i*) becomes $y = (0.99)\, x^2 + (20.2) \log_{10} x$.

Example 24.3. *The values of x and y obtained in an experiment are as follows :*

x	2.30	3.10	4.00	4.92	5.91	7.20
y	33.0	39.1	50.3	67.2	85.6	125.0

Th probable law is $y = ae^{bx}$. Test graphically the accuracy of this law and if the law holds good, find the best values of the constants.

Solution. Given law is $y = ae^{bx}$ ...(*i*)

Taking logarithms to base 10, we have $\log_{10} y = \log_{10} a + (b \log_{10} e)\, x$

Putting $x = X$ and $\log_{10} y = Y$, it becomes $y = (b \log_{10} e)\, X + \log_{10} a$ ...(*ii*)

Table for the values of X and Y is as under :

$X = x$	2.30	3.10	4.00	4.92	5.91	7.20
$Y = \log_{10} y$	1.52	1.59	1.70	1.83	1.93	2.1
Points	P_1	P_2	P_3	P_4	P_5	P_6

Scale : 1 small division along x–axis = 0.1

10 small divisions along y–axis = 0.1.

Plot these points and draw the line of best fit. As these points are lying almost along a straight line, the given law is nearly accurate (Fig. 24.3).

Now slope of this line $(= b \log_{10} e)$

$$= \frac{MN}{NM} = 0.12$$

whence $$b = \frac{0.12}{\log_{10} e} = 0.12 \times 2.303 = 0.276$$

Since the point L (4, 1.71) lies on (*ii*), therefore, $1.71 = 0.12 \times 4 + \log_{10} a$ whence $a = 17$ nearly.

Hence the curve of best fit is $y = 17\, e^{0.276x}$.

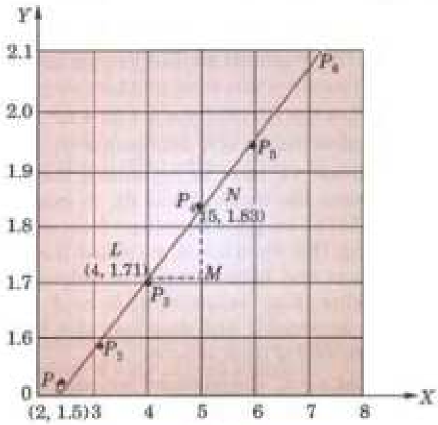


Fig. 24.3

PROBLEMS 24.1

1. If p is the pull required to lift the weight by means of a pulley block, find a linear law of the form $p = a + bw$, connecting p and w, using the following data :

w (lb) :	50	70	100	120
p (lb) :	12	15	21	25

Compute p, when w = 150 lb.

2. The resistance R of a carbon filament lamp was measured at various values of the voltage V and the following observations were made :

Voltage	V...	62	70	78	84	92
Resistance	R...	73	70.7	69.2	67.8	66.3

Assuming a law of the form $R = \frac{a}{V} + b$, find by graphical method the best value of a and b.

3. Verify if the values of x and y, related as shown in the following table, obey the law $y = a + b\sqrt{x}$. If so, find graphically the values of a and b.

x :	500	1,000	2,000	4,000	6,000
y :	0.20	0.33	0.38	0.45	0.51

4. The following values of T and l follow the law $T = al^n$. Test if this is so and find the best values of a and n.

T = 1.0	1.5	2.0	2.5
l = 25	56.2	100	1.56

5. Find the best value of a and b if $y = ax + b \log_{10} x$ is the curve which represents most closely the observed values given below :

x :	2	3	4	5	6
y :	3.74	5.99	7.47	8.92	9.86

6. Fit the curve $y = ae^{bx}$ to the following data :

x :	0	2	4
y :	5.1	10	31.1

(*Coimbatore, 1997*)

7. The following are the results of an experiment on friction of bearings. The speed being constant, corresponding values of the coefficient of friction and the temperature are shown in the table :

t :	120	110	100	90	80	70	60
μ :	0.0051	0.0059	0.0071	0.0085	0.00102	0.00124	0.00148

If μ and t are given by the law $\mu = ae^{bt}$, find the values of a and b by plotting the graph for μ and t.

24.4 PRINCIPLE OF LEAST SQUARES

The graphical method has the obvious drawback of being unable to give a unique curve of fit. *The principle of least squares, however, provides an elegant procedure for fitting a unique curve to a given data.*

Let the curve, $y = a + bx + cx^2 + \ldots + kx^{m-1}$ ...(1)

be fitted to the set of n data points $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$.

Now we have to determine the constants $a, b, c, \ldots k$ such that it represents the curve of best fit. In case $n = m$, on substituting the values (x_i, y_i) in (1), we get n equations from which a unique set of n constants can be found. But when $n > m$, we obtain n equations which are more than the m constants and hence cannot be solved for these constants. So we try to determine those values of $a, b, c, \ldots k$ which satisfy all the equations as nearly as possible and thus may give the best fit. In such cases, we apply the *principle of least squares*.

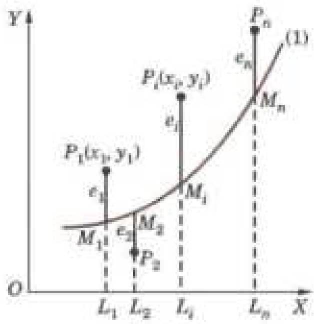


Fig. 24.4

At $x = x_i$, the *observed* (or *experimental*) *value* of the ordinate is $y_i = P_iL_i$ and the corresponding value on the fitting curve (1) is $a + bx_i + cx_i^2 + \ldots + kx_i^m = M_iL_i$ ($= \eta_i$, say) which is the *expected* (or *calculated*) *value* (Fig. 24.4). The difference of the observed and the expected values *i.e.* $y_i - \eta_i$ ($= e_i$) is called the *error* (or *residual*) at $x = x_i$. Clearly some of the errors $e_1, e_2, \ldots e_n$ will be positive and others negative. Thus to give equal weightage to each error, we square each of these and form their sum *i.e.* $E = e_1^2 + e_2^2 + \ldots e_n^2$.

The curve of best fit is that for which e's are as small as possible i.e., E, the sum of the squares of the errors is a minimum. This is known as the *principle of least squares* and was suggested by *Legendre** in 1806.

> Obs. The principle of least squares does not help us to determine the form of the appropriate curve which can fit a given data. It only determines the best possible values of the constants in the equation when the form of the curve is known before hand. The selection of the curve is a matter of experience and practical considerations.

24.5 (1) METHOD OF LEAST SQUARES

For clarity, suppose it is required to fit the curve

$$y = a + bx + cx^2 \quad \ldots(1)$$

to a given set of observations $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$. For any x_i, the observed value is y_i and the expected value is $\eta_i = a + bx_i + cx_i^2$ so that the error $e_i = y_i - \eta_i$.

$\therefore$ the sum of the squares of these errors is

$$E = e_1^2 + e_2^2 + \ldots + e_5^2$$

$$= [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 + \ldots + [y_5 - (a + bx_5 + cx_5^2)]^2 \quad \text{[See § 5.12 (3)]}$$

For E to be minimum, we have

$$\frac{\partial E}{\partial a} = 0 = 2[y_1 - (a + bx_1 + cx_1^2)] - 2[y_2 - (a + bx_2 + cx_2^2)] - \ldots - 2[y_5 - (a + bx_5 + cx_5^2)] \quad \ldots(2)$$

$$\frac{\partial E}{\partial b} = 0 = -2x_1[y_1 - (a + bx_1 + cx_1^2)] - 2x_2[y_2 - (a + bx_2 + cx_2^2)] - \ldots - 2x_5[y_5 - (a + bx_5 + cx_5^2)] \quad \ldots(3)$$

$$\frac{\partial E}{\partial c} = 0 = -2x_1^2[y_1 - (a + bx_1 + cx_1^2)] - 2x_2^2[y_2 - (a + bx_2 + cx_2^2)] - \ldots - 2x_5^2[y_5 - (a + bx_5 + cx_5^2)] \quad \ldots(4)$$

Equation (2) simplifies to

$$y_1 + y_2 + \ldots + y_5 = 5a + b(x_1 + x_2 + \ldots + x_5) + c(x_1^2 + x_2^2 + \ldots + x_5^2)$$

i.e., $$\Sigma y_i = 5a + b\Sigma x_i + c\,\Sigma x_i^2 \quad \ldots(5)$$

---

* See footnote on p. 311.

Equation (3) becomes

$$x_1y_1 + x_2y_2 + \ldots + x_5y_5 = a(x_1 + x_2 + \ldots + x_5) + b(x_1^2 + x_2^2 + \ldots + x_5^2) + c(x_1^3 + x_2^3 + \ldots + x_5^3)$$

i.e., $$\Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3 \quad \ldots(6)$$

Similarly (4) simplifies to $\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4$ ...(7)

The equations (5), (6) and (7) are known as *Normal equations* and can be solved as simultaneous equations in a, b, c. The values of these constants when substituted in (1) give the desired curve of best fit.

(2) Working procedure

(*a*) **To fit the straight line $y = a + bx$**

(*i*) Substitute the observed set of n values in this equation.

(*ii*) Form normal equations for each constant

i.e., $$\Sigma y = na + b\Sigma x, \quad \Sigma xy = a\Sigma x + b\Sigma x^2$$

[The normal equation for the unknown a is obtained by multiplying the equations by the coefficient of a and adding. The normal equation for b is obtained by multiplying the equations by the coefficient of b (*i.e.*, x) and adding.]

(*iii*) Solve these normal equations as simultaneous equations for a and b.

(*iv*) Substitute the values of a and b in $y = a + bx$, which is the required line of best fit.

(*b*) **To fit the parabola : $y = a + bx + cx^2$**

(*i*) Form the normal equations $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

and $$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

[The normal equation for c has been obtained by multiplying the equations by the coefficient of c (*i.e.*, x^2) and adding.]

(*ii*) Solve these as simultaneous equations for a, b, c.

(*iii*) Substitute the values of a, b, c in $y = a + bx + cx^2$, which is the required parabola of best fit.

(*c*) In general, the curve $y = a + bx + cx^2 + \ldots + kx^{m-1}$ can be fitted to a given data by writing m normal equations.

Example 24.4. *If P is the pull required to lift a load W by means of a pulley block, find a linear law of the form $P = mW + c$ connecting P and W, using the following data :*

$P = 12$	15	21	25
$W = 50$	70	100	120

where P and W are taken in kg-wt. Compute P when W = 150 kg. wt. (U.P.T.U., 2007 ; V.T.U., 2002)

Solution. The corresponding normal equations are

$$\left.\begin{aligned} \Sigma P &= 4c + m\Sigma W \\ \Sigma WP &= c\Sigma W + m\Sigma W^2 \end{aligned}\right\} \quad \ldots(i)$$

The values of ΣW etc. are calculated by means of the following table :

W	P	W^2	WP
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
Total = 340	73	31800	6750

$\therefore$ The equations (*i*) becomes $73 = 4c + 340m$ and $6750 = 340c + 31800m$

i.e., $$2c + 170m = 365 \quad \ldots(ii)$$

and $$34c + 3180m = 675 \quad \ldots(iii)$$

Multiplying (*ii*) by 17 and substracting from (*iii*), we get

$$m = 0.1879 \quad \therefore \text{ from } (ii), \quad c = 2.2785$$

Hence the line of best fit is

$$P = 2.2759 + 0.1879\ W$$

When $W = 150$ kg., $P = 2.2785 + 0.1879 \times 150 = 30.4635$ kg.

Obs. The calculations get simplified when the central values of x is zero. It is therefore, advisable to make the central value zero, if it be not so. This is illustrated by the next example.

Example 24.5. *Fit a second degree parabola to the following data :*

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

(P.T.U., 2006)

Solution. Let $u = x - 2$ and $v = y$ so that the parabola of fit $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \quad \text{...(i)}$$

The normal equations are

$$\Sigma v = 5A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 12.9 = 5A + 10C$$
$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 11.3 = 10B$$
$$\Sigma u^2 v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad 33.5 = 10A + 34C$$

Solving these as simultaneous equations, we get

$$A = 1.48, \qquad B = 1.13, \qquad C = 0.55.$$

$\therefore$ (i) becomes, $\quad v = 1.48 + 1.13u + 0.55u^2$

or $\quad y = 1.48 + 1.13\,(x - 2) + 0.55\,(x - 2)^2$

Hence $y = 1.42 - 1.07x + 0.55x^2$.

Example 24.6. *Fit a second degree parabola to the following data :*

$x = 1.0$	1.5	2.0	2.5	3.0	3.5	4.0
$y = 1.1$	1.3	1.6	2.0	2.7	3.4	4.1

(V.T.U., 2009 ; Bhopal, 2008)

Solution. We shift the origin to (2.5, 0) and take 0.5 as the new unit. This amounts to changing the variable x to X, by the relation $X = 2x - 5$.

Let the parabola of fit be $y = a + bX + cX^2$. The values of ΣX etc., are calculated as below :

x	X	y	Xy	X^2	X^2y	X^3	X^4
1.0	–3	1.1	–3.3	9	9.9	–27	81
1.5	–2	1.3	–2.6	4	5.2	–8	16
2.0	–1	1.6	–1.6	1	1.6	–1	1
2.5	0	2.0	0.0	0	0.0	0	0
3.0	1	2.7	2.7	1	2.7	1	1
3.5	2	3.4	6.8	4	13.6	8	16
4.0	3	4.1	12.3	9	36.9	27	81
Total	0	16.2	14.3	28	69.9	0	196

The normal equations are

$$7a + 28c = 16.2\ ; \qquad 28b = 14.3\ ; \qquad 28a + 196c = 69.9$$

Solving these as simultaneous equations, we get

$$a = 2.07,\ b = 0.511,\ c = 0.061$$

$\therefore \quad y = 2.07 + 0.511X + 0.061X^2$

Replacing X by $2x - 5$ in the above equation, we get

$$y = 2.07 + 0.511\,(2x - 5) + 0.061\,(2x - 5)^2$$

which simplifies to $y = 1.04 - 0.198x + 0.244x^2$. This is the required parabola of best fit.

Example 24.7. *Fit a second degree parabola to the following data :*

x	1989	1990	1991	1992	1993	1994	1995	1996	1997
y	352	356	357	358	360	361	361	360	359

(U.P.T.U., 2009)

Solution. Taking $u = x - 1993$ and $v = y - 357$, the equation $y = a + bx + cx^2$ becomes

$$v = A + Bu + Cu^2 \qquad ...(i)$$

x	$u = x - 1993$	y	$v = y - 357$	uv	u^2	u^2v	u^3	u^4
1989	−4	352	−5	20	16	−80	−64	256
1990	−3	360	−1	3	9	−9	−27	81
1991	−2	357	0	0	4	0	−8	16
1992	−1	358	1	−1	1	1	−1	1
1993	0	360	3	0	0	0	0	0
1994	1	361	4	4	1	4	1	1
1995	2	361	4	8	4	16	8	16
1996	3	360	3	9	9	27	27	81
1997	4	359	2	8	16	32	64	256
Total	$\Sigma u = 0$		$\Sigma v = 11$	$\Sigma uv = 51$	$\Sigma u^2 = 60$	$\Sigma u^2v = -9$	$\Sigma u^3 = 0$	$\Sigma u^4 = 708$

The normal equations are

$$\Sigma v = 9A + B\Sigma u + C\Sigma u^2 \quad \text{or} \quad 11 = 9A + 60C$$

$$\Sigma uv = A\Sigma u + B\Sigma u^2 + C\Sigma u^3 \quad \text{or} \quad 51 = 60B \quad \text{or} \quad B = \frac{17}{20}$$

$$\Sigma u^2v = A\Sigma u^2 + B\Sigma u^3 + C\Sigma u^4 \quad \text{or} \quad -9 = 60A + 708C$$

On solving these equations, we get $A = \dfrac{694}{231}, B = \dfrac{17}{20}, C = -\dfrac{247}{924}$

$\therefore$ (*i*) becomes
$$v = \frac{694}{231} + \frac{17}{20}u - \frac{247}{924}u^2$$

or
$$y - 357 = \frac{694}{231} + \frac{17}{20}(x - 1933) - \frac{247}{924}(x - 1933)^2$$

or
$$y = \frac{694}{231} - \frac{32861}{20} - \frac{247}{924}(1933)^2 + \frac{17}{20}x + \frac{247 \times 3866}{924}x - \frac{247}{924}x^2$$

or
$$y = 3 - 1643.05 - 998823.36 + 357 + 0.85x + 1033.44x - 0.267x^2$$

Hence
$$y = -1000106.41 + 1034.29x - 0.267x^2.$$

PROBLEMS 24.2

1. By the method of least squares, find the straight line that best fits the following data :

x :	1	2	3	4	5
y :	14	27	40	55	68

(U.P.T.U., 2008)

2. Fit a straight line to the following data :

Year x :	1961	1971	1981	1991	2001
Production y :	8	10	12	10	16

(in thousand tons)

and find the expected production in 2006.

3. A simply supported beam carries a concentrated load P (lb) at its mid-point. Corresponding to various values of P, the maximum deflection Y (in) is measured. The data are given below :

P:	100	120	140	160	180	200
Y:	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $Y = a + bP$.

4. The results of measurement of electric resistance R of a copper bar at various temperatures t°C are listed below :

t :	19	25	30	36	40	45	50
R :	76	77	79	80	82	83	85

Find a relation $R = a + bt$ where a and b are constants to be determined by you.

5. Find the best possible curve of the form $y = a + bx$, using method of least squares for the data :

x :	1	3	4	6	8	9	11	14
y :	1	2	4	4	5	7	8	9

(*V.T.U., 2011*)

6. Fit a straight line to the following data

(*a*)

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	5

(*Bhopal, 2008*)

(*b*)

x :	6	7	7	8	8	8	9	9	10
y :	5	5	4	5	4	3	4	3	3

(*J.N.T.U., 2008*)

7. Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations :

x :	−3	−2	−1	0	1	2	3
y :	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(*V.T.U., 2006 ; J.N.T.U., 2000 S*)

8. Fit a parabola $y = a + bx + cx^2$ to the following data :

x :	2	4	6	8	10
y :	3.07	12.85	31.47	57.38	91.29

(*V.T.U., 2003 S*)

9. Fit a second degree parabola to the following data :

x :	1	2	3	4	5	6	7	8	9	10
y :	124	129	140	159	228	289	315	302	263	210

(*U.P.T.U., 2009*)

10. The following table gives the results of the measurements of train resistances ; V is the velocity in miles per hour, R is the resistance in pounds per ton :

V :	20	40	60	80	100	120
R :	5.5	9.1	14.9	22.8	33.3	46.0

If R is related to V by the relation $R = a + bV + cV^2$, find a, b, and c. (*U.P.T.U., 2002*)

11. The velocity V of a liquid is known to vary with temperature according to a quadratic law $V = a + bT + cT^2$. Find the best values of a, b and c for the following table :

T :	1	2	3	4	5	6	7
V :	2.31	2.01	3.80	1.66	1.55	1.47	1.41

(*U.P.T.U., MCA, 2010*)

24.6 FITTING OF OTHER CURVES

(1) $y = ax^b$

Taking logarithms, $\log_{10} y = \log_{10} a + b \log_{10} x$

i.e., $Y = A + bX$ where $X = \log_{10} x$, $Y = \log_{10} y$ and $A = \log_{10} a$. ...(*i*)

∴ The normal equations for (*i*) are : $\Sigma Y = nA + b\Sigma X$, $\Sigma XY = A\Sigma X + b\Sigma X^2$

from which A and b can be determined. Then a can be calculated from $A = \log_{10} a$.

(2) $y = ae^{bx}$ (*Exponential curve*)

Taking logarithms, $\log_{10} y = \log_{10} a + bx \log_{10} e$

i.e., $Y = A + Bx$ where $Y = \log_{10} y$, $A = \log_{10} a$ and $B = b \log_{10} e$

Here the normal equations are : $\Sigma Y = nA + B\Sigma x$, $\Sigma xY = A\Sigma x + B\Sigma x^2$

from which A, B can be found and consequently a, b can be calculated.

(3) $xy^a = b$ (*or $pv^\gamma = k$*) (*Gas equation*)

Taking logarithms, $\log_{10} x + a \log_{10} y = \log_{10} b$ or $\log_{10} y = \dfrac{1}{a} \log_{10} b - \dfrac{1}{a} \log_{10} x$.

This is of the form $Y = A + BX$

where $X = \log_{10} x$, $Y = \log_{10} y$, $A = \frac{1}{a} \log_{10} b$, $B = -\frac{1}{a}$.

Here also the problem reduces to finding a straight line of best fit through the given data.

Example 24.8. *Find the least squares fit of the form $y = a_0 + a_1x^2$ to the following data :*

x :	-1	0	1	2
y :	2	5	3	0

(U.P.T.U., 2008)

Solution. Putting $x^2 = X$, we have $y = a_0 + a_1X$

$\therefore$ the normal equations are : $\Sigma y = 4a + a_1\Sigma X$; $\Sigma Y = a_0\Sigma X + a_1\Sigma X^2$.

The values of ΣX, ΣX^2 etc. are calculated below :

x	y	X	X^2	XY
-1	2	1	1	2
0	5	0	0	0
1	3	1	1	3
2	0	4	16	0
	$\Sigma y = 10$	$\Sigma X = 10$	$\Sigma X^2 = 18$	$\Sigma XY = 5$

$\therefore$ the normal equations become $\quad 10 = 400 + 6a_1$; $5 = 600 + 18a_1$

Solving these equations we get, $\quad a_0 = 4.167$, $a_1 = -1.111$.

Hence the curve of best fit is $\quad y = 4.167 - 1.111X$ *i.e.*, $y = 4.167 - 1.111x^2$.

Example 24.9. *An experiment gave the following values :*

v (ft/min)	:	350	400	500	600
t (min)	:	61	26	7	26

It is known that v and t are connected by the relation $v = at^b$. Find the best possible values of a and b.

Solution. We have $\quad \log_{10} v = \log_b a + b \log_{10} t$

or $\quad y = A + 6X$, where $X = \log_{10} t$, $y = \log_{10} v$, $A = \log_{10} a$

$\therefore$ the normal equations are

$$\Sigma Y = 4A + b\Sigma X \quad \text{...(i)}$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \quad \text{...(ii)}$$

Now ΣX etc. are calculated as in the following table :

v	t	$X = \log_{10} t$	$y = \log_{10} v$	XY	X^2
350	61	1.7853	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	2.6	0.4150	2.7782	1.153	0.172
Total		4.4604	10.6234	11.658	6.075

$\therefore$ Equations (*i*) and (*ii*) become

$$4A + 4.46b = 10.623 \; ; \; 4.46A + 6.075b = 11.658$$

Solving these, $A = 2.845$, $b = -0.1697$

$\therefore$ $a = \text{antilog } A = \text{antilog } 2.845 = 699.8$.

Example 24.10. *Predict the mean radiation dose at an altitude of 3000 feet by fitting an exponential curve to the given data :*

Altitude (x)	:	50	450	780	1200	4400	4800	5300
Dose of radiation (y)	:	28	30	32	36	51	58	69

(S.V.T.U., 2007 ; J.N.T.U., 2003)

Solution. Let $\quad y = ab^x$ be the exponential curve.

Then $\quad \log_{10} y = \log_{10} a + x \log_{10} b$

or $\quad Y = A + Bx$ where $Y = \log_{10} y, A = \log_{10} a, B = \log_{10} b$

$\therefore$ the normal equations are

$$\Sigma Y = 7A + B\, \Sigma x \quad ...(i)$$

$$\Sigma x\, Y = A\, \Sigma x + B\, \Sigma x^2 \quad ...(ii)$$

Now Σx etc. are calculated as follows :

x	y	$Y = \log_{10} y$	xY	x^2
50	28	1.447158	72.3579	2500
450	30	1.477121	664.7044	202500
780	32	1.505150	1174.0170	608400
1200	36	1.556303	1867.5636	1440000
4400	51	1.707570	7513.3080	19360000
4800	58	1.763428	8464.4544	23040000
5300	69	1.838849	9745.8997	28090000
$\Sigma = 16980$		11.295579	29502.305	72743400

$\therefore$ equations (i) and (ii) become

$$11.295579 = 7A + 16980B$$

$$29502.305 = 16980A + 72743400B$$

Solving these equations, we get $A = 1.4521015$, $B = 0.0000666289$

$\therefore \quad \log_{10} y = Y = 1.4521015 + 0.0000666289x$

Hence $\quad y$ (*at* $x = 3000$) $= 44.874$ *i.e.* 44.9 approx.

Example 24.11. *The pressure and volume of a gas are related by the equation* $pv^\gamma = k$, γ *and* k *being constants. Fit this equation to the following set of observations :*

p (kg/cm²)	:	0.5	1.0	1.5	2.0	2.5	3.0
v (litres)	:	1.62	1.00	0.75	0.62	0.52	0.46

(V.T.U., 2011)

Solution. We have $\log_{10} p + \gamma \log_{10} v = \log_{10} k$

or $$\log_{10} v = \frac{1}{\gamma} \log_{10} k - \frac{1}{\gamma} \log_{10} p \quad \text{or} \quad Y = A + BX$$

where $X = \log_{10} p, Y = \log_{10} v, A = \dfrac{1}{\gamma} \log_{10} k, B = -\dfrac{1}{\gamma}$.

$\therefore$ the normal equations are

$$\Sigma Y = 6A + B\Sigma X \quad ...(i)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad ...(ii)$$

Now ΣX etc. are calculated as follows :

p	v	$X = \log_{10} p$	$Y = \log_{10} v$	XY	X^2
.5	1.62	−0.3010	0.2095	−0.0630	0.0906
1.0	1.00	0.0000	0.0000	−0.0000	0.0000
1.5	0.75	0.1761	−0.1249	−0.0220	0.0310
2.0	0.62	0.3010	−0.2076	−0.0625	0.0906
2.5	0.52	0.3979	−0.2840	−0.1130	0.1583
3.0	0.46	0.4771	−0.3372	−0.1609	0.2276
Total		1.0511	−0.7442	−0.4214	0.5981

$\therefore$ equations (i) and (ii) become

$$6A + 1.0511B = -0.7442$$

$$1.0511A + 0.5981B = -0.4214$$

Solving these, we get $A = 0.0132$, $B = -0.7836$.

$\therefore \quad \gamma = -1/B = 1.1276$ and $k = \text{antilog}(A\gamma) = \text{antilog}(0.0168) = 1.039$.

Hence the equation of best fit is $pv^{1.276} = 1.039$.

PROBLEMS 24.3

1. If V (km/hr) and R (kg/ton) are related by a relation of the type $R = a + bV^2$, find by the method of least squares a and b with the help of the following table :

V:	10	20	30	40	50
R:	8	10	15	21	30

(*Indore, 2008*)

2. Using the method of least squares fit the curve $y = ax + bx^2$ to following observations :

x:	1	2	3	4	5
y:	1.8	5.1	8.9	14.1	19.8

3. Fit the curve $y = ax + b/x$ to the following data :

x:	1	2	3	4	5	6	7	8
y:	5.4	6.3	8.2	10.3	12.6	14.9	17.3	19.5

(*U.P.T.U., 2010*)

4. Estimate y at $x = 2.25$ by fitting the *indifference curve* of the form $xy = Ax + B$ to the following data :

x:	1	2	3	4
y:	3	1.5	6	7.5

(*J.N.T.U., 2003*)

5. Find the least square curve $y = ax + b/x$ for the following data :

x:	1	2	3	4
y:	-1.5	0.99	3.88	7.66

(*Madras, 2003*)

6. Predict y at $x = 3.75$, by fitting a *power curve* $y = ax^b$ to the given data :

x:	1	2	3	4	5	6
y:	298	4.26	5.21	6.10	6.80	7.50

(*J.N.T.U., 2003*)

7. Fit the curve of the form $y = ae^{bx}$ to the following data :

x:	77	100	185	239	285
y:	2.4	3.4	7.0	11.1	19.6

(*V.T.U., 2011 S ; J.N.T.U., 2006*)

8. Obtain the least squares fit of the form $f(t) = ae^{-3t} + be^{-2t}$ for the data :

x :	0.1	0.2	0.3	0.4
$f(t)$:	0.76	0.58	0.44	0.35

(*U.P.T.U., 2008*)

9. The voltage v across a capacitor at time t seconds is given by the following table :

t:	0	2	4	6	8
v:	150	63	28	12	5.6

Use the method of least squares to fit a curve of the form $v = ae^{kt}$ to this data.

10. Using method of least squares, fit a relation of the form $y = ab^x$ to the following data :

x:	2	3	4	5	6
y:	144	172.8	207.4	248.8	298.5

(*Tiruchirapalli, 2001*)

24.7 METHOD OF GROUP AVERAGES

Let the straight line, $\quad y = a + bx \quad$ ...(1)

fit the set of n observations

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ quite closely. (Fig. 24.5)

When $x = x_1$, the *observed* (*or experimental*) *value of* $y = y_1 = L_1P_1$ and from (1),

$$y = a + bx_1 = L_1M_1,$$

which is known as the *expected* (or *calculated*) *value* of y at L_1.

Then e_1 = observed value at L_1 – expected value at L_1

$$= y_1 - (a + bx_1) = M_1P_1,$$

which is called the *error* (or *residual*) at x_1. Similarly the errors for the other observations are

$$e_2 = y_2 - (a + bx_2) = M_2P_2$$

....

$$e_n = y_n - (a + bx_n) = M_nP_n$$

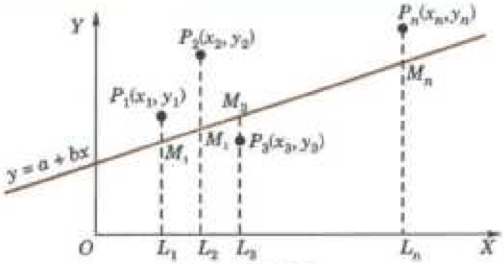


Fig. 24.5

Some of these errors may be positive and others negative.

The method of group averages is based on the assumption that the sum of the residuals is zero. To find the constants a and b is (1), we require two equations. As such we divide the data into two groups : the first containing k observations

$$(x_1, y_1), (x_2, y_2) \ldots (x_k, y_k) ;$$

and the second group having the remaining $n - k$ observations

$$(x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}), \ldots, (x_n, y_n).$$

Assuming that the sum of the errors in each group is zero, we get

$$\{y_1 - (a + bx_1)\} + \{y_2 - (a + bx_2)\} + \ldots \{y_k - (a + bx_k)\} = 0$$
$$\{y_{k+1} - (a + bx_{k+1})\} + \{y_{k+2} - (a + bx_{k+2})\} + \ldots + \{y_n - (a + bx_n)\} = 0$$

On simplification, we obtain

$$\frac{y_1 + y_2 + \ldots + y_k}{k} = a + b\,\frac{x_1 + x_2 + \ldots + x_k}{k} \quad \ldots(2)$$

$$\frac{y_{k+1} + y_{k+2} + \ldots + y_n}{n-k} = a + b\,\frac{x_{k+1} + x_{k+2} + \ldots + x_n}{n-k} \quad \ldots(3)$$

In (2), $\frac{1}{k}(x_1 + x_2 + \ldots + x_k)$ and $\frac{1}{k}(y_1 + y_2 + \ldots + y_k)$ are simply the average values of x's and y's of the first group. Hence the equations (2) and (3) are obtained from (1) by replacing x and y by their respective averages of the two groups. Solving (2) and (3), we get a and b.

Obs. The main drawback of this method is that a different grouping of the observations will give different values of a and b. In practice, we divide the data in such a way that each group contains almost an equal number of observations.

Example 24.12. *The latent heat of vaporisation of steam r, is given in the following table at different temperatures t :*

t :	*40*	*50*	*60*	*70*	*80*	*90*	*100*	*110*
r :	*1069.1*	*1063.6*	*1058.2*	*1052.7*	*1049.3*	*1041.8*	*1036.3*	*1030.8*

For this range of temperature, a relation of the form r = a + bt is known to fit the data. Find the values of a and b by the method of group averages. (Madras, 2003)

Solution. Let us divide the data into two groups each containing four readings. Then we have

t	r	t	r
40	1069.1	80	1049.3
50	1063.6	90	1041.8
60	1058.2	100	1036.3
70	1052.7	110	1030.8
$\Sigma t = 220$	$\Sigma r = 4243.6$	$\Sigma t = 380$	$\Sigma r = 4158.2$

Substituting the averages of t's and r's of the two groups in the given relation, we get

$$\frac{4243.6}{4} = a + b\,\frac{220}{4} \quad i.e., \; 1060.9 = a + 55b \qquad ...(i)$$

$$\frac{4158.2}{4} = a + b\,\frac{380}{4} \quad i.e., \; 1039.55 = a + 95b \qquad ...(ii)$$

Solving (*i*) and (*ii*), we obtain

$$a = 1090.26,\; b = -0.534.$$

24.8 FITTING A PARABOLA

We have applied the method of averages to *linear law* involving two constants only. To fit the parabola

$$y = a + bx + cx^2 \qquad ...(1)$$

which contains three constants, to a set of observations, we proceed as follows :

Let (x_1, y_1) be a point on (1) satisfying the given data so that

$$y_1 = a + bx_1 + cx_1^{\,2}$$

Then
$$y - y_1 = b\,(x - x_1) + c\,(x^2 - x_1^{\,2})$$

or
$$\frac{y - y_1}{x - x_1} = b + c\,(x + x_1)$$

Putting $x + x_1 = X$ and $(y - y_1)/(x - x_1) = Y$, it takes the linear form

$$Y = b + cX.$$

Now b and c can be found as before.

Example 24.13. *The corresponding values of x and y are given by the following table :*

x :	87.5	84.0	77.8	63.7	46.7	36.9
y :	292	283	270	235	197	181

Solution. Taking $x = 84$, $y = 283$ as a particular point on $y = a + bx + cx^2$,

we get
$$283 = a + b\,(84) + c\,(84)^2 \qquad ...(i)$$

$\therefore$
$$y - 283 = b\,(x - 84) + c\,[x^2 - (84)^2]$$

or
$$\frac{y - 283}{x - 84} = b + c\,(x + 84)$$

i.e.,
$$Y = b + cX \qquad ...(ii)$$

where $X = x + 84$, $Y = (y - 283)/(x - 84)$.

Now we have the following table of values :

x	y	$X = x + 84$	$Y = (y - 283)/(x - 84)$
87.5	292	171.5	2.571
84.0	283	—	—
77.8	270	161.8	2.097
		$\Sigma X = 333.3$	$\Sigma Y = 4.668$
63.7	235	147.7	2.364
46.7	197	130.7	2.306
36.9	181	120.9	2.166
		$\Sigma X = 399.3$	$\Sigma Y = 6.836$

Substituting the averages of X and Y in (*ii*), we get

$$\frac{4.668}{2} = b + c\,\frac{333.3}{2} \quad i.e., \; 2.33 = b + 166.65\,c \qquad ...(iii)$$

$$\frac{6.836}{3} = b + c\,\frac{399.3}{3} \quad i.e., \; 2.28 = b + 131.1\,c \qquad ...(iv)$$

(iv)–(iii) gives $c = 0.0014$

and (iii) gives $b = 2.0967$ i.e., 2.1 nearly

From (i), we get $a = 96.9988$ i.e., 97 nearly.

Hence the parabola of fit is

$$y = 97 + 2.1x + .0014x^2.$$

Example 24.14. *The train resistance R (lbs/ton) is measured for the following values of its velocity V (km/hr):*

V:	20	40	60	80	100
R:	5	9	14	25	36

If R is related to V by the formula $R = a + bV^n$, find a, b, and n.

Solution. To find a, we take the following three values of v which are in G.P. :

$$v_1 = 20, \quad v_2 = 40, \quad v_3 = 80$$

Then $R_1 = 5, \quad R_2 = 9, \quad R_3 = 25$

$\therefore \quad (R_1 - a)(R_3 - a) = (R_2 - a)^2$

whence $$a = \frac{R_1 R_3 - R_2^2}{R_1 + R_3 - 2R_2} = 3.67$$

Thus $R - 3.67 = bV^n$ or $\log_{10}(R - 3.67) = \log_{10} b + n \log_{10} V$

i.e., $$Y = k + nX \qquad \ldots(i)$$

where $X = \log_{10} V$, $Y = \log_{10}(R - 3.67)$, $k = \log_{10} b$.

Now we have the following table of values :

V	R	$X = \log_{10} V$	$Y = \log_{10}(R - 3.67)$
20	5	1.3010	0.1238
40	9	1.6021	0.7267
60	14	1.7782	1.0141
		$\Sigma X = 4.6813$	$\Sigma Y = 1.8646$
80	25	1.9031	1.3290
100	36	2.0000	1.5096
		$\Sigma X = 3.9031$	$\Sigma X = 2.8396$

Substituting the averages of X's and Y's in (i), we obtain

$$\frac{1.8646}{2} = k + n\,\frac{4.6813}{2} \qquad \text{i.e., } 0.6215 = k + 1.5604\,n \qquad \ldots(ii)$$

$$\frac{2.8386}{2} = k + n\,\frac{3.9031}{2} \qquad \text{i.e., } 1.4193 = k + 1.9516\,n \qquad \ldots(iii)$$

Solving (ii) and (iii), we get $n = 2.04$, $k = -2.56$ approx.

$$b = \text{antilog } k = \text{antilog }(-2.56) = 0.0028.$$

PROBLEMS 24.4

1. Fit a straight line of the form $y = a + bx$ to the following data by the method of group averages :

x:	0	5	10	15	20	25
y:	12	15	17	22	24	30

(Tiruchirapalli, 2001)

2. The weights of a calf taken at weekly intervals are given below :

Age :	1	2	3	4	5	6	7	8	9	10
Weight :	52.5	58.7	65.0	70.2	75.4	81.1	87.2	95.5	102.2	108.4

Find a straight line of best fit.

3. Using the method of averages, fit a parabola $y = ax^2 + bx + c$ to the following data :

x :	20	40	60	80	100	120
y :	5.5	9.1	14.9	22.8	33.3	46.0

4. While testing a centrifugal pump, the following data is obtained. It is assumed to fit the equation $y = a + bx + cx^2$, where x is the discharge in litre/sec and y, head in metres of water. Find the values of the constants a, b, c by the method of group averages.

x :	2	2.5	3	3.5	4	4.5	5	5.5	6
y :	18	17.8	17.5	17	15.8	14.8	13.3	11.7	9

5. By the method of averages, fit a curve of the form $y = ae^{bx}$ to the following data :

x :	5	15	20	30	35	40
y :	10	14	25	40	50	62

(*Madras, 2002*)

24.9 METHOD OF MOMENTS

Let $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$ be the set of n observations such that

$$x_2 - x_1 = x_3 - x_2 = \ldots = x_n - x_{n-1} = h \text{ (say)}$$

We define the *moments of the observed values* of y as follows :

m_1, the 1st moment $= h\Sigma y$

m_2, the 2nd moment $= h\Sigma xy$

m_3, the 3rd moment $= h\Sigma x^2 y$ and so on.

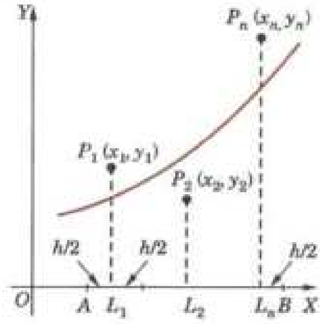


Fig. 24.6

Let the curve fitting the given data be $y = f(x)$. Then the *moments of the calculated values* of y are

μ_1, the 1st moment $= \int y dx$

μ_2, the 2nd moment $= \int xy\, dx$

μ_3, the 3rd moment $= \int x^2 y\, dx$ and so on.

This method is based on the assumption that the moment of the observed values of y are respectively equal to the moments of the calculated values of y i.e., $m_1 = \mu_1, m_2 = \mu_2, m_3 = \mu_3$ *etc. These equations (known as observation equations)* are used to determine the constants in $f(x)$.

m's are calculated from the tabulated values of x and y while μ's are computed as follows :

In Fig. 24.6, y_1 the ordinate of P_1 $(x = x_1)$, can be taken as the value of y at the mid-point of the interval $(x_1 - h/2, x_1 + h/2)$. Similarly, y_n, the ordinate of P_n $(x = x_n)$, can be taken as the value of y at the mid-point of the interval $(x_n - h/2, x_n + h/2)$. If A and B be the points such that

$$OA = x_1 - h/2 \text{ and } OB = x_n + h/2,$$

then
$$\mu_1 = \int y\, dx = \int_{x_1 - h/2}^{x_n + h/2} f(x)\, dx$$

$$\mu_2 = \int_{x_1 - h/2}^{x_n + h/2} x f(x)\, dx$$

and
$$\mu_3 = \int_{x_1 - h/2}^{x_n + h/2} x^2 f(x)\, dx.$$

Example 24.15. *Fit a straight line $y = a + bx$ to the following data by the method of moments :*

x :	1	2	3	4
y :	16	19	23	26

(*Madras, 2001 S*)

Solution. Since only two constants a and b are to be found, it is sufficient to calculate the first two moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(16 + 19 + 23 + 26) = 84$$

$$m_2 = h\Sigma xy = 1\,(1 \times 16 + 2 \times 19 + 3 \times 22 + 4 \times 26) = 227$$

To compute the moments of calculated values of $y = a + bx$, the limits of integration will be $1 - h/2$ and $4 + h/2$ *i.e.*, 0.5 to 4.5

$$\therefore \quad \mu_1 = 2\int_{0.5}^{4.5}(a + bx)\,dx = \left| ax + b\frac{x^2}{2} \right|_{0.5}^{4.5} = 4a + 10b$$

$$\mu_2 = \int_{0.5}^{4.5} x(a + bx)\,dx = 10a + \frac{91}{3}b.$$

Thus, the observation equations $m_r = \eta_r\ (r = 1, 2)$ are $4a + 10b = 84$; $10a + \frac{91}{3}b = 227$

Solving these, $a = 13.02$ and $b = 3.19$.

Hence the required equation is $y = 13.02 + 3.19x$.

Example 24.16. *Given the following data :*

x :	0	1	2	3	4
y :	1	5	10	22	38

find the parabola of best fit by the method of moments.

Solution. Let the parabola of best fit be $y = a + bx + cx^2$ ...(*i*)

Since three constants are to be found, we calculate the first three moments in each case. Here $h = 1$.

$$m_1 = h\Sigma y = 1\,(1 + 5 + 10 + 22 + 38) = 76$$
$$m_2 = h\Sigma xy = 1\,(0 + 5 + 20 + 66 + 152) = 243$$
$$m_3 = h\Sigma x^2y = 1\,(0 + 5 + 40 + 198 + 608) = 851$$

For computing the moments of calculated values of (*i*), the limits of integration will be $0 - h/2$ and $4 + h/2$ *i.e.*, -0.5 and 4.5.

$$\therefore \quad \mu_1 = \int_{-0.5}^{4.5}(a + bx + cx^2)dx = 5a + 10b + 30.4c$$

$$\mu_2 = \int_{-0.5}^{4.5} x\,(a + bx + cx^2)\,dx = 10a + 30.4b + 102.5c$$

$$\mu_3 = \int_{-0.5}^{4.5} x^2\,(a + bx + cx^2)\,dx = 30.4a + 102.5b + 369.1c$$

Thus the observation equations $m_r = \mu_r\ (r = 1, 2, 3)$ are

$$5a + 10b + 30.4c = 76\ ;\ 10a + 30.4 + 102.5c = 243\ ;\ 30.4a + 102.5b + 369.1c = 851$$

Solving these equations, we get $a = 0.4$, $b = 3.15$, $c = 1.4$.

Hence the parabola of best fit is $y = 0.4 + 3.15x + 1.4x^2$.

PROBLEMS 24.5

1. Use the method of moments to fit the straight line $y = a + bx$ to the data :

x :	1	2	3	4
y :	0.17	0.18	0.23	0.32

2. Fit a straight line to the following data, using the method of moments :

x :	1	3	5	7	9
y :	1.5	2.8	4.0	4.7	6.0

(*Madras, 2001*)

3. Fit a parabola of the form $y = a + bx + cx^2$ to the data :

x :	1	2	3	4
y :	1.7	1.8	2.3	3.2

by the method of moments.

4. By using the method of moments, fit a parabola to the following data :

x :	1	2	3	4
y :	0.30	0.64	1.32	5.40

(*Madras, 2000 S*)

24.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 24.6

Fill up the blanks or choose the correct answer in the following problems :

1. The law $y = ax^2 + bx$ converted to linear form is ....
2. The gas equation $pv^{\gamma} = k$ can be reduced to $y = a + bx$ where a = ... and b = ...
3. The principle of '*least squares*' states that ....
4. $y = ax^b + c$ in linear form is ....
5. To fit the straight line $y = mx + c$ to n observations, the normal equations are
   (*i*) $\Sigma y = n\,\Sigma x + \Sigma cm$, $\Sigma xy = c\,\Sigma x^2 + c\,\Sigma n$.
   (*ii*) $\Sigma y = m\,\Sigma x + nc$, $\Sigma xy = m\,\Sigma x^2 + c\,\Sigma x$.
   (*iii*) $\Sigma y = c\,\Sigma x + m\,\Sigma n$, $\Sigma xy = c\,\Sigma x^2 + m\,\Sigma x$.
6. To fit $y = ab^x$ by least square method, normal equations are ....
7. The observation equations for fitting a straight line by *method of moments* are ....
8. The *method of group averages* is based on the assumption that the sum of the residuals is ....
9. $y = ax^2 + b\log_{10} x$ reduced to linear law takes the form ....
10. Given $\begin{bmatrix} x: & 0 & 1 & 2 \\ y: & 0 & 1.1 & 2.1 \end{bmatrix}$ then the straight line of best fit is ....
11. The *method of moments* is based on the assumption that ....
12. In $y = a + bx$, $\Sigma x = 50$, $\Sigma y = 80$, $\Sigma xy = 1030$, $\Sigma x^2 = 750$ and $n = 10$, then a = ..., b = ...
13. $y = x/(ax + b)$ in linear form is ....
14. If $y = a + bx + cx^2$ and

x:	0	1	2	3	4
y:	1	1.8	1.3	2.5	7.3

then the first normal equation is :
(α) $15 = 5a + 10b + 29c$, (β) $15 = 5a + 10b + 31c$
(γ) $12.9 = 5a + 10b + 30c$ (δ) $34 = 5a + 10b + 27c$.

15. If $y = 2x + 5$ is the best fit for 8 pairs of values (x, y) by the method of least squares and $\Sigma y = 120$, then ΣX =
(*a*) 35 (*b*) 40 (*c*) 45 (*d*) 30.

CHAPTER 25

Statistical Methods

25.1 INTRODUCTION

Statistics deals with the methods for collection, classification and analysis of numerial data for drawing valid conclusions and making reasonable decisions. It has meaningful applications in production engineering, in the analysis of experimental data, etc. The importance of statistical methods in engineering is on the increase. As such we shall now introduce the student to this interesting field.

25.2 (1) COLLECTION OF DATA

The collection of data constitutes the starting point of any statistical investigation. Data may be collected for each and every unit of the whole lot (*population*), for it would ensure greater accuracy. But complete enumeration is prohibitively expensive and time consuming. As such out of a very large number of items, a few of them (a *sample*) are selected and conclusions drawn on the basis of this sample are taken to hold for the population.

(2) **Classification of data.** The data collected in the course of an inquiry is not in an easily assimilable form. As such, its proper classification is necessary for making intelligent inferences. The classification is done by dividing the raw data into a convenient number of groups according to the values of the variable and finding the frequency of the variable in each group.

Let us, for example, consider the raw data relating to marks obtained in Mechanics by a group of 64 students :

79	88	75	60	93	71	59	85
84	75	82	68	90	62	88	76
65	75	87	74	62	95	78	63
78	82	75	91	77	69	74	68
67	73	81	72	63	76	75	85
80	73	57	88	78	62	76	53
62	67	97	78	85	76	65	71
78	89	61	75	95	60	79	83

This data can conveniently be grouped and shown in a tabular form as follows :

Class	*Frequency*	*Cumulative frequency*
50—54	1	1
55—59	2	3
60—64	9	12
65—69	7	19
70—74	8	27
75—79	17	44
80—84	6	50
85—89	8	58
90—94	3	61
95—99	3	64
	Total = 64	

It would be seen from the above table that there is one student getting marks between 50—54, two students getting marks between 55—59, nine students getting marks between 60—64 and so on. Thus the 64 figure have been put into only 10 groups, called the **classes**. The width of the class is called the **class interval** and the number in that interval is called the **frequency.** The mid-point or the mid-value of the class is called the **class mark.** The above table showing the classes and the corresponding frequencies is called a *frequency table. Thus a set of raw data summarised by distributing it into a number of classes alongwith their frequencies is known as a* **frequency distribution**.

While forming a frequency distribution, the number of classes should not ordinarily exceed 20, and should not, in general, be less than 10. As far as possible, the class intervals should be of equal width.

(3) **Cumulative frequency.** In some investigations, we require the number of items less than a certain value. We add up the frequencies of the classes upto that value and call this number as the *cumulative frequency*. In the above table, the third column shows the cumulative frequencies, *i.e.*, the number of students, getting less than 54 marks, less than 59 marks and so on.

25.3 GRAPHICAL REPRESENTATION

A convenient way of representing a sample frequency distribution is by means of graphs. It gives to the eyes the general run of the observations and at the same time makes the raw data readily intelligible. We give below the important types of graphs in use :

(1) **Histogram.** A histogram is drawn by erecting rectangles over the class intervals, such that the areas of the rectangles are proportional to the class frequencies. If the class intervals are of equal size, the height of the rectangles will be proportional to the class frequencies themselves (Fig. 25.1).

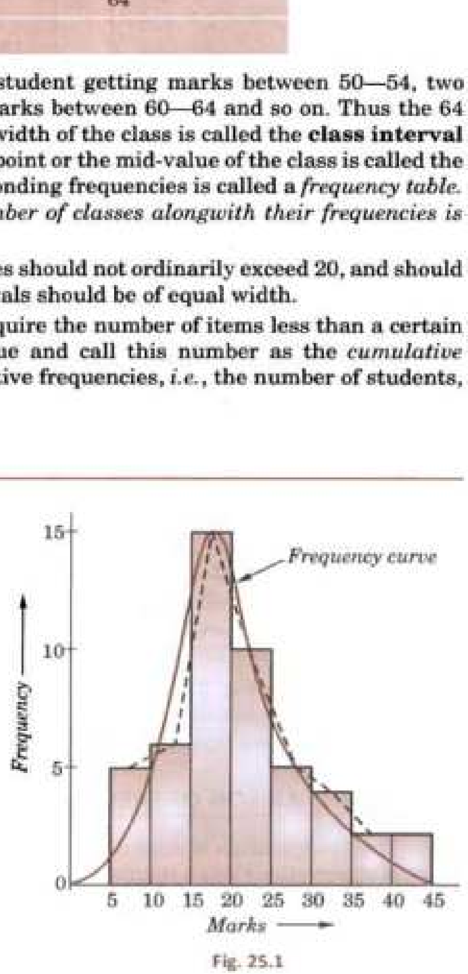


Fig. 25.1

(2) **Frequency polygon.** A frequency polygon for an ungrouped data can be obtained by joining points plotted with the variable values as the abscissae and the frequencies as the ordinates. For a grouped distribution, the abscissae of the points will be the mid-values of the class intervals. In case the intervals are equal, the frequency polygon can be obtained by joining the middle points of the upper sides of the rectangles of the histogram by straight lines (shown by dotted lines in Fig. 25.1). If the class intervals become very very small, the frequency polygon takes the form of a smooth curve called the *frequency curve*.

(3) **Cumulative frequency curve-Ogive.** Very often, it is desired to show in a diagrammatic form, not the relative frequencies in the various intervals, but the cumulative frequencies above or below a given value. For example, we may wish to read off from a diagram the number or proportions of people whose income is not less than any given amount, or proportion of people whose height does not exceed any stated value. Diagrams of

this type are known as *cumulative frequency curves* or *ogives*. These are of two kinds '*more than*' or '*less than*' and typically they look somewhat like a long drawn S (Fig. 25.2).

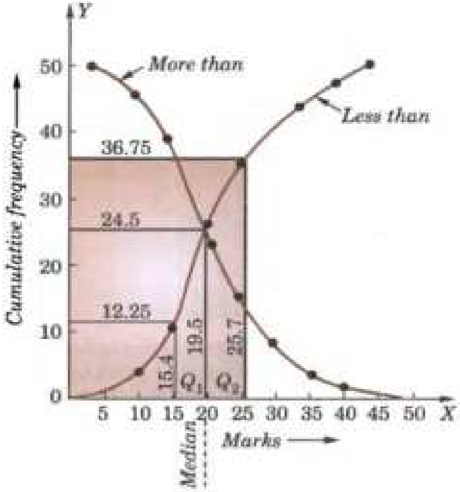


Fig. 25.2

Example 25.1. *Draw the histogram, frequency polygon, frequency curve and the ogive 'less than' and 'more than' from the following distribution of marks obtained by 49 students :*

Class (Marks group)	*Frequency (No. of students)*	*Cumulative frequency (Less than)*	*(More than)*
5—10	5	5	49
10—15	6	11	44
15—20	15	26	38
20—25	10	36	23
25—30	5	41	13
30—35	4	45	8
35—40	2	47	4
40—45	2	49	2

Solution. In Fig. 25.1, the rectangles show the *histogram*; the dotted polygon represents the *frequency polygon* and the smooth curve is the *frequency curve*.

The *ogives* 'less than' and 'more than' are shown in Fig. 25.2.

25.4 COMPARISON OF FREQUENCY DISTRIBUTIONS

The condensation of data in the form of a frequency distribution is very useful as far as it brings a long series of observations into a compact form. But in practice, we are generally interested in comparing two or more series. The inherent inability of the human mind to grasp in its entirety even the data in the form of a frequency distribution compels us to seek for certain constants which could concisely give an insight into the important characteristics of the series. The chief constants which summarise the fundamental characteristics of the frequency distributions are (*i*) *Measures of central tendency*, (*ii*) *Measures of dispersion and* (*ii*) *Measures of skewness*.

25.5 MEASURES OF CENTRAL TENDENCY

A frequency distribution in general, shows clustering of the data around some central value. Finding of this central value or the average is of importance, as it gives a most representative value of the whole group.

Different methods give different averages which are known as the *measures of central tendency*. The commonly used measures of central value are *Mean, Median, Mode, Geometric mean and Harmonic mean.*

(1) Mean. If $x_1, x_2, x_3, ..., x_n$ are a set of n values of a variate, then the *arithmetic mean* (or simply *mean*) is given by

$$\bar{\mathbf{x}} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n}, \text{ i.e. } \frac{\Sigma \mathbf{x}_i}{\mathbf{n}} \qquad ...(1)$$

In a *frequency distribution*, if $x_1, x_2, ..., x_n$ be the mid-values of the class-intervals having frequencies $f_1, f_2, ..., f_n$ respectively, we have

$$\bar{\mathbf{x}} = \frac{f_1x_1 + f_2x_2 + ... + f_nx_n}{f_1 + f_2 + ... + f_n} = \frac{\Sigma \mathbf{f}_i \mathbf{x}_i}{\Sigma \mathbf{f}_i} \qquad ...(2)$$

Calculation of mean. Direct method of computing especially when applied to grouped data involves heavy calculations and in order to avoid these, the following formulae are generally used :

I. Short-cut method $\qquad \bar{\mathbf{x}} = \mathbf{A} + \dfrac{\Sigma \mathbf{f}_i d_i}{\Sigma \mathbf{f}_i} \qquad ...(3)$

II. Step-deviation method $\quad \bar{\mathbf{x}} = \mathbf{A} + \mathbf{h}\dfrac{\Sigma \mathbf{f}_i \mathbf{u}_i}{\Sigma \mathbf{f}_i} \qquad ...(4)$

where $d = x - A$ and $u = (x - A)/h$, A being an arbitrary origin and h the equal class interval.

Proof. If $x_1, x_2, ..., x_n$ are the mid-values of the classes with frequencies $f_1, f_2, ..., f_n$, we have

$$\Sigma f_i x_i = \Sigma f_i (A + d_i) = A\Sigma f_i + \Sigma f_i d_i$$

$$\therefore \qquad \bar{x} = \frac{\Sigma f_i x_i}{\Sigma f_i} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

Further $u_i = d_i/h$ or $d_i = hu_i$. Substituting this value in (3), we get (4).

Obs. *The algebraic sum of the deviations of all the variables from their mean is zero, for*

$$\Sigma f_i(x_i - \bar{x}) = \Sigma f_i x_i - \bar{x}\,\Sigma f_i = \Sigma f_i x_i - \frac{\Sigma f_i x_i}{\Sigma f_i}\cdot \Sigma f_i = 0.$$

Cor. *If $\bar{x}_1, \bar{x}_2$ be the means of two samples of size n_1 and n_2, then the mean $\bar{x}$ of the combined sample of size $n_1 + n_2$ is given by*

$$\bar{\mathbf{x}} = \frac{\mathbf{n}_1\bar{\mathbf{x}}_1 + \mathbf{n}_2\bar{\mathbf{x}}_2}{\mathbf{n}_1 + \mathbf{n}_2}$$

For $\qquad n_1\bar{x}_1$ = sum of all observations of the first sample,

and $\qquad n_2\bar{x}_2$ = sum of all observations of the second sample.

$\therefore$ sum of the observations of the combined sample $= n_1\bar{x}_1 + n_2\bar{x}_2$.

Also number of the observations in the combined sample $= n_1 + n_2$.

$\therefore$ mean of the combined sample $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$.

Example 25.2. *The following is the frequency distribution of a random sample of weekly earnings of 509 employees :*

Weekly earnings :	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
No. of employees :	3	6	10	15	24	42	75	90	79	55	36	26	19	13	9	7

Calculate the average weekly earnings.

Solution. The calculations are arranged in the following table. The arbitrary origin is generally taken as the value corresponding to the maximum frequency.

By direct method, we have

$$\text{Mean } \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{13{,}315}{509} = 26.16$$

By step-deviation method, we have

$$\bar{x} = A + h\frac{\Sigma fu}{\Sigma f} = 25 + 2 \times \frac{295}{509}$$

$$= 25 + 1.16 = 26.16, \text{ which is same as found above.}$$

Weekly earnings	*Mid value* x	*No. of employees* f	$f \times x$	$u = (x-25)/2$	$f \times u$
10—12	11	3	33	−7	−21
12—14	13	6	78	−6	−36
14—16	15	10	150	−5	−50
16—18	17	15	255	−4	−60
18—20	19	24	456	−3	−72
20—22	21	42	882	−2	−84
22—24	23	75	1725	−1	−75
24—26	25	90	2250	0	−398
26—28	27	79	2133	1	79
28—30	29	55	1595	2	110
30—32	31	36	1116	3	108
32—34	33	26	858	4	104
34—36	35	19	665	5	95
36—38	37	13	481	6	78
38—40	39	9	351	7	63
40—42	41	7	287	8	56
					+693
		$\Sigma f = 509$	$\Sigma fx = 13{,}315$		$\Sigma fu = 295$

(2) Median. *If the values of a variable are arranged in the ascending order of magnitude, the median is the middle item if the number is odd and is the mean of the two middle items if the number is even.* Thus the median is equal to the mid-value, *i.e.*, the value which divides the total frequency into two equal parts.

For the grouped data,

$$\textbf{Median} = \mathbf{L} + \frac{(\frac{1}{2}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the median class, N = total frequency,

f = frequency of the median class, h = width of the median class,

and C = cumulative frequency upto the class preceding the median class.

Quartiles. Quartiles are those values which divide the frequency into four equal parts, when the values are arranged in the ascending order of magnitude. The **lower quartile** (Q_1) is mid-way between the lower extreme and the median. The **upper quartile** (Q_3) is midway between the median the upper extreme.

For the grouped data, these are calculated by the formulae :

$$\mathbf{Q_1} = \mathbf{L} + \frac{(\frac{1}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

and

$$\mathbf{Q_3} = \mathbf{L} + \frac{(\frac{3}{4}\mathbf{N} - \mathbf{C})}{\mathbf{f}} \times \mathbf{h}$$

where L = lower limit of the class in which Q_1 or Q_3 lies, f = frequency of this class, h = width of the class

and C = cumulative frequency upto the class preceding the class in which Q_1 or Q_3 lies.

The difference between the upper and lower quartiles, i.e., Q_3—Q_1 is called the **inter-quartile range.**

Obs. The ogives give a ready method of marking on the curve the values of the median and the quartiles. The two ogives 'less than' and 'more than' cut each other at the median (Fig. 25.2).

(3) Mode. *The mode is defined as that value of the variable which occurs most frequently, i.e., the value of the maximum frequency.*

For a grouped distribution, it is given by the formula

$$\textbf{Mode} = \mathbf{L} + \frac{\Delta_1}{\Delta_1 + \Delta_2}\,\mathbf{h}$$

where L = lower limit of the class containing the mode,

Δ_1 = excess of modal frequency over frequency of preceding class,

Δ_2 = excess of modal frequency over following class,

and h = size of modal class.

For a frequency curve (Fig. 25.1), the abscissa of the highest ordinate determines the value of the mode. There may be one or more modes in a frequency curve. Curves having a single mode are termed as *unimodal*, those having two modes as *bi-modal* and those having more than two modes as *multi-modal*.

Obs. In a symmetrical distribution, the mean, median and mode coincide. For other distributions, however, they are different and are known to be connected by the empirical relationship :

Mean – Mode = 3(Mean – Median).

Example 25.3. *Calculate median and the lower and upper quartiles from the distribution of marks obtained by 49 students of example 25.1. Find also the semi-interquartile range and the mode.*

Solution. *Median* (or 49/2) falls in the class (15—20) and is given by

$$15 + \frac{(49/2) - 11}{15} \times 5 = 15 + \frac{13.5}{3} = 19.5 \text{ marks.}$$

Lower quartile Q_1 (or 49/4) = 12.25) also falls in the class 15—20.

$$\therefore \quad Q_1 = 15 + \frac{(49/4) - 11}{15} \times 5 = 15 + \frac{12.5}{3} = 15.4 \text{ marks}$$

Upper quartile (or $\frac{3}{4} \times 49 = 36.75$) falls in the class 25—30.

$$\therefore \quad Q_3 = 25 + \frac{36.75 - 36}{5} \times 5 = 25.75 \text{ marks.}$$

Semi-interquartile range $= \frac{1}{2}(Q_3 - Q_1) = \dfrac{25.75 - 15.4}{2} = \dfrac{10.35}{2} = 5.175.$

Mode. It is seen that the mode value falls in the class 15—20. Employing the formula for the grouped distribution, we have

$$Mode = 15 + \frac{15 - 6}{(15 - 6) + (15 - 10)} \times 5 = 18.2 \text{ marks.}$$

Obs. In Fig. 25.2, the ogives meet at a point whose abscissa is 19.5 which is the *median* of the distribution. The values for the lower and upper quartiles are similarly seen to be 15.4 (for frequency 12.25) and 25.7 (for frequency 36.75).

Example 25.4. *Given below are the marks obtained by a batch of 20 students in a certain class test in Physics and Chemistry.*

Roll No. of students	*Marks in Physics*	*Marks in Chemistry*	*Roll No. of students*	*Marks in Physics*	*Marks in Chemistry*
1	53	58	11	25	10
2	54	55	12	42	42
3	52	25	13	33	15
4	32	32	14	48	46
5	30	26	15	72	50
6	60	85	16	51	64
7	47	44	17	45	39
8	46	80	18	33	38
9	35	33	19	65	30
10	28	72	20	29	36

In which subject is the level of knowledge of the students higher ?

Solution. The subject for which the value of the median is higher will be the subject in which the level of knowledge of the students is higher. To find the median in each case, we arrange the marks in ascending order of magnitude :

Sr. No.	Marks in Physics	Marks in Chemistry	Sr. No.	Marks in Physics	Marks in Chemistry
1	25	10	11	46	42
2	28	15	12	47	44
3	29	25	13	48	46
4	30	26	14	51	50
5	32	30	15	52	55
6	33	32	16	53	58
7	33	33	17	54	64
8	35	36	18	60	72
9	42	38	19	65	80
10	45	39	20	72	85

Median marks in Physics = A.M. of marks of 10th and 11th terms

$$= \frac{45 + 46}{2} = 45.5$$

Median marks in Chemistry = A.M. of marks of 10th and 11th items.

$$= \frac{39 + 42}{2} = 40.5$$

Since the median marks in Physics is greater than the median marks in Chemistry; the level of knowledge in Physics is higher.

Example 25.5. *An incomplete frequency distribution is given as below :*

Variable :	10—20	20—30	30—40	40—50	50—60	60—70	70—80
Frequency :	12	30	?	65	?	25	18

Given that the total frequency is 229 and median is 46, find the missing frequencies.

Solution. Let f_1, f_2 be the missing frequencies of the classes 30—40 and 50—60 respectively.

Since the median lies in the class 40—50,

$$\therefore \qquad 46 = 40 + \frac{229/2 - (12 + 30 + f_1)}{65} \times 10$$

which gives $f_1 = 33.5$ which can be taken as 34.

$$\therefore \qquad f_2 = 229 - (12 + 30 + 34 + 65 + 25 + 18) = 45.$$

(4) Geometric mean. If $x_1, x_2, \ldots, x_n$ are a set of n observations, then the *geometric mean* is given by

$$\text{G.M.} = (x_1 x_2 \ldots x_n)^{1/n}$$

or
$$\log \text{G.M.} = \frac{1}{n}(\log x_1 + \log x_2 + \ldots + \log x_n) \qquad \ldots(1)$$

In a frequency distribution, let $x_1, x_2, \ldots, x_n$ be the central values with corresponding frequencies $f_1, f_2, \ldots, f_n$, we have

$$\text{G.M.} = \left[(x_1)^{f_1} . (x_2)^{f_2} \ldots (x_n)^{f_n}\right]^{1/n} \quad \text{where } n = \Sigma f_i.$$

or
$$\log \text{G.M.} = \frac{1}{n}[f_1 \log x_1 + f_2 \log x_2 + \ldots + f_n \log x_n] \qquad \ldots(2)$$

Hence (1) and (2) show that logarithm of G.M. = A.M. of logarithms of the values.

(5) Harmonic mean. If $x_1, x_2, \ldots x_n$ be a set of n observations, then the *harmonic mean is defined as the reciprocal of the (arithmetic) mean of the reciprocals of the quantities.* Thus

$$\text{H.M.} = \frac{1}{\frac{1}{n}\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

In a frequency distribution, $\text{H.M.} = \dfrac{1}{\frac{1}{n}\left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \ldots + \frac{f_n}{x_n}\right)}$ where $n = \Sigma f_i$.

Example 25.6. *Three cities A, B, C are equidistant from each other. A motorist travels from A to B at 30 km/hr, from B to C at 40 km/hr, from C to A at 50 km/hr. Determine the average speed.*

Solution. Let $AB = BC = CA = s$ km

Time taken to travel from A to $B = s/30$

Time taken to travel from B to $C = s/40$

Time taken to travel from C to $A = s/50$

$\therefore$ average time taken $= \dfrac{1}{3}\left(\dfrac{s}{30} + \dfrac{s}{40} + \dfrac{s}{50}\right)$

Thus the average speed $= \dfrac{s}{\dfrac{1}{3}\left(\dfrac{s}{30} + \dfrac{s}{40} + \dfrac{s}{50}\right)}$

In other words, the average speed is the harmonic mean of 30, 40, 50 km/hr.

Hence the average speed $= \dfrac{1}{\dfrac{1}{3}\left(\dfrac{1}{30} + \dfrac{1}{40} + \dfrac{1}{50}\right)} = 38.3$ km/hr.

Obs. Of the various measures of central tendency, *the mean* is the most important for it can be computed easily. *The median,* though more easily calculable, cannot be applied with ease to theoretical analysis. Median is of advantage when there are exceptionally large and small values at the ends of the distribution.

The mode, though most easily calculated, has the least significance. It is particularly misleading in distributions which are small in numbers or highly unsymmetrical.

The geometrical mean though difficult to compute, finds application in cases like populations where we are concerned with a quantity whose changes tend to be directly proportional to the quantity itself.

The harmonic mean is useful in limited situations where time and rate or prices are involved.

PROBLEMS 25.1

1. Draw the histogram and frequency polygon for the following distribution. Also calculate the arithmetic mean :

Class interval :	0—99	100—199	200—299	300—399	400—499	500—599	600—699	700—799
Frequency :	10	54	184	264	246	40	1	1

2. The following marks were given to a batch of candidates :

66	62	45	79	32	51	56	60	51	49
25	42	54	54	58	70	43	58	50	52
38	67	50	59	48	65	71	30	46	55
82	51	63	45	53	40	35	56	70	52
67	55	67	30	63	42	74	58	44	55

Draw a cumulative frequency curve.

Hence find the proportion of candidates securing more than 50 marks. Also mark off the median, the first and third quartiles.

3. Find the mean, median and mode for the following :

Mid Value :	15	20	25	30	35	40	45	50	55
Frequency :	2	22	19	14	3	4	6	1	1

(*Kerala, 1990*)

4. Calculate mean, median and mode of the following data relating to weight of 120 articles :

Weight (in gm) :	0—10	10—20	20—30	30—40	40—50	50—60
No. of articles :	14	17	22	26	23	18

5. The population of a country was 300 million in 1971. It became 520 million in 1989. Calculate the percentage compound rate of growth per annum.

[**Hint.** Use $P_n = P_0(1 + r)^n$, r being the growth rate.]

6. The number of divorces per 1000 marriages in the United States increased from 84 in 1970 to 108 in 1990. Find the annual increase of the divorce rate for the period 1970 to 1990.

7. An aeroplane flies along the four sides of a square at speeds of 100, 200, 300 and 400 km/hr. respectively. What is the average speed of the plane in its flight around the square.

8. A man having to drive 90 km. wishes to achieve an average speed of 30 km/hr. For the first half of the journey, he averages only 20 km/hr. What must be his average speed for the second half of the journey if his overall average is to be 30 km/hr.

9. Following table gives the cumulative frequency of the age of a group of 199 teachers. Find the mean and median age of the group.

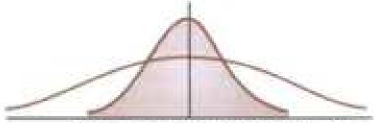
Age in years :	20—25	25—30	30—35	35—40	40—45	45—50	50—55	55—60	60—65	65—70
Cum. frequ. :	21	40	90	130	146	166	176	186	195	199

10. Recast the following cumulative table in the form of an ordinary frequency distribution and determine the median and the mode :

No. of days absent	*No. of students*	*No. of days absent*	*No. of students*
Less than 5	29	Less than 30	644
Less than 10	224	Less than 35	650
Less than 15	465	Less than 40	653
Less than 20	582	Less than 45	655
Less than 25	634		

25.6 MEASURES OF DISPERSION

Although measures of central tendency do exhibit one of the important characteristics of a distribution, yet they fail to give any idea as to how the individual values differ from the central value, *i.e.*, whether they are closely packed around the central value or widely scattered away from it. Two distributions may have the same mean and the same total frequency, yet they may differ in the extent to which the individual values may be spread about the average (*See* Fig. 25.3). The magnitude of such a variation is called *dispersion*. The important measures of dispersion are given below :



Same mean different dispersion

Fig. 25.3

(1) Range. *This is the simplest measure of dispersion and is given by the difference between the greatest and the least values in the distribution.* If the weekly wages of a group of labourers are

₹ 21 23 28 25 35 42 39 48

then *range* = Max. value – Min. value = 48 – 21 = ₹ 27.

(2) Quartile deviation or semi-interquartile range. One half of the interquartile range is called *quartile deviation*, or *semi-interquartile range*. If Q_1 and Q_3 are the first and third quartiles, the semi-interquartile range

$$Q = \tfrac{1}{2}(Q_3 - Q_1).$$

(3) Mean deviation. The mean deviation is the mean of the absolute differences of the values from the mean, median or mode. Thus *mean deviation* (*M.D.*)

$$= \frac{1}{n}\Sigma f_i\,|x_i - A|$$

where A is either the mean or the median or the mode. As the positive and negative differences have equal effects, only the absolute value of differences is taken into account.

(4) Standard deviation. The most important and the most powerful measure of dispersion is the *standard deviation* (*S.D.*) : generally denoted by σ. It is computed as the square root of the mean of the squares of the differences of the *variate* values from their mean.

Thus *standard deviation* (*S.D.*)

$$\sigma = \sqrt{\left[\frac{\Sigma \mathbf{f}_i(\mathbf{x}_i - \bar{\mathbf{x}})^2}{\mathbf{N}}\right]} \qquad ...(1)$$

where N is the total frequency Σf_i.

If however, the deviations are measured from any other value, say A, instead of $\bar{x}$, it is called the *root-mean-square deviation*.

The square of the standard deviation is known as the **variance.**

Calculation of S.D. The change of origin and the change of scale considerably reduces the labour in the calculation of standard deviation. The formulae for the computation of σ are as follows :

I. Short-cut method

$$\sigma = \sqrt{\left[\frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2\right]} \quad ...(2)$$

II. Step-deviation method

$$\sigma = h\sqrt{\left\{\frac{\Sigma f_i d_i'^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i'}{\Sigma f_i}\right)^2\right\}} \quad ...(3)$$

where $d_i = x_i - A$ and $d'_i = (x_i - A)/h$, being the assumed mean and h the equal class interval.

Proof. We know that $x_i - \bar{x} = (x_i - A) - (\bar{x} - A)$

$$\therefore \quad \Sigma f_i (x_i - \bar{x})^2 = \Sigma f_i [d_i - (\bar{x} - A)]^2 = \Sigma f_i d_i^2 + (\bar{x} - A)^2 \Sigma f_i - 2(\bar{x} - A)\Sigma f_i d_i$$

$$= \Sigma f_i d_i^2 - \frac{(\Sigma f_i d_i)^2}{\Sigma f_i} \qquad \left[\because \ \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right]$$

Hence
$$\sigma^2 = \frac{\Sigma f_i (x - \bar{x})^2}{\Sigma f_i} = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2$$

Further $d'_i = (x_i - A)/h = d_i/h$ or $d_i = hd'_i$, then substituting this value in (2), we get (3).

Obs. *The root mean square deviation is least when measured from the mean.*

The root mean square deviation is given by

$$s^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} \quad \text{and} \quad \frac{\Sigma f_i d_i}{\Sigma f_i} = \left[A + \frac{\Sigma f_i d_i}{\Sigma f_i}\right] - A = \bar{x} - A$$

$\therefore$ from (2), we have $\quad s^2 = \sigma^2 + (\bar{x} - A)^2 \quad ...(4)$

This shows that s^2 is always $> \sigma^2$ and the least value of $s^2 = \sigma^2$. This occurs when $A = \bar{x}$.

25.7 (1) COEFFICIENT OF VARIATION

The ratio of the standard deviation to the mean, is known as the *coefficient of variation*. As this is a ratio having no dimension, it is used for comparing the variations between the two groups with different means. It is often expressed as a percentage.

$\therefore$ *Coefficient of variation* $= \frac{\sigma}{\bar{x}} \times 100$

(2) Relations between measures of dispersion

(*i*) *Quartile deviation = 2/3 (standard deviation)*

(*ii*) *Mean deviation = 4/5 (standard deviation)*

25.8 STANDARD DEVIATION OF THE COMBINATION OF TWO GROUPS

If m_1, σ_1 be the mean and S.D. of a sample of size n_1 and *m_2, σ_2 be those for a sample of size n_2, then the S.D. σ of the combined sample of size $n_1 + n_2$ is given by*

$$(\mathbf{n_1 + n_2})\boldsymbol{\sigma}^2 = \mathbf{n_1}\boldsymbol{\sigma}_1^2 + \mathbf{n_2}\boldsymbol{\sigma}_2^2 + \mathbf{n_1 D_1^2} + \mathbf{n_2 D_2^2}$$

where $D_i = m_i - m$, m being the mean of combined sample.

From (4), we have $ns^2 = n\sigma^2 + n(\bar{x} - A)^2$ where n is the size of the sample.

i.e. sum of the squares of the deviations from $A = n\sigma^2 + n(\bar{x} - A)^2$.

Now let us apply this result to the first given sample taking A at m. Then, sum of the squares of the deviations of n_1 items from $m = n_1\sigma_1^2 + n_1(m_1 - m)^2$ ...(5)

Similarly for the second given sample taking A at m, sum of the squares of the deviations of n_2 items from $m = n_2\sigma_2^2 + n_2(m_2 - m)^2$ ...(6)

Adding (5) and (6), sum of the squares of the deviations of $n_1 + n_2$ items from m

$$= n_1\sigma_1^2 + n_2\sigma_2^2 + n_1(m_1 - m)^2 + n_2(m_2 - m)^2$$

$$\therefore \quad (n_1 + n_2)\,\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 D_1^2 + n_2 D_2^2$$

This result can be extended to the combination of any number of samples, giving a result of the form

$$(\Sigma n_i)\,\sigma^2 = \Sigma(n_i\sigma_i^2) + \Sigma(n_i D_i^2).$$

Example. 25.7. *Calculate the mean and standard deviation for the following :*

Size of item :	6	7	8	9	10	11	12
Frequency :	3	6	9	13	8	5	4

(V.T.U., 2001)

Solution. The calculations are arranged as follows :

Size of item x	*Frequency f*	*Deviation d = x − 9*	*f × d*	*f × d²*
6	3	−3	−9	27
7	6	−2	−12	24
8	9	−1	−9	9
9	13	0	0	0
10	8	1	8	8
11	5	2	10	20
12	4	3	12	36
	$\Sigma f = 48$		$\Sigma fd = 0$	$\Sigma fd^2 = 124$

$$\therefore \qquad mean = 9 + \frac{\Sigma fd}{\Sigma f} = 9$$

$$\text{Standard deviation} = \sqrt{\left[\frac{\Sigma f(x-\bar{x})^2}{\Sigma f}\right]} = \sqrt{\left(\frac{\Sigma fd^2}{\Sigma f}\right)} = \sqrt{\left(\frac{124}{48}\right)} = 1.607.$$

Example 24.8. *Calculate the mean and standard deviation of the following frequency distribution :*

Weekly wages in ₹	*No. of men*
4.5—12.5	4
12.5—20.5	24
20.5—28.5	21
28.5—36.5	18
36.5—44.5	5
44.5—52.5	3
52.5—60.5	5
60.5—68.5	8
68.5—76.5	2

Solution. The calculations are arranged in the table below :

Wages class ₹	*Mid value x*	*No. of men f*	*Step deviation* $d' = \frac{x-32.5}{8}$	fd'	fd'^2
4.5—12.5	8.5	4	−3	−12	36
12.5—20.5	16.5	24	−2	−48	96
20.5—28.5	24.5	21	−1	−21	21
28.5—36.5	32.5	18	0	0	0
36.5—44.5	40.5	5	1	5	5
44.5—52.5	48.5	3	2	6	12
52.5—60.5	56.5	5	3	15	45
60.5—68.5	64.5	8	4	32	128
68.5—76.5	72.5	2	5	10	50
		$\Sigma f = 90$		$\Sigma fd' = -13$	$\Sigma fd'^2 = 393$

$$\therefore \qquad mean\ wage = 32.5 + 8 \times \frac{\Sigma fd'}{\Sigma f} = 32.5 + 8\left(\frac{-13}{90}\right) = ₹\ 31.35$$

$$\text{Standard deviation} = 8\sqrt{\frac{\Sigma fd'^2}{\Sigma f} - \left(\frac{\Sigma fd'}{\Sigma f}\right)^2} = 8\sqrt{\frac{393}{90} - \left(\frac{-13}{90}\right)^2} = ₹\ 16.64.$$

Example 25.9. *The following are scores of two batsmen A and B in a series of innings :*

A :	12	115	6	73	7	19	119	36	84	29
B :	47	12	16	42	4	51	37	48	13	0

Who is the better score getter and who is more consistent ? (V.T.U., 2004)

Solution. Let x denote score of A and y that of B.

Taking 51 as the origin, we prepare the following table :

x	$d(=x-51)$	d^2	y	$\delta(=y-51)$	δ^2
12	−39	1521	47	−4	16
115	64	4096	12	−39	1521
6	−45	2025	16	−35	1225
73	22	484	42	−9	81
7	−44	1936	4	−47	2209
19	−32	1024	51	0	0
119	68	4624	37	−14	196
36	−15	225	48	−3	9
84	33	1089	13	−38	1444
29	−22	484	0	−51	2601
Total	−10	17508		−240	9302

For A, A.M. $\bar{x} = 51 + \dfrac{\Sigma d}{n} = 51 - \dfrac{10}{10} = 50$

$$\text{S.D. } \sigma_1 = \sqrt{\left\{\frac{\Sigma d^2}{n} - \left(\frac{\Sigma d}{n}\right)^2\right\}} = \sqrt{\{1750.8 - (-1)^2\}} = 41.8$$

$\therefore$ coefficient of variation $= \dfrac{\sigma_1}{\bar{x}} \times 100 = \dfrac{41.8}{50} \times 100 = 83.6\%$

For B, A.M. $\bar{y} = 51 + \dfrac{\Sigma \delta}{n} = 51 - \dfrac{240}{10} = 27$

$$\text{S.D. } \sigma_2 = \sqrt{\left\{\frac{\Sigma \delta^2}{n} - \left(\frac{\Sigma \delta}{n}\right)^2\right\}} = \sqrt{\{930.2 - (-24)^2\}} = 18.8$$

$\therefore$ coefficient of variation $= \dfrac{\sigma^2}{\bar{y}} \times 100 = \dfrac{18.8}{27} \times 100 = 69.6\%$

Since the A.M. of A > A.M. of B, it follows that A is a better score getter (*i.e.*, more efficient) than B.

Since the coefficient of variation of B < the coefficient of variation of A, it means that B is more consistent than A. Thus even though A is a better player, he is less consistent.

Example 25.10. *The numbers examined, the mean weight and S.D. in each group of examination by three medical examiners are given below. Find the mean weight and S.D. of the entire data when grouped together.*

Med. Exam.	*No. Examined*	*Mean Wt. (lbs.)*	*S.D. (lbs.)*
A	*50*	*113*	*6*
B	*60*	*120*	*7*
C	*90*	*115*	*8*

Solution. We have $n_1 = 50,\ \bar{x}_1 = 113,\ \sigma_1 = 6$

$n_2 = 60,\ \bar{x}_2 = 120,\ \sigma_2 = 7$

$n_3 = 90,\ \bar{x}_3 = 115,\ \sigma_3 = 8.$

If $\bar{x}$ is the mean of the entire data,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} = \frac{50 \times 113 + 60 \times 120 + 90 \times 115}{50 + 60 + 90} = \frac{23200}{200} = 116 \text{ lb.}$$

If σ is the S.D. of the entire data,

$$N\sigma^2 = n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1D_1^2 + n_2D_2^2 + n_3D_3^2$$

where $N = n_1 + n_2 + n_3 = 200$, $D_1 = \bar{x}_1 - \bar{x} = -3$, $D_2 = \bar{x}_2 - \bar{x} = 4$ and $D_3 = \bar{x}_3 - \bar{x} = -1$.

$$\therefore \quad 200\sigma^2 = 50 \times 36 + 60 \times 49 + 90 \times 64 + 50 \times 9 + 60 \times 16 + 90 \times 1$$
$$= 1800 + 2940 + 5760 + 450 + 960 + 90$$
$$\sigma^2 = \frac{12000}{200} = 60. \quad \text{Hence} \quad \sigma = \sqrt{60} = 7.746 \text{ lb.}$$

PROBLEMS 25.2

1. The crushing strength of 8 cement concrete experimental blocks, in metric tonnes per sq. cm., was 4.8, 4.2, 5.1, 3.8, 4.4, 4.7, 4.1 and 4.5. Find the mean crushing strength and the standard deviation.
2. Show that the variance of the first n positive integers is $\frac{1}{12}(n^2 - 1)$. *(V.T.U., 2003)*
3. The mean of five items of an observation is 4 and the variance is 5.2. If three of the items are 1, 2 and 6, then find the other two. *(V.T.U., 2002)*
4. For the distribution

x:	5	6	7	8	9	10	11	12	13	14	15
f:	18	15	34	47	68	90	80	62	35	27	11

find the mean, median and lower and upper quartiles, variance and the standard deviation.

5. The following table shows the marks obtained by 100 candidates in an examination. Calculate the mean, median and standard deviation :

Marks obtained :	1—10	11—20	21—30	31—40	41—50	51—60
No. of candidates :	3	16	26	31	16	8

(Osmania, 2003 S ; V.T.U., 2003 S)

6. Compute the quartile deviation and standard deviation for the following :

x:	100—109	110—119	120—129	130—139	140—149	150—159	160—169	170—179
f:	15	44	133	150	125	82	35	16

7. Calculate (*i*) mean deviation about the mean, (*ii*) mean deviation about the median for the following distribution :

Class :	3—4.9	5—6.9	7—8.9	9—10.9	11—12.9	13—14.9	15—16.9
f :	5	8	30	82	45	24	6

(Madras, 2002)

8. Two observers bring the following two sets of data which represent measurements of the same quantity :

I.	105.1	103.4	104.2	104.7	104.8	105.0	104.9
II.	105.3	105.1	104.8	105.2	106.7	102.9	103.1

Calculate the standard deviation in each case. Which set of data is more reliable ? Can the same conclusion be reached by calculating the mean deviation ?

Obs. The smaller the coefficient of variation, the greater is the *reliability or consistency* in the data.

9. The heights and weights of the 10 armymen are given below. In which characteristics are they more variable ?

Height in cm.	170	172	168	177	179	171	173	178	173	179
Weight in kg.	75	74	75	76	77	73	76	75	74	75

10. The index number of prices of two articles A and B for six consecutive weeks are given below :

A:	314	326	336	368	404	412
B:	330	331	320	318	321	330

Find which has a more variable price ?

11. The scores of two golfers A and B in 12 rounds are given below. Who is the better player and who is the more consistent player ?

A:	74	75	78	72	78	77	79	81	79	76	72	71
B:	87	84	80	88	89	85	86	82	82	79	86	80

12. The scores obtained by two batsmen A and B in 10 matches are given below :

A:	30	44	66	62	60	34	80	46	20	38
B:	34	46	70	38	55	48	60	34	45	30

Calculating mean, S.D. and coefficient of variation for each batsman, determine who is more efficient and who is more consistent.

13. Find the mean and standard deviation of the following two samples put together :

Sample No.	Size	Mean	S.D.
1	50	158	5.1
2	60	164	4.6

14. A distribution consists of three components with frequencies 200, 250 and 300 having means 25, 10 and 15 and S.Ds. 3, 4 and 5 respectively. Show that the mean of the combined distribution is 16 and its S.D. is 7.2 approximately.

25.9 (1) MOMENTS

The rth moment about the mean $\bar{x}$ of a distribution is denoted by μ_r and is given by

$$\mu_r = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^r \quad ...(1)$$

The corresponding moment about any point a is defined as

$$\mu_r' = \frac{1}{N}\Sigma f_i(x_i - a)^r \quad ...(2)$$

In particular, we have $\mu_0 = \mu_0' = 1$ ...(3)

$$\mu_1 = \frac{1}{N}\Sigma f_i(x_i - \bar{x}) = 0;\ \mu_1' = \frac{1}{N}\Sigma f_i(x_i - a) = \bar{x} - a = d, \text{ say} \quad ...(4)$$

$$\mu_2 = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^2 = \sigma^2. \quad ...(5)$$

(2) Moments about the mean in terms of moments about any point.

We have

$$\mu_r = \frac{1}{N}\Sigma f_i(x_i - \bar{x})^r = \frac{1}{N}\Sigma f[(x_i - a) - (\bar{x} - a)]^r$$

$$= \frac{1}{N}\Sigma f_i(X_i - d)^r \quad \text{where} \quad X_i = x_i - a,\ d = \bar{x} - a.$$

$$= \frac{1}{N}[\Sigma f_i X_i^r - {}^rC_1 d\Sigma f_i X_i^{r-1} + {}^rC_2 d^2 \Sigma f_i X_i^{r-2} - ...]$$

$$= \mu_r' - {}^rC_1\, d\mu_{r-1}' + {}^rC_2 d^2 \mu_{r-2}' - ...... \quad ...(6)$$

In particular,

$$\mu_2 = \mu'_2 - \mu'^2_1 \quad ...(7)$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 \quad ...(8)$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 \quad ...(9)$$

These three results should be committed to memory. It should be noted that in each of these relations, the sum of the coefficients of the various terms on the right side is zero. Also each term on the right side is of the same dimension as the term on the left.

25.10 SKEWNESS

Skewness measures the degree of asymmetry or the departure from symmetry. If the frequency curve has a longer 'tail' to the right, *i.e.*, the mean is to the right of the mode [as in Fig. 25.4 (*a*)], then the distribution is said to have *positive skewness*. If the curve is more elongated to the left, then it is said to have *negative skewness* [Fig. 25.4 (*b*)].

The following three measures of skewness deserve mention :

(*i*) *Pearson's* coefficient of skewness* $= \dfrac{\text{mean} - \text{mode}}{\sigma}$

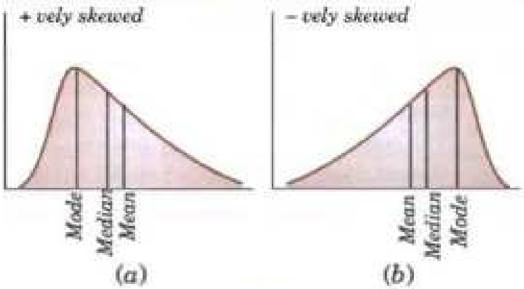


Fig. 25.4

* After the English statistician and biologist *Karl Pearson* (1857–1936) who did pioneering work and found the English school of statistics.

(ii) *Quartile coefficient of skewness* $= \dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$

Its value always lies between – 1 and + 1.

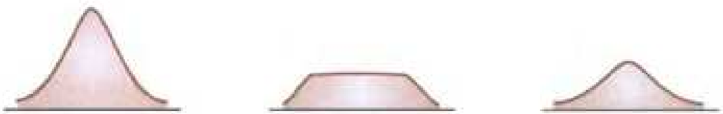
(iii) *Coefficient of skewness based on third moment* $\gamma_1 = \sqrt{\beta_1}$.

where $\beta_1 = \mu_3^2/\mu_2^3$

Thus $\gamma_1 = \sqrt{\beta_1}$ gives the simplest measure of *skewness*.

25.11 KURTOSIS

Kurtosis measures the degree of peakedness of a distribution and is given by $\beta_2 = \mu_4/\mu_2^2$.



(a) *Leptokurtic* (b) *Platykurtic* (c) *Mesokurtic*

Fig. 25.5

$\gamma_2 = \beta_2 - 3$ gives the excess of *Kurtosis*. The curves with $\beta_2 > 3$ are called *Leptokurtic* and those with $\beta_2 < 3$ as *Platykurtic*. The normal curve for which $\beta_2 = 3$, is called *Mesokurtic* [Fig. 25.5].

Example 25.11. *The first four moments about the working mean 28.5 of a distribution are 0.294, 7.144, 42.409 and 454.98. Calculate the moments about the mean. Also evaluate β_1, β_2 and comment upon the skewness and kurtosis of the distribution.* (V.T.U., 2...)

Solution. The first four moments about the arbitrary origin 28.5 are $\mu'_1 = 0.294$, $\mu'_2 = 7.144$, $\mu'_3 = 42.409$, $\mu'_4 = 454.98$.

$$\therefore \quad \mu'_1 = \frac{1}{N}\sum f_i(x_i - 28.5) = \frac{1}{N}\sum f_i x_i - 28.5 = \bar{x} - 28.5 = 0.294 \text{ or } \bar{x} = 28.794$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = 7.144 - (0.294)^2 = 7.058$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 = 42.409 - 3(7.144)(0.294) + 2(0.294)^3 = 36.151.$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$
$$= 454.98 - 4(42.409) \times (0.294) + 6(7.144)(0.294)^2 - 3(0.294)^4 = 408.738$$

Now
$$\beta_1 = \mu_3^2/\mu_2^3 = (36.151)^2/(7.058)^3 = 3.717$$
$$\beta_2 = \mu_4/\mu_2^2 = 408.738/(7.058)^2 = 8.205.$$

$\therefore$ $\gamma_1 = \sqrt{\beta_1} = 1.928$, which indicates considerable skewness of the distribution.

$\gamma_2 = \beta_2 - 3 = 5.205$ which shows that the distribution is *leptokurtic*.

Example 25.12. *Calculate the median, quartiles and the quartile coefficient of skewness from the following data:*

Weight (lbs) :	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
No. of persons :	12	18	35	42	50	45	20	8

Solution. Here total frequency $N = \Sigma f_i = 230$.

The cummulative frequency table is

Weight (lbs) :	70–80	80–90	90–100	100–110	110–120	120–130	130–140	140–150
f :	12	18	35	42	50	45	20	8
cum. f. :	12	30	65	107	157	202	222	230

Now $N/2 = 230/2 = 115$th item which lies in 110–120 group.

$$\therefore \quad \text{median or } Q_2 = L + \frac{N/2 - C}{f} \times h = 110 + \frac{115 - 107}{50} \times 10 = 111.6$$

Also $N/4 = 230/4 = 57.5$ *i.e.* Q_1 is 57.5th or 58th item which lies in 90–100 group.

$$\therefore \qquad Q_1 = L + \frac{N/4 - C}{f} \times h = 90 + \frac{57.5 - 30}{35} \times 10 = 97.85$$

Similarly, $3N/4 = 172.5$ *i.e.* Q_3 is 173rd item which lies in 120–130 group.

$$\therefore \qquad Q_3 = L + \frac{3N/4 - C}{f} \times h = 120 + \frac{172.5 - 157}{45} \times 10 = 123.44$$

Hence quartile coefficient of skewness $= \dfrac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$

$$= \frac{97.85 + 123.44 - 2 \times 111.6}{123.44 - 97.85} = -0.07 \text{ (approx.)}.$$

PROBLEMS 25.3

1. Calculate the first four moments of the following distribution about the mean :

x:	0	1	2	3	4	5	6	7	8
f:	1	8	28	56	70	56	28	8	1

Also evaluate β_1 and β_2. *(V.T.U., 2004 ; Madras, 2003)*

2. The following table gives the monthly wages of 72 workers in a factory. Compute the standard deviation, quartile deviation, coefficients of variation and skewness. *(V.T.U., 2001)*

Monthly wages (in ₹)	*No. of workers*	*Monthly wages (in ₹)*	*No. of workers*
12.5—17.5	2	37.5—42.5	4
17.5—22.5	22	42.5—47.5	6
22.5—27.5	19	47.5—52.5	1
27.5—32.5	14	52.5—57.5	1
32.5—37.5	3		

3. Find Pearson's coefficient of skewness for the following data :

Class :	10–19	20–29	30–39	40–49	50–59	60–69	70–79	80–89
Frequency :	5	9	14	20	25	15	8	4

(V.T.U., 2000 S)

4. Compute the quartile coefficient of skewness for the following distribution :

x:	3–7	8–12	13–17	18–22	23–27	28–32	33–37	38–42
f:	2	108	580	175	80	32	18	5

(Madras, 2002 ; V.T.U., 2000)

Also compute the measure of skewness based on the third moment.

5. The first three moments of a distribution about the value 2 of the variable are 1, 16 and – 40. Show that the mean = 3, the variance = 15 and $\mu_3 = -86$. *(V.T.U., 2003 S)*
6. Compute skewness and kurtosis, if the first four moments of a frequency distribution $f(x)$ about the value $x = 4$ are respectively 1, 4, 10 and 45. *(Coimbatore, 1999)*
7. In a certain distribution, the first four moments about a point are – 1.5, 17, – 30 and 108. Calculate the moments about the mean, β_1 and β_2 ; and state whether the distribution is leptokurtic or platykurtic ?

25.12 CORRELATION

So far we have confined our attention to the analysis of observations on a single variable. There are, however, many phenomenae where the changes in one variable are related to the changes in the other variable. For instance, the yield of a crop varies with the amount of rainfall, the price of a commodity increases with the reduction in its supply and so on. Such a simultaneous variation, *i.e.* when the changes in one variable are associated or followed by changes in the other, is called *correlation*. Such a data connecting two variables is called *bivariate population*.

If an increase (or decrease) in the values of one variable corresponds to an increase (or decrease) in the other, the correlation is said to be *positive*. If the increase (or decrease) in one corresponds to the decrease (or increase) in the other, the correlation is said to be *negative*. If there is no relationship indicated between the variables, they are said to be *independent or uncorrelated*.

To obtain a measure of relationship between the two variables, we plot their corresponding values on the graph, taking one of the variables along the x-axis and the other along the y-axis. (Fig. 25.6).

Let the origin be shifted to $(\bar{x}, \bar{y})$, where $\bar{x}, \bar{y}$ are the means of x's and y's that the new co-ordinates are given by

$$X = x - \bar{x}, \qquad Y = y - \bar{y}.$$

Now the points (X, Y) are so distributed over the four quadrants of XY-plane that the product XY is positive in the first and third quadrants but negative in the second and fourth quadrants. The algebraic sum of the products can be taken as describing the trend of the dots in all the quadrants.

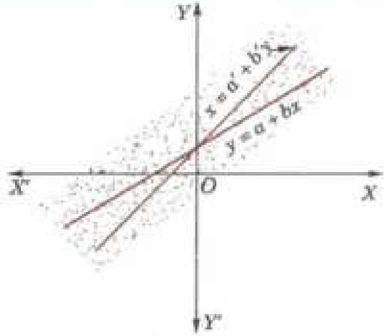


Fig. 25.6

$\therefore$ (*i*) If ΣXY is positive, the trend of the dots is through the first and third quadrants,

(*ii*) if ΣXY is negative the trend of the dots is in the second and fourth quadrants, and

(*iii*) if ΣXY is zero, the points indicate no trend *i.e.* the points are evenly distributed over the four quadrants.

The ΣXY or better still $\frac{1}{n}\Sigma XY$, *i.e.*, the average of n products may be taken as a measure of correlation. If we put X and Y in their units, *i.e.*, taking σ_x as the unit for x and σ_y for y, then

$$\frac{1}{n}\Sigma \frac{X}{\sigma_x}\cdot\frac{Y}{\sigma_y}, \text{ i.e., } \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

is the *measure of correlation*.

25.13 COEFFICIENT OF CORRELATION

The numerical measure of correlation is called the *coefficient of correlation* and is defined by the relation

$$r = \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

where X = deviation from the mean $\bar{x} = x - \bar{x}$, Y = deviation from the mean $\bar{y} = y - \bar{y}$,

σ_x = S.D. of x-series, σ_y = S.D. of y-series and n = number of values of the two variables.

Methods of calculation :

(*a*) *Direct method.* Substituting the value of σ_x and σ_y in the above formula, we get

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} \qquad ...(1)$$

Another form of the formula (1) which is quite handy for calculation is

$$r = \frac{n\Sigma xy - \Sigma x\Sigma y}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\} \times \{n\Sigma y^2 - (\Sigma y)^2\}]}} \qquad ...(2)$$

(*b*) *Step-deviation method.* The direct method becomes very lengthy and tedious if the means of the two series are not integers. In such cases, use is made of assumed means. If d_x and d_y are step-deviations from the assumed means, then

$$r = \frac{n\Sigma d_x d_y - \Sigma d_x \Sigma d_y}{\sqrt{[\{n\Sigma d_x^2 - (\Sigma d_x)^2\} \times \{n\Sigma d_y^2 - (\Sigma d_y)^2\}]}} \qquad ...(3)$$

where $d_x = (x - a)/h$ and $d_y = (y - b)/k$.

Obs. The change of origin and units do not alter the value of the correlation coefficient since r is a pure number.

(*c*) *Co-efficient of correlation for grouped data.* When x and y series are both given as frequency distributions, these can be represented by a two-way table known as the *correlation-table*. It is double-entry table with one series along the horizontal and the other along the vertical as shown on page 848. The co-efficient of correlation for such a *bivariate frequency distribution* is calculated by the formula.

$$r = \frac{n(\Sigma f d_x d_y) - (\Sigma f d_x)(\Sigma f d_y)}{\sqrt{[\{n\Sigma f d_x^2 - (\Sigma f d_x)^2\} \times \{n\Sigma f d_y^2 - (\Sigma f d_y)^2\}]}} \quad ...(4)$$

where d_x = deviation of the central values from the assumed mean of x-series,
d_y = deviation of the central values from the assumed mean of y-series,
f is the frequency corresponding to the pair (x, y)
and $n(= \Sigma f)$ is the total number of frequencies.

Example 25.13. *Psychological tests of intelligence and of engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (I.R.) and engineering ratio (E.R.). Calculate the co-efficient of correlation.*

Student	A	B	C	D	E	F	G	H	I	J
I.R.	105	104	102	101	100	99	98	96	93	92
E.R.	101	103	100	98	95	96	104	92	97	94

(*Andhra, 2000*)

Solution. We construct the following table :

Student	Intelligence ratio x	$x - \bar{x} = X$	Engineering ratio y	$y - \bar{y} = Y$	X^2	Y^2	XY
A	105	6	101	3	36	9	18
B	104	5	103	5	25	25	25
C	102	3	100	2	9	4	6
D	101	2	98	0	4	0	0
E	100	1	95	−3	1	9	−3
F	99	0	96	−2	0	4	0
G	98	−1	104	6	1	36	−6
H	96	−3	92	−6	9	36	18
I	93	−6	97	−1	36	1	6
J	92	−7	94	−4	49	16	28
Total	990	0	980	0	170	140	92

From this table, mean of x, *i.e.*, $\bar{x} = 990/10 = 99$ and mean of y, *i.e.* $\bar{y} = 980/10 = 98$.

$$\Sigma X^2 = 170,\ \Sigma Y^2 = 140 \text{ and } \Sigma XY = 92.$$

Substituting these values in the formula (1) p. 744, we have

$$r = \frac{\Sigma XY}{\sqrt{(\Sigma X^2 \Sigma Y^2)}} = \frac{92}{\sqrt{(170 \times 140)}} = 92/154.3 = 0.59.$$

Example 25.14. *The correlation table given below shows that the ages of husband and wife of 53 married couples living together on the census night of 1991. Calculate the coefficient of correlation between the age of the husband and that of the wife.* (*J.N.T.U., 2003*)

Age of husband	Age of wife 15–25	25–35	35–45	45–55	55–65	65–75	Total
15–25	1	1	–	–	–	–	2
25–35	2	12	1	–	–	–	15
35–45	–	4	10	1	–	–	15
45–55	–	–	3	6	1	–	10
55–65	–	–	–	2	4	2	8
65–75	–	–	–	–	1	2	3
Total	3	17	14	9	6	4	53

Solution.

Age of husband				Age of wife x-series						Total f	Suppose $d_x = \frac{x-40}{10}$, $d_y = \frac{y-40}{10}$		
				15–25	25–35	35–45	45–55	55–65	65–75				
Years			Mid pt. x	20	30	40	50	60	70				
				−20	−10	0	10	20	30				
Age group	Mid pt. y		d_y \ d_x	−2	−1	0	1	2	3		fd_y	fd_y^2	fd_xd_y
15–25	20	−20	−2	**4** 1	**2** 1					2	−4	8	**6**
25–35	30	−10	−1	**4** 2	**12** 12	**0** 1				15	−15	15	**16**
35–45	40	0	0		**0** 4	**0** 10	**0** 1			15	0	0	**0**
45–55	50	10	1			**0** 3	**6** 6	**2** 1		10	10	10	**8**
55–65	60	20	2				**4** 2	**16** 4	**12** 2	8	16	32	**32**
65–75	70	30	3					**6** 1	**18** 2	3	9	27	**24**
Total f				3	17	14	9	6	4	$53 = n$	16	92	**86**
fd_x				−6	−17	0	9	12	12	10	Thick figures in small sqs. stand for fd_xd_y		
fd_x^2				12	17	0	9	24	36	98			
fd_xd_y				**8**	**14**	**0**	**10**	**24**	**30**	**86**	**Check :** $\Sigma fd_xd_y = 86$ from both sides		

With the help of the above correlation table, we have

$$r = \frac{n(\Sigma fd_xd_y) - (\Sigma fd_x)(\Sigma fd_y)}{\sqrt{[\{n\Sigma fd_x^2 - (\Sigma fd_x)^2\} \times \{n\Sigma fd_y^2 - (\Sigma fd_y)^2\}]}}$$

$$= \frac{53 \times 86 - 10 \times 16}{\sqrt{[(53 \times 98 - 100) \times (53 \times 92 - 256)]}} = \frac{4398}{\sqrt{(5094 \times 4620)}} = \frac{4398}{4850} = 0.91 \text{ (approx.)}.$$

25.14 LINES OF REGRESSION

It frequently happens that the dots of the scatter diagram generally, tend to cluster along a well defined direction which suggests a linear relationship between the variables x and y. Such a line of best-fit for the given distribution of dots is called the *line of regression* (Fig. 25.6). In fact there are two such lines, one giving the best possible mean values of y for each specified value of x and the other giving the best possible mean values of x for given values of y. The former is known as the *line of regression of y on x* and the latter as the *line of regression of x on y*.

Consider first the line of regression of y on x. Let the straight line satisfying the general trend of n dots in a scatter diagram be

$$y = a + bx \quad ...(1)$$

We have to determine the constants a and b so that (1) gives for each value of x, the best estimate for the average value of y in accordance with the *principle of least squares* (page 816), therefore, the normal equations for a and b are

$$\Sigma y = na + b\Sigma x \quad ...(2)$$

and

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad ...(3)$$

(2) gives $\frac{1}{n}\Sigma y = a + b \cdot \frac{1}{n}\Sigma x$ *i.e.*, $\bar{y} = a + b\bar{x}$.

This shows that $(\bar{x}, \bar{y})$, *i.e.*, the means of x and y, lie on (1).

Shifting the origin to $(\bar{x}, \bar{y})$, (3) takes the form

$$\Sigma(x-\bar{x})(y-\bar{y}) = a\Sigma(x-\bar{x}) + b\Sigma(x-\bar{x})^2, \text{ but } a\Sigma(x-\bar{x}) = 0.$$

$$\therefore \quad b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{\Sigma XY}{\Sigma X^2} = \frac{\Sigma XY}{n\sigma_x^2} = r\frac{\sigma_y}{\sigma_x} \qquad \left[\because r = \frac{\Sigma XY}{n\sigma_x\sigma_y}\right]$$

Thus the line of best fit becomes $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$ ...(4)

which is the equation of *the line of regression of y on x*. Its slope is called the *regression coefficient of y on x*.

Interchanging x and y, we find that the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \quad ...(5)$$

Thus the *regression coefficient of y on x* $= r\sigma_y/\sigma_x$ ...(6)

and the *regression coefficient of x on y* $= r\sigma_x/\sigma_y$ ...(7)

Cor. *The correlation coefficient r is the geometric mean between the two regression co-efficients.*

For $r\frac{\sigma_y}{\sigma_x} \times r\frac{\sigma_x}{\sigma_y} = r^2$.

Example 25.15. *The two regression equations of the variables x and y are x = 19.13 – 0.87y and y = 11.64 – 0.50x. Find (i) mean of x's, (ii) mean of y's and (iii) the correlation coefficient between x and y.*

(V.T.U., 2004 ; Anna, 2003 ; Burdwan, 2003)

Solution. Since the mean of x's and the mean of y's lie on the two regression lines, we have

$$\bar{x} = 19.13 - 0.87\bar{y} \quad ...(i)$$

$$\bar{y} = 11.64 - 0.50\bar{x} \quad ...(ii)$$

Multiplying (*ii*) by 0.87 and subtracting from (*i*), we have

$$[1 - (0.87)(0.50)]\ \bar{x} = 19.13 - (11.64)(0.87) \text{ or } 0.57\ \bar{x} = 9.00 \text{ or } \bar{x} = 15.79$$

$$\therefore \quad \bar{y} = 11.64 - (0.50)(15.79) = 3.74$$

$\therefore$ regression coefficient of y on x is – 0.50 and that of x on y is – 0.87.

Now since the coefficient of correlation is the geometric mean between the two regression coefficients.

$$\therefore \quad r = \sqrt{[(-0.50)(-0.87)]} = \sqrt{(0.43)} = -0.66.$$

[– ve sign is taken since both the regression coefficients are – ve]

Example 25.16. *In the following table are recorded data showing the test scores made by salesmen on an intelligence test and their weekly sales :*

Salesmen	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Test scores	40	70	50	60	80	50	90	40	60	60
Sales (000)	2.5	6.0	4.5	5.0	4.5	2.0	5.5	3.0	4.5	3.0

Calculate the regression line of sales on test scores and estimate the most probable weekly sales volume if a salesman makes a score of 70.

Solution. With the help of the table below, we have

$$\bar{x} = \text{mean of } x \text{ (test scores)} = 60 + 0/10 = 60$$

$$\bar{y} = \text{mean of } y \text{ (sales)} = 4.5 + (-4.5)/10 = 4.05.$$

Regression line of sales (y) on scores (x) is given by

$$y - \bar{y} = r(\sigma_y / \sigma_x)(x - \bar{x})$$

where $$r\frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{\sigma_x \sigma_y} \times \frac{\sigma_y}{\sigma_x} = \frac{\Sigma XY}{(\sigma_x)^2} = \left[\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{n}\right] \Big/ \left[\Sigma d_x^2 - (\Sigma d_x)^2 / n\right]$$

$$= \frac{140 - \dfrac{0 \times (-4.5)}{10}}{2400 - 0^2/10} = \frac{140}{2400} = 0.06$$

$\therefore$ the required regression line is

$$y - 4.05 = 0.06(x - 60) \quad \text{or} \quad y = 0.06x + 0.45.$$

For $x = 70, y = 0.06 \times 70 + 0.45 = 4.65.$

Thus the most probable weekly sales volume for a score of 70 is 4.65.

Test scores x	*Sales* y	*Deviation of x from assumed mean (= 60)* d_x	*Deviation of y from assumed average (= 4.5)* d_y	$d_x \times d_y$	d_x^2	d_y^2
40	2.5	−20	−2	40	400	4
70	6.0	10	1.5	15	100	2.25
50	4.5	−10	0	0	100	0
60	5.0	0	0.5	0	0	2.25
80	4.5	20	0	0	400	0
50	2.0	−10	−2.5	25	100	6.25
90	5.5	30	1	30	900	1.00
40	3.0	−20	−1.5	30	400	2.25
60	4.5	0	0	0	0	0
60	3.0	0	−1.5	0	0	2.25
		$\Sigma d_x = 0$	$\Sigma d_y = -4.5$	$\Sigma d_x d_y = 140$	$\Sigma d_x^2 = 2400$	$\Sigma d_y^2 = 18.25$

Example 25.17. *If θ is the angle between the two regression lines, show that*

$$\tan\theta = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

Explain the significance when $r = 0$ and $r = \pm 1$. (U.P.T.U., 2007 ; V.T.U., 2007)

Solution. The equations to the line of regression of y on x and x on y are

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \text{ and } x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$$

$\therefore$ their slopes are $m_1 = r\sigma_y/\sigma_x$ and $m_2 = \sigma_y/r\sigma_x$

Thus $$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\sigma_y / r\sigma_x - r\sigma_y / \sigma_x}{1 + \sigma_y^2 / \sigma_x^2} = \frac{1 - r^2}{r} \cdot \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

When $r = 0$, $\tan\theta \to \infty$ or $\theta = \pi/2$ i.e. when the variables are independent, the two lines of regression are perpendicular to each other.

When $r = \pm 1$, $\tan\theta = 0$ i.e., $\theta = 0$ or π. Thus the lines of regression coincide i.e., there is perfect correlation between the two variables.

Example 25.18. *In a partially destroyed laboratory record, only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate $\bar{x}$, $\bar{y}$ and the coefficient of co-relation between x and y.* (S.V.T.U., 2009 ; U.P.T.U., 2009 ; V.T.U., 2005)

Solution. Since the regression lines pass through $(\bar{x}, \bar{y})$, therefore,

$$4\bar{x} - 5\bar{y} + 33 = 0, \quad 20\bar{x} - 9\bar{y} = 107.$$

Solving these equations, we get $\bar{x} = 13$, $\bar{y} = 17$.

Rewriting the line of regression of y on x as $y = \frac{4}{5}x + \frac{33}{5}$, we get

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{4}{5} \qquad ...(i)$$

Rewriting the line of regression of x on y as $x = \frac{9}{20}y + \frac{107}{9}$, we get

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{9}{20} \qquad ...(ii)$$

Multiplying (i) and (ii), we get

$$r^2 = \frac{4}{5} \times \frac{9}{20} = 0.36 \qquad \therefore \qquad r = 0.6$$

Hence $r = 0.6$, the positive sign being taken as b_{yx} and b_{xy} both are positive.

Example 25.19. *Establish the formula* $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$

Hence calculate r from the following data :

x :	21	23	30	54	57	58	72	78	87	90	
y :	60	71	72	83	110	84	100	92	113	135	(U.P.T.U., 2002)

Solution. (*a*) Let $z = x - y$ so that $\bar{z} = \bar{x} - \bar{y}$.

$$\therefore \qquad z - \bar{z} = (x - \bar{x}) - (y - \bar{y})$$

or $$(z - \bar{z})^2 = (x - \bar{x})^2 + (y - \bar{y})^2 - 2(x - \bar{x})(y - \bar{y})$$

Summing up for n terms, we have

$$\Sigma(z - \bar{z})^2 = \Sigma(x - \bar{x})^2 + \Sigma(y - \bar{y})^2 - 2\Sigma(x - \bar{x})(y - \bar{y})$$

or $$\frac{\Sigma(z - \bar{z})^2}{n} = \frac{\Sigma(x - \bar{x})^2}{n} + \frac{\Sigma(y - \bar{y})^2}{n} - 2\frac{\Sigma(x - \bar{x})(y - \bar{y})}{n}$$

i.e., $$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \qquad \left[\because r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}\right.$$

which is the required result.

(*b*) To fine r, we have to calculate σ_x, σ_y and σ_{x-y}. We make the following table :

x	$X = x - 54$	X^2	y	$Y = y - 100$	Y^2	$y - x$	$(x - y)^2$
21	− 33	1089	60	− 40	1600	39	1521
23	− 31	961	71	− 29	841	48	2304
30	− 24	576	72	− 28	784	42	1764
54	0	0	83	− 17	289	29	841
57	3	9	110	10	100	53	2809
58	4	16	84	− 16	256	26	676
72	18	324	100	0	0	28	784
78	24	576	92	− 8	64	14	196
87	33	1089	113	13	169	26	676
90	36	1296	135	35	1225	45	2025
Total	30	5936		− 80	5328	350	13596

$$\therefore \quad \sigma_x^2 = \frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2 = \frac{5636}{10} - \left(\frac{30}{10}\right)^2 = 593.6 - 9 = 584.6$$

$$\sigma_y^2 = \frac{\Sigma Y^2}{N} - \left(\frac{\Sigma Y}{N}\right)^2 = \frac{5328}{10} - \left(\frac{-80}{10}\right)^2 = 532.8 - 64 = 468.8$$

$$\sigma^2_{x-y} = \frac{\Sigma(x-y)^2}{N} - \left\{\frac{\Sigma(x-y)}{N}\right\}^2 = 1359.6 - 1225 = 134.6$$

From the above formula,

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{584.6 + 468.8 - 134.6}{2 \times 24.18 \times 23.85} = 0.876.$$

Example 25.20. *While calculating correlation coefficient between two variables x and y from 25 pairs of observations, the following results were obtained : $n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$.*

Later it was discovered at the time of checking that the pairs of values

x	y
8	12
6	8

were copied down as

x	y
6	14
8	6

Obtain the correct value of correlation coefficient. (*V.T.U., 2011 S ; S.V.T.U., 2009*)

Solution. To get the correct results, we subtract the incorrect values and add the corresponding correct values.

$\therefore$ The correct results would be

$$\Sigma n = 25,\ \Sigma x = 125 - 6 - 8 + 8 + 6 = 125,\ \Sigma x^2 = 650 - 6^2 - 8^2 + 8^2 + 6^2 = 650$$

$$\Sigma y = 100 - 14 - 6 + 12 + 8 = 100,\ \Sigma y^2 = 460 - 14^2 - 6^2 + 12^2 + 8^2 = 436$$

$$\Sigma xy = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{[\{n\Sigma x^2 - (\Sigma x)^2\}\{n\Sigma y^2 - (\Sigma y^2)\}]}} = \frac{25 \times 520 - 125 \times 100}{\sqrt{[\{25 \times 650 - (125)^2\}\{25 \times 436 - (100)^2\}]}}$$

$$= \frac{20}{\sqrt{(25 \times 36)}} = \frac{2}{3}.$$

25.15 STANDARD ERROR OF ESTIMATE

The sum of the squares of the deviations of the points from the line of regression of y on x is

$$\Sigma(y - a - bx)^2 = \Sigma(Y - bX)^2, \text{ where } X = x - \bar{x},\ Y = y - \bar{y}$$

$$= \sum\left(Y - r\frac{\sigma_y}{\sigma_x}X\right)^2 = \Sigma Y^2 - 2r(\sigma_y/\sigma_x)\,\Sigma XY + r^2(\sigma_y^2/\sigma_x^2)\,\Sigma X^2$$

$$= n\sigma_y^2 - 2r(\sigma_y/\sigma_x)\,r\,.\,n\sigma_x\sigma_y + r^2(\sigma_y^2/\sigma_x^2)\,.\,n\sigma_x^2 = n\sigma_y^2(1 - r^2).$$

Denoting this sum of squares by nS_y^2, we have $S_y = \sigma_y\sqrt{(1 - r^2)}$ ...(1)

Since S_y is the root mean square deviation of the points from the regression line of y on x, it is called the *standard error of estimate* of y. Similarly the standard error of estimate of x is given by

$$S_x = \sigma_x\sqrt{(1 - r^2)} \quad ...(2)$$

Since the sum of the squares of deviations cannot be negative, it follows that

$$r^2 \le 1 \quad \text{or} \quad -1 \le r \le 1.$$

i.e., correlation coefficient lies between – 1 and 1. (*J.N.T.U., 2006*)

If $r = 1$ or -1, the sum of the squares of deviations from either line of regression is zero. Consequently each deviation is zero and all the points lie on both the lines of regression. These two lines coincide and we say that the correlation between the variables is *perfect*. The nearer r^2 is to unity the closer are the points to the lines of regression. Thus *the departure of r^2 from unity is a measure of departure from linearity of the relationship between the variables.*

25.16 RANK CORRELATION

A group of n individuals may be arranged in order to merit with respect to some characteristic. The same group would give different orders for different characteristics. Considering the orders corresponding to two characteristics A and B, the correlation between these n pairs of ranks is called the *rank correlation* in the characteristics A and B for that group of individuals.

Let x_i, y_i be the ranks of the ith individuals in A and B respectively. Assuming that no two individuals are bracketed equal in either case, each of the variables taking the values 1, 2, 3, ..., n, we have

$$\bar{x} = \bar{y} = \frac{1+2+3+\ldots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

If X, Y be the deviations of x, y from their means, then

$$\Sigma X_i^2 = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 + n(\bar{x})^2 - 2\bar{x}\,\Sigma x_i = \Sigma n^2 + \frac{n(n+1)^2}{4} - 2\frac{n+1}{2} \cdot \Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)^2}{4} - \frac{n(n+1)^2}{2} = \frac{1}{12}(n^3 - n)$$

Similarly $\quad \Sigma Y_i^2 = \frac{1}{12}(n^3 - n)$

Now let $\quad d_i = x_i - y_i \quad$ so that $\quad d_i = (x_i - \bar{x}) - (y_i - \bar{y}) = X_i - Y_i$

$\therefore \quad \Sigma d_i^2 = \Sigma X_i^2 + \Sigma Y_i^2 - 2\Sigma X_i Y_i$

or $\quad \Sigma X_i Y_i = \frac{1}{2}(\Sigma X_i^2 + \Sigma Y_i^2 - \Sigma d_i^2) = \frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2.$

Hence the correlation coefficient between these variables is

$$r = \frac{\Sigma X_i Y_i}{\sqrt{(\Sigma X_i^2 \Sigma Y_i^2)}} = \frac{\frac{1}{12}(n^3 - n) - \frac{1}{2}\Sigma d_i^2}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\,\Sigma d_i^2}{n^3 - n}$$

This is called the *rank correlation coefficient* and is denoted by ρ.

Example 25.21. *Ten participants in a contest are ranked by two judges as follows :*

x :	1	6	5	10	3	2	4	9	7	8
y :	6	4	9	8	1	2	3	10	5	7

Calculate the rank correlation coefficient ρ. (V.T.U., 2002)

Solution. If $\quad d_i = x_i - y_i$, then $d_i = -5, 2, -4, 2, 2, 0, 1, -1, 2, 1$

$\therefore \quad \Sigma d_i^2 = 25 + 4 + 16 + 4 + 4 + 0 + 1 + 1 + 4 + 1 = 60$

Hence $\quad \rho = 1 - \frac{6\Sigma d_i^2}{n^3 - n} = 1 - \frac{6 \times 60}{990} = 0.6$ nearly.

Example 25.22. *Three judges, A, B, C, give the following ranks. Find which pair of judges has common approach*

A :	1	6	5	10	3	2	4	9	7	8
B :	3	5	8	4	7	10	2	1	6	9
C :	6	4	9	8	1	2	3	10	5	7

(J.N.T.U., 2003)

Solution. Here $n = 10$.

	Ranks by		d_1	d_2	d_3	d_1^2	d_2^2	d_3^2
$A\ (= x)$	$B\ (= y)$	$C\ (= z)$	$x - y$	$y - z$	$z - x$			
1	3	6	−2	−3	5	4	9	25
6	5	4	1	1	−2	1	1	4
5	8	9	−3	−1	4	9	1	16
10	4	8	6	−4	−2	36	16	4
3	7	1	−4	6	−2	16	36	4
2	10	2	−8	8	0	64	64	0
4	2	3	2	−1	−1	4	1	1
9	1	10	8	−9	1	64	81	1
7	6	5	1	1	−2	1	1	4
8	9	7	−1	2	−1	1	4	1
Total			0	0	0	200	214	60

$$\therefore \quad \rho(x, y) = 1 - \frac{6\Sigma d_1^2}{n(n^2 - 1)} = 1 - \frac{6 \times 200}{10 \times 99} = -0.2$$

$$\rho(y, z) = 1 - \frac{6\Sigma d_2^2}{n(n^2 - 1)} = 1 - \frac{6 \times 214}{10 \times 99} = -0.3$$

$$\rho(z, x) = 1 - \frac{6\Sigma d_3^2}{n(n - 1)} = 1 - \frac{6 \times 60}{10 \times 99} = 0.6$$

Since $\rho(z, x)$ is maximum, the pair of judges A and C have the nearest common approach.

PROBLEMS 25.4

1. Find the correlation co-efficient and the regression lines of y and x and x on y for the following data :

x :	1	2	3	4	5	
y :	2	5	3	8	7	*(V.T.U., 2010)*

2. Find the correlation coefficient between x and y from the given data :

x :	78	89	97	69	59	79	68	57	
y :	125	137	156	112	107	138	123	108	*(J.N.T.U., 2005)*

3. Find the co-efficient of correlation between industrial production and export using the following data and comment on the result.

Production (in crore tons) :	55	56	58	59	60	60	62	
Exports (in crore tons) :	35	38	38	39	44	43	45	*(Madras, 2000)*

4. Ten people of various heights as under, were requested to read the letters on a car at 25 yards distance. The number of letters correctly read is given below :

Height (in feet) :	5.1	5.3	5.6	5.7	5.8	5.9	5.10	5.11	6.0	6.1
No. of letters :	11	17	19	14	8	15	20	6	8	12

Is there any correlation between heights and visual power ?

5. Using the formula $r = \dfrac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y}$, find r from the following data :

x :	92	89	87	86	83	77	71	63	53	50
y :	86	88	91	77	68	85	52	82	37	57

6. Find the correlation between x (marks in Mathematics) and y (marks in Engineering Drawing) given in the following data :

x \ y	10—40	40—70	70—100	Total
0—30	5	20	—	25
30—60	—	28	2	30
60—90	—	32	13	45
Total	5	80	15	100

7. Find two lines of regression and coefficient of correlation for the data given below :
$n = 18$, $\Sigma x = 12$, $\Sigma y = 18$, $\Sigma x^2 = 60$, $\Sigma y^2 = 96$, $\Sigma xy = 48$. (*U.P.T.U., MCA, 2009*)
8. If the coefficient of correlation between two variables x and y is 0.5 and the acute angle between their lines of regression is $\tan^{-1}(3/8)$, show that $\sigma_x = \frac{1}{2}\sigma_y$. (*V.T.U., 2004*)
9. For two random variables x and y with the same mean, the two regression lines are $y = ax + b$ and $x = \alpha y + \beta$. Show that $\frac{b}{\beta} = \frac{1-a}{1-\alpha}$. Find also the common mean. (*U.P.T.U., 2010*)
10. Two random variables have the regression lines with equations $3x + 2y = 26$ and $6x + y = 31$. Find the mean values and the correlation coefficient between x and y. (*Madras, 2002*)
11. The regression equations of two variables x and y are $x = 0.7y + 5.2$, $y = 0.3x + 2.8$. Find the means of the variables and the coefficient of correlation between them. (*Osmania, 2002*)
12. In a partially destroyed laboratory data, only the equations giving the two lines of regression of y on x and x on y are available and are respectively, $7x - 16y + 9 = 0$, $5y - 4x - 3 = 0$. Calculate the co-efficient of correlation, $\bar{x}$ and $\bar{y}$.
13. The following results were obtained from records of age (x) and blood pressure (y) of a group of 10 men :

$$\left.\begin{array}{lcc} & x & y \\ \textit{Mean} & 53 & 142 \\ \textit{Variance} & 130 & 165 \end{array}\right\} \quad \text{and} \quad \Sigma(x-\bar{x})(y-\bar{y}) = 1220.$$

Find the appropriate regression equation and use it to estimate the blood pressure of a man whose age is 45.
14. Compute the standard error of estimate S_x for the respective heights of the following 12 couples :

Height x of husband (inches)	:	68	66	68	65	69	66	68	65	71	67	68	70
Height y of wife (inches)	:	65	63	67	64	68	62	70	66	68	67	69	71

15. Calculate the rank correlation coefficient from the following data showing ranks of 10 students in two subjects :

Maths	:	3	8	9	2	7	10	4	6	1	5
Physics	:	5	9	10	1	8	7	3	4	2	6

16. Find the rank correlation for the following data :

x	:	56	42	72	36	63	47	55	49	38	42	68	60
y	:	147	125	160	118	149	128	150	145	115	140	152	155

(*S.V.T.U., 2009 ; J.N.T.U., 2003*)

25.17 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 25.5

Select the correct answer or fill up the blanks in each of the following questions :

1. The median of the numbers 11, 10, 12, 13, 9 is
(*a*) 12.5 (*b*) 12 (*c*) 10.5 (*d*) 11.
2. The mode of the numbers 7, 7, 7, 9, 10, 11, 11, 11, 12 is
(*a*) 11 (*b*) 12 (*c*) 7 (*d*) 7 and 11.

3. S.D. is defined as

(a) $\sqrt{\dfrac{\Sigma f\,(x-\bar{x})^2}{\Sigma f}}$ (b) $\dfrac{\Sigma f\,(x-\bar{x})}{\Sigma f}$ (c) $\dfrac{\Sigma f\,(x-\bar{x})^2}{\Sigma f}$.

4. Coefficient of variation is

(a) $\dfrac{\sigma}{\bar{x}} \times 100$ (b) $\dfrac{\sigma}{x}$ (c) $\sqrt{\dfrac{\sigma^2}{x}} \times 100$.

5. Average scores of three batsman A, B, C are respectively 40, 45 and 55 and their S.D.s are respectively 9, 11, 16. Which batsman is more consistent ?

(a) A (b) B (c) C.

6. The equations of regression lines are $y = 0.5x + a$ and $x = 0.4y + b$. The correlation coefficient is

(a) $\sqrt{0.2}$ (b) 0.45 (c) $-\sqrt{0.2}$.

7. If the correlation coefficient is 0, the two regression lines are

(a) parallel (b) perpendicular (c) coincident (d) inclined at 45° to each other.

8. If r_1 and r_2 are two regression coefficients, then signs of r_1 and r_2 depend on ....
9. Regression coefficient of y on x is 0.7 and that of x on y is 3.2. Is the correlation coefficient r consistent ?
10. The standard deviation of the numbers 24, 48, 64, 36, 53 is ....
11. If $y = x + 1$ and $x = 3y - 7$ are the two lines of regression then $\bar{x}$ = ..., $\bar{y}$ = ... and r = ...
12. If the two regression lines are perpendicular to each other, then their coefficient of correlation is ....
13. Quartile deviation is defined as ....
14. The minimum value of correlation coefficient is ....
15. Prediction error of Y is defined as ....
16. If X and Y are independent, then the correlation coefficient between X and Y is ....
17. The point of intersection of the two regression lines is ....
18. The smaller the coefficient of variation, the greater is the .... in the data.
19. The moment coefficient of skewness is given by ....
20. Kurtosis measures the .... of a distribution.
21. The equation of the line of regression of y on x is ....
22. Coefficient of variation = ...
23. The angle between two regression lines is given by ....
24. A frequency curve is said to be Mesokurtic when β_2 is ....
25. Correlation coefficient is the geometrical mean between ....
26. When the variables are independent, the two lines of regression are ....
27. Arithmetic mean of the coefficients of regression is .... than the coefficient of correlation.
28. If two regression lines coincide then the coefficient of correlation is ....
29. The rank coefficient is given by ....
30. The ratio of the standard deviation to the mean is known as ....
31. The value of $\Sigma f(x - \bar{x})$ = ...
32. The value of coefficient of correlation lies between .... and ....
33. If the two regression coefficients are -0.4 and -0.9, then the correlation coefficient is ....
34. A distribution with the following constants is positively skew : $Q_1 = 25.8$, median $= 49.0$, $Q_3 = 64.2$. (True or False)
35. Quartile coefficient of skewness is $\dfrac{Q_3 + Q_1 - 2Q_2}{Q_3 - Q_1}$. (True or False)
36. Skewness indicates peakedness of the frequency distribution. (True or False)

CHAPTER

26

Probability and Distributions

26.1 (1) INTRODUCTION

We often hear such statements : 'It is likely to rain today', 'I have a fair chance of getting admission', and 'There is an even chance that in tossing a coin the head may come up'. In each case, we are not certain of the outcome, but we wish to assess the chances of our predictions coming true. The study of probability provides a mathematical framework for such assertions and is essential in every decision making process. Before defining probability, let us explain a few terms :

(2) Principle of counting. If an event can happen in n_1 ways and thereafter for each of these events a second event can happen in n_2 ways, and for each of these first and second events a third event can happen for n_3 ways and so on, then the number of ways these m event can happen is given by the product $n_1 \cdot n_2 \cdot n_3 \ldots n_m$.

(3) Permutations. A permutation of a number of objects is their arrangement in some definite order. Given three letters a, b, c, we can permute them two at a time as "bc, cb ; ca, ac; ab, ba" yielding 6 permutations. The combinations or groupings are only 3, *i.e.*, bc, ca, ab. Here the order is immetrial.

The number of permutations of n different thing taken r at a time is

$$n(n-1)(n-2)\ldots(n-r+1), \text{ which is denoted by } {}^nP_r.$$

Thus $${}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Permutations with repetitions. The number of permutations of n objects of which n_1 are alike, n_2 are alike and n_3 are alike is $\dfrac{n!}{n_1!\,n_2!\,n_3!}$.

(4) Combinations. *The number of combinations of n different objects taken r at a time is denoted by* ${}^nC_r$. If we take any one of the combinations, its r objects can be arranged in $r!$ ways. So the total number of arrangements which can be obtained from all the combinations is ${}^nP_r = {}^nC_r \cdot r!$.

Thus $${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

Also $${}^nC_{n-r} = {}^nC_r$$

e.g., $${}^{25}P_4 = 25 \times 24 \times 23 \times 22;\ {}^{25}C_{21} = {}^{25}C_4 = \frac{25 \times 24 \times 23 \times 22}{4 \times 3 \times 2 \times 1}.$$

Example 26.1. *In how many ways can one make a first, second, third and fourth choice among 12 firms leasing construction equipment.* (J.N.T.U., 2003)

Solution. First choice can be made from any of the 12 firms. Thereafter the second choice can be made from among the remaining 11 firms. Then the third choice can be made from the remaining 10 firms and the fourth choice can be made from the 9 firms.

Thus from the *principle of counting*, the number of ways in which first, second, third and fourth choice can be affected $= 12 \times 11 \times 10 \times 9 = 11880$.

Example 26.2. *Find the number of permutations of all the letters of the word (i) Committee (ii) Engineering.*

Solution. (*i*) $n = 9, n_1 (m, m) = 2, n_2 (t, t) = 2, n_3 (e, e) = 2$

$\therefore$ no. of permutations $= \dfrac{n!}{n_1!\,.\,n_2!\,.\,n_3!} = \dfrac{9!}{2!\,.\,2!\,.\,2!} = 45360.$

(*ii*) $n = 11, n_1(e's) = 3, n_2 (g, g) = 2, n_3 (i, i) = 2, n_4 (n's) = 3$

$\therefore$ no. of permutations $= \dfrac{11!}{3!\,2!\,2!\,3!} = 277200.$

Example 26.3. *From six engineers and five architects a committee is to be formed having three engineers and two architects. How many different committees can be formed if (i) there is no restriction. (ii) two particular engineers must be included. (iii) one particular architect must be excluded.*

Solution. (*i*) Number of committees ${}^6C_3 \times {}^5C_2 = \dfrac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \dfrac{5\,.\,4}{2\,.\,1} = 200.$

(*ii*) Here we have to choose one engineer from the remaining four engineers.

$\therefore$ no. of committees $= {}^4C_1 \times {}^5C_2 = 4 \times \dfrac{5\,.\,4}{2\,.\,1} = 40$

(*iii*) Here we have to choose two architects from the remaining four architects.

$\therefore$ no. of committees $= {}^6C_3 \times {}^4C_2 = \dfrac{6\,.\,5\,.\,4}{3\,.\,2\,.\,1} \times \dfrac{4\,.\,3}{2\,.\,1} = 120.$

PROBLEMS 26.1

1. If a test consists of 12 true-false questions, in how many different ways can a student make the test paper with one answer to each question. (J.N.T.U., 2003)
2. How many 4-digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9, without repetition ? How many of these are less than 500 ?
3. A student has to answer 9 out of 12 questions. How many choices has he (*i*) if he must answer first two questions (*ii*) if he must answer at least four of the first five questions.
4. How many car number plates can be made if each plate contains two different letters followed by three different digits ? Solve the problem (*a*) with repetitions and (*b*) without repetitions.

26.2 (I) BASIC TERMINOLOGY

(*i*) **Exhaustive events.** A set of events is said to be *exhaustive*, if it includes all the possible events. For example, in tossing a coin there are two exhaustive cases either head or tail and there is no third possibility.

(*ii*) **Mutually exclusive events.** If the occurrence of one of the events precludes the occurrence of all other, then such a set of events is said to be *mutually exclusive*. Just as tossing a coin, either head comes up or the tail and both can't happen at the same time, *i.e.*, these are two mutually exclusive cases.

(*iii*) **Equally likely events.** If one of the events cannot be expected to happen in preference to another then such events are said to be *equally likely*. For instance, in tossing a coin, the coming of the head or the tail is equally likely.

Thus when a *die** is thrown, the turning up of the six different faces of the die are exhaustive, mutually exclusive and equally likely.

(*iv*) **Odds in favour of an event.** If the number of ways favourable to an event A is m and the number of ways not favourable to A is n then *odds in favour of* $A = m/n$ and *odds against* $A = n/m$.

(2) Definition of probability. *If there are n exhaustive, mutually exclusive and equally likely cases of which m are favourable to an event A, then probability (p) of the happening of A is*

$$\mathbf{P(A) = m/n.}$$

As there are $n - m$ cases in which A will not happen (denoted by A'), *the chance of A not happening is q or P (A') so that*

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p$$

i.e., $$P(A') = 1 - P(A) \text{ so that } \mathbf{P(A) + P(A') = 1,}$$

i.e., if an event is certain to happen then its probability is unity, while if it is certain not to happen, its probability is zero.

> **Obs.** This definitions of probability fails when
> (*i*) number of outcomes is infinite (not exhaustive) and (*ii*) outcomes are not equally likely.

(3) Statistical (or Empirical) definition of probability. *If in n trials, an event A happens m times, then the probability (p) of happening of A is given by*

$$p = P(A) = \underset{n \to \infty}{Lt} \frac{m}{n}$$

Example 26.4. *Find the chance of throwing (a) four, (b) an even number with an ordinary six faced die.*

Solution. (*a*) There are six possible ways in which the die can fall and of these there is only one way of throwing 4. Thus the required chance $= \frac{1}{6}$.

(*b*) There are six possible ways in which the die can fall. Of these there are only 3 ways of getting 2, 4 or 6. Thus the required chance $= 3/6 = \frac{1}{2}$.

Example 26.5. *What is the chance that a leap year selected at random will contain 53 Sundays ?* (*Madras, 2003*)

Solution. A leap year consists of 366 days, so that there are 52 full weeks (and hence 52 Sundays) and two extra days. These two days can be (*i*) Monday, Tuesday (*ii*) Tuesday, Wednesday, (*iii*) Wednesday, Thursday (*iv*) Thursday, Friday (*v*) Friday, Saturday (*vi*) Saturday, Sunday (*vii*) Sunday, Monday.

Of these 7 cases, the last two are favourable and hence the required probability $= \frac{2}{7}$.

Example 26.6. *A five figure number is formed by the digits 0, 1, 2, 3, 4 without repetition. Find the probability that the number formed is divisible by 4.*

Solution. The five digits can be arranged in 5! ways, out of which 4! will begin with zero.

$\therefore$ total number of 5-figure numbers formed $= 5! - 4! = 96$.

Those numbers formed will be divisible by 4 which will have two extreme right digits divisible by 4, *i.e.*, numbers ending in 04, 12, 20, 24, 32, 40.

Now numbers ending in 04 $= 3! = 6$, numbers ending in 12 $= 3! - 2! = 4$,
numbers ending in 20 $= 3! = 6$, numbers ending in 24 $= 3! - 2! = 4$,
numbers ending in 32 $= 3! - 2! = 4$, and numbers ending in 40 $= 3! = 6$.

[The numbers having 12, 24, 32 in the extreme right are $(3! - 2!)$ since the numbers having zero on the extreme left are to excluded.]

* *Die* is a small cube. Dots 1, 2, 3, 4, 5, 6 are marked on its six faces. The outcome of throwing a die is the number of dots on its upper face.

$\therefore$ total number of favourable ways $= 6 + 4 + 6 + 4 + 4 + 6 = 30$.

Hence the required probability $= \dfrac{30}{96} = \dfrac{5}{16}$.

Example 26.7. *A bag contains 40 tickets numbered 1, 2, 3, ... 40, of which four are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4$). Find the probability of t_3 being 25?*

Solution. Here exhaustive number of cases $= {}^{40}C_4$

If $t_3 = 25$, then the tickets t_1 and t_2 must come out of 24 tickets numbered 1 to 24. This can be done in ${}^{24}C_2$ ways.

Then t_4 must come out of the 15 tickets (numbering 25 to 40) which can be done in ${}^{15}C_1$ ways.

$\therefore$ favourable number of cases $= {}^{24}C_2 \times {}^{15}C_1$

Hence the probability of t_3 being 25 $= \dfrac{{}^{24}C_2 \times {}^{15}C_1}{{}^{40}C_4} = \dfrac{414}{9139}$.

Example 26.8. *An urn contains 5 red and 10 black balls. Eight of them are placed in another urn. What is the chance that the latter then contains 2 red and 6 black balls ?*

Solution. The number of ways in which 8 balls can be drawn out of 15 is ${}^{15}C_8$.

The number of ways of drawing 2 red balls is ${}^5C_2$ and corresponding to each of these ${}^5C_2$ ways of drawing a red ball, there are ${}^{10}C_6$ ways of drawing 6 black balls.

$\therefore$ the total number of ways in which 2 red and 6 black balls can be drawn is ${}^5C_2 \times {}^{10}C_6$.

$\therefore$ the required probability $= \dfrac{{}^5C_2 \times {}^{10}C_6}{{}^{15}C_8} = \dfrac{140}{429}$.

Example 26.9. *A committee consists of 9 students two of which are from 1st year, three from 2nd year and four from 3rd year. Three students are to be removed at random. What is the chance that (i) the three students belong to different classes, (ii) two belong to the same class and third to the different class, (iii) the three belong to the same class ?* (V.T.U., 2002 S)

Solution. (*i*) The total number of ways of choosing 3 students out of 9 is ${}^9C_3$, *i.e.*, 84.

A student can be removed from 1st year students in 2 ways, from 2nd year in 3 ways and from 3rd year in 4 ways, so that the total number of ways of removing three students, one from each group is $2 \times 3 \times 4$.

Hence the required chance $= \dfrac{2 \times 3 \times 4}{{}^9C_3} = \dfrac{24}{84} = \dfrac{2}{7}$.

(*ii*) The number of ways of removing two from 1st year students and one from others

$$= {}^2C_2 \times {}^7C_1.$$

The number of ways of removing two from 2nd year students and one from others

$$= {}^3C_2 \times {}^6C_1.$$

The number of ways of removing 2 from 3rd year students and one from others

$$= {}^4C_2 \times {}^5C_1.$$

$\therefore$ the total number of ways in which two students of the same class and third from the others may be removed

$$= {}^2C_2 \times {}^7C_1 + {}^3C_2 \times {}^6C_1 + {}^4C_2 \times {}^5C_1 = 7 + 18 + 30 = 55.$$

Hence, the required chance $= \dfrac{55}{84}$.

(*iii*) Three students can be removed from 2nd year group in ${}^3C_3$, *i.e.* 1 way and from 3rd year group in ${}^4C_3$, *i.e.*, 4 ways.

$\therefore$ the total number of ways in which three students belong to the same class $= 1 + 4 = 5$.

Hence the required chance $= \dfrac{5}{84}$.

Example 26.10. *A has one share in a lottery in which there is 1 prize and 2 blanks ; B has three shares in a lottery in which there are 3 prizes and 6 blanks ; compare the probability of A's success to that of B's success.*

Solution. A can draw a ticket in ${}^3C_1 = 3$ ways.

The number of cases in which A can get a prize is clearly 1.

$\therefore$ the probability of A's success $= \dfrac{1}{3}$.

Again B can draw a ticket in ${}^9C_3 = \dfrac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84$ ways.

The number of ways in which B gets all blanks $= {}^6C_3 = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$

$\therefore$ the number of ways of getting a prize $= 84 - 20 = 64$.

Thus the probability of B's success $= 64/84 = 16/21$.

Hence A's probability of success : B's probability of success $= \dfrac{1}{3} : \dfrac{16}{21} = 7 : 16$.

26.3 PROBABILITY AND SET NOTATIONS

(1) Random experiment. Experiments which are performed essentially under the same conditions and whose results cannot be predicted are known as *random experiments. e.g.*, Tossing a coin or rolling a die are random experiments.

(2) Sample space. The set of all possible outcomes of a random experiment is called *sample space* for that experiment and is denoted by S.

The elements of the sample space S are called the *sample points*.

e.g., On tossing a coin, the possible outcomes are the head (H) and the tail (T). Thus $S = \{H, T\}$.

(3) Event. The outcome of a random experiment is called an *event*. Thus every subset of a sample space S is an *event*.

The null set ϕ is also an event and is called an *impossible event. Probability of an impossible event* is zero *i.e.*, $P(\phi) = 0$.

(4) Axioms

(*i*) *The numerical value of probability lies between 0 and 1.*

i.e., for any event A of S, $0 \le P(A) \le 1$.

(*ii*) *The sum of probabilities of all sample events is unity i.e.*, $P(S) = 1$.

(*iii*) *Probability of an event made of two or more sample events is the sum of their probabilities.*

(5) Notations

(*i*) *Probability of happening of events* **A** *or* **B** is written as $P(A + B)$ or $P(A \cup B)$.

(*ii*) *Probability of happening of* **both the events A and B** is written as $P(AB)$ or $P(A \cap B)$.

(*iii*) '*Event A implies* ($\Rightarrow$) *event B*' is expressed as $A \subset B$.

(*iv*) '*Events A and B are mutually exclusive*' is expressed as $A \cap B = \phi$.

(6) *For any two events A and B,*

$$P(A \cap B') = P(A) - P(A \cap B)$$

Proof. From Fig. 26.1,

$$(A \cap B') \cup (A \cap B) = A$$

$\therefore \quad P[(A \cap B') \cup (A \cap B)] = P(A)$

or $\quad P(A \cap B') + P(A \cap B) = P(A)$

or $\quad P(A \cap B') = P(A) - P(A \cap B)$

Similarly, $P(A' \cap B) = P(B) - P(A \cap B)$

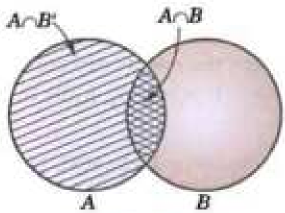


Fig. 26.1

26.4 ADDITION LAW OF PROBABILITY *or* THEOREM OF TOTAL PROBABILITY

(1) *If the probability of an event A happening as a result of a trial is P(A) and the probability of a* **mutually exclusive** *event B happening is P(B), then the probability of* **either** *of the events happening as a result of the trial is* $\mathbf{P(A + B)}$ **or** $\mathbf{P(A \cup B) = P(A) + P(B)}$.

Proof. Let n be the total number of equally likely cases and let m_1 be favourable to the event A and m_2 be favourable to the event B. Then the number of cases favourable to A or B is $m_1 + m_2$. Hence the probability of A or B happening as a result of the trial

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B).$$

(2) *If A, B, are any two events* **(not mutually exclusive),** *then*

$$\mathbf{P\,(A + B) = P\,(A) + P\,(B) - P\,(AB)}$$

or
$$\mathbf{P\,(A \cup B) = P\,(A) + P\,(B) - P\,(A \cap B)}$$

If the events A and B are any two events then, there are some outcomes which favour both A and B. If m_3 be their number, then these are included in both m_1 and m_2. Hence the total number of outcomes favouring either A or B or both is

$$m_1 + m_2 - m_3.$$

Thus the probability of occurrence of A or B or both

$$= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

Hence
$$P(A + B) = P(A) + P(B) - P(AB)$$

or
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Obs. *When A and B are mutually exclusive* $P(AB)$ or $P(A \cap B) = 0$ and we get

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B).$$

In general, for a number of **mutually exclusive** events $A_1, A_2, \ldots A_n$, we have

$$P(A_1 + A_2 + \ldots + A_n) \text{ or } P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

(3) *If A, B, C are any three events, then*

$$\mathbf{P\,(A + B + C) = P\,(A) + P\,(B) + P\,(C) - P\,(AB) - P\,(BC) - P\,(CA) + P\,(ABC).}$$

or
$$\mathbf{P\,(A \cup B \cup C) = P\,(A) + P\,(B) + P\,(C) - P\,(A \cap B) - P\,(B \cap C) - P\,(C \cap A) + P\,(A \cap B \cap C)}$$

Proof. Using the above result for any two events, we have

$$\begin{aligned}
P\,(A \cup B \cup C) &= P\,[(A \cup B) \cup C] \\
&= P\,(A \cup B) + P\,(C) - P\,[(A \cup B) \cap C] \\
&= [P\,(A) + P\,(B) - P\,(A \cap B)] + P\,(C) - P\,[(A \cap C) \cup (B \cap C)] \quad \text{(Distributive Law)} \\
&= P\,(A) + P\,(B) + P\,(C) - P\,(A \cap B) - [P\,(A \cap C) + P\,(B \cap C) - P\,(A \cap B \cap C)] \\
&\qquad [\because \ (A \cap C) \cap (B \cap C) = A \cap B \cap C] \\
&= P\,(A) + P\,(B) + P\,(C) - P\,(A \cap B) - P\,(B \cap C) - P\,(C \cap A) + P\,(A \cap B \cap C) \ [\because \ A \cap C = C \cap A.]
\end{aligned}$$

Example 26.11. *In a race, the odds in favour of the four horses* H_1, H_2, H_3, H_4 *are* $1 : 4, 1 : 5, 1 : 6, 1 : 7$ *respectively. Assuming that a dead heat is not possible, find the chance that one of them wins the race.*

Solution. Since it is not possible for all the horses to cover the same distance in the same time (a dead heat), the events are mutually exclusive.

If p_1, p_2, p_3, p_4 be the probabilities of winning of the horses H_1, H_2, H_3, H_4 respectively, then

$$p_1 = \frac{1}{1+4} = \frac{1}{5} \qquad [\because \text{ Odds in favour of } H_1 \text{ are } 1 : 4]$$

and
$$p_2 = \frac{1}{6}, p_3 = \frac{1}{7}, p_4 = \frac{1}{8}.$$

Hence the chance that one of them wins $= p_1 + p_2 + p_3 + p_4$

$$= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = \frac{533}{840}.$$

Example 26.12. *A bag contains 8 white and 6 red balls. Find the probability of drawing two balls of the same colour.*

Solution. Two balls out of 14 can be drawn in ${}^{14}C_2$ ways which is the total number of outcomes.

Two white balls out of 8 can be drawn in ${}^{8}C_2$ ways. Thus the probability of drawing 2 white balls

$$= \frac{{}^{8}C_2}{{}^{14}C_2} = \frac{28}{91}$$

Similarly 2 red balls out of 6 can be drawn in ${}^{6}C_2$ ways. Thus the probability of drawing 2 red balls

$$= \frac{{}^{6}C_2}{{}^{14}C_2} = \frac{15}{91}.$$

Hence the probability of drawing 2 balls of the same colour (either both white or both red)

$$= \frac{28}{91} + \frac{15}{91} = \frac{43}{91}.$$

Example 26.13. *Find the probability of drawing an ace or a spade or both from a deck of cards* ?*

Solution. The probability of drawing an ace from a deck of 52 cards = 4/52.

Similarly the probability of drawing a card of spades = 13/52, and the probability of drawing an ace of spades = 1/52.

Since the two events (*i.e.*, a card being an ace and a card being of spades) are not mutually exclusive, therefore, the probability of drawing an ace or a spade

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{4}{13}.$$

26.5 (1) INDEPENDENT EVENTS

Two events are said to be *independent*, if happening or failure of one does not affect the happening or failure of the other. Otherwise the events are said to be *dependent*.

For two dependent events A and B, the symbol $P(B/A)$ denotes the probability of occurrence of B, when A has already occurred. It is known as the **conditional probability** and is read as a '*probability of B given A*'.

(2) Multiplication law of probability *or* **Theorem of compound probability.** *If the probability of an event A happening as a result of trial is $P(A)$ and after A has happened the probability of an event B happening as a result of another trial (i.e.,* **conditional probability of B given A**) *is $P(B/A)$, then the probability of* **both** *the events A and B happening as a result of two trials is* $\mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) \,.\, P(B/A)}$.

Proof. Let n be the total number of outcomes in the first trial and m be favourable to the event A so that $P(A) = m/n$.

Let n_1 be the total number of outcomes in the second trial of which m_1 are favourable to the event B so that $P(B/A) = m_1/n_1$.

Now each of the n outcomes can be associated with each of the n_1 outcomes. So the total number of outcomes in the combined trial is nn_1. Of these mm_1 are favourable to both the events A and B. Hence

$$P(AB) \text{ or } P(A \cap B) = \frac{mm_1}{nn_1} = P(A) \,.\, P(B/A).$$

Similarly, the *conditional probability of A given B is $P(A/B)$.*

$\therefore \qquad P(AB)$ or $P(A \cap B) = P(B) \,.\, P(A/B)$

Thus $\qquad P(A \cap B) = P(A) \,.\, P(B/A) = P(B) \,.\, P(A/B).$

(3) *If the events A and B are* **independent,** *i.e.*, if the happening of B does not depend on whether A has happened or not, then $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

$\therefore \qquad \mathbf{P(AB)}$ **or** $\mathbf{P(A \cap B) = P(A) \,.\, P(B)}$.

In general, $\quad P(A_1A_2...A_n)$ or $P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) \,.\, P(A_2) .... P(A_n)$.

---

* *Cards* : A pack of cards consists of four suits *i.e.*, Hearts, Diamonds, Spades and Clubs. Each suit has 13 cards : an Ace, a King, a Queen, a Jack and nine cards numbered 2, 3, 4, ..., 10. Hearts and Diamonds are *red* while Spades and Clubs are *black*.

Cor. If p_1, p_2 be the probabilities of happening of two independent events, then

(*i*) the probability that the first event happens and the second fails is $p_1(1 - p_2)$.

(*ii*) the probability that both events fail to happen is $(1 - p_1)(1 - p_2)$.

(*iii*) the probability that at least one of the events happens is

$1 - (1 - p_1)(1 - p_2)$. This is commonly known as their **cumulative probability.**

In general, if $p_1, p_2, p_3, \ldots p_n$ be the chances of happening of n independent events, then their cumulative probability (i.e., the chance that at least one of the events will happen) is

$$1 - (1 - p_1)(1 - p_2)(1 - p_3) \ldots (1 - p_n).$$

Example 26.14. *Two cards are drawn in succession from a pack of 52 cards. Find the chance that the first is a king and the second a queen if the first card is (i) replaced, (ii) not replaced.*

Solution. (*i*) The probability of drawing a king $= \frac{4}{52} = \frac{1}{13}$.

If the card is replaced, the pack will again have 52 cards so that the probability of drawing a queen is 1/13.

The two events being independent, the probability of drawing both cards in succession $= \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

(*ii*) The probability of drawing a king $= \frac{1}{13}$.

If the card is not replaced, the pack will have 51 cards only so that the chance of drawing a queen is 4/51.

Hence the probability of drawing both cards $= \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$.

Example 26.15. *A pair of dice is tossed twice. Find the probability of scoring 7 points (a) once, (b) at least once (c) twice.* *(Kurukshetra, 2009 S ; V.T.U., 2004)*

Solution. In a single toss of two dice, the sum 7 can be obtained as (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1) *i.e.*, in 6 ways, so that the probability of getting 7 = 6/36 = 1/6.

Also the probability of not getting 7 = 1 – 1/6 = 5/6.

(*a*) The probability of getting 7 in the first toss and not getting 7 in the second toss = 1/6 × 5/6 = 5/36.

Similarly, the probability of not getting 7 in the first toss and getting 7 in the second toss = 5/6 × 1/6 = 5/36.

Since these are mutually exclusive events, addition law of probability applies.

$\therefore$ required probability $= \frac{5}{36} + \frac{5}{36} = \frac{5}{18}$.

(*b*) The probability of not getting 7 in either toss $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$\therefore$ the probability of getting 7 at least once $= 1 - \frac{25}{36} = \frac{11}{36}$.

(*c*) The probability of getting 7 twice $= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

Example 26.16. *There are two groups of objects : one of which consists of 5 science and 3 engineering subjects, and the other consists of 3 science and 5 engineering subjects. An unbiased die is cast. If the number 3 or number 5 turns up, a subject is selected at random from the first group, otherwise the subject is selected at random from the second group. Find the probability that an engineering subject is selected ultimately*

Solution. Prob. of turning up 3 or 5 $= \frac{2}{6} = \frac{1}{3}$.

Prob. of selecting an engg. subject from first group $= \frac{3}{8}$

$\therefore$ Prob of selecting an engg. subject from first group on turning up 3 or 5

$$= \frac{1}{3} \times \frac{3}{8} = \frac{1}{8} \qquad \ldots(i)$$

Now prob. of not turning 3 or 5 $= 1 - \frac{1}{3} = \frac{2}{3}$.

Prob. of selecting an engg. subject from second group $= \frac{5}{8}$

$\therefore$ prob. of selecting an engg. subject from second group on turning up 3 or 5

$$= \frac{2}{3} \times \frac{5}{8} = \frac{5}{12} \qquad \ldots(ii)$$

Thus the prob. of selecting an engg. subject

$$= \frac{1}{8} + \frac{5}{12} = \frac{13}{24}. \qquad \text{[From } (i) \text{ and } (ii)\text{]}$$

Example 26.17. *A box A contains 2 white and 4 black balls. Another box B contains 5 white and 7 black balls. A ball is transferred from the box A to the box B. Then a ball is drawn from the box B. Find the probability that it is white.* (V.T.U., 2004)

Solution. The probability of drawing a white ball from box B will depend on whether the transferred ball is black or white.

If a black ball is transferred, its probability is 4/6. There are now 5 white and 8 black balls in the box B. Then the probability of drawing white ball from box B is $\frac{5}{13}$.

Thus the probability of drawing a white ball from urn B, if the transferred ball is black

$$= \frac{4}{6} \times \frac{5}{13} = \frac{10}{39}.$$

Similarly the probability of drawing a white ball from urn B, if the transferred ball is white

$$= \frac{2}{6} \times \frac{6}{13} = \frac{2}{13}.$$

Hence required probability $= \frac{10}{39} + \frac{2}{13} = \frac{16}{39}$.

Example 26.18. *(a) A biased coin is tossed till a head appears for the first time. What is the probability that the number of required tosses is odd.* (Mumbai, 2006)

(b) Two persons A and B toss an unbiased coin alternately on the understanding that the first who gets the head wins. If A starts the game, find their respective chances of winning. (Madras, 2000 S)

Solution. (a) Let p be the probability of getting a head and q the probability of getting a tail in a single toss, so that $p + q = 1$.

Then probability of getting head on an odd toss

= Probability of getting head in the 1st toss

+ Probability of getting head in the 3rd toss

+ Probability of getting head in the 5th toss + ... ∞

$$= p + qqp + qqqqp + \ldots \infty$$

$$= p(1 + q^2 + q^4 + \ldots) = p \cdot \frac{1}{1-q^2} \quad (q < 1)$$

$$= p \cdot \frac{1}{(1-q)(1+q)} = p \cdot \frac{1}{p(1+q)} = \frac{1}{1+q}.$$

(b) Probability of getting a head = 1/2. Then A can win in 1st, 3rd, 5th, ... throws.

$\therefore$ the chances of A's winning

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \left(\frac{1}{2}\right)^6 \frac{1}{2} + \cdots$$

$$= \frac{1/2}{1-(1/2)^2} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.$$

Hence the chance of B's winning $= 1 - 2/3 = 1/3$.

Example 26.19. *Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd, if*

(i) the two cards are drawn together.

(ii) the two cards are drawn one after the other without replacement.

(iii) the two cards are drawn one after the other with replacement. (J.N.T.U., 2003)

Solution. (*i*) Two cards out of 10 can be selected in ${}^{10}C_2 = 45$ ways. The sum is odd if one number is odd and the other number is even. There being 5 odd numbers (1, 3, 5, ..., 9) and 5 even numbers (2, 4, 6, 8, 10), an odd and an even number is chosen in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25}{45} = \frac{5}{9}.$$

(*ii*) Two cards out of 10 can be selected one after the other *without replacement* in $10 \times 9 = 90$ ways.

An odd number is selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways

Thus $$p = \frac{25 + 25}{90} = \frac{5}{9}.$$

(*iii*) Two cards can be selected one after the other *with replacement* in $10 \times 10 = 100$ ways.

An odd number in selected in $5 \times 5 = 25$ ways and an even number in $5 \times 5 = 25$ ways.

Thus $$p = \frac{25 + 25}{100} = \frac{1}{2}.$$

Example 26.20. *Given $P(A) = 1/4$, $P(B) = 1/3$ and $P(A \cup B) = 1/2$, evaluate $P(A/B)$, $P(B/A)$, $P(A \cap B')$ and $P(A/B')$.*

Solution. (*i*) Since $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ $$\frac{1}{2} = \frac{1}{4} + \frac{1}{3} - P(A \cap B) \text{ or } P(A \cap B) = \frac{1}{12}$$

Thus $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/12}{1/3} = \frac{1}{4}.$$

(*ii*) $$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/12}{1/4} = \frac{1}{3}.$$

(*iii*) $$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}.$$

(*iv*) $$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/6}{1 - P(B)} = \frac{1/6}{1 - 1/3} = \frac{1}{4}.$$

Example 26.21. *The odds that a book will be reviewed favourably by three independent critics are 5 to 2, 4 to 3 and 3 to 4. What is the probability that of the three reviews, a majority will be favourable.* (V.T.U., 2003 S)

Solution. The probability that the book shall be reviewed favourably by first critic is 5/7, by second 4/7 and by third 3/7.

A majority of the three reviews will be favourable when two or three are favourable.

$\therefore$ prob. that the first two are favourable and the third unfavourable

$$= \frac{5}{7} \times \frac{4}{7} \times \left(1 - \frac{3}{7}\right) = \frac{80}{343}$$

Prob. that the first and third are favourable and second unfavourable

$$= \frac{5}{7} \times \frac{3}{7} \times \left(1 - \frac{4}{7}\right) = \frac{45}{343}$$

Prob. that the second and third are favourable and the first unfavourable

$$= \frac{4}{7} \times \frac{3}{7} \times \left(1 - \frac{5}{7}\right) = \frac{24}{343}$$

Finally, prob. that all the three are favourable $= \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} = \frac{60}{343}$

Since they are mutually exclusive events, the required prob.

$$= \frac{80}{343} + \frac{45}{343} + \frac{24}{343} + \frac{60}{343} = \frac{209}{343}.$$

Example 26.22. *I can hit a target 3 times in 5 shots, B 2 times in 5 shots and C 3 times in 4 shots. They fire a volley. What is the probability that (i) two shots hit, (ii) atleast two shots hit ?*

(*A.M.I.E.T.E., 2003 ; Madras, 2000 S*)

Solution. Prob. of A hitting the target = 3/5, prob. of B hitting the target = 2/5

Prob. of C hitting the target = 3/4.

(*i*) In order that two shots may hit the target, the following cases must be considered :

$$p_1 = \text{Chance that } A \text{ and } B \text{ hit and } C \text{ fails to hit} = \frac{3}{5} \times \frac{2}{3} \times \left(1 - \frac{3}{4}\right) = \frac{6}{100}$$

$$p_2 = \text{Chance that } B \text{ and } C \text{ hit and } A \text{ fails to hit} = \frac{2}{5} \times \frac{3}{4} \times \left(1 - \frac{3}{5}\right) = \frac{12}{100}$$

$$p_3 = \text{Chance that } C \text{ and } A \text{ hit and } B \text{ fails to hit} = \frac{3}{4} \times \frac{3}{5} \times \left(1 - \frac{2}{5}\right) = \frac{27}{100}$$

Since these are mutually exclusive events, the probability that any 2 shots hit

$$= p_1 + p_2 + p_3 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} = 0.45.$$

(*ii*) In order that at least two shots may hit the target, we must also consider the case of all A, B, C hitting the target [in addition to the three cases of (*i*)] for which

$$p_4 = \text{chance that } A, B, C \text{ all hit} = \frac{3}{5} \times \frac{2}{5} \times \frac{3}{4} = \frac{18}{100}$$

Since all these are mutually exclusive events, the probability of *atleast* two shots hit

$$= p_1 + p_2 + p_3 + p_4 = \frac{6}{100} + \frac{12}{100} + \frac{27}{100} + \frac{18}{100} = 0.63.$$

Example 26.23. *A problem in mechanics is given to three students A, B, and C whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved.* (*V.T.U., 2004*)

Solution. The probability that A can solve the problem is 1/2.

The probability that A cannot solve the problem is $1 - \frac{1}{2}$.

Similarly the probabilities that B and C cannot solve the problem are $1 - \frac{1}{3}$ and $1 - \frac{1}{4}$.

$\therefore$ the probability that A, B and C cannot solve the problem is $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$.

Hence the probability that the problem will be solved, *i.e.*, at least one student will solve it

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = \frac{3}{4}.$$

Example 26.24. *The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if*

(*i*) *the class consists of 4 boys and 3 girls.*

(*ii*) *the class consists of 3 boys and 3 girls.* (*J.N.T.U., 2003*)

Solution. (*i*) As there are 7 students in the class, the first examined must be a boy.

$\therefore$ prob. that first is a boy $= \dfrac{4}{7}$

Then the prob. that the second is a girl $= \dfrac{3}{6}$.

$\therefore$ prob. of the next boy $= \dfrac{3}{5}$

Similarly the prob. that the fourth is a girl $= \dfrac{2}{4}$,

the prob. that the fifth is a boy $= \dfrac{2}{3}$,

the prob. that the sixth is a girl $= \dfrac{1}{2}$

and the last is a boy $= \dfrac{1}{1}$.

Thus $$p = \frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{35}.$$

(*ii*) The first student is a boy and the first student is a girl are two mutually exclusive cases. If the first student is a boy, then the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}.$$

If the first student is a girl, then the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}.$$

Thus the required prob. $p = p_1 + p_2 = \dfrac{1}{20} + \dfrac{1}{20} = \dfrac{1}{10}$.

Example 26.25. (*Huyghen's problem*) *A and B throw alternately with a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, find his chance of winning.*

(*Madras, 2006 ; J.N.T.U., 2003*)

Solution. The sum 6 can be obtained as follows : (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), *i.e.*, in 5 ways.

The probability of A's throwing 6 with 2 dice is $\dfrac{5}{36}$.

$\therefore$ the probability of A's not throwing 6 is 31/36.

Similarly the probability of B's throwing 7 is 6/36, *i.e.*, $\dfrac{1}{6}$.

$\therefore$ the probability of B's not throwing 7 is 5/6.

Now A can win if he throws 6 in the first, third, fifth, seventh etc. throws.

$\therefore$ the chance of A's wining

$$= \frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \frac{31}{36}\times\frac{5}{6}\times\frac{31}{36}\times\frac{5}{6}\times\frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1 + \left(\frac{31}{36}\times\frac{5}{6}\right) + \left(\frac{31}{36}\times\frac{5}{6}\right)^2 + \left(\frac{31}{36}\times\frac{5}{6}\right)^3 + \cdots\right]$$

$$= \frac{5}{36}\cdot\frac{1}{1-(31/36)\times(5/6)} = \frac{5}{36}\times\frac{36\times 6}{61} = \frac{30}{61}.$$

PROBLEMS 26.2

1. (*i*) Given $P(A) = 1/2$, $P(B) = 1/3$ and $P(AB) = 1/4$, find the value $P(A + B)$. (*Burdwan, 2003*)

(*ii*) Let A and B be two events with $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A/B)$, $P(A \cup B)$, $P(A'/B')$.

(*Kurukshetra, 2009 ; V.T.U., 2003 S*)

2. In a single throw with two dice, what is the chance of throwing
   (a) two aces ? (b) 7 ? Is this probability the same as that for getting 7 in two throws of a single die ?
3. Compare the chances of throwing 4 with one dice, 8 with two dice and 12 with three dice.
4. Find the probability that a non-leap year should have 53 Saturdays ? (Madras, 2003)
5. When a coin is tossed four times, find the probability of getting (i) exactly one head, (ii) at most three heads and (iii) at least two heads ? (V.T.U., 2000 S)
6. Ten coins are thrown simultaneously. Find the probability of getting at least seven heads. (P.T.U., 2003)
7. If all the letters of word 'ENGINEER' be written at random, what is the probability that all the latters E are found together.
8. A ten digit number is formed using the digits from zero to nine, every digit being used only once. Find the probability that the number is divisible by 4.
9. Four cards are drawn from a pack of 52 cards. What is the chance that
   (i) no two cards are of equal value ? (ii) each belongs to a different suit ?
10. Suppose 5 cards are drawn at random from a pack of 52 cards. If all cards are red what is the probability that all of them are hearts? (Mumbai, 2005)
11. Out of 50 rare books, 3 of which are especially valuable, 5 are stolen at random by a thief. What is the probability that
   (a) none of the 3 is included ? (b) 2 of the 3 are included ?
12. Five men in a company of twenty are graduates. If 3 men are picked out of 20 at random, what is the probability that
   (a) they are all graduates ? (b) at least one is graduate ?
13. From 20 tickets marked from 1 to 20, one ticket is drawn at random. Find the probability that it is marked with a multiple of 3 or 5.
14. Five balls are drawn from a bag containing 6 white and 4 black balls. What is the chance that 3 white and 2 black balls are drawn ?
15. The probability of n independent events are $p_1, p_2, p_3, \ldots p_n$. Find the probability that at least one of the events will happen. Use this result to find the chance of getting at least one six in a throw of 4 dice.
16. Find the probability of drawing 4 white balls and 2 black balls without replacement from a bag containing 1 red, 4 black and 6 white balls.
17. A bag contains 10 white and 15 black balls. Two balls are drawn in succession. What is the probability that one of them is black and the other white ?
18. A purse contains 2 silver and 4 copper coins and a second purse contains 4 silver and 4 copper coins. If a coin is selected at random from one of the two purses, what is the probability that it is a silver coin ? (Osmania, 2002)
19. A box I contains 5 white balls and 6 black balls. Another box II contains 6 white balls and 4 black balls. A box is selected at random and then a ball is drawn from it : (i) what is the probability that the ball drawn will be white ? (ii) Given that the ball drawn is white, what is the probability that it came from box I. (Mumbai, 2006)
20. A party of n persons take their seats at random at a round table ; find the probability that two specified persons do not sit together.
21. A speaks the truth in 75% cases, and B in 80% of the cases. In what percentage of cases, are they likely to contradict each other in stating the same fact ? (V.T.U., 2002 S)
22. The probability that Sushil will solve a problem is 1/4 and the probability that Ram will solve it is 2/3. If Sushil and Ram work independently, what is the probability that the problem will be solved by (a) both of them, (b) at least one of them ?
23. A student takes his examination in four subjects, P, Q, R, S. He estimates his chances of passing in P as 4/5, in Q as 3/4, in R as 5/6 and in S as 2/3. To qualify, he must pass in P and at least two other subjects. What is the probability that he qualifies ? (Madras, 2000 S)
24. The probability that a 50 year old man will be alive at 60 is 0.83 and the probability that a 45 year old woman will be alive at 55 is 0.87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence ?
25. If on an average one birth in 80 is a case of twins, what is the probability that there will be at least one case of twins in a maternity hospital on a day when 20 births occur ?
26. Two persons A and B fire at a target independently and have a probability 0.6 and 0.7 respectively of hitting the target. Find the probability that the target is destroyed.
27. A and B throw alternately with a pair of dice. The one who throws 9 first wins. Show that the chances of their winning are 9 : 8.

26.6 BAYE'S THEOREM

An event A corresponds to a number of exhaustive events $B_1, B_2, ..., B_n$. If $P(B_i)$ and $P(A/B_i)$ are given, then

$$P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i) P(A/B_i)}.$$

Proof. By the multiplication law of probability,

$$P(AB_i) = P(A)\,P(B_i/A) = P(B_i)\,P(A/B_i) \quad ...(1)$$

$$\therefore \quad P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{P(A)} \quad ...(2)$$

Since the event A corresponds to $B_1, B_2, ..., B_n$, we have by the addition law of probability,

$$P(A) = P(AB_1) + P(AB_2) + ... + P(AB_n) = \Sigma P(AB_i) = \Sigma P(B_i)\,P(A/B_i) \quad \text{[By (1)]}$$

Hence from (2), we have $P(B_i/A) = \dfrac{P(B_i)\,P(A/B_i)}{\Sigma P(B_i)\,P(A/B_i)}$

which is known as the *theorem of inverse probability*.

Obs. The probabilities $P(B_i)$, $i = 1, 2, ..., n$ are called *apriori probabilities* because these exist before we get any information from the experiment.

The probabilities $P(A/B_i)$, $i = 1, 2, ..., n$ are called *posteriori probabilities*, because these are found after the experiment results are known.

Example 26.26. *Three machines M_1, M_2 and M_3 produce identical items. Of their respective output 5%, 4% and 3% of items are faulty. On a certain day, M_1 has produced 25% of the total output, M_2 has produced 30% and M_3 the remainder. An item selected at random is found to be faulty. What are the chances that it was produced by the machine with the highest output?*

Solution. Let the event of drawing a faulty item from any of the machines be A, and the event that an item drawn at random was produced by M_i be B_i. We have to find $P(B_i/A)$ for which we proceed as follows :

	M_1	M_2	M_3	*Remarks*
$P(B_i)$	0.25	0.30	0.45	$\because$ sum = 1
$P(A/B_i)$	0.05	0.04	0.03	
$P(B_i)\,P(A/B_i)$	0.0125	0.012	0.0135	sum = 0.38
$P(B_i/A)$	$\frac{0.0125}{0.038}$	$\frac{0.012}{0.038}$	$\frac{0.0135}{0.038}$	by Baye's theorem

The highest output being from M_3, the required probability = 0.0135/0.038 = 0.355.

Example 26.27. *There are three bags : first containing 1 white, 2 red, 3 green balls ; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.* (J.N.T.U., 2003)

Solution. Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A : the two balls are white and red.

Now $\quad P(B_1) = P(B_2) = P(B_3) = \dfrac{1}{3}$

$P(A/B_1) = P$ (a white and a red ball are drawn from first bag)

$$= ({}^1C_1 \times {}^2C_1)/{}^6C_2 = \frac{2}{15}$$

Similarly $\quad P(A/B_2) = ({}^2C_1 \times {}^3C_1)/{}^6C_2 = \dfrac{2}{5}$, $P(A/B_3) = ({}^3C_1 \times {}^1C_1)/{}^6C_2 = \dfrac{1}{5}$

By Baye's theorem, $\quad P(B_2/A) = \dfrac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)}$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{1}{3} \times \frac{2}{15} + \frac{1}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{1}{5}} = \frac{6}{11}.$$

PROBLEMS 26.3

1. In a certain college, 4% of the boys and 1% of girls are taller than 1.8 m. Further more 60% of the students are girls. If a student is selected at random and is found to be taller than 1.8 m., what is the probability that the student is a girl ?
2. In a bolt factory, machines A, B and C manufacture 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B or C ? (V.T.U., 2006 ; Rohtak, 2005 ; Madras, 2000 S)
3. In a bolt factory, there are four machines A, B, C, D manufacturing 20%, 15%, 25% and 40% of the total output respectively. Of their outputs 5%, 4%, 3% and 2% in the same order are defective bolts. A bolt is chosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine D ? (Hissar, 2007 ; J.N.T.U., 2003)
4. The contents of three urns are : 1 white, 2 red, 3 green balls ; 2 white, 1 red, 1 green balls and 4 white, 5 red, 3 green balls. Two balls are drawn from an urn chosen at random. These are found to be one white and one green. Find the probability that the balls so drawn came from the third urn. (Kurukshetra, 2007)

26.7 RANDOM VARIABLE

If a real variable X be associated with the outcome of a random experiment, then since the values which X takes depend on chance, it is called a *random variable* or a *stochastic variable* or simply a *variate*. For instance, if a random experiment E consists of tossing a pair of dice, the sum X of the two numbers which turn up have the value 2, 3, 4, ..., 12 depending on chance. Then X is the random variable. It is a function whose values are real numbers and depend on chance.

If in a random experiment, the event corresponding to a number a occurs, then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by $P(X = a)$. Similarly the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \le c$ is written as $P(X \le c)$.

If a random variable takes a finite set of values, it is called a *discrete variate*. On the other hand, if it assumes an infinite number of uncountable values, it is called a *continuous variate*.

26.8 (1) DISCRETE PROBABILITY DISTRIBUTION

Suppose a discrete variate X is the outcome of some experiment. If the probability that X takes the values x_i, is p_i, then

$$P(X = x_i) = p_i \text{ or } p(x_i) \text{ for } i = 1, 2, \ldots$$

where (*i*) $p(x_i) \ge 0$ for all values of i, (*ii*) $\Sigma p(x_i) = 1$

The set of values x_i with their probabilities p_i constitute a **discrete probability distribution** *of the discrete variate X.*

For example, the discrete probability distribution for X, the sum of the numbers which turn on tossing a pair of dice is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

[$\because$ There are $6 \times 6 = 36$ equally likely outcomes and therefore, each has the probability 1/36. We have $X = 2$ for one outcome, *i.e.* (1, 1) ; $X = 3$ for two outcomes (1, 2) and (2, 1); $X = 4$ for three outcomes (1, 3), (2, 2) and (3, 1) and so on.]

(2) Distribution function. *The distribution function F (x) of the discrete variate X is defined by*

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$ where x is any integer. The graph of $F(x)$ will be stair step form (Fig. 26.2). The distribution function is also sometimes called *cumulative distribution function.*

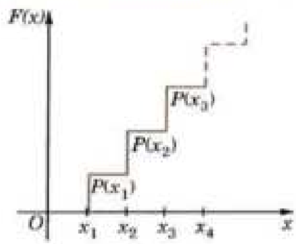


Fig. 26.2