Homogeneous System of Linear Equations :→

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \text{— (1)}$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0.$$

$\Rightarrow \quad AX = 0$

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

→ System (1) is always Consistent since $X = 0$ is always a Solution, Known as Trivial solution.

⇒ homogeneous system Can have Either
(a) Unique Solution i.e. Trivial solution
(b) Infinite no. of solutions.

→ System (1) has a unique solution if Rank(A) = no. of unknowns (n).

→ If Rank(A) < no. of unknowns (n), then System (1) has Infinite no. of solutions.

→ As Rank (A) $\leq$ min $\{m, n\}$
If $m < n$, Then System (1) always possesses a non-trivial Solution ⇒ Infinite no. of solutions.

Ex Solve
$$2x + y = 0$$
$$x - y = 0$$
$$3x + 2y = 0.$$

Sol

$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$

$$AX = 0$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$$

$R_2 \leftarrow R_2 - R_1/2$
$R_3 \leftarrow R_3 - 3R_1/2$

$$= \begin{bmatrix} 2 & 1 \\ 0 & -3/2 \\ 0 & 1/2 \end{bmatrix} \quad R_3 \leftarrow R_3 + \frac{R_2}{3} = \begin{bmatrix} 2 & 1 \\ 0 & -3/2 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank(A) = 2 = no. of unknowns
$\Rightarrow$ System has only a trivial solution.

Que Solve $x + y - z + w = 0$
$2x + 3y + z + 4w = 0$
$3x + 2y - 6z + w = 0$

Solⁿ

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 2 & -6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As A is 3×4 matrix. $\Rightarrow m < n$
$\Rightarrow$ System has Infinite no. of solutions.

$R_2 \leftarrow R_2 - 2R_1$
$R_3 \leftarrow R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$R_3 \leftarrow R_3 + R_2$

$$= \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Rank(A) = 2 < no. of unknowns.

$x + y - z + w = 0$

$y + 3z + 2w = 0$

$\Rightarrow y = -3z - 2w$

and $x = -y + z - w$

$\Rightarrow x = 3z + 2w + z - w = 4z + w.$

z and w are arbitrary.

Que Find the Solution of the following hom. System $Ax = 0$ where A is given by

(1) $\begin{bmatrix} 3 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 5 & -4 \end{bmatrix}$ Ans $[-\alpha, \alpha, \alpha]$; α arbitrary

(2) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 6 & 12 \end{bmatrix}$ Ans :- $[-2\alpha/3, 7\alpha/3, -8\alpha/3; \alpha]$; α arbitrary.

(3) $\begin{bmatrix} 3 & -11 & 5 \\ 4 & 1 & -10 \\ 4 & 9 & -6 \end{bmatrix}$ Ans :- $[0, 0, 0]$

(4) $\begin{bmatrix} 1 & 1 & +2 \\ 3 & 4 & -7 \\ -1 & -2 & 11 \end{bmatrix}$ Ans :- $[-15\alpha, 13\alpha, \alpha]$; α arbitrary.

Gauss Jordan Method:→ Let A be a non-singular matrix of order n.

Form the augmented matrix $[A|I]$

Using elementary row operations, we obtain

$$[A|I] \longrightarrow [I|B]$$

Thus $B = A^{-1}$

Que:→ Using Gauss Jordan method, find the inverse of the matrix $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

Sol^n Write the augmented matrix $[A|I]$.

$$[A|I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

Using elementary row operations, Convert $[A|I]$ into $[I/B]$.

$R_1 \leftarrow -R_1$ Then

$R_2 \leftarrow R_2 - 3R_1$

$R_3 \leftarrow R_3 + R_1$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2/2$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right]$$

$R_3 \leftarrow (R_3)/5$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_1 \leftarrow R_1 + R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_1 \leftarrow R_1 - \frac{3}{2} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$R_2 \leftarrow R_2 - \frac{7}{2} R_3$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right]$$

$$\therefore A^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

Que Find the Inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solⁿ

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 & 1 \end{array}\right]$$

$R_3 \leftarrow R_3 - 2R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

~~$R_3 \leftarrow R_3 + R_2$~~

$R_2 \leftarrow R_2 + 2R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right]$$

$R_3 \leftarrow (-R_3)$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$R_1 \leftarrow R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \text{ Ans}$$

Que

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Ans:

$$\begin{bmatrix} 3 & -5 & 6 \\ -2 & 4 & -5 \\ 1 & -2 & 3 \end{bmatrix}$$