Cauchy-Euler Equation:→

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \dots + a_{n-1} x D + a_n) y = Q(x) \quad -(1)$$

where $a_0, a_1 \dots a_n$ are constants.

Put $x = e^z$

$\log x = z$

$\frac{1}{x} dx = dz \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

Now $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$

Let $\frac{d}{dx} = D$, $\frac{d}{dz} = D_1$

$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow xD = D_1$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$$

$$= \frac{-1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow \cancel{x^2 \frac{d^2y}{dx^2}} = x^2 D^2 = D_1^2 - D_1 = D_1(D_1 - 1)$$

Similarly $x^3 D^3 = D_1(D_1-1)(D_1-2)$

$\vdots$

$$x^n D^n = D_1(D_1-1)(D_1-2)\dots(D_1-(n-1))$$

So $\left(a_0 [D_1(D_1-1)\dots(D_1-(n-1))] + a_1 \_\_\_ + a_{n-2} D_1(D_1-1) + a_{n-1} D_1 + a_n\right) y = Q(z) \quad -(2)$

So (2) becomes a linear diff Eqn with Constant Coefficients.

In short; to Solve

$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + \dots + a_{n-1}xD + a_n)\, y = Q(x) \quad -(1)$

(I) Put $x = e^z$ or $z = \log x$

(II) let $D = \frac{d}{dx}$ and $D_1 = \frac{d}{dz}$

Put $xD = D_1$, $x^2D^2 = D_1(D_1-1)$, ..., $x^nD^n = D_1(D_1-1)\dots(D_1-(n-1))$

(III) (1) becomes $f(D_1)\,y = Q(z) \quad -(2)$

Solve (2) using methods discussed Earlier.

and find Gen Sol^n $y = \phi(z)$

Put $z = \log x$ in the final sol^n.

Que $x^2y'' + xy' - 4y = 0$

Sol^n $(x^2D^2 + xD - 4)\,y = 0$

Put $xD = D_1$, $x^2D^2 = D_1(D_1-1)$

$(D_1(D_1-1) + D_1 - 4)\,y = 0$

$\Rightarrow (D_1^2 - 4)y = 0$

$\Rightarrow$ A.E. is $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

$y(z) = c_1e^{2z} + c_2e^{-2z}$

$\Rightarrow y(x) = c_1x^2 + c_2x^{-2} \quad (\because z = \log x$ or $x = e^z)$

Que $x^2y'' + y = 3x^2$

Sol^n $(x^2D^2 + 1)\,y = 3x^2$

Put $x = e^z$

and $x^2D^2 = D_1(D_1-1)$

$(D_1(D_1-1) + 1)\,y = 3e^{2z}$

$\Rightarrow (D_1^2 - D_1 + 1)\,y = 3e^{2z}$

A.E $m^2 - m + 1 = 0$

$\Rightarrow m = \frac{(1 \pm \sqrt{3}\,i)}{2}$

$y_c(z) = \cancel{c_1}\, e^{z/2}\left[c_1\cos\left(\frac{\sqrt{3}}{2}z\right) + c_2\sin\left(\frac{\sqrt{3}}{2}z\right)\right]$

Cauchy

P.I is $\frac{3e^{2z}}{D_1^2 - D_1 + 1}$

$\Rightarrow y_p(z) = \frac{\cancel{3}e^{2z}}{\cancel{3}} = e^{2z}$

$y(z) = y_c(z) + y_p(z)$

Put $z = \log x$

$y(x) = (x)^{1/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + x^2.$

Que $x^2y'' + 5xy' + 4y = x\log x.$

Que $(x^4D^3 + 2x^3D^2 - x^2D + x)y = 1.$

Que $x^3y''' + 2xy' - 2y = x^2\log x + 3x.$