Integrating factor:→

If $M(x,y)dx + N(x,y)dy = 0$ —(1) is a non-exact equation then sometimes Equ^n (1) can be made exact by multiplying it with some function $\phi(x,y)$. This $\phi(x,y)$ is called an Integrating factor (I.F).

i.e. If $M(x,y)dx + N(x,y)dy = 0$ is non-exact and if we multiply it with $\phi(x,y)$ s.t.

$$\phi(x,y)M(x,y)dx + \phi(x,y)N(x,y)dy = 0$$

is an exact diff Equ^n Then $\phi(x,y)$ is called an Integrating factor (I.F)

→ Integrating factor for a first order first degree differential Equation may or may not exist.

→ Consider the first order first degree differential Equ^n

$$xdy - ydx = 0. \quad —(1)$$ is non-Exact

Choose $\phi(x,y) = \frac{1}{x^2}$ s.t.

$$\frac{1}{x^2}(xdy - ydx) = 0 \Rightarrow \frac{1}{x}dy - \frac{y}{x^2}dx = 0$$ is an exact diff Equ^n.

$\because M = \frac{-y}{x^2}, \quad N = \frac{1}{x}$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2} \text{ and } \frac{\partial N}{\partial x} = \frac{-1}{x^2} \Rightarrow \text{Equ}^n \text{ is Exact}$$

$\therefore \phi(x,y) = \frac{1}{x^2}$ is an I.F for Equ^n (1) $\forall x \neq 0$.

Choose $\phi(x,y) = \frac{1}{y^2}$ s.t. $\frac{1}{y^2}(xdy - ydx) = 0 \Rightarrow \frac{x}{y^2}dy - \frac{1}{y}dx = 0$

$M = \frac{-1}{y}, \quad N = \frac{x}{y^2}$

$\partial M/\partial y = 1/y^2 \quad \partial N/\partial x = 1/y^2 \Rightarrow$ Equ^n is Exact

$\Rightarrow \phi(x,y) = \frac{1}{y^2}$ is an I.F. of Equ^n(1) $\forall\ y \neq 0$.

Choose $\phi(x,y) = \frac{1}{xy}$ St.

$$\frac{1}{xy}(x\,dy - y\,dx) = 0 \Rightarrow \frac{1}{y}dy - \frac{1}{x}dx = 0$$

$$M = \frac{-1}{x},\ N = \frac{1}{y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0 \Rightarrow \text{Equ}^n \text{ is Exact}$$

$\Rightarrow \phi(x,y) = \frac{1}{xy}$ is an I.F. of Equ^n (1) $\forall\ x \neq 0, y \neq 0$.

Choose $\phi(x,y) = \frac{1}{x^2+y^2}$ St.

$$\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx = 0$$

$$M = \frac{-y}{x^2+y^2};\quad N = \frac{x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\Rightarrow$ Equ^n is Exact

So $\phi(x,y) = \frac{1}{x^2+y^2}$ is an I.F. of Equ^n(1) $\forall\ x \neq 0, y \neq 0$.

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→ Let $Mdx + Ndy = 0$ —(1) is a non-exact differential Equation. Then ϕ is an I.F of (1) such that ϕ is function of v only where v is function of x and y.

$\Rightarrow$ $\phi(v)\,[Mdx + Ndy] = 0$ is Exact diff Eqⁿ

or $(\phi(v)M)\,dx + (\phi(v)N)\,dy = 0$ is Exact Eqⁿ

$$\Rightarrow \frac{\partial[\phi(v)M]}{\partial y} = \frac{\partial}{\partial x}[\phi(v)N]$$

$$\Rightarrow \frac{\partial}{\partial y}(\phi(v))M + \phi(v)\frac{\partial M}{\partial y} = \frac{\partial}{\partial x}(\phi(v))N + \phi(v)\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{d\phi}{dv}\cdot\frac{\partial v}{\partial y}\cdot M + \phi(v)\frac{\partial M}{\partial y} = \frac{d\phi}{dv}\cdot\frac{\partial v}{\partial x}\cdot N + \phi(v)\cdot\frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{d\phi}{dv}\left[\frac{\partial v}{\partial y}\cdot M - \frac{\partial v}{\partial x}\cdot N\right] + \phi(v)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = 0$$

$$\Rightarrow \phi'(v)\,[M\cdot v_y - Nv_x] = \phi(v)\,[N_x - M_y]$$

$$\Rightarrow \frac{\phi'(v)}{\phi(v)} = \frac{M_y - N_x}{Nv_x - Mv_y}$$

$$\Rightarrow \frac{M_y - N_x}{Nv_x - Mv_y} = \boxed{\frac{\phi'(v)}{\phi(v)} = f(v)}$$

$$\Rightarrow \int \frac{\phi'(v)}{\phi(v)}dv = \int f(v)\,dv$$

$$\Rightarrow \log\phi(v) = \int f(v)\,dv + c$$

$$\Rightarrow \phi(v) = \alpha\cdot e^{\int f(v)dv} \qquad \boxed{\alpha = e^c}$$

or $\boxed{\phi(v) = e^{\int f(v)dv}}$ is an I.F. of (1)

Hence we have shown that for $Mdx + Ndy = 0$ —(1) – a non Exact diff Equation, Then there Exist an Integrating factor ϕ which is function of v only iff $\dfrac{M_y - N_x}{Nv_x - Mv_y}$ is fun of v only

and $\phi(v) = e^{\int f(v)dv}$ is an I.F of (1) where $f(v) = \dfrac{M_y - N_x}{Nv_x - Mv_y}$.

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Ex

Solve $(5x^3+12x^2+6y^2)\,dx + 6xy\,dy = 0$ —(1)

Sol^n

$M = 5x^3+12x^2+6y^2$; $N = 6xy$.

$\frac{\partial M}{\partial y} = 12y$; $\frac{\partial N}{\partial x} = 6xy$

$\Rightarrow$ Equ^n (1) is not Exact

So let $V = \cancel{x}\,x$

$V_x = 1, \; V_y = 0$

Then $\frac{M_y - N_x}{NV_x - MV_y} = \frac{12y-6xy}{6xy(1)-0} = \frac{6y}{6xy} = \frac{1}{x}$ is fun of x only

$\Rightarrow \phi(x) = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$ is an I.F of (1)

$\therefore \; x(5x^3+12x^2+6y^2)\,dx + 6x^2y\,dy = 0$

$\frac{\partial M}{\partial y} = \cancel{6}\,12xy$; $\frac{\partial N}{\partial x} = 12xy$

$\Rightarrow$ Equ^n becomes Exact.

So Sol^n is $\int\limits_{y\text{-const}} M\,dx + \int(\text{Terms in } N \text{ not Containing } x)\,dy = C$

$\Rightarrow \int (5x^4 + 12x^3 + 6xy^2)\,dx + 0 = C$

$\Rightarrow \frac{5x^5}{5} + \frac{12x^4}{4} + 6y^2\left(\frac{x^2}{2}\right) = C$

$\Rightarrow \boxed{x^5 + 3x^4 + 3x^2y^2 = C}$

Ex $\rightarrow$ $(3x^2y^3e^y + y^3 + y^2)\,dx + (x^3y^3e^y - xy)\,dy = 0$ —(1)

Sol^n

$M = 3x^2y^3e^y + y^3 + y^2$; $N = x^3y^3e^y - xy$

$\frac{\partial M}{\partial y} = 9x^2y^2e^y + 3x^2y^3e^y + 3y^2 + 2y$

$\frac{\partial N}{\partial x} = 3x^2y^3e^y - y$

$\Rightarrow$ Equ^n (1) is not Exact.

Let $v = y$

$v_x = 0, \; v_y = 1$

So $\dfrac{M_y - N_x}{N v_x - M v_y} = \dfrac{9x^2y^2e^y + 3y^2 + 2y + y}{0 - (3x^2y^3e^y + y^3 + y^2)(1)}$

$= \dfrac{9x^2y^2e^y + 3y^2 + 3y}{-(3x^2y^3e^y + y^3 + y^2)}$

$= \dfrac{3\,(\cancel{3x^2y^2e^y + y^2 + y})}{-y\,(\cancel{3x^2y^2e^y + y^2 + y})} = -\dfrac{3}{y} = f(y)$

$\therefore \; \phi(y) = e^{\int -3/y \, dy} = e^{-3\log y} = y^{-3} = \dfrac{1}{y^3}$ is an I.F. of (1)

So $\dfrac{1}{y^3}\left[3x^2y^3e^y + y^3 + y^2\right] dx + \dfrac{1}{y^3}\left[x^3y^3e^y - xy\right] dy = 0$

$\Rightarrow \left(3x^2e^y + 1 + \dfrac{1}{y}\right) dx + \left(x^3e^y - \dfrac{x}{y^2}\right) dy = 0$

$\cancel{\dfrac{\partial M}{\partial y} = 3x^2e^y + x^2\left[\dfrac{y^3(0) - 1(3y^2)}{y^6}\right]}$

$\cancel{= 3x^2e^y - \dfrac{3x^2}{y^4}}$

$\cancel{\dfrac{\partial N}{\partial x} = 3x^2e^y - 1}$

$\dfrac{\partial M}{\partial y} = 3x^2e^y - \dfrac{1}{y^2}$

$\dfrac{\partial N}{\partial x} = 3x^2e^y - \dfrac{1}{y^2}$ $\Rightarrow$ Eq^n Reduces to Exact diff Eq^n

Sol^n is $\displaystyle\int_{y\text{-const}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\, dy = c$

$\Rightarrow \displaystyle\int \left(3x^2e^y + 1 + \frac{1}{y}\right) dx + \int 0 \, dy = c$

$\Rightarrow \boxed{x^3e^y + x + \dfrac{x}{y} = c}$.

Que $\left[y+xf(x^2+y^2)\right]dx+\left[yf(x^2+y^2)-x\right]dy=0$ —(1)

$M= y+xf(x^2+y^2)$; $N= yf(x^2+y^2)-x$

$$\frac{\partial M}{\partial y} = 1+xf'(x^2+y^2)(2y)$$

$$= 1+2xyf'(x^2+y^2)$$

$$\frac{\partial N}{\partial x} = yf'(x^2+y^2)(2x)-1 = 2xyf'(x^2+y^2)-1.$$

$\Rightarrow$ Eqn (1) is not Exact.

$$M_y-N_x = 1+\cancel{2xyf'(x^2+y^2)}-\cancel{2xyf'(x^2+y^2)}+1$$
$$=2$$

S.T. $\frac{1}{x^2+y^2}$ is an I.F.

$$\frac{1}{x^2+y^2}\left[y+xf(x^2+y^2)\right]dx+\frac{1}{x^2+y^2}\left[yf(x^2+y^2)-x\right]dy=0$$

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)\left[1+2xyf'(x^2+y^2)\right]-\left[y+xf(x^2+y^2)\right][2y]}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2+2xy(x^2+y^2)f'(x^2+y^2)-2y^2-2xyf(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2+2xy(x^2+y^2)f'(x^2+y^2)-2xyf(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)\left[yf'(x^2+y^2)(2x)-1\right]-(yf(x^2+y^2)-x)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2xy(x^2+y^2)f'(x^2+y^2)-x^2-y^2-2xyf(x^2+y^2)+2x^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2+2xy(x^2+y^2)f'(x^2+y^2)-2xyf(x^2+y^2)}{(x^2+y^2)^2}$$

Que $(x+y+1)\,dx + (2x+2y+1)\,dy = 0$

Solⁿ $\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = 2$

Not Exact diff Equⁿ.

Let $V = x+y$

$V_x = 1,\quad V_y = +1$

$$\frac{M_y - N_x}{NV_x - MV_y} = \frac{1-2}{(2x+2y+1)-(x+y+1)}$$

$$= \frac{-1}{x+y}$$

$$\phi(v) = e^{\int \frac{-1}{x+y}\, d(x+y)} = e^{-\log|x+y|} = \frac{1}{x+y}$$

$$\frac{1}{x+y}\left[(x+y+1)\,dx\right] + \left(\frac{2x+2y-1}{x+y}\right) dy = 0$$

$$M = \frac{x+y+1}{x+y}\ ;\quad N = \frac{2(x+y)+1}{x+y}$$

$$M = 1+\frac{1}{x+y}\ ;\quad N = 2+\frac{1}{x+y}$$

$$\frac{\partial M}{\partial y} = \frac{-1}{(x+y)^2}\ ;\quad \frac{\partial N}{\partial x} = \frac{-1}{(x+y)^2}$$

Solⁿ is $\int_{y\ \text{const}} \left(1+\frac{1}{x+y}\right) dx + \int 2\,dy = c$

$\Rightarrow$

$$x + \log|x+y| + 2y = c$$

$\Rightarrow$ $\boxed{(x+2y) + \log|x+y| = c.}$

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H.W (1) Solve $(x^2+y^2+x)dx + xy\,dy = 0$ Ans:- $4x^3+3x^4+6x^2y^2=c$.
$I.F. = x$

(2) $(2xy^4e^y + 2xy^3+y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$
$I.F. = 1/y^4$; Sol^n is $\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$

(3) $(x^4+y^4)dx - xy^3dy = 0$
Sol^n is $\boxed{y^4 = 4x^4\log x + cx^4.}$

(4) $y^2dx + (x^2-xy-y^2)dy = 0$
Sol^n is $\boxed{(x-y)y^2 = c(x+y)}$

(5) $y(2xy+1)dx + x(1+2xy-x^3y^3)dy = 0$
Sol^n is $\boxed{y = Ce^{-(3xy+1)/3x^3y^3}}$

(6) $(x^2+y^2)dx - 2xy\,dy = 0$.
Sol^n is $\boxed{x^2-y^2 = Cx}$

→ If $M(x,y)dx + N(x,y)dy = 0$ is a non-exact differential equation. The functions $M(x,y)$ and $N(x,y)$ are such that they are homogeneous function of degree n and $Mx+Ny \neq 0$ Then $\frac{1}{Mx+Ny}$ is an Integrating factor.

If $Mx+Ny = 0 \Rightarrow \frac{M}{N} = \frac{-y}{x}$

So $Mdx + Ndy = 0$

$\Rightarrow \frac{M}{N}dx + dy = 0$

$\Rightarrow \frac{-y}{x}dx + dy = 0$

$\Rightarrow -y\,dx + x\,dy = 0. \quad -(1)$

So $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$ are I.F of (1).