Legendre Equation and Legendre Polynomials

(1) $(1-x^2)y'' - 2xy' + n(n+1)y = 0$, where n is a real constant.
(In most application n is a positive Integer or whole number)
Equation (1) is Called Legendre differential Equation.

Let $n=0$ in (1)

$(1-x^2)y'' - 2xy' = 0$ → We Can observe that $y=$ Constant
↳ (2) is sol^n of (2).

So we Can choose $y(x)=1$.

let $n=1$ in (1)

$(1-x^2)y'' - 2xy' + 2y = 0$ ↳ We note that $y=x$ is sol^n of (3)
—(3) So we Can choose $y(x)=x$.

let $n=2$ in (1)

$(1-x^2)y'' - 2xy' + 6y = 0$ → we note that $y=x^2$ is one of
↳(4) the sol^n of (4). So we Select $y(x)=x^2$.

and so on.

⟹ The Sol^n of (1) Corresponding to different values of n are polynomials.

→ To Find the Sol^n of (1) about $x=0$.

Let the Sol^n of (1) about $x=0$ is of the form

$$y(x) = \sum_{r=0}^{\infty} C_r x^{n-r}$$

$$y'(x) = \sum_{r=1}^{\infty} (n-r) C_r x^{n-r-1}$$

$$y''(x) = \sum_{r=2}^{\infty} (n-r)(n-r+1) C_r x^{n-r-2}$$

Substitute $y(x)$, $y'(x)$ and $y''(x)$ in (1), we find the solution.

of the form $y(x) = C_1 y_1 + C_2 y_2$.

where $y_1 = a_0 \left(x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} + \dots \right)$

↳ Called Legendre polynomial of first Kind, and denoted by $\boxed{P_n(x)}$.

$$y_2 = b \left(x^{-n-1} + \frac{n(n+1)}{2(2n+3)} x^{-n-3} + \frac{n(n+1)(n+2)(n+3)}{2.4.(2n+3)(2n+5)} x^{-n-5} + \dots \right)$$

↳ Called Legendre Solution of Second Kind, and denoted by $\boxed{Q_n(x)}$

The value of a in y_1 is

$$a = \frac{1.3.5\dots(2n-1)}{n}$$

The general term of $P_n(x)$ will be

$$\frac{1.3.5\dots(2n-1)}{n} \left[\frac{(-1)^r n(n-1)(n-2)\dots(n-2r+1) \, x^{n-2r}}{(2.4.6\dots2r)(2n-1)(2n-3)\dots(2n-2r+1)} \right]$$

$$= \frac{(-1)^r (2n-2r)! \, x^{n-2r}}{2^n r! (n-r)! (n-2r)!}$$

$$\Rightarrow \boxed{P_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n-2r)! \, x^{n-2r}}{2^n r! (n-r)! (n-2r)!}} \longrightarrow \text{Legendre Polynomial.}$$

$n=0$; $P_0(x) = 1$

$n=1$; $P_1(x) = \frac{2! \, x^1}{2^1 0! \, 1! \, 1!} = x$

$n=2$; $P_2(x) = \sum_{r=0}^{[1]} \frac{(-1)^r (4-2r)! \, x^{2-2r}}{2^2 r! (2-r)! (2-2r)!}$

$$= \frac{4!\,x^2}{4\ 2!\ 2!} + \frac{(-1)\ 2!\ x^0}{2^2\,1!\ 1!\ 0!}$$

$$= \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \sum_{r=0}^{[3/2]} \frac{(-1)^r\,(2n-2r)!\ x^{n-2r}}{2^n\ r!\ (n-r)!\ (n-2r)!}$$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

Similarly $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Que→ Express $P(x) = 3P_3(x) + 2P_2(x) + 4P_1(x) + 5P_0(x)$ as a polynomial in x, where $P_n(x)$ is legenedre polynomial of order n.

Solⁿ→ $P_0(x) = 1$; $P_1(x) = x$; $P_2(x) = \frac{1}{2}(3x^2-1)$; $P_3(x) = \frac{1}{2}(5x^3-3x)$

$$P(x) = 3\left[\frac{1}{2}(5x^3-3x)\right] + \cancel{2}\left[\frac{1}{\cancel{2}}(3x^2-1)\right] + 4(x) + 5(1)$$

$$= \frac{15}{2}x^3 - \frac{9}{2}x + 3x^2 - 1 + 4x + 5$$

$$= \frac{15}{2}x^3 + 3x^2 - \frac{1}{2}x + 4.$$

$$= \frac{1}{2}\left[15x^3 + 6x^2 - x + 8\right].$$

Ques→ Express $f(x) = x^4 + 2x^3 - 6x^2 + 5x - 3$ in terms of legendre polynomials.

Solⁿ→ $1 = P_0(x)$; $x = P_1(x)$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$$\Rightarrow \frac{2P_2(x)+1}{3} = x^2 \quad \Rightarrow x^2 = \frac{1}{3}(2P_2 + P_0(x))$$

Date
Page

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$\Rightarrow 2P_3 = 5x^3 - 3P_1$$

$$\Rightarrow \frac{2P_3+3P_1}{5} = x^3$$

$$\Rightarrow \boxed{x^3 = \frac{1}{5}(2P_3+3P_1)}$$

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$\Rightarrow 8P_4(x) = 35x^4 - 30\left(\frac{1}{3}(2P_2+P_0)\right)+3P_0$$

$$\Rightarrow 8P_4 = 35x^4 - 10(2P_2+P_0)+3P_0$$

$$\Rightarrow 8P_4+20P_2+7P_0 = 35x^4$$

$$\Rightarrow \boxed{x^4 = \frac{1}{35}(8P_4+20P_2+7P_0)}$$

$$f(x) = x^4+2x^3-6x^2+5x-3$$

$$= \frac{1}{35}(8P_4+20P_2+7P_0) + 2\left(\frac{1}{5}(2P_3+3P_1)\right) - 6\left(\frac{1}{3}(2P_2+P_0)\right)$$

$$+5P_1 - 3P_0$$

$$= \frac{8}{35}P_4 + \frac{20}{35}P_2 + \frac{7}{35}P_0 + \frac{4}{5}P_3 + \frac{6}{5}P_1 - 4P_2 - 2P_0 + 5P_1 - 3P_0$$

$$= \frac{1}{35}(8P_4 + 28P_3 - 120P_2 + 217P_1 - 168P_0).$$

Que Express x^3+x+1 in Legendre polynomials.

Sol^n $1 = P_0$; $x = P_1$; $x^2 = \frac{1}{3}(2P_2+P_0)$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$$\Rightarrow 2P_3 = 5x^3 - 3P_1 \Rightarrow \frac{1}{5}(2P_3+3P_1) = x^3.$$

Date
Page

$$x^3+x+1 = \frac{1}{5}(2P_3+3P_1)+P_1+P_0$$

$$= \frac{2}{5}P_3+\frac{3}{5}P_1+P_1+P_0$$

$$= \frac{2}{5}P_3+\frac{8}{5}P_1+P_0$$

$$= \frac{1}{5}(2P_3+8P_1+5P_0).$$

H.W

Que (1) $3x^2+5x-6$
(2) $4x^3+3x^2+2x-6$
(3) $5x^4+3x^3-6x^2-2x+3$
→ Express In Legendre polynomials.

Que (1) $6P_3(x)-2P_1(x)+P_0(x)$
(2) $4P_3(x)+6P_2(x)-3P_1(x)-2P_0(x)$
(3) $8P_4(x)+2P_2(x)+P_0(x)$
(4) $5P_4(x)+10P_3(x)+2P_2(x)+P_1(x)$
→ Express In terms of polynomials of x.

Que:→ $(1-x^2)y''-2xy'+6y=0 \longrightarrow (1)$
If Solⁿ of (1) is $y(x)$ then find
$$\int_{-1}^{1} y(x)\,(x+x^2)\,dx$$

Solⁿ

(1) is legendre polynomial with $n=2$
So Solⁿ will be $P_2(x)=\frac{1}{2}(3x^2-1)=y(x)$

$$\frac{1}{2}\int_{-1}^{1}(3x^2-1)(x+x^2)\,dx = \frac{1}{2}\int_{-1}^{1}(3x^3+3x^4-x-x^2)\,dx$$

$$= \frac{1}{2}\int_{-1}^{1}(3x^4-x^2)\,dx$$

$$= \frac{2}{2}\int_{0}^{1}(3x^4-x^2)\,dx = \frac{3}{5}-\frac{1}{3}=\boxed{\frac{4}{15}}$$

Date
Page

Rodrigue's Formula:→

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n}$$

$$P_0(x) = 1$$

$$P_1(x) = \frac{1}{2^1} \frac{d}{dx}(x^2-1)$$

$$= \frac{1}{2}(2x) = x$$

$$P_2(x) = \frac{1}{2^2 \lfloor 2} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\left[2(x^2-1)(2x)\right]'$$

$$= \frac{1}{8}(4x^3-4x)'$$

$$= \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3 \lfloor 3} \frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{8\lfloor 3}\left[\frac{d^2}{dx^2}\left(3(x^2-1)^2(2x)\right)\right]$$

$$= \frac{1}{8}\left[\frac{d^2}{dx^2}(x^2-1)^2(x)\right]$$

$$= \frac{1}{8}\left[\frac{d}{dx}\left[(x^2-1)^2 + x(2(x^2-1))(2x)\right]\right]$$

$$= \frac{1}{8}\left[\frac{d}{dx}\left((x^2-1)^2 + 4x^2(x^2-1)\right)\right]$$

$$= \frac{1}{8}\left[2(x^2-1)(2x) + 16x^3 - 8x\right]$$

$$= \frac{1}{8}\left[4x^3 - 4x + 16x^3 - 8x\right] = \frac{20x^3 - 12x}{8}$$

$$= \frac{5x^3 - 3x}{2} = \frac{1}{2}(5x^3 - 3x)$$

Date
Page

Recurrence Relations for Legendre Polynomials :-

$$(n+1) P_{n+1}(x) = (2n+1)\, x P_n(x) - n P_{n-1}(x)$$

Eg. $P_0(x) = 1, \; P_1(x) = x$

Find $P_2(x), P_3(x), P_4(x)$.

$n=1; \quad 2P_2(x) = 3x P_1(x) - P_0(x)$

$= 3x(x) - 1 = \frac{3x^2-1}{2}$

$\Rightarrow P_2(x) = \frac{3x^2-1}{2}$

$n=2; \quad 3P_3(x) = 5x P_2(x) - 2P_1(x)$

$= 5x\left(\frac{3x^2-1}{2}\right) - 2x$

$= \frac{15x^3 - 5x}{2} - 2x = \cancel{\frac{9x^3 - x}{}} \; \frac{15x^3 - 9x}{2}$

$\Rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$n=3; \quad 4P_4(x) = 7x P_3(x) - 3P_2(x)$

$= 7x\left(\frac{5x^3-3x}{2}\right) - 3\left(\frac{3x^2-1}{2}\right)$

$= \frac{35x^4 - 21x^2}{2} - \left(\frac{9x^2-3}{2}\right)$

$= \frac{35x^4 - 30x^2 + 3}{2}$

$\Rightarrow P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Que Using the Recurrence Relations

$$(n+1) P_{n+1}(x) = (2n+1)\, x P_n(x) - n P_{n-1}(x).$$

Evaluate $P_2(1.5)$ and $P_3(2.1)$.

Solⁿ→ $n=1; \quad 2P_2(x) = 3x P_1(x) - P_0(x)$

$2P_2(1.5) = 3(1.5) P_1(1.5) - P_0(1.5) = 3(1.5)(1.5) - 1$

$= 5.75$

Date			
Page			

$\Rightarrow P_2(1.5) = 2.875.$

Similarly $n=2$

$$3P_3(x) = 5x P_2(x) - 2P_1(x)$$

$$3P_3(2.1) = 5(2.1)P_2(2.1) - 2P_1(2.1)$$

$$P_1(2.1) = 2.1$$

$$P_2(2.1) = \frac{1}{2}(3(2.1)^2-1) = 6.115$$

$$\Rightarrow \boxed{P_3(2.1) = 20.0025.}$$

Orthogonality property of Legendre polynomials:→

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0 & \text{If } m \neq n \\ \frac{2}{2n+1} & \text{If } m=n. \end{cases}$$

Que If $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{3}$ then n Equals

Sol^n

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1} = \frac{2}{3}$$

$$\cancel{\Rightarrow 2 = 30n+15 \Rightarrow n = \frac{-13}{30}}$$

$$\Rightarrow 2n+1 = 3 \Rightarrow \boxed{n=1}$$

Que Let $(1-x^2)y'' - 2xy' + n(n+1)y = 0$; let $y_n(x)$ be its solution

If $\int_{-1}^{1} (y_n^2 + y_{n+1}^2)\,dx = \frac{16}{15}$ then find n.

Sol^n

We Know that $\int_{-1}^{1} y_n^2(x)\,dx = \frac{2}{2n+1}$

$$\Rightarrow \int_{-1}^{1} y_{n+1}^2\,dx = \frac{2}{2(n+1)+1} = \frac{2}{2n+3}$$

Date
Page

$$\therefore \int_{-1}^{1} (y_n^2 + y_{n+1}^2)\,dx = \frac{2}{2n+1} + \frac{2}{2n+3} = \frac{16}{15}$$

$$\Rightarrow \frac{2n+1+2n+3}{(2n+1)(2n+3)} = \frac{8}{15}$$

$$\Rightarrow \frac{4n+4}{(2n+1)(2n+3)} = \frac{8}{15}$$

$$\Rightarrow (n+1)15 = 2(2n+1)(2n+3)$$

$$\Rightarrow 15n+15 = 8n^2+16n+6$$

$$\Rightarrow 8n^2+n-9=0$$

$$\Rightarrow 8n^2+9n-8n-9=0$$

$$\Rightarrow 8n(n-1)+9(n-1)=0$$

$$\Rightarrow (n-1)(8n+9)=0$$

$$\Rightarrow \boxed{n=1}$$

(1) $P_n(1) = 1$;

(2) $P_n(-1) = (-1)^n$

(3) $P_n(-x) = (-1)^n P_n(x)$

(4) $\int_{-1}^{1} P_n(x)\,dx = 0 \quad \forall\, n \geq 1.$