Exercise 3.2

Discuss whether V defined in problems **1** to **10** is a vector space. If V is not a vector space, state which of the properties are not satisfied.

1. Let V be the set of the real polynomials of degree $\leq m$ and having 2 as a root with the usual addition and scalar multiplication.
2. Let V be the set of all real polynomials of degree 4 or 6 with the usual addition and scalar multiplication.
3. Let V be the set of all real polynomials of degree ≥ 4 with the usual addition and scalar multiplication.
4. Let V be the set of all rational numbers with the usual addition and scalar multiplication.
5. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and usual scalar multiplication.
6. Let V be the set of all ordered pairs (x, y) in IR^2 with vector addition defined as $(x, y) + (u, v) = (x + u, y + v)$ and scalar multiplication defined as $\alpha(x, y) = (3\alpha x, y)$.
7. Let V be the set of all ordered triplets (x, y, z), $x, y, z \in \mathrm{IR}$, with vector addition defined as
$$(x, y, z) + (u, v, w) = (3x + 4u, y - 2v, z + w)$$
and scalar multiplication defined as
$$\alpha(x, y, z) = (\alpha x, \alpha y, \alpha z/3).$$
8. Let V be the set of all positive real numbers with addition defined as $x + y = xy$ and scalar multiplication defined as $\alpha x = x^{\alpha}$.
9. Let V be the set of all positive real valued continuous functions f on $[a, b]$ such that
(i) $\int_a^b f(x)\,dx = 0$ and (ii) $\int_a^b f(x)\,dx = 2$ with usual addition and scalar multiplication.
10. Let V be the set of all solutions of the
    (i) homogeneous linear differential equation $y'' - 3y' + 2y = 0$.
    (ii) non-homogeneous linear differential equation $y'' - 3y' + 2y = x$.

    under the usual addition and scalar multiplication.

Is W a subspace of V in problems **11** to **15**? If not, state why?

11. Let V be the set of all 3×1 real matrices with usual matrix addition and scalar multiplication and W consisting of all 3×1 real matrices of the from
$$\text{(i)} \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}, \quad \text{(ii)} \begin{bmatrix} a \\ a \\ a^2 \end{bmatrix}, \quad \text{(iii)} \begin{bmatrix} a \\ b \\ 2 \end{bmatrix}, \quad \text{(iv)} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}.$$
12. Let V be the set of all 3×3 real matrices with the usual matrix addition and scalar multiplication and W consisting of all 3×3 matrices $\mathbf{A}$ which
    (i) have positive elements, (ii) are non-singular,
    (iii) are symmetric, (iv) $\mathbf{A}^2 = \mathbf{A}$.
13. Let V be the set of all 2×2 complex matrices with the usual matrix addition and scalar multiplication and W consisting of all matrices with the usual addition and scalar multiplication and W consisting of all martices of the form $\begin{bmatrix} z & x+iy \\ x-iy & u \end{bmatrix}$, where x, y, z, u are real numbers and (i) scalars are real numbers, (ii) scalars are complex numbers.

14. Let V consist of all real polynomials of degree ≤ 4 with the usual polynomial addition and scalar multiplication and W consisting of polynomials of degree ≤ 4 having

(i) constant term 1, (ii) coefficient of t^2 as 0,

(iii) coefficient of t^3 as 1, (iv) only real roots.

15. Let V be the vector space of all triplets of the form (x_1, x_2, x_3) in IR^3 with the usual addition and scalar multiplication and W be the set of triplets of the form (x_1, x_2, x_3) such that

(i) $x_1 = 2x_2 = 3x_3$, (ii) $x_1 = x_2 = x_3 + 1$,

(iii) $x_1 \geq 0$, x_2, x_3 arbitrary, (iv) $x_1^2 + x_2^2 + x_3^2 \leq 4$. (v) x_3 is an integer.

16. Let $\mathbf{u} = (1, 2, -1)$, $\mathbf{v} = (2, 3, 4)$ and $\mathbf{w} = (1, 5, -3)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, where $\mathbf{x}$ is given by

(i) (4, 3, 10), (ii) (3, 2, 5) (iii) (−2, 1, −5).

17. Let $\mathbf{u} = (1, -2, 1, 3)$, $\mathbf{v} = (1, 2, -1, 1)$ and $\mathbf{w} = (2, 3, 1, -1)$. Determine whether or not $\mathbf{x}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, where $\mathbf{x}$ is given by

(i) (3, 0, 5, −1), (ii) (2, −7, 1, 11), (iii) (4, 3, 0, 3).

18. Let $P_1(t) = t^2 - 4t - 6$, $P_2(t) = 2t^2 - 7t - 8$, $P_3(t) = 2t - 3$, Write $P(t)$ as a linear combination of $P_1(t)$, $P_2(t)$, $P_3(t)$, when

(i) $P(t) = -t^2 + 1$, (ii) $P(t) = 2t^2 - 3t - 25$.

19. Let V be the set of all 3×1 real matrices. Show that the set

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ spans } V,$$

20. Let V be the set of all 2×2 real matrices. Show that the set

$$S = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix} \right\} \text{ spans } V.$$

21. Examine whether the following vectors in $\mathrm{IR}^3/\mathbb{C}^3$ are linearly independent.

(i) (2, 2, 1), (1, −1, 1), (1, 0, 1), (ii) (1, 2, 3), (3, 4, 5), (6, 7, 8),

(iii) (0, 0, 0), (1, 2, 3), (3, 4, 5), (iv) $(2, i, -1), (1, -3, i), (2i, -1, 5)$,

(v) (1, 3, 4), (1, 1, 0), (1, 4, 2), (1, −2, 1).

22. Examine whether the following vectors in IR^4 are linearly independent.

(i) (4, 1, 2, −6), (1, 1, 0, 3), (1, −1, 0, 2), (−2, 1, 0, 3),

(ii) (1, 2, 3,1), (2, 1, −1, 1), (4, 5, 5, 3), (5, 4, 1, 3),

(iii) (1, 2, 3, 4), (2, 0, 1, −2), (3, 2, 4, 2),

(iv) (1, 1, 0, 1), (1, 1, 1, 1), (−1, −1, 1, 1), (1, 0, 0, 1),

(v) (1, 2, 3, −1), (0, 1, −1, 2), (1, 5, 1, 8), (−1, 7, 8, 3).

23. If $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are linearly independent vectors in IR^3, then show that

(i) $\mathbf{x} + \mathbf{y}$, $\mathbf{y} + \mathbf{z}$, $\mathbf{z} + \mathbf{x}$; (ii) $\mathbf{x}$, $\mathbf{x} + \mathbf{y}$, $\mathbf{x} + \mathbf{y} + \mathbf{z}$

are also linearly independent in IR^3.

24. Write (− 4, 7, 9) as a linear combination of the elements of the set S: {(1, 2, 3), (− 1, 3, 4), (3, 1, 2)}. Show that S is not a spanning set in IR^3.

25. Write $t^2 + t + 1$ as a linear combination of the elements of the set S: $\{3t, t^2 - 1, t^2 + 2t + 2\}$. Show that S is the spanning set for all polynomials of degree 2 and can be taken as its basis.

26. Let V be the set of all vectors in $\mathbb{R}^4$ and S be a subset of V consisting of all vectors of the form
 (i) $(x, y, -y, -x)$, (ii) (x, y, z, w) such that $x + y + z - w = 0$,
 (iii) $(x, 0, z, w)$, (iv) (x, x, x, x).
 Find the dimension and the basis of S.

27. For what values of k do the following set of vectors form a basis in $\mathbb{R}^3$?
 (i) $\{(k, 1 - k, k), (0, 3k - 1, 2), (-k, 1, 0)\}$,
 (ii) $\{(k, 1, 1), (0, 1, 1), (k, 0, k)\}$,
 (iii) $\{(k, k, k), (0, k, k), (k, 0, k)\}$,
 (iv) $\{(1, k, 5), (1, -3, 2), (2, -1, 1)\}$.

28. Find the dimension and the basis for the vector space V, when V is the set of all 2×2 (i) real matrices (ii) symmetric matrices, (iii) skew-symmetric matrices, (iv) skew-Hermitian matrices, (v) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{22} = 0$, (vi) real matrices $\mathbf{A} = (a_{ij})$ with $a_{11} + a_{12} = 0$.

29. Find the dimension and the basis for the vector space V, when V is the set of all 3×3 (i) diagonal matrices (ii) upper triangular matrices, (iii) lower triangular matrices.

30. Find the dimension of the vector space V, when V is the set of all $n \times n$ (i) real matrices, (ii) diagonal matrices, (iii) symmetric matrices (iv) skew-symmetric matrices.

Examine whether the transformation T given in problems **31** to **35** is linear or not. If not linear, state why?

31. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = x + y + a$, $a \neq 0$, a real constant.

32. $T : \mathbb{R}^3 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x + z \end{pmatrix}$.

33. $T : \mathbb{R}^1 \to \mathbb{R}^2$; $T(x) = \begin{pmatrix} x^2 \\ 3x \end{pmatrix}$.

34. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} 0 & x \neq 0, y \neq 0 \\ 2y, & x = 0 \\ 3x, & y = 0. \end{cases}$

35. $T : \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + x + z$.

Find $ker(T)$ and $ran(T)$ and their dimensions in problems **36** to **42**.

36. $T : \mathbb{R}^3 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ z \\ x - y \end{pmatrix}$.

37. $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ y - x \\ 3x + 4y \end{pmatrix}$.

38. $T : \mathbb{R}^4 \to \mathbb{R}^3$; $T\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x + y + w \\ z \\ y + 2w \end{pmatrix}$.

39. $T : \mathbb{R}^2 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \end{pmatrix} = x + 3y$.

40. $T : \mathbb{R}^3 \to \mathbb{R}^1$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x + 3y$.

41. $T : \mathbb{R}^2 \to \mathbb{R}^2$; $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x - y \end{pmatrix}$.

42. $T : \mathrm{IR}^3 \to \mathrm{IR}^2$; $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x-y \\ 3x+z \end{pmatrix}$.

43. Let $T : \mathrm{IR}^3 \to \mathrm{IR}^2$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-z \end{pmatrix}$.

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathrm{IR}^3 \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right\} \text{ in } \mathrm{IR}^2.$$

44. Let V and W be two vector spaces in IR^3. Let $T : V \to W$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x+y \\ x+y+z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ in } V \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } W.$$

45. Let V and W be two vector spaces in IR^3. Let $T : V \to \mathrm{W}$ be a linear transformation defined by

$$T\begin{pmatrix} x \\ y \\ x \end{pmatrix} = \begin{pmatrix} x+z \\ x+y \\ x+y+z \end{pmatrix}.$$

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\} \text{ in } V \text{ and } \mathbf{y} = \left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right\} \text{ in } W$$

46. Let $T : \mathrm{IR}^3 \to \mathrm{IR}^4$ be a linear transformation defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ y+z \\ x+z \\ x+y+z \end{pmatrix}$.

Find the matrix representation of T with respect to the ordered basis

$$\mathbf{x} = \left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\} \text{ in } \mathrm{IR}^3 \text{ and } \mathbf{y} = \left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right) \text{ in } \mathrm{IR}^4$$

47. Let $T : \mathrm{IR}^2 \to \mathrm{IR}^3$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ be the matrix representation of the linear transformation T with respect to the ordered basis vectors $\mathbf{v}_1 = [1, 2]^T$, $\mathbf{v}_2 = [3, 4]^T$ in IR^2 and $\mathbf{w}_1 = [-1, 1, 1]^T$, $\mathbf{w}_2 = [1, -1, 1]^T$, $\mathbf{w}_3 = [1, 1, -1]^T$ in IR^3. Then, determine the linear transformation T.

48. Let $T : \mathrm{IR}^3 \to \mathrm{IR}^2$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -4 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis vectors $\mathbf{v}_1 = [1, -1, 1]^T$, $\mathbf{v}_2 = [2, 3, -1]^T$, $\mathbf{v}_3 = [1, 1, -1]^T$ in IR^3 and $\mathbf{w}_1 = [1, 1]^T$, $\mathbf{w}_2 = [2, 3]^T$ in IR^2. Then, determine the linear transformation T.

49. Let $T : P_1(t) \to P_2(t)$ be a linear transformation. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ -1 & 1 \end{bmatrix}$ be the matrix representation of the linear transformation with respect to the ordered basis $[1 + t, t]$ in $P_1(t)$ and $[1 - t, 2t, 2 + 3t - t^2]$ in $P_2(t)$. Then, determine the linear transformation T.

50. Let V be the set of all vectors of the form (x_1, x_2, x_3) in IR^3 satisfying (i) $x_1 - 3x_2 + 2x_3 = 0$; (ii) $3x_1 - 2x_2 + x_3 = 0$ and $4x_1 + 5x_2 = 0$. Find the dimension and basis for V.

3.4 Solution of General linear System of Equations

In section 3.2.5, we have discussed the matrix method and the Cramer's rule for solving a system of n equations in n unknowns, $\mathbf{Ax} = \mathbf{b}$. We assumed that the coefficient matrix $\mathbf{A}$ is non-singular, that is $|\mathbf{A}| \neq 0$, or the rank of the matrix $\mathbf{A}$ is n. The matrix method requires evaluation of n^2 determinants each of order $(n - 1)$, to generate the cofactor matrix, and one determinant of order n, whereas the Cramer's rule requires evaluation of $(n + 1)$ determinants each of order n. Since the evaluation of high order determinants is very time consuming, these methods are not used for large values of n, say $n > 4$. In this section, we discuss a method for solving a general system of m equations in n unknowns, given by

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{3.28}$$

where
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

are respectively called the *coefficient matrix*, *right hand side column vector* and the *solution vector*. The order of the matrices $\mathbf{A}$, $\mathbf{b}$, $\mathbf{x}$ are respectively $m \times n$, $m \times 1$ and $n \times 1$.

The matrix

$$(\mathbf{A} \mid \mathbf{b}) = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \tag{3.29}$$