Probability Theory

There are generally two types of experiment in our common life

(i) Deterministic experiment

(ii) Random experiment

(i) Deterministic experiment :→ The outcomes of such experiments are always certain, if the experiment is repeated again and again, the the result also remains approximately the same

For example :→ If we repeat the experiment of gas law $\boxed{PV=nRT}$ with different value of P, V, & T, we will always obtain the same value of R.

(ii) Random experiment :→ If such type of experiment is repeated again & again, the result always may not be the same that is there is no certainty about their result.

For example :→ If a coin is tossed again & again, it is not certain, when the head or tail will come.

Concept of probability :→ when the chance of happening an event is expressed numerically, it is called probability.

Def :→ According to Laplac "Probability is the ratio of favourable cases to the total number of likely cases."

① Trial :→ The performance of a random experiment for one time is called a trial, The outcome of trial is called event.

For example :→ Tossing of a coin is a trial Head (H) or tail (T) falls by tossing a coin. H & T are two events.

② Sample points :→ The outcomes of a random experiment are known as sample points.

③ Sample space :→ The set of all possible outcomes or sample points of random experiment is known as sample space.

We denote a sample space by S. Each element of the sample space is a sample point, if H & T come in tossing of a coin. Then S = {H, T}. where H & T are its two sample point.

④ Discrete sample space :→ A sample space, whose sample points (whose number is finite or countably infinite) can be arranged into a sample sequence according to possitive integers is known as discrete sample space.

⑤ Continuos sample space :→ A sample space whose sample points are uncountably infinite is known as continuos sample space.

⑥ Event :→ The subset of a sample space is known as event. The subset in which sample points are simple is known as simple or elementary event.

⑦ Equally likely events :→ Two or more events are said to be equally likely, if the chance of their happing are same.

For example :→ 1, 2, 3, 4, 5 & 6 are equally likely events in throwing of a dice.

⑧ Independent events :→ Two or more events are said to be independent events if the happing of one does not affect the happing of others.

For example :→ If two dice are thrown simultaneously, the number coming on first will not affect the number coming on second.

⑨ Dependent events :→ The events, which is not independent is called dependent events.

⑩ Mutually exclusive events :→ Two or more events are said to be mutually exclusive events if the happening of one prevents the happening of others

In other words, mutually exclusive events can not happen simultaneosly.

For example

(a) Tossing of a coin H & T are two mutually exclusive events.

(b) Throwing a dice 1, 2, 3, 4, 5 & 6 are mutually exclusive events.

(c) If one card is drawn from a deck of cards then there will be total 52 mutually exclusive events.

(11) Exhaustive events :→ All possible outcomes (events) of a random experiment are known as exhaustive events.

For example :→ If a bag contain 10 balls & one ball is drawn at random, then there will be 10 exhaustive events.

" In other words two or more events are said to be exhaustive events if their union is equal to sample space.

(12) Favourable cases :→ The outcomes from all equally likely outcomes of a random experiment which are favourable for the happing of a particular events are called favourable cases.

For example :→ Throwing a dice, there will be three favourable cases (2, 4, 6) for coming an even number.

Probability of the happening of the event

$$p = \frac{\text{Number of favourable cases}}{\text{Total number of equally likely cases}}$$

Similarly

Probability of not happening of the event

$$q = \frac{\text{No. of cases not favourable}}{\text{Total no. of equally likely cases}}$$

$$p = \frac{m}{m+n} \quad \& \quad q = \frac{n}{m+n}.$$

$$p+q = \frac{m+n}{m+n} \Rightarrow \boxed{p+q=1}$$

Example :→ A single letter is selected at random from the word "PROBABILITY" what is the probability that it is vowel.

Sol^n :→ Total letters $n = 11$

No. of vowels $m = 4$

Required probability $= \frac{m}{n} = \frac{4}{11}$

④ **Example** :→ What is the chance that a non leap year should have 53 Mondays.

Sol^n :→ There are 365 days in a non-leap year that is 52 complete weeks & one day.
Any day from seven days of week (Monday, Tuesday, Wednesday, Thursday, Friday, Saturday) may come 53^{rd} times.
So, equally likely cases = 7
Favourable ways for 53 Mondays = 1

$$\text{Required probability} = \frac{\text{Favourable cases}}{\text{Total cases}} = \frac{1}{7}$$

Example :→ Find the probability of 53 Saturdays in a leap year.

Sol^n :→ There are 366 days in a leap year
No. of days in 52 weeks = $52 \times 7 = 364$ days
There may be following pairs for remaining two days
1) Sunday & Monday 2) Monday & Tuesday
3) Tuesday & Wednesday 4) Wednesday & Thursday
5) Thursday & Friday 6) Friday & Saturday
7) Saturday & Sunday
Total permutations of two days = 7
Permutation of Saturday = 2

$\therefore$ Probability of 53 Saturdays = $\frac{2}{7}$ Ans.

Example :→ Two dice are drawn
(i) What is the prob. of 5 on both dice
(ii) What is prob. that the sum of numbers on the two dice is 9.

Sol^n :→ Total no. of cases = $6 \times 6 = 36$
(i) Favourable no. of cases for 5 on both dice = 1
$P(5)$ on both dice = $\frac{1}{36}$
(ii) Total no. on both will be 9 if the points are $(6,3), (5,4), (4,5)$ & $(3,6)$.
So, favourable cases = 4.
Prob. of getting 9 = $\frac{4}{36} = \frac{1}{9}$.

⑤

Law of probability :→ There are two main laws of probability

(i) Addition law of probability

(ii) Multiplication law of probability

(i) Addition law of probability :→ If events are not mutually exclusive that is there are some common elements.

Then the formula for addition law will be

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

(ii) Multiplication law of probability :→

If A, B & C are three independent events, then $\boxed{P(A \cap B \cap C) = P(A) . P(B) . P(C)}$

Example :→ Find the probability that a leap year selected at random will contain 53 Monday or 53 Tuesday?

Solⁿ :→ There are 366 days in a leap year, that is 364 days of 52 weeks & 2 days.

Hence all days occurs 52 times, Any one case may occur in remaining two days.

(i) Monday & Tuesday (ii) Tuesday & Wednesday

(iii) Wednesday & Thursday (iv) Thursday & Friday

(v) Friday & Saturday (vi) Saturday & Sunday

(vii) Sunday & Monday

Prob. of occuring 53 Mondays $P(A) = 2/7$

Prob. of " " Tuesdays $P(B) = \frac{2}{7}$

Prob. of occuring 53 Mondays & 53 Tuesdays

i, e $P(A \cap B) = \frac{1}{7}$

Req. Prob. i, e $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cup B) = \frac{2}{7} + \frac{2}{7} - \frac{1}{7} = \frac{4-1}{7} = \frac{3}{7}$ Ans.

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Example:→ The probability that a question can be solved by A is $\frac{1}{3}$ & B is $\frac{1}{4}$. What is the probability that the question will be solved by any one of them.

Sol^n:→ $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$

$P(A \cap B) = P(A) \cdot P(B)$ {∵ Both events are independent}

$= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

Therefore, the prob. that the question will be solved by any one of them.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12}$$

$$= \frac{4+3-1}{12} = \frac{6}{12} = \frac{1}{2} \text{ Ans.}$$

Conditional probability:→ When the happening of an event A depends on the happening of event B. The prob. of event A is called conditional prob. & it is denoted by $P(A/B)$.

Thus $\boxed{P(A/B) = \frac{P(A \cap B)}{P(B)}} \Rightarrow \boxed{P(A \cap B) = P(A/B) \cdot P(B)}$

Note:→ (i) If $P(A/B) = P(A)$, then A & B will be independent

(ii) If $P(B/A) = P(B)$, then B & A will be independent.

Example:→ A dice is thrown once, what is the conditional prob. that multiple of three obtained when it is known that it is an even number.

Sol^n:→ Let A be the event that a multiple of 3 is obtained. $A = [3, 6]$ $\therefore n(A) = 2$

& let B be the event that an even number is obtained

$B = [2, 4, 6]$ ∴ $n(B) = 3$, $A \cap B = \{6\}$ ∴ $n(A \cap B) = 1$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)/n(S)}{n(B)/n(S)} = \frac{1/6}{3/6}$$

$$= \frac{1}{6} \times \frac{6}{3} = \frac{1}{3} \text{ Ans.}$$

Example :- Find the prob. of drawing of two aces one after other without replacement from well shuffled 52 cards.

Sol^n :- Let A be an event that an ace will be drawn for first time & B be the event that an ace will be drawn for second time

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

Now, there are 3 aces in 51 cards in the deck

$P(B/A) = P(B) = \frac{3}{51}$ { $\because$ A & B are independent event }

P(An ace will be drawn for first & second time)

i, e $P(A \cap B) = P(A) \cdot P(B/A) = \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$.

Baye's theorem :- Let $B_1, B_2, \text{---} B_n$ be n mutually exclusive & exhaustive events, whose union be a sample space & assume A be an arbitrary event in space & $P(A) \neq 0$ & $P(B_1), P(B_2) \text{----} P(B_n)$ & $P(A/B_1), P(A/B_2) \text{----} P(A/B_n)$ are known.

Then

$$\boxed{P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}}$$

Example :- There are two urns. In the first urn there are 3 white, & 2 black balls & in the second 1 white & 4 black balls. One of the urn is selected & one ball is drawn. It turns out to be white, what is the probability that it came from the second urn.

Sol^n :- Let A = Event that a white ball is drawn

B_1 = Event that first urn is selected

B_2 = Event that second urn is selected

$P(B_1) = P(B_2) = \frac{1}{2}$, $P(A/B_1) = \frac{3}{5}$, $P(A/B_2) = 1/5$

(8) According to Baye's theorem

$$P(B_i/A) = \frac{P(B_i) \cdot P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

$$\Rightarrow P(B_2/A) = \frac{P(B_2) \cdot P(A/B_2)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{5}}{\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{1}{5}} = \frac{\frac{1}{10}}{\frac{3}{10} + \frac{1}{10}} = \frac{\frac{1}{\cancel{10}}}{\frac{4}{\cancel{10}}} = \frac{1}{4} \text{ Ans.}$$

Example: A box contains 3 blue & 2 red balls while another box contains 2 blue & 5 red balls. A ball is drawn at random from one of the boxes turns to be blue. What is the prob. that it came from the first box.

Soln: Let A be the event that a blue ball is drawn & let us assume B_1 & B_2 be the events that first & second box will be selected respectively.

$P(B_1) = P(B_2) = \frac{1}{2}$, $P(A/B_1) = \frac{3}{5}$, $P(A/B_2) = \frac{2}{7}$

By Baye's theorem, we get

$$P(B_i/A) = \frac{P(B_i)\, P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

$$\Rightarrow P(B_1/A) = \frac{P(B_1) \cdot P(A/B_1)}{P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{3}{5}}{\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{7}} = \frac{3/10}{\frac{3}{10} + \frac{1}{7}} = \frac{3/10}{\frac{21+10}{70}}$$

$$= \frac{3}{1\cancel{0}} \times \frac{7\cancel{0}}{31} = \frac{21}{31} \text{ Ans.}$$

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Example:- Four cards are drawn from a pack of cards. Find the probability that (i) all are diamonds, (ii) there is one card of each suit, and (iii) there are two spades and two hearts.

Sol^n:- 4 cards can be drawn from a pack of 52 cards in $^{52}C_4$ ways.

$\therefore$ Exhaustive number of cases $= {}^{52}C_4 = \frac{52 \times 51 \times 50 \times 49 \times 48!}{4 \times 3 \times 2 \times 1 \times 48!}$

$= 270725.$

(i) There are 13 diamonds in the pack and 4 can be drawn out of them in $^{13}C_4$ ways.

$\therefore$ favourable number of cases $= {}^{13}C_4 = \frac{13 \times 12 \times 11 \times 10 \times 9!}{4 \times 3 \times 2 \times 1 \times 9!} = 715.$

Required Probability $= \frac{715}{270725} = \frac{143}{54145} = \frac{11}{4165}$

(ii) There are 4 suits, each containing 13 cards.

$\therefore$ favourable number of cases $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$

$= 13 \times 13 \times 13 \times 13$

Required probability $= \frac{13 \times 13 \times 13 \times 13}{270725} = \frac{2197}{20825}.$

(iii) 2 spades out of 13 can be drawn in $^{13}C_2$ ways.
2 hearts out of 13 can be drawn in $^{13}C_2$ ways.
Favourable no of cases $= {}^{13}C_2 \times {}^{13}C_2$
$= 78 \times 78$
Required prob. $= \frac{78 \times 78}{270725} = \frac{468}{20825}$

Example:- A and B take turns in throwing dice, the first throw 10 being awarded the prize. Show that if A has the first throw, their chances of winning are in the ratio 12:11.

Sol^n:- The combinations of throwing 10 from two dice can be $(6+4)$, $(4+6)$, $(5+5)$

$\Rightarrow$ The prob. of getting $10 = \frac{3}{36} = \frac{1}{12} = p$

The number of combinations is 3.

Total combinations from two dice $= 6 \times 6 = 36$

$\Rightarrow$ The probability of not getting 10 $= 1 - \frac{1}{12} = \frac{11}{12} = q$

If A is to win, he should throw 10 in either the first, the third, the fifth ---- throws.

Their respective probabilities are p, q^2p, q^4p ----

$\left(\frac{1}{12}\right), \left(\frac{11}{12}\right)^2 \cdot \frac{1}{12}, \left(\frac{11}{12}\right)^4 \cdot \frac{1}{12}$ ----

A's total probability of winning

$= \left(\frac{1}{12}\right) + \left(\frac{11}{12}\right)^2 \cdot \frac{1}{12} + \left(\frac{11}{12}\right)^4 \cdot \frac{1}{12} + \text{----}$

$= \frac{\frac{1}{12}}{1 - \left(\frac{11}{12}\right)^2} = \frac{12}{23}$ $\left\{ \begin{array}{l} \text{This is infinite G.P series} \\ \text{Its sum} = \frac{a}{1-r} \end{array} \right\}$

B can win in either 2^{nd}, 4^{th}, 6^{th} ---- throws

So B's total chance of winning $= qp + q^3p + q^5p + \text{----}$

$= \left(\frac{11}{12}\right)\left(\frac{1}{12}\right) + \left(\frac{11}{12}\right)^3\left(\frac{1}{12}\right) + \left(\frac{11}{12}\right)^5\left(\frac{1}{12}\right) + \text{----}$

$= \frac{\left(\frac{11}{12}\right) \cdot \left(\frac{1}{12}\right)}{1 - \left(\frac{11}{12}\right)^2} = \frac{11}{23}.$

Hence A's chance to B's chance $= \frac{12}{23} : \frac{11}{23} = 12:11$ **Proved.**

Example :→ If the probability that the man aged 60 will live till 70 is 0.6, what is the probability that out of 10 men aged 60, 9 men will live upto 70.

Sol^n :→ Given probability of a man aged 60 will live upto 70 is $p = 0.6$ & $q = 1-p = 1-0.6 = 0.4$

$n = 10$, & $r = 9$

Also, using Binomial distribution

$P(X=r) = {}^nC_r\, p^r . q^{n-r}$

∴ Required probability = Prob. (9 men will live upto 70)

$= {}^{10}C_9 . (0.6)^9 . (0.4)^1$

$= 10 \times \left(\frac{6}{10}\right)^9 \times \frac{4}{10} = 0.0403$ Ans.

Example :→ An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, a car driver and a truck driver is 0.01, 0.03 & 0.15 respectively. One of the insured person meets with an accident. What is the prob. that he is a scooter driver.

Sol^n :→ Let E_1 = The event that a person chosen is a scooter driver

E_2 = The event that a person chosen is a car driver

E_3 = The event that a person chosen is a truck driver

A = The event that a person meets with an accident

Now, total no. of persons = 2000 + 4000 + 6000 = 12000

$P(E_1) = \frac{2000}{12000} = \frac{1}{6}$, $P(E_2) = \frac{4000}{12000} = \frac{1}{3}$, $P(E_3) = \frac{6000}{12000} = \frac{1}{2}$

Also, $P(A/E_1)$ = The prob. that a person meets with an accident given that he is a scooter driver $= 0.01$,

Similarly, $P(A/E_2) = 0.03$, & $P(A/E_3) = 0.15$,

we are asked to find $P(E_1/A)$ given that the person meets with an accident what is the prob. that he was a scooter driver

∴ By Baye's theorem

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + P(E_3).P(A/E_3)}$$

$$= \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15} = \frac{0.01}{6} \times \frac{6}{0.01 + 2 \times (0.03) + 3 \times (0.15)}$$

$$\Rightarrow P(E_1/A) = \frac{0.01}{0.01 + 0.06 + 0.45} = \frac{1}{1+6+45} = \frac{1}{52}$$

$$\Rightarrow \boxed{P(E_1/A) = \frac{1}{52}}$$

Example:→ The odds against A solving a certain problem are 5 to 7 and the odds in favour B solving the same problem are 3 to 4. What is the probability that if both of them try, the problem would be solved.

Sol^n:→ The prob. that A can not solve the problem is $= \frac{5}{5+7} = \frac{5}{12}$

The prob. that B can not solve the problem is $= \frac{5}{\cancel{12}_{3}} \times \frac{\cancel{4}}{7} = \frac{5}{21}$

Hence, the probability that the problem will be solved is

$= 1 - \frac{5}{21} = \frac{16}{21}$. Ans.

Example:→ In a toy factory, machines A, B and C manufacture respectively 25%, 35%, and 40% of the total, of their output 5, 4, 2 percents are respectively defective. A toy is drawn at random from the total production. What is the probability that the toy drawn is defective? Also find the probability that it was manufactured by machine A.

Sol^n:→ Let us suppose x toy is produced in a factory

No. of toys produced by $A = \frac{\cancel{25}x}{{}_4\cancel{100}} = \frac{x}{4}$

" " " " " $B = {}^{7}\frac{\cancel{35}x}{\cancel{100}_{20}} = \frac{7x}{20}$

" " " " " $C = {}^{2}\frac{\cancel{40}x}{{}_5\cancel{100}} = \frac{2x}{5}$

No. of defective toys produced by $A = \frac{\cancel{5}}{\cancel{100}_{20}} \times \frac{x}{4} = \frac{x}{80}$

" " " " " " $B = \frac{\cancel{4}}{100} \times \frac{7x}{{}_5\cancel{20}} = \frac{7x}{500}$

" " " " " " $C = \frac{2}{100} \times \frac{2x}{5} = \frac{4x}{500}$

$$P(\text{Toy drawn is defective}) = \frac{\text{Total no. of defective toys}}{\text{Total production}}$$

$$= \frac{\frac{x}{80} + \frac{7x}{500} + \frac{4x}{500}}{x} = \frac{500x + 560x + 320x}{40000/x} = \frac{1380\cancel{x}}{40000\cancel{x}}$$

$$= \frac{690}{20000} = \frac{345}{10000} = \frac{69}{2000} = 0.0345$$

$$\text{Prob.}(\text{Defective}/A) = \frac{P(\text{Defective}/A) \times P(\text{Machine } A)}{P(\frac{\text{Def.}}{A}) \times P(A) + P(\frac{\text{Def.}}{B}) \times P(B) + P(\frac{\text{Def.}}{C}) \times P(C)}$$

$$= \frac{\frac{5}{100} \times \frac{25}{100}}{\frac{5}{100} \times \frac{25}{100} + \frac{4}{100} \times \frac{35}{100} + \frac{2}{100} \times \frac{40}{100}} = \frac{125/10000}{\frac{125}{10000} + \frac{140}{10000} + \frac{80}{10000}}$$

$$= \frac{125}{1\cancel{0000}} \times \frac{1\cancel{0000}}{345} = \frac{{}^{25}\cancel{125}}{{}_{69}\cancel{345}} = \frac{25}{69} \text{ Ans.}$$

Example :→ In a bolt factory, there are four machines. A, B, C, D manufacturing 20%, 15%, 25%, and 40% of the total output respectively of their outputs 5%, 4%, 3%, and 2% in the same order, are defective bolts. A bolt is choosen at random from the factory's production and is found defective. What is the probability that the bolt was manufactured by machine A or machine D?

Sol^n :→ Let E_1, E_2, E_3 and E_4 denote the events that a bolt selected at random is manufactured by the machine A, B, C and D respectively and let A denote the event of its being defective. Then we have $P(E_1) = 0.20$, $P(E_2) = 0.15$, $P(E_3) = 0.25$ and $P(E_4) = 0.40$

The probability of drawing a defective bolt manufactured by machine A is $P(A/E_1) = 0.05$. Similarly $P(A/E_2) = 0.04$, $P(A/E_3) = 0.03$, and $P(A/E_4) = 0.02$, Hence the probability that a defective bolt manufactured by machine A is given by

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)} = \frac{0.20 \times 0.05}{0.20\times0.05 + 0.15\times0.04 + 0.25\times0.03 + 0.40\times0.02}$$

$$= \frac{100}{100+60+75+80} = \frac{100}{315} = 0.3175.$$

Similarly $$P(E_4/A) = \frac{P(E_4).\,P(A/E_4)}{\sum_{i=1}^{4} P(E_i).\,P(A/E_i)} = \frac{0.40 \times 0.02}{0.0315} = \frac{80}{315} = 0.254.$$ Ans.

Example :→ There are three bags, first containing 1 white, 2 red 3 green balls, second, 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn come from the second bag.

Sol^n :→ Let B_1, B_2, B_3 pertain to the first, second, third bags chosen and A, the two balls are white and red.

Now $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$

$P(A/B_1)$ = P(a white and a red ball are drawn from the first bag)

$$= \frac{{}^1C_1 \times {}^2C_1}{{}^6C_2} = \frac{2}{15}$$

Similarly, $$P(A/B_2) = \frac{{}^2C_1 \times {}^3C_1}{{}^6C_2} = \frac{2}{5}$$

$$P(A/B_3) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{1}{5}$$

By Baye's theorem

$$P(B_2/A) = \frac{P(B_2).\,P(A/B_2)}{P(B_1).P(A/B_1) + P(B_2).P(A/B_2) + P(B_3).P(A/B_3)}$$

$$= \frac{\frac{1}{3}\times\frac{2}{5}}{\frac{1}{3}\times\frac{2}{15} + \frac{1}{3}\times\frac{2}{5} + \frac{1}{3}\times\frac{1}{5}} = \frac{6}{11}$$ Ans.

Random variable :→ If such type of variable repeated again & again, but the result always may not be the same, that is there is no certainty about their result.

For example :→ If a coin is tossed again & again, it is not certain when the head or tail will come.

There are two types of random variable

1) Discrete random variable
2) Continuous random variable

→ If a random variable takes a finite set of values, then it is called discrete variable

→ On the other hand, if it takes a infinite set of values is called continuous variable.

Discrete probability distribution :→

Let a random variable x assume values $x_1, x_2, x_3 - - - x_n$ with probabilities $p_1, p_2, p_3 - - - p_n$ respectively, for each x_i

& $p_1 + p_2 + p_3 + - - - p_n = \sum_{i=1}^{n} p_i = 1$

Then x : $x_1 \quad x_2 \quad x_3 - - - x_n$

$p(x)$: $p_1 \quad p_2 \quad p_3 - - - p_n$

is called the discrete probability distribution for x.

(14) # The probability density function of a variable X is

X :	0	1	2	3	4	5	6
$P(X)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

i) Find $P(X<4)$ ii) $P(X\geq 5)$ iii) $P(3<X\leq 6)$

ii) What will be the minimum value of k so that $P(X\leq 2)>0.3$.

⇒ 1. i) If X is a random variable, then

$\sum_{i=0}^{6} P(x_i)=1$, i.e $k+3k+5k+7k+9k+11k+13k=1$

$\Rightarrow 49k=1 \Rightarrow k=\frac{1}{49}$

ii) $P(X<4)=k+3k+5k+7k=16k=\frac{16}{49}$

iii) $P(X\geq 5)=11k+13k=24k=24/49$

iv) $P(3<X\leq 6)=9k+11k+13k=33k=33/49$

2) $P(X\leq 2)=k+3k+5k=9k>0.3 \Rightarrow \boxed{k>\frac{1}{30}}$

Thus the minimum value of $k=\frac{1}{30}$

Example 1 : A random variable X has the following probability function

X :	0	1	2	3	4	5	6	7
$P(X)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Find the value of k ii) Evaluate $P(X<6)$, $P(X\geq 6)$

iii) $P(0<X<5)$.

Sol: ⇒ i) If X is random variable, then $\sum_{i=0}^{7} P(x_i)=1$, i.e

$0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1 \Rightarrow k=\frac{1}{10}$

$\Rightarrow 7k^2+9k-1=0$ i.e $(10-k)(k+1)=0$, i.e $k=\frac{1}{10}$

ii) $P(X<6)=0+k+2k+2k+3k+k^2=8k+k^2=\frac{8}{10}+\frac{1}{100}=\frac{81}{100}$

iii) $P(X\geq 6)=2k^2+7k^2+k=\frac{9}{100}+\frac{1}{10}=19/100$

iv) $P(0<X<5)=k+2k+2k+3k=8k=\frac{8}{10}=\frac{4}{5}$

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Imp (15)

Continuous probability distribution :→

Let X be a continuous random variable taking values in the interval $(-\infty, \infty)$. Let $f(x)$ be a function satisfying the following properties

i) $f(x)$ is integrable on $(-\infty, \infty)$

ii) $f(x) \geq 0$ for all x in $(-\infty, \infty)$ i.e $x \in R$,

iii) $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1.}$

Then $f(x)$ is called the probability distribution (or density) function (p.d.f) of the continuous random variable X.

Imp **Mean & variance of a random variables :→**

a) Let $X: x_1, x_2, x_3 --- x_n$

$p(X): p_1 \quad p_2 \quad p_3 --- p_n$

be a discrete probability distribution

Then we denote the mean by μ & define as $\mu = \frac{\Sigma p_i x_i}{\Sigma p_i} = \Sigma p_i x_i$, where $\Sigma p_i = 1$

& other names of the mean are average or expected value $E(X)$.

We denote the variance by σ^2 & define as $\sigma^2 = \Sigma p_i (x_i - \mu)^2$

standard deviation $\sigma = +\sqrt{\text{variance}}$

b) If X is a continuous random variable with probability density function $f(x)$

then $\boxed{\text{mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx}$

(16)

similarly $E[X^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$

In general $E[X^r] = \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx$

$$\boxed{\text{variance}(X) = \int_{-\infty}^{\infty} (x-\bar{x})^2 \cdot f(x)\, dx}$$

$$= E(X^2) - [E(X)]^2$$

ϕ mean deviation from the mean

$$= \int_{-\infty}^{\infty} |x-\bar{x}|\, f(x)\, dx$$

(17)

Example $\overset{i}{\rightarrow}$

If X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} kx & 0 \leq x \leq 5 \\ k(10-x) & 5 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

i) find the value of k, ii) mean of X

iii) $P(5 < X < 12)$

Sol^n $\rightarrow$

i) In order that $f(x)$ should be a probability density function $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\Rightarrow \int_0^5 kx\,dx + \int_5^{10} k(10-x)\,dx = 1.$$

$$\Rightarrow k\left[\frac{x^2}{2}\right]_0^5 + 10k\left[x\right]_5^{10} - k\left[\frac{x^2}{2}\right]_5^{10} = 1$$

$$\Rightarrow \frac{25k}{2} + 100k - 50k - \frac{100}{2}k + \frac{25}{2}k = 1$$

$$\Rightarrow \left(\frac{25}{2} + 100 - 50 - 50 + \frac{25}{2}\right)k = 1$$

$$\Rightarrow \left(\frac{25 + 200 - 100 - 100 + 25}{2}\right)k = 1$$

$$\Rightarrow \frac{50}{2}k = 1 \Rightarrow \boxed{k = \frac{1}{25}}$$

ii) mean of $\bar{x} = \int_0^{10} x\,f(x)\,dx$

$$= \int_0^5 x \cdot f(x)\,dx + \int_5^{10} x \cdot f(x)\,dx$$

$$= \int_0^5 x \cdot kx\,dx + \int_5^{10} x \cdot k(10-x)\,dx$$

$$= \frac{1}{25}\left[\frac{x^3}{3}\right]_0^5 + \frac{1}{25}\left[10 \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_5^{10}$$

18

$$= \frac{1}{25}\cdot\frac{125}{3} + \frac{1}{25}\left[500 - \frac{1000}{3} - 125 + \frac{125}{3}\right]$$

$$= \frac{1}{25}\left[\frac{125}{3} + 500 - \frac{1000}{3} - 125 + \frac{125}{3}\right]$$

$$= \frac{1}{25}\left[\frac{125 + 1500 - 1000 - 375 + 125}{3}\right] = \frac{1}{\cancel{25}} \times \frac{\overset{5}{\cancel{\cancel{375}\,125}}}{3}$$

$$\Rightarrow \boxed{\bar{X} = 5}$$

iii) $P(5 \le X \le 12) = \int_5^{12} f(x)\,dx$

$$= \int_5^{10} f(x)\,dx + \int_{10}^{12} f(x)\,dx = \int_5^{10} k(10-x)\,dx$$

$$= \frac{1}{25}\left[10x - \frac{x^2}{2}\right]_5^{10}$$

$$= \frac{1}{25}\left[100 - 50 - 50 + \frac{25}{2}\right] = \frac{1}{\cancel{25}} \times \frac{\cancel{25}}{2} = \frac{1}{2}$$

$$\boxed{P(5 \le X \le 12) = \frac{1}{2}}$$

\# **Example** → Is the function defined as follows a density function? $f(x) = \begin{cases} e^{-x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$

Sol^n → $f(x) \ge 0$, for every x in $(-\infty, \infty)$

$$\& \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_0^{\infty} e^{-x}\,dx = 0 + \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = -(0-1) = 1$$

$\Rightarrow f(x)$ satisfies the requirment of density function

$\Rightarrow f(x)$ is a density function

$$P(1 \le X \le 2) = \int_1^2 e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_1^2 = e^{-1} - e^{-2}$$

$$= 0.368 - 0.135$$

$$= 0.233$$

(19)

Example :→ X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} kx, & \text{if } 0 \leq x \leq 2 \\ 2k, & \text{if } 2 \leq x \leq 4 \\ -kx + 6k, & \text{if } 4 \leq x \leq 6. \end{cases}$$

Find k and mean value of X.

Sol^n :→ since total prob. = 1

we have $\int_0^6 f(x)\,dx = 1$

$\Rightarrow \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^6 f(x)\,dx = 1$

$\Rightarrow \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (-kx + 6k)\,dx = 1$

$\Rightarrow k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[x\right]_2^4 + \left[-k\frac{x^2}{2} + 6kx\right]_4^6 = 1$

$\Rightarrow k(2-0) + 2k(4-2) + (-18k + 36k) - (-8k + 24k) = 1$

$\Rightarrow 2k + 4k + 18k - 16k = 1 \Rightarrow 8k = 1 \Rightarrow \boxed{k = \frac{1}{8}}$

Mean value of $X = E(X) = \int_0^6 x \cdot f(x)\,dx$

$= \int_0^2 x \cdot kx\,dx + \int_2^4 x \cdot 2k\,dx + \int_4^6 x(-kx + 6k)\,dx$

$= k\left[\frac{x^3}{3}\right]_0^2 + 2k\left[\frac{x^2}{2}\right]_2^4 + \left[-k\frac{x^3}{3} + 6k \cdot \frac{x^2}{2}\right]_4^6$

$= k\left(\frac{8}{3}\right) + k(12) - \frac{k}{3}(152) + 3k(20)$
$\quad 3 \quad 3$

$= 24k = \cancel{24} \times \frac{1}{\cancel{8}} = 3$

$\Rightarrow \boxed{k = \frac{1}{8}}$ & $\boxed{\text{Mean} = 3}$

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Example:→ Determine the value of k, if the probability function of a random variable x is given by

$$p(x) = \begin{cases} \frac{kx}{20}, & x = 1, 2, 3, 4 \\ 0, & \text{for other integers} \end{cases}$$

Sol^n:→ $p(x) = \begin{cases} \frac{kx}{20}, & x = 1, 2, 3, 4 \\ 0 & \text{for other integers} \end{cases}$

By Discrete distribution function, we have

$\Rightarrow \sum_{i=0}^{\infty} p(x_i) = 1 \Rightarrow p(x_0) + p(x_1) + p(x_2) + p(x_3) + p(x_4) = 1$

$\Rightarrow 0 + \frac{k}{20} + \frac{2k}{20} + \frac{3k}{20} + \frac{4k}{20} = 1$

$\Rightarrow \frac{10k}{20} = 1 \Rightarrow \boxed{k = 2}$

Example:→ A random variable x has the following density function

x:	-2	-1	0	1	2	3
f(x):	0.1	k	0.2	2k	0.3	k

Find the value k and calculate mean & variance.

Sol^n:→ For the value of k $\Sigma p(x) = 1$.

$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\Rightarrow 4k = 0.4 \Rightarrow \boxed{k = 0.1}$

Mean $= \mu = \Sigma p_i x_i = -2 \times 0.1 - 1 \times 0.1 + 0 \times 0.2 + 2 \times 0.1 + 1 + 0.3 \times 2 + 3 \times 0.1 = -0.2 - 0.1 + 0.2 + 0.6 + 0.3 = 0.8$

Variance$(\sigma^2) = \Sigma (x_i - \mu)^2 . p_i(x)$

$= (-2 - 0.8)^2 \times 0.1 + (-1 - 0.8)^2 \times 0.1 + (0 - 0.8)^2 \times 0.2 + (1 - 0.8)^2 \times 0.2 + (2 - 0.8)^2 \times 0.3 + (3 - 0.8)^2 \times 0.1$

$= 0.1[(-2.8)^2 + (-1.8)^2 + (-0.8)^2 \times 2 + (0.2)^2 \times 2 + (1.2)^2 \times 3 + (2.8)^2] = 0.1[7.84 + 3.24 + 1.28 + 0.08 + 4.32 + 7.84]$

$= 2.23$

$\Rightarrow \boxed{k = 0.1}$ $\boxed{\mu = 0.8}$ & $\boxed{\sigma^2 = 2.23}$

(21) Example :→ The density function of a random variable X is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find (i) k (ii) Mean (iii) variance (iv) mean deviation about the mean.

Sol^n :→ (i) Since total probability = 1, we have

$$\int_0^2 f(x)\,dx = 1,$$

$$\Rightarrow \int_0^2 kx(2-x)\,dx = 1 \Rightarrow k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow k\left(4 - \frac{8}{3}\right) = 1 \Rightarrow k \cdot \frac{4}{3} = 1 \Rightarrow \boxed{k = \frac{3}{4}}$$

Hence $f(x) = \frac{3}{4}x(2-x)$, $0 \le x \le 2$.

(ii) Mean $= E(x) = \int_0^2 x f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2-x)\,dx$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left(\frac{16}{3} - 4\right) = \cancel{\frac{3}{4}} \cdot \cancel{\frac{4}{3}} = 1$$

(iii) Variance $(x) = E(x^2) - [E(x)]^2$

$$= \int_0^2 x^2 f(x)\,dx - (1)^2 = \frac{3}{4}\int_0^2 x^3(2-x)\,dx - 1$$

$$= \frac{3}{4}\left[2 \cdot \frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 - 1 = \frac{3}{4}\left(8 - \frac{32}{5}\right) - 1$$

$$\Rightarrow \frac{3}{4}\left[2 \cdot \frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 - 1 = \frac{3}{\cancel{4}} \cdot \frac{\overset{2}{\cancel{8}}}{5} - 1 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\Rightarrow \boxed{Var(x) = \frac{1}{5}}$$

(iv) Mean deviation about the mean

$$= \int_0^2 |x - \bar{x}|\, f(x)\,dx = \int_0^2 |x-1|\, f(x)\,dx$$

$$= \int_0^1 |x-1|\, f(x)\,dx + \int_1^2 |x-1| \cdot f(x)\,dx$$

$$= \int_0^1 (1-x) \cdot \frac{3}{4}x(2-x)\,dx + \int_1^2 (x-1) \cdot \frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^1 (2x - 3x^2 + x^3)\,dx + \frac{3}{4}\int_1^2 (-2x + 3x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1 + \frac{3}{4}\left[-x^2 + x^3 - \frac{x^4}{4}\right]_1^2$$

$$= \frac{3}{4}\left(\frac{1}{4}\right) + \frac{3}{4}\left(\frac{1}{4}\right) = \frac{3}{8}. \text{ Ans.}$$

(22)

Moment generating function (m.g.f) :→

The moment generating function of a random variable X about the point 'a' is defined as

$$M_a(t) = E\left[e^{t(x-a)}\right] = \int_{-\infty}^{\infty} e^{t(x-a)} \cdot f(x)\,dx$$

where t is a real number & x is continuous

Moment about the origin :→ we have

$$M_0(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \quad \text{if } x \text{ is continuous}$$

$$\& \; M_0(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} f(x) \quad \text{if } x \text{ is discrete}$$

Now,

$$M_a(t) = E\left[e^{t(x-a)}\right] = E\left[1 + t(x-a) + \frac{t^2}{2!}(x-a)^2 + \frac{t^3}{3!}(x-a)^3 + \cdots + \frac{t^r}{r!}(x-a)^r + \cdots\right]$$

$$= E[1] + t\,E(x-a) + \frac{t^2}{2!}(x-a)^2 + \frac{t^3}{3!}(x-a)^3 + \cdots + \frac{t^r}{r!}E[(x-a)^r + \cdots]$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots - \frac{t^r}{r!}\mu_r' + \cdots$$

where $\mu_r' = E[(x-a)^r]$ is the r^{th} moment about the point a.

Further :→ $\frac{d}{dt}\{M_a(t)\} = \mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2} + \cdots$

$$\Rightarrow \left|\frac{d}{dt}\{M_a(t)\}\right|_{at\ t=0} = \mu_1'$$

Again $\frac{d^2}{dt^2}\{M_a(t)\} = \mu_2' + \mu_3' t + \cdots$

$$\Rightarrow \left|\frac{d^2}{dt^2} M_a(t)\right|_{at\ t=0} = \mu_2'$$

Note :→ Where μ_r is moment about mean and μ_r' is moment about any number.

In general $\left|\frac{d^r}{dt^r} M_a(t)\right| = \mu_r'$

Remark :→ ①

$$\boxed{\begin{array}{l} Mean = E[X] = \mu_1' \\ \& \\ Var[X] = \mu_2' - (\mu_1')^2 \end{array}}$$

② **Properties** :→ (i) $M_{cx}(t) = M_x(ct)$

(ii) $M_a(t) = e^{-at} M_0(t)$

(iii) If X_1, X_2 are two independent random variable, then $M_{X_1+X_2} = M_{X_1}(t) \cdot M_{X_2}(t)$

(3) $\mu_2 = \mu_2' - (\mu_1')^2, \quad \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

(4) $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}$ if $\beta_1 = 0 \Rightarrow$ The distribution is symmetrical

$\beta_2 = \dfrac{\mu_4}{\mu_2^2}$

Example :→ Find the moment generating function of the random variable x having the density function

$$f(x) = \begin{cases} x, & 0 < x < 1. \\ 2-x, & 1 \leq x < 2. \\ 0 & \text{elsewhere} \end{cases}$$

Also determine μ_1', μ_2' & μ_2.

Solⁿ :→ We have $M_x(t) = \int_0^\infty e^{tx} \cdot f(x)\,dx$

$$M_x(t) = \int_0^1 x \cdot e^{tx}\,dx + \int_1^2 (2-x)e^{tx}\,dx + \int_2^\infty 0 \cdot e^{tx}\,dx$$

$$= \left\{\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2}\right\}_0^1 + \left\{\frac{2e^{tx}}{t} - \frac{x e^{tx}}{t} + \frac{e^{tx}}{t^2}\right\}_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \left[\left\{\cancel{\frac{2e^{2t}}{t}} - \cancel{\frac{2e^{2t}}{t}} + \frac{e^{2t}}{t^2}\right\} - \left\{\frac{2e^t}{t} - \frac{e^t}{t} + \frac{e^t}{t^2}\right\}\right]$$

$$= \frac{e^{2t}}{t^2} + \cancel{\frac{2e^t}{t}} - \cancel{\frac{2e^t}{t}} - \frac{2e^t}{t^2} + \frac{1}{t^2} = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$= \frac{(e^t - 1)^2}{t^2} = \frac{\left\{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots - 1\right\}^2}{t^2}$$

$\Rightarrow M_x(t) = 1 + t + t^2 + \cdots$

$\mu_1' = \left[\frac{d}{dt}\{M_x(t)\}\right]_{at\ t=0} = 1$

$\mu_2' = \left[\frac{d^2}{dt^2}\{M_x(t)\}\right]_{at\ t=0} = 2$

& $\mu_2 = \mu_2' - \mu_1'^2 = 2 - 1 = 1 \Rightarrow \boxed{\mu_2 = 1}$

$\Rightarrow \boxed{\mu_1' = 1}$ $\boxed{\mu_2' = 2}$ & $\boxed{\mu_2 = 1}$ Ans.

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Example:→ A random variable x has probability function $p(x) = \frac{1}{2^x}$, $x = 1, 2, 3 ----$ Find its moment generating function about origin.

Soln:→ $p(x) = \frac{1}{2^x}$, $x = 1, 2, 3 ----$

The moment generating function of x is

$$M_x(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$M_x(t) = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + -----$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + -----\right] \{G.P. \text{ series}\}$$

$$= \frac{e^t}{2}\left[\frac{1}{1-\frac{e^t}{2}}\right] = \frac{e^t}{\cancel{2}} \cdot \frac{1}{\frac{2-e^t}{\cancel{2}}} = \frac{e^t}{2-e^t}$$

$$M_x(t) = \frac{e^t}{2-e^t} \quad \text{——— (1)}$$

$$\frac{d}{dt}[M_x(t)] = \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right] = \frac{(2-e^t) \cdot e^t - e^t \cdot (-e^t)}{(2-e^t)^2}$$

$$= \frac{2e^t - \cancel{e^{2t}} + \cancel{e^{2t}}}{(2-e^t)^2} = \frac{2 \cdot e^t}{(2-e^t)^2}$$

$$\nu_1 = \left\{\frac{d}{dt}[M_x(t)]\right\}_{at\ t=0} = \frac{2}{1^2} = 2 \Rightarrow \boxed{\nu_1 = 2}$$

$\Rightarrow \boxed{\nu_1 = 2}$, where ν_1 = first moment about origin

Similarly

$$\frac{d^2}{dt^2}[M_x(t)] = 2\left[\frac{e^t \cdot (2-e^t)^2 + 2 \cdot e^t (2-e^t) \cdot e^t}{(2-e^t)^4}\right]$$

$$\Rightarrow \frac{d^2}{dt^2}[M_x(t)] = 2\left[\frac{e^t(2-e^t) + 2 \cdot e^{2t}}{(2-e^t)^3}\right] = \frac{4e^t - 2e^{2t} + 4e^{2t}}{(2-e^t)^3}$$

$$= \frac{4e^t + 2 \cdot e^{2t}}{(2-e^t)^3}$$

$$\nu_2 = \left[\frac{d^2}{dt^2}\{M_x(t)\}\right]_{at\ t=0} = \frac{6}{1} = 6$$

$$\Rightarrow \boxed{\nu_2 = 6}$$

(25)

Example:→ Find the moment generating function of the distribution $f(x) = \frac{1}{c} e^{-x/c}$, $0 \le x < \infty$, $c > 0$, about origin. Hence find its mean and standard deviation.

Solⁿ:→ $f(x) = \frac{1}{c} e^{-x/c}, \quad 0 \le x \le \infty, \quad c > 0$

The moment generating function about the origin is

$$M_0(t) = \int_0^\infty e^{tx} \cdot \frac{1}{c} \cdot e^{-x/c} \cdot dx = \frac{1}{c}\int_0^\infty e^{(t-\frac{1}{c})x} dx$$

$$= \frac{1}{c} \frac{\left| e^{(t-\frac{1}{c})x} \right|_0^\infty}{(t-\frac{1}{c})} = (1-ct)^{-1}$$

$$\Rightarrow M_0(t) = 1 + ct + c^2t^2 + c^3t^3 + \text{----}$$

$$\mu_1' = \left[\frac{d}{dt}\{M_0(t)\}\right]_{at\ t=0} = (c + 2c^2t + 3c^3t^2 + \text{---})_{at\ t=0}$$
$$= c$$

$$\Rightarrow \boxed{\mu_1' = c} \qquad \mu_2' = \left[\frac{d^2}{dt^2}\{M_0(t)\}\right]_{at\ t=0} = 2c^2$$

$$\& \ \mu_2 = \mu_2' - (\mu_1')^2 = 2c^2 - c^2 = c^2$$

Hence $\boxed{\text{Mean} = c}$ & $S.D = \sqrt{\mu_2} \Rightarrow \boxed{S.D = c}$

Example:→ For the continuous distribution $df = y_0 x(2-x)\,dx$, $0 \le x \le 2$.

(i) Find mean, variance, β_1, β_2 and hence show that the distribution is symmetrical

(ii) Also find mean deviation about mean.

Solⁿ:→ Since $\frac{df}{dx} = f(x) = y_0 x(2-x)$.

We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^2 y_0 x(2-x)\,dx = 1$

$\Rightarrow y_0 = \frac{3}{4}$, Hence the pdf is $f(x) = \frac{3}{4}x(2-x)$, $0 \le x \le 2$.

Now, we know rth moment $\mu_r' = E[x^r] = \int_0^2 x^r \cdot f(x)\,dx$

$$\Rightarrow \mu_r' = \frac{3}{4}\int_0^2 x^r \cdot x(2-x)\,dx \Rightarrow \mu_r' = \frac{3}{4}\int_0^2 x^{r+1}(2-x)\,dx$$

$$= \frac{3 \cdot 2^{r+1}}{(r+2)(r+3)}$$

Put $r = 1, 2, 3, 4$ ----- we obtain

$$\mu_1' = \frac{3 \cdot 2^2}{3 \cdot 4} = 1, \quad \mu_2' = \frac{3 \cdot 2^3}{4 \cdot 5} = \frac{6}{5}, \quad \mu_3' = \frac{3 \cdot 2^4}{5 \cdot 6} = \frac{8}{5}$$

$$\mu_4' = \frac{3 \cdot 2^5}{6 \cdot 7} = \frac{16}{7},$$

variance $= \mu_2 = \mu_2' - (\mu_1')^2 = \frac{6}{5} - (1)^2 = \frac{1}{5}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \frac{8}{5} - 3\cdot\frac{6}{5}\cdot 1 + 2\cdot 1^3 = 0$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= \frac{16}{17} - 4\cdot\frac{8}{5}\cdot 1 + 6\cdot\frac{6}{5}\cdot 1 - 3\cdot 1^4 = \frac{3}{35}$

Now $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \Rightarrow$ The distribution is symmetrical

$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3/35}{(1/5)^2} = \frac{15}{7}$

(ii) Mean deviation about mean $= E[|x-\mu_1'|]$

$\Rightarrow E[|x-\mu_1'|] = \int_0^2 |x-1| f(x)\,dx$

$= \int_0^1 |x-1| f(x)\,dx + \int_1^2 |x-1| f(x)\,dx$

$= \frac{3}{4}\left[\int_0^1 (1-x)\cdot x\cdot(2-x)\,dx + \int_1^2 (1-x)\cdot x\cdot(2-x)\,dx\right]$

$= \frac{3}{4}\left[\left|x^2 - x^3 + \frac{x^4}{4}\right|_0^1 + \left|x^3 - \frac{x^4}{4} - x^2\right|_1^2\right] = \frac{3}{8}.$

Example :- A variate x has the probability distribution

x :	-3	6	9
$P[X=x]$:	$1/6$	$1/2$	$1/3$

Find $E[X]$ & $E[X^2]$ and using the law of expectation, evaluate $E[2X+1]^2$.

Sol^n :- $E[X] = \Sigma x P(x) = -3\cdot\frac{1}{6} + 6\cdot\frac{1}{2} + 9\cdot\frac{1}{3} = \frac{11}{2}$

$E[X^2] = \Sigma x^2 P(x) = 9\cdot\frac{1}{6} + 36\cdot\frac{1}{2} + 81\cdot\frac{1}{3} = \frac{93}{2}$

$\therefore E[2X+1]^2 = E[4X^2 + 4X + 1]$

$= 4(E[X]^2) + 4E[X] + 1 = 4\times\frac{93}{2} + 4\times\frac{11}{2} + 1$

$= 209.$

Skewness and kurtosis

① **Symmetry** :→ A distribution is said to be symmetrical when its mean, median and mode are identical

i,e $\boxed{\text{Mean} = \text{Median} = \text{Mode}}$

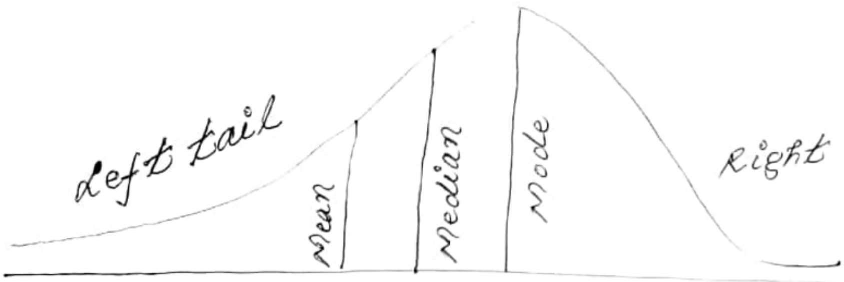
② **Skewness** :→ Skewness denote the opposite of symmetry. i,e Skewness means lack of symmetry.

Skew symmetrical distribution :→ A distribution which is not symmetrical is said to be skew symmetrical distribution.

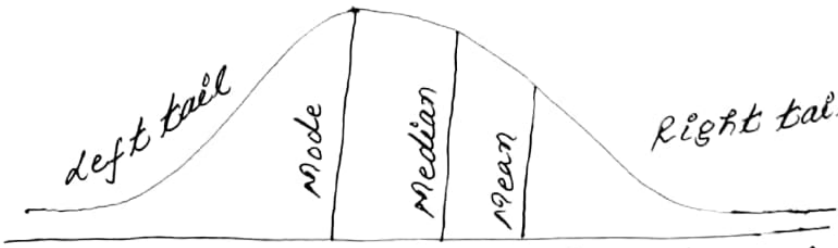
In skew symmetrical distribution, the left and the right tail are not of equal length.

One tail will be larger than the other.

(a) **Negative skew distribution** :→ The left tail is larger than the right tail.



(b) **Positive skew distribution** :→ The right tail of the curve will be longer than the left tail.



Uses of skewness :→ ① It gives the nature of the curve
② It gives nature and concetration of observation about the mean.

Measure of skewness :→ Measure of skewness is known as the measure of symmetry.

There are two types of measure of skewness.

1. Absolute measure :→ Absolute measure = |Mean − Mode|

2. Relative measure :→ There are four type of relative measure of skewness.

(i) Karl Pearson's coefficient of skewness

(ii) Bowley's coefficient of skewness

(iii) Kelly's coefficient of skewness

(iv) Measure of skewness based on the moments

$$\boxed{\text{Mode} = 3\,\text{Median} - 2\,\text{Mean}}$$

Moments :→ The rth moment of a variable x about the mean μ is usually denoted by μ_r is given by

$$\mu_r = \frac{1}{N} \Sigma (x_i - \mu)^r f_i \qquad \Sigma f_i = N$$

The rth moment of a variable x about any point 'a' is defined by

$$\mu'_r = \frac{1}{N} \Sigma (x_i - a)^r f_i$$

Moment about mean :→ Let μ be the mean

Then $\mu_r = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \mu)^r \qquad r = 1, 2, 3 - - - -$

If $r = 0$, $\mu_0 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \mu)^0 = 1$

If $r = 1$, $\mu_1 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \mu) = 0$

If $r = 2$ $\mu_2 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - \mu)^2$

If $r = 3$, $\mu_3 = \frac{1}{N} \sum_{i=1}^{n} (x_i - \mu)^3$

If $r = 4$ $\mu_4 = \frac{1}{N} \sum_{i=1}^{n} (x_i - \mu)^4$

Moment about point :→ Let a be any arbitrary point.

$$\mu'_r = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^r \qquad r = 0, 1, 2 - - - -$$

If $r = 0$ $\mu'_0 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a)^0 = 1$

If $r = 1$ $\mu'_1 = \frac{1}{N} \sum_{i=1}^{n} f_i (x_i - a) = \frac{1}{N} \Sigma f_i x_i - \frac{a}{N} \Sigma f_i$

$$\mu'_1 = \mu - a$$

(29) If $r=2$, $\mu_2' = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i-a)^2$

If $r=3$, $\mu_3' = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i-a)^3$

If $r=4$ $\mu_4' = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i-a)^4$

Moment about the origin :- $v_r = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^r$

If $r=0$, $v_0 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^0 = 1$, If $r=1$, $v_1 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i = \mu$

If $r=2$, $v_2 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^2$, If $r=3$, $v_3 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^3$

If $r=4$, $v_4 = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^4$

Relation between μ_r and μ_r' :- we have rth moment about the mean μ

$\mu_r = \frac{1}{N}\sum_{r=1}^{\infty} f_i (x_i-\mu)^r$ where $N = \sum f_i$

$= \frac{1}{N}\sum_{i=1}^{n} f_i[(x_i-a)-(\mu-a)]^r = \frac{1}{N}\sum_{i=1}^{n} f_i[(x_i-a)-\mu_1']^r$

$\because \mu_1' = \mu - a$

On expanding by Binomial theorem

$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i[(x_i-a)^r - {}^rC_1(x_i-a)^r\mu_1' + {}^rC_2(x_i-a)^{r-2}\mu_1'^2$
$+ - - - + (-1)^r \mu_1'^r]$

$\mu_r = \mu_r' - {}^rC_1 \mu_{r-1}' \mu_1' + {}^rC_2 \mu_{r-2}' \mu_1'^2 + - - - + (-1)^r \mu_1'^r$

put $r = 2, 3, 4$ - - - - - we get

$\mu_2 = \mu_2' - {}^2C_1 \mu_1' \mu_1' + {}^2C_2 \mu_0' \mu_1'^2$ $(\because \mu_0' = 1$

$= \mu_2' - 2\mu_1'^2 + \mu_1'^2 = \mu_2' - \mu_1'^2$

$\Rightarrow \boxed{\mu_2 = \mu_2' - \mu_1'^2}$

$\mu_3 = \mu_3' - {}^3C_1 \mu_2' \mu_1' + {}^3C_2 \mu_1' \mu_1'^2 - {}^3C_3 \mu_0' \mu_1'^3$

$= \mu_3' - 3\mu_2'\mu_1' + 3\mu_1'^3 - \mu_1'^3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - {}^4C_1 \mu_3' \mu_1' + {}^4C_2 \mu_2' \mu_1'^2 - {}^4C_3 \mu_1' \mu_1'^3 + {}^4C_4 \mu_0' \mu_1'^4$

$= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 4\mu_1'^4 + \mu_1'^4$

$= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

Thus

$$\boxed{\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= \mu_2' - \mu_1'^2 \\ \mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \\ \mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4. \end{aligned}}$$

Are first four moments

Measure of skewness based on moment :→

Measure of skewness is given by β_1

where $\boxed{\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}}$

Relation between μ_r and μ_r' :→

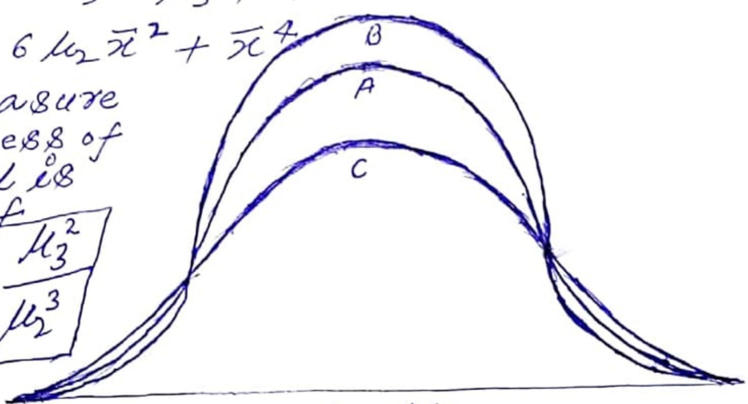
$\mu_1' = \mu_1 - a$ $\qquad$ $\mu_3' = \mu_3 + 3\mu_2\mu_1' + \mu_1'^3$

$\mu_2' = \mu_2 + \mu_1'^2$ $\qquad$ $\mu_4' = \mu_4 + 4\mu_3\mu_1' + 6\mu_2\mu_1'^2 + \mu_1'^4$

Relation between v :→ $v_1 = \bar{x}$, $v_2 = \mu_2 + \bar{x}^2$

$v_3 = \mu_3 + 3\mu_2\bar{x} + \bar{x}^3$

$v_4 = \mu_4 + 4\mu_3\bar{x} + 6\mu_2\bar{x}^2 + \bar{x}^4$



Kurtosis :→ It measure the degree of peakness of a distribution and is given by measure of kurtosis

$\boxed{\beta_2 = \dfrac{\mu_4}{\mu_2^2}}$ $\qquad$ $\boxed{\beta_1 = \dfrac{\mu_3^2}{\mu_2^3}}$

A → Mesokurtic
B → Leptokurtic
C → Platykurtic

1) If $\beta_2 = 3$, the curve is normal or mesokurtic
2) If $\beta_2 > 3$, the curve is peaked or Leptokurtic
3) If $\beta_2 < 3$, the curve is flat topped or platykurtic

Gamma coefficients :→ $\boxed{\gamma_1 = \pm\sqrt{\beta_1}}$ & $\boxed{\gamma_2 = \beta_2 - 3}$

Example → The first four moments of a distribution about the value 4 of the variate are -1.5, 17, -30, and 108. Calculate the first four moments about the mean and find β_1 and β_2.

Sol^n → We have $a=4$ $\mu_1' = -1.5$, $\mu_2' = 17$, $\mu_3' = -30$ & $\mu_4' = 108$

Moments about the mean

$\mu_1 = 0$, $\mu_2 = \mu_2' - \mu_1'^2 = 17 - (-1.5)^2 = 14.75$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -30 - 3.(17)(-1.5) + 2.(-1.5)^3 = 39.75$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 108 - 4(-30)(-1.5) + 6.(17)(-1.5)^2 - 3(-1.5)^4 = 142.31$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(39.75)^2}{(14.75)^3} = 0.4924$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{142.31}{(14.75)^2} = 0.6541$$

Also mean $\mu = \mu_1' + a = -1.5 + 4 = 2.5$

Example → The first three moments of a distribution about the value '2' of the variable are 1, 16, and -40. Show that the mean is 3, variance is 15 and $\mu_3 = -86$.

Sol^n → Given $a=2$, $\mu_1' = 1$, $\mu_2' = 16$, $\mu_3' = -40$

We know that $\mu_1' = \mu - a$, $\Rightarrow \mu = \mu_1' + a = 1+2 = 3$

$\Rightarrow$ Mean $= 3$, variance $= \mu_2' - \mu_1'^2 = 16 - (1)^2 = 15$.

and $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -40 - 3.(16).(1) + 2.(1)^3 = -86$.

Example → For a distribution the mean is 10, variance is 16, γ_1 is 1 and β_2 is 4. Find the first four moments about the origin.

Sol^n → When $\mu = 10$, $\mu_2 = 16$, $\gamma_1 = 1$, $\beta_2 = 4$

Now $\gamma_1 = 1 \Rightarrow \sqrt{\beta_1} = 1 \Rightarrow \beta_1 = 1$

Also $\beta_1 = 1$, $\Rightarrow \dfrac{\mu_3^2}{\mu_2^3} = 1 \Rightarrow \mu_3^2 = \mu_2^3 = (16)^3 = (4\times 4)^3 = (64)^2$

or $\mu_3^2 = 64^2 \Rightarrow \mu_3 = 64$ and $\beta_2 = 4 \Rightarrow \dfrac{\mu_4}{\mu_2^2} = 4$

$\Rightarrow \mu_4 = 4(\mu_2^2) = 4.(16)^2 = 1024$.

Moments about the origin → $\nu_1 = \mu = 10$

$\nu_2 = \mu_2 + \mu^2 = 16 + (10)^2 = 116$

$\nu_3 = \mu_3 + 3\mu_2\mu + \mu^3 = 64 + 3.(16).(10) + (10)^3 = 1544$

$\nu_4 = \mu_4 + 4\mu_3\mu + 6\mu_2\mu^2 + \mu^4$

$= 1024 + 4.(64)(10) + 6(16)(10)^2 + (10)^4 = 23184$.

(32)

Example:- Calculate the variance and third central moment from the following data:

x_i:	0	1	2	3	4	5	6	7	8
f_i:	1	9	26	59	72	52	29	7	1

Solⁿ:- Let us take assumed mean $a = 4$

x_i	f_i	$x_i - 4$	$f_i(x_i-4)$	$f_i(x_i-4)^2$	$f_i(x_i-4)^3$
0	1	−4	−4	16	−64
1	9	−3	−27	81	−243
2	26	−2	−52	104	−208
3	59	−1	−59	59	−59
4	72	0	0	0	0
5	52	1	52	52	52
6	29	2	58	116	232
7	7	3	21	63	189
8	1	4	4	16	64
	256		−7	507	−37

Since $\mu_r' = \dfrac{\Sigma f_i (x_i - a)^r}{\Sigma f_i}$

Then $\mu_1' = \dfrac{\Sigma f_i (x_i - a)}{\Sigma f_i} = \dfrac{-7}{256}$

$\mu_2' = \dfrac{\Sigma f_i (x_i - 4)^2}{\Sigma f_i} = \dfrac{507}{256}$

$\mu_3' = \dfrac{\Sigma f_i (x_i - 4)^3}{\Sigma f_i} = \dfrac{-37}{256}$

variance $\mu_2 = \mu_2' - \mu_1'^2 = \dfrac{507}{256} - \left(\dfrac{-7}{256}\right)^2$

$= 1.97972$

Third central moment

$\mu_3 = \mu_3' - 3\mu_2' \mu_1' + 2\mu_1'^3$

$= \dfrac{-37}{256} - 3\left(\dfrac{507}{256}\right)\left(\dfrac{-7}{256}\right) + 2\left(\dfrac{-7}{256}\right)^3 = 0.01789$ **Ans.**

(33) Example:- Find out kurtosis of the data given below

Class interval :	0 - 10	10 - 20	20 - 30	30 - 40
Frequency :	1	3	4	2

Solⁿ:-

Class	f_i	Mid Value x_i	x_i-25	$f_i(x_i-25)$	$f_i(x_i-25)^2$	$f_i(x_i-25)^3$	$f_i(x_i-25)^4$
0-10	1	5	-20	-20	400	-8000	160000
10-20	3	15	-10	-30	300	-3000	30000
20-30	4	25	0	0	0	0	0
30-40	2	35	10	20	200	2000	20000
	10			-30	900	-9000	210000

Let us assumed mean $a = 25$

Since $\mu_r' = \dfrac{\Sigma f_i (x_i - a)^r}{\Sigma f_i}$

Then $\mu_1' = \dfrac{\Sigma f_i (x_i - 25)}{\Sigma f_i} = \dfrac{-30}{10} = -3$

$\mu_2' = \dfrac{\Sigma f_i (x_i - 25)^2}{\Sigma f_i} = \dfrac{900}{10} = 90$

$\mu_3' = \dfrac{\Sigma f_i (x_i - 25)^3}{\Sigma f_i} = \dfrac{-9000}{10} = -900$

$\mu_4' = \dfrac{\Sigma f_i (x_i - 25)^4}{\Sigma f_i} = \dfrac{210000}{10} = 21000$

Now $\mu_2 = \mu_2' - \mu_1'^2 = 90 - (-3)^2 = 81$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -900 - 3(90)(-3) + 2(-3)^3 = -144.$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 21000 - 4(-900)(-3) + 6(90)(-3)^2 - 3(-3)^4$

$= 14817$

Kurtosis $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{14817}{(81)^2} = 2.258$

Also $\gamma_2 = \beta_2 - 3 = -0.742.$

(34)

Binomial distribution $\rightarrow$ A random variable X is said to have a binomial distribution if the discrete density function of X is given by

$$f_X(x) = \begin{cases} {}^nC_x\, p^x \cdot q^{n-x} & \text{for } x = 0, 1, 2 --- n \\ 0 & \text{otherwise} \end{cases}$$

Here n is a positive integer $0 \le p \le 1$, and $q = 1-p$
we write $X \sim B(n, p)$, where n and p are two parameters.
Noted that $\sum_{x=0}^{n} f_X(x) = \sum_{x=0}^{n} {}^nC_x\, p^x q^{n-x} = (p+q)^n = 1$

Theorem $\rightarrow$ If a random variable X has a binomial distribution, then $E[X] = np$, $Var[X] = npq$ and $M_X(t) = (q + p.e^t)^n$

Solⁿ $\rightarrow$ The mgf of X is given by

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \cdot f(x) = \sum_{x=0}^{n} e^{tx} \cdot {}^nC_x\, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x\, (pe^t)^x \cdot q^{n-x}$$

$$= {}^nC_0 (pe^t)^0 \cdot q^n + {}^nC_1 (pe^t)^1 \cdot q^{n-1} + {}^nC_2 (pe^t)^2 \cdot q^{n-2} + --- {}^nC_n (pe^t)^n q^0$$

$\Rightarrow$ $M_X(t) = (pe^t + q)^n$ $\left\{ \begin{array}{l} \text{By using} \\ [(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} \cdot b + --- {}^nC_n b^n] \end{array} \right\}$

Now $\frac{d}{dt}\{M_X(t)\} = n(pe^t + q)^{n-1} \cdot p.e^t$

$$\frac{d^2}{dt^2}\{M_X(t)\} = np[(n-1)(pe^t+q)^{n-2} \cdot pe^t + (pe^t+q)^{n-1} \cdot e^t]$$

At $t=0$, $\mu_1' = \frac{d}{dt}\{M_X(t)\}_{at\, t=0} = np(p+q)^{n-1} = np$

$$\mu_2' = \frac{d^2}{dt^2}\{M_X(t)\}_{at\, t=0} = np[(n-1) \cdot p + 1] = n(n-1)p^2 + np$$

Then $\boxed{E[X] = \mu_1' = np}$

$$Var[X] = \mu_2' - \mu_1'^2 = \cancel{n^2p^2} - np^2 + np - \cancel{(np)^2} = np(1-p) = npq$$

$\Rightarrow$ $\boxed{Var[X] = npq}$

Remark $\rightarrow$ ① Binomial distribution is given by $(p+q)^n$
② If we take $n=1$ in Binomial distribution, then random variable X is said to have Bernoulli distribution.
Its discrete density function is given by

$$f_X(x) = \begin{cases} p^x q^{1-x} & \text{for } x = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Mode of Binomial distribution → Mode is the value of the Binomial variate for which the probability is maximum. If r is the modal value, then $(n+1)p-1 \le r \le (n+1)p$

Case (i) → Let $(n+1)p$ be an integer, then there are two modes $(n+1)p$ and $(n+1)p-1$.

In this case the distribution is said to be bimodal.

Case (ii) → Let $(n+1)p$ be a fraction

The mode is the integral part of $(n+1)p$.

Example → The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P[X \ge 1]$.

Sol^n. → We are given that $E[X] = np = 4$ and $var(X) = npq = \frac{4}{3}$

$\Rightarrow npq = \frac{4}{3}$ or $4 \cdot q = \frac{4}{3} \Rightarrow q = \frac{1}{3}$ so $p = 1 - \frac{1}{3} = \frac{2}{3}$

and $n \cdot \frac{2}{3} = 4 \Rightarrow n = 6$, Hence $P[X \ge 1] = 1 - P[X=0] = 1 - {}^6C_0 p^0 q^6$

$\Rightarrow P[X \ge 1] = 1 - (\frac{1}{3})^6 = \frac{728}{729}$

Example → Determine the binomial distribution for which the mean is 4 and variance 3, and find its mode.

Sol^n. → We are given that $E[X] = np = 4$ and $var(X) = npq = 3$

$\Rightarrow npq = 3$ or $4q = 3 \Rightarrow q = \frac{3}{4}$ so $p = \frac{1}{4}$ and $n \cdot \frac{1}{4} = 4 \Rightarrow n = 16$.

Hence the binomial distribution is $(p+q)^n = (\frac{3}{4} + \frac{1}{4})^n$

Now, we have $(n+1)p = 17 \times \frac{1}{4} = 4.25$

Mode of Binomial distribution is integral part of $(n+1)p = 4$

Example → If the sum and the product of the mean and variance of a Binomial distribution are 24 and 128. Find the Binomial distribution.

Sol^n. → We are given $np + npq = 24$ —— ① $\Rightarrow np = \frac{24}{1+q}$

& $np \cdot (npq) = 128$ —— ② $\Rightarrow n^2p^2 = \frac{128}{q}$

$\Rightarrow \left(\frac{24}{1+q}\right)^2 = \frac{128}{q} \Rightarrow \frac{24 \times 24}{(1+q)^2} = \frac{128}{q} \Rightarrow 9q = 2(1+q)^2$

$\Rightarrow 2q^2 - 5q + 2 \Rightarrow (2q-1)(q-2) = 0 \Rightarrow q = \frac{1}{2}$ or $q = 2$ ✗ (Not possible)

Since $q = \frac{1}{2}$ so $p = \frac{1}{2} \Rightarrow n \cdot \frac{1}{2} = \frac{24}{1+\frac{1}{2}} \Rightarrow n = 32$

Hence the Binomial distribution is $(p+q)^n = \left(\frac{1}{2} + \frac{1}{2}\right)^{32}$.

Example → Six dice are thrown 729 times. How many times do you expect at least three dice to show a 5 or 6.

Sol^n. → Here $p = \frac{2}{6} = \frac{1}{3}$, $q = 1 - \frac{1}{3} = \frac{2}{3}$ & $n = 6$.

Then $P[X=x] = {}^6C_x p^x q^{n-x} = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x} = \frac{{}^6C_x \cdot 2^{6-x}}{3^6}$

$\Rightarrow P[\text{At least three success}] = P[X=3] + P[X=4] + P[X=5] + P[X=6]$

$= \frac{{}^6C_3 \cdot 2^3}{3^6} + \frac{{}^6C_4 \cdot 2^2}{3^6} + \frac{{}^6C_5 \cdot 2^1}{3^6} + \frac{{}^6C_6 \cdot 2^0}{3^6} = \frac{1}{3^6}\left[\frac{6.5.4}{3.2.1} \times 8 + \frac{6.5.4.3}{4.3.2.1} \times 4 + 6 \times 2 + 1\right]$

$= \frac{233}{729}$ **Ans.**

Hence the number of times at least 3 success occur
$= \cancel{729} \times \frac{233}{\cancel{729}} = 233.$

Example :- If on an average 1 vessel in every 10 is wrecked. find the probability that out of 5 vessel expected to arrive 4 at least will arrive safely.

Sol^n :- Let p = Probability that vessel to be safe $= 1 - \frac{1}{10} = \frac{9}{10}$

q = " " " " " wrecked $= \frac{1}{10}$

Also $n = 5$, $P[X=x] = {}^5C_x p^x q^{5-x} = {}^5C_x \left(\frac{9}{10}\right)^x \left(\frac{1}{10}\right)^{5-x} = \frac{{}^5C_x 9^x}{10^5}$

P[At least 4 will arrive safely] $= P[X=4] + P[X=5]$

$= \frac{{}^5C_4 . 9^4}{10^5} + \frac{{}^5C_5 . 9^5}{10^5} = \frac{9^4}{10^5}[(5+9)] = 0.91854.$

Note :- If $X \sim B(n_1, p)$ and $Y \sim B(n_2, p)$ are two independent binomial variates, show that $X+Y \sim B(n_1+n_2, p)$.

Example :- If X & Y are independent binomial variates $b(5, \frac{1}{2})$ and $b(7, \frac{1}{2})$. Find $P[X+Y=3]$.

Sol^n :- Since $X \sim b(5, \frac{1}{2})$ and $Y \sim b(7, \frac{1}{2})$ are independent binomial variate then $X+Y \sim b(5+7, \frac{1}{2})$

Thus $Z = X+Y$ is a binomial variate with $n=12$, $p = \frac{1}{2}$, $q = \frac{1}{2}$

$P[X+Y=3] = {}^{12}C_3 p^3 q^9 = \frac{12.11.10}{3.2.1}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^9 = 55\left(\frac{1}{2}\right)^{10}.$

Example :- Let X be a $B(2, p)$ and Y be a $B(4, p)$. If $P[X \geq 1] = \frac{5}{9}$ find $P[Y \geq 1]$.

Sol^n :- We have $P[X \geq 1] = 1 - P[X=0] = 1 - {}^2C_0 p^0 q^2 = 1 - q^2$ ($\because n=2$)

since $P[X \geq 1] = \frac{5}{9} \Rightarrow 1 - q^2 = \frac{5}{9} \Rightarrow q^2 = \frac{4}{9} \Rightarrow q = \frac{2}{3}$ & $p = 1 - \frac{2}{3} = \frac{1}{3}$

Hence $P[Y \geq 1] = 1 - P[Y=0] = 1 - {}^4C_0 p^0 q^4 = 1 - q^4$ ($\because n=4$)

$= 1 - \left(\frac{2}{3}\right)^4 = \frac{65}{81}.$

Example :- If X and Y are two independent binomial variates with parameter $n_1 = 6$, $p = \frac{1}{2}$ and $n_2 = 4$, $p = \frac{1}{2}$ respectively.

Evaluate (i) $P[X+Y=3]$ (ii) $P[X+Y \geq 3]$.

Example :- For a binomial distribution, show that

$$\mu_{r+1} = pq\left[nr\,\mu_{r-1} + \frac{d\mu_r}{dp}\right]$$

and deduce the values of μ_2, μ_3 and μ_4. Hence find β and γ coefficients for binomial distribution.

Example :- A perfect cubic die is thrown a large number of times in sets of 8, the occurence of 5 or 6 is called a success. In what proportion of the sets would you expect 3 success?

Ans: Required proportion $= 0.2731 = 27.31\%$.

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Poission Distribution :→ A random variable X is said to have a poission distribution if the discrete density function of X is given by

$$f_X(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & \text{for } x = 0, 1, 2 --- \text{ where } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

The variable X is called poission variate, denoted as $X \sim P(\lambda)$.

Remark :→ ① If $X \sim P[\lambda]$, then the probability of x success is given by

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!}$$

In particular, $P[X=0] = e^{-\lambda}$ and $P[X=1] = \lambda e^{-\lambda}$

② $\displaystyle\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}e^{\lambda} = 1.$

Theorem :→ If X is a poission distributed random variable, then $E[X] = \lambda$, $Var[X] = \lambda$ and $M_X(t) = e^{\lambda(e^t-1)}$.

Sol^n :→ Moment generating function of X is given by

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} f_X(x) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!} = e^{-\lambda}\left[1 + (\lambda e^t) + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + ---\right]$$

$$= e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)} \Rightarrow \boxed{M_X(t) = e^{\lambda(e^t-1)}}$$

Now $\dfrac{d}{dt}M_X(t) = e^{\lambda(e^t-1)} \cdot \lambda e^t$

$$\frac{d^2}{dt^2}M_X(t) = \lambda\left[e^{\lambda(e^t-1)} \cdot \lambda e^t e^t + e^{\lambda(e^t-1)} \cdot e^t\right]$$

At $t=0$ $\quad \mu_1' = \dfrac{d}{dt}M_X(t)\Big|_{at\ t=0} = \lambda$

$$\mu_2' = \frac{d^2}{dt^2}M_X(t)\Big|_{at\ t=0} = \lambda[\lambda+1] = \lambda^2+\lambda$$

Thus $E[X] = \mu_1' = \lambda$ and $var(X) = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

$$\Rightarrow \boxed{E(X) = var(X) = \lambda}$$

Theorem :→ Poission distribution as a limiting case of Binomial distribution :→

Derive Poission distribution as the limiting case of Binomial distribution, where $n \to \infty$, $p \to 0$ and $np = \lambda$ (a finite constant)

Proof :→ For a Binomial distribution, we have

$$P[X=x] = {}^nC_x\, p^x \cdot q^{n-x} \quad \text{and } p[0] = q^n$$

Also, $\lim_{n\to\infty} p[0] = \lim_{n\to\infty} (1-p)^n = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ —— ①

Now $\dfrac{p[x+1]}{p[x]} = \dfrac{{}^nC_{x+1}\, p^{x+1}\, q^{n-x-1}}{{}^nC_x\, p^x\, q^{n-x}}$ $\left\{\text{using } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^1\right\}$

$$= \frac{n-x}{x+1}\cdot\frac{p}{q} = \frac{n-x}{x+1}\,\frac{\frac{\lambda}{n}}{1-\frac{\lambda}{n}} = \frac{1-\frac{x}{n}}{1-\frac{\lambda}{n}}\cdot\frac{\lambda}{x+1}$$

$$\lim_{n\to\infty}\frac{p[x+1]}{p[x]} = \lim_{n\to\infty}\frac{1-\frac{x}{n}}{1-\frac{\lambda}{n}}\cdot\frac{\lambda}{x+1} = \frac{\lambda}{x+1}$$

$$\Rightarrow \frac{p[x]}{p[x-1]} = \frac{\lambda}{x} \text{ —— ②}$$

$$\text{Now } p[x] = \frac{p[x]}{p[x-1]}\cdot\frac{p[x-1]}{p[x-2]}\cdot\frac{p[x-2]}{p[x-3]} \cdots \frac{p[1]}{p[0]}\cdot p[0]$$

Taking $\lim n\to\infty$ and using ① & ②

$$\lim_{n\to\infty} p[x] = \frac{\lambda}{x}\cdot\frac{\lambda}{x-1}\,\frac{\lambda}{x-2} \cdots \frac{\lambda}{1}\cdot e^{-\lambda} = \frac{e^{-\lambda}\cdot\lambda^x}{x!}$$

Hence the probability of x success is given by

$$p[X=x] = \frac{e^{-\lambda}\cdot\lambda^x}{x!} \qquad x = 0, 1, 2 \text{ ----}$$

Theorem :→ Show that for the Poission distribution with λ $\mu_{r+1} = r\lambda\,\mu_{r-1} + \lambda\dfrac{d\mu_r}{d\lambda}$. Hence deduce the value of μ_2, μ_3 and μ_4. Also find β and γ - coefficient of the Poission distribution.

Proof. :→ We have $\mu_r = E[(x-\lambda)^r] = \sum_{x=0}^{\infty}(x-\lambda)^r\,\dfrac{e^{-\lambda}\lambda^x}{x!}$

$$\Rightarrow \mu_r = \sum_{x=0}^{\infty}\frac{1}{x!}\left[e^{-\lambda}\lambda^x(x-\lambda)^r\right]$$

$$\Rightarrow \frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty}\frac{1}{x!}\left[-e^{-\lambda}(x-\lambda)^r\cdot\lambda^x + x\lambda^{x-1}\cdot e^{-\lambda}(x-\lambda)^r - r(x-\lambda)^{r-1}e^{-\lambda}\lambda^x\right]$$

$$= \sum_{x=0}^{\infty}\frac{1}{x!}\left[e^{-\lambda}\lambda^{x-1}(x-\lambda)^r(x-\lambda) - r(x-\lambda)^{r-1}\cdot e^{-\lambda}\lambda^x\right]$$

$$= \frac{1}{\lambda}\sum_{x=0}^{\infty}(x-\lambda)^{r+1}e^{-\lambda}\cdot\frac{\lambda^x}{x!} - r\sum_{x=0}^{\infty}(x-\lambda)^{r-1}\cdot e^{-\lambda}\frac{\lambda^x}{x!}$$

$$= \frac{1}{\lambda}\,\mu_{r+1} - r\,\mu_{r-1} \Rightarrow \boxed{\mu_{r+1} = r\lambda\,\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda}} \text{ —— ①}$$

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(39) Put $r = 1, 2$ & (3) in (1) we get

$$\mu_2 = \lambda\mu_0 + \lambda\frac{d\mu_1}{d\lambda} = \lambda \quad (\because \mu_0 = 1,\ \mu_1 = 0)$$

$$\mu_3 = 2\lambda\mu_1 + \lambda\frac{d\mu_2}{d\lambda} = 0 + \lambda.1 = \lambda$$

$$\mu_4 = 3\lambda\mu_2 + \lambda\frac{d\mu_3}{d\lambda} = 3\lambda^2 + \lambda$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \quad \& \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\lambda^2+\lambda}{\lambda^2} = 3 + \frac{1}{\lambda}$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}} \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$

Example:→ If X is a Poisson variate such that $P[X=1] = 2P[X=2]$. Find mean and variance. Also find $P[X=0]$.

Sol^n:→ It is given that $P[X=1] = 2P[X=2]$

$$\frac{\lambda e^{-\lambda}}{1!} = 2\frac{\lambda^2}{2!}e^{-\lambda} \Rightarrow \boxed{\lambda = 1}$$

Hence Mean = variance = 1 & $P[X=0] = \frac{1 \cdot e^{-1}}{0!} = e^{-1}$

Example:→ If X is a Poisson variate such that $P[X=2] = 9P[X=4] + 90P[X=6]$ (i) Find $E[X]$.
(ii) β_1, the coefficient of skewness.

Sol^n:→ (i) It is given that $P[X=2] = 9P[X=4] + 90P[X=6]$

$$\Rightarrow \frac{e^{-\lambda}\cdot\lambda^2}{2!} = 9\cdot\frac{e^{-\lambda}\lambda^4}{4!} + 90\cdot\frac{e^{-\lambda}\lambda^6}{6!}$$

$$\frac{1}{2} = \frac{\overset{3}{\cancel{9}}\cdot\lambda^2}{4\cdot\cancel{3}\cdot2} + \frac{\cancel{90}\ \lambda^4}{\cancel{6\cdot5}\cdot4\cdot\cancel{3}\cdot2} \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow (\lambda^2+4)(\lambda^2-1) = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \boxed{\lambda = 1}$$

Hence $\boxed{E[X] = \lambda = 1}$

(ii) β_1 = Coefficient of skewness $= \frac{1}{\lambda} = 1$

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Example :→ Show that in a Poission distribution with unit mean, mean deviation about mean is $\left(\frac{2}{e}\right)$ times the standard deviation.

Sol^n :→ Here $\lambda = 1$ so that $\sigma_X = \sqrt{\lambda} = 1$

Poission distribution $P[X=x] = \frac{\lambda^x \cdot e^{-\lambda}}{x!} = \frac{e^{-1}}{x!}$, $x = 0, 1, 2 - - -$

Mean deviation about mean is

$$M.D = E[|x-1|] = \sum_{x=0}^{\infty} |x-1| \cdot \frac{e^{-1}}{x!} = |1| \frac{e^{-1}}{0!} + |1-1| \frac{e^{-1}}{1!} + \sum_{x=2}^{\infty} |x-1| \cdot \frac{e^{-1}}{x!}$$

$$= e^{-1} + \sum_{x=2}^{\infty} (x-1) \frac{e^{-1}}{x!} = e^{-1} + e^{-1} \sum_{x=2}^{\infty} \left\{ \frac{1}{(x-1)!} - \frac{1}{x!} \right\} = e^{-1} + e^{-1} = \frac{2}{e}$$

$$M.D = \frac{2}{e} \times 1 = \frac{2}{e} \sigma_X = \frac{2}{e} \times S.D$$

Example :→ An insurance company insures 4000 people against loss of both eyes in a car accident based on previous data, the rates were computed on the assumption that on the average 10 persons in 100000 will have car accident each year that result in this type of injury. what is the probability that more than 3 of the injured will collect on their policy in a given year? [Given $e^{-0.4} = 0.6703$].

Sol^n :→ Given $n = 4000$, p = prob. of loss of both eyes in a car accident $= \frac{10}{100000} = 0.0001$,

$\lambda = np = 4000 \times 0.0001 = 0.4$

Then by Poission distribution $P[X=x] = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.4} (0.4)^x}{x!}$ $x = 0, 1, 2, 3 - - -$

Prob. [More than 3 of the injured will collect on their policy in a given year]

$$= P[X>3] = 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3)\}$$

$$= 1 - e^{-0.4} \left\{ \frac{(0.4)^0}{0!} + \frac{(0.4)^1}{1!} + \frac{(0.4)^2}{2!} + \frac{(0.4)^3}{3!} \right\}$$

$$= 1 - e^{-0.4} (1 + 0.4 + 0.08 + 0.0107) = 1 - 0.6703 \times 1.4907$$
$$= 0.0008.$$

Example :→ In a book of 520 pages, 390 typo-graphical error occur. Assuming Poission law for the number of error per page, find the prob. that a random sample of 5 pages will contain no error.

Sol^n :→ The average no. of tyo-graphical error per page in book $= \lambda = \frac{390}{520} = 0.75$.

Then by Poission distribution $P[X=x] = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.75} (0.75)^x}{x!}$

P[5 pages contain no error] $= \{P(X=0)\}^5 = \left(e^{-0.75}\right)^5 = e^{-3.75}$

Ans.

(41)

Example :→ A manufacturer, who produces machine bottles, find that 0.1% of the bottles are deffective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottle. Using poission distribution. Find how many boxes will contain
(i) No defective (ii) At least two defectives {Given $e^{-0.5} = 0.6065$}

Solⁿ :→ Here $N = 100$, $n = 500$, $p = 0.1\% = 0.001$ so $\lambda = np$

$\lambda = 500 \times 0.001 = 0.5$

By poission distribution. Prob. [x defective bottles in box]

i.e $P[X=x] = \dfrac{e^{-0.5}.(0.5)^x}{x!}$

Hence in a consignment of 100 boxes, the number of boxes containing x defective bottles $= f(x) = N. P[X=x]$

$$= 100 \times \frac{e^{-0.5} \times (0.5)^x}{x!}$$

(i) No. of boxes containing no defective bottles

$= 100 \times P(X=0) = 100 \times \dfrac{e^{-0.5} \times (0.5)^0}{0!} = 100 \times 0.6065 = 60.65 \approx 61.$

(ii) No. of boxes containing at least two defective bottles

$= 100 \times P[X \geq 2] = 100 \times \{1 - P(X=0) - P(X=1)\} = 100 \times \left\{1 - \dfrac{e^{-0.5}(0.5)^0}{0!} - \dfrac{e^{-0.5}(0.5)^1}{1!}\right\}$

$= 100[1 - 0.6065 - 0.6065 \times 0.5] = 100 \times 0.0925 = 9.25 \approx 9.$ Ans.

Example :→ Six coins are tossed 6,400 times. Using the poission distribution. Find the approximate probability of getting six heads r times.

Solⁿ :→ Here $n = 6400$, p = Prob. of getting six heads in one throw of six coins $= \left(\frac{1}{2}\right)^6$.

$\lambda = np = 6400 \times \left(\frac{1}{2}\right)^6 = 100.$

By poission distribution,

$P[\text{getting 6 heads r times}] = P[X=r] = \dfrac{e^{-\lambda}\lambda^r}{r!} = \dfrac{e^{-100}.(100)^r}{r!}$

Example :→ In a poission frequency distribution, frequency corresponding to 3 success is $\frac{2}{3}$ times frequency corresponding to 4 success. Find the mean and standard deviation of the distribution.

Solⁿ :→ The frequency function is given by $f(x) = N.P(x) = N \times P(X=x)$

$= N \times \dfrac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2$ putting $x=3$ and $x=4$

$f(3) = N \times \dfrac{e^{-\lambda}\lambda^3}{3!}$ & $f(4) = N \times \dfrac{e^{-\lambda}\lambda^4}{4!}$

$f(3) = \frac{2}{3} f(4) \Rightarrow N.\dfrac{e^{-\lambda}\lambda^3}{3!} = \dfrac{2}{3} N \times \dfrac{e^{-\lambda}\lambda^4}{4!} \Rightarrow \dfrac{1}{6} = \dfrac{2}{3} \times \dfrac{\lambda}{24}$

$\Rightarrow \boxed{\lambda = 6}$ $\boxed{\text{Mean} = 6}$, & $\boxed{\text{varaiance} = 6}$

$\boxed{S.D = \sqrt{6}}$

Note :→ (1) Mode of the Poisson distribution with parameter λ. is given by $\lambda-1 \leq x \leq \lambda$.

Case (i) :→ If λ is a positive integer then $\lambda-1$ and λ are two modes of Poisson distribution. In this case the Poisson distribution is said to be bimodal.

Case (ii) :→ If λ is a fraction, then mode of P.D is the integral part of λ.

Example :→ A Poisson distribution has a double mode at $x=1$, and $x=2$. What is the probability that x will have one or the other of these two values?

Sol^n. :→ If the Poisson distribution is bimodal, then the two modes are at the points $x=\lambda-1$ and $x=\lambda$, where λ is the parameter of the Poisson distribution.

Therefore, since we are given that the two modes are at the points $x=1$ & $x=2$, we find that $\lambda=2$.

$$P[X=x] = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!} \quad x=0,1,2---$$

$$P[X=1] = e^{-2}.2 \ \& \ P[X=2] = \frac{e^{-2}.2^2}{2} = 2e^{-2}$$

Required probability $= P[X=1] + P[X=2] = 2e^{-2} + 2e^{-2} = 4e^{-2}$.

Example :→ If X and Y are independent Poisson variate such that $P[X=1] = P[X=2]$ and $P[Y=2] = P[Y=3]$. Find the variance of $X-2Y$.

Sol^n. :→ Let $X \sim P(\lambda)$ and $Y \sim P(\mu)$

then $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x=0,1,2-- \quad \lambda>0$

and $P[Y=y] = \frac{e^{-\mu}\mu^y}{y!} \quad y=0,1,2--- \quad \mu>0$

Using given condition,

$P[X=1] = P[X=2]$ & $P[Y=2] = P[Y=3]$

$$\frac{\lambda e^{-\lambda}}{1!} = \frac{\lambda^2 e^{-\lambda}}{2!} \quad \& \quad \frac{\mu^2 e^{-\mu}}{2!} = \frac{\mu^3 e^{-\mu}}{3!}$$

$\Rightarrow \boxed{\lambda=2}$ $\qquad \Rightarrow \boxed{\mu=3}$

Now $Var(X)=2$, $\quad Var(Y)=3$.

$\therefore Var(X-2Y) = 1^2 Var(X) + (-2)^2 Var(Y)$

$= 1.2 + 4.3 = 14$ Ans.

(43) **Example:-** Fit a binomial distribution to the following frequency data:

x :	0	1	2	3	4
f :	30	62	46	10	2

Soln.:- The table is as follows:

x	f	f.x
0	30	0
1	62	62
2	46	92
3	10	30
4	2	8
	$\Sigma f = 150$	$\Sigma fx = 192$

Mean of observations $= \frac{\Sigma fx}{\Sigma f}$

$= \frac{192}{150} = 1.28$

$\Rightarrow np = 1.28$

$\Rightarrow 4p = 1.28 \Rightarrow p = 0.32$

$\Rightarrow q = 1 - p = 1 - 0.32 = 0.68$

Also $N = 150 \Rightarrow \because \Sigma f = N$

Hence, the binomial distribution is $= N(q+p)^n$

$= 150(0.68 + 0.32)^4$. Ans.

Example:- Fit a Binomial distribution of the following data and compare the theoretical frequencies with the actual ones.

x :	0	1	2	3	4	5
f :	2	14	20	34	22	8

Soln:- To fit a Binomial distribution means we have to find the Binomial distribution given by $= N.(q+p)^n$.

$\Rightarrow$ Mean of the Binomial distribution $= np = \frac{\sum_{i=1}^{5} f_i x_i}{\sum_{i=1}^{5} f_i}$

$\Rightarrow np = \frac{0+14+40+132+88+40}{2+14+20+34+22+8} = \frac{284}{100} = 2.84$

$\Rightarrow np = 2.84, \Rightarrow p = \frac{2.84}{n} = \frac{2.84}{5} = 0.568$

Also, $q = 1 - p = 1 - 0.568 = 0.432$.

Hence the Binomial distribution $= N(q+p)^n = 100(0.432 + 0.568)^5$.

Further, the Binomial distribution is given by

$P(X=r) = N.\, {}^nC_r\, q^{n-r} p^r, \quad r = 0, 1, 2, 3, 4, 5.$

The theoretical frequencies are given by putting $r = 0, 1, 2, 3, 4, 5$.

when $r=0$, $P(X=0) = 100 \times {}^5C_0 (0.432)^5 (0.568)^0 = 100 \times 1 \times (0.0150) = 1.5045$.

when $r=1$, $P(X=1) = 100 \times {}^5C_1 (0.432)^4 (0.568)^1 = 100 \times 5 \times (0.0348) \times (0.568)$
$= 9.891$.

when $r=2$, $P(X=2) = 100 \times {}^5C_2 (0.432)^3 (0.568)^2 = 100 \times 10 \times (0.0806) \times (0.3226)$
$= 26$,

when $r=3$, $P(X=3) = 100 \times {}^5C_3 (0.432)^2 (0.568)^3 = 100 \times 10 \times (0.1866) \times (0.1832)$
$= 34.185$

when $r=4$, $P(X=4) = 100 \times {}^5C_4 (0.432)^1 (0.568)^4 = 100 \times 5 \times (0.432) \times (0.1040)$
$= 22.464$,

when $r=5$, $P(X=5) = 100 \times {}^5C_5 (0.432)^0 (0.568)^5 = 100 \times 1 \times (0.5912) = 5.9120$.

Tabulating all the above, we have the following Binomial distribution of best fit.

r	Actual frequencies	Theoretical frequencies
0	2	1.5045 = 2
1	14	9.891 = 10
2	20	26 = 26
3	34	34.185 = 34
4	22	22.464 = 22
5	8	5.9120 = 6
	Total = 100	Total = 100

Example → Fit a poisson distribution to the following data and calculate theoretical frequencies.

(i) x: 0 1 2 3 4
f: 120 60 15 2 1

& (ii) x: 0 1 2 3 4
f: 109 65 22 3 1

Sol^n → (i) Mean of given distribution $= \frac{\Sigma fx}{\Sigma f}$

$$\Rightarrow \lambda = \frac{60+30+6+4}{200} = 0.5$$

Required poisson distribution $= N.\frac{e^{-\lambda}.\lambda^r}{r!} = 200.\frac{e^{-0.5}.(0.5)^r}{r!} = (121.306).\frac{(0.5)^r}{r!}$

r	$N.P(r)$	Theoretical frequency
0	$121.306\times\frac{(0.5)^0}{0!} = 121.306$	121
1	$121.306\times\frac{(0.5)^1}{1!} = 60.653$	61
2	$121.306\times\frac{(0.5)^2}{2!} = 15.163$	15
3	$121.306\times\frac{(0.5)^3}{3!} = 2.527$	3
4	$121.306\times\frac{(0.5)^4}{4!} = 0.3159$	0

(ii) Mean of given distribution $= \frac{\Sigma fx}{\Sigma f} = \frac{65+44+9+4}{200} = \frac{122}{200} = 0.61$

This is the parameter (λ) of the poisson distribution

∴ Required poisson distribution $= N.\frac{e^{-\lambda}.\lambda^r}{r!} = 200.e^{-0.61}.\frac{(0.61)^r}{r!}$

$$= 108.67\times\frac{(0.61)^r}{r!}$$

r	$N.P(r)$	Theoretical frequency
0	$108.67\times\frac{(0.61)^0}{0!} = 108.67$	109
1	$108.67\times\frac{(0.61)^1}{1!} = 62.289$	66
2	$108.67\times\frac{(0.61)^2}{2!} = 20.218$	20
3	$108.67\times\frac{(0.61)^3}{3!} = 4.111$	4
4	$108.67\times\frac{(0.61)^4}{4!} = 0.6269$	1
		Total = 200.

Example :→ If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 1000 individuals more than 2 will be defective.

Sol :→ It follows a poisson distribution as the probability of occurrence is very small.

Mean$(m) = np = 1000 \times (0.001) = 1$.

Probability that more than 2 will be defective

= 1 − [Prob. that no one gets a bad reaction + Prob. that one gets a bad reaction + Prob. that two get a bad reaction]

$$= 1 - \left[e^{-m} + \frac{m e^{-m}}{1!} + \frac{m^2 e^{-m}}{2!}\right] = 1 - \left[\frac{1}{e} + \frac{1}{e} + \frac{1}{2e}\right]$$

$$= 1 - \left[\frac{2+2+1}{2e}\right] = 1 - \frac{5}{2e} = 1 - \frac{5}{2 \times 2.718} = 1 - \frac{5}{5.436}$$

$$= 1 - 0.91979 = 0.080206.$$ **Ans.**

Normal Distribution OR Gaussian Distribution

Def :→ A random variable X, whose probability density function is given by

$$\boxed{\phi(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}} \quad -\infty < x < \infty.$$

is called a normal variate with parameter μ (called mean) and σ^2 (called variance). It is denoted by $X \sim N(\mu, \sigma^2)$.

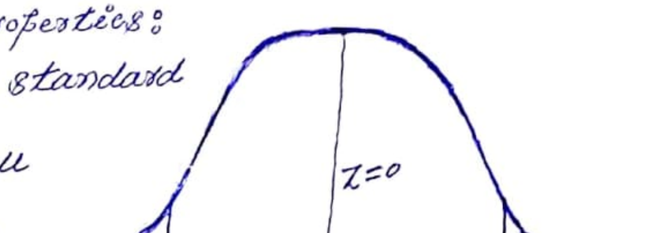
Remark :→ ① The cumulative distribution function $X \sim N(\mu, \sigma^2)$ is denoted as $\Phi(x)$. Thus

$$\Phi(x) = P[X \leq x] = \int_{-\infty}^{x} \phi(x)\,dx$$

② It can be verified that $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$.

③ The curve $y = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ is called the normal curve which passes the following properties:

1. The total area under the standard normal curve is 1.



2. The ordinate at mean i.e $x = \mu$ divides the area under the standard normal curve into two equal parts.

Symbolically,

$$P[-\infty \leq Z \leq 0] = P[0 \leq Z \leq \infty] = 0.5.$$

3. Since, the curve is symmetrical

Thus, $P[0 \leq Z \leq Z_1] = P[-Z_1 \leq Z \leq 0]$.

Standard Normal variate :→ If $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X-\mu}{\sigma}$ is called standard normal variate with $\boxed{E[Z]=0}$ and $\boxed{Var[Z]=1}$ and is written as $Z \sim N(0,1)$

The probability density function (pdf) of standard normal variate Z is given by $\phi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, $-\infty < z < \infty$.

Remark :→ ① The cumulative distribution function of $N(0,1)$ is given by $\Phi(z) = P[Z \le z] = \int_{-\infty}^{z} \phi(z)dz$

② Since $\int_{-\infty}^{\infty} \phi(z)dz = 1$ i.e $\int_{-\infty}^{\infty} \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot dz = 1$.

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \cdot dz = \sqrt{2\pi}} \qquad \left[\text{Also} \int_{-\infty}^{\infty} e^{-z^2} \cdot dz = \sqrt{\pi}\right].$$

Theorem :→ If $X \sim N(\mu, \sigma^2)$ then $E[X] = \mu$, $Var[X] = \sigma^2$ & $M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Proof :→ The m.g.f of X is given by $M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \cdot \phi(x)dx$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \cdot e^{-z^2/2} \sigma\, dz$$

$$\left\{ \because \frac{x-\mu}{\sigma} = z \Rightarrow \sigma z + \mu = x \Rightarrow dx = \sigma dz \right.$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2}z^2 + t\sigma z\right)} \cdot dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)} \cdot dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[z^2 - 2\sigma z t + (\sigma t)^2 - (\sigma t)^2]} dz = \frac{1}{\sqrt{2\pi}} e^{\mu t} \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(z-\sigma t)^2 - \sigma^2 t^2]} \cdot dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t} e^{\frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} \cdot dz = \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2} dz.$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dy \quad \{\text{Let } z - \sigma t = y \Rightarrow dz = dy$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}} e^{\mu t + \frac{1}{2}\sigma^2 t^2} \cdot \cancel{\sqrt{2\pi}} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Now $M_X'(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + t\sigma^2)$ &

$M_X''(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}(\mu + \sigma^2)^2 + \sigma^2 e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

At $t = 0$, $M_X'(t)_{at\, t=0} = \mu_1' = \mu$ & $M_X''(t)_{at\, t=0} = \mu_2' = \mu^2 + \sigma^2$

$\Rightarrow$ Mean $= E[X] = \mu_1' = \mu$ & $Var(X) = \mu_2' - \mu_1'^2$

$\Rightarrow Var(X) = \cancel{\mu^2} + \sigma^2 - \cancel{\mu^2} = \sigma^2$

$\boxed{E[X] = \mu}$ & $\boxed{Var(X) = \sigma^2} \Rightarrow \boxed{S.D = \sigma}$

Example: Show that the mean deviation from the mean of the general normal distribution is $\sigma\sqrt{\frac{2}{\pi}}$ or $\frac{4}{5}\sigma$ (Approx)

Proof: M.D about mean $\mu = \int_{-\infty}^{\infty} |x-\mu| f(x)\,dx$

$$\Rightarrow \mu = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} |x-\mu|\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} |\sigma z|\, e^{-\frac{1}{2}z^2}\,\sigma\,dz \quad \left\{ \text{Let } \frac{x-\mu}{\sigma} = z \Rightarrow x = \mu+\sigma z \Rightarrow dx = \sigma\,dz \right.$$

$$= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} |z|\, e^{-\frac{1}{2}z^2} dz = \frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty} z\cdot e^{-\frac{1}{2}z^2} dz \quad \{\text{Even function}\}$$

$$= \frac{2\sigma}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-t}\,dt \quad \left\{ \text{Take } \frac{z^2}{2} = t \Rightarrow dz = dt \right.$$

$$= \sqrt{\frac{2}{\pi}}\,\sigma\left[\frac{e^{-t}}{-1}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\,\sigma = \frac{4}{5}\sigma \text{ (Approx.)}$$

Example: If X is a normal variate with mean 30 and S.D 5. Find the probabilities that
(i) $26 \le X \le 40$ (ii) $X \ge 45$ & (iii) $|X-30| > 5$.

Sol^n: Given $\mu = 30$, $\sigma = 5$ Let $z = \frac{X-\mu}{\sigma} = \frac{X-30}{5}$

(i) $P[26 \le X \le 40] = P\left[\frac{26-30}{5} \le \frac{X-30}{5} \le \frac{40-30}{5}\right] = P[-0.8 \le z \le 2]$

$= P[-0.8 \le z \le 0] + P[0 \le z \le 2] = P[-0.8 \le z \le 0] + 0.4772$

$= P[0 \le z \le 0.8] + 0.4772$ {Due to symmetry}

$= 0.2881 + 0.4772 = 0.7653.$

(ii) $P[X \ge 45] = P\left[\frac{X-30}{5} \ge \frac{45-30}{5}\right] = P[z \ge 3] = 0.5 - P[0 \le z \le 3]$

$= 0.5 - 0.49865 = 0.00135$

(iii) Since $P[|X-30| \le 5] = P[25 \le X \le 35] = P\left[\frac{25-30}{5} \le \frac{X-30}{5} \le \frac{35-30}{5}\right]$

$= P[-1 \le z \le 1] = 2P[0 \le z \le 1] = 2 \times 0.3413 = 0.6826$

$\Rightarrow P[|X-30| > 5] = 1 - P[|X-30| \le 5] = 1 - 0.6826 = 0.3174.$

Example: If X is normally distributed with mean 11 & S.D 1.5. Find the number x_0 such that $P[X > x_0] = 0.3$

Sol^n: Given $0.3 = P[X > x_0] = 0.5 - P[11 < X < x_0]$

$= 0.5 - P\left[\frac{11-11}{1.5} < \frac{X-11}{1.5} < \frac{x_0-11}{1.5}\right]$

$= 0.5 - P\left[0 < z < \frac{x_0-11}{1.5}\right]$

$\Rightarrow P\left[0 < z < \frac{x_0 - 11}{1.5}\right] = 0.2$

$\Rightarrow \frac{x_0 - 11}{1.5} = 0.52$ (from table)

$\Rightarrow x_0 = 11 + 1.5 \times 0.52 = 11.78.$

<u>Example</u>:- The life of electric tubes of a certain type may be normally distributed with mean 155 hours & standard deviation 19 hours. What is the prob.

(i) That the life of randomly chosen tube is b/w 136 hours and 174 hours.

(ii) That the life of a randomly chosen tube is less than 117 hours.

(iii) That the total life of two randomly chosen tubes will be more than 395 hours.

<u>Solⁿ</u>:- Given $\mu = 155$, $\sigma = 19$, let $z = \frac{X - \mu}{\sigma} = \frac{X - 155}{19}$ where X denotes the life of electric.

(i) $P[136 < X < 174] = P\left[\frac{136-155}{19} < \frac{X-155}{19} < \frac{174-155}{19}\right]$

$= P[-1 < z < 1]$

$= 2P[0 < z < 1] = 2 \times 0.3413 = 0.6826.$

(ii) $P[X < 117] = P\left[\frac{X-155}{19} < \frac{117-155}{19}\right] = P[z < -2]$

$= P[z > 2]$ {By symmetry}

$= 0.5 - P[0 < z < 2] = 0.5 - 0.4772$

$= 0.0228.$

(iii) $P[X > 395] = P\left[\frac{X-155}{19} > \frac{395-155}{19}\right]$

$= P[z > 12.63] = 0.$

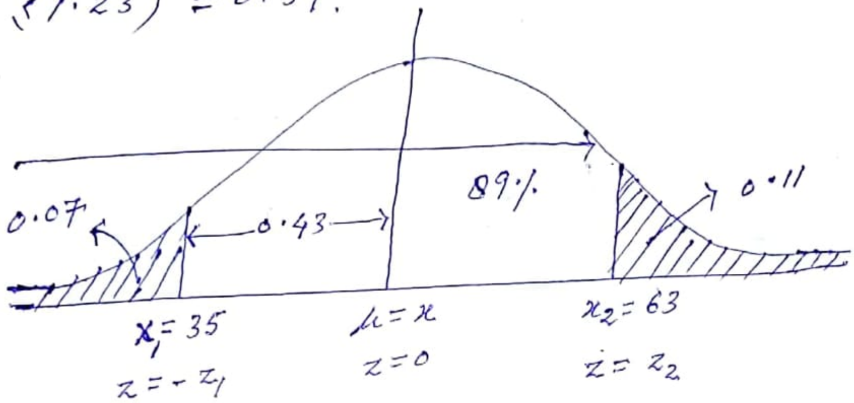
Imp Example:

1. In a normal distribution 7% of the items are under 35 and 89% are under 63. Find the mean and s.d of the distribution? Given that

Sol:

$P(0 \le Z \le 0.18) = 0.07$, $P(0 \le Z \le 1.48) = 0.43$,

$P(0 \le Z \le 1.23) = 0.39$.



Let μ & σ be the mean & standard deviation of the normal distribution

Here $P(x < 35) = 0.07$

when $x = 35$, $z = \frac{35 - \mu}{\sigma} = z_1$ (say) ——— ①

$P(x < 35) = 0.07$

$\Rightarrow P(z < z_1) = 0.07$

$\Rightarrow 0.5 - P(z_1 < z < 0) = 0.07$

$\Rightarrow P(z_1 < z < 0) = 0.50 - 0.07 = 0.43$

$\Rightarrow$ or $P(0 < z < z_1) = 0.43$

$\Rightarrow z_1$ corresponds to the value $1.48 \Rightarrow z_1 = -1.48$

from ① $35 - \mu = -1.48\sigma$ ——— ②

Again $P(x < 63) = 0.89$, when $x = 63$, $z = \frac{63 - \mu}{\sigma} = z_2$ (say) ——— ③

$\Rightarrow P(z < z_2) = 0.89$

$\Rightarrow 0.5 + P(0 < z < z_2) = 0.89$

$\Rightarrow P(0 < z < z_2) = 0.89 - 0.5 = 0.39$

$\Rightarrow z_2$ corresponds to the value 1.23, i.e $z_2 = 1.23$

from ③ $63 - \mu = 1.23\sigma$ ——— ④

On solving ② & ④, we get $\boxed{\sigma = 10.38}$

& $\boxed{\mu = 50.3}$

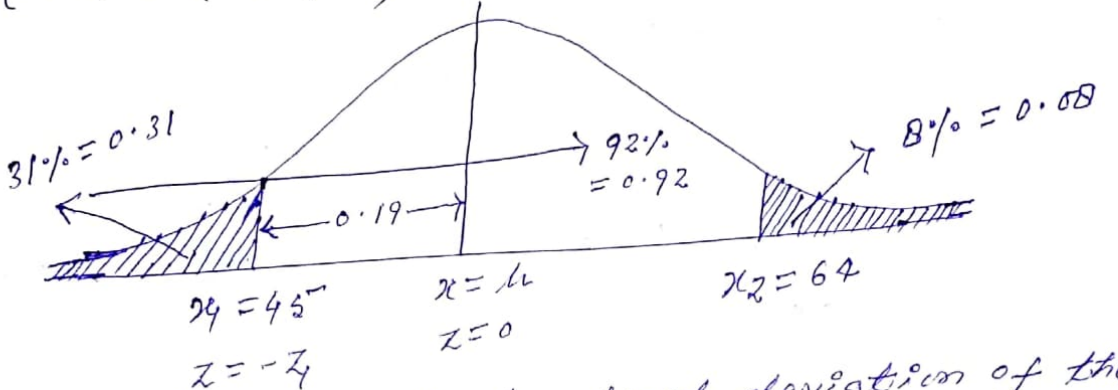
50 Example

2. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean & standard deviation of the distribution.

Given that $P(0 \leq z \leq 0.496) = 0.19$

& $P(0 \leq z \leq 1.405) = 0.42$.

Soln:→



Let μ & σ be the mean & standard deviation of the normal distribution. when $x = 45$ $z = \frac{45-\mu}{\sigma} = z_1 (\text{say})$ —— (1)

Here $P(x < 45) = 31\% = 0.31$, $\Rightarrow 0.5 - P(z_1 < z < 0) = 0.31$

$P(z < z_1) = 0.31 \Rightarrow 0.5 - P(z_1 < z < 0) = 0.31$

$\Rightarrow P(z_1 < z < 0) = 0.5 - 0.31 = 0.19$

$\Rightarrow P(0 < z < z_1) = 0.19 \Rightarrow z_1$ corresponds to the value 0.496

i.e $z_1 = -0.496$, from $45 - \mu = -0.496\sigma$ —— (2)

Again $P(x > 64) = 8\% = 0.08$ or $P(x < 64) = 92\% = 0.92$

when $x = 64 \Rightarrow z = \frac{64-\mu}{\sigma} = z_2 (\text{say})$ —— (3)

$\Rightarrow P(x < 64) = 0.92 \Rightarrow 0.5 + P(0 < z < z_2) = 0.92$

$\Rightarrow P(0 < z < z_2) = 0.92 - 0.5 = 0.42$

$\Rightarrow z_2$ corresponds to the value 1.405

from (3) $64 - \mu = 1.405\sigma$ —— (4)

On solving (2) & (4) we get

$\boxed{\mu = 50}$ & $\boxed{\sigma = 10}$

Example :→ Assuming the mean height of soldiers to be 68.22 inches with a variance of 10.8 inches², find how many soldiers in a regiment of 1000 would you expect to be over 6 feet tall. (Given: Area under the standard normal curve between z=0 & z=0.35 is 0.1368) and between z=0 and z=1.15 is 0.3746)

Solⁿ :→ Given $\mu = 68.22$, $\sigma^2 = 10.8 \Rightarrow \sigma = 3.286$, let $z = \frac{X-\mu}{\sigma}$

$\Rightarrow z = \frac{X - 68.22}{3.286}$, when $X = 6$ feet $= 72$ inches

$z = \frac{72-68.22}{3.286} = 1.15$

$P[X > 72] = P[Z > 1.15] = 0.5 - P[0 < Z < 1.15] = 0.5 - 0.3746 = 0.1254$

Hence, the number of soldiers out of 1000 who are over 6 feet tall

$= 1000 \times 0.1254 = 125.4 \approx 125$ nearly.

Example :→ The marks obtained by candidates in statistics in a certain examination are found to be normally distributed. If 12.5% of the candidates obtain 60 or more marks and 39% obtain less than 30 marks. Find the mean number of marks obtained by the candidates.

Solⁿ :→ Let $X \sim N(\mu, \sigma^2)$, where μ = mean, σ = S.D

It is given that $P[X \geq 60] = 0.125$, and $P[X < 30] = 0.39$

$\Rightarrow P[\mu < X < 60] = 0.5 - 0.125 = 0.375$ and $P[30 < X < \mu] = 0.5 - 0.39 = 0.11$

$\Rightarrow P\left[\frac{\mu-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{60-\mu}{\sigma}\right] = 0.375$ & $P\left[\frac{30-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu-\mu}{\sigma}\right] = 0.11$

$\Rightarrow P\left[0 < Z < \frac{60-\mu}{\sigma}\right] = 0.375$ & $P\left[\frac{30-\mu}{\sigma} < Z < 0\right] = 0.11$

$\Rightarrow \frac{60-\mu}{\sigma} = 1.15$ — ① & $P\left[0 < Z < \frac{\mu-30}{\sigma}\right] = 0.11$ (By symmetry)

& $\frac{\mu-30}{\sigma} = 0.28$ — ②

On solving both, we get $\boxed{\mu = 35.87}$ & $\boxed{\sigma = 20.97}$

Example :→ In a distribution exactly 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution.

Solⁿ :→ If $X \sim N(\mu, \sigma^2)$, where μ = mean & σ = S.D

we are given $P[X < 63] = 0.89$ and $P[X < 35] = 0.07$

$P[\mu < X < 63] = 0.89 - 0.5 = 0.39$ and $P[35 < X < \mu] = 0.5 - 0.07 = 0.43$

$P\left[\frac{\mu-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{63-\mu}{\sigma}\right] = 0.39$ and $P\left[\frac{35-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{\mu-\mu}{\sigma}\right] = 0.43$

$P\left[0 < Z < \frac{63-\mu}{\sigma}\right] = 0.39$ and $P\left[\frac{35-\mu}{\sigma} < Z < 0\right] = 0.43$

$\frac{63-\mu}{\sigma} = 1.23$ — ① and $P\left[0 < Z < \frac{\mu-35}{\sigma}\right] = 0.43$

$\Rightarrow \frac{\mu-35}{\sigma} = 1.48$ — ②

On solving ① & ②, we get

$\boxed{\mu = 50.3}$ & $\boxed{\sigma = 10.33}$