UNIT- 1.

1. FOURIER SERIES.

PERIODIC FUNCTION :- A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x+T) = f(x)$ for all x belonging to the domain of definition of the function.

The least number possessing this property is called the period of the function $f(x)$ and we often say that $f(x)$ is T-periodic.

For example; $\sin x$ and $\cos x$ are 2π- periodic, since

$$\sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots$$

$$\cos x = \cos(x+2\pi) = \cos(x+4\pi) = \cdots$$

Also, $\sin nx$ and $\cos nx$ are $\frac{2\pi}{n}$ - periodic, since

$$\sin n\left(x+\frac{2\pi}{n}\right) = \sin(nx+2\pi) = \sin nx$$

$$\cos n\left(x+\frac{2\pi}{n}\right) = \cos(nx+2\pi) = \cos nx.$$

A 2π- periodic function $f(x)$ can be expressed as a trigonometric series $\frac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \cdots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \cdots$ ⟶ ⊛

where $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are some constants, provided that $f(x)$ satisfies the following condition, called DIRICHLET'S CONDN

(i) $f(x)$ is periodic

(ii) $f(x)$ and its integrals are finite and single valued

(iii) $f(x)$ is bounded and can have a finite number of discontinuities.

(iv) $f(x)$ has a finite number of maximas and minimas over the range of time period (x).

Note :- ① If $f(x)$ is continuous in the range of definition, series ⊛ converges to $f(x)$

② If $f(x)$ has ordinary discontinuity at 'x', the series converges to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ for every x in the range of definition of $f(x)$.

This series ⊛ is called Fourier series of $f(x)$.

Examples :- . The function $f(x) = \frac{1}{x-3}$ does not satisfy the Dirichlet's condition in the interval $(0, 2\pi)$ as it is not defined at $x = 3$.

$\therefore$ $f(x) = \frac{1}{x-3}$ cannot posses Fourier ~~Poss~~ series in $(0, 2\pi)$.

To evaluate the unknowns, $a_0, a_1, a_2, \ldots b_1, b_2 \ldots$ we need to the following orthogonality properties :-

The trigonometric functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are ~~otho~~ orthogonal on the interval $(-\pi, \pi)$ and hence on $(c, c+2\pi)$. By $\boxed{\text{orthogonality}}$, we mean that the integral of the product of any two different such functions over that interval is zero.

(1) $$\int_c^{c+2\pi} \sin nx\,dx = \left[-\frac{1}{n}\cos nx\right]_c^{c+2\pi} = 0 \quad ; n \neq 0.$$

(2) $$\int_c^{c+2\pi} \cos nx\,dx = \left[\frac{1}{n}\sin nx\right]_c^{c+2\pi} = 0 \quad ; n \neq 0$$

(3) $$\int_c^{c+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_c^{c+2\pi} = 0 \; ; m \neq n$$

(4) $$\int_c^{c+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_c^{c+2\pi} = 0 \quad ; m \neq n$$

(5) $$\int_c^{c+2\pi} \sin mx \cos nx\,dx = -\frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right]_c^{c+2\pi} = 0.$$

Also,

Some more results are :-

(1) $\sin n\pi = 0$ (2) $\cos n\pi = (-1)^n$

(3) $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(4) $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

Evaluation of Fourier Coefficients

Let the function $f(x)$ be represented in the interval $c \le x \le c+2\pi$ by the Fourier series,

$$f(x) = \frac{1}{2}a_0 + \sum_{r=1}^{\infty}(a_r \cos rx + b_r \sin rx)$$

where
$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_r = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos rx\,dx$$

$$b_r = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin rx\,dx . \qquad r = 1, 2, \ldots$$

Corollary :- (1) If $c=0$, then the above integration becomes for $0 \leq x \leq 2\pi$, & Fourier coefficients are,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$$

Corollary :- (2) If $c = -\pi$, then the interval of integration becomes for $-\pi \leq x \leq \pi$, and Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Corollary :- (3) If $f(x)$ is an even function, i.e. $f(-x) = f(x)$ in $(-\pi, \pi)$ then by property of definite integrals,

$$\boxed{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx} \qquad \boxed{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx} \qquad \boxed{b_n = 0}$$

Corollary :- (4) If $f(x)$ is an odd function, i.e. $f(-x) = -f(x)$ in $(-\pi, \pi)$ then

$\boxed{a_n = 0}$ and $\boxed{b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$

Corollary (5) The Fourier expansion of the function $f(x)$ as $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$ is in two parts. The part $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ is an even ~~part~~ function of x and the part $\sum_{n=1}^{\infty} b_n \sin nx$ is odd function of x.

Also, we can write $f(x) = \frac{1}{2}\left[f(x) + f(-x)\right] + \frac{1}{2}\left[f(x) - f(-x)\right]$

Out of which $\frac{1}{2}\left[f(x) + f(-x)\right]$ is even function

and $\frac{1}{2}\left[f(x) - f(-x)\right]$ is odd function.

$\therefore$ We have,

$$\boxed{\frac{1}{2}\left[f(x) + f(-x)\right] = \frac{1}{2}a_0 + \sum_{n} a_n \cos nx}$$

and

$$\boxed{\frac{1}{2}\left[f(x) - f(-x)\right] = \cancel{\frac{a_0}{2}} \cdot \sum_{n} b_n \sin nx}$$

Ex. Find the Fourier Series for the function $f(x) = e^{ax}$ in the interval $(-\pi, \pi)$.

Solution Let the Fourier series for $f(x)$ be

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Here, $\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\,dx = \left[\frac{e^{ax}}{a\pi}\right]_{-\pi}^{\pi}$

$$= \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

$$= \frac{2\sinh(a\pi)}{a\pi}$$

$$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx = \left[\frac{e^{ax}}{\pi(a^2+n^2)}(a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}(a\cos n\pi + \pi\underbrace{\sin n\pi}_{0}) - e^{-a\pi}(a\cos n\pi - n\underbrace{\sin n\pi}_{0})\right]$$

$$= \frac{a\cos n\pi\,(e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{2a(-1)^n\sinh(a\pi)}{\pi(a^2+n^2)}.$$

$$\boxed{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,dx = \left[\frac{e^{ax}}{\pi(a^2+n^2)}(a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}(a\sin n\pi - \pi\cos n\pi) - e^{-a\pi}(-a\sin n\pi - n\cos n\pi)\right]$$

$$= \frac{-n\cos n\pi}{\pi(a^2+n^2)}\left(e^{a\pi}-e^{-a\pi}\right) = \frac{-2n(-1)^n \sinh(a\pi)}{\pi(a^2+n^2)}.$$

Substituting these values of a_0, b_n, a_n, we get

$$f(x) = e^{ax} = \frac{1}{a\pi}\sinh a\pi + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh a\pi}{\pi(a^2+n^2)}\left(a\cos nx - n\sin nx\right).$$

is the required Fourier Series.

Ex Find a Fourier series to represent $x - x^2$ from $x=-\pi$ to $x=\pi$ and hence show that

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \frac{\pi^2}{12}.$$

Solution let the Fourier series be given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

where $\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\,dx$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[-\frac{\pi^3}{3} - \frac{\pi^3}{3}\right]$$

$$= -\frac{2\pi^2}{3}.$$

$$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx$$

$$= \underset{(\text{odd})}{0} - \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= -\frac{2}{\pi}\left[\underbrace{\frac{x^2\sin nx}{n}}_{0} - \frac{(2x)(-\cos nx)}{n^2} + \underbrace{\frac{2(-\sin nx)}{n^3}}_{0}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{2\pi\cos n\pi}{n^2} - 0\right] = \frac{-4\pi(-1)^n}{\pi n^2} = \frac{-4(-1)^n}{n^2}.$$

$$\boxed{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2\sin nx}_{\text{odd}}\,dx \quad 0.$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n} - \underbrace{\frac{(1)(-\sin nx)}{n^2}}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n}\right] = \frac{-2(-1)^n}{n}.$$

$$\therefore\quad f(x) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{-4(-1)^n\cos nx}{n^2} - \frac{2(-1)^n\sin nx}{n}\right]$$

$$\Rightarrow f(x) = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} \cdots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \cdots\right]$$

Putting $x=0$ on both sides, we get

$$f(0) = 0 = -\frac{\pi^2}{3} + 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots\right]$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots = \frac{\pi^2}{12}.$$

—— x ——

Example 3:- Find the Fourier series to represent the function $f(x) = x\sin x$, $-\pi < x < \pi$.

Solution Since product of two odd functions is an even function. $\therefore$ $f(x)$ is an even function of x.

Hence, $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

Now, $\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\, dx$

$$= \frac{1}{\pi}\Big[x(-\cos x) - (1)(\sin x)\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\Big[-\pi\cos\pi - \sin\pi + (-\pi)\cos(-\pi) + \sin(-\pi)\Big]$$

$$= \frac{1}{\pi}\Big[\pi + \pi\Big] = 2.$$

$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\, dx$

$x\sin x \to$ odd * odd = even

$\cos nx \to$ even

$\therefore$ $x\sin x\cos nx \to$ even

$$= \frac{2}{\pi}\int_{0}^{\pi} x\sin x\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\Big[\sin(n+1)x - \sin(n-1)\cancel{x}\,x\Big]dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right\}-(1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right\}+\cancel{\{0\}}+0\right] \quad , \boxed{n\neq 1}$$

$$= (-1)^n\left[\frac{1}{n+1}-\frac{1}{n-1}\right] = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \quad \boxed{n\neq 1}$$

For $n=1$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx = \frac{2}{2\pi}\int_0^{\pi} 2x\sin x\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = \frac{1}{\pi}\left[-\frac{x\cos 2x}{2}+(1)\frac{(\sin 2x)}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}+0\right] = -\frac{1}{2}$$

$$\therefore \quad f(x) = 1-\frac{1}{2}\cos x+\sum_{n=2}^{\infty}\frac{2(-1)^{n+1}}{n^2-1}\cos nx.$$

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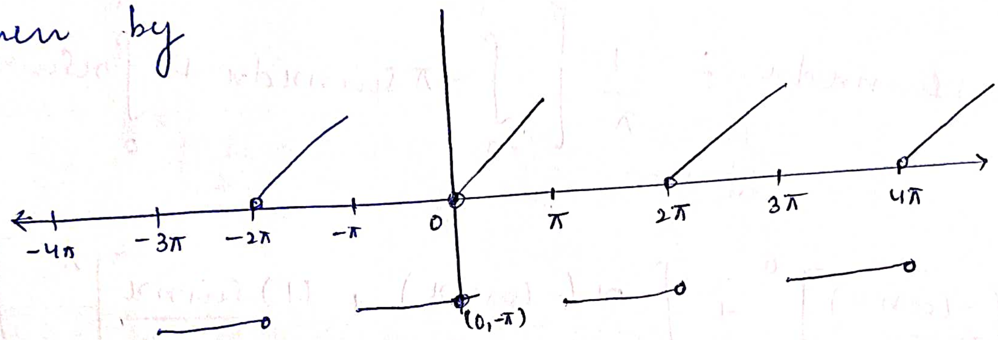
Example 4 :- The function $f(x)$ is given by

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Draw its graph and find its Fourier series and hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution. The graph of the 2π- periodic function is given by



Let the Fourier series of the given function be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

There $\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_{0}^{\pi} x\,dx\right]$

$$= \frac{1}{\pi}\left[\left[-\pi x\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + (-\pi)^2 + \frac{\pi^2}{2}\right] = -\frac{\pi^2}{2\pi} = \frac{-\pi}{2}$$

$$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= -\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x\sin nx}{n} + (1)\frac{\cos nx}{n^2}\right]_{0}^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right] = \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$\boxed{b_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\pi\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \left[x\frac{(-\cos nx)}{n} + (1)\frac{\sin nx}{n^2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi(1-(-1)^n)}{n} + \frac{\pi((-1)^{n+1}}{n} + 0\right]$$

$$= \frac{1-2(-1)^n}{n}.$$

$\therefore$ The Fourier series for $f(x)$ is

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n-1}{n^2}\right)\cos nx\right] + \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin nx \quad \circledast$$

Now, we need to prove the equality :-

The expansion of $f(x)$ at $x=0$, which is a point of discontinuity of $f(x)$. The value of $f(x)$ at $x=0$ is given by the mean of its left hand and right hand limits, equal to

$$\frac{f(0^-)+f(0^+)}{2} = \frac{-\pi+0}{2} = -\frac{\pi}{2}$$

Thus, we have (Putting $x=0$) in (*)

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \frac{1}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n^2}\right] + 0 .$$

$$\Rightarrow \quad \pi^2\left(\frac{-1}{2}+\frac{1}{4}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}$$

$$\Rightarrow \quad -\frac{2\pi^2}{8} = \frac{(-2)}{1^2}+\frac{(-2)}{3^2}+\frac{(-2)}{5^2}+\cdots$$

$\therefore$ we have

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}.$$

Example 5. Obtain the fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1+\frac{2x}{\pi} & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi} & 0 \le x \le \pi . \end{cases}$$

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Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}.$

Solution Clearly $f(-x) = f(x)$ and hence $f(x)$ is even in $(-\pi, \pi)$ and will have only cosine series. Also the continuous at $x = 0$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

Here $\boxed{a_0} = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = 0.$

$$\boxed{a_n} = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\underbrace{\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)}_{0} - \left(\frac{-2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-2}{\pi n^2} \cos nx\right]_0^{\pi} = \frac{-4}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{\pi^2 n^2} & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $f(x) = \sum_{n \text{ is odd}} \frac{8}{\pi^2 n^2} \cos nx = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \cdots\right]$

Putting $x = 0$, $\underset{(=1)}{f(0)} = \frac{8}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$

$\therefore \frac{\pi^2}{8} = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right].$

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x

Example: Find the Fourier series to represent the function $f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$

and Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution

$$\boxed{a_0} = \pi$$

$$\boxed{a_n} = \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ is even} \\ \frac{-4}{\pi n^2}. & n \text{ is odd} \end{cases}$$

$$\boxed{b_n} = 0.$$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n = \text{odd}} \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \cdots\right]$$

Putting $x = \pi$ on both sides,

$$\pi = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}.$$

x

Example Obtain the Fourier series for the function $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $[0, 2\pi)$ and hence show that

(1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(2) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \frac{\pi^2}{12}$.

Solution In this problem the function is defined in $(0, 2\pi)$. If we substitute $t = x - \pi$, then

$$f(x) = \phi(t) = \frac{t^2}{4} \quad \text{when } -\pi < t < \pi.$$

Clearly $\phi(t) = \frac{t^2}{4}$ is an even function of t.

let $\phi(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $\boxed{a_0} = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(t)\,dt = \frac{2}{\pi}\int_0^{\pi} \frac{t^2}{4}\,dt$

$$= \frac{2}{\pi}\left[\frac{t^3}{12}\right]_0^{\pi} = \frac{\pi^2}{6}.$$

$$\boxed{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \phi(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi} \frac{t^2}{4}\cos nt\,dt$$

$$= \frac{2}{\pi}\left[\frac{t^2}{4}\left(\frac{\sin nt}{n}\right) - \left(\frac{2t}{4}\right)\left(-\frac{\cos nt}{n^2}\right) + \frac{2}{4}\left(\frac{-\sin nt}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2t\cos nt}{4n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi\cos n\pi}{n^2} - 0\right] = \frac{(-1)^n}{n^2}.$$

$\therefore$ The fourier series is given by

$$\phi(t) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2}$$

Replacing t by $x-\pi$, we get

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n(x-\pi)$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left[\cos nx\cos n\pi + \sin nx\underbrace{\sin n\pi}_{0}\right]$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n\cos nx}{n^2}.$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}.$$

$$\Rightarrow \frac{(\pi-x)^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}.$$

Putting $x=0$, we get $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Putting $x=\pi$, we get $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$

Example. Obtain the Fourier expansion for $\sqrt{1-\cos x}$ in the interval $-\pi < x < \pi$.

Solution. $f(x) = \sqrt{1-\cos x} = \sqrt{2\sin^2\frac{x}{2}} = \pm\sqrt{2}\sin\frac{x}{2}$

$$f(x) = \begin{cases} \sqrt{2}\sin\frac{x}{2} & 0 < x < \pi \\ -\sqrt{2}\sin\frac{x}{2} & -\pi < x < 0 \end{cases}$$

is an even function of x.

Let its Fourier series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

where $\boxed{a_0} = \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\frac{x}{2}\,dx = \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right]_0^{\pi}$

$= \frac{4\sqrt{2}}{\pi}$.

$$\boxed{a_n} = \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\frac{x}{2}\cos nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} 2\sin\frac{x}{2}\cos nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\left(n+\frac{1}{2}\right)x}{\left(n+\frac{1}{2}\right)} + \frac{\cos\left(n-\frac{1}{2}\right)x}{\left(n-\frac{1}{2}\right)}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-1}{\left(n+\frac{1}{2}\right)}\cos\left(n\pi+\frac{\pi}{2}\right) + \frac{1}{\left(n-\frac{1}{2}\right)}\cos\left(n\pi-\frac{\pi}{2}\right) + \frac{1}{\left(n+\frac{1}{2}\right)} - \frac{1}{\left(n-\frac{1}{2}\right)}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{1}{(2n+1)}\underbrace{\sin n\pi}_{=0} + \frac{1}{(2n-1)}\underbrace{\sin n\pi}_{=0} + \frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}.$$

$$\therefore \quad f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{4n^2-1}.$$

x

Example :- Expand $f(x) = |\cos x|$ as a Fourier series in $(-\pi, \pi)$.

IPU 2010, 2014, 2016

Solution : $f(x) = |\cos x|$ is an even function of x in $(-\pi, \pi)$

let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $$\boxed{a_0} = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right]$$

$$= \frac{2}{\pi}\left[\sin x\right]_0^{\pi/2} - \frac{2}{\pi}\left[\sin x\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[1-0\right] - \frac{2}{\pi}\left[0-1\right] = \frac{4}{\pi}$$

$$\boxed{a_n} = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi} -\cos x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \cos(n+1)x + \cos(n-1)x\,dx\right]$$

$$- \frac{1}{\pi}\left[\int_{\pi/2}^{\pi} \cos(n+1)x + \cos(n-1)x\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2}$$

$$- \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{(n+1)} + \frac{\sin(n-1)\frac{\pi}{2}}{(n-1)}\right] \quad ; \quad (n \neq 1)$$

$$= \frac{2}{\pi(n^2-1)}\left[(n-1)\sin(n+1)\frac{\pi}{2} + (n+1)\sin(n-1)\frac{\pi}{2}\right] \quad ; (n \neq 1)$$

$$= \frac{2}{\pi(n^2-1)}\left[(n-1)\cos n\frac{\pi}{2} + (n+1)\left(-\cos n\frac{\pi}{2}\right)\right] \quad ; (n \neq 1)$$

$$= \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2} \quad , (n \neq 1)$$

Also, $a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos^2 x\,dx - \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos^2 x\,dx$

$$= \frac{1}{\pi}\int_0^{\pi/2}(1+\cos 2x)\,dx - \frac{1}{\pi}\int_{\pi/2}^{\pi}(1+\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2} - \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + 0\right] - \frac{1}{\pi}\left[\pi - \frac{\pi}{2}\right]$$

$$= \frac{1}{2} - \frac{1}{2} = 0.$$

$$\therefore \quad f(x) = |\cos x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\cos n\pi/2 \, \cos nx}{(n^2-1)}.$$