Probability

Result | outcome

deterministic | Predictable
(E.g. $v = u + at$)
for given u, a, at 't'
You can find v

Probabilistic | Unpredictable
(In our day to day life)
Possible, high chances are the words indicating degree of Uncertainity of an Event.

Theory of Probability — A mathematical branch to find out the numerical measure of Uncertainity.

Possible states

Certainty → 1
Impossibility → 0
Uncertainty → Ranges b/w 0 & 1.

Random Experiment :→ (E) While Performing an experiment, the outcome is not unique but may be any one of the possible outcomes, then an Experiment is called a random Experiment.

Sample Space :→ (S) The set of all possible outcomes of a given random Experiment is called the sample space associated with that Experiment.

Event :→ Every non-Empty Subset of S, which is a disjoint union of Single element Subsets of the Sample space S of a random Experiment E is called an Event.

→ As $\phi \subseteq S \Rightarrow \phi$ is also an event
So ϕ is called Impossible Event.

→ Let $A \subseteq S$ and A contains only single element of S then A is called an elementary event.

– In random toss of a single coin
$S = \{H, T\}$ $n(S) = 2$
In random toss of two coins
$S = \{HH, HT, TH, TT\}$ $n(S) = 2^2$
In random toss of three coins
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
$n(S) = 2^3$
E_1: Three heads $= \{HHH\}$
E_2: At least one head $= \{HTT, HHT, HTH, THH, THT, TTH, HHH\}$.

→ Exhaustive Events:→ Total no. of possible outcomes of a random experiment are called Exhaustive Events
OR no. of elements in a sample space (S) of a random experiment are called Exhaustive Events.

→ Favourable Events:→ no. of outcomes which entail the happening of the event.
E.g. no. of cases favourable to drawing of an ace $= 4$
of spade $= 13$.
of Red card $= 26$

→ Mutually Exclusive Event – If happening of any one of them excludes the happening of all the others.
i.e. If no two or more of the events can happen simultaneously.

Equally likely Events: ⇒ If there is no reason to Expect one in preference to the others.

Independent Events: ⇒ If happening of an event is not affected by the happening / non-happening of other events.

Def^n ⇒ If a random experiment results in n exhaustive, mutually Exclusive, and equally likely events, out of which m are favourable to the occurrence of an event E, then the Probability of occurrence of E is given by

$$P(E) = \frac{\text{no. of favourable events}}{\text{no. of Exhaustive Events}} = \frac{m}{n}$$

as $m, n \geq 0$ and $m \leq n$

$\Rightarrow P(E) \geq 0$ and $P(E) \leq 1$.

$\Rightarrow 0 \leq P(E) \leq 1$.

$P(\bar{E}) \rightarrow$ Probability of not happening of event E.

→ If $P(E) = 1$ then E is Called Certain event.
If $P(E) = 0$ then E is Called Impossible event.
If $0 < P(E) < 1$ Then E is Called uncertain event.

Ex What is the Probability that a leap year Consists of 53 Sundays?

Sol^n 366 days — 52 weeks + 2 days

(1) SM (2) MT (3) TW (4) WTh (5) ThF (6) FS (7) SS

$$P(53 \text{ Sundays}) = \frac{2}{7}.$$

Axioms of Probability :→

$$P : B \to [0,1]$$

where B is σ-field of the events generated by S.

[σ-field - is the Collection of subsets of S].

(1) $\forall A \in B$; $P(A)$ is definded and $P(A) \geq 0$

(2) $P(S) = 1$

(3) $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ where $A_1, A_2 \ldots A_n$ are disjoint Events in B.

— Function P defined above on σ-field B satisfying (1)-(3) is Called Prob. measure.

Thm $P(\phi) = 0$ ie. Prob of Impossible Event $= 0$

Solⁿ

$S = S \cup \phi$

$P(S) = P(S \cup \phi)$

$P(S) = P(S) + P(\phi)$

$\Rightarrow$ $P(\phi) = 0$.

Thm $P(\bar{A}) = 1 - P(A)$

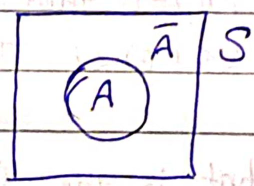
Solⁿ

$A \cup \bar{A} = S$

$P(A \cup \bar{A}) = P(S)$.

$P(A) + P(\bar{A}) = 1$

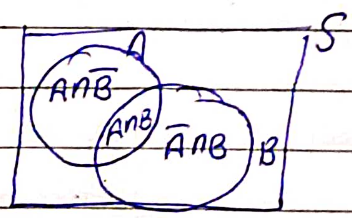
$\Rightarrow P(\bar{A}) = 1 - P(A)$



Thm (1) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

(2) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

for any two Events A & B.



Solⁿ $(\bar{A} \cap B) \cup (A \cap B) = B$

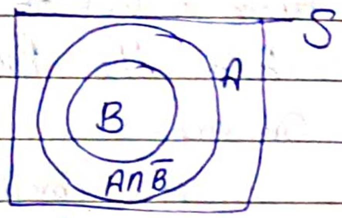
$\Rightarrow P(B) = P(\bar{A} \cap B) + P(A \cup B)$

$\because (\bar{A} \cap B)$ & $(A \cup B)$ are disjoint Events.

$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Que If $B \subset A$

(1) then $P(A \cap \bar{B}) = P(A) - P(B)$



Sol^n $A = B \cup (A \cap \bar{B})$

$\Rightarrow P(A) = P[B \cup (A \cap \bar{B})]$

$= P(B) + P(A \cap \bar{B})$ ($\because B$ & $(A \cap \bar{B})$ are disjoint Events)

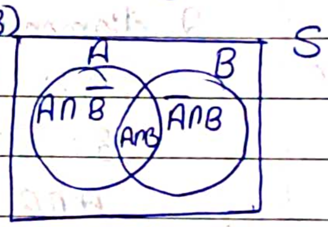
$\Rightarrow P(A \cap \bar{B}) = P(A) - P(B)$

(2) $P(B) \leq P(A)$

$P(B) = P(A) - P(A \cap \bar{B})$ $\quad \because P(A \cap \bar{B}) \geq 0$

$\leq P(A)$

Addition Thm of Probability:→ If A and B be any two Events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Sol^n $A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$

$\Rightarrow P(A \cup B) = P(A \cap \bar{B}) + P(A \cap B) + P(\bar{A} \cap B)$

$= P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$

Or $P(A \cup B) = \dfrac{n(A \cup B)}{n(S)} = \dfrac{n(A) + n(B) - n(A \cap B)}{n(S)}$

$= \dfrac{n(A)}{n(S)} + \dfrac{n(B)}{n(S)} - \dfrac{n(A \cap B)}{n(S)}$

$= P(A) + P(B) - P(A \cap B)$.

→ If A & B are disjoint Events i.e. $A \cap B = \phi$

Then $P(A \cup B) = P(A) + P(B)$.

→ For any three non-mutually Exclusive events A, B, C, we have

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Que From a Pack of well shuffled Cards, one Card is drawn. Find the Prob that the Card is either a King or an ace.

Sol^n

A: Card drawn is King

B: Card drawn is ace

$P(A) = \frac{4}{52}$; $P(B) = \frac{4}{52}$

$A \cap B = \phi$

$\Rightarrow P(A \cup B) = P(A) + P(B) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$.

Que From a Pack of well shuffled Cards, 4 Cards are drawn. Find the Prob. that there are 2 hearts and 2 diamonds.

Sol^n

~~A: Hearts~~

~~B: Diamonds~~

~~$A \cap B = \phi$~~

~~$P(A \cup B) = P(A) + P(B)$~~

Total Prob. $= {}^{52}C_4$

Two hearts are chosen in ${}^{13}C_2$ ways

Two diamonds are chosen in ${}^{13}C_2$ way

$$\text{Prob} = \frac{{}^{13}C_2 \cdot {}^{13}C_2}{{}^{52}C_4}.$$

Boole's Inequality $\Rightarrow$ $P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$; $A_1, A_2 \ldots A_n$ are n Events.

Sol^n

For $n=1$; $P(A_1) \leq P(A_1)$

$n=2$; $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$\leq P(A_1) + P(A_2)$

Let it be true for $n=k$

$$P\left(\bigcup_{i=1}^{k} A_i\right) \leq \sum_{i=1}^{k} P(A_i)$$

For $n = k+1$;

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right)$$

$$\leq P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1})$$

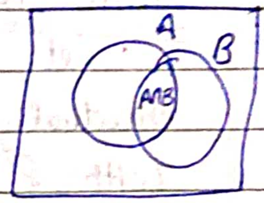
$$\leq \sum_{i=1}^{k} P(A_i) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i)$$

Conditional Prob^n → Prob. of the event A, given that event B has already happened.

$= P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

E.g. A: The card drawn is king.
→ B: The card drawn is red.

$P(A|B) = \dfrac{2}{26} = \dfrac{1}{13}$



Multiplication Thm of Prob^n → For two events A and B; $P(A\cap B) = P(A).P(B|A)$;

and $P(A\cap B) = P(B).P(A|B)$

→ If A and B and are two Independent events
$\Leftrightarrow P(A\cap B) = P(A).P(B)$

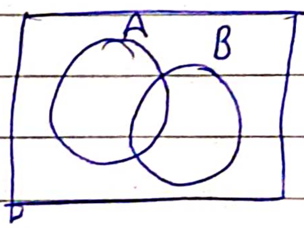
$\because$ A and B Independent $\Rightarrow P(A|B) = P(A)$
$\Rightarrow P(A\cap B) = P(A|B).P(B) = P(A).P(B)$

Conversely let $P(A\cap B) = P(A).P(B)$
$\Rightarrow P(A|B) = \dfrac{P(A).P(B)}{P(B)} = P(A)$

$\Rightarrow$ A & B are Independent.

→ If A and B are Independent events then
(i) A and $\bar{B}$ (ii) $\bar{A}$ & B (iii) $\bar{A}$ & $\bar{B}$ are also Independent.

Sol^n →



$P(A\cap B) = P(A).P(B)$
$A\cap\bar{B} = A-(A\cap B)$
$P(A\cap\bar{B}) = P(A)-P(A\cap B)$
$= P(A)-P(A)P(B) = P(A).P(\bar{B})$

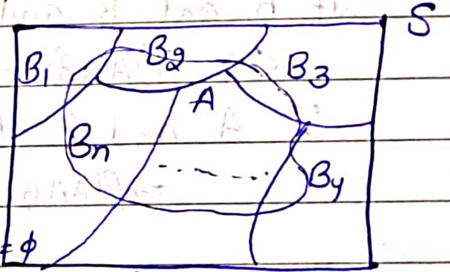
$P(\bar{A}\cap B) + P(A\cap B) = P(B)$
$\Rightarrow P(\bar{A}\cap B) = P(B)-P(A)P(B) = P(\bar{A})P(B)$
$P(A\cup B) = P(A)+P(B)-P(A).P(B)$
$= P(A)+P(B).P(\bar{A})$

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$$
$$= 1 - P(A \cup B)$$
$$= 1 - P(A) - P(B).P(\bar{A})$$
$$= P(\bar{A}) - P(B).P(\bar{A})$$
$$= P(\bar{A})\, P(\bar{B})$$

Total Probability:→

Let $B_1, B_2, \ldots, B_n$ be a set of Exhaustive and mutually Exclusive events of the sample space S with $P(B_K) \neq 0$; $K = 1, 2, \ldots n$ and let A be any event of S. Then

$$P(A) = \sum_{i=1}^{n} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i)\, P(A|B_i)$$



Sol^n→ $S = \bigcup_{i=1}^{n} B_i$ and $\bigcap_{i=1}^{n} B_i = \phi$

$\Rightarrow A \cap B_1 = \phi,\ A \cap B_2 = \phi \ldots A \cap B_n \neq \phi$

$\Rightarrow A = ((A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n))$

$\Rightarrow P(A) = P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n)$

$$= \sum_{i=1}^{n} P(A \cap B_i)$$

$$= \sum_{i=1}^{n} P(B_i)\, P(A/B_i)$$

Baye's Thm:→ $P(B_i|A) = \dfrac{P(B_i \cap A)}{P(A)}$

$$= \frac{P(B_i)\, P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\, P(A|B_i)} \quad ; i = 1, \ldots n.$$

Que A. factory has 4 units A B C D
which produce 40% 30% 20% 10% of the same items.
% of defective 2% 1% 0.5% 0.25%.
If an Item is Selected at Random, find the Prob. that the item is defective.

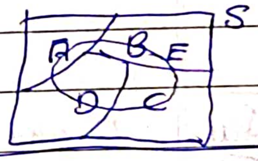
Sol^n Let E :— Item is defective.

$P(A) = 0.04;\ P(B) = 0.3;\ P(C) = 0.2;\ P(D) = 0.1$

$P(E/A) = 0.02;\ P(E|B) = 0.01;\ P(E|C) = 0.005;\ P(E|D) = 0.0025.$

So $P(E) = P(A) \cdot P(E|A) + P(B)\, P(E|B) + P(C) \cdot P(E|C) + P(D) \cdot P(E|D)$

$= 0.01225.$



→ Prob. that it is selected from box B.

$P(B|E) = \frac{P(B \cap E)}{P(E)} = \frac{P(B) \cdot P(E|B)}{P(E)} =$

Que A Box — 6 R, 4 B
B Box — 3R, 7B Balls. Two balls are drawn at random from box A and placed in box B. Now a ball is drawn at random from box B. Find the Prob. that it is a blue ball.

Sol^n $\underset{B_1}{2R};\ \underset{B_2}{1R, 1B};\ \underset{B_3}{2B}$

$P(B_1) = \frac{{}^6C_2}{{}^{10}C_2} = \frac{1}{3};\ P(B_2) = \frac{6 \times 4}{{}^{10}C_2} = \frac{8}{15};\ P(B_3) = \frac{{}^4C_2}{{}^{10}C_2} = \frac{2}{15}.$

E: Drawing blue ball from B box after transfer.

$P(E|B_1) = \frac{7}{12};\ P(E|B_2) = \frac{8}{12};\ P(E|B_3) = \frac{9}{12}$

~~All Balls is blue~~

$P(E) = P(B_1) \cdot P(E|B_1) + P(B_2) \cdot P(E|B_2) + P(B_3) \cdot P(E|B_3) = 0.65.$

Find the Prob. that Red balls have been transferred from A.

→ $P(B_1|E) = \frac{P(B_1 \cap E)}{P(E)} = \frac{P(B_1) \cdot P(E|B_1)}{0.65} = \frac{\frac{1}{3} \times \frac{7}{12}}{0.65} =$

Que E and F are two Independent Events. Find $P(F)$ If $P(E) = 0.4$ & $P(E \cup F) = 0.55$. Ans $\frac{1}{4}$

Que Given $P(A \cap \bar{B}) = \frac{1}{4}$; $P(A \cup B) = \frac{3}{4}$. Find (i) $P(A)$ (ii) $P(B)$ $\frac{1}{2}$

→ A die is thrown twice and the sum of the no. appearing is noted to be 8. What is the Conditional Prob. that no. 5 has appeared atleast once.

Solⁿ →
A: no. 5 appeared atleast once.
B: Sum is 8

$A = \{(5,1)(5,2)(5,3),(5,4)(5,5),(5,6),(1,5),(2,5)(3,5)(4,5)(6,5)\}$

$B = \{(2,6)(6,2)(3,5)(5,3),(4,4)\}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}.$$

→ Two Cards are drawn one after the other from a deck of 52 Cards. Find the Prob. that both are spade Cards if (i) first Card is replaced (ii) not replaced.

Solⁿ (i) $\frac{13}{52} \times \frac{13}{52}$ (ii) $\frac{13}{52} \times \frac{12}{51}$

Random Variable:→ A variable whose value is determined by a random Experiment and is denoted by X.

$X: S \to R$

Discrete Random variable – If a random variable X takes a finite no. or Countably Infinite no. of values

i.e. X takes values $x_1, x_2, x_3, \ldots, x_n, \ldots$

then X is Called a discrete random variable

Continuous Random variable:→ If X takes all possible values in a given Interval, then X is Called cts. Random variable

E.g. Height, weight, ~~age~~ etc. are cts. R.V.

In other words, If its different values Cannot be put in 1-1 Corr. with Z^+.

Discrete Prob. Dist.:→ A Table or formula that lists the prob. for each outcome of the Random variable X.

E.g. Toss / Flip 3 Coins. X – R.V. showing the no. of heads.

S = {HHH, HHT, HTH, THH, HTT, TTH, THT, TTT}

x	0	1	2	3
$P(X=x)$	1/8	3/8	3/8	1/8

$\frac{1}{8} + 3/8 + 3/8 + 1/8 = 1$

→ Throw 2 dice. X – Sum of 2 rolled dice.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$\Sigma P(X=x) = 1$

→ $P(X=x_i) = p_i$ is a prob. function If

$p_i \geq 0 \ \forall i$

$\sum_{i\geq 1} p_i = 1$

$P_X(x) = \begin{cases} P(X=x_i) = p_i ; & x = x_i \\ 0 ; & x \neq x_i ; i = 1, 2, \ldots \end{cases}$

→ Also called Prob. Mass function or point prob. function.

→ What is the prob. of getting a Sum = 7

Ans: 6/36

But If I asked, what is the prob. of getting a Sum ≤ 5

or

Sum ≤ 3

i.e. Sum may be 2 or 3

$= P(x=2) + P(x=3)$

$= \frac{1}{36} + \frac{2}{36} = \frac{3}{36}$

→ ~~let F(x) is a fun~~

→ let X be a random variable and F(x) is a function defined as $F(x) = P(X \leq x)$ is called a distribution function.

F(x) is also called Cummulative distribution function of X.

In Case of discrete R.V. X, we can define $F(x) = P(X \leq x) = \sum_{x_i \leq x} P(X = x_i)$

→ If F is dist. fun of R.V. X Then

(1) $0 \leq F(x) \leq 1$.

(2) $F(x) \leq F(y)$ If $x < y$.

→ If F is a distn fun of R.V. X and If $a < b$

Then $P(a < X \leq b) = F(b) - F(a)$

$P(a < X \leq b) + P(X \leq a) = P(X \leq b)$

$\Rightarrow P(a < X \leq b) = P(X \leq b) - P(X \leq a)$

$= F(b) - F(a)$

→ $P(a \leq X \leq b) = P(a < X \leq b) + P(X = a)$

$= F(b) - F(a) + P(X = a)$

→ $P(a \le X \le b) = P(a < X \le b) + P(X = a) - P(X = b)$
$= F(b) - F(a) + P(X = a) - P(X = b)$

→ $P(a < X < b) = P(a < X \le b) - P(X = b)$
$= F(b) - F(a) - P(X = b)$

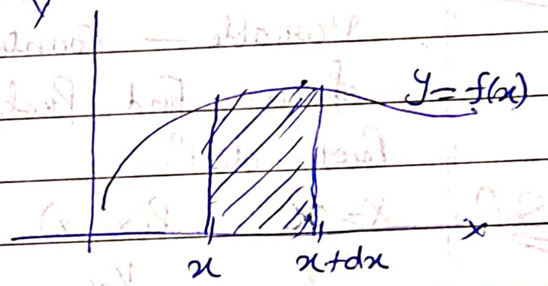
→ For discrete r.v. X, F - Dist fun of X,
$P(X = x_j) = F(x_j) - F(x_{j-1})$
$\because F(x_j) = P(X \le x_j) = \sum_{i=1}^{j} P(X = x_i)$

$F(x_{j-1}) = P(X \le x_{j-1}) = \sum_{i=1}^{j-1} P(X = x_i)$

$$\boxed{F(x_j) - F(x_{j-1}) = P(X = x_j)}$$

Prob. density function: → Consider a small interval $(x, x+dx)$

Let $f(x)$ be a cts fun of x
So that
$P(x \le X \le x+dx) = f(x)\,dx$



$f(x)\,dx$ — area under the curve $y = f(x)$, x-axis and $y = x$ and $y = x+dx$

The fun $f(x)$ — called Prob. density function.
$y = f(x)$ — Prob. density curve.

$f(x)\,dx$ = Prob. differential.

→ Pdf $f(x)$ is defined as
$$f(x) = \lim_{\delta x \to 0} \frac{P(x \le X \le x+\delta x)}{\delta x}$$

(1) $f(x) \ge 0 \ \forall \ x \in (x, x+\delta x)$
(2) $\int f(x) = 1.$

$\rightarrow$ $P(a \le X \le b) = \int_a^b f(x)\,dx.$

$P(X=a) = P(a \le X \le a) = \int_a^a f(x)\,dx = 0$

$\Rightarrow$ Prob at a pt is zero.
$\Rightarrow$ $P(a \le X \le b) = P(a < x < b) = P(a \le X < b) = P(a < X \le b).$

<u>Cummulative distⁿ fun:</u> $\Rightarrow$ $F(x) = P(X \le x)$
$= P(-\infty \le X \le x)$
$= \int_{-\infty}^{x} f(x)\,dx.$

$\rightarrow$ $0 \le F(x) \le 1 \;\forall\; -\infty < x < \infty$
$\rightarrow$ $F(x_1) < F(x_2)$ If $x_1 \le x_2$.

Que Two dice are rolled. Let X denote the random variable — Counts the total no. of points on the faces. Find Prob dist fun of X and Construct discrete Prob. distⁿ.

Solⁿ

$X = x$	$P(X=x)$	$F(x) = P(X \le x)$
2	1/36	1/36
3	2/36	3/36
4	3/36	6/36
5	4/36	10/36
6	5/36	15/36
7	6/36	21/36
8	5/36	26/36
9	4/36	30/36
10	3/36	33/36
11	2/36	35/36
12	1/36	36/36 = 1.

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Head run – Consecutive occurrence of atleast two heads.

Que Three Coins are tossed.

Let X – no. of heads

Y – no. of head runs ; Z – length of head runs

→ Find the Prob. fun of (i) X (ii) Y (iii) X+Y (iv) XY (v) Z

Sol^n

~~(X=x)~~	P(X=x)	Y	Z	X+Y	XY
HHH	3	1	3	4	3
HHT	2	1	2	3	2
HTH	2	0	0	2	0
THH	2	1	2	3	2
HTT	1	0	0	1	0
TTH	1	0	0	1	0
THT	1	0	0	1	0
TTT	0	0	0	0	0

Prob. dist^n of X

(X=x)	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

Prob. dist^n of Y

(Y=y)	0	1
P(Y=y)	5/8	3/8

Prob. dist^n of Z

(Z=z)	0	1	2	3
P(Z=z)	5/8	0	2/8	1/8

Prob. dist^n of (X+Y)=U

U=u	0	1	2	3	4
P(U=u)	1/8	3/8	1/8	2/8	1/8

Prob. dist^n of (XY=V)

V=v	0	1	2	3
P(V=v)	5/8	0	2/8	1/8

HW (1) From a lot of 12 items containing 3 defective items, a sample of 4 items are drawn at random without replacement. Let X – no. of defective items in the sample. Find Prob dist^n of X.

Sol^n 9 – non-defective
3 – defective
$X = 0, 1, 2, 3$

$P(x=0) = \frac{{}^9C_4}{{}^{12}C_4} = \frac{14}{55}$

$P(x=1) = \frac{{}^9C_3 \cdot {}^3C_1}{{}^{12}C_4} = \frac{28}{55}$

$P(x=2) = \frac{{}^9C_2 \; {}^3C_2}{{}^{12}C_4} = \frac{12}{55}$

$P(x=3) = \frac{{}^9C_1 \; {}^3C_3}{{}^{12}C_4} = \frac{1}{55}$.

$(X=x)$	$P(x=x)$
0	14/55
1	28/55
2	12/55
3	1/55

Que A r.v. X has the foll. Prob. dist^n

x	0	1	2	3	4
$P(x=x)$	c	$2c$	$2c$	c^2	$5c^2$

Find the value of c. Determine the dist^n fun of X.

Sol^n →

$\sum_{x=0}^{4} P(x) = 1 \Rightarrow c + 2c + 2c + c^2 + 5c^2 = 1$

$\Rightarrow 5c + 6c^2 = 1$

$\Rightarrow 6c^2 + 5c - 1 = 0$

$\Rightarrow 6c^2 + 6c - c - 1 = 0$

$\Rightarrow 6c(c+1) - 1(c+1) = 0$

$c = 1/6, -1 \Rightarrow c = 1/6$.

x	0	1	2	3	4
$P(x=x)$	1/6	2/6	2/6	1/36	5/36
$F(x) = P(x \le x)$	1/6	3/6	5/6	31/36	36/36 = 1.

Find $P(x < 3)$ & $P(0 < x < 4)$.