Ex: Let X — is assumed to be a continuous r.v. with p.d.f $(f(x)) = 6x(1-x)$; $0 \le x \le 1$.

(1) Check that $f(x)$ is a p.d.f

(2) Determine b s.t. $P(X<b) = P(X>b)$

Solⁿ:

$$f(x) = 6x(1-x); \quad 0 \le x \le 1$$

$$\Rightarrow f(x) \ge 0; \quad 0 \le x \le 1$$

and $\int_0^1 f(x)dx = 6\int_0^1 x(1-x)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$

$$= 6\left(\frac{1}{2} - \frac{1}{3}\right) = 1.$$

$\Rightarrow f(x)$ is a p.d.f of r.v. X.

(2) $P(X<b) = P(X>b)$

$$\int_0^b f(x)dx = \int_b^1 f(x)\,dx$$

$$\Rightarrow 6\int_0^b (x - x^2)\,dx = 6\int_b^1 (x - x^2)\,dx$$

$$\Rightarrow 6\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = 6\left(\frac{1}{2} - \frac{1}{3} - \frac{b^2}{2} + \frac{b^3}{3}\right)$$

$$\Rightarrow \frac{2b^2}{2} - \frac{2b^3}{3} = \frac{1}{6} \qquad \Rightarrow 2(3b^2 - 2b^3) = 1$$

$$\Rightarrow -4b^3 + 6b^2 - 1 = 0$$

$$\Rightarrow 2b^2\left(\frac{1}{2} - \frac{b}{3}\right) = \frac{1}{6} \qquad \Rightarrow 4b^3 - 6b^2 + 1 = 0$$

$$\Rightarrow (2b-1)(2b^2 - 2b - 1) = 0$$

$$\Rightarrow b = \frac{1}{2}, \frac{1 \pm \sqrt{3}}{2}$$

$$\Rightarrow b = \frac{1}{2} \in [0,1]$$

HW: A cont. R.V. X has p.d.f $f(x) = 3x^2$; $0 \le x \le 1$

Find a and b s.t. (1) $P(X \le a) = P(X > a)$

(2) $P(X > b) = 0.05$.

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Hint $P(x \le a) + P(x > a) = 1.$

$\Rightarrow P(x \le a) = \frac{1}{2}$ and $P(x > a) = \frac{1}{2}$ $\quad [\because P(x \le a) = P(x > a)]$

Using this $a = \left(\frac{1}{2}\right)^{1/3}$ Ans

Que Let X is a Cts R.V. with P.d.f

$$f(x) = \begin{cases} ax & ; 0 \le x \le 1 \\ a & ; 1 \le x \le 2 \\ -ax + 3a & ; 2 \le x \le 3 \\ 0 & ; \text{Elsewhere.} \end{cases}$$

(1) find Constant a (2) Compute $P(x \le 1.5)$

Sol^n

$$\int_{-\infty}^{\infty} f(x) = 1 \Rightarrow \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax+3a)dx = 1$$

$$\Rightarrow a\left(\frac{1}{2}\right) + a(2-1) + \left[-a\left(\frac{3^2}{2}\right) + 3a(3) + a\left(\frac{2^2}{2}\right) - 3a(2)\right] = 1$$

$$\Rightarrow \frac{a}{2} + a - \frac{9a}{2} + 9a + 2a - 6a = 1$$

$$\Rightarrow \frac{13a}{2} - \frac{9a}{2} = 1 \quad \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}.$$

$\Rightarrow \cancel{3a^2 - 13a + 2 = 0}$

(2) $$P(x \le 1.5) = \int_{-\infty}^{1.5} f(x)\,dx$$

$$= \int_0^1 f(x)dx + \int_1^{1.5} f(x)\,dx$$

$$= \int_0^1 ax\,dx + \int_1^{1.5} a\,dx$$

$$= a\left(\frac{1}{2}\right) + a(1.5-1)$$

$$= \frac{a}{2} + 0.5a = a = \frac{1}{2}.$$

Que The Prob. distⁿ of r.v. X is $f(x) = K \sin \frac{1}{5}\pi x;\ 0 \le x \le 5$.

Determine the Constant K. (Ans $K = \frac{\pi}{10}$)

Que Let X is a cts r.v. and its Pdf is $f(x) = Kx^2(1-x^3);\ 0 \le x \le 1$.
Find the value of K. (Ans $K=6$)

Que The time one has to wait at the busstand is a r.v. X with the P.d.f.

$$f(x) = \begin{cases} 0 & ; x < 0 \\ \frac{1}{9}(x+1) & ; 0 \le x < 1 \\ \frac{4}{9}(x-\frac{1}{2}) & ; 1 \le x < 3/2 \\ \frac{4}{9}(\frac{5}{2}-x) & ; \frac{3}{2} \le x < 2 \\ \frac{1}{9}(4-x) & ; 2 \le x < 3 \\ \frac{1}{9} & ; 3 \le x < 6 \\ 0 & ; x \ge 6. \end{cases}$$

Let the Events A: one waits b/w 0 to 2 min Incl
B: one waits b/w 1 to 3 min inclusive.
(1) Find $P(B|A)$
(2) $P(\bar{A} \cap \bar{B})$

Solⁿ $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$

$$P(A) = \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{3/2} f(x)\,dx + \int_{3/2}^2 f(x)\,dx$$

$$= \frac{1}{9}\int_0^1 (x+1)\,dx + \frac{4}{9}\int_1^{3/2}\left(x-\frac{1}{2}\right)dx + \frac{4}{9}\int_{3/2}^{2}\left(\frac{5}{2}-x\right)$$

$$\Rightarrow \frac{1}{9}\left[\frac{x^2}{2}+x\right]_0^1 + \frac{4}{9}\left[\frac{x^2}{2}-\frac{x}{2}\right]_1^{2} + \frac{4}{9}\left[\frac{5x}{2}-\frac{x^2}{2}\right]$$

$$= \frac{1}{9}\left(\frac{1}{2}+1\right) + \frac{4}{9}\left(2-1-\cancel{\tfrac{1}{2}}+\cancel{\tfrac{1}{2}}\right)$$

$$+ \frac{4}{9}\left[5-2-\frac{15}{4}+\frac{9}{8}\right] = \frac{1}{2}.$$

$P(A \cap B)$

$= P[(0 \le x \le 2) \cap (1 \le x \le 3)]$

$= P[1 \le x \le 2]$

$= \int_1^2 f(x)\,dx = \frac{1}{3}$ (Solve it)

$P(B|A) = \dfrac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$

(2) $\bar{A} \cap \bar{B}$: Waiting time is more than 3 minutes.

$\Rightarrow P(\bar{A} \cap \bar{B}) = P(X \geq 3)$

$= \int_3^6 \frac{1}{9}\,dx = \frac{1}{9}(3) = \frac{1}{3}.$

Que X – cts r.v.

$$f(x) = \begin{cases} kx; & 0 \le x < 5 \\ k(10-x); & 5 \le x < 10 \\ 0; & \text{otherwise.} \end{cases}$$

(1) Find the value of K s.t. $f(x)$ is a p.d.f. (Ans $k = \frac{1}{25}$)

(2) $A: P(5 \le X \le 10) = ?$

$B: P(X < 5) = ?$

$C: P(2.5 \le X \le 7.5) = ?$

Find $P(A \cap B)$ and $P(A \cap C)$

Check If A and B & A and C are Independent.

Que $$f(x) = \begin{cases} \frac{1}{20} e^{-x/20}; & x > 0 \\ 0; & x \le 0 \end{cases}$$ for X to be cts r.v.

Find (1) $P(X \le 10)$

(2) $P(16 \le x \le 24)$

(3) $P(x \geq 30)$.

Continuous distⁿ fun:→ Let X be a cts r.v. with P.d.f $f(x)$ then the function
$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt; \quad -\infty < x < \infty.$$
is called distⁿ fun or Cummulative distⁿ fun (c.d.f) of r.v. X.

(1) $F'(x) = \frac{d}{dx}(F(x)) = f(x) \geq 0 \quad (\because f(x) \text{ is P.d.f})$

$$P(a \le X \le b) = \int_{a}^{b} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx = F(b) - F(a)$$

Que Verify that the following is a distⁿ function.
$$F(x) = \begin{cases} 0 & ; \quad x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & ; \quad -a \le x \le a \\ 1 & ; \quad x > a. \end{cases}$$

Solⁿ→
$$\frac{d}{dx}(F(x)) = \begin{cases} \frac{1}{2a} & ; \quad -a \le x \le a \\ 0 & ; \quad x > a \end{cases} = f(x)$$

If $F(x)$ is a distⁿ function then $f(x)$ must be a P.d.f.

$\therefore \int_{-\infty}^{\infty} f(x) = 1$ (To prove)

Now $\int_{-\infty}^{\infty} f(x) = \int_{-a}^{a} f(x)\,dx = \frac{1}{2a}(a+a) = 1.$

$\Rightarrow F(x)$ is a distⁿ function.

Que X — r.v.

P.d.f is $f(x) = 6x(1-x); \; 0 \le x \le 1.$

(1) Find C.d.f. of X

(2) $P\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{3} \le X \le \frac{2}{3}\right)$

(3) find k s.t. $P(X < k) = P(X > k)$

Sol^n $f(x) = \begin{cases} 0 ; & x < 0 \\ 6x(1-x) ; & 0 \le x \le 1 \\ 0 ; & x > 1 \end{cases}$

For any x; $x \le 0$; $F(x) = P(X \le x)$

$$= \int_{-\infty}^{0} f(x)\,dx = 0$$

For any x; $0 < x \le 1$; $F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$

$$= 6\int_{0}^{x} x(1-x)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^x$$

$$= 3x^2 - 2x^3$$

For any x; $x > 1$; $F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\cancel{\infty}} f(x)\,dx$

or $1 < x < \infty$

$$= 1$$

$$\therefore F(x) = \begin{cases} 0 & ; x \le 0 \\ 3x^2 - 2x^3 & ; 0 < x \le 1 \\ 1 & ; x > 1. \end{cases}$$

Que:→ X — cts r.v. with P.d.f.

$$f(x) = \begin{cases} kx & ; 0 \le x < 1 \\ k & ; 1 \le x < 2 \\ -kx + 3k ; & 2 \le x < 3 \\ 0 & ; \text{o/w.} \end{cases}$$

Find k and
Find its C.d.f.

($k = \frac{1}{2}$) Solve.

Sol^n for any x, $x < 0$,

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = 0$$

for any x, $0 \le x < 1$

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= \int_{0}^{x} kx\,dx = \frac{kx^2}{2} = \frac{x^2}{4}.$$

For any x; $1 \le x < 2$

$$F(x) = P(X \le x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= \left(\frac{x^2}{4}\right)_0^1 + \int_1^x k\,dx$$

$$= \frac{1}{4} + \frac{1}{2}(x-1)$$

$$= -\frac{1}{4} + \frac{x}{2} = \frac{2x-1}{4}$$

for any x; $2 \le x < 3$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \int_2^x \left(\frac{-x}{2} + \frac{3}{2}\right) dx$$

$$= \frac{1}{4} + \frac{1}{2} + \left[-\frac{x^2}{4} + \frac{3x}{2}\right]_2^x$$

$$\cancel{= \frac{3}{4} + \left[\frac{-9}{4} + \frac{9x}{2} + 1 - 3\right]}$$

$$= \frac{3}{4} - \frac{x^2}{4} + \frac{3}{2}x + 1 - 3$$

$$= \frac{-5}{4} + \frac{3}{2}x - \frac{x^2}{2}$$

for any x; $x \ge 3$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$= 1$$

$$\therefore F(x) = \begin{cases} 0 & ; \quad x < 0 \\ x^2/4 & ; \quad 0 \le x < 1 \\ (2x-1)/4 & ; \quad 1 \le x < 2 \\ -x^2/2 + \frac{3}{2}x - 5/4 & ; \quad 2 \le x < 3 \\ 1 & ; \quad x \ge 3 \end{cases}$$

→ Expected value of a random variable :⇒

$$E(x) = \sum_x x f(x) \quad \text{— (for discrete R.V)}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x) \quad \text{— for Cts R.V.}$$

→ $E(a) = a$ — (a Constant)

→ $E(ax) = aE(x)$

→ $E(ax+b) = aE(x) + b$.

→ $var(x) = E(x-\bar{x})^2 = E(x-E(x))^2$

$= E(x^2+\bar{x}^2-2\bar{x}\cdot x)$

$= E(x^2) + E(\bar{x}^2) - 2\bar{x}E(x)$

$= E(x^2) + \bar{x}^2 - 2\bar{x}\cdot\bar{x}$

$= E(x^2) + \bar{x}^2 - 2\bar{x}^2$

$= E(x^2) - (E(x))^2$

→ $Var(ax) = E(a(x-\bar{x}))^2$

$= a^2E(x-\bar{x})^2 = a^2 var(x)$

→ $Var(b) = 0$ ($\because$ b is constant)

Ex Let X be R.V. with prob. $dist^n$

x	-3	6	9
$P(X=x)$	$1/6$	$1/2$	$1/3$

Find $E(x)$, $E(x^2)$ and $E(2x+1)^2$

Sol^n $E(x) = \sum x\, p(X=x)$

$= -3\times\frac{1}{6} + 6\times\frac{1}{2} + 9\times\frac{1}{3} = \frac{11}{2}$.

$E(x^2) = \sum x^2\cdot p(x) = 9\times\frac{1}{6} + 36\times\frac{1}{2} + 81\times\frac{1}{3}$

$= \frac{93}{2}$.

$E(2x+1)^2 = E(4x^2+1+4x) = 4E(x^2) + 1 + 4E(x)$

$= 4\times\frac{93}{2} + 1 + 4\times\frac{11}{2}$

$= 209$.

Que:→ Find the Exp. of the no. of on a die when thrown.

Sol^n X — no. on die (R.V)

$X = 1, 2, 3, 4, 5, 6$

$P(X = x_i) = \frac{1}{6} \quad \forall i$

$E(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \text{—} + 6 \times \frac{1}{6} = \frac{7}{2}.$

Que Two unbiased dice are thrown. Find the Exp value of Sum of no. of points on them

Sol^n

x	2	3	—	12
$P(X=x)$	1/36	2/36	—	1/36

$E(x) = \sum x_i p_i = 7.$

Que $P(x) = e^{-t}(1-e^{-t})^{x-1}$, for R.V. $X = 1, 2,$ —
(Prob fun of R.V. X)

$E(x) = \sum x . P(x) = e^{-t} \sum x . (1-e^{-t})^{x-1}$

$= e^{-t} \sum x\, a^{x-1}$; $a = 1-e^{-t}$

$= e^{-t}[1 + 2a + 3a^2 + \text{—}]$

$= e^{-t}(1-a)^{-2}$

$= e^{-t}(e^{-t})^{-2} = e^{t}$

$E(x^2) = \sum x^2 P(x) = e^{t}(2-e^{-t}) \cancel{A}\, e^{3t}$

$V(x) = e^{t}(e^{t}-1)$

Discrete Prob. Distⁿ :→

(1) **Uniform or Rectangular Distⁿ :→** A r.v. X is said to have a uniform distⁿ over the range [1,n] if its p.m.f is given by

$$P(X=x) = \begin{cases} \frac{1}{n} ; & x=1,2,\ldots n \\ 0 ; & o/w. \end{cases}$$

n – parameter of distⁿ

→ Mean $= E(x) = \sum x \cdot p(x) = \sum_{i=1}^{n} i \cdot P(i)$

$$= 1\times\frac{1}{n} + 2\times\frac{1}{n} + \ldots + n\times\frac{1}{n}$$

$$= \frac{1}{n}[1+2+\ldots+n] = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

$$E(x^2) = \sum x^2 \cdot p(x) = \frac{1}{n}\sum x^2 = \frac{1}{n}[1^2+\ldots+n^2]$$

$$= \frac{1}{\cancel{n}}\left[\frac{\cancel{n}(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6}.$$

$$V(x) = E(x^2) - (E(x))^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{(n+1)(n-1)}{12}.$$

(2) **Bernaulli Distⁿ :→** A r.v. X is said to have a Bernoulli Distⁿ if its P.m.f. is given by

$$P(X=x) = \begin{cases} p^x(1-p)^{(1-x)} ; & x=0,1 \\ 0 ; & o/w. \end{cases}$$

p – parameter of distⁿ

→ Mean $= E(x) = \sum x \cdot P(X=x)$

$$= \sum_{x=0,1} x \cdot p^x(1-p)^{1-x} = 0\cdot(1-p) + 1\cdot p = p.$$

$$E(x^2) = \sum_{x=0,1} x^2 \cdot P(X=x)$$

$$= 0\cdot p^0(1-p) + 1^2\cdot p(1-p)^0 = p.$$

$Var(x) = E(x^2) - (E(x))^2 = p - p^2 = pq.$

(3) **Binomial Distⁿ:** → Consider a set of n Independent Bernoulli trials — p is the prob. of Success

$(1-p) = q$ is the prob. of failure.

$P(x$ success, $(n-x)$ failures in n Independent trials$) =$

$$^nC_x\, p^x (1-p)^{n-x} = (p+q)^n$$

x — Can assume only non-negative values.

→ So A r.v. X is said to follow a binomial distⁿ if its p.m.f is given by

$$P(X=x) = \begin{cases} ^nC_x\, p^x (1-p)^{n-x} & ; x = 0,1,\ldots n \\ 0 & ; o/w. \end{cases}$$

n, p — parameters of distⁿ.

→ Imp (1) Each trial results in 2 Exhaustive and mutually Exclusive out comes. (S and F).

(2) n — Finite

(3) Trials are Independent of each other.

(4) p — Constant.

Ex. 10 Coins are thrown Simultaneously. Find the Prob. of getting atleast 7 heads.

Solⁿ

Prob of head $= 1/2 = p$

Prob of tail $= 1/2 = q.$

Prob of x heads in 10 trials $= \binom{10}{x} p^x . q^{10-x}$

$$P(x \geq 7) = P(x=7) + P(x=8) + P(x=9) + P(x=10)$$

$$= {}^{10}C_7\, p^7 q^3 + {}^{10}C_8\, p^8 q^2 + {}^{10}C_9\, p^9 q^1 + {}^{10}C_{10}\, p^{10}$$

$$= \left(\frac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + 1\right] = \frac{176}{1024}.$$

$$\text{Mean} = E(x) = \sum_{x=0}^{n} x \cdot p(x)$$

$$= \sum x \cdot {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

$$= \sum x \cdot \frac{n!}{x!\,(n-x)!}\, p^{x} (1-p)^{n-x}$$

$$= \sum \frac{n!}{(x-1)!\,(n-x)!}\, p^{x} (1-p)^{n-x}$$

$$= np \sum \frac{(n-1)!}{(x-1)!\,(n-x)!}\, p^{x-1} (1-p)^{n-x}$$

$$= np \cdot \sum \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}$$

$$= np \cdot (q+p)^{n-1} = np$$

$$E(x^2) = \sum x^2 \cdot p(x) = \sum x^2 \cdot {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

$$= \cancel{\sum \frac{x(n)!}{x(x-1)!}}\ \sum [x(x-1) + x]\, {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$$

$$V(x) = npq$$

$$= \sum \cancel{x(x-1)} \frac{n(n-1)(n-2)!}{\cancel{x(x-1)}(x-2)!}\, p^{x} \cdot (1-p)^{n-x}$$

$$\left[\because V(x) = E(x^2) - (E(x))^2 \right.$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$+ \sum x\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\left. = np(1-p) = npq \right]$$

$$= n(n-1)p^2 \left[\sum \binom{n-2}{x-2} p^{x-2} (1-p)^{n-x} \right] + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$= n(n-1)p^2 + np$$

Que A and B play a game in which their chances of winning are in ratio 3:2. Find A's chances of winning at least 3 games out of 5 games.

Sol^n $P(\text{A's winning}) = 3/5$; $P(\text{B's winning}) = 2/5$

$(\because n=5)$

So for A, $p = 3/5$, $q = 2/5$

$P(\text{winning } x \text{ games out of } 5) = 5C_x\, p^x q^{5-x}$

$P(x \geq 3) = P(x=3) + P(x=4) + P(x=5)$

$= 5C_3\, p^3 . q^2 + 5C_4\, p^4 q + 5C_5\, p^5$

$= 5C_3 . \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 + 5C_4 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right) + 5C_5 \left(\frac{3}{5}\right)^5$

$= \left(\frac{1}{5}\right)^5 \left[10 . 27 \times 4 + 5 . 3^4 \times 2 + 3^5\right]$

$\simeq 0.68$

Que Comment on the following:-

Mean of the Binomial dist^n is 3 and var = 4

Sol^n Mean $= np = 3$; var $= npq = 4$

So $\frac{npq}{np} = \frac{4}{3} \Rightarrow q = \frac{4}{3}$ X $(\because \text{Prob} \not> 1)$

Que Mean $= 4$, Var $= 4/3$

Find $P(x \geq 1)$.

Sol^n $np = 4$; $npq = 4/3$

$q = \frac{4}{12} = \frac{1}{3} \Rightarrow p = 2/3$

$\Rightarrow n = 6$

$P(x \geq 1) = 1 - P(x < 1) = 1 - P(x=0)$

$= 1 - 6C_0\, p^0 . q^6 = 1 - \left(\frac{1}{3}\right)^6 = 0.998$

Que The following data shows the result of throwing 12 fair dice 4096 times, a throw of 4, 5, or 6 being called success.
Fit a binomial distⁿ and find the Expected frequencies.

Que If $X \sim B(n,p)$ show that

$$E\left(\frac{X}{n}-p\right)^2 = \frac{pq}{n}$$

Soln $X \sim B(n,p) \Rightarrow E(X) = np;\ V(X) = npq$

$$E\left(\frac{X}{n}\right) = \frac{1}{n}\cdot np = p$$

$$V\left(\frac{X}{n}\right) = \frac{1}{n^2}V(X) = \frac{pq}{n}.$$

$$E\left(\frac{X}{n} - E\left(\frac{X}{n}\right)\right)^2 = V\left(\frac{X}{n}\right) = \frac{pq}{n}.$$

Que A set of three similar coins are tossed 100 times with the following results.

no. of heads	0	1	2	3	
frequency	36	40	22	2	$=\boxed{100}$

Fit a binomial distⁿ and estimate the expected frequencies.

Soln →

X	f	fx
0	36	0
1	40	40
2	22	44
3	2	6
	100	90

$n=3$
$N=100.$

$$\bar{X} = \frac{\Sigma fx}{\Sigma f} = 0.9.$$

$$np = \bar{X} \Rightarrow p = \frac{\bar{X}}{n} = \frac{0.9}{3} = 0.3.$$

$$q = 1-p = 0.7$$

classmate

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$$P(X=x) = {}^nC_x\, p^x (1-p)^{n-x}$$
$$= {}^3C_x (0.3)^x (0.7)^{n-x}$$
$$P(0) = {}^3C_0 (0.3)^0 (0.7)^{3-0} = 0.343$$

Now $$\frac{P(x+1)}{P(x)} = \frac{{}^nC_{x+1}\, p^{x+1} (1-p)^{n-(x+1)}}{{}^nC_x\, p^x (1-p)^{n-x}}$$
$$= \frac{\dfrac{n!}{(x+1)!\,(n-x-1)!}}{\dfrac{n!}{x!\,(n-x)!}} \cdot \frac{p}{q}$$
$$= \left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right)$$
$$\Rightarrow P(x+1) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right) \cdot P(x)$$

$$P(1) = \left(\frac{3-0}{0+1}\right)\left(\frac{p}{q}\right) P(0) = 3\left(\frac{0.3}{0.7}\right)(0.7)^3$$
$$= 3 \times 0.3 \times (0.7)^2 = 44.247$$

$$P(2) = \left(\frac{3-1}{1+1}\right)\left(\frac{p}{q}\right) P(1) = \left(\frac{0.3}{0.7}\right) \times 3 \times (0.3)(0.7)^2$$
$$= 19.03$$

$$P(3) = \left(\frac{3-2}{2+1}\right)\left(\frac{p}{q}\right) P(2) = \left(\frac{1}{3}\right)\left(\frac{0.3}{0.7}\right) \times 19.03 = 2.727.$$

X	Obs. Fr.	Exp. fr.
0	36	~~34~~ $N \times P(0) = 34$
1	40	$N \times P(1) = 44$
2	22	$N \times P(2) = 19$
3	2	$N \times P(3) = 3$
	100	100

Poisson distribution: → It is a limiting case of the binomial distⁿ under the following conditions:-

(1) $n \to \infty$ i.e. no. of trials are large.

(2) $p \to 0$ i.e. Prob of success is very small.

(3) $np = \lambda$ (say) – finite.

$$B(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \binom{n}{x}\left(\frac{p}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)!}}{x!\,\cancel{(n-x)!}}\left(\frac{p}{1-p}\right)^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}\,\frac{(\lambda/n)^x}{\left(1-\frac{\lambda}{n}\right)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\cancel{n^x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!}\,\frac{\frac{\lambda^x}{\cancel{n^x}}}{\left(1-\frac{\lambda}{n}\right)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty} B(x;n,p) = \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!}\,\frac{\lambda^x}{\left(1-\frac{\lambda}{n}\right)^x}\left(1-\frac{\lambda}{n}\right)^n$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}\ ; \quad x = 0,1,2,\ldots$$

→ A r.v. X is said to follow a poisson distⁿ if its P.m.f. is given by

$$P(X=x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} ; & x = 0,1,2,\ldots,\ \lambda > 0. \\ 0 ; & o/w \end{cases}$$

λ – Parameter of the distⁿ.

→ $$\sum_{x=0}^{\infty} P(X=x) = \sum_{x=0}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}\cdot e^{\lambda} = 1$$

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Mean :- $E(x) = \sum_{x=0}^{\infty} x.P(X=x)$

$$= \sum_{x=0}^{\infty} x.\frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}.e^{\lambda} = \lambda.$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2. P(X=x) = \sum_{x=0}^{\infty} x^2.\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x^2\lambda^x}{x!}$$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} \frac{[x(x-1)+x]\lambda^x}{x!}\right]$$

$$= e^{-\lambda}\left[\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}\right]$$

$$= e^{-\lambda}\left[\lambda^2.e^{\lambda} + \lambda e^{\lambda}\right]$$

$$= \lambda^2+\lambda$$

$$V(x) = E(x^2) - (E(x))^2 = \cancel{\lambda^2}+\lambda-\cancel{\lambda^2} = \lambda.$$

Que 3 Coins are tossed 800 times. Find the prob of getting 3 heads r times.

Sol^n →
$n = 800$

p = prob of head $= \frac{1}{2}$;

Prob. of getting 3 heads (when 3 Coins are tossed Single time)
$$= \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

So n – very large
p – very small

$\lambda = np = 800 \times \frac{1}{8} = 100.$

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$$P(X=r) = \frac{e^{-\lambda}\lambda^r}{r!} = \frac{e^{-100}(100)^r}{r!};\ r=0,1,2,\ldots$$

Que In a book of 520 pages, 390 typo Errors occur. Find the Prob. that a random sample of 5 pages has no error.

Sol^n $n = 520$ pages.

The average no. of typo errors per page $= \frac{390}{520} = 0.75$.

So $\lambda = 0.75$.

$$P(x \text{ errors per page}) = P(X=x) = \frac{e^{-0.75}(0.75)^x}{x!}$$

$$\Rightarrow P(X=0) = \frac{e^{-0.75}(0.75)^0}{0!} = e^{-0.75}$$

$$\Rightarrow P(\text{no error in 5 pages}) = [P(X=0)]^5 = (e^{-0.75})^5 = e^{-3.75}$$

Que $X \sim P(\lambda)$ s.t. $P(X=2) = 9P(X=4) + 90P(X=6)$.

Find (1) λ (2) $E(X)$.

Sol^n $X \sim P(\lambda) \Rightarrow P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!};\ x=0,1,2,\ldots,\ \lambda > 0$.

$$\Rightarrow P(X=2) = \frac{e^{-\lambda}\lambda^2}{\lfloor 2}$$

$$P(X=4) = \frac{e^{-\lambda}\lambda^4}{4!}$$

$$P(X=6) = \frac{e^{-\lambda}\lambda^6}{6!}$$

$$P(X=2) = 9P(X=4) + 90P(X=6)$$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = 9 \times \left(\frac{e^{-\lambda}\lambda^4}{4!}\right) + 90 \times \left(\frac{e^{-\lambda}\lambda^6}{6!}\right)$$

$$\Rightarrow \frac{1}{2} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!}$$

$$\Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$$\Rightarrow \boxed{\lambda = 1}$$

Fitting of Poisson distⁿ :-

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}; \quad x = 0, 1, \ldots \infty$$

$$P(x+1) = \frac{e^{-\lambda} \lambda^{x+1}}{(x+1)!}; \quad x = 0, 1, 2 \ldots \infty$$

$$\Rightarrow \frac{P(x+1)}{P(x)} = \frac{\lambda}{(x+1)} \Rightarrow P(x+1) = \frac{\lambda}{(x+1)} P(x).$$

$$\Rightarrow P(1) = \frac{\lambda}{1} \cdot P(0) = \lambda P(0)$$

$$P(2) = \frac{\lambda}{2} P(1) = \frac{\lambda^2}{2} P(0) \text{ and so on.}$$

Que Fit a Poisson distⁿ to the following data.

no. of mistakes per page	0	1	2	3	4	Total
no. of pages	109	65	22	3	1	$\boxed{200}$

x	f	fx
0	109	0
1	65	65
2	22	44
3	3	9
4	1	4
	200	122

$$\bar{x} = \lambda = \frac{\Sigma fx}{\Sigma f} = \frac{122}{200} = 0.61.$$

$$P(x \text{ mistakes per page}) = \frac{e^{-0.61} \cdot (0.61)^x}{x!}$$

$$P(0) = e^{-0.61} = 0.5432. \Rightarrow N \times P(0) \approx 108$$

$$P(1) = \lambda \cdot P(0) = 0.61 \times 0.5432 \Rightarrow N \times P(1) \approx 66$$

$$N \times P(2) \approx 20$$

$$N \times P(3) \approx 4$$

$$N \times P(4) \approx 1$$