Ex. $T: P_2(R) \to P_3(R)$ is a L.T s.t

$$T(f(x)) = 2f'(x) + \int_0^3 3f(t)\,dt$$

find $R(T)$. Prove that T is not onto but it is one-one.

Soln. $R(T) = \text{Span}(T(\beta))$ where $\beta = \{1, x, x^2\}$ is a standard basis of $V = P_2(R)$

$= \text{Span}(\{T(1), T(x), T(x^2)\})$

$= \text{Span}(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$

i.e, $\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$ form a basis of $R(T)$

$\therefore \dim R(T) = 3$

$\Rightarrow R(T) \neq W$ [as $\dim W = 4$]

$\therefore T$ is not onto.

$\text{Nullity}(T) + \text{Rank}(T) = \dim P_2(R)$

$\Rightarrow \text{Nullity}(T) = 0$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow N(T) = \{0\}$ alone

$\Rightarrow T$ is 1-1.

Ex. $T: F^2 \to F^2$ is a L.T s.t

$T(a_1, a_2) = (a_1 + a_2, a_1)$

$N(T) = \{(0,0)\}$

$\therefore T$ is 1-1

Here, $\dim F^2 = \dim F^2$

& F^2 is f.d.

$\therefore$ from theorem, T is onto.

Th. V and W are vector space over the same field F. Let V be ~~f.d~~ finite dimensional and let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $w_1, w_2, \ldots, w_n$ be elements in W, then $\exists$ a unique linear transformation $T: V \to W$ satisfying $T(v_i) = w_i$, $i = 1$ to n.

Cor. V and W are vector space both over field F. Let V be finite dim. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Suppose U & T are 2 L.T from V to W s.t

$U(v_i) = T(v_i)$, $i = 1$ to n

then $U = T$

(Contd)

Ex: $T: R^2 \to R^2$ is a L.T

$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$

Let $U: R^2 \to R^2$ be another L.T

Also given $U(1, 2) = (3, 3)$

& $U(1, 1) = (1, 3)$

show that $U = T$

Soln:

$T(1, 2) = (3, 3) = U(1, 2)$

& $T(1, 1) = (1, 3) = U(1, 1)$

we can check that $\{(1, 2), (1, 1)\}$ is a basis of

Hence, by the Cor. $U = T$.

Ex: $T: P_2[R] \to R^3$ is a L.T

s.t $T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$

Also T is 1-1

Soln. Let $S = \{2 - x_0 + 3x^2, x + x^2, 1 - 2x^2)$

then S is a subset of $P_2(R)$

Check whether S is L.I

Now $T(S) = \{T(2 - x - 3x^2), T(x + x^2), T(1 - 2x^2)\}$

$T(S) = \{(2, -1, 1), (0, 1, 1), (1, 0, -2)\}$.

check if $T(S)$ is L.I.

then S is also L.I.