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$$a_0(x)\, y^n + a_1(x)\, y^{n-1} + \_\_ + a_n(x) y = Q(x) \quad -(1)$$

where $a_0(x), a_1(x) \_\_, a_n(x)$ & $Q(x)$ are functions of x only. is called a linear O.D.E of order n.

→ If $Q(x) = 0$ Then Equ^n (1) is called homogeneous linear differential Equ^n otherwise it is called non-hom. Equ^n.

→ If $a_i(x)$; $i=1, \_\_ n$ are Constants then (1) is called linear diff. Equ^n with Constant Coefficients.

→ If $\exists\, a_i(x)$ s.t. $a_i(x)$ is a non-Constant function of x Then (1) is called Linear diff Equ^n with variable Coefficients.

E.g. $y'' + 4y' + 3y = x^2 e^x$ — Non-Hom, Second order Linear diff Equ^n with Constant

$x^2 y'' + xy' + (x^2 - 4)y = 0$ → Hom, Second order with variable coefficient.

$y'' + xy' = e^x$ → Non-hom, Second order with var. Coeff.

→ (1) Can be written as

$$F(D)\, y = Q(x)$$

Where $F(D) = a_0(x) D^n + a_1(x) D^{n-1} + \_\_ + a_{n-1}(x) D + a_n(x)$.

Here, D is a differential operator. , $D = \frac{d}{dx}$

$$D(f(x)) = \frac{df}{dx} = f'.$$

E.g. $D(x^n) = n x^{n-1}$. ; $D(\sin x) = \cos x$.

If f and g are two diff function Then

$$D(f+g) = D(f) + D(g)$$

$$D(\alpha f) = \alpha D(f).$$

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i.e. D is a linear operator.

So any Second order linear diff. Eqn

Eg $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = x^2$

Can be written as $(D^2 + 2D + 2)y = x^2$.

i.e. $F(D)y = x^2$

where $F(D) = D^2 + 2D + 2$.

Solution of 2nd order Hom linear diff Eqn with Constant Coeff:-

Consider $ay'' + by' + cy = 0$, a, b, c are Constants. — (1)

or $F(D)y = 0$ where $F(D) = (aD^2 + bD + c)$

So the auxiliary Eqn or Char. Eqn is

$am^2 + bm + c = 0$ (Replace D by m).

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \text{are char. Roots.}$$

(1) If $b^2 - 4ac > 0$ Then roots are real and distinct
(2) If $b^2 - 4ac = 0$, Then roots are real and equal.
(3) If $b^2 - 4ac < 0$, Then roots are Complex.

(1) If $b^2 - 4ac > 0$:- $m = m_1$ and $m = m_2$.

Then gen Soln of (1) is $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

$\left[\because \begin{array}{ll} y' - my = 0; & y' + my = 0 \\ y = e^{mx} + c & y = e^{-mx} + c \end{array} \right]$

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(2) $b^2-4ac=0$ $\quad m=m_1=m_2$.
Then gen sol^n is $y(x) = c_1e^{mx}+c_2xe^{mx}$.

(3) If $b^2-4ac<0$ $\quad m = p\pm iq$

gen sol^n is $y = e^{px}(c_1\cos qx + c_2\sin qx)$

$\because y = c_1e^{(p+iq)x} + c_2e^{(p-iq)x}$

$= c_1e^{px}.e^{iqx} + c_2e^{px}e^{-iqx}$

$= e^{px}[c_1(\cos qx+i\sin qx)+c_2(\cos qx - i\sin qx)$

$= e^{px}[c_1\cos qx + c_2\sin qx]$

Que $y''-y'-6y=0$
Sol^n $(D^2-D-6)y=0$

Char Equ^n is $m^2-m-6=0$

$\Rightarrow m^2-3m+2m-6=0$

$\Rightarrow m(m-3)+2(m-3)=0$

$\Rightarrow m=3,-2$

$y(x) = c_1e^{3x}+c_2e^{-2x}$

Que $4y''-8y'+3y=0$
Sol^n $(4D^2-8D+3)y=0$

Char equ^n is $4m^2-8m+3=0$

$\Rightarrow 4m^2-2m-6m+3=0$

$\Rightarrow 2m(2m-1)-3(2m-1)=0$

$\Rightarrow m=1/2, 3/2$

$y(x) = c_1e^{1/2x}+c_2e^{3/2x}$

Qu

Que $4y'' + 4y' + y = 0$

Solⁿ $(4D^2 + 4D + 1)y = 0$

Char. Equⁿ is $4m^2 + 4m + 1 = 0 \Rightarrow (2m+1)^2 = 0$

$\Rightarrow m = -1/2, -1/2$

$\Rightarrow y(x) = c_1 e^{-x/2} + c_2 x e^{-x/2}$

Que $y'' - 4y' - 5y = 0$

Solⁿ $(D^2 - 4D - 5)y = 0$

char. Equⁿ is $m^2 - 4m - 5 = 0$

$\Rightarrow m^2 + m - 5m - 5 = 0$

$\Rightarrow m(m+1) - 5(m+1) = 0$

$\Rightarrow m = 5, -1$

$y(x) = c_1 e^{5x} + c_2 e^{-x}$

Que $y'' + 2y' + 2y = 0$

Solⁿ $(D^2 + 2D + 2)y = 0$

Char Equⁿ is $m^2 + 2m + 2 = 0$

$\Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2}$

$= \frac{-2 \pm 2i}{2} = -1 \pm i$

$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$

Que $y'' + 4y' + 13y = 0$

Solⁿ $(D^2 + 4D + 13)y = 0$

char. Equⁿ is $m^2 + 4m + 13 = 0$

$m = \frac{-4 \pm \sqrt{16-52}}{2}$

$= -2 \pm 3i$

$y(x) = e^{-2x}[c_1 \cos 3x + c_2 \sin 3x]$

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Higher order hom. diff Equ^n with Constant Coeff :→

$$a_0 y^n + a_1 y^{n-1} + \_\_ + a_n y = 0.$$

(1) Real and distinct roots :→ let $m_1, m_2, \_\_ m_n$ be n roots.

Gen Sol^n is $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \_\_ + C_n e^{m_n x}$

(2) Real and Repeated roots :→ let $m = m_1 = m_2 = \_\_ = m_n$

then Gen Sol^n is $y(x) = C_1 e^{m_0 x} + C_2 x e^{mx} + C_3 x^2 e^{mx} + \_\_ + C_n x^n e$

(3) Complex Roots :→ $p_1 + iq_1, \; p_2 + iq_2, \_\_, p_k + iq_k$

then Gen Sol^n is $y(x) = e^{p_1 x}(C_1 \cos q_1 + C_2 \sin q_1 x)$

$+ e^{p_2 x}(C_3 \cos q_2 x + C_4 \sin q_2 x) + \_\_\_$

$+ e^{p_k x}(C_k \cos q_k x + C_k \sin q_k x)$

Que $y''' - 2y'' - 5y' + 6y = 0$

Sol^n $(D^3 - 2D^2 - 5D + 6)y = 0$

Char. equ^n is $m^3 - 2m^2 - 5m + 6 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$m^2 - 1m - 6 = 0$

$\cancel{\Rightarrow m = \frac{6 \pm \sqrt{36+24}}{2} = \frac{6 \pm \sqrt{60}}{2}}$

$m^2 - 3m + 2m - 6 = 0$

$m(m-3) + 2(m-3) = 0$

$m = 3, -2$

$$y(x) = C_1 e^{3x} + C_2 e^{-2x} + C_3 e^{x}.$$

Que $y''' - y'' - 4y' + 4y = 0.$

Sol^n $(D^3 - D^2 - 4D + 4)y = 0$

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Char. Equ^n is $m^3 - m^2 - 4m + 4 = 0$

$\Rightarrow m^2(m-1) - 4(m-1) = 0$

$\Rightarrow (m-1)(m^2-4) = 0$

$m = 1, 2, -2$

$y(x) = c_1e^x + c_2e^{2x} + c_3e^{-2x}$.

Qu $y^{4} - 5y^2 + 4y = 0$.

Sol^n $(D^4 - 5D^2 + 4)\, y = 0$

Ch. Equ^n is $m^4 - 5m^2 + 4 = 0$

$\Rightarrow (m^2-4)(m^2-1) = 0$

$\Rightarrow m = \pm 1, \pm 2$

$y(x) = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x}$.

Qu $4y^{iv} - 12y''' - y'' + 27y' - 18y = 0$

Sol^n $(4D^4 - 12D^3 - D^2 + 27D - 18)\, y = 0$.

Char. Equ^n is $4m^4 - 12m^3 - m^2 + 27m - 18 = 0$

$m = 1$,

1	4	-12	-1	27	-18
		4	-8	-9	18
	4	-8	-9	18	0

$4m^3 - 8m^2 - 9m + 18 = 0$

$4m^2(m-2) - 9(m-2) = 0$

$m = 2,\ m = \pm 3/2$

$y(x) = c_1e^x + c_2e^{2x} + c_3e^{3x/2} + c_4e^{-3/2x}$.

Qu $y''' - 3y' - 2y = 0$

Sol^n $(D^3 - 3D - 2)y = 0$

Char Equ^n is $m^3 - 3m - 2 = 0$

$m = -1$

-1	1	0	-3	-2
		-1	1	2
	1	-1	-2	$\boxed{0}$

$m^2 - m - 2 = 0 \Rightarrow m^2 - 2m + m - 2 = 0$

$m(m-2) + 1(m-2) = 0$

$\Rightarrow (m+1)(m-2) = 0$

$\Rightarrow m = -1, -1, 2$

$y(x) = c_1 e^{2x} + c_2 e^{-x} + c_3 x e^{-x}$.

$= c_1 e^{2x} + (c_2 + c_3 x) e^{-x}$.

Que $8y''' - 12y'' + 6y' - y = 0$

Solⁿ $(8D^3 - 12D^2 + 6D - 1) y = 0$

Char. Equⁿ $8D^3 - 12D^2 + 6D - 1 = 0$.

$\Rightarrow (2m-1)^3 = 0$

$\Rightarrow m = 1/2 ; 1/2, 1/2$.

$y(x) = (c_1 + c_2 x + c_3 x^2) e^{x/2}$.

Hw Que $y''' + 3y'' - 4y = 0$

Que $y^{iv} + 5y'' + 4y = 0$

Solⁿ $(D^4 + 5D^2 + 4) y = 0$

Char. Equⁿ is $m^4 + 5m^2 + 4 = 0$

$\Rightarrow m^4 + 4m^2 + m^2 + 4 = 0$

$\Rightarrow m^2(m^2+4) + 1(m^2+4) = 0$

$\Rightarrow m = \pm i, \pm 2i$

$y(x) = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$.

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HW Qu $y^{iv} + 2y''' + 11y'' + 18y' + 18 = 0$.

Qu $y^{iv} + 32y'' + 256y = 0$

Solⁿ $(D^4 + 32D^2 + 256)y = 0$

Char. Equⁿ is $m^4 + 32m^2 + 256 = 0$

$\Rightarrow (m^2+16)^2 = 0$

$\Rightarrow m = \pm 4i, \pm 4i$

$y(x) = (C_1 + C_2x)\cos 4x + (C_3 + C_4x)\sin 4x$.