Fourier series → Expansion of a function f(x) in a series of sine & cosines of multiplies of x. ~~was developed by~~ 19

Fourier series for the function f(x) in the interval $c < x < c+2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

This is Euler's formulae & a_0, a_n, b_n are called fourier coefficients of f(x).

If $c = 0$, the interval becomes $0 < x < 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

If $c = -\pi$, the interval becomes $-\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

when function is an odd function $a_0 = a_n = 0$

then $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ $\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

when function is even then $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{where } a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\& \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

Q. obtain the fourier series to represent $f(x) = \frac{1}{4}(\pi - x)^2$ in the interval $0 \le x \le 2\pi$.

Hence obtain the following relations:

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Soln let $f(x) = \frac{1}{4}(\pi - x)^2$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{4}(\pi - x)^2\,dx$$

$$= \frac{1}{4\pi}\left[-\frac{(\pi - x)^3}{3}\right]_0^{2\pi} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\left\{(\pi - x)^2 \frac{\sin nx}{n}\right\}_0^{2\pi} + \int_0^{2\pi} 2(\pi - x)\frac{\sin nx}{n}\,dx\right]$$

$$= -\frac{1}{2\pi n^2}(-\pi - \pi) = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\sin nx\,dx$$

$$= 0$$

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{\cos 3x}{3^2} \cdots \quad (2)$$

(i) Put $x = 0$ in (2)

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{--- (3)}$$ (20)

(ii) Put $x = \pi$ in (2)

$$0 = \frac{\pi^2}{12} + \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \cdots \right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \frac{\pi^2}{12} \quad \text{A} \quad \text{--- (4)}$$

(iii) eq (3) + eq (4)

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots \right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots$$

Q. Expand $f(x) = x \sin x$; $0 < x < 2\pi$ as a fourier series

Sol^n $f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$= \frac{1}{\pi}\left[x(-\cos x) - \sin x \right]_0^{2\pi} = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} 2x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left\{ \sin(n+1)x - \sin(n-1)x \right\} dx$$

$$= \frac{1}{2\pi}\left[x\left\{ -\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right\} - \left\{ -\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left\{ -\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1} \right\} \right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1}, \quad n \neq 1$$

when $n = 1$ we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - \left(-\frac{\sin 2x}{2}\right)\right]_0^{2\pi} = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\{\cos(n-1)x - \cos(n+1)x\}dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right\} - \left\{-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= 0 \quad , \quad n \neq 1$$

when $n = 1$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - \left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\cdot\left(2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right) = \pi$$

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty} a_n\cos nx + \sum_{n=2}^{\infty} b_n\sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx \quad \text{Ans}$$

Q. obtain the fourier series for function $f(x) = x^2$, $-\pi \le x \le \pi$. Hence show that

(i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots\cdots = \frac{\pi^2}{6}$

(ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots\cdots = \frac{\pi^2}{12}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots = \frac{\pi^2}{8}$

Solⁿ $f(-x) = (-x)^2 = x^2 = f(x)$

$\because f(x)$ is even function $\therefore b_n = 0$

(21)

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx = \frac{2}{\pi} \int_0^{\pi} x^2\, dx = \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \left[x^2 \left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) - 2\left(-\frac{\sin nx}{n^3}\right) \right]_0^{\pi}$$

$$= \frac{4}{n^2} (-1)^n$$

$$\therefore x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right) \quad \text{--- (1)}$$

Put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} - 4\left(-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} \cdots\right)$$

$$\frac{2\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \cdots\right)$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots = \frac{\pi^2}{6} \quad \text{Ans} \quad \text{--- (2)}$$

Put $x = 0$ in (1)

$$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots\right)$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots = \frac{\pi^2}{12} \quad \text{--- (3)} \quad \text{Ans}$$

eq (2) + eq (3) we gett

$$2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right) = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots = \frac{\pi^2}{8}$$

Fourier series for discontinuous functions

Q. Find the fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x \le 2\pi \end{cases}$$

deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots = \frac{\pi^2}{8}$

Sol^n let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right] = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_0^{\pi} + \left\{(2\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos nx}{n^2}\right)_0^{\pi} - \left(\frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi n^2}\left[(-1)^n - 1 - 1 + (-1)^n\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$= \begin{cases} -\frac{4}{\pi n^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right\}_0^{\pi} + \left\{(2\pi - x)\left(-\frac{\cos nx}{n^2}\right) - (-1)\left(-\frac{\sin nx}{n^3}\right)\right\}_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \frac{\pi}{n}\cos n\pi\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - - -\right)$$ (22)

Put $x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - - \right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots$$

Q. obtain fourier series for the function

$$f(x) = \begin{cases} x & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases}$$

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - - = \frac{\pi^2}{8}$

Soln $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\,dx + \int_{0}^{\pi} -x\,dx\right]$

$$= -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} -x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right\}_{-\pi}^{0} + \left\{-x\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\{1-(-1)^n}{n^2}\right\} = \frac{2}{\pi n^2}[1-(-1)^n]$$

$= 0$ if n is even

$\frac{4}{\pi n^2}$ n is odd

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\,dx + \int_{0}^{\pi} -x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right\}_{-\pi}^{0} + \left\{x\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right\}_{0}^{2\pi}\right]$$

$$= 0$$

$$\therefore f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - - - -\right]$$

at the point of discontinuity

$$f(0) = \frac{1}{2}\{f(0-0) + f(0+0)\} = \frac{1}{2}(0-0) = 0$$

$$\therefore 0 = -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots\right)$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \cdots\cdots = \frac{\pi^2}{8} \quad \text{Ans}$$

Change of interval

$c < x < c+2l$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx, \quad b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Put $c = 0$ interval becomes $0 < x < 2l$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Put $c = -l$ interval becomes $-l < x < l$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx, \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

If function is odd $a_0 = a_n = 0$ $\quad b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx$

function is even $b_n = 0$, $\quad a_0 = \frac{2}{l}\int_0^{l} f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

Q. Find fourier expansion for the function.

$f(x) = x - x^2 \quad , \quad -1 < x < 1$

Sol^n $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

$l = 1$

$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \int_{-1}^{1} x\,dx = -\frac{2}{3}$

$a_n = \int_{-1}^{1} (x - x^2)\cos n\pi x\,dx =$

$> \left[(x-x^2)\frac{\sin n\pi x}{n\pi} - (1-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_{-1}^{1}$

$= \int_{-1}^{1} x\cos n\pi x\,dx - \int_{-1}^{1} x^2 \cos n\pi x\,dx$

$= 0 - 2\left[x^2 \frac{\sin n\pi x}{n\pi} - 2x\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$

$= \frac{-4(-1)^n}{n^2\pi^2}$

$b_n = \int_{-1}^{1} (x - x^2)\sin n\pi x\,dx = \int_{-1}^{1} x \sin n\pi x\,dx - \int_{-1}^{1} x^2 \sin n\pi x\,dx$

$= 2\int_0^1 x \sin n\pi x\,dx - 0 = \frac{-2(-1)^n}{n\pi}$

$\therefore x - x^2 = -\frac{1}{3} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \frac{\cos 3\pi x}{3^2} - \cdots\right)$

$+ \frac{2}{\pi}\left(\frac{\sin \pi x}{1} - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right)$

Ans

Q. Obtain the fourier series expansion of

$f(x) = \left(\frac{\pi - x}{2}\right)$ for $\quad 0 < x < 2$

Sol^n $\quad l = 1$

$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^2 \left(\frac{\pi - x}{2}\right) dx = \pi - 1$

$a_n = \frac{1}{l}\int_0^{2l} \left(\frac{\pi - x}{2}\right)\cos \frac{n\pi x}{1}\,dx = 0$

$b_n = \frac{1}{l}\int_0^{2l} \left(\frac{\pi - x}{2}\right)\sin n\pi x\,dx = \frac{1}{n\pi}$

$\therefore f(x) = \frac{\pi - 1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\pi x$

Q. obtain Fourier series for $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$

Sol^n let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$

$l = 1$

Then $a_0 = \int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$

$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \pi$

$a_n = \int_0^2 f(x) \cos n\pi x\,dx = \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x) \cos n\pi x\,dx$

$= -\frac{4}{n^2\pi}$

$b_n = \int_0^2 f(x) \sin n\pi x\,dx = \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x) \sin n\pi x\,dx$

$= 0$

$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} \cdots\right)$

Half Range Series $\Rightarrow$ A function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct Half range series

Half range cosine series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

$a_0 = \frac{2}{l} \int_0^l f(x)\,dx$, $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,dx$

Half range sine series

$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ where $b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$

If range is $0 < x < \pi$

(i) Half range sine series

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

(ii) Half range cosine series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Q. Expand $\pi x - x^2$ in a half range sine series in the interval $(0, \pi)$ up to the first three terms.

Sol^n $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + \frac{2\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{4}{\pi n^3}\left[1-(-1)^n\right] = \begin{cases} 0 & n \text{ is even} \\ \frac{8}{\pi n^3} & n \text{ is odd} \end{cases}$$

$$\pi x - x^2 = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} \cdots\right)$$

Q. Develop $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in half range cosine series in the range $0 < x < l$.

Sol^n $a_0 = \frac{2}{l}\int_0^{l} \sin\frac{\pi x}{l}\,dx = \frac{4}{\pi}$.

$$a_n = \frac{2}{l}\int_0^{l} \sin\frac{\pi x}{l}\cdot\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l}\left[\sin(n+1)\frac{\pi x}{l} - \sin(n-1)\frac{\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}} + \frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^{l}$$

$$= \frac{1}{\pi}\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

when n is odd $\quad a_n = 0 \quad n \neq 1$

n is even $\quad a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$$= -\frac{4}{\pi(n+1)(n-1)}$$

when $n=1$

$$a_1 = \frac{2}{l}\int_0^l \sin\frac{\pi x}{l} \cdot \cos\frac{\pi x}{l}\, dx$$

$$= \frac{1}{l}\int_0^l \sin\frac{2\pi x}{l}\, dx = \frac{1}{l}\left[\frac{-\cos\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right]_0^l = 0$$

$$\therefore \sin\frac{\pi x}{l} = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos\frac{2\pi x}{l}}{1\cdot 3} + \frac{\cos\frac{4\pi x}{l}}{3\cdot 5} + \frac{\cos\frac{6\pi x}{l}}{5\cdot 7}\cdots\right]$$

Q. Expand $f(x) = x$ as a half range

(i) sine series in $0 < x < 2$.

(ii) cosine series in $0 < x < 2$.

Sol^n (i) $x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$

$$b_n = \frac{2}{2}\int_0^2 x \sin\frac{n\pi x}{2}\, dx.$$

$$= \left[x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - \left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi = -\frac{4}{n\pi}(-1)^n$$

$$\therefore x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}.$$

(ii) $\quad x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$

$$a_0 = \frac{2}{2}\int_0^2 x\, dx = 2$$

$$a_n = \frac{2}{2}\int_0^2 x \cos\frac{n\pi x}{2}\, dx$$

(25)

$$= \left[\frac{x \sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} + \frac{\cos \frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}} \right]_0^2$$

$$= \frac{4}{n^2\pi^2} (\cos n\pi - 1)$$

$$\therefore \; x = 1 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos(n\pi - 1)}{n^2} \cdot \cos \frac{n\pi x}{2} \quad \text{Ans}$$

Fourier Integrals ⇒

① Fourier integral for $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(t-x)\, du\, dt$

② Fourier sine integral for $f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin ut \sin ux \, du\, dt$

③ Fourier cosine integral for $f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos ut \cos ux \, du\, dt$

Q. Express the function $f(x) = \begin{cases} 1 & \text{when } |x| \le 1 \\ 0 & \text{when } |x| > 1 \end{cases}$

as a Fourier integral. Hence evaluate $\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$

Soln $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda$

$$= \frac{1}{\pi} \int_0^{\infty} \int_{-1}^{1} 1 \cdot \cos \lambda (t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \left\{ \frac{\sin \lambda (t-x)}{\lambda} \right\}_{-1}^{1} d\lambda$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{\sin \lambda (1-x) + \sin \lambda (1+x)}{\lambda} d\lambda$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$$

$$\int_0^{\infty} \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} f(x)$$

or $\begin{cases} \frac{\pi}{2} & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

for $|x| = 1$

which is a point of discontinuity of $f(x)$.

$$\int_0^\infty \frac{\sin\alpha \cos\alpha x}{\alpha} d\alpha = \frac{\frac{\pi}{2}+0}{2} = \frac{\pi}{4} \quad \underline{Ans}$$

Q. find the fourier sine integral for $f(x) = e^{-\beta x}$

Hence show that $\frac{\pi}{2} e^{-\beta x} = \int_0^\infty \frac{\alpha \sin\alpha x}{\beta^2+\alpha^2} d\alpha$

Sol^n The fourier sine ~~transform~~ integral of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\alpha x \, d\alpha \int_0^\infty f(t) \sin\alpha t \, dt \quad \text{---} (1)$$

on putting value of $f(x)$ in (1)

$$e^{-\beta x} = \frac{2}{\pi} \int_0^\infty \sin\alpha x \, d\alpha \int_0^\infty e^{-\beta t} \sin\alpha t \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \sin\alpha x \, d\alpha \left[\frac{e^{-\beta t}}{\alpha^2+\beta^2} (-\beta \sin\alpha t - \alpha\cos\alpha t) \right]_0^\infty$$

$$= \frac{2}{\pi} \int_0^\infty \sin\alpha x \, d\alpha \cdot \frac{\alpha}{\beta^2+\alpha^2}$$

$$\therefore \int_0^\infty \frac{\alpha \sin\alpha x}{\beta^2+\alpha^2} d\alpha = \frac{\pi}{2} e^{-\beta x} \quad \underline{Ans}$$

Q. using fourier cosine integral representation of an appropriate function show that

$$\int_0^\infty \frac{\cos wx}{k^2+w^2} dw = \frac{\pi}{2k} e^{-kx}$$

sol^n $f(x) = \frac{2}{\pi} \int_0^\infty \cos ux \, du \int_0^\infty f(t) \cos ut \, dt$ ——— (1)

Replace u by w in (1)

$$e^{-kx} = \frac{2}{\pi} \int_0^\infty \cos wx \, dw \int_0^\infty e^{-kt} \cos wt \, dt$$

$$= \frac{2}{\pi} \int_0^\infty \cos wx \, dw \left[\frac{e^{-kt}}{k^2+w^2} (-k\cos wt + w\sin wt) \right]_0^\infty$$

$$= \frac{2}{\pi} \int_0^\infty \cos wx \, dw \cdot \frac{k}{k^2+w^2} \Rightarrow \int_0^\infty \frac{\cos wx}{k^2+w^2} dw = \frac{\pi}{2k} e^{-kx} \quad \underline{Ans}$$

Fourier Transform ⇒

Fourier Transform $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$

inverse fourier Transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

fourier sine Transform $F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin st\, dt$

inverse fourier sine Transform.

Fourier Transform →

fourier transform $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$

inverse fourier transform $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$

Fourier sine Transform $F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin st\, dt$

inverse fourier sine Transform $f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \sin sx\, ds$

Fourier cosine Transform $F(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos st\, f(t)\, dt$

inverse fourier cosine Transform $f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(s) \cos sx\, dx$

Q. find the fourier Transform of

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Sol^n

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} 1\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{is} \left[e^{isa} - e^{-ias} \right] = \frac{1}{\sqrt{2\pi}} \frac{2}{s} \left[\frac{e^{isa} - e^{-ias}}{2i} \right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin as}{s} \quad \text{Ans}$$

Q. Find the Fourier Transform of

$$f(x) = \begin{cases} 1-x^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$$

Sol^n $F(S) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \frac{4}{s^3} [-s\cos s + \sin s]$

Q. Find the fourier sine & cosine transform of $f(x) = e^{-ax}$.

Sol^n $F(S) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \sin sx\, dx$

Sine Transform

$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \sin sx - s\cos sx) \right]_0^\infty$

$= \sqrt{\frac{2}{\pi}} \cdot \frac{s}{a^2+s^2}$

Fourier cosine Transform is

$F(S) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax} \cos sx\, dx$

$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a\cos sx + s \sin sx) \right]_0^\infty = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$

Q. Find fourier sine Transform of $\frac{1}{x}$

Sol^n $F(S) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{1}{x} \cdot \sin sx\, dx$

Put $sx = \theta$ $\quad dx = \frac{d\theta}{s}$

$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cancel{s} \cdot \frac{\sin\theta\, d\theta}{\theta\, \cancel{s}} = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2}$

$\left\{ \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2} \right.$

$= \sqrt{\frac{\pi}{2}}$ Ans

Q. Find the Fourier cosine Transform of

$$f(x) = \begin{cases} x & \text{for } 0 < x < \frac{1}{2} \\ 1-x & \frac{1}{2} < x < 1 \\ 0 & x > 1 \end{cases}$$

Sol^n - Fourier cosine Transform is

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1/2} x \cos sx \, dx + \sqrt{\frac{2}{\pi}} \int_{1/2}^{1} (1-x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos s}{s^2} + \frac{2 \cos s/2}{s^2} - \frac{1}{s^2} \right] \text{Ans}$$

Properties of Fourier Transform

1. Linear Property

$$F[a f_1(x) + b f_2(x)] = a F_1(s) + b F_2(s)$$

2. change of scale Property

$$F\{f(ax)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

3. Shifting Property $F\{f(x-a)\} = e^{isa} F(s)$

4. Modulation Theorem

$$F\{f(x) \cos ax\} = \frac{1}{2}[F(s+a) + F(s-a)]$$

5. if $F\{f(x)\} = F(s)$ then

$$F\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F(s)$$

6. $F\{e^{iax} f(x)\} = F(s+a)$

7. $F\{f'(x)\} = is F(s)$ if $f(x) \to 0$ as $x \to \pm\infty$

8. $F\left\{\int_a^x f(x)\, dx\right\} = \frac{F(s)}{(-is)}$

Convolution Theorem

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Q. Solve for $f(x)$ from the integral eq^n

~~Sol^n~~ $\int_0^\infty f(x) \cos sx \, dx = e^{-s}$

Sol^n $\int_0^\infty f(x) \cos sx \, dx = e^{-s}$ —— (1)

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \sqrt{\frac{2}{\pi}} e^{-s}$$

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} e^{-s}$$

$$f(x) = F_c^{-1}\left[\sqrt{\frac{2}{\pi}} e^{-s}\right]$$

$$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-s} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty e^{-s} \cos sx \, dx$$

$$= \frac{2}{\pi} \left[\frac{e^{-s}}{x^2+1} (\cos sx + s \sin sx)\right]_0^\infty = \frac{2}{\pi} \cdot \frac{1}{1+x^2}$$

Q. Find the function ~~if~~ its sine Transform is $\frac{e^{-s}}{s}$

Sol^n $F_s\{f(x)\} = \frac{e^{-as}}{s}$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{s} \sin sx \, ds$$

$$\frac{df}{dx} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx \, ds = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+x^2}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot a \int \frac{dx}{a^2+x^2} = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{a} + C \quad \text{—— (1)}$$

at $x=0$ $f(0)=0$ —— (2)

from (1) & (2)

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\frac{x}{a}$$

Put $a=0$ $F_s^{-1}\left(\frac{1}{s}\right) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$ Ans