Numerical Linear Algebra:

LU Decomposition or factorization of a matrix →

Lower-upper (LU) decomposition can be defined as the product of a lower and an upper triangular matrices.

Consider the system of eqns in three variables:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

These can be written in the form of $AX = B$ as;

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Steps to solve by LU decomposition method :-

Step 1: Generate a matrix $A = LU$ such that L is the lower triangular matrix with principal diagonal elements being equal to 1 and U is the upper triangular matrix. That means

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Step 2: Now we can write $AX = B$ as $\quad (\because A = LU)$

$$LUX = B \quad \cdots (1)$$

Step 3: Let us assume $UX = Y \quad \cdots (2)$

where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Step 4: From eqns (1) and (2), we have; $LY = B$

On solving this eqn, we get y_1, y_2, y_3.

Step 5: Substituting Y in eqn (2), we get $UX = Y$

By solving eqn, we get X i.e, x_1, x_2, x_3.

The above process is also called the process of triangularisation.

Ex. Solve the system of eqns $x_1+x_2+x_3=1$, $3x_1+x_2-3x_3=5$, $x_1-2x_2-5x_3=10$ by LU decomposition method.

Soln:
Given System of eqns are:

$$x_1+x_2+x_3=1$$
$$3x_1+x_2-3x_3=5$$
$$x_1-2x_2-5x_3=10$$

These eqns are written in the form $AX=B$ as;

$$\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\5\\10\end{bmatrix}$$

Step 1: Let us write the above matrix as $LU=A$. That means

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}=\begin{bmatrix}1&1&1\\3&1&-3\\1&-2&-5\end{bmatrix}$$

By expanding the left side matrices we get,

$$\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

Thus by equating the corresponding elements, we get;

$u_{11}=1$, $u_{12}=1$, $u_{13}=1$

$l_{21}u_{11}=3$, i.e, $l_{21}=3$

$l_{21}u_{12}+u_{22}=1$ i.e, $3\times1+u_{22}=1 \Rightarrow u_{22}=1-3=-2$

$l_{21}u_{13}+u_{23}=-3 \Rightarrow 3\times1+u_{23}=-3 \Rightarrow u_{23}=-6$

$l_{31}u_{11}=1 \Rightarrow l_{31}=1$

$l_{31}u_{12}+l_{32}u_{22}=-2 \Rightarrow 1\times1+l_{32}\times-2=-2 \Rightarrow 1-2l_{32}=-2$
$\Rightarrow -2l_{32}=-3 \Rightarrow l_{32}=\frac{3}{2}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=-5 \Rightarrow 1\times1+\frac{3}{2}\times\cancel{-6}^{\,3}+u_{33}=-5$
$\Rightarrow 1-9+u_{33}=-5 \Rightarrow 3$

Solving these eqns we get

$u_{22}=-2$, $u_{23}=-6$, $u_{33}=3$, $l_{21}=3$, $l_{31}=1$, $l_{32}=3/2$

Step 2: $LUX=B$

Step 3: Let $UX=Y$

Step 4: From the previous two steps, we have $LY=B$

Thus,

$$\begin{bmatrix}1&0&0\\3&1&0\\1&\frac{3}{2}&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\5\\10\end{bmatrix}$$

so,

$$y_1 = 1$$

$$3y_1 + y_2 = 5$$

$$y_1 + \frac{3}{2}y_2 + y_3 = 10$$

solving these eqns, we get;

$$y_1 = 1,\ y_2 = 2,\ y_3 = 6$$

Step5: Now consider, $UX = Y$. So,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

By expanding this eqn, we get

$$x_1 + x_2 + x_3 = 1$$

$$-2x_2 - 6x_3 = 2$$

$$3x_3 = 6$$

Solving these eqns, we can get

$$x_3 = 2,\ x_2 = -7,\ x_1 = 6$$

Therefore, the soln of the given system of eqns is $(6, -7, 2)$

Q. Find the soln of the system of eqns. by LU decomposition.

$x + 2y + 3z = 9,\ 4x + 5y + 6z = 24,\ 3x + y - 2z = 4.$

Cholesky Factorization : Similar to LU factorization method.
It is suitable for symmetric matrix $(A = A^T)$ and positive definite $(x^T A x > 0)$.

$A = L L^T$

Step 1 :
$$A = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L \text{ (lower triangular)}} \underbrace{\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}}_{L^T \text{ (upper triangular)}}$$

Step 2 : $AX = B$

$L \underbrace{L^T X}_{} = B$

$LY = B \rightarrow$ we solve for Y

Step 3 : $L^T X = Y \rightarrow$ we solve for X

Ex.
$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} \rightarrow \text{Find } L.$$

Soln. This is symmetric matrix

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

Expanding ~~we get~~ the leftside matrices and equating we get

$l_{11}^2 = 4 \quad \therefore l_{11} = 2$

$l_{11} \cdot l_{21} + 0 = 12 \quad \therefore l_{21} = 6$

$l_{11} l_{31} = -16 \quad \therefore l_{31} = -8$

$l_{21} l_{11} = 12 \quad \therefore l_{21} = 6$

$l_{21} l_{21} + l_{22} l_{22} = 37 \quad \therefore l_{21}^2 + l_{22}^2 = 37 \Rightarrow 36 + l_{22}^2 = 37 \Rightarrow l_{22} = 1$

$l_{31} l_{21} + l_{32} l_{22} = -43 \Rightarrow -8 \times 6 + l_{32} = -43 \Rightarrow l_{32} = -43 + 48 = 5$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 98 \Rightarrow (-8)^2 + (5)^2 + l_{33}^2 = 98 \Rightarrow 64 + 25 + l_{33}^2 = 98$

$\Rightarrow 89 + l_{33}^2 = 98$

$\Rightarrow l_{33}^2 = 98 - 89 = 9$

$\Rightarrow l_{33} = 3$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}$$

$$\cancel{L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix}}$$

Ex: Solve the system by cholesky method

$x+2y+3z=5,\ 2x+8y+22z=6,\ 3x+22y+82z=-10$

soln: → Let the given system is

$$Ax = B \quad \text{—(1)}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

Let $LL^T = A$ —(2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$\therefore l_{11}^2 = 1 \Rightarrow l_{11} = 1$

$l_{11} l_{21} + 0 = 2 \Rightarrow l_{21} = 2$

$l_{11} l_{31} = 3 \Rightarrow l_{31} = 3$

$l_{21} l_{11} = 2$

$l_{21}^2 + l_{22}^2 = 8 \Rightarrow l_{22}^2 = 8-4 = 4 \Rightarrow l_{22} = 2$

$l_{21} l_{31} + l_{22} l_{32} = 22 \Rightarrow 2\times 3 + 2 l_{32} = 22$

$\Rightarrow l_{32} = 8$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82 = 9 + 64 + l_{33}^2 = 82$

$\Rightarrow l_{33}^2 = 9 \Rightarrow l_{33} = 3$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix}$$

By (1) and (2) $LL^T X = B$ —(3)

Put $L^T X = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ —(4)

Then (3) becomes $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

ie, $y_1 = 5$

$2y_1 + 2y_2 = 6$

$3y_1 + 8y_2 + 3y_3 = -10$

$\therefore y_2 = -2,\ y_3 = -3$

$\therefore \boxed{y_1 = 5,\ y_2 = -2,\ y_3 = -3}$

By eqn (4) ie, $L^T X = Y$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$3z = -3 \Rightarrow z = -1.$

$2y + 8z = -2 \Rightarrow y = 3$

$x + 2y + 3z = 5 \Rightarrow x = 2$

$\therefore \boxed{x = 2,\ y = 3,\ z = -1}$