UNIT-2.

1. LAPLACE TRANSFORMATION

The Knowledge of 'Integral Transforms' provide an easy and effective means for the solutions of many problems arising in science and engineering.

For example, The Laplace transformation replaces a given function $F(t)$ by another function $f(s)$. Then laplace transformation converts an ordinary differential equation with some given initial conditions into an algebraic equation in terms of $f(s)$. Finally, using inverse Laplace transformation we recover the original function $F(t)$.

LAPLACE TRANSFORM

Definition Given a function $F(t)$ defined for all real $t \geq 0$, the laplace transform of $F(t)$ is a function of a new variable s & is given by

$$\boxed{L(F(t)) = f(s) = \bar{F}(s) = \int_0^\infty e^{-st} F(t)\, dt}$$

we can also write $L(f(t)) = \bar{f}(s)$.

NOTE A function $F(t)$ is called piecewise continuous or sectionally continuous in the closed interval $[a,b]$ if $\exists$ a finite number of points $t_1, t_2, \ldots, t_n$ $(a = t_1 < t_2 < t_3 < \ldots < t_n = b)$ such that $F(t)$ is continuous in each of open subinterval (t_i, t_{i+1}) and has finite right limit $(F(t_i + 0))$ and finite left limit $(F(t_{i+1} - 0))$.

EXISTENCE OF LAPLACE TRANSFORM.

Suppose a function f is piecewise continuous over R^+ and there are numbers M and k such that $|f(t)| \leq Me^{kt}$ for $t \geq 0$, then $\int_0^\infty e^{-st} f(t)\,dt$ converges for $s > k$ that is, $F(s)$ exists for $s > k$.

Further, we also define inverse Laplace transform.

If $f(t) = L^{-1}(\bar{f}(s))$.

Linearity Theorem :- If $L(f_1(t)) = \bar{f}_1(s)$ and $L(f_2(t)) = \bar{f}_2(s)$ then for any constants c_1 and c_2

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \bar{f}_1(s) + c_2 \bar{f}_2(s).$$

First Shifting Theorem.

If $L(f(t)) = \bar{f}(s)$ then $L(e^{at} f(t)) = \bar{f}(s-a)$ $(s > a)$

Proof

$$L(e^{at} f(t)) = \int_0^\infty e^{-st}(e^{at} f(t))\,dt$$

$$= \int_0^\infty e^{(a-s)t} f(t)\,dt$$

let $p = \cancel{a-s} = +(s-a)$

$$= \int_0^\infty e^{-pt} f(t)\,dt$$

$$= \bar{f}(p) = \bar{f}(s-a).$$

Laplace Transform of some elementary functions

(1) $f(t) = e^{at}$, $a > 0$

$$L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt \qquad (s > a)$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{-1}{s-a}\left[0-1\right]$$

$$= \frac{1}{s-a}$$

Thus $\boxed{L(e^{at}) = \dfrac{1}{s-a} \quad \text{where } s > a.}$

(2) $f(t) = \sin at$ or $\cos at$

We know $e^{ait} = \cos at + i \sin at$

$$\Rightarrow L(e^{ait}) = L(\cos at) + i\, L(\sin at)$$

Now $L(e^{ait}) = \dfrac{1}{s - ai} = \dfrac{s + ai}{s^2 + a^2}$

$$\therefore \quad L(\cos at) + i\, L(\sin at) = \frac{s}{s^2+a^2} + i \frac{a}{s^2+a^2}$$

$$\therefore \quad \boxed{L(\cos at) = \frac{s}{s^2+a^2} \qquad L(\sin at) = \frac{a}{s^2+a^2}}$$

(3) $f(t) = \sinh at$ or $\cosh at$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{1}{2} L(e^{at}) - \frac{1}{2} L(e^{-at})$$

$$= \frac{1}{2(s-a)} - \frac{1}{2(s+a)}$$

$$\boxed{L(\sinh at) = \frac{a}{s^2 - a^2}}$$

IIIly $\boxed{L(\cosh at) = \frac{s}{s^2 - a^2}}$

(4) $f(t) = t^n$

$$L(t^n) = \int_0^\infty e^{-st} \cdot t^n \, dt$$ Putting $st = x$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n \, dx$$

$$\boxed{L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}}$$

(5) $$L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$$

(6) $$L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$$ & $$L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$$

(7) $$L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$$ & $$L(e^{ab} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$$

Ex Find the Laplace transform of.

(1) $\sin^2 t$

(2) $\sin 2t \cos 3t$

(3) $\cos^3 t$

(4) $\sin t \cos t$

(5) $e^{-2t}(3\cos 4t - 2\sin 5t)$

(6) $\sinh at \cos at$.

Solution (1) $L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t)$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+4}\right)$$

$$= \frac{2}{s(s^2+4)}$$

(2) $L(\sin 2t \cos 3t) = L\left(\frac{1}{2}(\sin 5t - \sin t)\right)$

$$= \frac{1}{2}L(\sin 5t) - \frac{1}{2}L(\sin t)$$

$$= \frac{1}{2}\left(\frac{5}{s^2+25}\right) - \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2(s^2-5)}{(s^2+1)(s^2+25)}$$

(3) $L(\cos^3 t) = L\left(\frac{3\cos t + \cos 3t}{4}\right)$

$$= \frac{3}{4}L(\cos t) + \frac{1}{4}L(\cos 3t)$$

$$= \frac{3}{4}\left(\frac{s}{s^2+1}\right) + \frac{1}{4}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{s(s^2+7)}{(s^2+1)(s^2+9)}.$$

(4) $L(\sin t \cos t) = L\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2}L(\sin 2t)$

$$= \frac{1}{2}\left(\frac{2}{s^2+4}\right) = \frac{1}{s^2+4}$$

(5) $L\left(e^{-2t}(3\cos 4t - 2\sin 5t)\right)$

$$= 3L\left(e^{-2t}\cos 4t\right) - 2L\left(e^{-2t}\sin 5t\right)$$

$$= \frac{3\cdot(s+2)}{(s+2)^2+16} - 2\cdot\frac{5}{(s+2)^2+5^2}$$

$$= \frac{3(s+2)}{(s+2)^2+16} - \frac{10}{(s+2)^2+25}$$

(6) $L(\sinh at \cos at) = L\left(\left(\frac{e^{at}-e^{-at}}{2}\right)\cos at\right)$

$$= \frac{1}{2}L\left(e^{at}\cos at\right) - \frac{1}{2}L\left(e^{-at}\cos at\right)$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} - \frac{(s+a)}{(s+a)^2+a^2}\right]$$

—— x ——

Ex 1. Let $f(t) = \frac{t}{a}$ for $0<t<a$

$= 1$ for $t>a$

Find the Laplace transform of $f(t)$

Ex 2. Find the laplace transform of the function

(1) $f(t) = \begin{cases} 2 & : 0<t<\pi \\ 0 & : \pi<t<2\pi \\ \sin t & : 2\pi<t \end{cases}$

(2) $f(t) = \begin{cases} 2+t^2 & : 0<t<2 \\ 6 & : 2<t<3 \\ 2t-5 & : 3<t<\infty \end{cases}$

(3) $f(t) = |t-1| + |t+1|$, $t \geq 0$.

Solution 1. $L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^a e^{-st} \cdot \frac{t}{a}\,dt + \int_a^\infty e^{-st} \cdot 1\,dt$

$$= \frac{1}{a}\int_0^a t \cdot e^{-st}\,dt + \int_a^\infty e^{-st}\,dt$$

$$= \frac{1}{a}\left[\frac{t\,e^{-st}}{-s} - \int_0^\infty \frac{1 \cdot e^{-st}}{-s}\,dt\right]_0^a + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{1}{a}\left[\frac{-a\,e^{-sa}}{s} + \frac{e^{-st}}{-s^2}\Bigg|_0^a\right] + \left[\frac{-e^{-st}}{s}\right]_a^\infty$$

$$= -\frac{1}{s}e^{-as} - \frac{e^{-sa}}{as^2} + \frac{1}{as^2} + \frac{e^{-sa}}{s}$$

$$= \frac{-e^{-as}}{as^2} + \frac{1}{as^2}$$

Solution (2) (1) $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\pi 2e^{-st}dt + \int_\pi^{2\pi} 0 \cdot e^{-st}dt + \int_{2\pi}^\infty \sin t \cdot e^{-st}dt$$

$$\cancel{= \left[\frac{2e^{-st}}{-s}\right]_0^\pi + \left[\frac{\sin t\, e^{-st}}{-s}\bigg|_{2\pi}^\infty - \int_{2\pi}^\infty \frac{\cos t\, e^{-st}}{-s}dt\right]}$$

$$= \left[\frac{2e^{-st}}{-s}\right]_0^\pi + 0 + \left[\frac{e^{-st}}{1+s^2}(-s\sin t - 1\cos t)\right]_{2\pi}^\infty$$

$$= \frac{2}{s}\left[1 - e^{-s\pi}\right] + \frac{e^{-2\pi s}}{1+s^2}(s\sin 2\pi + \cos 2\pi)$$

$$= \frac{2}{s}(1 - e^{-s\pi}) + \frac{e^{-2\pi s}}{1+s^2}$$

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① Using :- $\lim_{t\to\infty} \frac{e^{-st}}{1+s^2}(-s\sin t) = 0$

(sin t and cos t are bounded functions. and $(\lim_{t\to\infty} = \text{zero}) \times$ bdd func. $= 0$)

② $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

(2). $L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$ ⑨

$$= \int_0^2 (2+t^2)e^{-st}\,dt + \int_2^3 6e^{-st}\,dt + \int_3^\infty (2t-5)e^{-st}\,dt$$

$$= \left[\frac{(2+t^2)e^{-st}}{-s}\right]_0^2 - \int_0^2 \frac{2t\,e^{-st}}{-s}\,dt + \left[\frac{6e^{-st}}{-s}\right]_2^3 + \left[\frac{(2t-5)e^{-st}}{-s}\right]_3^\infty$$

$$-2\int_3^\infty \frac{e^{-st}}{-s}\,dt$$

$$= -\frac{6}{s}e^{-2s} + \frac{2}{s} + \frac{2}{s}\int_0^2 t e^{-st}\,dt - \frac{6}{s}\left(e^{-3s} - e^{-2s}\right) + \frac{e^{-3s}}{s}$$

$$+ \frac{2}{s}\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \frac{2}{s} + \frac{2}{s}\left[\frac{te^{-st}}{-s} - \int_0^2 \frac{e^{-st}}{-s}\,dt\right]_0^2 - \frac{6e^{-3s}}{s} + \frac{e^{-3s}}{s} + \frac{2}{s^2}e^{-3s}$$

$$= \frac{2}{s} + \frac{2}{s^2}\left[-2e^{-2s} + 0\right] + \frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^2 - \frac{5e^{-3s}}{s} + \frac{2}{s^2}e^{-3s}$$

$$= \frac{2}{s} - \frac{4e^{-2s}}{s^2} - \frac{2}{s^3}\left[e^{-2s} - 1\right] - \frac{5}{s}e^{-3s} + \frac{2}{s^2}e^{-3s}$$

$$= \frac{2}{s} + \frac{2}{s^3} - e^{-2s}\left(\frac{4}{s^2} + \frac{2}{s^3}\right) - e^{-3s}\left(\frac{5}{s} - \frac{2}{s^2}\right).$$

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(3) $f(t) = |t-1| + |t+1|$, $t \geq 0$

$$= \begin{cases} 1-t+1+t = 2 & 0 < t < 1 \\ t-1+t+1 = 2t & t > 1 \end{cases}$$

$$\therefore L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 2e^{-st}\,dt + \int_1^\infty 2t\,e^{-st}\,dt$$

$$= \left[\frac{2e^{-st}}{-s}\right]_0^1 + 2\left[\frac{t\,e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^\infty$$

$$= \frac{-2}{s}\left[e^{-s} - 1\right] + 2\left[\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right]$$

$$= \frac{2}{s} + \frac{2e^{-s}}{s^2}.$$

X

PROPERTIES OF LAPLACE TRANSFORM

(1) Change of Scale $\boxed{L(f(at)) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)}$

Proof
$$L(f(at)) = \int_0^\infty e^{-st} f(at)\,dt$$
Putting $at = x$
$$= \int_0^\infty e^{-s\left(\frac{x}{a}\right)} f(x)\,d\left(\frac{x}{a}\right)$$
$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)\,dx = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

(2) $\boxed{L(t.f(t)) = -\frac{d}{ds}\bar{f}(s)}$

Proof We have $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

Differentiating both sides wrt s.
$$\frac{d}{ds}(\bar{f}(s)) = \int_0^\infty \frac{\partial}{\partial s}\left(e^{-st} f(t)\right) dt$$
$$= \int_0^\infty -te^{-st} f(t)\,dt = -L(t.f(t)).$$

(3) In general $\boxed{L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)}$ $n = 1, 2, 3 \ldots$

(4) $\boxed{L\left(\frac{1}{t}f(t)\right) = \int_s^\infty \bar{f}(s)\,ds}$

Proof. $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt.$

Integrating both sides w.r.t s under the limits s to ∞, we get

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\,dt \right] ds$$

changing the order of integration

$$= \int_0^\infty f(t) \left(\int_s^\infty e^{-st}\,ds \right) dt.$$

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-t} \right]_s^\infty dt = \int_0^\infty f(t) \left(0 - \frac{e^{-st}}{-t} \right) dt$$

$$= \int_0^\infty \frac{e^{-st} f(t)}{t}\,dt$$

$$= L\left(\frac{1}{t} f(t) \right).$$

(5) $\boxed{L(f'(t)) = s\bar{f}(s) - f(0)}$

Proof $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt.$

Integration by parts.

$$= \left[e^{-st} f(t) \right]_0^\infty - \int_0^\infty -s\, e^{-st} f(t)\,dt.$$

$$= \lim_{t \to \infty} e^{-st} f(t) - f(0) + s\bar{f}(s).$$

Since $f(t)$ is a continuous and bounded function

$$\therefore \lim_{t \to \infty} e^{-st} f(t) = 0.$$

Thus, we have $\boxed{L(f'(t)) = s\bar{f}(s) - f(0)}$

(5) If $L(f(t)) = \bar{f}(s)$ then

$$L\left(\int_0^t f(u)\, du\right) = \frac{1}{s}\bar{f}(s)$$

(6) $\boxed{L(f''(t)) = s^2\bar{f}(s) - sf(0) - f'(0)}$

Ex ① Show that

$$L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$$

$$L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

② find $L(\sin^3 2t)$

③ find $L(t^2 e^{-2t})$

④ find $L\left(\frac{e^{-t}\sin t}{t}\right)$

⑤ find $L(te^{-t}\sin 3t)$.

Solution ① $L(\sin at) = \frac{a}{s^2+a^2}$.

$$\Rightarrow L(t\sin at) = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{-a(2s)(-1)}{(s^2+a^2)^2}$$

$$= \frac{2as}{(s^2+a^2)^2}.$$

② $L(\sin^3 2t)$

Now $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

$$\Rightarrow \sin^3(2t) = \frac{1}{4}(3\sin 2t - \sin 6t)$$

$$\therefore L(\sin^3 2t) = \frac{3}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t)$$

$$= \frac{3}{4} \cdot \frac{2}{s^2+2^2} - \frac{1}{4} \cdot \frac{6}{s^2+6^2}$$

$$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right].$$

(3) We know if $L(f(t)) = \bar{f}(s)$

then $L(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n} \bar{f}(s)$.

Now $L(e^{-2t}) = \dfrac{1}{s+2}$

$$\Rightarrow L(t^2 e^{-2t}) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$= \frac{d}{ds}\left(\frac{(-1)}{(s+2)^2}\right) = \frac{(-1)(-2)}{(s+2)^3}.$$

$$= \frac{2}{(s+2)^3}.$$

(4) $L(e^{at} \sin bt) = \dfrac{b}{(s-a)^2+b^2}$

$$\therefore L(e^{-t}\sin t) = \frac{b=1}{(s+1)^2+1}$$

(6)

Also, $L\left(\frac{f(t)}{t}\right) = \int_s^{\infty} \bar{f}(s)\,ds$

$$\therefore L\left(\frac{1}{t}(\bar{e}^{t}\sin t)\right) = \int_s^{\infty} \frac{ds}{(s+1)^2+1} = \left[\tan^{-1}(s+1)\right]_s^{\infty}$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1).$$

Ex Find (a) $L\left(2e^{t}\sin 4t\cos 2t\right)$

(b) $L\left(\frac{\sin at}{t}\right)$

(c) Using Laplace transform, show that

$$L\int_0^t \frac{\cos at - \cos bt}{t}\,dt = \frac{1}{2s}\log\frac{s^2+b^2}{s^2+a^2}.$$

(d) Show that $L\int_0^t \frac{e^{t}\sin t}{t}\,dt = \frac{1}{s}\cot^{-1}(s-1).$

Solution (a) $L(2\sin 4t\cos 2t) = L(\sin 6t + \sin 2t)$

$$= \frac{6}{s^2+36} + \frac{2}{s^2+4}$$

By first shifting theorem

$$L(e^{t}\sin 6t + e^{t}\sin 2t) = L(e^{t}\sin 6t) + L(e^{t}\sin 2t)$$

$$= \frac{6}{(s-1)^2+36} + \frac{2}{(s-1)^2+4}$$

(b) $L(\sin at) = \dfrac{a}{s^2+a^2}.$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\,ds = \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

(c) $L\left(\cos at - \cos bt\right) = \dfrac{s}{s^2+a^2} - \dfrac{s}{s^2+b^2}$

$$\therefore \quad L\left(\frac{1}{t}(\cos at - \cos bt)\right) = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= 0 - \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

because $\lim\limits_{s\to\infty} \log\left(\dfrac{s^2+a^2}{s^2+b^2}\right) = \log 1 = 0.$

Now, using $L\left(\int_0^t f(t)\,dt\right) = \dfrac{1}{s}\bar{f}(s)$

we have

$$L\left[\int_0^t \frac{\cos at - \cos bt}{t}\,dt\right] = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right).$$

(d) $L\left[e^{t}\sin t\right) = \dfrac{1}{(s-1)^2+1}$

Hence $L\left(\dfrac{e^t \sin t}{t}\right) = \displaystyle\int_s^\infty \frac{ds}{(s-1)^2+1} = \tan^{-1}(s-1)\Big]_s^\infty$

$= \dfrac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1)$

$\therefore \quad L\left(\displaystyle\int_0^t \frac{e^t \sin t}{t}\, dt\right) = \dfrac{1}{s}\bar{f}(s) = \dfrac{1}{s}\cot^{-1}(s-1)$.

Ex Evaluate $\displaystyle\int_0^\infty t^3 e^{-t} \sin t\, dt$.

Solution

$L\left(t^3 \sin t\right) = (-1)^3 \dfrac{d^3}{ds^3}\left(\dfrac{1}{s^2+1}\right) = \dfrac{24s\,(s^2-1)}{(s^2+1)^4}$

That is $L\left(t^3 \sin t\right) = \displaystyle\int_0^\infty e^{-st}\, t^3 \sin t\, dt = \frac{24s(s^2-1)}{(s^2+1)^4}$

Putting $s=1$

we get $\displaystyle\int_0^\infty t^3 e^{-t} \sin t\, dt = 0$.

Ex Evaluate $L(\sinh^3 2t)$

Solution We know

$$\sinh 3t = 3\sinh t + 4\sinh^3 t$$

$$\Rightarrow \sinh^3 t = \frac{1}{4}\left[\sinh 3t - 3\sinh t\right].$$

$$\Rightarrow \sinh^3(2t) = \frac{1}{4}\left[\sinh 6t - 3\sinh 2t\right]$$

$$\therefore L(\sinh^3 2t) = \frac{1}{4}L(\sinh 6t) - \frac{3}{4}L(\sinh 2t)$$

$$= \frac{6}{4(s^2-6^2)} - \frac{3}{4}\cdot\frac{2}{(s^2-2^2)}$$

$$= \frac{48}{(s^2-4)(s^2-36)}.$$

Ex. If $L(f(t)) = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a), & t>a \\ 0 & 0<t<a \end{cases}$

then prove that $L(g(t)) = e^{-as}\bar{f}(s)$.

Solution $L(g(t)) = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$

$$= \underbrace{\int_0^a e^{-st}\cdot 0\,dt}_{0} + \int_a^\infty e^{-st} f(t-a)\,dt.$$

Putting $t-a=u$.

$\Rightarrow dt = du.$

$$= \int_0^\infty e^{-s(a+u)} f(u)\, du = e^{-as} \int_0^\infty e^{-su} f(u)\, du.$$

$$= e^{-as} \bar{f}(s).$$

Hence proved.

Ex Evaluate $L(\sin\sqrt{t})$

Solution $\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots$$

$$L(\sin\sqrt{t}) = L(t^{1/2}) - \frac{1}{3!} L(t^{3/2}) + \frac{1}{5!} L(t^{5/2}) - \cdots$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{6} \frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{120} \frac{\Gamma 7/2}{s^{7/2}}$$

as $L(t^n) = \frac{\Gamma n+1}{s^{n+1}}$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{s^{5/2}} + \frac{1}{120} \cdot \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}}{s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2 s^{3/2}} \left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 - \cdots \right]$$

$$= \frac{\sqrt{\pi}\, e^{-1/4s}}{2 s^{3/2}}$$

x

Ex Find the laplace transform of the function $f(t)$ given by

$$f(t) = \begin{cases} \sin(t-\alpha) & t > \alpha \\ 0 & t < \alpha \end{cases}$$

Solution $L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\alpha e^{-st}.0\,dt + \int_\alpha^\infty e^{-st}\sin(t-\alpha)\,dt$

Put $t-\alpha = u$

$$= \int_0^\infty e^{-s(u+\alpha)} \sin u\,du$$

$$= e^{-\alpha s} \int_0^\infty e^{-su} \sin u\,du = e^{-\alpha s} L(\sin u)$$

$$= \frac{e^{-\alpha s}}{s^2+1}$$

Ex Given that $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$

Show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$

Solution let $f(t) = 2\sqrt{\frac{t}{\pi}}$

Now $L(f(t)) = \frac{1}{s^{3/2}} = \bar{f}(s)$

Using property :-

$$L(f'(t)) = s\bar{f}(s) - f(0) = s \cdot \frac{1}{s^{3/2}} - 0 = \frac{1}{\sqrt{s}}$$

$$\therefore L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$$

Ex Using laplace transformation show that $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$

Solution We know that $L(\sin at) = \frac{a}{s^2+a^2}$

$$\therefore \quad L\left(\frac{\sin at}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2}\, ds$$

$$= \left[a \cdot \frac{1}{a} \tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Thus, by def^n,

$$\int_0^\infty \frac{e^{-st} \sin at}{t}\, dt = \cot^{-1}\left(\frac{s}{a}\right)$$

Taking $a=1$ and $s=0$ in above relation,

$$\int_0^\infty \frac{\sin t}{t}\, dt = \cot^{-1}(0) = \frac{\pi}{2}.$$

Ex Evaluate $\int_0^{\infty} t e^{-2t} \cos t \, dt$

Solution We know $L[t\cos t] = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]$$

$$= \frac{s^2-1}{(s^2+1)^2}.$$

By defn,

$$\int_0^{\infty} e^{-st} t\cos t \, dt = \frac{s^2-1}{(s^2+1)^2}$$

Taking $s=2$,

$$\int_0^{\infty} e^{-2t} t\cos t \, dt = \frac{3}{25}.$$

EXTRA QUESTIONS

Evaluate

(1) $L\left(\dfrac{\cos 2t - \cos 3t}{t}\right)$ $\qquad \dfrac{1}{2}\log\dfrac{s^2+9}{s^2+4}$

(2) $L\left(\dfrac{1-\cos t}{t^2}\right)$ $\qquad \cot^{-1}s - \dfrac{s}{2}\log(1+s^{-2})$

(3) $L(\sinh at \sin at)$ $\qquad \dfrac{2a^2 s}{s^4+4a^4}$

Ex If $L(\cos^2 t) = \dfrac{s^2+2}{s(s^2+4)}$, find $L(\cos^2 at)$.

Solution. (2). Let $f(t) = \dfrac{1-\cos t}{t^2}$

We let, $g(t) = t^2 f(t) = 1-\cos t$

$L(g(t)) = L(t^2 f(t)) = L(1-\cos t)$

$$\Rightarrow \quad (-1)^2 \frac{d^2}{ds^2}\bar{f}(s) = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\Rightarrow \quad \bar{f}(s) = \iint\left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds^2$$

$$= \int\left(\log s - \frac{1}{2}\log(s^2+1)\right) ds.$$

$$= \left(\log s\,(s) - \int \frac{1}{s}(s)\,ds\right) - \frac{1}{2}\left(s\log(s^2+1) - \int\left(\frac{2s^2}{s^2+1}\right)ds\right)$$

$$= -s + s\log s - \frac{s\log(s^2+1)}{2} + \int\left(\frac{s^2}{s^2+1}\right)ds$$

$$= -s + s\log s - \frac{s}{2}\log(s^2+1) + \int\left(1 - \frac{1}{s^2+1}\right)ds$$

$$= -s + s\log s - \frac{s}{2}\log(s^2+1) + s + \cot^{-1}s.$$

$$= s\log\left(\frac{s}{\sqrt{s^2+1}}\right) + \cot^{-1}s.$$

———— x ————