Bases and Dimension

Def^n: A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, as also say that the vectors of β form a bases of V.

Def^n: Let $u_1, u_2, \ldots, u_k$ be k vectors in R^n.

Let $V = L(u_1, u_2, \ldots, u_k)$. If $u_1, u_2, \ldots, u_k$ are linearly independent then $(u_1, u_2, \ldots, u_k)$ is called a basis of V. More generally, if $u_1, u_2, \ldots, u_k$ are k vectors in vector space V over R then $[u_1, u_2, \ldots, u_k]$ is called a basis of V if (i) $u_1, u_2, \ldots, u_k$ span V.

(ii) $u_1, u_2, \ldots, u_k$ are linearly idependent.

If $u_1, u_2, \ldots, u_k$ form a bases for a vector space V, then they must be distinct and non-zero otherwise $(u_1, u_2, \ldots, u_k)$ will be linearly dependent set.

Def^n: A bases β of vector space V is a linearly independent subset of V that generates V i.e, (i) β is linearly independent

(ii) $\text{Span}(\beta) = V$

Ex 1. $\text{Span}\ \phi = \{0\}$.

$\Rightarrow \phi$ is a basis of zero vector space.

Ex 2. The vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ form a basis of R^2.

As $\alpha_1 e_1 + \alpha_2 e_2 = 0$, $\alpha_1, \alpha_2 \in R$

$\Rightarrow \alpha_1(1, 0) + \alpha_2(0, 1) = (0, 0)$

or $(\alpha_1, 0) + (0, \alpha_2) = (0, 0)$

or $(\alpha_1, \alpha_2) = (0, 0)$

$\alpha_1 = 0 = \alpha_2$

$\therefore \{e_1, e_2\}$ is linearly independent set.

$\{e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), e_n = (0, 0, 0, \ldots, 1)\}$.

is a basis of R^n.

These basis are called standard or natural basis for $R^n = (F^n)$

Ex 3. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only non-zero entry is 1 in the i^{th} row and j^{th} column. Then $\{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis for $M_{m\times n}(F) \Rightarrow |\beta| = mn$.

Ex 4 Standard basis for $P_n(F)$ is

$$\beta = \{1, x, x^2, \ldots, x^{n-1}, x^n\}$$

Ex 5. Standard basis for $P(F)$

$$\beta = \{1, x, x^2, \ldots\}$$

Ex: Let $S = \{(2,-3,5), (8,-12,20), (1,0,-2), (0,2,-1), (7,2,0)\}$

claim : Span $S = \mathbb{R}^3$

Let $(a,b,c) \in \mathbb{R}^3$

then

$$(a,b,c) = \alpha(2,-3,5) + \beta(8,-12,20) + \gamma(1,0,-2) + \mu(0,2,-1) + \eta(7,2,0)$$

Now,

$$2\alpha + 8\beta + \gamma + 7\eta = a$$
$$-3\alpha - 12\beta + 2\mu + 2\eta = b$$
$$5\alpha + 20\beta - 2\gamma - \mu = c$$

$\alpha =$

$\beta =$

$\gamma =$

$\mu =$

$\eta =$

Hence span$(S) = \mathbb{R}^3$

Now to see if S is basis of $\mathbb{R}^3$, we need to check whether S is linearly idependent or not.

Consider,

$$\alpha(2,-3,5) + \beta(8,-12,20) + \gamma(1,0,-2) + \mu(0,2,-1) + \eta(7,2,0) = (0,0,0)$$

Solving we can see that $\alpha = \beta = \gamma = \eta = \mu = 0$ is not the only soln to the system. Hence S is not L.I set.

But since span $S = \mathbb{R}^3$, By theorem, (if a vector V is generated by a finite set S then some subset of S is a basis for V. Hence V has a finite basis). $\exists$ a subset S' of S such that S' is basis of $\mathbb{R}$. Now, we will proceed as described in theorem, we select a non-zero vector, now we will include one vector one by one till it remains L.I.

$S' = \{(2,-3,5), (8,-12,20)\}$ is not a L.I set as

$(8,-12,20) = 4(2,-3,5)$, ~~As (2,-3,5) is a multiple~~ $(8,-12,20)$ is a multiple of $(2,-3,5)$

Let $(1,0,-2) \in S'$

$\{(2,-3,5), (1,0,-2)$ clearly the set is L.I.

Let $(0,2,-1) \in S'$

$\{(2,-3,5), (1,0,-2), (0,2,-1)$ check for L.I.

$\alpha(2,-3,5) + \beta(1,0,-2) + \gamma(0,2,-1) = (0,0,0)$

$2\alpha + \beta = 0$ $\quad -3\alpha + 2\gamma = 0$ $\quad 5\alpha - 2\beta - \gamma = 0$

$2\alpha = -\beta$ $\quad -3\alpha = -2\gamma$ $\quad \gamma = 5\alpha - 2\beta$

$-3\alpha = -2(-9\alpha)$ $\quad \gamma = 5\alpha + 4\alpha$

$3\alpha = 18\alpha$ $\quad \gamma = 9\alpha$

$3 = 18$

Only soln is $\alpha = \beta = \gamma = 0$

Hence $\{(2,-3,5), (1,0,-2), (0,2,-1)$ are L.I.

Let $(7,2,0) \in S'$

$\{(2,-3,5), (1,0,-2), (0,2,-1), (7,2,0)$

$\alpha(2,-3,5) + \beta(1,0,-2) + \gamma(0,2,-1) + \eta(7,2,0) = (0,0,0)$

$2\alpha + \beta + 7\eta = 0$

$-3\alpha + 2\gamma + 2\eta = 0$

$5\alpha - 2\beta - \gamma = 0$

Since this set has more than trivial soln, S' is L.D.

Hence basis of $\mathbb{R}^3$ is $\{(2,-3,5), (1,0,-2), (0,2,-1)\}$.

Replacement Theorem.

Let V be a vector space generated by set G containing exactly n vectors. Let L be L.I subset of V containing exactly m vectors. Then $m \le n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generates V.

Cor. Let V be vector space having finite basis then every basis for V contains the same no. of vectors.

Ex. $S = \{(1,3), (1,-1)\}$ span R^2 or not ?

Soln. Let $\alpha(1,3) + \beta(1,-1) = (0,0)$

$\Rightarrow (\alpha, 3\alpha) + (\beta, -\beta) = (0,0)$

$\Rightarrow (\alpha+\beta, 3\alpha-\beta) = (0,0)$

or $\alpha+\beta = 0 \Rightarrow \alpha = -\beta$

$3\alpha - \beta = 0 \Rightarrow 3\alpha + \alpha = 0 \Rightarrow 4\alpha = 0 \Rightarrow \alpha = 0$

$\Rightarrow \beta = 0$

$\therefore \{(1,3), (1,-1)\}$ is L.I set in R^2

Hence $\{(1,3), (1,-1)\}$ is a basis of R^2.

Ex. $S = \{(0,0), (1,2), (2,4)\}$ span R^2 or not ?

Soln. Since S contains three vectors, therefore by theorem (Any set of vectors in R^n having more more than n vectors is linearly dependent and hence cannot be basis for R^n)

S cannot be basis of R^2.

Finite Dimensional Vector Space (FDVS)

A vector space is called finite dimens. if it has a basis consisting of finite no of vectors

The unique no of vectors in each basis for V is called dimension of V and is denoted by dim V.

A vector space that is not finite dimensional is called infinte dimensional vector space.

Example 7 $V = \{0\} \longrightarrow$ Finite Dim Vector Space

$\dim V = 0$ as $\beta = \phi$

Eg 8 $V = F^n$

$\beta = \{(1,0,0\ldots 0), (0,1,0\ldots 0) \ldots (0,0\ldots 1)\}$

$|\beta| = n$

$\dim V = n$ F^n is FDVS.

Eg 9 $V = M_{m \times n}(F)$

$\dim V = (m \times n = mn$

$\beta = \{A_{11}, A_{12}, A_{13} \ldots A_{1n}$

$A_{21}, \ldots \ldots A_{2n}$

$\vdots$

$A_{m1}, \ldots \ldots A_{mn}\}$

$\beta = \{A_{ij}$: the only non zero entry is 1 in ith row jth column $1 \le i \le n,$ $1 \le i \le m\}$

$|\beta| = mn$, $\dim V = mn$.

Eg 10 $V = P_n(F)$

$= \{ a_0 + a_1 x + \dots + a_n x^n \; ; \; a_i \in F$

$i = 1, 2 \dots n$

$\beta = \{1, x, x^2 \dots x^n\}$

$|\beta| = n+1 \Rightarrow \dim V = n+1$

Hence $P_n(F)$ is FDVS

Eg 12 $V = \mathbb{C}(\mathbb{R}) = \{ a + ib, \; a, b \in \mathbb{R} \}$

Basis

$a + ib = a(1 + 0i) + b(0 + i)$

$\beta = \{1, i\}$

$|\beta| = 2$

Hence $\dim V = 2$

$\therefore$ V is a FDVS.

Eg 11 $V = \mathbb{C}(\mathbb{C})$

$\beta = \{1\}$, $\dim V = 1$

$|\beta| = 1$

FD.VS

?

$\underset{\in F}{a + ib} = (a + ib) \underset{\in V}{1}$

Eg $V = \mathbb{R}(\mathbb{R})$

$a = \underset{\in F}{a} \cdot \underset{\in \beta}{1}$

$\beta = \{1\}$.

$\dim V = 1$

If $V = R(\phi)$ $\longrightarrow$ This is wrong.
$\downarrow$ should always be small.

Sub vector spaces are not defined.

* $V = P(F)$
$= \{ a_0 + a_1 x + a_2 x^2 + \dots \quad a_i \in F, \; i = 1, 2 \dots \}$

$\beta = \{ 1, x, x^2 \dots \}$

Since the basis is an infinite set. Hence $P(F)$ is a infinite vector dimensional space.

Corollary 2. Let V be a vector space with dimension n
$\dim(V) = n$

a.) Any finite generating set for V contains atleast n vectors and a generating set for V that contains exactly n vectors is a basis of V.

Proof: Let β be a basis of V.
Let G be a finite generating set for V.

By Theorem 1.9 [State] $\exists$ a subset H of G st H is a basis of V.

By corollary 1, H contains exactly n vec.

Since a subset of G contains n vectors
~~Since a subset of G contains~~

$\therefore$ G must contain atleast n vectors ; $|H| = n$
ie $|G| \geq n$.

Moreover, if G contains exactly n vectors.

$$H = G.$$

$\Rightarrow$ G is a basis for V.

(b) Any linearly independent set of V that contains exactly n vectors is a basis for V.

Proof Let L be a linearly independent subset of V. containing exactly n vectors.
Then by replacement theorem there exist a subset H of basis β which can always be regarded as the generaling set, such that

$$|H| = n - n = 0$$
$$L \cup H = L \cup \phi = L \quad \text{generates } V.$$

Since L generates V and by the given hypothesis L is linearly independent.

$\therefore$ L is a basis for V.

(c) Every linearly independent subset of V can be extended to form a basis of V.

Pf Let L be linearly independent subset of V containg m vectors. $|L| = m$

By replacement theorem, $\exists$ a subset H of β ($H \subseteq \beta$) with

$$|H| = n - m$$

containing $n-m$ vectors such that $L \cup H$ generates V.

Now $L \cup H$ contains atmost n vectors.

$\therefore$ By (a) $L \cup H$ contains exactly n elements,
Hence $L \cup H$ is basis for V.

Example 3 Consider $P_2(\mathbb{R}) = \{a_0, a_1x, a_2x^2, a_0, a_1, a_2 \in \mathbb{R}\}$

$\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$ generates $P_2(\mathbb{R})$

$\therefore$ β' is basis for $\mathbb{R}$.

Example 4 $M_{2\times 2}(\mathbb{R}) =$

$$\beta \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$|\beta| = 4$

$$\beta' = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

is a basis of $M_{2\times 2}(\mathbb{R})$.

e 5

$\mathbb{R}^4$

standard basis

$\beta = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$|\beta| = 4$

$\beta' = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$

is also a basis for $\mathbb{R}^4$.

$p_k(x) = x^k + x^{k+1} + \dots + x^n$

$k = 0, 1, 2, \dots, n$

$\{p_0(x), p_1(x), \dots p_n(x)\}$ is a basis.

Theorem 1.11 :-

Let W be subspace of a finite dimensional vector spaces V. Then W is finite dimensional and $\dim(W) \leq \dim V$. Moreover, if $\dim(W) = \dim V$ then $V = W$.

Eg 17

$V = F^5 = \{ (a_1, a_2, a_3, a_4, a_5) \; ; \; a_i \in F \}$

$W = \{ (a_1, a_2, a_3, a_4, a_5) \in F^5, \; a_1 + a_3 + a_5 = 0$
$a_2 = a_4 \}$

Clearly $0 \in W$ $\because (0,0,0,0,0)$ Here $a_1 + a_3 + a_5 = 0$
$a_2 = a_4 \}$

Also for

$x = (a_1, a_2, a_3, a_4, a_5)$ and
$y = (b_1, b_2, b_3, b_4, b_5) \quad \in W$

$x + y = (a_1, a_2, a_3, a_4, a_5) + (b_1, b_2, b_3, b_4, b_5)$
$= (a_1 + b_1, \; a_2 + b_2, \; a_3 + b_3, \; a_4 + b_4, a_5 + b_5)$

Now $a_1 + b_1 + a_3 + b_3 + a_5 + b_5 = 0$
$a_2 + b_2 = a_4 + b_4 \quad (\because x, y \in W)$

Hence $x + y \in W$

Also for $c \in W$

$cx = (ca_1, ca_2, ca_3, ca_4, ca_5)$

Now $ca_1 + ca_3 + ca_5 = 0$
& $ca_2 = ca_4$

Hence $cx \in W$

Hence W is a subspace of V.

$\dim(V) = 5 \qquad \beta = \{(1,0,0,0,0), (0,1,0,0,0),$
$(0,0,1,0,0)\ (0,0,0,1,0),$
$(0,0,0,0,1)\}$

[basis of w] $\beta' = \{(-1,0,1,0,0), (-1,0,0,0,1), (0,1,0,1,0)\}$.

Let $(a_1, a_2, a_3, a_4, a_5) \in W$

$a_1 + a_3 + a_5 = 0$

$a_2 = a_4$

$(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3, a_2, a_5)$

where $a_1 + a_2 + a_5 = 0$

$= (a_1, a_2, a_3, a_2, -a_1 - a_3)$

$\beta'' = (0,1,0,1,0), (1,0,0,0,-1), (1,0,-1,0,0)$

$a_1(0,1,0,1,0) + a_2(1,0,0,0,-1), a_3(1,0,-1,0,0)$

$= (a_2 + a_3, a_1, -a_3, a_1, -a_2) = (0,0,0,0,0)$

$a_1 = 0, \quad a_3 = 0, \quad a_2 = 0$

Hence scalars are all zero. Thus, they are linearly independent.

Eg 18 $M_{n\times n}(F) = V$ $\dim V = n^2$

W = set of all diagonal $n\times n$ matrices of $M_{n\times n}(F)$.

$\dim W = n$

$$\begin{pmatrix} a & 0 & \cdots & & \\ 0 & b & 0 & \cdots & \\ 0 & 0 & c & & 0 \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

$|\beta| = n$

$\beta = \{ A_{11}, A_{22} \cdots A_{nn} \}$

where E^{ij} is the matrix in which only nonzero entry is 1 in ith row & jth column.

Also,

clearly $\dim(W) \leq \dim(V)$.

Eg 19 $V = M_{n\times n}(F)$

W = set of symmetric matrices

$$W = \left\{ A = [a_{ij}] \; ; \; a_{ij} = a_{ji} \; ; \; \begin{matrix} 1 \leq j \leq n \\ 1 \leq j \leq n \end{matrix} \right\}$$

$\dim(V) = n^2$

$\dim(W) = \frac{1}{2} n(n+1)$ $\left.\begin{matrix} 3-6 \\ 4-10 \\ 5-15 \end{matrix}\right\}$

the minimum no of f elements required to generate this general element of W is reduced to the entries forming as upper triangular matrix as indicated.

Clearly, the no of elements require start from n, $n-1$ in second row & goes on till 1 in nth row. Hence total no of field element required to generate symm

Date : / /
Page No.

$\beta = \{ A_{ij} ; 1 \le i \le n, 1 \le j \le n,$ where A_{ij} is the $n \times n$ matrice having 1 in the ith row & jth column and jth row ith column & elsewhere 0 $\}$

Que 26 $P_{18}(F) = V$

$= \{ a_{18} x^{18} + a_{17} x^{17} + \dots\dots a_1 x + a_0 ; a_i \in F \}$ $i = 0, 1 \dots 18$

Consider a subset W of all polynomials of the form

$W = \{ a_{18} x^{18} + a_{16} x^{16} + \dots a_2 x^2 + a_0 ;$
$a_0, a_2, \dots a_{18} \in F \}$

$W \subseteq V$

Clearly, W is a subspace of V
Here $\dim W = 10$

&

$\beta = \{ 1, x^2, x^4, \dots x^{18} \}$

* Sum of Subspaces

Let V be a vector space
Let W_1 & W_2 be non empty subsets of V then

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1, w_2 \in W_2 \}$

Direct Sums

Let W_1 & W_2 be subspaces of V then V is said to be the direct sum of W_1 & W_2 denoted by $V = W_1 \oplus W_2$

Date: / /
Page No.

if $V = W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$

Sec 1.3

Q23 Let W_1 & W_2 be subspaces of a vector space V

a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 & W_2

(b) Prove that any subspace of V that contains both W_1 & W_2 must also contain $W_1 + W_2$

Proof (a) Clearly $0 \in W_1$, $0 \in W_2$ $\because W_1$ & W_2 are subspace of V

$\Rightarrow 0 = 0 + 0 \in W_1 + W_2$

Let $a+b, c+d \in W_1 + W_2$
$a, c \in W_1$ & $b, d \in W_2$

Consider, $(a+b) + (c+d) = (a+c) + (b+d)$ [$(V, +)$ is an abelian gp]
$\in W_1 + W_2$ [$\because W_1$ & W_2 are subgroups]

Consider, $\alpha(a+b)$ $\alpha \in F$
$= \alpha a + \alpha b$ $a \in W_1$, $b \in W_2$
$\in W_1 + W_2$ ($\because W_1$ & W_2 are subspace)

Now T.P. $W_1 \subseteq W_1 + W_2$ & $W_2 \subseteq W_1 + W_2$

Let $w_1 \in W_1$, Clearly $w_1 = w_1 + 0 \in W_1 + W_2$

||ly $w_2 = 0 + w_2 \in W_1 + W_2$

(b) Let U be a subspace of V containing both W_1 & W_2

i.e $W_1 \subseteq U \subseteq V$, $W_2 \subseteq U \subseteq V$

Let $a+b \in W_1+W$ be arbitrary , T.P $W_1+W_2 \subseteq U$

$a \in W_1 , b \in W_2$

$\Rightarrow a \in W_1 \subseteq U , b \in W_2 \subseteq U$

$\Rightarrow a+b \in U \Rightarrow W_1+W_2 \subseteq U$

Qu 30 Let W_1 & W_2 be subspaces of V.S. V; Prove that V is the direct sum of W_1 & W_2 iff each vector in V can be uniquely written as x_1+x_2 , $x_1 \in W_1$, $x_2 \in W_2$

Suppose $V = W_1 \oplus W_2$

$\Rightarrow V = W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$

So by defination every $v \in V$ is of the form $v = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$

Suppose $v = w_1 + w_2$, $w_1 , x \in W_1$
& $v = x + y$ $w_2 , y \in W_2$

$w_1 + w_2 = x + y$
$w_1 - x = y - w_2$
$\in W_1$ $\in W_2$

$\Rightarrow x - w_1 \in W_1 \cap W_2 \Rightarrow x = w_1$
& $y - w_2 \in W_1 \cap W_2$ $y = w_2$

Hence exp is unique

Conversly,

suppose that the expersion is unique.
to prove that, $V = W_1 \oplus W_2$.

since it's given that $V = W_1 + W_2$
we need to prove that $W_1 \cap W_2 = \{0\}$

Let $w \in W_1 \cap W_2$
$\Rightarrow w \in W_1$ and $w \in W_2$

Clearly $w \in V$
Now $w = w + 0$
$\in W_1 + W_2$

& $w = 0 + w$
$\in W_1 + W_2$

Hence, $w \in V$ has 2 expersion, ~~Hence $w=0$~~.
which is a contradiction
As w has unique expression.

Hence our Assumption is wrong.

And $\boxed{w = 0}$

Ques 29 a) Prove that if W_1 and W_2 are FDV subspaces of vector space V, then subspace $W_1 + W_2$ is finite dimensional, &

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proof Given :-

Also $W_1 \cap W_2$ has a finite basis
($\because W_1 \cap W_2$ is FD subsp.)

Let $\beta = \{u_1, u_2, \ldots, u_k\}$ be basis of $W_1 \cap W_2$.

we can extend the basis β to a basis β_1

$\beta_1 \{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_m\}$ for W_1

$\beta_2 = \{u_1, u_2 \ldots u_k, w_1, w_2 \ldots w_p\}$ for W_2

Let $\alpha = \{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_m, w_1, w_2 \ldots w_p\}$

Claim: α is basis for $W_1 + W_2$.

(i) α is linearly independent.

Let $a_1u_1 + a_2u_2 + \cdots + a_ku_k + b_1v_1 + b_2v_2 + \cdots b_mv_m + c_1w_1 + c_2w_2 + \cdots c_3w_3 = 0. \rightarrow (A)$

where a_i's b_j's c_k's are scalars

Then,

$$-b_1 v_1 - b_2 v_2 \ldots - b_m v_m = a_1 u_1 + a_2 u_2 + \ldots a_k u_k + c_1 w_1 + c_2 w_2 + \ldots c_p w_p$$

$\in W_1$ $\quad$ $\in W_2$

Hence,

the vector on LHS or equivalent RHS $\in W_1 \cap W_2$

Since β is basis for $W_1 \cap W_2$.

$$(-b_1) v_1 + (-b_2) v_2 + \ldots + (-b_m) v_m = d_1 u_1 + d_2 u_2 + \ldots d_k u_k$$

for some scalars d.

$\Rightarrow$ $d_1 u_1 + d_2 u_2 + \ldots + d_k u_k + b_1 v_1 + b_2 v_2 + \ldots b_m v_m = 0$

$\in$ span β_1

$\Rightarrow$ d_i's $= 0$ $\quad \forall\ i = 1$ to k $\quad$ $[\because \beta_1$ is LI$]$

b_j's $= 0$ $\quad j = 2$ to m

By (1)

$\Rightarrow$ $a_1 u_1 + \ldots + a_k u_k + c_1 w_1 + c_2 w_2 + \ldots c_p w_p = 0$

$\in$ span β_2

$\Rightarrow$ $a_1 = a_2 = a_3 \ldots = a_k = 0$ $\quad$ $(\because \beta_2$ is LI$)$

$c_1 = c_2 = c_3 \ldots c_p = 0$

Hence α is LI.

Date : / /
Page No.

(ii) $W_1 + W_2 = \text{span}(\alpha)$

Let $u = v + w \in W_1 + W_2$
; $v \in W_1$, $w \in W_2$

As β_1 is a basis for W_1 & β_2 is basis for W_2

$\Rightarrow \exists$ scalars such that

$$u = (x_1 u_1 + x_2 u_2 + \dots x_k v_k + y_1 v_1 + y_2 v_2 + \dots + y_m v_m)$$
$$+ (z_1 u_1 + z_2 u_2 + \dots z_k u_k + t_1 w_1 + t_2 w_2 + \dots t_p w_p)$$

$$u = (x_1 + z_1) u_1 + (x_2 + z_2) u_2 + \dots + y_1 v_1 + y_2 v_2 + \dots y_m v_m + t_1 w_1 + t_2 w_2 + \dots t_p w_p.$$

$\Rightarrow$ $u \in \text{span}\, \alpha$.
$W_1 + W_2 \subseteq \text{span}\, \alpha$
Also $\text{span}\, \alpha \subseteq W_1 + W_2$ (Retracing steps bck)
Hence $\Rightarrow$ $W_1 + W_2 = \text{span}\, \alpha$
ie α is a basis of $W_1 + W_2$

Now $\dim (W_1 + W_2) = |\alpha|$
$= k + m + p$
$= (k+m) + (k+p) - k$
$= \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$

Ques 33 (a) Let W_1 and W_2 be subspaces of vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2 respectively, show that

$\rightarrow$ $\beta_1 \cap \beta_2 = \phi$

$\rightarrow$ $\beta_1 \cup \beta_2$ is basis for V

Proof Let $v \in \beta_1 \cap \beta_2$

Then $v \in W_1 \cap W_2$

which is a contradiction

($\because V = W_1 \oplus W_2$

$\Rightarrow W_1 \cap W_2 = \{0\}$).

Hence $\beta_1 \cap \beta_2 = \phi$

We know $\beta_1 \cup \beta_2$ spans V since $V = W_1 + W_2$

Suppose,

$$a_1 v_1 + a_2 v_2 + \cdots a_m v_m + b_1 w_1 + \cdots b_n w_n = 0$$

where a_i's are not all zero & b_i's are not all zero.

Then $a_1 v_1 + \cdots a_m v_m = -b_1 w_1 - b_2 w_2 \cdots - b_n w_n$

& both are nonzero element in $W_1 \cap W_2$.

which is a contradiction $\because W_1 \cap W_2 = \{0\}$

Hence, $\beta_1 \cup \beta_2$ is a LI set.

and v is a basis for V.

(b) Conversely,
Let β_1, β_2 disjoint subspaces of W_1 & W_2 respectively of vector space V. Prove that

prove that if $\beta_1 \cup \beta_2$ is basis of V then $V = W_1 \oplus W_2$

Pf Suppose $\beta = \beta_1 \cup \beta_2$ is basis of V
& $\beta_1 \cap \beta_2 = \phi$

Let $v \in V$

$$v = a_1 v_1 + \cdots + a_m v_m + b_1 w_1 + \cdots b_n w_n$$

where v_i's $\in \beta_1$
w_i's $\in \beta_2$.

$$v = s + t \in W_1 + W_2$$

Hence, $V \subseteq W_1 + W_2$

Clearly, $W_1 + W_2 \subseteq V$

Hence $V = W_1 + W_2$

Let if possible suppose $W_1 \cap W_2 \neq \{0\}$

Then $\exists\ v \neq 0$ such that $v \in W_1$ & $v \in W_2$

$v = a_1 v_1 + a_2 v_2 + \cdots a_m v_m$
$v = b_1 w_1 + b_2 w_2 + \cdots b_n w_n$

where a_i's, b_i's are not 0. all.

$a_1v_1 + a_2v_2 + \dots a_mv_m = b_1w_1 + b_2w_2 + \dots b_mw_m$

$a_1v_1 + a_2v_2 + \dots a_mv_m - b_1w_1 - b_2w_2 - \dots - b_mw_m = 0$

Hence LHS $\in$ span $(\beta_1 \cup \beta_2)$

Hence all a_i's and b_i's must be zero which is a contradiction

Hence, $W_1 \cap W_2 = \{0\}$.

Ques 34 (a) Prove that if W_1 is any subspace of a FDVS, V then $\exists$ subspace W_2 of V such that

$$V = W_1 \oplus W_2$$

Proof: Let $\beta = \{u_1, u_2 \dots u_n\}$ be a basis of W_1

By Replacement theorem, we extend basis β of W_1 ~~for~~ to a basis

$$\alpha = \{u_1, u_2 \dots u_n, u_{n+1}, \dots u_m\} \text{ of } V.$$

Let $W_2 = \text{span}\{u_{n+1}, \dots u_m\}$

[$\forall$ $S \subseteq V$, span S always subspace of V]

(1) Claim $V = W_1 \oplus W_2$

(1) $V = W_1 + W_2$

(2) $W_1 \cap W_2 = \{0\}$

Let $v \in V$

Then
$$v = \sum_{i=1}^{m} a_i u_i$$

$$= \sum_{i=1}^{n} a_i u_i + \sum_{i=n+1}^{m} a_i u_i \in W_1 + W_2$$

$$\Rightarrow V \subseteq W_1 + W_2$$

Clearly $W_1 + W_2 \subseteq V$

Hence $V = W_1 + W_2$.

(2) $W_1 \cap W_2 = \{0\}$

Let $u \in W_1 \cap W_2$, $\Rightarrow u \in W_1$; $u \in W_2$.

$$\Rightarrow u = \sum_{i=1}^{n} b_i u_i \quad , \quad u = \sum_{i=n+1}^{m} c_i u_i$$

$$\Rightarrow \sum_{i=1}^{n} b_i u_i = \sum_{i=n+1}^{m} c_i u_i$$

$$\Rightarrow \sum_{i=1}^{n} b_i u_i - \sum_{i=n+1}^{m} c_i u_i = 0$$

$\Rightarrow$ vector on the LHS is spanned by vectors of α which are L.I.

$$\Rightarrow b_i = 0 \ \forall i \ \& \ c_i = 0 \ \forall i$$

$$\Rightarrow u = 0.$$