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Power Series Solution about an ordinary point:→

let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ —(1)

~~The Power~~ Let $x = x_0$ is an ordinary point of (1).

Then the power series solution of (1) about $x = x_0$ is of the form $y(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots$

where $c_0, c_1, c_2 \dots$ are constants.

Ex Find the power series solution about $x=0$ of the differential Equation $y' - 2y = 0$

Solⁿ The given diff. Equⁿ is $y' - 2y = 0$ —(1)

Let the power series solution of (1) about $x=0$ is of the form $y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$

$\Rightarrow y'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots$

Substitute $y(x)$ and $y'(x)$ in (1), we get

$(c_1 + 2c_2x + 3c_3x^2 + \dots) - 2(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots) = 0$

$\Rightarrow (c_1 - 2c_0) + 2(c_2 - c_1)x + (3c_3 - 2c_2)x^2 + \dots + [(m+1)c_m - 2c_m]x^m + \dots = 0$

Compare coeff of various powers of x, we get.

$c_1 - 2c_0 = 0 \Rightarrow c_1 = 2c_0$

$2c_2 - 2c_1 = 0 \Rightarrow c_2 = c_1 = 2c_0$

$3c_3 - 2c_2 = 0 \Rightarrow 3c_3 = 2c_2$

$\Rightarrow c_3 = \frac{2c_2}{3} = \frac{4c_0}{3} \dots$

$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$

$= c_0 + 2c_0x + 2c_0x^2 + \frac{4}{3}c_0x^3 + \dots$

$= c_0\left[1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots\right]$

$= c_0\left[1 + 2x + \frac{(2x)^2}{\lfloor 2} + \frac{(2x)^3}{\lfloor 3} + \dots\right] = c_0e^{2x}$

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Que Find the power Series Solution of $y''-y=0$ about $x=0$.

Solⁿ→ let the power Series solⁿ about $x=0$ is

$$y(x) = \sum_{m=0}^{\infty} c_m x^m$$

$$y'(x) = \sum_{m=1}^{\infty} m c_m x^{m-1}$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2}.$$

So $y''-y=0$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) c_m x^{m-2} - \sum_{m=0}^{\infty} c_m x^m = 0$$

$$\Rightarrow \underbrace{\sum_{t=0}^{\infty} (t+2)(t+1) c_{t+2} x^t}_{\text{Put } m-2=t} - \sum_{m=0}^{\infty} c_m x^m = 0$$

~~or~~ $\cancel{\sum_{m=0}}$

$$\text{or } \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m - \sum_{m=0}^{\infty} c_m x^m = 0$$

Comparing Coeff of like powers of x, we get

$$(m+2)(m+1) c_{m+2} - c_m = 0$$

$$\Rightarrow (m+2)(m+1) c_{m+2} = c_m$$

$$\Rightarrow c_{m+2} = \frac{c_m}{(m+1)(m+2)}$$

$$\Rightarrow c_2 = \frac{c_0}{(1)(2)} = \frac{1}{2} c_0 = \frac{1}{\lfloor 2} c_0$$

$$c_3 = \frac{c_1}{(2)(3)} = \frac{1}{6} c_1 = \frac{1}{\lfloor 3} c_1$$

$$c_4 = \frac{c_2}{(3)(4)} = \frac{1}{12} \times \frac{1}{2} c_0 = \frac{1}{\lfloor 4} c_0$$

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$$C_5 = \frac{C_3}{(4)(5)} = \frac{1}{\lfloor 3 \cdot 4 \cdot 5} C_1 = \frac{1}{\lfloor 5} C_1$$

and so on.

$$\therefore y(x) = C_0 + C_1 x + \frac{1}{\lfloor 2} C_0 x^2 + \frac{1}{\lfloor 3} C_1 x^3 + \frac{1}{\lfloor 4} C_0 x^4 + \frac{1}{\lfloor 5} C_1 x^5 + \dots$$

$$= C_0\left(1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \dots\right) + C_1\left(x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots\right)$$

where C_0 and C_1 are arbitrary constants.

Que Find the power series sol^n of $(1-x^2)y'' + 2xy' + y = 0$. about $x=0$. —(1)

Sol^n ⇒ let $y(x) = \sum_{m=0}^{\infty} C_m x^m$

$$y'(x) = \sum_{m=1}^{\infty} m C_m x^{m-1}$$

$$y''(x) = \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2}$$

Put $y(x)$, $y'(x)$ and $y''(x)$ in (1), we get

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} + 2x\left(\sum_{m=1}^{\infty} m C_m x^{m-1}\right) + \sum_{m=0}^{\infty} C_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) C_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) C_m x^m + 2\sum_{m=1}^{\infty} m C_m x^m + \sum_{m=0}^{\infty} C_m x^m = 0.$$

⇒ For $\sum_{m=2}^{\infty} m(m-1) C_m x^{m-2}$ Put $m-2=t$, $m = t+2$

$$\sum_{t=0}^{\infty} (t+2)(t+1) C_{t+2} x^t$$

or $\sum_{m=0}^{\infty} (m+2)(m+1) C_{m+2} x^m$ → We can replace t by m - just notation

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$$\therefore \sum_{m=0}^{\infty}(m+2)(m+1)c_{m+2}x^m - \sum_{m=2}^{\infty}m(m-1)c_m x^m + 2\sum_{m=1}^{\infty}mc_m x^m + \sum_{m=0}^{\infty}c_m x^m = 0$$

$$\Rightarrow 2c_2 + 6c_3x + \sum_{m=2}^{\infty}(m+2)(m+1)c_{m+2}x^m - \sum_{m=2}^{\infty}m(m-1)c_m x^m$$
$$+ 2c_1x + 2\sum_{m=2}^{\infty}mc_m x^m + c_0 + c_1x + \sum_{m=2}^{\infty}c_m x^m = 0$$

$$\Rightarrow 2c_2 + 6c_3x + 2c_1x + c_0 + c_1x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)c_{m+2} - m(m-1)c_m + 2mc_m + c_m\right]x^m = 0$$

$$\Rightarrow 2c_2 + c_0 = 0 \Rightarrow c_2 = \frac{-c_0}{2}$$

$$6c_3 + 3c_1 = 0 \Rightarrow c_3 = \frac{-1}{2}c_1$$

$$(m+2)(m+1)c_{m+2} - m(m-1)c_m + 2mc_m + c_m = 0$$

$$\Rightarrow (m+2)(m+1)c_{m+2} - (m^2 - 3m - 1)c_m = 0$$

$$\Rightarrow c_{m+2} = \frac{(m^2-3m-1)c_m}{(m+1)(m+2)} \;;\; m$$

$$\Rightarrow c_2 = \frac{-c_0}{2}, \quad c_3 = \frac{-1}{2}c_1, \quad c_4 = \frac{-3}{12}c_2 = \frac{1}{8}c_0 \dots$$

$$\therefore y(x) = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 \dots\right) +$$
$$c_1\left(x - \frac{x^3}{2} + \dots\right)$$