14. APPLICATIONS OF LAPLACE TRANSFORM.

Laplace Transform of a Periodic function

Let $f(t)$ be a periodic function with period T.

that is $f(t+T) = f(t)$. then by definition

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \cdots$$

$$= I_1 + I_2 + I_3 + \cdots$$

In I_2, putting $t = T+u$ so that $dt = du$.

we get $$I_2 = \int_T^{2T} e^{-st} f(t)\,dt = \int_0^T e^{-s(T+u)} f(T+u)\,du.$$

$(\because f(T+u) = f(u))$

$$= e^{-sT} \int_0^T e^{-su} f(u)\,du.$$

$$= e^{-sT} I_1$$

Similarly, In I_3, putting $t = 2T+u$ so that $dt = du$

we get $$I_3 = \int_{2T}^{3T} e^{-st} f(t)\,dt = \int_0^T e^{-s(2T+u)} f(2T+u)\,du.$$

$$= e^{-2sT} \int_0^T e^{-su} f(u)\,du = e^{-2sT} I_1$$

Proceeding the same way, we get

$$I_4 = e^{-3sT} I_1 \quad , \quad I_5 = e^{-4sT} I_1 \; ---$$

$$\therefore \; L(f(t)) = I_1 + e^{-sT} I_1 + e^{-2sT} I_1 + e^{-3sT} I_1 + ---$$

$$= \frac{I}{1-e^{-st}} \quad \left(\because \text{ sum of infinite gp with common ratio less than } 1 \right)$$

$$\Rightarrow \boxed{L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt}$$

__Ex__ If $f(t) = t^2$ for $0 < t < 2$ and $f(t+2) = f(t)$ for $t > 2$. find $L(f(t))$.

__Solution__ Since $f(t)$ is a periodic function with period $T = 2$

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt.$$

Here $f(t) = t^2$, Hence.

$$L(f(t)) = \frac{1}{1-e^{-2s}} \int_0^2 t^2 e^{-st}\, dt.$$

$$= \frac{1}{1-e^{-2s}} \left[\frac{t^2 e^{-st}}{-s} \Big|_0^2 - \int_0^2 \frac{2t\, e^{-st}}{-s}\, dt \right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4}{s}e^{-2s} + \frac{2}{s}\int_0^2 t e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{s} + \frac{2}{s}\left[\frac{te^{-st}}{-s}\Big|_0^2 + \int_0^2 \frac{1\cdot e^{-st}}{s}\,dt\right]\right]$$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{s} - \frac{4e^{-2s}}{s^2} - \frac{2e^{-2s}}{s^3} + \frac{2}{s^3}\right].$$

$$= \frac{-2e^{-2s}}{1-e^{-2s}}\left[\frac{2}{s} + \frac{2}{s^2} + \frac{1}{s^3} - \frac{e^{2s}}{s^3}\right].$$

X

Ex (1) For the periodic function $f(t)$ of period 4, defined by $f(t) = \begin{cases} 3t & 0<t<2 \\ 6 & 2<t<4 \end{cases}$. find $L(f(t))$.

Solution Given $f(t+4) = f(t)$

$\therefore$ For the periodic function $f(t)$ of periodic 4, we have

$$L(f(t)) = \frac{1}{1-e^{-4s}}\left[\int_0^4 e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\int_0^2 3te^{-st}\,dt + \int_2^4 6e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3te^{-st}}{-s}\Big|_0^2 + \frac{3e^{-st}}{s(-s)}\Big|_0^2 - \left[\frac{6}{s}e^{-st}\right]_2^4\right]$$

$$= \frac{1}{1-e^{-4s}}\left[-\frac{6}{s}e^{-2s} + \frac{3}{s^2}(1-e^{-2s}) - \frac{6}{s}e^{-4s} + \frac{6e^{-2s}}{s}\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3}{s^2}(1-e^{-2s}) - \frac{6e^{-4s}}{s}\right].$$

Ex Find the Laplace transform of a periodic function $f(t)$ given by

$$f(t) = \begin{cases} 1 & 0<t<L \\ -1 & L<t<2L. \end{cases}$$

Solution :-

$$L(f(t)) = \frac{1}{1-e^{-2sL}}\int_0^{2L} e^{-st} f(t)\,dt.$$

$$= \frac{1}{1-e^{-2sL}}\left[\int_0^{L} 1\cdot e^{-st}\,dt + \int_L^{2L}(-1)e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2sL}}\left[\left[\frac{e^{-st}}{-s}\right]_0^L - \left[\frac{e^{-st}}{-s}\right]_L^{2L}\right]$$

$$= \frac{1}{1-e^{-2sL}}\left[\frac{e^{-sL}}{-s} + \frac{1}{s} + \left(e^{-2sL} - e^{-sL}\right)\right]$$

$$= \frac{1}{1-e^{-2sL}}\left[\frac{1}{s} + \frac{e^{-2sL}}{s} - \frac{2e^{-sL}}{s}\right].$$

Ex Find the Laplace transform of 2c periodic function (9)

$$f(t) = \begin{cases} t & 0 < t < c \\ 2c - t & c < t < 2c \end{cases}$$

Solution

$$L(f(t)) = \frac{1}{1 - e^{-2cs}} \int_0^{2c} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2cs}} \left[\int_0^{c} t e^{-st}\,dt + \int_c^{2c} (2c-t) e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2cs}} \left[\left[\frac{t e^{-st}}{-s} - \frac{e^{-st}}{+s^2} \right]_0^c + \left[\frac{(2c-t) e^{-st}}{-s} - \frac{(-1) e^{-st}}{+s^2} \right]_c^{2c} \right]$$

$$= \frac{1}{1-e^{-2cs}} \left[\cancel{\frac{-c}{s} e^{-sc}} - \frac{e^{-sc}}{s^2} + \frac{1}{s^2} + \frac{e^{-2cs}}{s^2} + \cancel{\frac{c}{s} e^{-s(c)}} - \frac{e^{-cs}}{s^2} \right]$$

$$= \frac{1}{s^2(1-e^{-2cs})} \left[1 - e^{-sc} + e^{-2sc} - e^{-sc} \right]$$

$$= \frac{1}{s^2(1-e^{-2cs})} (1 + e^{-2sc} - 2e^{-sc}) = \frac{(1-e^{-sc})^2}{s^2(1-e^{-cs})(1+e^{-cs})}$$

$$= \frac{(1-e^{-sc})}{s^2(1+e^{-cs})}.$$

Ex The $\frac{2\pi}{\omega}$ -periodic function $f(t)$ is given by

$$f(t) = \begin{cases} a \sin \omega t & 0 < t < \pi/\omega \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

find its laplace transform.

Solution $$L(f(t)) = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} (a \sin \omega t)\, e^{-st}\, dt\right]$$

$$= \frac{a}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} (\sin \omega t)\, e^{-st}\, dt\right]$$

let $$I = \int \underset{1.}{\sin \omega t}\, \underset{2.}{e^{-st}}\, dt$$

$$= \frac{\sin \omega t\, e^{-st}}{-s} - \int \omega \cos \omega t\, \frac{e^{-st}}{-s}\, dt$$

$$= \frac{\sin \omega t\, e^{-st}}{-s} + \frac{\omega}{s}\int \underset{1.}{\cos \omega t}\, \underset{2.}{e^{-st}}\, dt$$

$$= \frac{\sin \omega t\, e^{-st}}{-s} + \frac{\omega}{s}\left[\frac{\cos \omega t\, e^{-st}}{-s} - \int -\sin \omega t\, (\omega)\, \frac{e^{-st}}{-s}\, dt\right]$$

$$= \frac{-\sin \omega t\, e^{-st}}{s} - \frac{\omega \cos \omega t\, e^{-st}}{s^2} - \frac{\omega^2}{s^2}\int \sin \omega t\, e^{-st}\, dt$$

$$\therefore \quad I = -\frac{\sin \omega t \, e^{-st}}{s} - \frac{\omega}{s^2}\cos \omega t \, e^{-st} - \frac{\omega^2}{s^2} I$$

$$\Rightarrow I \left(1 + \frac{\omega^2}{s^2}\right) = -\frac{\sin \omega t \, e^{-st}}{s} - \frac{\omega \cos \omega t \, e^{-st}}{s^2}$$

$$\therefore \quad L(f(t)) = \frac{a}{1 - e^{-2\pi s/\omega}} \left[\frac{-\sin \omega t \, e^{-st}}{s} - \frac{\omega \cos \omega t \, e^{-st}}{s^2} \right]_0^{\pi/\omega} \left[\frac{s^2}{s^2 + \omega^2} \right]$$

$$= \frac{a s^2}{(s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} \left[\frac{-\omega(-1) e^{-s\pi/\omega}}{s^2} + \frac{\omega e^{-s(0)}}{s^2} \right]$$

$$= \frac{a s^2}{s^2 (s^2 + \omega^2)(1 - e^{-2\pi s/\omega})} \left[\omega e^{-s\pi/\omega} + \omega \right]$$

$$= \frac{a\omega (1 + e^{-s\pi/\omega})}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})(1 + e^{-\pi s/\omega})}$$

$$= \frac{a\omega}{(s^2 + \omega^2)(1 - e^{-\pi s/\omega})} .$$
