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Engineering Mathematics - I BS - 111 Dr. Leena Mittal

Partial Differentiation and Its Applications

(1) Function : A symbol z which has a definite value for every pair of values of x and y is called a function of two variable x and y and written as $z = f(x,y)$ or $\phi(x,y)$

(2) Limits : The function $f(x,y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by point (x,y) as $x \to a$ and $y \to b$

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) = l$$

(3) Continuity : A function $f(x,y)$ is said to be continuous at the points (a,b) if

$$\lim_{\substack{x \to a \\ y \to b}} f(x,y) \text{ exsist } = f(a,b)$$

Analysis of Continuous and Discontinuous Function

Usually the limit is same irrespective of path along which the point (x,y) approaches (a,b) and

$$\lim_{x \to a}\left[\lim_{y \to b} f(x,y)\right] = \lim_{y \to b}\left[\lim_{x \to a} f(x,y)\right]$$

To check for discontinuous functions we can do subsitution according to the problem.

Linear subitution : $y = mx$ when $\begin{Bmatrix} x\to 0 \\ y \to 0 \end{Bmatrix}$

quadratic subitution : $y = mx^2$

For Eg: $\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(\frac{x-y}{x+y}\right) = \lim\limits_{\substack{x\to 0\\ y\to 0}} \left(\frac{x-mx}{x+mx}\right)$

$= \lim\limits_{x\to 0} \frac{x(1-m)}{x(1+m)} = \frac{1-m}{1+m}$

Here, the value depends on m which can differ the value of function. Hence discontinuous function

Homework Questions [Pg: 198]

Evaluate :

1. $\lim\limits_{\substack{x\to 1\\ y\to 2}} \left(\frac{2x^2y}{x^2+y^2+1}\right)$

$\lim\limits_{x\to 1}\left[\lim\limits_{y\to 2} \frac{2x^2y}{x^2+y^2+1}\right] = \lim\limits_{x\to 1}\left[\frac{4x^2}{x^2+4+1}\right] = \frac{4}{6} = \frac{2}{3}$

$\lim\limits_{y\to 2}\left[\lim\limits_{x\to 1} \frac{2x^2y}{x^2+y^2+1}\right] = \lim\limits_{y\to 2}\left[\frac{2y}{2+y^2}\right] = \frac{4}{6} = \frac{2}{3}$

2. $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \left(\frac{xy+1}{x^2+2y^2}\right)$

$\lim\limits_{y\to 2}\left[\lim\limits_{x\to\infty}\left(\frac{xy+1}{x^2+2y^2}\right)\right] = \lim\limits_{y\to 2}\left[\lim\limits_{x\to\infty} \frac{y/x + 1/x^2}{1+2(y/x)^2}\right] = 0$

$$\lim_{x\to\infty}\left[\lim_{y\to 2}\left(\frac{xy+1}{x^2+2y^2}\right)\right] = \lim_{x\to\infty}\left(\frac{2x+1}{x^2+8}\right) = \lim_{x\to\infty}\left(\frac{2/x+1/x^2}{1+8/x^2}\right)$$
$$= 0$$

3. $$\lim_{\substack{x\to 1\\ y\to 1}}\left(\frac{xy-x}{xy-y}\right) \qquad [y=mx]$$

$$\lim_{\substack{x\to 1\\ y\to 1}}\left(\frac{mx^2-x}{mx^2-mx}\right) = \lim_{\substack{x\to 1\\ y\to 1}}\left(\frac{mx-1}{mx-m}\right) = \lim_{y\to 1}\left(\frac{m-1}{0}\right)$$
$$= \infty$$

$$\lim_{\substack{x\to 1\\ y\to 1}} \cancel{mx^2}\left(\frac{y^2/m^2-y/m}{y^2/m-y}\right) = \lim_{\substack{x\to 1\\ y\to 1}} \frac{1/m^2-1/m}{1/m-1} = 0$$

4. Evaluate continuity of function

$$f(x) = \begin{cases} \dfrac{xy}{x^2+y^2} & f(x,y) \neq (0,0) \\ 0 & f(x,y) = (0,0) \end{cases}$$

$$\lim_{\substack{x\to 1\\ y\to 1}} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}$$

Function is discontinuous as value depends on m

Properties of limits

If $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = l$ and $\lim_{\substack{x\to a\\ y\to b}} g(x,y) = m$

(i) If $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x,y) \pm g(x,y)] = l \pm m$

(ii) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x,y) \cdot g(x,y)] = l \cdot m$

(iii) $\underset{\substack{x \to a \\ y \to b}}{Lt} [f(x,y) / g(x,y)] = l / m \quad (m \neq 0)$

Partial Derivatives

The derivative of z with respect to x treating y as constant is called partial derivative of z with respect to x and is denoted by symbols

$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x,y), D_x f$ Thus,

$$\frac{\partial z}{\partial x} = \underset{\delta x \to 0}{Lt} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$ [by first principle]

Similarly all results apply to partial derivative of z with respect to y.

$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x,y), D_y f$ Thus,

$$\frac{\partial z}{\partial y} = \underset{\delta y \to 0}{Lt} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

In general f_x and f_y can be further partially differentiated with respect to x and y

$$\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{xx}, f_{yy}, f_{yx}, f_{xy}$$

can also be written as

Note: $\frac{\partial z}{\partial x} = p \quad \frac{\partial z}{\partial y} = q \quad \frac{\partial^2 z}{\partial x^2} = r \quad \frac{\partial^2 z}{\partial y \partial x} = s \quad \frac{\partial^2 z}{\partial y^2} = t$

Note: $\frac{\partial^2 z}{\partial x . \partial y} = \frac{\partial^2 z}{\partial y . \partial x}$ [In ordinary cases]

For e.g. Given function $f(x,y) = z = 2x^2y + 3x + 4y^3$

$\frac{\partial z}{\partial x} = 4xy + 3 \qquad \frac{\partial^2 z}{\partial x^2} = 4y \qquad \frac{\partial z}{\partial x \partial y} = 4x$

$\frac{\partial z}{\partial y} = 2x^2 + 12y^2 \qquad \frac{\partial^2 z}{\partial y^2} = 24y \qquad \frac{\partial z}{\partial y \partial x} = 4x$

Homework Questions [Pg: 199, 200, 201]

1. Evaluate p, q, r, s, t of $z = \sin x \cos y$

$\frac{\partial z}{\partial x} = \cos x \cos y \qquad \frac{\partial^2 z}{\partial x^2} = -\sin x \cos y$

$\frac{\partial z}{\partial y} = -\sin x \sin y \qquad \frac{\partial^2 z}{\partial y^2} = -\sin x \cos y$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -\cos x \sin y$$

2. Evaluate p, q, r, s, t of $z = x^3 + y^3 - 3axy$

$\frac{\partial z}{\partial x} = 3x^2 - 3ay \qquad \frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial z}{\partial x \cdot \partial y} = -3a$

$\frac{\partial z}{\partial y} = 3y^2 - 3ax \qquad \frac{\partial^2 z}{\partial y^2} = 6y \qquad \frac{\partial x}{\partial y \cdot \partial x} = -3a$

3. If $x = f(x+ct) + \psi(x-ct)$, then prove that

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

$\frac{dz}{dx} = f'(x+ct) \cdot c + \psi'(x-ct) \cdot c$

$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) \cdot c^2 + \psi''(x-ct) \cdot c^2$

$\frac{\partial^2 z}{\partial t^2} = c^2\left[f''(x+ct) + \psi''(x-ct)\right]$ ------ (i)

$\frac{\partial^2 z}{\partial x^2} = f''(x+ct) + \psi''(x-ct)$ ------ (ii)

From (i) and (ii) $\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$

$\therefore$ Proved

4. If function $v(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot (2x) = -x(x^2+y^2+z^2)^{-3/2}$

$\frac{\partial^2 v}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} + \frac{3x \times 2x}{2}(x^2+y^2+z^2)^{-5/2}$

$\frac{\partial^2 v}{\partial x^2} = -(x^2+y^2+z^2)^{-5/2}\left[x^2+y^2+z^2-3x^2\right]$

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$$\frac{\partial^2 z}{\partial x^2} = +(x^2+y^2+z^2)^{-5/2}\boxed{2x^2-y^2-z^2}$$

$$\frac{\partial^2 z}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}\left[-x^2+2y^2-z^2\right]$$

$$\frac{\partial^2 V}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}\left[-x^2-y^2+2z^2\right]$$

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}\,[0] = 0$$

$\therefore$ Proved

Note: $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = 0$ is Laplace Equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)V = 0$$

$\Delta V = 0$ where

Δ : laplacian symbol

Δ : scalar quantity

$\bar{\Delta}$: del

If laplacian of any function (V) is 0 then is called harmonic function.

$\frac{\partial V}{\partial x}\hat{i}+\frac{\partial V}{\partial y}\hat{j}+\frac{\partial V}{\partial z}\hat{k} = 0$ is nabla Equation

$$\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)V = 0$$

$\nabla V = 0$ where

∇ : nablaian symbol

∇ : vector quantity

∇ : del

Note: $\nabla^2 = \nabla.\nabla = \Delta$

Homogeneous Equation

Any function $f(x,y)$ which can be expressed in the form of $x^n\phi(y/x)$ is called homogeneous equation

Euler Theorem on Homogeneous Functions

If u be a homogeneous function of degree n in x and y, then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Total Derivative

If $u = f(x,y)$ where $x = \phi(t)$ and $y = \psi(t)$ then we can express u as a function of t alone by subituting the values of x and y in $f(x,y)$

Chain Rule:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

Proof:

$$\frac{du}{dt} = \lim_{\delta t\to 0}\frac{\delta u}{\delta t} = \lim_{\delta t\to 0}\frac{u+\delta u-u}{\delta t}$$

$$u+\delta u = f(x+\delta x, y+\delta y)$$

$$u+\delta u-u = f(x+\delta x, y+\delta y) - f(x,y)$$

$$\frac{\delta u}{\delta t} = \frac{f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta t} + \frac{f(x+\delta x, y)-f(x,y)}{\delta t}$$

$$\lim_{\delta t\to 0}\frac{\delta u}{\delta t} = \lim_{\delta t\to 0}\frac{\partial}{\partial y}(f(x+\delta x), y)\cdot\frac{dy}{dt} + \lim_{\delta t\to 0}\frac{\delta}{\delta x}(f(x,y))\cdot\frac{dx}{dt}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial x}\cdot\frac{dx}{dt}$$

$\therefore$ Proved

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If $t \to x$ $\quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$

$$\frac{du}{dy} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dy} + \frac{\partial u}{\partial y}$$

Differentiation of Implicit function

If $f(x,y) = c$ be an implicit relation between x and y which defines as a differentiation function of x

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y} = \frac{-f_x}{f_y}$$

for first differential coefficients of an implicit function

Example: If $u = x\log(xy)$ where $x^3 + y^3 + 3xy = 1$ then find $\left(\frac{du}{dx}\right)$?

$\frac{\partial u}{\partial x} = \log(xy) + 1 \qquad \frac{dy}{dx} = -\left(\frac{x^2+1}{y^2+1}\right)$

$\frac{\partial u}{\partial y} = \frac{x}{y}$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= \log(xy) + 1 + \frac{x}{y}\left(-\frac{x^2+1}{y^2+1}\right)$$

$$= \log(xy) - \frac{x}{y}\left(\frac{x^2+1}{y^2+1}\right) + 1$$

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Example If $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t$ and $y = t^2$ then find $\left(\frac{du}{dt}\right)$

$$\frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)\cdot e^t + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)\cdot 2t$$

$$\frac{du}{dt} = \frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right)\cdot e^t - \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^4}\right)\cdot 2t$$

$$\frac{du}{dt} = \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - 2\cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^3}\right)$$

Change of Variables

If $u = f(r,s)$

where $r = \phi(x,y)$

$s = \psi(x,y)$

then
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

Example: If function $u = f(x-y,\ y-z,\ z-x)$ then prove that
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$p = x-y \quad q = y-z \quad r = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \quad \text{------(i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial q} \quad \text{------(ii)} \qquad \frac{\partial u}{\partial z} = \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} \quad \text{------(iii)}$$

adding (i) (ii) (iii)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Example: Transform $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates where u is a function in x and y

$x = r\cos\theta$

$y = r\sin\theta \quad r = \sqrt{x^2+y^2} \quad \theta = \tan^{-1}(y/x)$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta \qquad \frac{\partial \theta}{\partial x} = \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \cos\theta\left(\frac{\partial u}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial \theta}\right)$$

$$\frac{\partial u}{\partial x} = \cos\theta\left(\frac{\partial u}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial \theta}\right)$$

Similarly, $\frac{\partial u}{\partial y} = \sin\theta\left(\frac{\partial u}{\partial r}\right) + \frac{\cos\theta}{r}\left(\frac{\partial u}{\partial \theta}\right)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left[\cos\theta\left(\frac{\partial u}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial \theta}\right)\right]\left[\cos\theta\left(\frac{\partial u}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial u}{\partial \theta}\right)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta\left(\frac{\partial^2 u}{\partial x^2}\right) - \frac{2\sin\theta\cos\theta}{r}\left(\frac{\partial^2 u}{\partial r\,\partial \theta}\right) + \frac{\sin^2\theta}{r^2}\left(\frac{\partial^2 \theta}{\partial \theta^2}\right)$$

$$+\frac{\sin^2\theta}{r}\left(\frac{\partial u}{\partial r}\right) + \frac{2\sin\theta\cos\theta}{r^2}\left(\frac{\partial u}{\partial \theta}\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \theta}{\partial u}\right)$$

$$= \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\left(\frac{\partial^2 u}{\partial r \partial \theta}\right) + \frac{\cos^2\theta}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right)$$

$$+ \frac{\cos^2\theta}{r^2}\left(\frac{\partial u}{\partial r}\right) - \frac{2\sin\theta\cos\theta}{r^2}\left(\frac{\partial u}{\partial r}\right)$$

adding (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{1}{r}\left(\frac{\partial u}{\partial r}\right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{1}{r}\left(\frac{\partial u}{\partial r}\right) = 0$$

$\therefore$ Proved

Homework Questions

1. If function $z = u^2 + v^2$ where $u = at^2$ and $v = 2at$, then find $\left(\frac{dz}{dt}\right)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial t}$$

$$\frac{dz}{dt} = 2u \cdot 2at + 2v \cdot at$$

$$\frac{dz}{dt} = 2at^2 \cdot 2at + 4a^2t^2 = 4a^2t^3 + 4a^2t^2$$

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2. If function $u(x,y) = \tan^{-1}(y/x)$ where, $x = e^{t} - e^{-t}$ and $y = e^{t} + e^{-t}$ then find $\left(\frac{du}{dt}\right)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{du}{dt} = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) \cdot (e^{t}+e^{-t}) + \left(\frac{x^2}{x^2+y^2}\right)\frac{1}{y} \cdot (e^{t}-e^{-t})$$

$$\frac{du}{dt} = \frac{-(e^{t}+e^{-t})^2}{(e^{t}-e^{-t})^2+(e^{t}+e^{-t})^2} + \frac{(e^{t}-e^{-t})^3}{(e^{t}-e^{-t})^2+(e^{t}+e^{-t})^2}$$

3. If $f(x,y) = 0$ then show that

$$\frac{d^2y}{dx^2} = -\left(\frac{q^2r - 2pqs + p^2t}{q^3}\right)$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = \frac{-p}{q}$$

$$\frac{d^2y}{dx^2} = \frac{-d}{dx}\left(\frac{dy}{dx}\right) = \frac{-d}{dx}\left(\frac{p}{q}\right)$$

$$\frac{d^2y}{dx^2} = -\left(\frac{q(dp/dx) - p(dq/dx)}{q^2}\right)$$

$$\frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} \cdot \left(\frac{dy}{dx}\right) = r + s\left(\frac{-p}{q}\right) = \left(\frac{qr - ps}{q}\right)$$

$$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y} \cdot \left(\frac{dy}{dx}\right) = s + t\left(\frac{-p}{q}\right) = \left(\frac{qs - pt}{q}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{q^2}\left[q\frac{(qr-ps)}{(q)} - p\left(\frac{qs-pt}{q}\right)\right] = -\frac{(q^2r - 2qps + p^2t)}{q^3}$$

$\therefore$ Proved.

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4. At a given instant the sides of a rectangle are 4ft. and 3ft. respectively and they are increasing at rate of 1.5 ft/s and 0.5 ft/s respectively. Find the rate at which area is increasing at that instant.

$$A = xy$$

$$\frac{dA}{dt} = \frac{\partial A}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial A}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$\frac{dA}{dt} = 3(1.5) + 4(0.5) = 6.5 \text{ ft/s}$$

5. If function $z(x,y) = 2xy^2 - 3x^2y$ where x increases at rate of 2cm/s and it passes through the value of 3cm show that if y is passing through the value of $y = 1$cm, y must be decreasing $2\frac{2}{15}$ cm/s in order that z must remain constant

$$z = 2xy^2 - 3x^2y$$

$$\frac{dz}{dt} = 0 \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$0 = (2y^2 - 6xy) \cdot 2 + (4xy - 3x^2)\left(\frac{dy}{dt}\right)$$

$$0 = (2-18) \cdot 2 + (12-27)\frac{dy}{dx}$$

$$0 = -32 - 15\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{-32}{15} = -2\left(\frac{2}{15}\right) \text{ cm/s}$$

y must be decreasing $\therefore$ Proved.

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Homework Questions

1. If function $u = f(r,s)$ where $r = x+at$ and $s = y+bt$, x, y, t are independent variables show that

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial u}{\partial x}\right) + b\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial t}$$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial r}\right)a + \left(\frac{\partial u}{\partial s}\right)\cdot b \quad \text{------- (1)}$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} \quad \& \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\times 1 + \frac{\partial u}{\partial s}\times 0 = \frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\times\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\times\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\times 0 + \frac{\partial u}{\partial s}\times 1 = \frac{\partial u}{\partial s}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}} \qquad \boxed{\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}}$$

from (1)

$$\frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)a + \left(\frac{\partial u}{\partial y}\right)b$$

$\therefore$ Proved

2. If function $u = f(e^{y-z}, e^{z-x}, e^{x-y})$ then prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = -e^{z-x} + e^{x-y} \qquad \frac{\partial u}{\partial y} = e^{y-z} - e^{x-y} \qquad \frac{\partial u}{\partial z} = -e^{y-z} + e^{z-x}$$

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0}$$

$\therefore$ Proved

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3. If function $z = f(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$ then prove that,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \times \cos\theta + \frac{\partial z}{\partial y} \times \sin\theta \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \times (-r\sin\theta) + \frac{\partial z}{\partial y} \times (r\cos\theta) \quad \text{--- (2)}$$

adding (1) and (2) and squaring

$$\left(\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta + \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2\theta$$

$$\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta$$

$$\left(\frac{\partial z}{\partial r}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta$$

$$\boxed{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2}$$

$\therefore$ Proved

4. If function $u(x,y) = x^2 - y^2$ and function $v(x,y) = 2xy$ where $f(x,y) = \theta(u,v)$, then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

$u = x^2 - y^2$ $\qquad$ $v = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = 2x\left(\frac{\partial z}{\partial u}\right) + 2y\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial v}{\partial y} = -2y\left(\frac{\partial z}{\partial u}\right) + 2x\left(\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\theta(u,v)) = \frac{\partial \theta}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial \theta}{\partial v}\left(\frac{\partial v}{\partial x}\right)$$

$$= 2x\left(\frac{\partial \theta}{\partial u}\right) + 2y\left(\frac{\partial \theta}{\partial v}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) =$$

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Jacobians

u and v are two functions in $f_1(x,y)$ and $f_2(x,y)$ then jacobian of u,v w.r.t x,y is represented as

$$J\left(\frac{u,v}{x,y}\right) \quad \text{OR} \quad \frac{\partial(u,v)}{\partial(x,y)}$$

and is defined as $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly for three functions and three variables

Properties Of Jacobians

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Example In spherical polar coordinates, $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$ $z = r\cos\theta$ then show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$$\frac{\partial y}{\partial r} = r\sin\theta\sin\phi \qquad \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi \qquad \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi \qquad \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial z}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial\theta} = -r\sin\theta \qquad \frac{\partial z}{\partial\phi} = 0$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi(-r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(r\cos\theta\sin\theta\cos\phi)$$
$$- r\sin\theta\sin\phi(-r^2\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$
$$= r^2\sin\theta$$

I. If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $J.J' = 1$

Proof: Let $u = f(x,y)$ and $v = g(x,y)$ and on solving we get, x and y we get $x = \phi(u,v)$ and $y = \psi(u,v)$

Then,

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial v}{\partial v} = \emptyset I = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$J.J' = \begin{vmatrix} \partial v/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \cdot \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$J.J' = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial y/\partial x & \partial u/\partial y \end{vmatrix} \times \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix}$$

$$J.J' = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \therefore \text{Proved}$$

II Chain Rule for Jacobians :

If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial r & \partial u/\partial s \\ \partial v/\partial r & \partial v/\partial s \end{vmatrix} \times \begin{vmatrix} \partial r/\partial x & \partial s/\partial x \\ \partial r/\partial y & \partial s/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\cdot\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \qquad \therefore \text{Proved}$$

Example: In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$ then show that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$

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$$\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \quad \frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = -r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Example If $u = x^2 - y^2$ and $v = 2xy$ where $x = r\cos\theta$ and $y = r\sin\theta$ find, $\left[\frac{\partial(u,v)}{\partial(r,\theta)}\right]$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

Since $u = x^2 - y^2 \quad v = 2xy$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= 4(x^2+y^2)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = 4(x^2+y^2)\,r = (4r^2)r = \boxed{4r^3}$$

Jacobians of Implicit Functions

If $u_1\ u_2\ u_3$ instead of being given explicitly in terms of $x_1\ x_2\ x_3$ be connected with them by equations

$$f_1(u_1 u_2 u_3\ x_1 x_2 x_3) = 0 \qquad f_2(u_1 u_2 u_3\ x_1 x_2 x_3) = 0$$
$$f_3(u_1 u_2 u_3\ x_1 x_2 x_3) = 0$$

then,

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$$\frac{\partial(u_1 u_2 u_3)}{\partial(x_1 x_2 x_3)} = (-1)^3 \frac{\partial(f_1 f_2 f_3)}{\partial(x_1 x_2 x_3)} \div \frac{\partial(f_1 f_2 f_3)}{\partial(u_1 u_2 u_3)}$$

Example. If $u = xyz$, $v = x^2+y^2+z^2$ and $w = x+y+z$ find $\left[\frac{\partial(x,y,z)}{\partial(u,v,w)}\right]$

$f_1 = u - xyz \quad f_2 = v - (x^2+y^2+z^2) \quad f_3 = w - (x+y+z)$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\partial(f_1 f_2 f_3)}{\partial(x_1 x_2 x_3)} \div \frac{\partial(f_1 f_2 f_3)}{\partial(x\,y\,z)}$$

$$= (-1)^3 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \div \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$$= \frac{1}{2(x-y)(y-z)(z-x)}$$

Functional Relationship

If $u_1 u_2 u_3$ be function of $x_1 x_2 x_3$ then the necessary and sufficient conditions for the existence of a functional relationship of form $f(u_1 u_2 u_3) = 0$

$$J\left(\frac{u_1 u_2 u_3}{x_1 x_2 x_3}\right) = 0$$

xample. If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ and $v = \sin^{-1}x + \sin^{-1}y$ then show that u, v are functionally related and find the relationship

$$\frac{\partial u}{\partial x} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} \qquad \frac{\partial u}{\partial y} = \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= 1 - \frac{xy}{\sqrt{(1-x^2)(1-y^2)}} - 1 + \frac{xy}{\sqrt{(1-x^2)(1-y^2)}} = 0$$

Hence u and v are functionally related

We have, $v = \sin^{-1}x + \sin^{-1}y$

$$v = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$$

$$\boxed{u = \sin v}$$

Homework Questions

1. If $x = u(1-v)$ and $y = uv$ then prove that $J \cdot J' = 1$

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1 \qquad x+y = u$$

$$\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} \times \begin{vmatrix} 1/u & 1/u \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = LHS$$

$$u \times \left\{\frac{x+y}{(x+y)^2}\right\} = LHS$$

LHS = RHS

$$u \times \frac{1}{x+y} = \frac{u \times 1}{u} = 1$$

$\therefore$ Proved

2. If $u^3+v^3 = x+y$ and $u^2+v^2 = x^3+y^3$ then show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\left\{\frac{y^2-x^2}{uv(u-v)}\right\}$

$$u^3+v^3 = x+y$$

$$3u^2\left(\frac{\partial u}{\partial x}\right) + 3v^2\left(\frac{\partial v}{\partial x}\right) = 1$$

$$2u\left(\frac{\partial u}{\partial x}\right) + 2v\left(\frac{\partial v}{\partial x}\right) = 3x^2$$

$$\begin{bmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{bmatrix} \cdot \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial x \end{bmatrix} = \begin{bmatrix} 1 \\ 3x^2 \end{bmatrix}$$

Let $f_1 = u^3+v^3-x-y=0$ and $f_2 = u^2+v^2-x^3-y^3=0$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1 f_2)}{\partial(x,y)} \div \frac{\partial(f_1 f_2)}{\partial(u,v)}$$

$$= (-1)^2 \left\{3(y^2-x^2) \div 6uv(u-v)\right\}$$

$$= \frac{3(y^2-x^2)}{6uv(u-v)}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\left\{\frac{y^2-x^2}{uv(u-v)}\right\} \quad \therefore \text{Proved}$$

3. If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Are u and v functionally related. If so, find this relationship

$$v = \tan^{-1}x + \tan^{-1}y$$

$$v = \left(\frac{x+y}{1-xy}\right)\tan^{-1} \qquad v = \tan^{-1}u$$

$$\boxed{u = \tan v}$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)+(x+y)y}{(1-xy)^2} = \frac{(1+y^2)}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)+(x+y)x}{(1-xy)^2} = \frac{(1+y^2)}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(1-xy)^2}{(1-xy)^2+(x+y)^2} = \frac{(1-xy)^2}{(1+x^2)(1+y^2)} \times \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(1-xy)^2}{(1-xy)^2+(x+y)^2} = \frac{(1-xy)^2}{(1+x^2)(1+y^2)} \times \frac{1+x^2}{(1-xy)^2}$$

$$\frac{1+y^2}{(1-xy)^2} \cdot \frac{1}{(1+y^2)} - \frac{1+x^2}{(1-xy)^2} \times \frac{1}{(1+x^2)} = 0$$

$$\boxed{\frac{\partial (u,v)}{\partial (x,y)} = 0}$$

u and v are functionally related

Geometrical Interpretation of partial derivative

$\frac{\partial z}{\partial x}$ at point $P(x,y,z)$ is the tangent of the angle which the tangent at P to the curve of intersection of this surface $z=f(x,y)$ and the plane $y=b$ (which passes through point P) makes with the line parallel to the x-axis

$\frac{\partial z}{\partial y}$ at point $P(x,y)$ is the tangent of the angle which the tangent at P to the curve of intersection of this surface $z=f(x,y)$ and the plane $x=a$ (which passes through point P) makes with the line parallel to Y-axis

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equation of tangent plane to the surface $f(x,y,z)$ at $P(x,y,z)$ is

$$\frac{\partial f}{\partial x}(X-x_0)+\frac{\partial f}{\partial y}(Y-y_0)+\frac{\partial f}{\partial z}(Z-z_0)=0$$

equation of normal line to the surface $f(x,y,z)$ at $P(x,y,z)$ is

$$\frac{X-x_0}{\frac{\partial f}{\partial x}}=\frac{Y-y_0}{\frac{\partial f}{\partial y}}=\frac{Z-z_0}{\frac{\partial f}{\partial z}}$$

Maxima and Minima of functions of two variable

A function $f(x,y)$ is said to have maximum and minimum values of $x=a$ and $y=b$ according as

$$f(a+h, b+k) < \text{or} > f(a,b)$$

for all positive and negative values of a and b

The surface which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called saddle point

Stationary value: $f(a,b)$ is said to be stationary value of $f(x,y)$ if $f_x(a,b)=0$ and $f_y(a,b)=0$
the function is stationary at (a,b)

Working Rule to calculate minima and maxima

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate them to zero. Solve these simultaneous equations in x and y.

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2. $r = \frac{\partial^2 f}{\partial x^2}$ $s = \frac{\partial^2 f}{\partial x \partial y}$ $t = \frac{\partial^2 f}{\partial y^2}$

3. (i) If $rt - s^2 > 0$ and $r < 0$ at (a,b) is maximum value.
(ii) If $rt - s^2 > 0$ and $r > 0$ at (a,b) is minimum value
(iii) If $rt - s^2 < 0$ at (a,b) is saddle point
(iv) If $rt - s^2 = 0$ at (a,b) is doubtful and requires investigation

Example: Examine the following function for extreme values

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x = 0$$

$$x^3 - x + y = 0 \quad \text{------- (i)}$$

$$y^3 - y + x = 0 \quad \text{----- (ii)}$$

adding (i) and (ii)

$$x^3 + y^3 = 0 \qquad \boxed{y = -x}$$

$$(x+y)(x^2 - xy + y^2) = 0$$

put value of $y = -x$ in eq. 1

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0 \qquad \boxed{x = 0} \quad \boxed{x = \sqrt{2}} \quad \boxed{x = -\sqrt{2}}$$

Similarly values of y are $\boxed{y = 0}$ $\boxed{y = -\sqrt{2}}$ $\boxed{y = +\sqrt{2}}$

Case 1: At $(\sqrt{2}, -\sqrt{2})$,

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

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equation of tangent plane to the surface $f(x, y, z)$ at $P(x, y, z)$ is

$$\frac{\partial f}{\partial x}(X - x_0) + \frac{\partial f}{\partial y}(Y - y_0) + \frac{\partial f}{\partial z}(Z - z_0) = 0$$

equation of normal line to the surface $f(x, y, z)$ at $P(x, y, z)$ is

$$\frac{X - x_0}{\frac{\partial f}{\partial x}} = \frac{Y - y_0}{\frac{\partial f}{\partial y}} = \frac{Z - z_0}{\frac{\partial f}{\partial z}}$$

Maxima and Minima of functions of two variable

A function $f(x, y)$ is said to have maximum and minimum values of $x = a$ and $y = b$ according as
$f(a+h, b+k) <$ or $> f(a, b)$
for all positive and negative values of a and b

The surface which falls for displacement in certain directions and rises for displacements in other directions. Such a point is called saddle point

Stationary value: $f(a, b)$ is said to be stationary value of $f(x, y)$, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$
the function is stationary at (a, b)

Working Rule to calculate minima and maxima

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$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 - 4y + 4x = 0$$

$$x^3 - x + y = 0 \quad \text{-------- (i)}$$

$$y^3 - y + x = 0 \quad \text{----- (ii)}$$

adding (i) and (ii)

$$x^3 + y^3 = 0 \qquad \boxed{y = -x}$$

$$(x+y)(x^2 - xy + y^2) = 0$$

put value of $y = -x$ in eq. (1)

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0 \qquad \boxed{x = 0} \quad \boxed{x = \sqrt{2}} \quad \boxed{x = -\sqrt{2}}$$

Similarly values of y are $\boxed{y = 0}$ $\boxed{y = -\sqrt{2}}$ $\boxed{y = +\sqrt{2}}$

Case 1: At $(\sqrt{2}, -\sqrt{2})$;

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4 \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

at $(\sqrt{2}, -\sqrt{2})$

$rt - s^2 = (20 \times 20 - 4^2) =$ +ve (minimum value)

Case 2: At $(-\sqrt{2}, \sqrt{2})$

$rt - s^2 = (20 \times 20 - 4^2) =$ +ve (minimum value)

Case 3: At $(0,0)$

$rt - s^2 = 0$ (further investigation required)

If we take a point $(x,0)$ on x-axis where x is very close to origin

$f(x,y) = x^4 - 2x^2 = x^2(x^2 - 2) =$ -ve

If we take equation $y = x$ on plane then,

$f(x,y) = 2x^4 =$ +ve

It means $(0,0)$ is a saddle point

Lagrange's method of Undetermined Multiplier

Lagrange's method can be explained by,

Let $u = f(x,y,z)$

be a function of three variables (x,y,z) which are connected by the the relation

$$\phi(x,y,z) = 0$$

for u to have stationary values it is necessary that,

$$\left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)\partial y + \left(\frac{\partial u}{\partial z}\right)\partial z = du = 0$$

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = d\phi = 0$$

$$\left(\frac{\partial u}{\partial x}\right)+\lambda\left(\frac{\partial \phi}{\partial x}\right)+\left(\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)dy+\left(\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)dz=0$$

These three equations together with (2) will determine the values of x, y, z and λ for which u is independent

The values of x, y, z so obtained will give the stationary value of $f(x, y, z)$

Example: A rectangular box open at the top is to have volume of 32 cubic feet. Find the dimensions of the box requiring least material for its construction

$$\text{Surface Area} = xy + 2xz + 2yz$$

$$\therefore \text{Volume} = xyz - 32 = 0$$

$$F = (xy + 2yz + 2xz) + \lambda(xyz - 32)$$

$$\frac{\partial F}{\partial x} = (y + 2z) + \lambda yz = 0 \quad \text{----- (i)}$$

$$\frac{\partial F}{\partial y} = (x + 2z) + \lambda xz = 0 \quad \text{----- (ii)}$$

$$\frac{\partial F}{\partial z} = (2y + 2x) + \lambda xy = 0 \quad \text{----- (iii)}$$

$x \times (i) - y \times (ii)$ then $2z(x - y) = 0$

$z \neq 0$ $\quad \boxed{x = y}$

$y \times (ii) - z \times (iii)$ then $x(y - 2z) = 0$

$x \neq 0$ $\quad \boxed{y = 2z}$

$\boxed{x = y = 2z}$

Put in $xyz = 32$ $\quad z = 2 \quad y = 4 \quad$ Point $(4, 4, 2)$

$4z^3 = 32 \quad x = 4$

Surface area $= 48$ sq unit

Example Find the volume of the greatest parallelopiped (rectangular) that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Let the edges of parallelopiped be $2x$ $2y$ $2z$ then the volume is $8xyz$

We have to find the maximum value of V subject

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$

$$\frac{\partial f}{\partial x} = 8yz+\lambda\left(\frac{2x}{a^2}\right)=0 \qquad \frac{\partial F}{\partial z} = 8xy + \lambda\left(\frac{2z}{c^2}\right)=0$$

$$\frac{\partial f}{\partial y} = 8zx+\lambda\left(\frac{2y}{b^2}\right)=0$$

equating the value of λ we get $x^2/a^2 = y^2/b^2$

similarly $x^2/a^2 = y^2/b^2 = z^2/c^2$

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{c^2}{z^2}=\frac{1}{3} \qquad \boxed{x=a\sqrt{3}} \; \boxed{y=b\sqrt{3}} \; \boxed{z=c\sqrt{3}}$$

greatest volume of V that can come in ellipsoid is $= \boxed{\frac{8abc}{3\sqrt{3}}}$

Differentiation under Integral Sign

If a function $f(x,\alpha)$ of two variables x and α (called a parameter) be integrated with respect to x between the limits a and b then $\int_a^b f(x,\alpha)\,dx$ is a function of α.

$$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)\,dx\right] = \int_a^b \frac{\partial f(x,\alpha)}{\partial \alpha}\,dx$$

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Leibnitz rule for variable limits of integration

If $f(x,\alpha)$, $\dfrac{\partial f(x,\alpha)}{\partial \alpha}$

$$\frac{d}{d\alpha}\left\{\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right\} = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha} + \frac{d\psi}{d\alpha} f(\psi(\alpha)) - \frac{d\phi}{d\alpha} f(\phi(\alpha))$$

Example: Given $\int_0^{\pi} \dfrac{1}{a+b\cos x}\,dx = \dfrac{\pi}{\sqrt{a^2-b^2}}$ where $(a>b)$ then

(i) evaluate $\int_0^{\pi} \dfrac{dx}{(a+b\cos x)^2}$ (ii) evaluate $\int_0^{\pi} \dfrac{\cos x}{(a+b\cos x)^2}\,dx$

(i) $$\int_0^{\pi} \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$$

differentiating both sides of (i) w.r.t a

$$\int_0^{\pi} \frac{\partial}{\partial x}\left(\frac{1}{a+b\cos x}\right) dx = \frac{\partial}{\partial x}\left\{\frac{\pi}{\sqrt{a^2-b^2}}\right\}$$

$$\int_0^{\pi} \frac{-dx}{(a+b\cos x)^2} = \pi \cdot \left(\frac{-1}{2}\right)(a^2-b^2)^{-3/2} \cdot 2a$$

$$\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$$

(ii) differentiating both sides of (i) w.r.t b

$$\int_0^{\pi} -(a+b\cos x)^{-2} \cdot \cos x\,dx = \pi\left(\frac{-1}{2}\right)(a^2-b^2)^{-3/2}(-2b)$$

$$\int_0^{\pi} \frac{\cos x\,dx}{(a+b\cos x)^2} = \frac{\pi b}{(a^2-b^2)^{3/2}}$$

Example Evaluate $\int_0^{\alpha} \dfrac{\log(1+\alpha x)}{1+x^2}\,dx$ and hence show

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that $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log_e 2$

$F'(\alpha) = \frac{d}{d\alpha}\int_0^\alpha \frac{\log(1+\alpha x)}{x^2}$

$F'(\alpha) = \int_0^\alpha \frac{\partial}{\partial x}\left[\frac{\log(1+\alpha x)}{1+x^2}\right]dx + \frac{\log(1+\alpha^2)\cdot 1}{1+\alpha^2}$

$F'(\alpha) = \int_0^\alpha \left(\frac{1}{1+x^2}\right)\left(\frac{1}{1+\alpha x}\right)\cdot x\cdot dx + \frac{\log(1+\alpha^2)}{(1+\alpha^2)}$

solving integral by partial fraction

$\int_0^\alpha \left(\frac{Ax+B}{1+x^2} + \frac{C}{1+\alpha x}\right)x\cdot dx$

$\int_0^\alpha \frac{Ax + A\alpha x^2 + B + B\alpha x + C + Cx^2}{(1+x^2)(1+\alpha x)}$

$\frac{(A\alpha + C)x^2 + (A+B\alpha)x + (B+C)}{(1+x^2)(1+\alpha x)}$

$A\alpha + C = 0$

$A + B\alpha = 1 \qquad A - C\alpha = 1 \qquad A\alpha - C\alpha^2 = \alpha.$

$B + C = 0 \qquad A\alpha + C = 0 \qquad A\alpha + C = 0$

$\quad - \quad - \quad -$

$C = \left(\frac{-\alpha}{\alpha^2+1}\right) \qquad -C(\alpha^2+1) = \alpha$

$A = 1 - \frac{\alpha^2}{\alpha^2+1} = \frac{1}{\alpha^2+1}$

$B = \frac{\alpha}{\alpha^2+1}$

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$$F'(\alpha) = \int_0^{\alpha} \frac{\left(\frac{x^2}{\alpha^2+1}\right) + \left(\frac{\alpha x}{\alpha^2+1}\right)}{1+x^2} + \int_0^{\alpha} \frac{-\alpha x}{(\alpha^2+1)(1+\alpha x)} + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$F'(\alpha) = \int_0^{\alpha} \frac{x^2+\alpha x}{(\alpha^2+1)(1+x^2)}\, dx + \int_0^{\alpha} \frac{-\alpha x}{(\alpha^2+1)(1+\alpha x)} + \frac{\log(1+\alpha^2)}{1+\alpha^2}$$

$$F'(\alpha) = \frac{\log(1+\alpha^2)}{2(1+\alpha^2)} + \frac{\alpha \tan^{-1}\alpha}{1+\alpha^2}$$

$$F(\alpha) = \frac{1}{2}\int \log(1+\alpha^2) \cdot \frac{1}{1+\alpha^2}\, d\alpha + \int \frac{\alpha \tan^{-1}\alpha}{1+\alpha^2}\, d\alpha$$

$$F(\alpha) = \frac{1}{2}\left[\log(1+\alpha^2) . \tan^{-1}\alpha - \int \frac{2\alpha}{1+\alpha^2} \tan^{-1}\alpha \, d\alpha\right] + \int \frac{\alpha \tan^{-1}\alpha}{1+\alpha^2}\, d\alpha + C$$

$$F(\alpha) = \frac{1}{2}\log(1+\alpha^2) . \tan^{-1}\alpha + C$$

$\alpha = 0$, $F(0) = 0$ When $\alpha = 1$ we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = F(1) = \boxed{\frac{\pi}{8}\log_e 2}$$