Assignment 3

Q1. Solve $Ax = b$ by LU factorization where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 15 \\ 24 \end{bmatrix}$$

Q2. Find SVD of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

Q3. Solve the following by Cholesky Decomposition method

$$4x_1 - 2x_2 = 0$$
$$-2x_1 + 4x_2 - x_3 = 1$$
$$-x_2 + 4x_3 = 0$$

Assignment 4

Q1. Test the consistency of the following system of equations

$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$
$$9x_1 + 22x_2 + 79x_3 = 45$$

using the Gauss elimination method.

Q2. Find the first and second derivatives of $x = 1.6$ for the function represented by the following tabular data

x	1.0	1.5	2.0	3.0
$f(x)$	0.0	0.40547	0.69315	1.09861

Q3. Using Simpson's 1/3 rule, evaluate the integral $I = \int_0^1 \frac{dx}{x^2 + 6x + 10}$, with 2 and 4 subintervals. Compare with the exact solutions.

MATHS ASSIGNMENT-3

Q1 Solve $Ax=b$ by LU factorisation where

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 15 \\ 24 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = LU$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

$\boxed{u_{11}=1}$, $\boxed{u_{12}=2}$, $\boxed{u_{13}=4}$

$l_{21}u_{11}=4$

$\boxed{\Rightarrow l_{21}=4}$

$l_{31}u_{11}=7$

$\Rightarrow \boxed{l_{31}=7}$

$l_{21}u_{12}+u_{22}=5$

$4(2)+u_{22}=5$

$\Rightarrow \boxed{u_{22}=-3}$

$l_{31}u_{12}+l_{32}u_{22}=8$

$7(2)+l_{32}(-3)=8$

$\boxed{l_{32}=2}$

$l_{21}u_{13}+u_{23}=6$

$4(4)+u_{23}=6$

$\Rightarrow \boxed{u_{23}=-10}$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=9$

$7(4)+2(-10)+u_{33}=9$

$\boxed{u_{33}=1}$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

$AX = B$ let $UX = Y$

$LUX = B$

$LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ 24 \end{bmatrix}$$

$y_1 = 7$

$4y_1 + y_2 = 15$

$\boxed{y_2 = -13}$

$7y_1 + 2y_2 + y_3 = 24$

$49 - 26 + y_3 = 24$

$23 + y_3 = 24$

$\boxed{y_3 = 1}$

Now $UX = Y$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -10 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -13 \\ 1 \end{bmatrix}$$

$x + 2y + 4z = 7 \Rightarrow x = 1$

$-3y - 10z = -13 \Rightarrow y = 1$

$z = 1$

Q2 Find SVD of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B = AA^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 20 & 14 \\ 20 & 29 & 20 \\ 14 & 20 & 14 \end{bmatrix}$$

$$\det(B - \lambda I) = 0$$

$$\begin{vmatrix} 14-\lambda & 20 & 14 \\ 20 & 29-\lambda & 20 \\ 14 & 20 & 14-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (14-\lambda)\left((29-\lambda)(14-\lambda)-400\right) - 20\left(20\,(\cancel{14}-\lambda-\cancel{14})\right) + 14\left(400-14(29-\lambda)\right) = 0$$

$$\Rightarrow (14-\lambda)(406-43\lambda+\lambda^2-400) + 400\lambda + 14(14\lambda-6) = 0$$

$$\Rightarrow (14-\lambda)(\lambda^2-43\lambda+6) + 400\lambda + 196\lambda - 84 = 0$$

$$\Rightarrow 14\lambda^2 - 602\lambda + \cancel{84} - \lambda^3 + 43\lambda^2 - 6\lambda + 596\lambda - \cancel{84} = 0$$

$$\Rightarrow -\lambda^3 + 57\lambda^2 - 12\lambda = 0$$

$$\Rightarrow \lambda^3 - 57\lambda^2 + 12\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 57\lambda + 12) = 0$$

Using quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{57 \pm \sqrt{3201}}{2}$$

$$\lambda = 56.78,\ 0.211$$

$$\Rightarrow \lambda = 0,\ 56.78,\ 0.211$$

Eigen vector for $\lambda = 0$

$$\begin{bmatrix} 14 & 20 & 14 \\ 20 & 29 & 20 \\ 14 & 20 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Row Echelon form

$$R_3 \to R_3 - R_1 \quad \begin{bmatrix} 14 & 20 & 14 \\ 20 & 29 & 20 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to \cancel{R_2} - \frac{20}{14} R_1 \quad \begin{bmatrix} 14 & 20 & 14 \\ 0 & 0.42 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$14x_1 + 20x_2 + 14x_3 = 0$$

$$0.42x_2 = 0$$

$$x_2 = 0$$

let $x_3 = \lambda$

$$x_1 = -\lambda$$

$X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is eigen vector of $\lambda = 0$

Eigen vector for $\lambda = 56.78$

$$\begin{bmatrix} -42.78 & 20 & 14 \\ 20 & -27.78 & 20 \\ 14 & 20 & -42.78 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using Row Echelon form

$$R_2 \to R_2 + \frac{20}{42.78} R_1 \qquad R_3 \to R_3 + \frac{14}{42.78} R_1$$

$$\begin{bmatrix} -42.78 & 20 & 14 \\ 0 & -17.92 & 26.54 \\ 0 & 26.54 & -38.19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{26.54}{17.92} R_2 \qquad \begin{bmatrix} -42.78 & 20 & 14 \\ 0 & -17.92 & 26.54 \\ 0 & 0 & 1.11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$1.11 x_3 = 0$$

$$\boxed{x_3 = 0}$$

$$-17.92 x_2 + 26.54 x_3 = 0$$

$$\Rightarrow \boxed{x_2 = 0}$$

$$-42.78 x_1 + 20 x_2 + 14 x_3 = 0$$

$$\Rightarrow \boxed{x_1 = 0}$$

$$X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Eigen vector for $\lambda = 0.21$

$$\begin{bmatrix} 13.79 & 20 & 14 \\ 20 & 28.79 & 20 \\ 14 & 20 & 13.79 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - \frac{20}{13.79} R_1 \qquad \begin{bmatrix} 13.79 & 20 & 14 \\ 0 & -0.21 & -0.30 \\ 0 & -0.30 & -0.42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{14}{13.79} R_1$$

$$R_3 \to R_3 - \frac{0.30}{0.21} R_2 \qquad \begin{bmatrix} 13.79 & 20 & 14 \\ 0 & -0.21 & -0.30 \\ 0 & 0 & 0.0085 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 = 0$$

Also $x_1 = 0$

$x_2 = 0$

$$\sigma_1 = \sqrt{56.78} = 7.53$$

$$\sigma_2 = \sqrt{0.21} = 0.45$$

$$\Sigma = \begin{bmatrix} 7.53 & 0 & 0 \\ 0 & 0.45 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Now $\cancel{v_i = \frac{1}{\sigma_i} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}^T u_i}$

$$v_i = \frac{1}{\sigma_i} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}^T u_i$$

$$= \frac{1}{7.53} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve the following by Cholesky Decomposition Method

$$4x_1 - 2x_2 = 0$$
$$-2x_1 + 4x_2 - x_3 = 1$$
$$-x_2 + 4x_3 = 0$$

$AX = B$ where

$$A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$A = LL^T$

i.e.
$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating both sides row wise

$l_{11}^2 = 4 \Rightarrow \boxed{l_{11} = 2}$

$l_{11}l_{21} = -2 \Rightarrow \boxed{l_{21} = -1}$

$l_{11}l_{31} = 0 \Rightarrow \boxed{l_{31} = 0}$

$l_{21}l_{11} = -2 \Rightarrow l_{21} = -1$

$l_{21}^2 + l_{22}^2 = 4$

$\boxed{l_{22} = \sqrt{3}}$

$l_{21}l_{31} + l_{22}l_{32} = -1$

$-1(0) + \sqrt{3}\, l_{32} = -1$

$\boxed{l_{32} = \frac{-1}{\sqrt{3}}}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 4 \Rightarrow l_{33}^2 = 4 - \frac{1}{3} \Rightarrow l_{33} = \sqrt{\frac{11}{3}}$

$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & -1/\sqrt{3} & \sqrt{\frac{11}{3}} \end{bmatrix}$

$AX = B$ — ① $A = LL^T$ — ②

$\Rightarrow LL^T X = B$ — ③

Put $L^T X = Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Then ③ becomes $LY = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{3} & 0 \\ 0 & -1/\sqrt{3} & \sqrt{\frac{11}{3}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\boxed{y_1 = 0}$, $-y_1 + \sqrt{3} y_2 = 1$, $-\frac{1}{\sqrt{3}} y_2 + \sqrt{\frac{11}{3}} y_3 = 0$

$\boxed{y_2 = \frac{1}{\sqrt{3}}}$ $y_3 = \frac{1}{3} \frac{\sqrt{3}}{\sqrt{11}}$

$\boxed{\cancel{y_3 = \frac{1}{\sqrt{33}}}}$ $\boxed{y_3 = \frac{1}{\sqrt{33}}}$

Now $\begin{bmatrix} 2 & -1 & 0 \\ 0 & \sqrt{3} & -1/\sqrt{3} \\ 0 & 0 & \sqrt{\frac{11}{3}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{33} \end{bmatrix}$

$\therefore z \sqrt{\frac{11}{3}} = \frac{1}{\sqrt{33}}$

$\boxed{z = \frac{1}{11} = 0.091}$

$\sqrt{3} y - \frac{z}{\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\Rightarrow y = \frac{z+1}{3} = \frac{12}{33}$ $\Rightarrow \boxed{y = 0.363}$

$2x - y = 0 \Rightarrow x = \frac{y}{2}$

$\boxed{x = \frac{2}{11} = 0.181}$ Ans

Assignment - 4

Q1 Test the consistency of following system of equations

$$x_1 + 10x_2 - x_3 = 3$$
$$2x_1 + 3x_2 + 20x_3 = 7$$
$$9x_1 + 22x_2 + 79x_3 = 45$$

using Gauss elimination method

$$\Rightarrow \begin{bmatrix} 1 & 10 & -1 \\ 2 & 3 & 20 \\ 9 & 22 & 79 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 45 \end{pmatrix}$$

$$A : B = \left[\begin{array}{ccc:c} 1 & 10 & -1 & 3 \\ 2 & 3 & 20 & 7 \\ 9 & 22 & 79 & 45 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 1 & 10 & -1 & 3 \\ 0 & -17 & 22 & 1 \\ 0 & -68 & 88 & 18 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 9R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 10 & -1 & 3 \\ 0 & -17 & 22 & 1 \\ 0 & 0 & 0 & 14 \end{array}\right] \quad R_3 \to R_3 - \frac{68}{17} R_2$$

$\rho(A) = 2$

$\rho(A|B) = 3$

$\rho(A) < \rho(A|B)$.

The system is inconsistent. $\therefore$ It has no solution

Q. Find the first and second derivatives of $x=1.6$ for the function represented by following tabular data

x	1.0	1.5	2.0	3.0
$f(x)$	0.0	0.40547	0.69315	1.09861

Soln. Here values of x are not equally spaced so we have to use Newton's divided difference formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	0.0			
		0.81094		
1.5	0.40547		0.23558	
		0.57536		-0.17442
2.0	0.69315		-0.11326	
		0.40546		
3.0	1.09861			

$$f(x) = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0) + \dots$$

$$f'(x) = \Delta y_0 + (2x - x_0 - x_1)\Delta^2 y_0 + (3x^2 - 2x(x_0+x_1+x_2) + x_0x_1 + x_1x_2 + x_2x_0)\Delta^3 y_0 + \dots$$

Here $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2.0$, $x_3 = 3.0$, $x = 1.6$

$$f'(1.6) = 0.81094 + (3.2 - 1 - 1.5)\,0.23558 + (3(1.6)^2 - 3.2(1+1.5+2) + 1.5 + 3 + 2)(-0.17442)$$

$$= 0.81094 + 0.7(0.23558) + (7.68 - 14.4 + 1.5 + 3 \dots 0.34)$$

$$= 0.81094 + 0.164906 \quad -2.56$$

$$= \quad -1.584$$

We know

$$f'(x) = +\Delta y_0 + (2x - x_0 - x_1)\,\Delta^2 y_0 + (3x^2 - 2x(x_0 + x_1 + x_2) + x_0 x_1 + x_1 x_2 + x_2 x_0\,\Delta^3 y_0^3)$$

$$f''(x) = 2\,\Delta^2 y_0 + (6x - 2\,(x_0 + x_1 + x_2)\,\Delta^3 y_0)$$

$$= 2\,(-0.23558) + (6\,(1.6) - 2\,(4.5)\,(-0.17442))$$

$$= 10.69862$$

Sol 3 For $n=2$; $\Delta x = \frac{1}{2}$; $\int_0^1 \frac{1}{x^2+6x+10}\,dx$

$a = 0, \frac{1}{2}, 1, b$

$f(x_0) = f(0) = \frac{1}{10} = 0.1$

$4f(x_1) = 4f(\frac{1}{2}) = \frac{16}{53} = 0.301$

$f(x_2) = f(1) = \frac{1}{17} = 0.058$

$$\int_0^1 \frac{1}{x^2+6x+10} = \frac{0.5}{3}\left[0.1 + 0.301 + 0.058\right]$$

$= 0.0767$ Ans

For $n=4$

$a = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 = b$

$f(x_0) = f(0) = \frac{1}{10} = 0.1$

$4f(x_1) = 4f(\frac{1}{4}) = 0.345$

$2f(x_2) = 2f(\frac{1}{2}) = 0.15$

$4f(x_3) = 4f(\frac{3}{4}) = 0.265$

$f(x_4) = f(1) = 0.058$

$$\int_0^1 \frac{1}{x^2+6x+10}\,dx = \frac{0.25}{3}\left[0.1 + 0.345 + 0.15 + 0.265 + 0.058\right]$$

$= 0.0767$ Ans

Actual Value of $\int_0^1 \frac{1}{x^2+6x+10}\,dx = 0.0767$