Hypothesis Testing

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This part of statistics deals with testing statistically some statement about population distribution or population parameter.

- By CLT (Central Limit Theorem), we can assume Normal distribution for population. So basically here we will discuss hypothesis testing for some statement about population parameter (μ and σ^2).

Hypothesis: A statement about the parameters of one or more population.

For eg. $X \sim N(\mu, 25)$ here μ is unknown.

A statement is $\mu > 10.5$. (A claim that mean is greater than 10.5).

Hypothesis Testing: The decision making procedure to test hypothesis about a parameter is called Hypothesis Testing.

Reference — Douglas and George, Applied probability and statistics for Engineers

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Two types of hypothesis –

↳ Null hypothesis (denoted by H_0)

↳ Alternate hypothesis (denoted by H_1)

- In practical situations H_0, we take hypothesis which we intend to reject based upon sample. H_1 we take which we expect that it is true. (Any claim is put in H_1).

Example: If there is claim that the average no. of people who would voted for some political party A in the coming election has increased as compared to the true average no. of people who voted for the party in the last election.

X = no. of people who will favour party A in the coming election.

$E(X) = \mu \longrightarrow$ Not known.

Let $\mu_0 \rightarrow$ no. of people who favoured party A in the last election.

μ_0 is known.

Hence

$$\boxed{H_0 : \mu = \mu_0 \text{ and } H_1 : \mu > \mu_0}$$

Some points

- The claim is put in H_1.
- The null hypothesis H_0 is generally taken as equality.

$H_0 : \mu = \mu_0$. $\rightarrow$ No change in μ.

$H_1 : \mu > \mu_0$ $\rightarrow$ The claim : μ has increased.

- By Prof. R. A. Fisher (Statistician)

Null hypothesis is the hypothesis which is tested for possible rejection under the assumption that it is true.

Eg 2 Medical Claim:

Some medical measurements have decreased the number of deaths due to dengue this year. } Claim made by medical agency.

μ_0 = no. of deaths last year. (This is known).

X = no. of deaths this year.

$E(X) = \mu$ = Average no. of deaths this year.

$\mu \rightarrow$ Not known.

$$\boxed{H_0 : \mu = \mu_0 \text{ and } H_1 : \mu < \mu_0}$$

Eg 3 Criminal court proceeding

H_0 : Person is ~~guilty~~ innocent.

H_1 : Person is guilty.

Two Judgements

Given evidences are sufficient to reject H_0. } Based on evidences Person is declared guilty

OR

Given evidences are not sufficient to reject H_0. } Not guilty

- ~~Here~~ sample should be sufficient enough. (That is, evidence should be sufficient enough)

$\alpha \rightarrow$ level of significance $= P(\text{Type I error})$

α should be kept sufficiently low.

So that there should be less chances of "An innocent being punished by court".

- Level of significance → Probability of Type I error is called as level of significance. Generally, it is denoted by α.

- **Test statistics:** It is the statistics which we use for testing hypothesis.

~~For eg.~~ → Generally, if we want to perform test regarding parameter θ then we take statistics as some function of $\hat{\theta}$ (An estimator of θ.)

Z-test: If hypothesis testing involves Z distribution, then we call it Z-test.

t-test: If hypothesis testing involves t distribution, then we call it t-test.

χ^2-test: If hypothesis testing involves χ^2 distribution, then we call it χ^2-test.

- **Result of hypothesis testing**
  - ↳ Given sample rejects H_0. (Accept H_1).
  - ↳ Do not reject H_0.

- **Types of error**
  - ↳ Type I error ≡ Rejecting H_0 when H_0 is true.
  - ↳ Type II error ≡ Accepting H_0 when H_0 is not true.

Derivation of Test:

$X \sim N(\mu, \sigma^2)$ (Normal population with unknown Mean and variance is known)

$\mu \to$ Unknown

$\sigma^2 \to$ Known

- $H_0 : \mu = \mu_0$
  $H_1 : \mu > \mu_0$. (Hypothesis Testing on Mean)

H_1 is ~~that~~ $\mu - \mu_0 > 0$. $\bar{X}$ is an estimator of μ. So ~~if~~ $\bar{X} - \mu_0 \approx \mu - \mu_0 =$ significantly positive.

So Test will be to reject H_0 (accept H_1) when

$\bar{X} - \mu_0 > a$ (for some positive constant a.)

$\Rightarrow$ Reject H_0 when

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > a' \quad \text{for} \quad a' = \frac{a}{\sigma/\sqrt{n}} \text{ (constant)}$$

We also have $P(\text{Type I error}) = \alpha$.

$$\Rightarrow P(\text{Reject } H_0 / H_0 \text{ is true}) = \alpha$$

$$\Rightarrow P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > a' \,\Big/\, \mu = \mu_0\right) = \alpha.$$

$\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} = Z$. But when H_0 is true $\mu = \mu_0$

So $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} = Z$.

So we get $P(Z > a') = \alpha$.

by notation we get $a' = Z_\alpha$.

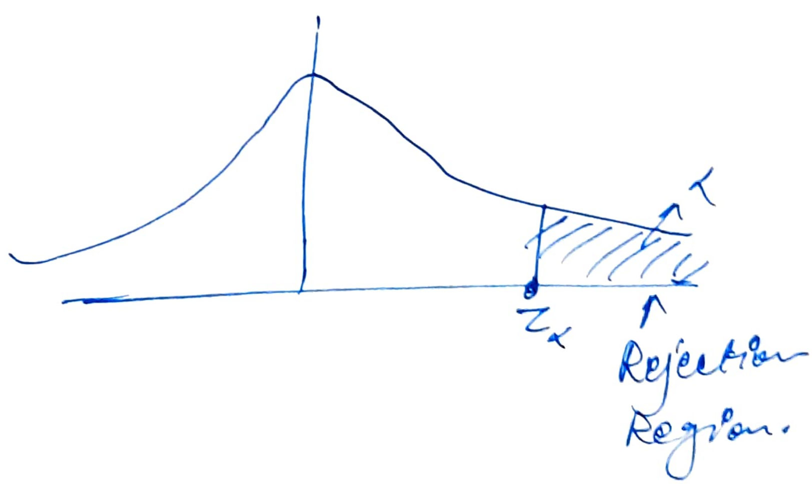
Thus for $H_0: \mu = \mu_0$
$H_1: \mu > \mu_0$ we get test as

$$\boxed{\text{Reject } H_0 \text{ when } \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > Z_\alpha .}$$

OR Do not reject H_0 if $\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq Z_\alpha$.

Test statistics $Z_0 = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$.

Rejection region:

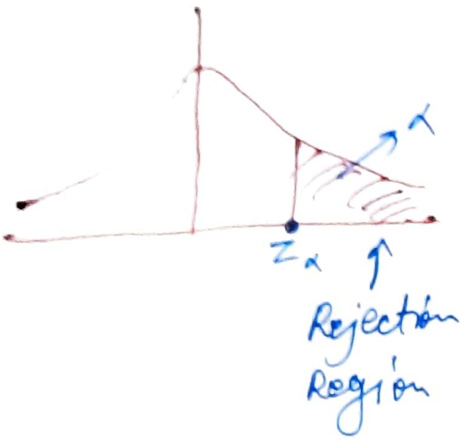


- If Test statistics has Z-distribution, then these tests are called Z-test.

Similarly we have tests for other types of alternatives :— $Z_0 = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$

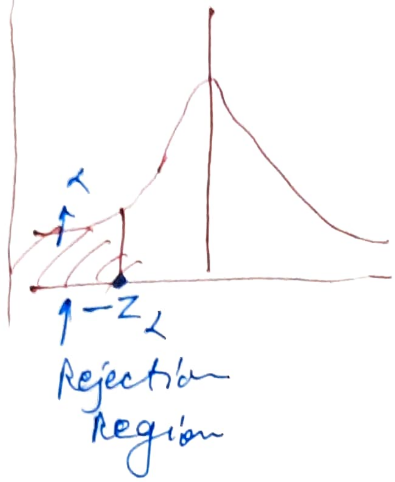
$H_0 : \mu = \mu_0$
$H_1 : \mu > \mu_0$

Reject H_0 when $Z_0 > Z_\alpha$



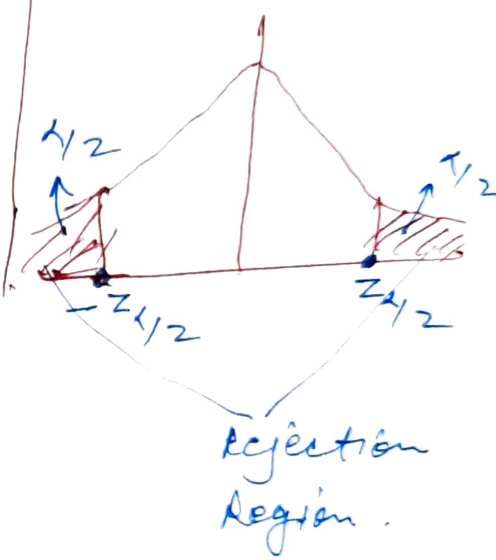
$H_0 : \mu = \mu_0$
$H_1 : \mu < \mu_0$

Reject H_0 when $Z_0 < -Z_\alpha$



$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

Reject H_0 when $|Z_0| > Z_{\alpha/2}$.
That is when $Z_0 > Z_{\alpha/2}$ or $Z_0 < -Z_{\alpha/2}$



- Large Sample Tests

Same above tests holds when population is Non-normal but sample size is sufficiently large ($n \geq 30$).

- For $H_1 : \mu > \mu_0$ or $H_1 : \mu < \mu_0$ → One sided Tests

For $H_1 : \mu \neq \mu_0$ → Two sided Test.

Hypothesis Testing on μ (Population $N(\mu, \sigma^2)$

μ is unknown.
σ^2 is also unknown.

T-tests

Test statistics $T_0 = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}}$

where S = sample standard deviation.

$= \sqrt{S^2}$

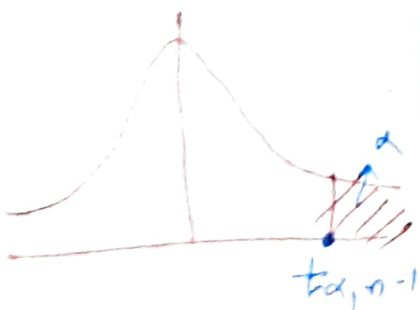
$= \sqrt{\dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

Reject H_0 when

$T_0 > t_{\alpha, n-1}$

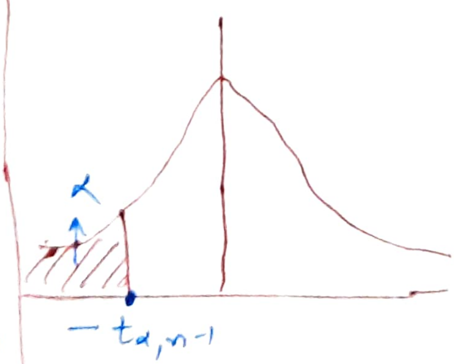


$H_0: \mu = \mu_0$

$H_1: \mu < \mu_0$

Reject H_0 when

$T_0 < -t_{\alpha, n-1}$



$H_0: \mu = \mu_0$

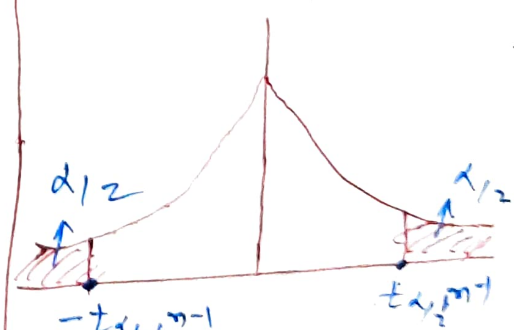
$H_1: \mu \neq \mu_0$

Reject H_0 when

$|T_0| > t_{\alpha/2, n-1}$

That is, when

$T_0 > t_{\frac{\alpha}{2}, n-1}$ or $T_0 < -t_{\frac{\alpha}{2}, n-1}$



Red shaded regions are rejection region.

Hypothesis Test on σ^2

(Normal Population.
- Same test will be there for μ known ~~or~~ unknown cases.)

Chi-square Tests

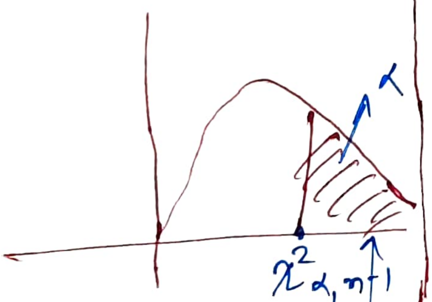
Test statistics $\chi_0^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 > \sigma_0^2$

Reject H_0 when

$\chi_0^2 > \chi^2_{\alpha, n-1}$



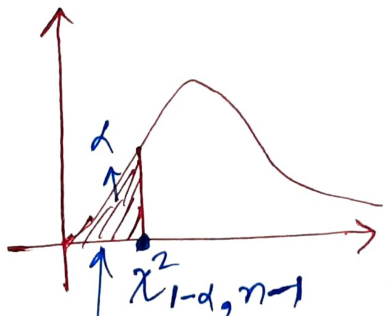
$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 < \sigma_0^2$

Reject H_0 ~~when~~

~~$\chi_0^2 < \chi$~~

$\chi_0^2 < \chi^2_{1-\alpha, n-1}$



$H_0: \sigma^2 = \sigma_0^2$

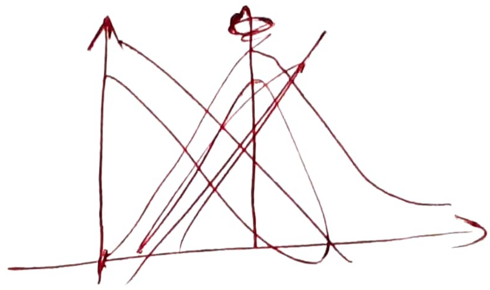
$H_1: \sigma^2 \neq \sigma_0^2$

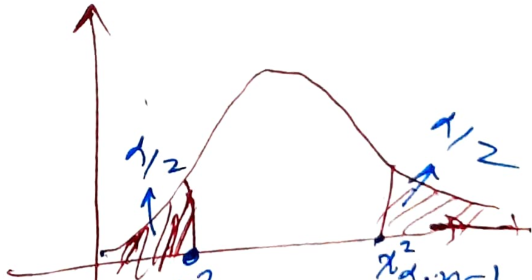
Reject H_0 when

$\chi_0^2 < \chi^2_{1-\frac{\alpha}{2}, n-1}$

or

$\chi_0^2 > \chi^2_{\frac{\alpha}{2}, n-1}$





Shaded regions are rejection region.

Tests on a population Proportion

population proportion — denoted by p

p = proportion of population that have some attribute.

For eg: p = proportion of students of 2nd semester who have marks more than 25 in the mid-semester examination.

let X = number of people out of n (random sample) that have that attributes.

For eg. Out of 50 randomly selected students 13 have more than 25 marks in mid-sem.

So $X = 13$, $n = 25$

Test statistics

$$Z_0 = \frac{X - np_0}{\sqrt{np_0(1-p_0)}}$$

$H_0 : p = p_0$ $H_1 : p > p_0$ Reject H_0 when $Z_0 > Z_\alpha$	$H_0 : p = p_0$ $H_1 : p < p_0$ Reject H_0 when $Z_0 < -Z_\alpha$	$H_0 : p = p_0$ $H_1 : p \neq p_0$ Reject H_0 when $Z_0 > Z_{\alpha/2}$ or $Z_0 < -Z_{\alpha/2}$

Example Example 9.10 (Douglas & George, Text book)

Statistical Inference for Two samples:

let $X \rightarrow$ Two populations
& $Y \nearrow$

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ = sample Mean for the 1st population

$\bar{Y} = \frac{1}{m}\sum_{i=1}^{m} y_i$ = Sample Mean for the 2nd population

let $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$

Hypothesis Testing on $\mu_X - \mu_Y$: (Here we assume σ_X and σ_Y are known)

Test statistics

~~$Z_0 = \bar{X} - \bar{Y} - (\mu_X$~~

$$Z_0 = \frac{\bar{X} - \bar{Y} - \mu_0}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$$

$$\left[\begin{array}{l} \bar{X} \sim N\left(\mu_X, \frac{\sigma_X^2}{n}\right) \\ \bar{Y} \sim N\left(\mu_Y, \frac{\sigma_Y^2}{m}\right) \\ \\ \bar{X} - \bar{Y} \sim N\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right) \end{array}\right.$$

Under H_0: $\mu_X - \mu_Y = \mu_0$.

$H_0: \mu_X - \mu_Y = \mu_0$ $H_1: \mu_X - \mu_Y > \mu_0$ Reject H_0 when $Z_0 > Z_\alpha$	$H_0: \mu_X - \mu_Y = \mu_0$ $H_1: \mu_X - \mu_Y < \mu_0$ Reject H_0 when $Z_0 < -Z_\alpha$	$H_0: \mu_X - \mu_Y = \mu_0$ $H_1: \mu_X - \mu_Y \neq \mu_0$ Reject H_0 when $Z_0 > Z_{\alpha/2}$ or $Z_0 < -Z_{\alpha/2}$.

Testing for Goodness of Fit (Section 9.7 Text Book - Douglas & George) (13)

Used for testing the hypothesis about the "distribution of population".

- The test procedure requires random sample of size n.
- These observations are arranged in frequency histogram, having k class intervals.
- let O_i = observed frequency.
  E_i = Expected frequency under the hypothesized probability distribution.

- Test statistics

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

- Under the hypothesized distribution. $\chi_0^2 \sim \chi^2(k-p-1)$

where p = no. of parameters of the hypothesized distribution.

- Test is to reject the hypothesis that "population has hypothesized distribution" if

$$\boxed{\chi_0^2 > \chi^2_{\alpha, k-p-1}}$$

- Example Example 9.12 (Text Book Douglas and George)