Series Solutions of Differential Equations:→

Analytic function:→ A function $f(x)$ is called analytic at point x_0 if it is differentiable at x_0 and at every point in the neighbourhood of x_0.

E.g. e^x, $\sin x$, $\cos x$, polynomial functions are analytic everywhere.

A rational function of the form $\frac{f(x)}{g(x)}$ is analytic at all x except at those values of x for which $g(x)=0$.

E.g. $\frac{x}{x^2-3x+2}$ is analytic everywhere except at x s.t. $x^2-3x+2=0$ i.e. $x=1$ and $x=2$.

Ordinary point:→ Let $y''+P(x)y'+Q(x)y=0$ —(1)

A point $x=x_0$ is called an ordinary point of (1) if $P(x)$ and $Q(x)$ are analytic functions at $x=x_0$.

If $x=x_0$ is not an ordinary point then it is called singular point of (1).

⇒ Singular points are of two types:-

(1) Regular singular point (2) Irregular singular point.

Regular Singular point:→ A singular point $x=x_0$ is called regular singular pt of (1) if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at $x=x_0$.

A singular point which is not regular is called irregular singular point of (1).

Que→ $2x^2 y'' + 7x(x+1)y' - 3y = 0 \quad -(1)$

$\Rightarrow y'' + \frac{7x(x+1)}{2x^2} y' - \frac{3y}{2x^2} = 0$

or $y'' + \frac{7(x+1)}{2x} y' - \frac{3}{2x^2} y = 0$

$P(x)$ and $Q(x)$ are analytic Everywhere Except at $x=0$.

$\Rightarrow x=0$ is a Singular point of (1)

$(x-0)\frac{7(x+1)}{2x} = \frac{7(x+1)}{2}$ is analytic at $x=0$

and $(x-0)^2\left(\frac{-3}{2x^2}\right) = \frac{-3}{2}$ is analytic at $x=0$

$\Rightarrow x=0$ is a regular Singular pt. of (1).

Que S.T. $x=0$ is an ordinary point of $(x^2-1)y'' + xy' - y = 0$ but $x=1$ is a regular Singular point.

Solⁿ→ $(x^2-1)y'' + xy' - y = 0$

$\Rightarrow y'' + \frac{x}{x^2-1} y' - \frac{y}{x^2-1} = 0$

$P(x) = \frac{x}{x^2-1}$ and $Q(x) = \frac{-1}{x^2-1}$ are analytic Everywhere Except at $x = \pm 1$

$\Rightarrow x=0$ is an ordinary pt and $x=1$ is a Singular pt.

Now $(x-1)p(x) = (x-1)\frac{x}{x^2-1} = \frac{x}{x+1}$

$(x-1)^2 Q(x) = (x-1)^2\left(\frac{-1}{(x-1)(x+1)}\right) = \frac{-(x-1)}{(x+1)}$

are analytic at $x=1 \Rightarrow x=1$ is a regular Singular pt

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Que $x^2(x+1)^2 y'' + (x^2-1)y' + 2y = 0$ — (1)

$\Rightarrow x y'' + \frac{(x-1)(x+1)}{x^2(x+1)^2} y' + \frac{2y}{x^2(x+1)^2} = 0$

$\Rightarrow y'' + \frac{(x-1)}{x^2(x+1)} y' + \frac{2y}{x^2(x+1)^2} = 0$

$P(x) = \frac{x-1}{x^2(x+1)}$; $Q(x) = \frac{2}{x^2(x+1)^2}$

are analytic Everywhere Except at $x=0$ and $x=-1$

$\Rightarrow x=0$ and $x=-1$ are Singular pts of (1)

$x P(x) = \frac{x(x-1)}{x^2(x+1)} = \frac{x-1}{x(x+1)} \rightarrow$ not analytic at $x=0$

and $x^2 Q(x) = x^2\left(\frac{2}{x^2(x+1)^2}\right) = \frac{2}{(x+1)^2} \rightarrow$ analytic at $x=0$

$\Rightarrow x=0$ is an irregular Singular pts.

$(x+1)P(x) = \frac{(x+1)(x-1)}{x^2(x+1)} = \frac{x-1}{x^2}$

$(x+1)^2 Q(x) = \cancel{(x+1)^2} \cdot \frac{2}{x^2\cancel{(x+1)^2}} = \frac{2}{x^2}$

are analytic at $x=-1$

$\Rightarrow x=-1$ is a regular Singular point.

H.W **Que** Determine the Singularities of the following Equations:

(1) $x(x-1)^3 y'' + 2(x-1)^3 y' + 3y = 0$ at $x=0$ and $x=1$.

(2) $3xy'' + 2x(x-1)y' + 5y = 0$

(3) $y'' + (1-x)y' + (1-x)^2 y = 0$.

(4) $x^3 y'' + y \sin x = 0$