Exact Differential Equation :-

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be a differentiable function.

Then $df(x,y) = 0$ is called an exact differential equation. and its general solution is $f(x,y) = c$ where c is an arbitrary constant.

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy \quad -(1)$$

Let a first order first degree diff Equ^n as

$$M(x,y)dx + N(x,y)dy = 0 \quad -(2)$$

Comparing (1) & (2), we get

$$M(x,y) = \frac{\partial f}{\partial x} \;;\; N(x,y) = \frac{\partial f}{\partial y}.$$

We assume that M and N have continuous partial derivatives Exist

$$\text{So } \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

Now If second order partial derivatives are continuous then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence an Equ^n $M(x,y)dx + N(x,y)dy$ is an exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\Rightarrow$ There Exist a function $f(x,y)$ st.

$$\frac{\partial f}{\partial x} = M(x,y) \quad -(A) \qquad \text{and} \qquad \frac{\partial f}{\partial y} = N(x,y) \quad -(B)$$

$f(x,y)$ can be determined from (A) & (B).

Integrate (A) w.r.t. x, we get

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$\Rightarrow$ ~~Put~~ $f(x,y) = K(x,y) + g(y)$ where $K(x,y) = \int M(x,y)\,dx$.

Put $f(x,y)$ in (B), we get

$$\frac{\partial f}{\partial y} = \frac{\partial K}{\partial y} + g'(y) = N(x,y)$$

$$\Rightarrow \quad g'(y) = N(x,y) - \frac{\partial K}{\partial y}$$

$$\Rightarrow \quad g(y) = \int\left(N(x,y) - \frac{\partial K}{\partial y}\right)dy + C$$

So $\boxed{f(x,y) = K(x,y) + \int\left(N(x,y) - \frac{\partial K}{\partial y}\right)dy + C}$

Que $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$.

Sol^n $M(x,y) = 3x^2 + 2e^y$; $N(x,y) = 2xe^y + 3y^2$

$$\frac{\partial M}{\partial y} = 2e^y \; ; \quad \frac{\partial N}{\partial x} = 2e^y$$

as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the given Eq^n is Exact.

$\Rightarrow$ There Exist $f(x,y)$ s.t. $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

or $\frac{\partial f}{\partial x} = 3x^2 + 2e^y$ —(A) ; $\frac{\partial f}{\partial y} = 2xe^y + 3y^2$ —(B)

from (A); $f(x,y) = \int (3x^2 + 2e^y)\,dx + g(y)$

$$f(x,y) = x^3 + 2xe^y + g(y)$$

Put in (B); $\frac{\partial f}{\partial y} = \cancel{2xe^y} + g'(y) = \cancel{2xe^y} + 3y^2$

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$$g'(y) = 3y^2$$
$$\Rightarrow \int g(y)dy = \int 3y^2 dy + c$$
$$\Rightarrow g(y) = y^3 + c$$
$$\therefore f(x,y) = x^3 + 2xe^y + y^3 + c.$$

Que
Sol^n

$$e^x(\cos y\, dx - \sin y\, dy) = 0. \quad -(1)$$
$$e^x \cos y\, dx - e^x \sin y\, dy = 0$$
$$M = e^x \cos y; \quad N = -e^x \sin y.$$
$$\frac{\partial M}{\partial y} = -e^x \sin y; \quad \frac{\partial N}{\partial x} = -e^x \sin y$$

$\Rightarrow$ (1) is Exact Diff. Eq^n.

$$\Rightarrow \exists f(x,y) \text{ s.t. } \frac{\partial f}{\partial x} = e^x \cos y \; -(A) \text{ and } \frac{\partial f}{\partial y} = -e^x \sin y \; -(B)$$

from (A); $f(x,y) = \int e^x \cos y\, dx + g(y)$

$$\Rightarrow f(x,y) = e^x \cos y + g(y)$$
$$\Rightarrow \frac{\partial f}{\partial y} = -e^x \sin y + g'(y)$$

Put in (B), we get
$$-e^x \cancel{\sin y} + g'(y) = -e^x \cancel{\sin y}$$
$$\Rightarrow g'(y) = 0$$
$$\Rightarrow g(y) = c$$
$$\boxed{\therefore f(x,y) = e^x \cos y + c} = c_1$$
$$\Rightarrow \boxed{e^x \cos y = c_2}$$

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Que $(y+x^3)dx + (ax+by^3)dy = 0$

Sol^n→ $M = y+x^3$; $N = ax+by^3$

$\frac{\partial M}{\partial y} = 1$; $\frac{\partial N}{\partial x} = a$

⇒ The Equ^n is Exact for $a=1$ no matters what b we choose.

∴ $(y+x^3)dx + (x+by^3)dy = 0$ is Exact

⇒ $\exists\, f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = y+x^3 \quad -(A) \; ; \quad \frac{\partial f}{\partial y} = x+by^3. \quad -(B)$$

$$f(x,y) = \int (y+x^3)dx + g(y)$$

$$\Rightarrow f(x,y) = xy + \frac{x^4}{4} + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = x + g'(y)$$

Put in (B), we get $\cancel{x} + g'(y) = \cancel{x} + by^3$

$$\Rightarrow g'(y) = by^3$$

$$\Rightarrow g(y) = \int by^3 dy + c$$

$$\Rightarrow g(y) = \frac{by^4}{4} + c$$

$$\therefore f(x,y) = xy + \frac{x^4}{4} + \frac{by^4}{4} + c.$$

Direct formula:→

$Mdx + Ndy = 0$ is Exact diff Equ^n if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The Sol^n $f(x,y) = c$ is given by

$$\int_{y\ \text{Constant}} M\,dx + \int (\text{terms in } N \text{ not containing } x)\, dy = c.$$

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Ex $(y+x^3)dx + (x+by^3)dy = 0$.

Sol^n $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Equ^n is Exact

Solution is given by

$$\int_{y\,const} M\,dx + \int (\text{terms in } N \text{ not Containing } x)\,dy = C$$

$$\int_{y-\text{Constant}} (y+x^3)dx + \int_{x-\text{Const}} (by^3)\,dy = C$$

$$\Rightarrow \quad xy + \frac{x^4}{4} + \frac{by^4}{4} = C.$$

Que $e^x \cos y\,dx - e^x \sin y\,dy = 0$

Sol^n Sol^n is given by

$$\int_{y\,const} M\,dx + \int (\text{terms in } N \text{ not Containing } x)\,dy = C.$$

$$\Rightarrow \int_{y-\text{Const}} (e^x \cos y)\,dy = C$$

$$\Rightarrow \boxed{e^x \cos y = C}$$

H.W (1) $(1+e^x)dx + y\,dy = 0$

(2) $(x e^{xy} + 2y)\,dy + y e^{xy}\,dx = 0$

(3) $2xy\,dx + (x^2+1)\,dy = 0$

(4) $(1+x^2)\,dy + 2xy\,dx = 0$

(5) $x\,dx + y\,dy = 2y\,(x^2+y^2)\,dy$.

Under what Conditions, the following diff Equ^n is Exact.

(6) $xy^3\,dx + ax^2y^2\,dy = 0$

(7) $(ax+y)dx + (Kx+by)dy = 0$.