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Lagrange's Method of Multipliers:→

Suppose we have to find the Critical / Stationary points of a function $f(x,y,z)=0$ Subject to the Constraint

$g(x,y,z)=0.$ — (1)

then The Critical points of the given function f are given by determining the critical points of F, where

$$F = f + \lambda g$$

or $F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$.

Here λ is Called Lagrange Multiplier.

The Critical points of F are determined by

$F_x = 0$, $F_y = 0$ and $F_z = 0$ — (2)

Ex:→ Find the Max and Min of x^2+y^2 s.t. the Constraint

$$3x^2+4xy+6y^2-140=0$$

Sol^n:→ $f(x,y) = x^2+y^2$

S.t. $3x^2+4xy+6y^2-140=0$ — (1)

Let $F(x,y) = x^2+y^2+\lambda(3x^2+6y^2+4xy-140)$.

Where λ is Lagrange's Multiplier.

$$\frac{\partial F}{\partial x} = 2x + \lambda(6x+4y) = 0$$

$$\frac{\partial F}{\partial y} = 2y + \lambda(12y+4x) = 0$$

$\Rightarrow$ $(1+3\lambda)x + 2\lambda y = 0$ — (2)

$2\lambda x + (1+6\lambda)y = 0.$ — (3)

$\because$ x and y are non-zero,

$$\begin{vmatrix} 1+3\lambda & 2\lambda \\ 2\lambda & 1+6\lambda \end{vmatrix} = 0 \Rightarrow (1+3\lambda)(1+6\lambda) - 4\lambda^2 = 0$$

$\Rightarrow 14\lambda^2 + 9\lambda + 1 = 0$

$\Rightarrow \boxed{\lambda = -\frac{1}{2}, -\frac{1}{7}}$

$\lambda = \frac{-1}{2} \Rightarrow \left(1-\frac{3}{2}\right)x + 2\left(\frac{-1}{2}\right)y = 0$

$\Rightarrow -\frac{1}{2}x - y = 0 \Rightarrow x = -2y$

From (1); $3x^2 + 4xy + 6y^2 = 140$

$\Rightarrow 3(4y^2) + 4y(-2y) + 6y^2 = 140$

$\Rightarrow 12y^2 - 8y^2 + 6y^2 = 140$

$\Rightarrow 10y^2 = 140 \Rightarrow y^2 = 14$

So $x = -2y \Rightarrow x^2 = 4y^2$

$\Rightarrow x^2 = 14 \times 4 = 56$

So $f(x,y) = x^2 + y^2 = 56 + 14 = 70.$

$\lambda = \frac{-1}{7} \Rightarrow y = 2x$

From (1); $3x^2 + 4x(2x) + 6(4x^2) = 140$

$\Rightarrow 35x^2 = 140 \Rightarrow x^2 = 4$

~~$\Rightarrow x = \pm 2$~~

$y = 2x \Rightarrow y^2 = 16$

So $f(x,y) = x^2 + y^2 = 4 + 16 = 20.$

$\Rightarrow$ Min. value of $f(x,y) = 20$ and Max value is 70.

Ex $f(x,y,z) = x^2+y^2+z^2$ s.t $lx + my + nz = p$.

Soln:→ Consider $f(x,y,z) = x^2+y^2+z^2+\lambda(lx+my+nz-p)$

$F_x = 2x + l\lambda = 0$

$\Rightarrow x = -\frac{l\lambda}{2}$

$F_y = 2y + m\lambda = 0 \Rightarrow y = \frac{-m\lambda}{2}$

$F_z = 2z + n\lambda = 0 \Rightarrow z = \frac{-n\lambda}{2}$

$lx + my + nz = p \Rightarrow -\frac{l^2\lambda}{2} - \frac{m^2\lambda}{2} - \frac{n^2\lambda}{2} = p$

$\Rightarrow -\lambda(l^2+m^2+n^2) = 2p$

$\Rightarrow \boxed{\lambda = \dfrac{-2p}{l^2+m^2+n^2}}$

$\Rightarrow x = \dfrac{lp}{l^2+m^2+n^2}$; $y = \dfrac{mp}{l^2+m^2+n^2}$; $z = \dfrac{np}{l^2+m^2+n^2}$ — (1)

is a critical points.

$F_{xx} = 2$, $F_{yy} = 2$, $F_{zz} = 2$

$F_{xy} = 0$, $F_{yz} = 0$, $F_{zx} = 0$

$$\Rightarrow J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$M_{11} = 2 > 0$

$M_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$M_{33} = |J| = 8 > 0$

$\Rightarrow$ J is positive definite matrix ~~at~~

$\Rightarrow$ (1) is point of Minima of $f(x,y,z)$

and Min value is

$$\frac{l^2p^2 + m^2p^2 + n^2p^2}{(l^2+m^2+n^2)^2} = \frac{p^2}{l^2+m^2+n^2}.$$

Ex Find the Minimum value of $x^2+y^2+z^2$ s.t. $x+y+z = 3a$

Ans: $3a^2$

Ex Find the Min. value of $x^2+y^2+z^2$ s.t. $yz+zx+xy = 3a^2$

Ans $3a^2$

Ex Find the Max and Min value of $x^2+y^2+z^2$

S.t. the Constraints $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $z=x+y$.

Sol^n

$$F(x,y,z) = x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(z-x-y)$$

$$F_x = 2x+\frac{\lambda_1 x}{2}-\lambda_2=0 \qquad -(1)$$

$$F_y = 2y+\frac{2}{5}y\lambda_1-\lambda_2=0 \qquad -(2)$$

$$F_z = 2z+\frac{2z}{25}\lambda_1+\lambda_2=0 \qquad -(3)$$

Mul (1) by x, (2) by y and (3) by z and adding, we get

$$2(x^2+y^2+z^2)+\lambda_1\left(\frac{x^2}{2}+\frac{2}{5}y^2+\frac{2z^2}{25}\right)+\lambda_2(z-x-y)=0$$

$$\Rightarrow 2(x^2+y^2+z^2)+2\lambda_1(1)+\lambda_2(0)=0$$

$$\Rightarrow \lambda_1 = -(x^2+y^2+z^2) = -u^2$$

Put λ_1 in (1), (2) and (3), we get

$$2x-\frac{x}{2}(x^2+y^2+z^2)-\lambda_2=0$$

$$\cancel{\Rightarrow \lambda_2 = 2x-\frac{u^2x}{2} = \frac{4x-u^2x}{2} = \left(\frac{4-u^2}{2}\right)x}$$

$$\Rightarrow x\left(2-\frac{1}{2}u^2\right)=\lambda_2$$

$$\Rightarrow x = \frac{2\lambda_2}{4-u^2}$$

$$2y+\frac{2}{5}y(-u^2)=\lambda_2$$

$$\Rightarrow y\left(2-\frac{2}{5}u^2\right)=\lambda_2 \Rightarrow y=\frac{5\lambda_2}{10-2u^2}$$

and $2z + \frac{2z(-x^2)}{25} + \lambda_2 = 0$

$\Rightarrow z\left(2 - \frac{2}{25}x^2\right) = -\lambda_2$

$\boxed{\cancel{\Rightarrow z = \frac{48\lambda_2}{10-2x^2}}}$ $\Rightarrow z = \frac{-25\lambda_2}{50-2x^2}$

Now $z = x+y$

$\Rightarrow \frac{-25\lambda_2}{50-2x^2} = \frac{2\lambda_2}{4-x^2} + \frac{5\lambda_2}{10-2x^2}$

$\cancel{\Rightarrow \frac{-10\lambda_2}{10-2x^2} = \frac{2\lambda_2}{4-x^2}}$

$\cancel{\Rightarrow -5(4-x^2) = 10-2x^2}$

$\cancel{\Rightarrow -20+5x^2 = 10-2x^2}$

$\cancel{\Rightarrow 7x^2 = 30 \Rightarrow x^2 = \frac{30}{7}}$

$\Rightarrow \frac{-25}{50-2x^2} = \frac{2}{4-x^2} + \frac{5}{10-2x^2} \quad ;(\because \lambda_2 \neq 0)$

$= \frac{2(10-2x^2)+5(4-x^2)}{(4-x^2)(10-2x^2)}$

$= \frac{20-4x^2+20-5x^2}{(4-x^2)(10-2x^2)}$

$\Rightarrow \frac{-25}{50-2x^2} = \frac{40-9x^2}{(4-x^2)(10-2x^2)}$

$\Rightarrow -25(4-x^2)(10-2x^2) = (40-9x^2)(50-2x^2)$

$\Rightarrow -25(40-18x^2+2x^4) = 2000-530x^2+18x^4$

$\Rightarrow -1000+450x^2-50x^4 = 2000-530x^2+18x^4$

$\Rightarrow 68x^4 - 980x^2 + 3000 = 0$

$\Rightarrow$

25×18: 200, 25, 450

~~Que~~ 1

→ A f

Que Find the absolute Max and Min values of $f(x,y) = 3x^2 + y^2 - x$ over the region $2x^2 + y^2 \le 1$.

Sol^n $f_x = 6x - 1 = 0 \Rightarrow x = 1/6$

$f_y = 2y = 0 \Rightarrow y = 0.$

point is $\left(\frac{1}{6}, 0\right) \rightarrow$

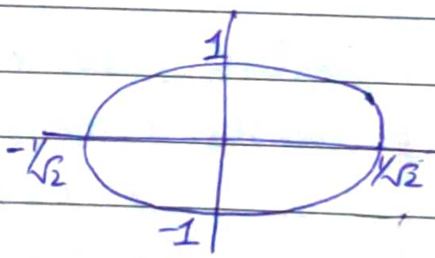
$f_{xx} = 6, \quad f_{yy} = 2, \quad f_{xy} = f_{yx} = 0$

$J = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$ positive definite

$\Rightarrow \left(\frac{1}{6}, 0\right)$ is pt of local min

and $f\left(\frac{1}{6}, 0\right) = \frac{3}{36} - \frac{1}{6} = \frac{3-6}{36} = \frac{-3}{36} = \frac{-1}{12}$

on the boundary $y^2 = 1 - 2x^2$



$f(x,y) = 3x^2 + 1 - 2x^2 - x$

$= x^2 - x + 1 = g(x)$

$g'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = 1/2$

$g''(x) = 2 > 0$

$\Rightarrow$ $x = 1/2$ is pt of Minima

at $x = 1/2$, $y = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \left(1/2, \pm\frac{1}{\sqrt{2}}\right)$ are pts of Min. and

$f\left(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}\right) = 3\left(\frac{1}{4}\right) + \frac{1}{2} - \frac{1}{2} = \frac{3}{4}$

At the vertices $f\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{3}{2} - \frac{1}{\sqrt{2}} = \frac{3-\sqrt{2}}{2}$.

$f\left(-\frac{1}{\sqrt{2}}, 0\right) = \frac{3+\sqrt{2}}{2}$, $f(0,1) = 1 = f(0,-1)$

$\Rightarrow$ abs. Min at $\left(\frac{1}{6},0\right)$ is $\frac{-1}{12}$

And abs. Max at $\left(\frac{-1}{\sqrt{2}},0\right)$ is $\frac{3+\sqrt{2}}{2}$

→ <u>Derivatives of Composite functions:⇒</u>

Let $z = f(x,y)$.

Let x and y are functions of some Independent variable 't'. i.e. $x = \phi(t)$ and $y = \psi(t)$

Then $z = f(\phi(t), \psi(t))$ is a composite function of the Independent variable t.

Then $\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}}$ – Total Differential of f w.r.t 't'.

Similarly, let x and y are functions of two Independent variables u and v

i.e. $x = \phi(u,v)$ and $y = \psi(u,v)$

Then $z = f(\phi(u,v), \psi(u,v))$ is a Composite function of two Independent variables u and v.

So $$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \text{— (A)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \text{— (B)}$$

The rules (A) and (B) are called the chain rules.

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Ex $f(x,y) = x\cos y + e^{x}\sin y$

$x = t^2+1 \; ; \; y = t^3+t$

Find $\frac{df}{dt}$ at $t=0$

Sol^n $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$

$\Rightarrow \frac{df}{dt} = (\cos y + e^{x}\sin y)(2t) + (-x\sin y + e^{x}\cos y)(3t^2+1)$

at $t=0$, $x=1$, $y=0$

$\Rightarrow \frac{df}{dt} = (1+0)\,2(0) + (e(1))(1)$

$= e$

Ex $f(x,y,z) = x^3 + xz^2 + y^3 + xyz$

$x = e^{t} \; ; \; y = \cos t \; ; \; z = t^3$ at $t=0$ find $\frac{df}{dt}$

$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$

$\Rightarrow \frac{df}{dt} = (3x^2 + z^2 + yz)(e^{t}) + (3y^2 + xz)(-\sin t) + (2xz + xy)(3t^2)$

at $t=0$, $x=1$, $y=1$, $z=0$

$\Rightarrow \frac{df}{dt} = 3 + 3(-0) + 1(0)$

$= 3$

Ex $z = f(x,y) \; ; \; x = e^{2u} + e^{-2v} \; ; \; y = e^{-2u} + e^{2v}$

S.T. $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$

Sol^n $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$

$= \frac{\partial f}{\partial x}(2e^{2u}) + \frac{\partial f}{\partial y}(-2e^{-2u})$

$= 2\frac{\partial f}{\partial x} e^{2u} - 2\frac{\partial f}{\partial y} \cdot e^{-2u}$

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$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x}(-2e^{-2v}) + \frac{\partial f}{\partial y}(2e^{2v})$$

$$= -2e^{-2v}\cdot\frac{\partial f}{\partial x} + 2e^{2v}\frac{\partial f}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2e^{2u}\frac{\partial f}{\partial x} - 2e^{-2u}\frac{\partial f}{\partial y} + 2e^{-2v}\frac{\partial f}{\partial x} - 2e^{2v}\frac{\partial f}{\partial y}$$

$$= 2(e^{2u}+e^{-2v})\frac{\partial f}{\partial x} - 2(e^{-2u}+e^{2v})\frac{\partial f}{\partial y}$$

$$= 2x\frac{\partial f}{\partial x} - 2y\frac{\partial f}{\partial y}.$$

<u>H.W</u> (1) If $z = \log(x^2+y)$

$x = e^{u+v^2}$; $y = u+v^2$

Then S.T. $2v\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = 0$

(2) If $w = \sqrt{x^2+y^2+z^2}$

$x = u\cos v$; $y = u\sin v$; $z = uv$

Then $u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v} = \frac{u}{\sqrt{1+v^2}}$

(3) If $w = f(x,y)$

$x = \sqrt{u^2+v^2}$; $y = \cot^{-1}\left(\frac{v}{u}\right)$

Then $\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \frac{1}{u^2+v^2}\left[(u^2+v^2)\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]$

Derivative of Implicit functions:→

The function $f(x,y)=0$ defines Implicit a function $y=\phi(x)$ of one Independent variable x.

Then $df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x(x,y)}{f_y(x,y)} \text{ provided } f_y(x,y)\neq 0$$

Ex $f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Find $\frac{dy}{dx}$

Soln

$$\frac{\partial f}{\partial x} = \frac{2x}{a^2} \; ; \; \frac{\partial f}{\partial y} = \frac{2y}{b^2}$$

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{x/a^2}{y/b^2}$$

$$= \frac{-b^2x}{a^2y}, \; y\neq 0.$$

Ex $f(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = 0$ Find $\frac{dy}{dx}$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{(2x-y)}{2y+x} = \frac{y-2x}{2y+x} \; ; \; 2y+x\neq 0 \; ; \; y\neq -x/2$$

HW (1) $x^y + y^x = \alpha$, α is any Constant, $x>0, y>0$

Find $\frac{dy}{dx}$

Sol^n $\frac{\partial f}{\partial x} = yx^{y-1} + y^x \log y$

$\frac{\partial f}{\partial y} = x^y \log x + xy^{x-1}$

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left(\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right)$$

(2) Find $\frac{dy}{dx}$ when $\cot^{-1}\left(\frac{x}{y}\right) + y^3 + 1 = 0$, $x>0, y>0$.