Moment Generating function:→ (m.g.f)

The m.g.f. of a random variable X is denoted by $M_X(t)$ and is defined as

$$M_X(t) = E(e^{tX}), \; t \in R$$

wherever this Expectation Exists.

→ If X is discrete R.V. then

$$M_X(t) = \sum_x e^{tx} \cdot P(x)$$

where $P(x)$ — Prob mass function (P.m.f)

→ If X is Continuous R.V. then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

where $f(x)$ — Prob. density function (P.d.f.)

→ m.g.f. is used to Compute moments of the distribution.

→ n^{th} moment about 0 is the n^{th} derivative of the m.g.f, at '0'.

i.e. $$\frac{d^n}{dt^n}(M_X(t))\Big|_{t=0} = n^{th} \text{ Moments.} \quad \text{— (1)}$$

For any positive Integer r, r^{th} moment about 0 (origin) is $\mu_r = E(X^r)$

So $\mu_1 = E(X) = \text{Mean}$

$\mu_2 = E(X^2)$

$\Rightarrow V(X) = E(X^2) - E(X)$

$\Rightarrow V(X) = \mu_2 - \mu_1$

From (1) $$\boxed{\mu_r = \frac{d^r}{dt^r}(M_X(t))\Big|_{t=0} = E(X^r)} \longrightarrow \text{Imp.}$$

→ Find the m.g.f of Binomial distⁿ and hence find its mean and variance.

Solⁿ The P.m.f. of binomial distⁿ is

$$P(x) = {}^{n}C_{x}\, p^{x} q^{n-x} \; ; x = 0, 1, \ldots n; \; p+q = 1$$

Its M.g.f is

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} . P(x)$$

$$= \sum_{x=0}^{n} e^{tx}\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} {}^{n}C_{x} (e^{t}p)^{x} q^{n-x}$$

$$= (e^{t}p + q)^{n} \; ; \text{ provided } |e^{t}p| < q$$

by using Binomial Expansion of $(a+x)^n = {}^{n}C_{0}\, a^{0} x^{n} + {}^{n}C_{1}\, a^{1} x^{n-1} + \ldots + {}^{n}C_{n}\, a^{n} x^{0}$

Provided $|x| < a$

So $M_x(t) = (e^{t}p + q)^{n}$ provided $|e^{t}p| < q$

$$\mu_1 = E(x) = \frac{d}{dt} M_x(t)$$

$$= \frac{d}{dt} (e^{t}p + q)^{n}$$

$$= n(e^{t}p + q)^{n-1} . e^{t}p \Big|_{t=0}$$

$$= n(p+q)^{n-1} . p = np \quad (\because p+q=1)$$

$$\mu_2 = E(x^2) = \frac{d^2}{dt^2} (e^{t}p + q)^{n}$$

$$= \frac{d}{dt} \left[n e^{t}p (e^{t}p + q)^{n-1} \right] \Big|_{t=0} = np\left[(n-1)p + 1\right]$$

$\Rightarrow V(x) = E(x^2) - E(x) = \mu_2 - \mu_1^2$

$= np[(n-1)p+1] - (np)^2$

$= np[\cancel{np} - p + 1 - \cancel{np}]$

$= np(1-p) = npq.$

→ Find the m.g.f. of Poisson distⁿ and hence find its mean and variance.

Solⁿ The p.m.f. of Poisson distⁿ is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} ; x = 0,1,2,\ldots$$

The m.g.f. is

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot P(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \ldots\right]$$

$$= e^{-\lambda} e^{\lambda e^t} \qquad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$$

$$= e^{\lambda(e^t - 1)}$$

$$\Rightarrow M_x(t) = e^{-\lambda(1-e^t)}$$

$$\mu_1 = E(x) = \frac{d}{dt}(M_x(t))\Big|_{t=0}$$

$$= \left[e^{-\lambda(1-e^t)} \cdot \lambda e^t\right]_{t=0}$$

$$= \lambda$$

$$\Rightarrow E(x) = \lambda$$

$$\mu_2 = E(x^2) = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\lambda e^t e^{-\lambda(1-e^t)}\right]_{t=0}$$

$$= \lambda\left[e^t e^{-\lambda(1-e^t)} + e^t \cdot e^{-\lambda(1-e^t)} \cdot \lambda e^t\right]_{t=0}$$

$$= \lambda\left[1+\lambda\right] = \lambda^2+\lambda$$

$$\Rightarrow V(X) = \mu_2 - \mu_1^2$$

$$= \cancel{\lambda^2}+\lambda-\cancel{\lambda^2} = \lambda$$

$$\Rightarrow V(X) = \lambda$$

→ Find the m.g.f. of geometric distⁿ and hence find its mean and variance.

Solⁿ → The p.m.f of geometric distⁿ is

$$P(x) = q^{x-1}p \; ; \; x = 1,2,\ldots$$

The m.g.f is

$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot q^{x-1} \cdot p$$

$$= \frac{p}{q}\sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + (qe^t)^3 + \ldots\right]$$

$$\Rightarrow M_X(t) = \frac{p}{q}\left[\frac{qe^t}{1-qe^t}\right] \quad \text{provided } qe^t < 1$$

$$\mu_1 = E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0}$$

$$= \frac{p}{q}\left[\frac{d}{dt}\left(\frac{qe^t}{1-qe^t}\right)\right]_{t=0}$$

$$= \frac{p}{q}\left[\frac{(1-qe^t)qe^t + qe^t(qe^t)}{(1-qe^t)^2}\right]_{t=0} = \frac{p}{q}\cdot\frac{q}{p^2} = \frac{1}{p}$$

$$\mu_2 = E(x^2) = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0}$$

$$= \frac{d^2}{dt^2}\left(\frac{pe^t}{(1-qe^t)^2}\right)\Big|_{t=0}$$

$$= \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}\Big|_{t=0}$$

$$= \frac{p+pq}{p^3} = \frac{1+q}{p^2}$$

$$V(x) = \mu_2 - \mu_1^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

Que: A perfect coin is tossed twice. Find the m.g.f of X where X – no. of heads.
Also find mean & variance.

$$S = \{HH, HT, TH, TT\}$$

Sol^n

x	$P(x)$
0	$1/4$
1	$2/4 = 1/2$
2	$1/4$

$$M_x(t) = E(e^{tX}) = \sum_{x=0}^{2} e^{tx} P(x)$$

$$= P(0) + e^t \cdot P(1) + e^{2t} P(2)$$

$$= \frac{1}{4} + \frac{1}{2} e^t + \frac{1}{4} e^{2t}$$

$$= \frac{1 + 2e^t + e^{2t}}{4} = \frac{(1+e^t)^2}{4}$$

$$\mu_1 = E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \frac{e^t}{4} 2(1+e^t) \Big|_{t=0} = 1$$

$$E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0} = \frac{d}{dt} \cdot \frac{e^t(1+e^t)}{2} \Big|_{t=0}$$

$$\boxed{\cancel{= \frac{1}{2}(e^t) \Big|_{t=0} = \frac{1}{2}}}$$

$$= \frac{e^t + 2e^{2t}}{2} \Big|_{t=0} = \frac{3}{2}$$

So $V(X) = \frac{3}{2} - 1 = \frac{1}{2}$.

Date ______
Page ______

Que Find the m.g.f. of the distⁿ

$$f(x) = \begin{cases} 2/3 ; & x=1 \\ 1/3 ; & x=2 \\ 0 ; & o/w. \end{cases}$$

Solⁿ

$$M_X(t) = \sum e^{tx} \cdot f(x)$$
$$= e^{t} \cdot f(1) + e^{2t} \cdot f(2)$$
$$= \frac{2}{3} e^{t} + \frac{1}{3} e^{2t} = \frac{2e^{t} + e^{2t}}{3}$$

Que The m.g.f. of r.v. X is given by $M_X(t) = e^{3(e^t - 1)}$. Find $P(X=1)$

Solⁿ

$M_X(t) = e^{3(e^t-1)}$ — m.g.f of Poisson distⁿ

$$\Rightarrow \lambda = 3$$

So $P(X=x) = \dfrac{e^{-\lambda} \lambda^x}{x!} ; x = 0,1,2 \ldots$

$$P(X=1) = \frac{e^{-3} \cdot 3^1}{1!} = \frac{3}{e^3} = 0.1494$$

Important Properties of M.g.f:→

(1) Translation
(2) Independence
(3) Shifting the origin and Scale
(4) Uniqueness Theorem.

(1) If $M_X(t)$ is the m.g.f of R.V. X then $M_X(ct)$ is the m.g.f. of the R.V. cX, where c is any constant.

Solⁿ:

$$M_X(t) = E(e^{tX})$$
$$\Rightarrow M_{cX}(t) = E(e^{tcX})$$
$$= E(e^{(ct)X}) = M_X(ct)$$

Ex If $M_x(t) = \frac{1}{1-t^2}$; $-1<t<1$

Then Find the m.g.f of $Y=3X$

Sol^n $M_Y(t) = M_{3X}(t) = M_X(3t)$

$$= \frac{1}{1-(3t)^2} = \frac{1}{1-9t^2}; \quad -1<3t<1$$

$$\Rightarrow M_Y(t) = \frac{1}{1-9t^2}; \quad \frac{-1}{3}<t<\frac{1}{3}.$$

(2) If X_1 and X_2 are Independent R.V. then

$$M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t)$$

Sol^n $M_{X_1+X_2}(t) = E(e^{t(X_1+X_2)})$

$$= E(e^{tX_1+tX_2}) = E(e^{tX_1} \cdot e^{tX_2})$$

$$= E(e^{tX_1}) \cdot E(e^{tX_2})$$

$$= M_{X_1}(t) \cdot M_{X_2}(t)$$

[$\because E(XY) = E(X) \cdot E(Y)$ If X and Y are Independent]

In general; If $X_1, X_2 \ldots X_n$ are Independent Random variables then $M_{X_1+X_2+\ldots+X_n} = \prod_{i=1}^{n} M_{X_i}(t)$

(i.e. M.g.f of Sum = Product of m.g.f of variables)

(3) If X is transformed into Y by changing both the Origin and scale i.e. $Y = \frac{X-a}{h}$

then $M_Y(t) = e^{-at/h} M_X\left(\frac{t}{h}\right)$

Date ______
Page ______

Sol^n

$$M_Y(t) = E(e^{tY})$$
$$= E\left[e^{t\left(\frac{X-a}{h}\right)}\right]$$
$$= E\left(e^{\frac{tX}{h}} \cdot e^{-at/h}\right)$$
$$= e^{-at/h}\, E\left(e^{\frac{tX}{h}}\right) \qquad [\because E(aX) = aE(X)]$$
$$= e^{-at/h} \cdot M_X\left(\frac{t}{h}\right)$$

Que If the m.g.f of X is $M_X(t) = \frac{1}{4}(1+e^t)^2$

Then find the m.g.f of $Y = \frac{X-4}{2}$

Sol^n

$$M_Y(t) = E(e^{Yt})$$
$$= E\left(e^{t\left(\frac{X-4}{2}\right)}\right) = E\left(e^{\frac{tX}{2}} \cdot e^{-2t}\right)$$
$$= e^{-2t}\, E\left(e^{\frac{t}{2}X}\right)$$
$$= e^{-2t} \cdot M_X\left(\frac{t}{2}\right)$$
$$= e^{-2t} \cdot \frac{1}{4}\left(1+e^{t/2}\right)^2$$
$$\Rightarrow M_Y(t) = \frac{1}{4}\frac{(1+e^{t/2})^2}{e^{2t}}$$

H.W (1) If m.g.f of X is $M_X(t) = \frac{1}{2t}(e^{2t}-1)^2$

Then find the m.g.f of $Y = \frac{X-6}{2}$

Ans:- $\frac{e^{-3t}}{t}(e^t-1)$

(2) If m.g.f of X is given by $M_X(t) = \frac{1}{1-t^2}$; $|t|<1$

Then find the M.g.f of $Y = \frac{X-4}{4}$

Ans:- $e^{-t}\left(\frac{16}{16-t^2}\right)$; $|t| < 4$.

(4) Uniqueness Theorem :→ The m.g.f of a distⁿ, if it exists, uniquely determines the distⁿ.

i.e. Corresponding to a given prob. distⁿ, there is only one m.g.f and corresponding to a given m.g.f., there is only one prob. distⁿ.

⇒ If $M_X(t) = M_Y(t)$ ⇒ X and Y are Identically distributed.

i.e. X and Y have same prob. distⁿ.

Note:- M.g.f may not Exist for all random variables:-

Ex:- Let X have the P.m.f.

$$f(x) = \begin{cases} \frac{6}{\pi^2 x^2} & ; x = 1, 2, \ldots \\ 0 & ; o/w \end{cases}$$

Find the M.g.f. of X.

Solⁿ $M_X(t) = E(e^{tX})$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot f(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{e^{tx}}{x^2}$$

$$= \frac{6}{\pi^2}\left[\frac{e^t}{1} + \frac{e^{2t}}{2^2} + \frac{e^{3t}}{3^2} + \ldots\right]$$

This Series is not cgt. $\forall t > 0$

⇒ $M_X(t)$ does not Exist.

Date
Page

→ Find the m.g.f. of the Exponential distⁿ and hence find its mean and variance.

Solⁿ The P.d.f of Exponential distⁿ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} ; & x \geq 0 \\ 0 ; & o/w. \end{cases}$$

$$M_x(t) = E(e^{tx})$$

$\because$ X is Continuous R.V.

$$\Rightarrow M_x(t) = \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^\infty e^{(t-\lambda)x} dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^\infty$$

Now $e^{-(\lambda-t)x} = 0$ If $\lambda - t > 0$

∞ If $\lambda - t < 0$

$$\therefore M_x(t) = \frac{\lambda}{t-\lambda} [0 - 1]$$

$$= \frac{\lambda}{\lambda - t}$$ Provided $\lambda - t > 0$ or $t < \lambda$

$$\mu_1 = E(x) = \frac{d}{dt}\left[M_x(t)\right]_{t=0} = \frac{d}{dt}\left[\frac{\lambda}{\lambda - t}\right]_{t=0}$$

$$= \lambda \left[\frac{+1}{(\lambda - t)^2} \right]_{t=0}$$

$$= \frac{1}{\lambda}$$

$$\mu_2 = E(x^2) = \frac{d^2}{dt^2}\left[M_x(t)\right]_{t=0} = \frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right]_{t=0} = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0}$$

$$\Rightarrow E(x^2) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\Rightarrow V(x) = \mu_2 - \mu_1^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

→ Find the m.g.f. of the uniform distⁿ over the range [a,b]. Hence find its mean and variance.

Solⁿ:

$$f(x) = \begin{cases} \frac{1}{b-a} & ; a \le x \le b \\ 0 & ; o/w \end{cases}$$

$$M_x(t) = E(e^{tx}) = \int_a^b e^{tx} \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$\Rightarrow M_x(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}$$