# UNIT-2: Complex Analysis - II (Continuation of unit 1...)

## 1) Laurent Series:

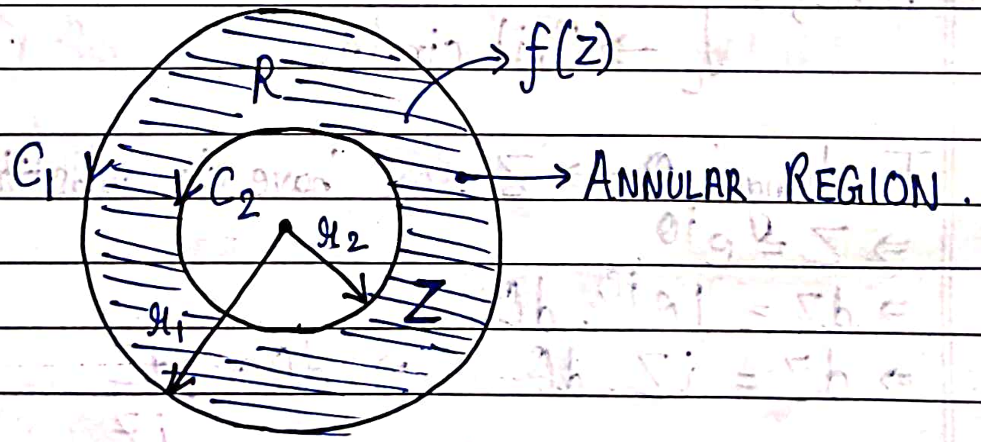
If $f(z)$ is analytic in annular (co-centric) region $R$ bounded by two co-centric circles $C_1$ and $C_2$ of radii $r_1$ and $r_2$ where, $r_1 \geq r_2$ and center at $a$, then for all $Z$ in $R$ we have,

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{General term}} + \underbrace{\sum_{n=0}^{\infty} b_n (z-a)^{-n}}_{\text{Principal term}}$$

here,

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{n+1}}\, dt \quad , \quad n = 0, 1, 2, 3, \ldots$$

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(t)}{(t-a)^{-n+1}}\, dt \quad , \quad n = 1, 2, 3, 4, \ldots$$




**Note:** Some important binomial expansions,

1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \infty$

2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots \infty$

## 2) Residue Integration Method :-

2.1) i) **Residue** → Residue of an analytic function $f(z)$ at $z=a$ is the coefficient of $\frac{1}{z-a}$, i.e., $b_n$ in the Laurentz series expansion $(z)$ for $n=1 \Rightarrow b_1$.

→ Residue of $f(z)$ is denoted by $\text{Res}\, f(z)$ or $\text{Res}\left[\{f(z)\}; a\right]$ at $z=a$.

Since,

$$\Rightarrow b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}\, dz.$$

$$\Rightarrow b_1 = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-1+1}}\, dz \qquad (n=1).$$

or,

$$\boxed{b_1 = \frac{1}{2\pi i}\int_C f(z).dz}.$$

★ Cauchy's Residue Theorem :-

$$\Rightarrow \int_C f(z).dz = 2\pi i\, b_1 = 2\pi i\,[\text{Residue}] = 2\pi i \sum_i R^+$$

ii) How to find out Residue :-

1) Residue at simple pole at $z=a$.

$$\underset{\text{at } z=a}{\text{Res}}\, f(z) = \lim_{z\to a} (z-a).f(z).$$

2) Residue at a pole of order 'm'.

$$\underset{\text{at } z=a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z)\right]_{z=a}.$$

3) If $f(z)$ is of the form,

$$f(z) = \frac{\phi(z)}{\psi(z)} \quad \text{at } z=a.$$

where, $\psi(a) \neq 0$ and, $\phi(a) \neq 0$.

then,

$$\underset{\text{at } z=a}{\text{Res}}\, f(z) = \frac{\phi(a)}{\psi(a)}$$

4) Residue of $f(z)$ at $z=\infty$

$$\underset{\text{at } z=a}{\text{Res}}\, (f(z)) = \lim_{z\to\infty} \left[-z \cdot f(z)\right]$$

OR

$$= -\left[\text{coefficient of } \frac{1}{z} \text{ in the expansion of } f(z)\right]$$

Q.1) Determine the residue of the function at each pole:

$$f(z) = \frac{z^2}{(z-1)(z-2)^2}.$$

Ans: First we calculate pole,

$\Rightarrow (z-1)(z-2)^2 = 0$

$\Rightarrow z = 1, 2, 2.$

or,

$z=1$ (simple pole) ; $z=2$ (order: 2).

- $$\underset{\text{at } z=1}{\text{Residue of}}\, f(z) = \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{z^2}{\cancel{(z-1)}(z-2)^2}$$

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$$= \lim_{z\to 1} \frac{z^2}{(z-2)^2} = \frac{1^2}{(1-2)^2} = \frac{1}{1} = 1 = R_1$$

- Residue of $f(z)$ at $z=2$ (order 2) $= \frac{1}{(2-1)!} \lim_{z\to 2} \frac{d^{2-1}}{dz^{2-1}} \left[ (z-2)^2 \cdot \frac{z^2}{(z-1)(z-2)^2} \right]$

$$\Rightarrow \operatorname{Res}_{z=2} (f(z)) = \frac{d}{dz}\left(\frac{z^2}{z-1}\right)_{z=2}$$

$$= \left[\frac{2z(z-1) - z^2(1)}{(z-1)^2}\right]_{z=2}$$

$$= \left[\frac{2z^2 - 2z - z^2}{(z-1)^2}\right]_{z=2}$$

$$= \left[\frac{z(z-2)}{(z-1)^2}\right]_{z=2}$$

$$\therefore \operatorname{Res}_{z=2} f(z) = \frac{2\times(2-2)}{(2-1)^2} = 0 = R_2$$

Hence,

$$\sum R^+ = R_1 + R_2 = 1 + 0 = \boxed{1} \rightarrow \text{Residue.}$$

Q.2) Find the residue of the tangent at its pole :-

$$f(z) = \frac{\sin z}{\cos z} = \tan z.$$

Ans : For pole,

$$\Rightarrow \cos z = 0 = \cos \pi/2$$

$$\therefore \boxed{z = \frac{\pi}{2}}$$

If, $f(z) = \frac{\phi(z)}{\psi(z)}$ at pole $z = a$.

$\Rightarrow$ Residue of $f(z)$ at $z=a$ $= \frac{\phi(a)}{\psi'(a)}$, if $\phi(a) \neq 0$, $\psi(a) = 0$

$$\Rightarrow \operatorname*{Res}_{z=a} f(z) = \frac{\sin \pi/2}{\cos \pi/2} = \frac{1}{0} = \infty.$$

**Q.3)** Find the residue of $f(z) = z^2 e^{1/z}$ ?

**Ans:** $f(z) = z^2 \cdot e^{1/z}$

$$\Rightarrow f(z) = z^2 \left[ 1 + \frac{1}{(1/z)^{-1}} + \frac{1}{z^2} \times \frac{1}{2!} + \frac{1}{z^3} \cdot \frac{1}{3!} + \dots \right]$$

$$\left( \because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right)$$

$$\Rightarrow f(z) = z^2 + z + \frac{1}{4} + \frac{1}{6z} + \dots$$

$\because$ Res $f(z) = -$ (coefficient of $1/z$ in the expansion) $= \boxed{-1/6}$

**Q.4)** Find the integral of $f(z)$ using Cauchy's residue theorem:-

$$f(z) = \int_C \frac{1-2z}{z(z-1)(z-2)} dz$$

where, 'C' is the circle $\rightarrow |z| = 1.5$ ?

**Ans** = $\displaystyle\int_C \frac{1-2z}{z(z-1)(z-2)} dz = 2\pi i \sum R^+$ (from Residue theorem)

$\rightarrow$ Clearly, poles are : $z = 0, 1, 2$.

But,

for the given circle $|z| = 1.5$, pole $= 2$ lies outside this circle, and only poles '0' and '1' lie inside C.

So, applying residue for $z=0$ and $z=1$ only,

$\Rightarrow \underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} (z-0). \left(\frac{1-2z}{(z)(z-1)(z-2)}\right)$

$\Rightarrow \underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)}$

$\therefore \underset{z=0}{\text{Res}}\, f(z) = \frac{1}{-1\times -2} = \boxed{\frac{1}{2}} \rightarrow R_1$

and,

$\Rightarrow \underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} (z-1) \left(\frac{1-2z}{z(z-1)(z-2)}\right)$

$\Rightarrow \underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} \frac{1-2z}{z(z-2)}$.

$\therefore \underset{z=1}{\text{Res}}\, f(z) = \frac{-1}{1\times -1} = \boxed{1} \rightarrow R_2$

hence,

$$\Sigma R^+ = R_1 + R_2 = \frac{1}{2} + 1 = \frac{3}{2}.$$

i.e.,

$\Rightarrow \int_C f(z).\,dz = 2\pi i\, \Sigma R^+ = 2\pi i \times \frac{3}{2} = \boxed{3\pi i}$.

## 2.2) Contour Integration : Residue Integration of Real Integrals :-

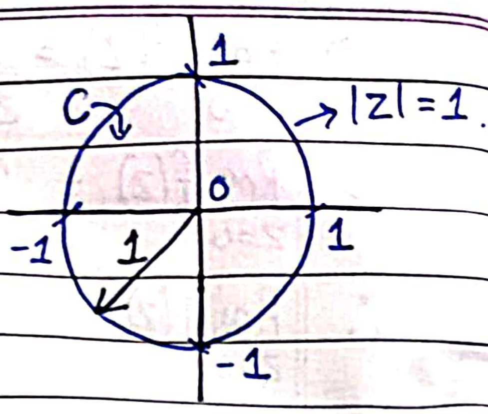
→ Integration of these types, can be solved by contour integration method only.

1) $\int_0^{2\pi} f(z).\,dz$ 2) $\int_{-\infty}^{\infty} f(z).\,dz$.

**Case 1)** Integration round the unit circle $|z| = 1$ of the type,

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta).\,d\theta.$$

→ For the region 'C' → a unit circle from 0 to $2\pi$,
'θ' will change to 'z' by;




Substituting,

$\Rightarrow z = e^{i\theta}$

$\Rightarrow z = e^{i\theta}$

$\Rightarrow dz = e^{i\theta} \cdot i\,d\theta = zi\,d\theta$

$\Rightarrow d\theta = \dfrac{dz}{iz}$ —— ①

For trigonometric terms,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z} \quad \text{—— ②}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2-1}{2zi} \quad \text{—— ③}$$

Use the equations ① - ③ to convert,

$$I = \int_0^{2\pi} f(\theta).d\theta \quad \text{to} \quad \int_{0\,C}^{2\pi} f(z).dz.$$

Q.1) Find the integral $f(z)$, $I = f(z) = \displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$.

Ans: Let,

$\Rightarrow z = e^{i\theta}$

$\Rightarrow dz = ie^{i\theta}.d\theta = iz.d\theta.$

$\Rightarrow d\theta = \dfrac{dz}{iz}$

and,

$$\Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2-1}{2zi}$$

On substituting in the integral, we have;

$$\Rightarrow I = \int_C \frac{1}{5-4\left(\frac{z^2-1}{2iz}\right)} \cdot \frac{dz}{iz}$$

$$\Rightarrow I = \int_C \frac{2\cancel{iz}}{10iz-4z^2+4} \cdot \frac{dz}{\cancel{iz}} = \int_C \frac{2 \cdot dz}{-4(z^2-5/2\,iz-1)}$$

$$\Rightarrow I = \frac{-1}{2}\int_C \frac{1}{(z^2-\frac{5}{2}iz-1)}\, dz = \frac{-1}{2}\int_C f(z) \cdot dz.$$

here,

$$f(z) = \frac{1}{z^2-\frac{5}{2}iz-1}$$

- For pole,

$$\Rightarrow z^2 - \frac{5}{2}iz - 1 = 0 \quad \text{or,} \quad 2z^2-5iz-2=0.$$

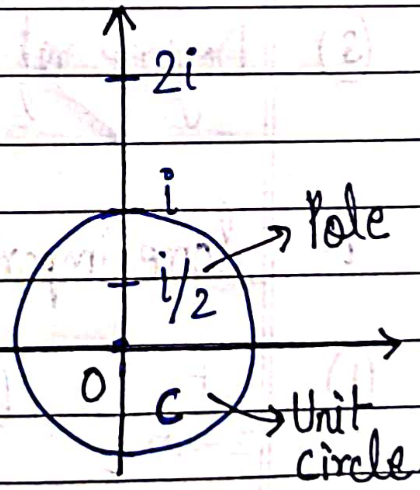
$$\Rightarrow z = \sqrt{(-5i)^2-4(2)(-2)} + 5i \,/\, 4.$$

or,

$$\Rightarrow z = \frac{5i \pm \sqrt{-25+16}}{4} = \frac{5i \pm 3i}{4} = \frac{i}{2} \text{ or } 2i$$

$$\therefore z = i/2 \text{ or } 2i.$$

But,
'2i' lies outside the region 'C' so it is not accounted in and, 'i/2' lies inside the unit circle 'C' region.




- Residue of $f(z)$ at $z = i/2$.

$$\Rightarrow \underset{\text{at } z=i/2}{\text{Res}}\, f(z) = \lim_{z \to i/2} \cancel{(z-i/2)} \frac{1}{(z-2i)\cancel{(z-i/2)}}$$

$$\therefore \underset{\text{at } z=i/2}{\text{Res}}\, f(z) = \lim_{z \to i/2} \frac{1}{z-2i} = \frac{2}{i-4i} = \boxed{\frac{-2}{3i}}$$

Now, by Cauchy's residue theorem;

$$\Rightarrow \int_C f(z).dz = 2\pi i \sum R^+ = 2\pi \cancel{i} \times \frac{-2}{3\cancel{i}} = \frac{-4\pi}{3}$$

hence,

$$\Rightarrow I = \frac{-1}{2}\int_C f(z).dz$$

$$\therefore I = \frac{-1}{2} \times \frac{-4\pi}{3} = \boxed{\frac{2\pi}{3}}$$

**Case 2)** Integration of the type, $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)} dx = 2\int_0^{\infty} \frac{f_1(x)}{f_2(x)} dx$.

★ Conditions to solve this integral:

1. No pole should be on X-axis, i.e., $f_2(x)$ = should have NO real roots.
2. $f_2(x)$ should have a polynomial degree of '2' atleast.
3. Degree of $f_2(x)$ > Degree of $f_1(x)$.

★ Some important points to solve this integral :-

1. $\lim_{z\to\infty} z.f(z) = 0$, then $\int_C f(z).dz = 0$
2. $\lim_{z\to\infty} f(z) = 0$, then $\lim_{R\to\infty}\int_C e^{imz}.f(z) = 0$ **JORDAN LEMMA**
3. If 'C' is the arc of the circle $|z-a| = r$ such that,

$\theta_1 \le \theta \le \theta_2$ and $\lim_{z \to a} (z-a) \cdot f(z) = K$ say, ($K$: constant)

then, $\lim_{r \to 0} \int_C f(z) \cdot dz = iK \, |\theta_2 - \theta_1|$.

(4) If 'C' is the arc of the circle $|z| = R$ such that $\theta_1 \le \theta \le \theta_2$ and, $\lim_{z \to \infty} z \cdot f(z) = K$, then;

$$\lim_{R \to \infty} \int_C f(z) \cdot dz = iK \, |\theta_2 - \theta_1|.$$

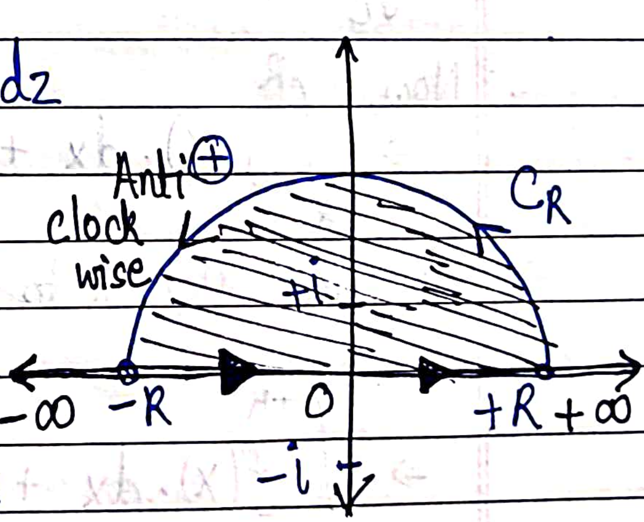
**Q.1)** Evaluate the integral $I = \int_0^{\infty} \frac{\cos mx}{x^2+1} \cdot dx$ (Important)

OR

Evaluate the integral $I = \int_{-\infty}^{\infty} \frac{\sin mx}{x^2+1} \cdot dx$

**Ans:** Consider the integral, $\int_C f(z) \cdot dz$

where, C: Is the curve consisting upper half of the circle, $|z| = R$ and the part of the real axis from $-R$ to $+R$ and,




$$f(z) = \frac{e^{imz}}{1+z^2}$$

For pole,

$\Rightarrow 1 + z^2 = 0$

$\Rightarrow z = \pm i$

But, pole $= -i$ lies outside the region 'C' so the only pole is $= +i$.

Here,

$$e^{imz} = \cos mz + i\sin mz.$$

and,

'x' is replaced by 'z' in the integral I.

$\Rightarrow$ Now, residue of $f(z)$ at $z=a$ where $a=i$,

$$\Rightarrow \operatorname*{Res}_{z=i} f(z) = \lim_{z\to i} (z-i)\frac{e^{imz}}{1+z^2}$$

$$= \lim_{z\to i} \cancel{(z-i)}\frac{e^{imz}}{(z+i)\cancel{(z-i)}}$$

$$= \lim_{z\to i} \frac{e^{imz}}{z+i}$$

$$\therefore \operatorname*{Res}_{z=i} f(z) = \frac{e^{im(i)}}{i+i} = \frac{e^{-m}}{2i}$$

So,

$$\int_C f(z).dz = 2\pi i \Sigma R^+ = 2\pi i \times \frac{e^{-m}}{2i} = \pi e^{-m}.$$

Now,

$$\Rightarrow \int_{-R}^{R} f(x).dx + \int_{C_R} f(z).dz = \pi e^{-m}.$$

$\Rightarrow$ At limit $\infty$, $-R$ to $+R$; $\int_{C_R} f(z).dz$ will be ALWAYS zero.

$$\Rightarrow \int_{-R}^{+R} f(x).dx + 0 = \pi e^{-m}.$$

$\Rightarrow$ Limit $(-R$ to $+R)$ will extend to $(-\infty$ to $+\infty)$;

$$\Rightarrow \int_{-\infty}^{\infty} f(x).dx = \pi e^{-m}.$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{1+x^2} dx = \pi e^{-m}.$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{\cos mx + i \sin mx}{1+x^2} dx = \pi e^{-m}$$

or,
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2} + i \int_{-\infty}^{\infty} \frac{\sin mx}{1+x^2} = \pi e^{-m} + i(0).$$

On comparing,
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{1+x^2} = \pi e^{-m}.$$

or,
$$\Rightarrow \int_{0}^{\infty} \frac{\cos mx}{1+x^2} = \frac{\pi}{2} e^{-m}.$$

and,
$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin mx}{1+x^2} = 0.$$

★ <u>Explanation: Why will be $C_R$ always zero for limit of $z$ to $\infty$:</u>

- By Jordan's Lemma Theorem,

if $\lim\limits_{z \to \infty} f(z) = 0$

then,
$$\lim_{R \to \infty} \int_C e^{imz} . f(z) . dz = 0$$

- In above question,
$$f(z) = \frac{1}{1+z^2}$$

($e^{imz}$ will be seperated out of $f(z)$...)

$$\Rightarrow \lim_{z \to \infty} \frac{1}{1+z^2} = 0 = \frac{1}{\infty} = 0.$$

So, $\lim\limits_{R \to \infty} \int_C e^{imz} . f(z) . dz = \lim\limits_{R \to \infty} \int e^{imz} (0) . dz = 0.$

**Q.2)** Evaluate the integral :

$$I = \int_0^\infty \frac{dx}{1+x^2}$$

**Ans :** Given,

$$I = \int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

here,

$$f(z) = \frac{1}{1+z^2}$$

Pole will be $= +i$

and,

$$\underset{at\ z=i}{\text{Res}}\ f(z) = \lim_{z\to i} \cancel{(z-i)}\frac{1}{(z+i)\cancel{(z-i)}} = \frac{1}{2i}$$

so,

$$\Rightarrow \int_C f(z).dz = 2\pi i \sum R^+ = 2\pi i \times \frac{1}{2i} = \pi.$$

Now,

$$\Rightarrow \int_{-R}^{+R} f(x).dx + \int_{C_R} f(z).dz = \pi.$$

By Jordan's Lemma theorem, at limit $\infty$ in $-R$ to $+R$,

$$\int_{C_R} f(z).dz = 0$$

$$\Rightarrow \int_{-R}^{+R} f(x).dx = \pi. \quad \Rightarrow \int_{-\infty}^{\infty} f(x).dx = \pi.$$

where, $f(x) = \dfrac{1}{1+x^2}$.

Therefore,

$$I = \frac{1}{2}\int_{-\infty}^{\infty} f(x).dx = \frac{1}{2}\times\pi = \boxed{\frac{\pi}{2}}.$$

Q.3) Apply the calculus of residue to evaluate the Integral I:

$$I = \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3}$$

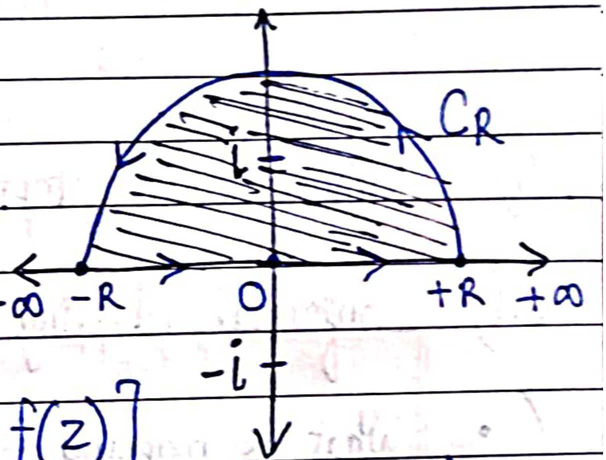
Ans: Consider the integral, $\int_c f(z).dz$ where, $f(z) = \frac{1}{(1+z^2)^3}$

and $C$ is the curve consisting the upper half $C_R$ of the circle $|z| = R$ and, part of the real axis from $-R$ to $+R$.

Pole of $f(z)$ is,

$$\Rightarrow (1+z^2)^3 = 0$$

$$\Rightarrow \boxed{z = +i} \text{ (order 3)}$$




Residue of $f(z)$ of order = 3,

$$= \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n . f(z)\right] \quad \text{(for order = } n\text{).}$$

$$= \frac{1}{2!} \times \lim_{z\to i} \frac{d^2}{dz^2}\left[(z-i)^3 . \frac{1}{(1+z^2)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left[\cancel{(z-i)^3} \times \frac{1}{[\cancel{(z-i)}(z+i)]^3}\right]$$

$$= \frac{1}{2} \lim_{z\to i} \frac{d^2}{dz^2}\left(\frac{1}{(z+i)^3}\right)$$

$$= \frac{1}{2} \lim_{z\to i} \left[-3(z+i)^{-5} \times -4\right]$$

$$= \frac{1}{2} \times \lim_{z\to i} \frac{12}{(z+i)^5}$$

$$= \frac{1}{2} \times \frac{12}{32 \times i^5} = \frac{3}{16i} \qquad (\because i^5 = i^{4+1} = i^5)$$

Now,

By Residue's theorem;

$$\Rightarrow \int_C f(z).dz = 2\pi i \Sigma R^+ = 2\pi i \times \frac{3}{16i} = \boxed{\frac{3\pi}{8}}$$

Hence,

$$= \int_{-R}^{R} f(x).dx + \int_{C_R} f(z).dz = \frac{3\pi}{8}.$$

or,

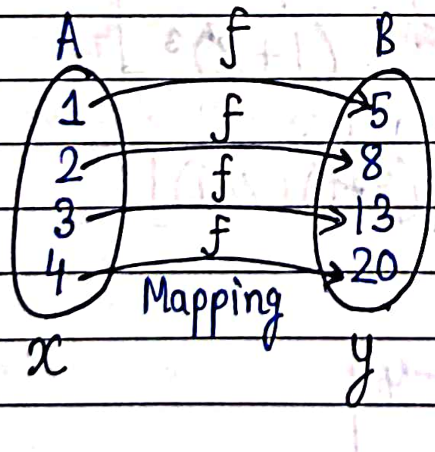
$$I = \int_{-\infty}^{\infty} f(x).dx = \boxed{\frac{3\pi}{8}}.$$

## 3) Geometrical Intepretation of Analytic Functions :-

### 3.1) Conformal Mapping :-

- What is mapping ?

→ An operator that is used to map or relate the elements of one set into another set is mapping.




Here,

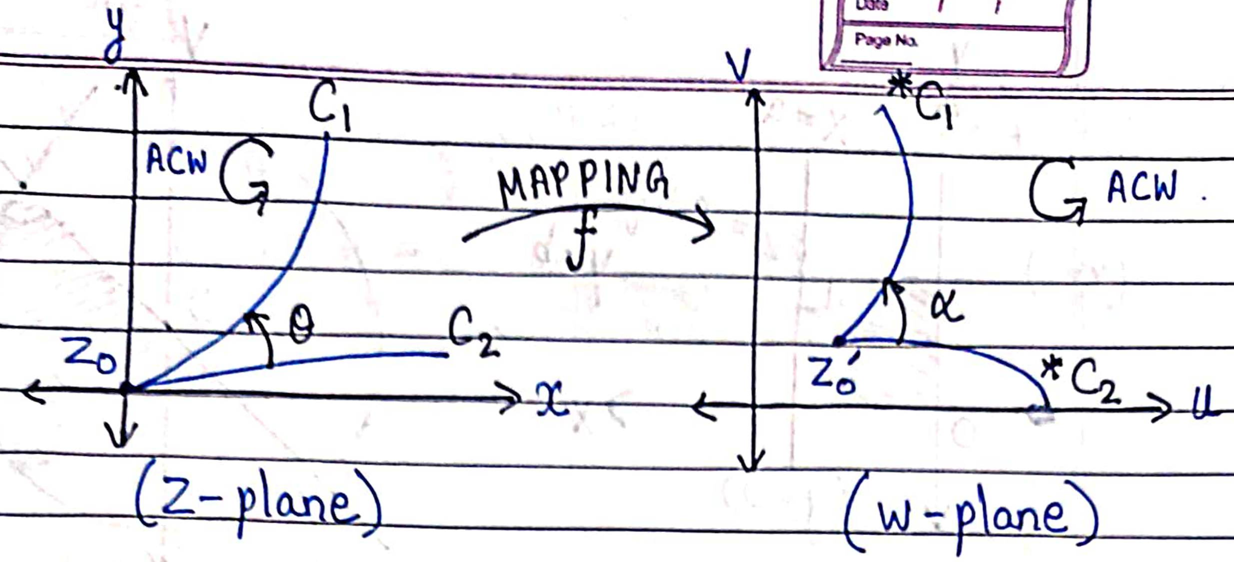
$$y = f(x) = x^2 + 4 . \forall x \in N.$$

In this case, element 1 of set 'x' is mapped with element 5 of set 'y' only.

- What is conformal mapping ?

→ The mapping of Z-plane into W-plane or W-plane into Z-plane by following relation,

- $u + iv = W = f(Z) = f(x+iy).$ $\left(\begin{array}{c} W = u + iv \\ Z = x + iy \end{array}\right)$




- For conformal mapping,
  1. $\theta = \alpha$.
  2. Sense of Rotation must be same.
     (here, ACW for both the sketches)

**Q.1)** Show that the mapping $w = e^z$ is conformal mapping in the whole z-plane?

**Ans:** We have,

$\Rightarrow$ $w = e^z$.

$\Rightarrow$ $u + iv = \exp(x+iy) = e^x . e^{iy} = e^x(\cos y + i \sin y)$

$\Rightarrow$ $u + iv = e^x \cos y + i e^x \sin y$.

So,
$$u = e^x \cos y \quad \text{and,} \quad v = e^x \sin y.$$

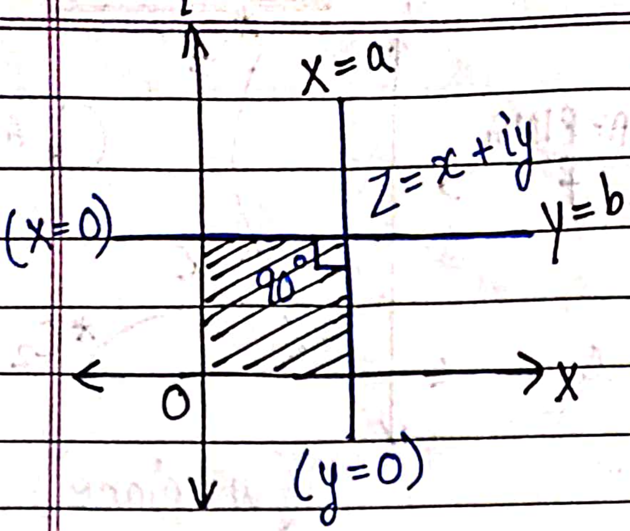
$\Rightarrow$ $(u)^2 + (v)^2 = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x}$.

or,
$$u^2 + v^2 = (e^x)^2. \longrightarrow \text{A circle.}$$

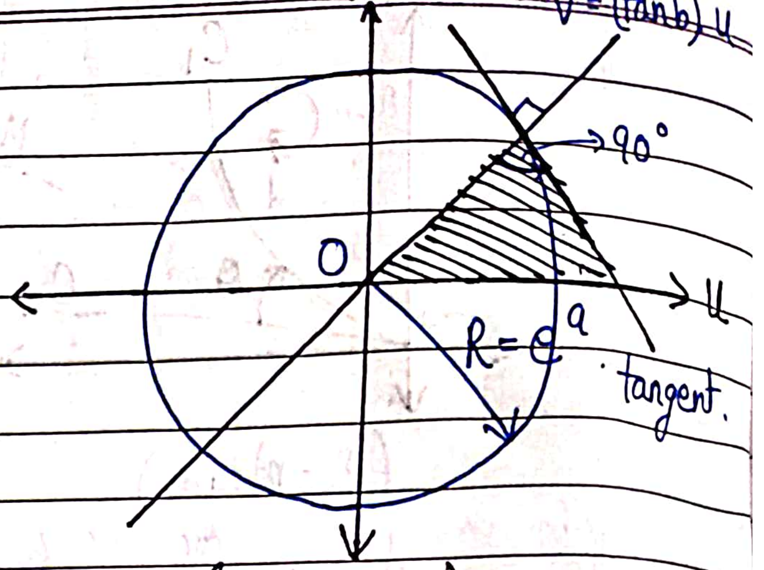
and,

$\Rightarrow$ $\dfrac{u}{v} = \dfrac{e^x \cos y}{e^x \sin y} = \cot y$

$\Rightarrow$ $\dfrac{v}{u} = \tan y$ or, $\Rightarrow v = (\tan y) . u \longrightarrow$ Straight line




(Z-plane)




(W-plane).

→ The rectangle in Z-plane is mapped as a triangle in W-plane, where $\theta = 90°$ and sense of rotation is same.

→ Both figures having the shaded areas are of equal areas.

∴ It is a conformal mapping of $w = e^{z}$.

**Note:** Polar coordinates of,

$$w = \rho e^{i\phi} \quad \text{and,} \quad z = re^{i\theta}.$$

**Q.2)** Show that the image on the real axis in the Z-plane on the w-plane by the transformation,

$$w = \frac{1}{z+i}$$

is a circle and, find radius 'R' and center of this circle?

**Ans:** $w = \dfrac{1}{z+i} \Rightarrow z + i = \dfrac{1}{w} \Rightarrow z = \dfrac{1}{w} - i$

$$\Rightarrow z = \frac{1 - iw}{w}$$

$$\Rightarrow x + iy = \frac{1 - i(u+iv)}{u+iv} \times \frac{u - iv}{u - iv}.$$

$$\Rightarrow x+iy = \left(\frac{1+v-ui}{u+iv}\right) \times \frac{u-iv}{u-iv}$$

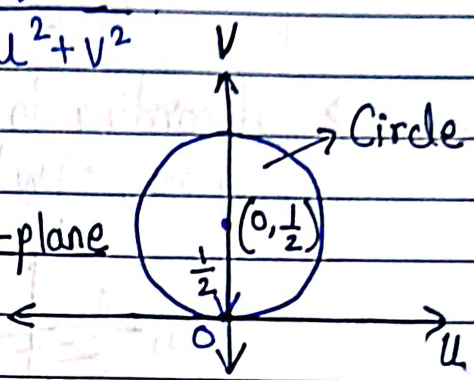
$$\Rightarrow x+iy = \left(\frac{u}{u^2+v^2}\right) - i\left(\frac{v}{u^2+v^2} - 1\right)$$

So,

$$x = \frac{u}{u^2+v^2} \quad \text{and,} \quad y = 1 - \frac{v}{u^2+v^2}$$

For image on the real axis,

$$y = 0$$




$$\Rightarrow \frac{1-v}{u^2+v^2} = 0$$

$$\Rightarrow u^2+v^2 = v$$

$$\Rightarrow u^2+v^2+0.u+(-1)v.+0 = 0 \longrightarrow \text{A circle}$$

$\because$ General equation of circle,

$$\Rightarrow x^2+y^2+2gx+2fy+c=0 \rightarrow C \equiv (-g,-f)$$

$$\hookrightarrow R = \sqrt{g^2+f^2-c}$$

i.e.,

$$C \equiv \left(0, \frac{1}{2}\right) \text{ and, } r = \frac{1}{2}.$$

**Q.3)** Under the transformation, $Z = \frac{4}{(w+1)^2}$. Find the region in the Z-plane corresponding to the interior of circle $|w|=1$?

**Ans:** $(w+1)^2 = \frac{4}{Z} \Rightarrow w+1 = \frac{2}{\sqrt{Z}} \Rightarrow w = \frac{2}{\sqrt{Z}} - 1.$

$\rightarrow$ On direct substitution, $Z = x+iy$, equation will be hard to solve, so putting polar coordinates,

$$\Rightarrow w = \frac{2}{\sqrt{re^{i\theta}}} - 1 = \frac{2}{\sqrt{r}} \cdot \frac{1}{e^{i\theta/2}} - 1 \quad (\because Z = re^{i\theta})$$

so,

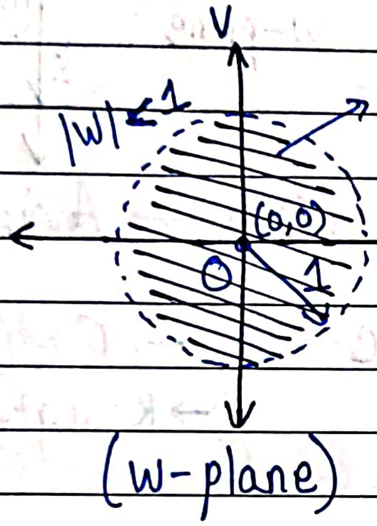
$$\Rightarrow u+iv = \frac{2}{\sqrt{r}} e^{-i\theta/2} - 1 = \frac{2e^{\frac{-i\theta}{2}} - \sqrt{r}}{\sqrt{r}}$$

$\Rightarrow \quad w = \frac{1}{\sqrt{r}}\left[2e^{-i\theta/2} - \sqrt{r}\right]$

$\Rightarrow \quad w = \frac{1}{\sqrt{r}}\left[2\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) - \sqrt{r}\right]$

$\Rightarrow \quad w = \frac{1}{\sqrt{r}}\left[\left(2\cos\frac{\theta}{2} - \sqrt{r}\right) - i\left(2\sin\frac{\theta}{2}\right)\right]$

$\Rightarrow$ According to Question, the region should be interior of circle, $|w| = 1$.




Only Interior $\Rightarrow |w| < 1$.
region or,

$u^2 + v^2 < 1$

where,

$u = \frac{2}{\sqrt{r}}\cos\frac{\theta}{2} - 1$

$v = -\frac{2}{\sqrt{r}}\sin\frac{\theta}{2}$.

$\Rightarrow \quad u^2 + v^2 < 1$

$\Rightarrow \quad \left(\frac{2}{\sqrt{r}}\cos\frac{\theta}{2} - 1\right)^2 + \left(\frac{-2}{\sqrt{r}}\sin\frac{\theta}{2}\right)^2 < 1$

$\Rightarrow \quad \frac{4}{r}\cos^2\frac{\theta}{2} + 1 - \frac{4}{\sqrt{r}}\cos\frac{\theta}{2} + \frac{4}{r}\sin^2\frac{\theta}{2} < 1$.

$\Rightarrow \quad \frac{4}{r} + \cancel{1} - \frac{4}{\sqrt{r}}\cos\frac{\theta}{2} < \cancel{1}$.

$\Rightarrow \quad \frac{4}{r}\left(1 - \sqrt{r}\cos\frac{\theta}{2}\right) < 0$.

$\Rightarrow \quad 1 < \sqrt{r}\cos\frac{\theta}{2} \quad \Rightarrow \sqrt{r} > \frac{1}{\cos\theta/2}$

or,

$\Rightarrow r > \frac{1}{\cos^2\frac{\theta}{2}} \quad \Rightarrow r > \frac{2}{2\cos^2\theta/2}$

$\Rightarrow r > \dfrac{2}{1+\cos\theta}$ $\quad (\because 1+\cos 2\theta = 2\cos^2\theta)$.

$\Rightarrow r(1+\cos\theta) > 2$

$\Rightarrow r + r\cos\theta > 2$.

$\because x = r\cos\theta$ and, $y = r\sin\theta$, i.e.,

on squaring,

$$\Rightarrow x^2 + y^2 = r^2.$$

$$\Rightarrow r = \sqrt{x^2+y^2}.$$

$\Rightarrow \sqrt{x^2+y^2} + x > 2$.

$\Rightarrow \sqrt{x^2+y^2} > 2 - x$.

$\Rightarrow x^2 + y^2 > (2-x)^2$.

$\Rightarrow y^2 > (2-x)^2 - x^2 = 4 + \cancel{x^2} - 4x - \cancel{x^2}$

or,

$\Rightarrow \boxed{y^2 > -4(x-1)} \longrightarrow$ A parabola.

**Q.4) Discuss the conformal mapping for $w = z^2$?**

Ans : $w = z^2$

$\Rightarrow u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$.

So,

$$u = x^2 - y^2 \text{ and, } v = 2xy.$$

Here,

$$z = (x, y) \quad ; \quad w = (u, v)$$

At $(0,0)$, $\quad u = 0$ and $v = 0$.

Hence, the origin gets mapped into the origin in w-plane.

At,

$u = 1$, $x^2 - y^2 = 1$

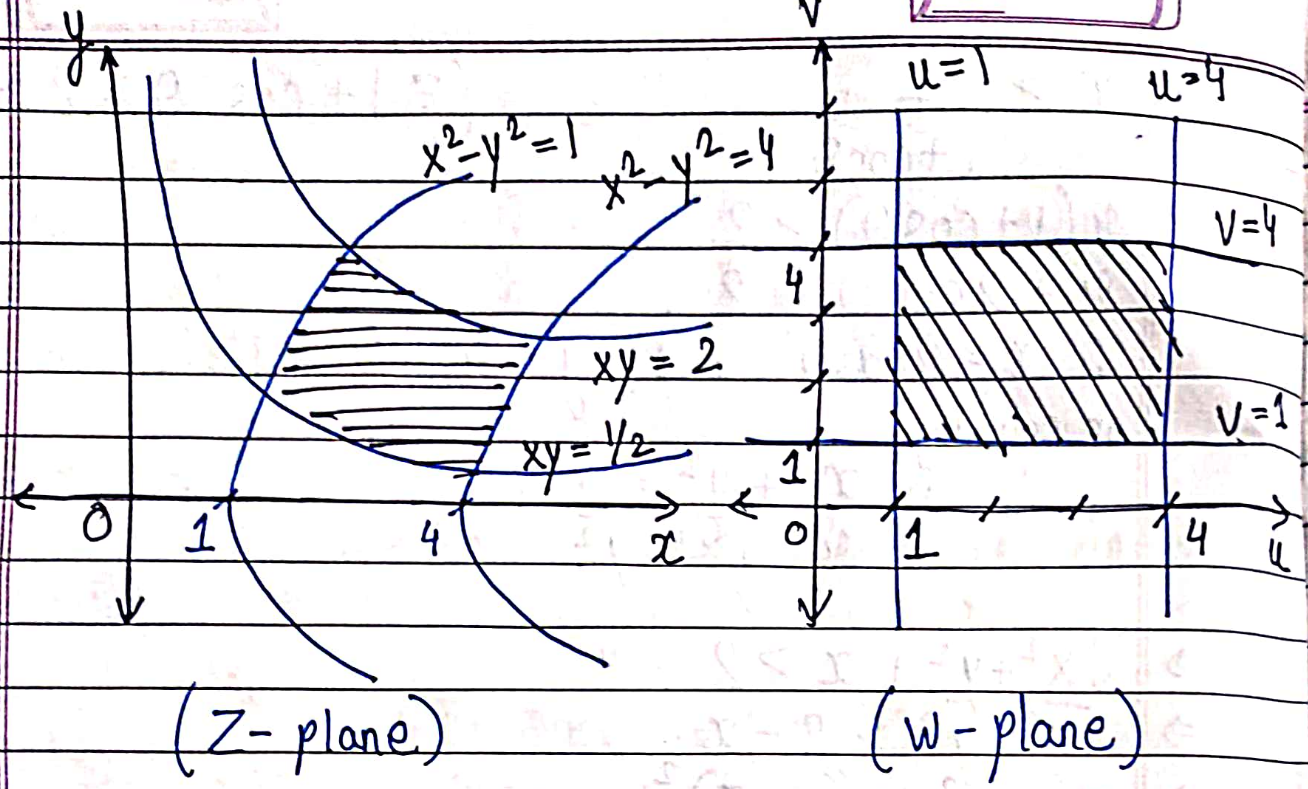
$u = 4$, $x^2 - y^2 = 4$

At,

$v = 1$, $xy = 1/2$

$v = 4$, $xy = 2$.

Geometrically,




**Q.5)** If $u = 2x^2+y^2$ and, $v = y^2/x$. Show that the curve $u$ = constant and $v$ = constant cut orthogonally but the transformation is not conformal ?

**Ans :** If transformation is not conformal $\longrightarrow$ Not Analytic $\rightarrow$ C-R Equations not satisfied.

We have,

$$u = 2x^2+y^2 \quad \text{--- (1)} \quad \text{and,} \quad v = \frac{y^2}{x} \quad \text{--- (2)}$$

$\Rightarrow \; 2x^2+y^2 = C_1$

$\Rightarrow \; \frac{y^2}{x} = C_2 \quad \Rightarrow y^2 = C_2 x$ ] $C_1$ and $C_2$ are constants.

$\because$ Both curves cut orthogonally, so;

$\Rightarrow \; 2y.\frac{dy}{dx} = 0 - 4x = -4x.$

$\Rightarrow \; y.\frac{dy}{dx} = -2x \quad \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$

and,

$\Rightarrow \; 2y.\frac{dy}{dx} = C_2 \quad \Rightarrow y.\frac{dy}{dx} = \frac{C_2}{2} \quad \Rightarrow \frac{dy}{dx} = \frac{C_2}{2y}.$

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$\because$ They cut orthogonally at $90°$,

$\Rightarrow$ product of the two slopes $= -1$.

$\Rightarrow m_1 \times m_2 = -1$.

i.e.,

$$\Rightarrow -\frac{2x}{y} \times \frac{C_2}{2y} = -1.$$

$$\Rightarrow xC_2 = y^2 \text{ — (3)} \Rightarrow \frac{y^2}{x} = C_2 = V \quad (\text{from (2)}).$$

$\therefore$ $\boxed{V \text{ is constant}}$.

also,

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{y} = \frac{C_2}{2y} \quad \Rightarrow x = -\frac{C_2}{4} = K \quad (K: \text{constant})$$

so putting in eq-(1),

$$\Rightarrow 2x^2 + y^2 = C_1 = u.$$

or,

$$\Rightarrow u = 2x^2 + y^2 = 2x^2 + xC_2 \quad (\text{from (3)})$$

i.e.,

$$\Rightarrow u = 2\left(-\frac{C_2}{4}\right)^2 + \left(-\frac{C_2}{4}\right)C_2$$

$$\Rightarrow u = 2K^2 + KC_2$$

$$\Rightarrow u = C_1 = \text{constant}$$

$\therefore$ $\boxed{u \text{ is constant}}$.

Q.6) Discuss the mapping $w = z^{-1}$, and show that the circle in z-plane match into the circle in the w-plane. Find the center and radius in w-plane?

Ans: $\Rightarrow w = \dfrac{1}{z} \quad \Rightarrow z = \dfrac{1}{w} \quad \Rightarrow x + iy = \dfrac{u - iv}{u^2 + v^2}$

$$\Rightarrow x + iy = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2}$$

On comparing,

$$x = \frac{u}{u^2+v^2} \quad \text{and,} \quad y = \frac{-v}{u^2+v^2}.$$

$\because$ General equation of circle is,

$$\Rightarrow x^2+y^2+2gx+2fy+c=0$$

$$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow \frac{\cancel{u^2+v^2}}{(u^2+v^2)^{\cancel{2}}} + 2\left(\frac{gu - fv}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow 1 + 2gu - 2fv + cu^2 + cv^2 = 0.$$

$$\Rightarrow cu^2 + cv^2 + 2gu - 2fv + 1 = 0.$$

or,

$$\Rightarrow \boxed{u^2+v^2+\frac{2g}{c}u - \frac{2f}{c}v + \frac{1}{c} = 0}. \longrightarrow \text{A circle in w-plane.}$$

here,

$$\Rightarrow C \equiv \left(\frac{-g}{c}, \frac{f}{c}\right)$$

and,

$$\Rightarrow R\,(\text{radius}) = \sqrt{\frac{g^2}{c^2}+\frac{f^2}{c^2}-\frac{1}{c^2}} = \frac{1}{c}\sqrt{g^2+f^2-1}$$

**Q.7)** Show that the transformation $w = \cosh z$, maps the lines parallel to X-axis in the Z-plane into the hyperbol in w-plane? Further the lines parallel to Y-axis in the Z-plane maps as ellipse into w-plane?

**Ans:** $2w = e^z + e^{-z} \qquad \left(\because \cosh z = \frac{e^z+e^{-z}}{2}\right)$

$$\Rightarrow 2w = e^{x+iy} + e^{-(x+iy)}$$

$$\Rightarrow 2w = e^{x}(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y).$$

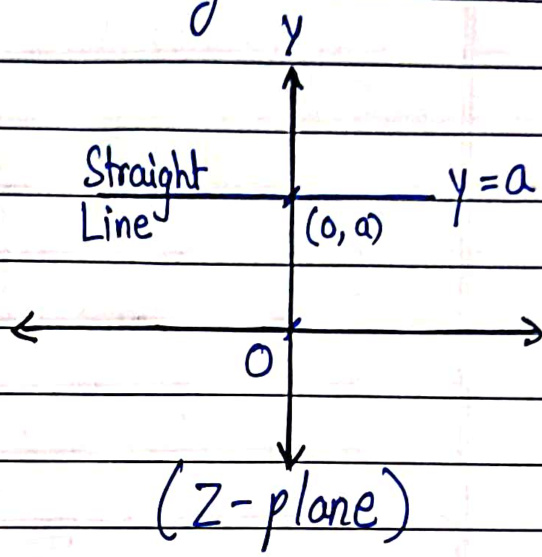
$\Rightarrow \quad 2w = (e^{x}\cos y + e^{-x}\cos y) + i\,(e^{x}\sin y - e^{-x}\sin y).$

$\Rightarrow \quad u + iv = \left(\dfrac{e^{x}+e^{-x}}{2}\right)\cos y + i\left(\dfrac{e^{x}-e^{-x}}{2}\right)\sin y.$

So,

$u = \cos y . \cosh x$ and, $v = \sin y . \sinh x.$

a) Lines parallel to X-axis →




$\Rightarrow \quad y = \text{Constant} = a$

So,

$u = \cos a . \cosh x.$

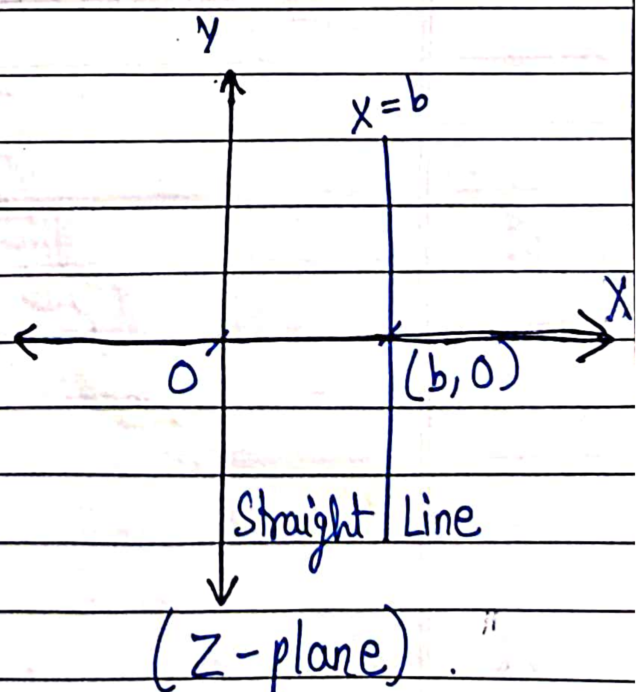
$v = \sin a . \sinh x$

$\because \quad \cosh^2 x - \sinh^2 x = 1.$

$\Rightarrow \quad \left(\dfrac{u}{\cos a}\right)^2 - \left(\dfrac{v}{\sin a}\right)^2 = 1.$

$\Rightarrow \quad \boxed{\dfrac{u^2}{\cos^2 a} - \dfrac{v^2}{\sin^2 a} = 1}.$ → Hyperbola in w-plane.

b) Lines parallel to Y-axis →




$\Rightarrow \quad x = \text{Constant} = b$

So,

$u = \cosh a . \cos y.$

$v = \sinh a . \sin y.$

$\because \quad \sin^2 y + \cos^2 y = 1$

$\Rightarrow \quad \left(\dfrac{v}{\sinh a}\right)^2 + \left(\dfrac{u}{\cosh a}\right)^2 = 1.$

$\Rightarrow \quad \boxed{\dfrac{u^2}{\cosh^2 a} + \dfrac{v^2}{\sinh^2 a} = 1}$ → Ellipse in w-plane.

## 3.2) Linear And Inverse Transformation :-

Here,

$W = az + b \rightarrow$ Linear transformation or Linear function where $a, b$ are real and complex constants.

$$W = f(z) = az + b.$$

Linear Transformation

① Translation $\rightarrow$ Shape, size and orientation same

② Magnification $\rightarrow$ Image can be magnified, same or diminished

③ Rotation $\rightarrow$ The angle of rotation will be determined...

$$\Rightarrow W = az + b = a\left(z + \frac{b}{a}\right) = aw_1$$

$$\Rightarrow W = |a| e^{i\theta}\left(z + \frac{b}{a}\right).$$

① Translation : $w_1 = z + \frac{b}{a} = z + c \qquad \left(c = \frac{b}{a}\right)$

② Magnification : $W_2 = aw_1 = a\left(z + \frac{b}{a}\right)$

③ Rotational : $W = e^{i\theta} W_2 = e^{i\theta}. aw_1 = ae^{i\theta}\left(z + \frac{b}{a}\right)$

**Q.1)** Let a rectangular region OABC with vertices $O(0,0)$, $A(1,0)$, $B(1,2)$ and $C(0,2)$ is defined in Z-plane. Find the image of this figure in w-plane under the mapping,

$$W = z + (2+i).$$

**Ans :** $W = z + (2+i)$ (here, $(2+i)$ is a complex constant)

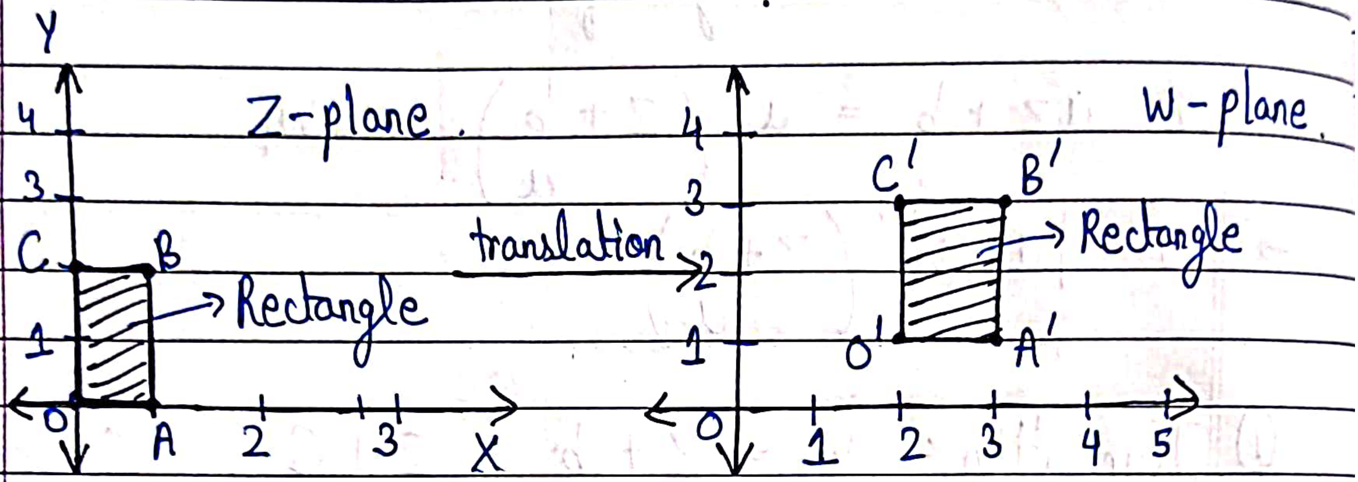
$\Rightarrow u + iv = (x+2) + i(y+1).$

or,

$$u = x + 2 \quad \text{and,} \quad v = y + 1.$$

$u = x + 2$ , $v = y + 1$ .

| $(x,y)$ | | $(u,v)$ |
|---|---|---|
| $O\ (0,0)$ | $\longrightarrow$ | $O'\ (2,1)$ |
| $A\ (1,0)$ | $\longrightarrow$ | $A'\ (3,1)$ |
| $B\ (1,2)$ | $\longrightarrow$ | $B'\ (3,3)$ |
| $C\ (0,2)$ | $\longrightarrow$ | $C'\ (2,3)$ . |
| (Z-plane) | | (W-plane) |




**Q.2)** Let a rectangular region ABCD with vertices $A(2,1)$, $B(3,1)$, $C(3,3)$, $D(2,3)$ be defined in Z-plane. Find the image in W-plane if $w = 2z$ ?

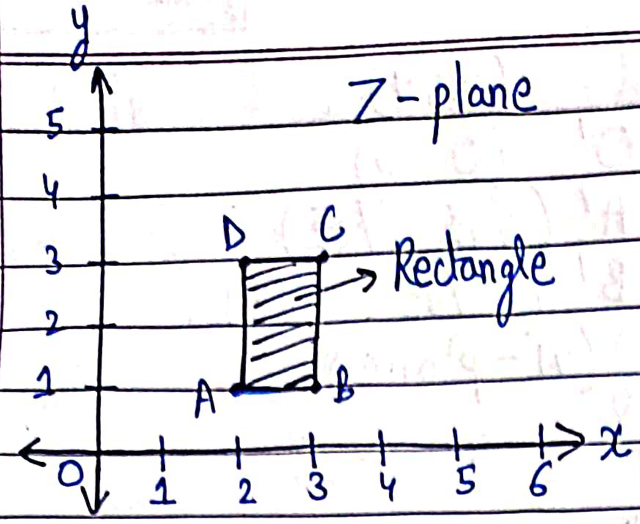
**Ans:** $u + iv = 2(x + iy) = 2x + i(2y)$.

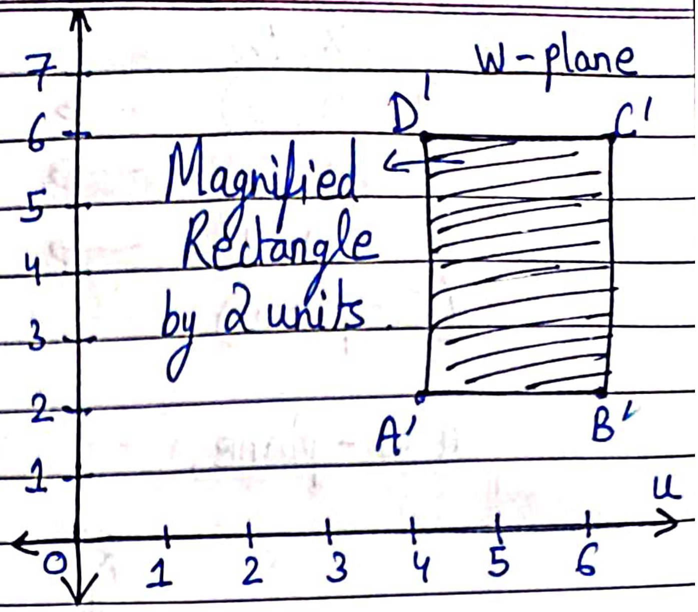
or,

$u = 2x$ and, $v = 2y$

| $(x,y)$ | | $(u,v)$ |
|---|---|---|
| $A\ (2,1)$ | $\longrightarrow$ | $A'\ (4,2)$ |
| $B\ (3,1)$ | $\longrightarrow$ | $B'\ (6,2)$ |
| $C\ (3,3)$ | $\longrightarrow$ | $C'\ (6,6)$ |
| $D\ (2,3)$ | $\longrightarrow$ | $D'\ (4,6)$ |
| (Z-plane) | | (W-plane) |

- The image will be magnified by relation $w = 2z$.




**Q.3)** Consider the transformation, $W = Ze^{\frac{i\pi}{4}}$. Determine the region R' in w-plane corresponding to triangular region R bounded by the lines $x=0$, $y=0$ and $x+y=1$.

**Ans:** $W = e^{\frac{i\pi}{4}} \cdot Z = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \cdot Z$

$\Rightarrow u + iv = \frac{(1+i)}{\sqrt{2}}(x+iy)$

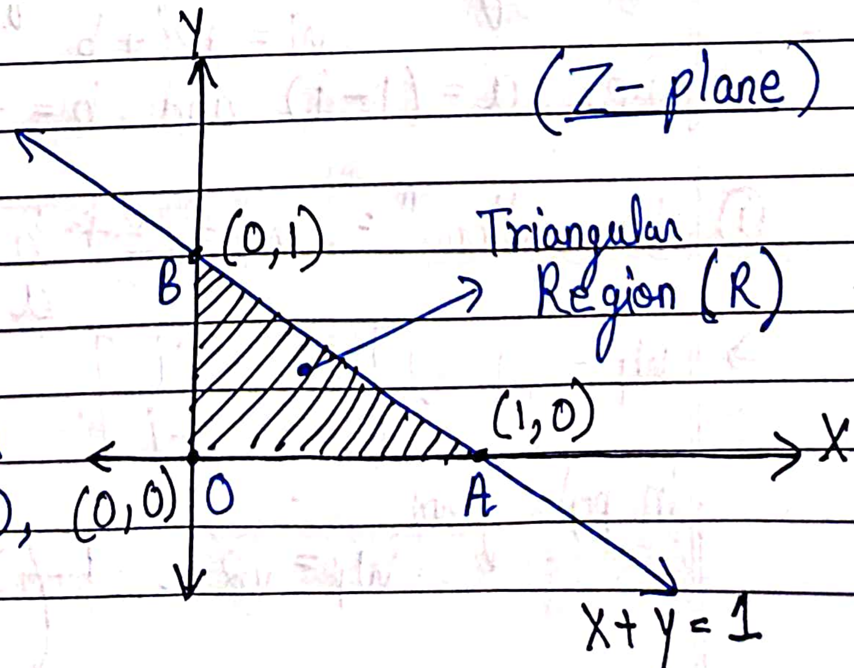
$\Rightarrow u + iv = \frac{(x-y)}{\sqrt{2}} + i\frac{(x+y)}{\sqrt{2}}$

So,

$$u = \frac{x-y}{\sqrt{2}} \quad \text{and} \quad v = \frac{x+y}{\sqrt{2}}$$

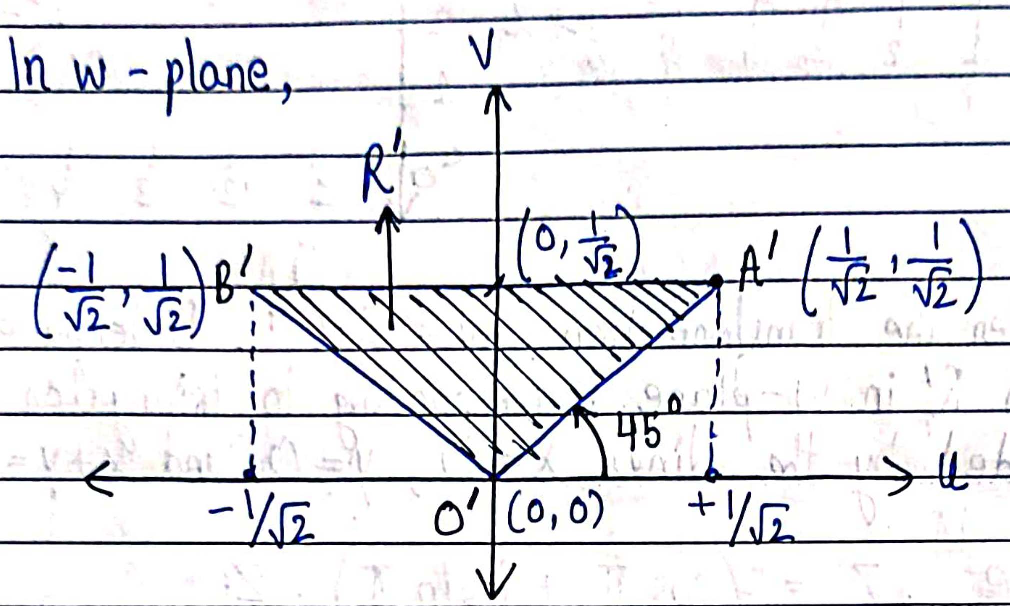
→ Region 'R' is bounded by:

1. $x=0 \rightarrow$ Y-axis
2. $y=0 \rightarrow$ X-axis
3. $x+y=1$ that passes through (0,1) & (1,0), a straight line.



| | $(x, y)$ | | | $(u, v)$ |
|---|---|---|---|---|
| $O$ | $(0, 0)$ | $\longrightarrow$ | $O'$ | $(0, 0)$ |
| $A$ | $(1, 0)$ | $\longrightarrow$ | $A'$ | $(1/\sqrt{2}, 1/\sqrt{2})$ |
| $B$ | $(0, 1)$ | $\longrightarrow$ | $B'$ | $(-1/\sqrt{2}, 1/\sqrt{2})$. |
| | (Z-plane) | | | (W-plane). |

In w-plane,




**Q.4)** Let a rectangular region with vertices OABC where, $O(0, 0)$, $A(1, 0)$, $B(1, 2)$ and $C(0, 2)$ in Z-plane. Find the image of this region in w-plane under the mapping,

$$W = (1-i)Z - 2i$$

**Ans:** Converting 'w' in the form of,

$$W = aZ + b$$

i.e., $a = (1-i)$ and $b = -2i$.

① Translational: $W_1 = Z + \frac{b}{a}$

$\Rightarrow$ $W_1 = (1-i)\left[Z - \frac{2i}{1-i}\right]$

In polar form,

$$W_1 = \sqrt{2}\, e^{+i\frac{\pi}{4}} \left(Z - \frac{2i}{1-i}\right) \ldots$$

This '$w$' $= \sqrt{2}\, e^{i\pi/4} \left( z - \frac{2i}{1-i} \right)$ has linear transformation.

i.e., $w_1 = z - \frac{2i}{1-i}$ (for translation).

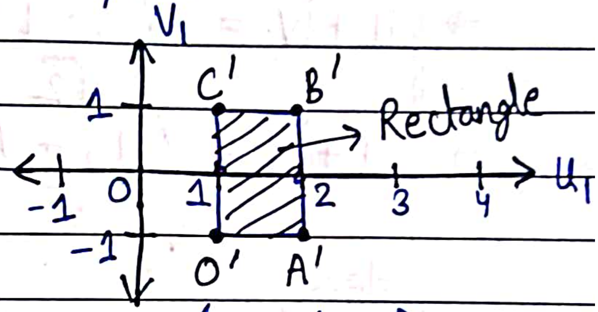
$\Rightarrow u_1 + iv_1 = (x+iy) - \frac{2i(1+i)}{2}$

$\Rightarrow u_1 + iv_1 = x + iy - i + 1. = (x+1) + i(y-1).$

or,

$$u_1 = (x+1) \quad \text{and,} \quad v_1 = (y-1)$$

| $(x,y)$ | | $(u,v)$ |
|---|---|---|
| O $(0,0)$ | $\longrightarrow$ | $O'$ $(1,-1)$ |
| A $(1,0)$ | $\longrightarrow$ | $A'$ $(2,-1)$ |
| B $(1,2)$ | $\longrightarrow$ | $B'$ $(2,1)$ |
| C $(0,2)$ | $\longrightarrow$ | $C'$ $(1,1)$ |
| (Z-plane) | | (w-plane) |




(w-plane).

② Magnification: $w_2 = a w_1$

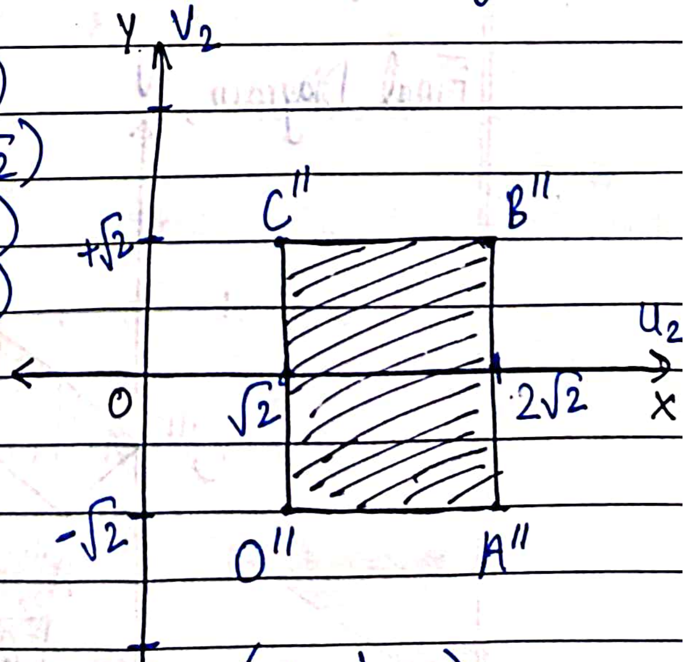
$\Rightarrow w_2 = \sqrt{2}(u_1 + iv_1)$

$\Rightarrow u_2 + iv_2 = \sqrt{2}u_1 + i\sqrt{2}v_1$

or,

$$u_2 = \sqrt{2}u_1 = \sqrt{2}(x+1) \quad \text{and,} \quad v_2 = \sqrt{2}v_1 = \sqrt{2}(y-1)$$

| $(x,y)$ | | $(u,v)$ |
|---|---|---|
| O $(0,0)$ | $\longrightarrow$ | $O''$ $(\sqrt{2}, -\sqrt{2})$ |
| A $(1,0)$ | $\longrightarrow$ | $A''$ $(2\sqrt{2}, -\sqrt{2})$ |
| B $(1,2)$ | $\longrightarrow$ | $B''$ $(2\sqrt{2}, \sqrt{2})$ |
| C $(0,2)$ | $\longrightarrow$ | $C''$ $(\sqrt{2}, \sqrt{2})$ |




(w-plane)

→ The image is a magnified rectangle by '$\sqrt{2}$' factor.

i.e.,

$$w_2 = \sqrt{2}.w_1.$$

③ **Rotational:**

$$\Rightarrow W = \sqrt{2}\, e^{-i\frac{\pi}{4}} \left[ Z + (1-i) \right]$$

or,

$$\Rightarrow W = e^{-i\frac{\pi}{4}} W_2 .$$

i.e.,

$$\Rightarrow u + iv = e^{-i\frac{\pi}{4}} (u_2 + iv_2) .$$

$$\Rightarrow u + iv = \left( \cos\frac{\pi}{4} - i \sin\frac{\pi}{4} \right) (u_2 + iv_2)$$

$$\Rightarrow u + iv = \frac{(1-i)}{\sqrt{2}} (u_2 + iv_2)$$

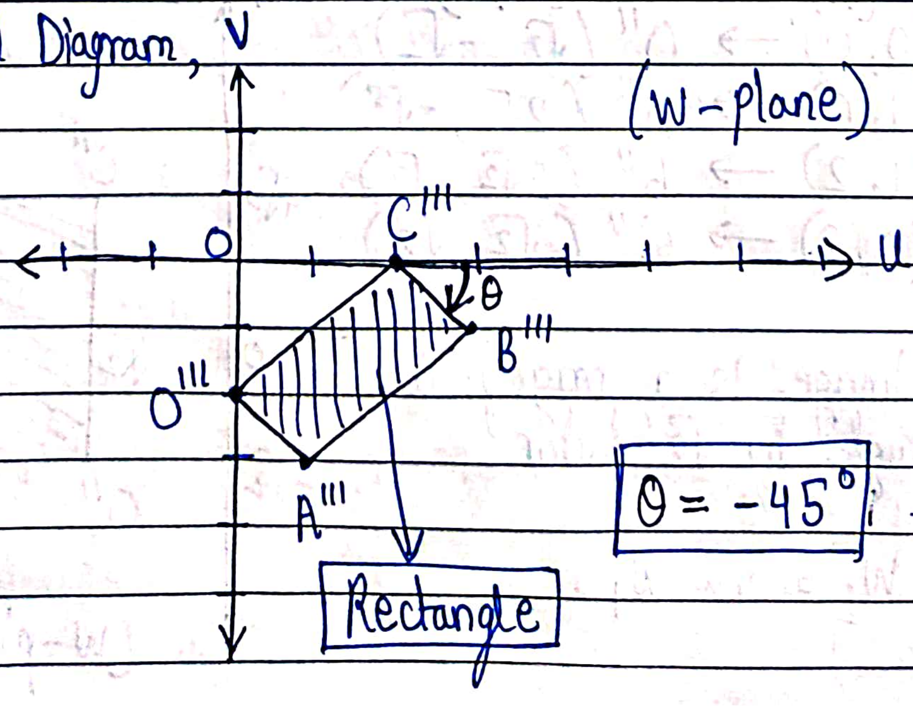
$$\Rightarrow u + iv = \left( \frac{u_2 + v_2}{\sqrt{2}} \right) + i \left( \frac{v_2 - u_2}{\sqrt{2}} \right)$$

Here,

$$u = \frac{u_2 + v_2}{\sqrt{2}} \quad \text{and,} \quad v = \frac{v_2 - u_2}{\sqrt{2}}$$

| w-plane $(u_2, v_2)$ | | w-plane $(u, v)$. |
|---|---|---|
| $O''(\sqrt{2}, -\sqrt{2})$ | $\longrightarrow$ | $O'''(0, -2)$ |
| $A''(2\sqrt{2}, -\sqrt{2})$ | $\longrightarrow$ | $A'''(1, -3)$ |
| $B''(2\sqrt{2}, \sqrt{2})$ | $\longrightarrow$ | $B'''(3, -1)$ |
| $C''(\sqrt{2}, \sqrt{2})$ | $\longrightarrow$ | $C'''(2, 0)$. |

Final Diagram,




$$\boxed{\theta = -45^\circ}.$$

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## 3.3) Inversion and Reflection :-

$$\boxed{w = \frac{1}{z}}$$ (general form)

In polar form,

$$z = re^{i\theta}$$
$$w = Re^{i\phi}$$

$$\Rightarrow Re^{i\phi} = \frac{1}{re^{i\theta}}$$

If $\theta = -\phi$,

$$\Rightarrow R\,\exp(i\phi) = \frac{1}{r}\exp(i\phi)$$

$$\therefore \boxed{R = \frac{1}{r}}.$$

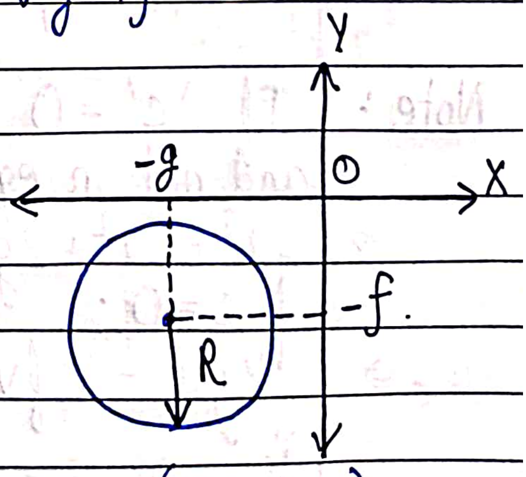
Q.1) Find the image of circle in Z-plane on the w-plane by the transformation $w = 1/z$ ?

Ans : In Z-plane,

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

whose,

$$C \equiv (-g, -f) \quad \text{and} \quad R = \sqrt{g^2 + f^2 - c}$$



(Z-plane).

$$\Rightarrow z = \frac{1}{w} \quad \text{(given)}$$

$$\Rightarrow x + iy = \frac{u - iv}{u^2 + v^2}, \quad (w = u + iv)$$

$$\Rightarrow x + iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right).$$

So,

$$x = \frac{u}{u^2+v^2} \quad \text{and,} \quad y = \frac{-v}{u^2+v^2}.$$

$\Rightarrow x^2+y^2+2gx+2fy+c=0$

$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2+2g\left(\frac{+u}{u^2+v^2}\right)+2f\left(\frac{-v}{u^2+v^2}\right)+c$

$\Rightarrow \frac{1+2gu-2fv}{u^2+v^2}+c=0.$

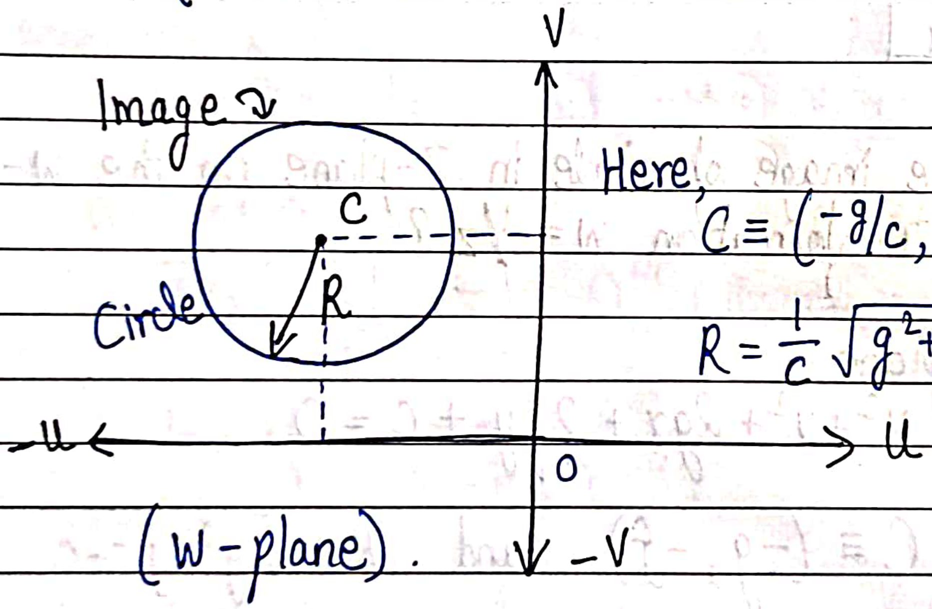
$\Rightarrow \boxed{cu^2+cv^2+2gu-2fv+1=0}$ —— ①

or,

$\Rightarrow \boxed{u^2+v^2+\frac{2g}{c}u-\frac{2f}{c}v+\frac{1}{c}=0}$ —— ②.

It is a equation of circle,

$$C \equiv \left(-\frac{g}{c}, \frac{f}{c}\right) \quad ; \quad R=\frac{1}{c}\sqrt{g^2+f^2-c}$$




Here,
$C \equiv (-g/c, f/c)$
$R=\frac{1}{c}\sqrt{g^2+f^2-c}.$

**Note:** If 'c' = 0 in above question then put c = 0 in eq-① and not in eq-②. So,

$\Rightarrow cu^2+cv^2+2gu+1-2fv=0$

If $c=0$;

$\Rightarrow 2gu-2fv+1=0.$

or,

$\Rightarrow \boxed{u=\frac{f}{g}v-\frac{1}{2g}} \rightarrow$ (A straight line of form: $y=mx+c$).

$m=f/g.$

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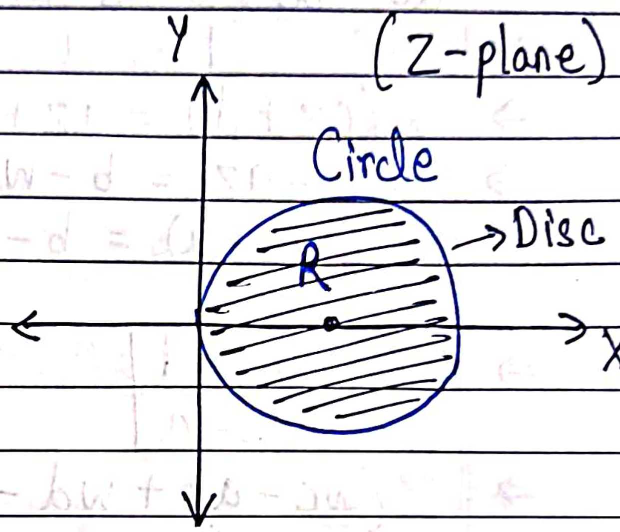
**Q.2)** Find the image in the w-plane of the disc in z-plane,

$$||z-|z-1|| \leq 1 \rightarrow |z-1| \leq 1.$$

under the mapping,

$$w = \frac{1}{z}.$$

**Ans:** Given, $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$




$\Rightarrow ||z-|z-1|| \leq 1.$

$\Rightarrow \left|\left|\frac{1}{w}\right| - \left|\frac{1}{w}-1\right|\right| \leq 1.$

$\Rightarrow \left|\frac{1}{w}-1\right| - \left|\frac{1}{w}\right| \geq |1|.$

$\Rightarrow \frac{|1-w|}{|w|} \geq \left|1+\frac{1}{w}\right|$

$\Rightarrow \frac{|1-w|}{|w|} \geq \left|\frac{w+1}{w}\right|.$

$\Rightarrow \left|\frac{1}{w}-1\right| \leq 1.$ (for $|z-1| \leq 1$).

$\Rightarrow |1-w| \leq |w|.$

$\Rightarrow |1-(u+iv)| \leq |u+iv|.$

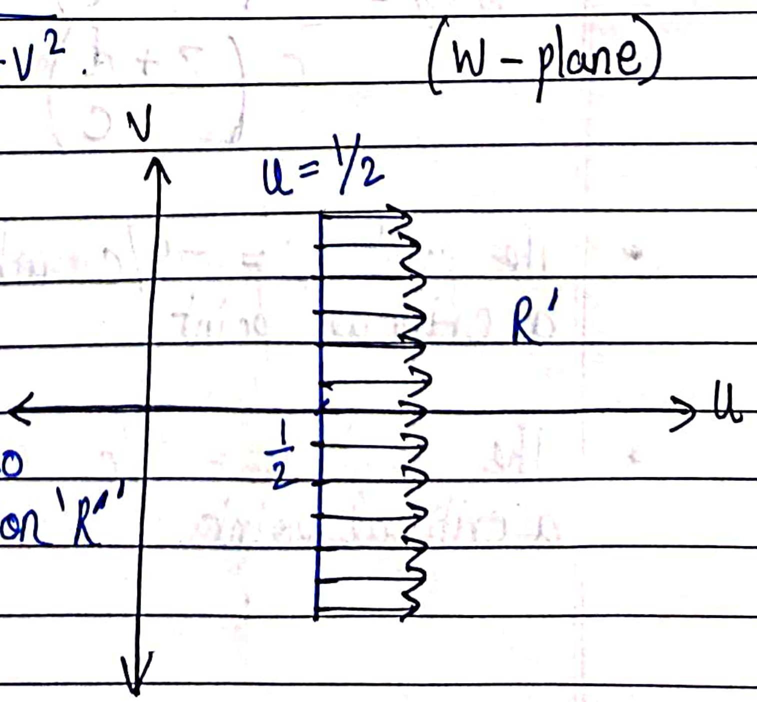
or,

$\Rightarrow \sqrt{(1-u)^2+v^2} \leq \sqrt{u^2+v^2}.$

$\Rightarrow (1-u)^2 + \cancel{v^2} \leq u^2 + \cancel{v^2}$

$\Rightarrow 1 \leq 2u.$

$\Rightarrow \boxed{u \geq \frac{1}{2}}$




→ The line $u = 1/2$ will also be counted in for the region 'R'' as, $u \geq \frac{1}{2}$.

## 3.4) Bilinear Transformation :-

⇒ $\boxed{w = \frac{az+b}{cz+d}}$, where $ad-bc \neq 0$.

- also called as functional transformation or Möbius transformation.
- $ad-bc \neq 0$ is called determinant of this transformation

⇒ $w(cz+d) = az+b$.
⇒ $wcz - az = b - wd$.
⇒ $z(wc-a) = b-wd$.

or

⇒ $\boxed{z = \frac{b-wd}{wc-a}}$

⇒ $z(wc-a) + wd - b = 0$. —— ①
⇒ $w(cz+d) - az - b = 0$ —— ②

here,

equations ① and ② both are linear in $z$ and $w$, hence both are bilinear.

also,

$$w = \frac{a\left(z+\frac{b}{a}\right)}{c\left(z+\frac{d}{c}\right)} \quad \text{and,} \quad z = \frac{d\left(\frac{b}{d}-w\right)}{c\left(\frac{w}{1}-\frac{a}{c}\right)}$$

- The point $z = -d/c$ where $c \neq 0 \rightarrow w = \infty$, it is a critical point.

- The point $w = a/c$ where $c \neq 0 \longrightarrow z = \infty$, it is a critical point.

**Q.1)** Find the fixed point of the transformation where,

$$w = \frac{2z-5}{z+4}.$$

$\Rightarrow w(z+4) = 2z-5$

$\Rightarrow wz - 2z = -5 - 4w$

$\Rightarrow z(w-2) = -5(-4w)$

or,

$\Rightarrow z = \frac{5+4w}{2-w}$

**Ans:** $z = f(w)$

$\Rightarrow z = \frac{5+4w}{2-w}$ (replace w by z)

or, $\Rightarrow z = \frac{2z-5}{z+4}$

$\Rightarrow z^2+4 = 2z-5$

$\Rightarrow z^2 - 2z + 9 = 0 \Rightarrow \boxed{z = -1 \pm 2i}$

→ The points which coincide with their transformation are called fixed points.

or,

$\Rightarrow z = \frac{az+b}{cz+d}$

$\Rightarrow cz^2 - (a-d)z - b = 0.$

$\Rightarrow z = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}$ → 2 fixed points.

- For one fixed point, put $D = 0$;

$$(a-d)^2 + 4bc = 0.$$

**3.5) Cross Ratio :-**

→ Let $z_1, z_2, z_3$ and $z_4$ are distinct points then the cross ratio will be,

$$= (z_1 - z_2)(z_3 - z_4) / (z_2 - z_3)(z_4 - z_1)$$

$z_1$ ② $z_2$ ③ $z_3$ ④ $z_4$

① $(z_1 - z_2)$ $(z_2 - z_3)$ $(z_3 - z_4)$ $(z_4 - z_1)$

numerator denominator

In terms of w,

$$= \frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)}$$

**Q.1)** Find the bilinear transformation which maps the point:

| | Z | W | |
|---|---|---|---|
| $Z_1$ | 0 | $-i$ | $W_1$ |
| $Z_2$ | $-1$ | 0 | $W_2$ |
| $Z_3$ | $i$ | $\infty$ | $W_3$ |

Ans :

$$\boxed{W = \frac{az+b}{cz+d}}$$

$\Rightarrow$ $Z \quad Z_1 \quad Z_2 \quad Z_3 \quad = \quad W \quad W_1 \quad W_2 \quad W_3$

$$\Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$\Rightarrow \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} = \frac{(w-w_1)\cancel{w_3}(w_2/w_3-1)}{(w_1-w_2)\cancel{w_3}(1-w/w_3)}$$

$$\Rightarrow \frac{(z-0)(-1-i)}{(0+1)(i-z)} = \frac{(w+i)(0-1)}{(-i-0)(1-0)}$$

$$\Rightarrow \frac{z(-1-i)}{i-z} = \frac{(w+i)}{i}$$

**Q.2)** Find the bilinear transformation which maps the point:

(a)

| | Z | W | |
|---|---|---|---|
| $Z_1$ | $1$ | $i$ | $W_1$ |
| $Z_2$ | $-i$ | $0$ | $W_2$ |
| $Z_3$ | $-1$ | $-i$ | $W_3$ |

**Ans:** For,

$$Z \quad Z_1 \quad Z_2 \quad Z_3 = W \quad W_1 \quad W_2 \quad W_3$$

$$\Rightarrow \frac{(Z-Z_1)(Z_2-Z_3)}{(Z_1-Z_2)(Z_3-Z)} = \frac{(W-W_1)(W_2-W_3)}{(W_1-W_2)(W_3-W)}$$

$$\Rightarrow \frac{(Z-1)(-i+1)}{(1+i)(-1-Z)} = \frac{(W-i)(0+i)}{(i-0)(-i-W)}$$

$$\Rightarrow \frac{(Z-1)(1-i)}{(1+i)(-1-Z)} = \frac{(W-i)\cancel{i}}{\cancel{i}\times(-W-i)}$$

$$\Rightarrow \frac{(Z-1)}{-(Z+1)} \times \frac{(1-i)^2}{2} = \frac{W-i}{-(W+i)} \times \frac{(W-i)}{(W-i)}$$

or,

$$\Rightarrow \frac{i-W}{i+W} = \frac{(Z-1)(i-1)}{(Z+1)(i+1)} = \frac{(i-1)Z+(1-i)}{(i+1)Z+1+i}.$$

$$\Rightarrow \frac{i-W}{i+W} = \frac{(i-1)Z+(1-i)}{(i+1)Z+(1+i)}$$

$$\Rightarrow \frac{(i-W)-(i+W)}{(i-W)+(i+W)} = \frac{[(i-1)Z+(1-i)]-[(i+1)Z+(1+i)]}{[(i-1)Z+(1-i)]+[(i+1)Z+(1+i)]}$$

$$\Rightarrow \boxed{W = \frac{iZ-1}{iZ+1}}.$$

(b) Also show that the transformation maps the region outside the region circle $|z| = 1$, into the half plane, $Re(w) \geq 0$.

Ans: $\because w = \frac{i(z-1)}{iz+1}$ —— (1)

$\Rightarrow u + iv = \frac{i(x+iy-1)}{i(x+iy)+1}$ → now solve and seperate $u$ & $v$.

or,

$\Rightarrow |w| = \left|\frac{i(z-1)}{iz+1}\right|$ (taking modulus on both sides of eq-(1))

or,

$\Rightarrow |z| = \left| i\left(\frac{w+1}{w-1}\right)\right| = 1$. ($\because |z| = 1$)

$\Rightarrow \left|\frac{w+1}{w-1}\right| = 1$.

$\Rightarrow |w+1| = |w-1|$

$\Rightarrow \sqrt{(u+1)^2+v^2} = \sqrt{(u-1)^2+v^2}$

$\Rightarrow u^2+1+2u = u^2+1-2u$

$\Rightarrow 4u \geq 0$. ($\because |z| \geq 1$).

$\Rightarrow \boxed{u \geq 0}$.