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Bessel Equation:→

$$x^2y'' + xy' + (x^2-n^2)y = 0 \quad —(1)$$

is called Bessel Equation. where n is a real Constant.

Gamma function:→ Generalization of the factorial function to non-Integral values.

Gamma function is defined as

$$\Gamma x = \int_0^\infty t^{x-1} e^{-t} dt.$$

$$\Gamma 1 = \int_0^\infty (t)^0 \cdot e^{-t} dt = \left(-e^{-t}\right)_0^\infty = 1$$

$$\Rightarrow \Gamma 1 = 1.$$

(1) $\Gamma(x+1) = x\Gamma x$

$\because \Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$

$$= \left[t^x\left(-e^{-t}\right)\right]_0^\infty - \int_0^\infty x t^{x-1}\left(\frac{e^{-t}}{-1}\right) dt$$

$$= 0 + \int_0^\infty x t^{x-1} e^{-t} dt$$

$$= x\Gamma x$$

$\Rightarrow \Gamma 2 = 1\Gamma 1 = 1$

$\Gamma 3 = \Gamma(2+1) = 2\Gamma 2 = 2\times 1 = \lfloor 2$

$\Gamma 4 = \Gamma(3+1) = 3\Gamma 3 = 3\times 2! = 3!$ and so on.

→ If x is a natural no. then $\Gamma x = \lfloor x-1$.

E.g. $\Gamma 4 = \lfloor 4-1 = 3! = 6$.

→ $\Gamma \frac{1}{2} = \sqrt{\pi}$

$\Rightarrow \Gamma 3/2 = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\frac{1}{2} = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}.$

→ Find $\Gamma\left(\frac{-1}{2}\right)$, $\Gamma\left(\frac{-3}{2}\right)$, $\Gamma\left(\frac{-5}{2}\right)$

We know $\Gamma(x) = \frac{\Gamma(x+1)}{x}$

$$\Rightarrow \Gamma\left(\frac{-1}{2}\right) = \frac{\Gamma\left(\frac{-1}{2}+1\right)}{-1/2} = -2\,\Gamma\left(\frac{1}{2}\right)$$
$$= -2\sqrt{\pi}$$

Illy $\Gamma\left(\frac{-3}{2}\right) = \frac{\Gamma\left(\frac{-3}{2}+1\right)}{-3/2} = \frac{\Gamma\left(\frac{-1}{2}\right)}{-3/2} = -2\sqrt{\pi} \times \left(\frac{-2}{3}\right)$
$$= \frac{4}{3}\sqrt{\pi}$$

$$\Gamma\left(\frac{-5}{2}\right) = \frac{\Gamma\left(\frac{-5}{2}+1\right)}{-5/2} = \frac{\Gamma\left(\frac{-3}{2}\right)}{-5/2} = \frac{-2}{5}\left(\frac{4}{3}\sqrt{\pi}\right)$$
$$= \frac{-8}{15}\sqrt{\pi}$$

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Bessel's Differential Equation and Bessel function:→

The Equation of the form

$$x^2y'' + xy' + (x^2-m^2)y = 0 \quad \text{— (1)}$$

is called Bessel's equation, ~~of~~ of order m, where n is a non-negative real number.

$$y'' + \frac{y'}{x} + \frac{x^2-m^2}{x^2}y = 0.$$

⇒ $x=0$ is a regular singular point as $P(x) = \frac{1}{x}$ and $Q(x) = \frac{x^2-m^2}{x^2}$ not analytic at $x=0$.

But $xP(x)$ and $x^2Q(x)$ are analytic at $x=0$.

The Sol^n of (1) about $x=0$ will be of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}.$$

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r)x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n.(n+r)(n+r-1)\, x^{n+r-2}$$

Substitute y, y', y'' in (1), we get

$$\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - m^2\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \left[(n+r)(n+r-1) + (n+r) - m^2\right] x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_n \left[(n+r)^2 - m^2\right] x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

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Equating Coeff of x^r and x^{r+1} to zero;

x^r; $r^2 - m^2 = 0 \Rightarrow r = \pm m. \rightarrow$ (Indicial Equation)

x^{r+1}; $c_1\left[(r+1)^2 - m^2\right] = 0$

$\Rightarrow c_1(1+2r) = 0 \quad (\because r = \pm m)$

$\Rightarrow c_1 = 0 \quad (\because r$ is positive or non-negative$)$

$\therefore$ Remaining terms of Summation are

$$\sum_{n=2}^{\infty} c_n\left((n+r)^2 - m^2\right) x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} c_{n+2}\left((n+r+2)^2 - m^2\right) x^{n+r+2} + \sum_{n=0}^{\infty} c_n x^{n+r+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left(\left[(n+r+2)^2 - m^2\right] c_{n+2} + c_n\right) x^{n+r+2} = 0$$

Comp. Coeff, we get

$$\left((n+r+2)^2 - m^2\right) c_{n+2} = -c_n$$

$$\Rightarrow c_{n+2} = \frac{-c_n}{(n+r+2)^2 - m^2}$$

For $r=m$ $\quad n=0$; $c_2 = \dfrac{-c_0}{(0+m+2)^2 - m^2} = \dfrac{-c_0}{(m+2)^2 - m^2}$

$$= \frac{-c_0}{(\cancel{m}+2-\cancel{m})(m+2+m)} = \frac{-c_0}{2^2(1+m)}$$

$c_3 = c_5 = c_7 = \text{\_\_\_} = 0 \quad (\because c_1 = 0)$

$n=2$; $c_4 = \dfrac{-c_2}{(2+m+2)^2 - m^2} = \dfrac{-c_2}{2^3(2+m)} = \dfrac{+c_0}{2^4\, \lfloor 2\, (1+m)(2+m)}$

Similarly $c_{2n} = \dfrac{(-1)^n \cdot c_0}{2^{2n} \lfloor n\, (1+m)(2+m) \text{\_\_} (n+m)}$; $n = 1, 2, 3 \ldots$

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$\therefore$ The Solⁿ of Bessel's Equation for $r = m$ is

$$y_1(x) = C_0 x^m \left(1 - \frac{x^2}{2^2(1+m)} + \frac{x^4}{2^4 \lfloor 2 (1+m)(2+m)} + \ldots\right)$$

where $C_0 = \dfrac{1}{2^m \Gamma(m+1)}$

$y_1(x)$ is called <u>Bessel's function of first</u> kind and is denoted by $J_m(x)$.

$$\therefore J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2n+m}.$$

Put $r = -m$ and solve in similar manner, we obtain

$$J_{-m}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-m+1)} \left(\frac{x}{2}\right)^{2n-m}.$$

So the general Solⁿ of (1) is

$$y(x) = A J_m(x) + B J_{-m}(x).$$

$J_m(x)$ and $J_{-m}(x)$ are called Bessel's functions.

<u>Properties:→</u>

(1) $[x^m J_m(x)]' = x^m J_{m-1}(x)$

<u>Pf:→</u> <u>LHS</u>

$$x^m J_m(x) = x^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2n+m}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\,(x)^{2n+2m}}{n!\,\Gamma(n+m+1)\cdot 2^{2n+m}}$$

$$[x^m J_m(x)]' = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2n+2m)\, x^{2n+2m-1}}{n!\,\Gamma(n+m+1)\; 2^{2n+m}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, 2(n+m)}{n!\,(n+m)\,\Gamma(n+m)}\, \frac{x^{2n+2m-1}}{2^{2n+m}}$$

$$= x^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+m)} \left(\frac{x}{2}\right)^{2n+m-1}$$

$$= x^m J_{m-1}(x)$$