Eigen Values and Eigen vectors

Defⁿ ⇒ Let A be an $n \times n$ matrix, A real number λ is called an eigenvalue of A if there exists a non-zero n-vector X such that $AX = \lambda X$. In this case, the vector X is called an eigen vector of A corresponding to λ.

Note: Eigenvalues and eigenvectors are sometimes called characteristic values and characteristic vectors or latent values and latent vectors.

Ex. Show that $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigen value 3.

Solⁿ. we have

$$AX = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5\times 1 + 2\times -1 \\ 2\times 1 + 5\times -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= 3X$$

i.e, $AX = 3X$

So $\lambda = 3$ is an eigen value of A and $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector corresponding to this eigen value.

Eigenspace

Defⁿ ⇒ Let A be an $n \times n$ matrix, and let λ be an eigen value of A. Then the eigen space of A for λ denoted by E_λ, is the set of all eigenvectors of A for λ, together with the inclusion of the zero vector. Thus $E_\lambda = \{ X : AX = \lambda X \}$

Th. Let A be a matrix and let λ be a real no. Then λ is an eigenvalue of A iff $|A - \lambda I| = 0$. The eigenvectors corresponding to λ are the non-trivial solns of the homogeneous system $(A - \lambda I)X = 0$. The eigen space E_λ is the complete soln set for this homogeneous system.

Characteristic Polynomial of A →

Defⁿ ⇒ Let A be an $n \times n$ matrix. The characteristic polynomial of A is defined to be the polynomial $P_A(x) = |A - xI|$

*. The eigen values of A are just real roots of the characteristic polynomial.

Ex. Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix}$. Also find all the Eigen values for A

Soln: $|A - \lambda I| = 0$

$$P_A(x) = \begin{vmatrix} 12-\lambda & -51 \\ 2 & -11-\lambda \end{vmatrix} = 0$$

$$= (12-\lambda)(-11-\lambda) + 102$$
$$= (\lambda-12)(\lambda+11) + 102$$
$$= \lambda^2 - 12\lambda + 11\lambda - 132 + 102$$
$$= \lambda^2 - \lambda - 30$$
$$= \lambda^2 - 6\lambda + 5\lambda - 30$$
$$= \lambda(\lambda-6) + 5(\lambda-6)$$
$$= (\lambda+5)(\lambda-6)$$

ie, $\lambda = -5, 6$

$\therefore$ Eigen values are -5 and 6.

Ex. Find the eigen values for the matrix $A = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$

Soln: $P_A(x) = |A - \lambda I| = \begin{bmatrix} 7-\lambda & 1 & -1 \\ -11 & -3-\lambda & 2 \\ 18 & 2 & -4-\lambda \end{bmatrix}$

Ans: $\lambda = -2, 4$

$$= -\lambda^3 + 12\lambda + 16$$
$$= -(\lambda+2)^2(\lambda-4)$$

or, $-(\lambda+2)^2(\lambda-4) = 0$

$\Rightarrow \lambda = -2, 4$

$\therefore$ Eigen values are -2 & 4.

Properties of eigen values and Eigen vectors

1. Let A be a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigen value of A.
2. Let A be a square matrix. Then A and A^T have the same characteristic polynomial and hence the same eigen values.
3. Let A be a square matrix, and let k be a positive integer. If λ is an eigen value of A, then λ^k is an eigenvalue of A^k.
4. Sum of eigenvalues = Sum of diagonal elements (Trace (A))
5. Product of eigenvalues = Determinant of matrix
6. Eigen values of triangular matrices are same as their diagonal elements

Orthogonal Matrix

A square matrix A is called orthogonal if $A \cdot A^T = A^T A = I$

$\Rightarrow A^{-1} = A^T$ is valid for orthogonal matrix

Q. Verify if the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Soln. we have

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow A$ is an orthogonal matrix.

Ex. Find the eigen values and the corresponding eigen vectors of the foll. matrices :

(i) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Soln: The characteristic eqn of A is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \text{ or, } \cancel{(1-\lambda)(2-\lambda)-(3-\lambda)(4-\lambda)=0}$$

$$(1-\lambda)(2-\lambda) - 12 = 0$$

$$\text{or, } \cancel{2 - \lambda - 2\lambda + \lambda^2 - 12 + 4\lambda + 4\lambda + \lambda^2 = 0}$$

$$\text{or, } 2 - 2\lambda - \lambda + \lambda^2 - 12 = 0$$

$$\text{or, } \lambda^2 - 3\lambda - 10 = 0$$

$$\text{or, } \lambda^2 - 5\lambda + 2\lambda - 10 = 0$$

$$\text{or, } \lambda(\lambda - 5) + 2(\lambda - 5) = 0$$

$$\text{or, } (\lambda + 2)(\lambda - 5) = 0$$

$$\text{or, } \lambda = -2, 5$$

Corresponding to the eigen value $\lambda = -2$, we have

$$(A + 2I)X = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or, } 3x_1 + 4x_2 = 0$$

$$\text{or, } x_1 = -\frac{4}{3}x_2$$

Hence the eigen vector X is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2/3 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$$

Since an eigen vector is unique upto a constant multiple, we can take the eigen vector as $[-4, 3]^T$.

Corresponding to $\lambda = 5$

$$(A - 5\lambda)X = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or, } x_1 - x_2 = 0 \text{ or, } x_1 = x_2$$

$$\therefore \text{ Eigen vector is } X = (x_1, x_2)^T = x_1(1,1)^T \text{ or } (1,1)^T$$

Ex. (ii). $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Soln: Ch. eqn is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 + 1 = 0 \Rightarrow 1 - 2\lambda + \lambda^2 + 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm i$$

Corr. to $\lambda = 1+i$, we have

$$[A - (1+i)I]X = \begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or, }$$

i.e., $-ix_1 + x_2 = 0$ i.e., $ix_1 = x_2$ i.e., $\frac{x_1}{1} = \frac{x_2}{i}$

$-x_1 - ix_2 = 0$

$\therefore$ Eigen vector for $\lambda = 1+i$ is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = [1, i]^T$$

For $\lambda = 1-i$

$$\begin{bmatrix} 1-(1-i) & 1 \\ -1 & (1-(1-i)) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e., $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $ix_1 + x_2 = 0$

$-x_1 - ix_2 = 0$

or $x_1 + ix_2 = 0$

or $ix_1 = -x_2$

or $\frac{x_1}{1} = \frac{x_2}{-i}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = [1 \quad -i]^T$$

$\therefore$ Eigen vector for $\lambda = 1-i$ is $[1, -i]^T$.

Remark: For a real matrix A, the eigen values and the corresponding eigen vectors can be complex.

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Soln: The characteristic eqn of A is given by

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] - 0 + 0 = (1-\lambda)(6-2\lambda-3\lambda+\lambda^2)$$

$$= (1-\lambda)(\lambda^2-5\lambda+6)$$

$$= (1-\lambda)(\lambda^2-3\lambda-2\lambda+6)$$

$$= (1-\lambda)[\lambda(\lambda-3)-2(\lambda-3)]$$

$$= (1-\lambda)(\lambda-2)(\lambda-3)$$

ie, $\lambda = 1, 2, 3$.

$\therefore$ Eigenvalues are $1, 2 \& 3$.

Corr. to $\lambda = 1$ eigen vector is

$[A-\lambda I]X = 0$

$$[A-\lambda I]X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_2 + x_3 = 0$

$2x_1 + 2x_3 = 0$

we obtain two eqns in three unknowns, one of the variable can be choosen arbitrarily. Taking $x_3 = 1$, $x_2 = -1$, $x_1 = -1$

$\therefore$ we obtain eigen vector as $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = [-1, -1, 1]^T$

For $\lambda = 2$

$$[A-2I]X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ~~$-1(0-0)-0+0=$~~ $-x_1 = 0 \Rightarrow x_1 = 0$, $x_3 = 0$, $2x_1 + x_3 = 0$

and x_2 is arbitrary. Taking $x_2 = 1$.

eigen vectors are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [0, 1, 0]^T$.

For $\lambda = 3$ we have

$$[A - 3I]X = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $-2x_1 = 0$ or $x_1 = 0$

$x_2 + x_3 = 0$ or $x_2 = -x_3$ or $\frac{x_2}{1} = \frac{x_3}{-1}$

$2x_1 + x_3 = 0$

$\therefore$ eigen vectors are $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = [0, 1, -1]^T$

Q. Find the Eigen value and Eigen vector of matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$\frac{x_1}{-0} = \frac{x_2}{\;} = \frac{x_3}{\;}$

Sol

Q. Find the Eigen values and Eigen vectors of the given matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln: The ch. eqn of matrix A is $|A-\lambda I| = 0$

$$\Rightarrow |A-\lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 7-\lambda \\ 2 & -4 \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16] + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda)] = 0$$

$$\Rightarrow (8-\lambda)[21-3\lambda-7\lambda+\lambda^2-16] + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0$$

$$\Rightarrow (8-\lambda)[\lambda^2-10\lambda+5] + 6[6\lambda-10] + 2[10+2\lambda] = 0$$

$$\Rightarrow 8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+20+4\lambda = 0$$

$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda = 0$$

$$\Rightarrow \lambda^3-18\lambda^2+45\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2-18\lambda+45) = 0$$

$$\Rightarrow \lambda(\lambda^2-15\lambda-3\lambda+45) = 0 \Rightarrow \lambda[\lambda(\lambda-15)-3(\lambda-15)] = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

$\therefore$ Eigen values are $\lambda = 0, 3, 15$.

To find eigen vectors find $[A-\lambda I]X = 0$

For $\lambda = 0$,

$$[A-\lambda I]X = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 \; 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

From eqn (1) & (2)

$$\begin{matrix} x_1 & x_2 & x_3 \\ 8 & -6 & 2 \\ -6 & 7 & -4 \end{matrix}$$

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} = k \text{ (let)}$$

$x_1 = 10k$, $x_2 = 20k$, $x_3 = 20k$

$\therefore$ Eigen vector for $\lambda = 0$ is $\begin{bmatrix} 10k \\ 20k \\ 20k \end{bmatrix}$ or, by taking $k = \frac{1}{10}$, we can find eigenvector $[1, 2, 2]^T$ or its scalar multiples will be eigen vector for it.

For $\lambda = 3$,

$[A - 3I]X = 0$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 5x_1 - 6x_2 + 2x_3 = 0 \\ -6x_1 + 4x_2 - 4x_3 = 0 \end{array}\right\} \qquad \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} = k \text{ (let)}$$

$$2x_1 - 4x_2 + 0x_3 = 0$$

$\therefore$ Eigenvector is $\begin{bmatrix} 16k \\ 8k \\ -16k \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$,

$$[A - 15I]X = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} -7x_1 - 6x_2 + 2x_3 = 0 \\ -6x_1 - 8x_2 - 4x_3 = 0 \end{array}\right\} \qquad \frac{x_1}{40} = \frac{-y}{40} = \frac{z}{20} = k$$

$$2x_1 - 4x_2 - 12x_3 = 0$$

$\therefore$ Eigenvector $= \begin{bmatrix} 40k \\ -40k \\ 20k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Q. Find the sum of the characteristic roots (Eigenvalues) of the matrix A^2, given that

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$$

Soln. The ch. eqn for given the matrix is $|A - \lambda I| = 0$

or $\begin{vmatrix} 3-\lambda & 0 & 0 \\ 8 & 4-\lambda & 0 \\ 6 & 2 & 5-\lambda \end{vmatrix} = 0$ whose roots as ch. roots are $\lambda = 3, 4 \; \& \; 5$ (or eigen value).

we know that the ch. root of the matrix A^2 are λ^2 as $9, 16, 25$ whose sum $= 9 + 16 + 25 = 50$.

Q. If matrix A is defined as $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$, then find the eigen-values of A^2.

Soln: → The ch. eqn of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -3-\lambda & 0 \\ 1 & 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)(-3-\lambda)(2-\lambda) = 0$

Hence $\lambda = -1$, $\lambda = -3$ and $\lambda = 2$ are eigen values of A.
We know that the eigen values of A^2 are the squares of the eigen values of A, i.e, 1, 9, 4.

Q. If the eigen values of matrix A are 2, 3, 6 then write the eigen values of A^T, A^{-1}, A^m $(m \in N)$, adj A and kA.

Soln → Eigen values of A are 2, 3, and 6 and since eigen values of A^T are same as these of A, hence eigen values of A^T are 2, 3 and 6.

Eigen value of A^{-1} are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$.

" " of A^m are $2^m, 3^m$ and 6^m $(m \in N)$.

Further eigen value of adj(A) are $\frac{1}{2}(2 \times 3 \times 6)$, $\frac{1}{3}(2 \times 3 \times 6)$ and $\frac{1}{6}(2 \times 3 \times 6)$
ie, 18, 12 and 6.

Also the eigen values of kA are 2k, 3k and 6k where k is const.

Q. If the eigen value of A are 1, 1, 1, find the eigen values of $A^2 + 2A + 3I$.

Soln: Eigen values of A^2 are $1^2, 1^2, 1^2$,

" " " 2A are 2, 2, 2

3I are 3, 3, 3.

∴ Eigen value of $A^2 + 2A + 3I = (1, 1, 1) + (2, 2, 2) + (3, 3, 3)$
$= 6, 6, 6$

Imp. Property

If A is a symmetric matrix the eigen vectors of the eqn $(A - \lambda I)X = 0$ are pairwise orthogonal.

If $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ are two eign vectors of a symmetric matrix A then $x_1 x_1' + x_2 x_2' + \dots x_n x_n' = 0$.

Some Useful deductions

If λ is a characteristic value of a square matrix A, then

(i) $\lambda + k$ is a ch. value of the matrix $A + kI$

(ii) $k\lambda$ is a ch. value of kA.

(iii) $\frac{1}{\lambda}$ is a ch. ~~value~~ value of A^{-1}. In otherwords $\frac{|A|}{\lambda}$ is the eigen value of the matrix $adj(A)$.

(iv) λ^2 is a ch. value of A^2. In general λ^k is an eigen value of A^k,

(ch. value is an eigen value)

Symmetric matrix → A $n \times n$ matrix is symmetric matrix if $A^T = A$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$$

Here $A = A^T$

Skew-Symmetric matrix → A square matrix is skew symmetric matrix if $A^T = -A$

$$B = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix} \Rightarrow B^T \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & -2 \\ 5 & 2 & 0 \end{bmatrix} = -B$$