Inner Product Spaces

Def^n. Let V be a vector space over R. An inner product on V is a function that assigns to each pair of vectors $u, v \in V$ a real number denoted by $\langle u, v\rangle$ satisfying:

(i) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

(ii) $\langle u, v\rangle = \langle v, u\rangle \; \forall\, u, v \in V$

(iii) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \; \forall\, u, v, w \in V$

(iv) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle \; \forall\, \alpha \in R,\ u, v \in V$

Properties (iii) & (iv) can be combined to give an equivalent property:

(v) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle \; \forall\, u, v, w \in V,\ \alpha, \beta \in R$.

Ex. We know $R^n = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in R\}$ is a vector space over R.

Define $\langle u, v\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$

where $u = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $v = (\beta_1, \beta_2, \ldots, \beta_n) \in R^n$. Then $\langle\,,\rangle$ is an inner product in R^n and is called standard inner product on R^n.

Soln: Now $\langle u, u\rangle = \alpha_1\alpha_1 + \alpha_2\alpha_2 + \cdots \alpha_n\alpha_n$

$= \alpha_1^2 + \alpha_2^2 + \cdots \alpha_n^2 \geq 0 \; \forall\, u = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$.

Also $\langle u, u\rangle = 0$

iff $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 0$

iff $\alpha_1^2 = 0, \alpha_2^2 = 0, \ldots, \alpha_n^2 = 0$

iff $\alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$

iff $u = (0, 0, \ldots, 0) = 0$

Now $\langle u, v\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$

$= \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_n\alpha_n$

$= \langle v, u\rangle$

$\forall\, u = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ v = (\beta_1, \beta_2, \ldots, \beta_n) \in R^n$

Further,

$\langle \alpha u + \beta v, w\rangle = \langle (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2, \ldots, \alpha\alpha_n + \beta\beta_n), (r_1, r_2, \ldots, r_n)\rangle$

$= (\alpha\alpha_1 + \beta\beta_1)r_1 + (\alpha\alpha_2 + \beta\beta_2)r_2 + \cdots + (\alpha\alpha_n + \beta\beta_n)r_n$

$= \alpha\alpha_1 r_1 + \alpha\alpha_2 r_2 + \cdots + \alpha\alpha_n r_n + \beta\beta_1 r_1 + \beta\beta_2 r_2 + \cdots + \beta\beta_n r_n$

$= \alpha(\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) + \beta(\beta_1 r_1 + \beta_2 r_2 + \cdots + \beta_n r_n)$

$= \alpha\langle u, w\rangle + \beta\langle v, w\rangle$

$\forall\, u = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ v = (\beta_1, \beta_2, \ldots, \beta_n),\ w = (r_1, r_2, \ldots, r_n) \in R^n$, $\alpha, \beta \in R$

$\therefore \langle\,,\rangle$ is an inner product on R^n. It is called standard inner product on R^n.

Remark.

Standard inner product on R^n is also called dot product and is denoted as

$u \cdot v = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$

where $u = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $v = (\beta_1, \beta_2, \cdots, \beta_n) \in R^n$.

In this notation

(i) $\langle u, u\rangle = u \cdot u \geq 0$, $u \cdot u = 0$ iff $u = 0$

(ii) $u \cdot v = v \cdot u \ \forall u, v \in R^n$

(iii) $(u+v)\cdot w = u\cdot w + v \cdot w \ \forall u, v, w \in R^n$.

(iv) $(\alpha u)\cdot v = \alpha(u \cdot v) \ \forall u, v \in R^n$.

(v) $(\alpha u + \beta v)\cdot w = \alpha(u\cdot w) + \beta(v \cdot w) \ \forall u, v, w \in R^n, \alpha, \beta \in R$.

Ex: Let $u = (\alpha_1, \alpha_2)$, $v = (\beta_1, \beta_2)$ be vectors in R^2. Define

$$\langle u, v\rangle = \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + 3\alpha_2\beta_2$$

Then this gives an inner product on R^2.

Soln: Now,

$$\langle u, u\rangle = \alpha_1\alpha_1 - \alpha_2\alpha_1 - \alpha_1\alpha_2 + 3\alpha_2\alpha_2$$
$$= \alpha_1^2 - 2\alpha_1\alpha_2 + 3\alpha_2^2$$
$$= (\alpha_1 - \alpha_2)^2 + 2\alpha_2^2 \geq 0$$

Also

$$\langle u, u\rangle = 0 \text{ iff } (\alpha_1 - \alpha_2)^2 + 2\alpha_2^2 = 0$$
$$\text{iff } \alpha_1 = \alpha_2 \text{ and } \alpha_2 = 0$$
$$\text{iff } \alpha_1 = \alpha_2 = 0$$
$$\text{iff } u = (0,0) = 0 \ \forall u = (\alpha_1, \alpha_2) \in R^2$$

Also, $\langle u, v\rangle = \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + 3\alpha_2\beta_2$

$$= \beta_1\alpha_1 - \beta_1\alpha_2 - \beta_2\alpha_1 + 3\beta_2\alpha_2$$
$$= \beta_1\alpha_1 - \beta_2\alpha_1 - \beta_1\alpha_2 + 3\beta_2\alpha_2$$
$$= \langle v, u\rangle \ \forall u = (\alpha_1, \alpha_2), v = (\beta_1, \beta_2) \in R^2$$

Further,

$$\langle \alpha u + \beta v, w\rangle = \langle(\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2), (r_1, r_2)\rangle$$
$$= (\alpha\alpha_1 + \beta\beta_1)r_1 - (\alpha\alpha_2 + \beta\beta_2)r_1 - (\alpha\alpha_1 + \beta\beta_1)r_2 + 3(\alpha\alpha_2 + \beta\beta_2)r_2$$
$$= \alpha\alpha_1 r_1 + \beta\beta_1 r_1 - \alpha\alpha_2 r_1 - \beta\beta_2 r_1 - \alpha\alpha_1 r_2 - \beta\beta_1 r_2 + 3\alpha\alpha_2 r_2 + 3\beta\beta_2 r_2$$
$$= \alpha\alpha_1 r_1 - \alpha\alpha_2 r_1 - \alpha\alpha_1 r_2 + 3\alpha\alpha_2 r_2 + \beta\beta_1 r_1 - \beta\beta_2 r_1 - \beta\beta_1 r_2 + 3\beta\beta_2 r_2$$

$$= \alpha(\alpha_1 r_1 - \alpha_2 r_1 + 3\alpha_2 r_2) + \beta(\beta_1 r_1 - \beta_2 r_1 - \beta_1 r_2 + 3\beta_2 r_2)$$

$$= \alpha\langle u, w\rangle + \beta\langle v, w\rangle$$

$\forall\, u = (\alpha_1, \alpha_2),\ v = (\beta_1, \beta_2),\ w = (r_1, r_2) \in R^2$

$\alpha, \beta \in R$.

Hence this defines a inner product on R^2.

Ex. Let $u = (1, 3)$, $v = (2, 1) \in R^2$

Find $w \in R^2$ such that $\langle w, u\rangle = 3$, $\langle w, v\rangle = -1$ where $\langle , \rangle$ is the standard inner product on R^2.

Soln: standard inner product on R^2 is given by

$$\langle(\alpha_1, \alpha_2), (\beta_1, \beta_2)\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 \quad \forall\, (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in R^2$$

Let $w = (\alpha, \beta) \in R^2$

Now, $\langle(\alpha, \beta), (1, 3)\rangle = 3$, $\langle(\alpha, \beta), (2, 1)\rangle = -1$

$\Rightarrow \alpha + 3\beta = 3$, $2\alpha + \beta = -1$

In matrix form

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$\therefore \beta = \frac{7}{5}$, $\alpha + 3 \times \frac{7}{5} = 3$, $\alpha = 3 - \frac{21}{5} = -\frac{6}{5}$

$$w = \left(-\frac{6}{5}, \frac{7}{5}\right).$$

Ex. Let $u \in R^2$. Show that

$$u = \langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2, \text{ where } e_1 = (1, 0), e_2 = (0, 1)$$

Soln Now let $u = (\alpha, \beta) \in R^2$, then

$\langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2$

$= \langle(\alpha, \beta), (1, 0)\rangle e_1 + \langle(\alpha, \beta), (0, 1)\rangle e_2$

$= (\alpha \cdot 1 + \beta \cdot 0)e_1 + (\alpha \cdot 0 + \beta \cdot 1)e_2$

$= \alpha(1, 0) + \beta(0, 1) = (\alpha, 0) + (0, \beta)$

$= (\alpha, \beta) = u$.

Ex:

Let $u=(1,1,1)$, $v=(1,2,3)$, $w=(1,3,4)\in R^3$. Find $u_1=(\alpha,\beta,\gamma)\in R^3$ such that

$$\langle u,u_1\rangle=7,\ \langle v,u_1\rangle=16,\ \langle w,u_1\rangle=22$$

where $\langle\ ,\ \rangle$ is standard inner product on R^3.

Soln: Now,

$$\langle u,u_1\rangle=7 \Rightarrow \alpha+\beta+\gamma=7$$

$$\langle v,u_1\rangle=16 \Rightarrow \alpha+2\beta+3\gamma=16$$

$$\langle w,u_1\rangle=22 \Rightarrow \alpha+3\beta+4\gamma=22$$

In matrix form

$$\begin{bmatrix}1&1&1\\1&2&3\\1&3&4\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}7\\16\\22\end{bmatrix}$$

Applying $R_2\to R_2-R_1$, $R_3\to R_3-R_1$

$$\begin{bmatrix}1&1&1\\0&1&2\\0&2&3\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}7\\9\\15\end{bmatrix}$$

Applying $R_3\to R_3-2R_2$

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&-1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}7\\9\\-3\end{bmatrix}$$

$\therefore \gamma=3$, $\beta+2\gamma=9 \Rightarrow \beta=3$

$\alpha+\beta+\gamma=7 \Rightarrow \alpha=1$

$\therefore u_1=(1,3,3)$

Length or Norm of a Vector

Let V be an inner product space and $u \in V$. The norm (or length) of u denoted by $\|u\|$ is defined as

$$\|u\| = \sqrt{\langle u, u\rangle} \quad \text{i.e. } \|u\|^2 = \langle u, u\rangle$$

Since $\langle u, u\rangle \geq 0$, so $\|\cdot\|$ is well defined

If $V = \mathbb{R}^n$ and $\langle , \rangle$ is standard inner product on $\mathbb{R}^n$, then

$$\|u\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2} \quad \forall\, u = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n.$$

If $u, v \in \mathbb{R}^n$. Then distance between u and v is defined as $\|u - v\|$

Let $u = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $v = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{R}^n$.

Then $u - v = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_n - \beta_n)$

$$\therefore \|u - v\| = \sqrt{(\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2 + \cdots + (\alpha_n - \beta_n)^2}$$

Ex. Let $u = (2, 3, 2, -1)$ and $v = (3, 2, 1, 3)$

Then $\|u\| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2}$

$$= \sqrt{4 + 9 + 4 + 1} = \sqrt{18}$$

$$\|v\| = \sqrt{3^2 + 2^2 + 1^2 + 3^2}$$

$$= \sqrt{9 + 4 + 1 + 9} = \sqrt{23}$$

$$\|u - v\| = \|(-1, 1, 1, -4)\|$$

$$= \sqrt{(-1)^2 + 1^2 + 1^2 + (-4)^2} = \sqrt{1 + 1 + 1 + 16} = \sqrt{19}.$$

Ex. Find the norm of the vector $u = (2, 3, 6) \in \mathbb{R}^3$

Show that $\frac{u}{\|u\|}$ is of unit length.

Soln. $\|u\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

Now, $\frac{u}{\|u\|} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$

$$\left\|\frac{u}{\|u\|}\right\| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = \sqrt{\frac{49}{49}} = 1.$$

$\therefore \frac{u}{\|u\|}$ is of unit length.

Cauchy-Schwarz Inequality

If u & v are in vectors in R^n, then

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

$|\,|$ on the L.H.S. stands for absolute value of real number.

Triangle Inequality :- If u & v are vectors in R^n, then

$$\|u+v\| \leq \|u\| + \|v\|$$

Th: $|\langle u, v\rangle| = \|u\|\|v\|$ if and only if u & v are linearly dependent

Angle between two vectors

If α is a real no in interval $[-1, 1]$ there exists unique θ in interval $[0, \pi]$ such that $\cos\theta = \alpha$

$$\therefore \cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

The angle θ is called the angle between u & v.

Orthogonal vectors

Def^n: Two non-zero vector u & v in inner product space V are said to be orthogonal if $\langle u, v\rangle = 0$. We assume $0 \in V$ is orthogonal to every $u \in V$

* Vectors in R^n are orthogonal if $\cos\theta = 0$
* u & v are parallel if $|\langle u, v\rangle| = \|u\|\|v\|$
* $\therefore$ vectors are parallel if $\cos\theta = \pm 1$.
* vectors are in same direction if $\cos\theta = 1$

Eg: Which of the foll. vectors are orthogonal

$u_1 = (4, 2, 6, -8)$ $\quad u_2 = (-2, 3, -1, -1)$

$u_3 = (-2, -1, -3, 4)$ $\quad u_4 = (1, 0, 0, 2)$

Soln:

$$\langle u_1, u_2\rangle = 4\times(-2) + 2\times 3 + 6\times(-1) + (-8)\times(-1)$$
$$= -8+6-6+8 = 0$$

$$\langle u_1, u_3\rangle = 4\times(-2) + 2\times(-1) + 6\times(-3) + (-8)\times 4$$
$$= -8 - 2 - 18 - 32 = -60$$

$$\langle u, u_4\rangle = 4\times 1 + 2\times 0 + 6\times 0 + (-8)\times 2 = 4+0+0-16 = -12$$

$$\langle u_2, u_3\rangle = (-2)\times(-2) + 3\times(-1) + (-1)\times(-3) + (-1)\times(4) = 4 - 3 + 3 - 4 = 0$$

$\langle u_2, u_4\rangle = (-2)\times 1 + 3\times 0 - 1\times 0 + (-1)\times 2 = -2+0-0-2 = -4$

$\langle u_3, u_4\rangle = (-2)\times 1 + (-1)\times 0 + (-3)\times 0 + 4\times 2 = -2+0+0+8$
$= 6.$

$\therefore$ u_1 and u_2 and u_2 and u_3 are orthogonal.

i.e., u_2 is orthogonal to u_1 and u_3.

Orthogonal and orthonormal Sets

Defn → Let $S = \{u_1, u_2, \ldots, u_k\}$ be set of k-vectors in V. S is said to be orthogonal set if $u_i \cdot u_j = \langle u_i, u_j\rangle = 0 \;\forall\; i \neq j$

The set S is said to be orthonormal set if

$$u_i \cdot u_j = \langle u_i, u_j\rangle = 0 \;\forall\; i \neq j$$
$$= 1 \text{ if } i = j$$

Defn: orthonormal Basis →
If an orthonormal set S is a basis of inner product space V then the set S is called an orthonormal basis of V.

Ex: Let $S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$
S is orthonormal basis of R^3.

Soln. → we know that S is basis of R^3.

Now, $\langle e_1, e_2\rangle = 1\times 0 + 0\times 1 + 0\times 0 = 0$

$\langle e_2, e_3\rangle = 0\times 0 + 1\times 0 + 0\times 1 = 0$

$\langle e_3, e_1\rangle = 0\times 1 + 0\times 0 + 1\times 0 = 0$

Also, $\langle e_1, e_1\rangle = 1\times 1 + 0\times 0 + 0\times 0 = 1$

$\langle e_2, e_2\rangle = 0\times 0 + 1\times 1 + 0\times 0 = 1$

$\langle e_3, e_3\rangle = 0\times 0 + 0\times 0 + 1\times 1 = 1$

$$e_i \cdot e_j = \langle e_i, e_j\rangle = 0 \;\forall\; i \neq j$$
$$= 1 \text{ if } i = j$$

$\therefore$ S is orthonormal basis of R^3.

Th. If $S = \{u_1, u_2, u_3\}$ is an orthogonal set of non-zero vectors then S is linearly independent.

Ex. Prove that

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), (0,1,0)\right\}$$

is an orthonormal basis of R^3.

Soln: Let $u_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $u_2 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $u_3 = (0,1,0)$

and

$$u_1 \cdot u_2 = \langle u_1, u_2 \rangle = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times 0 + \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

$$u_1 \cdot u_3 = \langle u_1, u_3 \rangle = \frac{1}{\sqrt{2}} \times 0 + 0 \times 1 + \frac{1}{\sqrt{2}} \times 0 = 0$$

$$u_2 \cdot u_3 = \langle u_2 \cdot u_3 \rangle = \frac{1}{\sqrt{2}} \times 0 + 0 \times 1 + -\frac{1}{\sqrt{2}} \times 0 = 0$$

Also

$$\|u_1\|^2 = \langle u_1, u_1 \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

$$\|u_2\|^2 = \langle u_2, u_2 \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\|u_3\|^2 = \langle u_3, u_3 \rangle = 0^2 + 1^2 + 0^2 = 1$$

$$\therefore u_i, u_j = \langle u_i, u_j \rangle = 0 \text{ if } i \neq j$$
$$= 1 \text{ if } i = j$$

$\therefore \langle u_1, u_2, u_3 \rangle$ is an orthonormal set in R^3.

Also every three orthonormal vectors being orthogonal vectors are linearly independent. Therefore (u_1, u_2, u_3) is linearly independent in R^3 and hence basis of R^3.

Gram Schmidt Process -

Let W be subspace of R^n of dimension k. Let $S = \{u_1, u_2, \dots, u_k\}$ be basis of W. Define $v_1 = u_1$

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right) v_1$$

$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_3 \cdot v_2}{\|v_2\|^2}\right) v_2$$

$$\cdots$$

$$v_k = u_k - \left(\frac{u_k \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_k \cdot v_2}{\|v_2\|^2}\right) v_2 - \cdots - \left(\frac{u_k \cdot v_{k-1}}{\|v_{k-1}\|^2}\right) v_{k-1}$$

Then $B = \{v_1, v_2, \dots, v_k\}$ is an orthogonal basis of W.

Ex. Let $S = \{(1,2,3), (1,1,1), (1,0,1)\}$ be a basis of R^3. Create a orthogonal basis of R^3 by Gram-Schmidt Process. (9)

Soln. Let $u_1 = (1,2,3), u_2 = (1,1,1), u_3 = (1,0,1)$

Now let $v_1 = u_1 = (1,2,3)$

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right) v_1$$

$$= (1,1,1) - \left(\frac{1\times1 + 2\times1 + 3\times1}{1^2+2^2+3^2}\right)(1,2,3)$$

$$= (1,1,1) - \frac{6}{14}(1,2,3)$$

$$= (1,1,1) - \left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right) = \left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right)$$

$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_3 \cdot v_2}{\|v_2\|^2}\right) v_2$$

$$= (1,0,1) - \left(\frac{1\times1 + 0\times2 + 1\times3}{1^2+2^2+3^2}\right)(1,2,3)$$

$$- \left(\frac{1\times\frac{4}{7} + 0\times\frac{1}{7} + 1\times\frac{-2}{7}}{\left(\frac{4}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{-2}{7}\right)^2}\right)\left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right)$$

$$= (1,0,1) - \frac{4}{14}(1,2,3) - \left(\frac{\frac{2}{7}}{\frac{21}{49}}\right)\left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right)$$

$$= (1,0,1) - \left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right) - \left(\frac{8}{21}, \frac{2}{21}, \frac{-4}{21}\right)$$

$$= \left(1 - \frac{2}{7} - \frac{8}{21}, 0 - \frac{4}{7} - \frac{2}{21}, 1 - \frac{6}{7} + \frac{4}{21}\right)$$

$$= \left(\frac{21-6-8}{21}, \frac{-12-2}{21}, \frac{21-18+4}{21}\right)$$

$$= \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

Now Consider

$$B = \left\{v_1 = (1,2,3), v_2 = \left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right), v_3 = \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)\right\}$$

$v_2 \cdot v_1 = 1\times\frac{4}{7} + 2\times\frac{1}{7} + 3\times\frac{-2}{7} = 0$

$v_2 \cdot v_3 = \frac{4}{7}\times\frac{1}{3} + \frac{1}{7}\times\frac{-2}{3} - \frac{2}{7}\times\frac{1}{3} = 0$

$v_3 \cdot v_1 = \frac{1}{3}\times1 - \frac{2}{3}\times2 + \frac{1}{3}\times3 = 0$

$\therefore$ B is an orthogonal basis of R^3

Ex. Show that

$S = \{u_1 = (2, -1, 3), u_2 = (-1, 1, 1), u_3 = (-4, -5, 1)\}$

is orthogonal basis of R^3. Find an orthonormal basis of R^3.

Soln: $u_1 \cdot u_2 = 2\times -1 + (-1)\times 1 + 3\times 1 = -2-1+3 = 0$

$u_2 \cdot u_3 = (-1)\times(-4) + (1)\times(-5) + 1\times 1 = 4-5+1 = 0$

$u_3 \cdot u_1 = -4\times 2 + (-5)\times(-1) + 1\times 3 = -8+5+3 = 0$

$\therefore$ S is orthogonal basis of R^3.

Also $\|u_1\| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{4+1+9} = \sqrt{14}$

$\|u_2\| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{1+1+1} = \sqrt{3}$

$\|u_3\| = \sqrt{(-4)^2+(-5)^2+1^2} = \sqrt{16+25+1} = \sqrt{42}$

$\therefore \left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\} = \left\{\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-4}{\sqrt{42}}, \frac{-5}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right)\right\}$

is orthonormal basis ~~for~~ R^3.

Orthogonal complements

Defn: Let W be subspace of R^n. Define $W^\perp = \{u \in R^n / u\cdot w = 0 \ \forall w \in W\}$.

$W^\perp$ is known as orthogonal complement of W.

Th: Let W be a subspace of R^n. $u \in W^\perp$ iff u is orthogonal to every vector in spanning set for W.

Ex: Let $W = \{(a, 0, 0) : a \in R\}$ we know W is subspace of R^3.

Find $W^\perp$. What is dimension of $W^\perp$?

Soln: Let $(x, y, z) \in W^\perp$ then $(x, y, z)\cdot(a, 0, 0) = 0 \ \forall a \in R$

$\Rightarrow ax + 0y + 0z = 0 \ \forall a \in R$

$\Rightarrow ax = 0 \ \forall a \in R$

$\Rightarrow x = 0$

$\therefore W^\perp = \{(0, y, z) : y, z \in R\}$

Basis of $W^\perp = \{(0,1,0), (0,0,1)\}$

$\dim W^\perp = 2$. Also basis of $W = \{(1,0,0)\}$, $\dim W = 1$

$\therefore \dim W + \dim W^\perp = 1+2 = 3 = \dim$

Orthogonal Projection

Defn. → Let W be a subspace of R^n having dimension k. Let $B = \{u_1, u_2, \cdots, u_k\}$ be an orthonormal basis of W. Let $v \in R^n$.

The orthogonal projection of v onto W

$$= \text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2 + \cdots + (v \cdot u_k) u_k$$

If $W = \{0\}$ then $\text{proj}_W v = 0$

Ex: Let $B = \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \right\}$ be orthonormal set in R^3.

Let $W = \text{span} B$. Now B is orthonormal basis of W. Let $v = (2, 3, 4) \in R^3$. Find orthogonal projection of v ~~onto~~ onto W i.e, $\text{proj}_W v$. ~~If you take B_1 to be another~~ orthonormal basis ~~of W, does $\text{proj}_W v$~~ remains same ?

Soln: → Now $\text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$

$$= \left[(2,3,4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right] \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$+ \left[(2,3,4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\right] \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$= \left(\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \left(\frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$= (3, 0, 3) + (-1, 0, 1) = (2, 0, 4)$$

Now Consider $\{(2\sqrt{2}, 0, 0), (0, 0, 3\sqrt{2})\}$

$$(2\sqrt{2}, 0, 0) = 2\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$(0, 0, 3\sqrt{2}) = 3\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) - 3\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$\therefore \{(2\sqrt{2}, 0, 0), (0, 0, 3\sqrt{2})\}$ is an orthogonal subset of W.

Hence $B_1 = \{(1, 0, 0), (0, 0, 1)\}$ is orthonormal basis of W.

Ex. Let $B = \left\{ \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right\}$ be an orthonormal subset of R^3. Let $W = \text{span } B$. Now B is an orthonormal basis of W. Express $v = (-3, 4, 5)$ as $w_1 + w_2$.

Soln.

Now $w_1 = \text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$

$$= \left[(-3, 4, 5) \cdot \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) \right] \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) + \left[(-3, 4, 5) \cdot \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$= \frac{5}{\sqrt{14}} \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) + \frac{12}{\sqrt{3}} \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$= \left(\frac{10}{14}, \frac{-5}{14}, \frac{15}{14} \right) + \left(\frac{-12}{3}, \frac{12}{3}, \frac{12}{3} \right)$$

$$= \left(\frac{10}{14} - 4, \frac{-5}{14} + 4, \frac{15}{14} + 4 \right) = \left(\frac{-46}{14}, \frac{51}{14}, \frac{71}{14} \right) \in W$$

$w_2 = v - \text{proj}_W v$

$$= (-3, 4, 5) - \left(\frac{-46}{14}, \frac{15}{14}, \frac{71}{14} \right) = \left(\frac{4}{14}, \frac{5}{14}, \frac{-1}{14} \right)$$

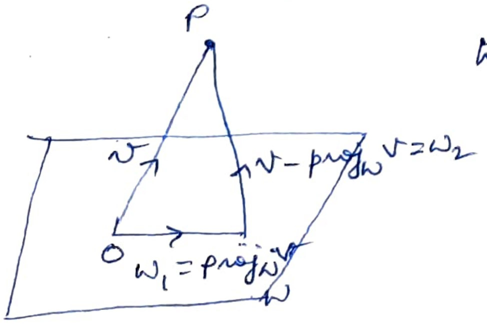
Now w_2 is orthogonal to both u_1 & u_2

$\therefore w_2 \in W^{\perp}$

$\therefore v = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^{\perp}$.

Now $v = \text{proj}_W v + v - \text{proj}_W v$.

The min^{m} distance of P from $W = \| v - \text{proj}_W v \|$



$$W = \| v - \text{proj}_W v \| = \| w_2 \| = \left\| \left(\frac{4}{14}, \frac{5}{14}, \frac{-1}{14} \right) \right\|$$

$$= \sqrt{\frac{42}{14 \times 14}} = \sqrt{\frac{3}{14}}.$$

Ex. Find the orthogonal projection of $v = \{-1, 4, 3\}$ onto the subspace W of R^3 spanned by the orthogonal vectors $v_1 = \{1, 1, 0\}$ and $v_2 = \{-1, 1, 0\}$

Soln The subspace W of R^3 is defined to be $W = span(\{(1,1,0), (-1,1,0)\})$

$$= span\left(\left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{2}}(-1,1,0)\right\}\right)$$

where we have normalized the vectors $(1,1,0)$ and $(-1,1,0)$ to get an orthonormal basis for W. Hence by defn of $proj_W v$, we have

$$proj_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$$

where $u_1 = \frac{1}{\sqrt{2}}(1,1,0)$, and $u_2 = \frac{1}{\sqrt{2}}(-1,1,0)$

$$proj_W v = \left((-1,4,3) \cdot \left(\frac{1}{\sqrt{2}}(1,1,0)\right)\right)\left(\frac{1}{\sqrt{2}}(1,1,0)\right)$$

$$+ \left((-1,4,3) \cdot \left(\frac{1}{\sqrt{2}}(-1,1,0)\right)\right)\left(\frac{1}{\sqrt{2}}(-1,1,0)\right)$$

$$= \frac{3}{\sqrt{2}}(1,1,0) + \frac{5}{2}(-1,1,0) = (-1,4,0).$$