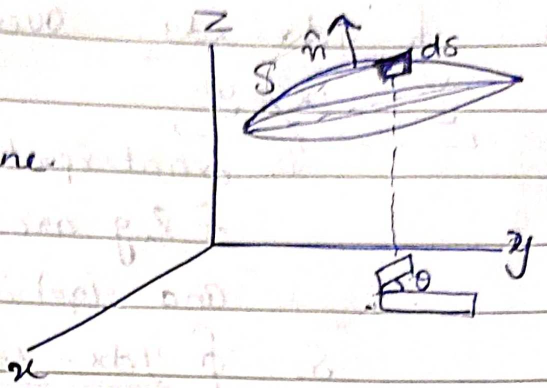
Surface Integral :→

Let S be a surface in xyz-plane



and $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Let $\hat{n}$ be the normal unit vector to the surface

$$\Rightarrow \hat{n} = \frac{\nabla S}{|\nabla S|}$$

$\vec{F}\cdot\hat{n}$ — will give the component of $\vec{F}$ in the direction of $\hat{n}$

$(\because \vec{F}\cdot\hat{n} = (\text{Proj of } \vec{F} \text{ on } \hat{n})(\text{magnitude of } \hat{n}) = \text{Proj of } \vec{F} \text{ on } \hat{n})$

θ is the angle b/w S and xy-plane

$\Rightarrow \theta$ is angle b/w their normals i.e. $\hat{n}$ and $\hat{k}$

$$\Rightarrow \cos\theta = \frac{\hat{n}\cdot\hat{k}}{|\hat{n}||\hat{k}|} = \hat{n}\cdot\hat{k}$$

$$\therefore ds\cos\theta = dx\,dy$$

$$\Rightarrow ds = \frac{dx\,dy}{\cos\theta}$$

Surface Integral is $\iint \vec{F}.\hat{n}\,ds$

$$= \iint \vec{F}.\hat{n}\,\frac{dx\,dy}{\cos\theta}$$

Que $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and $S = \{x^2+y^2+z^2=1 - \text{Hemisphere}\}$
Find the Surface Integral over S.

Sol^n $\hat{n} = \dfrac{\vec{\nabla f}}{|\nabla f|} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F}.\hat{n} = x^2+y^2+z^2$$

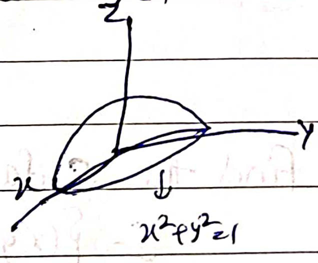
$$S.I. = \iint_S \vec{F}.\hat{n}\,ds = \iint_S (x^2+y^2+z^2).\frac{dx\,dy}{\cos\theta}$$

$$= \iint_S (x^2+y^2+z^2)\cdot\frac{dx\,dy}{\hat{n}.\hat{k}}$$

$$= \iint_S (x^2+y^2+z^2)\,\frac{dx\,dy}{z}$$

$$= \iint \frac{dx\,dy}{\sqrt{1-x^2-y^2}} \rightarrow \text{on } xy \text{ plane.}$$

Proj of $x^2+y^2+z^2=1$ on xy plane is circle $x^2+y^2=1$



So $S.I. = \iint \dfrac{dx\,dy}{\sqrt{1-x^2-y^2}}$ over $x^2+y^2=1$

$x = r\cos\theta$, $y = r\sin\theta$
$x^2+y^2 = r^2$

$$S.I = \int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{\sqrt{1-r^2}} = \int_0^1 \left(\frac{r\,dr}{\sqrt{1-r^2}}\right)2\pi$$

Put $\sqrt{1-x^2} = t$
$\dfrac{-2x}{2\sqrt{1-x^2}}\,dx = dt$

$$= 2\pi\left(-\sqrt{1-r^2}\right)\Big|_0^1$$

$$= -2\pi(-1) = \boxed{2\pi}$$

Gauss divergence Thm:→

Let S be a closed, Smooth Surface. let $f_1, f_2, f_3, \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$

are Continuous function on region 'D'. — where D is a closed and bdd region in 3-D space.

Then $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_D \operatorname{div}\vec{F}\, dx\, dy\, dz$

$\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

Que Find the Surface Integral of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ over the Surface $S = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$ — ~~[illegible]~~ sphere

Solⁿ $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_D \operatorname{div}\vec{F}\, dx\, dy\, dz$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 1+1+1 = 3$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = 3\iiint_D dx\, dy\, dz$$

$$= 3 \times \text{volume of sphere}$$

$$= 3 \times \frac{4}{3}\pi = 4\pi$$

Que Find the Surface Integral of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ Over the $S = \{(x,y,z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$

Solⁿ $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_D \operatorname{div}\vec{F}\, dx\, dy\, dz$

$$= 3\iiint_D dx\, dy\, dz = 3 \times \text{volume of ellipsoid.}$$

$$= 3 \times \frac{4}{3}\pi abc = 4\pi ac$$

Que $\vec{F}: \mathbb{R}^3 \to \mathbb{R}$ twice differentiable scalar field s.t.
$\text{div}(\nabla F) = 6$
let S be a surface $\{(x,y,z) \mid x^2+y^2+z^2=1\}$
Then Find $\iint_S \nabla F \cdot \hat{n}\, ds$

$$= \iiint_D \text{div}(\nabla F)\, dx\,dy\,dz$$

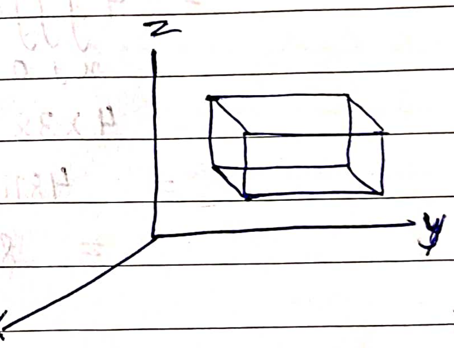
$$= 6 \iiint_D dx\,dy\,dz$$

$$= 6 \times \text{Volume of sphere}$$

$$= 6 \times \frac{4\pi r^3}{3} = 8\pi$$

Que→ Let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$. and S be a surface $\{(x,y,z) \mid 0 \le x \le 1,$ $0 \le y \le 2,\ 0 \le z \le 3\}$. Then find $\iint_S \vec{F} \cdot \hat{n}\, ds$

Solⁿ $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_D \text{div}\vec{F}\, dx\,dy\,dz$



$$= 3 \iiint_D dx\,dy\,dz$$

$$= 3 \times \text{Volume of Cuboid}$$

$$= 3 \times l \times b \times h$$

$$= 3 \times 1 \times 2 \times 3$$

$$= 18$$

Que→ $\vec{F} = x\hat{i} + y\hat{j} + 2z\hat{k}$
$S = \{(x-1)^2 + (y-2)^2 + (z-3)^2 = 1\}$ Find $\iint_S \vec{F} \cdot \hat{n}\, ds$

Solⁿ $\frac{16\pi}{3}$.

Que $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$; $S = \{(x,y,z) \mid x^2 + y^2 = 4 ;\ z = 0, z = 3\}$
Find $\iint_S \vec{F} \cdot \hat{n}\, ds$.

Sol^n

$$\text{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 4 - 4y + 2z$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \text{div}\vec{F}\,dx\,dy\,dz$$

$$= \iiint_D (4-4y+2z)\,dx\,dy\,dz$$

$$= \int_0^3\int_{-4}^{4}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-4y+2z)\,dy\,dx\,dz$$

$$= \iiint 4\,dy\,dx\,dz - \underbrace{\iiint 4y\,dy\,dx\,dz}_{0} + \iiint 2z\,dy\,dx\,dz$$

$$= 4\iint_{x\,y}\int_0^3 dz\,dy\,dx + \iint_{x\,y}\left[z^2\right]_0^3 dy\,dz$$

$$= 4\times 3\times \pi(2)^2 + 9\times\pi\times(2)^2$$

$$= 48\pi + 36\pi$$

$$= 84\pi \quad \text{Ans}$$

H.W.

Que Let D be the region bdd by the closed surface $x^2+y^2=16$, $z=0$ and $z=4$. Find $\iint_S \vec{F}\cdot\hat{n}\,ds$ when

$$\vec{F} = 3x^2\hat{i} + 6y^2\hat{j} + z\hat{k}$$

Ans 64π.

Que Find $\iint_S \left[\left(\frac{2x}{\pi} + \sin y^2\right)x + \left(e^z - \frac{y}{\pi}\right)y + \left(\frac{2z}{\pi} + \sin^2 y\right)z\right] ds$

where $S = \{(x,y,z) \mid x^2+y^2+z^2=1\}$.

Sol^n

$$\vec{\nabla F} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\vec{\nabla F}}{|\vec{\nabla F}|} = x\hat{i} + y\hat{j} + z\hat{k}$$

Date : / /201

$$\Rightarrow \vec{F} = \left(\frac{2x}{\pi} + \sin y^2\right)\hat{i} + \left(e^z - \frac{y}{\pi}\right)\hat{j} + \left(\frac{2z}{\pi} + \sin^2 y\right)\hat{k}.$$

$$\operatorname{div}\vec{F} = \frac{2}{\pi} - \frac{1}{\pi} + \frac{2}{\pi} = \frac{3}{\pi}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$= \frac{3}{\pi}\iiint_D dx\,dy\,dz$$

$$= \frac{\cancel{3}}{\cancel{\pi}} \times \frac{4}{\cancel{3}}\cancel{\pi} \times (1)^3 = 4 \text{ Ans}$$

Que $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}} = \dfrac{x\hat{i}}{(x^2+y^2+z^2)^{3/2}} + \dfrac{y\hat{j}}{(x^2+y^2+z^2)^{3/2}} + \dfrac{z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$S = \{1 \le x^2+y^2+z^2 \le 4\}$

Find $\iint_S \vec{F}\cdot\hat{n}\,ds$

Sol^n

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D \operatorname{div}\vec{F}\,dx\,dy\,dz$$

$$\operatorname{div}\vec{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\frac{\partial f_1}{\partial x} = \frac{(x^2+y^2+z^2)^{3/2}(1) - x\left(\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{1/2}(\cancel{2}x)\right)}{(x^2+y^2+z^2)^3}$$

$$= \frac{(x^2+y^2+z^2)^{1/2}\left[(x^2+y^2+z^2) - 3x^2\right]}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial f_1}{\partial x} = \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

Sim. $\dfrac{\partial f_2}{\partial y} = \dfrac{x^2+z^2-2y^2}{(x^2+y^2+z^2)^{5/2}}$ & $\dfrac{\partial f_3}{\partial z} = \dfrac{x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}}$

$$\Rightarrow \operatorname{div}\vec{F} = 0 \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = 0$$