Que S.T. $\int_C (yz-1)\,dx + (z+xz+z^2)\,dy + (y+xy+2yz)\,dz$ is Independent of the path of Integration from (1,2,2) to (2,3,4). Evaluate the Integral.

Solⁿ

$f(x,y,z) = yz-1$; $g(x,y,z) = z+xz+z^2$; $h(x,y,z) = y+xy+2yz$.

$f_y = z$; $f_z = y$

$g_x = z$; $g_z = 1+2z+x$

$h_x = y$; $h_y = 1+2z+x$

$f_y = g_x$; $f_z = h_x$; $g_z = h_y$

$\Rightarrow$ Integral is path Independent.

$\Rightarrow \exists\ \phi(x,y,z)$ s.t.

$d\phi = f\,dx + g\,dy + h\,dz$

But $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$

$\Rightarrow \frac{\partial \phi}{\partial x} = yz-1;\quad \frac{\partial \phi}{\partial y} = z+xz+z^2;\quad \frac{\partial \phi}{\partial z} = y+xy+2yz.$

$\phi(x,y,z) = xyz - x + h(y,z)$

$\frac{\partial \phi}{\partial y} = xz + \frac{\partial h}{\partial y}$

$\Rightarrow z + \cancel{xz} + z^2 = \cancel{xz} + \frac{\partial h}{\partial y}$

$\Rightarrow z+z^2 = \frac{\partial h}{\partial y}$

$\Rightarrow h(y,z) = zy + z^2y + s(z)$

$\therefore \phi(x,y,z) = xyz - x + yz + yz^2 + s(z)$

$\frac{\partial \phi}{\partial z} = xy + y + 2yz + s'(z)$

$\Rightarrow y + xy + 2yz = xy + y + 2yz + s'(z)$

$\Rightarrow s'(z) = 0 \Rightarrow s(z) = c$

$\Rightarrow \phi(x,y,z) = xyz - x + yz + yz^2 + C.$

$\therefore$ The value of Integral is

$$\int_C d\phi(x,y,z) = \int_{(1,2,2)}^{(2,3,4)} d(xyz - x + yz + yz^2)$$

$$= \left(xyz - x + yz + yz^2\right)_{(1,2,2)}^{(2,3,4)} = \underline{67}.$$

Que Find the value of $\int_C \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$ from $P(-1,2)$ to $Q(2,3)$

Ans: $-\sqrt{13}-\sqrt{5}$.

Que Show that $\int_P^Q (3x^2+2xyz)dx + (1+x^2z)dy + x^2y\,dz$; is path Independent from $P(1,1,1)$ and $Q(-2,-3,-4)$ and Evaluate the Integral.

Ans 34

Que S.T. $\int_P^Q (2xz+y)\,dx + (x+z)dy + (x^2+y)dz$ is path Independent and Evaluate the Integral where $P:(-1,2,3)$ and $Q:(3,3,4)$

Ans: 21

→ Let $F = fi + gj + hk$

If $\int_C F \cdot dr = \int_C f dx + g dy + h dz$

is path Independent then F is called ~~path~~ Conservative vector field.

Que Find whether $F = (2x + ye^{xy})i + (2y + xe^{xy})j$ is Conservative vector field. If it is, then find the potential function.

Sol^n $F = (2x + ye^{xy})i + (2y + xe^{xy})j$

$$\int F \cdot dr = \int (2x + ye^{xy})dx + (2y + xe^{xy})dy$$

$$\frac{\partial M}{\partial y} = yxe^{xy} + e^{xy}$$

$$\frac{\partial N}{\partial x} = xye^{xy} + e^{xy}$$

⇒ Eq^n is Exact.

$\int M dx + \int$ Terms In N not con. $x\,dy = c$

$$\Rightarrow \int 2x\,dx + \int ye^{xy}dx + \int 2y\,dy = c$$

$\Rightarrow x^2 + \frac{ye^{xy}}{y} + y^2 = c$

$\Rightarrow x^2 + e^{xy} + y^2 = c$

$\therefore \int F \cdot dr = \int d(x^2 + y^2 + e^{xy})$

$= x^2 + y^2 + e^{xy}$.

Que. $F = \cos(x+y)(i+j) = \cos(x+y)i + \cos(x+y)j$

Is F Conservative vector field? If Yes then find the potential function.

Ans: $\sin(x+y)$

Que. $F = yzi + xzj + xyk$

Is F Conservative vector field. If Yes then find the potential fun.

Sol^n $\int F \cdot dr = \int yz\,dx + xz\,dy + xy\,dz$

$f = yz;\ g = xz,\ h = xy$

$f_y = z;\ g_x = z$

$f_z = y;\ h_x = y$

$g_z = x;\ h_y = x$

$\Rightarrow$ Eq^n is Exact

$\Rightarrow$ F is Conservative vector field

$\Rightarrow \exists\ \phi(x,y,z)$ S.t

$\frac{\partial \phi}{\partial x} = yz;\ \frac{\partial \phi}{\partial y} = xz;\ \frac{\partial \phi}{\partial z} = xy$

$\phi(x,y,z) = xyz + h(y,z)$

$\frac{\partial \phi}{\partial y} = xz + \frac{\partial h}{\partial y}$

$\Rightarrow xz = xz + \frac{\partial h}{\partial y}$

$\Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h(y,z) = s(z)$

$\Rightarrow \phi(x,y,z) = xyz + s(z)$

$\frac{\partial \phi}{\partial z} = xy + s'(z)$

$\Rightarrow xy = xy + s'(z)$

$\Rightarrow s'(z) = 0$

$\Rightarrow s(z) = C$

$\Rightarrow \phi(x,y,z) = xyz + c$

$\therefore \int F\cdot dr = \int d(xyz) = xyz$

Que $F = 2xy\,i + (x^2+2yz)j + y^2k$

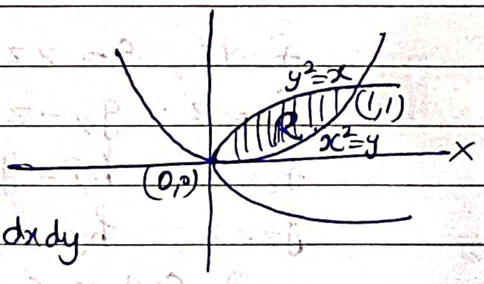
Is F Conservative? If yes then find the potential function.

Ans- $x^2y + y^2z$.

★ Green's Thm:→ Let C be a smooth closed curve bounding a region R. If $f, g, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}$ are continuous on R,

then $\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.$

Que Evaluate $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$ where C is the boundary of the region in the first quadrant that is bdd by the curves $y^2=x$ and $x^2=y$.



Sol^n $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy =$

$\iint_R \left(\frac{\partial}{\partial x}(y+2x) - \frac{\partial}{\partial y}(x^2+y^2)\right) dx\,dy$

$= \iint_R (2-2y)\,dx\,dy$

$\int_0^1\int_{y^2}^{\sqrt y}(2-2y)\,dx\,dy$ or $\int_0^1\int_{x^2}^{\sqrt x}(2-2y)\,dy\,dx$

$= \int_0^1 \left(2x-2xy\right)_{y^2}^{\sqrt y} dy$

$= \int_0^1 \left(2\sqrt y - 2y\sqrt y - 2y^2 + 2y^3\right) dy$

$= \boxed{\frac{11}{30}}$

Que $\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx - x\,dy$ where C is $\{(x,y) \mid x^2+y^2=1\}$ bounding a region R.

Evaluate the Integral.

Sol^n $\int_C y\,dx - x\,dy = \iint_R (-1-1)\,dx\,dy$

$= -2\iint_R dx\,dy = -2\cdot$ area of circle

$= -2\times\pi(1)^2$

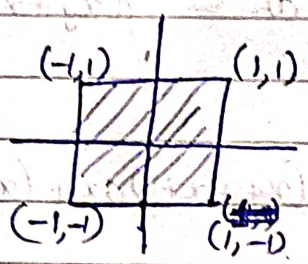
$= -2\pi.$

Que $\vec{F} = \sin x\,\hat{i} + e^{x}\hat{j}$

$C = \{(x,y) \mid x=\pm 1, y=\pm 1\} \rightarrow$ Rectangle.

Evaluate the line Integral.

Sol^n $\int_C \vec{F}\cdot d\vec{r} = \int_C \sin x\,dx + e^{y}dy$

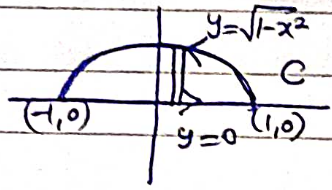


$= \iint_R \frac{\partial}{\partial y}(\sin x) - \frac{\partial}{\partial x}(e^y)$

$= 0$

Que $\vec{F} = (x^2-xy^2)\hat{i} + y^2\hat{j}$

C is the closed Curve bounded by Semi circle and x-axis. Find the Line Integral.



Sol^n $\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2-xy^2)\,dx + y^2 dy$

$= \iint_R \left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2-xy^2)\right) dx\,dy$

$= \iint_R 2xy\,dx\,dy$

$= 2\iint_R xy\,dx\,dy$

$= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (xy)\,dy\,dx$

$= 2\int_{-1}^{1} \frac{x(1-x^2)}{2}\,dx$

$= \int_{-1}^{1} x - x^3\,dx$

$= 0$

$\int \vec{F} \cdot \vec{dr} =$

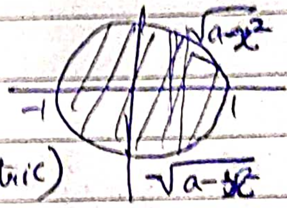
Que $\int (\cos x \sin y - xy)\,dx + (\sin x \cos y)\,dy$ and $C = \{(x,y) \mid x^2+y^2=a^2\}$ is the Curve bounding a region R. Then Find the Integral.

Sol^n $\int (\cos x \sin y - xy)\,dx + (\sin x \cos y)\,dy$

$$= \iint_R \big((\cos x \cos y) - \cos x \cos y + x\big)\,dx\,dy$$

$$= \iint_R x\,dx\,dy$$

$$= 0 \quad (\because \text{Curve } C \text{ is Symmetric})$$



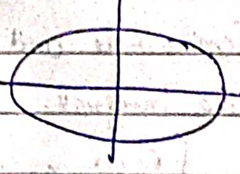
Que $\int \vec{F} \cdot \vec{dr} = \int (xy+x+y)\,dx + (xy+x-y)\,dy$ over the Region $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Sol^n $\int (xy+x+y)\,dx + (xy+x-y)\,dy = \iint_R \big((y+1)-(x+1)\big)\,dx\,dy$

$$= \iint_R (y-x)\,dx\,dy$$

$$= \iint_R y\,dx\,dy - \iint_R x\,dx\,dy$$

$$= 0 \quad \because$$



Que Find $\frac{1}{\pi} \oint_C \left(3y - e^{\cos x^2}\right)dx + \left(7x + \sqrt{y^4+11}\right)dy$

over positive oriented circle $x^2+y^2=9$.

Sol^n $\frac{1}{\pi} \oint_C \left(3y - e^{\cos x^2}\right)dx + \left(7x + \sqrt{y^4+11}\right)dy$

$$= \frac{1}{\pi} \iint_R (7-3)\,dx\,dy$$

$$= \frac{4}{\pi} \iint_R dx\,dy = \frac{4}{\pi} \times \pi (3)^2 = \boxed{36}$$

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Que: $\oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$; over the curve $C: x^2+y^2=4$.

$= \oint \frac{x\,dy}{x^2+y^2} - \frac{y}{x^2+y^2}\,dx$ → This is Exact Equation.

But ~~f~~ f and g are not continuous functions at (0,0).

$= \cancel{\iint}$

So we cannot apply green's Thm.

We'll use parametric Equations

$x = 2\cos t, \; y = 2\sin t; \; 0 \le t \le 2\pi$

$$\int_0^{2\pi} \frac{2\cos t\,(2\cos t)\,dt - (2\sin t)(-2\sin t)\,dt}{4}$$

$$= \int_0^{2\pi} \frac{4\cos^2 t + 4\sin^2 t}{4}\,dt = 2\pi.$$

Que $\oint_C \frac{y\,dx - x\,dy}{x^2+y^2}$ over the curve $C = \{(x,y) \mid (x-9)^2 + (y-8)^2 = 1\}$

⇓ Exact Equation.

f & g are continuous everywhere excep (0,0).

and (0,0) is excluded from the Region now.

So $\oint_C \frac{y\,dx - x\,dy}{x^2+y^2} = 0.$

Que $\oint_C \frac{x\,dy - y\,dx}{x^2+4y^2}$; $C: \{(x,y) \mid x^2+y^2=4\}$

$$= \iint_R \frac{\partial}{\partial x}\left(\frac{x}{x^2+4y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+4y^2}\right) dx\,dy$$

$$= \iint_R \left[\frac{(x^2+4y^2) - x(2x)}{(x^2+4y^2)^2} + \frac{(x^2+4y^2)(1) - y(8y)}{(x^2+4y^2)^2}\right] dx\,dy$$

$$= \iint_R \left(\frac{4y^2 - x^2}{(x^2+4y^2)^2} + \cancel{4y^2}\,\frac{x^2 - 4y^2}{(x^2+4y^2)^2}\right) dx\,dy$$

⇓

Exact but f & g are not cts at (0,0).

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So use $x = 2\cos t,\ y = 2\sin t$

$$\oint_C \frac{x\,dy - y\,dx}{x^2+4y^2} = \int_0^{2\pi} \frac{(2\cos t)(2\cos t)\,dt - (2\sin t)(-2\sin t)\,dt}{4\cos^2 t + 4(4\sin^2 t)}$$

$$= \int_0^{2\pi} \frac{4\,dt}{4\cos^2 t + 16\sin^2 t}$$

$$= \int_0^{2\pi} \frac{dt}{\cos^2 t + 4\sin^2 t}$$

$$= \int_0^{2\pi} \frac{\sec^2 t\,dt}{1+4\tan^2 t}$$

$\tan t = z$

$\sec^2 t\,dt = dz$

$$= 4\int_0^{\pi/2} \frac{\sec^2 t\,dt}{1+4\tan^2 t} = 4\int_0^{\infty} \frac{dz}{1+4z^2} \rightarrow \text{Solve } \boxed{\text{Ans} = \pi}$$