

UNIT-2 Part-I (Waves & Oscillations)

Wave!- A wave motion is a disturbance of some kind which moves from one place to another by means of a medium.

Oscillation!- oscillation is an effect expressible as a quantity that repeatedly & regularly fluctuate above and below some mean position.

I) Transverse wave!- Wave motion in which particle of medium vibrate about their mean position at right angle to the direction of propagation.

Eg- Light Waves.

II) Longitudinal waves!- Wave motion in which wave vibrate about their mean position along the same line as propagation of wave.

Eg. Sound waves.

Simple Harmonic Motion!- If the acceleration of a particle in a periodic motion is always directly proportional to its displacement from its equilibrium position.

Types of SHM!-

I) Linear Harmonic Simple Harmonic Motion!-

If the displacement of a particle executing SHM is linear is said to be linear simple harmonic motion.
Example- Simple Pendulum.

Angular Simple Harmonic Motion:-

If the Displacement of a Particle Executing SHM is angular. Example - compound Pendulum.

Essential Conditions for SHM:-

1) If f be the linear acceleration and x be the displacement from Equilibrium Position.

The essential condition is -

$$[f \propto -x]$$

2) If α be the angular Momentum & θ be the angular displacement then essential condition is -

$$[\alpha = -\theta]$$

Time Period:- The smallest time interval during which oscillation repeat itself is called Time Period, denoted by T . Its unit is seconds.

Frequency:- Number of oscillation that a body complete in one second is called frequency of Periodic Motion.

It is reciprocal of Time Period T and is given by -

$$\boxed{n = \frac{1}{T}}$$

Unit - hertz represented by Hz.

Ampitude:- Maximum displacement of a body from its Mean position.

Phase \Rightarrow It is a physical quantity that express the instantaneous position and direction of motion of an oscillating system.

Differential Equation of SHM & its solution:-

Let us consider a particle of mass m executing SHM along a straight line with x as displacement from the mean position at any time t . Then from the basic condition of SHM restoring force F will be proportional to displacement and will be directed opposite to it.

Therefore

$$F \propto -x$$

$$\boxed{F = -kx} \quad \text{--- (1)}$$

k is proportionality constant known as force constant

If $a = \frac{d^2x}{dt^2}$ be the acceleration at any instant of time

$$\text{From (1)} \quad F = -kx$$

$$ma = -kx$$

$$m \frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m} x \quad \text{--- (2)}$$

$$\text{Substituting } \frac{k}{m} = \omega^2$$

$$\frac{d^2x}{dt^2} = -\omega^2 x \Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \text{--- (2)}$$

Equation (1) & (2) are known as differential Equation of SHM.

Solution of Differential Equation:-

$$x = A \sin(\omega t + \phi)$$

This Equation gives the displacement of Particle Executing SHM at any instant of time.

$A \rightarrow$ Maximum Displacement of Particle

$$x = A \sin(\omega t + \phi) \quad \text{---(1)}$$

Replacing t by $(t + \frac{2\pi}{\omega})$, then we have -

$$x = A \sin\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right)$$

$$= A \sin(\omega t + \phi) \quad \text{---(2)}$$

① & ② are same this shows that motion is repeated after an interval of $\frac{2\pi}{\omega}$. So this interval will be

Time Period of SHM given by $T = \frac{2\pi}{\omega}$

Also we have -

$$\text{Time Period: } \omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

$$\text{Frequency } n = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

Phase:- The quantity $(\omega t + \phi)$ is known as phase of vibrating particle. If $t=0$ then $(\omega t + \phi) = \phi$, so that initial phase will be ϕ .

If the particle start from mean position, then $\phi=0$ for a particle start from extreme position they

$$\boxed{\phi = \pi/2}$$

Velocity & Acceleration - For a Particle Executing SHM.
Expression for Displacement is -

$$x = A \sin(\omega t + \phi)$$

Differentiating it wrt time we get -

$$v = \frac{dx}{dt} = A\omega \cos(\omega t + \phi) \quad \text{--- (1)}$$

$$= A\omega \sqrt{1 - \sin^2(\omega t + \phi)}$$

$$= \omega \sqrt{A^2 - A^2 \sin^2(\omega t + \phi)}$$

$$v = \omega \sqrt{A^2 - x^2}$$

This is the Expression for Velocity of Particle at any displacement x .

I) Maximum velocity is obtained by substituting

$$\begin{aligned} x &= 0 \\ \therefore [v_{\max}] &= \omega A \end{aligned}$$

because $x=0$ corresponds to mean position, so the particle has maximum velocity when it is at mean position.

II) At extreme position

At Extreme position $x=A$

on differentiating (1) wrt time we get -

$$f = \frac{dv}{dt} = \frac{d}{dt}(A \cos(\omega t + \phi))$$

$$= -A\omega^2 \sin(\omega t + \phi)$$

$$[f = -\omega^2 x] \quad (\text{using } x = A \sin(\omega t + \phi))$$

This is Standard Equation of SHM

$$[f_{\max} = \omega^2 A]$$

$$[f_{\min} = 0] \text{ at Mean Position}$$

Del operator

∇ operator is denoted by ∇ and is treated like a vector in Cartesian coordinates.

It is written as

$$\vec{\nabla} = \hat{i} \frac{d}{dx} + \hat{j} \frac{d}{dy} + \hat{k} \frac{d}{dz}$$

Gradient of Scalar field :-

Let $\phi(r)$ be a scalar field \vec{r} is the position of vector ($\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$)

where (x, y, z) are the coordinates.

ϕ be a function of three coordinates.

then

$$\text{grad } \phi = \nabla \cdot \phi$$

$$\text{or grad } \phi = \left(\hat{i} \frac{d}{dx} + \hat{j} \frac{d}{dy} + \hat{k} \frac{d}{dz} \right) \phi$$

$$\text{grad } \phi = \left(\hat{i} \frac{d\phi}{dx} + \hat{j} \frac{d\phi}{dy} + \hat{k} \frac{d\phi}{dz} \right)$$

Curl of a vector

Circulation of a closed field around a closed path is given by curl of a vector

Mathematically

$$\text{curl of } \vec{F} = \nabla \times \vec{F}$$

i.e cross product between $\nabla \times \vec{F}$

If $\nabla \times \vec{F} = 0$ then field is irrotational.

Numericals Problem

Q1 If $\sigma_{xx} = x^2 + y^2 + z^2$ find $\nabla \cdot \sigma$

$$\begin{aligned}\nabla \cdot \sigma_{xx} &= \left(i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz} \right) (x^2 + y^2 + z^2) \\ &= \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} \right) \\ &= 1 + 1 + 1 = 3\end{aligned}$$

Q2 If $\phi = 4x^3y^2z^4$ find $\nabla \phi$

$$\begin{aligned}\nabla \phi &= \left(i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz} \right) (4x^3y^2z^4) \\ &= \left(i \frac{\partial}{\partial x} (4x^3y^2z^4) + j \left(\frac{\partial}{\partial y} (4x^3y^2z^4) \right) \right. \\ &\quad \left. + k \frac{\partial}{\partial z} (4x^3y^2z^4) \right) \\ &= 12x^2y^2z^4 i + 8x^3y^2z^4 j + 16x^3y^2z^3 k\end{aligned}$$

Q3 If $A = 2xy + 2y + z^2$ find $\nabla \cdot A$

$$\begin{aligned}\nabla \cdot A &= \left(i \frac{d}{dx} + j \frac{d}{dy} + k \frac{d}{dz} \right) (2xy + 2y + z^2) \\ &= i \frac{d}{dx} (2xy + 2y + z^2) + j \left(\frac{d}{dy} (2xy + 2y + z^2) \right) \\ &\quad + k \left(\frac{d}{dz} (2xy + 2y + z^2) \right) \\ &= i(2y) + j(2x + 2) + k(2z)\end{aligned}$$

Q Find the curl of a field $F_1 = x\hat{i} + y\hat{j} + z\hat{k}$

prove that it is rotational or irrotational.

Sol: curl of a field $F_1 = \nabla \times F_1$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

where $\nabla = \hat{i} \frac{d}{dx} + \hat{j} \frac{d}{dy} + \hat{k} \frac{d}{dz}$

$$= \hat{i} \left(\frac{d^2}{dy^2} - \frac{dy}{dz} \right) - \hat{j} \left(\frac{dz}{dx} - \frac{dx}{dz} \right) + \hat{k} \left(\frac{dy}{dz} - \frac{dx}{dy} \right)$$

$$\Rightarrow \nabla \times F_1 \neq 0$$

so field is ~~irrotational~~.

Gauss Divergence Theorem:-

It States that, the volume integral of the divergence of vector field A taken over any volume V bounded by a closed surface S is equal to surface integral of A taken over the surface S.

Mathematically

$$\iiint_V \operatorname{div} A \, dV = \iint_S A \cdot dS$$

Stokes Theorem:-

It State that the surface integral of a curl of a vector field A taken over any surface S is equal to the line integral of A around a closed curve.

$$\iint_S (\operatorname{curl} A) \, dS = \oint A \cdot d\ell$$

OR

$$\iint_S (\nabla \times A) \, dS = \oint A \cdot d\ell.$$

Physical significance & derivation of Maxwell's Equations

To derive Maxwell Eqn we need to study following terms -

I) Gauss law of Electrostatics:-

Gauss law states that Electric flux through any closed surface is Equal to net charge Enclosed by the surface divided by Permittivity of vacuum.

$$\phi = \frac{q}{\epsilon_0} \text{ or } q = \epsilon_0 \phi$$

Gauss law of Electrostatics (Integrated form)

The number of lines of force passing through a small area element dS is given by.

$$d\phi = E \cdot dS = EdS \cos\theta$$

This is known as Electric flux of the field over the elementary surface dS .

For a closed Surface Flux of field is given by -

$$\phi = \oint E \cdot dS$$

In Integrated form

Total outward flux of Electric field over a closed surface is equal to $\frac{1}{\epsilon_0}$ times the total Net charge contained in a volume enclosed by the surface.

$$\boxed{\oint E \cdot dS = q/\epsilon_0}$$

E - Electric Field Intensity dS - Surface Element.

Gauss Law of Electrostatics! (Differential Form)

Let E be the electric field at the centre of an elementary area dS on the closed surface S . Let ρ be the volume charge density of volume enclosed by surface S . ρdV may be considered as a point charge contributing electric flux to the elementary area dS .

$$\Rightarrow q = \rho dV$$

Then, Total Flux of whole Surface -

$$\oint_S E \cdot dS = \frac{1}{\epsilon_0} \int \rho dV \quad \text{---(1)}$$

According to divergence theorem we have -

$$\oint_S E \cdot dS = \oint_V (\nabla \cdot E) dV \quad \text{---(2)}$$

Equating (1), (2) we have

$$\oint_V (\nabla \cdot E) dV = \frac{1}{\epsilon_0} \int \rho dV$$

$$\oint_V (\nabla \cdot E) dV = \int_V \frac{\rho}{\epsilon_0} dV \quad \text{---(3)}$$

Since dV is arbitrary volume Element (3) is valid for any volume

Hence

$$\boxed{\nabla \cdot E = \frac{\rho}{\epsilon_0}}$$

OR

$$\operatorname{div} E = \frac{\rho}{\epsilon_0}$$

Gauss Law in Magneto Statics:-

I) Differential Form:-

Since in magnetic field lines are continuous the magnetic field entering any region is equal to magnetic flux leaving it. so net flux over a volume is zero

Mathematically

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

This is Differential Form of gauss law.

II) Integral Form:-

Net Flux over a closed surface is -

$$\phi = \oint \mathbf{B} \cdot d\mathbf{s}$$

Applying divergence theory we have -

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = \int_V (\nabla \cdot \mathbf{B}) dV = 0$$

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0}$$

This is integral form of gauss law.

Faraday Law!:- 1) magnetic flux through a closed loop of conducting wire varies in time with EMF.

$$\begin{aligned} \oint \mathbf{E} \cdot d\mathbf{l} &= -\frac{d\phi}{dt} = \cancel{\oint_S \frac{d(\mathbf{B} \cdot d\mathbf{s})}{dt}} \\ &= \oint_S \frac{d\mathbf{B} \cdot d\mathbf{s}}{dt} \end{aligned}$$

ii) Negative time rate of variation of magnetic flux linked with circuit is equal to EMF induced within it.

$$\nabla \times E = -\frac{dB}{dt}$$

Ampere's circuital law:-

This law state that line integral of magnetic field (B) around any closed path or loop is equal to μ_0 times the total current enclosed by the loop.

$$\int B \cdot dI = \mu_0 I$$

Maxwell Equation (Differential Form)

First

I) $\nabla \cdot D = P$ (Gauss law of Electrostatics)
OR

II) $\nabla \cdot E = P/\epsilon_0$ (using $D = \epsilon_0 E$)

Second

III) $\nabla \cdot B = 0$ (Gauss law of Magnetostatics)

Third

IV) $\nabla \times E = -\frac{d\mathbf{B}}{dt}$ (Faraday law)

Fourth

V) $\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$ (Modified Ampere's Circuital Law)
OR

$\nabla \times H = \left(J + \frac{dD}{dt} \right)$ (Using $B = \mu_0 H$)

Maxwell Equation (Integral Form)

First

I) $\oint E \cdot dS = \frac{q}{\epsilon_0}$

Second

II) $\oint B \cdot dS = 0$

Third

IV) $\oint E \cdot dl = -\frac{d\phi_B}{dt}$

Fourth

V) $\oint B \cdot dl = \mu_0 \left(J + \frac{dD}{dt} \right) \cdot dl$

where

\mathbf{D} - Displacement or Electric Displacement
(coulomb/m²)

ρ - Charge density (coulom/m³)

\mathbf{B}_r - Magnetic Induction (wb/m²)
or Flux density

\mathbf{H} - Magnetic Field Intensity (Amp/m)

\mathbf{J} - Current Density (I/A) (current/ Amperes)

First Maxwell Equation:-

$$\nabla \cdot E = P/\epsilon_0$$

Physical significance:-

- It is based on gauss law of Electrostatics
- Net Electric Flux through a closed surface is Equal to $\frac{1}{\epsilon_0}$ the total charge Enclosed by the surface.

$$\nabla \cdot E = \frac{q}{\epsilon_0}$$

(Differential Form)

$$\oint E \cdot dS = \frac{1}{\epsilon_0} \int q \cdot dv$$

(Integrated Form)

- It relate Electric Flux with charge.
- Charge acts a source or sink for the line of Electric force

Derivation! - Consider a Surface bounded by a Volume V in a medium having charge density as P .

then

$$q = \int P dV$$

Using $\textcircled{1}$ Gauss law of Electrostatics we have

$$\oint E \cdot dS = \frac{q}{\epsilon_0}$$

$$\text{Substituting } q = \int P dV$$

then

$$\oint E \cdot dS = \frac{1}{\epsilon_0} \int P dV \quad \text{---} \textcircled{2}$$

From Gauss theorem we have -

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \int_V (\nabla \cdot \mathbf{E}) dv \quad \text{---(3)}$$

(Note Gauss Divergence theorem converts Surface Integral to Volume Integral)

Equating (2), (3) we get.

$$\int_V (\nabla \cdot \mathbf{E}) dv = \frac{1}{\epsilon_0} \int_V \rho dv$$

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\epsilon_0} \right) dv = 0$$

$$\Rightarrow \boxed{\nabla \cdot \mathbf{E} = \rho / \epsilon_0}$$

↓
Maxwell (1) Equation.

Maxwell's Second Equation!

Maxwell's Second Equation:-

$$\nabla \cdot \mathbf{B} = 0$$

Significance:-

- It is based on Gauss law of MagnetoStatics.
- It States that Magnetic flux through any closed surface is zero

Differential form

$$\nabla \cdot \mathbf{B} = 0$$

Integrated form

$$\oint \mathbf{B} \cdot d\mathbf{s} = 0$$

- Time independent Equation
- Magnetic Flux is zero
- According to this Equation, isolated magnetic poles don't exist.

Derivation:- we have by Gauss law of MagnetoStatics

$$\oint \mathbf{B} \cdot d\mathbf{s} = 0 \quad \text{---(1)}$$

By Gauss divergence theorem-we have,

$$\oint \mathbf{B} \cdot d\mathbf{s} = \int_V (\nabla \cdot \mathbf{B}) dv \quad \text{---(2)}$$

(Note:- By Gauss divergence theorem we have
Surface Integrated Equal to volume Integated)

Equating (1), (2)

$$\int_V (\nabla \cdot \mathbf{B}) dv = 0$$

$$\boxed{\nabla \cdot \mathbf{B} = 0} \rightarrow \text{Maxwell Second Equation}$$

Maxwell's Third Equation:-

$$\nabla \times E = -\frac{dB}{dt}$$

Significance:-

- This Equation represent Faraday law of Electro-Magnetic Induction.

Differential form

$$\nabla \times E = -\frac{dB}{dt}$$

Integral form

$$\oint E \cdot d\ell = -\frac{d\phi}{dt}$$

- It is time dependent Equation.
- Relates Space variation of E with variation of B
- Time variation of Magnetic field generates Electric Field.
- Negative sign Justify Lenz law.

Derivation:- we have by Faraday law.

$$\begin{aligned} \oint_{\text{c}} E \cdot d\ell &= -\frac{d\phi}{dt} \\ &= -\frac{d(B \cdot ds)}{dt} \quad (\phi = B \cdot ds) \end{aligned} \tag{①}$$

By Stokes theorem we have.

$$\oint_{\text{c}} E \cdot d\ell = \int_S (\nabla \times E) \cdot ds \tag{②}$$

$$\int_S (\nabla \times E) \cdot ds = -\frac{d}{dt} \int_S B \cdot ds$$

$$\int_S (\nabla \times E) \cdot ds = -\frac{d}{dt} \phi - \int_S \frac{dB}{dt} \cdot ds$$

$$\Rightarrow \boxed{\nabla \times E = -\frac{dB}{dt}} \quad \text{— Maxwell Third Law}$$

Maxwell fourth Equation

$$\text{curl } H = J + \frac{dD}{dt}$$

Significance

- It represents modified form of Amper's law.
- It is time dependent Equation
- It shows that Magnetic field can be generated by current density vector and time variation

Differential form

$$\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

Integrated Form

$$\oint B \cdot dl = \mu_0 \int \left(J + \frac{dD}{dt} \right) dl$$

Derivation:- By Amper's law we have.

$$\oint B \cdot dl = \mu_0 I$$

$$\oint (\mu_0 H) dl = \mu_0 I$$

$$\oint H \cdot dl = I$$

$$I = \int_S J \cdot dS$$

$$\Rightarrow \oint H \cdot dl = \int_S J \cdot dS \quad -①$$

By Stokes theorem we have.

$$\oint_C H \cdot dl = \int_S (\nabla \times H) \cdot dS \quad -②$$

Equating ①, ②

$$\int_S (\nabla \times H) \cdot dS = \int_S J \cdot dS$$

$$\int_S (\nabla \times H - J) \cdot dS = 0$$

$$\Rightarrow \nabla \times H - J = 0 \Rightarrow \boxed{\text{curl } H = J} \quad -③$$

using the equation of time varying field.

$$\operatorname{div} \mathbf{J} + \frac{dP}{dt} = 0$$

$$\operatorname{div} \mathbf{J} = -\frac{dP}{dt} \quad \text{--- (4)}$$

Maxwell added a current density \mathbf{J}_D to original current density \mathbf{J} i.e

$$\mathbf{C} = \mathbf{J} + \mathbf{J}_D$$

using (3) $\operatorname{curl} \mathbf{H} = \mathbf{J} + \mathbf{J}_D \quad \text{--- (5)}$

$$\operatorname{div} (\operatorname{curl} \mathbf{H}) = \operatorname{div} (\mathbf{J} + \mathbf{J}_D)$$

$$\operatorname{div} (\mathbf{J} + \mathbf{J}_D) = 0$$

($\operatorname{div} (\operatorname{curl} \mathbf{H}) = 0$ for time varying field.)

$$\Rightarrow \operatorname{div} \mathbf{J} + \operatorname{div} \mathbf{J}_D = 0$$

$$\operatorname{div} \mathbf{J}_D = -\operatorname{div} \mathbf{J}$$

using (4) $\operatorname{div} \mathbf{J}_D = +\frac{dP}{dt} \quad \text{--- (6)}$

But $\boxed{\operatorname{div} \mathbf{D} = P}$ by Maxwell's 1st eqn

$$\operatorname{div} \mathbf{J}_D = \frac{d}{dt} \operatorname{div} \mathbf{D}$$

$$= \frac{d}{dt} \operatorname{div} \left(\frac{d\mathbf{D}}{dt} \right)$$

$$\boxed{\mathbf{J}_D = \frac{d\mathbf{D}}{dt}}$$

Using (6) $\operatorname{curl} \mathbf{H} = \mathbf{J} + \mathbf{J}_D$

$$\boxed{\operatorname{curl} \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}}$$

Maxwell's (4) Equation

Conversion of Maxwell Equation from Differential to Integral Form!

① First Equation:-

$$\operatorname{div} \vec{E} = P/\epsilon_0 \quad \rightarrow 0$$

Integrating above Equation over the Volume V then-

$$\int_V \operatorname{div} \vec{E} dV = \int_V (P/\epsilon_0) dV$$

Applying divergence theorem $\oint_S \vec{E} \cdot d\vec{S} = \int_V \operatorname{div} \vec{E} dV$

Using ① $\oint_S \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_V P dV$

$$\boxed{\oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\epsilon_0}}$$

which is simple gauss law.

② Second Equation.

$$\operatorname{div} \vec{B} = 0 \quad \text{or} \quad \nabla \cdot \vec{B} = 0$$

Taking integral both sides.

$$\oint_S \vec{B} \cdot d\vec{S} = 0$$

i.e Magnetic flux through a closed Surface is always zero.

This implies absence of magnetic monopoles.

③ Third Equation we have -

$$\nabla \times E = -\frac{dB}{dt}$$

Taking Integral both sides.

$$\oint_S (\nabla \times E) \cdot dS = - \oint_S \frac{dB}{dt} \cdot dS \quad \text{--- (1)}$$

Applying Stokes theorem.

$$\int_C E \cdot dl = \oint_S (\nabla \times E) \cdot dS \quad \text{--- (2)}$$

Substituting in (1)

$$\begin{aligned} \oint C E \cdot dl &= - \oint_S \frac{dB}{dt} \cdot dS \\ &= - \frac{d}{dt} \int S B \cdot S = - \frac{d\phi_B}{dt} \end{aligned}$$

④ Fourth Equation:

$$\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

On Integrating both sides.

$$\oint_S (\nabla \times B) \cdot dS = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) \cdot dS$$

Using Stokes theorem we get.

$$\oint_S (\nabla \times B) \cdot dS = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) \cdot dS$$

$$\boxed{\oint C B \cdot dl = \mu_0 \oint_S \left(J + \frac{dD}{dt} \right) \cdot dS}$$

Integrated form of Fourth Equation

Electromagnetic Wave Equations - (Part 3)

I) In free space

II) In dielectric medium

III) In conducting medium

IV) Propagation of Electromagnetic Wave Equation in Free Space

It can be derived by using Maxwell's Equation.

Maxwell's Equation in general form are given by.

$$I) \nabla \cdot E = P/G_0$$

$$II) \nabla \cdot B = 0$$

$$III) \nabla \times E = -\frac{dB}{dt}$$

$$IV) \nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$$

Now in vacuum (free space) we have.

$$D = \epsilon_0 E, \quad B = \mu_0 H$$

conductivity $\sigma = 0$ (ie vacuum is non-conducting).

No free electron are there $J = 0, P = 0$

Using these conditions in I we get.

$$I) \nabla \cdot E = 0$$

$$II) \nabla \cdot H = 0$$

$$III) \nabla \times E = -\frac{\partial D}{\partial t} - \mu_0 \frac{dH}{dt}$$

$$IV) \nabla \times B = \epsilon_0 \frac{dE}{dt}$$

} - A

} - B

Now

- i) Equation for Electric field vector -
Taking curl on both side of (iii) of \textcircled{B}

$$\nabla \times (\nabla \times E) = \nabla \times \left(-\mu_0 \frac{dH}{dt} \right)$$

$$= -\mu_0 \frac{d}{dt} (\nabla \times H)$$

$$\text{or } \nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu_0 \frac{d}{dt} (\nabla \times H)$$

$$(\text{using } \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}))$$

Using (i, iv) of \textcircled{B}

$$0 - \nabla^2 E = -\mu_0 \epsilon_0 \frac{d^2 E}{dt^2}$$

$$\Rightarrow \boxed{\nabla^2 E = \mu_0 \epsilon_0 \frac{d^2 E}{dt^2}} \quad \text{--- (1)}$$

This is general Equation of Electromagnetic waves
in term of Electric field vector E .

- ii) Equation for magnetic field vector.

Taking curl on both side of Equation(iv) of \textcircled{B}

we have - $\nabla \times (\nabla \times H) = \epsilon_0 \left(\nabla \times \frac{dE}{dt} \right)$

$$\Rightarrow \nabla \times (\nabla \times H) = \epsilon_0 \left(\frac{d}{dt} (\nabla \times E) \right)$$

$$\text{or } \nabla \cdot (\nabla \cdot H) - \nabla^2 H = \epsilon_0 \left(\frac{d}{dt} (\nabla \times E) \right)$$

Using II, III of \textcircled{B}

$$0 - \nabla^2 H = -\mu_0 \epsilon_0 \frac{d^2 H}{dt^2}$$

$$\Rightarrow \boxed{\nabla^2 H = \mu_0 \epsilon_0 \frac{d^2 H}{dt^2}} \quad \text{--- (2)}$$

This is Electromagnetic wave Equation in term of
magnetic field.

Wave velocity :- The Electromagnetic wave Equations for E & H (Electric & magnetic vectors) are written as -

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{d^2 E}{dt^2} = 0$$

$$\nabla^2 H - \mu_0 \epsilon_0 \frac{d^2 H}{dt^2} = 0$$

These Equations can be written by using k -coordinates.

$$\frac{d^2 E_k}{dx^2} - \mu_0 \epsilon_0 \frac{d^2 E_k}{dt^2} = 0 \quad \text{--- (3)}$$

$$\frac{d^2 H_k}{dx^2} - \mu_0 \epsilon_0 \frac{d^2 H_k}{dt^2} = 0 \quad \text{--- (4)}$$

These Equations are of type -

$$\frac{d^2 y}{dx^2} - \frac{1}{c^2} \frac{d^2 y}{dt^2} = 0 \quad \text{--- (5)}$$

Equation (5) is general Equation of wave travelling with velocity c .

on comparing (3), (4), (5) we conclude that E & H travel in free space in term of wave with velocity c given by

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

$$\Rightarrow c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\text{But } \mu_0 = 4\pi \times 10^{-7} \text{ NA}^{-2}, \epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$$

$$\Rightarrow c = \frac{1}{\sqrt{4\pi \times 10^{-7} \text{ NA}^{-2} \times 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}}}$$

$$= 2.99792 \times 10^8 \text{ ms}^{-1} \approx 3 \times 10^8 \text{ ms}^{-1}$$

The velocity of Electromagnetic wave in free space equal to velocity of light.

② Electromagnetic Wave Equation in Dielectric

Maxwell's Equation in General Form is given by -

- I) $\nabla \cdot E = P/\epsilon_0$
- II) $\nabla \cdot B = 0$
- III) $\nabla \times E = -\frac{d\mathbf{B}}{dt}$
- IV) $\nabla \times B = \mu_0 \left(J + \frac{dD}{dt} \right)$

For dielectric conductivity $\boxed{\sigma=0}$ and $\boxed{J=\sigma E=0}$

When Dielectric is not charged then $\boxed{P=0}$

Then -

- I) $\nabla \cdot E = 0$
- II) $\nabla \cdot H = 0$
- III) $\nabla \times E = -\mu_0 \frac{dH}{dt}$
- IV) $\nabla \times H = \epsilon_0 \frac{dE}{dt}$

Now Replacing μ_0, ϵ_0 by $\mu \epsilon$ respectively in Equation ③ + ④ we get.

$$\nabla^2 E = -\mu \epsilon \frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 H = -\mu \epsilon \frac{\partial^2 H}{\partial t^2}$$

$$\text{or } \frac{\partial^2 E_x}{\partial x^2} - \mu \epsilon \frac{\partial^2 E_x}{\partial t^2} = 0$$

$$\frac{\partial^2 H_x}{\partial x^2} - \mu \epsilon \frac{\partial^2 H_x}{\partial t^2} = 0$$

Comparing with general wave Equation.

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = 0$$

$$\Rightarrow \frac{1}{c^2} = \mu \epsilon$$

$$\Rightarrow \boxed{c = \frac{1}{\sqrt{\mu \epsilon}}} \text{ - wave velocity}$$

Electromagnetic Wave Equation in conducting medium

Maxwell Equations in conducting medium with
permeability μ , permittivity ϵ and conductivity σ
can be written as -

$$\text{I) } \nabla \cdot E = 0 \quad \rightarrow (1)$$

$$\text{II) } \nabla \cdot H = 0 \quad \rightarrow (2)$$

$$\text{III) } \nabla \times E = -\mu \frac{dH}{dt} \quad \rightarrow (3)$$

$$\begin{aligned} \text{IV) } \nabla \times H &= J + \epsilon \frac{dE}{dt} \\ &= \sigma E + \epsilon \frac{dE}{dt} \end{aligned} \quad \rightarrow (4)$$

Taking curl of Equation (3) we have -

$$\begin{aligned} \nabla \times (\nabla \times E) &= \nabla \times \left(-\mu \frac{dH}{dt} \right) \\ &= -\mu \left(\nabla \times \frac{dH}{dt} \right) \\ &= -\mu \left(\frac{d}{dt} (\nabla \times H) \right) \\ &= -\mu \left(\frac{d}{dt} \left(\sigma E + \epsilon \frac{dE}{dt} \right) \right) \quad (\text{using (4)}) \end{aligned}$$

$$\nabla \times (\nabla \times E) = -\mu \sigma \frac{dE}{dt} - \mu \epsilon \frac{d^2 E}{dt^2}$$

$$\Rightarrow \nabla \cdot (\nabla \cdot E) - \nabla^2 E = -\mu \sigma \frac{dE}{dt} - \mu \epsilon \frac{d^2 E}{dt^2}$$

$$0 - \nabla^2 E = -\mu \sigma \frac{dE}{dt} - \mu \epsilon \frac{d^2 E}{dt^2}$$

$$\nabla^2 E = \mu \sigma \frac{dE}{dt} + \mu \epsilon \frac{d^2 E}{dt^2}$$

In case of Non-conducting Medium

$(\sigma = 0)$ Then

$$\boxed{\nabla^2 E = \mu \epsilon \frac{d^2 E}{dt^2}}$$

Now taking curl of $\text{Equation } ④$ we obtain

$$\begin{aligned}\nabla \times (\nabla \times H) &= \nabla \times \left(\sigma E + \epsilon \frac{dE}{dt} \right) \\ &= \sigma (\nabla \times E) + \epsilon \frac{d(\nabla \times E)}{dt}\end{aligned}$$

(using ③) $= \sigma \left(-\mu \frac{dH}{dt} \right) + \epsilon \frac{d}{dt} \left(-\mu \frac{dH}{dt} \right)$

$$-\nabla^2 H = -\mu \sigma \frac{dH}{dt} - \mu \epsilon \frac{d^2 H}{dt^2}$$

$$\nabla^2 H = \mu \sigma \frac{dH}{dt} + \mu \epsilon \frac{d^2 H}{dt^2}$$

If medium is non-conducting $\boxed{\sigma=0}$

$$\Rightarrow \boxed{\nabla^2 H = \mu \epsilon \frac{d^2 H}{dt^2}}$$

Solution of wave Equation:-

$$E(r,t) = E_0 e^{i(k \cdot r - \omega t)}$$

$$H(r,t) = H_0 e^{i(k \cdot r - \omega t)}$$

These two wave Equations represent solution of wave Equation

Skin depth (Depth of Penetration)

It may be defined as the depth in which strength of Electric field associated with the Electromagnetic waves reduce to $1/e$ times to its initial value.

The Amplitude of Strength of Electric Field of an Electromagnetic waves decrease by a factor $e^{-\alpha x}$, where α is attenuation constant.

If depth of Penetration is represented by s then

$$\alpha s = \frac{1}{s}$$

$$\text{or } s = \frac{1}{\alpha}$$

Thus Skin depth or Penetration depth

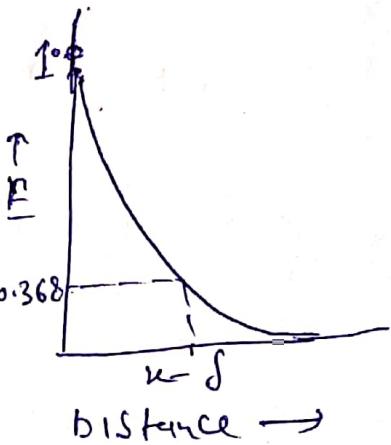
$$s = \frac{1}{\text{Attenuation Constant}}$$

Hence Reciprocal of Attenuation constant is called skin depth or penetration depth.

Where Attenuation constant is given by -

$$\alpha = \omega \left(\frac{\mu_r}{2} \left(\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon_0}} - 1 \right) \right)^{1/2}$$

$$\text{Skin Depth } s = \frac{1}{\omega \left(\frac{\mu_r}{2} \left(\sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon_0}} - 1 \right) \right)^{1/2}}$$



POYNTING THEOREM - Imp

The rate of Energy transport per unit area, is called Poynting vector. It is also termed as instantaneous energy flux density and is represented by S or P.

$$\boxed{\vec{S} = \vec{E} \times \vec{H}}$$

\vec{S} is perpendicular to both \vec{E} and \vec{H} .

Unit - W/m^2

Derivation:- We can calculate the energy density carried by Electromagnetic waves with the help of Maxwell's Equation given below-

$$i) \nabla \cdot \vec{B} = 0 \quad \text{--- (1)}$$

$$ii) \nabla \cdot \vec{D} = 0 \quad \text{--- (2)}$$

$$iii) \nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \quad \text{--- (3)}$$

$$iv) \nabla \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt} \quad \text{--- (4)}$$

Now taking (.) Dot product of (3) with H with (3)

and Dot product of E with (4)

$$\vec{H} \cdot (\nabla \times \vec{E}) = -\mu \vec{H} \cdot \frac{d\vec{H}}{dt} \quad \text{--- (5)} \quad (B = \mu \vec{H})$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \cdot \frac{d\vec{D}}{dt} \quad \text{--- (6)} \quad (D = \epsilon \vec{E})$$

Subtracting Equation (5) from Equation (6)

$$\Rightarrow \vec{E} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{E}) = \vec{E} \cdot \vec{J} + \epsilon \vec{E} \cdot \frac{d\vec{D}}{dt} - (-\mu \vec{H} \cdot \frac{d\vec{H}}{dt})$$

$$\text{Since } \vec{A} \cdot (\nabla \times \vec{B}) - \vec{B} \cdot (\nabla \times \vec{A}) = \vec{\nabla} \cdot (\vec{B} \times \vec{A}) = -\vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

$$\nabla \cdot (\vec{E} + \vec{H}) = \vec{J} \cdot \vec{E} + \frac{1}{2} \epsilon \frac{d \vec{E}}{dt}$$

$$\Rightarrow -\nabla \cdot (\vec{E} + \vec{H}) = \vec{J} \cdot \vec{E} + \frac{1}{2} \epsilon \frac{d \vec{E}^2}{dt} + \frac{1}{2} \mu \frac{d \vec{H}^2}{dt}$$

using $\left. \begin{aligned} \epsilon \frac{d \vec{E}}{dt} &= \frac{1}{2} \epsilon \frac{d \vec{E}^2}{dt} \\ \mu \frac{d \vec{H}}{dt} &= \frac{1}{2} \mu \frac{d \vec{H}^2}{dt} \end{aligned} \right\}$

$$\Rightarrow -\nabla \cdot \vec{S} = \vec{J} \cdot \vec{E} + \frac{1}{2} \frac{d}{dt} (\epsilon \vec{E}^2 + \mu \vec{H}^2)$$

or $\vec{J} \cdot \vec{E} + \frac{d}{dt} \left(\frac{\epsilon \vec{E}^2}{2} + \frac{\mu \vec{H}^2}{2} \right) + \vec{\nabla} \cdot \vec{S} = 0$

Taking volume Integral over the Volume V enclosed by the surface S,

$$\int_V \vec{\nabla} \cdot \vec{S} dV + \frac{d}{dt} \int_V \left(\frac{\epsilon \vec{E}^2}{2} + \frac{\mu \vec{H}^2}{2} \right) dV = - \int_V \vec{J} \cdot \vec{E} dV$$

using Divergence Theorem

$$\boxed{\int_S \vec{S} \cdot d\vec{S} + \frac{d}{dt} \int_V \left(\frac{\epsilon \vec{E}^2}{2} + \frac{\mu \vec{H}^2}{2} \right) dV = - \int_V \vec{J} \cdot \vec{E} dV}$$

$\rightarrow \int_S \vec{S} \cdot d\vec{S} \rightarrow$ Represent rate of flow of Energy or Power Flux.

$\rightarrow \frac{d}{dt} \int_V \left(\frac{\epsilon \vec{E}^2}{2} + \frac{\mu \vec{H}^2}{2} \right) dV =$ Rate of Change of Total Energy

$\rightarrow \frac{\epsilon \vec{E}^2}{2} + \frac{\mu \vec{H}^2}{2}$ represent Energy stored in Electric & Magnetic Field.

$\int \vec{J} \cdot \vec{E} dV =$ Rate of work done by the Electromagnetic field in displacing the charge within the volume

Hence

$$\vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} + \frac{d}{dt} \left(\frac{1}{2} \mu H^2 \right) + \frac{d}{dt} \left(\frac{1}{2} \epsilon E^2 \right) = 0$$

OR

$$\int_V \vec{\nabla} \cdot \vec{S} dV + \int_V \vec{J} \cdot \vec{E} dV + \frac{d}{dt} \int_V \left(\frac{\mu H^2}{2} + \frac{\epsilon E^2}{2} \right) dV = 0$$

Above Equation is known as Poynting Theorem or WORK-Energy Theorem.

According to Poynting theorem the power transformed into Electromagnetic field is Equal to the sum of the time rate of change of EM energy, within certain volume and the time rate of Energy flowing out through the boundary surface. This is also known as Energy conservation law in Electromagnetism.

Example 3.11 The electromagnetic wave intensity received on the surface of the earth from the sun is found to be 1.33 kW/m^2 . Find the amplitude of electric field vector associated with sunlight as received on earth surface. Assume Sun's light to be monochromatic ($\lambda = 6000 \text{ \AA}$).

[GGSIPU, Feb. 2012 (5 marks); Feb. 2008 (3 marks)]

Solution. The energy transported by an electromagnetic wave per unit area per second during propagation is represented by Poynting vector \mathbf{S} as

$$\mathbf{S} = (\mathbf{E} \times \mathbf{H})$$

The energy flux per unit area per second is

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin 90^\circ = EH$$

The energy flux per unit area per second at the earth surface.

$$|\mathbf{S}| = 1.33 \text{ kW/m}^2 = 1.33 \times 10^3 \text{ J m}^{-2} \text{ s}^{-1}$$

$$|\mathbf{S}| = 1330 \text{ J m}^{-2} \text{ s}^{-1}$$

(i)

We know that $Z_0 = \left| \frac{\mathbf{E}}{\mathbf{H}} \right| = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi \times 10^{-7} \text{ Wb/Am}}{8854 \times 10^{-12} \text{ C}^2/\text{Nm}^2}} = 376.72 \Omega$

$$\frac{E}{H} = 376.72 \Omega \quad \dots(ii)$$

Multiplying Eq. (i) and Eq. (ii), we get

$$EH \times \frac{E}{H} = 1330 \times 376.72$$

$$E^2 = 501037.6, E = 707.8 \text{ V/m}$$

Substituting this value in Eq. (ii)

$$H = \frac{707.8}{376.72} = 1.879 \text{ A/m}$$

Therefore, the amplitudes of electric and magnetic fields of radiation are

$$E_0 = E\sqrt{2} = 707.8\sqrt{2} = 1000.8292 = 1000 \text{ V/m}$$

and

$$H_0 = H\sqrt{2} = 1.879\sqrt{2} = 2.657 \text{ A.turn/m.}$$

Example 3.12 If the earth receives $2 \text{ Cal min}^{-1} \text{ cm}^{-2}$ solar energy, what are the amplitudes of electric and magnetic field of radiation ? [GGSIPU., June 2015 (5 Marks), May 2016 (4 marks)]

Solution. As Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = EH \sin 90^\circ = EH$$

$$\text{Solar energy} = 2 \text{ Cal min}^{-1} \text{ cm}^{-2}$$

$$= \frac{2 \times 4.18 \times 10^4}{60} \text{ Jm}^{-2} \text{ s}^{-1}$$

Both are energy flux per unit area per second

$$\text{Hence } EH = \frac{2 \times 4.18 \times 10^4}{60} \approx 1400$$

But

$$\frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi = 377$$

$$\therefore EH \times \frac{E}{H} = 1400 \times 377$$

$$E = \sqrt{1400 \times 377} = 726.5 \text{ V/m}$$

Now,

$$H = \frac{E}{377} = 1.927 \text{ A/m}$$

Amplitudes of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 1024.3 \text{ V/m}$$

$$H_0 = H\sqrt{2} = 2.717 \text{ A/m}$$

Example 3.13 Calculate the Poynting vector at the surface of the sun. Given that the energy radiated per second is $3.8 \times 10^{26} \text{ J}$ and the radius of the sun is $0.7 \times 10^9 \text{ m}$. Also calculate the amplitudes of electric and magnetic field vectors on the surface of the earth. The distance of the earth from the sun is $0.15 \times 10^{12} \text{ m}$

[GGSIPU., May 2014 (6 Marks)]

Solution. First case. The Poynting vector at the surface of sun

$$\begin{aligned}|S| &= |E \times H| = EH \sin 90^\circ \\&= \frac{P_0}{4\pi r_1^2} = \frac{3.8 \times 10^{26} \text{ J/s}}{4\pi \times (0.7 \times 10^9 \text{ m})^2} \\&= 6.17 \times 10^7 \text{ J m}^{-2} \text{ s}^{-1}\end{aligned}$$

Second case. The Poynting vector at the surface of earth

$$\begin{aligned}EH &= \frac{P_0}{4\pi r_1^2} \\&= \frac{3.8 \times 10^{26} \text{ J/s}}{4\pi \times (0.15 \times 10^{12} \text{ m})^2} = 1344.656 \text{ J m}^{-2} \text{ s}^{-1} \quad \dots(i)\end{aligned}$$

We also know $\frac{E}{H} = 120\pi\Omega = 377 \Omega$... (ii)

From Eqs. (i) and (ii)

$$\frac{E}{H} \times EH = 1344.656 \times 377$$

and

$$E^2 = \sqrt{1344.656 \times 377} = 711.99 \text{ V/m}$$

and from Eq. (ii)

$$\frac{E}{H} = 377 \Omega$$

$$H = \frac{E}{377\Omega} = \frac{711.9938}{377} \text{ A/m} = 1.888 \text{ A/m}$$

Amplitude of electric and magnetic field of radiation are

$$E_0 = E\sqrt{2} = 711.99 \times 1.414 = 1006.75 \text{ V/m}$$

and

$$H_0 = H\sqrt{2} = 1.888 \times 1.414 = 2.67 \text{ A/m}$$