

Classification of Partial Differential Equations :-

Consider the Equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0 \quad \text{--- (1)}$$

Where A, B, C may be constants or function of x and y .

Now the eqn (1) is

- (1) Elliptic if $B^2 - 4AC < 0$
- (2) Parabolic if $B^2 - 4AC = 0$
- (3) Hyperbolic if $B^2 - 4AC > 0$

Q.1 Classify the equation

$$2 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0$$

Solⁿ Now $A=2, B=4, C=3$

$$\therefore B^2 - 4AC = 16 - 24 = -8 < 0$$

i.e Elliptic

Q.2 $\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = 0$

Solⁿ $A=1, B=4, C=4$

$$\therefore B^2 - 4AC = 16 - 4 \times 4 = 0$$

i.e Parabolic

Q.3 $2 \frac{\partial^2 u}{\partial x^2} + 6 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0$

Solⁿ $A=2, B=6, C=3$

$$\therefore B^2 - 4AC = 36 - 4 \times 2 \times 3 = 36 - 24 = 12 > 0$$

i.e Hyperbolic.

Classification of Partial Differential Equation:

In practical problems, the following types of equations are generally used

(1) Wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

(2) One-dimensional heat flow : $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

(3) Two-dimensional heat flow : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Consider the equation.

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0 \rightarrow \textcircled{1}$$

Where A, B, C may be constants or functions of x and y .

Now the eqn $\textcircled{1}$ is

① Elliptic if $B^2 - 4AC < 0$

② Parabolic if $B^2 - 4AC = 0$

③ Hyperbolic if $B^2 - 4AC > 0$

Q.1 Classify the equation

$$2 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0$$

Sol: Given eqn compare with $A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + F(x, y, u, p, q) = 0$

$$\therefore A = 2, B = 4, C = 3$$

$$\therefore B^2 - 4AC = 16 - 24 = -8 < 0$$

i.e. Given eqn is Elliptic Ans

Q.2 $\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = 0$

Sol: $A = 1, B = 4, C = 4$

$$\therefore B^2 - 4AC = 16 - 16 = 0 \quad \text{i.e. Given eqn is Parabolic}$$

Q.3 $2 \frac{\partial^2 u}{\partial x^2} + 6 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0$

Sol: $A = 2, B = 6, C = 3$

$$\therefore B^2 - 4AC = 36 - 24 = 12 > 0$$

i.e. given eqn is Hyperbolic Ans

Method of Separation of Variable :-

In this method, we assume that the dependent variable is the product of two ~~independent~~ functions, each of which involves only one of the independent variable. So two ordinary differential equations are formed.

① Using the method of separation of variable, solve

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$$

$$\text{Where } u(x, t) = 6e^{-3x}$$

Sol. The given equation is:

$$\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \rightarrow ①$$

$$\text{Let } u = X \cdot T \rightarrow ②$$

Where x is function of x and T is function of t only

$$\therefore \frac{\partial u}{\partial x} = T \frac{\partial}{\partial x} X \cdot T$$

$$\frac{\partial u}{\partial x} = T \cdot \frac{\partial X}{\partial x}$$

$$\text{And } \frac{\partial u}{\partial t} = T \frac{\partial}{\partial t} X \cdot T$$

$$\frac{\partial u}{\partial t} = X \cdot \frac{\partial T}{\partial t}$$

Putting these values in eqn ①

$$\therefore T \cdot \frac{\partial X}{\partial x} = 2X \cdot \frac{\partial T}{\partial t} + X \cdot T$$

$$T \cdot X' = 2X \cdot T' + X \cdot T$$

divide by $X \cdot T$

$$\therefore \frac{X'}{X} = 2 \frac{T'}{T} + 1 = C (\text{say})$$

Take First Condition

$$\frac{x'}{x} = c$$

$$\frac{1}{x} \cdot \frac{dx}{dx} = c$$

$$\frac{dx}{x} = c dx$$

on Integrating

$$\log x = cx + \log a$$

$$\log \frac{x}{a} = cx$$

$$\frac{x}{a} = e^{cx}$$

$$x = ae^{cx}$$

Now taking second condition.

$$2\frac{T'}{T} + 1 = c$$

$$2\frac{T'}{T} = c - 1$$

$$\frac{T'}{T} = \frac{1}{2}(c-1)$$

$$\frac{1}{T} \cdot \frac{dT}{dt} = \frac{1}{2}(c-1)$$

$$\frac{dT}{T} = \frac{1}{2}(c-1) dt$$

on Integrating.

$$\log T = \frac{1}{2}(c-1)t + \log b$$

$$\log \frac{T}{b} = \frac{1}{2}(c-1)t$$

$$\frac{T}{b} = e^{\frac{1}{2}(c-1)t}$$

$$T = b e^{\frac{1}{2}(c-1)t}$$

Putting the values of x and T in ②

$$u = a e^{cx} \cdot b e^{\frac{1}{2}(c-1)t}$$

$$u = ab e^{cx + \frac{1}{2}(c-1)t}$$

$$\text{Given } u(x, 0) = 6e^{-3x}$$

$$\therefore 6e^{-3x} = ab e^{cx + \frac{1}{2}(c-1)0}$$

$$6e^{-3x} = ab e^{cx} \therefore 6 = ab$$

$$c = -3$$

$$\therefore u = 6 e^{-3x + \frac{1}{2}(-3-1)t}$$

$$\boxed{u = 6 e^{-3x - 2t}} \text{ OR}$$

Solve the following equation by method of separation of variables.

$$\frac{\partial^2 u}{\partial n \partial t} = e^{-t} \cos n, \text{ given that } u=0, \text{ when } t=0$$

and $\frac{\partial u}{\partial t} = 0, \text{ when } n=0$

Sol:

$$u = XT \rightarrow \textcircled{1}$$

where x is function of n and T is function of t only

$$\therefore \frac{\partial u}{\partial t} = X \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 u}{\partial n \partial t} = \frac{\partial X}{\partial n} \frac{\partial T}{\partial t}$$

Putting in given eqn

$$\frac{\partial X}{\partial n} \frac{\partial T}{\partial t} = e^{-t} \cos n$$

$$XT' = e^{-t} \cos n$$

$$e^t T' = \frac{\cos n}{x} = -p^2 \text{ say}$$

$$e^t T' = -p^2$$

$$T' = -p^2 e^{-t}$$

$$\frac{dT}{dt} = -p^2 e^{-t}$$

$$dT = -p^2 e^{-t} dt$$

on Integration

$$T = -p^2 \frac{e^{-t}}{-1} + c_1$$

$$T = p^2 e^{-t} + c_1 \rightarrow \textcircled{2}$$

And

$$\frac{\cos n}{x} = -p^2$$

$$\frac{\cos n}{p^2} = -\frac{dx}{dn}$$

$$\frac{-dx}{p^2} = 1 \text{ and } dn$$

$$x = -\frac{1}{p^2} \sin n + c_2 \rightarrow \textcircled{3}$$

Putting in \textcircled{1}

$$\therefore u = XT$$

$$u(n, t) = \left(-\frac{1}{p^2} \sin n + c_2 \right) (p^2 e^{-t} + c_1) \rightarrow \textcircled{4}$$

Now using condition

$$u=0, t=0$$

$$0 = \left(-\frac{1}{p^2} \sin n + c_2 \right) (p^2 e^0 + c_1)$$

$$0 = p^2 + c_1 \Rightarrow c_1 = -p^2$$

Again

$$\frac{\partial u}{\partial t} = \left(-\frac{1}{p^2} \sin n + c_2 \right) (-p^2 e^{-t} + 0)$$

$$0 = \left(-\frac{1}{p^2} \sin n + c_2 \right) (-p^2 e^0 + 0)$$

$$0 = c_2 (-p^2 e^0) \Rightarrow c_2 = 0$$

$$\therefore u(n, t) = \left(-\frac{1}{p^2} \sin n + 0 \right) (p^2 e^{-t} - p^2)$$

$$= -\frac{\sin n}{p^2} (p^2 e^{-t} - p^2)$$

$$= -\frac{\sin n}{p^2} p^2 (e^{-t} - 1)$$

$$u(n, t) = (1 - e^{-t}) \sin n$$

N

Q Solve the P.D.E. by separation of variable method
~~2010, 2011~~

$$u_{yy} = 4y + 24, u(0,y) = 0, \frac{\partial}{\partial y} u(0,y) = 1 + e^{-3y}$$

Sols'

$$u = xy \quad \text{--- (1)}$$

Where x is function of y only and y is function of y only

$$\frac{\partial u}{\partial x} = y \frac{\partial x}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = y \frac{\partial^2 x}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = x \frac{\partial y}{\partial y}$$

$$\therefore y \frac{\partial^2 x}{\partial x^2} = x \frac{\partial y}{\partial y} + 2xy$$

$$\frac{x''}{x} = \frac{y'}{y} + 2 = k(\text{say})$$

$$\frac{x''}{x} = k$$

$$x'' - kx = 0$$

$$(D^2 - k)x = 0$$

For C.F A.E is

$$m^2 - k = 0$$

$$m^2 = k$$

$$m = \pm \sqrt{k}$$

$$C.F = c_1 e^{+\sqrt{k}y} + c_2 e^{-\sqrt{k}y}$$

$$\therefore x = c_1 e^{\sqrt{k}y} + c_2 e^{-\sqrt{k}y}$$

$$\text{And } \frac{y'}{y} + 2 = k$$

$$\frac{y'}{y} = k - 2$$

$$y' = (k-2)y$$

$$\frac{dy}{dy} = (k-2)y$$

$$\frac{dy}{y} = (k-2)dy$$

$$\log y = (k-2)y + \log C_3$$

$$\therefore \frac{y}{C_3} = (k-2)y$$

$$y = C_3 e^{(k-2)y}$$

$$\therefore u = (c_1 e^{\sqrt{k}y} + c_2 e^{-\sqrt{k}y}) [C_3 e^{(k-2)y}]$$

$$\therefore u(x,y) = (c_1 e^{\sqrt{k}y} + c_2 e^{-\sqrt{k}y}) \{ C_3 e^{(k-2)y} \} \quad \text{--- (2)}$$

$$\text{Using condition } u(0,y) = 0$$

$$0 = \cancel{(c_1 + c_2)} \cdot (c_1 e^{\sqrt{k} \cdot 0} + c_2 e^{-\sqrt{k} \cdot 0}) C_3 e^{(k-2)y}$$

$$0 = (c_1 + c_2) C_3 e^{(k-2)y}$$

$$\therefore c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$$

on putting in (2)

$$u(x,y) = (c_1 e^{\sqrt{k}y} - c_1 e^{-\sqrt{k}y}) \{ C_3 e^{(k-2)y} \} \quad \text{--- (3)}$$

$$u(x,y) = c_1 C_3 (e^{\sqrt{k}y} - e^{-\sqrt{k}y}) e^{(k-2)y}$$

$$\therefore \frac{\partial u}{\partial y} = c_1 C_3 (\sqrt{k} e^{\sqrt{k}y} + \sqrt{k} e^{-\sqrt{k}y}) e^{(k-2)y}$$

$$1 + e^{-3y} = c_1 C_3 (\sqrt{k} e^0 + \sqrt{k} e^0) e^{(k-2)y}$$

$$1 + e^{-3y} = \sum_{n=1}^{\infty} b_n e^{(k-2)y}$$

$$1 + e^{-3y} = b_1 e^{(k-2)y} + b_2 e^{(k-2)y} + b_3 e^{(k-2)y}$$

$b_1 = 1$ and ~~$b_2 = b_3 = 0$~~

$$1 + e^{-3y} = b_1 e^{(k-2)y} + b_2 e^{(k-2)y} + \dots$$

$$b_1 = 1, k-2 = 0 \Rightarrow b_1 = 1, k = 2$$

$$b_2 = 1, k-2 = -3 \Rightarrow b_2 = 1, k = -1$$

$$b_3 = 0$$

If the method of separation of variable, solve Q. 20

$$\frac{dy}{dx} = 2 \frac{dy}{dx} + 4, \quad u(x_0) = 4e^{-3x}$$

Sol. The given equation is.

$$\frac{dy}{dx} = 2 \frac{dy}{dx} + 4 \longrightarrow \textcircled{1}$$

Let $u = x \cdot y \longrightarrow \textcircled{2}$

Where x is function of x only and y is function of y only.

$$\therefore \frac{dy}{dx} = y \frac{dx}{dx}$$

And $\frac{dy}{dx} = x \frac{dy}{dy}$

Putting these values in \textcircled{1}

$$y \frac{dx}{dx} = 2x \frac{dy}{dy} + xy$$

$$yx' = 2xy' + xy$$

Divide by xy

$$\frac{x'}{x} = 2 \frac{y'}{y} + 1 = c \text{ (say)}$$

$$\therefore \frac{x'}{x} = c$$

$$\frac{1}{x} \cdot \frac{dx}{dx} = c$$

$$\frac{dx}{x} = c dx$$

on Integration

$$\log x = cn + \log a$$

$$\log \frac{x}{a} = cn$$

$$\frac{x}{a} = e^{cn}$$

$$x = a e^{cn}$$

And $\frac{2y'}{y} = c - 1$

$$\frac{y'}{y} = \frac{1}{2}(c-1)$$

$$\frac{1}{y} \cdot \frac{dy}{dy} = \frac{1}{2}(c-1)$$

$$\frac{dy}{y} = \frac{1}{2}(c-1) dy$$

on Int.

$$\log y = \frac{1}{2}(c-1)y + \log b$$

$$\log \frac{y}{b} = \frac{1}{2}(c-1)x$$

$$\frac{y}{b} = e^{\frac{1}{2}(c-1)x}$$

$$y = b e^{\frac{1}{2}(c-1)x}$$

Putting the values of x and y in eqn ②

$$\therefore u = xy \\ = a e^{cx} \cdot b e^{\frac{1}{2}(c-1)x}$$

$$u = ab e^{cx + \frac{1}{2}(c-1)x} \quad \rightarrow ③$$

$$\text{Given } u(x, 0) = 4 e^{-3x}$$

$$\therefore u(x, 0) = ab e^{cx + \frac{1}{2}(c-1) \cdot 0}$$

$$4 e^{-3x} = ab e^{cx+0}$$

$$4 e^{-3x} = ab e^{cx}$$

$$\therefore ab = 4, c = -3$$

Putting these values in ③

$$u = 4 e^{-3x + \frac{1}{2}(-3-1)x}$$

$$\boxed{u = 4 e^{-3x - 2x}} \quad \text{Ans.}$$

Q. Solve

Solve by method of Separation of Variables:

$$4 \frac{dy}{dt} + \frac{dy}{dx} = 3y, y = 3e^x - e^{-5x}, t=0$$

Sol^b Let $y = xt \rightarrow ①$

where x is a function of x only and T is function of t only

$$\therefore \frac{dy}{dt} = x \frac{dT}{dt} = xt'$$

$$\frac{dy}{dx} = T \frac{dx}{dt} = TX'$$

Putting in ① given eqn

~~$$4xt' + TX' = 3xt$$~~

Divide by xt

$$4 \frac{T'}{T} + \frac{x'}{x} = 3$$

$$4 \frac{T'}{T} - 3 = -\frac{x'}{x} = P^2 \text{ (say)}$$

$$\therefore 4 \frac{T'}{T} = P^2 + 3$$

$$\frac{T'}{T} = \frac{P^2 + 3}{4}$$

$$\frac{T'}{T} = \left(\frac{P^2 + 3}{4}\right)$$

On Integration

$$\log T = \left(\frac{P^2 + 3}{4}\right)t + \log C$$

$$\log \frac{T}{C} = \left(\frac{P^2 + 3}{4}\right)t$$

$$T = C_1 e^{\left(\frac{P^2 + 3}{4}\right)t} \rightarrow ②$$

And $\frac{x'}{x} = P^2$

$$\frac{x'}{x} = -P^2$$

$$\frac{1}{x} \frac{dx}{dx} = -P^2$$

$$\frac{dx}{x} = -P^2 dx \text{ on } 71$$

$$\log x = -P^2 x + \log C_2$$

$$\log \frac{x}{C_2} = -P^2 x$$

$$x = C_2 e^{-P^2 x}$$

$$\therefore u = x \cdot T - p^2 x \cdot e^{-\frac{p^2 x}{4} t}$$

$$= c_1 e^{-p^2 x} \cdot c_2 e^{-\frac{(p^2 + 3)}{4} t}$$

$$u = c_1 c_2 e^{-p^2 x} \cdot e^{-\frac{(p^2 + 3)}{4} t}$$

$$u(x, t) = b_n e^{-p^2 x + \left(\frac{p^2 + 3}{4}\right)t}$$

most general solution is

$$u(x, t) = \sum_{n=1}^{\infty} b_n e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n e^{-p^2 x + \left(\frac{3+p^2}{4}\right)0}$$

$$\therefore 3e^{-x} - e^{-5x} = \sum_{n=1}^{\infty} b_n e^{-p^2 n + 0}$$

$$3e^{-x} - e^{-5x} = b_1 e^{-p^2 x} + b_2 e^{-p^2 x}$$

$$\therefore p^2 = +1, b_1 = 3 \text{ and } b_2 = -1, p^2 = 5$$

$$\therefore u(x, t) = 3e^{-x + \left(\frac{3+1}{4}\right)t} - e^{-5x + \left(\frac{3+5}{4}\right)t}$$

$$u(x, t) = 3e^{-x+t} - e^{-5x+2t}$$

$$u(x, t) = 3e^{-x+t} - e^{-5x+2t} \quad \text{Ans}$$

use the method of separation of variables to solve the equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x} + 2y$$

Solⁿ The given equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial x} + 2y \rightarrow \textcircled{1}$$

Let $u = xy \rightarrow \textcircled{2}$

Where x is a function of y only and y is a function of x only

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial(x \cdot y)}{\partial x}$$

$$\frac{\partial u}{\partial x} = y \frac{\partial x}{\partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = y \cdot \frac{\partial^2 x}{\partial x^2}$$

$$\frac{\partial u}{\partial y} = x \frac{\partial y}{\partial y}$$

Putting these values in \textcircled{1}

$$y \cdot \frac{\partial^2 x}{\partial x^2} = x \frac{\partial y}{\partial y} + 2x \cdot y$$

divide by $x \cdot y$

$$\frac{1}{x} \frac{\partial^2 x}{\partial x^2} = \frac{1}{y} \frac{\partial y}{\partial y} + 2$$

$$\frac{1}{x} \cdot x'' = \frac{1}{y} y' + 2 = -P^2 (say)$$

Take $\frac{x''}{x} = -P^2$

$$x'' = -P^2 x$$

$$\frac{d^2 x}{dx^2} = -P^2 x$$

$$\frac{d^2 x}{dx^2} + P^2 x = 0$$

which is ordinary diff. eqn

$$\therefore A.E. \text{ is } m^2 + P^2 = 0$$

$$m^2 = -P^2$$

$$m = \pm i P$$

$$C.F. = C_1 \cos Px + C_2 \sin Px \quad \text{and P.I.} = 0$$

$$\therefore x = C_1 \cos Px + C_2 \sin Px$$

And take relation

$$\frac{1}{y} y' + 2 = -P^2$$

$$\therefore \frac{y'}{y} = -(P^2 + 2)$$

$$\frac{y'}{y} = -(P^2 + 2)$$

~~$$\frac{y'}{y} = -(P^2 + 2) y$$~~

$$\frac{1}{y} \cdot \frac{dy}{dy} = -(P^2 + 2)$$

$$\frac{dy}{y} = -(P^2 + 2) dy$$

on Int.

$$\log y = -(P^2 + 2)y + \log a$$

$$\log \frac{y}{a} = -(P^2 + 2)y$$

$$\frac{y}{a} = e^{-(P^2 + 2)y}$$

$$y = a e^{-(P^2 + 2)y}$$

Hence $u = x \cdot y$

$$u = [C_1 \cos Pn + C_2 \sin Pn] + a e^{-\frac{(P^2 + 2)y}{2}}$$

$$u = [C_1 \cos Pn + C_2 \sin Pn] + [C_3 e^{-\frac{(P^2 + 2)y}{2}}] \text{ Ans.}$$

UNIT 5

Equation of A Stretched String, One Dimensional

$$\text{Wave Equation } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$\text{Solution of Wave Equation } \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow \textcircled{1}$$

Let $y = X \cdot T$ $\rightarrow \textcircled{2}$ be a solution of $\textcircled{1}$

Where X is a function of x and T is a function of t only.

$$\therefore \frac{\partial y}{\partial x} = T \cdot \frac{\partial X}{\partial x}$$

$$\frac{\partial^2 y}{\partial x^2} = T \cdot \frac{\partial^2 X}{\partial x^2}$$

$$\text{And } \frac{\partial y}{\partial t} = X \cdot \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 y}{\partial t^2} = X \cdot \frac{\partial^2 T}{\partial t^2}$$

Substituting these values in $\textcircled{1}$

$$\therefore X \cdot \frac{\partial^2 T}{\partial t^2} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$X T'' = c^2 T X''$$

Divide by $X \cdot T$

$$\frac{T''}{T} = c^2 \frac{X''}{X}$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} \rightarrow \textcircled{3}$$

Now L.H.S. of $\textcircled{3}$ is function of x only and R.H.S. is a function of t only. Since x and t are independent variables, this equation can hold only when both sides reduce to a constant say p^2 . Then equation $\textcircled{3}$ leads to ordinary linear differential equation.

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = p^2 (\text{say})$$

Case 1 When p^2 is positive

$$\therefore \frac{X''}{X} = p^2$$

$$\therefore X'' - p^2 X = 0$$

which is Linear ordinary diff eqn.

A.E. is $m^2 - p^2 = 0$

$$m^2 = p^2$$

$$m = \pm p$$

$$C.F. = C_1 e^{px} + C_2 e^{-px}$$

$$P.I. = 0$$

$$X = C_1 e^{px} + C_2 e^{-px}$$

$$\text{And } \frac{I''}{T} = C^2 p^2$$

$$T'' - C^2 p^2 T = 0$$

$$\therefore \text{A.E. is } m^2 - C^2 p^2 = 0 \quad \leftarrow TX = 0$$

$$m = \pm C p$$

$$\text{C.F.} = C_3 e^{Cpt} + C_4 e^{-Cpt}$$

$$\therefore T = C_3 e^{Cpt} + C_4 e^{-Cpt}$$

∴ complete solution

$$Y = X \cdot T$$

$$Y = [C_1 e^{px} + C_2 e^{-px}] [C_3 e^{Cpt} + C_4 e^{-Cpt}] \rightarrow ④$$

Case 2 - When p^2 is negative

$$\therefore \frac{X''}{X} = \frac{1}{C^2} \cdot \frac{I''}{T} = -P^2$$

$$X'' = -P^2 X$$

$$X'' + P^2 X = 0$$

$$\text{A.E. is } m^2 + P^2 = 0$$

$$m^2 = -P^2$$

$$m = \pm i P$$

$$\text{C.F.} = C_5 \cos Px + C_6 \sin Px$$

$$X = C_5 \cos Px + C_6 \sin Px$$

$$\text{And } \frac{1}{C^2} \cdot \frac{I''}{T} = -P^2$$

$$T'' = -P^2 C^2 T$$

$$T'' + P^2 C^2 T = 0$$

∴ A.E. is

$$m^2 + C^2 P^2 = 0$$

$$m = \pm i C P$$

$$\therefore \text{C.F.} = C_7 \cos Cpt + C_8 \sin Cpt$$

$$\therefore T = C_7 \cos Cpt + C_8 \sin Cpt$$

$$\therefore \text{solution is } Y = (C_5 \cos Px + C_6 \sin Px)(C_7 \cos Cpt + C_8 \sin Cpt) \rightarrow ⑤$$

When P^2 is zero.

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{I''}{T} = 0$$

$$\therefore \frac{X''}{X} = 0$$

$$\therefore X'' - 0X = 0$$

A.E. is $m^2 = 0$

$$m = 0, 0$$

$$\therefore C.F. = C_9 + C_{10}x$$

$$\therefore X = C_9 + C_{10}x$$

And

$$\frac{1}{c^2} \frac{I''}{T} = 0$$

$$\therefore T'' - 0T = 0$$

A.E. is $m^2 = 0$

$$m = 0, 0$$

$$C.F. = C_{11} + C_{12}t$$

$$T = C_{11} + C_{12}t$$

\therefore solution is

$$Y = X \cdot T$$

$$Y = (C_9 + C_{10}x)(C_{11} + C_{12}t) \rightarrow \textcircled{7}$$

of these three solutions, we have to choose that solution which is consistent with the physical nature of problem.

Here we dealing with the wave motion, so $P^2 < 0$ i.e.

P^2 is negative

Now applying boundary condition.

$$\text{When } Y = 0, n = 0 \rightarrow \textcircled{8}$$

$$\text{When } Y = 0, n = l$$

Here we have solution from $\textcircled{5}$

$$Y = (C_5 \cos px + C_6 \sin px)(C_7 \cos pt + C_8 \sin pt) \rightarrow \textcircled{8} \textcircled{9}$$

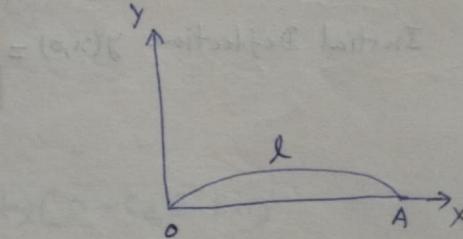
$$\text{use } Y = 0, n = 0$$

$$0 = (C_5 + C_6 \cdot 0)(C_7 \cos pt + C_8 \sin pt)$$

$$0 = C_5 [C_7 \cos pt + C_8 \sin pt]$$

$$\therefore C_5 = 0$$

Path is ~~is~~



And use condition $x=l, y=0$

$$\therefore 0 = c_5 \sin \theta$$

$$0 = (c_5 \cos \theta l + c_6 \sin \theta l)(c_7 \cos \theta t + c_8 \sin \theta t)$$

Putting $c_5 =$

$$0 = c_6 \sin \theta l (c_7 \cos \theta t + c_8 \sin \theta t)$$

which is satisfied when $c_6 \sin \theta l = 0$

$$c_6 \sin \theta l = \sin n\pi$$

$$\therefore \theta l = n\pi$$

$$\theta = \frac{n\pi}{l}, \text{ for } n=1, 2, 3, \dots$$

\therefore Since solution of the wave equation satisfies the boundary condition is

$$y = c_6 \left[c_7 \cos \frac{n\pi x}{l} + c_8 \sin \frac{n\pi x}{l} \right] \sin \frac{n\pi \theta}{l}$$

$$y = \left[c_6 c_7 \cos \frac{n\pi x}{l} + c_6 c_8 \sin \frac{n\pi x}{l} \right] \sin \frac{n\pi \theta}{l}$$

$$y = \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right] \sin \frac{n\pi \theta}{l}$$

$$y = \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right] \sin \frac{n\pi \theta}{l}$$

$$\text{Initial Velocity } \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0$$

$$\text{Initial Deflection } y(x, 0) = 0$$

A lightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y = b \sin^3 \frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x, t)$

Soln. The equation of string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow \textcircled{1}$$

And solution of eqn $\textcircled{1}$ is

$$y(x, t) = (C_1 \cos Cpt + C_2 \sin Cpt)(C_3 \cos px + C_4 \sin px) \rightarrow \textcircled{2}$$

Boundary conditions are

$$y(0, t) = 0 \rightarrow \textcircled{3}$$

$$y(l, t) = 0 \rightarrow \textcircled{4}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \rightarrow \textcircled{5} \quad [\text{Initial Transverse velocity of string/constant of the string is zero}]$$

$$y(x, 0) = b \sin^3 \frac{\pi x}{l} \rightarrow \textcircled{6}$$

Applying boundary condition in $\textcircled{2}$

From $\textcircled{3}$

$$y(0, t) = 0$$

$$y(0, t) = (C_1 \cos Cpt + C_2 \sin Cpt)(C_3 \cos px + 0)$$

$$0 = (C_1 \cos Cpt + C_2 \sin Cpt) C_3$$

$$C_3 = 0$$

∴ From $\textcircled{2}$

$$y(x, t) = (C_1 \cos Cpt + C_2 \sin Cpt)(0 + C_4 \sin px)$$

$$y(x, t) = (C_1 \cos Cpt + C_2 \sin Cpt)(C_4 \sin px) \rightarrow \textcircled{7}$$

$$\text{Again } y(l, t) = (C_1 \cos Cpt + C_2 \sin Cpt) C_4 \sin pl$$

$$0 = (C_1 \cos Cpt + C_2 \sin Cpt) C_4 \sin pl$$

$$\therefore \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$p = \frac{n\pi}{l}$$

Hence from ⑦

$$y(x,t) = [c_1 \cos \frac{n\pi c t}{l} + c_2 \sin \frac{n\pi c t}{l}] c_4 \sin \frac{n\pi x}{l}$$

$$\therefore \frac{\partial y}{\partial t} = \frac{n\pi c}{l} [-c_1 \sin \frac{n\pi c t}{l} + c_2 \cos \frac{n\pi c t}{l}] c_4 \sin \frac{n\pi x}{l}$$

At $t=0$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{n\pi c}{l} [-c_1 \sin \frac{n\pi c \cdot 0}{l} + c_2 \cos 0] c_4 \sin \frac{n\pi x}{l}$$

$$0 = \frac{n\pi c}{l} [0 + c_2] c_4 \sin \frac{n\pi x}{l}$$

$$0 = c_2$$

∴ From ⑧

$$y(x,t) = [c_1 \cos \frac{n\pi c t}{l} + 0] c_4 \sin \frac{n\pi x}{l}$$

$$= C_1 C_4 \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c t}{l}$$

$$y(x,t) = a_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c t}{l}$$

most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi c t}{l} \quad \rightarrow ⑨$$

But given condition

$$y(x,0) = b \sin^3 \frac{n\pi x}{l}$$

$$\therefore y(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cdot \cos 0$$

$$b \sin^3 \frac{n\pi x}{l} = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\frac{b}{4} \left[3 \sin \frac{n\pi x}{l} - \sin \frac{3n\pi x}{l} \right] = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l}$$

$$\frac{b}{4} \left[3 \sin \frac{n\pi x}{l} - \sin \frac{3n\pi x}{l} \right] = a_1 \sin \frac{n\pi x}{l} + a_2 \sin \frac{3n\pi x}{l} + a_3 \sin \frac{5n\pi x}{l} \dots$$

$$\frac{b}{4} 3 \sin \frac{n\pi x}{l} = a_1 \sin \frac{n\pi x}{l}$$

$$\therefore a_1 = \frac{3}{4} b$$

And $a_2 = 0$, And

$$\sin \frac{3\pi n}{l} = a_3 \sin \frac{3\pi n}{l}$$

$$-\frac{b}{4} = a_3$$

∴ From ⑨

$$y(n,t) = a_1 \sin \frac{\pi n}{l} \cos \frac{\pi c t}{l} + a_2 \sin \frac{2\pi n}{l} \cos \frac{2\pi c t}{l} \\ + a_3 \sin \frac{3\pi n}{l} \cos \frac{3\pi c t}{l}$$

$$y(n,t) = \frac{3}{4} b \sin \frac{\pi n}{l} \cos \frac{\pi c t}{l} + 0 - \frac{b}{4} \sin \frac{3\pi n}{l} \cos \frac{3\pi c t}{l}$$

which is required displacement.

String is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at a time $t=0$. Show that the displacement of any point at a distance x from one end at time t is given by

$$y(x, t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right)$$

Soln → Let us consider a string tightly stretched between two points O and A. Let y be the displacement at the point $P(x, y)$ at any time.

We know that The equation of string or wave equation is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow \textcircled{1}$$

And Solution of eqn $\textcircled{1}$ is.

$$y(x, t) = (C_1 \cosec pt + C_2 \sin cpt)(C_3 \cos px + C_4 \sin px) \rightarrow \textcircled{2}$$

As the end points of the string are fixed, for all time

$$y(0, t) = 0 \rightarrow \textcircled{3} \quad \text{Boundary conditions}$$

$$y(l, t) = 0 \rightarrow \textcircled{4}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \rightarrow \textcircled{5} \quad \begin{array}{l} \text{Initial Transverse velocity of any point of the} \\ \text{string is zero} \end{array}$$

$$\text{And given } y(x, 0) = a \sin \frac{\pi x}{l} \rightarrow \textcircled{6}$$

Applying boundary condition in eqn $\textcircled{2}$

$$\therefore y(0, t) = (C_1 \cosec pt + C_2 \sin cpt)(C_3 + C_4 \cdot 0)$$

$$0 = C_3(C_1 \cosec pt + C_2 \sin cpt)$$

$$C_3 = 0 \quad \text{Putting in (2)}$$

$$y(x, t) = (C_1 \cosec pt + C_2 \sin cpt) C_4 \sin px \rightarrow \textcircled{7}$$

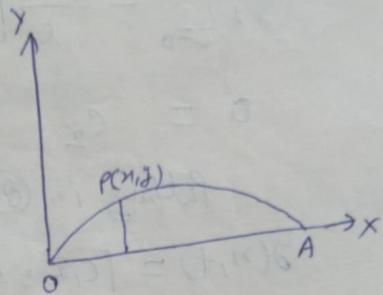
$$\therefore y(l, t) = (C_1 \cosec pt + C_2 \sin cpt) C_4 \sin pl$$

$$0 = C_4 \sin pl (C_1 \cosec pt + C_2 \sin cp)$$

$$\therefore 0 = \sin pl$$

$$\sin nx = \sin pl$$

$$\therefore n = \frac{p}{l}$$



equation ⑦ becomes

$$y(n, t) = \left[C_1 \cos \frac{n\pi c t}{l} + C_2 \sin \frac{n\pi c t}{l} \right] C_4 \sin \frac{n\pi x}{l}$$

$$\therefore \frac{\partial y}{\partial t} = \frac{n\pi c}{l} \left[-C_1 \sin \frac{n\pi c t}{l} + C_2 \cos \frac{n\pi c t}{l} \right] C_4 \sin \frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = \frac{n\pi c}{l} \left[0 + C_2 \cos \frac{n\pi c \cdot 0}{l} \right] C_4 \sin \frac{n\pi x}{l}$$

$$0 = C_2$$

Putting in ⑧

$$y(n, t) = \left[C_1 \cos \frac{n\pi c t}{l} + 0 \right] C_4 \sin \frac{n\pi x}{l}$$

$$= C_1 C_4 \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

$$y(n, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l} \rightarrow ⑨$$

But given $y(n_0, 0) = a \sin \frac{n_0 \pi x}{l}$

From ⑨ $\rightarrow a \sin \frac{n_0 \pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

$$a \sin \frac{n_0 \pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$a \sin \frac{n_0 \pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \dots$$

$$\therefore a = b_1$$

Hence

Putting in ⑨

$$y(n, t) = b_1 \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

$$y(n, t) = a \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

Ans.

(B) Solve completely the equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, representing the vibrations of a string of length l , fixed at both ends, given that $y(0, t) = 0$, $y(l, t) = 0$, $y(x, 0) = f(x)$ and $\frac{\partial y}{\partial t}(x, 0) = 0$, $0 < x < l$.

Sol. Given eqn $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$

We know that solⁿ of (1) is

$$y(x, t) = [c_1 \cos cpt + c_2 \sin cpt] [c_3 \cos px + c_4 \sin px] \rightarrow (2)$$

Given Boundary conditions are

$$y(0, t) = 0 \rightarrow (3)$$

$$y(l, t) = 0 \rightarrow (4)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \rightarrow (5) \quad \begin{matrix} \text{Initial Transverse Velocity of any point} \\ \text{of any string is zero} \end{matrix}$$

And $y(x, 0) = f(x) \rightarrow (6)$

Applying boundary condition in (2)

$$\therefore y(0, t) = [c_1 \cos cpt + c_2 \sin cpt] [c_3 \cos 0 + c_4 \sin 0]$$

$$0 = [c_1 \cos cpt + c_2 \sin cpt] c_3 + 0$$

$$0 = c_3$$

Putting in (2)

$$y(x, t) = [c_1 \cos cpt + c_2 \sin cpt] [c_4 \sin px] \rightarrow (7)$$

Again Using second boundary condition

$$y(l, t) = [c_1 \cos cpt + c_2 \sin cpt] [c_4 \sin pl]$$

$$0 = [c_1 \cos cpt + c_2 \sin cpt] c_4 \sin pl$$

which is satisfied when $\sin pl = 0$

$$\sin pl = \sin n\pi$$

$$p = \frac{n\pi}{l} \quad \text{for } n=1, 2, 3, \dots$$

$$\therefore y(x, t) = \left[c_1 \cos \frac{nh\pi t}{l} + c_2 \sin \frac{nh\pi t}{l} \right] c_4 \sin \frac{h\pi x}{l} \rightarrow (8)$$

$$\therefore \frac{\partial y}{\partial t} = \left[-c_1 \sin \frac{nh\pi t}{l} \cdot \left(\frac{nh\pi}{l} \right) + c_2 \cos \frac{nh\pi t}{l} \cdot \left(\frac{nh\pi}{l} \right) \right] c_4 \sin \frac{h\pi x}{l}$$

$$\therefore \left(\frac{\partial y}{\partial t} \right)_{t=0} = \left[-C_1 \cdot 0 + C_2 \left(\frac{c_1 n \pi}{l} \right) \right] c_4 \sin \frac{n \pi x}{l}$$

$$0 = c_2$$

Pulling in 8

$$y(n, t) = \left[C_1 \cos \frac{n \pi t}{l} + 0 \right] c_4 \sin \frac{n \pi x}{l}$$

$$y(n, t) = C_1 c_4 \cos \frac{n \pi t}{l} \cdot \sin \frac{n \pi x}{l} \quad \rightarrow \textcircled{9}$$

$$y(n, t) = \sum_{n=1}^{\infty} a_n \cos \frac{c_1 n \pi t}{l} \cdot \sin \frac{n \pi x}{l} \quad \rightarrow \textcircled{10}$$

But Given $y(n, 0) = f(x)$

$$\therefore y(n, 0) = \sum_{n=1}^{\infty} a_n \cos \frac{c_1 n \pi \cdot 0}{l} \cdot \sin \frac{n \pi x}{l} \quad \leftarrow \textcircled{11}$$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n \pi x}{l}$$

a_n can be calculated Using Fourier Sine Series as

$$a_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n \pi x}{l} dx$$

Hence complete sol's

$$y(n, t) = \sum_{n=1}^{\infty} a_n \cos \frac{c_1 n \pi t}{l} \cdot \sin \frac{n \pi x}{l} \quad \boxed{\text{Ans}}$$

If a string of length l is initially at rest in equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l}$, find the displacement $y(x, t)$

Sol: We know that the equation for the vibrations of the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \rightarrow \textcircled{1}$$

And solution of eqn $\textcircled{1}$ is

$$y(x, t) = (c_1 \cos cxt + c_2 \sin cxt)(c_3 \cos px + c_4 \sin px) \rightarrow \textcircled{2}$$

And Boundary conditions are

$$y(0, t) = 0 \rightarrow \textcircled{3}$$

$$y(l, t) = 0 \rightarrow \textcircled{4}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3 \frac{\pi x}{l} \rightarrow \textcircled{5}$$

$$\underline{y(x, 0)} = 0 \rightarrow \textcircled{6}$$

Using first Boundary condition

$$\therefore y(0, t) = (c_1 \cos cxt + c_2 \sin cxt)(c_3 + 0)$$

$$0 = c_3$$

Putting in $\textcircled{2}$

$$y(x, t) = (c_1 \cos cxt + c_2 \sin cxt)(c_4 \sin px) \rightarrow \textcircled{2}$$

Using second Boundary condition.

$$y(l, t) = (c_1 \cos cxt + c_2 \sin cxt) c_4 \sin pl$$

$$0 = (c_1 \cos cxt + c_2 \sin cxt) c_4 \sin pl$$

$$\text{Which satisfies only } \sin pl = 0 \Rightarrow pl = n\pi \\ p = \frac{n\pi}{l}$$

Putting in $\textcircled{2}$

$$y(x, t) = [c_1 \cos \frac{ch\pi}{l} t + c_2 \sin \frac{ch\pi}{l} t] c_4 \sin \frac{n\pi x}{l} \rightarrow \textcircled{8}$$

Now using fourth Boundary condition

$$y(x, 0) = [c_1 \cos \frac{ch\pi}{l} \cdot 0 + c_2 \sin \frac{ch\pi}{l} \cdot 0] c_4 \sin \frac{n\pi x}{l}$$

$$0 = c_1 c_4 \sin \frac{n\pi x}{l} \Rightarrow c_1 = 0$$

Putting in $\textcircled{8}$

$$y(x, t) = \underbrace{c_2 c_4}_{c_2} \sin \frac{ch\pi}{l} t \cdot \sin \frac{n\pi x}{l} \rightarrow \textcircled{9}$$

Now using ^{by} Third Boundary condition,

$$\frac{\partial y}{\partial t} = \cancel{\sum_{n=1}^{\infty}} b_n \cos \frac{ch\pi}{l} t \left(\frac{ch\pi}{l} \right) \cdot \sin \frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} b_n \frac{ch\pi}{l} \cdot \sin \frac{n\pi x}{l}$$

$$b \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{c\pi n}{l} \cdot \sin \frac{n\pi x}{l}$$

$$\frac{b}{4} \left[3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right] = b_1 \frac{c\pi}{l} \sin \frac{\pi x}{l} + b_2 \frac{2c\pi}{l} \sin \frac{2\pi x}{l} + b_3 \frac{3c\pi}{l} \sin \frac{3\pi x}{l}$$

$$\therefore \frac{3b}{4} = b_1 \frac{c\pi}{l} \Rightarrow b_1 = \frac{3bl}{4\pi c}, b_2 = 0$$

And

$$-\frac{b}{4} = b_3 \frac{3c\pi}{l} \Rightarrow b_3 = -\frac{bl}{12\pi c}, b_4 = 0, b_5 = 0$$

Putting these values in ④

$$y(x, t) = b_1 \sin \frac{c\pi}{l} t \cdot \sin \frac{\pi x}{l} + b_2 \sin \frac{2c\pi}{l} t \cdot \sin \frac{2\pi x}{l} + b_3 \sin \frac{3c\pi}{l} t \cdot \sin \frac{3\pi x}{l}$$

$$y(x, t) = \frac{3bl}{4\pi c} \sin \frac{c\pi}{l} t \cdot \sin \frac{\pi x}{l} + 0 - \frac{bl}{12\pi c} \sin \frac{3c\pi}{l} t \cdot \sin \frac{3\pi x}{l} \quad | \text{ Ans}$$

Q. 2002
A string is stretched and fastened to two points l apart.

Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time $t=0$, find the displacement of any point on the string at a distance a from one end at time t .

Sol^h We know that the vibrations of a stretched string are given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow ①$$

And sol^h of ① are

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 \cos px + c_4 \sin px) \rightarrow ②$$

And Boundary conditions are

$$y(0, t) = 0 \rightarrow ③$$

$$y(l, t) = 0 \rightarrow ④$$

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \rightarrow ⑤$$

$$y(x, 0) = k(lx - x^2) \rightarrow ⑥$$

Using first Boundary condition

$$\therefore y(0, t) = (c_1 \cos cpt + c_2 \sin cpt) (c_3 + 0)$$

$$0 = c_3 \Rightarrow c_3 = 0 \text{ Put this in } ②$$

$$y(x, t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px \rightarrow ⑦$$

Using Second Boundary Condition.

$$y(l, t) = (c_1 \cos c_1 pt + c_2 \sin c_1 pt)(c_4 \sin pl)$$

$$0 = (c_1 \cos c_1 pt + c_2 \sin c_1 pt)c_4 \sin pl$$

$$0 = \sin pl \Rightarrow pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

Putting in ⑦

$$y(x, t) = c_1 \cos c_1 pt + c_2 \sin c_1 pt$$

$$y(n, t) = (c_1 \cos \frac{cn\pi}{l} t + c_2 \sin \frac{cn\pi}{l} t) c_4 \sin \frac{n\pi x}{l} \quad \text{--- } ⑧$$

NOW Using Third boundary condition

$$\frac{\partial y}{\partial t} = \left[-c_1 \sin \frac{cn\pi}{l} t \left(\frac{cn\pi}{l} \right) + c_2 \cos \frac{cn\pi}{l} t \left(\frac{cn\pi}{l} \right) \right] c_4 \sin \frac{n\pi x}{l}$$

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 + c_2 \left(\frac{cn\pi}{l} \right) c_4 \sin \frac{n\pi x}{l}$$

$$0 = c_2 \quad \text{putting in } ⑧$$

$$y(n, t) = \left[c_1 \cos \frac{cn\pi}{l} t + 0 \right] c_4 \sin \frac{n\pi x}{l}$$

$$y(n, t) = c_1 c_4 \cos \frac{cn\pi}{l} t \cdot \sin \frac{n\pi x}{l}$$

$$y(n, t) = \sum_{n=1}^{\infty} b_n \cos \frac{cn\pi}{l} t \cdot \sin \frac{n\pi x}{l} \quad \text{--- } ⑨$$

Using ⑨ Last boundary condition

$$y(x, 0) = \sum_{n=1}^{\infty} b_n \cdot 1 \sin \frac{n\pi x}{l}$$

$$K(lx-x^2) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{--- } ⑩$$

obviously ⑨ represents the expansion of $f(x)$ in the form of a Fourier sine series, therefore

$$b_n = \frac{2}{l} \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l K(lx-x^2) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} K \left[\left((n-n^2) \left(-\cos \frac{n\pi x}{l} \right) \cdot \frac{l}{n\pi} \right)_0^l + \int_0^l (l-2x) \cos \frac{n\pi x}{l} \cdot \frac{l}{n\pi} dx \right]$$

$$= \frac{2}{l} K \left[(0+0) + \frac{l}{n\pi} \left[\left((l-2x) \sin \frac{n\pi x}{l} \cdot \frac{l}{n\pi} \right)_0^l + \int_0^l 2 \sin \frac{n\pi x}{l} \cdot \frac{l}{n\pi} dx \right] \right]$$

$$= \frac{2}{l} K \cdot \frac{l}{n\pi} \left[-l \sin \frac{n\pi l}{l} \cdot \frac{l}{n\pi} - 2 \frac{l^2}{n^2 \pi^2} \left(\cos \frac{n\pi l}{l} \right)_0^l \right]$$

Dimensional Heat Flow

$$\frac{\partial q}{\partial t} = C^2 \frac{\partial^2 q}{\partial x^2}$$

Solution of The Heat Equation:-

The Heat Equation is

$$\frac{\partial q}{\partial t} = C^2 \frac{\partial^2 q}{\partial x^2} \rightarrow \textcircled{1}$$

Let $u = x \cdot T \rightarrow \textcircled{2}$
 be a solution of $\textcircled{1}$, where x is a function of x only and
 T is a function of t only.

Then $\frac{\partial q}{\partial t} = x \cdot \frac{\partial T}{\partial t}$

$$\frac{\partial q}{\partial x} = T \frac{\partial x}{\partial x}, \frac{\partial^2 q}{\partial x^2} = T \frac{\partial^2 x}{\partial x^2}$$

Substituting these values in $\textcircled{1}$:

$$x \cdot \frac{\partial T}{\partial t} = C^2 \cdot T \frac{\partial^2 x}{\partial x^2}$$

$$x \cdot T' = C^2 T x''$$

Divide by $x \cdot T$

$$\frac{T'}{T} = C^2 \frac{x''}{x}$$

$$\therefore \frac{x''}{x} = \frac{1}{C^2} \cdot \frac{T'}{T} \rightarrow \textcircled{3}$$

Now L.H.S. of $\textcircled{3}$ is function of x only and the R.H.S. is function of t only. Since x and t are independent variables, this can hold only when both sides reduce to a constant say p^2 . The equation $\textcircled{3}$ leads to the ordinary differential equations.

$$\therefore \frac{x''}{x} = \frac{1}{C^2} \cdot \frac{T'}{T} = p^2 \rightarrow \textcircled{4}$$

Case 1 - When p^2 is positive

$$\therefore \frac{x''}{x} = p^2$$

$$x'' - p^2 x = 0$$

Which is ordinary diff. eqn.

$$\therefore A.E. \text{ is } m^2 - p^2 = 0$$

$$m = \pm p$$

$$C.F. = C_1 e^{px} + C_2 e^{-px}$$

$$\therefore u = x \cdot T = [C_1 e^{px} + C_2 e^{-px}] \cdot C_3 e^{C^2 p^2 t} \rightarrow \textcircled{5}$$

$$\frac{T'}{T} = C^2 p^2$$

$$\frac{1}{T} \frac{dT}{dt} = C^2 p^2$$

$$\frac{dT}{T} = C^2 p^2 dt$$

$$\log T = C^2 p^2 t + \log C_3$$

$$\log \frac{T}{C_3} = C^2 p^2 t$$

$$\therefore T = C_3 e^{C^2 p^2 t}$$

Case - 2 $\rightarrow P^2$ is negative

$$\therefore \frac{x''}{x} = -P^2$$

$$x'' + P^2 x = 0$$

ordinary diff. eqn.

$$\therefore A.E. \text{ is. } m^2 + P^2 = 0$$

$$m^2 = -P^2$$

$$m = \pm iP$$

$$C.F. = (C_1 \cos Px + C_2 \sin Px)$$

$$\therefore x = C_1 \cos Px + C_2 \sin Px$$

$$\star \frac{1}{c^2} \cdot \frac{T'}{T} = -P^2$$

$$\frac{T'}{T} = -P^2 c^2$$

$$\therefore \frac{dT}{dt} = -P^2 c^2 T$$

$$\frac{dT}{T} = -P^2 c^2 dt$$

on Int

$$\log T = -P^2 c^2 t + \log C_3$$

$$\frac{T}{C_3} = e^{-P^2 c^2 t}$$

$$\frac{T}{C_3} = C_3 e^{-P^2 c^2 t}$$

Solution is $u = x \cdot T$

$$u = [C_1 \cos Px + C_2 \sin Px] \cdot C_3 e^{-P^2 c^2 t} \rightarrow \textcircled{6}$$

Case 3 $\rightarrow P^2$ is zero.

$$\therefore \frac{x''}{x} = 0$$

$$x'' = 0$$

$$\therefore x = C_1 + C_2 x$$

$$\frac{1}{c^2} \cdot \frac{T'}{T} = 0$$

$$T' = 0$$

$$\frac{dT}{dt} = 0$$

$$T = C_3$$

Solution is $u = x \cdot T$

$$u = (C_1 + C_2 x) \cdot C_3 \rightarrow \textcircled{7}$$

of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. since u ~~decreases~~ decreases as time t increases, the only suitable solution of the heat equation is

$$u = [C_1 \cos Px + C_2 \sin Px] \cdot C_3 e^{-P^2 c^2 t}$$

2006 *

Q. Determine the solution of one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \text{ subject to the boundary}$$

Conditions $u(0, t) = 0, u(l, t) = 0, (t > 0)$ and the initial

conditions $u(x, 0) = x, l$ being the length of the bar.

Sol: Given Heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow ①$$

As we know that By Separation of Variable method it is solved given.

~~Unknown~~ \rightarrow ~~to the soln of~~ ①

$$u(x, t) = [C_1 \cos px + C_2 \sin px] C_3 e^{-c^2 p^2 t} \rightarrow ②$$

Boundary Condition are

$$u(0, t) = 0 \rightarrow ③$$

$$u(l, t) = 0 \rightarrow ④$$

$$u(x, 0) = x \rightarrow ⑤$$

Now using first condition i.e $u(0, t) = 0$

So putting $x=0$ in ②

$$\therefore u(0, t) = (C_1 + 0) C_3 e^{-c^2 p^2 t}$$

0 = C₁ putting in ②

$$\therefore u(x, t) = [C_2 \sin px] C_3 e^{-c^2 p^2 t} \rightarrow ⑥$$

Now using second condition i.e $u(l, t) = 0$

i.e putting $x=l$ in ⑥, we get

$$u(l, t) = [C_2 \sin pl] C_3 e^{-c^2 p^2 t}$$

$$0 = \sin pl$$

$$\sin nl = \sin pl \Rightarrow pl = n\pi \\ p = \frac{n\pi}{l}, \quad n = 1, 2, 3, \dots$$

∴ Putting the value of p in ⑥

$$u(x, t) = [C_2 \sin\left(\frac{n\pi x}{l}\right)] C_3 e^{-c^2 \frac{n^2 \pi^2}{l^2} t} \rightarrow ⑦$$

Now using third condition $u(x, 0) = x$

satisfying to sin(7)

$$\therefore u_{(n)} \in \{c_1 \sin \frac{n\pi y}{l}\}_{c_1}$$

Acceq

Putting in (7)

$$u(n,t) = c_1$$

$$\therefore u(n, t) = c_1 c_3 e^{-\frac{n^2 \pi^2 c^2}{l^2} t} \cdot \sin \frac{n\pi y}{l}$$

$$u(n, t) = b_n e^{-\frac{n^2 \pi^2 c^2}{l^2} t} \cdot \sin \frac{n\pi y}{l} \rightarrow (7)$$

Now using Third Condition

putting $t=0$

$$u(n, 0) = b_n$$

$$x = b_n \sin \frac{n\pi y}{l}$$

In General

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{l} \rightarrow (8)$$

Here b_n is given as by Half Range-Sinus

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \left[\left\{ x \left(-\frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) \right\}_0^l - \left\{ \left(-\frac{\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right\}_0^l \right]$$

$$= \frac{2}{l} \left[-\frac{l}{n\pi} \left\{ l \cos n\pi \right\} + \frac{l^2}{n^2 \pi^2} \left\{ \sin n\pi \right\} \right]$$

$$b_n = -\frac{2}{n\pi} l \cos n\pi + \frac{2l^2}{n^2 \pi^2} \sin n\pi$$

$$b_n = -\frac{2l \cos n\pi}{n\pi} + 0$$

putting this value of b_n in (7)

$$u(n, t) = \sum_{n=1}^{\infty} -\frac{2l \cos n\pi}{n\pi} \sin \frac{n\pi y}{l} \cdot e^{-\frac{n^2 \pi^2 c^2}{l^2} t} \quad \text{Ans.}$$

Solve the following boundary value problem which arises in the heat conduction in a rod

Exercise (30.3)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0, t) = u(l, t) = 0,$$

$$u(x, 0) = 100 \frac{x}{l}$$

(Q. No. 4)

OR

An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state condition prevails. If B is suddenly reduced to 0°C and maintained at 0°C , find the temperature at distance x from A at time t , solve the equation of heat $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, by method of separation of variables and obtain the solution.

Soln. Given heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow \textcircled{1}$$

$$\text{Let } u = X \cdot T \longrightarrow \textcircled{2}$$

be the solution of eqn $\textcircled{1}$, where X is function of x only and T is function of t only.

$$\therefore \frac{\partial u}{\partial t} = X \cdot \frac{\partial T}{\partial t}$$

$$\frac{\partial u}{\partial x} = T \frac{\partial X}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = T \frac{\partial^2 X}{\partial x^2}$$

Putting these values in $\textcircled{1}$

~~$$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 X}{\partial x^2}$$~~

$$X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$X T' = c^2 T X''$$

Divide by $X \cdot T$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T} \longrightarrow \textcircled{2}$$

$$\frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T'}{T} = -P^2 (\text{say}) \longrightarrow \textcircled{3}$$

$$\therefore \frac{X''}{X} = -P^2$$

$$X'' + P^2 X = 0$$

which is linear ordinary diff eqn.

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$m = \pm i p$$

$$C.F = C_1 \cos px + C_2 \sin px$$

$$P.I. = 0$$

$$\therefore X = C.F + P.I.$$

$$X = C_1 \cos px + C_2 \sin px$$

$$\text{And } \frac{T}{T_0} = -C^2 p^2$$

$$\frac{d}{dt} \frac{T}{T_0} = -C^2 p^2$$

$$\frac{dT}{T_0} = -C^2 p^2 dt$$

On Integration.

$$\log T = -C^2 p^2 t + \log C_3$$

$$\log \frac{T}{T_0} = -C^2 p^2 t$$

$$\frac{T}{T_0} = e^{-C^2 p^2 t}$$

$$T = C_3 e^{-C^2 p^2 t}$$

Put in

$$\therefore T = C_3 e^{-C^2 p^2 t}$$

$$\therefore u = X \cdot T$$

$$\therefore u(x, t) = [C_1 \cos px + C_2 \sin px] C_3 e^{-C^2 p^2 t} \quad \rightarrow (4)$$

Using Boundary condition.

$$u(0, t) = (C_1 \rightarrow 0) C_3 e^{-C^2 p^2 t}$$

$$0 = C_1 \quad \text{putting in (4)}$$

$$\therefore u(x, t) = [C_2 \sin px] C_3 e^{-C^2 p^2 t} \quad \rightarrow (5)$$

$$u(l, t) = [C_2 \sin pl] C_3 e^{-C^2 p^2 t}$$

$$0 = \sin pl$$

$$\sin n\pi = \sin pl \Rightarrow p = \frac{n\pi}{l}$$

$$\therefore u(x, t) = C_2 \sin \frac{n\pi x}{l} \cdot C_3 e^{-C^2 p^2 t}$$

$$u(n, t) = C_2 \sin \frac{n\pi x}{l} \cdot C_3 e^{-\frac{C^2 n^2 \pi^2}{l^2} t} \quad \rightarrow (6)$$

$\therefore u(x, t)$

$$u(n, t) = C_2 C_3 \cdot \sin \frac{n\pi x}{l} \cdot e^{-\frac{C^2 n^2 \pi^2}{l^2} t}$$

$$u(n, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{C^2 n^2 \pi^2}{l^2} t}$$

Boundary condition

$$u(0, t) = 0 \rightarrow (i)$$

$$u(l, t) = 0 \rightarrow (ii)$$

$$u(x, 0) = 100 \cdot \frac{x}{l} \rightarrow (iii)$$

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^0$$

$$100 \frac{x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

By Half-Range sine series

$$b_n = \frac{2}{l} \int_0^l 100 \frac{x}{l} \cdot \sin \frac{n\pi x}{l} dx$$

$$b_n = \frac{2}{l} \cdot \frac{100}{l} \left[\left(-x \frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right)_0^l + \frac{l^2}{h^2 \pi^2} \left(\sin \frac{n\pi x}{l} \right)_0^l \right]$$

$$= \frac{200}{l^2} \left[\frac{l}{n\pi} (-l \cos n\pi) + \frac{l^2}{h^2 \pi^2} (\sin n\pi - 0) \right]$$

$$= \frac{200}{l^2} \left[-\frac{l^2}{n\pi} \cos n\pi + 0 \right]$$

$$= \frac{200}{l^2} \left[-\frac{l^2}{n\pi} (-1)^n \right]$$

$$b_n = \frac{200}{n\pi} (-1)^{n+1}$$

$$\therefore u(x, t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{c^2 h^2 \pi^2}{l^2} t}$$

$$u(n, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2 h^2 \pi^2}{l^2} t}$$

Ans.

2007, 09 Q. Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is

$$\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$$

Sol. Let one dimensional heat flow eqn is.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \rightarrow ①$$

And boundary conditions are

$$u(0, t) = 0 \quad \rightarrow ②$$

$$u(2, t) = 0 \quad \rightarrow ③$$

And initial condition

$$u(x, 0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} \quad \rightarrow ④$$

We know that solⁿ of ① is

$$u(x, t) = (C_1 \cos \beta n + C_2 \sin \beta n) C_3 e^{-c^2 \beta^2 t} \quad \rightarrow ⑤$$

Using first condition $u(0, t) = 0$

$$0 = (C_1 + 0) C_3 e^{-c^2 \beta^2 t} \Rightarrow C_1 = 0$$

$$\therefore u(x, t) = (C_2 \sin \beta n) C_3 e^{-c^2 \beta^2 t} \quad \rightarrow ⑥$$

Using second condition

$$u(2, t) = 0 = (c_2 \sin p \cdot 2) c_3 e^{-c^2 p^2 t}$$

$$0 = \sin 2p \Rightarrow \sin n\pi = \sin 2p$$

$$2p = n\pi \Rightarrow p = \frac{n\pi}{2}$$

$$\therefore u(n, t) = (c_2 c_3 \sin \frac{n\pi x}{2}) e^{-c^2 \frac{n^2 \pi^2}{4} t}$$

$$u(n, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} e^{-c^2 \frac{n^2 \pi^2}{4} t} \rightarrow \textcircled{d}$$

using Third condition

$$u(n, 0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$$

$$\therefore \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} e^{-c^2 \frac{n^2 \pi^2}{4} t}$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} e^{-0}$$

$$\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} = b_1 \sin \frac{\pi x}{2} + b_2 \sin \frac{2\pi x}{2} + b_3 \sin \frac{3\pi x}{2} + b_5 \sin \frac{5\pi x}{2} + \dots$$

on comparing the coefficient of sin

$$(b_1 = 1), (b_5 = 3), b_2 = b_3 = b_4 = \dots = 0$$

$$\therefore u(n, t) = b_1 \sin \frac{\pi x}{2} e^{-\frac{c^2 \pi^2}{4} t} + b_2 \sin \frac{2\pi x}{2} e^{-\frac{c^2 4\pi^2}{4} t} + b_5 \sin \frac{5\pi x}{2} e^{-\frac{c^2 25\pi^2}{4} t},$$

$$u(n, t) = \sin \frac{\pi x}{2} e^{-\frac{c^2 \pi^2}{4} t} + 3 \sin \frac{5\pi x}{2} e^{-\frac{c^2 25\pi^2}{4} t}$$

$$\frac{\partial u}{\partial x} = \frac{\pi}{2} \cos \frac{\pi x}{2} + \frac{15\pi}{2} \cos \frac{5\pi x}{2}$$

on putting both equations in $\frac{\partial u}{\partial t}$

$$\textcircled{1} \leftarrow \frac{\partial^2 u}{\partial t^2} = \frac{\rho G}{L^2}$$

is satisfied by above both

$$\textcircled{2} \leftarrow 0 = (k, \omega) N$$

$$\textcircled{3} \leftarrow 0 = (k, \omega) N$$

$$\textcircled{4} \leftarrow \frac{\pi^2}{L^2} \cos^2 \frac{\pi x}{2} + \frac{25\pi^2}{L^2} \cos^2 \frac{5\pi x}{2} = (k, \omega) N$$

$$\textcircled{5} \leftarrow 25(\cos^2 \omega t + \cos^2 \omega t) = (k, \omega) N$$

A rod of length l with its insulated sides is initially at a uniform temperature U . Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temperature function $U(x,t)$ is given by

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2}{l^2} t}$$

Where b_n is determined from the equation

$$b_n = \frac{2}{l} \int_0^l u_0(x) \cdot \sin \frac{n\pi x}{l} dx$$

Solⁿ. Let temperature of the bar at any time t at a point x distance from the origin be $U(x,t)$.

Then the eqn of one dimensional heat flow is

$$\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2} \rightarrow ①$$

$$\text{Let } U = X \cdot T \rightarrow ②$$

Where X is function of x only and T is function of t only.

$$\therefore \frac{\partial U}{\partial t} = X \frac{\partial T}{\partial t} \quad \text{and} \quad \frac{\partial U}{\partial x} = T \cdot \frac{\partial X}{\partial x}$$

$$\frac{\partial^2 U}{\partial x^2} = T \cdot \frac{\partial^2 X}{\partial x^2}$$

Putting in ①

$$X \frac{\partial T}{\partial t} = c^2 T \frac{\partial^2 X}{\partial x^2}$$

$$X T' = c^2 T X''$$

Divide by $X T$

$$\frac{T'}{T} = c^2 \frac{X''}{X}$$

$$\therefore \frac{X''}{X} = \frac{1}{c^2} \frac{T'}{T} = -\rho^2 \sin f \rightarrow ③$$

$$\therefore \frac{X''}{X} = -\rho^2$$

$$X'' = -\rho^2 X$$

$$X'' + \rho^2 X = 0$$

which is ordinary diff. eqn

$$\text{A.E. is } m^2 + \rho^2 = 0$$

$$m^2 = -\rho^2$$

$$m = \pm i\rho$$

$$\text{C.F.} = C_1 \cos \rho n + C_2 \sin \rho n$$

$$\frac{1}{c^2} \frac{T'}{T} = -\rho^2$$

$$\frac{T'}{T} = -\rho^2 c^2$$

$$\frac{1}{T} \frac{dT}{dt} = -\rho^2 c^2$$

$$\frac{dT}{T} = -\rho^2 c^2 dt$$

on Int

$$\log T = -\rho^2 c^2 t + C_3$$

$$\log \frac{T}{C_3} = -\rho^2 c^2 t$$

$$T = C_3 e^{-P^2 C^2 t}$$

$$\therefore u = x \cdot T = x e^{-P^2 C^2 t}$$

$$u(x, t) = [C_1 \cos Pn + C_2 \sin Pn] C_3 e^{-P^2 C^2 t} \rightarrow ④$$

Using initial condition

$$\begin{aligned} u(0, t) &= 0 \\ u(l, t) &= 0 \end{aligned} \quad] \rightarrow ⑤$$

$$\therefore 0 = [C_1 \cos 0 + 0] C_3 e^{-P^2 C^2 t}$$

$$0 = C_1 \cancel{e^{-P^2 C^2 t}} \Rightarrow C_1 = 0$$

Putting in ④

$$u(x, t) = C_2 \sin Pn x C_3 e^{-P^2 C^2 t} \rightarrow ⑥$$

$$u(l, t) = C_2 \sin Pl C_3 e^{-P^2 C^2 t}$$

$$0 = C_2 \sin Pl C_3 e^{-P^2 C^2 t}$$

$$\cancel{C_2} = 0 = \sin Pl$$

$$\sin n\pi = \sin Pl \Rightarrow R = \frac{n\pi}{l}$$

$$\therefore u(n, t) = C_2 C_3 \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 C^2 t}{l^2}}$$

$$u(n, t) = b_n e^{\frac{-n^2 \pi^2 C^2 t}{l^2}} \cdot \sin \frac{n\pi x}{l} \quad \text{when } C_2 C_3 = b_n$$

$$\therefore u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 C^2 t}{l^2}}$$

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{n^2 \pi^2 C^2 t}{l^2}} \rightarrow ⑦$$

Putting $t = \infty$

~~$$u(x, \infty) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \rightarrow ⑧$$~~

$$\sum_{n=1}^{\infty} b_n = \frac{1}{l} \int_0^l f(x) dx$$

$$b_n = \frac{1}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_0^l f(x) \sin nx dx$$

An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state condition prevail. If B is suddenly reduced to 0°C and maintained at 0°C , find the temperature at a distance x from A at time t , solve the equation of heat

(2004, 05)

by method of separation of variable and obtain the solution.

Soln → Given heat equation

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{--- (1)}$$

We know that the solution of (1) is

$$u(x, t) = [c_1 \cos px + c_2 \sin px] c_3 e^{-C^2 p^2 t} \quad \text{--- (2)}$$

with steady state condition

$$u(0, t) = 0 \quad \text{--- (3)}$$

$$u(l, t) = 0 \quad \text{--- (4)}$$

And Initial condition is →

$$u(x, 0) = \frac{100}{l} x \quad \text{--- (5)}$$

Using steady state condition in (2)

$$\therefore u(0, t) = [c_1 + 0] c_3 e^{-C^2 p^2 t}$$

$$0 = c_1 \Rightarrow c_1 = 0$$

Putting in (2)

$$u(x, t) = c_2 \sin px \cdot c_3 e^{-C^2 p^2 t} \quad \text{--- (6)}$$

$$u(l, t) = c_2 \sin pl \cdot c_3 e^{-C^2 p^2 t}$$

$$0 = c_2 \sin pl \cdot c_3 e^{-C^2 p^2 t}$$

$$0 = \sin pt \Rightarrow \sin nl = \sin pl$$

$$\therefore p = \frac{n\pi}{l}$$

Putting in (6)

$$u(x, t) = c_2 \sin \frac{n\pi x}{l} \cdot c_3 e^{-\frac{C^2 n^2 \pi^2}{l^2} t}$$

$$u(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cdot c_3 e^{-\frac{C^2 n^2 \pi^2}{l^2} t}$$

$$u(n, t) = b_n \sin \frac{n\pi x}{l} e^{-\frac{c^2 \pi^2 n^2}{l^2} t}$$

$$u(n, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{c^2 \pi^2 n^2}{l^2} t} \rightarrow \textcircled{7}$$

Using initial condition

$$u(n, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Which is half-range sine series for $\frac{100x}{l}$

$$\therefore b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$\textcircled{8} = \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[x \cdot \left\{ \frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right\}_0^l + \int_0^l \frac{\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} dx \right]$$

$$= \frac{200}{l^2} \left[-x \frac{l}{n\pi} \left\{ \cos \frac{n\pi l}{l} \right\}_0^0 + \frac{l^2}{n^2 \pi^2} \left(\sin \frac{n\pi l}{l} \right)_0^0 \right]$$

$$= \frac{200}{l^2} \left[-\frac{l}{n\pi} \left\{ (-1)^n - 1 \right\} + 0 \right]$$

$$= -\frac{200}{n\pi} (-1)^n$$

$$b_n = (-1)^n \frac{200}{n\pi} (-1)^n = \frac{200}{n\pi} (-1)^{n+1}$$

Putting in $\textcircled{7}$

$$u(n, t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{c^2 \pi^2 n^2}{l^2} t}$$

$$u(n, t) = \boxed{\frac{200}{n\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2 \pi^2 n^2}{l^2} t}}$$

Ay.