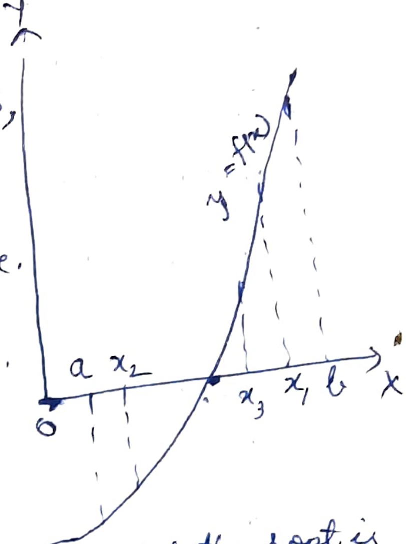
Methods for Solving non-linear equations →

To find the roots of an eqn $f(x)=0$, we start with a known approximate soln and apply the foll. methods:

1. Bisection Method →

The bisection method is used to find the roots of polynomial equation.
This method is based on intermediate value theorem for continuous fns.
- It works by narrowing the gap between the positive and negative intervals until it closes in on the correct answer.

This method narrows the gap between the positive and negative intervals.



- This method consists in locating the root of the eqn $f(x)=0$ b/w a & b. If $f(x)$ is continuous b/w a & b, and $f(a)$ and $f(b)$ are of opposite signs then there is a root b/w a and b.

For definiteness, let $f(a)$ be negative and $f(b)$ be +ve. Then the first approximation to the root is

$x_1 = \frac{1}{2}(a+b)$

If $f(x_1)=0$, then x_1 is the root of $f(x)=0$, otherwise, the root lies b/w a and x_1 or x_1 and b according as $f(x_1)$ is +ve or -ve. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

In fig. $f(x_1)$ is +ve, so that the root lies b/w a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a+x_1)$. If $f(x_2)$ is -ve, the root lies b/w x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1+x_2)$ and so on.

Ex: Find a real root of $x^3-x-1=0$, using the bisection method.

solⁿ →

$x^3-x-1=0$

$f(0) = 0^3-0-1 = -1$

$f(1) = 1^3-1-1 = -1$

$f(2) = 2^3-2-1 = 5$

$a=1,\ b=2$

*when
$f(x) = -ve$
$x = a$
$f(x) = f(a)$

*when
$f(x) = +ve$
$x = b$
$f(x) = f(b)$

approximation	a	b	f(a)	f(b)	$x = \frac{a+b}{2}$	f(x)	f(x)
1.	1	2	-1	5	1.5	0.875	+ve
2.	1	1.5	-1	0.875	1.25	-0.2968	$-$ve
3	1.25	1.5	-0.2968	0.875	1.375	0.2246	+ve
4	1.25	1.375	-0.2968	0.2246	1.3125	-0.0515	$-$ve
5	1.3125	1.375	-0.0515	0.2246	1.3437	0.0823	+ve
6	1.3125	1.3437	-0.0515	0.0823	1.3281	0.0144	+ve
7	1.3125	1.3281	-0.0515	0.0144	1.3203	-0.0187	$-$ve
8	1.3203	1.3281	-0.0187	0.0144	1.3242	-0.2078	$-$ve
9	1.3242	1.3281	-0.2078	0.0144	1.3261	0.0059	+ve
10	1.3242	1.3261	-0.2078	0.0059	1.3251	0.0016	+ve
	1.3242	1.3251	-0.2078	0.0016	1.3246	-0.0005	+ve

① when $f(x) = -$ve, $x = a$, $f(x) = f(a)$

② when $f(x) = +$ve, $x = b$, $f(x) = f(b)$

Hence the real root of the given eqn is 1.324 correct to three decimal.

Q. Find a root of the eqn $x^3 - 4x - 9 = 0$ using the bisection method in 4 stages.

Soln. $x^3 - 4x - 9 = 0$

$f(0) = -9$, $f(1) = -13$, $f(2) = -9$, $f(3) = (3)^3 - 4\times 3 - 9 = 27 - 12 - 9 = 6$

$\because$ $f(2)$ is $-$ve and $f(3)$ is +ve, a root lies b/w 2 and 3.

1^{st} approximation to the root is $x_1 = \frac{1}{2}(2+3) = 2.5$ —— (1)

Then $f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375$ i.e, $-$ve

$\therefore$ The root lies b/w x_1 and 3. Thus the 2^{nd} approximation to the root is

$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$ —— (2)

Then $f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ i.e, +ve

$\therefore$ The root lies b/w x_1 and x_2. Thus the 3^{rd} approx. to the root is

$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$ —— (3)

Then $f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ i.e, $-$ve

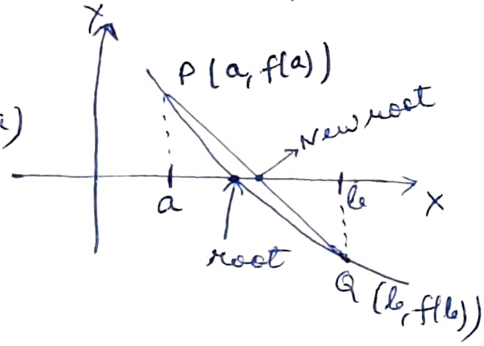
The root lies b/w x_2 and x_3. Thus the 4^{th} approximation to the root is

$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$ —— (4)

Hence the root is 2.6875 approximately.

2. Regula Falsi or False Position Method

It locates the root by line joining $P(a, f(a))$ and $Q(b, f(b))$ with a st. line.
The intersection of the line with the x-axis represent an improved estimate of the root.



The eqn of PQ is $y - f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$

This line touches x-axis, when $y=0$, thus

$$-f(a) = \frac{f(b) - f(a)}{b-a}(x-a)$$

$$\Rightarrow \quad x = a - \frac{(b-a)f(a)}{f(b) - f(a)}$$

$$\Rightarrow \quad x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$ is the new root.

False Position method

for finding root of $f(x) = 0$ also known as linear interpolation method or chord method or false position method

Let $f(x)$ be continuous where $x \in [a, b]$
with $f(a) f(b) < 0$
Assume $f(a) < 0$; $f(b) > 0$

Algorithm:

Step 1. For interval $[a, b]$, 1st approximation gives
$$x_k = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Step 2. check whether $f(x_k) > 0$ or $f(x_k) < 0$
If $f(x_k) > 0$ then root lies b/w $[a, x_k]$, then set $b = x_k$

If $f(x_k) < 0$ then root lies b/w $[x_k, b]$, then set $a = x_k$.

Step 3 If $[x_{k+1} - x_k] \leq \epsilon$ then x_k is the root and stop where $\epsilon > 0$ is the desired accuracy. If not go to Step 1.

Ex. Find the real root of the eqn $x^3 - 9x + 1 = 0$ by ~~Regulare~~-False position, correct upto four decimal place.

$\frac{1}{2} \times 10^{-n} \rightarrow$ Error.

Error upto 4 decimal places $= \frac{1}{2} \times 10^{-4} = .00005$

Soln. For $f(x) = x^3 - 9x + 1$

$f(0) = 1 > 0$

$f(1) = 1 - 9 + 1 = -9 < 0$

$[0, 1] \rightarrow$ Root lies b/w them.

$f(2) = 8 - 18 + 1 = -9 < 0$

$f(3) = 27 - 27 + 1 = 1 > 0$

Thus root lies b/w 2 and 3

Iteration No. (k)	a (−)	b (+)	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x)$	Error $\lvert x_{k+1} - x_k \rvert$	
1	2	3	2.9	−0.711		
2	2.9	3	2.9416	−0.0207	0.0416	
3	2.9416	3	2.9428	−0.0003	0.0012	$\not<$ 0.00005
4	2.9428	3	2.942817	−0.00005	0.00007	< 0.00005

Ans: 2.9428 correct to 4 decimal places

Ex. Find the real root of the eqn $xe^x - 3 = 0$ using Regula-falsi method correct to three decimal places.

Soln. For $f(x) = xe^x - 3$

$e = 2.718$

$2e^2$

~~$f(1) = -0.2817$~~

$f(0) = -3 < 0$

$f(1) = e^1 - 3 = < 0$

$f(2) = 2e^2 - 3 > 0$

$\therefore$ root lies b/w 1 & 2.

$f(1) = -0.2817 < 0$

$f(1.5) = 3.7225 > 0$

Thus root lies b/w 1 & 1.5

length is small so less no of iteration

or, $f(1) < 0$

$f(2) > 0$

$\therefore$ root lies b/w 1 & 2

length is larger so large no. of iteration.

Error up to 3 decimal places $= \frac{1}{2} \times 10^{-3} = 0.0005$

Iteration No. k	a (−)	b (+)	$x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x)$	Error $\lvert x_{k+1} - x_k \rvert$
1	1	1.5	1.0352	−0.0852	-
2	1.0352	1.5	1.0456	−0.0252	0.0104
3	1.0456	1.5	1.0487	−0.0071	0.0031
4	1.0487	1.5	1.0496	−0.0018	0.0009
5	1.0496	1.5	1.0498	−0.00064	0.0002

Thus, reqd root is 1.049 correct to three decimal places.

3. Secant Method (Chord method)

[* when $f(x_n) = f(x_{n-1})$ this method (secant method) fails].

(Improved Form of Regula Falsi Method)

working Rule :

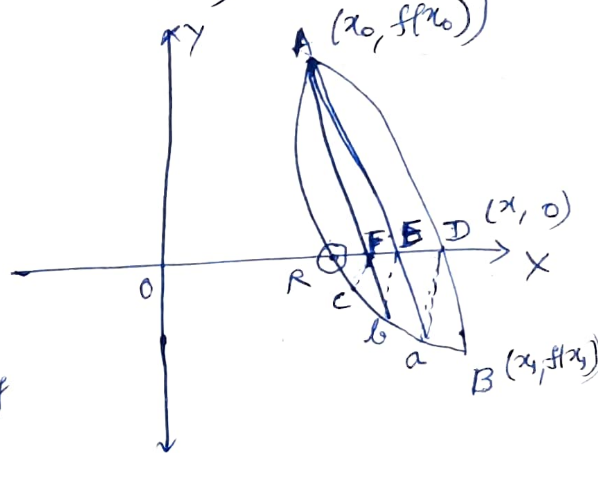
1. Let $f(x) = 0$ be the given eqn — (1)
2. Find x_0 and x_1 such that $f(x_0) < 0$ and $f(x_1) > 0$ (or $f(x_0) \cdot f(x_1) < 0$)
3. Find 1st approximate root by $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

   Find $f(x_2)$

4. Find 2nd approximate root by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

   Calculate $f(x_3)$



5. Find 3rd approximate root of (1) by

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

   Calculate $f(x_4)$

Repeat above process upto reqd approximate root.

General Formula

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$n = 1, 2, 3, 4, \ldots$

Q. Find the real root eqn $x^3 - x - 1 = 0$ by Secant method correct upto 4 decimal places.

Soln: Let $f(x) = x^3 - x - 1 = 0$ — (1)

To find x_0 and x_1 :- $f(0) = -1 < 0$

$f(1) = -1 < 0$

$f(2) = 8 - 2 - 1 = 5 > 0$

Roots lie b/w 1 & 2

$f(1.5) = 0.875 > 0$

$f(1.4) = 0.343 > 0$

$f(1.3) = -0.103 < 0$

Choosing $x_0 = 1.3$ and $x_1 = 1.4$

$f(x_0) = -0.103$, $f(x_1) = 0.343$

~~Hence the correct root upto 4 decimal place $x_0 = 1.3247$~~

1st approximation root by Secant method →

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 1.323042$$

$f(x_2) = -0.007136 < 0$

2nd approximate root →

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 1.324605$$

$f(x_3) = -0.000481 < 0$

3rd approximate root →

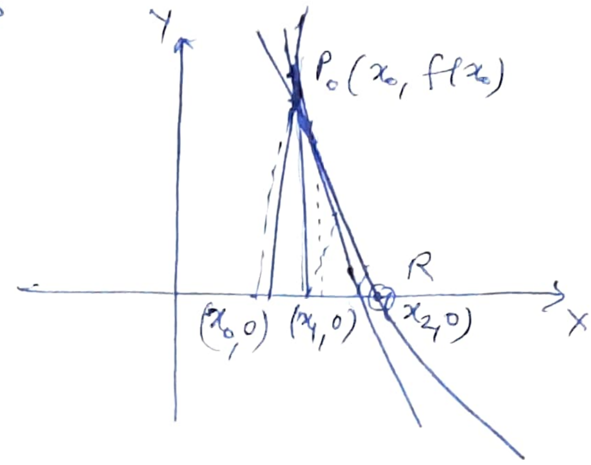
$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 1.324717$$

$f(x_4) = -0.000004$

Hence the correct root upto 4-decimal places

$x = 1.3247$.

4. Newton-Raphson Method →



Eqn of Tangent

$$Y - f(x_0) = f'(x_0)(x - x_0)$$

$$0 - f(x_0) = f'(x_0)(x_1 - x_0)$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0$$

Q. Find by Newton-Raphson method a root of eqn $x^3 - 2x - 5 = 0$ correct upto 3 decimal places.

Soln. →

$f(x) = x^3 - 2x - 5$, $f'(x) = 3x^2 - 2$

$f(0) = -5 < 0$

$f(1) = 1 - 2 - 5 = -6 < 0$

$f(2) = 8 - 4 - 5 = -1 < 0$ (−ve)

$f(3) = 27 - 6 - 5 = 27 - 11 = 16$ (+ve)

Since $f(2)$ & $f(3)$ have opposite sign so root lies b/w 2 and 3.

Let initial approximation $(x_0) = 2.5$

Now, $f(x) = x^3 - 2x - 5$, $f'(x) = 3x^2 - 2$

we know N-R formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

when $n = 0$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{(2.5)^3 - 2(2.5) - 5}{3(2.5)^2 - 2} = 2.1641$$

when $n = 1$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1641 - \frac{(2.1641)^3 - 2(2.1641) - 5}{3(2.1641)^2 - 2} = 2.0971$$

when $n = 2$,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0971 - \frac{(2.0971)^3 - 2(2.0971) - 5}{3(2.0971)^2 - 2} = 2.0945$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.0945 - \frac{(2.0945)^3 - 2(2.0945) - 5}{3(2.0945)^2 - 2} = 2.0945$$

Since the reqd root is $\underline{2.094}$. ($\because$ roots are repeated upto 3 decimal place).

Q. Use Newton Raphson method to find a real root of $\cos x - x e^{x} = 0$ corrected to four decimal places.

Soln. →
$f(x) = \cos x - x e^{x}$, $f'(x) = -\sin x - e^{x} - x e^{x}$
$f(0) = 1$ $= -\sin x - e^{x}(x+1)$
$f(1) = \cos 1 - 1 \cdot e^{1} = -2.1779$

$x_0 = 0.5$ | $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n + \frac{\cos x_n - x_n e^{x_n}}{\sin x_n + e^{x_n}(x_n+1)}$$

Put $n=0$

$$x_1 = x_0 + \frac{\cos x_0 - x_0 e^{x_0}}{\sin x_0 + e^{x_0}(x_0+1)} = 0.5 + \frac{\cos(0.5) - (0.5)\, e^{0.5}}{\sin(0.5) + e^{0.5}(0.5+1)}$$

$= 0.5182$

Put $n=1$

$$x_2 = x_1 + \frac{\cos x_1 - x_1 e^{x_1}}{\sin x_1 + e^{x_1}(x_1+1)} = 0.5182 + \frac{\cos(0.5182) - (0.5182)\, e^{(0.5182)}}{\sin(0.5182) + e^{(0.5182)}(0.5182+1)}$$

$= 0.5182$.

Hence root is 0.5182.

Q. Apply N-R method to solve the eqn $2(x-3) = \log_{10} x$

Soln. →
$f(x) = 2(x-3) = \log_{10} x$
$f(3) = 6 - 6 - \log_{10} 3 = -0.47712$
$f(4) = 8 - 6 \log_{10} 4 = 1.39794$

| $\log_{10} x = 0.4343 \log_e x$

Let $x_0 = 3.5$, $f(x) = 2x - 6 - 0.4343 \log_e x$

$f'(x) = 2 - \dfrac{0.4343}{x}$

Put $n = 0$, $x_1 = x_0 - \left(\dfrac{2x_0^2 - 6x_0 - 0.4343\, x_0 \log_e x_0}{2x_0 - 0.4343}\right)$

$x_1 = 3.25696$

Put $n = 1$

$x_2 = 3.256366$

put $n = 2$

$x_3 = 3.256$, Hence reqd root is 3.256

Methods for System of linear equations →

1. Gauss elimination →

In this method the coefficient matrix reduces to upper triangular matrix by elementary row operation.

Procedure for solving the system of linear equation by Gauss-elimination method is described below :-

Step I : Write the augmented matrix $[A : B]$ of the system.

Step II : Use the elementary row operations to reduce, the augmented matrix $[A | B]$ to a matrix $[C | D]$ in row echlon form.

Step III ; Write the linear system corresponding to the echlon matrix $[C | D]$ and use back substitution to obtain the solution.

Remark: If the final augmented matrix are of the form

$$[0 \; 0 \; \cdots \; 0 : C]$$

with all zeroes on the left and non-zero on the right, then the system $AX = B$ is inconsistent. ~~Otherwise~~ otherwise, the system $AX = B$ will be consistent. Moreover in this case, the system has either (a) a unique solution (b) infinitely many solutions.

The elimination procedure described above to determine the unknowns is called the Gauss elimination method.

Q. Solve the system of eqns by GEM

$$\left.\begin{aligned} x + y &= 1 \\ x + 2y &= 3 \end{aligned}\right\} \quad \text{—(1)}$$

Soln. →

$$[A : B] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow x + y = 1, \; y = 2$$

$\therefore x = -1, y = 2$ is the soln of the given eqns.

Q. $x_1 + x_2 + x_3 = 6$

$3x_1 + 3x_2 + 4x_3 = 20$

$2x_1 + x_2 + 3x_3 = 13$

Soln. →

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 13 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 3 & 4 & 20 \\ 2 & 1 & 3 & 13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

Now, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$ $\Rightarrow x + y + z = 6$

$\Rightarrow -y + z = 1$

$z = 2$

$\therefore z = 2, y = 1, x = 3$

Q. $x + 2y + 3z = 1$

$x + 3y + 5z = 2$

$2x + 5y + 9z = 3$

Soln. $\rightarrow$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The augmented matrix for the given system is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & 2 \\ 2 & 5 & 9 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore z = 0, \; y + 2z = 1, \; x + 2y + 3z = 1$

$\therefore z = 0, \; y = 1, \; x = -1.$

Solution of System of Linear Simultaneous Equations

Method to Solve

Direct Method (Exact Methods)

1. Gauss Elimination Method
2. Gauss Jordan Method
3. Crout's Method

Indirect Method (Iterative Methods)

1. Gauss - Jacobi Method
2. Gauss - Seidal Method
3. Relaxation Method

Gauss Jacobi Iteration Method

This method is applicable to the system of equation in which leading diagonal elements of co-efficient matrix are dominant (large in magnitude) in their respective rows.

Working Rule : Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Note : Diagonal dominance property must be satisfied

i.e, $|a_{11}| > |a_{12}| + |a_{13}|$

$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the eqn for x, y, z respectively.

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z)$$
$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z)$$
$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y)$$

Iteration 1 : Put $x = x_0,\ y = y_0,\ z = z_0$

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$$
$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_0 - a_{23}z_0)$$
$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_0 - a_{32}y_0)$$

Again substituting these values of x, y, z, the next approximation is obtained.

The above iteration process is continued until two successive approximations are equal.

Q. Solve the foll. system of equations by Gauss Jacobi Method

$$6x + 2y - z = 4$$
$$x + 5y + z = 3$$
$$2x + y + 4z = 27$$

Soln: Rewriting the eqns

$$\left.\begin{aligned} x &= \tfrac{1}{6}(4 - 2y + z) \\ y &= \tfrac{1}{5}(3 - x - z) \\ z &= \tfrac{1}{4}(27 - 2x - y) \end{aligned}\right\} \quad \text{--- (1)}$$

Iteration 1. Put $x = x_0 = 0$, $y = y_0 = 0$, $z = z_0 = 0$ in eqn (1)

$x_1 = \frac{1}{6}(4 - 2y_0 + z_0) = 0.6667$

$y_1 = \frac{1}{5}(3 - x_0 - z_0) = 0.6$

$z_1 = \frac{1}{4}(27 - 2x_0 - y_0) = 6.75$

Iteration 2: Put $x = x_1$, $y = y_1$, $z = z_1$, in eqn (1)

$x_2 = \frac{1}{6}(4 - 2y_1 + z_1) = 1.5917$

$y_2 = \frac{1}{5}(3 - x_1 - z_1) = -0.8833$

$z_2 = \frac{1}{4}(27 - 2x_1 - y_1) = 6.2666$

Iteration 3: Put $x = x_2$, $y = y_2$, $z = z_2$, in eqn (1)

$x_3 = \frac{1}{6}(4 - 2y_2 + z_2) = 2.0055$

$y_3 = \frac{1}{5}(3 - x_2 - z_2) = -0.9717$

$z_3 = \frac{1}{4}(27 - 2x_2 - y_2) = 6.1750$

Iteration 4: Put $x = x_3$, $y = y_3$, $z = z_3$ in eqn (1)

$x_4 = \frac{1}{6}(4 - 2y_3 + z_3) = 2.0197$

$y_4 = \frac{1}{5}(3 - x_3 - z_3) = -1.0361$

$z_4 = \frac{1}{4}(27 - 2x_3 - y_3) = 5.9902$

Iteration 5: Put $x = x_4$, $y = y_4$, $z = z_4$ in eqn (1)

$x_5 = \frac{1}{6}(4 - 2y_4 + z_4) = 2.0104$

$y_5 = \frac{1}{6}(3 - x_4 - z_4) = -1.0020$

$z_5 = \frac{1}{4}(27 - 2x_4 - y_4) = 5.9992$

Iteration 6: Put $x = x_5$, $y = y_5$, $z = z_5$ in eqn (1)

$x_6 = \frac{1}{6}(4 - 2y_5 + z_5) = 2.0005 \simeq 2.00$

$y_6 = \frac{1}{5}(3 - x_5 - z_5) = -1.0019 \simeq -1.00$

$z_6 = \frac{1}{4}(27 - 2x_5 - y_5) = 5.9953 \simeq 6.00$

Iteration 7:

Put $x = x_6$, $y = y_6$, $z = z_6$ in eqn (1)

$x_7 = \frac{1}{6}(4 - 2y_6 + z_6) = 1.9998 \simeq 2.00$

$y_7 = \frac{1}{5}(3 - x_6 - z_6) = -0.9992 \simeq -1.00$

$z_7 = \frac{1}{4}(27 - 2x_6 - y_6) = 6.0002 \simeq 6.00$

Since the 6th and 7th iteration values are same (upto 2 decimal places)

Hence the approximate solution is $x = 2$, $y = -1$, $z = 6$.

Gauss Siedel Iteration Method

Working Rule : Consider the system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

Diagonal dominance property must be satisfied

i.e. $|a_{11}| > |a_{12}| + |a_{13}|$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Rewriting the eqns for x, y and z respectively,

$$x = \frac{1}{a_{11}}(b_1 - a_{12}y - a_{13}z)$$

$$y = \frac{1}{a_{22}}(b_2 - a_{21}x - a_{23}z)$$

$$z = \frac{1}{a_{33}}(b_3 - a_{31}x - a_{32}y)$$

Iteration 1 : $x_1 = \frac{1}{a_{11}}(b_1 - a_{12}y_0 - a_{13}z_0)$

$$y_1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}z_0)$$

$$z_1 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}y_1)$$

Iteration 2 :

$$x_2 = \frac{1}{a_{11}}(b_1 - a_{12}y_1 - a_{13}z_1)$$

$$y_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_2 - a_{23}z_1)$$

$$z_2 = \frac{1}{a_{33}}(b_3 - a_{31}x_2 - a_{32}y_2)$$

The above iteration process is continued until two successive approximations are equal.

Q1. Solve the foll. system of eqns by Gauss Siedel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Soln: Rewriting the eqns and checking dominance property,

$|27| > |6| + |1|$
$|15| > |6| + |2|$
$|54| > |1| + |1|$

$$x = \frac{1}{27}(85 - 6y + z)$$
$$y = \frac{1}{15}(72 - 6x - 2z)$$
$$z = \frac{1}{54}(110 - x - y)$$

Iteration 1: $x_1 = \frac{1}{27}(85 - 6y_0 + z_0) = 3.1481$

$y_1 = \frac{1}{15}(72 - 6x_1 - 2z_0) = 3.5408$

$z_1 = \frac{1}{54}(110 - x_1 - y_1) = 1.9132$

Iteration 2: $x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.4322 \simeq 2.432$

$y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.5720 \simeq 3.572$

$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.9258 \simeq 1.926$

Iteration 3: $x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.4257 \simeq 2.426$

$y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.5729 \simeq 3.573$

$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.9260 \simeq 1.926$

Iteration 4: $x_4 = \frac{1}{27}(85 - 6y_3 - z_3) = 2.4256 \simeq 2.426$

$y_4 = \frac{1}{15}(72 - 6x_4 - 2z_3) = 3.5730 \simeq 3.573$

$z_4 = \frac{1}{54}(110 - x_4 - y_4) = 1.9260 \simeq 1.926$

$\therefore$ the 3rd and 4th iteration values are same upto 3 decimal places.

Hence the approximate soln is

$x = 2.426$, $y = 3.573$, $z = 1.926$.