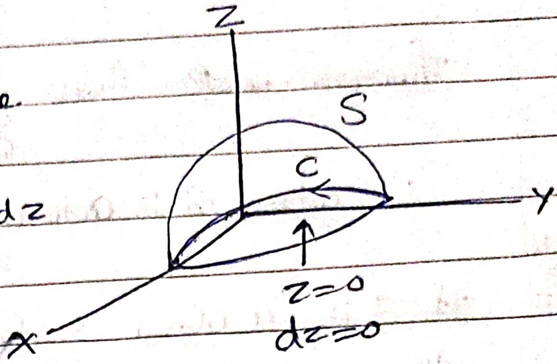
→ $\iint_S \vec{F}\cdot\hat{n}\,dS$ is also called Flux of a vector field F over a Surface S.

→ $S=\{x^2+y^2+z^2=1\}$ → hemisphere.



let $\vec{F}= x\hat{i}+y\hat{j}+z\hat{k}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C x\,dx+y\,dy+z\,dz$$

$$= \int_C x\,dx+y\,dy$$

↓ over $x^2+y^2=1$.

⇓

This can be easily solved now.

→ So we need to Evaluate the line Integral over Circle C, which is part of some Surface S, i.e. Evaluation of Line Integral in 3-dimension.

→ let $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$
and S be the upper half of the sphere i.e. hemisphere
Then find $\int \vec{F}\cdot d\vec{r}$.

Solⁿ → $\int \vec{F}\cdot d\vec{r} = \int_C (2x-y)dx + 0\,dy$ as $z=0$. in xy plane

where C is $x^2+y^2=1$.

$$= \iint dx\,dy = \pi \quad \text{(Using Green's Thm)}$$

Stokes' Thm :→ If S is an open Surface, $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector field. let c be a closed boundary of Surface 's'. $f_1, f_2, f_3, \frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$ are Continuous function. Then $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$

where $\hat{n}$ is outward unit normal vector to S.

Note

(1) If S is a Closed Surface Then we can apply Gauss divergence Thm.

$$\Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \iiint_D \text{div}(\nabla \times \vec{F}) \, dx\,dy\,dz$$

But $\text{div}(\nabla \times \vec{F}) = \text{div}(\text{Curl}\,\vec{F}) \equiv 0$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$$

Ex $\vec{F} = 2x\hat{i} + y\hat{j} + z\hat{k}$

and $S = \{x^2+y^2+z^2=1\}$ – upper half of the sphere.

$\Rightarrow$ S is open Surface.

and c be the closed boundary of S. Find $\int_C \vec{F} \cdot d\vec{r}$

Sol^n $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$

$$\nabla \times \vec{F} = \text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & y & z \end{vmatrix} = 0$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = 0$$

Que Find $\oint_C \sin z \, dx - \cos x \, dy + \sin y \, dz$ where

c is boundary of $0 \le x \le \pi$, $0 \le y \le 2$, $z = 4$.

Solⁿ (I) $z = 4 \Rightarrow dz = 0$

$$\int_C \sin 4\, dx - \cos x\, dy = \iint_R \frac{\partial(-\cos x)}{\partial x} - \frac{\partial(\sin 4)}{\partial y}\, dx\, dy$$

where R is the region Rectangle bdd by $0 \le x \le \pi$ and $0 \le y \le 2$.

$$= \int_0^2\int_0^\pi \sin x\, dx\, dy$$

$$= \int_0^\pi\int_0^2 \sin x\, dy\, dx$$

$$= \int_0^\pi \sin x\, (y)\Big|_0^2\, dx$$

$$= -2\left[\cos x\right]_0^\pi$$

$$= -2\left[-1-1\right] = 4.$$

(II) $$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= i\left[\cos y\right] - j\left[-\cos z\right] + \hat{k}\left[\sin x\right]$$

$$= \cos y\, \hat{i} + \cos z\, \hat{j} + \sin x\, \hat{k}$$

$\hat{n} = \hat{k}$ the outward unit normal to the given Surface S.

$\Rightarrow \text{Curl } \vec{F} \cdot \hat{n} = \sin x.$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl } \vec{F}\cdot \hat{n}\, ds = \iint_S \sin x\, ds$$

$$= \iint \sin x\, dx\, dy \quad \text{Over } \begin{matrix} 0 \le x \le \pi \\ 0 \le y \le 2 \end{matrix}$$

$$= 4 \quad (\text{as shown above})$$

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Que $\vec{F} = (z^2 - y)\hat{i} + (x - 2yz)\hat{j} + (2xz - y^2)\hat{k}$
over S – Surface of the sphere
$x^2 + y^2 + z^2 = 9$, $z \geq 0$. Find $\int_C \vec{F} \cdot d\vec{r}$ where C is $x^2 + y^2 = 9$, $z = 0$.

Solⁿ → The unit normal vector to the surface S is

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}}$$

$$= \frac{1}{3}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y & x - 2yz & 2xz - y^2 \end{vmatrix}$$

$$= \hat{i}(-2y + 2y) - \hat{j}(2z - 2z) + \hat{k}(1 + 1)$$

$$= 2\hat{k}$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$$

$$= \iint \frac{2}{3} z \cdot \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$= \frac{2}{3} \iint z \cdot \frac{dx\,dy}{(z/3)}$$

$$= 2 \iint_S dx\,dy$$

$$= 2 \times \text{Area of Circle}$$

$$= 2 \times \pi \times (3)^2$$

$$= 18\pi \text{ Ans}$$

Que

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Que. Find $\oint_C \vec{F}\cdot\vec{dr}$ for $\vec{F} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$

Where S is the surface of the sphere $x^2+y^2+z^2=16$, $z>0$.
and C is the circle $x^2+y^2=16$, $z=0$.

Ans:- 16π