# Rank - Nullity thm :

L.C of vectors (L.D of vectors and L.I of vectors)

Generating set of vectors

Span

$\boxed{\text{Basis}}$ — smallest generating set
— Largest L.I set

Dimension : $\mathbb{R}(\mathbb{R}) \to 1$

$\mathbb{C}(\mathbb{R}) \to 2$

$\mathbb{R}(\mathbb{C})$

$\mathbb{R}^n(\mathbb{R}) - n$

$M_{2\times 2}(\mathbb{R})$ or $\mathbb{R}^{2\times 2}(\mathbb{R}) - 4$

$z = x + iy$

Dimension
↓
order

$A = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \middle| \; x, y, z, w \in \mathbb{R} \right\}$

$B = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \; a_{12} = a_{21}, \quad a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$ – Symmetric Matrices

$C = \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \middle| \; a_{11}, a_{22} \in \mathbb{R} \right\}$

$D = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \; a_{12} = -a_{21}, \quad a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$

$\downarrow$

Skew-symmetric Matrices

$\begin{pmatrix} 0 & +a_{12} \\ -a_{12} & 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2×2

Order as well as dimension :

$\dim(A) = 4$

$\dim(B) = 3$

$\dim(C) = 2$

$\dim(D) = 1 \longrightarrow ??$ (all the diagonal elements are zero)

**Q.1:** let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^3$ be a L.T defined by

$$T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix},$$

$$T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \quad T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$

**Soln:**

**Note:** i) The standard / ordered basis of $\mathbb{R}^{2\times 2}(\mathbb{R})$ is

$$\beta_1 = \left\{ \overset{e_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \overset{e_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \overset{e_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \overset{e_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

*Another basis of $\mathbb{R}^{2\times 2}(\mathbb{R})$ $\quad (\beta_1 \neq \beta_2)$

$$\beta_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

* Another basis of $\mathbb{R}^{2\times 2}(\mathbb{R})$ is

$$\beta_3 = \left\{ \overset{v_1}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}, \overset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}, \overset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

Soln : To check: $\beta_3$ forms a basis of $\mathbb{R}^{2\times 2}(\mathbb{R})$.

Smallest Generating set :

$$\mathbb{R}^{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \middle| x, y, z, w \in \mathbb{R} \right\}.$$

$\therefore$ linear combination of $\beta_3$ is

$$\left\{ \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \middle| \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{pmatrix} \middle| \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R} \right\}$$ $\rightarrow$ this will generate complete $\mathbb{R}^{2\times 2}$

($\because$ $\mathbb{R}$ forms an abelian group under addition)

Now to check: smallest generating set:

Let us assume that $v_1, v_2, v_3$ form a generating set of $\mathbb{R}^{2\times 2}$

$\therefore \{v_1, v_2, v_3\} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \boxed{\alpha_1 + \alpha_2 + \alpha_3} & \boxed{\alpha_1 + \alpha_2 + \alpha_3} \end{pmatrix}$$

But $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ cannot be written as L.C of $\{v_1, v_2, v_3\}$. Hence $\{v_1, v_2, v_3\}$ is not a generating set of $\mathbb{R}^{2\times 2}$. Hence, $B_3$ is the smallest generating set of $\mathbb{R}^{2\times 2}$.

To check: Linear independence of $v_1, v_2, v_3$ and $v_4$

$\Rightarrow$ for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$$

$$\Rightarrow \begin{pmatrix} \alpha_1 & \alpha_1+\alpha_2 \\ \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

To check: Largest L.I set of vectors:

let us take any arbitrary vector $v_5$

for example: $v_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

let us consider the linear combination

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 + \alpha_5 v_5 = 0 \quad \text{--- } (*)$$

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_5 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 + \alpha_5 & \alpha_1 + \alpha_2 + \alpha_5 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} \alpha_1 + \alpha_5 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_5 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 &= 0 \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} &\cancel{\alpha_2 = 0} \\ &\cancel{\alpha_1 = 0} \\ &\cancel{\alpha_3 = 0} \\ &\cancel{\alpha_4 = 0} \\ &\cancel{\alpha_5 = 0} \end{aligned}}$$

$\alpha_1 = -\alpha_5$

$\alpha_2 = 0$

$$\boxed{\begin{aligned} &\alpha_2 = \alpha_4 = 0 \\ &\alpha_1 = -\alpha_3 = -\alpha_5 \end{aligned}}$$

$$\boxed{\left.\begin{aligned} \alpha_1 &= 1 \\ \alpha_2 &= 0 \\ \alpha_3 &= -1 \\ \alpha_4 &= 0 \\ \alpha_5 &= -1 \end{aligned}\right\} \Rightarrow v_1, v_2, v_3, v_4 \;\&\; v_5 \text{ are L.D}}$$

**Soln:** Since

$$v_1 = \begin{bmatrix}1&1\\1&1\end{bmatrix},\ v_2 = \begin{bmatrix}0&1\\1&1\end{bmatrix},\ v_3 = \begin{bmatrix}0&0\\1&1\end{bmatrix},\ v_4 = \begin{bmatrix}0&0\\0&1\end{bmatrix}$$

form a basis of $\mathbb{R}^{2\times 2}$

$\therefore \begin{bmatrix}4&5\\3&8\end{bmatrix}$ can be re-written as L.C of these four matrices

$\therefore$

$$\begin{bmatrix}4&5\\3&8\end{bmatrix} = \alpha_1\begin{bmatrix}1&1\\1&1\end{bmatrix} + \alpha_2\begin{bmatrix}0&1\\1&1\end{bmatrix} + \alpha_3\begin{bmatrix}0&0\\1&1\end{bmatrix} + \alpha_4\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}4&5\\3&8\end{bmatrix} = \begin{bmatrix}\alpha_1 & \alpha_1+\alpha_2\\ \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\alpha_4\end{bmatrix}$$

$$\Rightarrow \alpha_1 = 4,\ \alpha_2 = 1,$$
$$\alpha_3 = -2,\ \alpha_4 = 5$$

$$\therefore \begin{bmatrix}4&5\\3&8\end{bmatrix} = 4v_1 + v_2 - 2v_3 + 5v_4$$

$$\therefore T\begin{bmatrix}4&5\\3&8\end{bmatrix} = 4T(v_1) + T(v_2) - 2T(v_3) + 5T(v_4) \quad (\because T \text{ is L.T})$$

$$\Rightarrow T\begin{bmatrix}4&5\\3&8\end{bmatrix} = 4\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}1\\-2\\3\end{pmatrix} - 2\begin{pmatrix}-1\\-2\\3\end{pmatrix} + 5\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

$$= \begin{pmatrix}-2\\20\\36\end{pmatrix}$$

Q.2: let $T$ be a L.T from $\mathbb{R}^3$ to $\mathbb{R}^2$ over $\mathbb{R}$ where

$$Tx = Ax$$

where $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $x = [x\ y\ z]^T$

find $\text{Ker}(T)$ and range$(T)$ and their dimensions.

Soln: Null space $(T) = \{ v \in \mathbb{R}^3 \mid T(v) = 0 \}$

let $v = [v_1, v_2, v_3]^T$

$\therefore Tv = 0 \Rightarrow T[v_1, v_2, v_3]^T = 0$

$\Rightarrow A[v_1, v_2, v_3]^T = 0$

$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}_{2\times3} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{3\times1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} v_1 + v_2 = 0 \\ -v_1 + v_3 = 0 \end{matrix}$

$\Rightarrow v_1 = -v_2 = v_3$

$\therefore N(T)/\text{Ker}(T) = \{(v_1, -v_1, v_1) \mid v_1 \in \mathbb{R}\}$

$\therefore \text{Nullity}(T) = 1$

$= \{(1, -1, 1)\}$

$\mathbb{R}^2(\mathbb{R})$

$\therefore$ Range $(T) = \mathbb{R}^2$

dim $($Rang$(T)) =$ Rank$(T) = 2$

Aliter: Range$(T) = \{T(v) \mid v \in \mathbb{R}^3\}$

$= \{Av \mid v \in \mathbb{R}^3\}$

$\Rightarrow$ for $v = [v_1, v_2, v_3]^T$

$$Av = \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Also $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ dim$($Range$(T)) =$ Rank$(T) = \{(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}), (\begin{smallmatrix} 0 \\ 1 \end{smallmatrix})\}$

Q.3 let $T$ be a L.T, $Tx = Ax$ from $\mathbb{R}^2 \to \mathbb{R}^3$

(Task) -)) where $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}$ & $x = \begin{pmatrix} x \\ y \end{pmatrix}$

Find $\ker(T)$, range $(T)$ and their dimensions

Q.4 let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T defined by

$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}$

Determine the matrix of $T$ with respect to the ordered basis

$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$.

and $Y = \left\{ \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$ in $\mathbb{R}^2$.

<u>Soln</u> : <u>Rough Hint</u>

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \in \mathbb{R}^2 \quad = \underline{\boxed{(0)}\begin{pmatrix}1\\0\end{pmatrix} + \boxed{(0)}\begin{pmatrix}0\\1\end{pmatrix}} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{R}^2 \quad = \underline{\boxed{(1)}\begin{pmatrix}1\\0\end{pmatrix} + \boxed{(1)}\begin{pmatrix}0\\1\end{pmatrix}} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \in \mathbb{R}^2 = \underline{\boxed{(1)}\begin{pmatrix}1\\0\end{pmatrix} + \boxed{(-1)}\begin{pmatrix}0\\1\end{pmatrix}} = \underline{\begin{pmatrix}1\\-1\end{pmatrix}}$$

$\therefore A = \begin{pmatrix}0 & 1 & 1\\0 & 1 & -1\end{pmatrix}_{2\times3}$ is the matrix of L.T.

$\underline{\mathbb{R}}^3 \to \underline{\mathbb{R}}^2$