Bessel's Equations

Bessel's equations is defined as

$$x^2\left(\frac{d^2y}{dx^2}\right) + x\left(\frac{dy}{dx}\right) + (x^2-n^2)y = 0$$

To understand bessel's function equation we need to get Gamma function and Beta function cleared.

Beta function

The Beta function is defined as :

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \begin{cases} m>0 \\ n>0 \end{cases}$$

Putting $x = 1-y$ we get

$$\beta(m,n) = -\int_1^0 (1-y)^{m-1} y^{n-1}\,dy$$

$$\beta(n,m) = \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

Thus,

$$\beta(m,n) = \beta(n,m)$$

Subsituting $x = \sin^2\theta$ $dx = 2\sin\theta\cos\theta\,d\theta$

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\,d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$$

This is called Eulear's integral of first kind

Gamma function

The gamma function is defined as :

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx \qquad (n>0)$$

This is called Eulear's integral of second kind

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = \int_0^\infty (e^{-x}x^n)\,dx$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma(2) = \cancel{2}\ 1\times\Gamma(1) = 1!$$

$$\Gamma(3) = 2\times\Gamma(2) = 2!$$

$$\Gamma(4) = 3\times\Gamma(3) = 3!$$

In general, $\Gamma(n+1) = n!$

Relation between Beta and Gamma function

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(m) = \int_0^\infty e^{-t}t^{m-1}$$

Put $t = x^2$

$dt = 2x\,dx$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2}x^{2m-1}\,dx$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2}y^{2n-1}\,dy$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty e^{-x^2}x^{2m-1}\,dx\int_0^\infty e^{-y^2}y^{2n-1}\,dy$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty e^{-x^2}.x^{2m-1}\,e^{-y^2}y^{2n-1}\,dx\,dy$$

Change into polar coordinates $x = r\cos\theta$ $y = r\sin\theta$

$$\Gamma(m).\Gamma(n) = 4\int_0^\infty\int_0^{\pi/2} e^{-r^2}r^{2(m+n)-1}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta\,dr$$

$$\Gamma(m)\Gamma(n) = 2\left[\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta\right]\times 2\left[\int_0^\infty e^{-r^2}r^{2(m+n)-1}\,dr\right]$$

$$\Gamma(m)\Gamma(n) = \beta(m+n)\,\Gamma(m+n)$$

$$\left[\beta(m+n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta\right]$$

$$\left[\Gamma(m+n) = 2\int_0^{\infty} e^{-r^2}\,r^{2(m+n)-1}\,dr\right]$$

$$\beta(m+n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Note: Rule to evaluate $\int_0^{\pi/2} \sin^p x \cos^q x\,dx$

$$\int_0^{\pi/2} \sin^p x \cos^q x = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{\Gamma(p+1/2)\,\Gamma(q+1/2)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

If $q=0$, $p=n$ we have,

$$\int_0^{\pi/2} \sin^n x = \frac{\Gamma(n+1/2)}{\Gamma(n+2/2)} \cdot \frac{\sqrt{\pi}}{2}$$

If $p=0$, $q=n$ we have,

$$\int_0^{\pi/2} \cos^n dx = \frac{\Gamma(n+1/2)}{\Gamma(n+2/2)} \cdot \frac{\sqrt{\pi}}{2}$$

Example Show that $\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \quad (n>0)$

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\,dx \quad (n>0)$$

$$= \int_1^0 \left(\log\frac{1}{y}\right)^{n-1} y\left(\frac{-1}{y}\right) dy \qquad y = e^{-x},\; x = \log\left(\frac{1}{y}\right)$$

$$= \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy \qquad dx = \frac{-1}{y}dy$$

$$= \Gamma(n)$$

$\therefore$ Proved

Example Show that $\beta(p,q) = \int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$

$$= \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\,dx$$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx \qquad x = \frac{1}{1+y}$$

$$\beta(p,q) = \int_{\infty}^0 \frac{1}{(1+y)^{p-1}} \times \left(\frac{y}{y+1}\right)^{q-1} \frac{-1}{(1+y)^2}\,dy \qquad dx = \frac{-1}{(1+y)^2}dy$$

$$\beta(p,q) = \int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}\,dy$$

$\therefore$ Proved

$$\beta(p,q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}\,dy + \int_0^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}\,dy \qquad y = 1/x,\; dy = \frac{-1}{z^2}dz$$

$$\int_1^{\infty} \frac{y^{q-1}}{(1+y)^{p+q}}\,dy = \int_1^0 \frac{1}{z^{q-1}} \cdot \frac{1}{(1+1/z)^{p+q}}\left(\frac{-1}{z^2}\right)dz$$

$$\int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz$$

$$\beta(p,q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}} + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}\,dz = \int_0^1 \frac{z^{p-1} + z^{q-1}}{(1+z)^{p+q}}\,dx$$

$\therefore$ Proved

Bessel's equation and its solution

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$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2) = 0$$

bessel's equation of order n

Let solution of equation be :

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \dots\dots$$

$$\frac{dy}{dx} = ma_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \dots\dots$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + \dots\dots$$

$$x^2[m(m-1)a_0x^{m-2} + a_1m(m+1)x^{m-1} + a_2(m+1)(m+2)x^m + \dots\dots]$$
$$+ x[ma_0x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \dots\dots]$$
$$+ (x^2-n^2)y = 0$$

equating coeff. of x^m to zero

$$a_0m(m-1) + ma_0 - n^2 = 0$$
$$m^2 = n^2$$
$$\boxed{m = \pm n}$$

coeff. of x^{m+1} to zero

$$m(m+1)a_1 + a_1(m+1) = 0$$
$$a_1 = 0$$

coeff. of x^{m+2} to zero

$$-a_2n^2 + a_0 + a_2(m+1)(m+2) + a_2(m+2) = 0$$
$$-a_2n^2 + a_2(m+2)^2 = -a_0$$
$$a_2 = \frac{-a_0}{(m+2)^2-n^2}$$

$$a_1 = a_3 = a_5 = a_7 \dots\dots = 0$$

$$a_2 = \frac{-a_0}{(m+2)^2-n^2} \qquad a_4 = \frac{-a_2}{(m+4)^2-n^2}$$

if $m=n$, then y_1 solution :

$$y_1 = a_0x^n\left\{1 - \frac{1}{4(n+1)}x^2 + \frac{1}{4^2.2!(n+1)(n+2)}x^4 - \frac{1}{4^3.3!(n+1)(n+2)(n+3)}x^6 + \dots\dots\right\}$$

If $m=-n$, then y_2 solution :

$$y_2 = a_0x^{-n}\left\{1 - \frac{1}{4(-n+1)}x^2 + \frac{1}{4^2.2!(-n+1)(-n+2)}x^4 - \frac{1}{4^3.3!(-n+1)(-n+2)(-n+3)}x^6 + \dots\dots\right\}$$

When n is not integral then solution is $y = c_1y_1 + c_2y_2$

If we take $a_0 = \frac{1 \times 1}{2^n\,\Gamma(n+1)}$ then solution (y_1) is called bessel's function of order n and solution (y_2) is called bessel's function of order $(-n)$

$$J_n(x) = \left(\frac{x}{2}\right)^n\left\{\frac{1}{\Gamma(n+1)} - \frac{1}{\Gamma(n+2)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)}\left(\frac{x}{2}\right)^4 - \dots\right\}$$

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{r!\,\Gamma(-n+r+1)}$$

which is called Bessel's function of first kind of order n
complete solution becomes :

$$y = A J_n(x) + B J_{-n}(x)$$

When n is integer and not-zero
let solution be $y = v(x) J_n(x)$

$$\frac{dy}{dx} = v'(x) J_n(x) + v(x) J_n'(x)$$

$$\frac{d^2y}{dx^2} = v''(x) J_n(x) + 2v'(x) J_n'(x) + v(x) J_n''(x)$$

subituting in bessel's equation to obtain

$$x^2\left[v'' J_n + 2u' J_n' + v J_n''\right] + x\left[v' J_n + v J_n'\right] + (x^2-n^2) v J_n$$

$$u\left[x^2 J_n'' + x J_n' + (x^2-n^2) J_n\right] + x^2 u'' J_n + 2x^2 u' J_n' + xu' J_n = 0$$

J_n is a solution of (1) therefore,

$$x^2 J_n'' + x J_n' + (x^2-n^2) J_n = 0$$

Then $\quad x^2 u'' J_n + 2x^2 u' J_n' + xu' J_n = 0$

dividing throughout by $x^2 u' J_n$ it becomes,

$$\left(\frac{u''}{u'}\right) + 2\left(\frac{J_n'}{J_n}\right) + \frac{1}{x} = 0$$

$$\frac{d}{dx}(\log u') + 2\frac{d}{dx}(\log J_n) + \frac{d}{dx}(\log x) = 0$$

$$\frac{d}{dx}\left[\log(u' J_n^2 x)\right] = 0$$

Integrating both sides,

$$\log(u' J_n^2 x) = \log B$$

$$\boxed{xu' J_n^2 = B}$$

$$u' = \frac{B}{x J_n^2} \qquad u = B\int \frac{dx}{x J_n^2} + A$$

Thus,

$$y = A J_n(x) + B J_n(x) \int \frac{dx}{x[J_n(x)]^2}$$

Hence,
the complete solution of Bessel's equation

$$y = A J_n(x) + B Y_n(x)$$

where,

$$Y_n(x) = J_n(x) \int \frac{dx}{x[J_n(x)]^2}$$

Y_n is called Bessel's function of second kind of order n
or Neumann function.

Recurrence formulae for $J_n(x)$

We have, $\quad J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \times \frac{1}{r!\,\Gamma(n+r+1)}$

(i) prove that $\quad \frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

multiplying with x^n on both sides

$$x^n J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2(n+r)}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

differentiating both sides

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$$\frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r\, 2(n+r)\, x^{2(n+r)-1}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{\Gamma(n-1+r+1)}$$

$$\left\{\begin{aligned} &(n+r)\,\Gamma(n+r) = \Gamma(n+r+1) \\ &\frac{(n+r)}{\Gamma(n+r+1)} = \Gamma(n+r) \end{aligned}\right\}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!}\left(\frac{x}{2}\right)^{n+2r-1} \times \frac{1}{\Gamma(n+r)}$$

$$= x^n J_{n-1}(x)$$

∴ Proved

(ii) prove that $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \times \frac{1}{r!\,\Gamma(n+r+1)}$$

multiplying x^{-n} by $J_n(x)$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^r . 2r . x^{2r-1}}{2^{n+2r}\, r!\, \Gamma(n+r+1)}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} \frac{(-1)^{r-1} . x^{n+1+2(r-1)}\left[x^{-n}\right]}{2^{n+1+2(r-1)}\,(r-1)!\,\Gamma(n+r+1)}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} \sum_{r=0}^{\infty} \frac{(-1)^{r-1} . x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}\,(r-1)!\,\Gamma(n+r+1)}$$

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$K = r-1$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{n+1+2k}}{k!\,\Gamma(n+1+k+1)}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

∴ Proved

(iii) prove that $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$

from (i)

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

$$J_n'(x) + \left(\frac{n}{x}\right) J_n(x) = J_{n-1}(x)$$

from (ii)

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

$$-J_n'(x) + \frac{n}{x} J_n(x) = J_{n+1}(x)$$

adding equations

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

∴ Proved

(iv) prove that $J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$

subtracting equations

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

(v) $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$ prove that

from (ii),

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

$$-J_n'(x) + \left(\frac{n}{x}\right) J_n(x) = J_{n+1}(x)$$

$$\therefore J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

$\therefore$ Proved

(vi) $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$ prove that

from (iii)

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\frac{2n}{x} J_n(x) - J_{n-1}(x) = J_{n+1}(x)$$

$\therefore$ Proved

Expansion for J_0, J_1, $J_{1/2}$ and $J_{-1/2}$

Value of $\Gamma(1/2)$ we have,

$$\Gamma(1/2) = \int_0^\infty e^{-x} \cdot x^{-1/2} dx \qquad x = y^2, \quad dx = 2y\,dy$$

$$\Gamma(1/2) = 2\int_0^\infty e^{-y^2} dy \quad \text{which is} \quad 2\int_0^\infty e^{-x^2} dx$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dx\,dy$$

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,dr\,d\theta$$

$$\left[\Gamma(1/2)\right]^2 = \frac{4\cdot\pi}{2}\int_0^\infty e^{-r^2} dr.r = 2\pi\left[\left(\frac{-1}{2}\right)e^{-r^2}\right]_0^\infty = \pi$$

$$\boxed{\Gamma(1/2) = \sqrt{\pi}}$$

(i) $J_0 = \left[1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots\right]$

(ii) $J_1 = \frac{x}{2}\left[1 - \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^6 + \ldots\right]$

(iii) $J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$

$$= \left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3.1}{2\,2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{\frac{2.5.3.1}{2\,2\,2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \ldots\right\}$$

$$= \frac{\sqrt{x}}{\sqrt{2\Gamma\left(\frac{1}{2}\right)}}\left\{\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \ldots\right\}$$

$$= \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right\} = \sqrt{\frac{2}{\pi x}}\sin x$$

(iv) $J_{-1/2}(x) = \left(\frac{x}{2}\right)^{-1/2}\left\{\frac{1}{\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{\frac{3.1}{2\,2}\Gamma\left(\frac{1}{2}\right)}\right\}$

$$= \frac{\sqrt{x}}{\sqrt{2\Gamma\left(\frac{1}{2}\right)}}\left\{\frac{2x}{1!} - \frac{2x^3}{3!} + \frac{2x^5}{5!} - \ldots\right\}$$

$$= \frac{\sqrt{2}}{\sqrt{x}\sqrt{\pi}}\left\{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right\} = \sqrt{\frac{2}{\pi x}}\cos x$$

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[Assignment 3]

Legendre's Equation

Legendre's equation can be defined as:

$x=0$ is an ordinary point

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

solution of the equation can be given as:

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots\dots$$

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + a_2(m+2)x^{m+1} + \dots\dots$$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + m(m+1)a_1x^{m-1} + a_2(m+1)(m+2)x^m + \dots$$

$$(1-x^2)\left[m(m-1)a_0x^{m-2} + m(m+1)a_1x^{m-1} + a_2(m+1)(m+2)x^m \dots\right]$$
$$- 2x\left[ma_0x^{m-1} + (m+1)a_1x^m + a_2(m+2)x^{m+1} + \dots\right]$$
$$+ n(n+1)\left[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots\right]$$

Coeff. of x^{m-2} to zero

$$m(m-1)a_0 = 0$$

$$\boxed{m=0,1}$$

Coeff. of x^{m-1} to zero

$$m(m+1)a_1 = 0$$

$$m=0 \qquad [a_1 \neq 0]$$

Coeff. of x^{m+r} to zero

$$a_{r+2}(m+r+2)(m+r+1) - [(m+r)(m+r+1) - n(n+1)]a_r = 0$$

where $m=0$

then

$$a_2 = \frac{-n(n+1)a_0}{2!}$$

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$$a_3 = -\frac{(n+1)(n+2)a_1}{3!}$$

$$a_4 = -\frac{(n-2)(n+3)a_2}{4!} = \frac{n(n-2)(n+1)(n+3)a_0}{4!}$$

$$a_5 = -\frac{(n-3)(n+4)a_3}{5.4} = \frac{(n-1)(n-3)(n+2)(n+4)a_1}{5!}$$

hence the solution becomes

$$y_1 = a_0\left\{1 - \frac{n(n+1)x^2}{2!} + \frac{(n-2)n(n+1)(n+3)x^4}{4!} - \dots\dots\right\}$$

$$y_2 = a_1\left\{x - \frac{(n-1)(n+2)x^3}{3!} + \frac{(n-3)(n-1)(n+2)(n+4)x^5}{5!} - \dots\dots\right\}$$

when $m=1$ shows that $a_1=0$ $a_3=0$ $a_5=0$ ——

and $a_2 = -\frac{(n-1)(n+2)}{3!}a_0$

$$a_4 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_0$$

$y = y_1 + y_2$ is the general solution

If n is a positive even integer, the series terminates at x^{2n} and y_1 becomes a polynomial of degree n. Similarly, n is a positive odd integer the series terminates at x^{2n+1} and y_2 becomes a polynomial of degree n.

These polynomial solution with a_0 or a_1 so chosen that the value of polynomial is 1 for $x=1$ are called legendre polynomial of order n are denoted by $P_n(x)$

The infinite series solution with (a_0 or a_1) is called legendre's function of second kind and denoted by $Q_n(x)$

Rodrigue's formula

Let $V_n = (x^2-1)^n$ where V_n denotes the n^{th} derivative of V.

then, $V_1 = \frac{dV}{dx} = 2nx(x^2-1)^{n-1}$

$$V_1(1-x^2) = -2nxV_n$$

$$V_1(1-x^2) + 2nxV_n = 0 \quad \text{----- (i)}$$

$$V_2 = 2x\,2n(n-1)x^2(x^2-1)^{n-2} + 2n(x^2-1)^{n-1}$$

$$V_2 = 4nx^2(n-1)(x^2-1)^{n-2} + 2n(x^2-1)^{n-1}$$

differentiating (i) $(n+1)$ times by Leibnitz's theorem.

$$(1-x^2)V_{n+2} + (n+1)(-2x)V_{n+1} + \frac{1}{2!}\left[(n+1)n(-2V_n)\right]$$

$$+ 2n\left[xV_{n+1} + (n+1)V_n\right] = 0$$

$$(1-x^2)\left[\frac{d^2V_n}{dx^2}\right] - 2x\left[\frac{d(V_n)}{dx}\right] + n(n+1)V_n = 0$$

which is legendre equation with V_n as its solution CV_n in its solution and also a finite series solution in $P_n(x)$

$$P_n(x) = CV_n = \frac{c\,d^n}{dx^n}(x^2-1)^n$$

$$1 = c\left[\frac{d^n}{dx^n}\left[(x-1)^n (x+1)^n\right]\right]_{x=1}$$

$$1 = c\left[n!(x+1)^n\right] + \text{terms containing } (x-1) \text{ and its powers}$$

$$1 = c\left[n!\,2^n\right] + 0$$

$$c = \frac{1}{n!\,2^n}$$

$$P_n(x) = c\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}(x^2-1)^n \quad \therefore \text{Proved}$$

Legendre's polynomials of order 'n' is defined as

$$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

In general we have

$$P_n(x) = \sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}$$

where,

$N = n/2$ if n is even

$N = (n-1)/2$ if n is odd

By binomial theorem

$$(x^2-1)^n = \sum_{r=0}^{n} {}^nC_r (x^2)^{n-r}(-1)^r$$

$$= \sum_{r=0}^{n}(-1)^r\frac{n!}{r!(n-r)!}x^{2n-2r}$$

by polynomial's equation

$$P_n = \frac{1}{n!\,2^n}\sum_{n=0}^{r}\frac{(-1)^r n!}{r!(n-r)!}\frac{d^n}{dx^n}(x^{2n-2r})$$

$$P_n = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n r! (n-r)! (n-2r)!} x^{n-2r}$$

for $P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0}(x^2-1)^0 = 1$

$$P_1(x) = \frac{1}{2^1 1!} \frac{d^1}{dx^1}(x^2-1)^1 = \frac{1 \times 2x}{2} = x$$

$$P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1 \times 4(3x^2-1)}{8} = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3}(x^2-1)^3 = \frac{1 \times 24(5x^3-3x)}{48} = \frac{1}{2}(5x^3-3x)$$

$$P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4}(x^2-1)^4 = \frac{1 \cdot 48(35x^4-30x^2+3)}{384}$$

$$= \frac{1}{8}(35x^4-30x^2+3)$$

Unit - III Matrices and Determinants

Transpose of Matrix
The matrix obtained from any given matrix A by interchanging rows and columns is called transpose of A and denoted by A' or A^T

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \qquad A' = A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

Adjoint of a Square matrix
The determinant of the square matrix.
The matrix formed by the cofactors of the elements

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$ Then the transpose of the matrix i.e. $$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_2 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

is called the adjoint of the matrix A and is written as Adj. A

Inverse of a matrix
Inverse of A which is denoted by A^{-1} so that $A \cdot A^{-1} = I$

Also, $A^{-1} = \frac{\text{Adj. } A}{|A|}$

$$A. \frac{\text{Adj. } A}{|A|} = I$$

Rank of a Matrix

A matrix is said to be of rank r when
(i) it has at least one non-zero minor of order r
(ii) every minor of order higher than r vanishes

If a matrix has a non-zero minor of order r its rank $\geq r$
If all minors of a matrix of order $r+1$ are zero its rank $\leq r$

Elementary transformation of a matrix

(i) The interchange of any two rows or columns
(ii) The multiplication of any row or column by a constant
(iii) The addition of a number of the elements of any row or column to the corresponding elements of any other row or column

Notation

(i) R_{ij} for interchange of i^{th} and j^{th} row / column
(ii) KR_i for multiplication of i^{th} Row with K
(iii) $R_i + KR_j$ for addition to the i^{th} Row times the j^{th} row

Example: Determine the rank of following matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$R_2 - R_1$ $\qquad$ $R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 2 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2
Rank = 2

Gauss Jordan method of finding Inverse

Those elementary row transformation which reduce a given square matrix A to the unit matrix when applied to unit matrix I give inverse of A

Example: Using Gauss-Jordan Method find inverse of the matrix.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$ Operate (-2) common

$$-2\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & -\frac{1}{2} \end{array}\right]$$ operate $R_2 \to R_2 - R_3$, $R_3 \to R_3 - R_1$

$$-2\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -5 & 0 & 1 & +\frac{1}{2} \\ 0 & 1 & -1 & -1 & 0 & +\frac{1}{2} \end{array}\right]$$ Operate $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$

$$-2\left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 1 & -1 & -\frac{1}{2} \\ 0 & 1 & -5 & 0 & 1 & +\frac{1}{2} \\ 0 & 0 & 4 & -1 & -1 & -1 \end{array}\right]$$ Operate $R_1 \to R_1 - 2R_3$

$$\begin{array}{c} -2 \\ \\ \\ \end{array}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & -5 & 0 & 1 & 1/2 \\ 0 & 0 & 4 & -1 & -1 & -1 \end{array}\right]$$

4 common from R_3

$$\begin{array}{c} -2 \\ \times 4 \\ \\ \end{array}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & -5 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

operate $R_2 \to R_2 + 5R_3$

$$\begin{array}{c} -8 \\ \\ \\ \end{array}\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & 5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

Inverse of the matrix is $\begin{bmatrix} 3 & 1 & 3/2 \\ 5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$

Normal form of a matrix

Every non-zero matrix A of rank r can be reduced by a sequence of elementary transformations to the form

$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ called the normal form of A

The rank of matrix is r if and only if it can be reduced to normal form of A

Example: Reduce the following matrix into its normal form and hence find its rank

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ $R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 6R_1$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$ $C_2 \to C_2 + C_1$, $C_3 \to C_3 + 2C_1$, $C_4 \to C_4 + 4C_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$ $R_4 \to R_4 - R_2 - R_3$, $R_2 \to R_2 - R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $R_3 \to R_3 - 4R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $C_3 \to C_3 + 6C_2$, $C_4 \to C_4 + 3C_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} C_3 \to \frac{1}{33} C_3 \\ C_4 \to C_4 - 22C_3 \end{array}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank of matrix $\rho(A) = 3$
Identity matrix of order 3

Solution of linear system of equations

Method of Determinants — Cramer's Rule

Consider the equation

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

determinant of coefficients $(\Delta) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Thus,

$$x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{or} \quad x = \frac{\Delta_1}{\Delta}$$

$$y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{or} \quad y = \frac{\Delta_2}{\Delta}$$

$$z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{or} \quad z = \frac{\Delta_3}{\Delta}$$

Matrix Inversion Method

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

equation are equivalent to matrix equation

$$AX = D$$
$$A^{-1}AX = A^{-1}D$$
$$X = A^{-1}D$$

i.e.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Example: Solve the equation $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$

(i) determinants (ii) matrices

Here, (i) By determinants,

$$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8$$

$$x = \frac{1}{8}\begin{vmatrix} 3 & 1 & 2 \\ -3 & -3 & -1 \\ 4 & 2 & 1 \end{vmatrix} = 1 \qquad y = \frac{1}{8}\begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 2$$

$$z = \frac{1}{8}\begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix} = -1$$

$\boxed{x=1}$ $\boxed{y=2}$ $\boxed{z=-1}$

(ii) By matrices

$$\Delta = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 8$$

$A_1 = -1 \quad B_1 = -3 \quad C_1 = 7$

$A_2 = 3 \quad B_2 = 1 \quad C_2 = -5$

$A_3 = 5 \quad B_5 = 7 \quad C_3 = -11$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix} \times \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -3-9+20 \\ -9-3+28 \\ 21+15-44 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$x=1$
$y=2$
$z=-1$

Consistency of linear system of equations

The system of equations is consistent if and only if the coefficient matrix A and the augmented matrix K are of the same rank.

rank of A = r rank of K = r'

(i) If $r \neq r'$ the equation are inconsistent no solution

(ii) If $r=r'=n$ equation are cosistent unique solution

(iii) If $r=r'<n$ equation are consistent infinite solution

Example Test for consistency and solve

$5x+3y+7z=4 \quad 3x+26y+2z=9 \quad 7x+2y+10z=5$

We have

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$R_1 \to 3R_1$
$R_2 \to 5R_2$

$$\begin{bmatrix} 15 & 9 & 21 \\ 15 & 130 & 10 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 45 \\ 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$$\begin{bmatrix} 15 & 9 & 21 \\ 0 & 121 & -11 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 5 \end{bmatrix}$$

$R_1 \to 7/3\, R_1$
$R_3 \to 5R_3$
$R_2 \to 1/11\, R_2$

$$\begin{bmatrix} 35 & 21 & 49 \\ 0 & 11 & -1 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 28 \\ 3 \\ 25 \end{bmatrix}$$

$R_3 \to R_3 - R_1 + R_2$
$R_1 \to \frac{1}{7} R_1$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$5x+3y+7z=4 \qquad y = \frac{3}{11}+\frac{z}{11} \qquad x = \frac{7}{11}-\frac{16z}{11}$

$11y - z = 3$

$x = \frac{7}{11} \quad y = \frac{3}{11} \quad z=0$

Example Investigate the values of λ and μ so that the equations

$2x+3y+5z=9 \quad 7x+3y-2y=8 \quad 2x+3y+\lambda z=\mu$

(i) no solution

(ii) unique solution (iii) infinite solution

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The system admits of unique if and only if the rank is 3 which is by the following

$$\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5-\lambda) \neq 0 \qquad \boxed{\lambda \neq 5}$$

For a unique solution $\lambda \neq 5$ and μ may have any value
If $\lambda = 5$ then no solution for any value of μ for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \qquad K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

If $\lambda = 5$ and $\mu \neq 9$ then no solution
If $\lambda = 5$ and $\mu = 9$ then infinite number of solution

System of linear homogeneous equations

If $r = n$, the equations (homogeneous) have only a trivial solution zero

$$x_1 = x_2 = x_3 = x_4 = \cdots = x_n = 0$$

If $r < n$, the equation have $(n-r)$ linearly independent solution

Example : Find the values of K for which the system of equations

$$(3K-8)x + 3y + 3z = 0$$
$$3x + (3K-8)y + 3x = 0$$
$$3x + 3y + (3K-8)z = 0$$

$$\begin{vmatrix} 3K-8 & 3 & 3 \\ 3 & 3K-8 & 3 \\ 3 & 3 & 3K-8 \end{vmatrix} = 0 \qquad \begin{vmatrix} 3K-2 & 3 & 3 \\ 3K-2 & 3K-8 & 3 \\ 3K-2 & 3 & 3K-8 \end{vmatrix} = 0$$

$$(3K-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3K-8 & 3 \\ 1 & 3 & 3K-8 \end{vmatrix} = 0 \qquad (3K-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3K-11 & 0 \\ 0 & 0 & 3K-11 \end{vmatrix}$$

$$(3K-2)(3K-11)^2 = 0$$
$$K = 2/3,\ 11/3,\ 11/3$$

Example: Find the values of λ for which the equations

$$(\lambda-1)x + (3\lambda+1)y + (2\lambda)z = 0$$
$$(\lambda-1)x + (4\lambda-2)y + (\lambda+3)z = 0$$
$$2x + (3\lambda+1)y + (3\lambda-3)z = 0$$

are consistent and find the ratio of $x:y:z$ when λ has the smallest of these values.
The given equation will be consistent if,

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ \lambda-1 & 4\lambda-2 & \lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0 \qquad (\text{Operate } R_2 - R_1)$$

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 2\lambda \\ 0 & \lambda-3 & -\lambda+3 \\ 2 & 3\lambda+1 & 3\lambda-3 \end{vmatrix} = 0 \qquad (\text{Operate } C_3 + C_2)$$

$$\begin{vmatrix} \lambda-1 & 3\lambda+1 & 5\lambda+1 \\ 0 & \lambda-3 & 0 \\ 2 & 3\lambda+1 & 6\lambda-2 \end{vmatrix} = 0 \quad [\text{Expand by } R_2]$$

$$2(\lambda-3)[(\lambda-1)(3\lambda-1)-(5\lambda+1)] = 0$$

$$6\lambda(\lambda-3)^2 = 0$$

$$\boxed{\lambda=0} \quad \boxed{\lambda=3}$$

(a) When $\lambda=0$ the equation becomes $-x+y=0$

$$-x-2y+3z=0$$

$$2x+y-3z=0$$

solving equations we get

$$\frac{x}{6-3} = \frac{y}{6-3} = \frac{z}{-1+4} \quad \text{hence } x=y=z$$

(b) When $\lambda=3$ equations becomes identical

Eigen values

If a A square matrix of order n we can form the matrix $A-\lambda I$ where I is the n th order unit matrix. The determinant of the matrix equated to zero

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

is called the characteristic equations of λ on expanding the determinant

$$(-1)^n\lambda^n + K_1\lambda^{n-1} + K_2\lambda^{n-2} + \dots\dots + K_n = 0$$

Eigen Vectors

If $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

then the linear transformation $Y = AX$ carries the column vector X into the column vector Y by means of square matrix A

Then,

$$\lambda X = AX \quad \text{or}$$

$$AX = \lambda IX \quad \text{or}$$

$$|A-\lambda I|X = 0$$

$X = [x_1, x_2, \dots x_n]$ is called eigen vector

Example Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic equation is $|A-\lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \qquad \lambda^2-7\lambda+6=0$$

$$(\lambda-6)(\lambda-1)=0$$

$$\boxed{\lambda=6} \quad \boxed{\lambda=1}$$

Eigen values are 6, 1

Let eigen vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$ corresponding to eigen values of λ then

When $\lambda=6$, $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

equation formed is $-x+4y=0$

$\frac{x}{4} = \frac{y}{1}$ giving the eigen vector $(4,1)$

When $\lambda = 1$ $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

equation formed is $x+y = 0$

$\frac{x}{1} = \frac{y}{-1}$ giving the eigen vector $(1,-1)$

Example: Find the eigen values and eigen vectors of matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Characteristic equation is $|A-\lambda I|$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$$

$\lambda^3 - 7\lambda^2 + 36 = 0$

$(\lambda+2)(\lambda-3)(\lambda-6) = 0$

The eigen values of A are $\boxed{\lambda = -2}$ $\boxed{\lambda = 3}$ $\boxed{\lambda = 6}$

Let eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to the corresponding values of λ

When $\lambda = -2$ we have

$3x + y + 3z = 0$

$x + 7y + z = 0$ $\qquad \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$

$3x + y + 3z = 0$

Hence eigen vector is $(-1,0,1)$

When $\lambda = 3$ we have

$-2x + y + 3z = 0$

$x + 2y + z = 0$ $\qquad \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

$3x + y + (-2z) = 0$

Hence eigen vector is $(1,-1,1)$

When $\lambda = 6$ we have

$-5x + y + 3z = 0$

$x - y + z = 0$ $\qquad \frac{x}{1} = \frac{y}{2} = \frac{z}{1}$

$3x + y - 5z = 0$

Hence eigen vector is $(1,2,1)$

Properties of Eigen values

I. Any square matrix A and its transpose A' have same eigen values,

$|A-\lambda I| = |A' - \lambda I| = 0$

if and only if it is an eigen value of A'

II. The eigen values of a triangular matrix are just diagonal elements of the matrix

$|A-\lambda I| = (a_{11}-\lambda)(a_{22}-\lambda) \dots\dots (a_{nn}-\lambda)$

Roots of $|A-\lambda I|$ are $a_{11}, a_{22}, \dots\dots, a_{nn}$

III. The eigen values of an idempotent matrix are either zero or unity

$A^2 = A$ (idempotent)

$AX = \lambda X$

$A^2X = \lambda(AX)$

$$\lambda^2 X = \lambda X$$
$$(\lambda^2 - \lambda)X = 0$$
$$\boxed{\lambda = 0} \quad \boxed{\lambda = 1}$$

IV. The sum of the eigen values of a matrix is the sum of the elements of the principal elements

$$|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix}$$

If $\lambda_1 \lambda_2 \lambda_3$ are eigen values then

$$|A - \lambda I| = -\lambda_3 + \lambda^2 (a_{11} + a_{22} + a_{33}) - \ldots$$
$$|A - \lambda I| = (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$\boxed{\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}}$$

V. The product of eigen values of a matrix A is equal to its determinant

VI. If λ is an eigen value of A then $(\frac{1}{\lambda})$ is the eigen value of A^{-1}

VII. If λ is an eigen value of an orthogonal matrix then $(\frac{1}{\lambda})$ is also its eigen value

VIII. If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of matrix A then A^m has eigen values $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$ (m being a positive number)

Cayley Hamilton Theorem

Example: Verify Cayley Hamilton theorem for the matrix A $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and find its inverse

The characteristic equation of A is

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

or $\lambda^2 - 4\lambda - 5 = 0$

By cayley hamilton theorem, it must satisfy characteristics equation

$$A^2 - 4A - 5I = 0$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

multiplying A^{-1} we get

$$A - 4I - 5A^{-1} = 0$$
$$A^{-1} = \frac{1}{5}(A - 4I)$$

$$A^{-1} = \frac{1}{5}\left\{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$$

Example: Find the characteristic equation of the matrix A $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ and hence find its Inverse

The characteristic equation of A is :

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 3-\lambda & -3 \\ -2 & -4 & -4-\lambda \end{vmatrix} = 0 \quad \text{i.e.} \quad \lambda^3 - 20\lambda + 8 = 0$$

By Cayley Hamilton theorem,

$$A^3 - 20A + 8I = 0$$

$$A^{-1} = \frac{5I}{2} - \frac{1}{8}A^2$$

$$= \frac{5}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Reduction to Diagonal Form

If a square matrix A of order n has n linearly independent eigen vectors then a matrix P can be found that $D = P^{-1}AP$ is a diagonal matrix D

Let A be a square matrix of order 3. Let $\lambda_1 \lambda_2 \lambda_3$ be its eigen values and

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Denoting the square matrix $[X_1 X_2 X_3]$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \text{ by } P$$

The matrix P which diagonalises A is called modal matrix of A and the resulting diagonal matrix is D is known as spectral matrix of A

Working Procedure

1. Find the eigen values of square matrix A
2. Find the corresponding eigen vectors and write modal matrix P
3. Find the diagonal matrix D from $D = P^{-1}AP$
4. Obtain A^n from $A^n = PD^nP^{-1}$

Example Reduce the matrix $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to the diagonal form

The characteristic equation of A is

$$\begin{vmatrix} -1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} \quad \text{or} \quad \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$$

Solving we get $\lambda_1 = 1$ $\lambda_2 = \sqrt{5}$ $\lambda_3 = -\sqrt{5}$ as eigen values of A

When $\lambda = 1$, we get

$$-2x + 2y - 2z = 0$$
$$x + y + z = 0 \qquad \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$$
$$-x - y - z = 0$$

we get eigen vector as $(1, 0, -1)$

When $\lambda = \sqrt{5}$, we get

$$(-1-\sqrt{5})x + 2y + 2z = 0$$
$$x + (2-\sqrt{5})y + z = 0 \qquad \frac{x}{\sqrt{5}-1} = \frac{y}{1} = \frac{z}{-1}$$
$$-x - y - \sqrt{5}z = 0$$

we get eigen vector as $(\sqrt{5}-1, 1, -1)$

When $\lambda = -\sqrt{5}$, we get

$$(\sqrt{5}-1)x + 2x - 2z = 0$$
$$x + (2+\sqrt{5})y + z = 0 \qquad \frac{x}{\sqrt{5}+1} = \frac{y}{-1} = \frac{z}{1}$$
$$-x + (-y) + \sqrt{5}z = 0$$

we get eigen vectors $(\sqrt{5}+1, -1, -1)$

Writing three eigen vectors as three columns, we get the modal matrix

$$P = \begin{bmatrix} 1 & \sqrt{5}-1 & \sqrt{5}+1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Hence diagonal matrix is

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$$

Reduction of quadratic form to canonical form

A homogeneous expression of the second degree in any number of variables is called a quadratic form.

For instance if

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad X' = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$X'AX = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

which is quadratic form

Let $\lambda_1\ \lambda_2\ \lambda_3$ be the eigen values of the matrix A

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Then by creating matrix P by adding three columns $= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$

Hence the quadratic form is reduced to canonical form (or sum of square form or Principal form)

and P is matrix of transformation which is an orthogonal matrix

$$\begin{bmatrix} \text{coeff. of } x^2 & \frac{1}{2}\text{coeff. of } yz & \frac{1}{2}\text{coeff. of } zx \\ \frac{1}{2}\text{coeff. of } yz & \text{coeff. of } y^2 & \frac{1}{2}\text{coeff. of } xy \\ \frac{1}{2}\text{coeff. of } zx & \frac{1}{2}\text{coeff. of } xy & \text{coeff. of } z^2 \end{bmatrix}$$

Example: Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the canonical form and specify the matrix of transformation

The matrix of the given quadratic form is $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Its characteristics equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\boxed{\lambda = 2}$ $\boxed{\lambda = 3}$ $\boxed{\lambda = 6}$ as eigen values

canonical form is $(\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2)$

$$2x^2 + 3y^2 + 6z^2$$

for calculating eigen vectors we have,

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(3-\lambda)x - y + z = 0$$
$$-x + (5-\lambda)y - z = 0$$
$$x - y + (3-\lambda)z = 0$$

When $\lambda = 2$ eigen vector x_1 $(1, 0, -1)$ and normalised form is $(1/\sqrt{2}, 0, -1/\sqrt{2})$

When $\lambda = 3$ eigen vector x_2 $(1, 1, 1)$ and normalised form is $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

When $\lambda = 6$ eigen vector x_3 $(1, -2, 1)$ and normalised form is $(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6})$

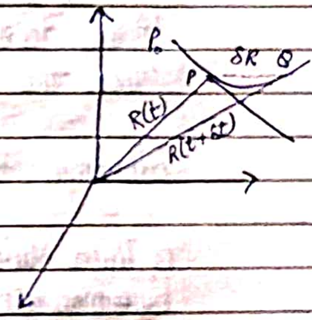
Matrix of transformation is $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

Unit IV — Vector calculus and its Application

Curves in Space

Tangent

Let $R(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the position vector of a point P



The vector $R' = \frac{dR}{dt}$ is a tangent to the space curve $R = F(t)$

Let P_0 be a fixed point of this curve corresponding to $t = t_0$. If s is the length of the arc P_0P, then

$$\frac{ds}{dt} = \left|\frac{dR}{dt}\right| \text{ or } R'(t)$$

If $R'(t)$ is continuous then arc P_0P is given by

$$s = \int_0^t R'(t) = \int_0^t \sqrt{(x')^2 + (y')^2 + (z')^2}$$

If we takes the parameter in place of t then the magnitude of the tangent vector i.e. $\left|\frac{dR}{ds}\right| = 1$. Thus, the denoting the unit tangent vector by T we have,

$$T = \left|\frac{dR}{ds}\right|$$

Principal Normal

Since T is a unit vector we have,

$$\frac{dT}{ds} \cdot T = 0$$

i.e. dT/ds is perpendicular to T. Or else $dT/ds = 0$ in which case T is a constant vector w.r.t the arc length s and so has a fixed direction i.e. the curve is a straight line

If we denote a unit normal vector to the curve at P by N then the plane of T and N is called osculating plane

Binormal

A third unit vector B defined by $B = T \times N$ is called the binormal at P.

Thus at each point P of a space curve there are three mutually perpendicular unit vectors, T, N and B which form a moving trihedral.

$T = N \times B$, $N = B \times T$ and $B = T \times N$

The osculating plane containing T and N

The normal plane containing N and B

The rectifying plane containing B and T

Curvature

The arc rate of turning of the tangent is called the curvature of the curve and is denoted by κ

Since $\frac{dT}{ds}$ is in the direction of principal normal N, therefore

$$\frac{dT}{ds} = \kappa N$$

Torsion

Since B is a unit vector, we have $\frac{dB}{ds} \cdot B = 0$

Also, $B \cdot T = 0$, therefore $\frac{dB}{ds} \cdot T + B \cdot \frac{dT}{ds} = 0$

$$\frac{dB}{ds} \cdot T + B \cdot (\kappa N) = 0$$

$$\frac{dB}{ds} \cdot T = 0$$

Hence $\frac{dB}{ds}$ is perpendicular to both B and T and is therefore parallel to N

The arc rate of turning of the binormal is called torsion denoted by τ

$$\frac{dB}{ds} = -\tau N$$

Finally to find $\frac{dN}{ds}$, we differentiate $N = B \times T$

$$\frac{dN}{ds} = \frac{dB}{ds} \times T + B \times \frac{dT}{ds}$$

$$\frac{dN}{ds} = -\tau N \times T + B \times \kappa N$$

$$\frac{dN}{ds} = \tau B - \kappa T$$

Example Find the curvature and torsion of the curve $x = a\cos t$, $y = a\sin t$, $z = bt$.

The vector equation of the curve $R = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$

$$\frac{dR}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}$$

$$s = \int_0^{R_p} \left| dR/dt \right| dt = \sqrt{a^2+b^2}\, t$$

$$\frac{ds}{dt} = \sqrt{a^2+b^2}$$

$$T = \frac{dR}{ds} = \frac{dR}{dt} \Big/ \frac{ds}{dt} = \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}}{\sqrt{a^2+b^2}}$$

$$\frac{dT}{dt} = \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{(\sqrt{a^2+b^2})^2}$$

$$\frac{dT}{ds} = \frac{dT}{dt} \Big/ \frac{ds}{dt} = \frac{-a(\cos t\,\hat{i} + \sin t\,\hat{j})}{(a^2+b^2)}$$

$$K = \left|\frac{dT}{ds}\right| = \frac{a}{a^2+b^2} \quad \text{and} \quad N = -(\cos t\,\hat{i} + \sin t\,\hat{j})$$

$$B = T \times N = \frac{(b\sin t\,\hat{i} - b\cos t\,\hat{j} + a\hat{k})}{\sqrt{a^2+b^2}}$$

$$\frac{dB}{ds} = \frac{dB}{dt} \Big/ \frac{ds}{dt} = \frac{b(\cos t\,\hat{i} + \sin t\,\hat{j})}{(a^2+b^2)} = -\tau N$$

$$\boxed{\tau = \frac{b}{a^2+b^2}} \quad \text{and} \quad \boxed{K = \frac{a}{a^2+b^2}}$$

Velocity and Acceleration

Let the position of a particle P at time t on a path curve (C) be $R(t)$.

$$\lim_{\delta t \to 0} \frac{\delta R}{\delta t} = \frac{dR}{dt} = V$$

is the tangent vector of C at P which is velocity (vector) V of the motion and is magnitude is given by $V = \frac{ds}{dt}$

The derivative of the velocity vector $V(t)$ is the acceleration vector $A(t)$ which is

$$A(t) = \frac{dV}{dt} = \frac{d^2R}{dt^2}$$

Tangential and Normal Acceleration

$$V(t) = \frac{dR}{dt} = \frac{dR}{ds} \times \frac{ds}{dt}$$

where, $\frac{dR}{ds}$ is a unit vector tangent of C

$$A(t) = \frac{dV}{dt} = \frac{d}{dt}\left[\frac{ds}{dt} \cdot \frac{dR}{ds}\right] = \frac{d^2s}{dt^2} \cdot \left(\frac{dR}{ds}\right) + \left(\frac{ds}{dt}\right)^2 \frac{d^2R}{ds^2}$$

$\frac{dR}{dt} \cdot \frac{d^2R}{dt^2} = 0$ implies that $\frac{d^2R}{dt^2} \perp \frac{dR}{dt}$

Tangential acceleration is $\frac{d^2s}{dt^2} \cdot \frac{dR}{ds}$

Normal acceleration is $\left(\frac{ds}{dt}\right)^2 \cdot \frac{d^2R}{ds^2}$

Example A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+3$ where t is the time. Find the component of its velocity and acceleration at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$.

$$\text{Velocity} = \frac{dR}{dt} = \frac{d}{dt}\left[(t^3+1)\hat{i} + t^2\hat{j} + (2t+3)\hat{k}\right]$$

$$\text{Velocity} = 3t^2\,\hat{i} + 2t\hat{j} + 2\hat{k}$$

$$\text{at 1s} = 3\hat{i} + 2\hat{j} + 2\hat{k} =$$

acclaration $= 6t\hat{i} + 2\hat{j} + 0\hat{k}$

at $t = 1s$ $= 6\hat{i} + 2\hat{j}$

Now unit vector in the direction of $\hat{i}+\hat{j}+3\hat{k}$ is

$$\frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{1^2+1^2+3^2}} = \frac{1}{\sqrt{11}} \times (\hat{i}+\hat{j}+3\hat{k})$$

component of velocity at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$

$$= \frac{(3\hat{i}+2\hat{j}+2\hat{k}).(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}} = \sqrt{11}$$

component of velocity at $t=1$ in the direction $\hat{i}+\hat{j}+3\hat{k}$

$$= \frac{(6\hat{i}+2\hat{j}).(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

Example A particle moves along the curve $R = (t^3-4t)\hat{i} + (t^2+4t)\hat{j} + (8t^2-3t^3)\hat{k}$ where t denotes time. Find the magnitude of acceleration along tangent and normal at time $t = 2s$

Velocity at $\left(\frac{dR}{dt}\right) = (3t^2-4)\hat{i} + (2t+4)\hat{j} + (16t-9t^2)\hat{k}$

acclaration $\left(\frac{d^2R}{dt^2}\right) = (6t)\hat{i} + 2\hat{j} + (16-8t)\hat{k}$

at $t = 2s$, velocity $V = 8\hat{i} + 8\hat{j} - 4\hat{k}$

and acclaration $A = 12\hat{i} + 2\hat{j} - 20\hat{k}$

Since the velocity is along the tangent of the curve, therefore the component of A along the tangent

$$= A . \frac{V}{|V|} = (12\hat{i}+2\hat{j}-20\hat{k}) . \frac{8\hat{i}+8\hat{j}-4\hat{k}}{\sqrt{64+64+16}}$$

$$= \frac{12\times8 + 2\times8 + (-20)\times(-4)}{12} = 16$$

Component of A along the normal

$= |A - \text{Resolved part of A along the tangent}|$

$$= \left|12\hat{i}+2\hat{j}-20\hat{k} - 16\left(\frac{8\hat{i}+8\hat{j}-4\hat{k}}{12}\right)\right|$$

$$= \frac{1}{3}\left|4\hat{i} - 26\hat{j} - 44\hat{k}\right| = 2\sqrt{73} \text{ units}$$

Scalar and vector point functions

(1) If to each point $P(R)$ of a region E in space there corresponds a definite scalar denoted by $f(R)$, then $f(R)$ is called a scalar point function E.

(2) If to each point $P(R)$ of region E in space there corresponds a definite vector denoted by $F(R)$, then $F(R)$ is called a vector point function in E.

Differentiation of vector point function follows the same rule as those of ordinary calculus. Thus if $F(x,y,z)$ be a vector point function, then

$$\frac{df}{dt} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}.\frac{\partial x}{\partial t}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}.dy + \frac{\partial f}{\partial z}.dz$$

$$df = \left[\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right]F$$

(3) Vector Operator del : The operator on the right side of the equation (i) is in the form of a scalar product of $\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ and

$$dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

If ∇ (read as del) be defined by the equation

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

then, may be written as

$$df = (\nabla . dR) f$$

Gradient of function (scalar point function)

The vector function ∇f is defined as the gradient of the scalar point function f and is written as grad. f

Thus, $\text{grad}. f = \nabla . f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$

∇f gives the maximum rate of change of f

Example Prove that $\nabla r^n = nr^{n-2}$, where $R = x\hat{i} + y\hat{j} + z\hat{k}$

We have $f(x,y,z) = r^n = (x^2+y^2+z^2)^{n/2}$

$$\frac{\partial f}{\partial x} = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1} . 2x = nxr^{n-2}$$

$$\frac{\partial y}{\partial y} = nyr^{n-2} \qquad \frac{\partial f}{\partial x} = nzr^{n-2}$$

$$\nabla r^n = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

$$\nabla r^n = nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2} R$$

Example If $\nabla u = 2r^4 R$, find u.

When, $\nabla u = 2(x^2+y^2+z^2)^2 R$

$$\nabla u = 2(x^2+y^2+z^2)(x\hat{i} + y\hat{j} + z\hat{k})$$

but, $\nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial x}\hat{k}$

comparing (i) and (ii) we get

$$\frac{\partial u}{\partial x} = 2x(x^2+y^2+z^2)^2 \qquad \frac{\partial u}{\partial y} = 2y(x^2+y^2+z^2)^2$$

$$\frac{\partial u}{\partial z} = 2z(x^2+y^2+z^2)^2$$

Also, $du(x,y,z) = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dx$

$$du(x,y,z) = 2(x^2+y^2+z^2)^2(x\,dx + y\,dy + z\,dz)$$

$$du = 2t^2 . \frac{dt}{2}$$

taking $x^2+y^2+z^2 = t$

$$2(x\,dx + y\,dy + z\,dz) = dt$$

Integrating both sides $u = \int t^2 dt + c = \frac{1}{3}t^3 + c$

$$\boxed{u = \frac{1}{3}r^{3/2} + c}$$

Example If $u = x+y+z$, $v = x^2+y^2+z^2$ and $w = xy+yz+zx$ prove that grad u, grad v and grad w are coplanar.

$$\text{grad } u = \nabla.u = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(x+y+z)$$

$$\text{grad } v = \nabla.v = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(x^2+y^2+z^2)$$

$$\text{grad } w = \nabla.w = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).(xy+yz+zx)$$

for coplanarity,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

$$\Delta = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix}$$

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0$$

$\boxed{\Delta = 0}$

Example: Find a unit vector normal to the surface $xy^2z^3 = 4$ at the point $(-1,-1,2)$

A vector normal to the given surface is $\nabla(xy^3z^2)$

$$= \frac{\partial}{\partial x}(xy^3z^2)\hat{i} + \frac{\partial}{\partial y}(xy^3z^2)\hat{j} + \frac{\partial}{\partial z}(xy^3z^2)\hat{k}$$

$$= y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$= -4\hat{i} - 12\hat{j} + 4\hat{k} \quad \text{at the point } (-1,-1,2)$$

Hence the desired unit normal to the surface

$$= \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + (-12)^2 + (4)^2}} = \frac{-1}{\sqrt{11}}(\hat{i} + 3\hat{j} + \hat{k})$$

Example Find the directional derivative of $f(x,y,z) = xy^3 + yz^3$ at the point $(2,-1,1)$ in the direction of vector $\hat{i} + 2\hat{j} + 2\hat{k}$

Here $\nabla f = y^3\hat{i} + (3xy^2 + z^3)\hat{j} + (3yz^2)\hat{k}$

$\nabla f = \hat{i} - 3\hat{j} - 3\hat{k}$

at the point $(2,-1,1)$

directional derivative of f in the direction $\hat{i} + 2\hat{j} + 2\hat{k}$

$$= (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1^2+2^2+2^2}} = (1.1 - 3.2 - 3.2)/3$$

$$= -3\tfrac{2}{3} \text{ (directional derivative of } f)$$

Example Find the directional derivative of $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$. Also calculate the magnitude of the maximum directional derivative

We have, $\nabla f = \left(\frac{\partial}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial y}\right)\hat{j} + \left(\frac{\partial}{\partial z}\right)\hat{k}\,(x^2 - y^2 + 2z^2)$

$\nabla f = 2x\hat{i} - 2y\hat{j} + 4z\hat{k}$

$\nabla f = 2\hat{i} - 4\hat{j} + 12\hat{k}$ at $P(1,2,3)$

$\vec{PQ} = \vec{OQ} - \vec{OP} = (5\hat{i} + 0\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$\vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

unit vector of $\hat{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$

The directional derivative of f in the direction of $\vec{PQ}$

$$\nabla f.\hat{PQ} = (2\hat{i} - 4\hat{j} + 12\hat{k})(4\hat{i} - 2\hat{j} + \hat{k})/\sqrt{21}$$

$$\nabla f \cdot \hat{PQ} = \frac{(8+8+12)}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

The directional derivative of its maximum in the direction of the normal to the surface.

Hence maximum value of this directional

$$= |\nabla f| = |2\hat{i} - 4\hat{j} + 12\hat{k}| = \sqrt{4+16+144}$$

$$\boxed{|\Delta f| = \sqrt{164}}$$

Example: Find the angle between the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$.

Let surfaces $f_1 = x^2+y^2+z^2-9=0$ and $f_2 = x^2+y^2-z-3=0$

Then,

$$N_1 = \nabla f_1 = (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$\text{at } (2,-1,2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$N_2 = \nabla f_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\text{at } (2,-1,2) = 4\hat{i} - 2\hat{j} - \hat{k}$$

Since the angle θ between the two normals is same as the angle between the two surfaces

$$\cos\theta = \frac{N_1 \cdot N_2}{n_1 n_2} = \frac{4\hat{i} - 2\hat{j} + 4\hat{k}}{\sqrt{16+4+16}} \cdot \frac{(4\hat{i} - 2\hat{j} - \hat{k})}{\sqrt{16+4+1}}$$

$$\cos\theta = \frac{4(4)+(-2)(-2)+4(-1)}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Divergence of function (vector-point function)

The divergence of a continuously differentiable vector point function F is denoted by div. F

$$\text{div}.f = \nabla.f = \left(\frac{\partial f}{\partial x}\right)\hat{i} + \left(\frac{\partial f}{\partial y}\right)\hat{j} + \left(\frac{\partial f}{\partial z}\right)\hat{k}$$

If $F = f\hat{i} + \phi\hat{j} + \psi\hat{k}$

then, $$\text{div}.f = \nabla.f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(f\hat{i} + \phi\hat{j} + \psi\hat{k})$$

$$\text{div}.f = \frac{\partial f}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial z}$$

The Curl of a continuously differentiable vector point function F is defined by the equation

$$\text{curl } F = \nabla \times F = \left(\frac{\partial F}{\partial x}\right)\times\hat{i} + \left(\frac{\partial F}{\partial y}\right)\times\hat{j} + \left(\frac{\partial F}{\partial z}\right)\times\hat{k}$$

If $F = f\hat{i} + \phi\hat{j} + \psi\hat{k}$

then curl F $$= \nabla \times F = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times(f\hat{i} + \phi\hat{j} + \psi\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & \phi & \psi \end{vmatrix} = \left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial z}\right)\hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial \psi}{\partial x}\right)\hat{j} + \left(\frac{\partial \phi}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

Example. If $R = x\hat{i} + y\hat{j} + z\hat{k}$, show that

(i) $\nabla . R = 3$ (ii) $\nabla \times R = 0$

(i) $\nabla . R = \left(\frac{\partial}{\partial x}\right)x + \left(\frac{\partial}{\partial y}\right)(y) + \left(\frac{\partial}{\partial z}\right)z = 3$

(ii) $\nabla \times R = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$

$\nabla \times R = 0$

Del Applied twice to point functions

(1) div. grad $f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(2) curl. grad $f = \nabla \times (\nabla . f) = 0$

(3) div. curl $f = \nabla . (\nabla \times f) = 0$

(4) curl curl $f = \nabla \times (\nabla \times f) = \nabla(\nabla . F) - \nabla^2 F$

(5) grad. ~~curl~~ div $F = \nabla . (\nabla . F) = \nabla \times (\nabla \times F) + \nabla^2 F$

Del applied to products of point functions

(1) $\text{grad}(fg) = f(\text{grad } g) + g(\text{grad } f)$

(2) $\text{div}(fg) = (\text{grad } f) . G + f(\text{div}. G)$

(3) $\text{curl}(fG) = (\text{grad } f) \times G + f(\text{curl } G)$

(4) $\text{grad}(F.G) = (F.\nabla)G + (G.\nabla)F + F \times (\text{curl } G) + (G \times \text{curl } F)$

(5) $\text{div}(F \times G) = G.(\text{curl } F) - (F.\text{curl } G)$

(6) $\text{Curl}(F \times G) = F(\text{div } G) - (G \text{ div} F) + (G.\nabla)F - (F.\nabla)G$

Example Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$

$\nabla^2(r^n) = \frac{\partial^2(r^n)}{\partial x^2} + \frac{\partial^2(r^n)}{\partial y^2} + \frac{\partial^2(r^n)}{\partial z^2}$

Now, $\frac{\partial(r^n)}{\partial x} = nr^{n-1}\left(\frac{\partial r}{\partial x}\right) = nr^{n-1}\left(\frac{x}{r}\right) = nr^{n-2}x$

$\frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2}\frac{\partial x}{\partial x} + (n-2)r^{n-3}\frac{\partial r}{\partial x}(x)\right]$

$\frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-3}\frac{x}{r}(x)\right]$

$\frac{\partial^2(r^n)}{\partial x^2} = n\left[r^{n-2} + (n-2)r^{n-4}x^2\right]$

Similarly $\frac{\partial^2 r^n}{\partial y^2} = n\left[r^{n-2} + (n-2)r^{n-4}y^2\right]$

$\frac{\partial^2 r^n}{\partial z^2} = n\left[r^{n-2} + (n-2)r^{n-4}z^2\right]$

Adding, $\nabla^2 r^n = n\left[3r^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)\right]$

$\nabla^2 r^n = n(n+1)r^{n-2}$

Example If r is the distance of a point (x, y, z) from the origin prove that $\text{curl}\left(K \times \text{grad}\frac{1}{r}\right) + \text{grad}\left(K . \text{grad}\frac{1}{r}\right) = 0$ where K is the unit vector in direction of OZ.

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$$\text{grad}\,\frac{1}{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(x^2+y^2+z^2\right)^{-1/2}$$

$$\text{grad}\,\frac{1}{r} = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right)$$

$$\text{grad}\,\frac{1}{r} = -\left(x^2+y^2+z^2\right)^{-3/2}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$\text{curl}\left(K\times\text{grad}\,\frac{1}{r}\right) = \nabla\times\left[-\left(x^2+y^2+z^2\right)^{-3/2}\left(x\hat{j}-y\hat{i}\right)\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{-x}{(x^2+y^2+z^2)^{3/2}} & 0 \end{vmatrix}$$

$$= \frac{\partial}{\partial z}\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\}\hat{i} + \frac{\partial}{\partial x}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\}\hat{j} - \frac{\partial}{\partial x}\left[\left\{\frac{x}{(x^2+y^2+z^2)^{3/2}}\right\} + \frac{\partial}{\partial y}\left\{\frac{y}{(x^2+y^2+z^2)^{3/2}}\right\}\right]$$

$$= \frac{-3xz\hat{i} - 3yz\hat{j} + (x^2+y^2+z^2)\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{grad}\left(K.\text{grad}\,\frac{1}{r}\right) = \nabla.\left\{-K.\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left\{\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$= \frac{3xz\hat{i}}{(x^2+y^2+z^2)^{5/2}} + \frac{3yz\hat{j}}{(x^2+y^2+z^2)^{5/2}} + \frac{(3z^2-x^2-y^2-z^2)\hat{k}}{(x^2+y^2+z^2)^{5/2}}$$

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Adding (i) and (ii) we get,

$$\text{curl}\left(K\times\text{grad}\,\frac{1}{r}\right) + \text{grad}\left(K.\text{grad}\,\frac{1}{r}\right) = 0$$

Integration of Vectors

If two vector functions $F(t)$ and $G(t)$ be such that

$$\frac{d[G(t)]}{dt} = F(t)$$

$$\int d[G(t)] = \int F(t)\,dt$$

$$\int_a^b F(t)\,dt = G(t) + C$$

$$F(t) = G(b) - G(a)$$

Line Integral

When the path of integration is a closed curve, this fact is denoted by using $\oint$ in place of $\int$

If, $F(R) = f(x,y,z)\hat{i} + \phi(x,y,z)\hat{j} + \psi(x,y,x)\hat{k}$

and $dR = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

then

$$\int_c F(R).dR = \int_c (f\,dx + \phi\,dy + \psi\,dz)$$

$$\int_c F(R).dR = \int_c (f\,dx + \phi\,dy + \psi\,dz)$$

Two other types of integral are $\int_c (F\times dR)$ and $\int_c f\,dR$

Example. If $F = 3xy\,\hat{i} - y^2\hat{j}$ evaluate $\int F.dR$ where C is the curve in the xy plane $y = 2x^2$ from (0,0) to (1,2)

$$\int_C F.dR = \int_C (3xy\,\hat{i} - y^2\hat{j})(dx\,\hat{i} + dy\,\hat{j})$$

$$\int_C = \int_C (3xy\,dx - y^2 dy)$$

substituting $y = 2x^2$ where x goes from 0 to 1

$$= \int_0^1 3x(2x^2)dx - (2x^2)^2 d(2x^2)$$

$$= \int_0^1 6x^3 dx - \int_0^1 16x^5 dx = -\frac{7}{6}$$

Example A vector field is given $F = \sin y\,\hat{i} + x(1+\cos y)\hat{j}$. Evaluate the line integral over a circular path given by $x^2+y^2 = a^2$, $z=0$

$$\oint F.dR = \oint_C [\sin y\,\hat{i} + x(1+\cos y)\hat{j}](dx\hat{i} + dy\hat{j})$$

$$\oint F.dR = \oint_C [(\sin y)dx + x\cdot(1+\cos y)dy]$$

$$\oint F.dR = \oint[(\sin y\,dx + x\cos y\,dy) + x\,dy]$$

$$\oint F.dR = \oint d(x\sin y) + x\,dy$$

$x = a\cos t$
$y = a\sin t$

$$\oint F.dR = \int_0^{2\pi} \oint d[a\cos t\,(a\sin t)\sin] + a^2\cos^2 t\,dt]$$

$$\oint F.dR = \left| a\cos t\,\sin(a\sin t)\right|_0^{2\pi} + \frac{a^2}{2}\int_0^{2\pi}(1+\cos 2t)dt$$

$$\oint F.dR = \frac{a^2}{2}\left| t + \frac{\sin 2t}{2}\right|_0^{2\pi} = \pi a^2$$

Surfaces

Representation: The parametric representation of S is of the form $R(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

The sum of all the sub-surfaces to form is called the normal surface integral of F(R) and S is denoted by

$$\int_S F.ds \quad \text{or} \quad \int_S F.N\,ds$$ where N is a unit outward normal at P to S

Other type of surface integral are $\int_S F \times ds$ or $\int_S f \times ds$ which are both vectors

Example Evaluate $\int_S F.N\,ds$ where $F = 2x^2y\,\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ and S is the closed surface of the region in the first octant bounded by cylinder $y^2+z^2 = 9$ and the plane $x=0$, $x=2$, $y=0$, $z=0$

$$\int_S F.Nds = \int_{S_1} F.Nds + \int_{S_2} F.Nds + \int_{S_3} F.Nds + \int_{S_4} F.Nds + \int_{S_5} F.Nds$$

Now, $\int_{S_1} F.ds = \int_{S_1} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}) . (-\hat{k}) ds$

$\int_{S_1} F.ds = -4\int_{S_1} xz^2 ds$ $\quad$ [$z=0$ in xy plane]

$\int_{S_1} F.Nds = 0$

Similarly, $\int_{S_2} F.Nds = 0$ and $\int_{S_3} F.Nds = 0$

$$\int_{S_4} F.Nds = \int_{S_4} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}) . \hat{i} ds$$

$$= \int_{S_4} (2x^2 y ds) = \int_0^3 \int_0^{\sqrt{9-y^2}} 8y\, dy\, dz$$

$$= 4\int_0^3 (9-z^2) dz = 72$$

Example Green's Theorem in the plane

If $\phi(x,y)$, $\psi(x,y)$, ϕ_y and ψ_x be continuous in a region E of the xy plane bounded by a closed curve C then,

$$\int_C (\phi dx + \psi dy) = \iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

Let E bounded by $x=a$, $y=\xi(x)$, $x=b$, $y=\eta(x)$ where $\eta \geq \xi$, so C can be divided into curve C_1 and C_2

$$\iint_E \left(\frac{\partial \phi}{\partial y}\right) dx\, dy = \int_a^b \left[\int_\psi^\eta \frac{\partial \phi}{\partial y} dy\right] dx = \int_a^b |\phi|_\psi^\eta dx$$

$$= \int_a^b [\phi(x,\eta) - \phi(x,\psi)] dx$$

$$= -\int_{C_2} \phi(x,y) dx - \int_{C_1} \phi(x,y) dx$$

$$= -\int_C \phi(x,y) dx$$

Similarly, it can be shown that

$$\iint_E \frac{\partial \psi}{\partial x} dx\, dy = \int_C \psi(x,y) dy$$

Example Verify Green's Theorem for $\int_C [(xy+y^2)dx + x^2 dy]$ where C is bounded by $y=x$ and $y=x^2$.

$$\phi = xy + y^2 \qquad \psi = x^2$$

$$\int_C (\phi dx + \psi dy) = \int_{C_1} + \int_{C_2}$$

along C_1 $y=x^2$ and x varies from 0 to 1

$$\int_{C_1} = \int_0^1 [x(x^2) + (x^2)^2] dx + (x^2)^2 d(x^2)$$

$$\int_{C_1} = \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}$$

along C_2 $y = x$ and x varies from 1 to 0

$$\int_{C_2} = \int_1^0 [x(x) + (x^2)]dx + x^2 d(x)]$$

$$\int_{C_2} = \int_1^0 3x^2\, dx = -1$$

Thus,

$$\int_C (\phi\, dx + \psi\, dy) = \frac{19}{20} - 1 = \frac{-1}{20}$$

$$\iint_E \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy = \iint_E \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right]$$

$$= \int_0^1 \int_{x^2}^{x} (2x - x - 2y)\, dy\, dx$$

$$= \int_0^1 [xy - y^2]_{x^2}^{x}\, dx = \int_0^1 (x^4 - x^3)\, dx = \frac{-1}{20}$$

Given theorem is proved

Example If C is a simple closed surface in the x-y plane not enclosing the origin show that

$$\int_C F.dR = 0 \quad \text{where} \quad F = \frac{y\hat{i} - x\hat{j}}{x^2 + y^2}$$

$$\int_C F.dR = \int \frac{y\hat{i} - x\hat{j}}{x^2 + y^2} (dx\,\hat{i} + dy\,\hat{j})$$

$$\int_C F.dR = \int_C \frac{y\,dx - x\,dy}{x^2 + y^2}$$

On comparing with $\int_C (\phi\, dx + \psi\, dy)$ where

$$\phi = \frac{y}{x^2 + y^2} \qquad \psi = \frac{-x}{x^2 + y^2}$$

$$\int_C F.dR = \iint_S \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy$$

$$\int_C F.dR = \iint_S \left[\frac{-(x^2 + y^2) + x(2x)}{(x^2 + y^2)^2} - \frac{(x^2 + y^2) - y(2y)}{(x^2 + y^2)^2}\right] dx\, dy$$

$$\int_C F.dR = \iint_S \left[\frac{x^2 - y^2}{(x^2 + y^2)^2} - \frac{x^2 - y^2}{(x^2 + y^2)^2}\right] dx\, dy = 0$$

Stokes' Theorem

If S be an open surface bounded by a closed surface S and $F = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be any continuously differentiable vector point function then,

$$\int_C F.dR = \int_S \text{curl}\, F.N\, ds$$

writing $dR = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ may be reduced to the form

$$\int_C (f_1 dx + f_2 dy + f_3 dz) =$$

$$= \int_S \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\cos\beta + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\cos\gamma$$

Example Verify Stoke's Law for $F = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y=0$ and $y=b$

Let ABCD be the given rectangle

$$\int_{ABCD} F.dR = \int_{AB} F.dR + \int_{BC} F.dR + \int_{CD} F.dR + \int_{DA} F.dR$$

and $F.dR = [(x^2+y^2)\hat{i} - 2xy\hat{j}].(\hat{i}dx + \hat{j}dy)$

$$F.dR = [(x^2+y^2)dx - 2xydy]$$

Along AB $x=a$ i.e. $dx=a$ and y varies from 0 to b

$$\int_{AB} F.dR = -2a\int_0^b y\,dy = -2a.\frac{b^2}{2} = -ab^2$$

$$\int_{BC} F.dR = \int_{-a}^{a} (x^2+b^2)dx = -\frac{2a^3}{3} - 3ab^2$$

$$\int_{CD} F.dR = 2a\int_b^0 y\,dy = -ab^2$$

$$\int_{DA} F.dR = \int_{-a}^{a} x^2 dx = \frac{2a^3}{3}$$

Thus, $\int_{ABCD} F.dR = -4ab^2$

Example Apply Stoke's Theorem to evaluate $\int_C (ydx + zdy + xdz)$ where C is the curve of intersection of $x^2+y^2+z^2 = a^2$ and $x+z=a$.

The curve C is evidently a circle lying in the plane $x+z=a$ and having $A(a,0,0)$ and $B(0,0,a)$ as the extremities of the diameter

$$\int_C (ydx + zdy + xdz) = \int_C (y\hat{i} + z\hat{j} + x\hat{k}).dR$$

$$= \int_S \text{curl}\,(y\hat{i} + z\hat{j} + x\hat{k}).N\,ds$$

where, S is the circle on AB as diameter and N

$$N = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$$

$$= \int_S -(\hat{i}+\hat{j}+\hat{k}).\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}\right)ds$$

$$= \frac{-2}{\sqrt{2}}\int_S ds = \frac{-2}{\sqrt{2}}\pi\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{-\pi a^2}{\sqrt{2}}$$

Example Using Stoke's Theorem evaluate $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$

$$F = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$$

$$\text{curl}F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\hat{i}+\hat{k}$$

Also equation of the plane through A, B and C

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1 \quad \text{or} \quad 3x+2y+z=6$$

Vector N Normal to this plane is

$$\nabla\cdot(3x+2y+z-6) = 3\hat{i}+2\hat{j}+\hat{k}$$

$$\int_C F.dR = \int_C \left[(x+y)dx+(2x-z)dy+(y+z)dz\right]$$

$$= \int_S \text{curl } F.\hat{N}\,ds$$

$$= \int_S (2\hat{i}+\hat{k})\left(\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right)ds$$

$$= \frac{17}{\sqrt{14}}\int_S ds = \frac{7}{\sqrt{14}}(\text{area of } \Delta ABC)$$

$$= \frac{7}{\sqrt{14}} \cdot 3\sqrt{14} = 21$$

Volume Integral

If $F(R) = f(x,y,z)\hat{i} + \phi(x,y,z)\hat{j} + \psi(x,y,z)\hat{k}$

so that $\delta v = \delta x\,\delta y\,\delta z$

$$\int_S F.dv = \hat{i}\iiint_E f\,dx\,dy\,dz + \hat{j}\iiint_E \phi\,dx\,dy\,dz$$

$$+ \hat{k}\iiint_E \psi\,dx\,dy\,dz$$

Example Verify Divergence theorem for $F=(x^2-yz)\hat{i}+(y^2-zx)\hat{j}+(z^2-xy)\hat{k}$ taken over the rectangular parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$

$$\text{As div } F = \frac{\partial}{\partial x}(x^2-yz)+\frac{\partial}{\partial y}(y^2-zx)+\frac{\partial}{\partial z}(z^2-xy)$$

$$\text{div. } F = 2(x+y+z)$$

$$\int_R \text{div. } F\,dv = 2\int_0^c\int_0^b\int_0^a (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c dz\int_0^b dy\left(\frac{a^2}{2}+ay+az\right)$$

$$= 2\int_0^c dz\left(\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right)$$

$$= 2\left[\left(\frac{a^2b}{2}\right)c+\frac{ab^2(c)}{2}+\frac{abc^2}{2}\right]$$

$$= abc(a+b+c)$$

Also, $\int_S F.Nds = \int_{S_1} F.Nds + \int_{S_2} F.Nds + \text{——} + \int_{S_6} F.Nds$

Now, $\int_{S_1} F.Nds = \int F.(-K).ds = -\int_0^b\int_0^a (0-xy)\,dx\,dy$

$$\int_{S_1} F.Nds = \frac{a^2b^2}{4}$$

Similarly we can find for other faces and verify theorem

Example Evaluate $\int_S F.ds$ where $F=4x\hat{i}-2y^2\hat{j}+z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$ $z=0$ and $z=3$

$$\int_S f.ds = \int_V \operatorname{div} f\,dv$$

$$= \int_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$

$$= \iiint_V (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left| 4z - 4yz + z^2 \right|_0^3 dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\,dy\,dx$$

$$= \int_{-2}^{2} \left| 21y - 6y^2 \right|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= 42\int_{-2}^{2} \sqrt{4-x^2}\,dx = 84\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left| \frac{x\sqrt{(4-x^2)}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right|_0^2 = 84\pi$$