# Second order Diff Eqn

**Linear Combination of functions:** → Let $f_1(x), f_2(x) \ldots f_n(x)$ be $n$ function. Then $c_1 f_1(x) + c_2 f_2(x) + \_\_ + c_n f_n(x)$ where $c_1, c_2 \_\_ c_n \in \mathbb{R}$ is called a linear combination of given functions.

**Linearly dependent / Independent functions:** → let $f_1(x), f_2(x) \_\_ f_n(x)$ be $n$ functions. Then If

$$c_1 f_1(x) + c_2 f_2(x) + \_\_ + c_n f_n(x) = 0$$

$$\Rightarrow c_1 = c_2 = \ldots = c_n = 0$$

$\Rightarrow f_1(x), \_\_ f_n(x)$ are linearly Independent.

If $\exists$ some $c_i \neq 0$ and $c_1 f_1(x) + \_\_ + c_n f_n(x) = 0$

Then $f_1(x), \_\_, f_n(x)$ are called linearly dependent.

$\because$ If $c_1 \neq 0$ Then

$$f_1(x) = -\frac{1}{c_1}\left[c_2 f_2(x) + \_\_ + c_n f_n(x)\right]$$

In other words, If any function can be expressed as linear combination of other functions, Then the given functions are linearly dependent.

**Ex** $f_1(x) = x^2, \; f_2(x) = x^3, \; f_3(x) = 6x^2 - x^3$

**Soln** let $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$

$\Rightarrow c_1 x^2 + c_2 x^3 + c_3(6x^2 - x^3) = 0$

$\Rightarrow (c_1 + 6c_3)x^2 + (c_2 - c_3)x^3 = 0$

$\Rightarrow c_1 + 6c_3 = 0, \; c_2 = c_3$

$\Rightarrow c_1 = -6c_3, \; c_2 = c_3$ $\quad$ If $c_3 = 1 \Rightarrow c_1 = -6, c_2 = 1$

$\Rightarrow 6x^2 - x^3 = 6f_1(x) - f_2(x)$

$\Rightarrow f_1, f_2, f_3$ are L.D.

Que S.T. $x^2-1,\ 3x^2,\ 2-5x^2$ are L.D.

Sol$^n$ $c_1(x^2-1)+c_2(3x^2)+c_3(2-5x^2)=0$

$\Rightarrow (c_1+3c_2-5c_3)x^2+(2c_3-c_1)=0$

$\Rightarrow c_1+3c_2-5c_3=0$

$2c_3-c_1=0 \Rightarrow c_1=2c_3$

$\Rightarrow 2c_3-5c_3=-3c_2$

$\Rightarrow c_2=c_3$

If $c_3=1,\ c_1=2,\ c_2=1$

$\Rightarrow$ given functions are L.D.

Que $f(x)=x,\ g(x)=|x|$, on $D=[0,\infty)$

$g(x)=x\ \forall\ x\geq 0$

$\Rightarrow g(x)=1 f(x)$

$\Rightarrow f$ & $g$ are L.D on D

If $D=(-\infty,0)$

$g(x)=-x\ \forall\ x<0$

$\Rightarrow g(x)=-f(x)\ \forall\ x\in(-\infty,0)$

$\Rightarrow f$ & $g$ are L.D on D.

If $D=(-\infty,\infty)$ or $\mathbb{R}$

$g(x)=x=f(x)\ \forall\ x\in[0,\infty)$

$g(x)=-x=-f(x)\ \forall\ x\in(-\infty,0)$

$\Rightarrow f$ & $g$ are L.I on D.

If $D=[-1,1]$

$\Rightarrow g(x)$ & $f(x)$ are L.I on D

→ If f and g are L.I. on D, then f & g may be L.I or L.D on $S\subseteq D$. i.e. Subset of D.

→ If f & g are L.I. on D Then f & g will be L.I on $S\supseteq D$ i.e. Superset of D.

Wronskian:→ Let $f_1(x), f_2(x) \ldots f_n(x)$ be $n$ functions.

Then $W(f_1(x), \ldots, f_n(x)) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ | & | & & | \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix} = W(x)$

(1) Let $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ —(1)

$a_0(x) \neq 0 \ \forall x$, $a_0(x), a_1(x), a_2(x)$ are continuous fun. $\forall x$. If $y_1$ & $y_2$ are Sol$^n$ of (1) then

$a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \, ] \times y_2$

$a_0 y_2'' + a_1 y_2' + a_2 y_1 = 0 \, ] \times y_1$

$\Rightarrow a_0(y_2 y_1'' - y_2'' y_1) + a_1(y_1' y_2 - y_2' y_1) = 0$ —(2)

$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

$= y_1(x) y_2'(x) - y_2(x) y_1'(x)$

i.e. $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

$\Rightarrow W'(y_1, y_2) = y_1 y_2'' + \cancel{y_1' y_2'} - \cancel{y_2' y_1'} - y_2 y_1''$

$= y_1 y_2'' - y_2 y_1''$

So (2) becomes $a_0(-w') + a_1(-w) = 0$

$\Rightarrow a_0(x) w' + a_1(x) w = 0$

$\Rightarrow \dfrac{dw}{dx} = \dfrac{-a_1(x)}{a_0(x)} w$

$\Rightarrow \displaystyle\int \frac{dw}{w} = \int \frac{a_1(x)}{a_0(x)} dx$

$\Rightarrow \log w = -\displaystyle\int \frac{a_1(x)}{a_0(x)} dx + \log \alpha$

$\Rightarrow \boxed{w = \alpha e^{-\int \frac{a_1(x)}{a_0(x)} dx}} \rightarrow$ Abel's formula.

★ So wronskian is a solution of first order linear diff Equation
$a_0(x) w' + a_1(x) w = 0$.

★ $e^{-\int \frac{a_1(x)}{a_0(x)} dx}$ is always positive.

So If $\alpha = 0$ Then $w = 0$

If $\alpha > 0$ Then $w > 0$

If $\alpha < 0$ Then $w < 0$.

$\Rightarrow w(x) = \alpha \cdot e^{-\int \frac{a_1(x)}{a_0(x)} dx}$ is Either throughout zero or nowhere zero.

i.e. $W(x) \equiv 0 \; \forall x$ or $W(x) \neq 0 \; \forall x$.

Thm:→ Let $y_1$ & $y_2$ be two solutions of (D). Then $W(y_1, y_2)(x) = 0$ Iff $y_1$ and $y_2$ are linearly dependent.

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**Solⁿ:→** Let $W(y_1, y_2)(x) = 0$

$$\Rightarrow \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0 \Rightarrow y_1 y_2' - y_2 y_1' = 0$$

$$\Rightarrow \int \frac{y_2'}{y_2} = \int \frac{y_1'}{y_1}$$

$$\Rightarrow \log y_2 = \log y_1 + \log c$$

$$\Rightarrow y_2 = c y_1$$

$\Rightarrow$ $y_1$ and $y_2$ are linearly dependent.

Conversely Let $y_1$ and $y_2$ are L.D.

$$\Rightarrow y_1 = c y_2 \text{ or } y_2 = c y_1$$

$$\Rightarrow W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & c y_1(x) \\ y_1'(x) & c y_1'(x) \end{vmatrix} = 0.$$

Hence:

$y_1$ and $y_2$ are L.I. $\Leftrightarrow$ $W(y_1, y_2) \neq 0, \ \forall x.$

→ Note that above results are not true for general functions.

Ex let $f(x) = x^2$ and $g(x) = x|x|$

then $g(x) = \begin{cases} x^2 & \text{if } x > 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

$\Rightarrow$ $g$ and $f$ are L.I on $(-\infty, \infty)$

But $W(f,g)(x) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix}$

$$= 2x^2|x| - 2x^2|x| = 0$$

$\Rightarrow$ $f$ and $g$ are L.D.

So Contradiction.

$\because \frac{d}{dx} x|x| = \begin{cases} 2x & \text{if } x > 0 \\ -2x & \text{if } x < 0 \end{cases}$

$= 2 \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Let $y'' - 2xy' + y = 0$ and $y_1, y_2$ be its two Solutions.

**Q:** Let $y_1(0) = 1$, $y_1'(0) = 0$, $y_2(0) = 1$, $y_2'(0) = -1$. Find $w(y_1, y_2)(1)$?

**Sol$^n$** →

$$w(y_1,y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1$$

But $w(y_1, y_2)(x) = \alpha . e^{-\int \frac{a_1(x)}{a_0(x)} dx}$

$$a_0(x) = 1, \; a_1(x) = -2x$$

$$\Rightarrow w(y_1, y_2)(x) = \alpha e^{-\int -2x\,dx} = \alpha e^{x^2}$$

$$\therefore w(y_1, y_2)(0) = \alpha e^0 = \alpha$$

$$\Rightarrow \boxed{\alpha = -1}$$

$$\Rightarrow w(y_1, y_2)(x) = -e^{x^2}$$

$$\Rightarrow \boxed{w(y_1, y_2)(1) = -e}$$

**Que** → Let $y_1(x)$ and $y_2(x)$ be two sol$^n$ of $(1-x^2)y'' - 2xy' + (\sec x)y = 0$, with $y_1(0) = 1$, $y_1'(0) = 0$, If $W(\frac{1}{2}) = \frac{1}{3}$ then find $y_2'(0)$.

**Sol$^n$**

$$(1-x^2)y'' - 2xy' + (\sec x)y = 0 \quad -(1)$$

$(1-x^2) \neq 0$ iff $x \neq \pm 1$.

$$W(x) = \alpha e^{-\int \frac{-2x}{1-x^2} dx}$$

$$= \alpha e^{\int \frac{2x}{1-x^2} dx} \qquad \text{Put } 1-x^2 = t$$

$$= \alpha e^{-\int dt/t} = \alpha e^{-\log t} \qquad -2x\,dx = dt$$

$$= \frac{\alpha}{t} = \frac{\alpha}{1-x^2}$$

$$w(0) = \frac{\alpha}{1} = \alpha \qquad \text{and } w\left(\frac{1}{2}\right) = \frac{\alpha}{1-\frac{1}{4}} = \frac{4\alpha}{3} = \frac{1}{3}$$

$$\Rightarrow \alpha = \frac{1}{4}$$

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \alpha \Rightarrow \begin{vmatrix} 1 & y_2(0) \\ 0 & y_2'(0) \end{vmatrix} = \alpha = \frac{1}{4}$$

$$\Rightarrow y_2'(0) = 1/4.$$