Q.1 Let T be a L.T defined by

$$T\left(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right)=\begin{pmatrix}1\\2\\3\end{pmatrix},\quad T\left(\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\right)=\begin{pmatrix}1\\-2\\3\end{pmatrix},\quad T\left(\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right)=\begin{pmatrix}1\\-2\\-3\end{pmatrix},$$

$$T\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

Find $T\left[\begin{pmatrix}4 & 5\\ 3 & 8\end{pmatrix}\right]$

Q.2 Let T be a L.T from $\mathbb{R}^3 \to \mathbb{R}^2$ where $\boxed{Tx = Ax}$, $A=\begin{bmatrix}1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}_{2\times 3}$, $x=(x\ y\ z)^T$

Find $\ker(T)$ and Range(T) and also find their dimensions.

$[A]_{2\times 3}$

$\mathbb{R}^3 \to \mathbb{R}^2$

Soln:

Aryan:

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(\underset{\substack{\downarrow \\ \text{domain}}}{V})$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{array}{l} \to R_1 \\ \to R_2 \end{array}$$

$$\rho(A) = 2$$

$$\text{Nullity}(T) + 2 = 3$$

$$\text{Nullity}(T) = 3 - 2 = \underline{1}$$

$$\text{Ker}(T)/N(T) = \{v \in \mathbb{R}^3 \mid T(v) = 0\}$$

$$\Rightarrow \{v \in \mathbb{R}^3 \mid T(v) = 0\} = \{v \mid Av = 0\}$$

$$\Rightarrow \text{let } v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad v_1, v_2, v_3 \in \mathbb{R}$$

$$\Rightarrow Av = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_1 + v_2 = 0$$
$$-v_1 + v_3 = 0$$

$$\Rightarrow v_1 = -v_2 = v_3$$

$$\Rightarrow \text{Ker}(T) = \left\{ \begin{pmatrix} v_1 \\ -v_1 \\ v_1 \end{pmatrix} \middle| v_1 \in \mathbb{R} \right\}$$

$\therefore$ dimension of $\text{Ker}(T) = v_1 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is the vector in $\text{Ker}(T)$

Hence dimension of $\text{Ker}(T) = 1$

$$\text{Range}(T) = \{T(v) \mid v \in \mathbb{R}^3\}$$

$$= \{T(v) = Av \mid v \in \mathbb{R}^3\}$$

$$= \left\{ \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix} \middle| \, v_1, v_2, v_3 \in \mathbb{R} \right\} = \mathbb{R}^2(\mathbb{R})$$

$$= v_1 \boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} + v_2 \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + v_3 \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

But $\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Hence $\text{Rank}(T) = 2$.

$$\mathbb{R}^2(\mathbb{R}) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Q.3 : (H.W) (Ans) let T be a L.T from $\mathbb{R}^2 \to \mathbb{R}^3$ defined as $Tx = Ax$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}_{3\times 2} \quad \& \quad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Find $\text{Ker}(T)$, $\text{Range}(T)$ and also find their dimensions.

Q.4 Imp

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}$$

Determine the matrix of T with respect to the ordered basis

$$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$ in $\mathbb{R}^3$ and

$$Y = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$ in $\mathbb{R}^2$.

$A_{2\times 3}$

Soln:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \boxed{(0)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boxed{(0)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{(1)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boxed{(1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \boxed{(1)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boxed{(-1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore$ The matrix of T is

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}_{2\times 3}$$