Groups:

Def^n. Consider a set G and an operation $\overset{\oplus}{\otimes} : G \times G \to G$ defined on G. Then $G := (G, \overset{\oplus}{\otimes})$ is called a Group if the following hold:

1. Closure of G und $\overset{\oplus}{\otimes}$: $\forall x, y \in G$: $x \overset{\oplus}{\otimes} y \in G$
2. Associativity: $\forall x, y, z \in G$: $(x \overset{\oplus}{\otimes} y) \overset{\oplus}{\otimes} z = x \overset{\oplus}{\otimes} (y \overset{\oplus}{\otimes} z)$
3. Identity element: $\exists\, e \in G\ \forall x \in G$: $x \overset{\oplus}{\otimes} e = x$ and $e \overset{\oplus}{\otimes} x = x$
4. Inverse element: $\forall x \in G\ \exists y \in G$: $\underset{x \oplus (-x) = e}{x \otimes y = e}$ and $\underset{-x \oplus (x) = e}{y \otimes x = e}$, where e is the identity element. we often write x^{-1} to denote the inverse element of x.

If there exist $\forall x, y \in G$: $x \overset{\oplus}{\otimes} y = y \overset{\oplus}{\otimes} x$, then $G = (G, \overset{\oplus}{\otimes})$ is an abelian Group (Commutative).

Ex: * $(\mathbb{Z}, +)$ is an abelian Group.

* $(\mathbb{N}_0, +)$ is not a group. Although $(\mathbb{N}_0, +)$ possesses a neutral (identity) element (0), the inverse elements are missing.

* $(\mathbb{Z}, \cdot)$ is not a group. Although $(\mathbb{Z}, \cdot)$ contains an identity element 1, the inverse elements are missing.

* $(\mathbb{R}, \cdot)$ is not a group, since 0 does not possess an inverse element

Vector Spaces:

Vector spaces are the basic settings in which linear algebra happens.

Definition $\rightarrow$
A real valued vector space $V = (V, +, \cdot)$ is a set V with two operations

$$+ : V \times V \to V$$
$$\cdot : \mathbb{R} \times V \to V$$

where

1. $(V, +)$ is an abelian Group
2. Distributivity:
   (i) $\forall \lambda \in \mathbb{R}, x, y \in V$: $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$
   (ii) $\forall \lambda, \psi \in \mathbb{R}, x \in V$: $(\lambda + \psi) \cdot x = \lambda \cdot x + \psi \cdot x$
3. Associativity: $\forall \lambda, \psi \in \mathbb{R}, x \in V$: $\lambda \cdot (\psi \cdot x) = (\lambda \psi) \cdot x$
4. ~~Identity element w.r.t~~. There exists a multiplicative identity (1) in $\mathbb{R}$ such that $1 \cdot x = x\ \forall x \in V$.

The elements $x \in V$ are called vectors. The identity element of $(V, +)$ is the

zero vector $0 = [0, \cdots, 0]^T$ and the operation $\oplus$ is called vector addition. The elements $\lambda \in \mathbb{R}$ are called scalars and the operation $\odot$ is a multiplication by scalars.

Ex 1. $\mathbb{R}^n = \{(x_1, \cdots, x_n), (y_1, y_2, \cdots y_n) : x_n, y_n \in \mathbb{R}^n\}$

$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots y_n) := (x_1+y_1, x_2+y_2, \cdots, x_n+y_n) \in \mathbb{R}^n$ (closed under addition)

For c, $c(x_1, x_2, \cdots x_n) = (cx_1, cx_2, \cdots, cx_n) \in \mathbb{R}^n$ (scalar multiplication)

Is multiplication linear?

$(a+b)(x_1, x_2, \cdots, x_n) = a(x_1, \cdots, x_n) + b(x_1, x_2, \cdots x_n)$

To show. $(a+b)(x_1, x_2, \cdots x_n) = ((a+b)x_1, \cdots, (a+b)x_n)$

$= (ax_1 + bx_1, \cdots, ax_n + bx_n)$

$= (ax_1, \cdots, ax_n) + (bx_1, \cdots, bx_n)$

$= a(x_1, \cdots, x_n) + b(x_1, \cdots, x_n)$ //

Ex 2. The scalars $\mathbb{R}$ is a vector space with vector addition as the usual addition of real nos and the scalar multiplication of real nos.

Ex 3. Let $V = \{0\}$.

Vector addition : $0 + 0 = 0$

Scalar multiplication : $c \cdot 0 = 0$

Check that the properties are satisfied. This vector space is called as the zero vector space.

Ex. 1. Let $R^n = \{(a_1, a_2, \dots, a_n) : a_i \in R, 1 \le i \le n\}$.
be the set of all n-tuples of real nos.

We introduce two operations as follows:

1. $(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) = (a_1+b_1, a_2+b_2+\dots a_n+b_n)$
$\forall\ u = (a_1, a_2, \dots a_n),\ v = (b_1, b_2, \dots b_n) \in R^n$
clearly $u+v \in R^n\ \forall u, v \in R^n$

2. $\alpha(a_1, a_2, \dots, a_n) = \alpha a_1, \alpha a_2, \dots, \alpha a_n)$
$\forall\ \alpha \in R,\ u = (a_1, a_2, \dots, a_n) \in R^n$
clearly, $\alpha u \in R^n\ \forall\ \alpha \in R, u \in R^n$

Now

A1. Let $u = (a_1, a_2, \dots, a_n)$
$v = (b_1, b_2, \dots, b_n)$ and
$w = (c_1, c_2, \dots, c_n) \in R^n$

$$(u+v)+w = (a_1+b_1, a_2+b_2, \dots, a_n+b_n) + (c_1, c_2, \dots c_n)$$
$$= ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \dots (a_n+b_n)+c_n)$$
$$= (a_1+(b_1+c_1), a_2+(b_2+c_2), \dots, a_n+(b_n+c_n))$$
$$= (a_1, a_2, \dots, a_n) + (b_1+c_1, b_2+c_2, \dots, b_n+c_n)$$
$$= u+(v+w)$$

A2. $\exists\ 0 = (0, 0, \dots 0) \in R^n$ such that
$(a_1, a_2, \dots, a_n) + (0, 0, \dots 0) = (a_1+a_2, \dots a_n)$
$\forall (a_1, a_2, \dots, a_n) \in R^n$

A3. $\forall\ u = (a_1, a_2, \dots, a_n) \in R^n$ there exists
$-u = (-a_1, -a_2, \dots, -a_n) \in R^n$
such that $u+(-u) = (a_1, a_2, \dots, a_n) + (-a_1, -a_2, \dots, -a_n)$
$= (0, 0, \dots 0)$
$= 0$

A4. $\forall\ u = (a_1, a_2, \dots, a_n); v = (b_1, b_2, \dots, b_n) \in R^n$
$$u+v = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$$
$$= (b_1+a_1; b_2+a_2, \dots, b_n+a_n)$$
$$= v+u$$

(Contd)

By virtue of above discussion R^n is abelian group w.r.t. addition defined above. [2(Contd)]

Now, if $\alpha, \beta \in \mathbb{R}$, $u = (a_1, a_2, \dots, a_n)$, $v = (b_1, b_2, \dots, b_n) \in R^n$, then

M1. $\alpha(u+v) = \alpha(a_1+b_1, a_2+b_2, \dots, a_n+b_n)$

$= (\alpha(a_1+b_1), \alpha(a_2+b_2), \dots, \alpha(a_n+b_n))$

$= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \dots, \alpha a_n + \alpha b_n)$

$= (\alpha a_1, \alpha a_2, \dots, \alpha a_n) + (\alpha b_1, \alpha b_2, \dots, \alpha b_n)$

$= \alpha(a_1, a_2, \dots, a_n) + \alpha(b_1, b_2, \dots, b_n)$

$= \alpha u + \alpha v$.

M2. $(\alpha+\beta)u = ((\alpha+\beta)a_1, (\alpha+\beta)a_2, \dots, (\alpha+\beta)a_n)$

$= (\alpha a_1 + \beta a_1, \alpha a_2 + \beta a_2, \dots \alpha a_n + \beta a_n)$

$= (\alpha a_1, \alpha a_2, \dots, \alpha a_n) + (\beta a_1, \beta a_2, \dots, \beta a_n)$

$= \alpha u + \beta u$

M3. $\alpha(\beta u) = \alpha(\beta a_1, \beta a_2, \dots, \beta a_n)$

$= (\alpha(\beta a_1), \alpha(\beta a_2), \dots, \alpha(\beta a_n))$

$= ((\alpha\beta)a_1, (\alpha\beta)a_2, \dots, (\alpha\beta)a_n)$

$= (\alpha\beta)u$

M4. $1u = 1(a_1, a_2, \dots, a_n)$

$= (1a_1, 1a_2, \dots, 1a_n)$

$= u$

So by def^n R^n is a vector space over $\mathbb{R}$.

Ex: Let $R^2 = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}$

Define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$

$\alpha(a_1, a_2) = (\alpha a_1, a_2) \forall (a_1, a_2), (b_1, b_2) \in R^2, \alpha \in \mathbb{R}$

Let $\alpha, \beta \in \mathbb{R}$ and $u = (a_1, a_2) \in R^2$. Then

$(\alpha+\beta)u = (\alpha+\beta)(a_1, a_2)$

$= ((\alpha+\beta)a_1, a_2) = (\alpha a_1 + \beta a_1, a_2)$

$= (\alpha a_1, a_2) + (\beta a_1, a_2)$

$= \alpha(a_1, a_2) + \beta(a_1, a_2) \neq \alpha u + \beta u$

Hence R^2 is not a vector space over R with compositions defined above.

Ex. The set $\mathbb{C}$ of complex nos. is a vector space over over $\mathbb{R}$ of real nos.

Soln. We know that $(\mathbb{C}, +)$ is an abelian group.

Let $a, b \in \mathbb{R}$ and $x, y, z \in \mathbb{C}$. Then $ax \in \mathbb{C}$ and

1. $a(x+y) = ax + ay$
2. $(a+b)x = ax + by$
3. $a(bx) = (ab)x$
4. $1 \cdot x = x$

$x = a+ib,\ y = c+id$
$z = e+if$.

Hence $\mathbb{C}$ is a vector space over $\mathbb{R}$.

Ex. The set $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ of all polynomials of degree 2 over $\mathbb{R}$ is a vector space over $\mathbb{R}$ w.r.t. the compositions:

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \quad \text{—(1)}$$

$$a(a_0 + a_1x + a_2x^2) = aa_0 + aa_1x + aa_2x^2 \quad \text{—(2)}$$

Soln. It is easy to verify that $(V, +)$ is an abelian group whose additive identity is $0 = 0 + 0x + 0x^2$ and the additive inverse of $a_0 + a_1x + a_2x^2 \in V$ is $(-a_0) + (-a_1)x + (-a_2)x^2 \in V$

Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in V$, By (2), $\alpha f \in V$ and

1. $\alpha(f+g) = \alpha f + \alpha g$
2. $(\alpha+\beta)f = \alpha f + \beta f$
3. $\alpha(\beta f) = (\alpha\beta)f$.
4. $1f = f$.

To verify the 1st property, let

$$f = a_0 + a_1x + a_2x^2,\ g = b_0 + b_1x + b_2x^2 \in V$$

Then $\alpha(f+g) = \alpha[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2]$, by (1)

$$= \alpha(a_0+b_0) + (\alpha a_1 + \alpha b_1)x + (\alpha a_2 + \alpha b_2)x^2$$

$$= (\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2) + (\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2)$$

$$= \alpha f + \alpha g.$$

Similarly, we can verify other three properties

Hence V is a vector space over $\mathbb{R}$.

Ex. The set $M_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ w.r.t. matrix addition and multiplication of a matrix by a scalar.

Soln. We know M_2 is an abelian group w.r.t. matrix addition.

Further $\alpha, \beta \in R$ and $A, B \in M_2$. Then $\alpha A \in M_2$ and

1. $\alpha(A+B) = \alpha A + \alpha B$
2. $(\alpha+\beta)A = \alpha A + \beta A$
3. $\alpha(\beta A) = (\alpha\beta)A$
4. $1A = A$.

Hence M_2 is a vector space over R.

Ex. Prove that the set $M_{m\times n}$ of all $m\times n$ matrices (where m and n are fixed positive integers) over the set R of real numbers is a vector space over R w.r.t. matrix addition and multiplication of matrices by a scalar (real no.)

Soln. Here we find that

1. $(M_{m\times n}, +)$ is a commutative group (abelian group w.r.t. matrix addition.
2. If $A \in M_{m\times n}$ and $a \in R$, then the matrix $aA \in M_{m\times n}$
3. Further we find that if $a, b \in R$ and $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n} \in M_{m\times n}$, then
4. $a(A+B) = a([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n})$

$$= a([a_{ij} + b_{ij}]_{m\times n})$$
$$= [a a_{ij} + a b_{ij}]_{m\times n}$$
$$= a[a_{ij}]_{m\times n} + a[b_{ij}]_{m\times n} = aA + aB$$

5. $(a+b)A = (a+b)[a_{ij}]_{m\times n} = [a a_{ij} + b a_{ij}]_{m\times n}$

$$= [a a_{ij}]_{m\times n} + [b a_{ij}]_{m\times n}$$
$$= a[a_{ij}]_{m\times n} + b[a_{ij}]_{m\times n} = aA + bA$$

6. $a(bA) = a(b[a_{ij}]_{m\times n} = a[b a_{ij}]_{m\times n} = [a b a_{ij}]_{m\times n} = (ab)[a_{ij}]_{m\times n} = (ab)A$
7. $1A = 1[a_{ij}]_{m\times n} = [1 a_{ij}]_{m\times n} = [a_{ij}]_{m\times n} = A$, where 1 is the multiplicative identity of $\mathbb{R}$

Hence all the properties of vector space are satisfied and $M_{m\times n}$ is a vector space over $\mathbb{R}$.

Ex. If $V = \{(a, b) : a, b \in R\}$. Show that R^2 is a vectorspace over $\mathbb{R}$ defined as $(a, b) + (c, d) = (a+c, b+d)$ — (I)

$k(a, b) = (ka, kb)$ — (II)

$\forall\ a, b, c, d, k \in \mathbb{R}$

Soln. Here we find that

1. $(V, +)$ is an abelian group, since

(i) Sum (as defined by (1)) of two ordered pairs in V is an ordered pair in V.

ie, V is closed w.r.t. +.

(ii) The operation + is associative in V, since the addition of real nos. obeys associative law.

(iii) There exists the additive identity $(0, 0)$ in V, such that

$$(0, 0) + (a, b) = (a, b) = (a, b) + (0, 0) \ \forall\ (a, b) \in V$$

(iv) For each $(a, b) \in V$ there exists its additive inverse $(-a, -b) \in V$, such that

$$(a, b) + (-a, -b) = (0, 0) = (-a, -b) + (a, b)$$

(v) The operation + is commutative in V as the addition of real nos. obeys commutative law.

2. $k[(a, b) + (c, d)] = k[a+c, b+d]$, from (I)

$= [k(a+c), k(b+d)]$, from (II)

$= (ka+kc, kb+kd)$ (by distributivity in $\mathbb{R}$)

$= (ka, kb) + (kc, kd)$ from I

$= k(a, b) + k(c, d)$ from II

3. $(k+m)(a, b) = [(k+m)a, (k+m)b]$, from II

$= (ka+ma, kb+mb)$, (by distributivity in $\mathbb{R}$)

$= (ka, kb) + (ma, mb)$, from I

$= k(a, b) + m(a, b)$, from II

4. $k[m(a, b)] = k(ma, mb)$, from II

$= (kma, kmb)$, from II

$= km(a, b)$, from II

5. $1(a, b) = (1a, 1b)$, from II

$= (a, b)$

Hence all the properties of vector space are satisfied and so V is a vector space.

Ex. If $V = \{(a, b) : a, b \in \mathbb{R}\}$, Show that R^2 is not a vector space under scalar addition and scalar multiplication defined as below :

(i) $(a, b) + (c, d) = (a+c, b+d)$

$k(a, b) = (0, kb)$

(ii) $(a, b) + (c, d) = (a+c, b+d)$

$k(a, b) = (ka, b)$

Soln. → 1. V is not a vector space under composition defined in (i) because

$1(a, b) = (0, 1b) = (0, b) \neq (a, b)$ ie, property for vectorspace is not satisfied.

2. V is not a vector space under the composition defined in (ii) because

$(k_1 + k_2)(a, b) = [(k_1 + k_2)a, b]$, $\because k(a, b) = (ka, b)$ (from (ii))

$= (k_1 a + k_2 a, 0 + b)$

$= (k_1 a, 0) + (k_2 a, b)$, $\because (a+c, b+d) = (a, b) + (c, d)$

$= k_1(a, 0) + k_2(a, b)$, $\because (ka, b) = k(a, b)$ (from (ii))

$\neq k_1(a, b) + k_2(a, b)$

Ex. If $V = \{(a, b) : a, b \in \mathbb{R}\}$ in which addition and scalar multiplication are defined as below, is not a vector space over $\mathbb{R}$:-

(i) $(a, b) + (c, d) = (a+d, b+c)$

$k(a, b) = (ka, kb)$

(ii) $(a, b) + (c, d) = (ac, bd)$

$k(a, b) = (ka, kb)$

Soln. → (i) Here we find that

$(k_1 + k_2)(a, b) = [(k_1 + k_2)a, (k_1 + k_2)b]$, $\because k(a, b) = (ka, kb)$

$= (k_1 a + k_2 a, k_1 b + k_2 b)$

$= (k_1 a, k_1 b) + (k_2 b, k_2 a)$, $\because (a, b) + (c, d) = (a+d, b+c)$

$= k_1(a, b) + k_2(b, a)$, $\because k(a, b) = (ka, kb)$

$\neq k_1(a, b) + k_2(a, b)$

Hence the property is not satisfied. Hence V is not a vector space under composition defined under (i).

(ii) Here we find that

$$k[(a,b)+(c,d)] = k(ac, bd), \quad \because (a,b)+(c,d) = (ac, bd)$$

$$= (kac, kbd), \quad \because k(a,b) = (ka, kb) \quad \text{—(1)}$$

Also $k(a,b) + k(c,d) = (ka, kb) + (kc, kd)$

$$\because k(a,b) = (ka, kb)$$

$$= (ka, kb) + (kc, kd) \quad \text{—(2)}$$

$\therefore$ From (1) & (2), we find that

$$k[(a,b)+(c,d)] \neq k(a,b) + k(c,d)$$

i.e, the property for vector space is not satisfied, hence V is not a vector space under composition defined in (2).

Eg 3 Let S be a ~~field~~ non empty set, F → Field and Let F(S, F) be set of all functions from S to f, then,

$$F(S, F) = \{ f : S \to F \}$$

vector space +, scalar multiplication

$f, g \in \mathcal{F}(S, f)$, $c \in f$

$$(f+g)(S) = f(S) + g(S)$$

$$cf(S) = c(f(S))$$

All properties clearly holds.

Eg 4: Consider the set of all polynomials with coeff from the field f, then this set P(f) forms the <u>vector space</u>, ~~addition~~ under operⁿ of addition & scalar multiplication

Eg 5 Let f be any field, a sequence in f, is a function $\sigma : \mathbb{Z}^+ \to F$ such that $\sigma(n) = a_n$ $n = 1, 2 \ldots$, where seq is denoted by $\{a_n\}$.
Let V consist of all these sequences $\{a_n\}$ inf

$$V = \{ \{a_n\} : a_n \in F \}$$

such that the sequence has only a finite no. of non-zero terms.

$a_n \neq 0$ for finite terms

Hence $\underline{a_n \to 0}$.

We define operation of addition of & scalar multiplication on V.

If $\{a_n\}, \{b_n\} \in V$, $t \in F$ then

$$\{a_n\} + \{b_n\} = \{a_n + b_n\}$$

$$t\{a_0\} = \{ta_n\}$$

[Clearly holds true].

Example 1.2.6. *Show that the set* $\mathbf{Q}(\sqrt{2}) = \{ a + b\sqrt{2} : a, b \in \mathbf{Q} \}$ *is a vector space over* $\mathbf{Q}$ *w.r.t. the compositions :*

$$(a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \qquad ...(i)$$

$$\alpha(a + b\sqrt{2}) = \alpha a + \alpha b\sqrt{2}, \alpha \in \mathbf{Q}. \qquad ...(ii)$$

Solution. We know $\{ \mathbf{Q}(\sqrt{2}), + \}$ is abelian group.

[See Illustration 7, p.2]

Properties of Scalar Multiplication.

Let $\alpha, \beta \in \mathbf{Q}$ and $u = a + b\sqrt{2}$, $v = c + d\sqrt{2} \in \mathbf{Q}(\sqrt{2})$.

From (ii), $\alpha u = \alpha a + \alpha b\sqrt{2} \in \mathbf{Q}(\sqrt{2})$.

1. $\alpha(u + v) = \alpha\{(a + c) + (b + d)\sqrt{2}\}$, by (i)

$= \{\alpha(a + c) + \alpha(b + d)\sqrt{2}\}$, by (ii)

$= \alpha a + \alpha c + (\alpha b + \alpha d)\sqrt{2}$, by distributive law in $\mathbf{Q}$

$= (\alpha a + \alpha b\sqrt{2}) + (\alpha c + \alpha d\sqrt{2})$.

$\therefore$ $\alpha(u + v) = \alpha u + \alpha v$.

2. $(\alpha + \beta)u = ((\alpha + \beta)a + (\alpha + \beta)b\sqrt{2})$, by (ii)

$= \alpha a + \beta a + (\alpha b + \beta b)\sqrt{2}$, by distributive law in $\mathbf{Q}$

$= (\alpha a + \alpha b\sqrt{2}) + (\beta a + \beta b\sqrt{2})$.

$\therefore$ $(\alpha + \beta)u = \alpha u + \beta u$, by (ii).

3. $\alpha(\beta u) = \alpha(\beta a) + \alpha(\beta b\sqrt{2})$, by (ii)

$= (\alpha\beta)a + (\alpha\beta)b\sqrt{2}$, by associative law in $\mathbf{Q}$

$\therefore$ $\alpha(\beta u) = (\alpha\beta)u$, by (ii).

4. $1u = 1(a + b\sqrt{2}) = a + b\sqrt{2} = u \ \forall \ u \in \mathbf{Q}(\sqrt{2})$.

Hence $\mathbf{Q}(\sqrt{2})$ is a vector space over $\mathbf{Q}$.

Example 1.2.7. *Let V be the set of all real valued continuous functions defined on the closed interval* $[0, 1]$ *and* $\mathbf{R}$ *be the field of real numbers. For any* $f, g \in V$ *and* $\alpha \in \mathbf{R}$, *define* $f + g$ *and* αf *by*

(i) $(f + g)(x) = f(x) + g(x)$, $\quad (ii)$ $(\alpha f)(x) = \alpha f(x)$, $x \in [0, 1]$.

Show that V is a vector space over $\mathbf{R}$.

Solution. Firstly, we show that $(V, +)$ is an abelian group.

Let $f, g, h \in V$.

1. Since $f : [0, 1] \to \mathbf{R}$ and $g : [0, 1] \to \mathbf{R}$ are continuous,

$\therefore \quad f + g : [0, 1] \to \mathbf{R}$ is also continuous and so $f + g \in V$.

2. $f + g = g + f$, since for any $x \in [0, 1]$,

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x).$$

(Note that + is commutative in $\mathbf{R}$)

3. $(f + g) + h = f + (g + h)$, since for any $x \in [0, 1]$

$$\begin{aligned}[(f + g) + h](x) &= (f + g)(x) + h(x) \\ &= \{ f(x) + g(x) \} + h(x) \\ &= f(x) + \{ g(x) + h(x) \}, \text{ by associative law in } (\mathbf{R}, +) \\ &= f(x) + (g + h)(x) \\ &= \{ f + (g + h) \}(x).\end{aligned}$$

4. The zero function $0 : [0, 1] \to \mathbf{R}$ defined as

$0(x) = 0 \ \forall \ x \in [0, 1]$ satisfies $f + 0 = 0 + f = f$.

5. For $f \in V$, we define $-f : [0, 1] \to \mathbf{R}$ as $(-f)(x) = -f(x) \ \forall \ x \in [0, 1]$. Then $f + (-f) = (-f) + f = 0$, since

$$\{ f + (-f) \}(x) = f(x) + (-f)(x) = f(x) - f(x) = 0 = 0(x).$$

II. *Properties of Scalar Multiplication.*

Let $\alpha, \beta \in \mathbf{R}$ and $f, g \in V$.

Since f is a continuous function on $[0, 1]$ and α is a real constant, therefore, αf is a continuous function on $[0, 1]$ and as such $\alpha f \in V$. Let $x \in [0, 1]$. Then $f(x), g(x) \in \mathbf{R}$.

1.
$$\begin{aligned}\{ \alpha(f + g) \}(x) &= \alpha \{ (f + g)(x) \}, \text{ by } (ii) \\ &= \alpha \{ f(x) + g(x) \}, \text{ by } (i) \\ &= \alpha f(x) + \alpha g(x), \text{ by distributive law in } \mathbf{R}. \\ &= (\alpha f)(x) + (\alpha g)(x), \text{ by } (ii) \\ &= (\alpha f + \alpha g)(x), \text{ by } (i).\end{aligned}$$

$\therefore \qquad \alpha(f + g) = \alpha f + \alpha g.$

2.
$$\begin{aligned}\{ (\alpha + \beta) f \}(x) &= (\alpha + \beta) f(x), \text{ by } (ii) \\ &= \alpha f(x) + \beta f(x) = (\alpha f)(x) + (\beta f)(x), \text{ by } (ii) \\ &= (\alpha f + \beta f)(x), \text{ by } (i).\end{aligned}$$

$\therefore \qquad (\alpha + \beta) f = \alpha f + \beta f.$

3.
$$\begin{aligned}\{ \alpha(\beta f) \}(x) &= \alpha \{ (\beta f)(x), \text{ by } (ii) \\ &= \alpha \{ \beta f(x) \}, \text{ by } (ii) \\ &= (\alpha\beta) f(x), \text{ by associative law in } \mathbf{R} \\ &= \{ (\alpha\beta) f \}(x), \text{ by } (ii)\end{aligned}$$

$\therefore \qquad \alpha(\beta f) = (\alpha\beta) f.$

4.
$$\begin{aligned}(1f)(x) &= 1f(x), \text{ by } (ii) \\ &= f(x) \text{ and so } 1f = f \quad \forall \ f \in V.\end{aligned}$$

Hence V is a vector space over $\mathbf{R}$.

1.3

SUBSPACES

Let V be vectorspace over field F, Let W be subset of V, Then W is subspace of V if V is vectorspace over F, with operation of addition and scalar multipl^n defined on V.

$\{0\}$ and V are the subspaces for vector space V.

$x+y \in W \quad \forall\ x, y \in W$

$cx \in W \quad c \in F,\ x \in W$

W has zero vector

Each vector in W has an additive inverse in W.

Theorem Let V be vector space, W be subset of V then W is subspace of V iff the foll. three condition hold true.

(a) $0 \in W$

(b) $x+y \in W \quad \forall\ x, y \in W$

(c) $cx \in W \quad c \in F,\ x \in W.$

for operation defined in V

Proof Suppose W is subspace of V, Then (b) and (c) must hold, because W is vector space

TP (a) holds.

Since ~~Some~~ W is a vectorspace over F.

$(\therefore)$ By VS3 $\exists\ 0' \in W$ such that

$x + 0' = x \longrightarrow (1) \quad \forall\ x \in W$

(coz $x \in V$)

Also $x + 0 = x$

$x + 0' = x + 0$

$0' = 0$

Hence $0' \in W$.

Hence, statement is true.

Converse

TP W is a subspace of V, ie. W is a subspace itself

By (b) and (c) addition and scalar multiplication is well defined in W

As W is a subset of V; ∴ (VS 1), (VS2), (VS 5-VS8) all hold true.

Also by (a) $0 \in W$, ie additive identity exists.
Now To prove (VS4).
Let $a \in W$, $-1 \in F$
$(-1)a \in W$ [scalar mult.]
$-a \in W$.
Also $a - a = 0$.
Hence inverses exists ∀ elements in W.
Hence W is a subspace of V.

Examples Of subspaces:

Consider $n \times n$ matrices in $M_{n\times n}(F)$. Let W be the set of all symm. matrices in $M_{n\times n}(F)$ then W is a subspace of $M_{n\times n}(F)$.

Consider the $n\times n$ zero matrix.

Clearly $0^t = 0$
$\Rightarrow 0 \in W$.
Now let $A, B \in W \Rightarrow A^t = A$, $B^t = B$
As A, B are $n\times n$ square matrices; $A+B$ is $n\times n$ square matrice

$\therefore$ $A+B$ is also a $(n \times n)$ square matrice.

Consider $(A+B)^T = A^T + B^T = A+B \Rightarrow A+B \in W$

$\Rightarrow$ ~~A~~ W is closed under addition

Also Consider

$(CA)^T = CA^T = CA \in W$.

Hence W is closed under scalar multiplict^n

Hence, By Thrm 1.3, W is subspace of $M_{n \times n}(F)$.

(2) $P(F) =$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

If $a_i = 0 \ \forall \ i = 0, 1, 2 \dots n$.

Then $f(x)$ is a zero polynomial

By convent^n $\deg f(x) = -1$

Consider the vectorspace $P(F)$ of all polyno- over the field 'F' and let n be non-neg. integer and $P_n(F)$ be set of all poly. in ~~P(F)~~ $P(F)$ having degree less than or equal to n.

We will prove that $P_n(F)$ is subspace of $P(F)$

Consider zero poly., Denote it by 0. Then by conven^n $\deg 0 = -1 < n$, $n \in \mathbb{Z}^+$.

Hence $0 \in P_n(F)$

Let $f(x), g(x) \in P_n(F)$

$\deg(f(x))$ and $\deg(g(x)) < n$

$f(x) + g(x) = h(x) \Rightarrow \deg(h(x)) < n$

Hence $h(x) \in P_n(F)$.

Coz,
Also sum of two poly with degree $\leq n$ is an another polynomial of degree $\leq n$
Hence, operation of addition is closed.

Also, product of scalar & poly. of degree $\leq n$ is a ~~ga~~ polynomial. of degree $\leq n$.
Hence, $P_n(F)$ is closed under scalar multiplicⁿ.

Hence, $P_n(F)$ is subspace of $P(F)$.

Eg(3)
Let $F(\mathbb{R},\mathbb{R})$ be set of all real valued functions
Let $C(\mathbb{R})$ denote the set of all continuous real valued function defined on $\mathbb{R}$.
Then $C(\mathbb{R})$ is subset of $F(\mathbb{R},\mathbb{R})$ & in particular it forms a subspace in $F(\mathbb{R},\mathbb{R})$

Clearly, the zero of $F(\mathbb{R},\mathbb{R})$ is $f(t)=0 \ \forall t$ as a constant funⁿ is always continuous funⁿ.

$$\therefore \quad f \in C(\mathbb{R})$$

Also, sum of 2 continuous function is continuous & product of scalar & continuous function is conti

$C(\mathbb{R})$ is closed under addition & scalar multi.
Hence,

$$C(\mathbb{R}) \text{ is subspace of } F(\mathbb{R},\mathbb{R})$$

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Example 4 : Consider vector space $M_{n\times n}(F)$ of $n\times n$ square matrices over the field F.

Let X be the set of diagonal matrices in $n\times n$.

$$X = \{ A = [a_{ij}] \in M_{n\times n}(F);\ a_{ij} = 0 \quad i \neq j \}$$

Then X is subspace of $M_{n\times n}(F)$.

(a) Clearly the zero matrix is a diagonal matrix. Hence $0 \in X$.

(b), (c) Also sum of 2 diagonal matrices is a diagonal matrice and product of a scalar and diagonal matrice is a diagonal matrice.
Hence, X is subspace of $M_{n\times n}(F)$.

Example -5 For a $n\times n$ matrix A, trace of A denoted by trA is the sum of its principal diagonal ~~axis~~ entries.

ie $A = [a_{ij}]$
$trA = a_{11} + a_{22} + \dots a_{nn}$

For vector space $M_{n\times n}(F)$. Consider the set of $n\times n$ matrices having trace $= 0$. then set of such matrices form subspace of $M_{n\times n}(F)$.

Let $S = \{ A = [a_{ij}] \in M_{n\times n}(F) : \sum_{i=1}^{n} a_{ii} = 0 \}$

Clearly,
0 matrix $\in S$
Let $A, B \in S$
$\exists$ $A = [a_{ij}]$; $\sum a_{ii} = 0$
& $B = [b_{ij}]$; $\sum b_{ii} = 0$

$tr(A+B) = tr(A) + tr(B) = 0$
$tr\, c(A) = c\, tr(A) = c \cdot 0 = 0$

Hence, S forms subspace of $M_{n\times n}(F)$.

Eg 6 Consider $M_{m\times n}(R)$, having non negative entries is not subspace of $M_{m\times n}(R)$, because it is not closed under scalar multiplicat^n

for $c = -1$ & $X \in M_{m\times n}(R)$
$cX \notin M_{m\times n}(R)$
As cX has negative entries.

Hence $M_{m\times n}(\mathbb{R})$ with non negative entries is not a subspace of $M_{m\times n}(R)$.

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Theorem 1.4
Any intersection of subspaces of vectorspace V is a subspace of V.

Pf Let C be the collection of subspaces of V.
Let W denote the intersection of subspaces in C.

$C = \{ V_i, \ i \in I \ \ V_i \text{ is a subspace of } V \ \forall i \in I \}$

& $W = \bigcap_{i \in I} V_i$

Since $0 \in V_i \quad \forall i \in I$ [V_i is subspace of V]
$\Rightarrow 0 \in \bigcap V_i$
$\Rightarrow 0 \in W$.

Let $x, y \in W$ and $c \in F$
ie $x, y \in \bigcap V_i$
$\Rightarrow x, y \in V_i \quad \forall i \in I$
Also $x+y \in V_i \quad \forall i$ [$\because V_i$ is subspace]
$\Rightarrow x+y \in \bigcap V_i$
$\Rightarrow x+y \in W$, Hence W is closed under (+).

As $x \in W \Rightarrow x \in \bigcap V_i \Rightarrow x \in V_i \ \forall i$
Now for $c \in F$
$cx \in V_i \quad \forall i$ [V_i is subspace]
$\Rightarrow cx \in \bigcap V_i$
$cx \in W$. Hence, W is closed under scalar multiplⁿ

Hence W is subspace of V.

REMARK:
1. Union of 2 subspaces may/may not be subspace of V.
2. Union of 2 subspaces of V is a subspace of V iff one of the subspace contains the other

Ex: Let $W = \{(a_1, a_2, 0) : a_1, a_2, \in \mathbb{R}\}$

Let $u = (a_1, a_2, 0)$; $v = (b_1, b_2, 0) \in W$ and $\alpha \in \mathbb{R}$

$u + v = (a_1 + b_1, a_2 + b_2, 0) \in W$

$\alpha u = (\alpha a_1, \alpha a_2, 0) \in W$

$\therefore W$ is a subspace of R^3.

Ex. Let $W = \{(a_1, a_2, 1) : a_1, a_2 \in R\}$

Let $u = (a_1, a_2, 1)$, $v = (b_1, b_2, 1) \in W$, $\alpha \in R$

$u + v = (a_1 + b_1, a_2 + b_2, 1 + 1)$

$= (a_1 + b_1, a_2 + b_2, 2) \notin W$

$\alpha u = (\alpha a_1, \alpha a_2, \alpha) \notin W$

$\therefore W$ is not a subspace of R^3.

Ex. Let $W = \{\alpha(1,1,1) : \alpha \in \mathbb{R}\} = \{(\alpha, \alpha, \alpha) : \alpha \in \mathbb{R}\}$

Let $u = (a, a, a)$, $v = (b, b, b) \in W$, $\alpha \in \mathbb{R}$

$u + v = (a + b, a + b, a + b) \in W$ as $a + b \in \mathbb{R}$

$\alpha u = (\alpha a, \alpha a, \alpha a) \in W$ as $\alpha a \in \mathbb{R}$

$\therefore W$ is a subspace of R^3.

Ex. Let $W = \{(a_1, a_2) : a_1 + a_2 = 0, a_1, a_2 \in R\}$.

Let $u = (a_1, a_2)$, $v = (b_1, b_2) \in W$ and $\alpha \in \mathbb{R}$

$u + v = (a_1 + b_1, a_2 + b_2)$

Now, $(a_1 + b_1) + (a_2 + b_2) = a_1 + a_2 + b_1 + b_2$

$= 0 + 0$ as $a_1 + a_2 = 0$, $b_1 + b_2 = 0$

$= 0$

$\therefore u + v \in W$

$\alpha u = (\alpha a_1, \alpha a_2)$

Now, $\alpha a_1 + \alpha a_2 = \alpha(a_1 + a_2) = \alpha \cdot 0 = 0$

$\therefore \alpha u \in W$

Hence W is a subspace of R^2.

Linear Combination

Def^n. A vector V in R^n is said to be linear combination of vector $u_1, u_2, \dots, u_k$ if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ in R such that

$$V = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k .$$

Ex. Every vector $(a_1, a_2) \in R^2$ is a linear combination of $(1,0)$ and $(0,1)$ as $(a_1, a_2) = a_1(1,0) + a_2(0,1)$

Ex. write $(2,2)$ as linear combination of $(1,2)$ and $(1,4)$

Soln. Let $(2,2) = \alpha_1(1,2) + \alpha_2(1,4)$

$$\Rightarrow (2,2) = (\alpha_1, 2\alpha_1) + (\alpha_2, 4\alpha_2)$$

$$\Rightarrow (2,2) = (\alpha_1 + \alpha_2, 2\alpha_1 + 4\alpha_2)$$

$$\Rightarrow \begin{array}{r} \cancel{\alpha_1} + \alpha_2 = 2 \\ -\ \cancel{\alpha_1} \pm 2\alpha_2 = 1 \\ \hline -\alpha_2 = 1 \end{array}$$

$$\Rightarrow \alpha_2 = -1$$

$$\therefore \alpha_1 = 2 + 1 = 3$$

$$\therefore (2,2) = 3(1,2) - 1(1,4)$$

Linear Span:

Def^n → Let $u_1, u_2, \dots u_k$ be k vectors in R^n. The collection of all linear combinations of $u_1, u_2, \dots, u_k$ is called linear span of $u_1, u_2, \dots, u_k$.
Linear span of a set S is denoted by $L(S)$ or $[S]$.

$$L(u_1, u_2, \dots, u_k) = [u_1, u_2, \dots, u_k]$$
$$= \{\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k ; \alpha_i \in R\}.$$

Span of vector →

Def^n The vectors $u_1, u_2, \dots, u_k$ in R^n are said to span R^n if every vector in R^n is a linear combination of $u_1, u_2, \dots u_k$ i.e,

$$L(u_1, u_2, \dots, u_k) = R^n.$$

Def^n. Let S be a non-empty subset of a vector space V. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S.
we define span$(\phi) = \{0\}$.

Ex. The set $\{(1,0,0), (0,1,0)\}$ consists of all vectors in R^3 that have the form $a(1,0,0) + b(0,1,0) = (a, b, 0)$ for some scalars a and b. Thus the span of $\{(1,0,0), (0,1,0)\}$ contains all the points in the xy-plane. In this case the span of the set is a subspace of R^3.

Ex: Which of the foll. vectors span R^2?

(a) $(1,2), (-1,1)$ (b) $(0,0), (1,1), (-2,-2)$

Soln (a) Let $(a,b) \in R^2$

Let if possible

$$(a,b) = \alpha_1(1,2) + \alpha_2(-1,1) \qquad \alpha_1, \alpha_2 \in R.$$

$$= (\alpha_1, 2\alpha_1) + (-\alpha_2, \alpha_2)$$

$$= (\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

$$\alpha_1 - \alpha_2 = a$$

$$2\alpha_1 + \alpha_2 = b.$$

In matrix form.

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Apply $R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b-2a \end{pmatrix}$$

$$\alpha_1 - \alpha_2 = a$$

$$3\alpha_2 = b - 2a$$

$$\Rightarrow \alpha_2 = \frac{b-2a}{3}$$

$$\alpha_1 = a + \alpha_2 = a + \frac{b-2a}{3} = \frac{a+b}{3}$$

Hence, $(a,b) = \frac{a+b}{3}(1,2) + \frac{b-2a}{3}(-1,1)$

Every element of R^2 can be written as linear combination of $(1,2)$ and $(-1,1)$, therefore these two vectors span R^2.

(b). Let $(a,b) \in R^2$

Let if possible

$$(a,b) = \alpha_1(0,0) + \alpha_2(1,1) + \alpha_3(-2,-2), \quad \alpha_1, \alpha_2, \alpha_3 \in R$$

$$\Rightarrow (a,b) = (\alpha_2 - 2\alpha_3, \alpha_2 - 2\alpha_3)$$

$$\Rightarrow \cancel{\alpha_2} - \cancel{2\alpha_3} = a$$

$$\cancel{\alpha_2} - \cancel{2\alpha_3} = b.$$

$$0 = a - b. \Rightarrow a = b.$$

$$\Rightarrow (a,b) = (a,a)$$

Every element of R^2 cannot be expressed as linear combination of $(0,0)$, $(1,1)$ and $(-2,-2)$ therefore they do not span R^2.

Bases and Dimension

Def^n. A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, as also say that the vectors of β form a basis of V.

Def^n. Let $u_1, u_2, \ldots, u_k$ be k vectors in R^n.

Let $V = L(u_1, u_2, \ldots, u_k)$. If $u_1, u_2, \ldots, u_k$ are linearly independent then $(u_1, u_2, \ldots, u_k)$ is called a basis of V. More generally, if $u_1, u_2, \ldots, u_k$ are k vectors in vector space V over R then $[u_1, u_2, \ldots, u_k]$ is called a basis of V if (i) $u_1, u_2, \ldots, u_k$ span V.

(ii) $u_1, u_2, \ldots, u_k$ are linearly independent.

If $u_1, u_2, \ldots, u_k$ form a basis for a vector space V, then they must be distinct and non-zero otherwise $(u_1, u_2, \ldots, u_k)$ will be linearly dependent set.

Def^n. A basis β of vector space V is a linearly independent subset of V that generates V i.e, (i) β is linearly independent

(ii) $\text{Span}(\beta) = V$

Ex 1. $\text{Span}\ \phi = \{0\}$.

$\Rightarrow \phi$ is a basis of zero vector space.

Ex 2. The vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ form a basis of R^2.

As $\alpha_1 e_1 + \alpha_2 e_2 = 0$, $\alpha_1, \alpha_2 \in R$

$\Rightarrow \alpha_1(1,0) + \alpha_2(0,1) = (0,0)$

or $(\alpha_1, 0) + (0, \alpha_2) = (0,0)$

or $(\alpha_1, \alpha_2) = (0,0)$

$\alpha_1 = 0 = \alpha_2$

$\therefore \{e_1, e_2\}$ is linearly independent set.

$\{e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), e_n = (0, 0, 0, \ldots 1)\}$.

is a basis of R^n.

These basis are called standard or natural basis for $R^n = (F^n)$

Ex 3. In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only non-zero entry is 1 in the ith row and j^{th} column. Then $\{E^{ij} : 1\le i\le m, 1\le j\le n\}$ is a basis for $M_{m\times n}(F) \Rightarrow |\beta| = mn$.

Ex 4 Standard basis for $P_n(F)$ is

$$\beta = \{1, x, x^2, \dots, x^{n-1}, x^n\}$$

Ex 5. Standard basis for $P(F)$

$$\beta = \{1, x, x^2, \dots\}$$

Ex: Let $S = \{(2,-3,5), (8,-12,20), (1,0,-2), (0,2,-1), (7,2,0)\}$

claim : Span $S = \mathbb{R}^3$

Let $(a,b,c) \in R^3$

then

$$(a,b,c) = \alpha(2,-3,5) + \beta(8,-12,20) + \gamma(1,0,-2) + \mu(0,2,-1) + \eta(7,2,0)$$

Now,

$$2\alpha + 8\beta + \gamma + 7\eta = a$$
$$-3\alpha - 12\beta + 2\mu + 2\eta = b$$
$$5\alpha + 20\beta - 2\gamma - \mu = c$$

$\alpha =$ $\gamma =$ $\eta =$

$\beta =$ $\mu =$

Hence span$(S) = R^3$

Now to see if S is basis of R^3, we need to check whether S is linearly idependent or not.

Consider,

$$\alpha(2,-3,5) + \beta(8,-12,20) + \gamma(1,0,-2) + \mu(0,2,-1) + \eta(7,2,0) = (0,0,0)$$

Solving we can see that $\alpha = \beta = \gamma = \eta = \mu = 0$ is not the only soln to the system. Hence S is not L.I set.

But since span $S = \mathbb{R}^3$, By theorem, (if a vector V is generated by a finite set S then some subset of S is a basis for V. Hence V has a finite basis). $\exists$ a subset S' of S such that S' is basis of $\mathbb{R}$. Now, we will proceed as described in theorem, we select a non-zero vector, now we will include one vector one by one till it remains L.I.

$S' = \{(2,-3,5), (8,-12,20)\}$ is not a L.I set as

$(8,-12,20) = 4(2,-3,5)$, ~~As (2,-3,5) is a multiple~~ $(8,-12,20)$ is a multiple of $(2,-3,5)$

Let $(1,0,-2) \in S'$

$\{(2,-3,5), (1,0,-2)$ clearly the set is L.I.

Let $(0,2,-1) \in S'$

$\{(2,-3,5), (1,0,-2), (0,2,-1)$ check for L.I.

$\alpha(2,-3,5) + \beta(1,0,-2) + \gamma(0,2,-1) = (0,0,0)$

$2\alpha + \beta = 0$ $\quad -3\alpha + 2\gamma = 0$ $\quad 5\alpha - 2\beta - \gamma = 0$

$2\alpha = -\beta$ $\quad -3\alpha = -2\gamma$ $\quad \gamma = 5\alpha - 2\beta$

$-3\alpha = -2(-9\alpha)$ $\quad \gamma = 5\alpha + 4\alpha$

$3\alpha = 18\alpha$ $\quad \gamma = 9\alpha$

$3 = 18$

Only soln is $\alpha = \beta = \gamma = 0$

Hence $\{(2,-3,5), (1,0,-2), (0,2,-1)$ are L.I.

Let $(7,2,0) \in S'$

$\{(2,-3,5), (1,0,-2), (0,2,-1), (7,2,0)$

$\alpha(2,-3,5) + \beta(1,0,-2) + \gamma(0,2,-1) + \eta(7,2,0) = (0,0,0)$

$2\alpha + \beta + 7\eta = 0$

$-3\alpha + 2\gamma + 2\eta = 0$

$5\alpha - 2\beta - \gamma = 0$

Since this set has more than trivial soln, S' is L.D.

Hence basis of $\mathbb{R}^3$ is $\{(2,-3,5), (1,0,-2), (0,2,-1)\}$.

Replacement Theorem.

Let V be a vector space generated by set G containing exactly n vectors. Let L be L.I subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n-m$ vectors such that $L \cup H$ generates V.

Cor. Let V be vector space having finite basis then every basis for V contains the same no. of vectors.

Ex. $S = \{(1,3), (1,-1)\}$ span R^2 or not ?

Soln. Let $\alpha(1,3) + \beta(1,-1) = (0,0)$

$\Rightarrow (\alpha, 3\alpha) + (\beta, -\beta) = (0,0)$

$\Rightarrow (\alpha+\beta, 3\alpha-\beta) = (0,0)$

or $\alpha+\beta = 0 \Rightarrow \alpha = -\beta$

$3\alpha - \beta = 0 \Rightarrow 3\alpha + \alpha = 0 \Rightarrow 4\alpha = 0 \Rightarrow \alpha = 0$

$\Rightarrow \beta = 0$

$\therefore \{(1,3), (1,-1)\}$ is L.I set in R^2

Hence $\{(1,3), (1,-1)\}$ is a basis of R^2.

Ex. $S = \{(0,0), (1,2), (2,4)\}$ span R^2 or not ?

Soln. Since S contains three vectors, therefore by theorem (Any set of vectors in R^n having more more than n vectors is linearly dependent and hence cannot be basis for R^n)

S cannot be basis of R^2.

Finite Dimensional Vector Space (FDVS)

A vector space is called finite dimens. if it has a basis consisting of finite no of vectors

The unique no of vectors in each basis for V is called dimension of V and is denoted by dim V.

A vector space that is not finite dimensional is called infinte dimensional vector space.

Example 7 $V = \{0\} \longrightarrow$ Finite Dim Vector space

$\dim V = 0$ as $\beta = \phi$

Eg 8 $V = F^n$

$\beta = \{(1,0,0 \ldots 0), (0,1,0 \ldots 0) \ldots (0,0 \ldots 1)\}$

$|\beta| = n$

$\dim V = n$ F^n is FDVS.

Eg 9 $V = M_{m \times n}(F)$

$\dim V = 1 \, m \times n = mn$

$$\beta = \left\{ \begin{matrix} A_{11}, & A_{12}, & A_{13} \ldots & A_{1n} \\ A_{21}, & \ldots & \ldots & A_{2n} \\ \vdots & & & \\ A_{m1}, & \ldots & & A_{mn} \end{matrix} \right\}$$

$\beta = \{A_{ij}$: the only non zero entry is 1 in ith row jth column $1 \le i \le n$, $1 \le i \le m\}$

$|\beta| = mn$, $\dim V = mn$.

Eg 10 $V = P_n(F)$

$= \{ a_0 + a_1 x + \dots + a_n x^n \; ; a_i \in F$

$i = 1, 2 \dots n$

$\beta = \{1, x, x^2 \dots x^n\}$

$|\beta| = n+1 \Rightarrow \dim V = n+1$

Hence $P_n(F)$ is FDVS

Eg 12 $V = \mathbb{C}(\mathbb{R}) = \{ a + ib, \; a, b \in \mathbb{R} \}$

Basis

$a + ib = a(1 + 0i) + b(0 + i)$

$\beta = \{1, i\}$

$|\beta| = 2$

Hence $\dim V = 2$

$\therefore$ V is a FDVS.

Eg 11 $V = \mathbb{C}(\mathbb{C})$ $\beta = \{1\}$, $\dim V = 1$

$|\beta| = 1$

FD.VS

? $a + ib = \underset{\in F}{(a + ib)} \; \underset{\in V}{1}$

Eg $V = \mathbb{R}(\mathbb{R})$ $a = \underset{\in F}{a} \cdot \underset{\in \beta}{1}$

$\beta = \{1\}$.

$\dim V = 1$

If $V = R(\mathbb{C})$ ⟶ This is wrong.
↓ should always be small.

Sub vector spaces are not defined.

* $V = P(F)$
$= \{ a_0 + a_1 x + a_2 x^2 + \dots \quad a_i \in F \quad i = 1, 2 \dots \}$

$\beta = \{ 1, x, x^2 \dots \}$

Since the basis is an infinite set. Hence $P(F)$ is a infinite vector dimensional space.

<u>Corollary</u> 2. Let V be a vector space with dimension n
$\dim(V) = n$

a.) Any finite generating set for V contains atleast n vectors and a generating set for V that contains exactly n vectors is a basis of V.

<u>Proof</u>: Let β be a basis of V.
Let G be a finite generating set for V.

By Theorem 1.9 [State] $\exists$ a subset H of G st H is a basis of V.

By corollary 1, H contains exactly n vec.

Since a subset of G contains n vectors
~~Since a subset of G contains~~

$\therefore$ G must contain atleast n vectors ; $|H| = n$

ie $|G| \geq n$.

Moreover, if G contains exactly n vectors.

$H = G$.

$\Rightarrow$ G is a basis for V.

(b) Any linearly independent set of V that contains exactly n vectors is a basis for V.

Proof Let α be a linearly independent subset of V containing exactly n vectors.
Then by replacement theorem there exist a subset H of basis β which can always be regarded as the generating set, such that

$|H| = n - n = 0$

$L \cup H = L \cup \phi = L$ generates V.

Since L generates V and by the given hypothesis L is linearly independent.

$\therefore$ L is a basis for V.

(c) Every linearly independent subset of V can be extended to form a basis of V.

Pf Let α be linearly independent subset of V containing m vectors. $|\alpha| = m$

By replacement theorem, $\exists$ a subset H of β $(H \subseteq \beta)$ with

$$|H| = n - m$$

containing $n-m$ vectors such that $L \cup H$ generates V.

New $L \cup H$ contains almost n vectors.

$\therefore$ By (a) $L \cup H$ contains exactly n elements, Hence $L \cup H$ is basis for V.

Example 3 Consider $P_2(\mathbb{R}) = \{a_0, a_1x, a_2x^2, a_0, a_1, a_2 \in \mathbb{R}\}$

$\{x^2 + 3x - 2,\ 2x^2 + 5x - 3,\ -x^2 - 4x + 4\}$ generates $P_2(\mathbb{R})$

$\therefore$ β' is basis for $\mathbb{R}$.

Example 4 $M_{2\times2}(\mathbb{R}) =$

$$\beta \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$|\beta| = 4$

$$\beta' = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

is a basis of $M_{2\times2}(\mathbb{R})$.

Ie 5

$\mathbb{R}^4$

standard basis

$\beta = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

$|\beta| = 4$

$\beta' = \{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$

is also a basis for $\mathbb{R}^4$.

$p_k(x) = x^k + x^{k+1} + \dots + x^n$

$k = 0, 1, 2 \dots, n$

$\{p_0(x), p_1(x), \dots p_n(x)\}$ is a basis.

Theorem 1.11 :-

Let W be subspace of a finite dimensional vector spaces V. Then W is finite dimensional and $\dim(W) \leq \dim V$. Moreover, if $\dim(W) = \dim V$ then $V = W$.

Eg 17 $V = F^5 = \{(a_1, a_2, a_3, a_4, a_5) ; a_i \in F\}$

$W = \{(a_1, a_2, a_3, a_4, a_5) \in F^5, a_1 + a_3 + a_5 = 0$
$a_2 = a_4\}$

Clearly $0 \in W$ $\because (0,0,0,0,0)$ Here $a_1 + a_3 + a_5 = 0$
$a_2 = a_4\}$

Also for
$x = (a_1, a_2, a_3, a_4, a_5)$ and
$y = (b_1, b_2, b_3, b_4, b_5) \in W$

$x + y = (a_1, a_2, a_3, a_4, a_5) + (b_1, b_2, b_3, b_4, b_5)$
$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$

Now $a_1 + b_1 + a_3 + b_3 + a_5 + b_5 = 0$
$a_2 + b_2 = a_4 + b_4$ $(\because x, y \in W)$

Hence $x + y \in W$

Also for $c \in W$

$cx = (ca_1, ca_2, ca_3, ca_4, ca_5)$

Now $ca_1 + ca_3 + ca_5 = 0$
& $ca_2 = ca_4$

Hence $cx \in W$

Hence W is a subspace of V.

$\dim(V) = 5$ $\quad \beta = \{(1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0)\ (0,0,0,1,0), (0,0,0,0,1)\}$

[basis of W] $\beta' = \{(-1,0,1,0,0), (-1,0,0,0,1), (0,1,0,1,0)\}$.

Let $(a_1, a_2, a_3, a_4, a_5) \in W$

$a_1 + a_3 + a_5 = 0$

$a_2 = a_4$

$(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3, a_2, a_5)$

where $a_1 + a_2 + a_5 = 0$

$= (a_1, a_2, a_3, a_2, -a_1 - a_3)$

$\beta'' = (0,1,0,1,0), (1,0,0,0,-1), (1,0,-1,0,0)$

$a_1(0,1,0,1,0) + a_2(1,0,0,0,-1), a_3(1,0,-1,0,0)$

$= (a_2 + a_3, a_1, -a_3, a_1, -a_2) = (0,0,0,0,0)$

$a_1 = 0, \quad a_3 = 0, \quad a_2 = 0$

Hence scalars are all zero. Thus, they are linearly independent.

Eg 18 $M_{n\times n}(F) = V$ $\dim V = n^2$

W = set of all diagonal $n\times n$ matrices of $M_{n\times n}(F)$.

$\dim W = n$

$$\begin{pmatrix} a & 0 & \cdots & & \\ 0 & b & 0 & \cdots & \\ 0 & 0 & c & & 0 \\ 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

$|\beta| = n$

$\beta = \{ A_{11}, A_{22} \ldots A_{nn} \}$

where E^{ij} is the matrix in which only non zero entry is 1 in ith row & jth column.

Also,

clearly $\dim(W) \leq \dim(V)$.

Eg 19 $V = M_{n\times n}(F)$

W = set of symmetric matrices

$$W = \left\{ A = [a_{ij}] \; ; \; a_{ij} = a_{ji} \; ; \; \begin{matrix} 1 \leq j \leq n \\ 1 \leq j \leq n \end{matrix} \right\}$$

$\dim(V) = n^2$

$\dim(W) = \frac{1}{2} n(n+1)$

$\left.\begin{matrix} 3-6 \\ 4-10 \\ 5-15 \end{matrix}\right\}$

the minimum no of f.elements required to generate this general element of W is reduced to the entries forming as upper triangular matrix as indicated.

Clearly, the no of elements require start from n, $n-1$ in second row & goes on till 1 in nth row. Hence total no of field element required to generate symm

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$\beta = \{ A^{ij} ; 1 \le i \le n, 1 \le j \le n,$ where A_{ij} is the $n \times n$ matrice having 1 in the ith row & jth column and jth row ith column & elsewhere $0\}$

Que 26

$P_{18}(F) = V$

$= \{ a_{18} x^{18} + a_{17} x^{17} + \dots a_1 x + a_0 ; a_i \in F\}$ $i = 0, 1 \dots 18$

Consider a subset W of all polynomials of the form

$W = \{ a_{18} x^{18} + a_{16} x^{11} + \dots a_2 x^2 + a_0 ;$ $a_0, a_2, \dots a_{18} \in F \}$

$W \subseteq V$

Clearly, W is a subspace of V
Here $\dim W = 10$
&

$\beta = \{ 1, x^2, x^4, \dots x^{18} \}$

* <u>Sum of Subspaces</u>

Let V be a vector space
Let W_1 & W_2 be non empty subsets of V then

$W_1 + W_2 = \{ w_1 + w_2 : w_1 \in W_1, w_2 \in W_2 \}$

<u>Direct Sums</u>

Let W_1 & W_2 be subspaces of V then V is said to be the direct sum of W_1 & W_2 denoted by $V = W_1 \oplus W_2$

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if $V = W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$

Sec 1.3

Q23 Let W_1 & W_2 be subspaces of a vector space V

a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 & W_2

(b) Prove that any subspace of V that contains both W_1 & W_2 must also contain $W_1 + W_2$

Proof (a) Clearly $0 \in W_1$, $0 \in W_2$ $\because W_1$ & W_2 are subspace of V

$\Rightarrow 0 = 0 + 0 \in W_1 + W_2$

Let $a+b, c+d \in W_1 + W_2$

$a, c \in W_1$ & $b, d \in W_2$

Consider, $(a+b) + (c+d) = (a+c) + (b+d) \in W_1 + W_2$ [$(V, +)$ is an abelian gp] [$\because W_1$ & W_2 are subgroups]

Consider, $\alpha(a+b) = \alpha a + \alpha b \in W_1 + W_2$ $\alpha \in F$, $a \in W_1$, $b \in W_2$ ($\because W_1$ & W_2 are subspace)

Now TP $W_1 \subseteq W_1 + W_2$ & $W_2 \subseteq W_1 + W_2$

Let $w_1 \in W_1$, Clearly $w_1 = w_1 + 0 \in W_1 + W_2$

||ly $w_2 = 0 + w_2 \in W_1 + W_2$

(b) Let U be a subspace of V containing both W_1 & W_2

i.e $W_1 \subseteq U \subseteq V$, $W_2 \subseteq U \subseteq V$

Let $a+b \in W_1+W$ be arbitrary , T.P $W_1+W_2 \subseteq U$

$a \in W_1$, $b \in W_2$

$\Rightarrow a \in W_1 \subseteq U$, $b \in W_2 \subseteq U$

$\Rightarrow a+b \in U \Rightarrow W_1+W_2 \subseteq U$

Qn 30 Let W_1 & W_2 be subspaces of V.S. V; Prove that V is the direct sum of W_1 & W_2 iff each vector in V can be uniquely written as x_1+x_2 , $x_1 \in W_1$, $x_2 \in W_2$

Suppose $V = W_1 \oplus W_2$

$\Rightarrow V = W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$

So by defination every $v \in V$ is of the form

$v = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$

Suppose $v = w_1 + w_2$, $w_1, x \in W_1$

& $v = x + y$ $w_2, y \in W_2$

$w_1 + w_2 = x + y$

$w_1 - x = y - w_2$

$\in W_1$ $\in W_2$

$\Rightarrow x - w_1 \in W_1 \cap W_2$ $\Rightarrow x = w_1$

$\Rightarrow y - w_2 \in W_1 \cap W_2$ $y = w_2$

Hence exp is unique

Conversly,

suppose that the expersion is unique.
to prove that, $V = W_1 \oplus W_2$.

Since it's given that $V = W_1 + W_2$
we need to prove that $W_1 \cap W_2 = \{0\}$

Let $w \in W_1 \cap W_2$
$\Rightarrow w \in W_1$ and $w \in W_2$

Clearly $w \in V$
Now $w = w + 0$
$\in W_1 + W_2$

& $w = 0 + w$
$\in W_1 + W_2$

Hence, $w \in V$ has 2 expersion, ~~Hence w=0~~.
which is a contradiction
As w has unique expression.

Hence our Assumption is wrong.

And $\boxed{w = 0}$

Ques 29 a) Prove that if W_1 and W_2 are FDV subspaces of vector space V, then subspace $W_1 + W_2$ is finite dimensonal, &

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proof Given :-

Also $W_1 \cap W_2$ has a finite basis
($\because W_1 \cap W_2$ is FD subsp.)

Let $\beta = \{u_1, u_2, \ldots, u_k\}$ be basis of $W_1 \cap W_2$.

We can extend the basis β to a basis β_1

$\beta_1 \{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_m\}$ for W_1

$\beta_2 = \{u_1, u_2 \ldots u_k, w_1, w_2 \ldots w_p\}$ for W_2

Let $\alpha = \{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_m, w_1, w_2 \ldots w_p\}$

Claim: α is basis for $W_1 + W_2$.

(i) α is unearly independent.

Let $a_1u_1 + a_2u_2 + \cdots + a_ku_k + b_1v_1 + b_2v_2 + \cdots b_mv_m + c_1w_1 + c_2w_2 + \cdots c_3w_3 = 0. \rightarrow (A)$

where a_i's b_j's c_k's are scalars

Then,

$$\underbrace{-b_1 v_1 - b_2 v_2 \ldots - b_m v_m}_{\in W_1} = \underbrace{a_1 u_1 + a_2 u_2 + \ldots a_k u_k + c_1 w_1 + c_2 w_2 + \ldots c_p w_p}_{\in W_2}$$

Hence,

the vector on LHS or equivalent RHS $\in W_1 \cap W_2$

Since β is basis for $W_1 \cap W_2$.

$$(-b_1) v_1 + (-b_2) v_2 + \ldots + (-b_m) v_m = d_1 u_1 + d_2 u_2 + \ldots d_k u_k$$

for some scalars d.

$$\Rightarrow \underbrace{d_1 u_1 + d_2 u_2 + \ldots + d_k u_k + b_1 v_1 + b_2 v_2 + \ldots b_m v_m = 0}_{\in \text{ span } \beta_1}$$

$\Rightarrow$ d_i's $= 0$ $\forall\, i = 1$ to k
b_j's $= 0$ $\quad j = 2$ to m $\quad [\because \beta_1$ is LI$]$

By (1)

$$\Rightarrow \underbrace{a_1 u_1 + \ldots + a_k u_k + c_1 w_1 + c_2 w_2 + \ldots c_p w_p}_{\in \text{ span} \beta_2} = 0$$

$\Rightarrow$ $a_1 = a_2 = a_3 \ldots = a_k = 0$ $\quad (\because \beta_2$ is LI$)$
$c_1 = c_2 = c_3 \ldots c_p = 0$

Hence α is <u>LI</u>.

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(ii) $W_1 + W_2 = \text{span}(\alpha)$

Let $u = v + w \in W_1 + W_2$
; $v \in W_1$, $w \in W_2$

As β_1 is a basis for W_1 & β_2 is basis for W_2

$\Rightarrow$ $\exists$ scalars such that

$$u = (x_1 u_1 + x_2 u_2 + \dots x_k v_k + y_1 v_1 + y_2 v_2 + \dots y_m v_m)$$
$$+ (z_1 u_1 + z_2 u_2 + \dots z_k u_k + t_1 w_1 + t_2 w_2 + \dots \dots t_p w_p)$$

$$u = (x_1 + z_1) u_1 + (x_2 + z_2) u_2 + \dots + y_1 v_1 + y_2 v_2 + \dots y_m v_m + t_1 w_1 + t_2 w_2 + \dots t_p w_p.$$

$\Rightarrow$ $u \in \text{span}\,\alpha$.

$W_1 + W_2 \subseteq \text{span}\,\alpha$

Also $\text{span}\,\alpha \subseteq W_1 + W_2$ (Retracing stepsbck)

Hence = $W_1 + W_2 = \text{span}\,\alpha$

ie α is a basis of $W_1 + W_2$

Now $\dim(W_1 + W_2) = |\alpha|$

$= k + m + p$

$= (k+m) + (k+p) - k$

$= \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$

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Ques 33 (a) Let W_1 and W_2 be subspaces of vector space V such that $V = W_1 \oplus W_2$
If β_1 and β_2 are bases for W_1 and W_2 respectively, show that

$\rightarrow$ $\beta_1 \cap \beta_2 = \phi$

$\rightarrow$ $\beta_1 \cup \beta_2$ is basis for V

Proof Let $v \in \beta_1 \cap \beta_2$
Then $v \in W_1 \cap W_2$
which is a contradiction
($\because V = W_1 \oplus W_2 \Rightarrow W_1 \cap W_2 = \{0\}$).

Hence $\beta_1 \cap \beta_2 = \phi$

We know $\beta_1 \cup \beta_2$ spans V since $V = W_1 + W_2$
Suppose,

$$a_1 v_1 + a_2 v_2 + \cdots a_m v_m + b_1 w_1 + \cdots b_n w_n = 0$$

where a_i's are not all zero & b_i's are not all zero.

Then $a_1 v_1 + \cdots a_m v_m = -b_1 w_1 - b_2 w_2 \cdots - b_n w_n$

& both are nonzero element in $W_1 \cap W_2$.

which is a contradiction $\because W_1 \cap W_2 = \{0\}$

Hence, $\beta_1 \cup \beta_2$ is a LI set.

and v is a basis for V.

(b) Conversely,
Let β_1, β_2 disjoint subspaces of W_1 & W_2 respectively of vector space V. Prove that prove that if $\beta_1 \cup \beta_2$ is basis of V then $V = W_1 \oplus W_2$

Pf Suppose $\beta = \beta_1 \cup \beta_2$ is basis of V
& $\beta_1 \cap \beta_2 = \phi$

Let $v \in V$

$$v = a_1 v_1 + \dots + a_m v_m + b_1 w_1 + \dots b_n w_n$$

where v_i's $\in \beta_1$
w_i's $\in \beta_2$.

$$v = s + t \in W_1 + W_2$$

Hence, $V \subseteq W_1 + W_2$

Clearly, $W_1 + W_2 \subseteq V$

Hence $V = W_1 + W_2$

Let if possible suppose $W_1 \cap W_2 \neq \{0\}$

Then $\exists\ v \neq 0$ such that $v \in W_1$ & $v \in W_2$

$v = a_1 v_1 + a_2 v_2 + \dots a_m v_m$
$v = b_1 w_1 + b_2 w_2 + \dots b_n w_n$

where a_i's, b_i's are not all. 0.

$a_1 v_1 + a_2 v_2 + \dots a_m v_m = b_1 w_1 + b_2 w_2 + \dots b_m w_m$

$a_1 v_1 + a_2 v_2 + \dots a_m v_m - b_1 w_1 - b_2 w_2 - \dots - b_m w_m = 0$

Hence LHS $\in$ span $(\beta_1 \cup \beta_2)$

Hence all a_i's and b_i's must be zero which is a contradiction

Hence, $W_1 \cap W_2 = \{0\}$.

Ques 34 (a) Prove that if W_1 is any subspace of a FDVS, V then $\exists$ subspace W_2 of V such that

$$V = W_1 \oplus W_2$$

Proof Let $\beta = \{u_1, u_2 \dots u_n\}$ be a basis of W_1

By Replacement theorem, we extend basis β of W_1 ~~for~~ to a basis

$$\alpha = \{u_1, u_2 \dots u_n, u_{n+1}, \dots u_m\} \text{ of } V.$$

Let $W_2 = \text{span}\{u_{n+1}, \dots u_m\}$

[$\because S \subseteq V$, span S always subspace of V]

(1) Claim $V = W_1 \oplus W_2$

(1) $V = W_1 + W_2$

(2) $W_1 \cap W_2 = \{0\}$

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Let $v \in V$

Then

$$v = \sum_{i=1}^{m} a_i u_i$$

$$= \sum_{i=1}^{n} a_i u_i + \sum_{i=n+1}^{m} a_i u_i \in W_1 + W_2$$

$$\Rightarrow V \subseteq W_1 + W_2$$

Clearly $W_1 + W_2 \subseteq V$

Hence $V = W_1 + W_2$.

(2) $W_1 \cap W_2 = \{0\}$

Let $u \in W_1 \cap W_2$, $\Rightarrow u \in W_1$; $u \in W_2$.

$$\Rightarrow u = \sum_{i=1}^{n} b_i u_i \quad , \quad u = \sum_{i=n+1}^{m} c_i u_i$$

$$\Rightarrow \sum_{i=1}^{n} b_i u_i = \sum_{i=n+1}^{m} c_i u_i$$

$$\Rightarrow \sum_{i=1}^{n} b_i u_i - \sum_{i=n+1}^{m} c_i u_i = 0$$

$\Rightarrow$ vector on the LHS is spanned by vectors of α which are L.I.

$\Rightarrow b_i = 0 \ \forall i$ & $c_i = 0 \ \forall i$

$\Rightarrow u = 0$.

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(b) $V = \mathbb{R}^2$

$W_1 = \{ (a_1, 0) : a_1 \in \mathbb{R} \}$

Give examples of 2 diff subspaces W_2 and W_2' such that

$$V = W_1 \oplus W_2$$
$$\& \quad V = W_1 \oplus W_2'$$

Solution

(a) Let $W_2 = \{ (0, a_2) ; a_2 \in \mathbb{R} \}$

W_2 is a subspace as $(0,0) \in W_2$,

Also, $w_1 + w_2 \in W_2$ $[(0, a_2) + (0 + a_2') = (0, a_2 + a_2')$.

~~Now~~

Now claim : $V = W_1 \oplus W_2$ (1) ~~w_1~~ $V = W_1 + W_2$

(2) $W_1 \cap W_2 \{0\}$.

(1) $V = W_2 + W_1$.

$(0, a_2) + (a_1, 0) = (a_1, a_2) \in V = V$;

(2) $W_1 \cap W_2 = \{0\}$

Let $(x, y) \in W_1 \cap W_2$

$(x, y) \in W_1$ and $(x, y) \in W_2$. $\Rightarrow x = 0, y = 0$; $(0,0) \in W_1 \cap W_2$

(b) $W_2' = \{ (t, t) : t \in \mathbb{R} \}$

Clearly, W_2' is subspace $(0,0) \in W_2'$

Now (1) $V = W_2' + W_1$

$\alpha (a_1, 0) + \beta (t, t) = (a_1, a_3)$

$(\alpha a_1 + \beta t, \beta t) = (a_1, a_3)$.

$\alpha a_1 + a_3 = a_1$

$\beta t = a_3$

Quotient subspace

$V/W = \{ v + W : v \in V \}$

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$(v+w)+(v'+w) = v + v' + w$ & $\alpha(v+w) = (\alpha v + w)$.

$\alpha = \dfrac{a_1 - a_3}{a_1}$, $\beta = \dfrac{a_3}{t}$

Hence $W_1 + W_2' = V$.

(2) $W_1 \cap W_2 = \{0\}$;

Let $(x,y) \in W_1 \cap W_2$

$y = 0$ & $x = y$ $\Rightarrow$ $(0,0) \in W_1 \cap W_2$

Ques 35 Let W be subspace of FDVS, V.

Let $\{u_1, u_2, \ldots, u_k\}$ be basis of W.

Let $\gamma = \{u_1, u_2 \ldots u_k, u_{k+1} \ldots u_n\}$ be basis of V

(a) Prove that

$\alpha = \{u_{k+1} + W, u_{k+2} + W, \ldots u_n + W\}$

is basis of V/W.

$V/W = \{ v + W, v \in V \}$

Proof (a) α is basis for V/W

(1) α is linearly independent

Consider

$a_1(u_{k+1} + W) + a_2(u_{k+2} + W) + \ldots a_n(u_n + W) = W$

$(a_1 u_{k+1} + W) + (a_2 u_{k+2} + W) + \ldots (a_n u_n + W) = W$

(by operⁿ of scalar.)

$(a_1 u_{k+1} + a_2 u_{k+2} + \ldots a_n u_n) + W = W$.

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$a_1 u_{k+1} + a_2 u_{k+2} + \dots a_n u_n \in W$.

$a_1 u_{k+1} + a_2 u_{k+2} + \dots a_n u_n = b_1 u_1 + b_2 u_2 + \dots b_k u_k$

$b_i \in F$

$\{\because \beta$ is basis of W

$\Rightarrow a_1 u_{k+1} + \dots a_n u_n - b_1 u_1 - b_2 u_2 \dots - b_k u_k) = 0$

$\in \gamma$.

Hence, the above vector must be L.I.

$\Rightarrow a_i = 0 \quad \& \; b_i = 0$

Hence α is LI.

(b) (2) $V/W \in \text{span}\,\alpha$.

Let $v + W \in V/W$

As $v \in V \Rightarrow v = a_1 u_1 + \dots a_k u_k + b_1 u_{k+1} + \dots + b_n u_n$

$\Rightarrow v + W = (a_1 u_1 + a_2 u_2 + \dots a_k u_k + b_1 u_{k+1} + \dots b_n u_n) + W$

$= (b_1 u_{k+1} + \dots b_n u_n) + (a_1 u_1 + a_2 u_2 \dots a_k u_k) + W$

$= (b_1 u_{k+1} + \dots b_n u_n) + W$.

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$= b_1(u_{k+1} + w) + b_2(u_{k+2} + w) + \dots b_n(u_n + w)$

$\Rightarrow v + w \in \text{span}\{u_{k+1} + w, \dots, u_n + w\}$

$\Rightarrow$ Basis for V/W is $\{ \quad ,, \quad \}$.

Now $\dim(V/W) = n - k$.

$\dim(V/W) = \dim V - \dim W$..

Theorem 5.5 (*a*) *Any set of vectors in R^n having more than n vectors is linearly dependent and hence cannot be basis for R^n.*

(*b*) *Any set of vectors that span R^n must contain atleast n vectors therefore any set of vectors in R^n having less than n vectors cannot be basis for R^n.*

(*c*) *Any linearly independent set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ spans R^n and hence is a basis of R^n.*

(*d*) *Any set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ spanning R^n is linearly independent and hence a basis of R^n.*

(*e*) *If $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly independent set of vectors in R^n and $\mathbf{v} \notin L(S)$, then*

$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{v}\}$ is also linearly independent.

Example 5.46. *Which of the following sets of vectors are bases for R^2 ?*

(*a*) $\{(1, 3), (1, -1)\}$ (*b*) $\{(0, 0), (1, 2), (2, 4)\}$

(*c*) $\{(1, 2), (2, -3), (3, 2)\}$ (*d*) $\{(1, 3), (-2, 6)\}$

Solution. (*a*) Let $\alpha_1 (1,3) + \alpha_2 (1,-1) = (0, 0)$

$$\Rightarrow \quad (\alpha_1 + \alpha_2, 3\alpha_1 - \alpha_2) = (0, 0)$$

or

$$\alpha_1 + \alpha_2 = 0$$

$$3\alpha_1 - \alpha_2 = 0$$

$$4\alpha_1 = 0$$

$$\alpha_1 = 0, \ \alpha_2 = 0$$

$\therefore$ $\{(1,3), (1,-1)\}$ is linearly independent set in R^2

By theorem 5.5(*c*) $\{(1, 3), (1, -1)\}$ is a basis of R^2.

(*b*) Let $S = \{(0, 0), (1, 2), (2, 4)\}$

Since S contains 3 vectors therefore by Theorem 5.5(*a*) S cannot be basis of R^2.

(c) $S = \{(1, 2), (2, -3), (3, 2)\}$

Again as S contains 3 vectors (more than 2)

by theorem 5.5(a) S cannot be basis of R^2.

(d) Let $S = \{(1, 3), (-2, 6)\}$

Let $(a, b) \in R^2$

Let if possible $(a, b) = \alpha_1 (1,3) + \alpha_2 (-2,6)$

or $$(a, b) = (\alpha_1 - 2\alpha_2, 3\alpha_1 + 6\alpha_2)$$

or $$\alpha_1 - 2\alpha_2 = a$$

$$3\alpha_1 + 6\alpha_2 = b$$

In matrix form

$$\begin{pmatrix} 1 & -2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\begin{pmatrix} 1 & -2 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b-3a \end{pmatrix}$$

$\Rightarrow$ $$\alpha_1 - 2\alpha_2 = a$$

$$12\alpha_2 = b - 3a$$

$$\alpha_2 = \frac{b-3a}{12},$$

$$\alpha_1 = a + 2\alpha_2 = a + \frac{2b-6a}{12} = \frac{6a+2b}{12}$$

Hence, $L(S) = R^2$

$\therefore$ By Theorem 5.5(d) S is a basis of R^2.

Example 5.47. *Which of the following sets of vectors are bases of R^3 ?*

(a) $\{(1, 2, 0), (0, 1, -1)\}$

(b) $\{(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)\}$

(c) $\{(3, 2, 2), (-1, 2, 1), (0, 1, 0)\}$

(d) $\{(1, 0, 0), (0, 2, -1), (3, 4, 1), (0, 1, 0)\}$

Solution. (*a*) Let $S = \{(1, 2, 0), (0, 1, -1)\}$

Since S contains two vectors (less than 3) and any basis of R^3 must contain 3 vectors therefore by theorem 5.5(*b*) S cannot be basis of R^3.

(*b*) Let $S = \{(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)\}$

Since S contains four vectors (more than 3) and any basis of R^3 must contain 3 vectors therefore by theorem 5.5(*a*) S cannot be basis of R^3.

(*c*) Let $S = \{(3, 2, 2), (-1, 2, 1), (0, 1, 0)\}$

Let $\alpha_1, \alpha_2, \alpha_3 \in R$ be such that

$$\alpha_1 (3,2,2) + \alpha_2 (-1,2,1) + \alpha_3 (0,1,0) = (0, 0, 0)$$

$$\Rightarrow \quad (3\alpha_1 - \alpha_2, 2\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2) = (0, 0, 0)$$

or

$$3\alpha_1 - \alpha_2 = 0$$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$

$$2\alpha_1 + \alpha_2 = 0$$

In matrix form

$$\begin{pmatrix} 3 & -1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{2}{3} R_1,\ R_2 \to R_3 - \frac{2}{3} R_1$

$$\begin{pmatrix} 3 & -1 & 0 \\ 0 & \frac{8}{3} & 1 \\ 0 & \frac{5}{3} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 3\alpha_1 - \alpha_2 = 0$$

$$\frac{8}{3}\alpha_2 + \alpha_3 = 0$$

$$\frac{5}{3}\alpha_2 = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$\therefore$ S is linearly independent. Hence by theorem 5.5(*c*) S is a basis of R^3.

(d) Let $S = \{(1, 0, 0), (0, 2, -1), (3, 4, 1), (0, 1, 0)\}$

Since S contains 4 vectors (more than 3) therefore by theorem 5.5(a) S is not a basis of R^3.

Example 5.48. *Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for the vector space R^3. Prove that* $\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2 + \mathbf{u}_3, \mathbf{u}_3 + \mathbf{u}_1\}$

and $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3\}$ are also basis of R^3.

Solution. Let $\alpha_1, \alpha_2, \alpha_3 \in R$ such that

$$\alpha_1(\mathbf{u}_1 + \mathbf{u}_2) + \alpha_2(\mathbf{u}_2 + \mathbf{u}_3) + \alpha_3(\mathbf{u}_3 + \mathbf{u}_1) = (0, 0, 0)$$

$$\Rightarrow \quad \alpha_1\mathbf{u}_1 + \alpha_3\mathbf{u}_1 + \alpha_1\mathbf{u}_2 + \alpha_2\mathbf{u}_2 + \alpha_2\mathbf{u}_3 + \alpha_3\mathbf{u}_3 = (0, 0, 0)$$

$$\Rightarrow \quad (\alpha_1 + \alpha_3)\mathbf{u}_1 + (\alpha_1 + \alpha_2)\mathbf{u}_2 + (\alpha_2 + \alpha_3)\mathbf{u}_3 = (0, 0, 0)$$

$$\Rightarrow \quad \alpha_1 + \alpha_3 = 0$$

$$\alpha_1 + \alpha_2 = 0$$

$$\alpha_2 + \alpha_3 = 0$$

as $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ being basis of R^3 is linearly independent set.

In matrix form

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or

$$\alpha_1 + \alpha_3 = 0$$

$$\alpha_2 - \alpha_3 = 0$$

$$2\alpha_3 = 0$$

$$\Rightarrow \qquad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Hence, $\{\mathbf{u}_1 + \mathbf{u}_2,\ \mathbf{u}_2 + \mathbf{u}_3,\ \mathbf{u}_3 + \mathbf{u}_1\}$ is linearly independent set in R^3. Therefore, by theorem 5.5(*c*) it is a basis of R^3. Now suppose

$$\alpha_1 \mathbf{u}_1 + \alpha_2 (\mathbf{u}_1 + \mathbf{u}_2) + \alpha_3 (\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) = (0, 0, 0)$$

$$\Rightarrow \quad (\alpha_1 + \alpha_2 + \alpha_3)\mathbf{u}_1 + (\alpha_2 + \alpha_3)\mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = 0$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = 0$$

$$\alpha_2 + \alpha_3 = 0$$

$$\alpha_3 = 0$$

(because $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ being basis of R^3 is linearly independent.)

$$\Rightarrow \quad \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Hence, $\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2,\ \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3\}$ is linearly independent

$\therefore$ By theorem 5.5(*c*) it is a basis of R^3.

Example 5.49. *Extend the set* $\{\mathbf{u}_1 = (1, 1, 1),\ \mathbf{u}_2 = (1, 0, 0)\}$ *to form a basis of* R^3.

Solution. Let $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 = \mathbf{0}$

or $\qquad (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 0, 0) = (0, 0, 0)$

$$\Rightarrow \qquad (\alpha_1 + \alpha_2, \alpha_1, \alpha_1) = (0, 0, 0)$$

$$\Rightarrow \quad \alpha_1 + \alpha_2 = 0,\ \alpha_1 = 0$$

$$\Rightarrow \quad \alpha_1 = \alpha_2 = 0.$$

$\therefore \quad \{\mathbf{u}_1,\ \mathbf{u}_2\}$ is linearly independent set

Now,

$$L(\mathbf{u}_1, \mathbf{u}_2) = \{\alpha_1 (1,1,1) + \alpha_2 (1,0,0) : \alpha_1, \alpha_2 \in R\}$$

$$= \{(\alpha_1 + \alpha_2, \alpha_1, \alpha_1) : \alpha_1, \alpha_2 \in R\}$$

Clearly,

$$\mathbf{u}_3 = (0, 1, 0) \notin L(\mathbf{u}_1, \mathbf{u}_2)$$

$\therefore$ By theorem 5.5(*e*) $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{u}_3\}$ is linearly independent and hence a basis of R^3.

Example 5.50. *Extend the set $S=\{(1, 1, 0)\}$ to form two different basis of R^3.*

Solution. Since $(1, 1, 0) \neq (0, 0, 0)$ $\therefore$ S is linearly independent.

$$L(S)=\{\alpha(1,1,0) : \alpha \in R\}$$
$$=\{(\alpha,\alpha,0) : \alpha \in R\}$$

Now, $(1, 0, 0) \notin L(S)$

$\therefore$ $S_1=\{(1,1,0),(1,0,0)\}$ is linearly independent.

Now,

$$L(S_1)=\{\alpha_1(1,1,0)+\alpha_2(1,0,0) : \alpha_1, \alpha_2 \in R\}$$
$$=\{(\alpha_1+\alpha_2, \alpha_1, 0) : \alpha_1, \alpha_2 \in R\}$$

Now, $(0, 0, 1), (1, 1, 1) \notin L(S_1)$

$\therefore$ $S_2=\{(1, 1, 0), (1, 0, 0), (0, 0, 1)\}$ and

$S_3=\{(1, 1, 0), (1, 0, 0), (1, 1, 1)\}$

are linearly independent and hence S_2 and S_3 are two basis of R^3.

Example 5.51. *Let $S=\{\mathbf{u}_1=(4, 2, 1), \mathbf{u}_2=(2, 6, -5), \mathbf{u}_3=(1, -2, 3)\}$. Is S basis of R^3. If $W=L(S)$ what is dimension of W ?*

Solution. Let $\alpha_1(4,2,1)+\alpha_2(2,6,-5)+\alpha_3(1,-2,3)=(0, 0, 0)$

$\Rightarrow (4\alpha_1+2\alpha_2+\alpha_3, 2\alpha_1+6\alpha_2-2\alpha_3, \alpha_1-5\alpha_2+3\alpha_3)=(0, 0, 0)$

$$\Rightarrow \quad 4\alpha_1+2\alpha_2+\alpha_3=0$$
$$2\alpha_1+6\alpha_2-2\alpha_3=0$$
$$\alpha_1-5\alpha_2+3\alpha_3=0$$

Now determinant of coefficient matrix is

$$\begin{vmatrix} 4 & 2 & 1 \\ 2 & 6 & -2 \\ 1 & -5 & 3 \end{vmatrix}=4(18-10)-2(6+2)+1(-10-6)$$

$$=32-16-16=0$$

$\therefore$ The system has infinite solutions.

Hence, S is linearly dependent. $\therefore$ S is not a basis of R^3.

We express one of the vectors as linear combination of other two.

Let $(1, -2, 3) = \alpha_1 (4,2,1) + \alpha_2 (2,6,-5)$

$$= (4\alpha_1 + 2\alpha_2, 2\alpha_1 + 6\alpha_2, \alpha_1 - 5\alpha_2)$$

$$\Rightarrow \quad 4\alpha_1 + 2\alpha_2 = 1$$

$$2\alpha_1 + 6\alpha_2 = -2$$

$$\alpha_1 - 5\alpha_2 = 3$$

Solving last two, we get

$$16\alpha_2 = -8 \Rightarrow \alpha_2 = -\frac{1}{2}$$

$$\alpha_1 = 3 + 5 \times \left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = -\frac{1}{2}$$

also satisfies the first equation.

$$\therefore \quad (1, -2, 3) = \frac{1}{2}(4,2,1) - \frac{1}{2}(2,6,-5)$$

Also, If $\alpha_1 (4,2,1) + \alpha_2 (2,6,-5) = (0,0,0)$

Then $\alpha_1 = \alpha_2 = 0$

$$\therefore \quad S_1 = \{(4, 2, 1), (2, 6, -5)\}$$

is linearly independent and $(1, -2, 3) \in L(S_1)$

$$\therefore \quad L(S_1) = L(S) = W \qquad \therefore \mathbf{dim}\, W = 2$$

Definition 5.7 *Let V be non-zero subspace of R^n. The* ***dimension*** *of V is the number of vectors in a basis of V.* **dim** *V is notation for dimension of V. Since $\{\mathbf{0}\}$ is linearly dependent the dimension of vector space $\{\mathbf{0}\}$ is defined to be 0.*

It is to be understood that if **dim** $V = k$ then V has a basis containing k vectors say $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$

S is linearly independent and $\mathbf{v} \in V$ can be uniquely expressed as

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots + \alpha_k \mathbf{u}_k$$

Examples 5.38. dim $R^2 = 2$ since $\{(1, 0), (0, 1)\}$ is a basis of R^2.

Every basis of R^2 has two vectors.

dim $R^3 = 3$ since $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of R^3.

Every basis of R^3 has three vectors.

dim $R^n = n$ since $\{(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 0, \ldots, 1)\}$ is a basis of R^n.

Every basis of R^n has n–vectors.

Example 5.39. Let $W = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$

We have shown that W is subspace of R^3

Now $(a_1, a_2, 0) = a_1 (1,0,0) + a_2 (0,1,0)$

It can be verified that $S = \{(1, 0, 0), (0, 1, 0)\}$ is linearly independent.

Hence S is basis of W. **dim** $W = 2$

Example 5.40. $W = \{\alpha (1,1,1) : \alpha \in R\}$

W is a subspace of R^3

$\{(1, 1, 1)\}$ is basis of W.

dim $W = 1$

Example 5.41. $W = \{(a_1, a_2) : a_1 + a_2 = 0, a_1, a_2 \in R\}$

We have shown W is subspace of R^2.

$$(a_1, a_2) \in W \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

Any vector in W is of type $(a_1, -a_1) = a_1(1, -1)$

$\therefore$ $\{(1, -1)\}$ is a basis for W

dim $W = 1$

Example 5.42. $W = \{(0, a) : a \in R\}$

We have shown W is a subspace of R^2.

$\{(0, 1)\}$ is a basis for W.

dim $W = 1$

Example 5.43. $W = \{(x, y) : 3x - 4y = 0\}$

We have shown W is a subspace of R^2.

If $(x, y) \in W$ then $3x - 4y = 0$ or $y = \frac{3}{4}x$

Any element of W is of type

$$\left(x, \frac{3}{4}x\right) = x\left(1, \frac{3}{4}\right)$$

$\therefore$ $\left\{\left(1, \frac{3}{4}\right)\right\}$ is a basis of W.

dim $W = 1$

Example 5.44. Let $W = \{(x_1, x_2, x_3) : x_1 - x_2 - x_3 = 0\} \subseteq R^3$

We have shown W is a subspace of R^3.

All elements of W are of type

$$(x + y, x, y), \ x, y \in R$$

Now, $(1,0,1)$ and $(1,1,0) \in W$

If $\alpha_1(1,0,1) + \alpha_2(1,1,0) = (0, 0, 0)$

then $(\alpha_1, 0, \alpha_1) + (\alpha_2, \alpha_2, 0) = (0, 0, 0)$

$$\Rightarrow \quad (\alpha_1+\alpha_2,\ \alpha_2,\ \alpha_1)=(0,0,0)$$

$$\Rightarrow \quad \alpha_1=0,\ \alpha_2=0$$

$\therefore$ $\{(1,0,1),(1,1,0)\}$ is linearly independent.

Let $(x+y,\ x,\ y)\in W$

Then $(x+y,\ x,\ y)=x(1,1,0)+y(1,0,1)$

$\therefore$ $(x+y,\ x,\ y)\in L\{(1,1,0),\ (1,0,1)\}$

So $L((1,1,0),(1,0,1))=W$

$\therefore$ $\{(1,1,0),\ (1,0,1)\}$ is a basis of W **dim** $W=2$.

Example 5.45. $W=\{(a,b,c):a-2b+3c=0\}\subset R^3$

Find a basis of W

Let $\mathbf{u}=(a_1,b_1,c_1)$, $\mathbf{v}=(a_2,b_2,c_2)\in W$ and $\alpha\in R$.

Then $(a_1,\ b_1,\ c_1)+(a_2,\ b_2,\ c_2)=(a_1+a_2,\ b_1+b_2,\ c_1+c_2)$

Now $(a_1+a_2)-2(b_1+b_2)+3(c_1+c_2)$

$$=(a_1-2b_1+3c_1)+(a_2-2b_2+3c_2)$$

$$=0+0=0$$

Also, $\alpha(a_1,\ b_1,\ c_1)=(\alpha a_1,\ \alpha b_1,\ \alpha c_1)$

$$\alpha a_1-2\alpha b_1+3\alpha c_1=\alpha(a_1-2b_1+3c_1)=\alpha\cdot 0=0$$

$\therefore$ $\mathbf{u}+\mathbf{v},\ \alpha\mathbf{u}\in W$

Hence, W is a subspace of R^3.

Now every member of W is of type $(2x-3y,\ x,\ y)$

Now $(2,1,0),\ (-3,0,1)\in W$

Let $\alpha_1(2,1,0)+\alpha_2(-3,0,1)=(0,0,0)$

Then $(2\alpha_1-3\alpha_2,\ \alpha_1,\ \alpha_2)=(0,0,0)$

$$\Rightarrow \quad \alpha_1=0,\ \alpha_2=0$$

$\Rightarrow$ $\{(2,1,0),\ (-3,0,1)\}$ is linearly independent set in R^3.

Also, $(2x-3y,\ x,\ y)=x(2,1,0)+y(-3,0,1)$

$\therefore$ $L((2,1,0),(-3,0,1))=W$

$\therefore$ $\{(2,1,0),\ (-3,0,1)\}$ is a basis of W

dim $W=2$.

Linear Transformation

Let V & W be two vector spaces over F. A function $T: V \to W$ is said to be a linear transformation from ~~W~~ V to W if

$\forall\ x, y \in W$ & $\forall\ c \in F$

1.) $T(x+y) = T(x) + T(y)$

2.) $T(cx) = cT(x)$

Properties of Linear Transformation

1.) $T(0) = 0$

Proof:- $0 + 0 = 0$ $\quad 0 \in V$

$T(0+0) = T(0)$

$\Rightarrow T(0) + T(0) = T(0)$ $\quad T(0) \in$

$\Rightarrow T(0) = 0 \in W.$

2.) T is linear iff $T(cx+y) = cT(x) + T(y)$ $\forall\ x, y \in V$, $c \in F$

Proof:- Let T be linear transformation

Consider $T(cx+y)$

$= T(cx) + T(y)$ $\quad$ [$\because$ T is OP wrt (+)

$= cT(x) + T(y)$ $\quad$ (T preserves scalar multiplication)

$\Rightarrow T(cx+y) = cT(x) + T(y)$

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Conversely Let $T(cx+y) = cT(x) + T(y)$ $\forall x, y \in V$, $c \in F$

T.P (i) $T(x+y) = T(x) + T(y)$

(ii) $T(cx) = cT(x)$

Proof: Let $c=1$ in (1)

Then $T(1.x+y) = 1.T(x) + T(y)$

$= T(x) + T(y)$

which proves (i)

Now, let $y=0$ in (1)

$T(cx+0) = cT(x) + T(0)$

$= c.T(x) + 0 = c.T(x)$

which proves (ii)

3. T is linear then $T(x-y) = T(x) - T(y)$ $\forall x, y \in V$

Proof: Consider

$T(x-y)$

$= T(x+(-y)) = T(x) + T(-y)$ [$\because$ T is linear]

$= T(x) + (-1).T(y)$ [$\because$ T is linear]

$= T(x) - T(y)$

$\Rightarrow T(x-y) = T(x) - T(y)$

4. T is linear iff $\forall x_1, x_2 \ldots x_n \in V$, $\forall a_1, a_2 \ldots a_n \in F$

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

L.H.S $= T\left(\sum_{i=1}^{n} a_i x_i\right)$

$\Leftrightarrow T(a_1 x_1 + a_2 x_2 + \cdots a_n x_n)$

$\Leftrightarrow T(a_1 x_1) + T(a_2 x_2) + \cdots T(a_n x_n)$

$\Leftrightarrow a_1 T(x_1) + a_2 T(x_2) + \cdots a_n (T(x_n))$

$\Leftrightarrow \sum_{i=1}^{n} a_i T(x_i) = \underline{\underline{R.H.S}}$

Hence, the result holds both ways.

<u>Example-1</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(a_1, a_2) = (2a_1 + a_2, a_1)$

Check if T is a linear transformation

Let $c \in \mathbb{R}$ & $x, y \in \mathbb{R}^2$

where $x = (b_1, b_2)$, $y = (d_1, d_2)$

Now $(cx + y)$

$= c(b_1, b_2) + (d_1, d_2)$

$= (cb_1 + d_1, cb_2 + d_2)$

$T(cx + y) = T(cb_1 + d_1, cb_2 + d_2)$

$= (2(cb_1 + d_1) + (cb_2 + d_2), cb_1 + d_1)$

$= ((2cb_1 + cb_2) + 2d_1 + d_2, cb_1 + d_1)$

$= (2cb_1 + cb_2, cb_1) + (2d_1 + d_2, d_1)$

$= c(2b_1 + b_2, b_1) + (2d_1 + d_2, d_1)$

$= c\,T(b_1, b_2) + T(d_1, d_2)$

$= c\,T(x) + T(y)$

Hence T is linear.

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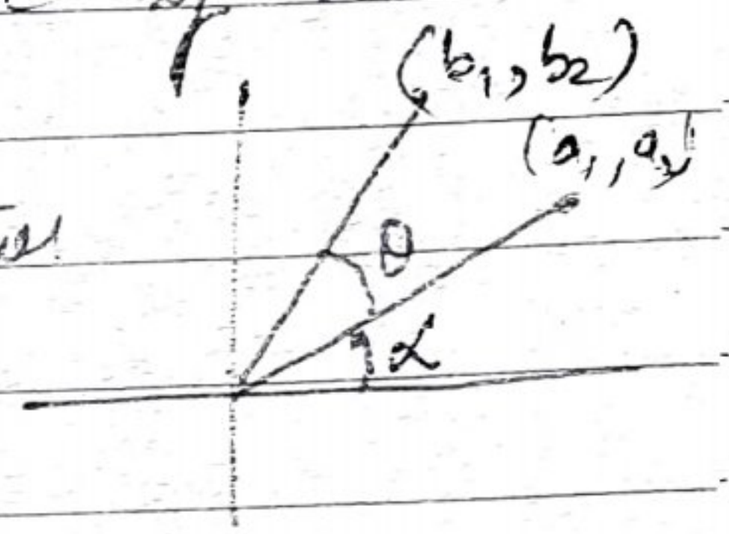
Example-2 For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$
$T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counter clockwise by θ if $(a_1, a_2) \neq (0,0)$ & $T_\theta(0,0) = (0,0)$

Let (a_1, a_2) be a non-zero vector of $\mathbb{R}^2$ that makes an angle of α with the +ve ~~axis~~ x-axis



Now, magnitude of this vector
$r = \sqrt{a_1^2 + a_2^2}$.
then $a_1 = r\cos\alpha$,
$a_2 = r\sin\alpha$.

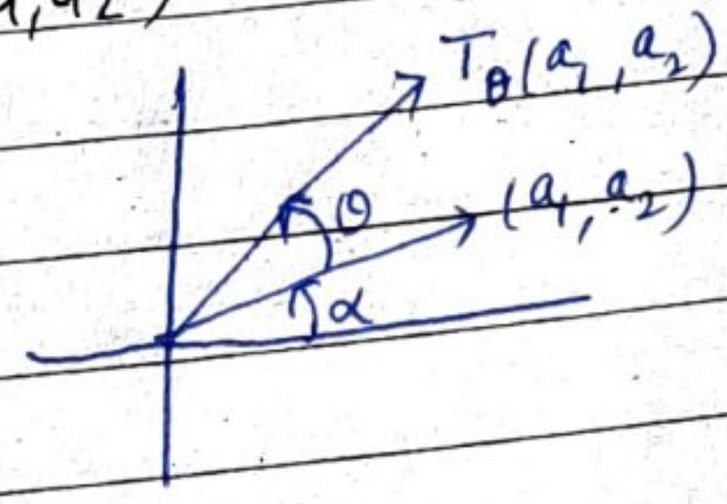
Now,
$T_\theta(a_1, a_2) = (b_1, b_2)$ is the vector in $\mathbb{R}^2$ ~~obt~~ with same magnitude r with an angle of $\alpha + \theta$ with +ve direction of x-axis.

$T_\theta(a_1, a_2) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$
$= (r(\cos\theta.\cos\alpha - \sin\theta.\sin\alpha), r(\sin\theta.\cos\alpha + \cos\theta\sin\alpha))$
$= (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$
$T_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$
$T_\theta(0,0) = (0,0)$
Check that T is linear transformation.

Rotation by θ

$T_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_2 \cos\theta - a_1 \sin\theta)$

T.P $T_\theta(c(a_1,a_2)+(b_1,b_2)) = c\, T_\theta(a_1,a_2) + T_\theta(b_1,b_2)$



L.H.S

$T_\theta[ca_1+b_1, ca_2+b_2]$

$= [(ca_1+b_1)\cos\theta - (ca_2+b_2)\sin\theta, (ca_2+b_2)\cos\theta - (ca_1+b_1)\sin\theta]$

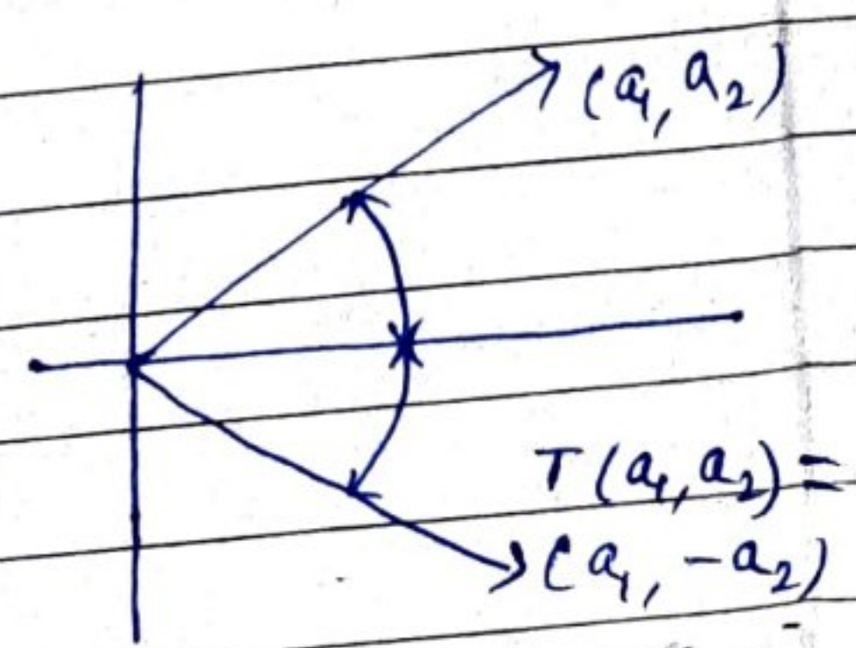
$= [ca_1\cos\theta - ca_2\sin\theta, ca_2\cos\theta - ca_1\sin\theta] + [b_1\cos\theta - b_2\sin\theta, b_2\cos\theta - b_1\sin\theta]$

$= c[a_1\cos\theta - a_2\sin\theta, a_2\cos\theta - a_1\sin\theta] + [b_1\cos\theta - b_2\sin\theta, b_2\cos\theta - b_1\sin\theta]$

$= c\, T_\theta(a_1,a_2) + T_\theta(b_1,b_2) = R.H.S$

Hence T_θ is L.T.

Reflection about x-axis



$T(a_1,a_2) = (a_1, -a_2)$

T.P $\Rightarrow T[c(a_1,a_2)+(b_1,b_2)] = c\,T(a_1,a_2) + T(b_1,b_2)$

L.H.S $T[c(a_1,a_2)+(b_1,b_2)]$

$= T(ca_1+b_1, ca_2+b_2)$

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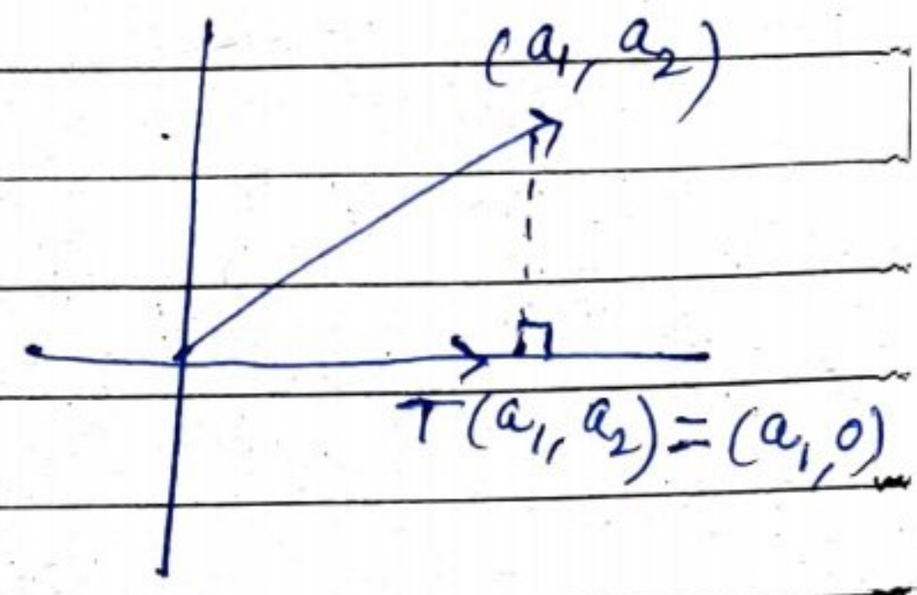
$= [\, ca_1 + b_1,\ -(ca_2 + b_2) \,]$

$= [\, (ca_1 - ca_2) + (b_1 - b_2) \,]$

$= c\,(a_1 - a_2) + (b_1 - b_2)$

$= c\,T(a_1, a_2) + T(b_1, b_2) =$ R.H.S

Ref Map is a L.T



* Projection on x-axis

$T(a_1, a_2) = (a_1, 0)$

T.P $T[\, c(a_1, a_2) + (b_1, b_2) \,] = c\,T(a_1, a_2) + T(b_1, b_2)$

LHS $T[\, c(a_1, a_2) + (b_1, b_2) \,]$

$= T(ca_1 + b_1,\ ca_2 + b_2) = [\, ca_1 + b_1,\ 0 \,]$

$= (ca_1, 0) + (b_1, 0)$

$= c\,(a_1, 0) + (b_1, 0)$

$= c\,T(a_1, a_2) + T(b_1, b_2) =$ RHS

* $T: M_{m\times n}(F) \longrightarrow M_{n\times m}(F)$ s.t. $T(A) = A^t$

Let $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m\times n}(F)$, $c \in F$

T.P $T[\, cA + B \,] = c\,T(A) + T(B)$

LHS $T[\, c[a_{ij}] + [b_{ij}] \,] = T[\, ca_{ij} + b_{ij} \,]$

$= [\, c[a_{ij}] + b_{ij} \,]^t$

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$= [C\, a_{ij}]^t + [b_{ij}]^t$ $\quad [(A+B)^t = A^t + B^t]$

$= C\,[a_{ij}]^t + [b_{ij}]^t$

$= C\,T(A) + T(B) = \underline{R.H.S}$

* $T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ s.t. $T(f(x)) = f'(x)$

T.P $T(c\,f(x) + g(x)) = c\,T(f(x)) + T(g(x))$

$(f(x), g(x) \in P_n(\mathbb{R}), c \in \mathbb{R})$

LHS $T[c\,f(x) + g(x)]$

$= [c\,f(x) + g(x)]' = [c\,f(x)]' + g'(x)$

$= c\,f'(x) + g'(x)$

$= c\,T(f(x)) + T[g(x)] = R.H.S$

* $V = C(\mathbb{R}) \to$ V.S of cts real valued f^n's on $\mathbb{R}$

Let $a, b \in \mathbb{R}$, $a < b$ $T: V \to \mathbb{R}$ s.t. $T(f) = \int_a^b f(t)\,dt$

T.P $T(cf + g) = c\,T(f) + T(g)$
where $f, g \in V$, $c \in \mathbb{R}$

LHS $T(cf+g) = \int_a^b (cf(t) + g(t))\,dt$

$= \int_a^b c\,f(t)\,dt + \int_a^b g(t)\,dt$

$= c\int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

$= c\,T(f) + T(g) = \underline{RHS}$

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* Identity Transformation

$I : V(F) \to V(F)$ s.t. $I(x) = x \ \forall x \in V$

Consider, $I(c x_1 + x_1)$ $\quad x_1, x_2 \in V$, $c \in F$

$= \cancel{c(x_1 + x_2)}\ c x_1 + x_2$

$= c I(x_1) + I(x_2)$

* Zero Transformation

$T_0 : V \to W$, $T_0(v) = 0_W$

Consider, $T_0(c v_1 + v_2)$ $\quad v_1, v_2 \in W$

$= 0_W$

$= c 0_W + 0_W$

$= c T_0(v_1) + T_0(v_2)$ $\Rightarrow T_0$ is L.T

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Eg-7. $V = C(\mathbb{R})$ → vector space of continuous real valued functions on $\mathbb{R}$.

Let $a, b \in \mathbb{R}$, $a < b$. $T : V \to \mathbb{R}$ s.t

$$T(f) = \int_a^b f(t).dt$$

Let $f(t)$ & $g(t) \in C(\mathbb{R})$

$c \in \mathbb{R}$

T.P

$$T(cf+g) = cT(f) + T(g)$$

L.H.S $T(cf+g) = \int_a^b (c f(t) + g(t))\, dt$

$$= \int_a^b c f(t)\, dt + \int_a^b g(t)\, dt$$

$$= c \cdot \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$$

$$= cT(f) + T(g) = R.H.S.$$

Hence Proved

8. Identity Transformation

Let $I : V \to V$ s.t $I(x) = x \ \forall x \in V$.

$x_1, x_2 \in V$, $c \in F$

Consider $I(cx_1 + x_2)$

$= cx_1 + x_2$

$= c(x_1) + (x_2)$

$= c\, I(x_1) + I(x_2)$

Hence I is an linear transformation

Eg. 1. Zero transformation:-

$T_0 : V \to W$ s.t. $T_0(v) = 0_w$

Consider $T_0(cv_1 + v_2)$

$= 0_W$

$= c \cdot 0_W + 0_W$

$= c \cdot T(v_1) + T(v_2)$

Thus, T_0 is a linear transformation.

Defn: Let V & W be vector spaces. $T: V \to W$ be linear

Null space $N(T)$ of T or Kernel of T:

$= \{x \in V : T(x) = 0_w\}$.

Range $R(T)$ of T or image of T:

$R(T) = \{T(x) : x \in V\}$

In zero transformation

$N(T_0) = \{x \in V : T_0(v) = 0_w\} = V$

$R(T_0) = \{0_w\}$

$N(I) = \{x \in V : I(v) = 0_w\} = \{0_v\}$.

$R(I) = V$

Eg: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation s.t. $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$T(c(a_1, a_2, a_3) + (b_1, b_2, b_3))$

$= T(ca_1+b_1, ca_2+b_2, ca_3+b_3)$

$= (ca_1+b_1 - ca_2-b_2, 2ca_3+2b_3)$

$= [c(a_1-a_2)+(b_1-b_2), c(2a_3)+(2b_3)]$

$= c(a_1-a_2, 2a_3) + (b_1-b_2, 2b_3)$

$= c\,T(a_1,a_2,a_3) + T(b_1,b_2,b_3)$

Hence T is a linear transformation.

$N(T) = \{(a_1,a_2,a_3) \in \mathbb{R}^3$ s.t $T(a_1,a_2,a_3) = (0,0)\}$

$\Rightarrow (a_1-a_2, 2a_3) = (0,0)$

$\Rightarrow a_1-a_2=0, \quad 2a_3=0$

$a_1=a_2, \quad a_3=0$

$\Rightarrow N(T) = \{(a_1,a_1,0) : a_1 \in \mathbb{R}\}$.

$R(T) = \{T(a_1,a_2,a_3) : (a_1,a_2,a_3) \in \mathbb{R}^3\}$

$= \{(a_1-a_2, 2a_3), \quad a_1,a_2,a_3 \in \mathbb{R}\}$

$= \mathbb{R}^2$

THM-2.1. Let V & W be vector spaces. $T: V \to W$ is a linear transformation. Then $N(T) \to$ Null space of T, ~~R(T)~~ is a subspace of V & $R(T) \to$ range of T is a subspace of W.

By defn $N(T) \subseteq V$

Proof: Let 0_v & 0_w denote the zero vectors of v & w respectively..

As $T(0_v) = 0_w$, therefore $0_v \in N(T)$

Now consider $T(x+y)$, $x,y \in N(T)$

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T.P. $(x+y) \in N(T)$, $cx \in N(T)$, $x, y \in N(T)$, $c \in F$

Consider $T(x+y)$

$= T(x) + T(y)$ [T is linear]

$= 0_W + 0_W$ [$x, y \in N(T)$]

$= 0_W$

$\Rightarrow x+y \in N(T)$

Consider $T(cx)$

$= c\,T(x)$ = [T is linear]

$= c.\,0_W$ [$x \in N(T)$]

$= 0_W$

$\Rightarrow cx \in N(T)$

Hence $N(T)$ is a subspace of V

Now

By defn, $R(T) \subseteq W$

As $T(0_V) = 0_W \Rightarrow 0_W \in R(T)$

Let $x, y \in R(T)$, $c \in F$

T.P $x+y \in R(T)$, $cx \in R(T)$

~~Consider~~ Since $x, y \in R(T)$

$\Rightarrow \exists\, v, w \in V$ s.t

$T(v) = x$, $T(w) = y$.

Consider ~~$T(v+w)$~~

$x+y$

$= T(v) + T(w)$

$= T(v+w) \in R(T)$

$\because v, w \in V$, as $v, w \in V$

$\Rightarrow x+y \in R(T)$

Consider cx

$= cT(v)$

$\in R(T)$ [$\because cv \in V$ as $v \in V$ & $c \in F$].

* Thus, $R(T)$ is a subspace of W.

Thm-2.2 Let V & W be vector spaces and $T: V \to W$ be linear transformation. Then.

$\beta = \{v_1, v_2, \ldots v_n\}$ be the basis for V.

Then $R(T) = \text{span}(T(\beta))$

$= \text{span}(\{T(v_1), T(v_2), \ldots T(v_n)\})$.

Proof: Clearly $T(v_i) \in R(T)$ $\forall i$. $\therefore$

As $R(T)$ is a subspace

$\therefore R(T)$ contains $\text{span}(\{T(v_1), T(v_2), \ldots T(v_n)\})$

$= \text{span}(T(\beta))$.

[$\because$ span of any subspace S of a vector space is a subspace of V].

i.e. $\text{span}(T(\beta)) \subseteq R(T)$

Now To prove $R(T) \subseteq \text{span}(T(\beta))$

Let $w \in R(T)$

$\Rightarrow \exists\, v \in V$ s.t. $w = T(v)$

As β is a basis of V

$\therefore v = \sum_{i=1}^{n} a_i v_i$, $a_i \in F$, $i = 1, 2, \ldots n$

$T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right)$

$\Rightarrow w = T(v) = \sum_{i=1}^{n} a_i T(v_i)$ [T is linear

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$$w = T(v_i) = \sum_{i=1}^{n} a_i T(v_i)$$

$$\in \text{span}(T(\beta))$$

Hence $R(T) \subseteq \text{span}(T(\beta))$

Thus $R(T) = \text{span}(T(\beta))$

Remark: The above theorem is true even if β is finite.

Eg-10 $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is a linear transformation given by.

$$T(f(x)) = \begin{bmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{bmatrix}$$

Since $\beta = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$

$$R(T) = \text{span}[T(\beta)]$$
$$= \text{span}(\{T(1), T(x), T(x^2)\})$$

$$T(1) = \begin{bmatrix} 1-1=0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(x^2) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R(T) = \text{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right.$$

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$$R(T) = \text{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right)$$

Since, the spanning set of $R(T)$ is now a linearly independent set. $\therefore$ It is a basis of $R(T)$.

$\therefore \dim R(T) = 2.$

Defn: If V & W are vector spaces & ~~then~~ $T: V \to W$ is linear & then $N(T)$ & $R(T)$ are finite dimensional. Then dimension of $N(T)$ is called nullity of T denoted by nullity(T) & dimension of $R(T)$ is called rank of T denoted by rank(T).

THM 2.3 DIMENSION THEOREM -

Let V & W be two vector spaces. $T: V \to W$ be linear. If V is finite dimensional. Then nullity(T) + rank(T) = dim(V)

Proof:- Suppose $\dim(V) = n$ and $\dim(N(T)) = k$.

Let $\{v_1, v_2, \ldots v_k\}$ be a basis for $N(T)$

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As $N(T)$ is a subspace of V.
$\therefore$ we may extend $\{v_1, v_2, \ldots v_k\}$ to a basis for $\beta = \{v_1, v_2, \ldots v_n\}$ of V

<u>Claim</u>: $S = \{T(v_{k+1}), T(v_{k+2}), \ldots T(v_n)\}$ is a basis for $R(T)$.

As $T(v_i) = 0 \quad \forall \quad 1 \le i \le k$ $\quad$ [$\because v_i \in N(T)$ for $i = 1, \ldots k$ $\therefore T(v_i) = 0$]

Using thm 2.2,

$$R(T) = \text{span}\{T(v_1), T(v_2), \ldots T(v_k), T(v_{k+1}), \ldots T(v_n)\}$$

$$= \text{span}\{T(v_{k+1}), T(v_{k+2}), \ldots T(v_n)\}.$$

$$= \text{span}(S).$$

Now, To <u>prove</u> S is linearly independent.

Let $\sum_{i=k+1}^{n} b_i T(v_i) = 0$, $b_i \in F$, $i = k+1, \ldots n$

$$\Rightarrow \cancel{T\left(\sum_{i=k+1}^{n} b_i T(v_i)\right) = T(0)}$$

$$\Rightarrow b_{k+1} T(v_{k+1}) + b_{k+2} T(v_{k+2}) + \ldots b_n T(v_n) = 0$$

$$\Rightarrow T(b_{k+1} v_{k+1} + \ldots b_n v_n) = 0 \quad [\because T \text{ is linear}]$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$$

$$\Rightarrow \sum_{i=k+1}^{n} b_i v_i \in N(T)$$

$\therefore \exists$ scalars $c_1, c_2, \ldots c_k \in F$ s.t.

$$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$$

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$$\Rightarrow \sum_{i=1}^{k} -c_i v_i + \sum_{i=k+1}^{n} b_i v_i = 0$$

Since β is a basis for V.

$\therefore$ we have $b_i = 0 \ \forall i$.

$\Rightarrow S$ is linear independent.

$\Rightarrow S$ is a basis for $R(T)$.

$\Rightarrow \dim(R(T)) = n-k = \text{rank}(T)$

$\dim(N(T)) = k = \text{nullity}(T)$.

Hence, $\text{nullity}(T) + \text{rank}(T)$

$= k + (n-k)$

$= n = \dim(V)$

Thm-2.4 V & W are vector space. $T: V \to W$ is linear. T is one-one if & only if $N(T) = \{0\}$.

Proof Suppose T is one-one.

T.P $N(T) = \{0\}$.

Let $x \in N(T)$

$\Rightarrow T(x) = 0_W = T(0_V)$

$\Rightarrow x = 0_V$ [$\because T$ is one-one]

$\Rightarrow N(T) = \{0_V\}$. [$\because x \in N(T)$ was arbitrary]

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Conversely:- Suppose that $N(T) = \{0\}$.

T.P T is one-one

Let $T(x) = T(y)$

$\Rightarrow T(x) - T(y) = 0_W$

$\Rightarrow T(x-y) = 0_W$ ($\because$ T is linear).

$\Rightarrow x-y \in N(T) = \{0\}$

$\Rightarrow x-y = 0$

$\Rightarrow x = y$.

Thus $T(x) = T(y)$

$\Rightarrow x = y$.

Hence, T is one-one.

Thm 2.5 Let V & W be vector spaces of equal finite dimension and let $T: V \to W$ be linear transformation. Then, the following statements are equivalent:-

(i) T is one-one (iii) $\text{rank}(T) = \dim V$

(ii) T is onto.

Proof:- By dim thm,

$\text{nullity}(T) + \text{rank}(T) = \dim(V)$

If (i) holds $\Rightarrow$ T is one-one

$\Leftrightarrow N(T) = \{0\}$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow \text{nullity}(T) = 0$

$\Rightarrow$ Hence,

$\text{rank}(T) = \dim(V)$

$\dim(V) = \text{rank}(T)$

$\Leftrightarrow \dim(W) = \text{rank}(T) = \dim(R(T))$

$\Leftrightarrow \dim(R(T)) = \dim W$

Also $R(T) \subseteq W$

$\Leftrightarrow W = R(T)$

co-domain range

$\Leftrightarrow$ T is onto

Hence, the result ~~is~~ is both ways true.

(Contd)
Rank+Nullity

Ex. $T: P_2(R) \to P_3(R)$ is a L.T s.t

$$T(f(x)) = 2f'(x) + \int_0^3 3f(t)\,dt$$

find $R(T)$. Prove that T is not onto but it is one-one.

Soln. $R(T) = \text{Span}(T(\beta))$ where $\beta = \{1, x, x^2\}$ is a standard basis of $V = P_2(R)$

$= \text{Span}(\{T(1), T(x), T(x^2)\})$

$= \text{Span}(\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\})$

i.e, $\{3x, 2+\frac{3}{2}x^2, 4x+x^3\}$ form a basis of $R(T)$

$\therefore \dim R(T) = 3$

$\Rightarrow R(T) \neq W$ [as $\dim W = 4$]

$\therefore T$ is not onto.

$\text{Nullity}(T) + \text{Rank}(T) = \dim P_2(R)$

$\Rightarrow \text{Nullity}(T) = 0$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow N(T) = \{0\}$ alone

$\Rightarrow T$ is 1-1.

Ex. $T: F^2 \to F^2$ is a L.T s.t

$T(a_1, a_2) = (a_1 + a_2, a_1)$

$N(T) = \{(0,0)\}$

$\therefore T$ is 1-1

Here, $\dim F^2 = \dim F^2$

& F^2 is f.d.

$\therefore$ from theorem, T is onto.

Th. V and W are vector space over the same field F. Let V be ~~f.d.~~ finite dimensional and let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Let $w_1, w_2, \ldots, w_n$ be elements in W. then $\exists$ a unique linear transformation $T: V \to W$ satisfying $T(v_i) = w_i$, $i = 1$ to n.

Cor. V and W are vector space both over field F. Let V be finite dim. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V. Suppose U & T are 2 L.T from V to W s.t

$U(v_i) = T(v_i)$, $i = 1$ to n

then $U = T$

(Contd)

Ex. $T: R^2 \to R^2$ is a L.T

$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$

Let $U: R^2 \to R^2$ be another L.T

Also given $U(1,2) = (3,3)$

& $U(1,1) = (1,3)$

show that $U = T$

Soln.

$T(1,2) = (3,3) = U(1,2)$

& $T(1,1) = (1,3) = U(1,1)$

we can check that $\{(1,2), (1,1)\}$ is a basis of

Hence, by the Cor. $U = T$.

Ex. $T: P_2[R] \to R^3$ is a L.T

s.t $T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$

Also T is 1-1

Soln. Let $S = \{2 - x_0 + 3x^2, x + x^2, 1 - 2x^2)$

then S is a subset of $P_2(R)$

check whether S is L.I

Now $T(S) = \{T(2 - x - 3x^2), T(x + x^2), T(1 - 2x^2)\}$

$T(S) = \{(2, -1, 1), (0, 1, 1), (1, 0, -2)\}$.

check if $T(S)$ is L.I.

then S is also L.I.

Q. Which of the foll. are linear transformations?

(a) $T(x, y, z) = (x - y, x^2, 2z)$

(b) $T(x, y) = (x - y, 2x + 2)$

Soln: (a) Let $(x_1, y_1, z_1); (x_2, y_2, z_2) \in R^3$ and $\alpha \in R$

Now, $T((x_1, y_1, z_1) + (x_2, y_2, z_2))$

$= T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= ((x_1 + x_2) - (y_1 + y_2), (x_1 + x_2)^2, 2(z_1 + z_2))$

$= (x_1 - y_1 + x_2 - y_2, x_1^2 + x_2^2 + 2x_1x_2, 2z_1 + 2z_2)$

$= (x_1 - y_1, x_1^2, 2z_1) + (x_2 - y_2, x_2^2, 2z_2) + (0, 2x_1x_2, 0)$

$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) + (0, 2x_1x_2, 0)$

$\neq T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$\therefore T$ is not L.T.

(b) Let $(x_1, y_1), (x_2, y_2) \in R^2$

$T((x_1, y_1) + (x_2, y_2)) = L(x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 - y_1 - y_2, 2(x_1 + x_2) + 2)$

$= (x_1 - y_1 + x_2 - y_2, 2x_1 + 2 + 2x_2)$

$= (x_1 - y_1, 2x_1 + 2) + (x_2 - y_2, 2x_2)$

$= T(x_1, y_1) + T(x_2, y_2) + (0, -2)$

$\therefore T((x_1, y_1) + (x_2, y_2)) \neq T(x_1, y_1) + L(x_2, y_2)$

$\therefore T$ is not linear transformation.

Q. Show that $f: R^4 \to R^4$ given

$$f(x, y, z, t) = (2x, 3y, 0, 0)$$

is a linear transformation. Find its rank and nullity.

Soln: Let $v_1 = (x_1, y_1, z_1, t_1)$, $v_2 = (x_2, y_2, z_2, t_2) \in R^4$

and $\alpha, \beta \in R$

Then $\alpha v_1 + \beta v_2 = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2, \alpha t_1 + \beta t_2)$

As given, $f(\alpha v_1 + \beta v_2) = (2(\alpha x_1 + \beta x_2), 3(\alpha y_1 + \beta y_2), 0, 0)$

$$= \alpha(2x_1, 3y_1, 0, 0) + \beta(2x_2, 3y_2, 0, 0)$$

$$= \alpha f(v_1) + \beta f(v_2)$$

Hence f is a linear transformation.

We have $\text{Ker} f = \{(x, y, z, t) \in R^4 : f(x, y, z, t) = (0, 0, 0, 0)\}$

$\therefore (x, y, z, t) \in \text{Ker} f \iff f(x, y, z, t) = (0, 0, 0, 0)$

$\iff (2x, 3y, 0, 0) = (0, 0, 0, 0)$

$\iff x = 0, y = 0$

$\iff (x, y, z, t) = (0, 0, z, t)$

$$\text{Ker} f = \{(0, 0, z, t) : z, t \in R\}$$

Since $(0, 0, z, t) = z(0, 0, 1, 0) + t(0, 0, 0, 1)$ so

Ker f is spanned by the set $S = \{e_3 = (0, 0, 1, 0),$

$e_4 = (0, 0, 0, 1)\}$.

It is easy to verify that e_3, e_4 are L.I.

Thus S is a basis of Ker f and so dim Ker $f = 2$

Hence nullity $f = 2$

By, rank-nullity theorem,

rank f + nullity f = dim $R^4 = 4$

Hence rank $f = 4 - 2 = 2$.

Q. Find the range, rank, and nullity of the linear transformation $T: R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$ —(1)

Soln: Let $(a, b) \in$ Range T be arbitrary. Then

$(a, b) = T(x_1, x_2, x_3)$, for some $(x_1, x_2, x_3) \in R^3$.

or, $(a, b) = (x_1 + x_2, 2x_3 - x_1) = (x_1 + x_2 + 0x_3, -x_1 + 0x_2 + 2x_3)$

or, $(a, b) = x_1(1, -1) + x_2(1, 0) + 2x_3(0, 1)$ —(2)

we see that $S = \{(1,0),(0,1)\}$ is L.I and

$$(1,-1) \in L(S) \quad [\because (1,-1) = (1,0) - 1(0,1)]$$

Hence by (2), Range $T = L(S) = ((1,0,(0,1))$

$\therefore$ rank T = dim Range $T = 2$

Let $(a,b,c) \in$ Ker T be arbitrary. Then

$T(a,b,c) = (0,0)$ i.e $(a+b, 2c-a) = (0,0)$, by (1)

$\Rightarrow a+b=0,\ 2c-a=0 \Rightarrow a=2, b=-2, c=1$

We see that Ker T is spanned by $(2,-2,1)$

Hence Ker $T = \{(2,-2,1)\}$ and dim Ker $T = 1$ i.e, nullity $T = 1$.

Q. Find the range, rank, kernel and nullity of the linear transf $T: R^2 \to R^3$ defined by

$$T(x_1, x_2) = (x_1+x_2, x_1-x_2, x_2) \quad \text{--- (1)}$$

Soln: Let $(a,b,c) \in$ Range T be arbitrary. Then

$(a,b,c) = T(x_1,x_2)$, for some $(x_1,x_2) \in R^2$.

or $(a,b,c) = (x_1+x_2, x_1-x_2, x_2)$, by (1)

$= x_1(1,1,0) + x_2(1,-1,1)$; $x_1, x_2 \in R$

This shows that Range T is spanned by $S = \{(1,1,0),(1,-1,1)\}$

Further $(1,1,0),(1,-1,1)$ are L.I. Since

$$\alpha(1,1,0) + \beta(1,-1,1) = (0,0,0) \ ; \alpha, \beta \in R$$

$\Rightarrow \alpha+\beta=0,\ \alpha-\beta=0,\ \beta=0 \Rightarrow \alpha=0, \beta=0$

Hence S is a basis of Range T and so dim Range $T = 2$

i.e, Range $T = L(S) = ((1,1,0),(1,-1,1))$ and rank $T = 2$

Let $(a,b) \in$ Ker T be arbitrary. Then $T(a,b) = (0,0,0)$

or $(a+b, a-b, b) = (0,0,0)$, by (1)

$\Rightarrow a+b=0,\ a-b=0,\ b=0 \Rightarrow a=0, b=0$

$\therefore$ Ker $T = \{(0,0)\}$ and dim Ker $T = 0$ i.e, nullity $T = 0$.

Q. Let $T: R^3 : R^3$ be defined as

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

verify rank nullity theorem for T.

Soln. Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker T$

$$\Rightarrow T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + z = 0$

$y + z = 0$

$\Rightarrow x = -z, y = -z$

$\therefore x = -a, y = -a, z = a$ is a soln.

$$\therefore \ker T = \left\{ a\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} : a \in R \right.$$

As $\ker T \neq \{0\}$

$\therefore T$ is not one-one

$\dim(\ker T) =$ nullity of $T = 1$

Let $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in R^3$

Suppose there exists $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3$

such that $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = u$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b-a \\ c-2a \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b-a \\ c-2a-b+a \end{pmatrix}$$

The soln exists only if

$c - 2a - b + a = 0$ (T is onto iff $r(T) = \dim V$)

or $c = a + b$

$\therefore$ T is not onto and any element of Range T is of type $\begin{pmatrix} a \\ b \\ a+b \end{pmatrix}$ ($\because \dim T = 1$, $\text{rank} T = 3$, $\dim T \neq \text{rank} T$)

$$\text{Range} T = \left\{ \begin{pmatrix} a \\ b \\ a+b \end{pmatrix} : a, b \in R \right\}$$

A basis of Range $T = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\therefore \dim(\text{Range} T) = \text{Rank of } T = 2$

Now Nullity of T + Rank of T $= 1 + 2 = 3 = \dim R^3$.

Eigen Values and Eigen vectors

Def^n: Let A be an $n \times n$ matrix. A real number λ is called an eigenvalue of A if there exists a non-zero n-vector X such that $AX = \lambda X$. In this case, the vector X is called an eigenvector of A corresponding to λ.

Note: Eigenvalues and eigenvectors are sometimes called characteristic values and characteristic vectors or latent values and latent vectors.

Ex. Show that $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to the eigen value 3.

Soln. we have

$$AX = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5\times 1 + 2\times -1 \\ 2\times 1 + 5\times -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 3X$$

i.e, $AX = 3X$

So $\lambda = 3$ is an eigen value of A and $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector corresponding to this eigen value.

Eigenspace

Def^n: Let A be an $n \times n$ matrix, and let λ be an eigen value of A. Then the eigen space of A for λ denoted by E_λ, is the set of all eigenvectors of A for λ, together with the inclusion of the zero vector. Thus $E_\lambda = \{ X : AX = \lambda X \}$

Th: Let A be a matrix and let λ be a real no. Then λ is an eigenvalue of A iff $|A - \lambda I| = 0$. The eigenvectors corresponding to λ are the non-trivial solns of the homogeneous system $(A - \lambda I)X = 0$. The eigen space E_λ is the complete soln set for this homogeneous system.

Characteristic Polynomial of A →

Def^n: Let A be an $n \times n$ matrix. The characteristic polynomial of A is defined to be the polynomial $P_A(x) = |A - xI|$

* The eigenvalues of A are just real roots of the characteristic polynomial.

Ex. Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 12 & -51 \\ 2 & -11 \end{bmatrix}$. Also find all the Eigen values for A.

Soln: $|A - \lambda I| = 0$

$$P_A(x) = \begin{vmatrix} 12-\lambda & -51 \\ 2 & -11-\lambda \end{vmatrix} = 0$$

$$= (12-\lambda)(-11-\lambda) + 102$$
$$= (\lambda-12)(\lambda+11) + 102$$
$$= \lambda^2 - 12\lambda + 11\lambda - 132 + 102$$
$$= \lambda^2 - \lambda - 30$$
$$= \lambda^2 - 6\lambda + 5\lambda - 30$$
$$= \lambda(\lambda-6) + 5(\lambda-6)$$
$$= (\lambda+5)(\lambda-6)$$

i.e, $\lambda = -5, 6$

$\therefore$ Eigen values are -5 and 6.

Ex. Find the eigen values for the matrix $A = \begin{bmatrix} 7 & 1 & -1 \\ -11 & -3 & 2 \\ 18 & 2 & -4 \end{bmatrix}$

Soln. $P_A(x) = |A - \lambda I| = \begin{bmatrix} 7-\lambda & 1 & -1 \\ -11 & -3-\lambda & 2 \\ 18 & 2 & -4-\lambda \end{bmatrix}$

Ans.: $\lambda = -2, 4$

$$= -\lambda^3 + 12\lambda + 16$$
$$= -(\lambda+2)^2(\lambda-4)$$

or $-(\lambda+2)^2(\lambda-4) = 0$

$\Rightarrow \lambda = -2, 4$

$\therefore$ Eigen values are -2 & 4

Properties of Eigenvalues and Eigen vectors

1. Let A be a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigen value of A.
2. Let A be a square matrix. Then A and A^T have the same characteristic polynomial and hence the same eigen values.
3. Let A be a square matrix, and let k be a positive integer. If λ is an eigen value of A, then λ^k is an eigenvalue of A^k.
4. Sum of eigenvalues = Sum of diagonal elements (Trace (A))
5. Product of eigenvalues = Determinant of matrix
6. Eigen values of triangular matrices are same as their diagonal elements.

Orthogonal Matrix

A square matrix A is called orthogonal if $AA^T = A^TA = I$

$\Rightarrow A^{-1} = A^T$ is valid for orthogonal matrix.

Q. Verify if the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal.

Soln. we have

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow A$ is an orthogonal matrix.

Ex. Find the eigen values and the corresponding eigen vectors of the foll. matrices :

(i) $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Soln: The characteristic equn of A is given by

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \text{ or } \cancel{(1-\lambda)(2-\lambda)-(3-\lambda)(4-\lambda)=0}$$

$$(1-\lambda)(2-\lambda) - 12 = 0$$

$$\cancel{\text{or } 2-\lambda-2\lambda+\lambda^2-12+4\lambda+4\lambda-\lambda^2=0}$$

$$\text{or } 2-2\lambda-\lambda+\lambda^2-12=0$$

$$\text{or } \lambda^2-3\lambda-10=0$$

$$\text{or } \lambda^2-5\lambda+2\lambda-10=0$$

$$\text{or } \lambda(\lambda-5)+2(\lambda-5)=0$$

$$\text{or } (\lambda+2)(\lambda-5)=0$$

$$\text{or } \lambda=-2, 5$$

Corresponding to the eigen value $\lambda=-2$, we have

$$(A+2I)X = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } 3x_1+4x_2=0$$

$$\text{or } x_1 = -\frac{4}{3}x_2$$

Hence the eigen vector X is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_2/3 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$$

Since an eigen vector is unique upto a constant multiple, we can take the eigen vector as $[-4, 3]^T$.

Corresponding to $\lambda=5$

$$(A-5\lambda)X = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{or } x_1 - x_2 = 0 \text{ or } x_1 = x_2$$

$$\therefore \text{Eigen vector is } X = (x_1, x_2)^T = x_1(1,1)^T \text{ or } (1,1)^T$$

Ex. (ii). $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Soln: Ch. eqn is given by

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2+1=0 \Rightarrow 1-2\lambda+\lambda^2+1=0$$

$$\Rightarrow \lambda^2-2\lambda+2=0 \Rightarrow \lambda = 1 \pm i$$

Corr. to $\lambda = 1+i$, we have

$$[A-(1+i)I]X = \begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \cancel{1-(1+i)(1-i)-(-1)}$$

$$\cancel{\text{or } 1+1+}$$

ie, $-ix_1 + x_2 = 0$ ie, $ix_1 = x_2$ ie, $\frac{x_1}{1} = \frac{x_2}{i}$

$-x_1 - ix_2 = 0$

$\therefore$ Eigen vector for $\lambda = 1+i$ is

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = [1, i]^T$$

For $\lambda = 1-i$

$$\begin{bmatrix} 1-(1-i) & 1 \\ -1 & (1-(1-i)) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie, $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $ix_1 + x_2 = 0$

$-x_1 - ix_2 = 0$

or $x_1 + ix_2 = 0$

or $ix_1 = -x_2$

or $\frac{x_1}{1} = \frac{x_2}{-i}$.

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = [1 \ -i]^T$$

$\therefore$ Eigen vector for $\lambda = 1-i$ is $[1, -i]^T$.

Remark: For a real matrix A, the eigen values and the corresponding eigen vectors can be complex.

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

Soln: The characteristic eqn of A is given by

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)[(2-\lambda)(3-\lambda)] - 0 + 0 = (1-\lambda)(6 - 2\lambda - 3\lambda + \lambda^2)$$
$$= (1-\lambda)(\lambda^2 - 5\lambda + 6)$$
$$= (1-\lambda)(\lambda^2 - 3\lambda - 2\lambda + 6)$$
$$= (1-\lambda)[\lambda(\lambda-3) - 2(\lambda-3)]$$
$$= (1-\lambda)(\lambda-2)(\lambda-3)$$

i.e, $\lambda = 1, 2, 3$.

$\therefore$ Eigenvalues are 1, 2 & 3.

Corr. to $\lambda = 1$, eigen vector is

$[A - \lambda I]X = 0$

$$[A - \lambda I]X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $x_2 + x_3 = 0$

$2x_1 + 2x_3 = 0$

we obtain two eqns in three unknowns. one of the variable can be choosen arbitrarily. Taking $x_3 = 1$, $x_2 = -1$, $x_1 = -1$

$\therefore$ we obtain eigen vector as $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = [-1, -1, 1]^T$

For $\lambda = 2$

$$[A - 2I]X = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ~~$+(0-0) - 0 + 0 =$~~ $-x_1 = 0 \Rightarrow x_1 = 0$, $x_3 = 0$, ~~& $x_1 + x_3 = 0$~~

and x_2 is arbitrary. Taking $x_2 = 1$.

eigen vectors are $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [0, 1, 0]^T$.

For $\lambda = 3$ we have

$$[A - 3I]X = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $-2x_1 = 0$ or $x_1 = 0$

$x_2 + x_3 = 0$ or $x_2 = -x_3$ or $\frac{x_2}{1} = \frac{x_3}{-1}$

$2x_1 + x_3 = 0$

$\therefore$ eigen vectors are $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = [0, 1, -1]^T$

Q. Find the Eigen value and Eigen vector of matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$\frac{x_1}{0} = \frac{x_2}{\cdot} = \frac{x_3}{\cdot}$

Q. Find the Eigen values and Eigen vectors of the given matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Soln. The ch. eqn of matrix A is $|A-\lambda I| = 0$

$$\Rightarrow |A-\lambda I| = \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)\begin{vmatrix} 7-\lambda & -4 \\ -4 & 3-\lambda \end{vmatrix} - (-6)\begin{vmatrix} -6 & -4 \\ 2 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} -6 & 7-\lambda \\ 2 & -4 \end{vmatrix} = 0$$

$$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16] + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda)] = 0$$

$$\Rightarrow (8-\lambda)[21-3\lambda-7\lambda+\lambda^2-16] + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0$$

$$\Rightarrow (8-\lambda)[\lambda^2-10\lambda+5] + 6[6\lambda-10] + 2[10+2\lambda] = 0$$

$$\Rightarrow 8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+20+4\lambda = 0$$

$$\Rightarrow -\lambda^3+18\lambda^2-45\lambda = 0$$

$$\Rightarrow \lambda^3-18\lambda^2+45\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2-18\lambda+45) = 0$$

$$\Rightarrow \lambda(\lambda^2-15\lambda-3\lambda+45) = 0 \Rightarrow \lambda[\lambda(\lambda-15)-3(\lambda-15)] = 0$$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

$\therefore$ Eigen values are $\lambda = 0, 3, 15$.

To find eigen vectors find $[A-\lambda I]X = 0$

For $\lambda = 0$,

$$[A-\lambda I]X = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 \quad 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

From eqns (1) & (2)

$$\begin{array}{ccc} x_1 & x_2 & x_3 \\ 8 & -6 & 2 \\ -6 & 7 & -4 \end{array}$$

$$\frac{x_1}{24-14} = \frac{-x_2}{-32+12} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20} = k \text{ (let)}$$

$x_1 = 10k$, $x_2 = 20k$, $x_3 = 20k$

$\therefore$ Eigen vector for $\lambda = 0$ is $\begin{bmatrix} 10k \\ 20k \\ 20k \end{bmatrix}$ or by taking $k = \frac{1}{10}$, we can find eigen vector $[1, 2, 2]^T$ or its scalar multiples will be eigen vector for it.

For $\lambda = 3$,

$[A - 3I]X = 0$

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 5x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 4x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 0x_3 &= 0 \end{aligned}\right\} \qquad \frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16} = k \text{ (let)}$$

$\therefore$ Eigenvector is $\begin{bmatrix} 16k \\ 8k \\ -16k \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

For $\lambda = 15$,

$$[A - 15I]X = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -7x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 - 8x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 - 12x_3 &= 0 \end{aligned}\right\} \qquad \frac{x_1}{40} = \frac{-y}{40} = \frac{z}{20} = k$$

$\therefore$ Eigenvector $= \begin{bmatrix} 40k \\ -40k \\ 20k \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Q. Find the sum of the characteristic roots (Eigenvalues) of the matrix A^2, given that

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 2 & 5 \end{bmatrix}$$

Soln. The ch. eqn for given the matrix is $|A - \lambda I| = 0$

or $\begin{vmatrix} 3-\lambda & 0 & 0 \\ 8 & 4-\lambda & 0 \\ 6 & 2 & 5-\lambda \end{vmatrix} = 0$ whose roots as ch. roots are $\lambda = 3, 4$ & 5 (or eigen value).

we know that the ch. root of the matrix A^2 are λ^2 as $9, 16, 25$ whose sum $= 9 + 16 + 25 = 50$.

Q. If matrix A is defined as $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{bmatrix}$, then find the eigen values of A^2.

Soln. $\rightarrow$ The ch. eqn of A is $|A - \lambda I| = 0$

or $\begin{vmatrix} -1-\lambda & 0 & 0 \\ 2 & -3-\lambda & 0 \\ 1 & 4 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)(-3-\lambda)(2-\lambda) = 0$

Hence $\lambda = -1$, $\lambda = -3$ and $\lambda = 2$ are eigen values of A.
we know that the eigen values of A^2 are the squares of the eigen values of A, i.e, 1, 9, 4.

Q. If the eigen values of matrix A are 2, 3, 6 then write the eigen values of A^T, A^{-1}, A^m $(m \in N)$, adj A and kA.

Soln. $\rightarrow$ Eigen values of A are 2, 3, and 6 and since eigen values of A^T are same as these of A, hence eigen values of A^T are 2, 3 and 6.

Eigen value of A^{-1} are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{6}$.

" " of A^m are $2^m, 3^m$ and 6^m $(m \in N)$.

Further eigen value of adj(A) are $\frac{1}{2}(2 \times 3 \times 6)$, $\frac{1}{3}(2 \times 3 \times 6)$ and $\frac{1}{6}(2 \times 3 \times 6)$
i.e, 18, 12 and 6.

Also the eigen values of kA are 2k, 3k and 6k where k is const.

Q. If the eigen value of A are 1, 1, 1, find the eigen values of $A^2 + 2A + 3I$.

Soln. Eigen values of A^2 are $1^2, 1^2, 1^2$,
" " " 2A are 2, 2, 2
3I are 3, 3, 3.

$\therefore$ Eigen value of $A^2 + 2A + 3I = (1,1,1) + (2,2,2) + (3,3,3)$
$= 6, 6, 6$.

Imp. Property

If A is a symmetric matrix the eigen vectors of the eqn $(A-\lambda I)X=0$ are pairwise orthogonal.

If $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$ are two eigen vectors of a symmetric matrix A then $x_1x_1' + x_2x' + \cdots x_nx' = 0$.

Some Useful deductions

If λ is a characteristic value of a square matrix A, then

(i) $\lambda + k$ is a ch. value of the matrix $A + kI$

(ii) $k\lambda$ is a ch. value of kA.

(iii) $\frac{1}{\lambda}$ is a ch. value of A^{-1}. In otherwords $\frac{|A|}{\lambda}$ is the eigen value of the matrix adj(A).

(iv) λ^2 is a ch. value of A^2. In general λ^k is an eigen value of A^k.

(ch. value is an eigen value)

Symmetric matrix → A $n \times n$ matrix is symmetric matrix if $A^T = A$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}, \quad A^T = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$$

Here $A = A^T$

Skew-Symmetric matrix → A square matrix is skew symmetric matrix if $A^T = -A$

$$B = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix} \Rightarrow B^T \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & -2 \\ 5 & 2 & 0 \end{bmatrix} = -B$$

Inner Product Spaces.

Def^n. Let V be a vector space over R. An inner product on V is a function that assigns to each pair of vectors $u, v \in V$ a real number denoted by $\langle u, v\rangle$ satisfying:

(i) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$

(ii) $\langle u, v\rangle = \langle v, u\rangle \; \forall \; u, v \in V$

(iii) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \; \forall \; u, v, w \in V$

(iv) $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle \; \forall \; \alpha \in R, \; u, v \in V$

Properties (iii) & (iv) can be combined to give an equivalent property:

(v) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle \; \forall \; u, v, w \in V, \alpha, \beta \in R$.

Ex. We know $R^n = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) : \alpha_i \in R\}$ is a vector space over R.

Define $\langle u, v\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$

where $u = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $v = (\beta_1, \beta_2, \cdots, \beta_n) \in R^n$. Then $\langle , \rangle$ is an inner product in R^n and is called standard inner product on R^n.

Soln. Now $\langle u, u\rangle = \alpha_1\alpha_1 + \alpha_2\alpha_2 + \cdots \alpha_n\alpha_n$

$$= \alpha_1^2 + \alpha_2^2 + \cdots \alpha_n^2 > 0 \; \forall \; u = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in R^n.$$

Also $\langle u, u\rangle = 0$

iff $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 0$

iff $\alpha_1^2 = 0, \alpha_2^2 = 0, \cdots, \alpha_n^2 = 0$

iff $\alpha_1 = 0, \alpha_2 = 0, \cdots, \alpha_n = 0$

iff $u = (0, 0, \cdots, 0) = 0$

Now $\langle u, v\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$

$$= \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_n\alpha_n$$

$$= \langle v, u\rangle$$

$\forall \; u = (\alpha_1, \alpha_2, \cdots, \alpha_n), \; v = (\beta_1, \beta_2, \cdots, \beta_n) \in R^n$

Further,

$$\langle \alpha u + \beta v, w\rangle = \langle (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2, \cdots, \alpha\alpha_n + \beta\beta_n), (\gamma_1, \gamma_2, \cdots, \gamma_n)\rangle$$

$$= (\alpha\alpha_1 + \beta\beta_1)\gamma_1 + (\alpha\alpha_2 + \beta\beta_2)\gamma_2 + \cdots + (\alpha\alpha_n + \beta\beta_n)\gamma_n$$

$$= \alpha\alpha_1\gamma_1 + \alpha\alpha_2\gamma_2 + \cdots + \alpha\alpha_n\gamma_n + \beta\beta_1\gamma_1 + \beta\beta_2\gamma_2 + \cdots + \beta\beta_n\gamma_n$$

$$= \alpha(\alpha_1\gamma_1 + \alpha_2\gamma_2 + \cdots + \alpha_n\gamma_n) + \beta(\beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_n\gamma_n)$$

$$= \alpha\langle u, w\rangle + \beta\langle v, w\rangle$$

$\forall \; u = (\alpha_1, \alpha_2, \cdots, \alpha_n), \; v = (\beta_1, \beta_2, \cdots, \beta_n), \; w = (\gamma_1, \gamma_2, \cdots, \gamma_n) \in R^n$

$\alpha, \beta \in R$

$\therefore \langle , \rangle$ is an inner product on R^n. It is called standard inner product on R^n.

Remark.

Standard inner product on R^n is also called dot product and is denoted as

$$u \cdot v = \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$$

where $u = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $v = (\beta_1, \beta_2, \cdots, \beta_n) \in R^n$.

In this notation

(i) $\langle u, u\rangle = u \cdot u \geq 0$, $u \cdot u = 0$ iff $u = 0$

(ii) $u \cdot v = v \cdot u \ \forall u, v \in R^n$

(iii) $(u+v) \cdot w = u \cdot w + v \cdot w \ \forall u, v, w \in R^n$.

(iv) $(\alpha u) \cdot v = \alpha(u \cdot v) \ \forall u, v \in R^n$.

(v) $(\alpha u + \beta v) \cdot w = \alpha(u \cdot w) + \beta(v \cdot w) \ \forall u, v, w \in R^n, \alpha, \beta \in R$.

Ex: Let $u = (\alpha_1, \alpha_2)$, $v = (\beta_1, \beta_2)$ be vectors in R^2. Define

$$\langle u, v\rangle = \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + 3\alpha_2\beta_2$$

Then this gives an inner product on R^2.

Soln: Now,

$$\begin{aligned}\langle u, u\rangle &= \alpha_1\alpha_1 - \alpha_2\alpha_1 - \alpha_1\alpha_2 + 3\alpha_2\alpha_2 \\ &= \alpha_1^2 - 2\alpha_1\alpha_2 + 3\alpha_2^2 \\ &= (\alpha_1 - \alpha_2)^2 + 2\alpha_2^2 \geq 0\end{aligned}$$

Also

$$\begin{aligned}\langle u, u\rangle = 0 &\text{ iff } (\alpha_1 - \alpha_2)^2 + 2\alpha_2^2 = 0 \\ &\text{ iff } \alpha_1 = \alpha_2 \text{ and } \alpha_2 = 0 \\ &\text{ iff } \alpha_1 = \alpha_2 = 0 \\ &\text{ iff } u = (0,0) = 0 \ \forall u = (\alpha_1, \alpha_2) \in R^2\end{aligned}$$

Also,

$$\begin{aligned}\langle u, v\rangle &= \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + 3\alpha_2\beta_2 \\ &= \beta_1\alpha_1 - \beta_1\alpha_2 - \beta_2\alpha_1 + 3\beta_2\alpha_2 \\ &= \beta_1\alpha_1 - \beta_2\alpha_1 - \beta_1\alpha_2 + 3\beta_2\alpha_2 \\ &= \langle v, u\rangle \ \forall u = (\alpha_1, \alpha_2), v = (\beta_1, \beta_2) \in R^2\end{aligned}$$

Further,

$$\begin{aligned}\langle \alpha u + \beta v, w\rangle &= \langle (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2), (\lambda_1, \lambda_2)\rangle \\ &= (\alpha\alpha_1 + \beta\beta_1)\lambda_1 - (\alpha\alpha_2 + \beta\beta_2)\lambda_1 - (\alpha\alpha_1 + \beta\beta_1)\lambda_2 \\ &\quad + 3(\alpha\alpha_2 + \beta\beta_2)\lambda_2 \\ &= \alpha\alpha_1\lambda_1 + \beta\beta_1\lambda_1 - \alpha\alpha_2\lambda_1 - \beta\beta_2\lambda_1 - \alpha\alpha_1\lambda_2 - \beta\beta_1\lambda_2 \\ &\quad + 3\alpha\alpha_2\lambda_2 + 3\beta\beta_2\lambda_2 \\ &= \alpha\alpha_1\lambda_1 - \alpha\alpha_2\lambda_1 - \alpha\alpha_1\lambda_2 + 3\alpha\alpha_2\lambda_2 + \beta\beta_1\lambda_1 - \beta\beta_2\lambda_1 \\ &\quad - \beta\beta_1\lambda_2 + 3\beta\beta_2\lambda_2\end{aligned}$$

$$= \alpha(\alpha_1 r_1 - \alpha_2 r_1 + 3\alpha_2 r_2) + \beta(\beta_1 r_1 - \beta_2 r_1 - \beta_1 r_2 + 3\beta_2 r_2)$$

$$= \alpha\langle u, w\rangle + \beta\langle v, w\rangle$$

$\forall\, u = (\alpha_1, \alpha_2),\ v = (\beta_1, \beta_2),\ w = (r_1, r_2) \in R^2$

$\alpha, \beta \in R$.

Hence this defines a inner product on R^2.

Ex. Let $u = (1, 3),\ v = (2, 1) \in R^2$

Find $w \in R^2$ such that $\langle w, u\rangle = 3,\ \langle w, v\rangle = -1$

where $\langle , \rangle$ is the standard inner product on R^2.

Soln: Standard inner product on R^2 is given by

$$\langle (\alpha_1, \alpha_2), (\beta_1, \beta_2)\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 \quad \forall\, (\alpha_1, \alpha_2), (\beta_1, \beta_2) \in R^2$$

Let $w = (\alpha, \beta) \in R^2$

Now, $\langle (\alpha, \beta), (1, 3)\rangle = 3,\ \langle (\alpha, \beta), (2, 1)\rangle = -1$

$$\Rightarrow \alpha + 3\beta = 3,\quad 2\alpha + \beta = -1$$

In matrix form

$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$$

$$\therefore \beta = \frac{7}{5},\quad \alpha + 3\times\frac{7}{5} = 3,\quad \alpha = 3 - \frac{21}{5} = -\frac{6}{5}$$

$$w = \left(-\frac{6}{5}, \frac{7}{5}\right).$$

Ex. Let $u \in R^2$. Show that

$$u = \langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2, \text{ where } e_1 = (1, 0),\ e_2 = (0, 1)$$

Soln: Now let $u = (\alpha, \beta) \in R^2$, then

$$\langle u, e_1\rangle e_1 + \langle u, e_2\rangle e_2$$

$$= \langle (\alpha, \beta), (1, 0)\rangle e_1 + \langle (\alpha, \beta), (0, 1)\rangle e_2$$

$$= (\alpha\cdot 1 + \beta\cdot 0)e_1 + (\alpha\cdot 0 + \beta\cdot 1)e_2$$

$$= \alpha(1, 0) + \beta(0, 1) = (\alpha, 0) + (0, \beta)$$

$$= (\alpha, \beta) = u.$$

Ex:

Let $u = (1,1,1)$, $v = (1,2,3)$, $w = (1,3,4) \in R^3$. Find $u_1 = (\alpha, \beta, \lambda) \in R^3$ such that

$$\langle u, u_1 \rangle = 7, \langle v, u_1 \rangle = 16, \langle w, u_1 \rangle = 22$$

where $\langle , \rangle$ is standard inner product on R^3.

Soln: Now,

$$\langle u, u_1 \rangle = 7 \Rightarrow \alpha + \beta + \lambda = 7$$

$$\langle v, u_1 \rangle = 16 \Rightarrow \alpha + 2\beta + 3\lambda = 16$$

$$\langle w, u_1 \rangle = 22 \Rightarrow \alpha + 3\beta + 4\lambda = 22$$

In matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \lambda \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -3 \end{bmatrix}$$

$\therefore \lambda = 3$, $\beta + 2\lambda = 9 \Rightarrow \beta = 3$

$\alpha + \beta + \lambda = 7 \Rightarrow \alpha = 1$

$\therefore u_1 = (1, 3, 3)$

Length or Norm of a Vector

Let V be an inner product space and $u \in V$. The norm (or length) of u denoted by $\|u\|$ is defined as

$$\|u\| = \sqrt{(u,u)} \quad \text{i.e, } \|u\|^2 = \langle u, u\rangle$$

Since $\langle u, u\rangle \geq 0$, so $\|\cdot\|$ is well defined

If $V = R^n$ and $(\,,\,)$ is standard inner product on R^n, then

$$\|u\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2} \quad \forall\, u = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n.$$

If $u, v \in R^n$. Then distance between u and v is defined as $\|u - v\|$

Let $u = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $v = (\beta_1, \beta_2, \ldots, \beta_n) \in R^n$.

Then $u - v = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \ldots, \alpha_n - \beta_n)$

$$\therefore \|u - v\| = \sqrt{(\alpha_1 - \beta_1)^2 + (\alpha_2 - \beta_2)^2 + \cdots + (\alpha_n - \beta_n)^2}$$

Ex: Let $u = (2, 3, 2, -1)$ and $v = (3, 2, 1, 3)$

Then $\|u\| = \sqrt{2^2 + 3^2 + 2^2 + (-1)^2}$

$$= \sqrt{4 + 9 + 4 + 1} = \sqrt{18}$$

$$\|v\| = \sqrt{3^2 + 2^2 + 1^2 + 3^2}$$

$$= \sqrt{9 + 4 + 1 + 9} = \sqrt{23}$$

$$\|u - v\| = \|(-1, 1, 1, -4)\|$$

$$= \sqrt{(-1)^2 + 1^2 + 1^2 + (-4)^2} = \sqrt{1 + 1 + 1 + 16} = \sqrt{19}.$$

Ex. Find the norm of the vector $u = (2, 3, 6) \in R^3$

Show that $\frac{u}{\|u\|}$ is of unit length.

Soln. $\|u\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

Now, $\frac{u}{\|u\|} = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$

$$\left\|\frac{u}{\|u\|}\right\| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = \sqrt{\frac{49}{49}} = 1.$$

$\therefore \frac{u}{\|u\|}$ is of unit length.

Cauchy - Schwarz Inequality

If u & v are in vectors in R^n, then

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

$|\cdot|$ on the L.H.S. stands for absolute value of real number.

Triangle Inequality :- If u & v are vectors in R^n, then

$$\|u+v\| \leq \|u\| + \|v\|$$

Th: $|\langle u, v\rangle| = \|u\|\|v\|$ if and only if u & v are linearly dependent

Angle between two vectors

If α is a real no in interval $[-1, 1]$ there exists unique θ in interval $[0, \pi]$ such that $\cos\theta = \alpha$

$$\therefore \cos\theta = \frac{\langle u, v\rangle}{\|u\|\|v\|}$$

The angle θ is called the angle between u & v.

Orthogonal vectors

Def^n: Two non-zero vector u & v in inner product space V are said to be orthogonal if $\langle u, v\rangle = 0$. We assume $0 \in V$ is orthogonal to every $u \in V$

* Vectors in R^n are orthogonal if $\cos\theta = 0$
* u & v are parallel if $|\langle u, v\rangle| = \|u\|\|v\|$
* $\therefore$ vectors are parallel if $\cos\theta = \pm 1$.
* vectors are in same direction if $\cos\theta = 1$

Ex: Which of the foll. vectors are orthogonal

$u_1 = (4, 2, 6, -8)$ $\quad u_2 = (-2, 3, -1, -1)$

$u_3 = (-2, -1, -3, 4)$ $\quad u_4 = (1, 0, 0, 2)$

Soln: $\langle u_1, u_2\rangle = 4\times(-2) + 2\times 3 + 6\times(-1) + (-8)\times(-1)$

$= -8 + 6 - 6 + 8 = 0$

$\langle u_1, u_3\rangle = 4\times(-2) + 2\times(-1) + 6\times(-3) + (-8)\times 4$

$= -8 - 2 - 18 - 32 = -60$

$\langle u_1, u_4\rangle = 4\times 1 + 2\times 0 + 6\times 0 + (-8)\times 2 = 4 + 0 + 0 - 16 = -12$

$\langle u_2, u_3\rangle = (-2)\times(-2) + 3\times(-1) + (-1)\times(-3) + (-1)\times(4) = 4 - 3 + 3 - 4 = 0$

$\langle u_2, u_4\rangle = (-2)\times 1 + 3\times 0 - 1\times 0 + (-1)\times 2 = -2+0-0-2 = -4$

$\langle u_3, u_4\rangle = (-2)\times 1 + (-1)\times 0 + (-3)\times 0 + 4\times 2 = -2+0+0+8$
$= 6.$

$\therefore$ u_1 and u_2 ~~$\langle u_1, u_2\rangle$~~ and u_2 and u_3 ~~$\langle u_2, u_3\rangle$~~ are orthogonal.

i.e, u_2 is orthogonal to u_1 and u_3.

Orthogonal and orthonormal Sets

Def^n → Let $S = \{u_1, u_2, \ldots, u_k\}$ be set of k-vectors in V. S is said to be orthogonal set if $u_i \cdot u_j = \langle u_i, u_j\rangle = 0 \ \forall\ i \neq j$

The set S is said to be orthonormal set if

$$u_i \cdot u_j = \langle u_i, u_j\rangle = 0 \ \forall\ i \neq j$$
$$= 1 \text{ if } i = j$$

Def^n orthonormal Basis →
If an orthonormal set S is a basis of inner product space V then the set S is called an orthonormal basis of V.

Ex. Let $S = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$
S is orthonormal basis of R^3.

Soln. we know that S is basis of R^3.

Now, $\langle e_1, e_2\rangle = 1\times 0 + 0\times 1 + 0\times 0 = 0$

$\langle e_2, e_3\rangle = 0\times 0 + 1\times 0 + 0\times 1 = 0$

$\langle e_3, e_1\rangle = 0\times 1 + 0\times 0 + 1\times 0 = 0$

Also, $\langle e_1, e_1\rangle = 1\times 1 + 0\times 0 + 0\times 0 = 1$

$\langle e_2, e_2\rangle = 0\times 0 + 1\times 1 + 0\times 0 = 1$

$\langle e_3, e_3\rangle = 0\times 0 + 0\times 0 + 1\times 1 = 1$

$e_i \cdot e_j = \langle e_i, e_j\rangle = 0 \quad \forall\ i \neq j$
$= 1 \text{ if } i = j$

$\therefore$ S is orthonormal basis of R^3.

Th. If $S = \{u_1, u_2, u_3\}$ is an orthogonal set of non-zero vectors then S is linearly independent.

Ex. Prove that

$$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right), (0, 1, 0) \right\}$$

is an orthonormal basis of R^3.

Soln. Let $u_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$, $u_2 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, $u_3 = (0, 1, 0)$

and

$$u_1 \cdot u_2 = \langle u_1, u_2 \rangle = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times 0 + \frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

$$u_1 \cdot u_3 = \langle u_1, u_3 \rangle = \frac{1}{\sqrt{2}} \times 0 + 0 \times 1 + \frac{1}{\sqrt{2}} \times 0 = 0$$

$$u_2 \cdot u_3 = \langle u_2 \cdot u_3 \rangle = \frac{1}{\sqrt{2}} \times 0 + 0 \times 1 + -\frac{1}{\sqrt{2}} \times 0 = 0$$

Also

$$\|u_1\|^2 = \langle u_1, u_1 \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

$$\|u_2\|^2 = \langle u_2, u_2 \rangle = \left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$\|u_3\|^2 = \langle u_3, u_3 \rangle = 0^2 + 1^2 + 0^2 = 1$$

$$\therefore u_i, u_j = \langle u_i, u_j \rangle = 0 \text{ if } i \neq j$$
$$= 1 \text{ if } i = j$$

$\therefore \langle u_1, u_2, u_3 \rangle$ is an orthonormal set in R^3.

Also every three orthonormal vectors being orthogonal vectors are linearly independent. Therefore (u_1, u_2, u_3) is linearly independent in R^3 and hence basis of R^3.

Gram Schmidt Process -

Let W be subspace of R^n of dimension k. Let $S = \{u_1, u_2, \ldots, u_k\}$ be basis of W. Define $v_1 = u_1$

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right) v_1$$

$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_3 \cdot v_2}{\|v_2\|^2}\right) v_2$$

$$\cdots\cdots$$

$$v_k = u_k - \left(\frac{u_k \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_k \cdot v_2}{\|v_2\|^2}\right) v_2 \cdots \left(\frac{u_k \cdot v_{k-1}}{\|v_{k-1}\|^2}\right) v_{k-1}$$

Then $B = \{v_1, v_2, \ldots, v_k\}$ is an orthogonal basis of W.

Ex. Let $S = \{(1,2,3), (1,1,1), (1,0,1)\}$ be a basis of R^3. Create a orthogonal basis of R^3 by Gram-Schmidt Process. (9)

Soln. Let $u_1 = (1,2,3)$, $u_2 = (1,1,1)$, $u_3 = (1,0,1)$

Now let $v_1 = u_1 = (1,2,3)$

$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right) v_1$$

$$= (1,1,1) - \left(\frac{1\times1 + 2\times1 + 3\times1}{1^2+2^2+3^2}\right)(1,2,3)$$

$$= (1,1,1) - \frac{6}{14}(1,2,3)$$

$$= (1,1,1) - \left(\frac{3}{7}, \frac{6}{7}, \frac{9}{7}\right) = \left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right)$$

$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_3 \cdot v_2}{\|v_2\|^2}\right) v_2$$

$$= (1,0,1) - \left(\frac{1\times1 + 0\times2 + 1\times3}{1^2+2^2+3^2}\right)(1,2,3)$$

$$- \left(\frac{1\times\frac{4}{7} + 0\times\frac{1}{7} + 1\times\frac{-2}{7}}{\left(\frac{4}{7}\right)^2 + \left(\frac{1}{7}\right)^2 + \left(\frac{-2}{7}\right)^2}\right)\left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right)$$

$$= (1,0,1) - \frac{4}{14}(1,2,3) - \left(\frac{\frac{2}{7}}{\frac{21}{49}}\right)\left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right)$$

$$= (1,0,1) - \left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right) - \left(\frac{8}{21}, \frac{2}{21}, \frac{-4}{21}\right)$$

$$= \left(1 - \frac{2}{7} - \frac{8}{21}, 0 - \frac{4}{7} - \frac{2}{21}, 1 - \frac{6}{7} + \frac{4}{21}\right)$$

$$= \left(\frac{21-6-8}{21}, \frac{-12-2}{21}, \frac{21-18+4}{21}\right)$$

$$= \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

Now consider

$$B = \left\{v_1 = (1,2,3), v_2 = \left(\frac{4}{7}, \frac{1}{7}, \frac{-2}{7}\right), v_3 = \left(\frac{1}{3}, \frac{-2}{3}, \frac{1}{3}\right)\right\}$$

$v_2 \cdot v_1 = 1\times\frac{4}{7} + 2\times\frac{1}{7} + 3\times\frac{-2}{7} = 0$

$v_2 \cdot v_3 = \frac{4}{7}\times\frac{1}{3} + \frac{1}{7}\times\frac{-2}{3} - \frac{2}{7}\times\frac{1}{3} = 0$

$v_2 \cdot v_1 = \frac{1}{3}\times1 - \frac{2}{3}\times2 + \frac{1}{3}\times3 = 0$

$\therefore$ B is an orthogonal basis of R^3

Ex. Show that

$S = \{u_1 = (2,-1,3),\ u_2 = (-1,1,1),\ u_3 = (-4,-5,1)\}$

is orthogonal basis of R^3. Find an orthonormal basis of R^3.

Soln. $u_1 \cdot u_2 = 2\times -1 + (-1)\times 1 + 3\times 1 = -2 - 1 + 3 = 0$

$u_2 \cdot u_3 = (-1)\times(-4) + (1)\times(-5) + 1\times 1 = 4 - 5 + 1 = 0$

$u_3 \cdot u_1 = -4\times 2 + (-5)\times(-1) + 1\times 3 = -8 + 5 + 3 = 0$

$\therefore$ S is orthogonal basis of R^3.

Also $\|u_1\| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{4+1+9} = \sqrt{14}$

$\|u_2\| = \sqrt{(-1)^2+1^2+1^2} = \sqrt{1+1+1} = \sqrt{3}$

$\|u_3\| = \sqrt{(-4)^2+(-5)^2+1^2} = \sqrt{16+25+1} = \sqrt{42}$

$\therefore \left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\} = \left\{\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-4}{\sqrt{42}}, \frac{-5}{\sqrt{42}}, \frac{1}{\sqrt{42}}\right)\right\}$

is orthonormal basis ~~for~~ R^3.

Orthogonal complements

Def^n. Let W be subspace of R^n. Define $W^\perp = \{u \in R^n \mid u\cdot w = 0\ \forall w \in W\}$

$W^\perp$ is known as orthogonal complement of W.

Th: Let W be a subspace of R^n. $u \in W^\perp$ iff u is orthogonal to every vector in spanning set for W.

Ex. Let $W = \{(a,0,0) : a \in R\}$ we know W is subspace of R^3.

Find $W^\perp$. What is dimension of $W^\perp$?

Soln. Let $(x,y,z) \in W^\perp$ then $(x,y,z)\cdot(a,0,0) = 0\ \forall a \in R$

$\Rightarrow ax + 0y + 0z = 0\ \forall a \in R$

$\Rightarrow ax = 0\ \forall a \in R$

$\Rightarrow x = 0$

$\therefore W^\perp = \{(0,y,z) : y, z \in R\}$

Basis of $W^\perp = \{(0,1,0), (0,0,1)\}$

$\dim W^\perp = 2$. Also basis of $W = \{(1,0,0)\}$, $\dim W = 1$

$\therefore \dim W + \dim W^\perp = 1+2 = 3 = \dim$

Orthogonal Projection

Def^n: Let W be a subspace of R^n having dimension k. Let $B = \{u_1, u_2, \ldots, u_k\}$ be an orthonormal basis of W. Let $v \in R^n$.

The orthogonal projection of v onto W

$$= \text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2 + \cdots + (v \cdot u_k) u_k$$

If $W = \{0\}$ then $\text{proj}_W v = 0$

Ex: Let $B = \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \right\}$ be orthonormal set in R^3.

Let $W = \text{span} B$. Now B is orthonormal basis of W. Let $v = (2, 3, 4) \in R^3$. Find orthogonal projection of v onto W i.e, $\text{proj}_W v$. ~~If you take B_1 to be another orthonormal basis of W, does $\text{proj}_W v$ remains same?~~

Soln: Now $\text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$

$$= \left[(2,3,4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right] \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$+ \left[(2,3,4) \cdot \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)\right] \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$= \left(\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + \left(\frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$= (3, 0, 3) + (-1, 0, 1) = (2, 0, 4)$$

Now consider $\{(2\sqrt{2}, 0, 0), (0, 0, 3\sqrt{2})\}$

$$(2\sqrt{2}, 0, 0) = 2\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$$(0, 0, 3\sqrt{2}) = 3\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) - 3\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$\therefore \{(2\sqrt{2}, 0, 0), (0, 0, 3\sqrt{2})\}$ is an orthogonal subset of W,

Hence $B_1 = \{(1, 0, 0), (0, 0, 1)\}$ is orthonormal basis of W.

Ex. Let $B = \{ (\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}), (\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \}$ be an orthonormal subset of R^3. Let $W = \text{span } B$. Now B is an orthonormal basis of W. Express $v = (-3, 4, 5)$ as $w_1 + w_2$.

Soln.

Now $w_1 = \text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$

$$= \left[(-3, 4, 5) \cdot \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) \right] \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$$

$$+ \left[(-3, 4, 5), \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) \right] \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$= \frac{5}{\sqrt{14}} \left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) + \frac{12}{\sqrt{3}} \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$= \left(\frac{10}{14}, \frac{-5}{14}, \frac{15}{14} \right) + \left(\frac{-12}{3}, \frac{12}{3}, \frac{12}{3} \right)$$

$$= \left(\frac{10}{14}, -4, \frac{-5}{14} + 4, \frac{15}{14} + 4 \right) = \left(\frac{-46}{14}, \frac{51}{14}, \frac{71}{14} \right) \in W$$

$w_2 = v - \text{proj}_W v$

$$= (-3, 4, 5) - \left(\frac{-46}{14}, \frac{15}{14}, \frac{71}{14} \right) = \left(\frac{4}{14}, \frac{5}{14}, \frac{-1}{14} \right)$$

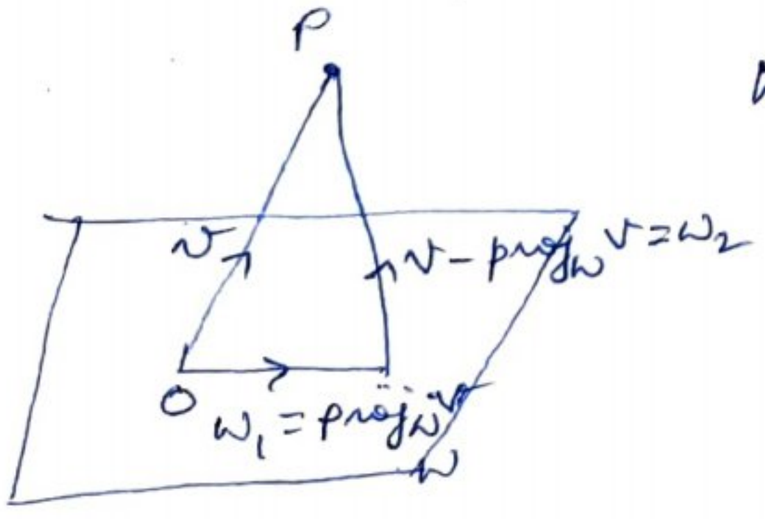
Now w_2 is orthogonal to both u_1 & u_2

$\therefore w_2 \in W^\perp$

$\therefore v = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$.

Now $v = \text{proj}_W v + v - \text{proj}_W v$.

The min^m distance of P from W $= \| v - \text{proj}_W v \|$



$$W = \| v - \text{proj}_W v \| = \| w_2 \| = \left\| \left(\frac{4}{14}, \frac{5}{14}, \frac{-1}{14} \right) \right\|$$

$$= \sqrt{\frac{42}{14 \times 14}} = \sqrt{\frac{3}{14}}.$$

Ex. Find the orthogonal projection of $v = \{-1, 4, 3\}$ onto the subspace W of R^3 spanned by the orthogonal vectors $v_1 = \{1, 1, 0\}$ and $v_2 = \{-1, 1, 0\}$

Soln: The subspace W of R^3 is defined to be $W = \text{span}(\{(1,1,0), (-1,1,0)\})$

$$= \text{span}\left(\left\{\tfrac{1}{\sqrt{2}}(1,1,0), \tfrac{1}{\sqrt{2}}(-1,1,0)\right\}\right)$$

where we have normalized the vectors $(1,1,0)$ and $(-1,1,0)$ to get an orthonormal basis for W. Hence by defn of $\text{proj}_W v$, we have

$$\text{proj}_W v = (v \cdot u_1)\, u_1 + (v \cdot u_2)\, u_2$$

where $u_1 = \frac{1}{\sqrt{2}}(1,1,0)$, and $u_2 = \frac{1}{\sqrt{2}}(-1,1,0)$

$$\text{proj}_W v = \left((-1,4,3) \cdot \left(\tfrac{1}{\sqrt{2}}(1,1,0)\right)\right) \cdot \left(\tfrac{1}{\sqrt{2}}(1,1,0)\right)$$
$$+ \left((-1,4,3) \cdot \left(\tfrac{1}{\sqrt{2}}(-1,1,0)\right)\right)\left(\tfrac{1}{\sqrt{2}}(-1,1,0)\right)$$

$$= \frac{3}{\sqrt{2}}(1,1,0) + \frac{5}{2}(-1,1,0) = (-1,4,0).$$