Algebra: → composition

By composition, we mean the concept of two object coming together to form a new one.

for example, adding two numbers, multiplying 2 numbers or composing real valued single variable functions.

* The concept of unity. The number 1.

$\mathbb{N} := \{1, 2, 3 \ldots\}$. $\mathbb{N}$ comes equipped with two natural operations $+$ and $\times$.

$\mathbb{Z} = \{\ldots -2, -1, 0, 1, 2, \ldots\}$, the integers

$\mathbb{Z}$ also comes with $+$ and $\times$.

Addition on $\mathbb{Z}$ has particularly good properties

eg. additive inverses exist.

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}$$

$\mathbb{Q}$ are also equipped with $+$ and $\times$.

Non-zero elements have multiplicative inverses.

For ex. $\quad a+b = b+a \quad \forall a, b \in \mathbb{Q}$

or $\quad a*(b+c) = a*b + a*c \quad \forall a, b, c \in \mathbb{Q}$

* $(\mathbb{Z}, +) \to$ Groups

$(\mathbb{Z}, +, \times) \to$ Rings

$(\mathbb{Q}, +, \times) \to$ Fields

In linear algebra the analogous idea is
$(\mathbb{R}^n, +, \text{scalar mult}^n) \to$ V.S over $\mathbb{R}$.

Functions:

$$f : S \to T$$
$$x \to f(x)$$

1. $S = T = \mathbb{N}$
$$f : \mathbb{N} \to \mathbb{N}$$
$$a \to a^2$$

2. $S = \mathbb{Z} \times \mathbb{Z}$, $T = \mathbb{Z}$
$$f : \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$$
$$(a,b) \to a+b$$

Composition:

$f : R \to S$, $g : S \to T$
we compose them to give a new f^n
$$g \circ f : R \to T$$

This is only possible if the domain of g is the same as co-domain of f.

$*$ let f be a map from S to T i.e $f : S \to T$

1. A left inverse for f is a map
$g : T \to S$ such that
$g \circ f = Id_S$

2. A right inverse for f is a map
$g : T \to S$ such that
$f \circ g = Id_T$

Algebra is the general study of laws of composition or binary operations.

Defn: let S be a set. Then a binary operation $*$ on S is a function
$$* : S \times S \to S$$

We ~~denotes~~ often write $a * b$ in place of $*(a,b)$ for $a, b \in S$

Ex: 1. let S be either $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or $M_n(\mathbb{R})$.
Then both $+$ and $\times$ define binary operations

2. If S is $\mathbb{R}$ or $\mathbb{Z}$, then $x * y = x^2 + y + 1$ defines a B.O on S. Note that this is not associative.

3. If $S = \{-1, 0, 1, 2, 3 \_\_\}$ then addition does not define a binary operation of S
$\because$ $-1 + (-1) = -2$ is not in S

Basic defns:

Defn: let $n \in G$ be a set. A B.O is a map of sets

$$*: G \times G \to G$$

Any B.O on G gives a way of combining elements

Fundamental defn: A group is a n.e set G, together with B.O $*$ such that the following hold

1. closure property : $a*b \in G \ \forall \ a,b \in G$
2. Associativity : $a*(b*c) = (a*b)*c \ \forall a,b,c \in G$.
3. Additive identity: $\exists \ e \in G$ such that $a*e = e*a = a \ \forall a \in G$.
4. Additive inverse: For every $a \in G$, $\exists \ a^{-1} \in G$ such that $a*a^{-1} = a^{-1}*a = e$

Examples: We have seen 5 different examples $(\mathbb{Z},+)$, $(\mathbb{Q},+)$, $(\mathbb{Q}-\{0\},\times)$, $(\mathbb{Z}/m\mathbb{Z},+)$ & $(\mathbb{Z}/m\mathbb{Z}-\{[0]\},\times)$ if m is prime.

Note: $(\mathbb{Z},\times)$ is not a group. $(\mathbb{N},+)$ is not a group

These are examples of groups which are both finite and infinite.

Informally, vectors are objects that can be added together and multiplied by a scalar and they remain objects of the same type.

Definition:

A group $(G,*)$ is called Abelian if it also satisfies

$$a*b = b*a \quad \forall \ a,b \in G$$

i.e the given binary operation is commutative.

* A group which is not abelian is called non-abelian.

Q: Show that the set $\mathbb{Z}$ of all integers ---- $-4,-3,-2,-1,0,1,2,3,4,$ ---- is a group w.r.t operation of addition of integers.

Soln: closure property: We know that the sum of two integers is also an integer i.e $a+b \in \mathbb{Z} \ \forall \ a,b \in \mathbb{Z}$

$\therefore \mathbb{Z}$ is closed w.r.t addition.

Associativity: We know that addition of integers is an associative composition.

$\therefore \ a+(b+c) = (a+b)+c \quad \forall \ a,b,c \in \mathbb{Z}$

Existence of identity: The number $0 \in \mathbb{Z}$. Also we have $0+a = a = a+0 \ \forall \ a \in \mathbb{Z}$. $\therefore$ 0 is the additive identity.

Existence of inverse: If $a \in \mathbb{Z}$, then $-a \in \mathbb{Z}$. Also we have $(-a)+a = 0 = a+(-a)$.

$\therefore$ Every integer possesses additive inverse.

Also in $(\mathbb{Z},+)$, addition of integers is a commutative composition.

$\therefore (\mathbb{Z},+)$ is an abelian group.

$\therefore \mathbb{Z}$ contains infinite no. of elements.

$\therefore (\mathbb{Z},+)$ is an abelian group of infinite order.

Order of a group: The no. of elements in a finite group is called the order of the group. An infinite group is said to be of infinite order.

Q.2 Show that the set $\mathbb{N}$ of all natural nos. 1,2,3,4,5 ---- is not a group w.r.t addition.

Soln: Addition is obviously a binary composition in $\mathbb{N}$ i.e $\mathbb{N}$ is closed w.r.t addition.
Also addition of natural nos is associative composition
But there exist no natural no. $e \in \mathbb{N}$ s.t

$$e+a = a+e = a \quad \forall\, a \in \mathbb{N}.$$

For the addition of nos; the number 0 is the identity and $0 \notin \mathbb{N}$.

$\therefore$ $(\mathbb{N}, +)$ is not a group. $(\mathbb{R}-\{0\}, \cdot)$ is abelian.

More examples: $(\mathbb{Z}, \cdot)$ is not a group
$(\mathbb{R}, \cdot)$ is not a group ($\because$ 0 doesnot possess inverse elt.)

* Group is an algebraic structure equipped with one binary operation. Ring is an algebraic structure equipped with two binary operations.

Ring definition: Suppose R is a non-empty set equipped with 2 binary operations called addition and multiplication & denoted by '+' and '.' resp. i.e for all $a, b \in R$ we have $a+b \in R$ and $a \cdot b \in R$. Then this algebraic structure $(R, +, \cdot)$ is called a ring if the following axioms are satisfied:

1. Addition is closed.
2. Addition is associative i.e $(a+b)+c = a+(b+c)$ $\forall\, a,b,c \in R$
3. $\exists$ an identity element denoted by 0 in R s.t $0+a = a = a+0$ $\forall\, a \in R$
4. To each element $a \in R$ $\exists$ an elt. $-a$ in R s.t $(-a)+a = a+(-a) = 0$ $a \cdot b \in R$ $\forall\, a, b \in R$
5. Multiplication is associative i.e $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ $\forall\, a, b, c \in R$
6. Multiplication is distributive w.r.t addition i.e
$$\left.\begin{aligned} a\cdot(b+c) &= a\cdot b + a\cdot c \;\text{ Left dist. law} \\ (b+c)\cdot a &= b\cdot a + c \cdot a \;\text{ Right dist. law}\end{aligned}\right\} \forall\, a,b,c \in R$$

 $a+b = b+a$ $\forall\, a, b \in R$

i.e R is an abelian group w.r.t '+' and R is closed under an associative operation '.'.
Axiom 6 serves to interrelate the two operations of R.

* If there is an element 1 in R such that $a \cdot 1 = 1 \cdot a = a$ $\forall\, a \in R$, then R is a ring with unit element.

* If the multiplication of R is such that $a \cdot b = b \cdot a$ $\forall\, a, b \in R$, then we call R a commutative ring.

Examples of ring:

Ex.1: The set R consisting of a single element 0 with two B.O defined by $0+0=0$ and $0 \cdot 0 = 0$ is a ring. This ring is called the null ring or the zero ring.

Ex.2: The set $\mathbb{Z}$ of all integers is a ring w.r.t addition and multiplication of integers as the two ring compositions. This ring is called ring of integers.

Soln: In group, we proved $\mathbb{Z}$ is an abelian group w.r.t addition of integers.
Further we observe that
i) The product of 2 integers is also an integer. Therefore $\mathbb{Z}$ is closed w.r.t mult^n of integers.
ii) Mult^n of integers is an associative composition.
iii) " " " dist. with r.t addition of integer
i.e $a \cdot (b+c) = a \cdot b + a \cdot c$, $(b+c) \cdot a = b \cdot a + c \cdot a$ $\forall\, a, b, c \in \mathbb{Z}$

$\therefore$ $\mathbb{Z}$ is a ring with respect to addition & mult^n of integers.

0 is the zero element of this ring.

Also the multiplicative identity exists and is the integer 1. We have

$$1 \cdot a = a \cdot 1 = a \;\; \forall\, a \in \mathbb{Z}.$$

Thus ring of integers is a ring with unity. Also it is commutative ring.

Ex.3: The set $2\mathbb{Z}$ of all even integers is a comm. ring without unity, the addition and mult^n of integers being the two ring compositions.

Ex.4: The set $\mathbb{Q}$ of all rational nos. is a comm. ring with unity, the add^n & mult^n of rational nos. being the 2 ring compositions

Ex.5: The set $\mathbb{R}$ of all real nos. is a CRU, the add^n & mult^n of real nos. being the 2 ring compositions

Ex.6: The set $\mathbb{C}$ of all complex nos. is a comm. ring with unity, the add^n & mult^n of complex nos. being the 2 ring compositions.

Field Defn:

A ring R with atleast two elements is called a field if it

i) is commutative
ii) has unity
iii) is such that each non-zero element possesses multiplicative inverse.

Ex.: $(\mathbb{Q}, +, \cdot)$ is a field under the usual addition and mult^n of rational numbers.

Internal and external composition:

let A be any set. If $a * b \in A \;\forall\, a,b \in A$, and $a*b$ is unique then $*$ is s.t.b an internal composition in the set A.

let V and F be any two sets. If $a \circ \alpha \in V \;\forall\, a \in F$ and $\forall\, \alpha \in V$ and $a \circ \alpha$ is unique, then $\circ$ is s.t.b an external composition in V over F.

Vector space:

let $(F, +, \cdot)$ be a field. The elements of F will be called scalars. let V be a non-empty set whose elements will be called vectors. Then V is called a vector space over the field F, if

1. There is defined an internal comp. in V called addition of vectors and denoted by '$+$'. Also for this composition V is an abelian group.

2. There is an external comp. in V over F called scalar mult^n and denoted multiplicatively i.e $a\alpha \in V$ $\forall\, a \in F$ and $\forall\, \alpha \in V$.
   In other words, V is closed w.r.t scalar mult^n.

3. The two compositions i.e scalar mult^n & add^n of vectors satisfy the foll. axioms:
   i) $a(\alpha + \beta) = a\alpha + a\beta \;\; \forall\, a \in F$ and $\forall\, \alpha, \beta \in V$
   ii) $(a+b)\alpha = a\alpha + b\alpha \;\; \forall\, a,b \in F$ and $\forall\, \alpha \in V$
   iii) $(ab)\alpha = a(b\alpha) \;\; \forall\, a,b \in F$ and $\forall\, \alpha \in V$
   iv) $1\alpha = \alpha \;\; \forall\, \alpha \in V$ and 1 is the unity element of the field F.

When V is a v.s over the field F, we say that $V(F)$ is a V.S. or simply V is a V.S.

More examples:

* $(\mathbb{R}^n,+)$, $(\mathbb{Z}^n,+)$, $n \in \mathbb{N}$ are abelian if $+$ is defined componentwise

i.e $(x_1, x_2 - x_n) + (y_1, y_2 - y_n) = (x_1+y_1, - x_n+y_n)$

$(x_1, x_2 - x_n)^{-1} = (-x_1, -x_2 -, -x_n)$ is the inverse elt.

and $e = (0,0-0)$ is the neutral / identity element.

* $(\mathbb{R}^{m\times n}, +)$, the set of $m\times n$ matrices is abelian (with componentwise addition)

* $(\mathbb{R}^{n\times n}, \cdot)$ i.e the set of $n\times n$ matrices with matrix mult^n

- closure and associativity follows
- Identity element - The identity matrix I_n is the identity element with r.t matrix mult^n
- Inverse element: If the inverse exist (A is non-singular) then A^{-1} is the inverse element of $A \in \mathbb{R}^{n\times n}$ and in this case $(\mathbb{R}^{n\times n}, \cdot)$ is a group, called general linear group. $(GL(n,\mathbb{R}))$.

Vector space examples:

* $\mathbb{R}^n(\mathbb{R})$ is a vector space with operations defined as
- addition: $\boldsymbol{x}+\boldsymbol{y} = (x_1, x_2 - x_n) + (y_1, - y_n) = (x_1+y_1, -, x_n+y_n)$ $\forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
- mult^n by scalars: $\lambda \boldsymbol{x} = \lambda(x_1 - x_n) = (\lambda x_1, -, \lambda x_n)$ $\forall\ \lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^n$

* $\mathbb{R}^{m\times n}(\mathbb{R})$ is a V.S with
- Addition: $A+B = \begin{bmatrix} a_{11}+b_{11} & - & a_{1n}+b_{1n} \\ a_{m1}+b_{m1} & - & a_{mn}+b_{mn} \end{bmatrix}$ is defined elementwise $\forall\ A, B \in \mathbb{R}^{m\times n}$
- Mult^n by scalars: $\lambda A = \begin{bmatrix} \lambda a_{11} & - \lambda a_{1n} \\ \lambda a_{m1} & - \lambda a_{mn} \end{bmatrix}$

* $\mathbb{C}(\mathbb{R})$ is V.S with add^n of complex nos.

* $\mathbb{R}^n$, $\mathbb{R}^{n\times 1}$ are same but $\mathbb{R}^{n\times 1}$ & $\mathbb{R}^{1\times n}$ are diff^n.
n-tuples (column vectors)

Vector Subspaces:

Let $W \subseteq V$. Then the necessary and sufficient condition for a non empty subset W of a V.S $V(F)$ to be a subspace of V is

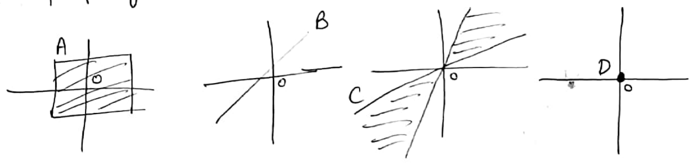
1. $W \neq \phi$, i.e in particular: $\mathbf{0} \in W$
2. Closure of W:
   a) w.r.t the external composition: $\forall\ \lambda \in F, \forall\ \boldsymbol{x} \in W$: $\lambda \boldsymbol{x} \in W$
   b) w.r.t the internal composition: $\forall\ \boldsymbol{x}, \boldsymbol{y} \in W$: $\boldsymbol{x}+\boldsymbol{y} \in W$.

OR

$\forall\ a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$.

Examples: i) for every V.S V, the trivial subspaces are V and $\{0\}$.

ii) Only example in D is a subspace of $\mathbb{R}^n$ with usual operations (+ and ·). In A&C, the closure property is violated; B does not contain 0.



iii) The solution set of a homogeneous system $Ax=0$ with n unknowns of linear equations $x = [x_1, x_2 \ldots x_n]^T$ is a subspace of $\mathbb{R}^n$.
($x=0$ is always a soln., if x_1 & x_2 are solns of $Ax=0$, then so is any linear combination of $\gamma_1 x_1 + \gamma_2 x_2$ of x_1 & x_2)

iv) The solution of a non-homogeneous system of linear equations $Ax=b$, $b \neq 0$ is not a subspace of $\mathbb{R}^n$.
(Since $b \neq 0$ $\therefore$ $0 \notin W$.)

v) The intersection of arbitrarily many subspaces is a subspace itself.

Linear dependence of vectors:

let $V(F)$ be a V.S.
If $\alpha_1, \alpha_2 \ldots \alpha_n \in V$, then any vector

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \quad \text{where } a_1, a_2 \ldots a_n \in F$$

is called a <u>linear combination</u> of the vectors $\alpha_1, \alpha_2 \ldots \alpha_n$.

<u>Generating set and span</u>: Consider a V.S $V=(\mathcal{V},+,\cdot)$ and set of vectors $A = \{x_1, x_2 \ldots x_k\} \subseteq \mathcal{V}$. If every vector $v \in \mathcal{V}$ can be expressed as a L.C of $x_1, x_2 \ldots x_k$, A is called a generating set of V. The set of all L.Cs of vectors in A is called the span of A. If A spans the V.S V, we write $V = \text{span}[A]$ or $V = \text{span}[x_1, x_2 \ldots x_k]$.

<u>Smallest generating set</u>:

<u>Basis</u>: Consider a V.S $V = (\mathcal{V},+,\cdot)$ and $A \subseteq \mathcal{V}$.
A generating set A of V is called minimal if there exists no smaller set $\tilde{A} \subseteq A \subseteq \mathcal{V}$ that spans V. Every L.I generating set of V is minimal and is called a basis of V.

<u>* Imp</u>
let $V = (\mathcal{V},+,\cdot)$ be a V.S and $\beta \subseteq \mathcal{V}, \beta \neq \phi$.
Then, the following statements are equivalent.

i) β is a basis of V.
ii) β is a minimal generating set
iii) β is a maximal L.D set of vectors in V i.e adding any other vector to this set will make it L.D.
iv) Every vector $x \in V$ is a L.C of vectors from β & every L.C is unique i.e. with

$$x = \sum_{i=1}^{k} \lambda_i b_i = \sum_{i=1}^{k} \psi_i b_i$$

& $\lambda_i, \psi_i \in F$, $b_i \in \beta$ it follows that $\lambda_i = \psi_i \ \forall i=1, \ldots k$

<u>Linear dependence and independence of Vectors</u>:
let us consider a V.S V with $k \in \mathbb{N}$ and $x_1, \ldots x_k \in V$. If there is a non-trivial linear combination, such that $0 = \sum_{i=1}^{k} \lambda_i x_i$, with atleast one $\lambda_i \neq 0$, the vectors $x_1, \ldots x_k$ are L.D. If only the trivial solution exists, i.e, $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$, the vectors $x_1, \ldots x_k$ are L.I.

Examples:

1) In $\mathbb{R}^3$, the canonical / standard basis is

$$B = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}$$

2) Different bases in $\mathbb{R}^3$ are

$$B_1 = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\}$$

$$B_2 = \left\{ \begin{bmatrix}0.5\\0.8\\0.4\end{bmatrix}, \begin{bmatrix}1.8\\0.3\\0.3\end{bmatrix}, \begin{bmatrix}-2.2\\-1.3\\3.5\end{bmatrix} \right\}$$

3) The set

$$A = \left\{ \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\\4\end{bmatrix} \right\}$$

is L.I but not a generating set (and not basis) of $\mathbb{R}^4$ since the vector $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ can not be obtained by a L.C of elements in A.

Note: i) Every V.S V possesses a basis B.

ii) Basis is not unique.

iii) All the basis (bases) possesses the same number of elements. called the basis vectors.

For a finite-dimensional vector V, the dimension of V is the no. of basis vectors of V and we write $\dim(V)$.

* If $U \subseteq V$ is a subspace of V,

$$\dim(U) \leq \dim(V)$$

and $(U) = \dim(V)$ iff $U = V$.

Example: For a vector subspace $U \subseteq \mathbb{R}^5$, spanned by the vectors

$$x_1 = \begin{bmatrix}1\\2\\-1\\-1\\-1\end{bmatrix}, \quad x_2 = \begin{bmatrix}2\\-1\\1\\2\\-2\end{bmatrix}, \quad x_3 = \begin{bmatrix}3\\-4\\3\\5\\-3\end{bmatrix}, \quad x_4 = \begin{bmatrix}-1\\8\\-5\\-6\\1\end{bmatrix} \in \mathbb{R}^5$$

which of the vectors $x_1, - x_4$ are a basis for U.

Soln: We solve

$$\sum_{i=1}^{4} \lambda_i x_i = 0$$

which leads to a homogeneous system of eqns. with matrix

$$[x_1, x_2, x_3, x_4] = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 3 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

By row/col transformations

$$\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ x_1, x_2, x_4 are L.I

s.t $\quad \lambda_1 x_1 + \lambda_2 x_2 + \lambda_4 x_4 = 0$

can only be solved with $\lambda_1 = \lambda_2 = \lambda_4 = 0$

$\therefore$ $\{x_1, x_2, x_4\}$ is a basis of U

* ($\{x_1, x_3, x_4\}$ is also a basis).

Rank:

No. of L.I rows or columns of a matrix $A \in \mathbb{R}^{m \times n}$ is called Rank of A (No. of L.I rows = " " " columns)

Important Notes:

i) $\rho(A) = \rho(A^T)$

ii) $A_{n \times n}$ is non-singular iff $\rho(A) = n$

iii) $\forall$ $A \in \mathbb{R}^{n \times n}$, A is non-singular iff $\rho(A) = n$.

iv) $\forall$ $A \in \mathbb{R}^{m \times n}$ and $\forall$ $b \in \mathbb{R}^m$, the system of linear equations $Ax = b$ can be solved or has solution iff $\rho(A) = \rho(A|b)$

Image or range:

v) The columns of $A \in \mathbb{R}^{m \times n}$ span a subspace $U \subseteq \mathbb{R}^m$ with $\dim(U) = \rho(A)$. This subspace U is called as the image or range.

vi) For $A \in \mathbb{R}^{m \times n}$, the subspace of solutions for $Ax = 0$ possesses dim $n - \rho(A)$. This subspace is called as the Kernel or the null space.

vii) The rows of $A \in \mathbb{R}^{m \times n}$ span a subspace $W \subseteq \mathbb{R}^n$ with $\dim(W) = \rho(A)$.

Assignment/Practice Questions:

Q.1: let $S = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \text{ and}$$
$$c(a_1, a_2) = (ca_1, ca_2)$$

Check whether $S(\mathbb{R})$ forms a vector space or not. If not, give reasons.

Q.2: let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \_ + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \_ + b_1 x + b_0$$

be polynomials with coefficients from a field F.

Suppose that $m \le n$, and define $b_{m+1} = b_{m+2} = \dots = b_n = 0$. Then $g(x)$ can be written as

$$g(x) = b_n x^n + b_{n-1} x^{n-1} + \_ + b_1 x + b_0$$

Define

$$f(x) + g(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \_ + (a_1 + b_1) x + (a_0 + b_0)$$

and for any $c \in F$, define

$$cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + \_ + ca_1 x + ca_0$$

Does set of all polynomial with coefficients from F form a vector space or not? Give reasons.

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Q.3 let $S = \{(a_1, a_2): a_1, a_2 \in \mathbb{R}\}$
for $(a_1, a_2), (b_1, b_2) \in S$ and $c \in \mathbb{R}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0)$ and
$c(a_1, a_2) = (ca_1, 0)$

Does $S(\mathbb{R})$ form a vecter space or not?
Give reasons

Q.4 let V denote the set of ordered pair of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in \mathbb{R}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2)$ and
$c(a_1, a_2) = (ca_1, a_2)$

Is V a vector space over $\mathbb{R}$ with these operations?
Justify your answer.

Q.5

Linear transformations.

The special functions defined on vector spaces that in some sense "preserve" the structure are called linear transformations.

Ex. i) operations of differentiation and integration.
ii) In geometry, rotations, reflections and projections

Defn: let V and W be V.S over the same field F. We call a function
$T: V \to W$ a L.T from V to W if
$\forall\ x, y \in V$ & $c \in F$, we have

a) $T(x+y) = T(x) + T(y)$
b) $T(cx) = cT(x)$

Example 1: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by (over field $\mathbb{R}$)
$T(a_1, a_2) = (2a_1 + a_2, a_1)$

To show T is a L.T.

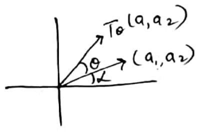
let $c \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$
let $x = (b_1, b_2)$, $y = (c_1, c_2)$

$T(x+y) = T(b_1 + c_1, b_2 + c_2) = (2(b_1 + c_1) + b_2 + c_2, b_1 + c_1)$

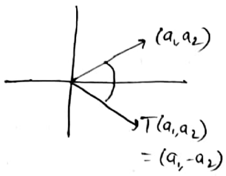
Also $T(x) + T(y) = T(b_1, b_2) + T(c_1, c_2)$
$= (2b_1 + b_2, b_1) + (2c_1 + c_2, c_1)$
$= (2b_1 + b_2 + 2c_1 + c_2, b_1 + c_1)$

Also we can verify
$T(cx) = cT(x)$.
Hence T is L.T.

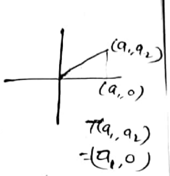
... images (under T) of vectors in V ...



rotation



Reflection



Projection

<u>Reflection</u> : Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by
$T(a_1,a_2) = (a_1,-a_2)$ – reflection about x-axis.

<u>Projection</u> : Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by
$T(a_1,a_2) = (a_1,0)$ – proj. on the x-axis.

<u>Ex.</u> Define $T: M_{m\times n}(F) \to M_{n\times m}(F)$ by
$T(A) = A^t$.
T is L.T

<u>Ex</u>. Define $T: V \to V$ by
$T(f) = f'$, (derivative of f).
T is linear.
where V is ~~be~~ the set of all real valued functions defined on real line that have derivatives of all order. V is V.S (prove).

<u>Ex</u>. let $V = C(R)$, V.S of cts. real valued f^ns on $\mathbb{R}$.
let $a, b \in \mathbb{R}$, $a < b$.
Define $T: V \to \mathbb{R}$ by
$T(f) = \int_a^b f(t)\,dt \quad \forall f \in V$.
Then T is L.T.

<u>Ex</u>. Identity transformation & zero transf.
$I_V: V \to V$
$I_V(x) = x \quad \forall x \in V$

$T_0: V \to W$
$T_0(x) = 0 \quad \forall x \in V$.

<u>Range and null space</u>.

let $T: V \to W$ be L.T where V & W are V.S
We define null space (or kernel) $N(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$
$N(T) = \{x \in V : T(x) = 0\}$
We define the range (or image) of T to be the subset of W consisting of all images (under T) of vectors in V: $R(T) = \{T(x) : x \in V\}$

Example: i) let V and W be vector spaces. let $I: V \to V$ and $T_0: V \to W$ be the identity and zero transformation.

$N(I) = \{0\}$, $R(I) = V$

$N(T_0) = V$ and $R(T_0) = \{0\}$

ii) let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be L.T defined by

$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$N(T) = \{x \in V \mid T(x) = 0\}$

$\Rightarrow N(T) = \{x \in \mathbb{R}^3 \mid T(x) = (0,0)\}$

$= \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid T(a_1, a_2, a_3) = (0,0)\}$

$= (a_1 - a_2, 2a_3) = (0,0)$

$\Rightarrow a_1 - a_2 = 0$

$a_3 = 0$

$\Rightarrow a_1 = a_2$ & $a_3 = 0$

$\therefore N(T) = \{(a_1, a_1, 0) : a_1 \in \mathbb{R}\}$

$R(T) = \mathbb{R}^2$.

Result: 1) The null space $N(T)$ and range space $R(T)$ for $T: V \to W$ where V & W are v.s and T is L.T, are subspaces of V & W, resp.

2) let V & W be v.s, & let $T: V \to W$ be L.T. If $\beta = \{v_1, v_2 \ldots v_n\}$ is a basis for V, then

$R(T) = \text{span}(T(\beta)) = \text{span}(T(v_1), T(v_2) \ldots T(v_n))$

Definitions: let V & W be v.s and let $T: V \to W$ be L.T. If $N(T)$ & $R(T)$ are finite-dimensional, then we define the nullity of T and the rank of T, to be the dimensions of $N(T)$ and $R(T)$, resp.

Dimension thm / Rank-Nullity Thm.: let V & W be v.s, & let $T: V \to W$ be L.T. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Q. If A is a 5×6 matrix with rank 2, what is the dimension of null space of A?

Soln: $\text{Nullity}(A) = 6 - 2 = 4$

$\therefore$ Its null space is a 4-dim. subspace of $\mathbb{R}^6$.

Matrix of a linear transformation.

let $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$ where rows of $A \in \mathbb{R}^n(\mathbb{C}^n)$ and columns of $A \in \mathbb{R}^m(\mathbb{C}^m)$

If $x \in \mathbb{R}^n$, then $Ax \in \mathbb{R}^m$.

$\therefore$ A matrix $A_{m\times n}$ maps the elements in $\mathbb{R}^n$ into the elements in $\mathbb{R}^m$.

$\therefore$ $T = A: \mathbb{R}^n \to \mathbb{R}^m$ with $Tx = Ax$ defines a linear transformation

Proof: let $x_1, x_2 \in \mathbb{R}^n$

$$T(x_1+x_2) = A(x_1+x_2) = Ax_1 + Ax_2 = T(x_1)+T(x_2)$$

let $c \in \mathbb{R}$

$$T(cx_1) = Acx_1 = cAx_1 = cT(x_1)$$

$\therefore$ T is L.T

Also $R(T)$ is a subspace of $\mathbb{R}^m$ & $N(T)$ is a subspace of $\mathbb{R}^n$.

Examples: 1) let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T defined by $Tx = Ax$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

find Tx when x is given by $[3\ 4\ 5]^T$.

$$Tx = Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 26 \\ 62 \end{bmatrix}$$

2) let T be a L.T defined by

$T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$

The matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are L.I. therefore form a basis in 2×2 matrices.

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_1+\alpha_2 \\ \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \end{pmatrix}$$

$\Rightarrow \alpha_1 = 4,\ \alpha_2 = 1,\ \alpha_3 = -2,\ \alpha_4 = 5.$

$$\therefore T\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1 T\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 T\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 T\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4 T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 20 \\ 36 \end{pmatrix}$$

Ex. let T be a LT from $\mathbb{R}^3 \to \mathbb{R}^2$ where $Tx = Ax$,

$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $x = [x\ y\ z]^T$.

Find $\ker(T)$, $\operatorname{ran}(T)$ and their dimensions.

Soln: $Tv = 0 \Rightarrow Av = 0 \Rightarrow \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}\underset{3\times 1}{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow v_1 + v_2 = 0$ & $-v_1 + v_3 = 0$

$\therefore v = [1\ -1\ 1]^T$

$\therefore$ dimension of $\ker(T)$ is 1.

$\operatorname{rank}(T)$ $\{T(v) \mid v \in V\}$

$$T(v) = Av = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_3 \end{bmatrix} = v_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + v_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\therefore \dim|\operatorname{ran}(T)| = 2$

Ex. let T be a LT $Tx = Ax$ from $\mathbb{R}^2 \to \mathbb{R}^3$ where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix} \quad \& \quad x = \begin{pmatrix} x \\ y \end{pmatrix}$$

Find $\ker(T)$, $\operatorname{ran}(T)$ & their dimensions.

Soln: $T(v) = Av = 0$ for $v = (v_1, v_2)^T$

$$\Rightarrow \begin{aligned} 2v_1 + v_2 &= 0 \\ v_1 - v_2 &= 0 \\ 3v_1 + 2v_2 &= 0 \end{aligned}$$

$$\Rightarrow v_1 = v_2 = 0$$

$\therefore \dim \ker(T) = 0$

$$\operatorname{ran}(T) = T(v) = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ are L.I.

$\therefore$ rank(T) is 2.

Matrix of LT:

Ex. let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a LT defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}$$

Determine the matrix of T w.r.t the ordered basis

i) $X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$

& $Y = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$. $\quad TX = YA$

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(0)$$

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(1)$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(-1)$$

$$\therefore A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Matrix Representations of Linear transformation:

let V and W be vector spaces over same field F. let $\beta_1 = (v_1, \ldots v_n)$ be an ordered basis for V and let $\beta_2 = (w_1, w_2 \ldots w_m)$ be an ordered basis for W.

let $T: V \to W$ be a linear transformation we can give a matrix representation of T as follows.

for each $j \in \{1, 2, \ldots n\}$, $T(v_j)$ is a vector in W.

$\therefore$ we can write $T(v_j)$ as a L.C of $w_1, w_2 \ldots w_m$.

$\therefore$ $\exists$ scalars $a_{1j}, a_{2j}, \ldots a_{mj}$ such that

$$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \ldots + a_{mj} w_m$$

And for an arbitrary $v \in V$, $\exists$ scalars $\lambda_1, \lambda_2 \ldots \lambda_n$ such that $v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n$

so $[v]_{\beta_1} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

$$\begin{aligned} \therefore T(v) &= T(\lambda_1 v_1 + \ldots + \lambda_n v_n) \\ &= \lambda_1 T(v_1) + \ldots + \lambda_n T(v_n) \\ &= \sum_{j=1}^{n} \lambda_j T(v_j) \\ &= \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{m} a_{ij} w_i \right) \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} (a_{ij} \lambda_j) w_i \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \lambda_j \right) w_i \end{aligned}$$

$$\therefore [T(v)]_{\beta_2} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}\lambda_j \\ \sum_{j=1}^{n} a_{2j}\lambda_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}\lambda_j \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

So if we let A be the $m \times n$ matrix such that $A_{ij} = a_{ij}$ then $[T(v)]_{\beta_2}$ is precisely $A[v]_{\beta_1}$

Hence given any $u \in V$, we can obtain the tuple representation of $T(u)$ w.r.t β_2 by computing $A[u]_{\beta_1}$.

Then matrix A is called the matrix representation of T and is denoted as $[T]_{\beta_1}^{\beta_2}$

<u>Note</u> that i column of $[T]_{\beta_1}^{\beta_2}$ is given by $(T(u_i))_{\beta_2}$.

(43(i))

Q. let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a L.T defined by

$$\begin{pmatrix} -5 & -8 & -7 \\ 3 & 5 & 4 \end{pmatrix}$$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x-z \end{pmatrix}$$

find the matrix rep. of T w.r.t ordered basis

$$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ in } \mathbb{R}^3$$

$$\& \ Y = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ in } \mathbb{R}^2.$$

(44)

<u>Q</u>. let V & W be V.S in $\mathbb{R}^3$.

let $T: V \to W$ be a L.T defined by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ x+y \\ x+y+z \end{pmatrix}$$

find matrix rep. of T wrt ordered basis

(i) $$X = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ in } V$$

$$\& \ Y = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ in } W.$$

$$A = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & 0 & -1/2 \\ 1 & 1 & 1/2 \end{bmatrix}$$