**Quadratic forms:** $\Rightarrow$ Let $X = (x_1, x_2, --, x_n) \in \mathbb{R}^n$ be an arbitrary vector. A real Quadratic form is an homogeneous expression of the form

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i x_j \quad -(1)$$

in which the total power in each term is 2.

We can write (1) as

$$\begin{aligned} Q = & \; a_{11}x_1^2 + a_{12}x_1x_2 + --- a_{1n}x_1x_n \\ & + a_{21}x_2x_1 + a_{22}x_2^2 + ---+ a_{2n}x_2x_n \\ & \;\vdots \\ & + a_{n1}x_nx_1 + a_{n2}x_nx_2 + --- + a_{nn}x_n^2 \\ = & \; a_{11}x_1^2 + (a_{12}+a_{21})x_1x_2 + ---+ (x_1x_n)(a_{1n}+a_{n1}) \\ & + a_{22}x_2^2 + (a_{23}+a_{32})x_2x_3 + --- + (a_{2n}+a_{n2})x_2x_n \\ & + --- + a_{nn}x_n^2 \\ = & \; X^TAX \quad \text{(By def}^n \text{ of Matrix multiplication)} \end{aligned}$$

$\hookrightarrow (2)$

Now set $b_{ij} = \dfrac{(a_{ij}+a_{ji})}{2}$

$\Rightarrow B = (b_{ij})$ Matrix is Symmetric matrix ($\because b_{ij} = b_{ji}$)

So (2) can be written as

$$Q = X^TBX$$

where $B$ is a Symmetric matrix and $b_{ij} = \dfrac{a_{ij}+a_{ji}}{2}$.

**Que** Obtain the Symmetric matrix $B$ for the Quadratic form

(1) $Q = 2x_1^2 + 3x_1x_2 + x_2^2$

$B = (b_{ij})$ and $b_{ij} = \dfrac{a_{ij}+a_{ji}}{2}$

$a_{11} = 2;\; a_{12}+a_{21} = 3;\; a_{22} = 1$

$b_{12} = \dfrac{3}{2} = b_{21}$

Hence $B = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$

**Que** $Q = x_1^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3 - 5x_2^2 + 4x_3^2$

$$B = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$$

$\left[ \because a_{11} = 1,\ a_{22} = -5,\ a_{33} = 4 \right.$
$\left. a_{12} + a_{21} = 2;\ a_{13} + a_{31} = -4;\ a_{23} + a_{32} = 6. \right.$

Conversely If a Symmetric matrix is given then we can find the Corresponding Quadratic form.

**Ex** $A = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$

then Quadratic form is $Q = X^T A X$ where $X \in \mathbb{R}^2$

$$Q = (x_1\ x_2) \begin{pmatrix} 2 & 3/2 \\ 3/2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 + \frac{3x_2}{2} & \frac{3}{2}x_1 + x_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= 2x_1^2 + \frac{3}{2}x_1x_2 + \frac{3}{2}x_1x_2 + x_2^2$$

$$Q = 2x_1^2 + 3x_1x_2 + x_2^2$$

**H.W Que** Find the Sym. matrix A for the Quadratic forms

(1) $Q = x_1^2 - 2x_1x_2 + 4x_2x_3 - x_2^2 + x_3^2$

(2) $Q = 3x_1^2 + 2x_1x_2 - 4x_1x_3 + 8x_2x_3 + x_2^2.$

Ans (1) $\begin{bmatrix} 1 & -1 & 2 \\ -1 & -1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 3 & 1 & -2 \\ 1 & 1 & 4 \\ -2 & 4 & 0 \end{bmatrix}$

and Find their Corresponding Quadratic forms (For practice)