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Q1 Let, $S = \{(a_1, a_2) ; a_1, a_2 \in R\}$
for $(a_1, a_2)$ and $(b_1, b_2) \in S$ and $c \in R$, define
$$(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2 - b_2)$$
and $c(a_1, a_2) = (ca_1, ca_2)$
Check whether $S(R)$ forms a vector space or not justify your answer.

Solution: for $S(R)$ to be a vector space it must staisfy the following property

(i) Internal composition w.r.t + & forms abelian group
Here $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$
~~so~~ & $a_1 + b_1 \in R$ and $a_2 - b_2 \in R$
as $a_1, a_2 \in R$
So it satisfy the first condition.

(ii) External composition w.r.t × (Scalar multiplication)
$$c(a_1, a_2) = (ca_1, ca_2) \text{ [given]}$$
$$\forall c \in R$$

(iii) Associativity w.r.t addition
let three vector $V_1 = (x_1, y_1)$, $V_2 = (x_2, y_2)$ & $(x_3, y_3) = V_3$
~~LHS~~ $= (V_1 + V_2) + V_3$
$= ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$
$= (x_1 + x_2, y_1 - y_2) + (x_3, y_3)$
$= (x_1 + x_2 + x_3, y_1 - y_2 - y_3$ — (1)

& Now,
~~RHS~~ $= V_1 + (V_2 + V_3)$
$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$
$(x_1, y_1) + (x_2 + x_3, y_2 - y_3)$
$(x_1 + x_2 + x_3, y_1 - y_2 + y_3)$

~~for~~ In order to be associative
$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$
But it fails this case of associativity as $(V_1 + V_2) + V_3$ ~~is~~ $\neq V_1 + (V_2 + V_3)$ so it $S(R)$ doesn't forms a vector space.

Q2) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots b_1 x' + b_0$. ~~Then g(x) can be written~~ as be the polynomial with coefficients from a field F. Suppose that $m \leq n$ and define $b_{m+1} = b_{m+2} = \cdots = b_n = 0$. Then $g(x)$ can be written as $g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$. define $f(x) + g(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots (a_1 + b_1) x + (a_0 + b_0)$. & for any $c \in F$, define $cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + \cdots ca_1 x + ca_0$. Does set of all polynomial with coefficients from F form a vector space or not? Gives reasons.

Solution In order to be a vector space following conditions must be followed:
~~(I) Internal composition w.r.t Addition (+)~~
Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$, $g(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots b_1 x + b_0$ & $h(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots c_1 x + c_0$ be the polynomial with coefficients from field F

(i) $f(x) + g(x) \in V$
$f(x) + g(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + (a_0 + b_0)$
and all coefficients belongs to F so it so it satisfy follows the condition of Internal composition

~~(ii) Multiplication with scalars scalars.~~

(ii) $f(x) + g(x) = g(x) + f(x)$ [commutative]
~~Since F is the set of~~
$f(x) + g(x) = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots + (a_1 + b_1) x + a_0 + b_0$ — (i)
$g(x) + f(x) = (b_n + a_n) x^n + (b_{n-1} + a_{n-1}) x^{n-1} + \cdots (b_1 + a_1) x + b_0 + a_0$
$= (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \cdots (a_1 + b_1) x + a_0 + b_0$ — (ii)

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from eqn (I) & (II) we can say that $f(x)+g(x) = g(x)+f(x)$

(III) $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$

Now, $(f(x)+g(x))+h(x) =$

$= ((a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \dots (a_1+b_1)x + (a_0+b_0)) + c_n x^n + c_{n-1}x^{n-1} \dots$
$c_1 x + c_0$

$= (a_n+b_n+c_n)x^n + (a_{n-1}+b_{n-1}+c_{n-1})x^{n-1} + \dots (a_1+b_1+c_1)x + (a_0+b_0+c_0)$

Now,

$f(x) + (g(x)+h(x)) =$

$= a_n x^n + a_{n-1}x^{n-1} + \dots a_1 x + a_0 + (b_n+c_n)x^n + (b_{n-1}+c_{n-1})x^{n-1} + \dots (b_1+c_1)x + (b_0+c_0)$

$= (a_n+b_n+c_n)x^n + (a_{n-1}+b_{n-1}+c_{n-1})x^{n-1} + \dots\dots + (a_1+b_1+c_1)x + (a_0+b_0+c_0)$

(IV) Exististence of ~~zero~~ additive Identity in V.
Here 0 is the additive identity when all coefficients of ~~ex~~ $f(x)$ is zero.

(V) Existince of additive Inverse
additive inverse also exist here when all coefficient of $f(x)$ and $g(x)$ are equal & opposite of each other.

(VI) Scalar multiplication.
Since $cf(x) = ca_n x^n + ca_{n-1}x^{n-1} + \dots + ca_1 x + ca_0$
~~so~~ and $cf(x)$ also is a polynomial so it belongs to V

(VII) Existence of multiplicative inverse.
Here also

(VIII) Right ~~Left~~ distributive

$c(f(x)+g(x)) = cf(x) + cg(x)$

LHS $= c((a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \dots + (a_1+b_1)x + (a_0+b_0))$

$= c(a_n+b_n)x^n + c(a_{n-1}+b_{n-1})x^{n-1} + \dots c(a_1+b_1)x + c(a_0+b_0)$

$= c(a_n x^n + a_{n-1}x^{n-1} + \dots a_0) + c(b_n x^n + b_{n-1}x^{n-1} + \dots + b_0)$

$= cf(x) + cg(x)$

$=$ RHS

~~Right~~ Left distributive

$(f(x)+g(x))c = f(x)\cdot c + g(x)\cdot c$

$LHS = ((a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \dots + (a_0+b_0))c$

$= c(a_n+b_n)x^n + c(a_{n-1}+b_{n-1})x^{n-1} + \dots + c(a_0+b_0)$

$= c(a_n x^n + a_{n-1}x^{n-1} + \dots a_0 x^0) + c(b_n x^n + b_{n-1}x^{n-1} + \dots + b_0)$

$= c\cdot f(x) + c\cdot g(x)$

So the given polynomial forms a vector space.

Q3 Let $S = \{(a_1, a_2) : a_1, a_2 \in R\}$ for $(a_1,a_2), (b_1,b_2)$ and $c \in R$, define $(a_1,a_2)+(b_1,b_2) = (a_1+b_1, 0)$ & $c(a_1,a_2) = (ca_1, 0)$. Does $S(R)$ forms a vector space or not? Give reasons.

For $S(R)$ to be a vector space, it must follow the below conditions:

Let $V_1 = (a_1, a_2)$, $V_2 = (b_1, b_2)$ & $V_3 = (c_1, c_2)$ ~~& $\alpha$ be the const~~ be the vectors from S & $\alpha$ be the constant belonging to the field R.

Condn 1: Internal Composition w.r.t addition & forms a abelian group w.r.t addition.

(i) Closed: since $V_1+V_2 = (a_1,a_2)+(b_1,b_2) = (a_1+b_1, 0)$ & $(a_1+b_1, 0) \in S$ so it is closed.

(ii) Commutative: $V_1+V_2 = V_2+V_1$

$LHS = (a_1,a_2)+(b_1,b_2) = (a_1+b_1, 0)$

$RHS = (b_1,b_2)+(a_1,a_2) = (b_1+a_1, 0)$

$= (a_1+b_1, 0) = LHS$

(iii) Associavity: ~~$(a+b)+c = a+(b+$~~ $(V_1+V_2)+V_3 = V_1+(V_2+V_3)$

$LHS = ((a_1,a_2)+(b_1,b_2)) + (c_1,c_2)$

$= (a_1+b_1, 0) + (c_1, c_2)$

$= (a_1+b_1+c_1, 0)$

$RHS = (a_1,a_2) + ((b_1,b_2)+(c_1,c_2))$

$= (a_1,a_2) + (b_1+c_1, 0)$

$= (a_1+b_1+c_1, 0) = LHS$

(iv) Existance of zero element in V: Since $S = (a_1, a_2)$ & $a_1, a_2 \in R$ so $(0,0)$ is also an element of S

Now, $(a_1, a_2) + (0,0)$
$= (a_1 + 0, 0) = (a_1, 0) \in S$

so ~~there~~ a zero element exist here but fails to satisfy $a + 0 = a$

(v) Existance of additive inverse in S: Since $S = (a_1, a_2)$ & $a_1, a_2 \in R$, so $(-a_1, -a_2)$ also belongs to S

Now,
$(a_1, a_2) + (-a_1, -a_2) = (a_1 - a_1, 0) = (0,0)$

$\therefore$ it satisfy the above condition.

Cond^n 2: External composition w.r.t $\cdot$ (multiplication)

$\because \alpha v_1 = \alpha(a_1, a_2) = (\alpha a_1, 0)$ [given]

so, & $(\alpha a_1, 0) \in S$ so it satisfies the above condition

(i) Existence of multiplicative identity:
$\because \left(\frac{1}{a_1}, \frac{1}{b_1}\right) \in S$

so
$\frac{1}{a_1}, \frac{1}{b_1}$

(ii) Left distributive law:
$(\alpha_1 + \alpha_2) v_1 = \alpha_1 v_1 + \alpha_2 v_1$

LHS $= (\alpha_1 + \alpha_2)(a_1, b_1) = ((\alpha_1 + \alpha_2) a_1, 0)$

RHS $= \alpha_1 (a_1, b_1) + \alpha_2 (a_1, b_1)$
$= (\alpha_1 a_1, 0) + (\alpha_2 a_1, 0)$
$= (\alpha_1 a_1 + \alpha_2 a_1, 0) = ((\alpha_1 + \alpha_2) a_1, 0) =$ LHS

(iii) Right distributive law:- $\alpha(v_1 + v_2) = \alpha v_1 + \alpha v_2$

LHS $= \alpha(a_1 + b_1, 0) = (\alpha(a_1 + b_1), 0)$

RHS $= (\alpha a_1, 0) + (\alpha b_1, 0) = (\alpha a_1 + \alpha b_1, 0)$
$= (\alpha(a_1 + b_1), 0) =$ LHS.

So it is not a vectorspace.

Q4: Let V denote the set of ordered pair of real numbers. If $(a_1, a_2)$ & $(b_1, b_2)$ are elements of V & $c \in R$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_2, a_2 b_2)$ & $c(a_1, a_2) = (ca_1, a_2)$. Is V a vector space over R with these operation? Justify your answer.

Solution:- For V(R) be a vecter space, it must obeys the following conditions:

① Existance of zero element & $0 + V = V$

Since ~~V = a~~, V is a set of ordered pair of real numbers so $(0, 0)$ is also a element of V, let $V_1 = (a_1, a_2)$

Now,
$$\cancel{V_1 + 0} \cdot (a_1, a_2) + (0, 0)$$
$$= (a_1 + 0 + a_2 \cdot 0)$$
$$= (a_1, 0)$$
$$\neq \cancel{V_1}(a_1, a_2)$$

So V(R) doesn't forms a vector space.

Q5: Check whether the following set of vectors forms a basis of $R^3$ or not?

a) $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution for $B_1$ to form a basis $a_1 V_1 + a_2 V_2 + a_3 V_3$ must represents all vectors in $R^3$.

Now,
$$a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 \\ a_2 + a_3 \\ a_3 \end{bmatrix}$$

$\because a_1, a_2, \& a_3 \in R$ so $a_1 + a_2 + a_3$, $a_2 + a_3$ & $a_3$ also belongs to R so the given set of vectors is a basis of $R^3$.

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(11) $B_2 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$

Solution: for $B_2$ to be basis, it must represents all ~~to~~ vectors in $R^3$

Now,

$$\alpha \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix} + \beta \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix} + \gamma \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5\alpha + 1.8\beta - 2.2\gamma \\ 0.8\alpha + 0.3\beta - 1.3\gamma \\ 0.4\alpha + 0.3\beta + 3.5\gamma \end{bmatrix}$$

$\because$ $0.5\alpha + 1.8\beta - 2.2\gamma$, $0.8\alpha + 0.3\beta - 1.3\gamma$ & $0.4\alpha + 0.3\beta + 3.5\gamma$ $\in R$ $\forall \alpha, \beta, \gamma \in R$.

So it Represents all vectors in $R^3$ therefore it forms a basis.

Q.6 Does the following set of vectors forms a basis of $R^4$

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix} \right\}$$

Solution: for $B_1$ to be a basis it must be able to represent whole $R^4$ ~~in~~.

Now,

$$\alpha \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta + \gamma \\ 2\alpha - \beta + \gamma \\ 3\alpha \\ 4\alpha + 2\beta + 4\gamma \end{bmatrix}$$

Q 7 Let V be the set of all ordered pairs $(x, y)$, $\forall x, y \in R$, let $a = (x_1, y_1)$ & $b = (x_2, y_2)$. Define addition as $a + b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$ & scalar multiplication as $\alpha(x_1, y_1) = \left(\frac{\alpha x_1}{3}, \frac{\alpha y_1}{3}\right)$ Show that V is not a vector space

Which of the properties is not satisfied?

Solution For ~~a~~ V to be vector space it must ~~follow the~~ be associative

Let $V_1 = (x_1, y_1)$, $V_2 = (x_2, y_2)$ & $V_3$ ~~be~~ $= (x_3, y_3)$ be the vectors in V.

Now, $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$

$LHS = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$

$= (2x_1 - 3x_2, y_1 - y_2) + (x_3, y_3)$

$= 4x_1 - 6x_2 - 3x_3, y_1 - y_2 - y_3$

$RHS = (x_1, y_1) + ((x_2, y_2) + (x_3, y_3))$

$= (x_1, y_1) + (2x_2 - 3x_3, y_2 - y_3)$

$= 2x_1 - 6x_2 + 9x_3, y_1 - y_2 + y_3$

$\because$ LHS $\neq$ RHS so, V fails to satisfy the cond$^n$ of associativity so it is not a vector space.

Q 8 Let V be the set of all 2x2 real matrices Show that the sets

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Solution Let a, b, c & D be the constants belonging to R.

If $aV_1 + bV_2 + cV_3 + dV_4$ is able to represent all $2\times 2$ matrix then ~~∀~~ S can spans V.

$$a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

Since all elements of matrix is diffirent so it spans V.

Q9. Let V be the set of all polynomials of degree $\leq 3$. Determine whether or not the set $S = \{t^3, t^2+t, t^3+t+1\}$ spans V.

Solution:- ~~for~~ Let a polynomial with 3 degree be $ax^3+bx^2+cx+d$ and let $\alpha, \beta, \gamma$ be the some constant belonging to R.

Now, $ax^3+bx^2+cx+d = \alpha t^3 + \beta(t^2+t) + \gamma(t^3+t+1)$

$a = \alpha + \gamma$ —— (i)

$b = \beta$ —— (ii)

$c = \beta + \gamma$ —— (iii)

$d = \gamma$ —— (iv)

from eq$^n$ (ii), (iii) & (iv)

$c = b + d$

∵ there is a relation between b, c & d ~~is~~ so it doesnt spans V.

Q10) Let $V = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3 = (0,2,1)$ & $V_4 = (1,0,3)$ be elements of $R^3$. show that the set of vectors $\{U_1, V_2, V_3, V_4\}$ is linearly dependent.

Solution In order to be linearly dependent ~~there must~~ if there exists some scalars, $\alpha, \beta, \gamma$ & $\delta$ not all zero such that

$$\alpha V_1 + \beta V_2 + \gamma V_3 + \delta V_4 = 0 \quad \text{—— (a)}$$

Substituting the values of $V_1, V_2, V_3$ & $V_4$ we get

$\alpha+\delta=0$ — (i), $-\alpha+\beta+2\gamma=0$ — (ii)

$~~\alpha~~ -\beta+\gamma+3\delta=0$ — (iii)

Now,

$\alpha=-\delta$ — (iv)

putting the value of $\delta$ in eq$^n$ 3 from eq$^n$ (iv), and adding eq$^n$ (ii) to it we get

$$-\alpha+\beta+2\gamma=0$$
$$-3\alpha-\beta+\gamma=0$$
$$-4\alpha+3\gamma=0 \Rightarrow \alpha=\frac{3\gamma}{4}$$

using eq$^n$ (ii)

$$\beta=-\frac{5\alpha}{3}$$

Now, substituting values of $\alpha, \beta, \gamma$ & $\delta$ in eq$^n$ (a)

$$\alpha V_1 - \frac{5\alpha}{3}V_2 + \frac{4}{3}\alpha V_3 - \alpha V_4 = 0$$

$$V_1 - \frac{5}{3}V_2 + \frac{4}{3}V_3 - V_4 = 0$$

Hence there exist scalars not all zero such that $\alpha V_1 + \beta V_2 + \gamma V_3 + \delta V_4 = 0$. So it is linearly dependent

Q.11) Determine whether the following set of vectors $\{u, v, w\}$ forms a basis in $R^3$, where $u=(0,1,-1)$, $v=(-1,0,-1)$, & $w=(3,1,3)$.

Given set of vectors can forms a basis in $R^3$ iff ~~and only iff~~ it is able to represents whole $R^3$.

Now, for $\alpha, \beta$ & $\gamma$ ~~& $\delta$~~ $\in R$

$= \alpha u + \beta v + \gamma w$

$= (-\beta+3\gamma, \alpha+\gamma, -\alpha-\beta+3\gamma)$

As $\alpha, \beta$ & $\gamma \in R$ so, $-\beta+3\gamma$, $\alpha+\gamma$ & $-\alpha-\beta+3\gamma$ also belongs to $R$ so given set of vectors form a basis.

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Q12) Is W a subspace of V in the following? Justify your answer

(i) $V = R^{3\times1}(R)$ and W consists $3\times1$ real matrices of the form $\begin{bmatrix} a \\ a \\ a^2 \end{bmatrix}$

Solution: for W be a subspace of it must satisfy internal composition w.r.t addition (+).

Let $W_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_2^2 \end{bmatrix}$ & $W_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_2^2 \end{bmatrix}$

Now, $W_1 + W_2 = \begin{bmatrix} a_1 + b_1 \\ a_1 + b_1 \\ a_2^2 + b_2^2 \end{bmatrix} \notin W$

Since $W_1 + W_2 \notin W$ so it doesn't forms a vector subspace

(ii) $V = R^{3\times3}(R)$ with usual matrix addition and scalar multiplication & W consisting of all $3\times3$ matrices A which are non-singular.

As sum of two non-singular matrices need not to be singular, so it fails the condition of internal composition, therefore it doesn't forms a vector ~~space~~ subspace

(iii) Let V consists of all real polynomials of degree $\leq 4$ with usual polynomial addition and scalar multiplication & W consisting of polynomials of degree $\leq 4$ having coefficient of $t^2$ as 0.

vector space V can be representated as $ax^4 + bx^3 + cx^2 + dx + e$ & W as ~~$ax^4+bx^3+cx+d$~~ $\alpha x^4 + \beta x^3 + \gamma x + \delta$

In order to be a vector subspace W must follow

(i) $W \subseteq V \rightarrow$ as coefficient of $t^2$ is zero & W is a polynomial of degree 4 so it is a subset of V

(ii) Internal composition:

Let us assume $W_1$ as $\alpha_1 x^4 + \beta_1 x^3 + \gamma_1 x + \delta_1$ & $W_2$ as $\alpha_2 x^4 + \beta_2 x^3 + \gamma_2 x + \delta_2$

$W_1 + W_2 = (\alpha_1 + \alpha_2)x^4 + (\beta_1 + \beta_2)x^3 + (\gamma_1 + \gamma_2)x + (\delta_1 + \delta_2)$

[as it ~~fol~~ follows polynomial simple add$^n$]

~~aa~~ $\because W_1 + W_2 \in W$ so it follows internal composition

(iii) External Composition:-

Let us assume a scalar c now,

$cW_1 = c(\alpha_1 x^4 + \beta_1 x^3 + \gamma_1 x^2 + \delta_1)$

$= c\alpha_1 x^4 + c\beta_1 x^3 + c\gamma_1 x^2 + c\delta_1$

$\because cW_1$ also belongs to W so it follows external composition w.r.t multiplication

Hence it is a vector subspace.

Q13) Let V be the set of all 3×1 real matrices. Show that the set $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans V

Solution Let us assume some scalars as a, b & c

Now,

$$a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \\ c \end{pmatrix}$$

$\because a, b, c \in R$ so $a+b, a-b$ & $c \in R$ $\therefore$ we can say that set S spans V.

Q14) Check whether the following vectors are linearly dependent or Independent

$(1,1,0,1), (1,1,1,1),$ ~~(-1,~~ $(-1,-1,1,1), (1,0,0,1)$

Solution:- Let us assume $\alpha, \beta, \gamma$ & $\delta$ as scalars

Now, $V_1, V_2, V_3$ & $V_4$ are said to linearly Independent if $\alpha, \beta, \gamma$ & $\delta$ all are zero simultaneously else dependent.

so, $\alpha V_1 + \beta V_2 + \gamma V_3 + \delta V_4 = 0$

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Substituting $V_1, V_2, V_3$ & $V_4$ we get

$\alpha+\beta-\gamma+\delta=0$ — (I)

$\alpha+\beta-\gamma=0$ — (II)

$\beta+\gamma=0$ — (III)

$\alpha+\beta+\gamma+\delta=0$ — (IV)

Using (I) & (II), $\delta=0$ — (V)

Using (III), (IV) ~~& $\delta=0$~~ & (V) $\Rightarrow \alpha=0$ — (VI)

Using (VI), (III) & (II) $\Rightarrow \alpha=0, \beta=0$

~~Hence~~ Since $\alpha=\beta=\gamma=\delta=0$ so given set of vectors are linearly independent.

Q15) Let $u=(1,-2,1,3), v=(1,2,-1,1)$ & $w=(2,3,1,-1)$

Determine whether or not $x$ is a linear combination of $u, v$ & $w$ where $x=(2,-7,1,11)$

Let us assume that $x$ is a linear combination of $u, v$ & $w$ so we can write $x$ as.

$x=\alpha u+\beta v+\gamma(w)$

Substituting $x, u, v$ & $w$ in above eq$^n$, we get

$\alpha+\beta+2\gamma=2$ — (I)

$-2\alpha+2\beta+3\gamma=-7$ — (II)

$\alpha-\beta+\gamma=1$ — (III)

$3\alpha+\beta-\gamma=11$ — (IV)

Adding ~~Using~~ eq$^n$ (III) & (IV) we get

$\alpha-\beta+\gamma=1$

$3\alpha+\beta-\gamma=11$

$4\alpha=12 \Rightarrow \alpha=3$

Adding eq$^n$ (I) & (III) we get

$2\alpha+3\gamma=3 \Rightarrow \gamma=-1$

putting value of $\alpha$ & $\gamma$ in eq$^n$ (III), we get

$\beta=1$

so our assumption is correct therefore $x$ can be represented in a linear combination of $u, v$ & $w$

Q16) for what values of k do the following vectors forms a basis in $R^3$?
$\{(k,1,1),(0,1,1),(k,0,k)\}$

Solution: Let us assume $\alpha, \beta, \gamma$ as scalars
Since the given set is a basis therefore we can write it as.
$= \alpha(k,1,1) + \beta(0,1,1) + \gamma(k,0,k)$
$= (\alpha k + \gamma k, \alpha + \beta, \alpha + \beta + \gamma k)$
Since it forms a basis so all rows & column must be unique ~~and~~ and non zero
Here all columns are unique but
$\alpha k + \gamma k \neq 0$
$k \neq 0$ [$\because$ ~~$\alpha\beta\gamma$~~ $\alpha, \beta$ & $\gamma$ can be ~~are non~~ zero]
Hence $k \in R - \{0\}$.

Q17) find the dimension and basis for the vector space V, when V is the set of all 3X3 diagonal matrices.

Solution: Basis for a 3x3 diagonal matrix can be given as
$\{(diag(1,0,0), diag(0,1,0), diag(0,0,1)\}$

Let us assume V be $diag(a,b,c)$
rank ~~can~~ for diagonal matrix $diag(a,b,c) = 3$
Since it has 3 independent rows & column and nullspace is zero as no row or column has all element zero therefore by rank nullity theorem
dimension = nullity(v) + rank(v)
$= 0 + 3 = 3$
Hence dimension of given vector space is 3.

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Q18) Let $T: R \to R^2$ over $R$ where
$$T(x) = \begin{pmatrix} x^2 \\ 3x \end{pmatrix}$$, is T is a L.T

Solution! for T to be a linear transformation.

Condⁿ 1:
$$T(\alpha) + T(\beta) = T(\alpha+\beta)$$
$$LHS = \begin{pmatrix} \alpha^2 \\ 3\alpha \end{pmatrix} + \begin{pmatrix} \beta^2 \\ 3\beta \end{pmatrix} = \begin{pmatrix} \alpha^2+\beta^2 \\ 3\alpha+3\beta \end{pmatrix}$$ [Using matrix algebra]

$$RHS = \begin{pmatrix} (\alpha+\beta)^2 \\ 3(\alpha+\beta) \end{pmatrix} \neq LHS$$
Hence T is not a linear Transformation.

Q19) Find Nullspace N(T) & Rank & R(T) & their dimension in the following!

i) $T: R^2 \to R^3$
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ y-x \\ 3x+y \end{pmatrix}$$