Probability and Statistics

Random Variable

A random variable is a function that asigns a real number to each outcome in the sample space of a random experiment. Generally denoted by capital letters X.
After the experiment is conducted the measured value of the random experiment is denoted by lowercase letters such as x.

X is a function from

$$X : \Omega \longrightarrow \mathbb{R}$$
$$\omega \in \Omega \longrightarrow x(\omega) \in \mathbb{R}$$

Two types of random variable

1. Discrete Random Variable : Random Variable which takes discrete / countable values (may be finite or infinite)

2. Continuous Random Variable : Random variable which takes values in an interval. It will always be infinite number of values. However, internal length may be infinite or finite.

Probability Mass Function

For a discrete random variable X with possible values $x_1\ x_2\ x_3 \dots x_n$ a probability mass function is a function such that

(1) $f(x_i) \geq 0$

(3) $f(x_i) = P(X = x_i)$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

Cumulative Distribution Function

The cumulative distribution function of a discrete random variable X, denoted as $F(x)$ is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

Properties

(i) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

(ii) $0 \leq F(x) \leq 1$

(iii) If $x \leq y$ then, $F(x) \leq F(y)$

(iv) $P(X \leq a) = F_X(a)$

(v) $P(X > a) = 1 - F_X(a)$

(vi) $P(c \leq x \leq d) = F_X(d) - F_X(c)$

Example: Consider the following CDF of $X = x$

$$F_X(x) = \begin{cases} 0, & x<0 \\ x+1/2 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \end{cases}$$

find $P(X>1/7)$

$$P(X>1/7) = 1 - F_X(1/7)$$
$$P(X>1/7) = 1 - P(X \le 1/7)$$
$$P(X>1/7) = 1 - \left(\frac{1}{2}+\frac{1}{7}\right) = \frac{5}{14}$$

Probability Distribution function

Let X be a continuous random variable taking values in an given interval say (a,b) then probability distribution function $f_x(x)$ is defined as

$$f_x(x) = \frac{d}{dx} F_x(x)$$

Properties

(i) $f_x(x) \ge 0$

(ii) $\int_{-\infty}^{x} f_x(t)\,dt = F_x(x)$

(iii) Let $B \subseteq R$, then $P(X \in B) = \int_B f_x(x)\,dx$

(iv) Probability of taking a single or finite value is 0

(v) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Example Given that $f_x = \begin{cases} A(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

then find A and $P(X > 1.2)$

for A,

$$\int_0^2 A(2x - x^2) = 1$$

$$A \left| x^2 - \frac{x^3}{3} \right| = 1$$

$$A\left(\frac{4-8}{3}\right) = 1 \qquad \boxed{A = \frac{3}{4}}$$

$$f_x = \int_{1.2}^{2} \frac{3}{4}(2x - x^2) = \frac{3}{4}x^2 - \frac{3}{4}\frac{x^3}{3} \Big|_{1.2}^{2}$$

$$= \frac{3x^2}{4} - \frac{x^3}{4} \Big|_{1.2}^{2} = 3 - 1.08 - 2 + 0.432$$

$$= 0.352$$

Mean of random variable x

Also called average of X, or Expectation of x or expected value of x. Denoted by $E(X)$ or $\mu(x)$. It is given by :

for discrete $= E(x) = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n$

$$= \sum_{i=1}^{n} x_i p_i$$

for continuous $\equiv E(x) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

Variance of a Random Variable X

Variance of random variable can be denoted as Var(x) or σ_X^2

for discrete $\equiv \sigma_X^2 = \sum_{i=1}^{n} (x_i - \mu_X)^2 p_i$

for continuous $\equiv \int_{-\infty}^{\infty} (X-\mu_X)^2 f_X(x)\,dx = \sigma_X^2$

variance of X (Var(X)) $= E(X-\mu_X)^2$

$$= E(X^2 + \mu_X^2 - 2\mu_X x)$$
$$= E(X^2) + E(\mu_X^2) - E(2\mu_X x)$$
$$= E(X^2) + \mu_X^2 - 2\mu_X E(X)$$
$$= E(X^2) + \mu_X^2 - 2\mu_X^2$$

$$\text{Var}(X) = E(X^2) - \mu_X^2$$

for $E(X^2)$ discrete $\equiv \sum_{i=1}^{n} x_i^2 p_i$

continuous $\equiv \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

Example Find mean and variance where PDF is given by :

$$f_X(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$E(x) = \int_{0}^{\infty} x\,2e^{-2x}\,dx = 2\int_{0}^{\infty} xe^{-2x}\,dx$$

$$E(x) = 2\left[x\left(\frac{e^{-2x}}{-2}\right)\Big|_0^{\infty} - \int_0^{\infty} 1\cdot \frac{e^{-2x}}{-2}\,dx \right]$$

$$E(x) = 2\left[0 + \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx \right]$$

$$E(x) = \frac{e^{-2x}}{-2}\Big|_0^{\infty} = \boxed{\frac{1}{2}}$$

$$Var(X) = \int_{-\infty}^{\infty} (X-\mu_X)^2 f_X(x)\,dx$$

$$Var(X) = \int_0^{\infty} \left(x-\frac{1}{2}\right)^2 2e^{-2x}\,dx$$

$$Var(X) = 2\int_0^{\infty} \left(x^2+\frac{1}{4}-x\right) e^{-2x}\,dx$$

$$Var(X) = 2\int_0^{\infty} x^2 e^{-2x} + \frac{1}{2}\int_0^{\infty} e^{-2x}\,dx + 2\int_0^{\infty} -xe^{-2x}\,dx$$

$$Var(X) = 2\int_0^{\infty} x^2 e^{-2x} + \frac{1}{4} - \frac{1}{2}$$

$$Var(X) = \frac{1}{2} + \frac{1}{4} - \frac{1}{2} = \boxed{\frac{1}{4}}$$

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Standard Deviation

Defined as square root of variance and is the measure of average deviation of x from μ_x.

$$S.D. = \sqrt{var(X)} = \sigma_X$$

Types of Discrete Random Variable

(a) Uniform
(b) Binomial
(c) Poisson
(d) Geometric
(e) Negative Binomial
(f) Hypergeometric

Binomial Distribution

The random variable X that equals the number of trials that results in a success has a binomial random variable with parameters $0<p<1$ and $n = 0,1,2,\dots$
where p is probability of success of 1 trial and n is no. of trials.
X has binomial distribution with parameters n and p and ranges from $0,1,2 \dots n$

$$PMF = P(X=K) = {}^nC_K\, p^K (1-p)^{n-K}$$

$$PMF = [p + (1-p)]^K$$
$$PMF = 1^K$$
$$PMF = 1$$

$$\left[\sum_{K=0}^{n} {}^nC_K\, a^K\, b^{n-K} = (a+b)^K \right]$$

$$\text{Mean} = E(x) = \sum x_k P_k$$

$$= \sum_{K=0}^{n} K . P(X=K)$$

$$= \sum_{K=0}^{n} K . {}^nC_K \, p^K (1-p)^{n-K}$$

$$= \sum_{K=0}^{n} K \times \frac{n!}{(K-1)!(n-K)!} p^K (1-p)^{n-K}$$

$$= np \sum_{K=1}^{n} \frac{(n-1)!}{(K-1)![n-1-(K-1)]!} p^{K-1} (1-p)^{n-1-(K-1)}$$

$$= np \sum_{r=0}^{n} \frac{(n-1)!}{r!(n-1-r)!} p^r (1-p)^{n-1-r}$$

$$= np \, [p + (1-p)]^r$$

$$E(x) = np \, (1)^r \qquad \boxed{E(x) = np}$$

$$Var(x) = E(x^2) - n^2p^2$$

$$= \sum_{K=0}^{n} K^2 . P(X=K) - n^2p^2$$

$$= \sum_{K=0}^{n} K^2 . {}^nC_K \, p^K (1-p)^{n-K} - n^2p^2$$

$$= \sum_{K=1}^{n} K^2 . \frac{n!}{K!(n-K)!} p^K (1-p)^{n-K} - n^2p^2$$

$$= \sum_{K=1}^{n} K \frac{(n-1)!}{(K-1)!\,[n-1-(K-1)]!} p^{K-1} (1-p)^{[n-1-(K-1)]} - n^2p^2$$

$$= np \sum_{K=1}^{n} \frac{(n-1)!}{(K-1)!\,[n-1-(K-1)]!} p^{K-1} (1-p)^{[n-1-(K-1)]} - n^2p^2$$

$$+ np \sum_{K=1}^{n} \frac{(n-1)!}{(K-2)!\,[(n-1)-(K-1)]!} p^{K-1} (1-p)^{[n-1-(K-1)]}$$

$$= np + n(n-1)p^2 \sum_{K=1}^{n} \frac{(n-2)!}{(K-2)!\,[(n-2)-(K-2)]!} p^{K-2} (1-p)^{[n-2-(K-2)]} - n^2p^2$$

$$= np + n(n-1)p^2 \, [p + (1-p)]^{K-2} - n^2p^2$$

$$= np + n(n-1)p^2 - n^2p^2$$

$$Var(X) = np(1-p)$$

$$\boxed{Var(X) = npq} \qquad \text{where, } q = 1-p$$

Mean $= np$ and variance $= npq$

Geometric Distribution

Let the Random variable X denote the number of trials until the first success. Then X is a geometric random variable with parameter $0 < p < 1$ and

$$f_x = (1-p)^k p$$

$$PMF = \sum_{K=0}^{\infty} (1-p)^K p = p \sum_{K=0}^{\infty} (1-p)^K$$

$$= p \left[\frac{1}{1-(1-p)}\right] = 1$$ [sum of infinite geometric series]

$$\boxed{PMF = 1}$$

$$\text{Mean} = E(x) = \sum_{K=0}^{\infty} (1-p)^K p \cdot K$$

$$E(x) = p \sum_{K=0}^{\infty} (1-p)^K \cdot K = Sp$$

Let $S = \sum_{K=0}^{\infty} (1-p)^K \cdot K$

$$S = (1-p) + 2(1-p)^2 + 3(1-p)^3 + \dots$$

$$S(1-p) = (1-p)^2 + 2(1-p)^3 + \dots$$

$$Sp = (1-p) + (1-p)^2 + (1-p)^3 + \dots$$

$$Sp = \frac{1-p}{1-(1-p)} = \frac{1-p}{p}$$

$$\boxed{E(x) = \frac{1-p}{p}}$$

$$\text{Variance} = Var(x) = E(x^2) - \left(\frac{1-p}{p}\right)^2$$

$$= \sum_{K=0}^{\infty} (1-p)^K \cdot p \cdot K^2 - \left(\frac{1-p}{p}\right)^2$$

$$= p(p-1)^2 \sum_{K=0}^{\infty} (1-p)^{K-2} \cdot K^2 - \left(\frac{1-p}{p}\right)^2$$

$$= p(1-p)^2 \sum_{K=0}^{\infty} K.(1-p)^{K-2}(K-1) + \sum_{K=0}^{\infty} (1-p)^K pK - \left(\frac{1-p}{p}\right)^2$$

$$= p(1-p)^2 \sum_{K=0}^{\infty} K.(K-1)(1-p)^{K-2} + \sum_{K=0}^{\infty} (1-p)^K pK - \left(\frac{1-p}{p}\right)^2$$

$$= p(1-p)^2 \sum_{K=0}^{\infty} \frac{d^2}{dp^2}(1-p)^K + \left(\frac{1-p}{p}\right) - \left(\frac{1-p}{p}\right)^2$$

$$= p(1-p)^2 \frac{d^2}{dp^2} \sum_{K=0}^{\infty} (1-p)^K + \frac{p(1-p)-(1-p)^2}{p^2}$$

$$= p(1-p)^2 \frac{d^2}{dp^2}\left(\frac{1}{p}\right) + \left[\frac{(1-p)(2p-1)}{p^2}\right]$$

$$= p(1-p)^2\left(\frac{2}{p^3}\right) + \frac{(1-p)(2p-1)}{p^2}$$

$$= \frac{2(1-p)^2}{p^2} + \frac{(1-p)(2p-1)}{p^2}$$

$$= \frac{1-p}{p^2}\left[2-2p+2p-1\right] = \frac{1-p}{p^2}$$

$$\boxed{Var(x) = \frac{1-p}{p^2}}$$

Negative Binomial Distribution

The negative binomial distribution is defined in terms of a random variable X = number of failures before r^{th} success. It is defined as

$$f_x = {}^{K+r-1}C_K \, p^r (1-p)^K$$

$$\text{Mean} = E(X) = \sum_{K=0}^{\infty} K \cdot {}^{r+K-1}C_K \, p^r (1-p)^K$$

$$= \sum_{K=1}^{\infty} \frac{(r+K-1)!}{(K-1)!\,(r-1)!} p^r (1-p)^K$$

$$= \sum_{K=1}^{\infty} \frac{r(1-p)}{p} \, {}^{r+K-1}C_{K-1} \, p^{r+1} (1-p)^{K-1}$$

$$= \frac{r(1-p)}{p} \sum_{x=0}^{\infty} {}^{r+1+x-1}C_x \, p^{r+1} (1-p)^x$$

$$= \frac{r(1-p)}{p} \qquad \text{Mean} = \boxed{E(X) = \frac{r(1-p)}{p}}$$

$$\text{Variance} = \text{Var}(X) = \sum_{K=0}^{\infty} K^2 \, {}^{K+r-1}C_K \, p^r (1-p)^K$$

$$= \sum_{K=0}^{\infty} K^2 \frac{(K+r-1)!}{K!\,(r-1)!} p^r (1-p)^K - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \sum_{K=0}^{\infty} \frac{r(1-p)}{p} K \frac{(K+r-1)!}{(K-1)!\,r!} p^{r+1} (1-p)^{K-1} - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \frac{r(1-p)}{p} \sum_{K=0}^{\infty} \frac{(K+r-1)!}{(K-1)!\,r!} p^{r+1} (1-p)^{K-1} + \frac{r(1-p)}{p} \sum_{K=0}^{\infty} \frac{(K+r-1)!}{(K-2)!\,r!} p^{r+1} (1-p)^{K-1} - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \frac{r(1-p)}{p} + \frac{r^2(1-p)^2}{p} \sum_{K=0}^{\infty} \frac{(K-2+r+1)!}{(K-2)!\,(r+1)!} (1-p)^{K-2}$$

$$\text{Mean} = E(x) = \sum_{K=0}^{\infty} K \cdot {}^{K+r-1}C_K \, p^r (1-p)^K$$

$$= \sum_{K=1}^{\infty} \frac{(r+K-1)!}{(K-1)!\,(r-1)!} p^r (1-p)^K$$

$$= \sum_{K=1}^{\infty} \frac{r(1-p)}{p} \, {}^{r+K-1}C_{K-1} \, p^{r+1} (1-p)^{K-1}$$

$$= \frac{r(1-p)}{p} \sum_{z=0}^{\infty} {}^{r+1+z-1}C_z \, p^{r+1} (1-p)^z$$

$$= \frac{r(1-p)}{p} \qquad \text{Mean} = \boxed{E(x) = \frac{r(1-p)}{p}}$$

$$\text{Variance} = \mathrm{Var}(X) = \sum_{K=0}^{\infty} K^2 \, {}^{K+r-1}C_K \, p^r (1-p)^K$$

$$= \sum_{K=0}^{\infty} K^2 \frac{(K+r-1)!}{K!\,(r-1)!} p^r (1-p)^K - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \sum_{K=0}^{\infty} \frac{r(1-p)}{p} K \frac{(K+r-1)!}{(K-1)!\,r!} p^{r+1} (1-p)^{K-1} - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \frac{r(1-p)}{p} \sum_{K=0}^{\infty} \frac{(K+r-1)!}{(K-1)!\,r!} p^{r+1} (1-p)^{K-1} + \frac{r(1-p)}{p} \sum_{K=0}^{\infty} \frac{(K+r-1)!}{(K-2)!\,r!} p^{r+1} (1-p)^{K-1} - \left[\frac{r(1-p)}{p}\right]^2$$

$$= \frac{r(1-p)}{p} + \frac{r^2(1-p)^2}{p} \sum_{K=0}^{\infty} \frac{(K-2+r+1)!}{(K-2)!\,(r+1)!} (1-p)^{K-2}$$

$$Var(X) = \frac{r(1-p)}{p^2}$$

Hypergeometric distribution

A set of N objects, K classified as successes and N-K objects classified as failures. A sample of size n objects is selected randomly from the N objects where $K \leq N$ and $n \leq N$.

Let the random variable X denotes the number of successes in the sample. Then X is a hypergeometric random variable and

$$f_x(x) = \frac{{}^K C_x \; {}^{N-K}C_{n-x}}{{}^N C_n}$$

$$\text{Mean} = E(X) = \sum_{X=0}^{n} x P(X=x)$$

$$= \sum_{X=0}^{n} x \frac{{}^K C_x \; {}^{N-K}C_{n-x}}{{}^N C_n}$$

$$= \frac{1}{{}^N C_n} \sum_{X=0}^{n} \frac{x \; K!}{x!\,(K-x)!} \; {}^{N-K}C_{n-x}$$

$$= \frac{1}{\frac{N!}{n!\,(N-n)!}} \sum_{X=0}^{n} \frac{x \; K!}{x(x-1)!\,(K-x)!} \; {}^{N-K}C_{n-x}$$

$$= \frac{1}{\frac{N(N-1)!}{n(n-1)!\,[N-1(n-1)]!}} \sum_{X=0}^{n} \frac{K(K-1)!}{(x-1)!\,(K-x)!} \; {}^{N-K}C_{n-x}$$

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$$= \frac{K}{\frac{N}{n}} \times \frac{1}{{}^{N-1}C_{n-1}} \sum_{X=0}^{n} \frac{(K-1)!}{(X-1)!\,[K-1-(X-1)]!} \; {}^{N-K}C_{n-x}$$

$$= \frac{nK}{N} \sum_{X=0}^{n} \left[\frac{{}^{K-1}C_{X-1} \; {}^{N-K}C_{n-x}}{{}^{N-1}C_{n-1}} \right]$$

$$= \frac{nK}{N} \sum_{X=0}^{n} \left(\frac{{}^{N-1}C_{n-1}}{{}^{N-1}C_{n-1}} \right) = \frac{nK}{N}$$

$$\boxed{E(X) = \frac{NK}{n}}$$

$$\text{Variance} = Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{X=0}^{n} x^2 P(X=x) = \sum_{X=0}^{n} (X^2 - X + X) P(X=x)$$

$$= \sum_{X=0}^{n} X^2 - X \; P(X=x) + \sum_{X=0}^{n} x P(X=x)$$

$$= \sum_{X=0}^{n} X(X-1) \frac{{}^{K}C_{X} \; {}^{N-K}C_{n-x}}{{}^{N}C_{n}} + \frac{nK}{N}$$

$$= \frac{1}{\frac{N!}{n!(N-n)!}} \sum_{X=0}^{n} \frac{x(X-1) \; K(K-1)(K-2)!}{X(X-1)(X-2)!\,(K-X)!} \; {}^{N-K}C_{n-x} + \frac{nK}{N}$$

$$= \frac{1}{\frac{N(N-1)(N-2)!}{n(n-1)(n-2)!(N-n)!}} \sum_{X=0}^{n} \frac{K(K-1)(K-2)!}{(X-2)!\,(K-X)!} \; {}^{N-K}C_{n-x} + \frac{nK}{N}$$

$$= \frac{K(K-1)(K-2)}{\frac{N(N-1)}{n(n-1)}} \times \frac{1}{{}^{N-2}C_{n-2}} \sum_{x=0}^{n} {}^{K-2}C_{x-2}\, {}^{N-K}C_{n-x} + \frac{nK}{N}$$

$$= \frac{K(K-1)}{\frac{N(N-1)}{n(n-1)}} \sum_{x=0}^{n} \frac{{}^{K-2}C_{x-2}\, {}^{N-K}C_{n-x}}{{}^{N-2}C_{n-2}} + \frac{nK}{N}$$

$$= \frac{nK(K-1)(n-1)}{N(N-1)} \sum_{x=0}^{n} \frac{{}^{N-2}C_{n-2}}{{}^{N-2}C_{n-2}} + \frac{nK}{N}$$

$$= \frac{nK(K-1)(n-1)}{N(N-1)} + \frac{nK}{N} = \frac{nK(K-1)(n-1) + nK(N-1)}{N(N-1)}$$

$$Var(x) = \frac{nK(K-1)(n-1) + nK(N-1)}{N(N-1)} - \frac{n^2K^2}{N^2}$$

$$Var(x) = \frac{N\,nK(K-1)(n-1) + nK(N-1) - (N-1)n^2K^2}{N^2(N-1)}$$

$$= \frac{nK}{N}\left[\frac{(K-1)(n-1) + (N-1)}{N-1} - \left(\frac{nK}{N}\right)\right]$$

$$= \frac{nK}{N}\left[\frac{(Kn-K-n+1+N-1)N - nK(N-1)}{(N-1)N}\right]$$

$$= \frac{nK}{N}\left[\frac{\cancel{NKn} - NK - NN \cancel{+N} + N^2 \cancel{-N} - \cancel{nKN} + nK}{N(N-1)}\right]$$

$$= \frac{nK}{N}\left[\frac{K(n-N) - N(n-N)}{N(N-1)}\right]$$

$$= \boxed{\frac{nK}{N}\left(\frac{N-K}{N}\right)\left(\frac{N-n}{N-1}\right)}$$

Poisson Distribution

The random variable X that equals the number of events in a poisson process is a poisson random variable with $0<\lambda$ and probability mass function is

$$f_x(x) = \frac{e^{-\lambda}\lambda^K}{K!}$$

$\boxed{\text{PMF} = 1}$ $\equiv \sum_{K=0}^{\infty} \frac{e^{-\lambda}\lambda^K}{K!} = e^{-\lambda} \sum_{K=0}^{\infty} \frac{\lambda^K}{K!}$

$$= e^{-\lambda} \cdot e^{\lambda} = 1$$

Mean $= E(X) = \sum_{X=0}^{\infty} x P(X=x)$

$$= \sum_{X=0}^{\infty} x\left(\frac{e^{-\lambda}\lambda^x}{K!}\right)$$

$$= e^{-\lambda} \sum_{K=0}^{\infty} K\left[\frac{e^{-\lambda}\lambda^K}{K(K-1)!}\right]$$

$$= e^{-\lambda}\lambda \sum_{K=0}^{\infty} \left[\frac{e^{-\lambda}\lambda^{K-1}}{(K-1)!}\right] = e^{-\lambda} \cdot \lambda e^{\lambda}$$

$\boxed{E(X) = \lambda}$

Variance $= \text{Var}(X) = E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_{X=0}^{\infty} X^2 P(X=x)$$

$$= \sum_{X=0}^{\infty} x^2 - x + x \left(\frac{e^{-\lambda}\lambda^x}{x!}\right)$$

$$= \sum_{X=0}^{\infty} x^2 - x \left(\frac{e^{-\lambda}\lambda^x}{x!}\right) + \sum_{X=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{X=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x-2+2}}{x(x-1)(x-2)!} + \lambda$$

$$= e^{-\lambda}\lambda^2 \sum_{X=0}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= e^{-\lambda} . \lambda^2 . e^{\lambda} + \lambda$$

$$= \lambda^2 + \lambda$$

$$Var(X) = \lambda^2 + \lambda - \lambda^2 \Rightarrow \boxed{\lambda = Var(X)}$$

Negative Binomial Proof for Variance

$$Var(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{X=0}^{\infty} x^2 p(X=x) = \sum_{X=0}^{\infty} x^2 - x + x \; P(X=x)$$

$$= \sum_{X=0}^{\infty} (x^2 - x) P(X=x) + \sum_{X=0}^{\infty} x P(X=x)$$

$$= \sum_{X=0}^{\infty} x(x-1) \, {}^{x+r-1}C_x \, p^r (1-p)^x + E(X)$$

$${}^{x+r-1}C_x = (-1)^x \, {}^{-r}C_x$$

$$= p^r \sum_{x=0}^{\infty} x(x-1)(-1)^x p^r (1-p)^x + \frac{rq}{p}$$

$$= p^r \sum_{x=0}^{\infty} x(x-1)\, {}^{-r}C_x (-q)^x + \frac{rq}{p}$$

$$= p^r \sum_{x=0}^{\infty} x(x-1) \frac{-r(-r-1)(-r-2)!}{x(x-1)(x-2)!\,(-r-x)!} (-q)^{x-2+2} + \frac{rq}{p}$$

$$= p^r q^2\, r(r+1) \sum_{x=0}^{\infty} \frac{(-r-2)!}{(x-2)!\,[-r-2-(x-2)]!} (-q)^{x-2} + \frac{rq}{p}$$

$$= p^r q^2 (r^2+r) \sum_{x=0}^{\infty} {}^{-r-2}C_{x-2} (-q)^{x-2} + \frac{rq}{p}$$

$$= p^r q^2 (r^2+r) \left[(1-q)^{-r-2}\right] + \frac{rq}{p}$$

$$= (p^r q^2 r^2 + r p^r q^2)(p)^{-r-2} + \frac{rq}{p}$$

$$= r^2 q^2 p^{-2} + r p^{-2} q^2 + \frac{rq}{p}$$

$$= r^2\left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)$$

$$Var(x) = r^2\left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right)^2 + r\left(\frac{q}{p}\right) - r^2\left(\frac{q}{p}\right)^2$$

$$Var(x) = r\left(\frac{q^2}{p^2}\right) + r\left(\frac{q}{p}\right) = r\left(\frac{q}{p}\right)\left[\frac{q}{p}+1\right]$$

$$\boxed{Var(x) = \frac{rq}{p^2}}$$

Types of Continuous Random Variable

1. Uniform
2. Exponential
3. Gamma
4. Erlang
5. Normal
6. Log Normal
7. Weibull
8. Beta

Probability Density function

A probability density function $f(x)$ can be used to describe the probability distribution of a continuous random variable X.

For a continuous random variable X, a probability density function is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b

for any a and b

Cumulative distribution function

The cumulative distribution function of a continuous random variable X is

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$$

for $-\infty < x < \infty$

Mean and Variance of a continuous random variable

Suppose X is a continuous random variable with probability density function $f(x)$. The mean or expected value of X denoted as μ or $E(X)$

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

The variance of X denoted as $V(X)$ or σ^2

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

The standard deviation of X is $\sigma = \sqrt{\sigma^2}$

Uniform Random Variable

A continuous random variable X with probability density function

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

is a continuous uniform random variable

$$\mu = E(X) = \frac{a+b}{2} \qquad \sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

Deriving mean of function

$$E(x) = \int_a^b x f(x)\,dx = \int_a^b \frac{x\,dx}{b-a}$$

$$E(x) = \frac{1}{b-a}\left|\frac{x^2}{2}\right|_a^b = \left(\frac{b^2-a^2}{2}\right)\frac{1}{b-a}$$

$$\boxed{E(x) = \frac{a+b}{2}}$$

Deriving variance of function

$$V(X) = \int_a^b x^2 f(x)\,dx - \mu^2 = \int_a^b x^2 f(x)\,dx - \left(\frac{a+b}{2}\right)^2$$

$$V(X) = \frac{1}{b-a}\left|\frac{x^3}{3}\right|_a^b - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

$$\boxed{V(X) = \sigma^2 = \frac{(b-a)^2}{12}}$$

PDF = 1

$$\int_a^b f(x)\,dx = \frac{1}{b-a}\int_a^b dx = \frac{1}{b-a}(b-a) = 1$$

$\therefore$ Proved

Normal Random Variable

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}(\sigma)} e^{-(x-\mu)^2/2\sigma^2}$$

$\mu = E(X)$ and $V(X) = \sigma^2$

$f(x)$ is a function with parameters μ and σ where $-\infty < \mu < \infty$ and $\sigma > 0$ and $-\infty < x < \infty$

and the notation $N(\mu, \sigma^2)$ is used to denote distribution

Deriving mean of function

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$E(X) = \int_{-\infty}^{\infty} (\sigma y + \mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2} \sigma\,dy$$

$$\frac{x-\mu}{\sigma} = y$$

$$dx = \sigma\,dy$$

$$x = \sigma y + \mu$$

$$E(X) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy + \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

$$E(X) = 0 + \frac{\mu}{\sqrt{2\pi}} \times \sqrt{2\pi} = \mu$$

$$\boxed{E(X) = \mu}$$

$V(X)$ = Deriving variance of function

$$V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

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$$V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$V(X) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx - \mu^2$$

$$V(x) = \int_{-\infty}^{\infty} (\sigma y+\mu)^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2}} \sigma\,dy - \mu^2$$

$$\frac{X-\mu}{\sigma} = Y$$

$$X = \sigma Y + \mu$$

$$dx = \sigma dy$$

$$V(X) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2} dy + \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy + \frac{2\sigma\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y e^{-y^2/2} dy - \mu^2$$

$$V(X) = \frac{\sigma^2}{\sqrt{2\pi}} (\sqrt{2\pi}) + \frac{\mu^2}{\sqrt{2\pi}} (\sqrt{2\pi}) + 0 - \mu^2$$

$$\boxed{V(X) = \sigma^2}$$

PDF = 1

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\frac{X-\mu}{\sigma} = Y \qquad X = \sigma Y + \mu$$

$$dx = \sigma dy$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/2} \sigma\,dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi}$$

$$= 1$$

Note: Standard Normal Random Variable

A normal random variable with $\mu = 0$ and $\sigma^2 = 1$ is called a standard normal random variable and is denoted by Z. The cummulative distribution function of a standard normal random variable is denoted as

$$\phi(z) = P(Z \le z)$$

Some properties of Z

$f_Z(x)$ is symmetric about y-axis

1. $P(Z>a) = P(Z<-a)$
2. $P(A^c) = 1 - P(A)$
3. $P(Z>a) = 1 - P(Z \le a)$

Example Evaluate the following

(i) $P(Z > 1.26) = 1 - P(Z \le 1.26) = 1 - 0.89616$
$= 0.10384$

(ii) $P(Z < -0.86) = 0.19490 = P(Z > 0.86)$

(iii) $P(Z > -1.37) = P(Z < 1.37) = 0.91465$

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X-\mu}{\sigma}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$

That is, Z is a standard normal random variable.

Standardising to calculate a probability

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z)$$

where, Z is a standard normal random variable and $z = (x-\mu)/\sigma$ is the z-value obtained by standardising X. The probability is obtained by using $z = \frac{x-\mu}{\sigma}$

Example The diameter of a shaft in an optical storage drive is normally distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specification on the shaft are 0.2500 ± 0.0015 inch. What proportion of shafts conforms to specifications

Let X denote the shaft diameter in inches.

$$P(0.2485 < X < 0.2515)$$

$$= P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right)$$

$$= P(-4.6 < Z < 1.4)$$

$$= P(Z < 1.4) - P(Z < -4.6)$$

$$= P(Z < 1.4) - [1 - P(Z < -4.6)]$$

$$= P(Z < 1.4) + P(Z < -4.6) - 1$$

$$= 0.91924 + 0.0000 - 1$$

$$= 0.91924$$

Normal Distribution to the Binomial Distribution

If X is a random variable with parameters n and p,

$$Z = \frac{X - np}{\sqrt{np(1-p)}}$$

is approximately a standard normal random variable. To approximate a binomial probability with a normal distribution a continuity correction is applied as follows:

$$P(X \le x) = P(X \le x + 0.5) \cong P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and,

$$P(x \le X) = P(x - 0.5 \le X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z\right)$$

The approximation is good for $np > 5$ and $n(1-p) > 5$

Probabilities involving X can be approximated by using a standard normal distribution. The approximation is good when n is large relative to p

Normal Approximation to the Poisson Distribution

If X is a poisson distribution random variable with $E(X) = \lambda T$ and $V(X) = \lambda T$

$$Z = \frac{X - \lambda T}{\sqrt{\lambda T}}$$

is approximately a standard normal random variable. The same continuity correction used for binomial distribution can also be applied. The approximation is good for $(\lambda > 5)$

Example Assume that the number of asbestos particles in a squared meter of dust on a surface follows a poisson distribution with a mean of 1000. If a squared meter of dust is analysed, what is the probability that 950 or fewer particles are found?

$$P(X \le 950) = \sum_{x=0}^{950} \frac{e^{-1000}(1000)^x}{x!}$$

$$P(X \le 950) = P(X \le 950.5) \approx P\left(z \le \frac{950.5 - 1000}{\sqrt{1000}}\right)$$

$$P(X \le 950) = P(z \le -1.57)$$

$$P(X \le 950) = 0.058$$

To approximate a poisson probability with a normal distribution a continuity correction is as follows:

$$P\left(\frac{X - \lambda T}{\sqrt{\lambda T}} < \frac{x + 0.5 - \lambda T}{\sqrt{\lambda T}}\right)$$

$$P\left(z < \frac{x + 0.5 - \lambda T}{\sqrt{\lambda T}}\right)$$

Exponential Distribution

The random variable X that equals the distance between successive events of a Poisson process with mean number of events $\lambda > 0$ per unit interval is an exponential random variable with parameter λ. The probability density function of X is

$-\lambda x$ for $0 \le x \le \infty$

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Example Assume that the number of asbestos particles in a squared meter of dust on a surface follows a poisson distribution with a mean of 1000. If a squared meter of dust is analysed, what is the probability that 950 or fewer particles are found?

$$P(X \le 950) = \sum_{x=0}^{950} \frac{e^{-1000}(1000)^x}{x!}$$

$$P(X \le 950) = P(X \le 950.5) \approx P\left(Z \le \frac{950.5-1000}{\sqrt{1000}}\right)$$

$$P(X \le 950) = P(Z \le -1.57)$$

$$P(X \le 950) = 0.058$$

Note: To approximate a poison probability with a normal distribution a continuity correction is as follows:

$$P\left(\frac{X-\lambda T}{\sqrt{\lambda T}} < \frac{x+0.5-\lambda T}{\sqrt{\lambda T}}\right)$$

$$P\left(Z < \frac{x+0.5-\lambda T}{\sqrt{\lambda T}}\right)$$

Exponential Distribution

The random variable X that equals the distance between successive events of a Poisson process with mean number of events $\lambda > 0$ per unit interval is an exponential random variable with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } 0 \le x \le \infty$$

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PDF = 1

$$\int_0^\infty x f(x)\,dx = \int_0^\infty \lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty e^{-\lambda x}\,dx = \lambda\left|\frac{e^{-\lambda x}}{\lambda}\right|_0^\infty$$

$$= \frac{\lambda \times 1}{\lambda} = 1 \quad \therefore \text{Proved}$$

$$E(X) = \cancel{\lambda}\ ^{\frac{1}{}}$$

Deriving formula of mean for exponential distribution

$$E(x) = \int_0^\infty x f(x)\,dx = \int_0^\infty x\lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty x e^{-\lambda x}\,dx$$

$$= \lambda\left[\left(\frac{xe^{-\lambda x}}{-\lambda}\right)\Big|_0^\infty - \frac{1}{\lambda}\left(\frac{e^{-\lambda x}}{-\lambda}\right)\Big|_0^\infty\right]$$

$$= \lambda\left[0 + \frac{1}{\lambda^2}\right] = \frac{1}{\lambda} \qquad \boxed{E(x) = \frac{1}{\lambda}}$$

$$Var(X) = \frac{1}{\lambda^2}$$

Deriving formula of variance for exponential distribution

$$Var(X) = \int_0^\infty x^2 f(x)\,dx - \mu^2 = \int_0^\infty x^2 e^{-\lambda x}\,dx - \left(\frac{1}{\lambda}\right)^2$$

$$Var(X) = x^2\left|\frac{e^{-\lambda x}}{-\lambda}\right|_0^\infty - \int_0^\infty \frac{2x\,e^{-\lambda x}}{-\lambda}\,dx - \left(\frac{1}{\lambda}\right)^2$$

$$Var(X) = 0 + \left(\frac{2}{\lambda^2}\right) - \left(\frac{1}{\lambda^2}\right) = \frac{1}{\lambda^2}$$

$$\boxed{Var(X) = 1/\lambda^2}$$

Lognormal Distribution

Let X be a random variable having normal distribution with mean μ and variance σ^2 then Y be a random variable which is said to have log-normal distribution if

$$Y = e^x$$

possible values of $Y \equiv (0, \infty)$

$$f_Y(Y) = \frac{1}{\sqrt{2\pi}\,\sigma y} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2}$$

mean $\equiv E(X) = e^{\mu + \frac{1}{2}\sigma^2}$ variance $\equiv V(X) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$

Deriving mean of lognormal distribution

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} y f(y)\,dy$$

$$= \int_0^{\infty} y \frac{1}{\sqrt{2\pi}\,\sigma Y} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2} dy$$

$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^{\infty} e^{\frac{-1}{2\sigma^2}\left[(\ln y)^2 + \mu^2 - 2\ln y\,\mu\right]} dy$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^{\infty} e^{\frac{-(\ln y)^2}{2\sigma^2}} \cdot e^{\frac{-\mu^2}{2\sigma^2}}\, e^{\frac{\ln y\,\mu}{\sigma^2}} dy$$

Joint Probability Distribution

If X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a joint probability distribution function.

The joint probability distribution is also sometimes referred to as bivariate probability distribution or bivariate distribution.

Joint Probability Mass Function

The joint probability mass function of the discrete random variables X and Y denoted as $f_{XY}(x,y)$ satisfies

1. $f_{XY}(x,y) \geq 0$

2. $\sum_{x=1}^{n}\sum_{y=1}^{m} f_{XY}(x,y) = 1$

3. $f_{XY}(x,y) = P(X=x, Y=y)$

Joint Probability Density Function

The joint probability density function of the continuous random variable X and Y denoted $f_{XY}(x,y)$ satisfies

1. $f_{XY}(x,y) \geq 0 \quad \forall (x,y)$

2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

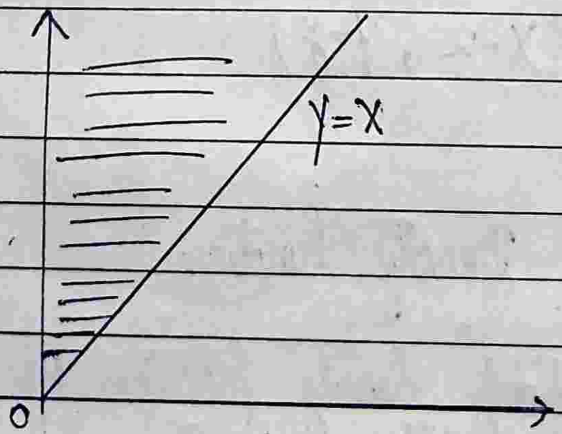
3. for any region R of two dimensional space,

$$P(X,Y) = \iint f_{XY}(x,y)\,dx\,dy$$

Example Let the Random Variable X denote the time until a computer server connects to your machine (in milliseconds) and let Y denote the time until the server authorises a valid user. Each of these random variables measures the wait from a common starting time $X<Y$. function is

$$f_{XY}(x,y) = 6\times10^{6}\exp(-0.001x-0.002y)$$

(i) for $[x<y]$. Prove $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dy\,dx = 1$



The property that this joint probability density function integrates to 1 can be verified by the integral of $f_{xy}(x,y)$ over the region is as follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = \int_{0}^{\infty}\int_{x}^{\infty} 6\times10^{-6} e^{-0.001x-0.002y}\,dy$$

$$= 6\times10^{-6}\int_{0}^{\infty}\left(\int_{x}^{\infty} e^{-0.002y}\,dy\right)e^{-0.001x}\,dx$$

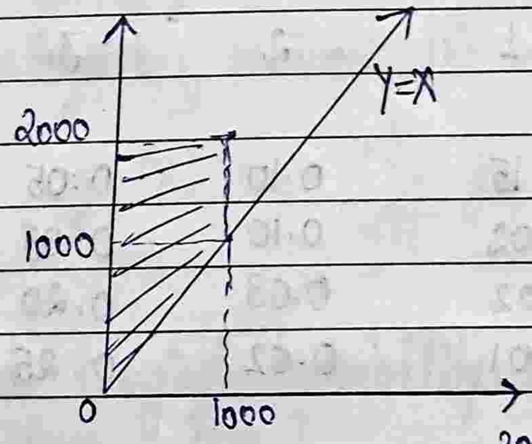
$$= 6\times10^{-6}\int_{0}^{\infty}\left(\frac{e^{-0.002x}}{0.002}\right)e^{-0.001x}\,dx$$

$$= 0.003\int_{0}^{\infty} e^{-0.003x}\,dx$$

$$= 0.003\times\frac{1}{0.003} = 1$$

$\therefore$ Proved

(ii) Calculate the probability for $X<1000$ and $Y<2000$



$$P(X\leq 1000,\ Y\leq 2000) = \int_{0}^{1000}\int_{x}^{2000} f_{XY}(x,y)\,dy\,dx$$

$$= 6\times10^{-6}\int_{0}^{1000}\left(\int_{x}^{2000} e^{-0.002y}\,dy\right)e^{-0.001x}\,dx$$

$$= 6\times10^{-6}\int_{0}^{1000}\left(\frac{e^{-0.002x}-e^{-4}}{0.002}\right)e^{-0.001x}\,dx$$

$$= 0.003 \int_0^{1000} \left(e^{-0.003x} - e^{-4}\right) e^{-0.001x}\, dx$$

$$= 0.003 \left[\left(\frac{1-e^{-3}}{0.003} \right) - e^{-4} \left(\frac{1-e^{-1}}{0.001} \right) \right]$$

$$= 0.003\ (316.738 - 11.578)$$

$$= 0.915$$

Example Calls are made to check the air line schedule at your departure city. You monitor the number of bars of signal strength on your cell phone and the number of times you have to state the name of your departure city before the voice system recognizes the name.

y\x	1	2	3
4	0.15	0.10	0.05
3	0.02	0.10	0.05
2	0.02	0.03	0.20
1	0.01	0.02	0.25

(i) Prove $\sum_{x=1}^{n} \sum_{y=1}^{m} f_{xy}(x,y) = 1$

$\sum_{x=1}^{3} \sum_{y=1}^{4} f_{xy}(x,y) =$ summation of all values of the table

$$= 0.15 + 0.02 + 0.02 + 0.01 + 0.10 + 0.10 + 0.03 + 0.02 + 0.05 + 0.05 + 0.20 + 0.25$$

$$= 1$$

(ii) find probability $P(X-2, Y=1)$

$$P(X=2, Y=1) = 0.02$$

Marginal Probability Distribution

The individual probability distribution of a random variable is referred to as its marginal probability distribution.

Marginal Probability mass function

If we consider random variables X and Y and to determine $P(X=x)$ we sum $P(X=x$ and $Y=y)$ over all points in the range of (X,Y) for which $X=x$.

Subscripts on the probability mass function distinguish between the random variables

$$f_x(x) = \sum_{y=1}^{m} f_{xy}(x,y)$$

Example Find marginal probability mass function when $X=3$ [Signal Bars]

$$f_x(3) = P(X=3) = P(X=3, Y=1) + P(X=3, Y=2) + P(X=3, Y=3) + P(X=3, Y=4)$$

$$= 0.25 + 0.20 + 0.05 + 0.05$$

$$= 0.55$$

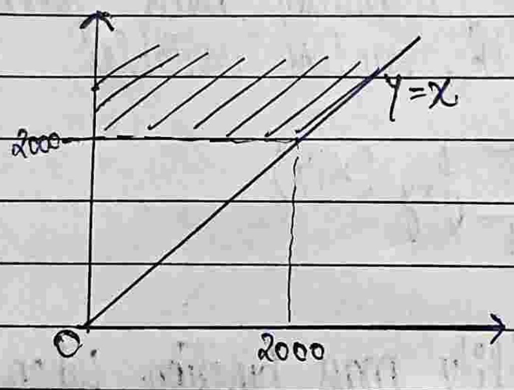
The marginal probability distribution for X is found by summing the probabilities in each column whereas the marginal probability distribution of Y is found by summing the probabilities in each row.

Marginal Probability Density Function

If the joint probability density function of random variables X and Y is $f_{XY}(x, y)$ the marginal probability density function of X and Y are

$$f_X(x) = \int_y f_{XY}(x, y)\, dy \quad \text{and} \quad f_Y(y) = \int_x f_{XY}(x, y)\, dx$$

Example Find the probability that Y exceeds 2000 ($Y \geq 2000$) [Server Access Time]



$y > x > 0$

The probability can be calculated from the marginal probability distribution of Y as follows

$P(Y > 2000) =$

$$f_Y(y) = P(Y > 2000) = \int_{2000}^{\infty} \left(\int_0^y (6 \times 10^{-6})\, e^{-0.001x - 0.002y}\, dx \right)$$

$$= 6 \times 10^{-6}\, e^{-0.002y} \int_{2000}^{\infty} \left(\int_0^y e^{-0.001x}\, dx \right) dy$$

$$= 6 \times 10^{-6}\, e^{-0.002y} \int_{2000}^{\infty} \left| \frac{e^{-0.001x}}{-0.001} \right|_0^y dy$$

$$= 6\times10^{-6}\, e^{-0.002y} \int_{2000}^{\infty} \left(\frac{1 - e^{-0.001y}}{0.001} \right) dy$$

$$= 6\times10^{-3} \int_{2000}^{\infty} e^{-0.002y} \left(1 - e^{-0.001y}\right)$$

$$= 6\times10^{-3} \left[\left(\frac{e^{-0.002y}}{-0.002} \Big|_{2000}^{\infty} \right) - \left(\frac{e^{-0.003y}}{-0.003} \Big|_{2000}^{\infty} \right) \right]$$

$$= 6\times10^{-3} \left(\frac{e^{-4}}{0.002} - \frac{e^{-6}}{0.003} \right) = 0.05$$

Conditional Probability Distribution

The definition of conditional probability for events A and B is $P(B|A) = P(A \cap B) / P(A)$.
This definition can be applied with the event A defined to $X = x$ and event B defined to be $Y = y$

Example Find the conditional probability when
$P(X=3 \mid X=3)$ [Signal Bars]
$P(Y=2 \mid X=3)$

(i) ~~$P(X=3 \mid Y=1$~~ $P(Y=1 \mid X=3) = \frac{P(X=3, Y=1)}{P(X=3)}$

$$P(Y=1 \mid X=3) = \frac{f_{xy}(3,1)}{f_x(3)}$$

$$P(Y=1 \mid X=3) = \frac{0.25}{0.55} = 0.454$$

(ii) $P(Y=2 \mid X=3) = P(X=3, Y=2) / P(X=3)$

$$= \frac{f_{XY}(3,2)}{f_X(3)} = \frac{0.2}{0.55} = 0.364$$

Conditional Probability Density Function

Given continuous random variables X and Y with joint probability density function $f_{XY}(x,y)$ the conditional probability density function of Y given $X=x$ is

$$f_{Y|X}(y) = \frac{f_{XY}(x,y)}{f_X(x)} \quad \text{for } f_X(x) > 0$$

Following properties are satisfied

(1) $f_{Y|X}(y) \geq 0$

(2) $\int f_{Y|X}(y)\,dy = 1$

(3) $P(Y \in B \mid X=x) = \int_B f_{Y|X}(y)\,dy$

Example: Determine the conditional probability density function for Y given that $X=x$. Determine the value when $Y>2000$ and $X=1500$.

$$f_X(x) = \int_x^\infty 6 \times 10^{-6} e^{-0.001x - 0.002y}\,dy$$

$$= 6\times10^{-6}\, e^{-0.001x}\left(\frac{e^{-0.002y}}{-0.002}\Big|_{x}^{\infty}\right)$$

$$= 6\times10^{-6}\, e^{-0.001x}\left(\frac{e^{-0.002x}}{0.002}\right)$$

$$= 0.003\, e^{-0.0003x}$$

$$f_{Y|X} = f_{XY}(x,y)\,/\,f_X(x) = \frac{6\times10^{-6}\, e^{-0.001x-0.002y}}{0.003\, e^{-0.003x}}$$

$$= 0.002\, e^{0.002x-0.002y}$$

for $(0<x)$ and $(x<y)$

$$P(Y>2000\,|\,X=1500) = \int_{2000}^{\infty} f_{Y|1500}(y)\,dy$$

$$= \int_{2000}^{\infty} 0.002\, e^{0.002(1500)-0.002y}\,dy$$

$$= 0.002e^{3}\left(\frac{e^{-0.002y}}{-0.002}\Big|_{2000}^{\infty}\right)$$

$$= 0.002e^{3}\left(\frac{e^{-4}}{0.002}\right) = 0.368$$

Deriving mean for lognormal derivation

Let $t = \dfrac{\ln(x) - \mu}{\sigma}$

$$dt = \frac{1}{\sigma x} dx \qquad \sigma x \, dt = dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \, \sigma \, e^{(\mu + \sigma t)} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2}(t^2 - 2\sigma t + \sigma^2)\right)} \, e^{\left(\mu + \frac{1}{2}\sigma^2\right)} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2}(t-\sigma)^2\right)} dt \; \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$= \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

Deriving variance for lognormal distribution

$$Var(X) = E(X^2) - \left[\exp\left(\mu + \frac{1}{2}\sigma^2\right)\right]^2$$

$$Var(X) = \int_{0}^{\infty} x^2 \frac{1}{x\sqrt{2\pi}\,\sigma} e\left[-\frac{1}{2} \frac{(\ln x - \mu)^2}{\sigma^2}\right] dx$$

$$Var(X) = \int_{0}^{\infty} x \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2} \frac{(\ln(x) - \mu)^2}{\sigma^2}\right] dx$$

Let $t = \dfrac{\ln(x) - \mu}{\sigma}$ $\qquad dt = \dfrac{1}{\sigma x} dx$

$$Var(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(t^2 - 4\sigma t + 4\sigma^2)\right) \exp\left(2\sigma^2 + 2\mu\right) dt$$

$$= \exp(2\sigma^2 + 2\mu) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[\frac{-1}{2}(t-2\sigma)^2\right]$$

$$= \exp(2\sigma^2 + 2\mu) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(t-2\sigma)^2\right]$$

$$= \exp[(2\sigma^2)+2\mu] - \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$= e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$$

Beta Distribution

The random variable X with probability density function

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

is a beta random variable with parameters $\alpha > 0$ $\beta > 0$

$$E(X) = \frac{\alpha(\cancel{\sigma^2})}{\alpha+\beta} \qquad \sigma^2 = V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Deriving mean for Beta distribution

$$E(X) = \int_{-\infty}^{\infty} x f_x(x)\,dx = \int_0^1 \frac{x\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$E(X) = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{(\alpha+1)-1}(1-x)^{\beta-1}\,dx$$

$$E(X) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\left[\frac{\Gamma(\alpha+1)\,\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\right]$$

$$E(X) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}$$

$$E(X) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta).(\alpha+\beta)} \frac{\Gamma(\alpha).\alpha}{\Gamma(\alpha)}$$

$$\boxed{E(x) = \frac{\alpha}{\alpha+\beta}}$$

<u>Deriving variance of Beta Distribution</u>

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx$$

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+2)-1}(1-x)^{\beta-1}dx$$

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)}$$

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)} \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}$$

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1).(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+1).(\alpha+1)}{\Gamma(\alpha)}$$

$$E(X^2) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta).(\alpha+\beta+1).(\alpha+\beta)} \frac{\Gamma(\alpha).(\alpha+1).\alpha}{\Gamma(\alpha)}$$

$$E(X^2) = \frac{(\alpha+1).\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$Var(X) = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2$$

$$Var(X) = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$Var(X) = \frac{(\alpha+1)\alpha(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$Var(X) = \frac{\alpha^3 + \alpha^2\beta + \alpha^2 + \alpha\beta - \alpha^3 - \alpha^2\beta - \alpha^2}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

$$\boxed{Var(X) = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}}$$

Gamma Distribution

The random variable X with probability density function

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)} \quad \text{for } x>0$$

has a gamma random variable with parameters $\lambda>0$ and $r>0$. If r is an integer X has an erlang distribution

If X is a gamma random variable with parameters λ and r,

$$\mu = E(X) = \frac{r}{\lambda} \qquad \sigma^2 = V(X) = \frac{r}{\lambda^2}$$

Weibull Distribution

The random variable X with probability density functions

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right]$$

is a weibull random variable with scale perimeter $\delta>0$ and perimeter $\beta>0$

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$\mu = E(X) = \delta\Gamma\left(1+\frac{1}{\beta}\right)$ $\quad \sigma^2 = V(X) = \delta^2\Gamma\left(1+\frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$

Note

Co-variance and Correlation

Expected value of a function of two random variables

$$E[h(X,Y)] = \begin{cases} \sum\sum h(x,y) f_{XY}(x,y) \\ \iint h(x,y) f_{XY}(x,y)\,dx\,dy \end{cases}$$

$E(h(x,y))$ can be thought of as weighted average $h(x,y)$ for each point in the range of (X,Y). The value of $E(h(x,y))$ represents the average value of $h(x,y)$ that is expected in along sequence of repeated trials

The covariance between the random variable X and Y dented as cov (X,Y) or σ_{XY} is

$$\sigma_{XY} = E(XY) - E(X)E(Y)$$

The correlation between random variable X and Y, denoted as ρ_{XY} is

$$\rho_{XY} = \frac{\text{Cov.}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

For any two random variables $-1 \le \rho_{XY} \le 1$

Note: $\rho = +1$ Perfectly Positively correlated

$0.5 < \rho < +1$ heavily positively correlated

$\rho = 0$ No linear relation

$\rho < 0$ negatively correlated

If X and Y are independent random variable

$$\sigma_{XY} = \rho_{XY} = 0$$

Conditional mean and conditional variance

For discrete, conditional mean and conditional variance

$$E(X/y) = \sum x_i \, P(X=x_i / Y=y_i)$$

$$Var(X/y) = E(X^2/Y=y_i) - [E(X/Y=y_i)]^2$$

For continuous, conditional mean and conditional variance

$$E(X/y=y_i) = \int x_i \, f_{X/Y=y_i}(x) \, dx$$

$$Var(X/Y=y_i) = \int x_i^2 \, f_{X/Y=y_i}(x) \, dx - [E(X/Y=y_i)]^2$$

Moment Generating Function Of Binomial Distribution

$$M(t) = E(e^{xt}) = \sum_{K=0}^{n} e^{tK} P(X=K)$$

$$M_x(t) = \sum_{x=0}^{n} e^{xt} \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$M_x(t) = \sum_{x=0}^{n} (pe^t)^x \frac{n!}{x!\,(n-x)!} (1-p)^{n-x}$$

$$M_x(t) = (1-p+pe^t)^n$$

If we differentiate the moment generating function with respect to t using function of a function rule

$$\frac{dM_x(t)}{dt} = n(1-p+pe^t)^{n-1} pe^t$$

$$= npe^t (1-p+pe^t)^{n-1}$$

evaluating this at $t=0$ gives

$$E(x) = np \qquad \text{(mean)}$$

$$\frac{d^2 M_x(t)}{dt^2} = npe^t \{(n-1)(q+pe^t)^{n-2} pe^t\}$$
$$+ (1-p+pe^t)^{n-1}(npe^t)$$

$$\frac{d^2 M_x(t)}{dt^2} = npe^t(1-p+pe^t)^{n-2}\{(n-1)pe^t + (1-p+pe^t)\}$$

evaluating this at $t=0$ gives,

$$E(x^2) = np(1-p+np)$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = np - np^2 + n^2p^2 - n^2p^2 = np(1-p)$$

$$Var(X) = np(1-p)$$

Moment Generating function for poisson distribution

$$M_x(t) = \sum_{x=0}^{\infty} e^{xt} \left(\frac{\lambda^k \exp(-\lambda)}{k!} \right)$$

$$M_x(t) = \sum_{x=0}^{\infty} e^{xt} \frac{\lambda^x \exp(-\lambda)}{x!}$$

$$M_x(t) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$M_x(t) = e^{-\lambda} e^{e^t\lambda}$$

$$M_x(t) = \exp[\lambda(e^t - 1)]$$

differentiate with respect to t

$$\frac{dM_x(t)}{dt} = \exp[\lambda(e^t - 1)] + \lambda e^t$$

when $t = 0$

$$\frac{dM_x(t)}{dt} = \lambda \qquad E(x) = \lambda \text{ (mean)}$$

$$\frac{d^2M_x(t)}{dt} = \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^t e^{\lambda(e^t-1)} = \lambda + \lambda^2$$

$$E(X^2) = \lambda + \lambda^2$$

$$Var(X) = E(X^2) - [E(X)]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$var(X) = \lambda$$

Moment Generating function for normal distribution

$$N = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$$

$$M_z(t) = E(e^{zt}) = \int e^{zt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$