Unit 3

Vector Space :→ Let $V \neq \phi$ set of certain objects, which may be vectors, matrices, functions or some other objects. Each object is an element of V and is called a vector.

Let $+ : V \times V \to V$ a binary operation, called vector addition

$\cdot : F \times V \to V$ an operation, called scalar multiplication

Then $(V, +, \cdot)$ is called a vector space if $\forall\, a, b, c \in V$, $\alpha, \beta \in F$ the following properties are satisfied.

(1) $a + b \in V$
(2) $a + b = b + a$
(3) $(a+b)+c = a+(b+c)$
(4) $a + 0 = a = 0 + a$ where 0 is the zero element in V
(5) $a + (-a) = 0$

(6) $\alpha \cdot a \in V$
(7) $(\alpha+\beta)\cdot a = \alpha \cdot a + \beta \cdot a$
(8) $(\alpha \cdot \beta)\cdot a = \alpha \cdot (\beta \cdot a)$
(9) $\alpha \cdot (a+b) = \alpha \cdot a + \alpha \cdot b$
(10) $1 \cdot a = a$ where 1 is unity element in F.

Note that F is the set of real numbers or set of complex numbers, called field of scalars.

→ $V = \{0\}$ is called a trivial vector space.

Ex (1) $V = \{$Set of real numbers$\}$ is a vector space with usual addition and scalar multiplication over $F = \mathbb{R}$

(2) $V = \{$ Set of all continuous functions i.e. $f : [a,b] \to \mathbb{R}\}$
V is a vector space over $\mathbb{R}$.

(3) $V = \{$ Set of polynomials of degree $\leq n\}$
is a Vector space over $F = \mathbb{R}$.

(4) $V = \{$ Set of all $m \times n$ matrices$\}$
V is a vector space over $F = \mathbb{R}$.
Here $0 =$ Null matrix.

(5) let $V = \{$ Set of all polynomials of degree $n\}$.
Then V is not a vector space.
$\because$ let $p_n = a_0 + a_1 x + \_\_\_ + a_{n-1} x^{n-1} + x^n \in V$
$q_n = b_0 + b_1 x + \_\_\_ + b_{n-1} x^{n-1} - x^n \in V$
$p_n + q_n = (a_0 + b_0) + (a_1 + b_1)x + \_\_\_ + (a_{n-1} + b_{n-1})x^{n-1}$
is a polynomial of deg $(n-1)$
$\Rightarrow p_n + q_n \notin V$.

(6) $V = \{$ Set of all polynomials of degree $n\}$
for $a, b \in V$, $a + b = ab$
and usual scalar multiplication.
Then V is not a vector space.
$\because$ let $p_n = a_0 + a_1 x + \_\_\_ + a_n x^n \in V$
$q_n = b_0 + b_1 x + \_\_\_ + b_n x^n \in V$
Then $p_n + q_n = p_n \cdot q_n$ is a polynomial of degree
$2n \Rightarrow p_n + q_n \notin V$
$\Rightarrow$ V is not a vector space.

(7) $V = \{\mathbb{R}^2$ i.e. Set of all ordered pairs (x, y), $x, y \in \mathbb{R}$
Vector addition is defined as
$a + b = (x_1, y_1) + (x_2, y_2) = (2x_1 - 3x_2, y_1 - y_2)$
$\alpha \cdot a = \alpha(x_1, y_1) = \left(\frac{\alpha x_1}{3}, \frac{\alpha y_1}{3}\right)$

Check whether V is a vector space or not over $F = \mathbb{R}$.

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Solⁿ:→ $a+b \neq b+a$

$\because$ let $a=(x_1,y_1)$, $b=(x_2,y_2)$

Then $a+b = (x_1,y_1)+(x_2,y_2) = (2x_1-3x_2, y_1-y_2)$

$b+a = (x_2,y_2)+(x_1,y_1) = (2x_2-3x_1, y_2-y_1)$

Similarly $(a+b)+c \neq a+(b+c)$

Also $1.(x_1,y_1) = (x_1,y_1) = \left(\frac{x_1}{3},\frac{y_1}{3}\right) \neq (x_1,y_1)$

$\therefore \Rightarrow 1.a \neq a$ where 1 is the unity Element in $\mathbb{R}$.

$\Rightarrow V$ is not a vector space over $\mathbb{R}$

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let $V=\mathbb{R}^2$ i.e. set of all ordered pairs.

let $a=(x_1,y_1)$, $b=(x_2,y_2) \in V$

$a+b = (x_1,y_1)+(x_2,y_2) = (x_1x_2, y_1y_2)$

and $\alpha a = \alpha(x_1,y_1) = (\alpha x_1, \alpha y_1)$

Then show that V is not a V.S over $\mathbb{R}$.

→ Subspace :→ let $(V,+,\cdot)$ be a vector space over field F.
let W be a non-Empty subset of V. i.e. $W \subseteq V$.
Then W is called a subspace of V if W is a vector space under the vector addition '+' and scalar multiplication '$\cdot$'.

Two Step test to check the Subspace:→

let $(V,+,\cdot)$ be a vector space over F

and $W \neq \phi$, $W \subseteq V$

Then W is subspace of V if

$a+b \in W \ \forall \ a,b \in W$

and $\alpha a \in W \ \forall \ \alpha \in F, a \in W$.

One step Test:→ let $(V,+,\cdot)$ be a V.S. over F

and $W \neq \phi$, $W \subseteq V$

Then W is subspace of V if

$\alpha a + \beta b \in W \ \forall \ \alpha,\beta \in F, \ a,b \in W$.

Ex. Let $V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\}$; $F = \mathbb{R}$

V is a V.S. with usual addition & scalar multiplication.

(1) let $W = \{(x_1, \ldots, x_n) \mid x_1 = 0\} \subseteq V$

Then W is subspace of V

$\because$ let $a = (0, x_2, \ldots, x_n)$; $b = (0, y_2, \ldots, y_n) \in W$

Then $a + b = (0, 2x_2, \ldots, 2x_n) \in W$

And $\alpha \in F$; $\alpha a \in W$.

(2) let $W = \{(x_1, \ldots, x_n) \mid x_1 \geq 0\}$

Then W is not a subspace of V.

$\because$ let $\alpha = -1$ let $a = (x_1, \ldots, x_n) \in W$

Then $\alpha a = -a = -(x_1, \ldots, x_n)$

$= (-x_1, -x_2, \ldots, -x_n) \notin W$.

(3) let $W = \{(x_1, \ldots, x_n) \mid x_2 = x_1 + 1\}$

Then W is not a subspace of V

$\because$ let $a = (x_1, x_2, \ldots, x_n)$; $b = (y_1, y_2, \ldots, y_n) \in W$

$\Rightarrow x_2 = x_1 + 1$; $y_2 = y_1 + 1$.

Then $a + b = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

And $x_2 + y_2 = (x_1 + 1) + (y_1 + 1)$

$= x_1 + y_1 + 2 \neq (x_1 + y_1) + 1$

$\Rightarrow$ W is not a subspace.

Ex. let $V = \{p(x) = a_0 + a_1 x + \ldots + a_m x^m$ – polynomials of deg. $\leq m\}$

is a V.S. with usual addition and scalar multiplication

$F = \mathbb{R}$

Then

(1) let $W = \{$polynomials of deg $\leq m \mid p(0) = 0\}$

Then W is a subspace of W

$\because$ let $p(x), q(x) \in W \Rightarrow p(0) = 0 = q(0)$

$p(x) + q(x) \in W$ as $p(0) + q(0) = 0$

And $\alpha p(x) \in W$ as $\alpha p(0) = 0$, $\alpha \in F$.

(2) $W = \{ P(x) \in V \mid P(0) = 1 \}$
is not a Subspace
$\because$ let $P(x), q(x) \in W$
$\Rightarrow P(0) = 1 = q(0)$
Then $P(x) + q(x) \notin W$ $(\because P(0) + q(0) = 2 \neq 1)$

(3) $W = \{ P(x) \in V \mid$ Coefficients are Positive $\}$
Then W is not a Subspace
$\because$ let $P(x) = a_0 + a_1 x + \_\_ + a_m x^m \in W$, $a_0, a_1 \_\_ a_m > 0$.
let $\alpha = -1 \in F$
Then $\alpha P(x) = -a_0 - a_1 x - \_\_ - a_m x^m \notin W$

HW $V = \{$Set of all ~~m~~ $n \times n$ real Square matrices$\}$ is a V.S. with usual matrix addition & scalar multiplication. $F = \mathbb{R}$
(1) $W = \{$Set of all Symmetric matrices$\}$
(2) $W = \{$Set of all upper Triangular matrices$\}$
(3) $W = \{$Set of all $n \times n$ matrices having real positive Elements$\}$.

Check that W is a Subspace of V or not.