Linear Independence of vectors:→

Let V be a vector space, $V_1, V_2 \ldots V_n$ be n vectors of V. Then $\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n$ is called a Linear Combination of vectors where $\alpha_1, \alpha_2 \ldots \alpha_n \in F (= \mathbb{R}$ or $\mathbb{C})$

→ The Set of vectors $\{V_1, V_2, \ldots V_n\}$ is said to be linearly dependent if there Exist scalars $\alpha_1, \alpha_2 \ldots \alpha_n$, not all zero such that

$$\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_n V_n = 0.$$

→ The Set of Vectors $\{V_1, V_2, \ldots V_n\}$ is said to be Linearly Independent if $\alpha_1 V_1 + \ldots + \alpha_n V_n = 0$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0.$$

In other words, the set of vectors $\{V_1 \ldots V_n\}$ is Linearly dependent iff atleast one element / vector Can be written as a Linear Combination of the other vectors.

i.e. Let $\alpha_i \neq 0$

$$\text{Then } V_i = -\frac{1}{\alpha_i}\left[\alpha_1 V_1 + \alpha_2 V_2 + \ldots + \alpha_{i-1} V_{i-1} + \alpha_{i+1} V_{i+1} + \ldots + \alpha_n V_n\right].$$

Ex Let $V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3 = (0,0,1) \in \mathbb{R}^3$.
S.T. $\{V_1, V_2, V_3\}$ is Linearly Independent.

Sol^n Let $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$.

$\Rightarrow \alpha_1 (1,-1,0) + \alpha_2 (0,1,-1) + \alpha_3 (0,0,1) = 0$

$\Rightarrow (\alpha_1, -\alpha_1 + \alpha_2, -\alpha_2 + \alpha_3) = 0 = (0,0,0)$

$\Rightarrow \alpha_1 = 0, \quad -\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = 0$

$\quad -\alpha_2 + \alpha_3 = 0 \Rightarrow \alpha_3 = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow \{V_1, V_2, V_3\}$ is L.I.

Ex Let $V_1 = (1,-1,0)$, $V_2 = (0,1,-1)$, $V_3 = (0,2,1)$, $V_4 (1,0,3) \in \mathbb{R}^3$
Then S.T. $\{V_1, V_2, V_3, V_4\}$ is L.D.

Sol^n Consider the Equation $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

$\Rightarrow \alpha_1(1,-1,0) + \alpha_2(0,1,-1) + \alpha_3(0,2,1) + \alpha_4(1,0,3) = 0$

$\Rightarrow (\alpha_1 + \alpha_4, -\alpha_1 + \alpha_2 + 2\alpha_3, -\alpha_2 + \alpha_3 + 3\alpha_4) = 0$

$\Rightarrow \alpha_1 + \alpha_4 = 0 \Rightarrow \alpha_1 = -\alpha_4$

$-\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \Rightarrow \alpha_2 + 2\alpha_3 = -\alpha_4$

$-\alpha_2 + \alpha_3 + 3\alpha_4 = 0 \Rightarrow \alpha_2 - \alpha_3 = 3\alpha_4$

$3\alpha_3 = -4\alpha_4$

$\alpha_3 = \frac{-4}{3}\alpha_4$

$\Rightarrow \alpha_2 = \alpha_3 + 3\alpha_4$

$= \left(\frac{-4}{3} + 3\right)\alpha_4 = 5/3\,\alpha_4$

$\therefore \alpha_1 = -\alpha_4, \; \alpha_2 = \frac{5}{3}\alpha_4, \; \alpha_3 = \frac{-4}{3}\alpha_4.$

If $\alpha_4 = 1, \; \alpha_1 = -1, \; \alpha_2 = \frac{5}{3}, \; \alpha_3 = \frac{-4}{3}$

So not all scalars are zero

$\Rightarrow \{v_1, v_2, v_3, v_4\}$ are L.D.

→ Singleton Set is L.I. Set

$\because$ let $A = \{a\}$ be a Singleton Set, $A \neq \phi$, $a \neq 0$

let $\alpha a = 0$; α is a Scalar

Either $\alpha = 0$, or $a = 0$

$a \neq 0 \Rightarrow \alpha = 0$

$\Rightarrow A = \{a\}$ is Singleton Set

→ Every Subset of L.I. Set is L.I.

i.e. Let $S = \{v_1, v_2 \_\_ v_n\}$ be a L.I. Set

Then S.T. $S_1 = \{v_1, v_2 \_\_ v_k\}, k < n$ is also L.I.

Sol^n let $\alpha_1 v_1 + \_\_ + \alpha_k v_k = 0$

$\Rightarrow \alpha_1 v_1 + \_\_ + \alpha_k v_k + 0.v_{k+1} + \_\_ + 0v_n = 0$

And $\{v_1 \_\_ v_n\}$ is L.I.

$\Rightarrow \alpha_1 = 0 = \alpha_2 = \_\_ = \alpha_k = 0$

$\Rightarrow S_1$ is also L.I. Set.

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→ ϕ is subset of every set.
⇒ Empty set is Linearly Independent.

→ Superset of Every Linearly dependent set is Linearly dependent.
8 i.e. Let $S = \{v_1, \ldots v_k\}$ be a LD Set
Then S.T. $S' = \{v_1, \ldots v_n\}$ $n > k$ is also L.D. Set.

Pf If $S' = \{v_1 \ldots v_n\}$, $n > k$ is Linearly Independent
⇒ Every Subset of S' is also Linearly Independent
⇒ S is also L.I.
— Contradiction.
⇒ S' is L.D. Set.

Que Examine whether the following set of vectors is L.I. or L.D.
(1) (1,2,3,4), (2,0,1,−2), (3,2,4,2)
(2) (1,1,0,1), (1,1,1,1), (−1,1,1,1), (1,0,0,1)

Elementary Row and Column operations:→
(1) Interchange of any two rows ($R_i \sim R_j$)
(2) Multiplication / division of any row by a non-zero scalar (αR_i)
(3) Adding / Subtracting a scalar multiple of any row to another row ($R_i \leftarrow R_i + \alpha R_j$)

Echelon form of a matrix:→ An $m \times n$ matrix is called a row Echelon matrix if the no. of zeros preceeding the first non-zero entry of a row increases row by row until a row having all zero entries is obtained.
↳ i.e. (1) If ith row contains all zeros, then it is true for all subsequent ~~zeros~~ rows.
(2) If a column contains a non-zero entry of any row then every subsequent entry in this column is zero.
(3) Rows containing all zeros occur only after all non-zero rows.

for Ex $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are the matrices in row Echelon form.

Similary we can define the Column echelon form of a matrix.

Rank of a matrix :→ The number of Linearly Independent rows ~~or~~ ~~Columns~~ of a matrix gives the ranks of the matrix.

→ If A is $m \times n$ matrix then rank of $A \leq \min\{m, n\}$.

→ In other words, the no. of non-zero rows in the row echelon form of a matrix gives the rank of the matrix.

Ex Reduce the following matrices to row Echelon form and find their ranks.

(i) $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 + 2R_1$

$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix}$ $R_3 \leftarrow R_3 + 2R_2 = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

⇒ # of non-zero rows in row Echelon form is 2

⇒ Rank of $A = 2$.

H.W (1) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$ (2) $A = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$

→ Examine whether the following set of vectors is L.I. or not.

(1) (1, 2, 3, 4) ; (2, 0, 1, -2); (3, 2, 4, 2)

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix}$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 - 3R_1$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & -4 & -5 & -10 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

⇒ # of non-zero rows in $A = 2$

⇒ Rank $(A) = 2$.

⇒ given set of vectors are linearly dependent.