Linear Transformation

Let V & W be two vector spaces over F. A function $T : V \to W$ is said to be a linear transformation from ~~W~~ V to W if $\forall\ x, y \in W$ & $\forall\ c \in F$

1) $T(x+y) = T(x) + T(y)$
2) $T(cx) = cT(x)$

Properties of Linear Transformation

1) $T(0) = 0$

Proof:– $0 + 0 = 0 \qquad 0 \in V$

$T(0+0) = T(0)$

$\Rightarrow T(0) + T(0) = T(0) \qquad \because T(0) \in W$

$\Rightarrow T(0) = 0 \in W.$

2) T is linear iff $T(cx+y) = cT(x) + T(y)$ $\forall\ x, y \in V$, $c \in F$

Proof– Let T be linear transformation

Consider $T(cx+y)$

$= T(cx) + T(y) \qquad [\because T \text{ is OP w.r.t } (+)$

$= cT(x) + T(y) \qquad$ (T preserves scalar multiplication)

$\Rightarrow T(cx+y) = cT(x) + T(y)$

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Conversely Let $T(cx+y) = cT(x) + T(y)$ $\forall x, y \in V$, $c \in F$

T.P (i) $T(x+y) = T(x) + T(y)$

(ii) $T(cx) = cT(x)$

Proof Let $c = 1$ in (1)

Then $T(1.x+y) = 1.T(x) + T(y)$

$= T(x) + T(y)$

which proves (i)

Now, let $y = 0$ in (1)

$T(cx+0) = cT(x) + T(0)$

$= c.T(x) + 0 = c.T(x)$

which proves (ii)

3. T is linear then $T(x-y) = T(x) - T(y)$ $\forall x, y \in V$

Pf: Consider

$T(x-y)$

$= T(x+(-y)) = T(x) + T(-y)$ [$\because$ T is linear]

$= T(x) + (-1).T(y)$ [$\because$ T is linear]

$= T(x) - T(y)$

$\Rightarrow T(x-y) = T(x) - T(y)$

4. T is linear iff $\forall x_1, x_2 \ldots x_n \in V$, $\forall a_1, a_2 \ldots a_n \in F$

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

L.H.S $= T\left(\sum_{i=1}^{n} a_i x_i\right)$

$\Leftrightarrow T(a_1 x_1 + a_2 x_2 + \cdots a_n x_n)$

$\Leftrightarrow T(a_1 x_1) + T(a_2 x_2) + \cdots T(a_n x_n)$

$\Leftrightarrow a_1 T(x_1) + a_2 T(x_2) + \cdots a_n(T(x_n))$

$\Leftrightarrow \sum_{i=1}^{n} a_i T(x_i) = \underline{\underline{\text{R.H.S}}}$

Hence, the result holds both ways.

Example-1 $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(a_1, a_2) = (2a_1 + a_2, a_1)$

Check if T is a linear transformation

Let $c \in \mathbb{R}$ & $x, y \in \mathbb{R}^2$

where $x = (b_1, b_2)$, $y = (d_1, d_2)$

Now $(cx + y)$

$= c(b_1, b_2) + (d_1, d_2)$

$= (cb_1 + d_1, cb_2 + d_2)$

$T(cx + y) = T(cb_1 + d_1, cb_2 + d_2)$

$= (2(cb_1 + d_1) + (cb_2 + d_2), cb_1 + d_1)$

$= ((2cb_1 + cb_2) + 2d_1 + d_2, cb_1 + d_1)$

$= (2cb_1 + cb_2, cb_1) + (2d_1 + d_2, d_1)$

$= c(2b_1 + b_2, b_1) + (2d_1 + d_2, d_1)$

$= c\,T(b_1, b_2) + T(d_1, d_2)$

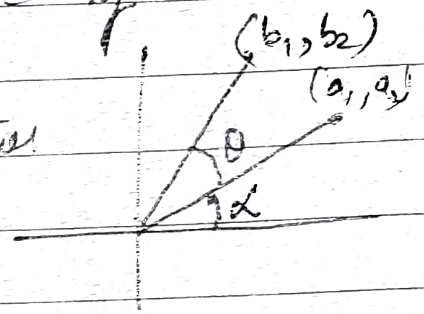
$= c\,T(x) + T(y)$

Hence T is linear.

Example-2 For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counter clockwise by θ if $(a_1, a_2) \neq (0,0)$ & $T_\theta(0,0) = (0,0)$

Let (a_1, a_2) be a non-zero vector of $\mathbb{R}^2$ that makes an angle of α with the +ve ~~axis~~ x-axis



Now, magnitude of this vector

$$r = \sqrt{a_1^2 + a_2^2}.$$

then $a_1 = r\cos\alpha$,

$a_2 = r\sin\alpha$.

Now,

$T_\theta(a_1, a_2) = (b_1, b_2)$ is the vector in $\mathbb{R}^2$ ~~abt~~ with same magnitude r with an angle of $\alpha + \theta$ with +ve direction of x-axis.

$$T_\theta(a_1, a_2) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha))$$

$$= (r(\cos\theta.\cos\alpha - \sin\theta\sin\alpha), r(\sin\theta.\cos\alpha + \cos\theta\sin\alpha))$$

$$= (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$$

$$T_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$$

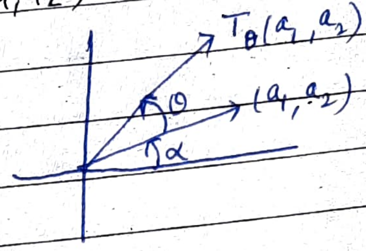
$$T_\theta(0,0) = (0,0)$$

Check that T is linear transformation

Rotation by θ

$$T_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta,\ a_2\cos\theta - a_1\sin\theta)$$

T.P $\quad T_\theta(c(a_1,a_2)+(b_1,b_2)) = c\,T_\theta(a_1,a_2) + T_\theta(b_1,b_2)$



LHS

$$T_\theta[ca_1+b_1,\ ca_2+b_2]$$

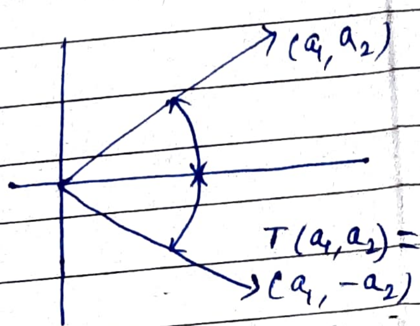
$$= [(ca_1+b_1)\cos\theta - (ca_2+b_2)\sin\theta,\ (ca_2+b_2)\cos\theta - (ca_1+b_1)\sin\theta]$$

$$= [ca_1\cos\theta - ca_2\sin\theta,\ ca_2\cos\theta - ca_1\sin\theta] + [b_1\cos\theta - b_2\sin\theta,\ b_2\cos\theta - b_1\sin\theta]$$

$$= c[a_1\cos\theta - a_2\sin\theta,\ a_2\cos\theta - a_1\sin\theta] + [b_1\cos\theta - b_2\sin\theta,\ b_2\cos\theta - b_1\sin\theta]$$

$$= c\,T_\theta(a_1,a_2) + T_\theta(b_1,b_2) = RHS$$

Hence T_θ is L.T.



Reflection about x-axis

$$T(a_1,a_2) = (a_1, -a_2)$$

T.P $\Rightarrow$ $T[c(a_1,a_2)+(b_1,b_2)] = c\,T(a_1,a_2) + T(b_1,b_2)$

LHS $\quad T[c(a_1,a_2)+(b_1,b_2)]$

$$= T(ca_1+b_1,\ ca_2+b_2)$$

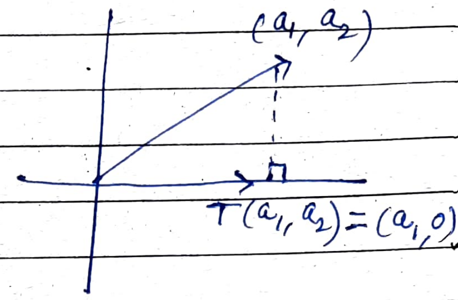
$$= [ca_1 + b_1, \ -(ca_2 + b_2)]$$

$$= [(ca_1 - ca_2) + (b_1 - b_2)]$$

$$= c(a_1 - a_2) + (b_1 - b_2)$$

$$= c\,T(a_1, a_2) + T(b_1, b_2) = \underline{R.H.S}$$

Ref Map is a L.T



* Projection on x-axis

$$T(a_1, a_2) = (a_1, 0)$$

T.P $\quad T[c(a_1, a_2) + (b_1, b_2)] = c\,T(a_1, a_2) + T(b_1, b_2)$

LHS $\quad T[c(a_1, a_2) + (b_1, b_2)]$

$$= T(ca_1 + b_1, \ ca_2 + b_2) = [ca_1 + b_1, 0]$$

$$= \cancel{T} \ (ca_1, 0) + (b_1, 0)$$

$$= c(a_1, 0) + (b_1, 0)$$

$$= c\,T(a_1, a_2) + T(b_1, b_2) = RHS$$

* $T: M_{m\times n}(F) \longrightarrow M_{n\times m}(F)$ s.t. $T(A) = A^t$

Let $A = [a_{ij}]$, $B = [b_{ij}] \in M_{m\times n}(F)$, $c \in F$

T.P $\quad T[cA + B] = c\,T(A) + T(B)$

LHS $\quad T[c[a_{ij}] + [b_{ij}]] = T[ca_{ij} + b_{ij}]$

$$= [c[a_{ij}] + b_{ij}]^t$$

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$= [C\,a_{ij}]^t + [b_{ij}]^t \qquad [(A+B)^t = A^t + B^t]$

$= C\,[a_{ij}]^t + [b_{ij}]^t$
$= C\,T(A) + T(B) \quad = \underline{R.H.S}$

* $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ s.t. $T(f(x)) = f'(x)$

T.P $\quad T(c\,f(x) + g(x)) = c\,T(f(x)) + T(g(x))$

$[f(x), g(x) \in P_n(\mathbb{R}), c \in \mathbb{R}]$

LHS $\quad T[c\,f(x) + g(x)]$

$= [c\,f(x) + g(x)]' = [c\,f(x)]' + g'(x)$
$= c\,f'(x) + g'(x)$
$= c\,T(f(x)) + T[g(x)] = R.H.S$

* $V = C(\mathbb{R}) \to$ V.S of cts real valued f^n's on $\mathbb{R}$

Let $a, b \in \mathbb{R}$, $a < b$ $\quad T : V \to \mathbb{R}$ s.t. $T(f) = \int_a^b f(t)\,dt$

T.P $\quad T(cf + g) = c\,T(f) + T(g)$
where $f, g \in V$, $c \in \mathbb{R}$

LHS $\quad T(cf+g) = \int_a^b (c f(t) + g(t))\,dt$

$= \int_a^b c\,f(t)\,dt + \int_a^b g(t)\,dt$

$= c \int_a^b c\,f(t)\,dt + \int_a^b g(t)\,dt$

$= c\,T(f) + T(g) \quad = \underline{RHS}$

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* **Identity Transformation**

$I : V(F) \to V(F)$ s.t. $I(x) = x \ \forall x \in V$

Consider, $I(cx_1 + x_2)$ $\quad x_1, x_2 \in V, \ c \in F$

$= \cancel{c(x_1 + x_2)} \ cx_1 + x_2$

$= cI(x_1) + I(x_2)$

* **Zero Transformation**

$T_0 : V \to W, \quad T_0(v) = 0_w$

Consider, $T_0(cv_1 + v_2)$ $\quad v_1, v_2 \in W$

$= 0_w$

$= c0_w + 0_w$

$= cT_0(v_1) + T_0(v_2)$ $\quad \Rightarrow T_0$ is L.T

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Eg-7. $V = C(\mathbb{R}) \to$ vector space of continuous real valued functions on $\mathbb{R}$.

Let $a, b \in \mathbb{R}$, $a < b$. $T : V \to \mathbb{R}$ s.t

$$T(f) = \int_a^b f(t).dt$$

Let $f(t)$ & $g(t) \in C(\mathbb{R})$

$c \in \mathbb{R}$

T.P

$$T(cf+g) = cT(f) + T(g)$$

L.H.S $\quad T(cf+g) = \int_a^b (c f(t) + g(t))\, dt$

$$= \int_a^b c f(t)\, dt + \int_a^b g(t)\, dt$$

$$= c.\int_a^b f(t)\, dt + \int_a^b g(t)\, dt$$

$$= cT(f) + T(g) = R.H.S.$$

Hence Proved.

8. Identity Transformation

Let $I : V \to V$ s.t $I(x) = x \;\; \forall x \in V$.

$x_1, x_2 \in V$, $c \in F$

Consider $I(cx_1 + x_2)$

$$= cx_1 + x_2$$

$$= c(x_1) + (x_2)$$

$$= c\, I(x_1) + I(x_2)$$

Hence I is an linear transformation

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Eg. Zero transformation:-

$T_0 : V \to W$ s.t. $T_0(v) = 0_w$

Consider $T_0(cv_1 + v_2)$
$= 0_w$
$= c.0_w + 0_w$
$= c.T(v_1) + T(v_2)$

Thus, T_0 is a linear transformation.

Defn: Let V & W be vector spaces. $T: V \to W$ be linear.

Null space $N(T)$ of T or Kernel of T:
$= \{x \in V : T(x) = 0_w\}$.

Range $R(T)$ of T or image of T:

$R(T) = \{T(x) : x \in V\}$

In zero transformation
$N(T_0) = \{x \in V : T_0(v) = 0_w\} = V$
$R(T_0) = \{0_w\}$

$N(I) = \{x \in V : I(v) = 0_w\} = \{0_v\}$.
$R(I) = V$

Eg:- Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation
s.t. $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$T(c(a_1, a_2, a_3) + (b_1, b_2, b_3))$

$= T(ca_1+b_1, ca_2+b_2, ca_3+b_3)$

$= (ca_1+b_1 - ca_2 - b_2, \ 2ca_3+2b_3)$

$= [c(a_1-a_2) + (b_1-b_2), \ c(2a_3) + (2b_3)]$

$= c(a_1-a_2, 2a_3) + (b_1-b_2, 2b_3)$

$= c\,T(a_1,a_2,a_3) + T(b_1,b_2,b_3)$

Hence T is a linear transformation.

$N(T) = \{(a_1,a_2,a_3) \in \mathbb{R}^3$ s.t. $T(a_1,a_2,a_3) = (0,0)\}$

$\Rightarrow (a_1-a_2, 2a_3) = (0,0)$

$\Rightarrow a_1 - a_2 = 0, \quad 2a_3 = 0$

$a_1 = a_2, \quad a_3 = 0$

$\Rightarrow N(T) = \{(a_1,a_1,0) : a_1 \in \mathbb{R}\}$.

$R(T) = \{T(a_1,a_2,a_3) : (a_1,a_2,a_3) \in \mathbb{R}^3\}$

$= \{(a_1-a_2, 2a_3), \ a_1,a_2,a_3 \in \mathbb{R}\}$

$= \mathbb{R}^2$

THM-2(1). Let V & W be vector spaces. $T: V \to W$ is a linear transformation. Then $N(T) \to$ Null space of T, ~~R(T)~~ is a subspace of V & $R(T) \to$ range of T is a subspace of W.

By defn $N(T) \subseteq V$

Proof: Let 0_V & 0_W denote the zero vectors of V & W respectively.

As $T(0_V) = 0_W$, therefore $0_V \in N(T)$

Now consider $T(x+y)$, $x, y \in N(T)$

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T.P. $(x+y) \in N(T)$, $cx \in N(T)$, $x, y \in N(T)$, $c \in F$

Consider $T(x+y)$
$= T(x) + T(y)$ [T is linear]
$= O_W + O_W$ [$x, y \in N(T)$]
$= O_W$
$\Rightarrow x+y \in N(T)$

Consider $T(cx)$
$= c\,T(x)$ = [T is linear]
$= c \cdot O_W$ [$x \in N(T)$]
$= O_W$

$\Rightarrow cx \in N(T)$
Hence $N(T)$ is a subspace of V

Now

By defn, $R(T) \subseteq W$
As $T(O_V) = O_W$ $\Rightarrow O_W \in R(T)$

Let $x, y \in R(T)$, $c \in F$
T.P. $x+y \in R(T)$, $cx \in R(T)$

~~Consider~~ Since $x, y \in R(T)$
$\Rightarrow \exists\, v, w \in V$ s.t
$T(v) = x$, $T(w) = y$.
Consider ~~$T(v+w)$~~
$x+y$
$= T(v) + T(w)$
$= T(v+w) \in R(T)$
$\because v, w \in V$, as $v, w \in V$
$\Rightarrow x+y \in R(T)$

Consider cx

$= cT(v)$

$\in R(T)$ $[\because cv \in V$ as $v \in V$ & $c \in F]$.

* Thus, $R(T)$ is a subspace of W.

Thm-2.2 Let V & W be vector spaces and $T: V \to W$ be linear transformation. Then $\beta = \{v_1, v_2, \ldots v_n\}$ be the basis for V. Then $R(T) = \text{span}(T(\beta))$

$= \text{span}(\{T(v_1), T(v_2), \ldots T(v_n)\})$.

Proof: Clearly $T(v_i) \in R(T)$ $\forall i$.

As $R(T)$ is a subspace

$\therefore R(T)$ contains $\text{span}(\{T(v_1), T(v_2), \ldots T(v_n)\})$

$= \text{span}(T(\beta))$

[$\because$ span of any subspace S of a vector space is a subspace of V]

i.e. $\text{span}(T(\beta)) \subseteq R(T)$

Now To Prove $R(T) \subseteq \text{span}(T(\beta))$

Let $w \in R(T)$

$\Rightarrow \exists\, v \in V$ s.t $w = T(v)$

As β is a basis of V

$\therefore v = \sum_{i=1}^{n} a_i v_i$, $a_i \in F$, $i = 1, 2, \ldots n$

$T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right)$

$\Rightarrow w = T(v) = \sum_{i=1}^{n} a_i T(v_i)$ [T is linear

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$$w = T(v_i) = \sum_{i=1}^{n} a_i T(v_i)$$

$$\in \text{span}(T(\beta))$$

Hence $R(T) \subset \text{span}(T(\beta))$

Thus $R(T) = \text{span}(T(\beta))$

Remark: The above theorem is true even if β is finite.

Eg-10 $T: P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is a linear transformation given by.

$$T(f(x)) = \begin{bmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{bmatrix}$$

since $\beta = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$

$$R(T) = \text{span}\,[T(\beta)]$$
$$= \text{span}(\{T(1), T(x), T(x^2)\})$$

$$T(1) = \begin{bmatrix} 1-1=0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T(x^2) = \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R(T) = \text{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right)$$

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$$R(T) = \text{span}\left(\left\{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\right)$$

Since, the spanning set of $R(T)$ is now a linearly independent set. $\therefore$ It is a basis of $R(T)$.

$\therefore \dim R(T) = 2.$