→ $f: \mathbb{R}^2 \to \mathbb{R}$ is a function

If $\forall (x,y) \in \mathbb{R}^2$, $\exists$ unique $z \in \mathbb{R}$ s.t.

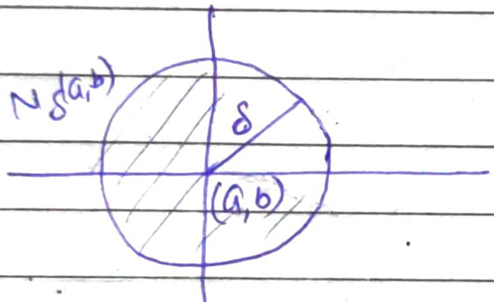
$$z = f(x,y)$$

→ <u>Neighbourhood of (a,b)</u>:→

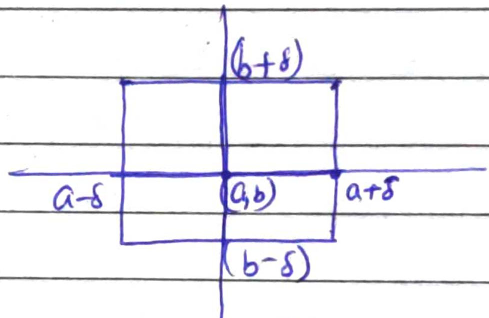
$$N_\delta(a,b) = \{(x,y) \mid \sqrt{(x-a)^2 + (y-b)^2} < \delta\}$$

or

$$N_\delta(a,b) = \{(x,y) \mid |x-a| < \delta,\ |y-b| < \delta\}$$



Circular Nbd



Square Nbd.

→ <u>Limit of a function</u>:→

If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ be a function

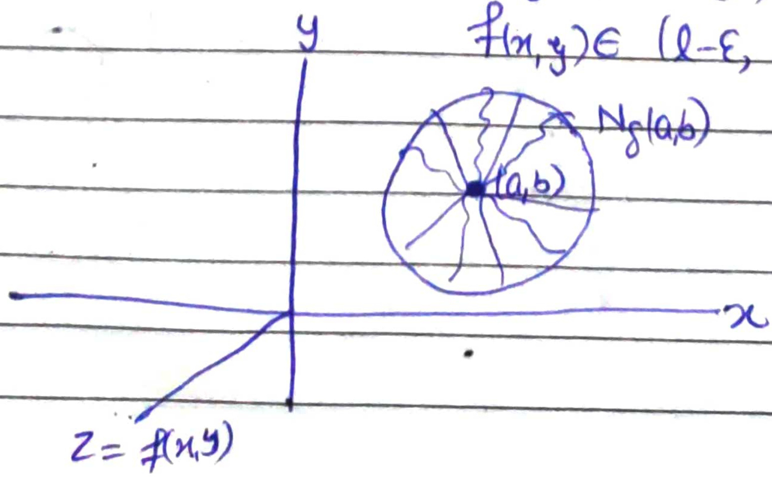
Then $l \in \mathbb{R}$ is called limit of f

as $(x,y) \to (a,b)$ if

$\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t.

~~f(x~~ If $(x,y) \in N_\delta(a,b) - \{(a,b)\}$ Then

$f(x,y) \in (l-\varepsilon, l+\varepsilon)$



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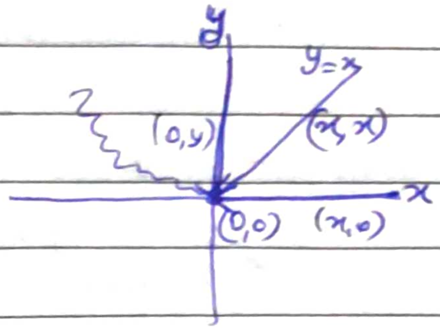
Ex $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2 = 0 \end{cases}$ $f: \mathbb{R}^2 \to \mathbb{R}$

Then at $(0,0)$ $\lim f$ Exist or not.



Sol^n along x-axis

$$\lim_{(x,0)\to(0,0)} f(x,y) = 1$$

along y-axis

$$\lim_{(0,y)\to(0,0)} f(x,y) = -1$$

along line $y=x$

$$\lim_{(x,x)\to(0,0)} f(x,y) = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) \text{ does not Exist.}$$

($\because$ As we are approaching towards different values in different directions)

Ex→ $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} ; & x-y \neq 0 \text{ or } x \neq y \\ 0 ; & x=y \end{cases}$ Then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

along x-axis; $\lim\limits_{(x,0)\to(0,0)} f(x,y) = 0$

along y-axis; $\lim\limits_{(0,y)\to(0,0)} f(x,y) = 0$

along $y=mx$ line; $\lim\limits_{(x,mx)\to(0,0)} f(x,y) = \lim\limits_{(x,mx)\to(0,0)} \dfrac{x^3(1+m^3)}{x(1-m)} = 0$

Put $x = r\cos\theta$; $y = r\sin\theta$; $r = \sqrt{x^2+y^2}$

$(x,y)\to(0,0)$
$\Rightarrow r \to 0$
and θ is any angle.

$$f(x,y) = f(r,\theta) = \frac{r^3(\cos^3\theta + \sin^3\theta)}{r(\cos\theta - \sin\theta)}$$

$$= \frac{r^2(\cos^3\theta + \sin^3\theta)}{(\cos\theta - \sin\theta)}$$

$\cos\theta = \sin\theta \Rightarrow \tan\theta = 1$
$\Rightarrow \theta = \pi/4$

$\Rightarrow$ for all those curves whose slope is 1 at (0,0), $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

may not exist.

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2 = 0 \end{cases}$; $\lim\limits_{(x,y)\to(0,0)} f(x,y) = ?$

$x = r\cos\theta, \quad y = r\sin\theta$

$f(r,\theta) = \dfrac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta\sin\theta$

different values for different θ's.

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Continuity: $\Rightarrow$ If $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$

Then f is called continuous function at $P = (a,b) \in D$ if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2 = 0 \end{cases}$ at $P(0,0) \in \mathbb{R}^2$

$\because \lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\Rightarrow f(x,y)$ is not cts at (0,0)

$f: \mathbb{R}^2 \to \mathbb{R}$

Ex:→ $f(x,y) = \begin{cases} x\sin\left(\frac{1}{x}\right) + y\sin\left(\frac{1}{y}\right); & xy \neq 0 \\ 0 ; & xy = 0 \end{cases}$

Then at $(0,0)$, f is cts or not?

Sol^n $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ $\left(\because \sin\frac{1}{x}, \sin\frac{1}{y} \text{ bdd}\right)$

$= f(0,0)$

Ex:→ $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2 = 0 \end{cases}$ at $(0,0)$

Sol^n $x = r\cos\theta$; $y = r\sin\theta$

$(x,y) \to 0 \Rightarrow r \to 0$ and θ is any angle

$f(r,\theta) = \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} = r(\cos^3\theta - \sin^3\theta)$

$\to 0$ as $r \to 0$

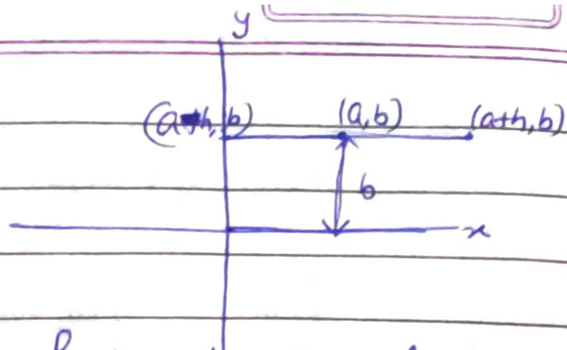
and $(\cos^3\theta - \sin^3\theta)$ is bdd

and $f(0,0) = 0$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Partial Derivatives :→

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$



$$f_y(a,b) = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

f_x and f_y are called first order partial derivatives of f w.r.t. x and y respectively.

Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; x^2+y^2 \neq 0 \\ 0 & ; x^2+y^2 = 0 \end{cases}$. $P(0,0)$ be any point.

find $f_x(0,0)$ and $f_y(0,0)$.

Solⁿ $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

$f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0.$

But $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

$\Rightarrow$ $f(x,y)$ is not continuous at $(0,0)$.

Ex $f: \mathbb{R}^2 \to \mathbb{R}$; $f(x,y) = \begin{cases} x\sin\left(\frac{1}{x}\right) + y\sin\left(\frac{1}{y}\right) & ; x\neq 0, y\neq 0 \\ x\sin\left(\frac{1}{x}\right) & ; x\neq 0, y=0 \\ y\sin\left(\frac{1}{y}\right) & ; x=0, y\neq 0 \\ 0 & ; x=0, y=0. \end{cases}$

Find $f_x(0,0)$ & $f_y(0,0)$

Solⁿ $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{h\sin\frac{1}{h}}{h}$

$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right) \to$ does not exist.

$f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \dfrac{k\sin\left(\frac{1}{k}\right)}{k} \to$ does not exist

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But $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$

$\Rightarrow f(x,y)$ is continuous fun. at $(0,0)$

\# If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$,
s.t. f has both first order Partial derivatives Exist at Point $(a,b) \not\Rightarrow f$ is cts at point (a,b)

\# If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$
s.t. f is continuous at Point $(a,b) \not\Rightarrow f$ has both first order partial derivatives exist at point (a,b).

\# If $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$
s.t. $f_x(a,b)$ and $f_y(a,b)$ Exist and are bdd in the neighbour-hood of $(a,b) \Rightarrow f$ is cts function at point (a,b).

H.W (1) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} 1 ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$ Find $f_x(0,0)$ & $f_y(0,0)$.

(2) $$f(x,y) = \begin{cases} \sin\left(\frac{xy}{x^2+y^2}\right) ; & \text{If } (x,y) \neq (0,0) \\ 0 ; & \text{If } (x,y) = (0,0) \end{cases}$$

Find $f_x(0,0)$ and $f_y(0,0)$. Is f Continuous at $(0,0)$.

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Que $f(x,y) = \sqrt{|xy|}$

Find $f_x(0,0)$, $f_y(0,0)$ and Cont. at $(0,0)$

Solⁿ $f(0,0) = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \sqrt{|r^2 \cos\theta \sin\theta|}$$

$$= \lim_{r\to 0} |r| \sqrt{|\cos\theta \sin\theta|} \to 0 = f(0,0)$$

$\Rightarrow f$ is cts at $(0,0)$

Que
$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & ; x^2+y^2 \neq 0 \\ 0 & ; x^2+y^2 = 0 \end{cases}$$

is cts at $(0,0)$; $f_x(0,0)$, $f_y(0,0)$?

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$f_{yx}(a,b) = \lim_{k\to 0} \frac{f_x(a,b+k) - f_x(a,b)}{k}$$

$$f_{xy}(a,b) = \lim_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$

$z = f(x,y)$

$$\frac{\partial z}{\partial x} = f_x \; ; \; \frac{\partial z}{\partial y} = f_y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx} \; ; \; \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$$

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$$f(x,y) = \begin{cases} \frac{x^3y}{x^2+y^2} ; & x^2+y^2 \neq 0 \\ 0 ; & x^2+y^2=0 \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \frac{\frac{h^3k}{h^2+k^2}}{h} = \lim_{h\to 0} \frac{h^2k}{h^2+k^2} = 0$$

$$f_{yx}(0,0) = 0$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{h^3k}{(h^2+k^2)k}$$

$$= \lim_{k\to 0} \frac{h^3}{h^2+k^2} = h$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

→ $f(x,y) = \log(xy + 2y^2 - 2x)$

Find $f_x(2,3)$ & $f_y(2,3)$

$$f_y = \frac{1}{xy+2y^2-2x}(x+4y)$$

$$f_x = \frac{1}{xy+2y^2-2x}(y-2)$$

$$f_y(2,3) = \frac{14}{6+18-4} = \frac{14}{20}$$

$$= 7/10$$

$$f_x(2,3) = \frac{1}{6+18-4}(1) = \frac{1}{20}$$