Unit-II

Dependent/Independent variables:→ The variable whose value is assigned is called Independent variable and the variable whose value is obtained w.r.t. assigned value is called dependent variable.

If $f: A \rightarrow B$ be any function
then $\forall\, x \in A$, $\exists$ unique $y \in B$ s.t.

$$\underbrace{y}_{\text{dependent variable}} = f(\underbrace{x}_{\text{Independent variable}})$$

Differential Equation:→ An equation containing dependent variables, Independent variables and the derivatives of dependent variables w.r.t Independent variables.

If $y = f(x)$

Then $\frac{dy}{dx} + \sin x = y$

$\frac{dy}{dx} = y - x$

Ordinary Diff. Equⁿ (ODE)

If $y = f(x,t)$

$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = 2\sin t$

$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = x + t$

Partial Diff. Equⁿ (PDE)

Order of ODE:→ The highest order derivative in the differential equation is called the order of differential equation.

Degree of ODE:→ The highest power of the highest derivative in the differential Equⁿ provided all the derivatives are in natural power.

Eg. $y' + \sqrt{y} = \sin x$ order - 1, degree - 1

$y'' + \sqrt{y'} = x$ order - 2

$\Rightarrow y'' - x = -\sqrt{y'}$

$\Rightarrow (y'' - x)^2 = (-\sqrt{y'})^2$

Date
Page

$\Rightarrow (y'')^2 + x^2 - 2xy'' = y'$

$\Rightarrow$ Order $=2$, degree $=2$

$$-\begin{cases} y'' + e^{y'} = x & \text{order}-2 \\ y'' + \left(1 + y' + \frac{(y')^2}{\underline{|2}} + \dots\right) = x \end{cases}$$

$\downarrow$ These two are not same as only finitely many terms should be there in an equation.

$\Rightarrow$ degree of this ODE is not deffined.

$y' + |y| = x$ order $=1$, degree $=1$

$|y'| + y = x$ Order $=1$, degree not defined.
($\because$ Power of y' is not natural power)

Ques $(y'')^{3/2} = (y'')^{2/3}$ Find degree and order.

Sol^n $\Rightarrow (y'')^9 = (y'')^4$ order $=2$, degree $=9$

If we do $\frac{(y'')^9}{(y'')^4} = 1$

$\Rightarrow (y'')^5 = 1$ $\Rightarrow$ degree $=5$ $\rightarrow$ This is wrong.

Linear and Non-linear ODE:→

The differential equation $f(x, y, y', y'', \dots, y^{(n)}) = 0$ is called linear ODE if

(1) all the derivatives and dependent variables are of degree 1.

(2) There does not Exist any ~~product~~ term which contains product of dependent variables and derivatives or two derivatives.

E.g. $y \cdot y'$ or $y \cdot y''$ or $y' \cdot y''$ should not be present.

Date
Page

Ques Which of following is/are linear Diff. Equ^n

(a) $y'' + \sin y = x$

(b) $y'' + y.y' = x$

✓(c) $y'' + xy' = \sin x$

(d) $y'' + xy = y'.y$

→ An ODE is linear diff Equ^n $\Rightarrow$ deg of this diff Equ^n = 1

But Converse need not to be true.

i.e. If deg of any diff Equ^n = 1 $\not\Rightarrow$ It is linear diff Equ^n.

E.g. $y.y' = x$. Order = 1, degree = 1

↳ But it is Non-Linear.

→ Solution of diff Equ^n $f(x, y, y', y'', \ldots, y^{(n)}) = 0$

The function $\phi(x)$ is called solution of $f = 0$ defined on domain D If

(1) $\phi(x), \phi'(x), \phi''(x), \ldots, \phi^{(n)}(x)$ Exists $\forall\, x \in D$

(2) ϕ satisfies the diff. Equ^n $f = 0$

i.e. $f(x, \phi, \phi', \phi'', \ldots, \phi^{(n)}) = 0.$

First order first degree ODE:→

Any of these two forms.

$$\left[\begin{array}{l} \dfrac{dy}{dx} = f(x,y) \quad -(1) \\[2ex] M(x,y)\,dx + N(x,y)\,dy = 0 \quad -(2) \end{array}\right]$$

Date
Page

Formation of differential Equⁿ →

Ex $y = A\cos 2x + B\sin 2x$

$y' = -A(\sin 2x)(2) + B\cos 2x(2)$

$= -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x$

$= -4y$

$\Rightarrow y'' + 4y = 0.$ → Second order, Linear Diff Equⁿ.

Ex $y = cx + \frac{1}{c}; \quad c \neq 0$

$y' = c$

$\Rightarrow y = y'(x) + \frac{1}{y'}$

$\Rightarrow yy' = x(y')^2 + 1$

$\Rightarrow x(y')^2 - yy' + 1 = 0$ — First order, Second degree, Non-Linear.

H.W (1) $y = \frac{a}{x^2} + bx$; a, b arbitrary Constants

(2) $y = ce^{qx}$; q: fixed constant

(3) $y = C\cos(Pt - a)$: P: fixed Constant.

Find order and degree of following ODE's.

(4) $[1 + (y')^2]^{1/2} = x^2 + y$

(5) $y' = \sin y$

(6) $(1 + y')^{1/2} = y''$.

Date			
Page			

Separable ODEs:→

$$y' = f(x,y)$$

If $f(x,y) = g(x)$ – function of one variable x alone

then $\int \frac{dy}{dx} = \int g(x) + c$

$$\Rightarrow y = F(x) + c$$

If $f(x,y) = g(x) \cdot h(y)$ – Separable form

Then $\frac{dy}{dx} = g(x)h(y)$

$$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)dx$$

$$\Rightarrow A(y) = B(x) + c.$$

Que $y' - 2y + a = 0$, a is fixed Constant. Find gen sol^n

Sol^n $\frac{dy}{dx} = 2y - a \Rightarrow \frac{dy}{2y-a} = dx$, $y \neq \frac{a}{2}$ —(1)

For $y = \frac{a}{2}$; $\frac{dy}{dx} = 0$ and the given diff Equ^n is satisfied

$\Rightarrow y = \frac{a}{2}$ is a solution.

But It is not a General solution as this doesnot Contain any arbitrary Constant.

So Integrating (1), we get

$$\int \frac{dy}{2y-a} = \int dx + c$$

$$\Rightarrow \frac{1}{2}\log|2y-a| = x + c$$

$$\Rightarrow \log|2y-a| = 2x + 2c$$

$$\Rightarrow |2y-a| = e^{2x} \cdot e^{2c_1}$$

$$\Rightarrow 2y - a = k \cdot e^{2x} \text{ where } k = \pm e^{2c}.$$

OR $y = Ce^{2x} + \frac{a}{2}$; where $c = \frac{k}{2}$.

Que $y' + xy = x$

Sol^n $\frac{dy}{dx} = x - xy$

$\Rightarrow \frac{dy}{dx} = x(1-y)$

$\Rightarrow \int \frac{dy}{1-y} = \int x\,dx$; $\cancel{y \neq 1}$

$\Rightarrow -\log(1-y) = x + c$

$\Rightarrow \log|1-y| = -x - c$

$\Rightarrow |1-y| = e^{-x-c}$

$\Rightarrow |1-y| = e^{-x} \cdot e^{-c}$

$\Rightarrow |1-y| = ke^{-x}$, $\cancel{k = e^{-c}}$

$\Rightarrow 1-y = ke^{-x}$ $\quad (k = \pm e^{-c})$

$\Rightarrow$

$\boxed{\cancel{\text{Que} \; y\log y \frac{dy}{dx}} =}$

Que $x\frac{dy}{dx} = y\log y$

$\int \frac{dy}{y\log y} = \int \frac{dx}{x} + c$

Put $\log y = t$

$\frac{1}{y}dy = dt$

$\Rightarrow \int \frac{dt}{t} = \int \frac{dx}{x} + c$

$\Rightarrow \log t = \log x + \log c$

$\Rightarrow \underline{t = cx}.$

Que. Using $xy = v$, reduce $xy' = e^{-xy} - y$ into Separable

Sol^n

$$xy = v$$
$$xy' + y = \frac{dv}{dx}$$

$$\Rightarrow \quad xy' = e^{-xy} - y$$
$$\Rightarrow \quad xy' + y = e^{-xy}$$
$$\Rightarrow \quad \frac{dv}{dx} = e^{-v}$$

$$\Rightarrow \int \frac{dv}{e^{-v}} = \int dx$$

$$\Rightarrow \quad e^{v} = x + c$$
$$\Rightarrow \quad e^{xy} = x + c \quad \text{or} \quad xy = \log|x+c|$$

HW (1) $xy' = y + x^2 \tan\left(\frac{y}{x}\right)$; $\frac{y}{x} = t$

(2) $(xy' - y) \cos\left(\frac{y}{x}\right) + x = 0$; $\frac{y}{x} = t$.

→ Equations Reducible to Separable form:→

$$\frac{dy}{dx} = f(ax+by+c)$$

Put $ax + by + c = z$

$$a + b\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$$\Rightarrow \quad \frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$$

$$\Rightarrow \quad \frac{dz}{dx} = a + bf(z)$$

$$\Rightarrow \int \frac{dz}{a + bf(z)} = \int dx + c$$

Date
Page

Ex

$y' = \cancel{(2y-y^2)^{1/2}}$

$\cancel{\;} = (y(2-y))^{1/2}$

$y' = \sqrt{y}\sqrt{2-y}$

$\Rightarrow \dfrac{dy}{\;}$

Ex

$y' = (4x+y)^2$

Put $4x+y = t$

$4 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\Rightarrow \dfrac{dt}{dx} - 4 = t^2 \Rightarrow \dfrac{dt}{dx} = t^2+4$

$\Rightarrow \displaystyle\int \frac{dt}{t^2+4} = \int dx + c$

$\Rightarrow \dfrac{1}{2}\tan^{-1}\left(\dfrac{t}{2}\right) = x + c$

$\Rightarrow \dfrac{1}{2}\tan^{-1}\left(\dfrac{4x+y}{2}\right) = x + c$

$\Rightarrow \tan^{-1}\left(\dfrac{4x+y}{2}\right) = 2(c+x)$

H.W. Ex

$\dfrac{dy}{dx} = (2x-y+1)^2$

Que

$y' = \sqrt{2y-y^2}$

Put $y-1 = t$

$\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\dfrac{dt}{dx} = \sqrt{2(t+1) - (t+1)^2}$

$= \sqrt{2t + 2 - t^2 - 1 - 2t}$

$= \sqrt{1-t^2}$

$\displaystyle\int \frac{dt}{\sqrt{1-t^2}} = \int dx + c$

$\sin^{-1} t = x + c$

$\sin^{-1}(y-1) = x + c$

Date			
Page			

Homogeneous function :⇒

A function $f(x,y)$ is said to be homogeneous function of deg. n if by substituting $x = \lambda x$, $y = \lambda y$ produces

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

Here n can be an Integer or any real number.

E.g $f(x,y) = x^2+y^2$

$f(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2(x^2+y^2) = \lambda^2 f(x,y)$

$\Rightarrow$ homogeneous fun of deg 2.

E.g $f(x,y) = \dfrac{x^2+y^2+xy}{x^2-y^2}$, $x \neq y$

$f(\lambda x, \lambda y) = \dfrac{\lambda^2 x^2 + \lambda^2 y^2 + \lambda^2 xy}{\lambda^2 x^2 - \lambda^2 y^2} = \lambda^0 f(x,y)$

$\Rightarrow$ hom. function of degree 0.

E.g $f(x,y) = x^3 \log\left[\dfrac{\sqrt{x+y}}{\sqrt{x-y}}\right]$

$f(\lambda x, \lambda y) = \lambda^3 x^3 \left[\log\left(\dfrac{\sqrt{\lambda x+\lambda y}}{\sqrt{\lambda x-\lambda y}}\right)\right] = \lambda^3 x^3 \log\left[\sqrt{\dfrac{x+y}{x-y}}\right]$

$= \lambda^3 f(x,y)$

$\Rightarrow$ hom. function of degree 3.

Homogeneous first order differential equation :⇒

$y' = f(x,y)$ is called a hom. equation if $f(x,y)$ is a hom. function of degree 0.

Eg. $y' = \dfrac{x^3+y^3+x^2y}{x^3+y^3}$ is a homogeneous equation.

Method to solve hom. Equation:→

Put $y = vx$ or $x = vy$
and Reduce the given ODE to a separable form.

Ex $(x^2+4y^2+xy)\,dx - x^2\,dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+4y^2+xy}{x^2}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x^2+4x^2v^2+x^2v}{x^2}$$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = 1+4v^2+\cancel{v}$$

$$\Rightarrow \int\frac{dv}{1+4v^2} = \int\frac{dx}{x} + c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(2v) = \log|x| + \log c$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{2y}{x}\right) = \log|cx|$$

$$\Rightarrow \tan^{-1}\left(\frac{2y}{x}\right) = 2\log|cx|$$

$$\Rightarrow 2y = x\tan[2\log|cx|].$$

Ex $xy' = xe^{-y/x} + y$

$$y' = e^{-y/x} + \frac{y}{x}$$

Put $\frac{y}{x} = v \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \cancel{v} + x\frac{dv}{dx} = e^{-v} + \cancel{v} \qquad \Rightarrow \int\frac{dv}{e^{-v}} = \int\frac{dx}{x} + \log c$$

Date			
Page			

$\Rightarrow e^v = \log x + \log c$

$\Rightarrow e^v = \log|cx|$

$\Rightarrow e^{y/x} = \log|cx|$

Ex $(x+y)(xy'-y)y = x^3.$

$\Rightarrow (x^2y' - xy + xyy' - y^2)\, y = x^3$

$\Rightarrow x^2yy' - xy^2 + xy^2y' - y^3 = x^3$

$\Rightarrow y'(x^2y + xy^2) = x^3 + y^3 + xy^2$

$$\Rightarrow y' = \frac{x^3 + y^3 + xy^2}{x^2y + xy^2}$$

Put $y = vx$

$$\text{So } v + x\frac{dv}{dx} = \frac{x^3 + v^3x^3 + v^2x^3}{vx^3 + v^2x^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 + v^3 + v^2}{v + v^2} - v$$

$$= \frac{1 + \cancel{v^3} + \cancel{v^2} - \cancel{v^2} - \cancel{v^3}}{v + v^2}$$

$$= \frac{1}{v(v+1)}$$

$$\Rightarrow \int (v + v^2)\, dv = \int \frac{dx}{x} + \log|c|$$

$$\Rightarrow \left(\frac{v^2}{2} + \frac{v^3}{3}\right) = \log|cx|$$

$$\Rightarrow \frac{1}{2}\left(\frac{y}{x}\right)^2 + \frac{1}{3}\left(\frac{y}{x}\right)^3 = \log|cx|$$

$\Rightarrow 3xy^2 + 2y^3 = 6x^3 \log|cx|$ Ans.

HW (1) $xy' = y + x\sec\left(\frac{y}{x}\right)$

(2) $(2xy + x^2)\, y' = 3y^2 + 2xy.$