Date
07-10-2022
FRIDAY

Binary operations → groups → Ring → field → vector space

Date: / /
Page No.

LINEAR ALGEBRA :: MATRIX

(i) $\rho(A) = 0, 1, 2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

(ii) $\rho(B) = 0, 1, 2$

$$B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

$f: S \to T$

$f: \mathbb{N} \to \mathbb{R}$ $\quad f: \mathbb{N} \to \mathbb{C}$

$\langle x_n \rangle = \frac{n}{n+1}$ or $(-1)^n$ ∴ Sequences

$f: \mathbb{R} \to \mathbb{R}$
$f(x) = y$
func of one variable

$\underbrace{f: \mathbb{R}^2 \to \mathbb{R} \quad f: \mathbb{R}\times\mathbb{R} \to \mathbb{R} \quad f(x,y) = z}_{\text{function of two variables}}$

$\mathbb{R}^{n\times n} = M_n(\mathbb{R}) \to$ Matrix of order 'n' have Real no.'s

TOPICS:- Countable, Uncountable, Countable infinite

(i) COUNTABLE INFINITE

$N = \{1, 2, 3, 4, 5, \ldots\}$ $\quad Z = \{\ldots -2, -1, 0, 1, 2 \ldots\}$

$Q = \left\{ \frac{a}{b} \mid a, b \in Z, b \neq 0 \right\}$

UNCOUNTABLE

$R = Q \cup Q^c \Rightarrow$ Rational $\cup$ Irrational

* Algebra is composition and by composition we mean two objects coming together to form a new object.

2022.08.12

Scanned with CamScanner

Date: / /
Page No.

* $N = \{1, 2, 3, \ldots\}$ set of Natural numbers comes equipped with 2 natural operations $+$ & $*$.

* $Z = \{\ldots -2, -1, 0, 1, 2, \ldots\}$ the set of integers, comes equipped with 2 natural operations $+$ & $\times$

* Also Z has a particular property of ADDITIVE inverses i.e. for $a \in Z$, $\exists \boxed{-a} \in Z$ such that
$\therefore$ $a + (-a) = (-a) + a = 0$
Where $\boxed{0}$ is $\boxed{\text{ADDITIVE IDENTITY}}$

* $Q = \left\{ \frac{a}{b} \mid a, b \in Z, b \neq 0 \right\}$ the set of rationals,

also comes equipped with 2 natural operations $+$ & $\times$

* Also they've property of multiplicative inverses i.e. for $q \in Q$ $\exists \boxed{\frac{1}{q} \in Q}$ such that $q \times \frac{1}{q} = 1$
where $\boxed{1}$ is the multiplicative identity.

* Functions: $f: S \to T$ is defined as $x \to f(x)$
1. let $S = T = IN$
then $f: IN \to IN \Rightarrow x \to x^2$

2. let $S = IN \times IN$, $T = IN$
Then $f: IN \times IN \to IN$ st $(a, b) \to (a+b)$ $\forall a, b \in IN$
or $(a, b) \in IN \times IN$

NOTE: $(Z, +) \to$ Group
$(Z, +, \cdot) \to$ Ring
$(Q, +, \cdot) \to$ field

Properties (axioms or postulates)

Date: / /
Page No.

BINARY OPERATIONS (B.O.)
Let S be a non-empty set. Then a Binary operation
* On S is defined as
* $S \times S \to S$

We often write $a * b$ forall $(a, b \in S)$ instead of $*(a, b)$

EXAMPLES:
(i) The sets IN, Q, Z, IR, C and $M_n(IR)$ / $IR^{m \times n}$ $\times$
All the sets are B.O. under the operation $+$ and $\times$

(ii) For the sets Z & IR, $x * y = x^2 + 2y + 1$ is B.O.
(iii) The set $S = \{-1, 0, 1, 2, \ldots\}$ is not a B.O. under addition since $-1 \in S$ st $(-1) + (-1) = -2 \notin S$

GROUPS (Base of Abstract Algebra)
Any BO on a set G gives a way of combining the elements.
FUNDAMENTAL DEFINITION: A Group G is a non-empty set together with BO '$*$' st the following axioms hold:

(i) CLOSURE PROPERTY: $\forall a, b \in G$
$a * b \in G$

(ii) ASSOCIATIVITY: $\forall a, b, c \in G$
$(a * b) * c = a * (b * c)$

(iii) EXISTENCE OF IDENTITY: $\forall a \in G$, $\exists e \in G$ st
$e =$ identity element
$a * e = e * a = a$

(iv) EXISTENCE OF INVERSE: $\forall a \in G$ $\exists a' \in G$ st
$a * a^{-1} = a^{-1} * a = e$

Date: / /
Page No.

EXAMPLES

(i) $(N, +)$ not a grp (iv) $(Q, +)$ grp (vii) $(\mathbb{R}, +)$ group
(ii) $(Z, +)$ grp (v) $(Q, \times)$ not a grp (viii) $(\mathbb{R}, \times)$ not a group
(iii) $(Z, \times)$ not a grp (vi) $(Q - \{0\}, \times)$ grp (ix) $(\mathbb{R} - \{0\}, \times)$ grp
(X) $(\mathbb{R}^n, +)$ under addition defined as for $x, y \in \mathbb{R}^n$
$x + y = \{x_1 + y_1, x_2 + y_2, \ldots + x_n + y_n\}$
where $x = (x_1, x_2, \ldots, x_n)$ Belongs to group
$y = (y_1, y_2, \ldots, y_n)$ idenity element $= 0$

(xi) $(\mathbb{R}^{m \times n}, +)$ under componentwise addition
null Matrix

(i) Additive identity & Additive inverse not exist
(ii) all 4 postulates satisfied
(iii) Multiplicative inverse not exist 4^{th} not satisfied
(iv) All 4 postulates satisfied.
(v) Since 0, belongs to Q Inverse of 0 (zero) not exist
(viii) " Same Reason

(X) $(x_1 + x_2 + \ldots + x_n) + (e_1 + e_2 + \ldots + e_n) = (x_1 + x_2 + \ldots x_n)$
$(x_1 + e_1, x_2 + e_2, + \ldots + x_n + e_n) = (x_1 + x_2 + \ldots + x_n)$
$\Rightarrow x_1 + e_1 = x_1$ or $x_2 + e_2 = x_2 \ldots$
$e_1 = e_2 = e_3 = \ldots e_n = 0$

(xi)

10-10-2022
Monday

Date: / /
Page No.

ABELIAN GROUPS

If a group G with BO $*$ satisfies
$a * b = b * a \quad \forall a, b \in G$
i.e commutative property, then G is said to be a (ABELIAN GROUP)

ORDER OF A GROUP

For a finite Group G, the number of elements in G is said to be the order of the group.
For an infinite group, the order is infinite.

RINGS → Suppose R is a non-empty set equipped with 2 BO '+' and '.' resp, i.e.
$\forall a, b \in R, \quad a + b \in R$ and $a \cdot b \in R$
Then this algebraic structure $(R, +, .)$ is called a ring if the following axioms are satisfied.
(i) $(R, +)$ is an abelian group
(ii) In $(R, .)$ multiplication is associative, i.e.
$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in R$
(iii) Multiplication is distributive wrt addition i.e.
$$\left.\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c \\ (b + c) a &= b \cdot a + c \cdot a \end{aligned}\right\} \forall a, b, c \in R$$

NOTE → 1. If there is an element 1 in R st $a \cdot 1 = 1 \cdot a = a \ \forall a \in R$ then R is a ring with unity (RU)

2. If multiplication in R is st $a \cdot b = b \cdot a \ \forall a, b \in R$
then we call R is a commutative ring (CR)

FIELD: A ring R with atleast two elements is called a field if (i) it is commutative (ii) has unity (iii) is such that each non-zero element possess multiplicative inverse
CRU where each non-zero element has multiplicative inverse.

Scanned with CamScanner

CR+RU = CRU = Commutative Ring with unity

Date:

Page No.

GROUP (single operation)	RING (Two operations)	Field $(R,+,\cdot)$
$*: G\times G \to G$	$(R,+,\cdot)$	+ Multiplicative Inverse
$(G,*)$	$(R,+)$. Abelian Grp	All the properties of Ring
I Closure	(i) Closure	
II Associative	(ii) Associative	
III Identity	(iii) Multiplicative Identity (RU)	
IV Inverse	(iv) $a\cdot b = b\cdot a$ (Commutative Ring) } CRU	
V Commutative (Abelian)		

Examples (i) The set $\mathbb{Z}$ of all integers with two Binary operation + and . i.e addition and multiplication, written as $(\mathbb{Z},+,\cdot)$ → This is called as ring of integers

Homework (ii) The set $2\mathbb{Z}$ of all even integers → CR

(iii) The set $(\mathbb{R},+,\cdot)$ → Field

(iv) The set $(\mathbb{Q},+,\cdot)$ → Field

Note * (i) CRYPTOGRAPHY extensively uses group and ring theory.

(ii) vector spaces are generally consider as n-tuples (in which manner data is stored).

$\circ: F\times V \to V$, i.e. multiply a scalar with vector, we get a vector.

Date 14-10-2022 Friday

* = Union of all operations exist in the world

o = operator

Date:

Page No.

HW (Imp)

$M_n(\mathbb{R}),+,\cdot)$ – Field | Ring – ??

(i) If the set of all the invertible matrices – field | Ring?

INTERNAL COMPOSITION (on same set): Let A be any set. If $a*b \in A \ \forall\ a,b \in A$ (diff. from B.O.) and $a*b$ is unique, then $\boxed{*}$ is said to be an internal composition in A. vector addition

EXTERNAL COMPOSITION (on diff sets): Let V and F be any two sets. If $a\circ\alpha \in V \ \forall\ \alpha\in V$ and $\forall\ a\in F$ (not a dot) and $a\circ\alpha$ is unique, then o is stb an external composition in V over F (scalar multiplication)

Imp
$$\boxed{\begin{array}{l} *: A\times A \to A \\ \circ: F\times V \to V \end{array}}$$

$G \to 1$ (no. of operations)

$R|F \to 2$

Vector space $\to 4 \begin{cases} 2 \text{ vector space operations} \\ 2 \text{ field operations} \end{cases}$

NOTE: $(F,+,\cdot)$, $(V,*,\circ)$, $(F,+,\cdot)$, $(V,+,\cdot)$

VECTOR SPACE: Let $(F,+,\cdot)$ be a field. The elements of F will be scalars. Let V be a non-empty set whose elements will be called vectors. Then V is called a vector space over a field F, if:

(i) There is defined an internal composition in V called addition of vectors and denoted by '+'. Also for this combination V forms an abelian group.

⇒ $+: V\times V \to V$

(ii) There is an external composition in V over F called scalar multiplication and denoted multiplicatively i.e $a\cdot\alpha \in V \ \forall\ a\in F,\ \alpha\in V$

The other words V is closed with respect to scalar multiplication.

(iii) The 2 composition, i.e. vector addition and scalar multiplication satisfy the following axioms:-

(a) $a\cdot(\alpha+\beta) = a\cdot\alpha + a\cdot\beta \quad \forall\ \alpha,\beta\in V,\ a\in F$

2022.08.12 06:

$\boxed{VS = +, \cdot, *, \circ}$

Date : / / Page No.

(b) $(a+b).\alpha = a.\alpha + b.\alpha \ \forall\ a, b \in F,\ \alpha \in V$

(c) $(a.b)\alpha = a.(b\alpha) \ \forall\ a, b \in F,\ \alpha \in V$

(d) $1.\alpha = \alpha \ \forall\ \alpha \in V$ where 1 is the unity element of field F.

Then V is called a vector space over F, denoted as $V(F)$ or simply V.

EXAMPLES: (i) $\mathbb{R}(\mathbb{Q}) \rightarrow$ Vector Space (VS)

(ii) $\mathbb{Q}(\mathbb{R}) \rightarrow$ not a VS (scalar multiplication fails)

(iii) $\mathbb{R}(\mathbb{C}) \rightarrow$ not a VS "

(iv) $\mathbb{C}(\mathbb{R}) \rightarrow$ VS

(v) $\mathbb{R}^n(\mathbb{R}) \rightarrow$ VS

(wrt componentwise addition in $\mathbb{R}^n$ and scalar multiplication)

(vi) $\mathbb{R}^{m \times n}(\mathbb{R}) \rightarrow$

17-10-22 Monday

VECTOR SUBSPACES: $\boxed{\text{have same field like vector}}$ Let $W \subseteq V$. Then the necessary and sufficient condition for a non-empty subset W of a vector space $V(F)$ to be a subspace is

(i) $W \neq \phi$ i.e. in particular $0 \in W$. (additive identity)

(ii) CLOSURE OF W: (a) with respect to the external composition: $\forall\ \lambda \in F\ \forall\ x \in W \Rightarrow \lambda x \in W$

(b) wrt internal composition, $\forall\ x, y \in W$ $x + y \in W$ (these are vectors)

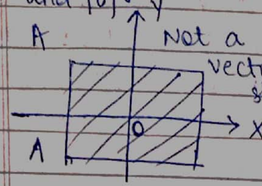
or $\forall\ a, b \in F,\ x, y \in W$ $ax + by \in W$

Vector Space { Scalar multiplication, Vector addition

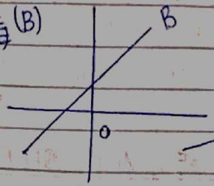
Date : / / Page No.

(i) For every vector space V, the trivial subspaces are V and $\{0\}$

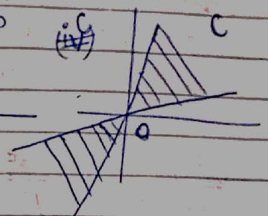
(ii) A



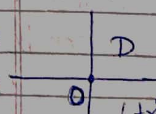
(B)



(C)



(D)



(trivial subspace)

Suppose, field $\rightarrow \mathbb{R}$

(iii) The solution set of Homogenous system of LE $Ax = 0$ with n unknowns $x = [x_1, x_2, \ldots, x_n]^T$ is a subspace of $\mathbb{R}^n$

(iv) The solution of a non-homogenous system of LE is not a subspace of $\mathbb{R}^n$ where the system is $Ax = B,\ b \neq 0 \Rightarrow 0 \notin W$

Date
28-10-2022
Friday

Date: / /
Page No.

LINEAR COMBINATION OF VECTORS:

Let V(F) be a VS. If $\alpha_1, \alpha_2 \ldots \alpha_n \in V$, then any vector $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ where $a_1, a_2, \ldots, a_n \in F$ is a linear combination of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$

GENERATING SET AND SPAN

Consider a VS $V = (V, +, \cdot)$ and a set of vectors $A = \{x_1, x_2, \ldots, x_n\} \subseteq \mathcal{V}$. If every vectors $v \in \mathcal{V}$ can be expressed as a linear combination of $x_1, x_2, \ldots, x_n$, A is called a generating set of V.

The set of all linear combinations of vectors in A is called the span of A.

If A span the VS V, then we write

$$V = \text{span}[A] \quad \text{or} \quad V = \text{span}(x_1, x_2, \ldots, x_n)$$

Smallest generating set

BASIS: Consider a VS, $V = (\mathcal{V}, +, \cdot)$ and $A \in \mathcal{V}$. A generating set A of V is called minimal / smallest if $\exists$ no smaller set $\tilde{A} \subset A \subseteq V$ that spans V.

Note:- Every linearly independent generating set of V is minimal & is called as BASIS of V.

VI result: Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\beta \subset \mathcal{V}$, $\beta \neq \phi$. Then the following statements are equivalent.

(i) β forms a Basis of V.

(ii) β is a minimal generating set.

(iii) β is a maximal linearly independent set of vectors in V.

(iv) Every vector $x \in V$ is a linear combination of vectors from β and every linear combination is unique i.e.

$$x = \sum_{i=1}^{K} \lambda_i b_i = \sum_{i=1}^{K} \psi_i b_i \;\&\; \lambda_i, \psi_i \in F \;\&\; b_i \in \beta, \text{ then}$$

$$\lambda_i = \psi_i \quad \forall\, i = 1, 2, \ldots, K$$

Date : / /
Page No.

LINEAR DEPENDENCE OF VECTORS

Let us consider a VS V with $R \in IN$ and $x_1, x_2, \ldots x_K \in V$. If there is a non-trivial linear combination such that

(zero) $0_2 = \sum_{i=1}^{K} \lambda_i x_i$ with atleast one $\lambda_i \neq 0$, then the vectors $x_1, x_2, \ldots, x_K$ are LD otherwise LI, i.e. when $0 = \sum_{i=1}^{K} \lambda_i x_i$ has only trivial solution such that $\lambda_i = 0 \ \forall \ i = 1, 2, \ldots, K$, then the vectors $x_1, x_2, \ldots, x_K$ are stb LI

Examples:

Ques 1 (i) In $\mathbb{R}^3(\mathbb{R})$ (3-D), the standard basis in $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ $\forall \ x, y, z \in \mathbb{R}$

$V_1 \quad V_2 \quad V_3$

Sol 1.
$$xV_1 + yV_2 + zV_3 = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad x = 0 \quad y = 0 \quad z = 0$

$\mathbb{R}^3 = \{ (x, y, z) \mid (x, y, z \in \mathbb{R}) \}$

(ii) In $\mathbb{R}^3(\mathbb{R})$, the basis are $\beta_2 \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

2022.08.12 06:43

(Imp)

NOTE: (i) Every VS $V(F)$ possess a basis.

(ii) A basis is not unique but the elements in bases remains same.

** (iii) Bases of $\mathbb{R}^n(\mathbb{R})$ always has n numbers of elements

(iv) For a finite-dimensional VS $V(F)$, the dimension of V is the number of basis vectors of V, and we denote it as $\dim(V)$

(* For example, $\mathbb{R}^3(\mathbb{R})$ has dimension 3

or $\dim(\mathbb{R}^3(\mathbb{R})) = 3$

(v) If $U \subseteq V$ is a subspace of V.

$\dim(U) \leq \dim(V)$

and $\dim(U) = \dim(V)$ iff $U = V$

RANK: Number of linearly independent rows or columns of a Matrix $A \in \mathbb{R}^{m \times n}$ is called rank of A or $\rho(A)$

NOTE: No. of LI rows in A

No. of LI columns in A

Example. $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 3 & 6 & 7 & 8 \end{bmatrix} \in \mathbb{R}^{2\times 4}$ $\quad \rho(A) = 2$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 3}$, $\rho(B) = 3$

Date : / /
Page No.

NOTE: (i) $\rho(A) = \rho(A^T)$

(ii) $A_{n \times n}$ is non-singular iff $\rho(A) = n$

(iii) The system of LE $AX = B$, $b \neq 0$ is consistent or always has a solution iff $\rho(A) = \rho(A|b)$ where $(A|b)$ is called as argumented matrix.

LINEAR TRANSFORMATION: The special func() defined on VS that in some sense "preserve" the structure are called linear transformations.

Example: (i) In calculus, operations of differentiation and integration

(ii) In geometry, rotations, reflections & projections are linear transformation.

Definition: Let V and W be two VS over the same field F. Then a function

$T: V \to W$ is called a linear transformation from V to W if $\forall\, x, y \in V$ and $c \in F$, it satisfies

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(cx) = cT(x)$ OR (i) $T(ax+by) = aT(x) + bT(y)$ $\forall\, a, b \in F$

EXAMPLE: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ over the field of $\mathbb{R}$ by
$T(a_1, a_2) = (2a_1 + a_2, a_1)$
Show that T is a linear transformation

Sol For any $x, y \in \mathbb{R}^2$ let $x = (b_1, b_2)$
$y = (c_1, c_2)$

$x + y = (b_1, b_2) + (c_1, c_2)$

$T(x+y) = T(b_1 + c_1, b_2 + c_2)$

$= (2(b_1 + c_1) + b_2 + c_2,\ b_1 + c_1)$

$T(x) + T(y) = (2b_1 + b_2, b_1) + (2c_1 + c_2, c_1)$

$(2b_1 + b_2 + 2c_1 + c_2;\ b_1 + c_1)$

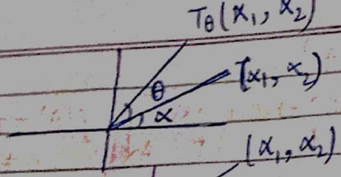
$T(cx) = cT(x)$

$\cancel{2b_1 + b_2 + 2c_1 + c_2}$

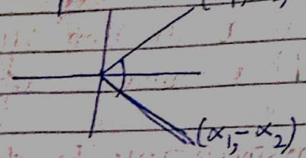
Hence, T is a LT.

Date 04/11/22 Friday

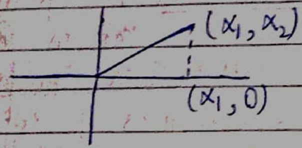
ROTATIONS :



REFLECTION :



PROJECTION :



Q1.
Reflection : Define $T: \mathbb{R}^2 \to \mathbb{R}$ over the field of Real nos $\mathbb{R}$ such that $T(\alpha_1, \alpha_2) = (\alpha_1, -\alpha_2) \ \forall (\alpha_1, \alpha_2) \in \mathbb{R}^2$
Then, T is called reflection on x-axis and T is LT (Prove)

Q2.
Projection : Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ over the field of $\mathbb{R}$, such that $T(\alpha_1, \alpha_2) = (\alpha, 0) \ \forall (\alpha_1, \alpha_2) \in \mathbb{R}^2$
Then T is called a projection on X-axis & T is LT (prove)

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ $\quad \alpha_1 = (a, b)$
$\alpha_2 = (c, d)$

$\alpha_1 + \alpha_2 = (a, b) + (c, d) = (a+c), (b+d)$

$T(\alpha_1 + \alpha_2) = T[(a+c), (b+d)]$
$T[a+c, -(b+d)]$

$T(\alpha_1) + T(\alpha_2) = (a, -b) + (c, -d)$
$(a+c), -(b+d)$

$T(\alpha_1 + \alpha_2) = T(\alpha_1 + \alpha_2)$

$T(c\alpha) = cT(\alpha)$

PROJECTION : Let $x, y \in \mathbb{R}^2$
$x = a_1, a_2$
$y = b_1, b_2$

$x + y = (a_1, a_2) + (b_1, b_2) = (a_1 + b_1), (b_2 + a_2)$
$T(x+y) = (a_1 + b_1, 0)$
$T(x) + T(y) = (a_1, 0) + (b_1, 0)$
$((a_1 + b_1), 0)$

$\therefore T(x+y) = T(x) + T(y)$
$T(cx) = cT(x)$

Let $x = (a, b)$
$T(cx) = T[ca, cb] = [ca, 0]$
$cT(x) = cT(a, 0) = ca, 0$

Q3. Define $T: M_{m\times n}(F) \to M_{n\times m}(F)$ such that $T(A) = A^t \ \forall A \in M_{m\times n}(F)$ where t denotes transpose and F is the field of Real numbers $\mathbb{R}$. Prove that T is LT.

For any $A, B \in M_{m\times n}(\mathbb{R})$
$T(A+B) = (A+B)^t = A^t + B^t = T(A) + T(B)$
$T(CA) = (CA)^t = C(A)^t = CT(A)$, for any $C \in \mathbb{R}$
$\therefore$ T is LT

Q4. Define $T: V \to V$ such that $T(f) = f' \ \forall f \in V$ where f' is derivative of f. Prove T is LT.
(Here, V is the set of all real valued f() defined on real line which have derivatives of all order. Then $V(\mathbb{R})$ is a VS ??)

Q5. Let $V = C(\mathbb{R})$ the set of all continuous real valued functions defined on $(\mathbb{R})$. Define $T: V \to \mathbb{R}$ by
$T(f) = \int_a^b f(t)\,dt$, for $a, b \in \mathbb{R}$, $a < b$
Prove T is LT.

Date : / /
Page No.

(6) Identity transformation: Define $I_V : V \to V$ by $I_V(x) = x$ $\forall\ x \in V$.

Then I_V is LT (prove)

(7) Zero Transformation: Define $T_0 : V \to W$ by $T_0(x) = 0\ \forall\ x \in V$ where V & W are VS. Prove T is LT

RANGE AND NULL SPACE.

Let $T : V \to W$ be a LT where V & W are vector spaces. We define $\boxed{\text{NULL space}}$ (or Kernel), N(T) of T to be the set of all vectors 'x' in V such that
$T(x) = 0$

i.e. $N(T) = \{x \in V \mid T(x) = 0\}$

We define the $\boxed{\text{RANGE space}}$ (or image) of T to be the subset of W consisting of all images (under T) of vectors in V i.e.

$$R(T) = \{T(x) \mid x \in V\}$$

(i) Identity Transf: $N(I_V) = \{0\}$
$R(I_V) = V$

(ii) Zero Transf: $N(T_0) = V$
$R(T_0) = \{0\}$

(iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ over $\mathbb{R}$ be LT defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$N(T) = \{x \in \mathbb{R} \mid T(x) = 0\}$

Let $x = (a_1, a_2, a_3)$

$\Rightarrow T(a_1, a_2, a_3) = (0,0) \Rightarrow (a_1 - a_2, 2a_3) = (0,0)$

$\Rightarrow a_1 - a_2 = 0 \qquad 2a_3 = 0$

$\Rightarrow a_1 = a_2 \qquad a_3 = 0$

$\Rightarrow N(T) = \{(a_1, a_1, 0)\}$

$\Rightarrow R(T) = \mathbb{R}^2$

Result (i) The null space & range sparce for $T : V \to W$ where V and W are VS, are the subspaces of V & W resp

(ii) Let V & W be VS, let $T : V \to W$ be LT. If $\beta = \{v_1, v_2 \ldots v_n\}$

Date: / /
Page No.

is a basis for V, then $R(T) = \text{span}(T(B)) = \text{span}\{T(v_1), T(v_2), \ldots T(v_n)\}$ i.e. any LT can be analysed by its action on basis of V.

Definitions Let V and W be VS and let $T: V \to W$ be LT. If $N(T)$ and $R(T)$ are finite dimensional, then we define the Nullity of T and Rank of T as dimension of $N(T)$ & $R(T)$ respectively.

RANK NULLITY THEOREM: Let V and W be 2 VS and let $T: V \to W$ be LT. If V is finite dimensional, then

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V)$$

2022.08.12 06:43

07-11-22

Q1. If $A \in \mathbb{R}^{5\times 6}$ with $\rho(A) = 2$ what is the dimension of null space of A?

Sol: $\rho(A) + \text{Nullity}(A) = \dim(\mathbb{R}^6)$ using rank-nullity theorem

$2 + \text{nullity}(A) = 6$

$\boxed{\text{nullity}(A) = 4}$

Matrix of a LT: Let $A \in \mathbb{R}^{m\times n}$ or $\mathbb{C}^{m\times n}$ where
$$\begin{array}{c} C_1 \; C_2 \; \cdots \; C_n \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array} \begin{array}{c} R_1 \\ R_2 \\ \vdots \\ R_m \end{array}$$
rows of $A \in \mathbb{R}^n$ or $\mathbb{C}^n$ & columns of $A \in \mathbb{R}^m$ or $\mathbb{C}^m$. If $x \in \mathbb{R}^n$ then $Ax \in \mathbb{R}^m$ $\therefore$ a matrix $A_{m\times n}$ maps the elements in $\mathbb{R}^n$ into the elements in $\mathbb{R}^m$ $\therefore$ $\boxed{T = A}: \mathbb{R}^n \to \mathbb{R}^m$ with $\boxed{Tx = Ax}$. Here, T defines a linear transformation.

$\boxed{Ax = b}$ $\qquad A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$

PROOF: Let $x_1, x_2 \in \mathbb{R}^n$ $\quad T(x_1 + x_2) = A(x_1 + x_2)$

$A(x_1) + A(x_2) = T(x_1) + T(x_2)$

For any $c \in \mathbb{R}$ or $\mathbb{C}$

$T(cx_1) = Acx_1 = c(Ax_1) = cT(x_1)$

$\therefore$ T is a linear transformation.

Eg 1. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a LT defined by $Tx = Ax$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, find Tx, when x is given by $[3\ 4\ 5]^T$

Sol 1. $Tx = Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}_{3\times1} = \begin{bmatrix} 3+8+15 \\ 12+20+30 \end{bmatrix} = \begin{bmatrix} 26 \\ 62 \end{bmatrix}_{2\times1}$

Domain dim   Dimension Co-domain

Eg 2. Write down a basis of $\mathbb{R}^{2\times2}(\mathbb{R})$

$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ it will generate all the matrices of order 2×2

$x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + w \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

linear combinations

$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \boxed{x = y = 0 = z = w}$

Hence, it is LI

Eg 3. Let T be a LT defined by $T\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

$T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ Find $T\left[\begin{pmatrix} 4 & 3 \\ 5 & 8 \end{pmatrix}\right]$

The matrices $\underset{v_1}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$ are LI & generate $\mathbb{R}^{2\times2}$

$\therefore \begin{bmatrix} 4 & 5 \\ 3 & 8 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\alpha_1 = 4$

$\alpha_1 + \alpha_2 = 5 \Rightarrow \alpha_2 = 1$

$\alpha_1 + \alpha_2 + \alpha_3 = 3 \Rightarrow \alpha_3 = -2$

$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 8 \Rightarrow \alpha_4 = 5$

$\therefore \begin{bmatrix} 4 & 3 \\ 5 & 8 \end{bmatrix} = 4v_1 + v_2 - 2v_3 + 5v_4$

$T\begin{bmatrix} 4 & 3 \\ 5 & 8 \end{bmatrix} = 4\,T(v_1) + T(v_2) - 2(T(v_3)) + 5(T(v_4))$

11-11-22 Friday

REVISION

$A = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \middle| w, x, y, z \in \mathbb{R} \right\}$ $\dim(A) = 4$

$B = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{12} = a_{21},\ a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$ Symmetric Matrices, $\dim(B) = 3$

$C = \left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \middle| a_{11}, a_{22} \in \mathbb{R} \right\}$ $\dim(C) = 2$

$D = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{12} = -a_{21},\ a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$ Skew symmetric, $\dim(D) = 1$ ~~or 2~~ all diagonal elements of matrices are zero

Order as well as dimensions

$\beta = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ check it forms a basis of $\mathbb{R}^{2\times2}(\mathbb{R})$

Smallest generating set

$\mathbb{R}^{2\times2}(\mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \middle| x, y, z, w \in \mathbb{R} \right\}$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$

$\alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}$ $\therefore$ This will generate complete $\mathbb{R}^{2\times2}$

$\because$ ($\mathbb{R}$ forms abelian group under addition)

Now to check, smallest generating set

Let us assume that v_1, v_2, v_3 form a generating set of $\mathbb{R}^{2\times2}$.

$\therefore \{v_1\ v_2\ v_3\} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \alpha_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 \end{pmatrix}$ get same elements in last row

Scanned with CamScanner

Date: / /
Page No.

This means, β is the smallest generating set & no other set $\{v_2, v_2, v_3\}$ is smaller than β.
like $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ cannot be written as a LC of v_1, v_2, v_3

~~Largest~~ LI of v_1, v_2, v_3, v_4 for any $\alpha_1, \alpha_2, \alpha_3, \alpha_4$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$$

$$\Rightarrow \begin{pmatrix} \alpha_1 & \alpha_1+\alpha_2 \\ \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

To check: Largest LI set of vectors
Let any other vector v_5 i.e. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 + \alpha_5 v_5 = 0$$

$$\alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_5 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} \alpha_1+\alpha_5 & \alpha_1+\alpha_2+\alpha_5 \\ \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\alpha_2 = \alpha_4 = 0$$

$$\alpha_1 = -\alpha_3 = -\alpha_5$$

$\Rightarrow$ We have a non-trivial solution for the vectors v_1, v_2, v_3, v_4 & v_5.

$$T\begin{bmatrix} 4 & 3 \\ 5 & 8 \end{bmatrix} = 4\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 4+1-2-5 \\ 8-2+4+10 \\ 12+3+6+15 \end{pmatrix} = \begin{pmatrix} -2 \\ 20 \\ 36 \end{pmatrix}$$

Date: / /
Page No.

Q2. Let T be a LT from $\mathbb{R}^3 \to \mathbb{R}^2$ over $\mathbb{R}$ where $Tx = Ax$ where $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $x = [x\ y\ z]^T$. find $\ker(T)$, & Null space (T) ~~dimension~~ range(T) and their dimensions.

Sol2. Null space $(T) = \{ v \in \mathbb{R}^3 \mid T(v) = 0 \}$
Let $v = [v_1, v_2, v_3]^T$

$$Tv = 0 \Rightarrow T[v_1, v_2, v_3]^T = 0$$

$$A[v_1, v_2, v_3] = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 + v_2 = 0$$

$$-v_1 + v_3 = 0$$

$\because N(T) = \{(v_1, -v_1, v_1) \mid v_1 \in \mathbb{R}\} = \ker(T)$
$\therefore$ Nullity $(T) = 1 = \{(1, -1, 1)\}$
$\therefore$ Range $(T) = \mathbb{R}^2$
dim (Range (T)) = Rank (T) = 2

Alter Range $(T) = \{T(v) \mid v \in \mathbb{R}^3\} = \{Av \mid v \in \mathbb{R}^3\}$
$\Rightarrow$ for $v = [v_1, v_2, v_3]^T$

$$Av = \begin{bmatrix} v_1+v_2 \\ -v_1+v_3 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + v_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Also, $\begin{bmatrix} 1 \\ -1 \end{bmatrix} = (1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

dim (Range (T)) = Rank (T) = $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Q3. Let T be a LT, $Tx = Ax$ from $\mathbb{R}^2 \to \mathbb{R}^3$ where $A \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix}$ & $x = \begin{pmatrix} x \\ y \end{pmatrix}$ find $\ker(T)$, range(T) & their dimension
HW

Q4. Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a LT defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y+z \\ y-z \end{pmatrix}$ Determine the matrix of T wrt ordered basis $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ & $Y = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

2022.08.12 06

Hint 4: $T\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ $\quad T\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$ $\quad T\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

Linear combination

$0\begin{pmatrix}1\\0\end{pmatrix}+0\begin{pmatrix}0\\1\end{pmatrix}$

$1\begin{pmatrix}1\\0\end{pmatrix}+1\begin{pmatrix}0\\1\end{pmatrix}$

$1\begin{pmatrix}1\\0\end{pmatrix}-1\begin{pmatrix}0\\1\end{pmatrix}$

NOTE: Every domain has its image in f(U)

$$\begin{array}{c} T(v_1)\ T(v_2)\ T(v_3) \\ \begin{bmatrix}0&1&1\\0&1&-1\end{bmatrix}_{2\times3} \end{array}\begin{array}{l}\to w_1\\ \to w_2\end{array}$$

Rows = T(V)

Column = co-domain (y)

Q2. Let V & W be VS in $\mathbb{R}^3$ let $T: V\to W$ be defined by $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\x+y\\x+y+z\end{pmatrix}$ find the matrix representation of T wrt. the ordered ~~pairs~~ basis

$X=\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$ (V) X = {

$Y=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$ (W)

element of a Domain, images

Sol2. $T\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}$ $\quad T\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\2\\2\end{pmatrix}$ $\quad T\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}$

Linear combination

$\boxed{0}\begin{pmatrix}1\\0\\0\end{pmatrix}+\boxed{1}\begin{pmatrix}0\\1\\0\end{pmatrix}+\boxed{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{0}\begin{pmatrix}1\\0\\0\end{pmatrix}+\boxed{2}\begin{pmatrix}0\\1\\0\end{pmatrix}+\boxed{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$\boxed{0}\begin{pmatrix}1\\0\\0\end{pmatrix}+\boxed{1}\begin{pmatrix}0\\1\\0\end{pmatrix}+\boxed{2}\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\begin{bmatrix}0&0&0\\1&2&1\\2&2&2\end{bmatrix}_{3\times3}$$

Q3. Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be a LT defined by $T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}x+y\\x-z\end{pmatrix}$ find the matrix of T wrt ordered basis.

$X=\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}\in\mathbb{R}^3$ $\quad Y=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right\}\in\mathbb{R}^2$

Sol3. $T\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\+0\end{pmatrix}$ $\quad T\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$ $\quad T\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

LC $\boxed{1}\begin{pmatrix}1\\0\end{pmatrix}+\boxed{0}\begin{pmatrix}1\\1\end{pmatrix}$

$\boxed{1}\begin{pmatrix}1\\0\end{pmatrix}+\boxed{1}\begin{pmatrix}1\\1\end{pmatrix}$

$\boxed{2}\begin{pmatrix}1\\0\end{pmatrix}+\boxed{-1}\begin{pmatrix}1\\1\end{pmatrix}$

$$\begin{bmatrix}1&1&2\\0&1&-1\end{bmatrix}_{2\times3}=T$$

Matrix representation of LT

Let V & W be VS over same field F and let $\beta_1=(v_1, v_2, \dots, v_n)$ be an ordered basis for V and $\beta_2=(\omega_1,\omega_2,\dots,\omega_n)$ be an ordered basis for W. Let $T: V\to W$ be a LT. We can give a matrix representation

2022.08.12 06:43

Date
18|8|22
Friday

Date : / /
Page No.

UNIT – 3

Examples of group

(i) The set $GL_n(\mathbb{R})$ of all $n \times n$ invertible matrices wrt usual multiplication forms a group. This group is called as General linear Group.

(ii) $SL_n(\mathbb{R})$ – Special linear Group is the gp of all $n \times n$ invertible matrices under usual matrix multiplication, consisting of all the matrices whose det = 1.

Q1. The set of matrices
$S = \left\{ e\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ forms a gp wrt usual matrix multiplication.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Composition table
Every row and column consists of all the ~~matrices~~ elements & no ~~matrix~~ elements is repeated.

Q2. The set of $\mathbb{C}$ $S = \{1, -1, i, -i\}$ forms a group wrt usual multiplication

*	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

(iv) Quaternion group : Let $S = \{\pm 1, \pm i, \pm j, \pm K\}$ be the set of 8 elements and the binary operation of multiplication is set as $i^2 = j^2 = K^2 = -1$

$ij = -ji = K$, $jK = -Kj = i$, $Ki = -iK = j$

S forms a group.

*	1	-1	i	-i	j	-j	K	-K
1	1	-1	i	-i	j	-j	K	-K
-1	-1	1	-i	i	-j	j	-K	K
i	i	-i	-1	+1	K	K	j	j
-i	-j	i	1	-1	K	K	j	j
j	j	-j	K	K	-1	1	i	i
-j	-j	j	K	K	1	-1	i	i
K	K	-K	-j	j	i	i	-1	1
-K	-K	K	j	j	i	i	1	-1

e = identity element (1)

This forms a group wrt usual multiplication of numbers. But it is not an abelian group.

(V) The set sym(X) of one-one & onto functions on the n-element set X, with multiplication defined to be the composition of functions. Then sym(X) forms a group.

* The elements of sym(X) are called <u>permutations</u> and sym(X) is called the symmetric group on X.

<u>PERMUTATIONS</u> : Suppose S is a finite having n distinct elements. Then a one-one mapping of S onto ~~myself~~ itself is called a permutation of <u>degree n</u>.

<u>Symbol for a permutation</u> : Let $S = \{a_1, a_2, \dots, a_n\}$ be a finite set of n distinct elements. If $f : S \to S$ is 1-1 and onto function such that $f(a_1) = b_1, f(a_2) = b_2, \dots f(a_n) = b_n$ where the set $\{b_1, b_2, \dots, b_n\} = \{a_1, a_2, \dots, a_n\}$ i.e. $\{b_1, b_2, \dots b_n\}$ is just an <u>arrangement</u> of $\{a_1, a_2, \dots, a_n\}$ then f is written as or denoted as

$$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

Total number of distinct $\boxed{\text{permutations of degree } n}$. If S is a finite set of n distinct elements then total number of n! or $\lfloor n$ arrangements of elements of S are possible ∴ There will be <u>n! distinct permutation</u> of degree n.

If P_n be the set of all permutations of degree n then P_n will have n! elements. The set P_n is called the symmetric set of permutations of degree n.

$S = \{1, 2, 3\}$ or $\{a_1, a_2, a_3\}$

$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

<u>Example</u> Let S be a finite set with 3 distinct elements $\{1, 2, 3\}$ or $\{a_1, a_2, a_3\}$. Then the set of permutation of degree 3 will be

Composition of $f_1 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $f(1)\, f(23)$

$f_2 \circ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $f(132)$

$f_3 \circ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $f(1)(23)$

$f_5 \circ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $f(1)(2)(3)$

Date 22-11-22

Q1. Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix}$ be 2 permutations of degree 5. Find $f \circ g$ & $g \circ f$ and check whether $f \circ g = g \circ f$ or not!

Sol1. $f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}$

$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 3 & 1 \end{pmatrix}$

$\Rightarrow f \circ g \neq g \circ f$

$\therefore$ Symmetric groups are in general non-abelian wrt composition of function.

Task: Prove sym(S) is a grp wrt composition of functions where S is a set of n elements)

Date 25-11-22 friday

Symmetric Group

Sym(S), where $S = \{1, 2, 3\}$ i.e. a set of 3 forms a group wrt. It is denoted by S_3.

$\therefore$ Sym(S) or $S_3 = \{ I, (12), (13), (23), (123), (132) \}$

where

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $\quad (12) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12)(3)$

$(13) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $\quad (13)(2)$

$(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $\quad (1)(23)$

$(123) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad (132) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Q1. $f = (1\,2\,3)$ – a permutation of degree 3

$g = (1, 2, 3)$ ————————— 5

Sol1 $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$

$\Rightarrow f \neq g$

Q2. $f = (1\,2\,3)$ – a permutation of degree 3

$g = (1\,2\,3)$ – ————————— 5

$h = (1\,2\,3)(4\,5)$ – ————————— 5

Sol2. $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$ $\quad h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$

$f \neq g \neq h$

Scanned with CamScanner

COMPOSITION TABLE FOR S_3

Date : / / Page No.

*	I	(12)	(13)	(23)	(123)	(132)
I	I	(12)	(13)	(23)	(123)	(132)
(12)	(12)	I	(123)	(132)	(13)	(23)
(13)	(13)	(132)	I	(123)	(23)	(12)
(23)	(23)	(123)	(132)	I	(12)	(13)
(123)	(123)	(23)	(12)	(13)	(132)	I
(132)	(132)	(13)	(23)	(12)	I	(123)

Q1. Find the inverse of the following permutation

(i) $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 5 & 4 & 1 & 2 & 3 & 6 & 8 \end{pmatrix}$

(ii) (1435) – a permutation degree of 9

(iii) (123) – ——— 3

NOTE : (i) The inverse 'f' of a permutation of degree n, is given by f^{-1} where

$$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix}$$

then

$$f^{-1} = \begin{pmatrix} b_1 & b_2 & \dots & b_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

(ii) A group is stb finite | infinite if it has finite | infinite no. of elements in it and is called as a finite | infinite group.

ORDER OF AN ELEMENT

Let $(G, *)$ be a group. Then an element $a \in G$ has order n if n is the smallest +ve integer such that

$$\underbrace{a * a * \dots * a}_{n-\text{times}} = e$$ where e is the identity element of $(G, *)$

Date : / / Page No.

Sol 1. (i) $f^{-1} = \begin{pmatrix} 7 & 5 & 4 & 1 & 2 & 3 & 6 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 3 & 2 & 7 & 1 & 8 \end{pmatrix}$

(ii) $f^{-1} = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 4 & 1 & 3 & 6 & 7 & 8 & 9 \end{pmatrix}$

(iii) $f^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

A 2 cycle is always transposition, like $(2,5) = (5\ 2)$

* Order of gp $(Z, +)$ = infinite
$(Z, \cdot)$ = not a group

$(\pm 1, \pm i, \pm j, \pm K)$ Quaternion group = 8

* $\{(0,1,2,3,4) +_5\}$ group of addition modulo 5

$a +_m b = r$

$\frac{0+0}{5}, \frac{1+1}{5}, \frac{2+2}{5}, \frac{3+3}{5}, \frac{4+4}{5}$ $\qquad \frac{3+2}{5} = 0$

Remainder 0, 2, 4, 1, 3 $\qquad \frac{0+0}{5} = 0$

$O(a)$

For eg:- (i) If B.O. is taken to be multiplication then is stb n if $a^n = e$, the identity element and where n is the smallest +ve integer.

(ii) If B.O. is taken to be addition then $O(a)$ is stb n if $na = e$ (the identity) & where n is smallest positive integer.

NOTE:- If no such n exist, then $O(a)$ is stb ∞.

Q1. If the additive group of integers $O(1)$, $O(-1)$ order of 0, $O(2)$
∞, ∞, 1, ∞

Q2. In the multiplication group of non-zero rational numbers

$O(2) = \infty$ $O(1) = 1$ $O(-1) = 2$

Q3. addition modulo 5: $\{\{0,1,2,3,4\}, +_5\}$: $o(0) = 5$ $o(1) = 5$ $o(2) = 5$ $o(3) = 5$ $o(4) = 5$

Q4. Quaternion grp: $\{\pm 1, \pm i, \pm j, \pm K\}$
$o(1) = 1$ $o(-1) = 2$ $o(\pm i) = 4$ $o(\pm j) = 4$ $o(\pm K) = 4$

RESULT (i) Order of an element of a finite group is always finite & is less than or equal to the order of the group.

(ii) Order of (ab) is always equal to order of (ba) where a and b are elements of a gp.

(iii) In the additive group of integers, every element has infinite order except 0.

(iv) In the multiplicative group of non-zero rational numbers, only 1 and -1 have finite order.

Subgroups: A subset S of a group $(G,*)$ is said to be a subgroup of G if the set S itself forms a group wrt $*$.

Result:- (i) A subset S of a group $(G,*)$ forms a subgrp if

(a) Identity element $\in S$

(b) for $a \in S \Rightarrow a^{-1} \in S$

(c) for $a,b \in S \Rightarrow a*b \in S$

(ii) A subset S of a group $(G,*)$ forms a subgroup of G iff S is non-empty & for $a,b \in S \Rightarrow a * b^{-1} \in S$

$3! = S_3 = \{I, (123), (13), (12), (23), (132)\}$

Examples:- (i) The set $S = \{1,-1\}$ forms a subgroup of $G = \{1,-1,i,-i\}$ wrt multiplication.

(ii) The set of even integers will form a subgroup of additive group of integers.
Check for a set of odd integers that whether it forms a subgroup of additive group of integers or not ?? (No, because, '0' is not in the set).

(iii) In S_3 check which of the following forms a subgroup?

(a) $A = \{(I)(12)\}$: It forms a subgroup

(b) $B = \{I, (13)\}$: ✓

(c) $C = \{I, (23)\}$: ✓

(d) $D = \{I, (123)\}$: $(132) \notin D$ (inverse)

(e) $E = \{I, (123), (132)\}$: ✓

(f) $F = \{(12), (13)\}$: $I \notin F$

(g) $G = \{I, (13), (12)\}$: composition of $(13)(12) \notin G$

29-11-22 Tuesday

$H = \{(123)\}$: Identity $\notin H$

$I = \{I\}$: Trivial Subgroups

$J = S_3$: Improper Subgroups

Sol (a) $I \in A$ (identity element), $(21) \in A$ (Inverse)

Some practice questions

Q1 Write down the order of each element of symmetric grp S_3.

Q2. Check whether the set $S = \{\{1,2,3,4,5,6\}, \times_7\}$ wrt multiplication modulo 7 forms a group or not. If yes, write down the order of each element.

Q3. Check whether the set $S = \{\{1,2,3,4,5,6,0\}, +_6\}$ forms a group or not. If yes, write down the order of the group

Cyclic groups : Let G be a group with BO $*$ G is stb cyclic if $\exists$ an element $a \in G$ such that every element $x \in G$ can be written in the form $a * a * \ldots * a$ or a^n (if $*$ is multi-plication) for some integer n

Also 'a' is called a generator of G.

For Example

(i) The group $G = \{1, -1, i, -i\}$ is a cyclic group?

G can be written as $G = \{i^1, i^2, i^3, i^4\}$

also, $G = \{(-i), (-i)^2, (-i)^3, (-i)^4\}$ $\therefore i, -i$ are generators

Hence, it forms a cyclic group

(ii) The group of integers with addition forms a cyclic group

$a + a + \ldots + a = na$ $\therefore n \in \mathbb{Z}$

$a = 1$ generator

(iii) Check whether the following groups are cyclic or not.

(a) $\{1, \omega, \omega^2\}$ wrt multiplication generators are $\underline{\omega^2}$

(b) S_3

(c) $(\mathbb{R}, +)$

(d) $(\{0, 1, 2, 3, 4, 5\}, +_6)$ Generators are $(1, 5)$

$\{1, 1^2, 1^3, 1^4, 1^5, 1^6\}$ $1 = \{1, 2, 3, 4, 5, 0\}$

Date
11/12/22

Date : / /
Page No.

HOMOMORPHISM & ISOMORPHISM

* The concept of homomorphism & isomorphism is the most common notation in abstract algebra. Here we are interested in talking about the like algebraic structures.

Particularly, in homomorphism. We say a group $(G, *)$ is homomorphic to another group $(G', \cdot)$ when $\exists$ a mapping $f: G \to G'$ such that f preserves the structure.

$$f(a * b) = f(a) f(b) \quad \forall\, a, b \in G$$

Isomorphism of groups

Isomorphic mapping: Suppose $(G, *)$ and $(G', \cdot)$ be two groups. A mapping f of $\boxed{G \text{ into } G'}$ is stb isomorphic ~~groups~~ mapping of G into G' if

i) f is one-one

ii) $f(a * b) = f(a) + cf(b) \quad \forall\, a, b \in G$

If f is an isomorphic mapping of G into G', then f is called an isomorphism of $\boxed{G \text{ into } G'}$.

ISOMORPHIC GROUPS

Suppose $(G, *)$ and (G', o) are two groups. We say G is isomorphic to the group 'G'', if $\exists$ a 1-1 mapping of $\boxed{G \text{ onto } G'}$ such that

$$f(a * b) = f(a) \cdot f(b) \quad \forall\, a, b \in G$$

Symbolically we write it as $\boxed{G \cong G'}$ or simply

Date : / /
Page No.

Q1. If $\mathbb{R}$ is an additive grp of real numbers and $\mathbb{R}_+$, the multiplicative group of +ve real no's, prove that the mapping $f: \mathbb{R} \to \mathbb{R}_+$ defined by $f(x) = e^x \ \forall\, x \in \mathbb{R}$ is an isomorphism of R onto $\mathbb{R}_+$

Sol. $f: \mathbb{R} \to \mathbb{R}_+$ such that $f(x) = e^x \ \forall\, x \in \mathbb{R}$

i) Let $x_1, x_2 \in \mathbb{R} \Rightarrow f(x_1 + x_2) = e^{x_1 + x_2} = e^{x_1} e^{x_2}$
$f(x_1) f(x_2)$

ii) Let us consider $f(x_1) = f(x_2)$
$e^{x_1} = e^{x_2}$

taking log on both the sides

$\Rightarrow \log e^{x_1} = \log e^{x_2}$

$\Rightarrow \boxed{x_1 = x_2}$

Q1(ii) To show f is 'onto'.

Let $y \in \mathbb{R}_+ \Rightarrow \log y \in \mathbb{R}$

$\Rightarrow f(\log y) = e^{\log y} = y$

This holds true for any $y \in \mathbb{R}_+$

$\Rightarrow f$ is onto

Hence, f is an isomorphism from G onto G'

Task (1)

Q2. Let $\mathbb{R}_+$ be the multiplication group of all +ve real no's of $\mathbb{R}$ be the additive group of all real no's. Show that the mapping $g: \mathbb{R}_+ \to \mathbb{R}$ defined by $g(x) = \log(x) \ \forall\, x \in \mathbb{R}$ is an isomorphism.

Q3. Show that the additive groups of integers $G = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$ is isomorphic to the additive group
... m, 2m, 3m, ...

2022.08.12 06:43

Date : / /
Page No.

is any fixed integer not equal to zero.

Some Important properties of isomorphic groups

Let f be a mapping from G onto G' where $G \cong G'$

(a) The f-image of the ~~rest~~ identity e of G is the identity of G'.

(b) The f-image of the inverse of an element a of G is the inverse of the f-image of a i.e. $f(a^{-1}) = [f(a)]^{-1}$

(iii) The order of an element a of G is equal to the order of image of a i.e.

$$O(a) = O(f(a))$$

Sol3:- (a) To show f is 'onto'.

Let $y \in G' \Rightarrow \frac{y}{m} \in G$

$$\Rightarrow f\left(\frac{y}{m}\right) = m\left(\frac{y}{m}\right) = y \;\forall\; y \in G'$$

Hence, f is onto.

(b) To show f is one-one

Let $f(x_1) = f(x_2) \Rightarrow mx_1 = mx_2$

$\Rightarrow x_1 = x_2 \quad \because$ (onto)

This exists $\forall\; x_1, x_2 \in G$

$\therefore f$ is one-one

Date
27/12/22
Tuesday

classmate
Date
Page

MATHS

COSETS Let H be a subgroup of a group $(G, \times)$ assuming the B.O to be multiplication, then the set Hx ⊂ G
$Hx = \{hx : h \in H, x \in G\}$ is said to be a right coset of 'H' in 'G', for any $x \in G$ sim, xH is called as the left coset of 'H' in 'G' for any $x \in G$. Cosets may not be subgroups of G but they are the complexes of G, also Hx may not be equal to xH $(Hx \neq xh)$.

Complexes: A subset H of group $(G, *)$ is called a complex. It may or may not be a subgroup.

Remark (i) For an abelian group G $Hx = xH$, for any subgroup H of G.
(ii) For $e \in G$, $eH = He = H$.

Example:- Let $G = (\mathbb{Z}, +)$ and let $H = (3\mathbb{Z}, +) \Rightarrow H \subset G$ Write down the distinct right cosets of H & G.

$H = \{\dots -9, -6, -3, 0, 3, 6, 9, \dots\}$

Identity = $H+0 = \{\dots -9, -6, -3, 0, 3, 6, 9, \dots\} = H$
$H+1 = \{\dots -8, -5, -2, 1, 4, 7, 10, \dots\} \neq H$
$H+2 = \{\dots -7, -4, -1, 2, 5, 8, 11, \dots\} \neq H$
$H+3 = \{\dots -6, -3, 0, 3, 6, 9, 12, \dots\}$ $\therefore 0 \in H+3$

$\therefore$ $H, H+1, H+2$ are the distinct right cosets of H in G.
$(\mathbb{Z}, +) = (H) \cup (H+1) \cup (H+2)$ (right cosets)
$(H) \cup (1+H) \cup (2+H)$ ($\because$ G is abelian group)
(left cosets)

Result: No. of distinct right cosets = No. of distinct left cosets
No matter Group is abelian

Q1: For the symmetric group S_3 & the subgroup $H = \{I, (12)\}$. Write down all the right cosets.

Sol 1. Let $S_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Scanned with CamScanner

find left cosets

where $f_1 = I$, $f_2 = (12)$, $f_3 = (23)$, $f_4 = (13)$, $f_5 = (123)$, $f_6 = (1$

Let $H = \{f_1, f_2\}$

right cosets

$H \circ f_1 = \{f_1 \circ f_1, f_2 \circ f_1\} = \{f_1, f_2\}$

$H \circ f_2 = \{f_1 \circ f_2, f_2 \circ f_2\} = \{f_2, f_1\}$

$H \circ f_3 = \{f_1 \circ f_3, f_2 \circ f_3\} = \{f_3, f_5\}$

$H \circ f_4 = \{f_1 \circ f_4, f_2 \circ f_4\} = \{f_4, f_6\}$

$H \circ f_5 = \{f_1 \circ f_5, f_2 \circ f_5\} = \{f_5, f_3\}$

$H \circ f_6 = \{f_1 \circ f_6, f_2 \circ f_6\} = \{f_6, f_4\}$

$S_3 = (H \circ f_1) \cup (H \circ f_3) \cup (H \circ f_6)$

i.e. there are 3 distinct right co of 'H' in 'G'.

Q2. For symmetric group S_3, find the distinct right cosets of $H = \{I, (12), (132)\}$

Sol $S_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

$f_1 = I$ $f_4 = (13)$

$f_2 = (12)$ $f_5 = (123)$

$f_3 = (23)$ $f_6 = (132)$

$H = \{f_1, f_2, f_6\}$

$H \circ f_1 = \{I \circ I, (12) \circ I, (132) \circ I\} = \{I, (12), (132)\} = H$ ✓

$H \circ f_2 = \{I \circ (12), (12) \circ (12), (132) \circ (12)\} = \{(12), I, (13)\}$

$H \circ f_3 = \{I \circ (23), (12) \circ (23), (132) \circ (23)\} = \{(23), (132), (12)\}$

$H \circ f_4 = \{I \circ (13), (12) \circ (13), (132) \circ (13)\} = \{(13), (123), (23)\}$ ✓

$H \circ f_5 = \{I \circ (123), (12) \circ (123), (132) \circ (123)\} = \{(123), (13), I\}$

$H \circ f_6 = \{I \circ (132), (12) \circ (132), (132) \circ (132)\} = \{(132), (23), (123)\}$

2022.08.12 06:43

INDEX OF A SUBGROUP: Let H be a subgroup of a group G, then the index of H in G is equal to number of distinct right or left coset of H in G.
It is denoted as $[G:H]$ or $i_G(H) = \frac{O(G)}{O(H)}$

Lagrange's Theorem: The order of a subgroup of finite group 'G' is a divisor of order of the group.

NOTE: But the converse of the thm is not true.
$S_4 = \{I, (12), (13), (14), (23)$

Factors of $24 = 1, 2, 3, 4, 6, 8, 12, 24\}$ subgroup of order 6 don't have cosets.
Abelian groups are always cyclic ⇒ they must have the grp of order 6\}

** **ALTERNATING GROUPS:** The cycles in a permutations are called odd (even) acc. as the no. of transpositions are odd (even) resp.
The alternating group A_n of S_n is the collection of all even cycles of S_n. (length of cycle on basis of transposition.)
$S_3 = \{I, (12), (13), (23), (123), (132)\}$
$A_3 = \{I, (132), (123)\}$

NOTE The order of A_n is always half of the S_n

Cayley's Theorem:- Every finite group G is isomorphic to ~~its~~ a group of permutation.
Cayley's theorem even hold true for ∞ group and we restate it as: every group G is isomorphic to a group of one-one, onto f(.)

Normal Subgroups:- A subgroup H(G) is said to be a normal subgroup of G, if $\forall x \in G$ and $\forall h \in H$,
$xhx^{-1} \in H$

Remarks: (iii) Every subgrp of a cyclic grp is normal.
(iv) A grp having no proper normal subgroups is a simple grp.
(iii) A subgroup H of a group 'G' is normal iff $xHx^{-1} = H \; \forall x \in G$
(iv) $xH = Hx \; \forall x \in G$. A subgrp 'H' of a G is a normal subgroup of 'G' iff every right coset of H in G is equal to the every left coset of H in G.
(v) Intersection of 2 normal subgroup is always normal.
(vi) If 'G' is a gp & H is a subgp of G with index 2, H is always a normal subgp of G, but converse not true.

Q3. Show that $H = \{1, -1\}$ is normal subgp of 'G' $= \{1, -1, i, -i\}$

Sol3. Let $x = 1$ $\quad xHx^{-1} = 1\{1,-1\}1 = \{1,-1\} = H$

$x = -1$ $\quad xHx^{-1} = -1\{1,-1\}-1 = \{1,-1\} = H$

$x = i$ $\quad xHx^{-1} = i\{1,-1\}-i = \{i,-i\}-i = \{1,-1\} = H$

$x = -i$ $\quad xHx^{-1} = -i\{1,-1\}i = \{-i, i\}i = \{1,-1\} = H$

$\forall x \in G, \; xHx^{-1} = H$

Q4. Show that in S_3, $H_1 = \{I, (12)\}$ is not a normal subgp while $H_2 = \{I, (123), (132)\}$ is a normal subgp (HW)

Q5. Name a non-abelian group & find whose every subgroup is normal

Q6. find the permutation group isomorphic to the multiplicative group $G = \{1, \omega, \omega^2\}$, $G = \{1, -1, i, -i\}$

Q1. Verify cayley's thm for a cyclic group of order 3

Sol1. Let G be the cyclic group of order 3 $\therefore G = \{a, a^2, a^3 = e\}$

Then the permutation gp of G is given by $\bar{G} = \{f_a, f_{a^2}, f_{e}\}$

$f_a(e) = a.e = a$

$f_a(a) = a.a = a^2$

$f_a(a^2) = a.a^2 = e$

$= \begin{pmatrix} a & a^2 & a^3 = e \\ a^2 & e & a \end{pmatrix}$

$f_{a^2}(a) = e$

$f_{a^2}(a^2) = e.a = a$

$f_{a^2}(e) = a^2$

$f_{a^2} = \begin{pmatrix} a & a^2 & a^3 = e \\ e & a & a^2 \end{pmatrix}$

$f_e(a) = a = e.a$

$f_e(a^2) = a^2 = e.a^2$

$f_e(a^3) = e = a^3 = e.a^3$

$f_{a^3} = \begin{vmatrix} a & a^2 & a^3 \\ a & a^2 & a^3 \end{vmatrix}$

Scanned with CamScanner

Date : / /

Page No.

Ans 4. Let $S_3 = \{f_1, f_2, f_3, f_4, f_5, f_6\} = \{I, (12), (23), (13), (123), (132)\}$

(i) $H_1 = \{f_1, f_2\}$

$f_2 \circ H \circ f_2^{-1} = (12)\{I, (12)\}(12) = \{(12), I\}(12)$
$= \{I, (12)\} = H$

$f_1 \circ H \circ f_1^{-1} = I\{I, (12)\}I = \{I, (12)\}I = \{I, (12)\} = H$

$f_5 \circ H \circ f_5^{-1} = (123)\{I, (12)\}(132) = \{(123), (23)\}(132) = \{I, (12)\} = H$

$f_3 \circ H \circ f_3^{-1} = (23)\{I, (12)\}(23) = \{(23), (123)\}(23) = \{I, (13)\} \neq H$

$H_1 = \{I, (12)\}$ does not forms a normal subgroup

$\because f_3 H f_3^{-1} \neq H$

(ii) $H_2 = \{I, f_5, f_6\}$

$f_1 H f_1^{-1} = I\{I, f_5, f_6\}I = \{I, f_5, f_6\} = H$

$f_2 H f_2^{-1} = (12)\{I, \underset{(123)}{f_5}, \underset{(132)}{f_6}\}(12) = \{(12), (13), (23)\}(12)$
$= \{I, (132),$

$f_3 H f_3^{-1} = (23)\{I, (123), (132)\}(23) = \{(23), (12), (13)\}(23)$
$\{I, (132), (123)\} = H$

$f_4 H f_4^{-1} = (123)\{I, (123), (132)\}(132) = \{(123), (132), I\}(132)$
$\{I, (123), (132)\} = H$

$f_5 H f_5^{-1} = (13)\{I, (123), (132)\}(13) = \{(13), (23), (12)\}(13)$
$\{I, (132), (123)\} = H$

$f_6 H f_6^{-1} = (132)\{I, (123), (132)\}(132) = \{(132), I, (123)\}(132)$
$\{(123), (132), I\} = H$

Therefore, H_2 forms a normal subgroup.

Ans. Let G' is a regular isomorphic to G, $G' = \{f_1, f_2, f_3, f_4\}$

$f_1(1) = 1$	$f_2(1) = -1$	$f_3(1) = i$	$f_4(1) = -i$
$f_1(-1) = -1$	$f_2(-1) = 1$	$f_3(-1) = -i$	$f_4(-1) = i$
$f_1(i) = i$	$f_2(i) = -i$	$f_3(i) = -1$	$f_4(i) = 1$
$f_1(-i) = -i$	$f_2(-i) = i$	$f_3(-i) = 1$	$f_4(-i) = -1$

$$f_1 = \begin{pmatrix} 1 & -1 & i & -i \\ 1 & -1 & i & -i \end{pmatrix} \qquad f_2 = \begin{pmatrix} 1 & -1 & i & -i \\ -1 & 1 & -i & i \end{pmatrix}$$

$$f_3 = \begin{pmatrix} 1 & -1 & i & -i \\ i & -i & -1 & 1 \end{pmatrix} \qquad f_4 = \begin{pmatrix} 1 & -1 & i & -i \\ -i & i & 1 & -1 \end{pmatrix}$$

$G' = \{f_1, f_\omega, f_{\omega^2}\}$

$f_1(1) = 1$	$f_\omega(1) = \omega$	$f_{\omega^2}(1) = \omega^2$
$f_1(\omega) = \omega$	$f_\omega(\omega) = \omega^2$	$f_{\omega^2}(\omega) = 1$
$f_1(\omega^2) = \omega^2$	$f_\omega(\omega^2) = 1$	$f_{\omega^2}(\omega^2) = \omega$

$$f_1 = \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{pmatrix} \qquad f_\omega = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \end{pmatrix} \qquad f_{\omega^2} = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \end{pmatrix}$$

Date
29/12/22
Thursday

UNIT - 4

Date : / /

$i, -i$
$\Rightarrow \{3$
$\{4\}$

RINGS

(i) The set S consisting of a single element 0 with the binary operations defined as $0+0=0$ and $0.0=0$ forms a ring. This ring is called a null ring or zero ring.

(ii) The set $R = (\{0,1,2,3,4,5\}, +_6, \times_6)$
$S = (\{1,2,3,4,5\} \times_6)$

Ring
RU
CR
CRU
Field

Some special types of rings

* Rings with & without zero divisors

Definition :- A non-zero element of a ring R is called a zero divisor or a divisor of zero if ∃ an element $b \neq 0 \in R$ such that either $ab = 0$ or $ba = 0$

Ring without zero divisors

A ring R is without zero divisors if the product of no two non-zeros element of R is zero. i.e.
if $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$.
Else, a ring is called as ring with zero divisors

Ex1. $M_2(Z)$ wrt usual multiplication & addition
(This is a ring with zero divisors)

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$A.B = 0$ $A \neq 0, B \neq 0$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

2022.08.12 06:43

Date: __/__/__

$(\mathbb{Z}, +, \cdot)$ is a ring without zero divisors

Integral Domains (ID) :- A ring is called an integral domain if : (i) It is commutative
(ii) It has unit elements i.e. it is R

without zero divisor is ID

it is without zero divisors.

Ans: Every field is an integral domain (ID) but not conversely.

$(\mathbb{Z}, +, \cdot)$: Inverse doesn't exist that's why doesn't form a field but it ID because it CRU without zero divisors.

Example (ii) $R = \{\{0,1,2,3,4\}, +_5, \times_5\}$

This is a field hence also an ID

the following sets form ID's wrt addition & Multiplication, state if they are fields.

set of numbers of the form $b\sqrt{2}$ where $b \in \mathbb{Q}$

set of even integers

set of +ve integers.

Date 30/12/22 Friday

Date: __/__/__

UNIT - 2

INNER PRODUCT SPACE

The inner product or dot product of $\mathbb{R}^n$ is a function $<,>$ defined as two entries in the functions $<u,v> = a_1b_1 + a_2b_2 + a_3b_3 + \dots + a_nb_n$ where $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$ $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \in \mathbb{R}^n$

The inner product $<,>$ satisfies the following axioms

(i) Linearity $<au+bv, w> = a<u,w> + b<v,w>$ where $a, b \in \mathbb{R}$ or F

(ii) Symmetric $<u,v> = <v,u>$

(iii) Positive definite property : For any $u \in V$, this is always true $<u,u> \geq 0$ and more precisely

$<u,u> = 0$ iff $u = 0$

$(V \times V \rightarrow F)$

Inner Product Spaces, Let V be a VS over F and inner product on V is a function that assigns, to every ordered pair of $(x,y) \in V$ and a scalar in F, denoted by $<x,y>$, such that $\forall\, x,y,z \in V$ and $a,b \in F$ we have

(i) $<ax+by, z> = a<x,z> + b<y,z>$

(ii) $\overline{<x,y>} = <y,x>$, where bar represents the complex conjugate.

(iii) $<x,x> \geq 0$ and $<x,x> = 0$ iff $x = 0$

Q1. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$ define $<x,y> = 2x_1y_1 - \cancel{x_1y_2}\, x_2y_1 - x_1y_2 + 5x_2y_2$

Prove that $<,>$ is an inner product on $\mathbb{R}^2$

Ans. LINEARITY: Let $a, b \in F$ & let $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathbb{R}^2$ then

$<ax+by, z> = 2(ax_1+by_1)z_1 - z_2(ax_1+by_1) - z_1(ax_2+by_2) + 5(ax_2+by_2)z_2$

2022.08.12 06:43

$tr(AB) = tr(BA)$

Date

$\begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \end{bmatrix}$, $z = [z_1, z_2]^T$

RHS: $a\langle x, z\rangle + b\langle y, z\rangle$

$a(2x_1z_1 - x_2z_1 - x_1z_2 + 5x_2z_2) +$

$b(2y_1z_1 - y_2z_1 - y_1z_2 + 5y_2z_2)$

(iii) $\langle x, x\rangle = 2x_1^2 - 2x_1x_2 + 5x_2^2$

$(x_1 - x_2)^2 + x_1^2 + 4x_1 \geq 0$

$\therefore (x_1 - x_2)^2 + x_1^2 + 4x_2^2 = 0$

$\Leftrightarrow x_1 - x_2 = 0, \; x_1 = 0, \; x_2 = 0$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\langle x, y\rangle = 2x_1y_1 - x_2y_1 - x_1y_2 + 5x_2y_2$

$2y_1x_1 - y_1x_2 - y_2x_1 + 5y_2x_2$

$\langle y, x\rangle$

Defn: Let $A \in M_{m\times n}(F)$. We define the conjugate, transpose, or adjoint of A to be the $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}} \; \forall \, i, j$

Conjugate transpose

Q2. Let $V = M_{m\times n}(F)$ & define $\langle A, B\rangle = \text{trace}(B^*A) \; \forall \, A, B \in$

Prove that $\langle , \rangle$ is IPS on $M_{m\times n}(F)$.

Inner Product space

$A = \begin{bmatrix} i & 2+i \\ 5i & 5 \end{bmatrix}$ $\quad A^* = \begin{bmatrix} -i & -5i \\ 2-i & 5 \end{bmatrix}$

Sol 2. $\langle A+B, C\rangle = tr(C^*(A+B))$

Linearity

$= tr(C^*A + C^*B)$

$tr(C^*A) + tr(C^*B)$

$= \langle A, C\rangle + \langle B, C\rangle$

$\langle \eta A, C\rangle = tr(C^*(\eta A)) = \eta\, tr(C^*A) = \eta\langle A, C\rangle$

$(AB)^T = B^T A^T$ | $(B^*A)^* = A^*B$

$(AB)^* = B^*A^*$ | $tr(A^*B) = tr(B^*A)$ $\langle A, B\rangle = \langle B, A\rangle$ IPS

Date / /

Page No.

Symmetric: $\langle A, B\rangle = tr(B^*A)$

$\overline{\langle A, B\rangle} = tr(A^*B)$ $\because \overline{AB} = \overline{B}\,\overline{A}$

$= \langle B, A\rangle$

$\Rightarrow \langle A, B\rangle = \langle B, A\rangle$

Hence, symmetric is proved.

(iii) $\langle A, A\rangle \geq 0$

$\langle A, A\rangle = tr(A^*A) = \sum_{i=1}^{n} (A^*A)_{ij}$

$\sum_{i=1}^{n}\sum_{k=1}^{n} (A^*)_{ik} A_{ki}$

$\sum_{i=1}^{n}\sum_{k=1}^{n} (\overline{A_{ki}}) A_{ki}$

$\sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2$

Now, $\sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2 = 0$ iff $A = 0$ and $\sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ki}|^2 > 0$
$i=1$ & $k=1$ otherwise

NOTE

(i) For each vector $u \in V$, the norm (or the length) of vector u is defined as norm of $u = \sqrt{\langle u, u\rangle}$ in the terms of real numbers.
$\hookrightarrow \|u\|$

(ii) If $\|u\| = 1$, then u is called as a unit vector

(iii) for any non-zero vector $v \in V$, $v = \frac{v}{\|v\|}$
this process is called the process of normalisation of a vector.

Let V be an inner product space over the field F

Properties then $\forall\, x, y \in V$ and $c \in F$ we have

(a) $\|cx\| = |c|\,\|x\|$

(b) $\|x\| = 0$ iff $x = 0$ in any case $\|x\| \geq 0$

2022.08.12 06:43

Date :
Page No.

(c) Cauchy- Schwartz Inequality
$$|\langle x,y\rangle| \leq \|x\| \, \|y\|$$
(d) Triangle inequality
$$\|x+y\| \leq \|x\| + \|y\|$$

02-01-2023

Ques:- In $C[0,1]$. Let $f(t)=t$ and $g(t)=e^t$ compute $\langle f,g\rangle$, $\|f\|$, $\|g\|$, $\|f+g\|$. Then verify cauchy schwarts inequality and triangle inequality.

linearity

Sol. $\langle f,g\rangle = \int_0^1 f(t)g(t)dt = \int_0^1 \underset{I}{t}\underset{II}{e^t} dt = \left[e^t[t-1]\right]_0^1 = 1$

$\langle g,f\rangle = \int_0^1 g(t)f(t)dt = \int_0^1 \underset{II}{e^t}\underset{I}{t}dt = \left[e^t(t-1)\right]_0^1 = 1$

$\langle f,f\rangle = \int_0^1 f^2(t)dt = \int_0^1 t^2 dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3} = 0.333$

$\langle g,g\rangle = \int_0^1 g^2(t)dt = \int_0^1 e^{2t}dt = \left[\frac{e^{2t}}{2}\right]_0^1 = \frac{e^2}{2} - \frac{1}{2} = 3.194$

$\|f\| = \sqrt{\langle f,f\rangle} = \frac{1}{\sqrt{3}} = 0.577$ $\quad \|g\| = \sqrt{\langle g,g\rangle} = \frac{\sqrt{e^2-1}}{\sqrt{2}} = 1.786$

$f+g = t+e^t$

$\langle f(t)+g(t), f(t)+g(t)\rangle = \int_0^1 t^2+e^{2t}+2te^t \, dt$

$\left[\frac{t^3}{3} + \frac{e^{2t}}{2} + 2\left[e^t(t-1)\right]\right]_0^1 = \frac{e^2}{2} - \frac{13}{6}$

$\|f+g\| = \sqrt{\langle f+g, f+g\rangle} = \sqrt{\frac{3e^2-13}{6}} = 1.527$

Δ inequality $\|x+y\| \leq \|x\|+\|y\|$

$1.527 \leq 0.577 + 1.786$

$1.527 \leq 2.363$ Hence verified

Cauchy swartz $|\langle f,g\rangle| \leq \|f\|\,\|g\|$

$1 \leq 1.03$ Hence verified

Date
02/01/2023
Monday

Date : / /
Page No.

Cauchy swartz inequality
$$\underbrace{|\underbrace{\langle x,y\rangle}_{IPS}|}_{Mode} \leq \underbrace{\|x\|}_{norm}.\|y\|$$

$$\boxed{\|x\| = \sqrt{\underbrace{\langle x,x\rangle}_{IPS}}}$$
Definition of norm

Q1. For $x=(a_1,a_2,\ldots,a_n)$ & $y=(b_1,b_2,\ldots,b_n)$ in F^n define $\langle x,y\rangle = \sum_{i=1}^{n} a_i\overline{b_i}$ Prove that $\langle,\rangle$ is IPS on F^n

Sol 1. Linearity: $\langle \alpha x+\beta y, z\rangle = \sum_{i=1}^{n}(\alpha x_i+\beta y_i)\overline{z_i}$

$= \sum_{i=1}^{n} \alpha x_i\overline{z_i} + \sum_{i=1}^{n}\beta y_i\overline{z_i} = \alpha\sum_{i=1}^{n} x_i\overline{z_i} + \beta\sum_{i=1}^{n} y_i\overline{z_i}$

$= \alpha\langle x,z\rangle + \beta\langle y,z\rangle$

Hence, linearity proved.

Symmetric: $\overline{\langle x,y\rangle} = \overline{\sum_{i=1}^{n} x_i\overline{y_i}} = \sum_{i=1}^{n}\overline{x_i}\,y_i = \langle y,x\rangle$

$\overline{\langle x,y\rangle} = \langle y,x\rangle$

Hence, symmetric proved

Positive definite: $\langle x,x\rangle \geq 0$

$\sum_{i=1}^{n} x_i\overline{x_i} = \sum_{i=1}^{n}|x_i|^2 \geq 0$ $\quad \therefore x\overline{x} = |x|^2$

$\sum_{i=1}^{n}|x_i|^2 = 0$ iff $x_i = 0$

Hence positive definite proved

Q2. In $V=C[0,1]$, the VS of real valued continuous function on $[0,1]$ For $f,g \in V$, define $\langle f,g\rangle = \int_0^1 f(t)g(t)dt$ Prove that $\langle,\rangle$ is IPS on V.

Sol 2. $V = C[0,1]$

Linearity:- $\langle af(t)+bg(t), h(t)\rangle$

$= \int_0^1 (af(t)+bg(t))\,h(t)\,dt$

$$\int_0^1 a f(t)\, h(t)\, dt + \int_0^1 b\, g(t)\, h(t)\, dt$$

$$a \langle f, h \rangle + b \langle g, h \rangle$$

<u>Symmetric</u> :- $\langle f(t), g(t) \rangle = \int_0^1 f(t) g(t)\, dt$

$$= \int_0^1 g(t) f(t)\, dt = \langle g(t), f(t) \rangle$$

Hence, symmetric proved.

<u>Positive definite</u> :- $\langle f(t), f(t) \rangle = \int_0^1 f(t) f(t)\, dt = \int_0^1 f^2(t)\, dt \geq 0$

$$\int_0^1 f^2(t)\, dt = 0 \quad \text{iff} \quad f(t) = 0$$

NOTE: In real valued continuous function over the domain $[0,1]$ no need to take conjugate complex, to prove symmetric.

ORTHOGONAL AND ORTHONORMAL

Let V be an inner product space (IPS). The vectors $x, y \in V$ are said to be orthogonal (or $\perp$) if $\langle x, y \rangle = 0$. A subset S of V is orthogonal if any two distinct vectors in S are orthogonal. A vector $x \in V$ is a unit vector if $\|x\| = 1$. So, finally a subset S of V is orthonormal if S is orthogonal and consists of entirely unit vectors.

Q1. In F^3 prove that $S = \{\underset{x}{(1,1,0)}, \underset{y}{(1,-1,1)}, \underset{z}{(-1,1,2)}\}$ is an

(i) orthogonal subset.

Sol1. $\langle x, y \rangle = x \cdot y = 1 \cdot 1 + 1 \cdot (-1) + 0 \cdot 1 = 1 - 1 + 0 = 0$

$\langle y, z \rangle = y \cdot z = 1 \cdot (-1) + (-1) \cdot 1 + 1 \cdot 2 = -1 + 2 - 1 = 0$

$\langle z, x \rangle = z \cdot x = -1 \cdot 1 + 1 \cdot 1 + 2 \cdot 0 = -1 + 1 + 0 = 0$

Hence S is an orthogonal subset.

(ii) Check whether S is an orthonormal set.

$\|x\| = \sqrt{(1)^2 + (1)^2 + (0)^2} = \sqrt{2}$

$\|y\| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$

$\|z\| = \sqrt{(-1)^2 + (1)^2 + (2)^2} = \sqrt{6}$

$\|x\| = \|y\| = \|z\| \neq 1$ None of the norm is 1

Hence S is not an orthonormal set.

(iii) In F^3 prove that $S = \left\{ \underset{x}{\frac{1}{\sqrt{2}}(1,1,0)}, \underset{y}{\frac{1}{\sqrt{3}}(1,-1,1)}, \underset{z}{\frac{1}{\sqrt{6}}(-1,1,2)} \right\}$

$\langle x, y \rangle = x \cdot y = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} \cdot \left(\frac{-1}{\sqrt{3}}\right) + 0 \cdot \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} + 0 = 0$

$\langle y, z \rangle = y \cdot z = \frac{1}{\sqrt{3}} \cdot \frac{-1}{\sqrt{6}} + \left(\frac{-1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right) + 1 \cdot \frac{2}{\sqrt{3}\sqrt{6}} = \frac{-2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}} = 0$

$\langle z, x \rangle = z \cdot x = \frac{-1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{6}} \cdot 0 = \frac{-1}{2\sqrt{3}} + \frac{1}{2\sqrt{3}} + 0 = 0$

$\|x\| = \frac{1}{\sqrt{2}} \sqrt{(1)^2 + (1)^2 + (0)^2} = 1$

$\|y\| = \frac{1}{\sqrt{3}} \sqrt{(1)^2 + (-1)^2 + (1)^2} = 1$

$\|z\| = \frac{1}{\sqrt{6}} \sqrt{(-1)^2 + (1)^2 + (2)^2} = 1$

Hence the $\langle x, y \rangle, \langle y, z \rangle, \langle z, x \rangle = 0$ and $\|x\|, \|y\|, \|z\|$ all are equal to 1. Hence the S is orthonormal subset.

Q2. Let $x = (\overset{x_1}{2}, \overset{x_2}{1+i}, \overset{x_3}{i})$, $y = (\overset{y_1}{2-i}, \overset{y_2}{2}, \overset{y_3}{1+2i})$ be the vectors in $\mathbb{C}^3$. Compute $\langle x, y \rangle$, $\|x\|$, $\|y\|$, $\|x+y\|$.

Sol2. ~~Linearity: $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$~~

$\langle x, y \rangle = x_1 \cdot y_1 + x_2\, y_2 + x_3 \cdot y_3$

$= 2(2-i) + (1+i)2 + i(1+2i)$

$4 - \cancel{2i} + 2 + \cancel{2i} + i - 2 = 4 + i$

$\|x\| = \sqrt{(2)^2 + (1+i)^2 + (i)^2} = \sqrt{4 + 1 - 1 + 2i - 1} = \sqrt{3 + 2i}$

$2 + i$

Date
Page No.

$$\|y\| = \sqrt{(2-i)^2 + (2)^2 + (1+2i)^2}$$
$$= \sqrt{4+i^2-4i+4+1+4i^2+4i} = \sqrt{4} = 2$$

$$\|x+y\| = \sqrt{\langle x+y \rangle, \langle x+y \rangle}$$

$$x+y = 2+2-i,\ 1+i+2,\ i+1+2i$$
$$4-i,\ 3+i,\ 1+3i$$
$$\langle x+y, x+y \rangle = (4-i)^2 + (3+i)^2 + (1+3i)^2$$
$$= 16+i^2-8i+9+i^2+6i+1+9i^2+6i$$
$$26+11i^2+4i = 4i+15$$

$$\|x+y\| = \sqrt{\langle x+y \rangle, \langle x+y \rangle} = \sqrt{4i+15} = 4+i$$

Q3. In $\mathbb{C}^3$ show that $\langle x,y \rangle = xAy^*$ is an IP where $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$ compute $\langle x,y \rangle$ for $x = (1-i, 2+3i)$
$y = (2+i, 3-2i)$

Sol3. $\langle x,y \rangle = x_1 y_1 + x_2 y_2 \cancel{+ x_3 y_3}$
$(1-i)(2+i) + (2+3i)(3-2i)$
$2+i-2i-i^2+6-4i+9i-6i^2 = 15+4i$

Date 06/01/23 Friday

Date: / /
Page No.

GRAM-SCHMIDT ORTHOGONALISATION PROCESS

Let V be an IPS a subset of V is an orthonormal basis of V if it is an ordered basis i.e. orthonormal

Result 1. Let V be an inner product space and $S = \{w_1, w_2 \ldots w_n\}$ be a LI subset of V define $S' = \{v_1, v_2 \ldots v_n\}$ such that $v_1 = w_1$ and $v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} \cdot v_j$

Q1. In $\mathbb{R}^4$ let $w_1 = (1,0,1,0)$ and $w_2 = (1,1,1,1)$ and $w_3 = (0,1,2,1)$ then check that w_1, w_2, w_3 is LI and use the gram schmidt orthogonalisation process to compute the orthogonal vectors v_1, v_2, v_3 and further normalise these vectors to obtain the orthonormal set.

Soln. $v_1 = w_1 = (1,0,1,0)$

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} \cdot v_1$$

$$(1,1,1,1) - \frac{(1,0,1,0)\cdot(1,1,1,1)}{(\sqrt{(1)^2+(1)^2})^2}(1,0,1,0)$$

$$(1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1)$$

$$v_3 = w_3 - \sum_{j=1}^{2} \frac{\langle w_3, v_2 \rangle}{\|v_2\|^2} \cdot v_2 = (0,1,2,1) - \sum_{j=1}^{2} \frac{(0,1,2,1)(0,1,0,1)}{(\sqrt{2})^2}$$

$$(0,1,2,1) - \left[\frac{2}{2}(0,1,0,1) + \frac{2}{2}(1,0,1,0)\right] = 0,1,2,1 - 1,1,1,1$$
$$(-1,0,1,0)$$

The orthonormal set of vectors is $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right\}$

ordered basis of $P_2(R) = \{1, x, x^2\}$

Date:

Page No.

Q2. Let V be P(R) IP with $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(x)g(x)\,dx$ and consider the subspace $P_2(R)$ with the standard ordered basis β. Use the gram schmidt orthogonalisation process to obtain the orthonormal basis for $P_2(R)$.

Sol 2. As we know, $\beta = \{1, x, x^2\}$

$\therefore \boxed{v_1 = w_1 = 1}$

$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = x - \frac{\langle x, 1\rangle}{\|v_1\|^2} . 1$

$x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx} . 1 = x - \frac{\left|\frac{x^2}{2}\right|_{-1}^{1}}{|x|_{-1}^{1}} = x - \frac{0}{2} = x$

$\boxed{v_2 = x}$

$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$

$= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} - \frac{\langle x^2, x\rangle}{\langle x, x\rangle} . x$

$= x^2 - \frac{2/3}{2} - \frac{0}{2/3} . x = \boxed{x^2 - \frac{1}{3} = v_3}$

$\|v_1\| = \sqrt{\langle 1, 1\rangle} = \left(\int_{-1}^{1} dx\right)^{1/2} = |x|_{-1}^{1} = \sqrt{1-(-1)} = \sqrt{2}$

$\|v_2\| = \sqrt{\langle x, x\rangle} = \left(\int_{-1}^{1} x^2\,dx\right)^{1/2} = \left|\frac{x^3}{3}\right|_{-1}^{1} = \sqrt{\frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{2}{3}}$

$\|v_3\| = \sqrt{\langle (x^2 - \frac{1}{3}), (x^2 - \frac{1}{3})\rangle} = \left(\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx\right)^{1/2} = \left(\int_{-1}^{1} x^4 + \frac{1}{9} - \frac{2x^2}{3}\,dx\right)^{1/2}$

$\left(\left|\frac{x^5}{5}\right|_{-1}^{1} + \frac{1}{9}|x|_{-1}^{1} - \frac{2}{3}\left|\frac{x^3}{3}\right|_{-1}^{1}\right)^{1/2} = \left(\frac{2}{5} + \frac{2}{9} - \frac{4}{9}\right)^{1/2} = \frac{2}{5} - \frac{2}{9} = \frac{2\sqrt{2}}{3\sqrt{5}}$

Orthogonal set of vectors $= \left(\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right) = \left(\frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}, \frac{\sqrt{5}(3x^2-1)}{\sqrt{8}}\right)$

Date: / /

Page No.

* PROOF OF CAUCHY SCHWARTZ INEQUALITY

$|\langle x, y\rangle| \leq \|x\|\|y\|$

Proof:- Case 1: If $y = 0$, obviously result hold true

Case 2: If $y \neq 0$, then we have $0 \leq \|x - cy\|^2 = \langle x - cy, x - cy\rangle$

$\langle x, x\rangle + \langle x, -cy\rangle + \langle -cy, x\rangle + \langle -cy, -cy\rangle$ — *

Without loss of Generality (WLOG), let

$C = \frac{\langle x, y\rangle}{\langle y, y\rangle}$

$\because \langle cx, y\rangle = c\langle x, y\rangle$

$\langle x, cy\rangle = \bar{c}\langle x, y\rangle$

if F of $\mathbb{C}$

By (*)

$= \langle x, x\rangle + (-\bar{c})\langle x, y\rangle + (-c)\langle y, x\rangle + c\bar{c}\langle y, y\rangle$

$\|x\|^2 + \left(-\frac{\overline{\langle x, y\rangle}}{\langle y, y\rangle}\right)\langle x, y\rangle - \left(\frac{\langle x, y\rangle}{\langle y, y\rangle}\right)\langle y, x\rangle + c\bar{c}\|y\|^2$

$= \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2} \Rightarrow 0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|}{\|y\|^2}$

$\Rightarrow 0 \leq \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$

$\Rightarrow |\langle x, y\rangle| \leq \|y\|\|x\|$

* Proof of triangle inequality $\|x+y\| \leq \|x\| + \|y\|$

$\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2$

$= \|x\|^2 + 2\,\mathrm{Re}(\langle x, y\rangle) + \|y\|^2$

$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$ using cauchy schwarts

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2$

$\|x+y\| \leq \|x\| + \|y\|$

Date : /
Page No.

Remark Let V be an IPS then caucy schwartz inequality for any 2 vectors x, y in V gets converted into equality i.e. $|\langle x, y\rangle| = \|x\|\|y\|$ iff

CS (i) one of the vectors x and y is a multiple of the another one

(ii) If the 2 vectors x and y are orthogonal (or $\perp$) to each other the Δ inequality gets converted into equality.