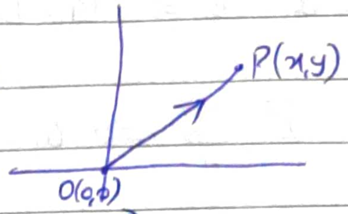
Unit - IV

Vector:→ A quantity that has magnitude as well as direction is called a vector.

Position vector:→ The vector $\overrightarrow{OP}$ having O and P as initial and terminal points, respectively is called position vector.



$\overrightarrow{OP} = x\hat{i} + y\hat{j}$

Zero Vector:→ A vector whose initial and terminal points coincide is called a zero vector. E.g. $\overrightarrow{AA}$, $\overrightarrow{BB}$ etc. are zero vectors.

Unit vector:→ A vector whose magnitude is unity is called unit vector. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$. $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Coinitial vectors:→ Vectors having same initial point are called coinitial vectors.

Equal vectors:→ Two vectors $\vec{a}$ and $\vec{b}$ are called equal if they have the same magnitude and direction and written as $\vec{a} = \vec{b}$.

Negative of a vector:→ A vector whose magnitude is the same as that of a given vector but direction is opposite to that of it; is called negative of a vector. E.g. $\overrightarrow{BA}$ is negative of the vector $\overrightarrow{AB}$ and written as $\overrightarrow{BA} = -\overrightarrow{AB}$.

Length of vector:→ Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2+y^2+z^2}$.

→ Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then

(1) $\vec{a}+\vec{b} = (a_1+b_1)\hat{i} + (a_2+b_2)\hat{j} + (a_3+b_3)\hat{k}$.

(2) $\vec{a}-\vec{b} = (a_1-b_1)\hat{i} + (a_2-b_2)\hat{j} + (a_3-b_3)\hat{k}$

(3) $\vec{a} = \vec{b} \iff a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.

(4) $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$ where λ is any scalar.

Ex Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i}+\hat{j}$. Are the vectors $\vec{a}$ and $\vec{b}$ Equal?

Solⁿ
$|\vec{a}| = \sqrt{1+4} = \sqrt{5}$
$|\vec{b}| = \sqrt{4+1} = \sqrt{5}$
$\Rightarrow |\vec{a}| = |\vec{b}|$
But $\vec{a} \neq \vec{b}$ as their Corresponding Components are not Equal.

Ex Find unit vector in the direction of $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k}$.

Solⁿ Unit vector is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$$|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$$

$$\Rightarrow \hat{a} = \frac{1}{\sqrt{14}}(2\hat{i}+3\hat{j}+\hat{k}) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}.$$

Ex→ Find a vector in the direction of the vector $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 unit.

Solⁿ Unit vector in the direction of $\vec{a}$ is $\hat{a}$

$$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}-2\hat{j}}{\sqrt{1+4}} = \frac{\hat{i}}{\sqrt{5}} - \frac{2}{\sqrt{5}}\hat{j}$$

Now the vector having magnitude $7 = 7\hat{a}$

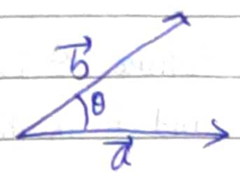
$$= 7\left(\frac{\hat{i}}{\sqrt{5}} - \frac{2}{\sqrt{5}}\hat{j}\right) = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j}.$$

<u>Scalar product or dot product</u> :→ The dot product of two vector $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and defined as

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$$

where θ is the angle b/w $\vec{a}$ and $\vec{b}$.

$0 \le \theta \le \pi$.



→ $\vec{a}\cdot\vec{b}$ is a scalar.

(1) $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are perpendicular to each other ie. $\theta = \pi/2$.

(2) If $\theta = 0$ then $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|$

<u>Ex</u> Find angle θ b/w the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$.

<u>Soln</u>

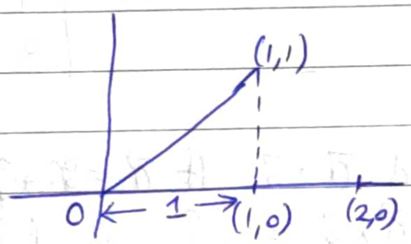
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{1-1-1}{\sqrt{3}\sqrt{3}} = \frac{-1}{3}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{3}\right).$$

→ $\vec{a}\cdot\vec{b}$ actually represents (projection of $\vec{a}$ on $\vec{b}$) × (magnitude of $\vec{b}$)

<u>E.g.</u> $\vec{a} = \hat{i}+\hat{j}$

$\vec{b} = \hat{i}$



Proj of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \dfrac{1}{1} = 1.$

If $\vec{b} = 2\hat{i}$

then Proj. of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{2}{2} = 1.$

<u>Que</u> Find the Projection of $\vec{a} = 2\hat{i}+3\hat{j}+2\hat{k}$ on $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$.

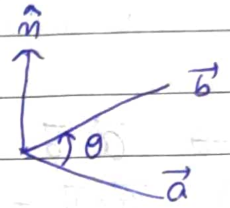
<u>Soln</u>

Proj of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{2+6+2}{\sqrt{1+4+1}} = \dfrac{10}{\sqrt{6}} = \dfrac{5\sqrt{6}}{3}.$

Vector product or cross product: $\rightarrow$ The cross product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and is defined as

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta \cdot \hat{n}$$

where θ is the angle b/w $\vec{a}$ and $\vec{b}$ and $0\le\theta\le\pi$ and $\hat{n}$ is the unit vector $\perp$ to $\vec{a}$ and $\vec{b}$ both.



(1) $\vec{a}\times\vec{b}$ is a vector.

(2) $\vec{a}\times\vec{b} = 0 \iff \vec{a}$ and $\vec{b}$ are Parallel to each other i.e. $\theta = 0$ or $\theta = \pi$.

(3) If $\theta = \frac{\pi}{2}$ then $\vec{a}\times\vec{b} = |\vec{a}|\cdot|\vec{b}|$

(4) $\hat{i}\times\hat{i} = 0$, $\hat{j}\times\hat{j} = 0$, $\hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = K$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$.

$\rightarrow$ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Ex Find $\vec{a}\times\vec{b}$ If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$

Sol^n $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$

$= \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3) = -17\hat{i} + 13\hat{j} + 7\hat{k}$.

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Vector valued Function:→ A function $f: D \to \mathbb{R}^n$, $n \geq 1$ is Called vector valued function.

E.g. $f: \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = x\hat{i} + x^2\hat{j} \quad \text{or}$$

$$f(x) = (x, x^2)$$

Then f is a vector valued function.

$f: D \to \mathbb{R}^3$ by, $D \subseteq \mathbb{R}$

$$f(x) = x\hat{i} + x^2\hat{j} + x^3\hat{k}. \text{ or}$$

$f(x) = (x, x^2, x^3)$ is vector valued function.

Scalar valued function or Scalar point function:→ A function f whose Co-domain is subset of Real no.s or $\mathbb{R}$ itself is called Scalar point function.

E.g. $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = xy^2$ is Scalar point function

$f: \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = xy^2z^3$ is Scalar valued function.

→ $f(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$

$f: \mathbb{R} \to \mathbb{R}^3$

Domain of f $(D_f) = D_{f_1} \cap D_{f_2} \cap D_{f_3}$.

Limit:→ $\lim\limits_{t \to a} f(t)$ Exist Iff limit of Component functions $f_1(t), f_2(t), f_3(t)$ Exist as $t \to a$.

And If $\lim\limits_{t\to a} f_1(t) = l_1$, $\lim\limits_{t\to a} f_2(t) = l_2$, $\lim\limits_{t\to a} f_3(t) = l_3$

Then $\lim\limits_{t\to a} f(t) = l$ where

$$l = l_1\hat{i} + l_2\hat{j} + l_3\hat{k}.$$

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Continuity:→ A vector function $f(t)$ is continuous at $t=a$ iff the component functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous at $t=a$.

Differentiability:→ A vector function $f(t)$ is differentiable at $t=a$ if the component functions $f_1(t)$, $f_2(t)$ and $f_3(t)$ are diff. at $t=a$.

and $f'(t) = f_1'(t)\hat{i} + f_2'(t)\hat{j} + f_3'(t)\hat{k}$.

→ $(f(t) \cdot g(t))' = f(t) \cdot g'(t) + f'(t) \cdot g(t)$

→ $(f(t) \times g(t))' = f(t) \times g'(t) + f'(t)\, g(t)$

where $f(t)$ and $g(t)$ are vector valued functions.

→ $f''(t) = f_1''(t)\hat{i} + f_2''(t)\hat{j} + f_3''(t)\hat{k}$.

→ $(f(t) \cdot u(t))' = f'(t)u(t) + f(t)u'(t)$ where $u(t)$ is scalar valued fun.

Que Let $v(t) = (\cos t + t^2)(t\hat{i} + \hat{j} + 2\hat{k})$. Find $v'(t)$

Sol^n
$$v'(t) = (\cos t + t^2)'(ti + j + 2k) + (\cos t + t^2)(t\hat{i} + j + 2k)'$$
$$= (-\sin t + 2t)(ti + j + 2k) + (\cos t + t^2)(i)$$
$$= (-t\sin t + 2t^2)i + (\sin t + 2t)j + (-2\sin t + 4t)k + (\cos t + t^2)i$$
$$= (3t^2 - t\sin t + \cos t)i + (2t - \sin t)(j + 2k)$$

Que $V(t) = (3ti + 5t^2j + 6k) \cdot (t^2i - 2tj + tk)$

Find $v'(t)$

Sol^n
$$V'(t) = (3ti + 5t^2j + 6k)' \cdot (t^2i - 2tj + tk) + (3ti + 5t^2j + 6k) \cdot (t^2i - 2tj + tk)'$$
$$= (3i + 10tj) \cdot (t^2i - 2tj + tk) + (3ti + 5t^2j + 6k) \cdot (2ti - 2j + k)$$
$$= (3t^2 - 20t^2) + (6t^2 - 10t^2 + 6)$$
$$= 9t^2 - 30t^2 + 6 = -21t^2 + 6.$$

H.W $v(t) = (ti + e^tj - t^2k) \times (t^2i + j + t^3k)$,

Find $v'(t)$.

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Que S.T. $(v(t)\times v'(t))' = v(t)\times v''(t)$

Sol^n L.H.S $(v(t)\times v'(t))' = v'(t)\times(v'(t))' + (v(t))'\times v'(t)$

$$= v(t)\times v''(t) + v'(t)\times v'(t)$$

$$= v(t)\times v''(t)$$

$(\because v'(t)\times v'(t) = 0)$

due to $i\times i = 0 = j\times j = k\times k$.

Gradient and directional derivative:→

Let f be any scalar valued function. ie. $f: D \to \mathbb{R}, D\subseteq \mathbb{R}^n, n\geq 1$
Then we know total derivative of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\cdot dy$$

$$\Rightarrow df = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right)$$

Then $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \nabla f$ is called gradient of f

denoted by ∇f or grad(f).
where $\nabla f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\right) f$

$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$ is called vector differential operator.

→ • The directional derivative of f in the direction of $\vec{b}$ is defined as $\nabla f\cdot\hat{b} = \nabla f\cdot\frac{\vec{b}}{|\vec{b}|}$

* So the directional der. of f in direction of x-axis

$$= \nabla f\cdot\hat{i} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\hat{i} = f_x = \frac{\partial f}{\partial x}$$ → Partial der. of f W.r.t. x

* dir. der. of f in the direction of y-axis

$$= \nabla f\cdot\hat{j} = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right)\cdot\hat{j} = \frac{\partial f}{\partial y}$$ → Partial der. of f W.r.t y.

Que $f(x,y) = y^2 - 4xy$. Find ∇f at $(1,2)$

Sol^n $\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$$= -4y\hat{i} + (2y-4x)\hat{j}$$

$$\Rightarrow \nabla f(1,2) = -8\hat{i} + 0\hat{j} = -8\hat{i}$$

Que $f(x,y,z) = x^2y^2 + xy^2 - z^2$.

Find $\nabla f(x,y,z)$ at $(3,1,1)$

Ans $7\hat{i} + 24\hat{j} - 2\hat{k}$.

Que If $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$; $|\vec{V}| = r$. Then find $grad\left(\frac{1}{r}\right)$

Sol^n $|\vec{V}| = r = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow r^2 = x^2+y^2+z^2$$

$$grad\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[i\left(\frac{x}{r}\right) + j\left(\frac{y}{r}\right) + k\left(\frac{z}{r}\right)\right]$$

$$= \frac{-1}{r^2}\left[\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right]$$

$$= \frac{-1}{r^2}\left[\frac{\vec{V}}{|\vec{V}|}\right] = \frac{-1}{r^2}\hat{V} = \frac{-\hat{V}}{r^2}.$$

Properties of ∇f:→

let f and g be two diff scalar valued functions.

Then $\nabla(f+g) = \nabla f + \nabla g$.

$\nabla(c_1 f + c_2 g) = c_1 \nabla f + c_2 \nabla g$. , c_1, c_2 arbitrary constants.

$\nabla(fg) = f\nabla g + g\nabla f$.

$\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$, $g \neq 0$.

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Que Find the directional der. of $f(x,y,z) = xy^2 + 4xyz + z^2$ at $(1,2,3)$ in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$.

Sol^n let $\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

Dir. der. of f in direction of $\vec{b} = \nabla f \cdot \hat{b}$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{3\hat{i} + 4\hat{j} - 5\hat{k}}{\sqrt{9+16+25}} = \frac{1}{5\sqrt{2}}(3\hat{i} + 4\hat{j} - 5\hat{k})$$

So $D_b(f) = (\nabla f) \cdot \hat{b}$

$$\nabla f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= (y^2 + 4yz)\hat{i} + (2xy + 4xz)\hat{j} + (4xy + 2z)\hat{k}$$

Now $D_b(f) = \frac{1}{5\sqrt{2}}\left[3(y^2 + 4yz) + 4(2xy + 4xz) - 5(4xy + 2z)\right]$

$$= \frac{1}{5\sqrt{2}}(3y^2 + 12yz + 8xy + 16xz - 20xy - 10z)$$

at $(1,2,3)$

$$D_b(f)(1,2,3) = \frac{1}{5\sqrt{2}}\left[12 + 72 + 16 + 48 - 40 - 30\right]$$

$$= \frac{78}{5\sqrt{2}}.$$

Que Find the dir. der. of $(x^2y - xy^2z - xyz)$ in the direction of $\hat{i} - \hat{j} + 2\hat{k}$ at point $(1,-1,0)$.

Ans $\frac{-3}{\sqrt{6}}$ or $-\sqrt{\frac{3}{2}}$

Que $f(x,y,z) = (x^2+y^2+z^2)^{3/2}$. Find dir. der. of f at $(-1,1,2)$ in the direction of $\hat{i} - 2\hat{j} + \hat{k}$.

Ans -3.

Que Find the dir. der of $f(x,y) = x^2 + y^2$ in the direction of $\vec{a} = i + j$ at $(1,1)$.

Sol^n dir. der of f in the direction of $\vec{a} = \nabla f \cdot \hat{a}$

$$\nabla f = 2x\hat{i} + 2y\hat{j}$$

$$\hat{a} = \frac{i+j}{\sqrt{2}}$$

$$\Rightarrow D_{\vec a}(f) = \nabla f \cdot \hat a$$
$$= (2x\hat i + 2y\hat j)\left(\frac{\hat i + \hat j}{\sqrt 2}\right)$$
$$= \frac{2x+2y}{\sqrt 2} = \sqrt 2 x + \sqrt 2 y.$$

at $(1,1)$, $\sqrt 2 + \sqrt 2 = 2\sqrt 2$.

Level Surfaces: $\rightarrow$ Let $f: \mathbb{R}^3 \to \mathbb{R}$ or $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^3$ be a scalar valued function.

Then $f(x,y,z) = c$ defines the equation of a surface and is called a Level Surface of the function.

For different values of c, we obtain different surfaces, no two of which intersect.

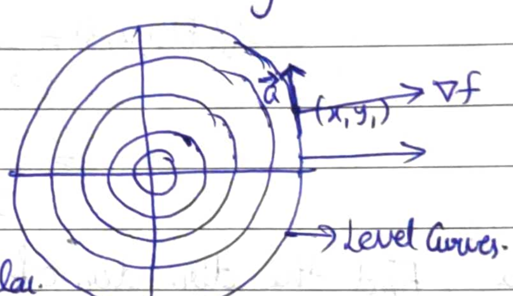
Eg. $f(x,y) = x^2+y^2 = c$ — represents a family of circles.

direction derivative of f in the direction of $\vec a = \nabla f \cdot \hat a = 0$

$(\because f(x,y) = \text{Constant})$

$\Rightarrow \nabla f$ and $\vec a$ are perpendicular.

$\Rightarrow \nabla f$ is normal vector to the given surface.



Que Find a ~~unit~~ normal vector to the surface $xy^2+2yz=8$ at the point $(3,-2,1)$.

Sol^n let $f(x,y,z) = xy^2+2yz = 8$. —(1)

$$\nabla f = \frac{\partial f}{\partial x}\hat i + \frac{\partial f}{\partial y}\hat j + \frac{\partial f}{\partial z}\hat k$$

$$\Rightarrow \nabla f = y^2\hat i + (2xy+2z)\hat j + (2y)\hat k$$

$\nabla f(3,-2,1) = 4\hat i - 10\hat j - 4\hat k$ is a normal vector to (1).

$\rightarrow$ $\nabla f(x_1,y_1) \cdot ((x-x_1)\hat i + (y-y_1)\hat j) = 0$ is the Equ^n of tangent line to the curve $f(x,y) = c$.

Que Find the normal vector to the surface $z = \sqrt{x^2+y^2}$ at the pt. (3,4,5).

Solⁿ→ let $f(x,y,z) = z - \sqrt{x^2+y^2} = 0$ be the surface.

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \frac{-1(2x)}{2\sqrt{x^2+y^2}}\hat{i} - \frac{1(2y)}{2\sqrt{x^2+y^2}}\hat{j} + \hat{k}$$

$$= -\frac{x}{z}\hat{i} - \frac{y}{z}\hat{j} + \hat{k}, \; (z \neq 0)$$

at (3,4,5); $\nabla f(3,4,5) = -\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j} + \hat{k}$.

Que Find the angle b/w the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the pt. (1,1,1).

Solⁿ→ let $f_1(x,y,z) = x\log z - y^2 + 1 = 0$

$f_2(x,y,z) = x^2 y - 2 + z = 0.$

$\nabla f_1(x,y,z) = \log z\,\hat{i} - 2y\hat{j} + \frac{x}{z}\hat{k}$

$\Rightarrow \nabla f_1(1,1,1) = -2\hat{j} + \hat{k}$

$\nabla f_2(x,y,z) = 2xy\hat{i} + x^2\hat{j} + \hat{k}$

$\Rightarrow \nabla f_2(1,1,1) = 2\hat{i} + \hat{j} + \hat{k}$

Thus ~~Cos~~ $\nabla f_1 \cdot \nabla f_2 = |\nabla f_1|\,|\nabla f_2| \cos\theta$

$\Rightarrow \cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1||\nabla f_2|} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$

HW Find the unit normal vector to the given surfaces / curves at a given pt.

(1) $x^2+y^2 = 25$ at (3,4)

Ans:- $\frac{3\hat{i}+4\hat{j}}{5}$

(2) $x^2+2y^2+z^2 = 4$ at (1,1,1)

Ans:- $\frac{\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}}$

(3) $z^2 = x^2 - y^2$ at $(2,1,\sqrt{3})$

Ans:- $\frac{2\hat{i}-\hat{j}-\sqrt{3}\hat{k}}{\sqrt{8}}$.

Maximum and Minimum Directional derivatives

We know that for a given scalar valued function f, directional der. of f in the direction of $\vec{a}$ is

$$D_{\vec{a}}f = \nabla f \cdot \hat{b}$$
$$= |\nabla f||\hat{b}| \cos\theta$$
$$= |\nabla f| \cdot \cos\theta \qquad (\because |\hat{b}|=1)$$

where θ is angle b/w ∇f and $\hat{b}$, $0 \le \theta \le \pi$

$$\Rightarrow -1 \le \cos\theta \le 1.$$

$$\Rightarrow \cancel{D_a f} \quad -|\nabla f| \le |\nabla f| \cos\theta \le |\nabla f|$$

$$\Rightarrow -|\nabla f| \le D_a f \le |\nabla f|$$

$\Rightarrow$ Max value of dir. der. is $|\nabla f|$ and it occurs when $\theta = 0$.
& Min value of dir der is $-|\nabla f|$ and it occurs when $\theta = \pi$.

Imp
$\theta = 0 \Rightarrow$ ∇f and $\hat{b}$ have same direction and ∇f and $\hat{b}$ are Parallel.
$\theta = \pi \Rightarrow$ ∇f and $\hat{b}$ have opp. directions and Parallel.

Hence dir. der is max in the direction of (∇f)
and dir. der is min in the direction of $-(\nabla f)$.
Dir. der. is 0 when $\nabla f \cdot \hat{b} = 0$
$\Rightarrow$ ∇f and $\hat{b}$ are Perpendicular.

Que Find a vector that gives the direction of max. rate of increase and find the max rate.

(i) $e^{2y}\cos x$ at $\left(\frac{\pi}{4}, 0\right)$

$$f(x,y) = e^{2y}\cos x$$
$$\nabla f = -e^{2y}\sin x\,\hat{i} + 2e^{2y}\cos x\,\hat{j}.$$
$$|\nabla f| = \sqrt{(e^{2y}\sin x)^2 + (2e^{2y}\cos x)^2}$$
$$= \sqrt{e^{4y}(\sin^2 x + 4\cos^2 x)}$$
$$\nabla f(\pi/4, 0) = -\frac{1}{\sqrt{2}}\hat{i} + \frac{2}{\sqrt{2}}\hat{j} = \frac{-\hat{i} + 2\hat{j}}{\sqrt{2}}$$

$$|\nabla f(\tfrac{\pi}{4},0)| = \sqrt{\tfrac{5}{2}}$$

$\therefore$ Max Rate $= \sqrt{\frac{5}{2}}$

and vector that gives the dir. of max rate Increase $= \frac{-i+2j}{\sqrt{2}}$.

H.W (1) $3x^2+y^2+2z^2$ at $(0,1,2)$

Ans $2(\hat{j}+4\hat{k})$ and $2\sqrt{17}$.

(2) $6xyz$ at $(-1,2,1)$

Ans $6(2\hat{i}-\hat{j}-2\hat{k})$ and 18.

Que Find the vector that gives the dir of min rate of Increase and find the min rate.

(1) $x^3-xy^2+y^3$ at $(-2,1)$

Ans $-(11\hat{i}+7\hat{j})$ and $-\sqrt{170}$

(2) $x^2-y^2+z^2$ at $(1,2,1)$

Ans $-2(\hat{i}-2\hat{j}+\hat{k})$ and $-2\sqrt{6}$.

Que Find the angle b/w the two Surfaces at given point.

(1) $z=x^2+y^2$; $z=2x^2-3y^2$ at $(2,1,5)$.

Ans $\cos^{-1}\left(\sqrt{21/101}\right)$

(2) $x^2+y^2+z^2=9$; $z+3=x^2+y^2$ at $(-2,1,2)$

Ans. $\cos^{-1}\left(8/3\sqrt{21}\right)$

Que Find the Equ^n of tangent plane to $f(x,y,z)=x^2-3y^2-z^2=2$ at $(3,1,2)$.

Sol^n Equ^n of tangent plane is $\nabla f(3,1,2)\cdot\left((x-3)\hat{i}+(y-1)\hat{j}+(z-2)\hat{k}\right)=0$

$$\Rightarrow (2x\hat{i}-6y\hat{j}-2z\hat{k})\cdot\left((x-3)\hat{i}+(y-1)\hat{j}+(z-2)\hat{k}\right)=0$$

$$\Rightarrow (6\hat{i}-6\hat{j}-4\hat{k})\cdot\left((x-3)\hat{i}+(y-1)\hat{j}+(z-2)\hat{k}\right)=0$$

$$\Rightarrow 6(x-3)-6(y-1)-4(z-2)=0$$

$$\Rightarrow 6x-6y-4z-4=0$$

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Que Find the Equ^n of tangent plane to : $z = 16 - x^2 - y^2$ at $(1,3,6)$.

Ans $2x + 6y + z = 26$

Que Find the Equ^n of tangent plane to $xy + yz + zx = -1$ at $(1,-1,2)$.

Ans $x + 3y + 2 = 0$.