STEP TOWARDS SUCCESS

AKASH'S

Guru Gobind Singh Indra Prastha University Series

SOLVED PAPERS

[PREVIOUS YEARS SOLVED QUESTION PAPERS]

[B.Tech]

FIRST SEMESTER

Applied Mathematics-I

(BS-111)

Rs.81.00/-

AKASH BOOKS

NEW DELHI

Common Scheme for B.Tech. First Year
From Academic Session 2021-22 Onwards

FIRST SEMESTER

Group	Code	Paper	L	P	Credits
Theory Papers					
ES BS	ES101 BS103	*Any one of the following: Programming in 'C' Applied Chemistry	3	–	3
BS	BS105	Applied Physics-I	3	–	3
ES BS	ES107 BS109	*Any one of the following: Electrical Science Environmental Studies	3	–	3
BS	BS111	Applied Mathematics-I	4	–	4
HS	HS113	**Group 1 or Group 2 shall be offered: Group 1: Communications Skills Or	3	–	3
HS HS	HS115 HS117	Group 2: Indian Constitution Human Values and Ethics	2 1	–	2 1
ES	ES119	Manufacturing Process	4		4

SECOND SEMESTER

Group	Code	Paper	L	P	Credits
Theory Papers					
ES BS	ES102 BS104	*Any one of the following: Programming in 'C' Applied Chemistry	3	–	3
BS	BS106	Applied Physics-II	3	–	3
ES BS	ES108 BS110	*Any one of the following: Electrical Science Environmental Studies	3	–	3
BS	BS112	Applied Mathematics-II	4	–	4
HS	HS114	**Group 1 or Group 2 shall be offered: Group 1: Communications Skills OR	3	–	3
HS HS	HS116 HS118	Group 2: Indian Constitution Human Values and Ethics	2 1		2 1
ES	ES114	Engineering Mechanics	3	–	3

*For a particular batch of a programme of study one out of these two papers shall be taught in the first semester while the other shall be taught in the 2nd semester. Students who have to re-appear can only reappear in the odd semester. If originally offered to the student in the 1st semester and similarly for the students who study the paper in the second semester. The institution shall decide which paper to offer in which semester.

**For a particular batch of a programme of study either the paper on "Communications Skills" (Group 1), or Group 2 : papers ("Indian Constitution" and "Human values and ethics") shall be taught in the first semester while the other group shall be taught in the 2nd semester. Students who have to re-appear can only reappear in the odd semester if originally offered to the student in the 1st semester and similarly for the students who study the paper(s) in the second semester. The institution shall decide which paper group to offer in which semester.

SYLLABUS (From Academic Session 2021-22)

Applied Mathematics-I [BS-111]

Marking Scheme:

(a) Teacher Continuous Evaluation: 25 marks

(b) Term End Theory Examination : 75 marks

UNIT I

Partial derivatives, Chain rule, Differentiation of Implicit functions, Exact differentials. Maxima, Minima and saddle points, Method of Lagrange multipliers. Differentiation under Integral sign, Jacobians and transformations of coordinates.

[8 Hrs.] [T2]

UNIT II

Ordinary Differential Equations (ODEs): Basic Concepts. Geometric Meaning of $y' = f(x, y)$. Direction Fields, Euler's Method, Separable ODEs. Exact ODEs. Integrating Factors, Linear ODEs. Bernoulli Equation. Population Dynamics, Orthogonal Trajectories. Homogeneous Linear ODEs with Constant Coefficients. Differential Operators. Modeling of Free Oscillations of a Mass-Spring System, Euler-Cauchy Equations. Wronskian, Nonhomogeneous ODEs, Solution by Variation of Parameters.

Power Series Method for solution of ODEs: Legendre's Equation. Legendre Polynomials, Bessel's Equation, Bessels's functions Jn(x) and Yn(x). Gamma Function.

[12 Hrs.] [T1]

UNIT III

Linear Algebra: Matrices and Determinants, Gauss Elimination, Linear independence. Rank of a Matrix. Vector Space. Solutions of Linear Systems and concept of Existence, Uniqueness, Determinants. Cramer's Rule, Gauss-Jordan Elimination. The Matrix Eigenvalue Problem.

Determining Eigenvalues and Eigenvectors, Symmetric, Skew-Symmetric, and Orthogonal Matrices. Eigenbases, Diagonalization, Quadratic Forms. Cayley-Hamilton Theorem (without proof)

[10 Hrs.] [T1]

UNIT IV

Vector Calculus: Vector and Scalar Functions and Their Fields, Derivatives, Curves, Arc Length. Curvature. Torsion, Gradient of a Scalar Field. Directional Derivative, Divergence of a Vector Field, Curl of a Vector Field, Line integrals, Path Independence of Line Integrals, Double Integrals, Green's Theorem in the Plane, Surfaces for Surface Integrals, Surface Integrals, Triple Integrals, Stokes Theorem. Divergence Theorem of Gauss.

[10 Hrs.] [T1]

NEW TOPIC ADDED FROM ACADEMIC SESSION 2021-22
FIRST SEMESTER
APPLIED MATHEMATICS - I (BS-111)
UNIT-I : PARTIAL DIFFERENTIATION

Geometrically

Let $z = f(x, y)$ be a function of two independent variables x and y defined for all pairs of values of x and y which belong to an area A of the xy-plane, then to each point (x, y) of this area corresponds a value of z given by the relation $z = f(x, y)$. Representing all these values (x, y, z) by points in space we get a surface

Hence $z = f(x, y)$ represents a surface.

Partial Derivative of First Order

Let $z = f(x, y)$ be a function of two independent variable x and y.

Now $\frac{\partial z}{\partial x}$ is partial derivative of z w.r.t 'x' keeping y as constant. Also denoted by f_x.

Similarly $\frac{\partial z}{\partial y}$ is partial derivative of z w.r.t 'y' keeping x as constant, denoted by f_y.

Higher Order Derivatives

(*i*) $f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

(*ii*) $f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

(*iii*) $f_{xy} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

(*iv*) $f_{yx} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$

(*v*) $f_{xxx} = \frac{\partial^3 z}{\partial x^3}$

(*vi*) $f_{xxy} = \frac{\partial^3 z}{\partial x^2 \partial y}$ and so on.

Composite Function

If u is a function of the variables x and y and these variables x, y are also a function of the other variable t, then u is said to be a composite function of the variable t. It is given by $\frac{du}{dt}$.

Differentiation of composite function

(*i*) If $u = f(x, y)$, where $x = \phi_1(t)$ and $y = \phi_2(t)$ then

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

(*ii*) If $f(x, y) = c$, then $u = f(x, y) \Rightarrow u = c$

$$\Rightarrow \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

Jacobians

It u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v w.r.t x, y and is denoted by $J\left(\frac{u, v}{x, y}\right)$ or $\frac{\partial(u, v)}{\partial(x, y)}$

Properties of Jacobian

(i) **Chain rule -** If u, v are functions of r, s where r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

(ii) If J_1 is Jacobian of u, v w.r.t x, y and J_2 is Jacobian of x, y w.r.t u, v then $J_1 J_2 = 1$

Jacobian of Implicit Functions

If u and v are implicit functions of x and y denoted by relations $f_1(u, v, x, y) = 0$ and $f_2 = (u, v, x, y) = 0$ then,

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(f_1, f_2)}{\partial(x,y)} \Bigg/ \frac{\partial(f_1, f_2)}{\partial(u,v)}$$

Partial Derivatives from Implicit Functions using Jacobians

Let u, v are implicit functions of independent variables x, y connected by functional relation $f_1(u, v, x, y) = 0$ and $f_2(u, v, x, y) = 0$ then

$$\frac{\partial u}{\partial x} = -\frac{\partial(f_1, f_2)}{\partial(x,v)} \Bigg/ \frac{\partial(f_1, f_2)}{\partial(u,v)}$$

$$\frac{\partial v}{\partial x} = \frac{-\partial(f_1, f_2)}{\partial(v,x)} \Bigg/ \frac{\partial(f_1, f_2)}{\partial(u,v)}$$

Jacobians to Determine Functional Dependence

Let $f(x, y)$ and $\phi(x, y)$ are two functions. Then $f(x, y)$ and $\phi(x, y)$ are functionally dependent if their Jacobian vanishes identically *i.e.*,

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix} = 0$$

Taylor's Theorem for Functions of Two Variables

Let $f(x, y)$ be a function of 2 variables then

$$f(x + h, y + k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)$$

$$+\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}\right)+\ldots$$

Corollary 1. Putting $x = a$ and $y = b$.

i.e., Taylor's expansion in neighbourhood of (a, b)

$$f(a+h, b+k) = f(a, b) + [h\, f_x(a, b) + k\, f_y(a, b)]$$
$$+\frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \ldots$$

Mc'laurin's Theorem for 2 Variables

Putting $a = 0$, $b = 0$ in above

$$f(x, y) = f(0, 0) + [x\, f_x(0, 0) + y\, f_y(0, 0)]$$
$$+\frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy\, f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \ldots$$

Differentiation Under Integral Sign

If a function $f(x, \alpha)$ of two variables x and α, α being a parameter, be integrated w.r.t 'x' between the limits a and b, $\int_a^b f(x,\alpha)dx$ is a function of x.

i.e.,
$$\int_a^b f(x,\alpha)dx = F(\alpha)$$

To find derivative of $F(\alpha)$, when it exists, it is not always possible to first evaluate this integral and then to find the derivative.

We have Leibnitz's Rule for this.

Leibnitz's Rule

If $f(x, \alpha)$ and $\frac{\partial}{\partial x}[f(x,\alpha)]$ be continuous functions of x and α then,

$\frac{d}{d\alpha}\left[\int_a^b f(x,\alpha)dx\right] = \int_a^b \frac{\partial}{\partial x}[f(x,\alpha)]dx$ where a and b are constants independent of α. (*i.e.*, the order of differentiation and integration can be interchanged).

Note: (1) Leibnitz rule enables us to derive form the value of a simple definite integral, the value of another definite integral which it may otherwise be difficult, or even impossible to evaluate.

(2) The rule for differentiation under the integral sign of an infinite integral is same as for a definite integral.

Q.1. Evaluate the integral $\int_0^1 \frac{x^{\alpha}-1}{\log x}dx$ by applying differentiation under the integral sign ($\alpha \geq 0$).

Ans. Let
$$F(\alpha) = \int_0^1 \frac{x^{\alpha}-1}{\log x}dx \quad \ldots(1)$$

Differentiating both sides w.r.t 'α'

$$F'(\alpha) = \int_0^1 \frac{\partial}{\partial x}\left[\frac{x^{\alpha}-1}{\log x}\right]dx \quad \left[\frac{d}{dx}a^x = a^x \log a\right]$$

$$= \int_0^1 \frac{1}{\log x} x^\alpha \log x \, dx = \int_0^1 x^\alpha \, dx$$

$$= \left| \frac{x^{\alpha+1}}{\alpha+1} \right|_0^1 \text{ for } \alpha > -1 = \frac{1}{1+\alpha}$$

Integrating both sides w.r.t 'α'

$$F(\alpha) = \log(1+\alpha) + c \qquad ...(2)$$

By (1), when $\alpha = 0$, $F(0) = \int_0^1 \frac{1-1}{\log x} dx = 0$

$\therefore$ By (2),

$$0 = \log 1 + c$$

$\Rightarrow$

$$c = 0$$

Thus,

$$F(x) = \log(1+\alpha), \text{ where } \alpha > -1.$$

$\therefore$

$$\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1+\alpha)$$

Q.2. Evaluate $\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx$ by applying differentiate under the integral sign.

Ans. Let

$$F(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx \qquad ...(1)$$

Differentiating both sides w.r.t 'a'

$$F'(a) = \int_0^\infty \frac{\partial}{\partial a} \left[\frac{\tan^{-1} ax}{x(1+x^2)} \right] dx$$

$$= \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{1+a^2x^2} dx$$

$\Rightarrow$

$$F'(a) = \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

Do Partial fraction by putting $x^2 = t$

$\Rightarrow$

$$F'(a) = \frac{1}{1-a^2} \int_0^\infty \frac{dx}{1+x^2} - \frac{a^2}{1-a^2} \int_0^\infty \frac{dx}{1+a^2x^2}$$

$$= \frac{1}{1-a^2} \left[\tan^{-1} x \right]_0^\infty - \frac{a^2}{1-a^2} \cdot \frac{1}{a^2} \int_0^\infty \frac{dx}{x^2 + \frac{1}{a^2}}$$

$$= \frac{1}{1-a^2} [\tan^{-1} \infty - \tan^{-1} 0] - \frac{a}{1-a^2} \left[\tan^{-1} \frac{x}{1/a} \right]_0^\infty$$

$$F'(a) = \frac{1}{1-a^2} \cdot \frac{\pi}{2} - \frac{a}{1-a^2} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a.\frac{\pi}{2}\right] = \frac{\pi(1-a)}{2(1-a^2)}$$

$$F'(a) = \frac{\pi}{2(1+a)}$$

Integrating both sides w.r.t. 'a'.

$$F(a) = \frac{\pi}{2}\log(1+a)+c \qquad \text{...(2)}$$

By (1), when $a = 0$, $F(0) = 0$

$\therefore$ from (2), $$0 = \frac{\pi}{2}\log 1 + c$$

$\Rightarrow$ $$c = 0$$

Thus $$F(a) = \frac{\pi}{2}\log(1+a)$$

Hence, $$\int_0^\infty \frac{\tan^{-1}ax}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$$

UNIT-II

Population Dynamics

Population dynamics is the type of mathematics used to model and study the size and age compositions of population as dynamical systems.

Q.1. If the population of a country doubles in 50 years. In how many years will it triple assuming that the rate of increase is proportional to then number of inhabitants?

Ans. Let
$$t = \text{time in years}$$
$$y = \text{population after } t \text{ years}$$
$$p = \text{original population (when } t = 0)$$

The rate of increase of population is proportional to the population, so that $\frac{dy}{dt} = ky$ where k is constant of proportionality

$$\Rightarrow \quad \int\frac{dy}{y} = \int kdt$$

$$\Rightarrow \quad \log y = kt + c$$

when $t = 0, y = p \Rightarrow c = \log p$

when $t = 50, y = 2p \Rightarrow 50k + c = \log 2p$

$$\Rightarrow \quad 50k + \log p = \log 2 + \log p$$

$$\Rightarrow \quad k = \frac{\log 2}{50}$$

The value of t when population has triple is obtained by putting $y = 3p$, we get

$$\log 3p = kt + c$$

$$\Rightarrow \quad \log 3 + \log p = kt + \log p$$

$$\Rightarrow \quad t = \frac{\log 3}{k} = \frac{50}{\log 2}.\log 3$$

Q.2. In a certain bacteria culture the rate of increase in the number of bacteria is proportional to the number present.

(a) If the number triples in 5 hrs, how many will be present in 10 hrs?

Ans. Let
$$t = \text{time in hrs}$$
$$y = \text{population after } t \text{ hrs}$$
$$p = \text{original population (when } t = 0)$$

The rate of increase of population is proportional to the population, so that

$$\frac{dy}{dt} = ky$$

$$\Rightarrow \quad \log y = kt + c$$

when $t = 0, y = p \Rightarrow c = \log p$

when $t = 5, y = 3p \Rightarrow \log 3p = 5k + \log p$

$$\Rightarrow \quad \log 3 = 5k$$

$$\Rightarrow \quad k = \frac{\log 3}{5}$$

When $t = 10$, Let $y = xp$

$$\therefore \quad \log xp = \frac{\log 3}{5} \times 10 + \log p$$

$$\Rightarrow \quad \log x + \log p = 2 \log 3 + \log p$$

$$\Rightarrow \quad \log x = 2 \log 3 = 0.95424$$

$$\Rightarrow \quad x = 10^{\wedge} (0.95424) = 8.999 = 9$$

(b) when will the number presnt be 10 times the number initially present?

Ans.

$$C = \log p$$

$$K = \frac{\log 3}{5}$$

We have to calcualte $t = ?$

when $y = 10\,p$

$$\therefore \quad \log 10\,p = \frac{\log 3}{5} t + \log p$$

$$\Rightarrow \quad 5 \log 10 = (\log 3)t$$

$$\Rightarrow \quad t = \frac{5 \log 10}{\log 3} = \frac{5}{0.4771}$$

$$= 10.479 \text{ hrs.}$$

Wronskian

Consider a second order differential equation $a_0(x)\dfrac{d^2y}{dx^2} + a_1(x)\dfrac{dy}{dx} + a_2(x)y = 0$

Let $y_1(x)$ and $y_2(x)$ be two solutions of the second order differential equation

Then Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Result 1. If $y_1(x)$ and $y_2(x)$ are any two solutions of , then the linear combination $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants is also a solution of the given equation.

Result 2. Two solutions $y_1(x)$ and $y_2(x)$ of second order differential equation are linearly dependent iff their Wronskian is identically zero.

Q.1. Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants C_1 and C_2 so that $\sin x + 3 \cos x = c_1y_1(x) + c_2y_2(x)$.

Ans.

$$y'' + y = 0 \qquad \ldots(1)$$

Let

$$y_1(x) = \sin x,$$

$$y'_1 x = \cos x \text{ and } y''_1(x) = -\sin x$$

Since $y''_1 + y_1 = -\sin x + \sin x = 0$

$\therefore$ hence $y_1(x)$ is solution of (1).

Similarly, $y_2(x)$ is also solution of (1).

The Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x(\cos x + \sin x) - \cos x(\sin x - \cos x)$$

$$= 1 \neq 0$$

$\therefore$ $y_1(x)$ and $y_2(x)$ are linearly independent solution of (1).

Given $\sin x + 3\cos x = c_1 y_1(x) + c_2 y_2(x)$

$$= c_1 \sin x + c_2 (\sin x - \cos x)$$

Comparing coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$c_1 + c_2 = 1 \text{ and } -c_2 = 3$$

$$\Rightarrow \quad c_1 = 4, c_2 = -3.$$

Q.2. Show that e^{2x} and e^{3x} are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solution $y(x)$ with the property that $y(0) = 0$ and $y^1(0) = 1$.

Ans. $$y^{11} - 5y^1 + 6y = 0 \qquad ...(1)$$

Let $$y_1(x) = e^{2x} \text{ and } y_2(x) = e^{3x}$$

As $$y_1(x) = e^{2x}$$

$$y^1_1(x) = 2e^{2x}, y^{11}_1(x) = 4e^{2x}$$

$$\therefore y''_1(x) - 5y^1_1(x) + 6y_1(x) = 4e^{2x} - 10e^{2x} + 6e^{2x} = 0$$

Similarly $y_2(x)$ is a solution of (1)

The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$w(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1^1(x) & y_2^1(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x}$$

$$= e^{5x} \neq 0$$

$\therefore$ $y_1(x)$ and $y_2(x)$ are linearly independent Solutions of (1)

$\therefore$ General solution of (1) is

$$y(x) = c_1 e^{2x} + c_2 e^{3x} \qquad ...(2)$$

$$\Rightarrow \quad y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x} \qquad ...(3)$$

For $x = 0, y = 0$

By (2), $$0 = c_1 + c_2 \qquad ...(4)$$

For $$x = 0, y' = 1$$

By (3) $$1 = 2c_1 + 3c_2 \qquad ...(5)$$

Solving (4) and (5), we get

$$c_1 = -1, c_2 = 1$$

$$\therefore \quad y(x) = -e^{2x} + e^{3x} = e^{3x} - e^{2x}.$$

Oscillation of a Spring System

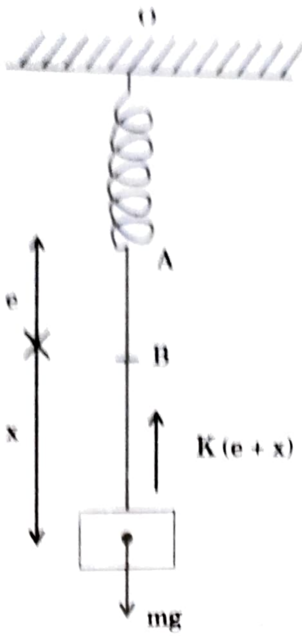
(1) Free Oscillations: Consider a spring OA suspended vertically from a fixed support at O. Let a body of mass m be suspended from the end A.

Let $e(= AB)$ be the elongation produced by the mass m hanging in equillibrium.

Let k be the stiffness of the spring.

The equation of motion of mass m is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \text{ where } \omega^2 = \frac{k}{m}.$$



2. Damped Free Oscillations: If the motion of mass m is subjected to an additional force of resistance, proportional to the instantaneous velocity of the mass, say $\lambda \frac{dx}{dt}$.

The equation of motion of the mass m is

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + w^2x = 0$$

(3) Forced Oscillations (Without Damping) : If the point of the support of the string is also vibrating with same external periodic force, then the resulting motion is called forced oscillatory motion.

Consider motion of the mass m with an external periodic force Q cos nt impressed upon the system. Now support is not steady

The equation of motion of mass m is

$$\frac{d^2x}{dt^2} + w^2x = E \cos nt$$

(4) Forced Oscillations (with Damping) : If in addition there is a damping force, which is proportional to the instantaneous velocity of the mass, say $\lambda \frac{dx}{dt}$,

The equation of motion of mass m is

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + w^2x = E \cos nt$$

Q.1. A spring of negligible weight which stretches 1 inch under tension of 2 *lb* is fixed at one end and is attached to a weight of wlb at the other. It is found that resonance occurs when an axial periodic force 2 cos 2*t lb* acts on the weight. Show that when the free vibrations have died out, the forced vibrations have died out, the forced vibrations are given by $x = ct \sin 2t$ and find the values of ω and c.

Ans. When a weight of 2 *lb* is attached to A, spring stretches by $\frac{1}{12}$ ft.

$$\therefore \quad 2 = k.\frac{1}{12}$$

$$\Rightarrow \quad k = 24 \text{ lb/ft}$$

Let B be the equation position of the weight w attached to A then,

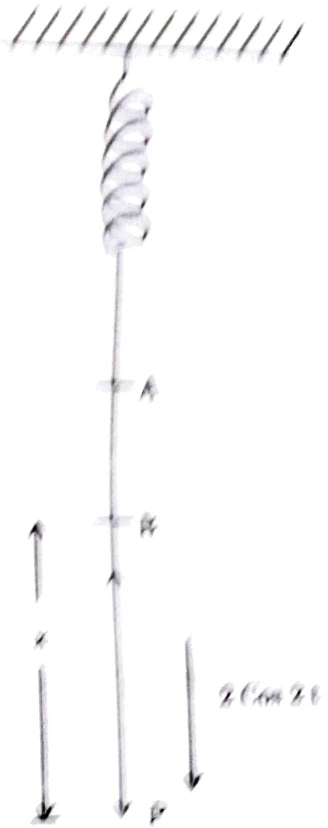
$$w = k \times AB$$

$$\Rightarrow \quad AB = \frac{w}{24} \text{ ft}$$

At any time t, let the weight be at P where $BP = x$

Tension at P, $T_p = k \times AP$

$$= 24\left(\frac{w}{24} + x\right) = w + 24x$$



Its equation of motion is

$$\frac{w}{g}\frac{d^2x}{dt^2} = -T + w + 2\cos 2t$$

$$\Rightarrow \quad \frac{w}{g}\frac{d^2x}{dt^2} = -w - 24x + w + 2\cos 2t$$

$$\Rightarrow \quad \frac{wd^2x}{dt^2} + 24gx = 2g\cos 2t \qquad \ldots(1)$$

The phenomenon of resonance occurs when the period of free oscillations is equal to the period of forced oscillations.

Form (1), $\frac{d^2x}{dt^2} + \mu^2 x = \frac{2g}{w}\cos 2t$ (where $\mu^2 = \frac{24g}{w}$)

∴ The period of free oscillation is $\frac{2\pi}{\mu}$ and the period of the force $\left(\frac{2g}{w}\right)\cos 2t$ is π.

$$\therefore \quad \frac{2\pi}{\mu} = \pi \Rightarrow \mu = 2 \qquad \ldots(2)$$

Hence $$4 = \frac{24g}{w} \Rightarrow w = 6g \qquad \ldots(3)$$

Again (1) $\Rightarrow$ $$\frac{d^2x}{dt^2} + 4x = \frac{1}{3}\cos 2t$$

We know that the free Oscillations are given by the C.F and the forced oscillation are given by P.I. Thus, when the free oscillations have died out, the forced oscillation are given by the P.I of (3)

$$PI = \frac{1}{3}\left(\frac{1}{D^2+4}\cos 2t\right)$$

$$= \frac{1}{3}t\frac{1}{2D}\cos 2t$$

$$= \frac{t}{12}\sin 2t$$

Hence $$C = \frac{1}{12}$$

Q.2. A mass M suspended from the end of a helical spring is subjected to a periodic force $f = F \sin wt$ in the direction of its length. The force F is measured positive vertically downwards and at zero time, M is at rest. If the spring stiffness is S, prove that the displacement of M at time to from the commencement of motion is given by

$$x = \frac{F}{M(p^2 - w^2)}\left(\sin wt - \frac{w}{p}\sin pt\right)$$

where $p^2 = \dfrac{S}{M}$ and damping effects are neglected.

Ans. Let x be the displacement from the equillibrium position at any time t, then the equal of motion of the mass M is

$$M\frac{d^2x}{dt^2} = -Sx + F\sin \omega t$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{S}{M}x = \frac{F}{M}\sin wt$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + p^2x = \frac{F}{M}\sin wt \qquad ...(1)$$

AE is $m^2 + p^2 = 0$

$$\Rightarrow \quad m = \pm ip$$

$$\therefore \quad cF = c_1 \cos t + c_2 \sin pt$$

$$\text{PI} = \frac{F}{M}\cdot\frac{1}{D^2 + p^2}\sin wt = \frac{F}{M}\frac{1}{p^2 - w^2}\sin wt$$

$\therefore$ Complete solution of (1) is

$$x = C_1 \cos pt + C_2 \sin pt + \frac{F}{M}\frac{1}{p^2 - w^2}\sin wt \qquad ...(2)$$

Initially when $t = 0, x = 0$

$$\therefore \quad c_1 = 0$$

Differentiating (2) w.r.t 't'

$$\frac{dx}{dt} = -pc_1 \sin pt + pc_2 \cos pt + \frac{F}{M}\frac{w}{p^2 - w^2}\cos wt$$

Since $$\frac{dx}{dt} = 0 \text{ when } t = 0$$

$$\therefore \quad pC_2 + \frac{F}{M}\cdot\frac{w}{p^2 - w^2} = 0$$

or $$C_2 = \frac{-w}{p}\frac{F}{M(p^2 - w^2)}$$

Substituting the values of C_1 and C_2 in (2), the displacement of the mass at any time t is given by

$$x = \frac{-w}{p}\frac{F}{M(p^2 - w^2)}\sin pt + \frac{F}{M}\frac{1}{(p^2 - w^2)}\sin wt$$

or $$x = \frac{F}{M(p^2 - w^2)}\left(\sin wt - \frac{w}{p}\sin pt\right)$$

Orthogonal Trajectories

Trajectory: A curve which cuts every member of a given family of curves according to some definite law is called trajectory of the family.

Orthogonal Trajectory: A curve which cuts every member of a given family of curves at a right angles is called an orthogonal trajectory of the family.

Orthogonal Trajectories: Two families of curves are said to be orthogonal if every member of either family cuts each member of other family at right angle.

Working Rules to Find Equation of Orthogonal Trajectories

(a) Cartesian Cruves $f(x, y, c) = 0$

$$f(x, y, c) = 0 \quad \text{...(1)}$$

Diffⁿ (1) and eliminate the arbitrary consonant c bet (1) ans resulting equation. That diffⁿ equation of the family (1)

Let it be $F(x, y, \frac{dy}{dx}) = 0$ ...(2)

Replace $\frac{dy}{dx}$, by $-\frac{dx}{dy}$

The diffⁿ eqⁿ of orthogonal trajectory is $F(x, y, -\frac{dy}{dx}) = 0$

Solve (3) to get eqⁿ of the required orthogonal trajectory.

(b) Polar curves $f(r, \theta, c) = 0$

$$f(r, \theta, c) = 0 \quad \text{...(1)}$$

Differentiate (1) and eliminate the arbitrary consonant c between (1) and resulting equation that given diffⁿ eqⁿ of are family (1).

Let it be $F(r, \theta, \frac{dr}{d\theta}) = 0$

Relace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (2)

The diffⁿ eqⁿ of the orthogonal tracjection is E

$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0 \quad \text{...(3)}$$

Q.1. Find orthogonal trajectories of the family of parabolas $y^2 = 4ax$.

Ans. The eqⁿ of family of given parabolas is

$$y^2 = 4ax \quad \text{...(1)}$$

Diffⁿ (1) w.r.t x

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad y\frac{dy}{dx} = 2a \quad \text{...(2)}$$

Eliminating 'a' bet (1) and (2), we get

$$y^2 = 4\frac{y}{2}\frac{dy.x}{dx}$$

$$\Rightarrow \quad y^2 = 2xy\frac{dy}{dx}$$

$$\Rightarrow \quad y - 2x\frac{dy}{dx} = 0$$

This is diff^n eq^n of (1)

Replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ in (3), we get

$$y + 2x\frac{dx}{dy} = 0 = ydy = -2xdx \quad ...(4)$$

This is diff^n eq^n of orthogonal trajectories

On Integr ating we get

$$\Rightarrow \quad \int ydy = -\int 2x2dx.$$

$$\Rightarrow \quad \frac{y^2}{2} = -x^2 + c \Rightarrow y^2 + 2x^2 = 2c = c^1$$

Which is required eq^n of orthogonal trajectories of (1).

Q.2. Find orthogonal trajectories of family of the curves

$$\boldsymbol{r = a(1 + \cos\theta)}$$

Ans. The eq^n of family of given curve is

$$r = a(1 + \cos\theta) \quad ...(1)$$

Diff^n (1) w.r.t. 'θ' we get $\frac{dr}{d\theta} = -a\sin\theta$ ...(2)

Eliminating (a) between (1) and (2)

$$\frac{dr}{d\theta} = \frac{-r}{1+\cos\theta}.\sin\theta$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \frac{-2\sin\theta/2\cos\theta/2}{2\cos\theta^2\theta/2}$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan\theta/2 \quad ...(3)$$

Which is diff^n eq^n of given family (1)

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in (3)

$$\Rightarrow \quad -\frac{r^2}{r}\frac{d\theta}{dr} = -\tan\theta/2$$

$$\Rightarrow \quad r\frac{d\theta}{dr} = \tan\theta/2$$

$$\Rightarrow \quad \frac{dr}{} - \cot / d\theta = O \quad ...(4)$$

which is diffn eq^n of family of orthogonal trajectories

Integrating (4) $\log r - 2\log \sin \theta/_2 = \log c$.

$\Rightarrow$ $\log r = \log c + \log \sin^2 \theta/_2$

$\Rightarrow$ $r = c\sin^2 \theta/_2$

$\Rightarrow$ $r = \frac{c}{2}(1-\cos\theta)$

$\Rightarrow$ $r = c(1-\cos\theta)$

which is required eq^n of orthogonal trajectories of (1).

Q.3. Find orthogonal trajectories of the family of parabolas $y = ax^2$

Ans. The eq^n of given family of curve is

$$y = ax^2 \quad ...(1)$$

Diff^n (1) w.r.t. 'x'

$$\frac{dy}{dx} = 2ax \quad ...(2)$$

Eliminating 'a' bet (1) and (2)

$$\frac{dy}{dx} = 2x\frac{y}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2y}{x} \Rightarrow \frac{dy}{dx} - \frac{2y}{x} = 0 \quad ...(3)$$

This is diff eq^n of (1)

Replacing by $\frac{-dx}{dy}$ in (3) we get

$$\Rightarrow \quad \frac{-dx}{dy}\frac{-2y}{x} = 0$$

$$\Rightarrow \quad \frac{-2y}{x} = \frac{dx}{dy} \Rightarrow -2ydy = xdx$$

$$\Rightarrow \quad xdx + 2y\,dy = 0 \quad ...(4)$$

This is diff^n eq^n of orthogonal triajectory

on integrating we get $\frac{x^2}{2}\,y^2 = C$

$$\Rightarrow \quad x^2 + 2y^2 = C$$

which is required eq^n of orthogonal trajectories of (1)

UNIT I

Partial derivatives, Chain rule, Differentiation of Implicit functions, Exact differentials. Maxima, Minima and saddle points, Method of Lagrange multipliers. Differentiation under Integral sign, Jacobians and transformations of coordinates. **[8 Hrs.] [T2]**

Q. If J_1 is the Jacobian if u, v with respect to x, y and J_2 is the Jacobian of x, y with respect to u, v, then prove that $J_1 J_2 = 1$ (3)

Ans. Let $u = u(x, y)$ and $v = v(x, y)$, so that u and v are functions of x, y.

Differentiating partially w.r.t u and v, we get

$$1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Now $$\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Interchanging rows and columns in second determinant

$$\Rightarrow \quad = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial x}{\partial v} & \frac{\partial u}{\partial x}\frac{\partial y}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial x}{\partial v} & \frac{\partial v}{\partial x}\frac{\partial y}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Q. If $u = F(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ (4)

Ans. Let $X = x - y,\ Y = y - z, Z = z - x$

Here $u = F(X, Y, Z)$

$\therefore$ u is a composite function of x, y, z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$= 1.\frac{\partial u}{\partial X}+0.\frac{\partial u}{\partial Y}+(-1)\frac{\partial u}{\partial Z}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z} \quad ...(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$

$$= \frac{-\partial u}{\partial X}+\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}.0$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial Y}-\frac{\partial u}{\partial X} \quad ...(2)$$

Similarly
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}.0-\frac{\partial u}{\partial Y}+\frac{\partial u}{\partial Z}=\frac{\partial u}{\partial Z}\frac{-\partial u}{\partial Y} \quad ...(3)$$

Consider,
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}+\frac{\partial u}{\partial Y}-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Z}\frac{-\partial u}{\partial Y}$$

$$= 0.$$

Q.) If $u = f(r)$ where $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r)$. **(5)**

Ans. As
$$r^2 = x^2+y^2$$

Differentiating partially w.r.t x we get

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly,
$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Now,
$$u = f(r)$$

$$\frac{\partial u}{\partial x} = f'(r).\frac{\partial r}{\partial x} = \frac{x}{r}f'(r)$$

Differentiating w.r.t 'x', we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}f'(r) + x\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)f'(r) + \frac{x}{r}f''(r).\frac{\partial r}{\partial x}$$

$$= \frac{1}{r}f'(r) - \frac{x}{r^2}.\frac{x}{r}f'(r) + \frac{x^2}{r^2}f''(r)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{r}f'(r)\frac{-x^2}{r^3}f'(r) + \frac{x^2}{r^2}f''(r) \quad ...(1)$$

Similarly $$\frac{\partial u}{\partial y} = \frac{y}{r}f'(r)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r}f'(r) + y\left(\frac{-1}{r^2}\frac{\partial r}{\partial y}\right)f'(r) + \frac{y}{r}f''(r)\frac{\partial r}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r}f'(r) - \frac{y^2}{r^3}f'(r) + \frac{y^2}{r^2}f''(r) \quad ...(2)$$

Adding (1) and (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2}{r}f'(r) - \frac{(x^2+y^2)}{r^3}f'(r) + \left(\frac{x^2+y^2}{r^2}\right)f''(r)$$

$$= \frac{2}{r}f'(r) - \frac{1}{r}f'(r) + f''(r)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) - \frac{1}{r}f'(r) + f''(r) = f''(r) + \frac{1}{r}f'(r)$$

Q..) Expand $e^x \cos y$ in power of x and y as far as the terms of third degree by Taylor's series **(5)**

Ans.

$$f(x,y) = e^x \cos y,\ f(0,0) = 1$$

$$f_x(x,y) = e^x \cos y,\ f_x(0,0) = 1$$

$$fy(x,y) = -e^x \sin y,\ fy(0,0) = 0$$

$$fxx(x,y) = e^x \cos y,\ fxx(0,0) = 1$$

$$fxy(x,y) = -e^x \sin y,\ fxy(0,0) = 0$$

$$fyy(x,y) = -e^x \cos y,\ fyy(0,0) = -1$$

$$fxxx(x,y) = e^x \cos y,\ fxxx(0,0) = 1$$

$$fxxy(x,y) = -e^x \sin y,\ fxxy(0,0) = 0$$

$$fxyy(x,y) = -e^x \cos y,\ fxyy(0,0) = -1$$

$$fyyy(x,y) = e^x \sin y,\ fyyy(0,0) = 0$$

$$\therefore \quad f(x,y) = e^x \cos y$$

$$= f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right]$$

$$+ \frac{1}{2!}\left[x^2 fxx(0,0) + 2xy\, f\, xy(0,0) + y^2 fyy(0,0)\right]$$

$$+\frac{1}{3!}\left[x^3 fxxx(0,0) + 3x^2 y\, fxyy(0,0) + y^3 fyyy(0,0) + 3xy^2 fx_{yy}(0,0)\right] + \ldots\ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{-y^2}{2} + \frac{x^3}{6} - \frac{3x^2 y}{6} + \ldots\ldots$$

Q. **Show that the function $u = x + y + z$, $v = x^3 + y^3 + z^3 - 3xyz$, $w = x^2 + y^2 + z^2 - xy - yz - zx$ are functionally dependent. Also find the relation between them.** (5)

Ans. u, v, w are functionally dependent if

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

$$\Rightarrow \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \\ 2x - y - z & 2y - x - z & 2z - y - x \end{vmatrix}$$

Now $c_2 \to C_2 - C_1$, $c_3 \to c_3 - c_1$

$$\sim \begin{vmatrix} 1 & 0 & 0 \\ 3x^2 - 3yz & 3\left(y^2 - x^2 - xz + yz\right) & 3\left(z^2 - xy - x^2 + yz\right) \\ 2x - y - z & 3(y - x) & 3(z - x) \end{vmatrix}$$

Taking 3 common from R_2

$$\sim 3\begin{vmatrix} 1 & 0 & 0 \\ x^2 - yz & y^2 - x^2 - xz + yz & z^2 - x^2 - xy + yz \\ 2x - y - z & 3(y - x) & 3(z - x) \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 0 & 0 \\ x^2 - yz & (y - x)(x + y + z) & (z - x)(x + y + z) \\ 2x - y - z & 3(y - x) & 3(z - x) \end{vmatrix}$$

Taking $(y-x)$ and $(z-x)$ common from c_2 and c_3

$$= 3(y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x^2-yz & x+y+z & x+y+z \\ 2x-y-z & 3 & 3 \end{vmatrix}$$

Expanding along R_1

$$= 3(y-x)(z-x)[3(x+y+z)-3(x+y+z)] = 0$$

$\therefore$ u, v, w are functionally dependent.

Now
$$v = x^3+y^3+z^3-3xyz$$
$$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$
$$= uw$$

$\Rightarrow$ $v = uw.$

Q. A rectangular box open at top it to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction. (5)

Ans. Let x, y, z be length, breadth and height.

Let v be given volume and s be surface.

$\Rightarrow$
$$xyz = 32 \text{ or } z = \frac{32}{xy}$$

and
$$s = xy + 2xz + 2yz.$$

$\Rightarrow$
$$s = xy + 2\times\frac{32}{y} + 2\times\frac{32}{x}$$

$$= xy + \frac{64}{y} + \frac{64}{x} = f(x, y)$$

$\Rightarrow$
$$fx = y - \frac{64}{x^2}$$

$$fy = x - \frac{64}{y^2}$$

Also
$$r = fxx = \frac{128}{x^3},\ s = fxy = 1$$

$$t = fyy = \frac{128}{y^3}$$

For maximum or minimum $fx = 0, fy = 0.$

$\Rightarrow$
$$y - \frac{64}{x^2} = 0, \Rightarrow y = \frac{64}{x^2}$$

and
$$x - \frac{64}{y^2} = 0\ x = \frac{64}{y^2}$$

$\Rightarrow$
$$x = 64\cdot\frac{x^4}{64\times 64}$$

$\Rightarrow \quad 64 = x^3 \Rightarrow x = (64)^{1/3} = 4.$

and $\quad y = \frac{64}{16} = 4$

$x = y = 4$

At this point, $\quad r = \frac{128}{64} = 2 > 0$

$s = 1$

and $\quad t = \frac{128}{64} = 2 > 0$

Thus least material is verified.

$\therefore rt - s^2 = 4 - 1 = 3 > 0$

and $\quad x = y = 4, z = \frac{32}{16} = 2.$

Q. **If $u = xy\,z, v = x^2 + y^2 + z^2, w = x + y + z$, find $\frac{\partial x}{\partial u}$** (5)

Ans. Let

$$f_1 = u - xyz = 0$$
$$f_2 = v - x^2 - y^2 - z^2 = 0$$
$$f_3 = w - x - y - z = 0$$

$$\frac{\partial x}{\partial u} = \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}} \quad \ldots(1)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -xz & -xy \\ 0 & -2y & -2z \\ 0 & -1 & -1 \end{vmatrix}$$

$$= 2y - 2z = 2(y - z)$$

$$\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= \begin{vmatrix} -yz & z(y-x) & y(z-x) \\ -2x & 2(x-y) & 2(x-z) \\ -1 & 0 & 0 \end{vmatrix}$$

Taking $(x-y)$ and $(z-x)$ Common from c_2 and c_3

$$= (x-y)(z-x)\begin{vmatrix} -yz & -z & y \\ -2x & 2 & -2 \\ -1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3

$$= (x-y)(z-x)[-1(2z-2y)]$$

$$= 2(x-y)(y-z)(z-x)$$

Now by (1), $$\frac{\partial x}{\partial u} = \frac{2(y-z)}{2(x-y)(y-z)(z-x)}$$

$$= \frac{1}{(x-y)(z-x)}$$

Q. Find the shortest distance between the line $2x + y - 10 = 0$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **(6)**

Ans. Let (x, y) be the points of $d\,(x, y)$.

Now. $$f = \frac{2x+y-10}{\sqrt{4+1}} = \frac{2x+y-10}{\sqrt{5}}$$

and $$\mu = 4x^2 + 9y^2 - 36 = 0$$

By Lagrangian.

$$F = \frac{2x+y-10}{\sqrt{5}} + \lambda\left(4x^2+9y^2-36\right)$$

$$\frac{\partial F}{\partial x} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{5}} \Rightarrow [2 + 8\lambda x] = 0$$

$$\Rightarrow \quad 1 + 4\lambda x = 0$$

$$\Rightarrow \quad 1 = -4\lambda x$$

$$\Rightarrow \quad x = \frac{-1}{4\lambda}$$

and
$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{5}}[1 + 18\lambda y] = 0$$

$$\Rightarrow \quad y = \frac{-1}{18\lambda}$$

Consider
$$4x^2 + 9y^2 = 36$$

$$\Rightarrow \quad 4.\frac{1}{16\lambda^2} + 9\frac{1}{324\lambda^2} = 36$$

$$\Rightarrow \quad \frac{1}{4} + \frac{1}{36} = 36\lambda^2$$

$$\Rightarrow \quad \frac{10}{36} = 36\lambda^2$$

$$\Rightarrow \quad \lambda^2 = \frac{10}{36^2}$$

$$\Rightarrow \quad \lambda. = \pm\frac{\sqrt{10}}{36}$$

Now
$$x = \mp\frac{9}{\sqrt{10}}\ y = \mp\frac{2}{\sqrt{10}}$$

Points are

$$\left(\frac{9}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) \text{ and } \left(\frac{-9}{\sqrt{10}}, \frac{-2}{\sqrt{10}}\right)$$

$$\text{distance} = \frac{1}{\sqrt{5}}\left(2\times\frac{9}{\sqrt{10}} + \frac{2}{\sqrt{10}} - 10\right)$$

$$= \frac{1}{\sqrt{5}}\left(\frac{20}{\sqrt{10}} - 10\right)$$

$$= \frac{1}{5}\left(\frac{20}{\sqrt{2}} - 10\right)$$

$$= 2\left(\frac{10}{\sqrt{2}} - 1\right)$$

Q. If $u = (e^{x-y}, e^{y-z}, e^{z-x})$, find the value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ **(2.5)**

Ans. Let $u = (X, Y, Z)$
where consider $X = e^{x-y}, Y = e^{y-z}, Z = e^{z-x}$

$$\frac{\partial u}{\partial X} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial x}$$

$$= e^{x-y}\frac{\partial u}{\partial X}+0.\frac{\partial u}{\partial Y}-e^{z-x}\frac{\partial u}{\partial z}$$

$$\frac{\partial u}{\partial x} = e^{x-y}\frac{\partial u}{\partial X}-e^{z-x}\frac{\partial u}{\partial z} \qquad ...(1)$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}\frac{\partial Z}{\partial y}$$

$\Rightarrow$
$$\frac{\partial u}{\partial Y} = -e^{x-y}\frac{\partial u}{\partial X}+e^{y-z}\frac{\partial u}{\partial Y} \qquad ...(2)$$

$$\frac{\partial u}{\partial Z} = \frac{\partial u}{\partial X}\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$\Rightarrow$
$$\frac{\partial u}{\partial z} = -e^{y-z}\frac{\partial u}{\partial Y}+e^{z-x}\frac{\partial u}{\partial z} \qquad ...(3)$$

Adeding equation (1), (2) and (3)

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = e^{x-y}\frac{\partial u}{\partial X}-e^{z-x}\frac{\partial u}{\partial Z}-e^{y-z}\frac{\partial u}{\partial Y}$$

$$+\ e^{z-x}\frac{\partial u}{\partial Z}-e^{x-y}\frac{\partial u}{\partial X}+e^{y-z}\frac{\partial u}{\partial Y}.$$

$$= 0$$

Q. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $\frac{x^2\partial^2 u}{\partial^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u$ **(5)**
$(1-4\sin^2 u)$.

Ans.
$$u = \tan^{-1}\frac{x^3+y^3}{x-y}$$

u is not a homogeneous function.

$\Rightarrow$
$$\tan u = \frac{x^3+y^3}{x-y} = f(u)$$

$\Rightarrow$
$$\tan u = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left[1-y/x\right]} = x^2 f(y/x)$$

tan u is a homogeneous function of degree 2 in x and y.

By Euler's theorem

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$\Rightarrow \quad x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}.\cos^2 u = 2\sin u\cos u$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \sin 2u \qquad ...(1)$$

Partially differentiating (1) $w.r.t$ 'x'

$$\Rightarrow \quad \frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y} = 2x\cos 2u\frac{\partial u}{\partial x} \qquad ...(2)$$

Partially differentiating (1) w.r. t 'y'.

$$\Rightarrow \quad x\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2} = 2\cos 2u\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad xy\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = 2y\cos 2u\frac{\partial u}{\partial y} \qquad ...(3)$$

Adding (2) and (3), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y} = 2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

By (1), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y} = (2\cos 2u-1)\sin 2u.$$

$$= [2(1-2\sin^2 u)-1]\sin 2u$$

$$= (1-4\sin^2 u)\sin 2u.$$

Q. By changing the independent variables u and v to x and y by means of the relations $x = u\cos\alpha - v\sin\alpha$, $y = u\sin\alpha + v\cos\alpha$, show that $\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$ transforms into $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$. (5)

Ans. $x = u\cos\alpha - v\sin\alpha, \quad y = u\sin\alpha + v\cos\alpha$

$\therefore$ z is a composite function of (u, v)

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= \cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}$$

$\Rightarrow$ $$\frac{\partial z}{\partial u} = \left(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right)z$$

$\Rightarrow$ $$\frac{\partial}{\partial u} = \frac{\partial}{\partial x}\cos\alpha + \frac{\partial}{\partial y}\sin\alpha \qquad ...(1)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= -\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}$$

$\Rightarrow$ $$\frac{\partial Z}{\partial v} = \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)z$$

$\Rightarrow$ $$\frac{\partial}{\partial v} = -\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y} \qquad ...(2)$$

Now, by (1)

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)$$

$$= \left(\cos\alpha\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\sin\alpha\right)\left(\cos\alpha\frac{\partial z}{\partial x} + \sin\alpha\frac{\partial z}{\partial y}\right)$$

$\Rightarrow$ $$\frac{\partial^2 z}{\partial u^2} = \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + \cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2} = \cos^2\alpha\frac{\partial^2 z}{\partial x^2} + 2\cos\alpha\sin\alpha\frac{\partial^2 z}{\partial x\partial y} + \sin^2\alpha\frac{\partial^2 z}{\partial y^2} \qquad ...(3)$$

Similarly, by (2) we get

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

$$= \left(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}\right)\left(-\sin\alpha\frac{\partial z}{\partial x} + \cos\alpha\frac{\partial z}{\partial y}\right)$$

$$= \sin^2\alpha\frac{\partial^2 z}{\partial x^2} - \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial x\partial y} - \sin\alpha\cos\alpha\frac{\partial^2 z}{\partial y\partial x} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2}$$

$\Rightarrow$ $$\frac{\partial^2 z}{\partial v^2} = \sin^2\alpha\frac{\partial^2 z}{\partial x^2} + \cos^2\alpha\frac{\partial^2 z}{\partial y^2} - 2\sin\alpha\cos\alpha\frac{\partial^2 z}{\partial y\partial x} \qquad ...(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}\left(\cos^2\alpha+\sin^2\alpha\right)+\frac{\partial^2 z}{\partial y^2}\left(\sin^2\alpha+\cos^2\alpha\right)$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}=\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}$$

Q. **If u, v, w are the roots of the equation $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ in λ, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.** **(5)**

Ans. As $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3 = 0$

$$\Rightarrow \lambda^3-x^3-3x\lambda^2+3x^2\lambda+\lambda^3-y^3-3y\lambda^2+3y^2\lambda+\lambda^3-z^3-3z\lambda^2+3z^2\lambda=0$$

$$\Rightarrow 3\lambda^3-3(x+y+z)\lambda^2+3(x^2+y^2+z^2)\lambda-(x^3+y^3+z^3)=0$$

since u, v, w, are its roots

$$\Rightarrow \quad u+v+w = x+y+z$$

$$uv+vw+wu = x^2+y^2+z^2$$

$$uvw = \frac{x^3+y^3+z^3}{3}$$

$$\Rightarrow \quad f_1(u,v,w,x,y,z) = u+v+w-x-y-z=0$$

$$f_2(u,v,w,x,y,z) = uv+vw+wu-x^2-y^2-z^2=0$$

$$f_3(u,v,w,x,y,z) = uvw-\frac{x^3+y^3+z^3}{3}=0$$

Now

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}\Bigg/\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & 0 & 0 \\ -2x & 2(x-y) & 2(x-z) \\ -x^2 & (x^2-y^2) & x^2-z^2 \end{vmatrix}$$

Taking $(x-y)$ and $(x-z)$ common from C_2 & C_3 respectively

$$= (x-y)(x-z)\begin{vmatrix} -1 & 0 & 0 \\ -2x & 2 & 2 \\ -x^2 & x+y & x+z \end{vmatrix}$$

Expand along R_1

$$\Rightarrow (x-y)(x-z)(-1)[2(x+z)-2(x+y)] = -2(x-y)(z-y)(x-z)$$

$$= -2(x-y)(y-z)(z-x)$$

and $$\frac{\partial(f_1,f_2,f_3)}{\partial(v,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & v+u \\ vw & uw & uv \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$\Rightarrow \quad = \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}$$

Taking $(u-v)$ and $(u-w)$ common from C_2 and C_3

$$= (u-v)(u-w)\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

Expand along R_1

$$= (u-v)(u-w)(v-w)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(-1)^3(-2)(x-y)(y-z)(z-x)}{(u-v)(u-w)(v-w)}$$

$$= \frac{2(x-y)(y-z)(z-x)}{(u-v)(u-w)(v-w)}$$

Q. If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$, find the value of $\frac{dz}{dx}$, where $x = y = a$. **(4)**

Ans.

$$z = f(x,y), \quad \phi(x,y) = C.$$

$$\Rightarrow \quad z = \sqrt{x^2+y^2}, x^3+y^3+3axy = 5a^2.$$

Now $$\frac{dz}{dz} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} \qquad ...(1)$$

Now $$\frac{\partial z}{\partial x} = \frac{1}{2}.2x.(x^2+y^2)^{-1/2} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

Consider $x^3+y^3+3axy = 5a^2$

Differentiate w.r.t. (x).

$$3x^2+3y^2\frac{dy}{dx}+3ay+3ax\frac{dy}{dx} = 0$$

$$(y^2+ax)\frac{dy}{dx} = -(x^2+ay)$$

$\therefore$

$$\frac{dy}{dx} = \frac{-(x^2+ay)}{y^2+ax}.$$

By equation (1)

$$\frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}}\frac{(x^2+ay)}{(y^2+ax)}$$

At $x = y = a$

$$\frac{dz}{dx} = \frac{a}{\sqrt{2a^2}} - \frac{a}{\sqrt{2a^2}}\frac{(a^2+a^2)}{(a^2+a^2)} = 0$$

Q. If $u = f(r, s, t)$, where $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$. Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Ans. As

$$u = f(r, s, t)$$

and

$$r = \phi(x, y, z) \quad s = \psi(x, y, z), \quad t = g(x, y, z)$$

$\therefore$ u is composite function of x, y, z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}.$$

$$= \frac{1}{y}\frac{\partial u}{\partial r} + 0 - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t} \qquad \text{...(A)}$$

Now

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{-x}{y^2}\cdot\frac{\partial u}{\partial r} + \frac{1}{z}\cdot\frac{\partial u}{\partial s} + 0$$

$\Rightarrow$

$$\frac{\partial u}{\partial y} = \frac{-x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\cdot\frac{\partial u}{\partial s} \qquad \text{...(B)}$$

similarly

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}.$$

$$= 0 - \frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\cdot\frac{\partial u}{\partial t}.$$

$$\frac{\partial u}{\partial z} = \frac{1}{x}\cdot\frac{\partial u}{\partial t} - \frac{y}{z^2}\cdot\frac{\partial u}{\partial s} \qquad \text{...(C)}$$

consider

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} - \frac{-x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} - \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t}$$

$$= 0.$$

Q. Find the maximum and minimum distances from the origin to the cuve $5x^2 + 6xy + 5y^2 - 8 = 0$. **(6)**

Ans. Let (x, y) be any point on the curve Distance of pt. (0, 0) from (x, y) is

$$\sqrt{x^2+y^2} = D.$$

Let $f(x, y) = D^2 = x^2 + y^2$ ...(1)

and $\phi(x, y) = (5x^2 + 6xy + 5y^2 - 8)$ ...(2)

Lagrange's function is

$$F(x, y) = x^2 + y^2 + \lambda(5x^2 + 6xy + 5y^2 - 8)$$

for stationary values dF = 0

$$[2x + (10x + 6y)\lambda]\,dx + [2y + \lambda(6x + 10y)]\,dy = 0$$

$\Rightarrow$ $2x + (10x + 6y)\lambda = 0$ ...(3)

$2y + \lambda(6x + 10y) = 0$ ...(4)

$3 \Rightarrow \quad x = -\lambda(5x + 3y)$

$4 \Rightarrow \quad y = -\lambda(3x + 5y)$

$$\Rightarrow \quad \lambda = -\frac{x}{5x+3y}$$

and
$$\lambda = -\frac{4}{3x+5y}$$

Now
$$\frac{x}{5x+3y} = \frac{y}{3x+5y}$$

$$\Rightarrow \quad 3x^2 + 5xy = 5xy + 3y^2$$

$$\Rightarrow \quad x^2 = y^2$$

$$\Rightarrow \quad x = \pm y$$

Points are (x, y) and $(x, -y)$

$$\text{Distance } D = \sqrt{x^2+x^2} = 2\sqrt{x}$$

Minimum and Maximum distance is $2\sqrt{x}$.

Q. Expand $e^x \sin y$ in powers of x and y as far as the terms of second degree by Taylor's series.
(2.5)

Ans. Let $f(x, y) = e^x \sin y$

By taylor's series

$$f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)]$$
$$+ \frac{1}{2!}[x^2 f_{xx}(0, 0) + y^2 f_{yy}(0, 0) + 2xy\, f_{xy}(0, 0)] + \ldots$$

Now
$$\begin{aligned}
f(x, y) &= e^x \sin y, & f(0, 0) &= 0\\
f_x(x, y) &= e^x \sin y, & f_x(0, 0) &= 0\\
f_y(x, y) &= e^x \cos y, & f_y(0, 0) &= 1\\
f_{xx}(x, y) &= e^x \sin y, & f_{xx}(0, 0) &= 0\\
f_{yy}(x, y) &= -e^x \sin y, & f_{yy}(0, 0) &= 0\\
f_{xy}(x, y) &= e^x \cos y, & f_{xy}(0, 0) &= 1
\end{aligned}$$

$$\Rightarrow \quad e^x \sin y = 0 + [x.0 + y.1] + \frac{1}{2!}[x^2.0 + 2xy.1 + y^2.0] + \ldots$$

$$= y + xy + \ldots\ldots$$

Q. If $u^3+v^3=x+y,\ u^2+v^2=x^3+y^3$, show that $\dfrac{\partial(u,v)}{\partial(x,y)}=\dfrac{y^2-x^2}{2uv(u-v)}$ (5)

Ans. Let

$$f_1 = u^3+v^3-x-y=0$$
$$f_2 = u^2+v^2-x^3-y^3=0$$

Now

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}} \qquad ...(1)$$

Consider

$$\frac{\partial(f_1,f_2)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y}\end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2\end{vmatrix}$$

$$= 3y^2-3x^2 = 3(y^2-x^2)$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v}\end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v\end{vmatrix} = 6u^2v-6uv^2$$

$$= 6uv\,(u-v)$$

Now, by equation (1), we get

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{3(y^2-x^2)}{6uv(u-v)} = \frac{y^2-x^2}{2uv(u-v)}$$

Q. Expand $f(x,y)=x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$ using Taylor's Expansion (6.5)

Ans. $f(x,y)=x^2y+3y-2$

By Taylor's expansion

$$f(x,y) = f(a,b)+[(x-a)f_x+(y-b)f_y]+\frac{1}{2}\left[(x-a)^2f_{xx}+2(x-a)(y-b)f_{xy}+(y-b)^2f_{yy}\right]$$
$$+\frac{1}{3!}\left[\begin{array}{l}(x-a)^3f_{xxx}+3(x-a)^2(y-b)f_{xxy}\\ +3(x-a)(y-b)^2f_{xyy}+(y-b)^3f_{yyy}\end{array}\right] \qquad ...($$

Here

$$a = 1,\ b=-2$$
$$f(x,y) = x^2y+3y-2,$$
$$f(1,-2) = -10$$

$$\begin{aligned}
f_x &= 2xy, & f_x(1,-2) &= -4\\
f_y &= x^2+3, & f_y(1,-2) &= 4\\
f_{xx} &= 2y, & f_{xx}(1,-2) &= -4\\
f_{xy} &= 2x, & f_{xy}(1,-2) &= 2\\
f_{yy} &= 0, & f_{yy}(1,-2) &= 0\\
f_{xxx} &= 0, & f_{xxx}(1,-2) &= 0\\
f_{xxy} &= 2, & f_{xxy}(1,-2) &= 2\\
f_{xyy} &= 0, & f_{xyy}(1,-2) &= 0\\
f_{yyy} &= 0, & f_{yyy}(1,-2) &= 0
\end{aligned}$$

$$f(x,y) = -10 - 4(x-1) + 4(y+2) + \frac{1}{2}\left[-4(x-1)^2 + 4(x-1)(y+2)\right] + \frac{1}{3!}\left[6(x-1)^2(y+2)\right] + \ldots$$

$$f(x,y) = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2) + \ldots$$

Q. Show that the volume of the greatest rectangular parallel piped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{8}{3\sqrt{3}}abc$. **(6.5)**

Ans. Let x, y, z be a vertex of the parallelopiped, then it lies on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \qquad \ldots(1)$$

Also its dimensions are $2x, 2y, 2z$ so that the volume v is given by $v = 2x.\ 2y.\ 2z = 8xyz$

$$\Rightarrow \qquad v^2 = 64x^2y^2z^2 = 64x^2y^2c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$v^2 = 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right) = f(x,y)$$

Now
$$f_x = 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right)$$

$$f_y = 64c^2\left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right)$$

$$r = f_{xx} = 64c^2\left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right)$$

$$s = f_{xy} = 64c^2\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right)$$

$$t = f_{yy} = 64c^2\left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right)$$

Now
$$f_x = 0 \text{ and } f_y = 0$$

$\Rightarrow$ $$128c^2xy^2\left(1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}\right)=0$$

and $$128c^2x^2y\left(1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}\right)=0$$

$\Rightarrow$ $$1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{...(2)}$$

and $$1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}=0 \quad \text{...(3)}$$

Solving (2) and (3)

$$-\frac{x^{2+}}{a^{20}}+\frac{y^2}{b^2}=0 \Rightarrow y=\frac{bx}{a}$$

$\therefore$ from (2), $$1-\frac{2x^2}{a^2}-\frac{x^2}{a^2}=0 \Rightarrow x^2=\frac{a^2}{3}.$$

$\Rightarrow$ $$x = \frac{a}{\sqrt{3}}$$

$\therefore$ $$y = \frac{b}{a}.\frac{a}{\sqrt{3}}=\frac{b}{\sqrt{3}}$$

and $$z^2 = c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$$= c^2\left(1-\frac{1}{3}-\frac{1}{3}\right)=\frac{c^2}{3}$$

$\therefore$ $$z = \frac{c}{\sqrt{3}}$$

Thus $x = \frac{a}{\sqrt{3}}$, $y=\frac{b}{\sqrt{3}}$ is a stationary point.

this point, $$r = 64c^2\left[\frac{2b^2}{3}-\frac{12}{a^2}\frac{a^2}{3}\frac{b^2}{3}-\frac{2}{b^2}\frac{b^4}{9}\right]$$

$$= \frac{-512}{9}b^2c^2<0$$

$$s = 64c^2\left[\frac{4.a}{\sqrt{3}}\frac{b}{\sqrt{3}}-\frac{8}{a^2}\frac{a^3}{3\sqrt{3}}.\frac{b}{\sqrt{3}}-\frac{8}{b^2}\frac{a}{\sqrt{3}}\frac{y^3}{3\sqrt{3}}\right]$$

$$= -\frac{256}{9}abc^2$$

$$t = 64c^2\left[2.\frac{a^2}{3} - \frac{2}{a^2}.\frac{a^4}{9} - \frac{12}{b^2}.\frac{a^2}{3}.\frac{b^2}{3}\right]$$

$$= -\frac{512}{9}a^2c^2$$

$$rt - s^2 = \left(\frac{512}{9}\right)^2 a^2b^2c^4 - \frac{(256)^2}{9}a^2b^2c^4$$

$$= \left(\frac{256}{9}\right)^2 a^2b^2c^4(4-1) > 0.$$

Also $\quad r < 0$

$\therefore v^2$ and hence, v is maximum

when $\quad x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}.$

Max. volume $\quad = \frac{8abc}{3\sqrt{3}}$

UNIT II

Ordinary Differential Equations (ODEs): Basic Concepts. Geometric Meaning $Y' = f(x, y)$. Direction Fields, Euler's Method, Separable ODEs. Exact ODEs. Integrating Factors, Linear ODEs. Bernoulli Equation. Population Dynamics, Orthogonal Trajectories. Homogeneous Linear ODEs with Constant Coefficients. Differential Operators. Modeling of Free Oscillations of a Mass-Spring System, Euler-Cauchy Equations. Wronskian, Nonhomogeneous ODEs, Solution by Variation of Parameters.

Power Series Method for solution of ODEs: Legendre's Equation. Lagendre Polynomials, Bessel's Equation, Bessels's functions Jn(x) and Yn(x). Gamma Function.

[12 Hrs.] [T1]

Q. Solve $(y^3 - 3xy^2)\,dx + (2x^2y - xy^2)\,dy = 0$

Sol. Here

$$M = y^3 - 3xy^2$$
$$N = 2x^2y - xy^2$$
$$\frac{\partial M}{\partial y} = 3y^2 - 6xy$$
$$\frac{\partial N}{\partial x} = 4xy - y^2$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ (Not exact)}$$

Consider

$$Mx + Ny = xy^3 - 3x^2y^2 + 2x^2y^2 - xy^3$$
$$= -x^2y^2 = 0$$
$$\text{I.F.} = \frac{-1}{x^2y^2}$$

Equation changes to

$$\left(\frac{-y}{x^2} + \frac{3}{x}\right)dx + \left(\frac{-2}{y} + \frac{1}{x}\right)dy = 0$$

$$M = \frac{-y}{x^2} + \frac{3}{x},\ N = \frac{1}{x} \frac{-2}{y}$$

Now

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2},\ \frac{\partial N}{\partial x} = \frac{-1}{x^2}$$

∴ equation is exact

$$\text{consider } \int_{y\text{ constt}} Mdx = \int\left(-\frac{y}{x^2} + \frac{3}{x}\right)dx$$
$$= \frac{y}{x} + 3\log x.$$
$$\int N^* dy = \int\left(-\frac{2}{y}\right)dy$$
$$= -2\log y$$

Complete solution is

$$\frac{y}{x} + 3\log x - 2\log y = c$$

Q. Show that $J_{3/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$

Sol. As we know

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x \quad ...(1)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad ...(2)$$

By recurrence relation

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x}J_n(x)$$

Replacing n by $\frac{1}{2}$

$$J_{3/2}(x) + J_{-1/2}(x) = \frac{1}{x}J_{1/2}(x)$$

$$\Rightarrow \quad J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{\sin x}{x} - \cos x\right]$$

Q. Solve $\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = xe^x \cos x$ **(6.5)**

Sol. Given equation is

$$(D^2 - 2D + 1)y = x e^x \cos x$$

A.E.

$$D^2 = -2D + 1 = 0$$

$$\Rightarrow \quad D = 1, 1$$

$$C.F = (c_1 + c_2 x)e^x$$

P.I.

$$= \frac{1}{(D^2 - 2D - 1)}(x e^x \cos x)$$

$$= \frac{1}{(D-1)^2}(xe^x \cos x)$$

$$\Rightarrow \quad e^x \frac{1}{D^2} x \cos x$$

$$P.I. = e^x \frac{1}{D}\int x \cos x \, dx$$

$$= e^x \frac{1}{D}[x \sin x + \cos x]$$

$$= e^x \int (x \sin x + \cos x) dx$$

$$= e^x [x(-\cos x) + \sin x + \sin x]$$

$$= e^x[-x \cos x + 2 \sin x].$$

Complete solution is

$$y = (c_1 + c_2 x)e^x + e^x(2 \sin x - x \cos x)$$

Q. Solve by M.O.V.P. **(6)**

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + y = e^x \log x$$

Sol. Given equation is

$$(D^2 - 2D + 1)y = e^x \log x$$

A.E.

$$D^2 - 2D + 1 = 0$$

$$D = 1, 1$$

C.F.

$$(c_1 + c_2 x)e^x$$

$$= c_1 e^x + c_2 xe^x$$

Let

$$y_1 = e^x, y_2 = xe^x$$

$$X = e^x \log x$$

$$y'_1 = e^x, y'_2 = xe^x + e^x$$

$$W = y_1 y'_2 - y'_1 y_2$$

$$W = e^x (x e^x + e^x) - e^x . xe^x$$

$$W = e^{2x}$$

$$\text{P.I.} = uy_1 + Vy_2$$

$$u = -\int \frac{y_2 x}{W} dx$$

$$= -\int \frac{xe^x . e^x \log x}{e^2 x} dx$$

$$= -\int x \log x \, dx$$

$$= -\left[\log x \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx\right]$$

$$= -\left[\frac{x^2 \log x}{2} - \int \frac{x}{2} dx\right]$$

$$= -\left[\frac{x^2 \log x}{2} - \frac{x^2}{4}\right]$$

$$= \frac{x^2}{4} - \frac{x^2 \log x}{2}$$

$$V = \int \frac{y_1 x}{W} dx$$

$$= \int \frac{e^x \cdot e^x \log x}{e^{2x}}\, dx$$

$$= \int \log x \, dx$$

$$= \log x \,.\, x - \int \frac{1}{x} \cdot x \, dx$$

$$= x \log x - x$$

$$\text{P.I.} = \frac{x^2 e^x}{4} - \frac{x^2 e^x \log x}{2} + x^2 e^x \log x - x^2 e^x$$

$$= \frac{-3x^2 e^x}{4} + \frac{x^2 e^x \log x}{2}$$

$$= \frac{x^2 e^x}{2}\left(\log x - \frac{3}{2}\right)$$

Complete solution is

$$y = (c_1 + c_2 x)e^x + \frac{x^2 e^x}{2}\left(\log x - \frac{3}{2}\right)$$

Q. **Show that**

$$\frac{d}{dx}[x J_n J_{n+1}] = x[J_n{}^2 - J_{n+1}{}^2]$$

Sol.

$$\frac{d}{dx}[x J_n J_{n+1}] = J_n J_{n+1} + x\{J'_n J_{n+1} + J'_{n+1} J_n\}$$

$$\Rightarrow \quad \frac{d}{dx}[x J_n J_{n+1}] = J_n J_{n+1} + J_{n+1}\{x J'_n\} + J_n\{x J'_{n+1}\}$$

As we know by relation

$$x J'_n(x) = n J_n(x) - x J_{n+1}(x) \qquad \text{...(2)}$$

also,

$$J'_n(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \qquad \text{...(3)}$$

Changing n to $(n + 1)$ in (3), we get

$$J'_{n+1} + \frac{(n+1)}{x} J_{n+1} = J_n$$

$$\Rightarrow \quad x J'_{n+1} = x J_n - (n+1) J_{n+1} \qquad \text{...(4)}$$

Putting (2) and (4) in (1), we get

$$\frac{d}{dx}\{x J_n J_{n+1}\} = J_n J_{n+1} + J_{n+1}\{n J_n - x J_{n+1}\} + J_n\{x J_n - (n+1) J_{n+1}\}$$

$$= J_n J_{n+1} + n J_n J_{n+1} - x J_{n+1}^2 + x J_n^2 - (n+1) J_n J_{n+1}$$

$$= x\{J_n{}^2 - J_{n+1}^2\}$$

Q. Prove that $\int_{-1}^{1} x^2\, p_{n+1}(x)\, p_{n-1}(x)\, dx$ ...(6)

$$= \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

Sol. By recurrence relation

$$(2n+1)\,x\,P_n(x) = (n+1)P_{n+1} + n\,P_{n-1} \quad ...(1)$$

Replace n by $n-1$

$$(2n-1)x\,P_{n-1} = n\,P_n + (n-1)\,P_{n-2}$$

$\Rightarrow$

$$x\,P_{n-1} = \frac{n}{2n-1}P_n + \frac{n-1}{2n-1}P_{n-2} \quad ...(2)$$

Replace n by $(n+2)$ is (2)

$\Rightarrow$

$$x\,p_{n+1} = \frac{n+2}{2n+3}P_{n+2} + \frac{(n+1)}{2n+3}P_n \quad ...(3)$$

Multiply (2) and (3), we get

$$x^2 P_{n-1} P_{n+1} = \left[\frac{n}{2n-1}P_n + \frac{n-1}{2n-1}P_{n-2}\right] \times \left[\frac{n+2}{2n+3}P_{n+2} + \frac{n+1}{2n+3}P_n\right]$$

$$= \frac{1}{(2n-1)(2n+3)}[n(n+1)P_n^2 + n(n+2)P_nP_{n+2} + (n-1)(n+2)P_{n-2}P_{n+2} + (n^2-1)P_{n-2}P_n]$$

As by orthogonality

$$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases} \quad ...(5)$$

Integrating (4) w.r.t. x, from $a-1$ to 1 and using (5), we get

$$\int_{-1}^{1} x^2 P_{n-1}(x)\,P_{n+1}(x)dx = \frac{n(n+1)}{(2n-1)(2n+3)}\int_{-1}^{1} P_n^2(x)dx$$

$$= \frac{(n+1)2n}{(2n-1)(2n+3)(2n+1)}$$

Q. Solve $(D-1)^2(D+1)^2y = \sin^2\frac{x}{2} + e^x + x$. **(2)**

Ans. $$(D-1)^2(D+1)^2y = \sin^2\frac{x}{2} + e^x + x$$

A.E. $$(D-1)^2(D+1)^2 = 0$$

$$\Rightarrow \qquad D = 1, 1, -1, -1$$

C.F is $(C_1 + C_2x)e^x + (C_3 + C_4x)e^{-x}$

P.I. $$\frac{1}{(D-1)^2(D+1)^2} = \sin^2\frac{x}{2} + e^x + x$$

$$\Rightarrow \frac{1}{(D-1)^2(D+1)^2}\sin^2\frac{x}{2} + \frac{1}{(D-1)^2(D+1)^2}e^x + \frac{1}{(D-1)^2(D+1)^2}x$$

$$\Rightarrow \frac{1}{[(D-1)(D+1)]^2}\left[\frac{1}{2} - \frac{\cos x}{2} + e^x + x\right]$$

$$\Rightarrow \frac{1}{(D^2-1)^2}\left(e^x - \frac{\cos x}{2} + x + \frac{1}{2}\right)$$

$$\Rightarrow \frac{1}{(D^4-2D^2+1)}e^x - \frac{1}{D^4-2D^2+1}\frac{\cos x}{2} + \frac{1}{(D^4-2D^2+1)}\left(x + \frac{1}{2}\right)$$

Consider $$\frac{1}{D^4-2D^2+1}e^x = \frac{1}{(1-2+1)}e^x[D=1] \text{ Case of failure}$$

$$= x\frac{1}{4D^3-4D}e^x = x\frac{1}{4-4}e^x\ [D=4]$$

gain case of failure

$$= x^2\frac{1}{12D^2-4}e^x = x^2\frac{1}{12-4}e^x = \frac{x^2}{8}e^x$$

Consider $$\frac{1}{D^4-2D^2+1}\left(\frac{\cos x}{2}\right) = \frac{1}{2[(-1)^2-2(-1)+1]}\cos x \qquad [D^2=1]$$

$$= \frac{\cos x}{2[4]} = \frac{\cos x}{8}$$

$$\frac{1}{D^4-2D^2+1}\left(x+\frac{1}{2}\right) = \frac{1}{1-2D^2+D^4}\left(x+\frac{1}{2}\right)$$

$$= [1-(2D^2-D^4)]^{-1}\left(x+\frac{1}{2}\right)$$

$$= [1+2D^2]\left[x+\frac{1}{2}\right]$$

[Neglecting higher power terms]

$$= x+\frac{1}{2}+2D^2\left(x+\frac{1}{2}\right)$$

$$= x+\frac{1}{2}$$

$$\text{P.I.} = \frac{x^2e^x}{8}-\frac{\cos x}{8}+x+\frac{1}{2}$$

∴ Complete solution is.

$$y = (c_1+c_2x)e^x+(c_3+c_4x)e^{-x}+\frac{x^2e^x}{8}-\frac{\cos x}{8}+\left(x+\frac{1}{2}\right).$$

Q. **Solve** $x^2\dfrac{d^2y}{dx^2}-4x\dfrac{dy}{dx}+6y=x^4\sin x$.

Ans. It is homogeneous form of equation.

Put
$$Z = \log x \Rightarrow x = e^z.$$

$$x\frac{dy}{dx} = Dy \quad \text{where} \quad D=\frac{d}{dz}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Now $D(D-1)y-4Dy+6y = e^{4z}\sin e^z.$

$$(D^2-5D+6)y = e^{4z}\sin e^z.$$

A.E. $D^2-5D+6 = 0$

$\Rightarrow$ $(D-2)(D-3) = 0$

$\Rightarrow$ $D = 2, 3 \Rightarrow \text{CF}: C_1e^{2z}+C_2e^{3z}$

$$PI = \frac{1}{D^2-5D+6}e^{4z}\sin e^z.$$

$$= e^{4z}\frac{1}{(D+4)^2-5(D+4)+6}\sin e^z.$$

$$= e^{4z}\frac{1}{D^2+16+8D-5D-20+6}\sin e^z.$$

$$= e^{4z}\frac{1}{D^2+3D+2}\sin e^z.$$

$$= e^{4z}\frac{1}{(D+1)(D+2)}\sin e^z = e^{4z}\left[\frac{1}{D+1}-\frac{1}{D+2}\right]\sin e^z$$

$$= e^{4z}\left[\frac{1}{D+1}\sin e^z-\frac{1}{D+2}\sin e^z\right]$$

Using $\quad \frac{1}{D-a}X = e^{az}\int e^{-az}X\,dz$

$$\frac{1}{D+1}\sin e^z = e^{-z}\int e^z\sin e^z\,dz \qquad [\text{Put } e^z = t]$$

$$= e^{-z}\int \sin t\,dt$$

$$= -e^{-z}\cos e^z$$

$$\frac{1}{D+2}\sin e^z = e^{-2z}\int e^{2z}\sin e^z\,dz$$

$$= e^{-2z}\int t\sin t\,dt$$

$$= e^{-2z}[t(-\cos t)-(-\sin t)]$$

$$= e^{-2z}[-e^z\cos e^z+\sin e^z]$$

$\therefore$

$$\text{PI} = e^{4z}[-e^{-z}\cos e^z+e^{-z}\cos e^z-e^{-2z}\sin e^z]$$

$$= -e^{2z}\sin e^z \text{ Complete solution.}$$

$$y = c_1e^{2z}+c_2e^{3z}-e^{2z}\sin e^z$$

$$y = c_1x^2+c_2x^3-x^2\sin x$$

Q. **Show that** $\int_0^\infty x^{m-1}\cos(ax)d_x = \frac{\Gamma m}{a^m}\cos\frac{m\pi}{2}$. **(5)**

Ans. Consider $\int_0^\infty x^{m-1}e^{-iax}dx$

let $\quad iax = t$

$\Rightarrow \quad ia\,dx = dt$

$$\Rightarrow \int_0^\infty \frac{t^{m-1}}{i^{m-1}a^{m-1}}e^{-t}\frac{dt}{ia}$$

$$= \frac{1}{i^m a^m}\int_0^{\infty} e^{-t} t^{m-1} dt$$

$$= \frac{\Gamma m}{i^m a^m} \quad \text{(by def}^n\text{)}$$

$$\Rightarrow \int_0^{\infty} x^{m-1} e^{-iax} dx = \frac{\Gamma m}{i^m a^m}$$

$$\Rightarrow \int_0^{\infty} x^{m-1}(\cos ax - i\sin ax)\,dx = \frac{\Gamma m}{i^m a^m} \quad \text{(A)}$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Now

$$i^m = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^m$$

$$\Rightarrow \quad = \cos\frac{m\pi}{2} + i\sin\frac{m\pi}{2}$$

By (A), we get

$$\int_0^{\infty} x^{m-1}(\cos ax - i\sin ax)\,dx = \frac{\Gamma m}{a^m} i^{-m}$$

$$= \frac{\Gamma m}{a^m}\left(\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right)$$

Comparing imaginary parts both sides, we get

$$\int_0^{\infty} x^{m-1}\cos ax\,dx = \frac{\Gamma m}{a^m}\cos\frac{m\pi}{2}$$

Q. Solve the differential equation $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$. (3)

Ans. Here $M = x^2y - 2xy^2,\ N = 3x^2y - x^3$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \quad \frac{\partial N}{\partial x} = 6xy - 3x^2$$

This is not exact.

Consider

$$Mx + Ny = x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$= x^2y^2 \neq 0$$

$$\text{I.F.} = \frac{1}{x^2y^2}$$

Multiply throughout by $\frac{1}{x^2y^2}$, we get

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy = 0$$

Again, here $$M' = \frac{1}{y}-\frac{2}{x},\ N' = \frac{3}{y}-\frac{x}{y^2}$$

Now $$\frac{\partial M'}{\partial y} = \frac{-1}{y^2},\ \frac{\partial N'}{\partial x} = -\frac{1}{y^2}$$

This is exact.

Consider $$\int_{y\text{ constant}} M'dx = \int\left(\frac{1}{y}-\frac{2}{x}\right)dx$$

$$= \frac{x}{y}-2\log x$$

and $$\int N'^{*}\,dy = \int\frac{3}{y}dy = 3\log y$$

$\therefore$ required solution is

$$\frac{x}{y}-2\log x+3\log y = C.$$

Q. Prove that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0, m \neq n.$ **(2)**

Ans. Consider the differential equation

$$(1-x^2)u''-2xu'+m(m+1)u = 0 \quad ...(1)$$

$$(1-x^2)v''-2xv'+n(n+1)v = 0 \quad ...(2)$$

$P_m(x)$ and $P_n(x)$ are required solution of equation (1).

(1) and (2) i.e. $u = P_m(x)$ and $v = P_n(x)$.

Multiply (1) by v and (2) by u and subtracting, we get

$$(1-x^2)(u''v-v''u)-2x(u'v-v'u)+[m(m+1)-n(n+1)]uv = 0$$

$$\Rightarrow \frac{d}{dx}[(1-x^2)(u'v-v'u)]+(m^2+m-n^2-n)uv = 0$$

$$\Rightarrow \frac{d}{dx}[(1-x^2)(u'v-v'u)]+(m-n)(m+n+1)uv = 0$$

$$\Rightarrow (n-m)(m+n+1)uv = \frac{d}{dx}[(1-x^2)(u'v-v'u)]$$

Integrating w.r.t. 'x' from -1 to 1, we get

$$(n-m)(m+n+1)\int_{-1}^{1} uv\,dx = \left|(1-x^2)(u'v-v'u)\right|_{-1}^{1}$$
$$= 0$$
$$\int_{-1}^{1} uv\,dx = 0 \qquad (\text{as } m \neq n)$$
$$\Rightarrow \quad \int_{-1}^{1} P_m(x)P_n(x)\,dx = 0.$$

Q. Solve $\frac{dy}{dx} + \left(\frac{y}{x}\right)\log y = \frac{y}{x^2}(\log y)^2$. **(6)**

Ans. Consider $\frac{dy}{dx} + \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$

Multiply equation by $\frac{1}{y(\log y)^2}$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x^2}$$

Let $\frac{1}{\log y} = z$

$$\Rightarrow \quad \frac{-1}{(\log y)^2}\cdot\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$$

∴ Equation becomes

$$\frac{-dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$$
$$\Rightarrow \quad \frac{dz}{dx} - \frac{z}{x} = \frac{-1}{x^2}, \text{ which is linear}$$

Now,
$$\text{I.F.} = e^{-\int 1/x\,dx} = e^{-\log x} = e^{\log x^{-1}}$$
$$\text{I.F.} = \frac{1}{x}$$

∴ Required solution is

$$z\cdot\frac{1}{x} = \int -\frac{1}{x^2}\cdot\frac{1}{x}dx + C$$
$$\Rightarrow \quad z\cdot\frac{1}{x} = \frac{-x^{-2}}{-2} + C$$
$$\Rightarrow \quad \frac{1}{x\log y} = \frac{1}{2x^2} + C$$
$$\Rightarrow \quad \frac{1}{\log y} = \frac{1}{2x} + Cx$$

Q. Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$.

Ans. Consider recurrence relation

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow \quad x^n J_n'(x) + nx^{n-1}J_n(x) = x^n J_{n-1}(x)$$

$$\Rightarrow \quad J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x) \quad \text{...(A)}$$

And from $\quad \frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

$$\Rightarrow \quad x^{-n} J_n'(x) - nx^{-n-1}J_n(x) = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow \quad -J_n'(x) + \frac{n}{x} J_n(x) = J_{n+1}(x) \quad \text{...(B)}$$

Subtracting (*B*) from (*A*), we get

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \quad \text{...(1)}$$

Differentiating (1) w.r.t. '*x*', we get

$$J_n''(x) = \frac{1}{2}\left[J_{n-1}'(x) - J_{n+1}'(x)\right]$$

$$J_n''(x) = \frac{1}{2}J_{n-1}'(x) - \frac{1}{2}J_{n+1}'(x).$$

Again applying (1), we get

$$J_n''(x) = \frac{1}{2}\left[\frac{1}{2}\{J_{n-2}(x) - J_n(x)\}\right] - \frac{1}{2}\left[\frac{1}{2}\{J_n(x) - J_{n+2}(x)\}\right]$$

$$= \frac{1}{4}[J_{n-2}(x) - J_n(x) - J_n(x) + J_{n+2}(x)]$$

$$= \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

$$\Rightarrow \quad 4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x).$$

Q. Prove that $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n} (x^2 - 1)^n$. **(6.5)**

Ans. Let us consider $V = (x^2 - 1)^n$

$$V' = \frac{dV}{dx} = n(x^2 - 1)^{n-1} \cdot 2x$$

$\Rightarrow$

Multiplying both sides by $(x^2 - 1)$, we get

$$(x^2 - 1)V_1 = 2nx(x^2 - 1)^n$$

$$(x^2 - 1)V_1 = 2nxV$$

$$\Rightarrow \quad (1 - x^2)V_1 + 2nxV = 0$$

$\Rightarrow$

Differentiating $(n + 1)$ times by leibnitz theorem, we get

$$[(1 - x^2)V_{n+2} + (n+1)(-2x)V_{n+1} + \frac{n(n+1)}{2!}(-2)V_n] + 2n[xV_{n+1} + (n+1)V_n] = 0$$

$$\Rightarrow (1 - x^2)V_{n+2} - 2xV_{n+1} + n(n+1)V_n = 0$$

$$\Rightarrow (1 - x^2)\frac{d^2V_n}{dx^2} - 2x\frac{d(V_n)}{dx} + n(n+1)V_n = 0$$

Which is Legendre's equation and V_n is its solution. As solution of Legendre's equation are $P_n(x)$ and $Q_n(x)$ and $V_n = \dfrac{d^n}{dx^n}(x^2 - 1)^n$ contains only +ve powers of x, it must be constant multiple of $P_n(x)$.

$$\Rightarrow \quad V_n = C\,P_n(x)$$

$$\Rightarrow \quad CP_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n \qquad ...(1)$$

$$= \frac{d^n}{dx^n}[(x-1)^n (x+1)^n]$$

$$= (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^nC_1 n(x-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n$$

$$+ \dots + (x+1)^n \frac{d^n}{dx^n}(x-1)^n$$

$$= (x-1)^n n! + {}^nC_1\, n(x-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \dots + (x+1)^n n!$$

$$= n!(x+1)^n + \text{terms containing powers of } (x-1).$$

Putting $x = 1$ on both sides, we get

$$\Rightarrow \quad CP_n(1) = n!\, 2^n$$

$$C = n!\, 2^n \qquad [\because P_n(1) = 1]$$

By (1), we get

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

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Q. Solve $(x^4 e^x - 2mxy^2)\,dx + 2mx^2 y\,dy = 0$

Ans.

$$\text{Here } M = x^4 e^x - 2m\,xy^2$$

$$N = 2mx^2 y$$

$$\frac{\partial M}{\partial y} = -4mx\,y,\quad \frac{\partial N}{\partial x} = 4m\,xy$$

Eqution is not exact.

Now

$$\frac{\frac{\partial M}{\partial x} - \frac{\partial N}{\partial x}}{N} = \frac{-4m\,xy - 4m\,xy}{2mx^2 y}$$

$$= \frac{-8\,m\,xy}{2m\,x^2 y} = \frac{-4}{x} = f(x).$$

$$\text{I.F} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = x^{-4} = \frac{1}{x^4}$$

Multiply given equation by $\frac{1}{x^4}$

$$\Rightarrow \quad \left(e^x - \frac{2m\,y^2}{x^3}\right)dx + \left(\frac{2m\,y}{x^2}\right)dy = 0$$

$$\text{New} \quad M' = e^x - \frac{2my^2}{x^3},\ N' = \frac{2my}{x^2}$$

$$\frac{\partial M'}{\partial y} = \frac{-4my}{x^3},\ \frac{\partial N'}{\partial x} = \frac{-4m\,y}{x^3}$$

Now equation is exact.

$$\text{Consider} \int_{y\text{ constt}} M'\,dx = \int_{y\text{ constt}} \left(e^x - \frac{2my^2}{x^3}\right)dx$$

$$\Rightarrow \quad e^x + \frac{2my^2}{2x^2} = e^x + \frac{my^2}{x^2}$$

and

$$\int (N')^+ dy = \int 0\,dy = 0$$

Required. solution

$$e^x + \frac{my^2}{x^2} = C$$

Q. Solve $\frac{d^2y}{dx^2}+y=-\cot x$ by method of variation of parameters. (6.5)

Ans. Given $(D^2+1)y=-\cot x$

A.E. $$D^2+1=0$$

$$D^2=-1 \Rightarrow D=\pm i$$

$\Rightarrow$ $$CF=C_1\cos x+C_2\sin x$$

$\therefore$ Here $y_1=\cos x, y_2=\sin x, X=-\cot x$

Now $$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W=\cos^2 x+\sin^2 x+1$$

$P.I=uy_1+vy_2$

where $$u=-\int\frac{y_2X}{W}dx=-\int\frac{-\sin x\cot x}{1}dx$$

$$=\int\sin x\cdot\frac{\cos x}{\sin x}dx=\int\cos x\,dx$$

$$=\sin x$$

$$V=\int\frac{y_1x}{W}dx=-\int\frac{\cos x\cot x}{1}dx$$

$$=-\int\frac{\cos^2 x}{\sin x}dx=-\int\frac{1-\sin^2 x}{\sin x}dx_2-\int(\operatorname{cosec} x-\sin x)dx$$

$$=-\log(\operatorname{cosec} x-\cot x)-\cos x$$

$$P.I=\sin x\cos x-[\log(\operatorname{cosec} x-\cot x)+\cos x]\sin x$$

$$=-\sin x[\log(\operatorname{cosec} x-\cot x]$$

$\therefore$ Complete soln is

$$y=C_1\cos x+C_2\sin x-\sin x[\log(\operatorname{cosec} x-\cot x]$$

Q. Solve $(x^2D^2+3x+D+1)y=\frac{1}{(1-x)^2}$

Ans. $$(x^2D^2+3xD+1)y=\frac{1}{(1-x)^2}$$

Putting $x=e^z$, we get

$$[D(D-1)+3D+1]y=\frac{1}{(1-x)^2}$$

$\Rightarrow$ $$(D+1)^2y=\frac{1}{(1-x)^2},\text{ where } D=\frac{d}{dz}=\frac{xd}{dx}$$

$$A.E=(D+1)^2=0$$

$$D = -1, -1$$

$\Rightarrow$

$$C.F = (C_1 + C_2 z)e^{-z} = (C_1 + C_2 \log x)\frac{1}{x}$$

$$P.I. = \frac{1}{(\theta+1)^2}\frac{1}{(1-x)^2}, \theta = x\frac{d}{dx}$$

Let $\frac{1}{\theta+1}\left[\frac{1}{(1-x)^2}\right] = v$ So that $(\theta + 1)\, u = \frac{1}{(1-x)^2}$

$$= x\frac{du}{dx} + u = \frac{1}{(1-x)^2} \text{ or}$$

$$\frac{du}{dx} + \frac{u}{x} = \frac{1}{x(1-x)^2}$$

I.F. of (2) $= e^{\int dx/x} = e^{\log x} = x$

Sol of (2) $u \cdot x = \int \frac{x\, dx}{x(1-x)^2} = \frac{1}{1-x}$

$$\text{or } u = \frac{1}{x(1-x)}$$

By (*i*), we get

$$P.I = \frac{1}{\theta+1} \cdot u = V(\text{say}), \text{Then } U = (\theta + 1)V$$

or $\frac{1}{x(1-x)} = x\frac{dV}{dx} + V$ or $\frac{dV}{dx} + \frac{V}{x} = \frac{1}{x^2(1-x)}$ ...(*iii*)

I.F. of (*iii*) is x and solution of (*iii*) is

$$V.x = \int \frac{x\, dx}{x^2(1-x)}$$

$$= \int \frac{dx}{x(1-x)} = \int\left(\frac{1}{1-x} + \frac{1}{x}\right)dx$$

or

$$Vx = \log\frac{x}{1-x}$$

$\Rightarrow$

$$V = \frac{1}{x}\log\frac{x}{1-x} = P.I$$

$$\text{Hence } y = x^{-1}(C^1 + C^2 \log x) + \frac{1}{x}\log\frac{x}{1-x}$$

Solve

d. $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3, y(0) = \dfrac{\pi}{4}$. (3)

Ans. $\sec^2 y \dfrac{dy}{dx} + 2x \tan y = x^3$

Let
$$Z = \tan y$$
$$\frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$$

$\Rightarrow$
$$\frac{dz}{dx} + 2xz = x^3 \text{ (linear in } z).$$
$$\text{I.F} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ Required solution is

$$Ze^{x^2} = \int x^3 e^{x^2} dx$$
$$Ze^{x^2} = \frac{x^3 e^{x^2}}{2x} - \int 3x^2 \frac{e^{x^2}}{2x} dx \qquad \{x^2 = t\}$$
$$= \frac{x^2 e^{x^2}}{2} - \frac{3}{2}\int x e^{x^2} dx$$
$$Ze^{x^2} = \frac{x^2 e^{x^2}}{2} - \frac{3}{4}\int e^t\, dt$$
$$= \frac{x^2 e^{x^2}}{2} - \frac{3}{4} e^{x^2} + c$$
$$Z = \frac{x^2}{2} - \frac{3}{4} + c e^{-x^2}$$

$\Rightarrow$
$$\tan y = \frac{x^2}{2} - \frac{3}{4} + c e^{-x^2}$$

At
$$x = 0$$
$$\tan \frac{\pi}{4} = \frac{-3}{4} + c$$

$\Rightarrow$
$$c = 1 + \frac{3}{4} = \frac{7}{4}$$

$\therefore$
$$\tan y = \frac{x^2}{2} - \frac{3}{4} + \frac{7}{4} e^{-x^2}$$

Q. Show that $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$. (3)

Ans. Consider $(1-2xz+z+z^2)|^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$ ...(1)

Differentiating (1) both sides partially w.r.t 'z', we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=1}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow \quad (x-z)(1-2xz+z^2)^{-1/2} = (1-2xz+z^2)\sum_{n=1}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0} nz^{n-1}P_n(x)$$

Equating Co-efficients of z^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2x\,nP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow \quad (2n+1)x.P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$\Rightarrow \quad (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

Q. If $\boldsymbol{\Gamma n\ \Gamma 1-n = \dfrac{\pi}{\sin n\pi}}$, then show that $\boldsymbol{\int_0^\infty \dfrac{x^{n-1}}{1+x}dx = \dfrac{\pi}{\sin n\pi}}$. **(6.5)**

Ans. As $$B(n, m) = \frac{\Gamma n\,\Gamma m}{\Gamma n+m}$$

Put $$m = 1-n$$

$$\Rightarrow \quad B(n, 1-n) = \frac{\Gamma n\,\Gamma 1-n}{\Gamma 1}$$

$$\Rightarrow \quad \Gamma n\,\Gamma 1-n = B(n, 1-n) = B(1-n, n) \text{ (By symmetry)} \quad ...(1)$$

Consider $$\beta(1-n, n) = \int_0^1 y^{-n}(1-y)^{n-1}dy$$

Let $$y = \frac{1}{1+x} \Rightarrow dy = \frac{-1}{(1+x)^2}dx$$

Now $$\beta(1-n, n) = \int_\infty^0 \left(\frac{1}{1+x}\right)^{-n}\left(\frac{x}{1+x}\right)^{n-1}\left(\frac{-1}{(1+x)^2}\right)dx$$

$$= -\int_\infty^0 \frac{x^{n-1}}{(1+x)^{-n+n-1+2}}dx$$

$$\beta(1-n,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)} dx$$

$\Rightarrow$
$$\overline{|n}\,\overline{|1-n} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)} dx = \frac{\pi}{\sin n\pi}$$

By (1)

Q. Solve $(D^2+1)y = \tan x$ by using variation of parameter method. **(6.5)**

Ans.

$$(D^2+1)y = \tan x$$

A.E.
$$D^2+1 = 0$$

$\Rightarrow$
$$D = \pm i$$

$$y = c_1 \cos x + c_2 \sin x$$

Let
$$y_1 = \cos x,\ y_2 = \sin x,\ X = \tan x$$

$$y_1' = -\sin x,\ y_2' = \cos x$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

Now
$$PI = uy_1 + vy_2$$

where
$$u = -\int \frac{y_2 X}{w} dx$$

$$= -\int \frac{\sin x . \tan x}{1.} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{(1-\cos^2 x)}{\cos x} dx$$

$$= -\int (\sec x - \cos x) dx$$

$$u = -[\log(\sec x + \tan x) - \sin x]$$

$$v = \int \frac{y_1 X}{w} dx = \int \cos x \tan x \, dx$$

$$= \int \sin x \, dx = -\cos x$$

$$P.I = \cos x[\sin x - \log(\sec x + \tan x)] - \sin x \cos x$$

$$P.I = -\cos x \log(\sec x + \tan x)$$

complete solution is
$$y = c_1 \cos x + c_2 \sin x - \cos x[\log(\sec x + \tan x)]$$

UNIT III

Linear Algebra: Matrices and Determinants, Gauss Elimination, Linear independence. Rank of a Matrix. Vector Space. Solutions of Linear Systems and concept of Existence, Uniqueness, Determinants. Cramer's Rule. Gauss-Jordan Elimination. The Matrix Eigenvalue Problem.

Determining Eigenvalues and Eigenvectors, Symmetric, Skew-Symmetric, and Orthogonal Matrices. Eigenbases, Diagonalization, Quadratic Forms. Cayley-Hamilton Theorem (without proof) **[10 Hrs.] [T1]**

Q. Reduce the matrix $\begin{bmatrix} 8 & 1 & 36 \\ 0 & 3 & 22 \\ -8 & -1 & -34 \end{bmatrix}$ **to normal form and hence, find its rank.**

Sol.

$$\text{Let } A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$$

Operating $c_1 \to \dfrac{c_1}{8}$

$$\sim A = \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -1 & -1 & -3 & 4 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_1$

$$\sim A = \begin{bmatrix} 1 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Operating $c_2 \to c_2 - c_1, c_3 \to c_3 - 3c_1, c_4 \to c_4 - 6c_1$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Operating $c_2 \to \dfrac{1}{3}c_2, c_3 \to \dfrac{1}{2}c_2, c_4 \to \dfrac{1}{2}c_4$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Operating $c_3 \to c_3 \to c_2, c_4 \to c_4 - c_2$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$c_4 \to \frac{1}{5}c_4$$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Operating $c_3 \leftrightarrow c_4$

$$\sim A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A = [I_3 : 0]$$

$$\text{Rank of } A = 3.$$

Q. Find the values of a and b such that the system of equations
$3x - 2y + z = 6, 5x - 8y + 9z = 3,$
$2x + y + az = -b$ may have a unique solution.

Sol. Consider $AX = B$

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -b \end{bmatrix}$$

Operating $R_2 \to R_2 - \frac{5}{3} R_1, R_3 \to R_3 - \frac{2}{3} R_1$

$$\sim \begin{bmatrix} 3 & -2 & 1 \\ 0 & -14/3 & 22/3 \\ 0 & 7/3 & a - 2/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ -b-4 \end{bmatrix}$$

Operating $R_3 \to R_3 + 1/2\, R_2$

$$\sim \begin{bmatrix} 3 & -2 & 1 \\ 0 & -14/3 & 22/3 \\ 0 & 0 & a+3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ b - 15/2 \end{bmatrix}$$

For unique solution

$$\rho(A) = \rho(A : B) = 3$$

$a \neq -3$, b can have any value.

Q. The eigen vectors of a 3 × 3 matrix A corresponding to the eigen values 1, 2, 3 are $[1, 2, 1]^T$, $[2, 3, 4]^T$, $[1, 4, 9]^T$ respectively. Find the matrix A.

Sol. Modal matrix of A is $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix} = B$

and diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Now as we know

$$D = B^{-1}AB$$
$$BDB^{-1} = A$$
$$A = BDB^{-1}$$

$$\text{As} \rightarrow B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}$$

$$|B| = 1(27-16) - 2(18-4) + 1(8-3) = -12$$

$$B^{-1} = \frac{-1}{12}\begin{bmatrix} 11 & -14 & 5 \\ -14 & 8 & -2 \\ 5 & -2 & -1 \end{bmatrix}$$

$$A = -\frac{1}{12}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 11 & -14 & 5 \\ -14 & 8 & -2 \\ 5 & -2 & -1 \end{bmatrix}$$

$$A = -\frac{1}{12}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} 11 & -14 & 5 \\ -28 & 16 & -4 \\ 15 & -6 & -3 \end{bmatrix}$$

$$= \frac{-1}{12}\begin{bmatrix} -30 & 12 & -6 \\ -2 & -4 & -14 \\ 34 & -4 & -38 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 15 & -6 & 3 \\ .1 & .2 & 7 \\ 17 & 2 & 19 \end{bmatrix}$$

Q. Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$. **Hence find** A^{-1}. **(6.5)**

Sol. Characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 4 & 1-\lambda & 0 \\ 8 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda)^2 + 1[4-4\lambda] + 1[4-8+8\lambda] = 0$$

$$\Rightarrow \quad (1-\lambda)^3 + 4 - 4\lambda - 4 + 8\lambda = 0$$

$$\Rightarrow \quad 1-\lambda^3 - 3\lambda + 3\lambda^2 + 4\lambda = 0$$

$$\Rightarrow \quad -\lambda^3 + 3\lambda^2 + \lambda + 1 = 0$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - \lambda - 1 = 0$$

Its characteristic equation is

$$A^3 - 3A^2 - A - I = 0$$

Multiply by A^{-1}

$$A^2 - 3A - AA^{-1} - A^{-1}I = 0$$

$$\Rightarrow \quad A^2 - 3A - I - A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = A^2 - 3A - I$$

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} .5 & -1 & 2 \\ 8 & -3 & 4 \\ 20 & -6 & 9 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 5 & -1 & 2 \\ 8 & -3 & 4 \\ 20 & -6 & 9 \end{bmatrix} - 3\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$$

Find the eigen values and eigen-vectors of the matrix.

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ ...(6)

Sol. Characteristic equation

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)[(7-\lambda)(3-\lambda)-16]+6[-18+6\lambda+8]+2[24-14+2\lambda]=0$

$\Rightarrow (8-\lambda)(21-10\lambda+\lambda^2-16)+6(6\lambda-10)+2(10+2\lambda)=0$

$\Rightarrow (8-\lambda)(5-10\lambda+\lambda^2)+36\lambda-60+20+4\lambda=0$

$\Rightarrow 40-80\lambda+8\lambda^2-5\lambda+10\lambda^2-\lambda^3-40+40\lambda=0$

$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$

$\Rightarrow -\lambda(\lambda^2-18\lambda+45)=0$

$\Rightarrow \lambda=0, (\lambda^2-15\lambda-3\lambda+45)=0$

$(\lambda-15)(\lambda-3)=0$

$\Rightarrow \lambda = 0, 3, 15$ (eigen values)

For $\lambda = 0$, eigen vector

$$[A - OI][X] = 0$$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operating $R_3 \to R_3 - \frac{R_1}{4}, R_2 \to R_2 + \frac{3}{4}R_1$

$$\sim\begin{bmatrix} 8 & -6 & 2 \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$

$$\sim\begin{bmatrix} 8 & -6 & 2 \\ 0 & 5/2 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3$$

One variable be given arbitrary value.

$$8x_1 - 6x_2 + 2x_3 = 0$$

$$\frac{5}{2}x_2 - \frac{5}{2}x_3 = 0$$

$$4x_1 - 3x_2 + x_3 = 0$$

$\Rightarrow$ $$x_2 - x_3 = 0$$

$$x_2 = x_3$$

$\Rightarrow$

Now $$4x_1 - 3x_2 + x_2 = 0$$

$\Rightarrow$ $$4x_1 - 2x_2 = 0$$

$\Rightarrow$ $$2x_1 = x_2$$

$\Rightarrow$ $$\frac{x_1}{1} = \frac{x_2}{2}$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda = 3$, eigenuector is

$$[A - 3I]X = 0$$

$\Rightarrow$ $$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Operation $$R \to R_2 + \frac{6}{5}R_1,\ R_3 \to R_3 - \frac{2}{5}R_1$$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & -8/5 & -4/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16/5 & -8/5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r(A) = 2 < 3 \text{ (infinite many solution)}$$

As $$5x_1 - 6x_2 + 2x_3 = 0$$

and $$\frac{-16}{5}x_2 - \frac{8}{5}x_3 = 0$$

$\Rightarrow$ $$2x_2 + x_3 = 0$$

$\Rightarrow$ $$2x_2 = -x_3$$

$\Rightarrow$ $$\frac{x_2}{-1} = \frac{x_3}{2}$$

$$x_1 = -2$$

$$X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda = 15$ eigenvector is

$$[A - 15\,I][X] = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$

Operating $R_2 \to \frac{1}{2}R_2,\ R_3 \to \frac{1}{2}R_3$

$$\sim \begin{bmatrix} -7 & -6 & 2 \\ -3 & -4 & -2 \\ 1 & -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 - 6x_2 + 2x_3 = 0$$

$$-3x_1 - 4x_2 - 2x_3 = 0$$

$$x_1 - 2x_2 - 6x_3 = 0$$

on solving last two equations.

$$\frac{x_1}{24-4} = \frac{x_2}{-2-18} = \frac{x_3}{6+4}$$

$\Rightarrow$

$$\frac{x_1}{20} = \frac{x_2}{-20} = \frac{x_3}{10}$$

$\Rightarrow$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Q. Reduce the quadratic form

$12x_1^2 + 4x_2^2 + 5x_3^2 - 4x_2x_3 + 6x_1x_3 - 6x_1x_2$ to Canonical form. Also find the nature of quadratic form. **(2.5)**

Ans. It can be written as $X'AX$ where

$$X' = [x_1 \ \ x_2 \ \ x_3] \text{ and } A = \begin{bmatrix} 12 & -3 & 3 \\ -3 & 4 & -2 \\ 3 & -2 & 5 \end{bmatrix}$$

As

$$A = I_3 A I_3$$

$$\begin{bmatrix} 12 & -3 & 3 \\ -3 & 4 & -2 \\ 3 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_2 \to R_2 + \frac{1}{4}R_1, R_3 \to R_3 - \frac{1}{4}R_1$

$$\sim \begin{bmatrix} 12 & -3 & 3 \\ 0 & 13/4 & -5/4 \\ 0 & -5/4 & 17/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_2 \to C_2 + \frac{1}{4}C_1, C_3 \to C_3 - \frac{1}{4}C_1$

$$\sim \begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & -5/4 \\ 0 & -5/4 & 17/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -1/4 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + \frac{5}{13}R_2$

$$\begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & -5/4 \\ 0 & 0 & 49/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -2/13 & 5/13 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + \frac{5}{13}C_2$

$$\sim \begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & 0 \\ 0 & 0 & 49/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -2/13 & 5/13 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/4 & -2/13 \\ 0 & 1 & 5/13 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Diag}\left(12, \frac{13}{4}, \frac{49}{13}\right) = P' A P$$

Canonical form is

$$Y'(P'AP)Y = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 12 & 0 & 0 \\ 0 & 13/4 & 0 \\ 0 & 0 & 49/13 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 12y_1^2 + \frac{13}{4}y_2^2 + \frac{49}{13}y_3^2$$

rank $(r) = 3$, index $(p) = 3$, Dignature $= 3$, $n = 3$. Nature is positive definite.

Q. Determine the values of λ for which the following system of equation have non-trivial solution $3x_1 + x_2 - \lambda x_3 = 0$, $4x_1 - 2x_2 - 3x_3 = 0$, $2\lambda x_1 + 4x_2 + \lambda x_3 = 0$. For each permissible value of λ, find the solution. (5)

Ans. In matrix notation, it can be written as

$$AX = 0$$

$$\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

For non trivial solution $|A| = 0$

$$\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$$

$\Rightarrow$

$$\Rightarrow 3(-2\lambda + 12) - 1(4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$$

$$-6\lambda + 36 - 10\lambda - 16\lambda - 4x^2 = 0$$

$$\Rightarrow -4\lambda^2 - 32\lambda + 36 = 0$$

$$\Rightarrow \lambda^2 + 8\lambda - 9 = 0$$

$$\Rightarrow (\lambda + 9)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = 1, -9$$

for $\lambda = 1$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \to R_2 - \frac{4}{3}R_1, R_3, \to R_3 \frac{-2}{3}R_1$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & -10/3 & -5/3 \\ 0 & 10/3 & 5/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 3 & 1 & -1 \\ 0 & -10/3 & -5/3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_1 + x_2 - x_3 = 0$$

$$-\frac{10}{3}x_2 - \frac{5}{3}x_3 = 0$$

Let

$\Rightarrow$

$$x_2 = k \Rightarrow 5x_3 = -10k \Rightarrow x_3 = -2k.$$

$$3x_1 + 2k + k = 0$$

$$x_1 = k.$$

$$x_1 = k, x_2 = k, x_3 = -2k.$$

For $\lambda = -9$

$$\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + 6R_1,\ R_2 \to R_2 - \frac{4}{3} R_1$$

$$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & 10/3 & -5 \\ 0 & 10 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_2 \to 3R_2$$

$$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 10 & 45 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$R_3 \to R_3 + R_2$$

$$\sim \begin{bmatrix} 3 & 1 & 9 \\ 0 & -10 & -45 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$3x_1 + x_2 + 9x_3 = 0$$

$$-10x_2 - 45x_3 = 0$$

Let $\quad x_2 = k$

$\Rightarrow \quad 45x_3 = -10k.$

$$x_3 = \frac{-2}{9}k$$

$\Rightarrow \quad 3x_1 + k - \dfrac{9 \times 2}{9}k = 0.$

$\Rightarrow \quad 3x_1 = k \Rightarrow x_1 = \dfrac{k}{3}$

$\Rightarrow \quad x_1 = \dfrac{k}{3}, x_2 = k, x_3 = -\dfrac{2}{9}k.$

Q. **Express the matrix** $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ **as sum of a symmetric and a skew symmetric matrix.**

Ans. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

i.e., A can be written as

$$A = \frac{1}{2}(A+A') + \frac{1}{2}(A-A')$$

$$A' = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$A+A' = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 3 \\ 0 & 8 & 3 \\ 3 & 3 & -4 \end{bmatrix}$$

$$A - A' = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 1 \\ 0 & 2 & -2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

$\therefore$

$$A = \frac{1}{2}\begin{bmatrix} 4 & 0 & 3 \\ 0 & 8 & 3 \\ 3 & 3 & -4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$$

Q. Reduce the qudratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$ to the canonical form. Also write the nature of the quadratic form. **(3)**

Ans. The given quadratic form can be written as $X'AX$, where $X' = [x_1, x_2, x_3]$ and symmetric matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

As $$A = I_3 A I_3$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $R_3 \to R_3 + \frac{1}{3}R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Operating $C_3 \to C_3 + \frac{1}{3} C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Diag.} \left(1, 3, \frac{8}{3}\right) = P'AP.$$

$\therefore$ The canonical form of given quadratic form is

$$Y'(P'AP)Y = [y_1\ y_2\ y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1\ \ y_2\ \ y_3] \begin{bmatrix} y_1 \\ 3y_2 \\ \frac{8}{3} y_3 \end{bmatrix} = y_1^2 + 3y_2^2 + \frac{8}{3} y_3^2$$

Here index, $p = 3$, rank $= 3$, signature

$$= 2p - r = 2 \times 3 - 3 = 3$$

It is positive definite.

Q. Use Guass-Jordan method to find the inverse of the matrix

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}.$$ (6)

Ans. It can be written as

$$A = I_3 A$$

$$\Rightarrow \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operating $R_2 \to R_2 - \frac{2}{3} R_1$

$$\sim \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Operating $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operating $R_1 \to R_1 - 3R_2$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operating $R_2 \to R_2 + 4R_3$

$$\sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ 2 & -3 & 4/3 \\ 2/3 & -1 & 1 \end{bmatrix} A$$

Operating $R_2 \to -R_2, R_3 \to -3R_3, R_1 \to \frac{1}{3}R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4/3 \\ -2 & 3 & -3 \end{bmatrix} A$$

$$I = A^{-1}A$$

$\Rightarrow$

Thus, inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4/3 \\ -2 & 3 & -3 \end{bmatrix}$$

Q. Use Guass-Jordan method to find the inverse of the following matrice. (5)

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

Ans. Let $A = IA$

$$\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

operate $R_2 \to R_2 + 2R_1, R_3 \to R_3 + 2R_1$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A$$

operate $R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} -1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} A$$

operate $R_2 \to \frac{R_2}{3}, R_3 \to \frac{R_3}{-9}$

$$\sim\begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1/3 & 0 \\ 2/9 & 2/9 & -1/9 \end{bmatrix} A$$

Operate $R_1 \to R_1 - 2R_2$

$$\sim\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/3 & -2/3 & 0 \\ 2/3 & 1/3 & 0 \\ 2/9 & 2/9 & -1/9 \end{bmatrix} A$$

Operate $R_1 \to R_1 + 2R_3, R_2 \to R_2 - 2R_3$

$$\sim\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix} A$$

Operate $R_1 \to -R_1$

$$\sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/9 & 2/9 & 2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix} A$$

$\Rightarrow$ $$I = A^{-1}A$$

$\Rightarrow$ $$A^{-1} = \begin{bmatrix} -1/9 & 2/9 & 2/9 \\ 2/9 & -1/9 & 2/9 \\ 2/9 & 2/9 & -1/9 \end{bmatrix}$$

Q. For what value of K does the following system of equations $x + y +$ $z = 1, x + 2y + 4z = K, x + 4y + 10z = K^2$ have a solution and solve them completely in each case. (5)

Ans. Given $x + y + z = 1$

$x + 2y + 4z = K$

$x + 4y + 10z = K^2$

In matrix form $AX = B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K \\ K^2 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1, R_3 \to R_3 \to R_1$

$$\sim\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K-1 \\ K^2-1 \end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$

$$\sim\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ K-1 \\ K^2-3K+2 \end{bmatrix}$$

$\Rightarrow r(A) = 2$

and
$$r(A:B) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & K-1 \\ 0 & 0 & 0 & K^2-3K+2 \end{bmatrix}$$

System is consistent iff $r(A) = r(A:B)$

$\Rightarrow K^2 - 3K + 2 = 0$

$\Rightarrow K = 1, 2$

for $K = 1, 2, r(A) = r(A:B) <$ no. of unknowns

$\therefore$ system has infinite many solutions.

For $K = 1$, equation (i) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x + y + z = 1$$
$$\Rightarrow \quad y + 3z = 0$$
$$\Rightarrow \quad y = -3z$$
$$x = 1 + 2z$$

Let $z = t$

$\therefore x = 1 + 2t, y = -3t$ and $z = t$

For $K = 2$, equation (i) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow x + y + z = 1$

$y + 3z = 1$

Let $z = t \Rightarrow y = 1 - 3t$

$\Rightarrow x + 1 - 3t + t = 1$

$\Rightarrow x = 2t$

Thus $x = 2t, y = 1 - 3t, z = t$.

Q. Test whether the vectors (1, 1, 1, 3), (1, 2, 3, 4) and (2, 3, 4, 9) are linearly dependent or not. If depentent, find the relations between them. (2.5)

Ans. Consider

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = (0, 0, 0, 0)$

$\Rightarrow \lambda_1(1, 1, 1, 3) + \lambda_2(1, 2, 3, 4) + \lambda_3(2, 3, 4, 9) = (0, 0, 0, 0)$ ...(i)

$$\Rightarrow \left.\begin{aligned} \lambda_1 + \lambda_2 + 2\lambda_3 &= 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \\ \lambda_1 + 3\lambda_2 + 4\lambda_3 &= 0 \\ 3\lambda_1 + 4\lambda_2 + 9\lambda_3 &= 0 \end{aligned}\right\} \quad ...(ii)$$

This is a homogeneous system of equation

Matrix form is AX = B

$$\begin{bmatrix}1 & 1 & 2\\1 & 2 & 3\\1 & 3 & 4\\3 & 4 & 9\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1, R_4 \to R_4 - 3R_1$

$$\sim\begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 2 & 2\\0 & 1 & 3\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_4 \to R_4 - R_2, R_3 \to \frac{1}{2}R_3$

$$\sim\begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 1 & 1\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim\begin{bmatrix}1 & 1 & 2\\0 & 1 & 1\\0 & 0 & 2\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Since $r(A) = 3 =$ no. of unknowns

$\therefore$ System has zero or trivial solution.

Thus all scalars $\lambda_1, \lambda_2, \lambda_3$ are zero

$\therefore x_1, x_2, x_3$ are linearly independent

Also $\therefore \lambda_1 + \lambda_2 + 2\lambda_3 = 0$

$\lambda_2 + \lambda_3 = 0$

$2\lambda_3 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0.$

Q. Show that $\frac{d}{dx}\left\{J_n^2(x)\right\} = \frac{x}{2n}\left\{J_{n-1}^2(x) - J_{n+1}^2(x)\right\}$ **(2.5)**

Ans. As we know

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right] \quad ...(i)$$

$$Jn'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right] \quad ...(ii)$$

Consider $\frac{d}{dx}\left[J_n^2(x)\right] = 2J_n(x)J_n'(x)$

$$= 2\cdot\frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]\cdot\frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

[Using (i) and (ii)]

$$= \frac{x}{2^n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$$

Q. Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ **by reducting it to echelon form. (6)**

Ans.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to \frac{1}{3}R_2, R_3 \to -\frac{1}{2}R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2, R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & -4/3 \end{bmatrix}$$

$R_3 \to -3R_3,\ R_4 \to \frac{-3}{4}R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - \frac{1}{3}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix is reduced to echelon form

$\therefore r(A) = 3$

Q. Find the modal matrix of $A\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ **and diagonalize it.** **(6.5)**

Ans. The characterstic equation of A is

$$|A-\lambda I| = \begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)[(3-\lambda)(1-\lambda)-8] - 2[5\lambda-5+4] - 2[-20+6-2\lambda] = 0$

$\Rightarrow (4-\lambda)(3-4\lambda+\lambda^2-8) - 2(5\lambda-1) - 2(-14-2\lambda) = 0$

$(4-\lambda)(\lambda^2-4\lambda-5) - 10\lambda + 2 + 28 + 4\lambda = 0$

$\Rightarrow 4\lambda^2 - 16\lambda - 20 - \lambda^3 + 4\lambda^2 + 5\lambda - 6\lambda + 30 = 0$

$\Rightarrow -\lambda^3 + 8\lambda^2 - 17\lambda + 10 = 0$

$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$\Rightarrow (\lambda-1)(\lambda^2-7\lambda+10) = 0$

$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-5) = 0$

$\Rightarrow \lambda = 1, 2, 5$ **(eigen values)**

For $\lambda = 1$

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 + \frac{5}{3}R_1,\ R_3 \to R_3 + \frac{2}{3}R_1$

$$\sim\begin{bmatrix} 3 & 2 & -2 \\ 0 & 16/3 & -4/3 \\ 0 & 16/3 & -4/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\sim\begin{bmatrix} 3 & 2 & -2 \\ 0 & 16/3 & -4/3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x_1 + 2x_2 - 2x_3 = 0$$

$$\frac{16}{3}x_2 - \frac{4}{3}x_3 = 0$$

Since $r(A) = 2 < 3$

$\Rightarrow n - r = 3 - 2 = 1$ Variable can be given arbitrary value.

$$\Rightarrow \quad 16x_2 = 4x_3$$

$$4x_2 = x_3$$

$$\Rightarrow \quad \frac{x_2}{1} = \frac{x_3}{4}$$

Also. $3x_1 + 2 - 8 = 0$

$\Rightarrow$ $3x_1 = 6$

$x = 2$

$\therefore$ eigen vector is $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$.

For $\lambda = 2$

$$\begin{bmatrix} 2 & 2 & -2 \\ -5 & 1 & 2 \\ -2 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + \frac{5}{2} R_1, R_3 \to R_3 + R_1$

$$\sim \begin{bmatrix} 2 & 2 & -2 \\ 0 & 6 & -3 \\ 0 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 2 & 2 & -2 \\ 0 & 6 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_2 - 2x_3 = 0$

$\Rightarrow$ $x_1 + x_2 - x_3 = 0$

and $6x_2 - 3x_3 = 0$

$\Rightarrow$ $2x_2 - x_3 = 0$

$\Rightarrow$ $\frac{x_2}{1} = \frac{x_3}{2}$

Now $2x_1 + 2 - 4 = 0$ $\Rightarrow 2x_1 = 2$

$\Rightarrow x_1 = 1$

$\therefore$ eigen vector is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda = 5$.

$$\begin{bmatrix} -1 & 2 & -2 \\ -5 & -2 & 2 \\ -2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1, R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} -1 & 2 & -2 \\ 0 & -12 & 12 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + 2x_2 - 2x_3 = 0$$
$$-12x_2 + 12x_3 = 0$$

Now

$$\frac{x_2}{1} = \frac{x_3}{1}$$

$\Rightarrow$

$$-x_1 + 2 - 2 = 0$$
$$x_1 = 0$$

Also,

$\Rightarrow$

$\therefore$ Eigen vector is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ Modal matrix $M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

$\Rightarrow |M| = 1$

$$M^{-1} = \frac{1}{1}\begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}$$

Now $D = M^{-1}AM = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

$\Rightarrow D = \begin{bmatrix} -1 & -1 & 1 \\ 3 & 2 & -2 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 5 \\ 4 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Q.4. Invertigate whether the set of equations $2x - y - z = 2, x + 2y + z = 2,$ $4x - 7y - 5z = 2$ is consistent or not, if consistent, solve it. (6.5)

Ans. Given system of equations, can be written as

$$AX = B$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

operating $R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 4 & -7 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

operating $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\sim\begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & -15 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}$$

$$R_3 \to \frac{-1}{3}R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

operating $R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\text{Since } r(A) = 2, r(A:B) = 2.$$
$$i.e., r(A) = r(A:B)$$

$\therefore$ System is consistent

As $r(A) = r(A:B) = 2 < 3$ (no. of unknowns)

Thus system has inifnite many solutions.

We give $n - r = 3 - 2 = 1$ variable arbitrary value

$$\text{Let } z = k$$
$$x + 2y + z = 2$$
$$-5y - 3z = -2$$
$$\Rightarrow \quad -5y - 3k = -2$$
$$\Rightarrow \quad y = \frac{2-3k}{5}$$
$$\text{Also} \quad x = 2 - 2y - z$$
$$\Rightarrow \quad x = 2 - \frac{(4-6k)}{5} - k$$
$$\Rightarrow \quad x = \frac{10-4+6k-5K}{5} = \frac{6+K}{5}$$
$$\therefore \quad x = \frac{6+K}{5}, y = \frac{2-3K}{5}, Z = K.$$

Q. For the matrix $A = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & -i \\ -2-3i & -i & 0 \end{bmatrix}$

Show that $\bar{A}$ is skew - hermitian matrix. (2.5)

Ans. Let
$$\bar{A} = B = \begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & i \\ -2+3i & i & 0 \end{bmatrix}$$

Now to show
$$\bar{B}' = -B.$$

$$\bar{B} = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & -i \\ -2-3i & -i & 0 \end{bmatrix}$$

$$\bar{B}' = \begin{bmatrix} i & -1+i & -2-3i \\ 1+i & 2i & -i \\ 2-3i & -i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -i & 1-i & 2+3i \\ -1-i & -2i & i \\ -2+3i & i & 0 \end{bmatrix} = -B$$

$\therefore \bar{A}$ is skew hermitian matrix.

Q. Find eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ **(5)**

Ans. Characterstic equation of given matrix is $|A - \lambda I| = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(2-\lambda)(5-\lambda) = 0$$

$\Rightarrow$ $\lambda = 2, 3, 5$ (eigen values).

$\Rightarrow$ $\lambda = 2$, eigen vector is given by

For $(A+2I)X = 0$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $r(A) = 2 < 3$

Let $x_3 = 0$

$\Rightarrow$ $x_1 + x_2 + 4x_3 = 0$

$\Rightarrow$ $x_1 + x_2 = 0$

$x_1 = -x_2$

$\therefore$ eigen vector is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda = 5$ eigen vector is given by $(A + 3I)X = 0$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As $\qquad r(A) = 2 < 3$

$\Rightarrow \qquad x_2 + 4x_3 = 0, \; 10x_3 = 0$

$\Rightarrow \qquad x_3 = 0, \; x_2 = 0$

Let $\qquad x_1 = 1$

$\therefore$ eigenvector is $\qquad X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

for $\lambda = 5$, eigenvector is given by

$$(A + 5I)X = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + 4x_3 = 0$$

$$-3x_2 + 6x_3 = 0$$

$$x_2 = 2x_3$$

$$\frac{x_2}{2} = \frac{x_3}{1}$$

and $\qquad -2x_1 + 2 + 4 = 0$

$\Rightarrow \qquad x_1 = 3$

$\therefore$ eigenvector is $\qquad X_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Q. Show that the matrix $A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ **is a Hermitian matrix.** **(2)**

Ans. To show

$$\bar{A}' = A.$$

Given

$$A = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 3 & 7+4i & -2-5i \\ 7-4i & -2 & 3-i \\ -2+5i & 3+i & 4 \end{bmatrix}$$

$$\bar{A}' = \begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix} = A$$

∴ A is a hermitian matrix.

Q. If λ be an eigen value of a non-singular matrix A, then show that λ^{-1} is an eigen value of A^{-1}. **(3)**

Ans. Since λ is eigenvalue of A then

$$(A - \lambda I)X = 0 \quad ...(1)$$

Premultiply (1) by A^{-1}

$$A^{-1}(A - \lambda I)X = 0$$

⇒

$$(I - \lambda A^{-1} I)X = 0 \quad ...(2)$$

⇒

Multiply (2) throughout by λ^{-1}.

$$(\lambda^{-1} I - A^{-1})X = 0$$

⇒

$$-(A^{-1} - \lambda^{-1} I)X = 0$$

$$(A^{-1} - \lambda^{-1} I)X = 0$$

∴ λ^{-1} is an eigenvalue of A^{-1}.

Q. Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ **by elementary row operations.** **(6)**

Ans. Let consider

$$A = IA$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$R_3 \leftrightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$$R_3 \rightarrow R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$$R_3 \rightarrow \frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$$R_1 \rightarrow R_1 + R_3 \quad , \quad R_2 \rightarrow R_2 - 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$$\Rightarrow \qquad I = A^{-1}A$$

$$\therefore \qquad A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Q. Using matrix method, show that the equations $3x + 4y + 2z = 1, x + 2y = 4, 10y + 3z = -2, 2x - 3y - z = 5$ are consistent and hence obtain solutions for x, y and z. **(6.5)**

Ans.

$$\begin{aligned} 3x + 4y + 2z &= 1 \\ x + 2y &= 4 \\ 10y + 3z &= -2 \\ 2x - 3y - z &= 5 \\ AX &= B \end{aligned}$$

In matrix notation

$$\begin{bmatrix} 3 & 4 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}.$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 2 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 5 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1, R_4 \to R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 10 & 3 \\ 0 & -7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -2 \\ -3 \end{bmatrix}$$

$$R_2 \to -2R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 10 & 3 \\ 0 & -7 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -2 \\ -3 \end{bmatrix}$$

$$R_3 \to R_3 - 10R_2, \quad R_4 \to R_4 + 7R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 13 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -57 \\ 71/2 \end{bmatrix}$$

$$R_3 \to \frac{1}{13}R_3,\ R_4 \to \frac{-1}{8}R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -57/13 \\ -71/16 \end{bmatrix}$$

$$R_4 \to R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 11/2 \\ -57/13 \\ -11/208 \end{bmatrix}$$

$$r(A) = 3,\ r(A:B) = 4$$

$\therefore$ System has inconsistent solution.

Q. Use Cayley – Hamilton theorem to express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A where $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$. (6.5)

Ans. Characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = 0$$

By cayley Hamilton theorem

$$A^2 - 4A - 5I = 0 \qquad \text{...(1)}$$

Consider

$$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$$

$$\Rightarrow \quad A^3(A^2 - 4A - 5I) + 5A^3 - 7A^3 + 11A^2 - A - 10I$$

$$\Rightarrow \quad -2A^3 + 11A^2 - A - 10I \qquad \text{[By (1)]}$$

$$\Rightarrow \quad -2A(A^2 - 4A - 5I) - 8A^2 - 10A + 11A^2 - A - 10I$$

$$\Rightarrow \quad 3A^2 - 11A - 10I$$

$$\Rightarrow \quad 3(A^2 - 4A - 5I) - 11A - 10I + 12A + 15I$$

$$\Rightarrow \quad A + 5I.$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$$

Find the value of a, b, c if matrix.

$$A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$$

is orthogonal. **(2)**

Ans. For matrix A to be orthogonal $AA^T = I$

$$\text{Consider } AA^T = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}\begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix}$$

$$= \begin{bmatrix} 4b^2+c^2 & 2b^2-c^2 & -2b^2+c^2 \\ 2b^2-c^2 & a^2+b^2+c^2 & a^2-b^2-c^2 \\ -2b^2+c^2 & a^2-b^2-c^2 & a^2+b^2+c^2 \end{bmatrix}$$

$$\text{Since } AA^T = I$$

$$\text{On solving } 2b^2-c^2 = 0$$

$$a^2-b^2-c^2 = 0$$

and

$$\Rightarrow \quad c = \pm\sqrt{2}b$$

$$a^2 = b^2+c^2$$

and

$$a^2 = 3b^2$$

$$\Rightarrow \quad a = \pm\sqrt{3}b$$

$$\Rightarrow \quad 4b^2+c^2 = 1$$

Comparing

$$6b^2 = 1$$

$$\Rightarrow \quad b = \pm\sqrt{\frac{1}{6}}$$

$$\Rightarrow \quad c = \pm\sqrt{2}\cdot\frac{1}{\sqrt{6}} = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad a = \pm\sqrt{3}\cdot\frac{1}{\sqrt{6}} = \pm\frac{1}{\sqrt{2}}, b = \pm\frac{1}{\sqrt{6}}$$

UNIT IV

Vector Calculus: Vector and Scalar Functions and Their Fields, Derivatives, Curves, Arc Length. Curvature. Torsion, Gradient of a Scalar Field. Directional Derivative, Divergence of a Vector Field, Curl of a Vector Field, Line integrals, Path Independence of Line Integrals, Double Integrals, Green's Theorem in the Plane, Surfaces for Surface Integrals, Surface Integrals, Triple Integrals, Stokes Theorem. Divergence Theorem of Gauss.

Q. If $\vec{F} = xyi - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to

1. Evaluate the line integral of $\int \vec{F} \times \vec{dr}$ **(4)**

Ans. If
$$x = t^2, y = 2t, z = t^3$$
$$\frac{dx}{dt} = 2t,$$
$$\frac{dy}{dt} = 2, \frac{dz}{dt} = 3t^2$$

Let
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Then.
$$\vec{F} \times \vec{dr} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xy & -z & x^2 \\ dx & dy & dz \end{vmatrix}$$
$$= (-zdz - x^2\,dy)i$$
$$- (xydz - x^2\,dx)\,j + (xy\,dy + zdx)\,\hat{k}$$

In terms of t
$$\vec{F} \times \vec{dr} = \left(-t^3.3t^2dt - t^4.2dt\right)\hat{i}$$
$$- \left(2t^3.3t^2dt - t^4.2t\,dt\right)\hat{j}$$
$$+ \left(2t^3.2dt + t^3.2tdt\right)\hat{k}$$
$$= \left[\left(-3t^5 - 2t^4\right)\hat{i} - \left(6t^5 - 2t^5\right)\hat{j} + \left(4t^3 + 2t^4\right)\hat{k}\right]dt.$$

$$\int_C \vec{F} \times \vec{dr} = \int_o^1 \left[\left(-3t^5 - 2t^4\right)\hat{i} - 4t^5\hat{j} + \left(4t^3 + 2t^4\right)\hat{k}\right]dt$$
$$= \left|\frac{-t^6}{2} - \frac{2t^5}{5}\right|_0^1 \hat{i} - \left|\frac{4t^6}{6}\right|_o^1 \hat{j} + \left|t^4 + \frac{2}{5}t^5\right|_0^1 \hat{k}$$
$$= \left(\frac{-1}{2} \frac{-2}{5}\right)\hat{i} - \frac{2}{3}\hat{j} + \left(1 + \frac{2}{5}\right)\hat{k}$$
$$= \left(\frac{-9}{10}\right)\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k}$$

Q. Find the directional derivative of the function $\phi = x^2 z + 2xy^2 + yz^2$ at the point (1, 2, –1) in the direction of the vector $\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$. (3)

Ans.

$$\nabla\phi = \left(2xz + 2y^2\right)\hat{i} + \left(4xy + z^2\right)\hat{j} + \left(x^2 + 2yz\right)\hat{k}$$

At point P (1, 2, – 1)

$$(\nabla\phi)_P = 7\hat{i} + 9\hat{j} - 3\hat{k}$$

In direction of $\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$.

Required directional derivative

$$= \left(7\hat{i} + 9\hat{j} - 3\hat{k}\right) \cdot \frac{\left(2\hat{i} + 3\hat{j} - 4\hat{k}\right)}{\sqrt{4+9+16}}$$

$$= \frac{14+27+12}{\sqrt{29}} = \frac{53}{\sqrt{29}}$$

Q. Find the directional derivative of $\bar{\nabla}.(\nabla\phi)$ at the point (1, – 2, 1) in the direction of normal to the surface $xy^2z = 3x + z^2$, where $\phi = 2x^3y^2z^4$. (6)

Ans. Let

$$f = xy^2z - 3x - z^2$$

$$\phi = 2x^3y^2z^4.$$

Now

$$\bar{\nabla}.(\nabla\phi) = \nabla^2\phi = g$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Now

$$g = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2.$$

Consider

grad g.

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}\hat{k}\right)\left(12xy^2z^4 + 4x^3x^4 + 24x^3y^2z^2\right)$$

$$= (12y^2z^4 + 12x^2z^4 + 72\,x^2y^2z^2)\,i + (24xyz^4 + 48x^3yz^2)\,\hat{j}$$

$$+ (48xy^2z^3 + 16x^3z^3 + 48x^3y^2z)\hat{k}$$

$$(\text{grad}\, g)_{(\log 2,\, 1)} = 348\,\hat{i} - 144\hat{j} + 400\hat{k}$$

$$\vec{w} = \text{normal to surface } f$$

$$\nabla f = (y^2 z - 3)\hat{i} + 2xyz\,\hat{j} + (xy^2 - 2z)\hat{k}$$

At point $(1, -2, 1)$

$$\therefore \quad \nabla f = \hat{i} - 4\hat{j} + 2\hat{k} = \vec{w}$$

Thus directional derivative is $(348\hat{i} - 144\hat{j} + 400\,\hat{k}).\dfrac{\hat{i} - 4\hat{j} + 2\hat{k}}{\sqrt{1+16+4}}$

$$= \frac{348 + 576 + 800}{\sqrt{21}} = \frac{1724}{\sqrt{21}}.$$

Q. Use divergence theorem to evaluate $\iint_s \vec{F}.\vec{N}\,ds$, where $\vec{F} = x^2 zi + y\hat{j} - xz^2\hat{k}$ and s is the surface of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4y$ **(6.5)**

Ans. By divergence them.

$$\iint_s \vec{F}.\hat{N}\,ds = \iiint_v div\,\vec{F}\,dv \qquad \text{...(1)}$$

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(xz^2)$$

$$= 2x^2 + 1 - 2xz = 1.$$

$$\Rightarrow \quad \iint_s \vec{F}.\vec{N}\,ds = \iiint_v 1.\,dx\,dy\,dz.$$

z varies from. $4y$ to $x^2 + y^2$.

As $\quad x^2 + y^2 = z$ and $4y = z$

$\Rightarrow \quad x^2 + y^2 = 4y$

In polar Co-ordinates

A circle $\quad r^2 = 4r\sin\theta$

$\Rightarrow \quad$ r $= 4\sin\theta.$

Thus r varies from 0 to $4\sin\theta$ and θ from 0 to π.

$$\therefore \quad \iint_s \vec{F}.\vec{N}\,ds = \int_0^{\pi}\int_0^{4\sin\theta} r \int_{4y}^{x^2+y^2} dz\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{4\sin\theta} r\,|z|_{4y}^{x^2+y^2}\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{4\sin\theta} r\,(x^2+y^2-4y)\,dr\,d\theta$$

$$= \int_0^{\pi}\int_{r=0}^{4\sin\theta} (r^2-4r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{4\sin\theta} r^3-4r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi}\left|\frac{r^4}{4}-\frac{4r^3}{3}\sin\theta\right|_0^{4\sin\theta} d\theta$$

$$= \int_0^{\pi}\left[4^3\sin^4\theta-\frac{4^4}{3}\sin^4\theta\right]d\theta$$

$$= \int_0^{\pi}\frac{-64}{3}\sin^4\theta\,d\theta = \frac{-64}{3}\int_0^{\pi}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^4 d\theta$$

$$= \frac{-64\times 16}{3.}\int_0^{\pi}\sin^4\frac{\theta}{2}\cos^4\frac{\theta}{2}\,d\theta.$$

put $\quad \dfrac{\theta}{2} = t. \Rightarrow d\theta = 2dt.$

$\theta = 0 \Rightarrow t = 0$ annd $\theta = \pi \Rightarrow t = \pi/2$

$$= \frac{-64\times 16}{3}\int_0^{\pi/2} 2\sin^4 t\cos^4 t\,dt$$

$$= \frac{-64\times 16}{3}\, I_{4,4}$$

$$I_{4,4} = \frac{3}{8}\, I_{4,2}$$

$$I_{4,2} = \frac{1}{6}\, I_{4,0}$$

$$I_{4,0} = \int_0^{\pi/2} \sin^4 t \, dt$$

$$= \frac{3.1}{4.2} \cdot \frac{\pi}{2}$$

$$= \frac{-2048}{3} \frac{3}{8} \times \frac{1}{6} \times \frac{3.1}{4.2} \frac{\pi}{2}$$

$$= -8\pi$$

$$= 8\pi \text{ (in magnitude)}$$

Q. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\vec{r}| = r$. Show that div $r^n \vec{r} = (n+3)r^n$.

Ans.

$$\text{div}\left(r^n \vec{r}\right) = \text{div}\left(\phi\vec{A}\right)\phi \, div \, \vec{A} + grad\phi \cdot \vec{A}$$

$$= r^n div \, \vec{r} + grad \, r^n \cdot \vec{r}$$

$$= 3r^n + nr^{n-2}\vec{r} \cdot \vec{r}$$

$$= 3r^n + nr^{n-2}r^2$$

$$= 3r^n + nr^n$$

$$= (n+3) r^n.$$

Q. Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ by Stoke's theorem, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x-z)\hat{k}$ and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0). **(6.5)**

Ans. By stokes theorem

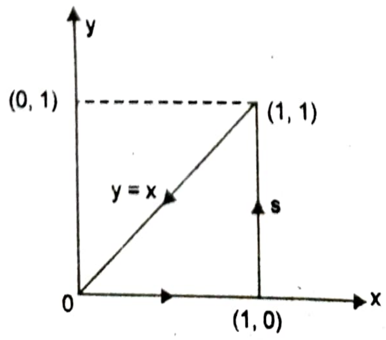
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_s curl \, \vec{F} . \hat{n} \, ds$$

Curl

$$\vec{F} = \begin{vmatrix} i & j & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= 0\hat{i} - [-1]\hat{j} + [2x - 2y]\hat{k}$$

$$= \hat{j} + 2(x-y)\hat{k}$$



Since z–coordinates of each vertex of triangle is zero i.e. triangles lies in xy plane

$$\Rightarrow \qquad \hat{n} = \hat{k}$$

$$\therefore \text{Curl } \bar{F}.\hat{n} = \hat{j} + 2(x-y)\,\hat{k}\,.\,\hat{k} = 2(x-y)$$

Now

$$\oint_C \bar{F}\cdot d\vec{r} = \int_0^1\int_0^x 2(x-y)\,dy\,dx$$

$$= 2\int_0^1 \left(xy - \frac{y^2}{2}\right)\Bigg|_0^x dx = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right)dx$$

$$= \int_0^1 x^2\,dx = \left|\frac{x^3}{3}\right|_0^1 = \frac{1}{3}.$$

Q. **Evaluate the following integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$.** (6)

Ans.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{2}xy\left|z^2\right|_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dydx$$

$$= \frac{1}{2}\int_0^1 x\left|\frac{y^2}{2} - \frac{x^2y^2}{2} - \frac{y^4}{4}\right|_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x}{2}(1-x^2) - \frac{x^3}{2}(1-x^2) - \frac{x}{4}(1-x^2)^2\right]dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x}{2} - \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^5}{2} - \frac{x}{4}(1+x^4-2x^2)\right]dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x}{2} - x^3 + \frac{x^5}{2} - \frac{x}{4} + \frac{x^5}{4} + \frac{2x^3}{4}\right]dx$$

$$= \frac{1}{2}\int_0^1 \left[\frac{x}{4} - \frac{x^3}{2} + \frac{x^5}{4}\right]dx$$

$$= \frac{1}{2}\left[\frac{x^2}{8} - \frac{x^4}{8} + \frac{x^6}{24}\right]_0^1 = \frac{1}{2}\left[\frac{1}{8} - \frac{1}{8} + \frac{1}{24}\right]$$

$$= \frac{1}{48}$$

Q. Apply divergence theorem to find $\iint_S \vec{F}\cdot\hat{n}\,ds$, $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$. and S is the surface of the sphere having centre at (3, -1, 2) and radius 3. **(6.5)**

Ans. Let v be the volume enclosed by surface S. Then by Guass Divegence theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV = \iiint_V (2-1+2)\,dV$$

$$= 3\iiint_v dV = 3V$$

Where

$$V = \frac{4}{3}\pi r^3 = 36\pi$$

$\therefore$

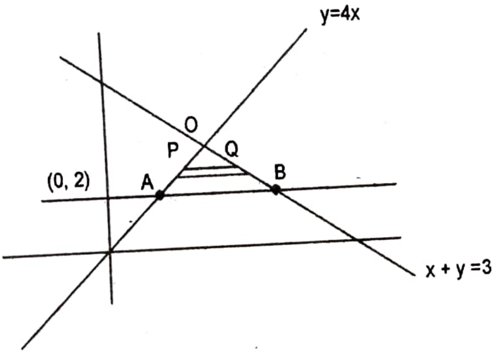
$$\iint_S \vec{F}\cdot\hat{n}\,ds = 108\pi.$$

Q. Evaluate $\iint(x^2+y^2)dx\,dy$ throughout the area enclosed by the curves $y = 4x, x + y = 3, y = 0$ and $y = 2$. **(6.5)**

Ans. $y = 4x, y = 0, y = 2$ and $x + y = 3$.

Draw lines parallel to x-axis PQ.

At $P, x = \dfrac{y}{4}$ and Q, $x = 3 - y$



Limits for y are 0 to 2

Integral becomes

$$\iint(x^2+y^2)dx\,dy = \int_0^2\int_{y/4}^{3-y}(x^2+y^2)\,dx\,dy$$

$$= \int_0^2\left[\frac{x^3}{3} + xy^2\right]_{y/4}^{3-y} dy$$

$$= \int_0^2 \frac{(3-y)^3}{3} + (3-y)y^2 - \frac{1}{3}\frac{y^3}{64} - \frac{y^3}{4}\,dy$$

$$= \int_0^2 (3-y)\left[\frac{(3-y)^2}{3} + y^2\right] - \frac{y}{4}\left[\frac{y^2}{48} + y^2\right]dy$$

$$= \int_0^2 (3-y)\frac{(4y^2+9-6y)}{3} - \frac{49y^3}{192}\,dy$$

$$= \int_0^2 \frac{12y^2+27-18y-4y^3-9y+6y^2}{3} - \frac{49y^3}{192}\,dy$$

$$= \int_0^2 \frac{18y^2-4y^3-27y+27}{3} - \frac{49y^3}{192}\,dy$$

$$= \int_0^2 6y^2 - \frac{4}{3}y^3 - 9y + 9 - \frac{49y^3}{192}\,dy$$

$$= \left.\frac{6y^3}{3} - \frac{4}{3}\frac{y^4}{4} - \frac{9y^2}{2} + 9y - \frac{49}{192}\frac{y^4}{4}\right]_0^2$$

$$= 2\times 8 - \frac{16}{3} - 18 + 18 - \frac{49}{192}\times 4$$

$$= 16 - \frac{16}{3} - \frac{49}{48}$$

$$= \frac{768-256-49}{48}$$

$$= \frac{463}{48}$$

Q. Use Green's theorem to evaluate $\int_C (x^2+xy)dx + (x^2+y^2)dy$ where C is the square formed by the lines $y = \pm 1$, $x = \pm 1$ (6

Ans. By Green's theorem

$$\int M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here
$$M = x^2+xy,\ N = x^2+y^2$$

$$\frac{\partial M}{\partial y} = x,\quad \frac{\partial N}{\partial x} = 2x$$

y varies from −1 to 1

x varies from −1 to 1

Now

$$\int M dx + N dy = \int_{-1}^{1}\int_{-1}^{1}(2x - x)dx\,dy$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy$$

$$= \int_{-1}^{1}\left[\frac{x^2}{2}\right]_{-1}^{1} dy$$

$$= \int_{-1}^{1}\frac{1}{2} - \frac{1}{2}dy$$

$$= \int_{-1}^{1} 0\,dy$$

$$= 0.$$

Q.) Apply stoke's theorem to calculate $\int_c 4y\,dx + 2z\,dy + 6y\,dz$, where c is the curve of integration $x^2 + y^2 + z^2 = 6z$ and $z = x + 3$ **(6.5)**

Ans. By Stokes theorem

$$\int_c F.dr = \iint \nabla \times \vec{F}.\hat{n}\,ds$$

Here

$$\vec{F}.dr = 4y\,dx + 2z\,dy + 6y\,dz$$

$$\vec{F} = 4yi + 2zj + 6y\hat{k}$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4y & 2z & 6y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}6y - \frac{\partial}{\partial z}2z\right)i - \left(\frac{\partial}{\partial x}6y - \frac{\partial}{\partial z}4y\right)j$$

$$+\left(\frac{\partial}{\partial x}2z - \frac{\partial}{\partial y}4y\right)\hat{k}$$

$$= 4\hat{i} - 0\hat{j} - 4\hat{k}$$

$$= 4\hat{i} - 4\hat{k}$$

(04-2001)

Surface S is $x^2 + y^2 + z^2 = 6z$, $z = x + 3$.

$$x^2 + y^2 + (x + 3)^2 = 6(x + 3)$$

$\Rightarrow$

$$\Rightarrow x^2 + y^2 + x^2 + 9 + 6x - 6x - 18 = 0$$

$$2x^2 + y^2 - 9 = 0$$

$\Rightarrow$

Projection is in xy plane.

$$ds = dx.$$

$$\iint \nabla \times \vec{F}.\hat{n}\, ds = \iint 4\, dy\, dz - 4dx\, dy$$

$$= 0$$

Q. If a force $\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces in the xy - plane from (0, 0) to (1, 4) along the curve $y = 4x^2$, find the work done (6)

Ans.

$$\vec{F} = 2x^2yi + 3xy\hat{j}$$

Work done

$$= \int \vec{F}.\overline{dr}$$

$$= \int_A^B \vec{F}.\overline{dr}$$

$$\vec{F}.\overline{dr} = \left(2x^2y\hat{i} + 3xy\hat{j}\right).\left(dx\hat{i} + dy\hat{j}\right)$$

$$= 2x^2y\, dx + 3xy\, dy$$

A is (0, 0) to B (1, 4)

As

$$y = 4x^2$$

$$dy = 8x\, dx.$$

Now x varies from 0 to 1

$$\int \vec{F}.\overline{dr} = \int_0^1 2x^2y\, dx + 3xy.8x\, dx$$

$$= \int_0^1 2x^2(4x^2) + 24x^2(4x^2)\, dx$$

$$= \int_0^1 8x^4 + 96x^4\, dx$$

$$= \int_0^1 104x^4\, dx = 104\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{104}{5}.$$

Q. If $\nabla\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$ find ϕ. (5)

Ans. Given
$$\nabla\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}.$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}.$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = y^2 - 2xyz^3$$

$$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3$$

$$\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2$$

Now partially differentiate w.r.t x, y, z

$$\phi = xy^2 - x^2yz^3 + f_1(y_1 z) \quad ...(1)$$
$$\phi = 3y + xy^2 - x^2yz^3 + f_2(x, z) \quad ...(2)$$
$$\phi = \frac{6z^4}{4} - x^2yz^3 + f_3(x,y). \quad ...(3)$$

Partially differentiate (1) w.r.t 'y' and 'z'.

$$\frac{\partial\phi}{\partial y} = 2xy - x^2z^3 + \frac{\partial f_1}{\partial y}$$

On comparing

$$\frac{\partial f_1}{\partial y} = 3. \quad \Rightarrow \quad f_1 = 3y$$

and
$$\frac{\partial\phi}{\partial z} = -3x^2yz^2 + \frac{\partial f_1}{\partial z}$$

On comparing, we get

$$\frac{\partial f_1}{\partial z} = 6z^3$$

$$\Rightarrow \quad f_1 = \frac{6z^4}{4} = \frac{3z^4}{2}$$

Now
$$\phi = xy^2 - x^2yz^3 + 3y + \frac{3z^4}{2}.$$

Q. Use Divergence theorem to evaluate $\iint_S \vec{F}.\hat{n}\, ds$ where $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (6.5)

Ans. Here S is the closed surface. By Divergence theorem

$$\iint_S \vec{F}.\hat{n}\, ds = \iiint_V \vec{\nabla}.\vec{F}\, dV$$

$$= \iiint_V \nabla.(x^3\hat{i} + y^3\hat{j} + z^3\hat{k})dx\, dy\, dz$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)dx\, dy\, dz$$

Transforming it into spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

Now θ varies from 0 to π, ϕ from 0 to 2π and $r = 0$ to a

$$dx\, dy\, dz = r^2 \sin\theta\, d\theta\, d\phi\, dr$$

Thus, integral becomes

$$\iint_S \vec{F}.\vec{ds} = 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^2.r^2\, dr\, d\theta\, d\phi$$

$$= 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^4\, dr\, d\theta\, d\phi \sin\theta.$$

$$= 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta \left|\frac{r^5}{5}\right|_0^a d\theta\, d\phi$$

$$= 3\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{a^5}{5}\sin\theta\, d\theta\, d\phi.$$

$$= \frac{3a^5}{5}\int_{\phi=0}^{2\pi}(-\cos\theta)_0^{\pi}\, d\phi$$

$$= \frac{3a^5}{5}\int_{\phi=0}^{2\pi}(1+1)\, d\phi = \frac{6a^5}{5}\int_{\phi=0}^{2\pi} d\phi$$

$$= \frac{6a^5}{5}(2\pi - 0) = \frac{12\pi a^5}{5}.$$

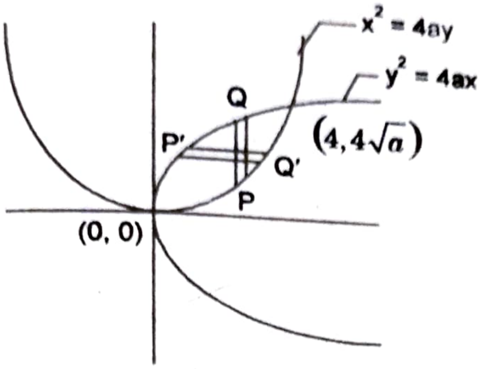
Q. **Change the order of integration in the following integral and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$.** (4)

Ans. Given strip is PQ

i.e. from $\quad y = \frac{x^2}{4a} \quad \Rightarrow x^2 = 4ay$

and $\quad y^2 = 4ax$

Point of intersection are (0, 0) and $(4, 4\sqrt{a})$.



To change order of integration, consider P'Q'

Where P' is $\quad y^2 = 4ax \Rightarrow x = \dfrac{y^2}{4a}$ and

Q' is $\quad x^2 = 4ay \Rightarrow x = 2\sqrt{ay}$

and y varies from 0 to $4\sqrt{a}$.

$$\int_0^4 \int_{x^2/4a}^{2\sqrt{ax}} dydx = \int_0^{4\sqrt{a}} \int_{y^2/4a}^{2\sqrt{ay}} dxdy$$

$$= \int_0^{4\sqrt{a}} [x]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \int_0^{4\sqrt{a}} \left(2\sqrt{a}\sqrt{y} - \frac{y^2}{4a}\right) dy.$$

$$= \left[2\sqrt{a}\cdot\frac{2}{3}y^{3/2} - \frac{y^3}{12a}\right]_0^{4\sqrt{a}} \qquad ...(1)$$

$$= \frac{4}{3}\sqrt{a}\cdot 8a^{3/4} - \frac{64a^{3/2}}{12a}$$

$$= \frac{32}{3}a^{5/4} - \frac{16}{3}a = \frac{16}{3}a\left(2a^{1/4} - 1\right)$$

Q. Apply Stoke's theorem to evaluate $\int\int_c [(x+y)dx + (2x-z)dy + (y+z)dz]$ where c is the boundary of the triangle with vertices (2, 0, 0), (0, 3, 0) and (0, 0, 6) **(6)**

Ans. As by Stoke's theorem

$$\oint_c \vec{F}.\vec{dr} = \iint_s \text{curl}\,\vec{F}.\hat{x}\,ds = \iint_s (\nabla\times\vec{F})\,.\hat{n}\,ds$$

$$\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}.$$

Here

$$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

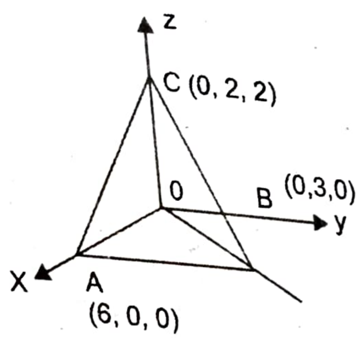
S is the surface of the plane $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$, therefore $\hat{x}$ is normal to the plane ABC,

i.e.

$$\hat{n} = \frac{\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1\right)}{\left|\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{2}+\frac{y}{3}+\frac{z}{6}-1\right)\right|}$$

$$= \frac{\frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{6}\hat{k}}{\sqrt{\frac{1}{4}+\frac{1}{9}+\frac{1}{36}}} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{9+4+1}}$$

$$= \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$



Now

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(0) + \hat{k}(2-1)$$

$$= 2\hat{i}+\hat{k}$$

$\therefore$

$$(\vec{\nabla}\times\vec{F}).\hat{n} = (2\hat{i}+\hat{k}).\left[\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right]$$

$$= \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}}$$

Thus

$$\oint_C [(x+y)dx + (2x-z)dy + (y+z)dz] = \iint_S \frac{7}{\sqrt{14}}\, ds$$

$$= \frac{7}{\sqrt{14}} \iint_R \frac{dx\,dy}{|\hat{n}.\hat{k}|}$$

where R is the projection of S on xy–plane

$$= \frac{7}{\sqrt{14}} \iint_R \frac{dx\,dy}{\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}} \cdot \hat{k}}$$

$$= 7\iint_R dx\,dy. = 7 \times \text{area of } \Delta OAB$$

$$= 7\left[\frac{1}{2}\times 2\times 3\right] = 21$$

Q. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$ **(6)**

Ans. The given integral is

$$I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y}.e^z\, dz\, dy\, dx$$

$$= \int_0^a \int_0^x e^{x+y} \left| e^z \right|_0^{x+y} dy\, dx$$

$$= \int_0^a \int_0^x e^{x+y}(e^{x+y} - 1)\, dy\, dx$$

$$= \int_0^a \int_0^x (e^{2x}e^{2y} - e^x e^y)\, dy\, dx$$

$$= \int_0^a \left[e^{2x}.\frac{e^{2y}}{2} - e^x e^y \right]_0^x dx$$

$$= \int_0^a \left(\frac{e^{2x}.e^{2x}}{2} - e^{2x} - \frac{e^{2x}}{2} + e^x \right) dx$$

$$= \int_0^a \left(\frac{e^{4x}}{2} - \frac{3e^{2x}}{2} + e^x \right) dx$$

$$= \left[\frac{e^{4x}}{8} - \frac{3e^{2x}}{4} + e^{x}\right]_0^a$$

$$= \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{1}{8} + \frac{3}{4} - 1$$

$$I = \frac{e^{4a}}{8} - \frac{3e^{2a}}{4} + e^{a} - \frac{3}{8}$$

and
$$\frac{\partial\phi}{\partial z} = (6z^3 - 3x^2yz^2)$$

Now, By degree of total differentiation

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + [6z^3 - 3x^2yz^2]dz$$

On Integrating; we get;

$$\phi = [xy^2 - x^2yz^3 + f(y,z)] + [3y + xy^2 - x^2yz^3] + f(x,z)] + \left[\frac{3}{2}z^4 - x^2yz^3 + f(x,y)\right]$$

$$\Rightarrow \qquad \phi = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 \text{ Ans.}$$

Q. Using divergence theorem to evaluate $\iint \vec{F}\cdot d\vec{s}$, where $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ (6)

Ans. Here S is the closed surface hence by divergence theorem

$$\iint \vec{F}\cdot\overline{dx} = \iiint_v \nabla\cdot\vec{F}\,dv$$

$$= \iiint_v \nabla\cdot(x^3\hat{i} + y^3\hat{j} + z^3\hat{k})dx\,dy\,dz$$

$$= \iiint (3x^2 + 3y^2 + 3z^2)dx\,dy\,dz \qquad ...(1)$$

Now,

Transforming it into spherical polar co-ordinates

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi.$$

and
$$z = r\cos\theta.$$

We get
$$\iint \vec{F}\cdot d\vec{s} = 3\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{a} r^2 r^2 \sin\theta\, dr\, d\phi\, d\theta$$

$$\left\{2s\,dx\,dy\,dz = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}dr\,d\theta\,d\phi = r^2\sin\theta dr\,d\theta\,d\phi\right\}$$

$$= 3\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{\phi=0}^{2\pi}d\phi = \int_{r=0}^{a} r^2 dr = 3.2.2\pi\frac{a^2}{5} = 12\pi\frac{a^5}{5} \text{ Ans.}$$

Q.9. (a) Using Green's theorem in xy - plane, evaluate $\int_C [(xy^2 - 2xy)\,dx + (x^2y + 3)\,dy]$ around the curve C of the region enclosed by $y^2 = 8x$ and $x = 2$ **(6.5)**

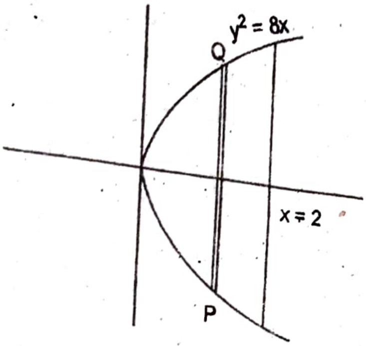
Ans. By Green's theorem

$$\oint Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

Now

$$M = xy^2 - 2xy, N = x^2y + 3$$

$$\therefore \frac{\partial N}{\partial x} = 2xy, \quad \frac{\partial M}{\partial y} = 2xy - 2x$$



Consider strip PQ.

Where

$$y = -\sqrt{8x} \text{ to } \sqrt{8x}$$

and

$$x = 0 \text{ to } 2$$

$$\therefore \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_0^2 \int_{-\sqrt{8x}}^{\sqrt{8x}} (2xy - 2xy + 2x)\,dydx$$

$$= \int_0^2 \int_{-\sqrt{8x}}^{\sqrt{8x}} 2x\,dydx$$

$$= 4\int_0^2 \int_0^{\sqrt{8x}} x\,dydx = 4\int_0^2 x\,|y|_0^{\sqrt{8x}}\,dx$$

$$= 4\int_0^2 x\sqrt{8x}\,dx = 4\sqrt{8}\int_0^2 x^{3/2}dx$$

$$= 4\sqrt{8}\left|\frac{x^{5/2}}{5/2}\right|_0^2 = 8\sqrt{2} \times \frac{2}{5}\cdot 2^{5/2} = \frac{128}{5}$$

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Q.9. (b) Using stoke's theorem, evaluate $\int \text{curl}\, \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and surface S is part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane. (6)

Ans. By definition $$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\, \vec{F}\cdot\hat{n}\, ds$$

As S is upper half of sphere $x^2 + y^2 + z^2 = 1$ i.e. $z = 0$ and $x^2 + y^2 = 1$

Also $$\vec{F}\cdot d\vec{r} = (y\hat{i} + z\hat{j} + x\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dx\hat{k})$$
$$= ydx.$$

$\therefore$ $$\oint_C \vec{F}\cdot d\vec{r} = \oint_C ydx = \int_{x^2+y^2=1} ydx$$

Let $$x = \cos t, y = \sin t$$

$\Rightarrow$ $$dx = -\sin t\, dt, dy = \cos t\, dt$$

t varies from 0 to 2π

$$= \int_0^{2\pi} (\sin t)(-\sin t)dt = -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)dt$$

$$= -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = -\frac{1}{2}(2\pi) = -\pi$$