Linear Combination

Def^n. A vector V in R^n is said to be linear combination of vector $u_1, u_2, \dots, u_k$ if there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ in R such that

$$V = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k .$$

Ex. Every vector $(a_1, a_2) \in R^2$ is a linear combination of $(1,0)$ and $(0,1)$ as $(a_1, a_2) = a_1(1,0) + a_2(0,1)$

Ex. write $(2,2)$ as linear combination of $(1,2)$ and $(1,4)$

Soln. Let $(2,2) = \alpha_1(1,2) + \alpha_2(1,4)$

$$\Rightarrow (2,2) = (\alpha_1, 2\alpha_1) + (\alpha_2, 4\alpha_2)$$

$$\Rightarrow (2,2) = (\alpha_1 + \alpha_2, 2\alpha_1 + 4\alpha_2)$$

$$\Rightarrow \begin{array}{r} \alpha_1 + \alpha_2 = 2 \\ \alpha_1 \pm 2\alpha_2 = 1 \\ \hline -\alpha_2 = 1 \end{array}$$

$$\Rightarrow \alpha_2 = 1$$

$$\therefore \alpha_1 = 2 + 1 = 3$$

$$\therefore (2,2) = 3(1,2) - 1(1,4)$$

Linear Span:

Def^n → Let $u_1, u_2, \dots u_k$ be k vectors in R^n. The collection of all linear combinations of $u_1, u_2, \dots, u_k$ is called linear span of $u_1, u_2, \dots, u_k$.

Linear span of a set S is denoted by $L(S)$ or $[S]$.

$$L(u_1, u_2, \dots, u_k) = [u_1, u_2, \dots, u_k]$$

$$= \{\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k \,;\, \alpha_i \in R\}.$$

Span of vector →

Def^n The vectors $u_1, u_2, \dots, u_k$ in R^n are said to span R^n if every vector in R^n is a linear combination of $u_1, u_2, \dots u_k$ i.e,

$$L(u_1, u_2, \dots, u_k) = R^n.$$

Def^n Let S be a non-empty subset of a vector space V. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. We define span$(\phi) = \{0\}$.

Ex. The set $\{(1,0,0), (0,1,0)\}$ consists of all vectors in R^3 that have the form $a(1,0,0) + b(0,1,0) = (a, b, 0)$ for some scalars a and b. Thus the span of $\{(1,0,0), (0,1,0)\}$ contains all the points in the xy-plane. In this case the span of the set is a subspace of R^3.

Ex: Which of the foll. vectors span R^2?

(a) $(1,2), (-1,1)$ (b) $(0,0), (1,1), (-2,-2)$

Soln (a) Let $(a,b) \in R^2$

Let if possible

$$(a,b) = \alpha_1(1,2) + \alpha_2(-1,1) \quad \alpha_1, \alpha_2 \in R.$$

$$= (\alpha_1, 2\alpha_1) + (-\alpha_2, \alpha_2)$$

$$= (\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

$$\alpha_1 - \alpha_2 = a$$

$$2\alpha_1 + \alpha_2 = b.$$

In matrix form.

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Apply $R_2 \to R_2 - 2R_1$

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b-2a \end{pmatrix}$$

$$\alpha_1 - \alpha_2 = a$$

$$3\alpha_2 = b - 2a$$

$$\Rightarrow \alpha_2 = \frac{b-2a}{3}$$

$$\alpha_1 = a + \alpha_2 = a + \frac{b-2a}{3} = \frac{a+b}{3}$$

Hence, $(a,b) = \frac{a+b}{3}(1,2) + \frac{b-2a}{3}(-1,1)$

Every element of R^2 can be written as linear combination of $(1,2)$ and $(-1,1)$, therefore these two vectors span R^2.

(b). Let $(a,b) \in R^2$

Let if possible

$$(a,b) = \alpha_1(0,0) + \alpha_2(1,1) + \alpha_3(-2,-2), \quad \alpha_1, \alpha_2, \alpha_3 \in R$$

$$\Rightarrow (a,b) = (\alpha_2 - 2\alpha_3, \alpha_2 - 2\alpha_3)$$

$$\Rightarrow \cancel{\alpha_2} - \cancel{2\alpha_3} = a$$

$$\cancel{\alpha_2} - \cancel{2\alpha_3} = b.$$

$$0 = a - b. \Rightarrow a = b.$$

$$\Rightarrow (a,b) = (a,a)$$

Every element of R^2 cannot be expressed as linear combination of $(0,0)$, $(1,1)$ and $(-2,-2)$ therefore they do not span R^2.