VECTOR SPACE

INTERNAL COMPOSITION – Let A be any set. If $a*b \in A$ $\forall a, b \in A$ and $a*b$ is unique then $*$ is s.t.b an internal composition in set A

EXTERNAL COMPOSITION – Let V and F be two sets, if $a \cdot \alpha \in V$ $\forall a \in F, \alpha \in V$ and $a \cdot \alpha$ is unique then $\circ$ is s.t.b an ext. composition ~~of~~ in V over F.

VECTOR SPACE – Let $(F, +, \circ)$ be a field. The elements of F will be called scalars. Let V be a non empty set whose elements will be called vectors. Then V is a vector space over field F, if

i) $(V, +)$ under internal composition is an abelian group

ii) External composition of V over F called scalar multiplication. V is closed w.r.t scalar multiplication $[a\alpha \in V \ \forall a \in F, \alpha \in V]$

iii) scalar multiplication and add^n of vectors satisfies

$a(\alpha+\beta) = a\alpha + a\beta \quad \forall a \in F \quad \alpha, \beta \in V$

$(a+b)\alpha = a\alpha + b\alpha \quad \forall a, b \in F \quad \alpha \in V$

$(ab)\alpha = a(b\alpha) \quad \forall a, b \in F \quad \alpha \in V$

$1 . \alpha = \alpha \quad \forall \alpha \in V$ 1 is unit element of F

$(V, +)$ abelian

$\alpha \in F, x \in V, \alpha x \in V$

$\alpha(x+y) = \alpha x + \alpha y$

$(\alpha+\beta)x = \alpha x + \beta x$

$(\alpha\beta)x = \alpha(\beta x)$

$1 . x = x$

Eg – $\mathbb{R}^n(\mathbb{R})$

GENERAL PROPERTIES OF A VECTOR SPACE

Let $V(F)$ be a vector space over field F $(F, +, \circ)$ ~~then~~, 0 be the additive identity of V. $\alpha, \beta \in V$, $a \in F$ 0 be the additive identity of F

i) $a0 = 0$

$a0 = a(0+0)$ distributive law

$0 + a0 = a0 + a0$

$\because (V, +)$ is an abelian group → cancell^n law

$\cancel{a0} + 0 = a0 + \cancel{a0}$

$0 = a0$

ii) $0\alpha = 0$

$0\alpha = (0+0)\alpha$

$\cancel{0\alpha} + 0 = 0\alpha + \cancel{0\alpha}$

$0 = 0\alpha$

iii) $a(-\alpha) = (-a\alpha)$

If $\alpha \in V$ $-\alpha \in V$

$(V, +)$ is abelian

$a \in F$

$a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$

$a[\alpha - \alpha] = a\alpha + a(-\alpha)$

$a0 = a\alpha + a(-\alpha)$

$-(a\alpha) = a(-\alpha)$

iv) $(-a)\alpha = -(a\alpha)$

If $a \in F$ $-a \in F$

$\because (F, +)$ is an abelian group

$\alpha(a + (-a)) = a\alpha + (-a)\alpha$

$0 = a\alpha + (-a)\alpha$

$-(a\alpha) = (-a)\alpha$

v) $a(\alpha - \beta) = a\alpha - a\beta$.

$a(\alpha + (-\beta)) = a\alpha + a(-\beta)$

$= a\alpha + (-a\beta)$

$= a\alpha - a\beta$

vi) $a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

Let $a\alpha = 0$, $a \neq 0$

$a^{-1} \in F$

$a\alpha = 0$

$a^{-1}(a\alpha) = a^{-1} . 0$

$(a^{-1}a)\alpha = a^{-1} . 0$

$1 . \alpha = 0$

$\alpha = 0$

$a\alpha = 0$, $\alpha \neq 0$

$a . \alpha = 0$

Let $a \neq 0 \Rightarrow a^{-1} \in F$

$a . \alpha = 0$

$a^{-1}(a . \alpha) = 0$

$a^{-1}a . \alpha = 0$

$1 . \alpha = 0$

$\alpha = 0$

contradicts our initial statement

$a = 0$

contained in

VECTOR SUBSPACE – Let V be a vector space over the field F and let $W \subseteq V$. Then W is called a vector space over F ~~w.r.t. opern~~ if itself is a vector space over F w.r.t opern of vector addn and scalar multiplicn

Theorem 1 A subset W of a vector space $V(F)$ is a subspace iff $\forall \alpha, \beta \in W$ and $a, b \in F \Rightarrow a\alpha + b\beta \in W$

considering W subspace of V

if $a \in F$ $\alpha \in W \Rightarrow a\alpha \in W$ (scalar multiplicn)

if $b \in F$ $\beta \in W \Rightarrow b\beta \in W$ (" ")

Now $(W, +)$ is an abelian group

$\therefore a\alpha + b\beta \in W$

given $\rightarrow a\alpha + b\beta \in W$ $\forall a$

$V(F)$ is a vector space

$W \subseteq V$

W is a non empty subset of V

W is closed under vector addn and scalar multiplic

$\forall a, b \in F$ if $a = 1$ $b = 1$

$\alpha + \beta \in W$ (W is closed under addn)

if $a = 0$ $b = 0$

$0(\alpha) + 0(\beta) \in W$

$0 \in W$ (additive identity)

if $a = 0$ $b = -1$

$0(\alpha) + \beta(-1) \in W$

$-\beta \in W$ (additive inverse)

As $W \subseteq V$ (vector addn is associative) and commutative)

$(W, +)$ is abelian

Taking $b = 0$

$a\alpha \in W$ $\forall a \in F$

W is closed w.r.t. scalar multiplic

So remaining postulates shall also hold

Theorem 2 The necessary and sufficient condn for a non-empty subset W of a vector space $V(F)$ to be a subspace of V are

(i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F$, $\alpha \in W \Rightarrow a\alpha \in W$

if W is a subspace and $\beta \in W$ then $-\beta \in W$

$(W, +)$ is an abelian grp.

Let $\alpha \in W$

$\alpha + (-\beta) \in W$ (by closure)

$a \in F$ $\alpha \in W$

$a\alpha \in W$ (scalar multiplicn)

Now using condn

(i) $\alpha \in W$ $\beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F$ $\alpha \in W \Rightarrow a\alpha \in W$

$W \subseteq V$

$\alpha \in W$ $\alpha \in W$

$\alpha - \alpha \in W$

$0 \in W$ (additive identity)

$0 \in W$ $\beta \in W$

$-\beta \in W$ (additive inverse)

$\alpha \in W$ $-\beta \in W$

$\alpha - (-\beta) \in W$

$\alpha + \beta \in W$ (closed)

Since $W \subseteq V$ the vector addn is commutative and associative

$(W, +)$ is an abelian grp.

$a \in F$ $\alpha \in W \Rightarrow a\alpha \in W$ (scalar multiplicn)

W itself is a vectorspace

Theorem 3 An arbitrary intersecn of subspaces i.e the intersection of any family of subspaces is a subspace

Suppose W_1 and W_2 are two subspaces of $V(F)$

$0 \in W_1$ $0 \in W_2$ (Additive identity)

$0 \in W_1 \cap W_2$

$W_1 \cap W_2 \neq \phi$

if $a, b \in F$ $\alpha, \beta \in W_1 \cap W_2$

$a\alpha + b\beta \in W_1 \cap W_2$

Theorem The union of two subspace is a subspace only if one is contained in another

w_1 and w_2 are two subspace

considering $w_1 \subseteq w_2$ and $w_1 \cup w_2 = w_2$

since w_2 is a subspace

$w_1 \cup w_2$ will be a subspace

Let $w_1 \cup w_2$ be a sub space of $V(F)$

To prove $\rightarrow$ $w_1 \subseteq w_2$

Let $w_1 \nsubseteq w_2$ $\exists\, \alpha \in w_1, \alpha \notin w_2$

$w_2 \nsubseteq w_1$ $\exists\, \beta \in w_2$ $\beta \notin w_1$

$\alpha \in w_1 \Rightarrow \alpha \in w_1 \cup w_2$

$\beta \in w_2 \Rightarrow \beta \in w_1 \cup w_2$

Since $w_1 \cup w_2$ is a subspace

$\alpha + \beta \in w_1 \cup w_2$ (closure)

$\alpha + \beta \in w_1$ or $\alpha + \beta \in w_2$

if $(\alpha + \beta) \in w_1$

also $\alpha \in w_1$ then $-\alpha \in w_1$

$\alpha + \beta - \alpha \in w_1$

$\beta \in w_1$

which contradicts $\beta \notin w_1$

LINEARLY DEPENDENT – Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ of vectors of V is s.t.b linearly dependent if $\exists$ scalar $a_1, a_2, \ldots a_n \in F$ not all of them 0

$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \ldots + a_n\alpha_n = 0$

LINEARLY INDEPENDENT – Let $V(F)$ be a vector space. A finite set $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ of vectors of V is s.t.b linearly independent if $\exists$

$a_1\alpha_1 + a_2\alpha_2 + \ldots a_n\alpha_n = 0$

$\forall$ ~~a∈~~ $a = 0$

Eg find whether the set of vectors are dependent or independent $v_1 = (1,2,1)$ $v_2 = (3,1,5)$ $v_3 = (3,-4,7)$

Let a_1, a_2, a_3 be 3 scalars such that

$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

$a_1(1,2,1) + a_2(3,1,5) + a_3(3,-4,7) = 0$

$(a_1 + 3a_2 + 3a_3,\ 2a_1 + a_2 - 4a_3,\ a_1 + 5a_2 + 7a_3) = 0$

$a_1 + 3a_2 + 3a_3 = 0$

$2a_1 + a_2 - 4a_3 = 0$

$a_1 + 5a_2 + 7a_3 = 0$

$-2a_2 - 4a_3 = 0$

$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$

$|A| = 0$

Linearly dependent if $|A| = 0$

Rank = 2

If no. of vectors = Rank then linearly independent

No. of unknowns

SYSTEM OF .....

BASIS - Any subset $V(F)$ of a vector space $V(F)$ is called a basis of $V(F)$ if
S is linearly independent
S generates V i.e $V = \langle S \rangle$ {S is span of V}
each vector can be expressed as a linear combinⁿ of basis vector

Does $(1,0,0)$ $(1,1,0)$ $(1,1,1)$ forms a basis of R^3?

$$\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c \\ c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a+b+c \\ 0+b+c \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c=0$ $b=0$ $a=0$

LI

DIMENSION → The no. of elements in finite basis of a vector space $V(F)$ is called the dimension of vector space. It is denoted by $\dim(V) = n$, then V is called n-DIMENSIONAL VECTOR SPACE.

FINITE DIMENSIONAL VECTOR SPACE - If a vector space has a finite basis

INFINITE DIMENSIONAL VECTOR SPACE - If a vector has ∞ basis

Eg $C(R)$ is a vector space of dimension 2 $\{1, i\}$
Dimension of $F(F)$ i.e $\{R\}$, $R(R)$ is 1 $\{1\}$.
Dimension of $F^n(F)$ is n. std basis
$e_1 = (1,0,0\ldots)$ $e_2 = (0,1,0\ldots)$
$e_3 = (0,0,1,0\ldots)$
Dimension of $A_{mn}(R) = mn$

LINEAR TRANSFORMATION - Let $U(F)$ and $V(F)$ be two vector spaces over field F. Then mapping $U(F) \to V(F)$ is called linear transformⁿ or homomorphism from U to V if

$$f(\alpha+\beta) = f(\alpha) + f(\beta)$$
$$f(a\alpha) = af(\alpha) \quad \forall a \in F \; \alpha, \beta \in U$$

KERNEL OF HOMOMORPHISM - Let f be a homomorphism of a vector space $U(F)$ into vector space $V(F)$. Then the set K of all the elements of U which are mapped into zero element of V is called kernel of homomorphism

$$K = \{\alpha \in U : f(\alpha) = 0'\}, \text{ where } 0' \text{ is the zero vector of } V$$

RANGE AND NULL SPACE OF LINEAR TRANSFORMATION
Let $U(F)$ and $V(F)$ be two vector spaces and let T be a transformⁿ from U to V. Then the image or range of T denoted by $R(T)$ is the set of all vectors $\beta \in V$ such that

$$T(\alpha) = \beta \text{ for some } \alpha \in U$$

$$\text{RANGE} = R(T) = \{\beta \in V \mid f(\alpha) = \beta, \text{ for some } \alpha \in U\}$$

$$\dim(\text{RANGE}) = \text{RANK} = \rho(T)$$

Null space of T is denoted as $N(T)$ is a set of all vectors $\alpha \in U$ s.t $T(\alpha) = 0$

$$\text{NULL SPACE} = N(T) = \{\alpha \in U : T(\alpha) = 0 \quad 0 \in V\}$$

$$\dim(\text{NULL SPACE}) = \text{Nullity} = \nu(T)$$

SYSTEM OF LINEAR EQUATIONS

$\rho[A] \neq \rho[A \vdots B]$ Inconsistent No soln

$\rho[A] = \rho[A \vdots B] = $ No. of unknown Unique soln

$\rho[A] = \rho[A \vdots B] < $ No. of unknown ∞ soln

$\left.\begin{matrix}\text{Unique soln} \\ \infty \text{ soln}\end{matrix}\right\}$ consistent