→ A two dimensional random variable (X, Y) is said to be cts if there exist a non-negative function $f: \mathbb{R}^2 \to \mathbb{R}$ such that for every $(x, y) \in \mathbb{R}^2$ we have,

$$F(x,y) = P(X \le x, Y \le y)$$

Where F is the distribution function of (X, Y).
and The function f is called joint density function, if $f(x,y) \geq 0$ and

$$\int_x \int_y f(x,y)\, dy\, dx = 1.$$

Marginal density functions:→ For two random variables X and Y.

- the marginal function of X, denoted by $f(x)$, is given as

$$f(x) = \int_y f(x,y)\, dy$$

- the marginal function of Y, denoted by $f(y)$ is given as

$$f(y) = \int_x f(x,y)\, dx.$$

Conditional density function:→ The conditional density function for X given Y, denoted by $f(X/Y)$ or $f_{X/Y}$ is defined as

$$f(X/Y) = \frac{f(x,y)}{f(y)} \begin{array}{l} \text{— Joint density fun} \\ \text{— Marginal of } Y. \end{array}$$

The conditional density function for Y given X, denoted by $f(Y/X)$ or $f_{Y/X}$ is defined as

$$f(Y|X) = \frac{f(x,y)}{f(x)} \quad \begin{array}{l} \text{— Joint density function} \\ \text{— Marginal of X.} \end{array}$$

→ Two random variables X and Y are said to be Independent if $f(x,y) = f(x) \cdot f(y)$ for all $(x,y) \in \mathbb{R}^2$.

Que The joint density function of random var. X and Y is given by

$$f(x,y) = \begin{cases} kxy \; ; & 0<x<1; \; 0<y<1 \\ 0 \; ; & o/w. \end{cases}$$

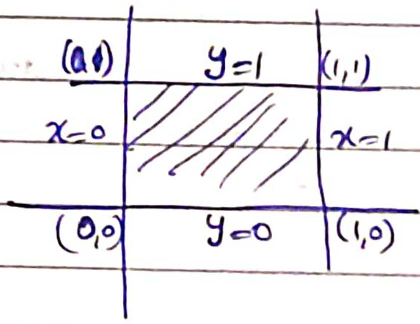
Find (1) K

(2) Marginal density function of X and Y.

(3) Check whether X and Y are Independent or not?

(4) $P(X+Y \le 1)$.

Sol^n (1) $\int_0^1 \int_0^1 f(x,y)\, dx\, dy = 1$



$\Rightarrow \int_0^1 \int_0^1 Kxy\, dx\, dy = 1$

$\Rightarrow \int_0^1 \left(\frac{Kx^2}{2} y\right)_0^1 dy = 1$

$\Rightarrow \frac{K}{2}\left(\frac{y^2}{2}\right)_0^1 = 1 \Rightarrow \frac{K}{4} = 1 \Rightarrow \boxed{K=4.}$

(2) $f(x) = \int_y f(x,y)\, dy = 4\int_0^1 xy\, dy$

$= 4\left(\frac{xy^2}{2}\right)_0^1 = \frac{4x}{2} = 2x \; ; 0<x<1$

$$f(y) = \int_0^1 f(x,y)\,dx = \int_0^1 kxy\,dx$$

$$= \left(\frac{k}{2}x^2 y\right)_0^1 = \frac{k}{2}y$$

$$= 2y;\ 0<y<1.$$

(3) So $f(x,y) = \begin{cases} 4xy; & 0<x<1;\ 0<y<1 \\ 0; & o/w \end{cases}$

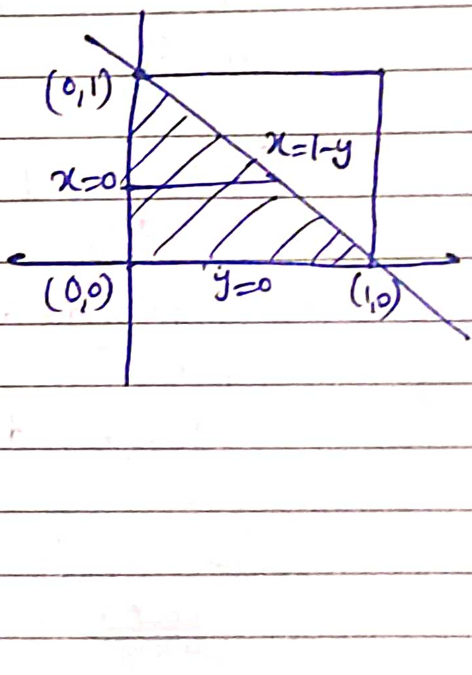
$f(x) = 2x;\ 0<x<1$

$f(y) = 2y;\ 0<y<1$

Since $f(x,y) = f(x)\cdot f(y)$

$\Rightarrow$ X and Y are Independent.

(4) $P(X+Y \leq 1)$



$$= \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$$

$$= \int_0^1 4x\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 2x\cdot(1-x)^2\,dx$$

$$= 2\int_0^1 (x+x^3-2x^2)\,dx$$

$$= 2\left(\frac{x^2}{2}+\frac{x^4}{4}-\frac{2x^3}{3}\right)_0^1 = 2\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)$$

$$= 2\left(\frac{6+3-8}{12}\right) = \frac{1}{6}$$

H.W

Que The joint density function of X and Y is given by

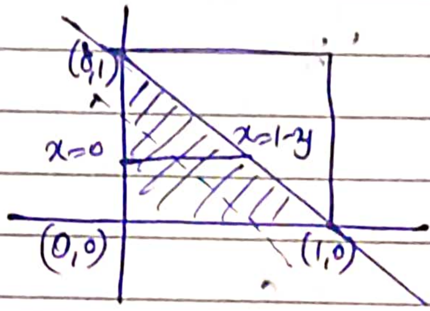
$$f(x,y) = \begin{cases} kxy; & 0<x<1;\ 0<y<1;\ x+y\le 1 \\ 0; & o/w \end{cases}$$

find (1) k

(2) marginal density fun. of X and Y

(3) $P(X+Y \le 1/2)$

(4) check about Independence of X and Y.



(1) $\int_y \int_x f(x,y)\,dx\,dy = 1$

$$\int_0^1 \int_0^{1-y} kxy\,dx\,dy = 1$$

$$\boxed{k = 24}$$

(2) $f(x) = \int_y f(x,y)\,dy = \int_0^{1-x} 24xy\,dy$

$$= 24x\left(\frac{y^2}{2}\right)_0^{1-x}$$

$$= 12x(1-x)^2;\ 0<x<1$$

Similarly $f(y) = \int_0^{1-y} f(x,y)\,dx = 12y(1-y)^2;\ 0<y<1$

(3) $\because f(x,y) \ne f(x)\cdot f(y) \Rightarrow X$ & Y are not Independent.

(4) $P(X+Y \le \frac{1}{2}) = \int_0^{1/2}\int_0^{1/2-x} 24xy\,dy\,dx = \frac{1}{16}$ (Solve it)

Que $f(x,y) = \begin{cases} \alpha & ; 0<x<1; \ 0<y<x \\ 0 & ; \text{o/w} \end{cases}$

find (1) α

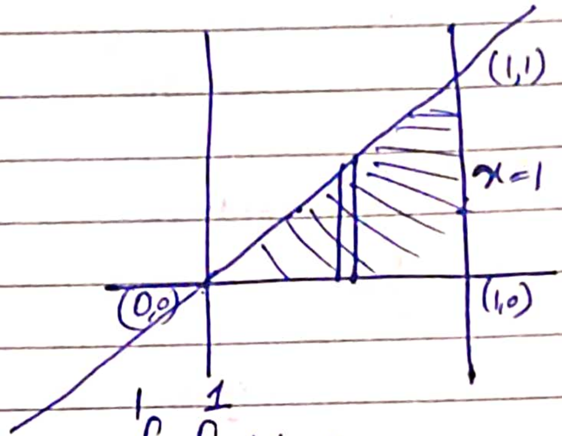
(2) Marginal density fun of X and Y

(3) X and Y are Independent or not?

Sol^n (1) $\int_y \int_x f(x,y)\,dx\,dy = 1$

or

$\int_x \int_y f(x,y)\,dy\,dx = 1$



$\int_0^1 \int_0^x \alpha\,dy\,dx = 1$ or $\int_0^1 \int_y^1 \alpha\,dx\,dy = 1$

$\int_0^1 (\alpha x)\,dx = 1$ $\qquad$ $\int_0^1 \alpha(1-y)\,dy = 1$

$\frac{\alpha}{2} = 1 \Rightarrow \boxed{\alpha = 2}$ $\qquad$ $\Rightarrow \left[\frac{\alpha(1-y)^2}{-2}\right]_0^1 = 1$

$\Rightarrow \frac{\alpha}{2} = 1 \Rightarrow \boxed{\alpha = 2}$

(2) $f(x) = \int_0^x f(x,y)\,dy = 2x; \ 0<x<1$

$f(y) = \int_y^1 f(x,y)\,dx = 2(1-y); \ 0<y<1$

(3) $\because f(x,y) \neq f(x) \cdot f(y)$

$\Rightarrow$ X and Y are not Independent.

Conditional $dist^n$ of Y given X=x is

$$f(y/x=x) = \frac{f(y,x)}{f(x)}$$

$$= \frac{\alpha}{\alpha x} = \frac{2}{2x} = \frac{1}{x};\ 0<x<1,\ 0<y<x$$

$$f(x/y=y) = \frac{f(x,y)}{f(y)}$$

$$= \frac{2}{2(1-y)} = \frac{1}{1-y};\quad 0<y<x<1.$$

Que $f(x,y) = \begin{cases} e^{-x-y}; & x\geq 0,\ y\geq 0 \\ 0; & o/w. \end{cases}$

Find $P(X>1)$; $E(X)$; $E(Y)$, $E(XY)$, $E(X+Y)$.

Check If X and Y are Independent or not?

Sol^n

$$f(x) = \int_0^\infty f(x,y)\,dy$$

$$= \int_0^\infty e^{-x} \cdot e^{-y}\,dy$$

$$= -\left(e^{-x-y}\right)_0^\infty = e^{-x};\ x\geq 0.$$

|||ly $f(y) = \int_0^\infty e^{-x-y}\,dx = e^{-y};\ y\geq 0.$

$\because\ f(x,y) = f(x) \cdot f(y)$

$\Rightarrow$ X and Y are Independent.

$$E(X) = \int_0^\infty x \cdot f(x)\,dx$$

$$= \int_0^\infty x\,e^{-x}\,dx = 1.$$

$$E(Y) = \int_0^\infty y \cdot e^{-y}\, dy = 1$$

$$P(X>1) = \int_1^\infty f(x)\,dx = \int_1^\infty e^{-x}\,dx$$

$$= 1/e$$

$$E(XY) = \int_0^\infty \int_0^\infty xy\, f(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^\infty xy\, e^{-x-y}\,dx\,dy$$

$$= \int_0^\infty x e^{-x} \left(\int_0^\infty y e^{-y}\,dy \right) dx$$

$$= 1$$

Or $E(XY) = E(X) \cdot E(Y)$ ($\because$ X and Y are Independent)

$$E(X+Y) = \int_0^\infty \int_0^\infty (X+Y)\, f(x,y)\,dx\,dy$$

$$\int_0^\infty \int_0^\infty (x+y) e^{-x-y}\,dx\,dy = 2$$

Or

$$E(X+Y) = E(X) + E(Y)$$

$$= 1+1 = 2$$

Transformation of two dimensional random variables

Let $U = u(x,y)$ and $V = v(x,y)$ for given random variables X and Y.

If $f(x,y)$ is the joint density function of X and Y then joint density function of U and V is given by

$$g(u,v) = f(x,y)\,|J|$$

where

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$ is called jacobian

→ For one dim r.v. X, If $Y = f(x)$ then p.d.f of y is given by

$$g(y) = f(x)\left|\frac{dx}{dy}\right|$$

Methodology $u = u(x,y)$; $v = v(x,y)$ (given)

(1) Find X and Y in terms of u and v and find J

(2) Compute $g(u,v) = f(x,y)\,|J|$

(3) Domain of u and v from given domains of X and Y.

Ex Let $f(x,y) = \begin{cases} \frac{1}{\alpha^2} e^{-(x+y)/\alpha} & ; x,y>0;\ \alpha>0 \\ 0 & ; o/w \end{cases}$

Find the distⁿ of : $\frac{1}{2}(X-Y)$

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<u>Sol^n</u> Let $U = \frac{1}{2}(x-y)$ and V can be choosen either x or y or $x+y$ etc.

Let $V = y$

So $x = 2u + v$ and
$y = v$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$$

So $g(u,v) = f(x,y) \cdot |J|$
$= \frac{2}{\alpha^2} e^{-(x+y)/\alpha} = \frac{2}{\alpha^2} e^{-(2u+2v)/\alpha}$

<u>Domains</u> $x, y > 0$
$x > 0 \Rightarrow 2u + v > 0 \Rightarrow v > -2u$
$y > 0 \Rightarrow v > 0$

So $$g(u,v) = \begin{cases} 2/\alpha^2 \, e^{-(2u+2v)/\alpha} ; & -\infty < u < \infty \\ & v > 0 ; v > -2u \\ 0 ; & o/w. \end{cases}$$

<u>Ex</u> $f(x,y) = 2e^{-(x+y)}$; $0 < x < y < \infty$
$U = 2x_0$; ~~$V = X_2$~~ x_1. $V = Y - X$,
Find the joint density function of U and V.

<u>Sol^n</u> $g(u,v) = f(x,y) \, |J|$

$U = 2x$ or $x = \frac{U}{2}$

$V = Y - x \Rightarrow Y = V + x = V + \frac{U}{2}$

So $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} 1/2 & 1/2 \\ 0 & 1 \end{vmatrix} = 1/2$

$\therefore g(u,v) = \cancel{2} e^{-\left(\frac{u}{2} + v + \frac{u}{2}\right)} \cdot \frac{1}{\cancel{2}}$

$= e^{-(u+v)}$

when $0 < x < y < \infty$

$\Rightarrow \underbrace{0 < \frac{u}{2}}_{u>0} \underbrace{< v + \frac{u}{2}}_{v>0} \underbrace{< \infty}_{v<\infty}$

$\Rightarrow u > 0;\ 0 < v < \infty$

So $g(u,v) = \begin{cases} e^{-(u+v)} & ; u>0; 0<v<\infty \\ 0 & ; o/w \end{cases}$

Ex $f(x,y) = 8xy;\ 0 < x < y \leq 1$

$u = \frac{x}{y};\ v = y$. Find $g(u,v)$?

Solⁿ $v = y;\ u = \frac{x}{y} \Rightarrow x = uv$

$J = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$

$g(u,v) = f(x,y) \cdot |J|$

$= 8(uv) \cdot v \cdot v$

$= 8uv^3$

when $0 < x < y < 1$

$\Rightarrow \underbrace{0 < uv}\underbrace{< v} < 1$

$uv > 0 \quad \Rightarrow u > 0,\ v > 0$

$v(1-u) > 0 \quad \Rightarrow v > 0,\ 1-u > 0$ or $u < 1$

$v < 1 \quad \Rightarrow v < 1$

So $0 < u < 1$; $0 < v < 1$

$$\therefore g(u,v) = \begin{cases} 8uv^3; & 0<u<1;\ 0<v<1 \\ 0; & o/w \end{cases}$$

Que The pdf of X and Y is

$f(x,y) = \frac{1}{2} x e^{-y};\quad 0 < x < 2;\ y > 0$

Find the distn of ~~X~~ X+Y.

Sol^n let $U = X+Y$ and $V = X$

so $y = u - v$; and $x = v$

$$J = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

$g(u,v) = f(x,y)\,|J|$

$= \frac{1}{2} v e^{v-u} (1)$

$= \frac{1}{2} v e^{-(u-v)}$

$0 < x < 2 \Rightarrow 0 < v < 2$

$y > 0 \Rightarrow u - v > 0 \Rightarrow u > v$

So $g(u,v) = \begin{cases} \frac{1}{2} v e^{-(u-v)} & ; u > v;\ 0 < v < 2. \\ 0 & ; \text{o/w} \end{cases}$

H.W Que If $f(x,y) = e^{-x-y}$; $x, y > 0$

$u = x - y$; $v = x + y$.

Find the joint Pdf of u & v.

Ans $g(u,v) = \begin{cases} \frac{e^{-v}}{2} ; & v > 0;\ -v < u < v \\ 0 ; & \text{o/w}. \end{cases}$