Confidence Intervals

— Dr. Arti Singh

Definition

A $(1-\alpha)100\%$ confidence interval (C.I.) for an unknown parameter θ is the interval $[a,b]$ such that

$$P(a \leq \theta \leq b) = 1-\alpha .$$

- Eg 90% C.I. for $\theta = [a,b]$

  $$P(a \leq \theta \leq b) = 0.9$$

  $(1-\alpha)100\% = 90\%$

  so $1-\alpha = 0.9$

- It means that we are 90% confident that parameter θ lies in the interval $[a,b]$.

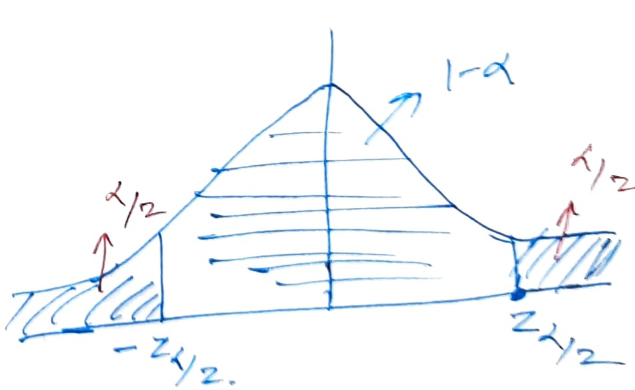
- **$(1-\alpha)100\%$ Confidence Interval (C.I.) for μ.** σ is known.

  $X \sim$ Normal

  let $X_1, X_2, \dots X_n$ be random sample.

  We know $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) = 1-\alpha$$



(Due to symmetry)

(Total area is 1.)

So,

$$P\left(-z_{\alpha/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1-\alpha$$

$\downarrow$ solving for μ

$$P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} < \mu < \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right) = 1-\alpha$$

So $(1-\alpha)100\%$ C.I. for μ,

$$= \left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \,,\; \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right]$$

$$\left[\begin{array}{l} \dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2} \\ \bar{X}-\mu < \dfrac{\sigma}{\sqrt{n}} z_{\alpha/2} \\ \bar{X} - \dfrac{\sigma}{\sqrt{n}} z_{\alpha/2} < \mu \end{array}\right]$$

Eg Example 8.1 (Douglas & George - Text Book).

- Large sample C.I. for μ

  let X has any general distribution. (Not necessarily Normal)

  σ is known

  n is sufficiently large. $(n \geq 30)$

So $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ has approximate Z. (By CLT)

$(1-\alpha)100\%$ C.I. for μ $= \left[\bar{X} - \dfrac{\sigma}{\sqrt{n}} z_{\alpha/2} \,,\; \bar{X} + \dfrac{\sigma}{\sqrt{n}} z_{\alpha/2}\right]$

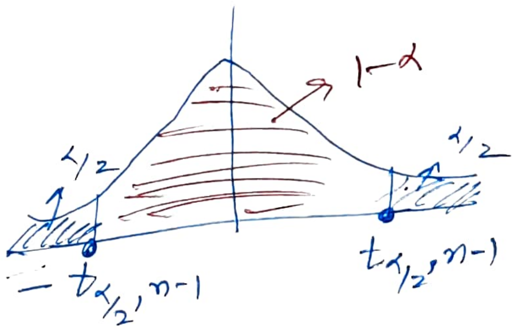
Same as above.

$(1-\alpha)100\%$ C.I. for μ → X → Normal. → σ^2 UNKNOWN

we know $\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$

By symmetry

$$P\left(-t_{\frac{\alpha}{2},n-1} \le t(n-1) \le t_{\alpha/2,n-1}\right) = 1-\alpha$$



So

$$P\left(-t_{\alpha/2,n-1} \le \frac{\bar{X}-\mu}{S/\sqrt{n}} \le t_{\alpha/2,n-1}\right) = 1-\alpha$$

Solving above inequalities for μ, we get

$$P\left(\bar{X} - \frac{S}{\sqrt{n}} t_{\alpha/2,n-1} \le \mu \le \bar{X} + \frac{S}{\sqrt{n}} t_{\alpha/2,n-1}\right) = 1-\alpha$$

Here

$$C.I. = \left[\bar{X} - \frac{S}{\sqrt{n}} t_{\alpha/2,n-1}, \; \bar{X} + \frac{S}{\sqrt{n}} t_{\alpha/2,n-1}\right]$$

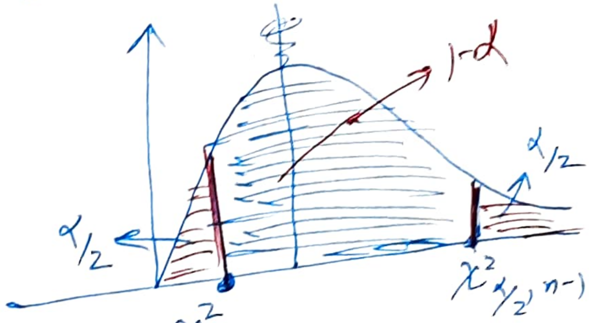
Eg Example 8.6 (Douglas & George - Text book)

• Note: For $n-1 \geq 30$ $t_{\alpha/2,n-1} \equiv Z_{\alpha/2}$

$(1-\alpha)100\%$ C.I. for σ^2 [for both cases μ known or unknown.]

we know $\dfrac{(n-1)s^2}{\sigma^2} \equiv \chi^2(n-1)$

$$P\left(\chi^2_{1-\frac{\alpha}{2},n-1} \le \chi^2(n-1) \le \chi^2_{\frac{\alpha}{2},n-1}\right) = 1-\alpha$$



(Area on right of this point is $1-\frac{\alpha}{2}$.)

$$P\left(\chi^2_{1-\frac{\alpha}{2},n-1} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{\frac{\alpha}{2},n-1}\right) = 1-\alpha.$$

Solving for σ^2, we get

$$P\left(\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right) = 1-\alpha$$

So C.I. $= \left[\dfrac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2},n-1}}, \dfrac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2},n-1}}\right]$

Eg: Example 8.7 (Douglas & George, Text book)