→ Divergence of a vector field :→

Let $f = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$. be a vector field or vector valued function.

Then $\text{div } f = \nabla \cdot f$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k})$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Que Find the divergence of $V = (x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}$

Solⁿ:→ $\text{div } V = \frac{\partial}{\partial x}(x^2y^2 - z^3) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}(e^{xyz})$

$$= 2xy^2 + 2xz + e^{xyz} \cdot xy.$$

Que P.T. $\text{div}(fv) = f(\text{div } v) + (\text{grad } f) \cdot v$, where f is a scalar function.

where $V = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$

Solⁿ:→ $\text{div}(fv) = \nabla \cdot (fv)$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (fv_1 \hat{i} + fv_2 \hat{j} + fv_3 \hat{k})$$

$$= \frac{\partial}{\partial x}(fv_1) + \frac{\partial}{\partial y}(fv_2) + \frac{\partial}{\partial z}(fv_3)$$

$$= \left(f \cdot \frac{\partial v_1}{\partial x} + v_1 \cdot \frac{\partial f}{\partial x}\right) + \left(f \cdot \frac{\partial v_2}{\partial y} + v_2 \cdot \frac{\partial f}{\partial y}\right) + \left(f \cdot \frac{\partial v_3}{\partial z} + v_3 \cdot \frac{\partial f}{\partial z}\right)$$

$$= f\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) + \left(v_1 \cdot \frac{\partial f}{\partial x} + v_2 \cdot \frac{\partial f}{\partial y} + v_3 \cdot \frac{\partial f}{\partial z}\right)$$

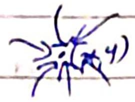
$$= f(\text{div } v) + (\text{grad } f) \cdot v$$

→ div (v) > 0 → Source

div (v) < 0 → Sink

div (v) = 0 → v is called Solenoidal vector.

 diverge.

 Converge

Curl of vector field :→ Let $V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\text{Curl } v = \nabla \times V = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Que $V = (x^2y^2 - z^3)\hat{i} + 2xyz\hat{j} + e^{xyz}\hat{k}$

Find Curl v

Solⁿ

$$\text{Curl } v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2 - z^3 & 2xyz & e^{xyz} \end{vmatrix}$$

$$= i(xze^{xyz} - 2xy) - \hat{j}(yze^{xyz} + 3z^2) + \hat{k}(2yz - 2x^2y)$$

→ Curl of divergence i.e. Curl(div f) = $\nabla \times \nabla \cdot f$ is not defined as $\nabla \cdot f$ is scalar valued function, But Curl is evaluated for vector valued function.

→ Curl (grad f) = 0 or $\nabla \times \nabla f = 0$.

$$\nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \hat{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial y \partial z}\right) - \hat{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial x \partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial x \partial y}\right) = 0.$$

H.W Div (Curl v) = $\nabla \cdot (\nabla \times V) = 0$

~~Grad (div v) = $\nabla(\nabla \cdot \vec{V})$ = $\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2}$ where $v = V_1\hat{i} + V_2\hat{j}$.~~

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Que $\vec{F} = 2x^2\hat{i} + y^2\hat{j} + \alpha z\hat{k}$

at $(1,2,1)$ $\vec{F}$ is solenoidal vector. Find α.

Solⁿ $\text{div}\vec{F} = 4x + 2y + \alpha$

at $(1,2,1)$

$\text{div}\vec{F} = 0$

$\Rightarrow 4 + 4 + \alpha = 0 \Rightarrow \alpha = -8$

$\rightarrow$ $\text{div}(\text{grad} f) = \nabla^2 f$ where f = Scalar valued function.

Pf $\text{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$$\text{div}(\text{grad} f) = \nabla\cdot(\nabla f) = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

$$= \nabla^2 f. \quad (\nabla^2 - \text{Laplacian operator})$$

Que Compute div(v), Curl(v) and verify that div (Curl v) = 0.

(1) $V = x\hat{i} + 2y\hat{j} + z\hat{k}$

Ans 4, 0 vector.

(2) $V = xy\hat{i} + yz\hat{j} + zx\hat{k}$

Ans:- $x+y+z$; $-(iy + jz + kx)$

(3) $V = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

Ans:- $2(x+y+z)$; 0 vector.

(4) Show that $V = (2x+3y)\hat{i} + (x-y)\hat{j} - (x+y+z)\hat{k}$.

is Solenoidal vector.

(5) If $V = -(x+y+2)\hat{i} - 2\hat{j} + (x+y)\hat{k}$. Show that $V\cdot \text{Curl}\, V = 0$.

→ If Curl(f) = 0 Then f is called Conservative vector field.

Line Integral:→ Let f be a vector valued function. Let C be a simple smooth curve and (x,y,z) be any point on the curve. $r = x\hat{i} + y\hat{j} + z\hat{k}$

Then $$\int_C f \cdot dr = \int_C (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

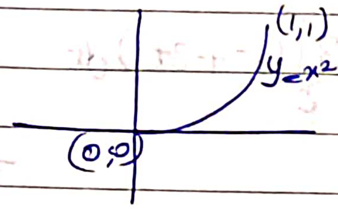
$$= \int_C f_1 dx + f_2 dy + f_3 dz$$

defines a line Integral of f over C.

Que Find the line Integral of $F(x,y) = x^2\hat{i} + y\hat{j}$ along the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

Sol^n $$\int_C F\cdot dr = \int_C (x^2\hat{i} + y\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= \int_C (x^2 dx + y\,dy)$$



$y = x^2$

$\Rightarrow dy = 2x\,dx$

$$\Rightarrow \int_C F\cdot dr = \int_0^1 x^2 dx + x^2(2x)\,dx$$

$$= \int_0^1 (x^2 + 2x^3)\,dx$$

$$= \left(\frac{x^3}{3} + \frac{x^4}{2}\right)_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

Que Find the line Integral of $F = xy\hat{i} + y^2\hat{j} + e^z\hat{k}$ over the curve C whose parametric representation is given by $x = t^2,\ y = 2t,\ z = t;\ 0 \le t \le 1$.

Sol^n $$\int_C F\cdot dr = \int_C xy\,dx + y^2 dy + e^z dz$$

$$= \int_0^1 2t^3(2t\,dt) + 4t^2(2\,dt) + e^t dt$$

$$= \int_0^1 (4t^4 + 8t^2 + e^t)\,dt$$

$$= \left(\frac{4t^5}{5} + \frac{8}{3}t^3 + e^t\right)_0^1 = \frac{4}{5} + \frac{8}{3} + e - 1$$

$$= \frac{37}{15} + e$$

Que Find $\int_C (x^2 + yz)\,dz$ where C is given by $x = t$, $y = t^2$, $z = 3t$;

$1 \le t \le 2$.

Sol^n:→

$x = t \Rightarrow dx = dt$

$y = t^2 \Rightarrow dy = 2t\,dt$

$z = 3t \Rightarrow dz = 3\,dt$

$$3\int_C (t^2 + 3t^3)\,dt = 3\int_1^2 (t^2 + 3t^3)\,dt$$

$$= 3\left[\frac{t^3}{3} + \frac{3t^4}{4}\right]_1^2$$

$$= 3\left[\frac{8}{3} + \frac{48}{4} - \frac{1}{3} - \frac{3}{4}\right]$$

$$= 3\left[\frac{7}{3} + \frac{45}{4}\right] = \frac{163}{4}.$$

Que $\int_C (x+y)\,dx - x^2\,dy + (y+z)\,dz$

where C is $x^2 = 4y$, $z = x$, $0 \le x \le 2$

Ans $\frac{10}{3}$

Que Evaluate the line Integral $\int_C F\cdot dr$

(i) $F = xi + (\sin y)j + k$; C is given by $x = t^2$, $y = t$, $z = 2t$;
$0 \le t \le 1$

Ans $\frac{7}{2} - \cos(1)$

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(2) $F = x^2y\hat{i} - xy^2\hat{j}$; $r(t) = t\hat{i} + t^2\hat{j}$; $0 \le t \le 3$.

Ans:- $-20169/35$

(3) $F = e^x\hat{i} + xe^{xy}\hat{j} + \hat{k}$; $r(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$; $0 \le t \le 2$.

Ans:- $(2e^8 + 3e^2 + 19)/3$.

Line Integrals Independent of path:→ When a differential equation $f_1dx + f_2dy + f_3dz$ is an Exact differential Equation then line Integral $\int_C F\cdot dr$ is path Independent. and Conversely.

Que $F = x\hat{i} + y\hat{j}$. Find $\int F\cdot dr$ along $y = x^2$ from $(0,0)$ to $(1,1)$.

Solⁿ→ $\int F\cdot dr = \int (x\hat{i} + y\hat{j})\cdot(dx\hat{i} + dy\hat{j})$

$= \int x\,dx + y\,dy$

$x\,dx + y\,dy = 0$ → is Exact differential Equⁿ

$x\,dx + y\,dy = d\left(\frac{x^2+y^2}{2}\right)$

$\therefore \int F\cdot dr = \int_{(0,0)}^{(1,1)} d\left(\frac{x^2+y^2}{2}\right) = \left(\frac{x^2+y^2}{2}\right)_{(0,0)}^{(1,1)}$

$= \frac{2}{2} - 0 = 1$.

Que:→ $\int_P^Q 2xy^2dx + (2x^2y+1)\,dy$; $P:(-1,2)$; $Q:(2,3)$

Solⁿ→ $M = 2xy^2$, $N = 2x^2y+1$

$\frac{\partial M}{\partial y} = 4xy$; $\frac{\partial N}{\partial x} = 4xy$

$\int M\,dx + \int (\text{terms in N not Cont } x)\,dy = C$

$\Rightarrow \int 2xy^2\,dx + \int 1\,dy = C$

$\Rightarrow x^2y^2 + y = C$ $\Rightarrow f(x,y) = x^2y^2 + y$

$$\int_P^Q d(x^2y^2+y) = (x^2y^2+y)_P^Q$$

$$= (x^2y^2+y)_{(-1,2)}^{(2,3)} = (36+3)-(4+2)$$

$$= 33.$$

Que $\int_P^Q (1-\sin x \sin y)\,dx + \int (1+\cos x \cos y)\,dy$; $P:\left(\frac{\pi}{4},\frac{\pi}{4}\right)$; $Q:\left(\frac{\pi}{2},0\right)$

Ans:- $-1/2$

Que $\int_P^Q (y^3+2xy^2)\,dx + (3xy^2+2x^2y+1)\,dy$; $P:(-2,1)$; $Q:(1,2)$

Ans: 11.

Que $\int_P^Q (2xz+y)\,dx + (x+z)\,dy + (x^2+y)\,dz$; $P:(-1,2,3)$; $Q:(2,2,4)$