Singular Value Decomposition

Theorem. Singular Value Theorem for Linear Transformation →

Let V and W be finite-dimensional inner product spaces, and let $T: V \to W$ be a L.T of rank r. Then there exists orthonormal bases $\{v_1, v_2, \dots, v_n\}$ for V and $\{u_1, u_2, \dots, u_m\}$ for W and positive scalars $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$ such that

$$T(v_i) = \begin{cases} \sigma_i u_i & \text{if } 1 \leq i \leq r \\ 0 & \text{if } i > r \end{cases}$$

Conversely, suppose that the preceding conditions are satisfied. Then for $1 \leq i \leq n$, v_i is an eigen vector of T^*T with corresponding eigenvalue σ_i^2 if $1 \leq i \leq r$ and 0 if $i > r$. Therefore the scalars $\sigma_1, \sigma_2, \dots, \sigma_r$ are uniquely determined by T.

Singular Values of Transformation

Defn → The unique scalars $\sigma_1, \sigma_2, \dots, \sigma_r$ are called the singular values of T. If r is less than both m and n, then the term singular value is extended to include $\sigma_{r+1} = \dots = \sigma_k = 0$, where k is the minimum of m and n.

Singular Values of matrix

Defn → Let A be a $m \times n$ matrix. We define the singular values of A to be the singular values of the linear transformation L_A.

Singular Value Decomposition Theorem for Matrices :

Let A be an $m \times n$ matrix of rank r with the positive singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$, and let Σ be the $m \times n$ matrix defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists an $m \times m$ unitary matrix U and an $n \times n$ unitary matrix V such that $A = U \Sigma V^*$.

Singular Value Decomposition of A

Def^n → Let A be an $m \times n$ matrix of rank r with positive singular value $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$. A factorization $A = U \Sigma V^*$ where U and V are unitary matrices and Σ is the $m \times n$ matrix defined by

$$\Sigma_{ij} = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

is called a singular value decomposition of A.

Let A be an $m \times n$ matrix. Then $A = U \Sigma V^T$ is the singular value decomposition of A.

- U is $m \times m$ orthogonal matrix with columns equal to the unit eigenvectors of AA^T.

$$U = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{u_1} & \vec{u_2} & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

- V is an $n \times n$ orthogonal matrix whose columns are unit eigenvectors $A^T A$;

$$\cancel{A^T A}\ V = \begin{bmatrix} \vdots & \vdots & \vdots \\ \vec{v_1} & \vec{v_2} & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

- Σ is an $m \times n$ matrix with the singular values of A on the main diagonal and all other entries of zero.

$$\Sigma = \begin{bmatrix} \vec{\sigma_1} & 0 & 0 \\ 0 & \vec{\sigma_2} & \vdots \\ 0 & 0 & \vdots \end{bmatrix}$$

Eigen values → $|A - \lambda I| = 0$

Eigen vectors → $[A - \lambda I] X = 0$

$$A_{2\times 3} = U_{2\times 2}\, \Sigma_{2\times 3}\, V^T_{3\times 3}$$

Singular values of a matrix →

Let A be $m \times n$ matrix

- The singular values of A are the square roots of the positive eigen-values of $A^T A$ or $A A^T$.
- $A^T A$ and $A A^T$ have the same positive eigenvalues.

steps →

1. Determine V and then V^T
2. Determine the singular values σ_i and then Σ
3. Determine U using $A = U\Sigma V^T \rightarrow AV = U\Sigma$ since V is orthogonal to V^T, we know $VV^T = I$.

Q: $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ find the singular values of A.

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$$A^TA = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

For eigen values. $|A - \lambda I| = 0$

$$\begin{vmatrix} 10-\lambda & 2 & 6 \\ 2 & 2-\lambda & -2 \\ 6 & -2 & 10-\lambda \end{vmatrix} = 0$$

ch. eqn. is $\lambda(\lambda-16)(\lambda-6) = 0 \Rightarrow \lambda_1 = 16, \lambda_2 = 6, \lambda_3 = 0$
(order from greatest to least)

Corr. eigen vectors for $\lambda_1 = 16$
$[A - \lambda I]X = 0$

$$\begin{bmatrix} 10-16 & 2 & 6 \\ 2 & 2-16 & -2 \\ 6 & -2 & 10-16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -6 & 2 & 6 \\ 2 & -14 & -2 \\ 6 & -2 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - x_3 = 0 \\ x_2 = 0 \\ x_3 = x_3 \end{matrix} \quad \begin{matrix} x_1 = t \\ x_2 = 0 \\ x_3 = t \end{matrix} \quad X_1 = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$t \neq 0$

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(normalizing the vector X_1)

For $\lambda_2 = 6$, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 10-6 & 2 & 6 \\ 2 & 2-6 & -2 \\ 6 & -2 & 10-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 2 & 6 \\ 2 & -4 & -2 \\ 6 & -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 = x_3 \end{matrix} \quad \begin{matrix} x_1 = -t \\ x_2 = -t \\ x_3 = t \end{matrix} \quad X_2 = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$t \neq 0$

$$\vec{v}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

For $\lambda_3 = 0$, $[A - \lambda I]X = 0$

$$\begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_2 - 2x_3 = 0 \\ x_3 = x_3 \end{matrix} \quad \begin{matrix} x_1 = -t \\ x_2 = 2t \\ x_3 = t \end{matrix} \quad X_3 = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$\vec{v}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

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Find SVD for

Q $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$

$$\begin{array}{ccc} v_1 & v_2 & v_3 \end{array}$$

$\lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 0, \therefore \sigma_1 = \sqrt{6}, \Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$, $U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

$V = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

$$\therefore V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \text{ and } V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The singular values of AA^T in order from greatest to least are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{6}$$

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}$$

[Σ = singular values = $\sqrt{(\text{Eigenvalues})}$]

Now we need to find U.

$$A = U\Sigma V^T$$

$$AV = U\Sigma V^T V$$

$$AV = U\Sigma I$$

Since V and V^T are orthogonal $V^T V = I$

$$AV = U\Sigma$$

It follows: $A\vec{v}_1 = \sigma_1 \vec{u}_1$

$$A\vec{v}_2 = \sigma_2 \vec{u}_2$$

$$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{bmatrix}$$

$$Av_2 = \sigma_2 u_2 \rightarrow u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$\therefore$ ~~To check~~ $A = U\Sigma V^T$

$$\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}}_{V^T} = \underbrace{\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}}_{A}$$

Q. Find SVD for the given matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

Soln: $A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$

For eigenvalue $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 2 & -2 \\ 2 & 2-\lambda & -2 \\ -2 & -2 & 2-\lambda \end{vmatrix} = 0$$

ch. eqn is

$(2-\lambda)[(2-\lambda)^2 - 4] - 2[2(2-\lambda) - 4] - 2[-4 + 2(2-\lambda)] = 0$

$\Rightarrow (2-\lambda)[4 - 4\lambda + \lambda^2 - 4] - 2[4 - 2\lambda - 4] - 2[-4 + 4 - 2\lambda] = 0$

$\Rightarrow (2-\lambda)(\lambda^2 - 4\lambda) + 4\lambda + 4\lambda = 0$

$\Rightarrow 2\lambda^2 - \cancel{8\lambda} - \lambda^3 + 4\lambda^2 + \cancel{8\lambda} = 0$

$\Rightarrow -\lambda^3 + 6\lambda^2 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 = 0$

$\Rightarrow \lambda^2(\lambda - 6) = 0$

$\Rightarrow \lambda = 0, 0, 6$

$\therefore \lambda_1 = 6, \lambda_2 = 0, \lambda_3 = 0$

Corr. eigen vector for $\lambda_1 = 6$ is $[A - \lambda I]X = 0$

$\Rightarrow \begin{bmatrix} -4 & 2 & -2 \\ 2 & -4 & -2 \\ -2 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, normalizing the vector X_1 we get

$v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Eigenvector Corr. to $\lambda_2 = 0 = \lambda_3$

$\begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, normalizing the vector X_2

$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, normalizing the vector X_3, and $v_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\therefore V = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}, \quad V^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

The singular values of A^TA in order from greatest to least are :-
$\sigma_1 = \sqrt{6}$ is the only non-zero singular value of A.

$$\therefore \Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\because$ V and V^T are orthogonal $V^TV = I$.

$$AV = U\Sigma$$

It follows : $Av_1 = \sigma_1 u_1$
$Av_2 = \sigma_2 u_2$

$$Av_1 = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sigma_1} Av_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\therefore u_1 = \frac{1}{\sqrt{6\times 3}} \begin{bmatrix} 3 \\ 3 \end{bmatrix}_{2\times 1} = \frac{3}{3\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

next choose $u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, a unit vector orthogonal to u_1,

to obtain orthonormal basis $U = \{u_1, u_2\}$ for R^2 and set

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Then $A = U\Sigma V^T$ is the desired SVD.

$$\therefore \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

Pseudo inverse of A =

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}$$

The Pseudoinverse of a Matrix $\rightarrow$ (Generalization of Matrix inverse)

Let A be a $m \times n$ matrix. Then there exists a unique $n \times m$ matrix B such that $(L_A)^+ : F^m \to F^n$ is equal to the left multiplication transformation L_B. We call B the pseudo inverse of A and denote it by $B = A^+$. Thus $(L_A)^+ = L_{A^+}$.

- Th. Let A be an $m \times n$ matrix of rank r with a singular value decomposition $A = U \Sigma V^T$ and non-zero singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$. Let Σ^+ be the $n \times m$ matrix defined by

$$\Sigma^+_{ij} = \begin{cases} \frac{1}{\sigma_i} & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases}$$

Then $A^+ = V \Sigma^+ U^T$, and this is a SVD of A^+.

Note: Σ^+ is pseudo inverse of Σ.

~~SVD to Pseudo inverse~~

~~$AX = b$~~

~~$A = U\Sigma V^T \Rightarrow A^+ = V\Sigma^+ U^T$~~

Q. Find A^+ for the matrix ~~$A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$~~ $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Soln. $\Rightarrow$

$$A^T A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}$$

Eigen values $|A - \lambda I| = 0$ are, $\lambda_1 = 16$, $\lambda_2 = 6$, $\lambda_3 = 0$

Corr. Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

$$\therefore v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad v_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad v_3 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \quad V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Singular values of AA^T are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{6}$

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}, \quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = U \Sigma V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

we have

$$A^+ = V\Sigma^+U^T = \begin{bmatrix} \frac{1}{\sqrt2} & -\frac{1}{\sqrt3} & -\frac{1}{\sqrt6} \\ 0 & -\frac{1}{\sqrt3} & \frac{2}{\sqrt6} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{\sqrt6} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \end{bmatrix}$$

$$= \frac{1}{98}\begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}$$

Least Squares and SVD

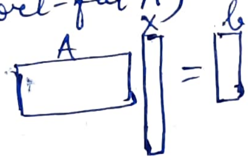
$A\underset{\text{solve}}{x} = b$ linear system of eqns.

A known, b known

SVD allows us to generalize to non-square A.

underdetermined, $n < m$ (short-fat A)

- ∞ many solns x given b



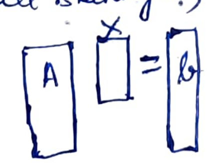
$\min \|\tilde{x}\|$ s.t. $A\tilde{x} = b$

(minimum norm soln)

$$Ax = b$$
$$U\Sigma V^T x = b$$
$$V\Sigma^{-1}U^TU\Sigma V^T x = \underbrace{V\Sigma^{-1}U^T}_{A^+} b$$
$$\tilde{x} = V\Sigma^{-1}U^T b$$
$$= A^+ b$$

Overdetermined, $n > m$ (tall skinny A)

- zero soln x for given b.



(least squares soln)

$\min \|A\tilde{x} - b\|$

Q. $A = \begin{bmatrix} 1 & 1 \\ .0001 & 0 \\ 0 & .0001 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ .0001 \\ .0001 \end{bmatrix}$

$U = \begin{pmatrix} -1 & 0 \\ 0 & -0.7071 \\ 0 & 0.7071 \end{pmatrix}$, $\Sigma = \begin{bmatrix} 1.4142 & 0 \\ 0 & 0.0001 \end{bmatrix}$, $V = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$

$$\tilde{x} = V\Sigma^{-1}U^T b$$
$$= A^+ b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Low Rank Approximation →

SVD provide a very simple solution to low rank approximation problem

Suppose $A \in \mathbb{R}^{m \times n}$, has SVD

$$A = U \Sigma V^t = \sum_{i=1}^{n} u_i \sigma_i v_i^t$$

then the k- approximation to A is given by

$$A_k = \sum_{i=1}^{k} u_i \sigma_i v_i^t \qquad \text{where } k \le \text{rank} A.$$

let A be 5×5 matrix.

$\sigma_1 = 3, \quad \sigma_2 = 1, \quad \sigma_3 = 0.5, \quad \sigma_4 = 0.2, \quad \sigma_5 = 0.05$

$$A_{5\times5} = U_{5\times5} \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \end{bmatrix} V^t_{5\times5}.$$

$$A_3 = U_{5\times3} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V^t_{3\times5}$$

$$= \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 \\ | & | & | & | & | \end{bmatrix} \Sigma_{3\times3} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ | & | & | & | & | \end{bmatrix}^t$$

$$A_3 = \begin{bmatrix} u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}_{5\times 3} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}_{3\times 3} \begin{bmatrix} v_1 & v_2 & v_3 \\ & & \end{bmatrix}^t_{3\times 5}.$$

A measure of quality of the approximation is given by

$$\frac{\|A_k\|_F^2}{\|A\|_F^2} = \frac{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2}{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}.$$

Ex Find rank 2 approximation of

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U \Sigma V^t$$

$$A = \begin{pmatrix} 0.91 & 0.42 & 0.02 \\ 0.41 & -0.87 & -0.26 \\ 0.09 & -0.24 & 0.97 \end{pmatrix} \begin{pmatrix} 4.04 & 0 & 0 \\ 0 & 1.20 & 0 \\ 0 & 0 & 0.87 \end{pmatrix} \begin{pmatrix} 0.67 & 0.73 & 0.08 \\ 0.65 & -0.53 & -0.53 \\ 0.35 & -0.41 & 0.84 \end{pmatrix}^t$$

$$A_2 = \begin{pmatrix} 0.91 & 0.42 \\ 0.41 & -0.87 \\ 0.09 & -0.24 \end{pmatrix} \begin{pmatrix} 4.04 & 0 \\ 0 & 1.20 \end{pmatrix} \begin{pmatrix} 0.67 & 0.73 \\ 0.65 & -0.53 \\ 0.35 & -0.41 \end{pmatrix}^t$$

$$= \begin{pmatrix} 2.99 & 2.01 & 0.98 \\ 0.02 & 1.88 & 1.1 \\ -0.07 & 0.45 & 0.29 \end{pmatrix} \rightarrow$$