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Ex:→ $(2xy+x^2)\dfrac{dy}{dx} = 3y^2+2xy \quad -(1)$

Sol^n $\Rightarrow (2xy+x^2)\,dy - (3y^2+2xy)\,dx = 0$

$M = -(3y^2+2xy)$; $N = 2xy+x^2$.

$\dfrac{\partial M}{\partial y} = -6y-2x$; $\dfrac{\partial N}{\partial x} = 2y+2x$

$\Rightarrow$ (1) is not Exact.

M and N are hom. functions of degree 2.

$\therefore \dfrac{1}{Mx+Ny}$ will be an I.F.

i.e. $\dfrac{1}{(2xy+x^2)y+(-3y^2-2xy)x} = \dfrac{1}{2xy^2+x^2y-3xy^2-2x^2y}$

$= \dfrac{1}{-x^2y-xy^2} = \dfrac{-1}{xy(x+y)}$

Using this I.F (1) becomes

$\dfrac{1}{xy(x+y)}(3y^2+2xy)\,dx - \dfrac{1}{xy(x+y)}(2xy+x^2)\,dy = 0$

$\Rightarrow \dfrac{3y+2x}{x(x+y)}dx - \dfrac{(2y+x)}{y(x+y)}dy = 0.$

Now $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (check)

Sol^n is $\displaystyle\int \frac{3y+2x}{x(x+y)}dx + \int \frac{-1}{y}dy = c$

$\Rightarrow \displaystyle\int \frac{3y\,dx}{x(x+y)} + 2\int \frac{dx}{x+y} \overset{-\log|y|}{} = c$

$\Rightarrow \displaystyle\int \frac{3y\,dx}{x(x+y)} + 2\log|x+y| - \log y = \log c$

$\displaystyle\int \frac{3y\,dx}{x(x+y)} = \frac{A}{x} + \frac{B}{x+y}$ $\therefore \displaystyle\int \frac{3y\,dx}{x(x+y)} = \int \frac{3}{x}dx - \int \frac{3}{x+y}dx$

$3y = A(x+y) + Bx$

$x=0$ $3y = Ay \Rightarrow A=3$ | $x=-y$; $3y = B(-y)$, $B=-3$

$= 3\log|x| - 3\log|x+y|$

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$\therefore f(x,y) = 3\log|x| - 3\log|x+y| + 2\log|x+y| - \log y = \log c$

$\Rightarrow \log x^3 - \log(x+y) - \log y = \log c$

$\Rightarrow \log \frac{x^3}{y(x+y)} = \log c$

$\Rightarrow \frac{x^3}{y(x+y)} = c \Rightarrow \boxed{x^3 = cy(x+y)}$

Ques. $(2x+y)\,dy - (x+2y)\,dx = 0 \quad -(1)$

Solⁿ $M = -(x+2y)\ ;\ N = 2x+y$

$\frac{\partial M}{\partial y} = -2\ ;\ \frac{\partial N}{\partial x} = 2 \Rightarrow$ (1) is not Exact.

I.F is $\frac{1}{Mx+Ny}$ ($\because$ M & N are hom fun of deg. 1)

$= \frac{1}{-(x+2y)x+(2x+y)y} = \frac{1}{-x^2 \cancel{-2xy} + \cancel{2xy} + y^2} = \frac{1}{y^2-x^2}$

$\frac{1}{y^2-x^2}(2x+y)\,dy - \frac{1}{y^2-x^2}(x+2y)\,dx = 0$

$M = \frac{-1}{y^2-x^2}(x+2y)\ ;\ N = \frac{1}{y^2-x^2}(2x+y)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (check)

$\int \frac{x+2y}{x^2-y^2}\,dx + \int 0\,dy = c$

$\Rightarrow \int \frac{x+y+y}{(x-y)(x+y)}\,dx = c$

$\Rightarrow \int \frac{1}{x-y}\,dx + \int \frac{y}{x^2-y^2}\,dx = c$

$\Rightarrow \log|x-y| + \frac{y}{2y}\log\left|\frac{x-y}{x+y}\right| = \log c$

$\Rightarrow \log|x-y| + \frac{1}{2}\log\left|\frac{x-y}{x+y}\right| = \log c$

$\Rightarrow \log|x-y| + \log\left|\frac{x-y}{x+y}\right|^{1/2} = \log c$

$\Rightarrow \log\left((x-y)\frac{(x-y)^{1/2}}{(x+y)^{1/2}}\right) = \log c$

$\Rightarrow (x-y)^{3/2} = c\sqrt{x+y}$

$\Rightarrow \boxed{(x-y)^3 = c(x+y)}$

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H.W.

(1) $3x^2y^4\,dx + 4x^3y^3\,dy = 0$ — Exact

(2) $3y\,dx + 2x\,dy = 0$ $\quad$ I.F. $= \sqrt{x}$

(3) $xy\,dx - (x^2+y^2)\,dy = 0$ $\quad$ I.F. $= 1/y^3$.

~~(4)~~

Linear Differential Equation:→

$$\frac{dy}{dx} + P(x)y = r(x) \qquad -(1)$$

$$\Rightarrow dy + (P(x)y - r(x))\,dx = 0 \quad -$$

$$M = P(x)y - r(x) \;;\; N = 1$$

$$\frac{\partial M}{\partial y} = P(x), \quad \frac{\partial N}{\partial x} = 0$$

$$\cancel{\text{IF}} \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{NV_x - MV_y} \qquad \text{Chose } V = x,\; V_x = 1,\; V_y = 0$$

$$\therefore \frac{M_y - N_x}{N} = \frac{P(x)}{1} = P(x) \quad \text{— fun of } x \text{ only.}$$

$$\text{I.F.} = e^{\int P(x)\,dx}.$$

Multiply Equ (1) with the I.F., we get

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x).y.e^{\int P(x)\,dx} = e^{\int P(x)\,dx}.r(x)$$

$$\Rightarrow \frac{d}{dx}\left[y\,e^{\int P(x)\,dx}\right] = e^{\int P(x)\,dx}.r(x)$$

Integrating both sides w.r.t x, we get

$$\boxed{y.\,e^{\int P(x)\,dx} = \int r(x).\,e^{\int P(x)\,dx}\,dx + c}$$

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Que $x\frac{dy}{dx} = 2y + x^4 + 6x^2 + 2x, \quad x \neq 0.$

Solⁿ $\frac{dy}{dx} - \frac{2y}{x} = x^3 + 6x + 2$

$P(x) = \frac{-2}{x}, \quad Q(x) = x^3 + 6x + 2$

$I.F. = e^{\int -2/x\, dx} = e^{-2\log|x|} = \frac{1}{x^2}$

$y \cdot \frac{1}{x^2} = \int \frac{(x^3 + 6x + 2)}{x^2} dx + c$

$= \int \left(x + \frac{6}{x} + \frac{2}{x^2}\right) dx + c$

$\Rightarrow \frac{y}{x^2} = \frac{x^2}{2} + 6\log|x| + 2\left(\frac{x^{-1}}{-1}\right) + c$

$\Rightarrow y = \frac{x^4}{4} + 6x^2 \log|x| - 2x + cx^2$

Que $(x-a)\frac{dy}{dx} + 3y = 12(x-a)^3$

Solⁿ $\frac{dy}{dx} + \left(\frac{3}{x-a}\right)y = 12(x-a)^2$

$I.F. = e^{\int \frac{3}{x-a} dx} = e^{3\log|x-a|} = (x-a)^3$

$y \cdot (x-a)^3 = \int (12)(x-a)^2 \cdot (x-a)^3\, dx + c$

$= 12\frac{(x-a)^6}{6} + c$

$= 2(x-a)^6 + c$

$\Rightarrow y = 2(x-a)^3 + c(x-a)^{-3}.$

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Que $\cot 3x \frac{dy}{dx} - 3y = \cos 3x + \sin 3x \; ; \; 0 < x < \frac{\pi}{2}$

Sol^n $\frac{dy}{dx} - \frac{3}{\tan 3x} y = \sin 3x + \frac{\sin^2 3x}{\cos 3x}$

$$I.F. = e^{\int -3/\tan 3x \, dx} = e^{-3\int \cot 3x \, dx}$$
$$= e^{\log|\cos 3x|} = \cos 3x.$$

$$y \cdot \cos 3x = \int \left(\sin 3x + \frac{\sin^2 3x}{\cos 3x}\right) \cos 3x \, dx + C$$
$$= \frac{1}{2}\int \sin 6x + (1 - \cos 6x) \, dx + c$$
$$= \frac{1}{2}\left[-\frac{\cos 6x}{6} + x - \frac{\sin 6x}{6}\right] + c$$

$$\Rightarrow y \cos 3x = \frac{-1}{12}\cos 6x + \frac{x}{2} - \frac{1}{12}\sin 6x + c$$

H.W (1) $(1+x^2)y' + 2xy = x \sin x$ Ans:- $y = (c + \sin x - x\cos x) / (1+x^2)$

(2) $y' + 3y = e^{2x} + 6$ Ans:- $y = \frac{1}{5}e^{2x} + 2 + Ce^{-3x}$

(3) $xy' + (1+2x)y = 1 + xe^{-2x}$

Ans:- $2xe^{2x}y = x^2 + e^{2x} + c$

I.F. of Separable Equation:→ Let $f(x)g(y)\,dx + F(x)\,G(y)\,dy = 0$ —(1) is a non-Exact differential equation.

Then $\frac{1}{g(y)F(x)}$ is an I.F. of (1)

$$\therefore \frac{1}{g(y)F(x)}\left[f(x)g(y)\,dx + F(x)G(y)\,dy\right] = 0$$

$$\Rightarrow \frac{f(x)}{F(x)}\,dx + \frac{G(y)}{g(y)}\,dy = 0$$

$\Rightarrow \frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0$ ⇒ Equ^n becomes Exact diff Equ^n.

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Determination of $M(x,y)$ and $N(x,y)$ Such that Equation is Exact:→

(1) $(x^3 + xy^2)\,dx + N(x,y)\,dy = 0 \quad -(1)$ is Exact. Find $N(x,y)$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (\because \ (1) \text{ is Exact})$$

$$M = x^3 + xy^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = 2xy$$

$$\therefore \quad \frac{\partial N}{\partial x} = 2xy$$

$$\Rightarrow \int \frac{\partial N}{\partial x}\,dx = \int 2xy\,dx + g(y)$$

$$\Rightarrow \quad N = x^2y + g(y)$$

(2) $(x^{-2}y^{-2} + xy^{-3})\,dx + N(x,y)\,dy = 0$ is Exact. Find $N(x,y)$

$$\because \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M = x^{-2}y^{-2} + xy^{-3}$$

$$\frac{\partial M}{\partial y} = x^{-2}(-2y^{-3}) + x(-3)y^{-4}$$

$$= \frac{-2}{x^2y^3} - \frac{3x}{y^4}$$

$$\therefore \quad \frac{\partial N}{\partial x} = \frac{-2}{x^2y^3} - \frac{3x}{y^4}$$

$$\text{So} \quad N = \int \frac{-2}{x^2y^3}\,dx - \int \frac{3x\,dx}{y^4} + g(y)$$

$$\Rightarrow N = \frac{-2}{y^3}\left(\frac{x^{-2+1}}{-2+1}\right) - \frac{3x^2}{2y^4} + g(y)$$

$$\Rightarrow N = \frac{2}{xy^3} - \frac{3x^2}{2y^4} + g(y)$$

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(3) $M(x,y)\,dx + (2x^2y^3 + x^4y)\,dy = 0$. —(1) Find $M(x,y)$

Sol^n As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ($\because$ (1) is Exact)

$$N = 2x^2y^3 + x^4y$$

$$\frac{\partial N}{\partial x} = 4xy^3 + 4x^3y$$

$$\therefore \frac{\partial M}{\partial y} = 4xy^3 + 4x^3y.$$

$$\Rightarrow M = \int (4xy^3 + 4x^3y)\,dy + g(x)$$

$$\Rightarrow M = xy^4 + 2x^3y^2 + g(x)$$

(4) $M(x,y)\,dx + (2ye^x + y^2e^{3x})\,dy = 0$; Find $M(x,y)$.
—(1)

Sol^n:→ as $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ ($\because$ (1) is Exact)

$$\frac{\partial N}{\partial x} = 2ye^x + 3y^2e^{3x}.$$

$$\therefore \frac{\partial M}{\partial y} = 2ye^x + 3y^2e^{3x}$$

$$M = \int (2ye^x + 3y^2e^{3x})\,dy + g(x)$$

$$\Rightarrow M = y^2e^x + y^3e^{3x} + g(x).$$