P.D.E.

★ **Chain rule**

$u = f(x, y)$, $x = f_1(t)$, $y = f_2(t)$

(Total derivative) $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

For Eg:- $u = f(\underbrace{y-z}_{t_1}, \underbrace{z-x}_{t_2}, \underbrace{x-y}_{t_3}) = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} + \frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial x}$$

★ **Jacobian**:- $\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

- $\frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1$
- $(u, v) \to (r, s) \to (x, y)$

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$$

- For x, y, z be dependent

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

- Implicit :-

$f_1(x, y, u, v) = 0$, $f_2(x, y, u, v) = 0$

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^n \frac{\frac{\partial(f_1, f_2)}{\partial(x, y)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

★ **Maxima & Minima**

$\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$. (a, b) are saddle points

$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$, $s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}$, $t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}$

1. $rt - s^2 > 0$ & $(r > 0)$, $f(x, y)$ is minima at (a, b)
- $rt - s^2 < 0$, Neither maxima nor minima.
- $rt - s^2 > 0$, $r < 0$, $f(x, y)$ is maxima at (a, b).
- $rt - s^2 = 0$, Case is doubtful

$A+B+C=\pi$ in a triangle

$S = xy + 2yz + 2zx$

$xyz = 32$

Put saddle point in equation.

- Angles wale me $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ hi saddle point ata hai

★ Lagrange's Method

Given, $f = abc$, $g = a+b+c$

$L = f + \lambda g$.

$\frac{\partial L}{\partial a} = 0, \frac{\partial L}{\partial b} = 0, \frac{\partial L}{\partial c} = 0, \frac{\partial L}{\partial \lambda} = 0.$

Find a, b, c.

$$\begin{vmatrix} L_{11}-k & L_{12} & L_{13} & g_1 \\ L_{21} & L_{12}-k & L_{23} & g_2 \\ L_{31} & L_{32} & L_{13}-k & g_3 \\ g_1 & g_2 & g_3 & 0 \end{vmatrix}$$

$L_{11} = \frac{\partial^2 L}{\partial a^2}, L_{12} = \frac{\partial L}{\partial a \partial b}, g_1 = \frac{\partial g}{\partial a}$..

If $k = -ve$, then maxima.

$k = +ve$, then minima.

$k =$ +ve & -ve, neither maxima nor minima

★ For polar coordinates.

$x = r\cos\theta, y = r\sin\theta, r^2 = x^2+y^2, \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}.$

★ Equation of tangent plane.

$$\frac{\partial F}{\partial x}(X-x) + \frac{\partial F}{\partial y}(Y-y) + \frac{\partial F}{\partial z}(Z-z) = 0$$

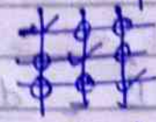
Equation of Normal.

$$\frac{X-x}{\partial F/\partial x} = \frac{Y-y}{\partial F/\partial x} = \frac{Z-z}{\partial F/\partial z}$$

★ $$\frac{d}{d\alpha}\left[\int_{\phi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx\right] = \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\partial x + \frac{d\psi}{d\alpha} f[\psi(\alpha),\alpha] - \frac{d\phi}{d\alpha} f[\phi(\alpha),\alpha]$$

$F(\alpha) = \int_0^{\alpha} \frac{\log(1+\alpha x)}{1+x^2}$. $F'(\alpha) = \int_0^{\alpha} \frac{\partial}{\partial \alpha}\left(\frac{\log(1+\alpha x)}{1+x^2}\right)dx + \frac{d(\alpha)}{d\alpha}\cdot \log\frac{(1+\alpha^2)}{1+\alpha^2} - 0$

Matrices

- Symmetric matrix, $A^T = A$
- Skew Symmetric matrix, $A^T = -A$
- Inverse of matrix, $A^{-1} = \frac{Adj\,A}{|A|}$ [Adj A is transpose of Cofactor matrix]
- Cofactors :- Eg :- $A_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$
- Gauss Elimination :-  $Ax = B$, $C = [A : B]$
- Gauss Jordan :- 
- Rank of matrix :- No. of non zero rows = Rank (in Echelon form)

  $\rho(A) + \eta(A) = n$
- Normal form :- $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$
- Consistent → Solution Exist → Unique / → Infinitely many

Inconsistent → Soln doesn't exist → No soln.

$\rho(A) = r$, $\rho(A : B) = r'$ & n = no. of variables

$r \neq r'$ (No soln), $r = r' = n$ (Unique), $r = r' < n$ (Infinitely many)

- Cramer's Rule

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Delta = |A|$, $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$

$$\Delta_1 = \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}$$

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- Linear Dependent / Independent.
  $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$
  If a_1, a_2, a_3 are zero, then LI
  No. of vectors = Rank, then LI
- Homogeneous linear equations.
  $d_1, d_2, d_3 = 0$, There will no (No soln) case

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Use

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

- Eigen values and Vectors
  $\underbrace{(A - \lambda I)x = 0}_{\text{characteristic equation}}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are eigen vectors
  λ_1, λ_2 are eigen values
  Largest eigen value is called Spectral.
- AM :- No. of times eigen value repeats.
  $\lambda = 1, 1, 3$ $(\lambda - 1)^2 (\lambda - 3)$ AM = 2 for 1, AM = 1 for 3
- GM :- No. of distinct LI eigenvector for a particular Eigen Value
  $AM \geq GM$.
- Orthogonal matrix :- $A \cdot A^T = I$
- Sum of eigenvalues are sum of diagonal entries (Trace of matrix)
- Idempotent, $A^2 = A$.
- Involutary, $A^2 = I$; Unitary, $AA^* = I$
- Product of eigen values is determinant of matrix
- Cayley Hamilton theorem.
  In char eq., use A.
  Like $A^2 - 4A - 5I = 0$.

To find inverse, Multiply char. equation by A^{-1}

- Diagonalisation :-
  P matrix will form of eigenvectors.

and if $P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ then diagonisable.

- Inverse of matrix using elementary operations.

  $A = I\,A$ (Right wale A ko A hi rehne do)

  Then using Elementary transformations.

  $I = B \cdot A$ will form.

  $\boxed{A^{-1} = B}$

- Quadratic to Canonical form.

  Eg:- $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$

  $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ $\quad \frac{-2}{2}x_1x_2 = -1$ $\quad \frac{2}{2}x_1x_3 = 1$

  Find char. equation, then find eigen vectors

  Form P matrix of eigen vectors.

  Now, $P^{-1}AP =$ Diagonisable (D)

  So, canonical form is $\quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

  $Q = Y^{-1}DY$

  Index = no. of +ve eigen value

  Nature = +ve definite

  Signature = No. of +ve & negative eigen value.

  Eg:- Agar eigen values are 1, 2, 3

  Canonical form = $1x^2 + 2y^2 + 3z^2$

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Vectors Calculus

- ★ $\nabla = \hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}$
- ∇f (grad) $= \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right) f$
- $\nabla \cdot f$ (div) $= \left(\hat{i}\frac{d}{dx} + \hat{j}\frac{d}{dy} + \hat{k}\frac{d}{dz}\right) \cdot (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})$
- $\nabla \cdot f = 0$ (solinoidal vector), $\nabla \times f = 0$ (irrotational vector)
- $\nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$, Normal Vector $(\vec{N})$ = grad f, Unit Normal Vector = $\frac{\text{grad } f}{|\text{grad } f|}$
- Direction Derivative in $\vec{A}$ direction = (grad f). $\hat{A}$
- Angle b/w 2 scalars:- (grad f) and (grad g)

  Angle = $\frac{(\text{grad } f) \cdot (\text{grad } g)}{|\text{grad } f|\,|\text{grad } g|}$
- Tangent vector = $\frac{d\vec{r}}{dt}$, Max D.D = |grad f|
- Orthogonal means 90° angle

★ Line Integral

- $\int_C f \cdot dr = ?$, $d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$
- In circle, $x = a\cos\theta$, $y = a\sin\theta$, $dy = a\cos\theta\, d\theta$
- $\int_C f \cdot dr = \int_{AB} f \cdot dr + \int_{BC} f \cdot dr + \int_{CD} f \cdot dr$.
- Find $\int f \cdot dr$ along each curve or line

★ Application of Line integral

- Work done $(W) = \int_A^B \vec{f} \cdot d\vec{r}$, $F = \text{grad } \phi$ (ϕ = scalar potential)
- $\nabla \times \vec{F} = 0$, the conservative vector field (independence of path)
- Then, find ϕ by $d\phi = (2xy + z^3)dx + x^2 dy + 3z^2 x\, dz$

  $= 2xy\,dx + x^2 dy + 3z^2 x\, dz + z^3 dx$

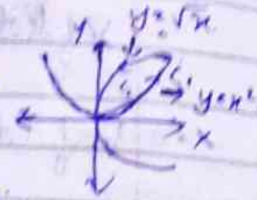
  $\int d\phi = \int d(x^2 y) + d(z^3 x)$

  $\phi = x^2 y + xz^3 + C$
- $W = [\phi]_A^B$

* Double integrals give Area. Eg:- $\int_0^a \int_0^{\sqrt{a^2-x^2}} dx\,dy$
* Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

* Eg:- For limits in line integral (Proof)



Along C_1, $x = 0$ to 1, $dy = 2x\,dx$
$y = x^2$

$$\int_{C_1} f\cdot dr = \int_0^1 (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

Along C_2, $y = 1$ to 0, $y = \sqrt{x}$, $2y\,dy = dx$

$$\int_{C_1} f\cdot dr = \int_1^0 (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

* Surface Integral :-

$$\iint_S \vec{F}\cdot ds = \iint_S \vec{F}\cdot\hat{n}\,ds$$

* In X-Y plane, $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$; X-Z, $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{j}|}$
* Y-Z plane, $ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{i}|}$
* First find $\hat{n}$, then $\hat{n}\cdot\hat{k}$, then $\vec{F}\cdot\hat{n}$, then $\iint_S \vec{F}\cdot\hat{n}\,ds$.
  In X-Y plane, $z=0$

* Stokes Theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds$$

* In XY plane, $\hat{n} = \hat{k}$; Y-Z, $\hat{n} = \hat{i}$; X-Z, $\hat{n} = \hat{j}$

* Gauss Divergence Theorem.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot F\,dV$$

$$\iint_S f_1\,dx\,dz + f_2\,dz\,dx + f_3\,dx\,dy = \iiint_V \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dx\,dy\,dz$$

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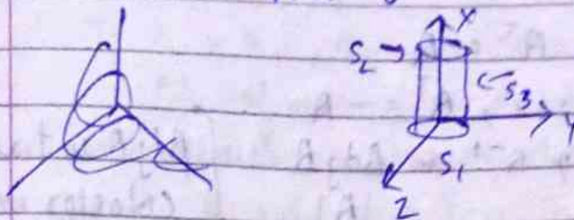
Eg:- $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}}\int_{0}^{1} z\,dx\,dy\,dz$



base S_1, $x^2+y^2=4$, $z=0$
S_2, $x^2+y^2=4$, $z=1$
S_3, $x^2+y^2=4$

* Curvature :- $\left|\frac{d\hat{T}}{ds}\right| = k$. T, tangent $= \frac{dR}{ds} = \frac{dR}{dt}\frac{dt}{ds}$

$\frac{dT}{ds} = k.N$

$B = T \times N$

τ(Torsion) $= \frac{d\hat{B}}{ds} = \frac{dB}{dt}\frac{dt}{ds}$

Q Find curvature & torsion for curve $x = a\cos t$, $y = a\sin t$, $z = bt$.

Ans $\vec{r} = (x, y, z)$

$\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$

$\frac{dr}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}$

$\left|\frac{dr}{dt}\right| = \sqrt{a^2+b^2}$, $s = \int_0^t \left|\frac{dr}{dt}\right| dt = t\sqrt{a^2+b^2}$

$\frac{ds}{dt} = \sqrt{a^2+b^2}$, $T = \frac{dR}{ds}$

$T = \frac{dR}{ds} = \frac{dR/dt}{ds/dt} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}\right)$

$\hat{T} = 1$

$\frac{dT}{ds} = \frac{dT/dt}{ds/dt} = \frac{a}{a^2+b^2}\left[-\cos t\,\hat{i} - \sin \hat{j}\right]$

$k = \frac{a}{a^2+b^2}$ and $N = -\cos t\,\hat{i} - \sin\hat{j}$

$B = T \times N$ then Find $\frac{dB}{ds}$, then $\tau = \left|\frac{dB}{ds}\right|$