UNIT - III

Laplace Transform ⇒ If $f(t)$ is a function of t defined for all $t \geq 0$, then $\int_0^\infty e^{-st} f(t)\, dt$ is defined as laplace Transform of $f(t)$, and is denoted by $L\{f(t)\}$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s) \quad \{s \text{ is dummy parameter}\}$$

Sufficient condition for the existence of Laplace Transform :-

The Laplace Transform of function $f(t)$ exists when the following sufficient conditions are satisfied:

(i) $f(t)$ is piecewise continuous ie $f(t)$ is continuous in every subinterval and $f(t)$ has finite limits at the end points of each subinterval.

(ii) $f(t)$ is of exponential order of α ie there exists M, α such that $|f(t)| \leq M e^{\alpha t}$ for all $t \geq 0$.
in other words $\lim_{t \to \infty} \{e^{-\alpha t} f(t)\} =$ finite quantity

Laplace Transform of some standard functions

(1) $L\{t^n\} = \dfrac{n!}{s^{n+1}} = \dfrac{\Gamma(n+1)}{s^{n+1}}$

(2) $L\{K\} = \dfrac{K}{s}$ K is const.

3. $L\{\sin at\} = \dfrac{a}{s^2+a^2}$

4. $L\{\cos at\} = \dfrac{s}{s^2+a^2}$

5. $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$ (6) $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

Q: find the laplace transform of

(i) $f(t) = \begin{cases} 3 & 0<t<5 \\ 0 & t>5 \end{cases}$ (ii) $f(t) = \begin{cases} t & 0<t<a \\ b & t>a \end{cases}$

Sol^n $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

i) $L\{f(t)\} = \int_0^5 e^{-st} \cdot 3\,dt + \int_5^\infty e^{-st} \cdot 0\,dt$

$= 3\left[\frac{e^{-st}}{-s}\right]_0^5 = 3\left[\frac{e^{-5s}}{-s} + \frac{1}{s}\right]$

$= \frac{3}{s}(1 - e^{-5s})$ Ans

(ii) $L\{f(t)\} = \int_0^a e^{-st} \cdot t\,dt + \int_a^\infty e^{-st} \cdot b\,dt$

$= \left[\frac{e^{-st}}{-s} \cdot t - \frac{e^{-st}}{s^2}\right]_0^a + b\left[\frac{e^{-st}}{-s}\right]_a^\infty$

$= e^{-as}\left(-\frac{a}{s} - \frac{1}{s^2}\right) - e^0\left(0 - \frac{1}{s^2}\right) - \frac{b}{s}\left(0 - e^{-as}\right)$

$= \frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right] e^{-as}$

(iii) $f(t) = \begin{cases} \cos t & 0<t<\pi \\ \sin t & t>\pi \end{cases}$ (iv) $f(t) = \begin{cases} 1 & 0<t<1 \\ e^t & 1<t<4 \\ 0 & 4>t \end{cases}$

$L\{f(t)\} = \int_0^\pi e^{-st} \cos t\,dt + \int_\pi^\infty e^{-st} \sin t\,dt$

$$\boxed{\begin{aligned} &\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] \\ &\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) \end{aligned}}$$

$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_\pi^\infty$

$= \frac{1}{s^2+1}\left[e^{-\pi s}(-s\cos\pi) - (-s\cos 0) + 0 - e^{-\pi s}(-\cos\pi)\right]$

$= \frac{1}{s^2+1}\ [e^{-\pi s}(s-1) - s]$ Ans

(iv) $L\{f(t)\} = \int_0^1 e^{-st}.1\,dt + \int_1^4 e^{-st}.e^{t}dt + \int_4^\infty e^{-st}.0\ldots$

$= \left[\frac{e^{-st}}{-s}\right]_0^1 + \left[\frac{e^{t(1-s)}}{1-s}\right]_1^4 + 0$

$= \frac{e^{-s}-1}{-s} + \frac{e^{4(1-s)} - e^{(1-s)}}{1-s}$

$= \frac{1-e^{-s}}{s} + \frac{e^{(1-s)} - e^{4(1-s)}}{s-1}$ Ans

Properties of Laplace Transform ⇒

Linearity

$L\{f_1(t)\} = F_1(s)$, $L\{f_2(t)\} = F_2(s)$

then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

where a & b are constant

Q. find the Laplace Transform of the functions

(i) $4t^2 + \sin 3t + e^{2t}$ (ii) $\cosh^5 t$

(iii) $\sin\sqrt{t}$ (iv) $\cos t \cos 2t \cos 3t$

(v) $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$

Solⁿ (i) $L\{4t^2 + \sin 3t + e^{2t}\}$

$= 4L\{t^2\} + L\{\sin 3t\} + L\{e^{2t}\}$

$= 4 \cdot \frac{2}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2} = \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{1}{s-2}$

(ii) $L\{\cosh^5 t\} = L\left\{\frac{e^{t}+e^{-t}}{2}\right\}^5$

$= L\{\frac{1}{25}(e^{5t} + e^{-5t} + 5e^{4t}e^{-t} + 10e^{3t}e^{-2t} + 10e^{2t}e^{-3t}$
$+ 5e^{t}e^{-4t})\}$

$= \frac{1}{32} L\{ (e^{5t} + e^{-5t}) + 5(e^{3t} + e^{-3t}) + 10(e^{t} + e^{-t}) \}$

$= \frac{1}{32} L\{ 2\cosh 5t + 2\times 5 \cosh 3t + 10\times 2\cosh t \}$

$= \frac{1}{16} [L\{\cosh 5t\} + 5L\{\cosh 3t\} + 10L\{\cosh t\}]$

$= \frac{1}{16} \left[\frac{s}{s^2-25} + \frac{5s}{s^2-9} + \frac{10s}{s^2-1} \right]$ As

(iii) we know that

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

$\sin\sqrt{t} = \sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} + \cdots$

$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!} L\{t^{3/2}\} + \frac{1}{5!} L\{t^{5/2}\} + \cdots$

$= \frac{\Gamma\frac{3}{2}}{s^{3/2}} - \frac{1}{3!} \frac{\Gamma\frac{5}{2}}{s^{5/2}} + \frac{1}{5!} \frac{\Gamma\frac{7}{2}}{s^{7/2}} + \cdots$

$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} - \frac{1}{3!} \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \frac{1}{5!} \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{7/2}} + \cdots$

$= \frac{\Gamma\frac{1}{2}}{2s^{3/2}} \left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \cdots \right]$

$= \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-\frac{1}{4s}}$ As

(iv) $L\{ (\sqrt{t} - \frac{1}{\sqrt{t}})^3 \} = L\{ t^{3/2} - 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}} - t^{-3/2} \}$

$= L\{t^{3/2}\} - 3L\{t^{1/2}\} + 3L\{t^{-1/2}\} - L\{t^{-3/2}\}$

$= \frac{\Gamma\frac{5}{2}}{s^{5/2}} - \frac{3\Gamma\frac{3}{2}}{s^{3/2}} + \frac{3\Gamma\frac{1}{2}}{s^{1/2}} - \frac{\Gamma-\frac{1}{2}}{s^{-1/2}}$

$= \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} - \frac{3\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} + \frac{3\sqrt{\pi}}{s^{1/2}} - \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}\cdot s^{-1/2}}$ $\left\{ \begin{array}{l} \Gamma n+1 = n\Gamma n \\ \Gamma n = \frac{\Gamma n+1}{n} \end{array} \right.$

$= \sqrt{\frac{\pi}{s}} \cdot \left(\frac{3}{4s^2} - \frac{3}{2s} + 3 + 2s \right)$ As

(v) $L\{\cos t \cos 2t \cos 3t\} = L\{\frac{1}{2}(\cos 3t + \cos t)\cos 3t\}$

$= \frac{1}{2} L\{\cos^2 3t + \cos t \cos 3t\}$

$= \frac{1}{2} L\{(\frac{1+\cos 6t}{2}) + \frac{\cos 4t + \cos 2t}{2}\}$

$= \cancel{\frac{1}{2}} L\{\frac{1}{4} + \frac{1}{4}\cos 6t + \frac{1}{4}\cos 4t + \frac{1}{4}\cos 2t\}$

$= L\{\frac{1}{4}\} + \frac{1}{4}L\{\cos 6t\} + \frac{1}{4}L\{\cos 4t\} + \frac{1}{4}L\{\cos 2t\}$

$= \frac{1}{4s} + \frac{s}{4(s^2+36)} + \frac{s}{4(s^2+36)} + \frac{s}{4(s^2+4)}$ Ans

Change of scale Property –

If $L\{f(t)\} = F(s)$ then

$L\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$

Q. If $L\{f(t)\} = \log(\frac{s+3}{s+1})$, find $L\{f(2t)\}$

Solⁿ $L\{f(t)\} = \log(\frac{s+3}{s+1})$

then $L\{f(2t)\} = \frac{1}{2}\log\left(\frac{\frac{s}{2}+3}{\frac{s}{2}+1}\right) = \frac{1}{2}\log\left(\frac{s+6}{s+2}\right)$

Q. If $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{4s}}$, find $L\{\sin 2\sqrt{t}\}$

Solⁿ $L\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-\frac{1}{4s}}$

$L\{\sin 2\sqrt{t}\} = L\{\sin\sqrt{4t}\} = \frac{\sqrt{\pi}}{2\,4\cdot\frac{s}{4}\sqrt{\frac{s}{4}}} \cdot e^{-\frac{1}{4\cdot\frac{s}{4}}} = \frac{\sqrt{\pi}}{2s\sqrt{s}} \cdot e^{-\frac{1}{s}}$ Ans

Q. If ~~$L\{e^{at}\}$~~ $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$, find $L\{J_0(3t)\}$

Solⁿ $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

$L\{J_0(3t)\} = \frac{1}{3}\frac{1}{\sqrt{\frac{s^2}{9}+1}} = \frac{1}{\sqrt{s^2+9}}$

Q. If $L\{f(t)\} = \frac{8(s-3)}{(s^2-6s+25)^2}$ find $L\{f(2t)\}$

Sol^n $L\{f(t)\} = \frac{8(s-3)}{(s^2-6s+25)^2}$

$L\{f(2t)\} = \frac{1}{2} \cdot \frac{8\left(\frac{s}{2}-3\right)}{\left(\frac{s^2}{4} - 6\cdot\frac{s}{2} + 25\right)^2} = \frac{2(s-6)}{\frac{1}{16}(s^2-12s+100)^2}$ Ans

Q. If $L\{f(t)\} = \frac{2}{s^3}e^{-s}$ find $L\{f(3t)\}$

Sol^n $L\{f(t)\} = \frac{2}{s^3}e^{-s}$

$L\{f(3t)\} = \frac{1}{3}\frac{2}{\left(\frac{s}{3}\right)^3} \cdot e^{-s/3} = \frac{18}{s^3}e^{-s/3}$ Ans

First shifting Property —

If $L\{f(t)\} = F(s)$

then $L\{e^{-at}f(t)\} = F(s+a)$

Q. Find the Laplace Transform of the following functions :

(i) $e^{-3t}t^4$ (ii) $e^{4t}\sin^3 t$ (iii) $(t+1)^2 e^t$

(iv) $e^{-3t}\cosh 4t \sin 3t$ (v) $\frac{\cos 2t \sin t}{e^t}$

Sol^n (i) $L\{t^4\} = \frac{4!}{s^5}$

$L\{e^{-3t}t^4\} = \frac{4!}{(s+3)^5}$ Ans

(ii) $L\{\sin^3 t\} = L\{\frac{1}{4}(3\sin t - \sin 3t)\}$

$= \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$

$$L\{e^{4t}\sin^3 t\} = \frac{3}{4[(s-4)^2+1]} - \frac{3}{4[(s-4)^2+9]}$$

$$= \frac{3}{4(s^2-8s+17)} - \frac{3}{4(s^2-8s+25)}$$

$$= \frac{6}{(s^2-8s+17)(s^2-8s+25)} \quad \text{Ans}$$

(iii) $L\{(t+1)^2\} = L\{t^2+2t+1\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$

$$L\{(t+1)^2 e^t\} = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)}$$

(iv) $e^{-3t}\cosh 4t \sin 3t = e^{-3t}\left(\frac{e^{4t}+e^{-4t}}{2}\right)\sin 3t$

$$= \frac{1}{2}(e^{t}\sin 3t + e^{-7t}\sin 3t)$$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-3t}\cosh 4t \sin 3t\} = \frac{1}{2}\left[\frac{3}{(s-1)^2+9} + \frac{3}{(s+7)^2+9}\right]$$

$$= \frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)} \quad \text{Ans}$$

(v) $\frac{\cos 2t \sin t}{e^t} = e^{-t}\left(\frac{\sin 3t - \sin t}{2}\right)$

$$= \frac{1}{2}(e^{-t}\sin 3t - e^{-t}\sin t) \quad \text{--- (1)}$$

$$L\{\sin t\} = \frac{1}{s^2+1} \qquad L\{\sin 3t\} = \frac{3}{s^2+9}$$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1} \qquad L\{e^{-t}\sin 3t\} = \frac{3}{(s+1)^2+9}$$

from (1)

$$= \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

Second shifting Theorem $\Rightarrow$ If $L\{f(t)\} = F(s)$

and $g(t) = f(t-a)$, $t > a$

$= 0$, $t < a$

then $L\{g(t)\} = e^{-as} F(s)$

Q. Find the Laplace transform of the following functions :

(a) $g(t) = \cos(t-a)$, $t > a$
$= 0$, $t < a$

(b) $g(t) = e^{t-a}$, $t > a$
$= 0$, $t < a$

(c) $g(t) = \sin\left(t - \frac{\pi}{4}\right)$, $t > \frac{\pi}{4}$
$= 0$, $t < \frac{\pi}{4}$

(d) $g(t) = (t-1)^3$, $t > 1$
$= 0$, $t < 1$

Solⁿ (a) ~~L{f~~ $f(t) = \cos t$

$$L\{f(t)\} = F(s) = \frac{s}{s^2+1}$$

By second shifting Prop.

$$L\{g(t)\} = e^{-as} \cdot \frac{s}{s^2+1}$$

(b) $f(t) = e^t$, $L\{f(t)\} = F(s) = \frac{1}{s-1}$

$$L\{g(t)\} = e^{-as} \frac{1}{s-1}$$

(c) $f(t) = \sin t \Rightarrow$ $L\{\sin t\} = F(s) = \frac{1}{s^2+1}$

$$L\{g(t)\} = e^{-\frac{\pi s}{4}} \cdot \frac{1}{s^2+1}$$

(d) Let $f(t) = t^3$

$$L\{f(t)\} = F(s) = \frac{3!}{s^4}$$

$$L\{g(t)\} = e^{-s} \cdot \frac{3!}{s^4}$$

Ans.

Multiplication by t

If $L\{f(t)\} = F(s)$, then

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

Q. Find the Laplace transform of the following functions:

(a) $t \sin at$ (b) $t e^{3t} \sin t$ (c) $t\sqrt{1+\sin t}$

(d) $t^2 \cos at$ (e) $t\left(\frac{\sin t}{e^t}\right)^2$ ~~(f) $t^2 \cos at$~~

Soln (a) $L\{\sin at\} = \frac{a}{s^2+a^2} = F(s)$

$$L\{t \sin at\} = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

(b) $L\{\sin t\} = \frac{1}{s^2+1}$

$$\cancel{L\{e^{3t}\sin t\}}\; L\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$L\{e^{3t}\, t \sin t\} = \frac{2(s-3)}{[(s-3)^2+1]^2} = \frac{2(s-3)}{(s^2-6s+10)^2}$$

(c) $$L\{\sqrt{1+\sin t}\} = L\left\{\sin\frac{t}{2} + \cos\frac{t}{2}\right\}$$

$$= \frac{\frac{1}{2}}{s^2+\frac{1}{4}} + \frac{s}{s^2+\frac{1}{4}} = \frac{4s+2}{4s^2+1}$$

$$L\{t\sqrt{1+\sin t}\} = -\frac{d}{ds}\left(\frac{4s+2}{4s^2+1}\right) = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

(d) $L\{\cos at\} = \frac{s}{s^2+a^2} = F(s)$

$$L\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{(s^2+a^2)^2\cdot(-2s) - (a^2-s^2)\, 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^3}$$

(e) $f(t) = t\left(\frac{\sin t}{e^t}\right)^2 = t\,e^{-2t}\sin^2 t$

$= t\,e^{-2t}\frac{(1-\cos 2t)}{2} = \frac{1}{2}t\,e^{-2t}(1-\cos$

$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

$L\{t(1-\cos 2t)\} = -\frac{d}{ds}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$

$= -\left\{-\frac{1}{s^2} - \frac{(s^2+4) - s\cdot 2s}{(s^2+4)^2}\right] = \frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}$

$L\{\frac{t}{2}e^{-2t}(1-\cos 2t)\} = \frac{1}{2}\left[\frac{1}{(s+2)^2} + \frac{4-(s+2)^2}{\{(s+2)^2+4\}^2}\right]$

<u>Division by t</u> $\Rightarrow$ If $L\{f(t)\} = F(s)$ then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

Q. Find the Laplace Transform of the following functions:

(a) $\frac{1-e^{-t}}{t}$ (b) $\frac{\cos at - \cos bt}{t}$ (c) $\frac{e^{-t}\sin t}{t}$

(d) $\frac{\sin^2 t}{t^2}$ (e) $\frac{1-\cos t}{t}$

<u>Sol^n</u> (a) $L(1-e^{-t}) = \frac{1}{s} - \frac{1}{s+1} = F(s)$

$L\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+1}\right)ds$

$= [\log s - \log(s+1)]_s^\infty$

$= \left[\log\frac{s}{s+1}\right]_s^\infty = \left[\log\frac{\cancel{s}}{\cancel{s}(1+\frac{1}{s})}\right]_s^\infty$

$= \left[\log 1 - \log\frac{s}{s+1}\right] = \log\frac{s+1}{s}$ Ans

(b) $L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\}ds$

$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty = \frac{1}{2}\left[\log \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log \frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right] = -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2} \quad \text{As}$$

(c) $L\{\sin t\} = \frac{1}{s^2+1}$

$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty L\{e^{-t}\sin t\}\,ds = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1)$$

(d) $L\left\{\frac{\sin^2 t}{t^2}\right\} = L\left\{\frac{1-\cos 2t}{2t^2}\right\} = \frac{1}{2}L\left\{\frac{1-\cos 2t}{t^2}\right\}$

$L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds = \left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty = \log\frac{1}{\sqrt{1+\frac{4}{s^2}}}$$

$$= \log 1 - \log\frac{1}{\sqrt{1+\frac{4}{s^2}}} = -\log\frac{s}{\sqrt{s^2+4}} = \frac{1}{2}\log\frac{s^2+4}{s^2}$$

$$L\left\{\frac{1-\cos 2t}{t^2}\right\} = \frac{1}{2}\int_s^\infty\left(\log\frac{s^2+4}{s^2}\right)ds$$

$$= \frac{1}{2}\left[\left\{s\cdot\log\frac{s^2+4}{s^2}\right\}_s^\infty - \int_s^\infty s\cdot\frac{s^2}{s^2+4}\cdot\left\{\frac{2s\cdot s^2 - 2s(s^2+4)}{s^4}\right\}ds\right]$$

$$= \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} - \int_s^\infty -\frac{8}{s^2+4}\,ds\right]$$

$$= \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} + \frac{8}{2}\left\{\tan^{-1}\frac{s}{2}\right\}_s^\infty\right]$$

$$= \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} + 4\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[-s\log\frac{s^2+4}{s^2} + 4\cot^{-1}\frac{s}{2}\right]$$

$$L\left\{\frac{\sin^2 t}{t^2}\right\} = \frac{1}{2}L\left\{\frac{1-\cos 2t}{t^2}\right\} = \frac{1}{4}\left[-s\log\frac{s^2+4}{s^2} + 4\cot^{-1}\frac{s}{2}\right]$$

Ans

(e) $L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log(s^2+1) - \log s^2\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[\log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}\log\left(1+\frac{1}{s^2}\right)$$

$$= \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$ Ans

Laplace Transform of Derivative ⇒

If $L\{f(t)\} = F(s)$

then $L\{f'(t)\} = sF(s) - f(0)$

$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

In general

$$L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) \cdots \cdots f^{(n-1)}(0)$$

Q. Find $L\{f(t)\}$ and $L\{f'(t)\}$ of following functions

(i) $f(t) = \frac{\sin t}{t}$ (ii) $f(t) = \begin{cases} t & 0 \le t < 3 \\ 6 & t > 3 \end{cases}$

(iii) $e^{-5t}\sin t$

Sol^n $L\{f(t)\} = F(s) = L\{\frac{\sin t}{t}\}$.

$L\{\sin t\} = \frac{1}{s^2+1}$

$L\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{s^2+1}\,ds = [\tan^{-1}s]_s^\infty = \frac{\pi}{2} - \tan^{-1}s$

$= \cot^{-1}s$

$L\{f'(t)\} = sF(s) - f(0)$

$= s\cot^{-1}s - \lim_{t\to 0}\frac{\sin t}{t} = s\cot^{-1}s - 1$ Ans

(ii) $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^3 t e^{-st}\,dt + \int_3^\infty e^{-st}\, 6\,dt$

$= -\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6}{s}e^{-3s} = \frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right)$

$L\{f'(t)\} = sF(s) - f(0)$

$= \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right) - 0 = \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$

(iii) $f(t) = e^{-5t}\sin t$

$L\{\sin t\} = \frac{1}{s^2+1} = F(s)$

$L\{e^{-5t}\sin t\} = \frac{1}{(s+5)^2+1} = \frac{1}{s^2+10s+26}$

$L\{f'(t)\} = sF(s) - f(0)$

$= \frac{s}{s^2+10s+26} - 0 = \frac{s}{s^2+10s+26}$

Q. Laplace Transform of Integrals $\Rightarrow$ If $L\{f(t)\} = F(s)$

Sol then $L\{\int_0^t f(t)\,dt\} = \frac{F(s)}{s}$

Q. Find Laplace Transform of following functions:

(i) $\int_0^t e^{-2t} t^3\,dt$ (ii) $\int_0^t t e^{-4t}\sin 3t\,dt$ (iii) $e^{-4t}\int_0^t t\sin 3t\,dt$

(iv) $t\int_0^t e^{-4t}\sin 3t\,dt$

Sol^n i) $L\{e^{-2t}t^3\} = \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$

$L\{\int_0^t e^{-2t} t^3\,dt\} = \frac{6}{s(s+2)^4}$ Ans

(ii) $L\{t\sin 3t\} = -\frac{d}{ds}\{L(\sin 3t)\}$

$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{3\cdot 2s}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$

$L\{e^{-4t}t\sin 3t\} = \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$

$L\{\int_0^t t e^{-4t}\sin 3t\, \cancel{dt}\} = \frac{1}{s}\cdot\frac{6(s+4)}{(s^2+8s+25)^2}$

(iii) $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{t\sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$

$L\{\int_0^t t\sin 3t\, \cancel{dt}\} = \frac{1}{s}\cdot\frac{6s}{s^2+9} = \frac{6}{(s^2+9)^2}$

$L\{e^{-4t}\int_0^t t\sin 3t\} = \frac{6}{[(s+4)^2+9]^2} = \frac{6}{(s^2+8s+25)^2}$

(iv) $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{e^{-4t}\sin 3t\} = \frac{3}{(s+4)^2+9} = \frac{3}{s^2+8s+25}$

$L\{\int_0^t e^{-4t}\sin 3t\} = \frac{3}{s(s^2+8s+25)} = \frac{3}{s^3+8s^2+25s}$

$L\{t\int_0^t e^{-4t}\sin 3t\} = -\frac{d}{ds}\frac{3}{(s^3+8s^2+25s)} = \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$

Evaluation of an integral using Laplace Transform

Q. Evaluate $\int_0^\infty e^{-2t} \sin^3 t \, dt$

Sol^n $\int_0^\infty e^{-st} \sin^3 t \, dt = L\{\sin^3 t\}$

$= L\left\{\frac{3\sin t - \sin 3t}{4}\right\} = \frac{3}{4(s^2+1)} - \frac{3}{4(s^2+9)}$

$= \frac{6}{(s^2+1)(s^2+9)}$ —— (1)

Put $s=2$ in (1)

$\int_0^\infty e^{-2t} \sin^3 t \, dt = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$ Ans

Q. Evaluate $\int_0^\infty e^{-4t} \cosh^3 t \, dt$

Sol^n $\int_0^\infty e^{-st} \cosh^3 t \, dt = L\left\{\frac{\cosh 3t + 3\cosh t}{4}\right\}$

$= \frac{1}{4}\frac{s}{s^2-9} + \frac{3}{4}\frac{s}{s^2-1} = \frac{s(s^2-7)}{(s^2-9)(s^2-1)}$ —— (1)

Put $s=4$ in (1)

$\int_0^\infty e^{-4t} \cosh^3 t \, dt = \frac{4(16-7)}{(16-9)(16-1)} = \frac{12}{35}$ Ans

Q. Evaluate $\int_0^\infty e^{-3t} t^5 \, dt$

Sol^n $\int_0^\infty e^{-st} t^5 \, dt = L\{t^5\} = \frac{120}{s^6}$ —— (1)

Put $s=3$ in (1)

$\int_0^\infty e^{-3t} \cdot t^5 \, dt = \frac{120}{3^6} = \frac{40}{243}$ Ans

Q. Evaluate $\int_0^\infty e^{-3t} t \sin t \, dt$.

Sol^n $\int_0^\infty e^{-st} t \sin t \, dt = L\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$= \frac{2s}{(s^2+1)^2}$ —— (1)

Put $s=3$

$\int_0^\infty e^{-3t} t \sin t \, dt = \frac{6}{(9+1)^2} = \frac{3}{50}$ Ans

Q. If $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$, Prove that $\int_0^\infty e^{-3t}\, t\, J_0(4t)\,dt = \frac{3}{125}$

Sol^n $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

By change of scale prop.

$$L\{J_0(4t)\} = \frac{1}{4}\,\frac{1}{\sqrt{\left(\frac{s}{4}\right)^2+1}} = \frac{1}{\sqrt{s^2+16}}$$

$$\int_0^\infty e^{-st}\cdot t\; J_0(4t)\,dt = L\{t\cdot J_0(4t)\} = -\frac{d}{ds}L\{J_0(4t)\}$$

$$= -\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+16}}\right) = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}s}{(s^2+16)^{3/2}} \quad \text{--- (1)}$$

Put $s = 3$ in (1)

$$\int_0^\infty e^{-3t}\cdot t\, J_0(4t)\,dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125} \quad \text{Ans}$$

Q. show that $\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log 5$

Sol. $\int_0^\infty e^{-st}\,\frac{\sin^2 t}{t}\,dt = L\left\{\frac{\sin^2 t}{t}\right\}$

$$= L\left\{\frac{1-\cos 2t}{2t}\right\}$$

$$= \frac{1}{2}\int_s^\infty\left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+4}}\right]_s^\infty = \frac{1}{2}\log\sqrt{\frac{s^2+4}{s}}$$

Put $s = 1$

$$\int_0^\infty e^{-t}\,\frac{\sin^2 t}{t}\,dt = \frac{1}{2}\log\frac{\sqrt{5}}{1} = \frac{1}{4}\log 5 \quad \text{Ans}$$

Heaviside's Unit step function ⇒

(i) Laplace Transform of unit step function $u(t)$

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$L\{u(t)\} = \int_0^\infty \frac{1}{s}$$

(ii) Laplace Transform of unit step function $u(t-a)$.

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

$$L\{u(t-a)\} = \frac{1}{s} e^{-as}$$

(iii) Laplace Transform of function $f(t-a)\,u(t-a)$

$$f(t-a)\,u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$$

$$= e^{-as} \cdot F(s)$$

(iv) $L\{f(t)\,u(t-a)\}$
$= e^{-as} L\{f(t+a)\}$

(Q.) Find the Laplace Transform of $t^2 u(t-2)$

Sol^n $L\{f(t)\,u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$L\{t^2 u(t-2)\} = e^{-2s} L\{(t+2)^2\}$$

$$= e^{-2s} L\{t^2 + 4 + 4t\}$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s} + \frac{4}{s^2}\right)$$

Q. Find the Laplace Transform of $e^{-t} \sin t \; u(t-\pi)$

Sol^n $L\{f(t)\,u(t-a)\} = e^{-as} L\{f(t+a)\}$

$$L\{e^{-t} \sin t \, u(t-\pi)\} = e^{-\pi s} L\{e^{-(t+\pi)} \sin(t+\pi)\}$$

$$= -e^{-\pi s} e^{-\pi} L\{e^{-t} \sin t\}$$

$$= -e^{-\pi(s+1)} \frac{1}{(s+1)^2 + 1} = e^{-\pi(s+1)} \frac{1}{s^2 + 2s + 2}$$

Q. find the Laplace Transform of $\sin t\, u(t-\frac{\pi}{2}) - u(t-\frac{3\pi}{2})$

Sol^n $L\{\sin t\, u(t-\frac{\pi}{2}) - u(t-\frac{3\pi}{2})\} = L\{\sin t\, u(t-\frac{\pi}{2})\} - L\{u(t-\frac{3\pi}{2})\}$

$$= e^{-\frac{\pi}{2}s} L\{\sin(t+\frac{\pi}{2})\} - \frac{1}{s} e^{-\frac{3\pi}{2}s}$$

$$= e^{-\frac{\pi}{2}s} L\{\cos t\} - \frac{1}{s} e^{-\frac{3\pi}{2}s}$$

$$= e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1} - e^{-\frac{3\pi}{2}s} \cdot \frac{1}{s} \quad \text{As}$$

Q. find the Laplace transforms of the following functions:

i) $f(t) = t^2 \quad 0<t<1$

$\quad = 4t \quad t>1$

(ii) $f(t) = \sin 2t \quad 2\pi < t < 4\pi$

$\quad = 0 \quad$ otherwise

(iii) $f(t) = \begin{cases} \cos t & 0<t<\pi \\ \sin t & t>\pi \end{cases}$

(iv) $f(t) = \begin{cases} \cos t & 0<t<\pi \\ \cos 2t & \pi<t<2\pi \\ \cos 3t & t>2\pi \end{cases}$

Sol^n (i) $f(t) = t^2 u(t) - t^2 u(t-1) + 4t\, u(t-1)$

$L\{f(t)\} = L\{t^2 u(t) - t^2 u(t-1) + 4t\, u(t-1)\}$

$$= L\{t^2 u(t)\} - L\{t^2 u(t-1)\} + 4L\{t\, u(t-1)\}$$

$$= \frac{2}{s^3} - e^{-s} L\{(t+1)^2\} + 4e^{-s} L\{(t+1)\}$$

$$= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) + 4e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$= \frac{2}{s^3} + e^{-s}\left(-\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right) \quad \text{As}$$

(ii) $f(t) = \sin 2t\, u(t-2\pi) - \sin 2t\, u(t-4\pi)$

$L\{f(t)\} = L\{\sin 2t\, u(t-2\pi)\} - L\{\sin 2t\, u(t-4\pi)\}$

$$= e^{-2\pi s} L\{\sin 2(t+2\pi)\} - e^{-4\pi s} L\{\sin 2(t+4\pi)\}$$

$$= e^{-2\pi s} L\{\sin 2t\} - e^{-4\pi s} L\{\sin 2t\}$$

$$= e^{-2\pi s} \cdot \frac{2}{s^2+4} - e^{-4\pi s} \cdot \frac{2}{s^2+4} \quad Ans$$ (11)

iii) $f(t) = \cos t \, u(t) - \cos t \, u(t-\pi) + \sin t \, u(t-\pi)$

$$L\{f(t)\} = L\{\cos t \, u(t)\} - L\{\cos t \, u(t-\pi)\} + L\{\sin t \, u(t-\pi)\}$$

$$= \frac{s}{s^2+1} - e^{-\pi s} L\{\cos(t+\pi)\} + e^{-\pi s} L\{\sin(t+\pi)\}$$

$$= \frac{s}{s^2+1} - e^{-\pi s} L\{-\cos t\} + e^{-\pi s} L\{-\sin t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$$

$$= \frac{1}{s^2+1} [s + e^{-\pi s}(s-1)] \quad Ans$$

iv) $f(t) = [\cos t \, u(t) - \cos t \, u(t-\pi)] + [\cos 2t \, u(t-\pi) - \cos 2t \, u(t-2\pi)] + \cos 3t \, u(t-2\pi)$

~~$L\{f(t)\} = L\{\cos t \, u(t)\} -$~~

$$= \cos t \, u(t) + \{\cos 2t - \cos t\} u(t-\pi) + (\cos 3t - \cos 2t) u(t-2\pi)$$

$$L\{f(t)\} = L\{\cos t \, u(t)\} + L\{(\cos 2t - \cos t) u(t-\pi)\} + L\{(\cos 3t - \cos 2t)\} u(t-2\pi)$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2(t+\pi) - \cos(t+\pi)\} + e^{-2\pi s} L\{\cos 3(t+2\pi) - \cos 2(t+2\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} L\{\cos 2t + \cos t\} + e^{-2\pi s} L\{\cos 3t - \cos 2t\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s} \left(\frac{s}{s^2+4} + \frac{s}{s^2+1} \right) + e^{-2\pi s} \left(\frac{s}{s^2+9} - \frac{s}{s^2+4} \right)$$

Ans

Laplace Transform of Dirac Delta functions :-

(i) $L\{\delta(t)\} = 1$

(ii) $L\{\delta(t-a)\} = e^{-as}$

(iii) $L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

Q. Find the Laplace Transform of the following functions

(i) $\sin 2t\,\delta(t-\frac{\pi}{4}) - t^2\delta(t-2)$

(ii) $t\,u(t-4) + t^2\delta(t-4)$

(iii) $t^2 u(t-2) - \cosh t\,\delta(t-2)$

Soln (i) $L\{f(t)\,\delta(t-a)\} = e^{-as} f(a)$

$$L\{\sin 2t\,\delta(t-\tfrac{\pi}{4}) - t^2\delta(t-2)\} = e^{-\frac{\pi}{4}s}\sin 2\left(\tfrac{\pi}{4}\right) - e^{-2s}(2)^2$$

$$= e^{-\frac{\pi}{4}s} - 4e^{-2s} \quad \text{Ans}$$

(ii) $L\{t\,u(t-4) + t^2\delta(t-4)\}$

$$= e^{-4s} L\{(t+4)\} + L\{t^2\delta(t-4)\}$$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + e^{-4s}\cdot(4)^2$$

$$= e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s} + 16\right) \quad \text{Ans}$$

(iii) $L\{t^2 u(t-2) - \cosh t\,\delta(t-2)\}$

$$= L\{t^2 u(t-2)\} - L\{\cosh t\,\delta(t-2)\}$$

$$= e^{-2s} L\{(t+2)^2\} - e^{-2s}\cosh 2$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) - e^{-2s}\cosh 2$$

$$= e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} - \cosh 2\right) \quad \text{Ans}$$

Q. Evaluate the following integrals :-

(i) $\int_0^\infty \cos 2t\, \delta(t-\frac{\pi}{4})\, dt$ (ii) $\int_0^\infty t^2 e^{-t} \sin t\, \delta(t-2)\,dt$

(iii) $\int_0^\infty t^m (\log t)^n\, \delta(t-3)\, dt$

Sol^n (i) $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\int_0^\infty \cos 2t\, \delta(t-\frac{\pi}{4})\, dt = \cos\frac{2\pi}{4} = 0$$

(ii) $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

$$\int_0^\infty t^2 e^{-t} \sin t\, \delta(t-2)\, dt = 2^2 \cdot e^{-2} \sin 2 = 4e^{-2}\sin 2$$

(iii) $$\int_0^\infty t^m (\log t)^n\, \delta(t-3)\, dt = 3^m (\log 3)^n \quad \text{Ans}$$

Laplace Transform of Periodic function ⇒

Let $f(t)$ be a periodic function with period T

i.e $f(t+T) = f(t)$

then $$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Q. Find the Laplace Transform of $f(t) = k\frac{t}{T}$, $0<t<T$

if $f(t) = f(t+T)$.

Sol^n $$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \cdot \frac{k}{T} t\, dt$$

$$= \frac{k}{T}\frac{1}{1-e^{-sT}} \left\{ t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right\}_0^T$$

$$= \frac{k}{T(1-e^{-sT})} \left\{ -\frac{T}{s} e^{-sT} + \frac{1}{s^2}(1-e^{-sT}) \right\}$$

$$= \frac{k}{Ts^2} - \frac{ke^{-sT}}{s(1-e^{-sT})} \quad \text{Ans}$$

Q. Find the Laplace Transform of

$$f(t) = \begin{cases} t & 0<t<1 \\ 0 & 1<t<2 \end{cases}$$

if $f(t) = f(t+T)$

Sol^n $L\{f(t)\} = \frac{1}{1-e^{-s2}} \int_0^2 e^{-st} f(t)\, dt$

$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot t\, dt + \int_1^2 0 \right]$

$= \frac{1}{1-e^{-2s}} \left[-\frac{e^{-st}}{s} t - \frac{e^{-st}}{s^2} \right]_0^1$

$= \frac{1}{(1-e^{-2s})\, s^2} (1 - e^{-s} - s e^{-s})$ Ans

Q. Find the Laplace Transform of

$$f(t) = \begin{cases} t^2 & 0<t<2 \\ f(t+2) & \end{cases}$$

Sol^n $L\{f(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} \cdot t^2\, dt$

$= \frac{1}{1-e^{-2s}} \left[t^2 \frac{e^{-st}}{-s} - \frac{2t}{s^2} e^{-st} + \frac{2}{-s^3} e^{-st} \right]_0^2$

$= \frac{1}{1-e^{-2s}} \left(-\frac{4}{s} e^{-2s} - \frac{4}{s^2} e^{-2s} - \frac{2}{s^3} e^{-2s} + \frac{2}{s^3} \right)$

$= \frac{1}{1-e^{-2s}} \cdot \frac{1}{s^3} \left(2 - 2e^{-2s} - 4s e^{-2s} - 4s^2 e^{-2s} \right)$ Ans

Q. Find the Laplace Transform of

$f(t) = e^t \qquad 0<t<2\pi$

$f(t) = f(t+2\pi)$

Sol^n $L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} \cdot e^{t}\, dt$

$= \frac{1}{1-e^{-2\pi s}} \left[\frac{e^{(1-s)t}}{1-s} \right]_0^{2\pi}$

$= \frac{e^{(1-s)2\pi} - 1}{(1-e^{-2\pi s})(1-s)}$ Ans

Inverse Laplace Transform $\Rightarrow$ If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace Transform of $F(s)$ & written as

$$f(t) = L^{-1}\{F(s)\}$$

(1) $L^{-1}\left(\frac{1}{s}\right) = 1$

2. $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$, $n = 1, 2, \dots$

3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

4. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

5. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

6. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

7. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

8.

Q. Find the inverse Laplace Transform of following function:

(i) $\frac{s^2-3s+4}{s^3}$ (ii) $\frac{3s+4}{s^2+9}$ (iii) $\frac{4s+15}{16s^2-25}$ (iv) $\frac{2s+3}{s^2+2s+2}$

(v) $\frac{3s+1}{(s+1)^4}$ (vi) $\frac{s}{(2s+1)^2}$ (vii) $\frac{3s+1}{(s+1)^4}$ (viii) $\frac{1}{\sqrt{2s+3}}$

Sol^n (i) $F(s) = \frac{s^2-3s+4}{s^3} = \frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}$

$L^{-1}\{F(s)\} = 1 - 3t + 2t^2$ Ans

(ii) $F(s) = \frac{3s+4}{s^2+9} = \frac{3s}{s^2+9} + \frac{4}{s^2+9}$

$L^{-1}\{F(s)\} = 3\cos 3t + \frac{4}{3}\sin 3t$ Ans

(iii) $F(s) = \frac{4s+15}{16s^2-25}$

$\{F(s)\} = \frac{4s}{16\left(s^2-\frac{25}{16}\right)} + \frac{15}{16\left(s^2-\frac{25}{16}\right)}$

$L^{-1}\{F(s)\} = \frac{1}{4}.\cosh\frac{5}{4}t + \frac{3}{4}\sinh\frac{5}{4}t$

(iv) $F(s) = \dfrac{2s+3}{s^2+2s+10} = \dfrac{2s+2+1}{s^2+2s+10} = \dfrac{2(s+1)+1}{(s+1)^2+1}$

$= \dfrac{2(s+1)}{(s+1)^2+1} + \dfrac{1}{(s+1)^2+1}$

$L^{-1}\{F(s)\} = 2e^{-t}\, L^{-1}\left\{\dfrac{s}{s^2+1}\right\} + e^{-t}\, L^{-1}\left\{\dfrac{1}{s^2+1}\right\}$

$= \cancel{2e^{-t}\cos t} \cdot 2e^{-t}\cos t + e^{-t}\sin t$

(v) $F(s) = \dfrac{3s+1}{(s+1)^4} = \dfrac{3(s+1)-2}{(s+1)^4} = \dfrac{3}{(s+1)^3} - \dfrac{2}{(s+1)^4}$

$L^{-1}\{F(s)\} = 3e^{-t}\, L^{-1}\left\{\dfrac{1}{s^3}\right\} - 2e^{-t}\, L^{-1}\left\{\dfrac{1}{t^4}\right\}$

$= 3e^{-t}\dfrac{t^2}{2!} - 2e^{-t}\dfrac{t^3}{3!} = \dfrac{3}{2}e^{-t}t^2 - \dfrac{1}{3}e^{-t}t^3$ Ans

(vi) $F(s) = \dfrac{s}{(2s+1)^2} = \dfrac{1}{4}\dfrac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2} = \dfrac{1}{4}\left[\dfrac{1}{s+\frac{1}{2}} - \dfrac{1}{2\left(s+\frac{1}{2}\right)^2}\right]$

$L^{-1}\{F(s)\} = \dfrac{1}{4}L^{-1}\left\{\dfrac{1}{s+\frac{1}{2}}\right\} - \dfrac{1}{8}e^{-\frac{1}{2}}\, L^{-1}\left\{\dfrac{1}{s^2}\right\}$

$= \dfrac{1}{4}e^{-t/2} - \dfrac{1}{8}e^{-t/2}\,t$ Ans

(vii) $F(s) = \dfrac{3s+1}{(s+1)^4} = \dfrac{3(s+1)-2}{(s+1)^4} = \dfrac{3}{(s+1)^3} - \dfrac{2}{(s+1)^4}$

$L^{-1}\{F(s)\} = 3e^{-t}\, L^{-1}\left\{\dfrac{1}{s^3}\right\} - 2e^{-t} L^{-1}\left\{\dfrac{1}{s^4}\right\}$

$= 3e^{-t}.\dfrac{t^2}{2} - 2e^{-t}.\dfrac{t^3}{6} = \dfrac{3}{2}e^{-t}t^2 - \dfrac{1}{3}e^{-t}t^3$ A

(viii) $F(s) = \dfrac{1}{\sqrt{2s+3}} = \dfrac{1}{\sqrt{2}}\dfrac{1}{\left(s+\frac{3}{2}\right)^{1/2}}$

$L^{-1}\{F(s)\} = \dfrac{1}{\sqrt{2}}e^{-\frac{3t}{2}}\, L^{-1}\left\{\dfrac{1}{s^{1/2}}\right\} = \dfrac{1}{\sqrt{2}}e^{-\frac{3t}{2}}\dfrac{t^{-\frac{1}{2}}}{\Gamma\frac{1}{2}}$

$= \dfrac{1}{\sqrt{2\pi}}e^{-\frac{3t}{2}}.t^{-1/2}$ A

Partial fraction ⇒

Case - I factors are linear & distinct

$$F(s) = \frac{P(s)}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b}$$

Case - II factors are linear & repeated

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{B_1}{(s+b)^2} \cdots \frac{B_n}{(s+b)^n}$$

Case - III factors are quadratic & distinct

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)} = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case - IV factors are quadratic & repeated

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

$$= \frac{As+B}{s^2+as+b} + \frac{C_1s+D_1}{s^2+cs+d} + \frac{C_2s+D_2}{(s^2+cs+d)^2} \cdots \frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Q find the inverse Laplace Transform of following functions -:

(i) $\frac{s+2}{s(s+1)(s+3)}$ (ii) $\frac{s+2}{s^2(s+3)}$ (iii) $\frac{s^2-15s-11}{(s+1)(s-2)^3}$

(iv) $\frac{3s+1}{(s+1)(s^2+2)}$ (v) $\frac{s}{(s^2+1)(s^2+4)}$ (vi) $\frac{s+2}{(s^2+4s+8)(s^2+4s+13)}$

Sol^n $F(s) = \frac{s+2}{s(s+1)(s+3)} = \cancel{\frac{2}{3s}} + \cancel{\frac{1}{2}}$

$$= \frac{2}{3s} - \frac{1}{2(s+1)} - \frac{1}{6(s+3)}$$

$$L^{-1}\{F(s)\} = \frac{2}{3} L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2} L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6} L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t} \quad \text{Ans}$$

(ii) $F(s) = \dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3}$

on solving

$F(s) = \dfrac{1}{9s} + \dfrac{2}{3s^2} - \dfrac{1}{9(s+3)}$

$L^{-1}\{F(s)\} = \dfrac{1}{9} L^{-1}\{\dfrac{1}{s}\} + \dfrac{2}{3} L^{-1}\{\dfrac{1}{s^2}\} - \dfrac{1}{9} L^{-1}\{\dfrac{1}{s+3}\}$

$= \dfrac{1}{9} + \dfrac{2}{3} t - \dfrac{1}{9} e^{-3t}$ Ans

(iii) $F(s) = \dfrac{5s^2-15s-11}{(s+1)(s-2)^2} = \dfrac{A}{s+1} + \dfrac{B}{s-2} + \dfrac{C}{(s-2)^2}$

on solving

$F(s) = \dfrac{1}{s+1} + \dfrac{4}{s-2} - \dfrac{7}{(s-2)^2}$

$L^{-1}\{F(s)\} = e^{-t} + 4e^{2t} - 7e^{2t} L^{-1}\{\dfrac{1}{s^2}\}$

$= e^{-t} + 4e^{2t} - 7e^{2t} t$

(iv) $F(s) = \dfrac{3s+1}{(s+1)(s^2+2)} = \dfrac{A}{s+1} + \dfrac{Bs+C}{s^2+2}$

on solving

$F(s) = -\dfrac{2}{3(s+1)} + \dfrac{2}{3}\dfrac{s}{(s^2+2)} + \dfrac{7}{3(s^2+2)}$

$L^{-1}\{F(s)\} = -\dfrac{2}{3} L^{-1}\left(\dfrac{1}{s+1}\right) + \dfrac{2}{3} L^{-1}\left(\dfrac{s}{s^2+2}\right) + \dfrac{7}{3} L^{-1}\{\dfrac{1}{s^2+2}\}$

$= -\dfrac{2}{3} e^{-t} + \dfrac{2}{3} \cos\sqrt{2}t + \dfrac{7}{3}\cdot\dfrac{1}{\sqrt{2}} \sin\sqrt{2}t$ Ans

(v) $F(s) = \dfrac{s}{(s^2+1)(s^2+4)} = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+4}$

on solving

$F(s) = \dfrac{1}{3}\left[\dfrac{s}{s^2+1} - \dfrac{s}{s^2+4}\right]$

$L^{-1}\{F(s)\} = \dfrac{1}{3}\{\cos t - \cos 2t\}$

(vi) $F(s) = \dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \dfrac{As+B}{s^2+4s+8} + \dfrac{Cs+D}{s^2+4s+13}$

on solving $F(s) = \dfrac{1}{5}\left[\dfrac{s+2}{s^2+4s+8} - \dfrac{s+2}{s^2+4s+13}\right]$

$$= \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} - \frac{s+2}{(s+2)^2+9}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{5}\, e^{-2t} L^{-1}\left\{\frac{s}{s^2+4}\right\} - e^{-2t} L^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= \frac{1}{5}\left[e^{-2t}\cos 2t - e^{-2t}\cos 3t\right] \text{ Ans}$$

Convolution Theorem ⇒ If $L^{-1}\{F_1(s)\} = f_1(t)$ and $L^{-1}\{F_2(s)\} = f_2(t)$ then

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

Q. Find the inverse laplace Transform of following functions :-

(i) $\frac{1}{(s+2)(s-1)}$ (ii) $\frac{1}{s^2(s+1)^2}$ (iii) $\frac{1}{s\sqrt{s^2+4}}$ (iv) $\frac{1}{s(s^2+a^2)}$

(v) $\frac{1}{s^2(s^2+1)}$ (vi) $\frac{(s+2)^2}{(s^2+4s+8)^2}$ (vii) $\frac{1}{(s^2+4s+13)^2}$

Ans (i) $F(s) = \frac{1}{(s+2)(s-1)}$

let $F_1(s) = \frac{1}{s+2}$ $F_2(s) = \frac{1}{s-1}$

$f_1(t) = e^{-2t}$ $f_2(t) = e^{t}$

By convolution theorem

$$L^{-1}\{F(s)\} = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$= \int_0^t e^{-2u}\cdot e^{(t-u)}\, du = e^t \int_0^t e^{-3u}\, du$$

$$= e^t \left[\frac{e^{-3u}}{-3}\right]_0^t = -\frac{1}{3} e^t \left(e^{-3t} - 1\right)$$

$$= \frac{e^t}{3}\left(1 - e^{-3t}\right) \text{ Ans}$$

(ii) $F(s) = \frac{1}{s^2(s+1)^2}$

let $F_1(s) = \frac{1}{(s+1)^2}$ $F_2(s) = \cancel{\frac{1}{(s+1)^2}}\ \frac{1}{s^2}$

$f_1(t) = e^{-t}\cdot t$ $f_2(t) = t$

By convolution theorem

$$L^{-1}\{F(s)\} = \int_0^t u e^{-u}(t-u)\,du$$

$$= t\int_0^t e^{-u}(ut-u^2)\,du$$

$$= t\left[(ut-u^2)\frac{e^{-u}}{-1} - (t-2u)e^{-u} + (-2)\frac{e^{-u}}{-1}\right]_0^t$$

$$= te^{-t} + 2e^{-t} + t - 2 \text{ Ans}$$

(iii) $F(s) = \dfrac{1}{s\sqrt{s+4}}$

let $F_1(s) = \dfrac{1}{\sqrt{s+4}}$ $\qquad F_2(s) = \dfrac{1}{s}$

$f_1(t) = e^{-4t}\dfrac{t^{-1/2}}{\Gamma\frac{1}{2}}$ $\qquad f_2(t) = 1$

$= e^{-4t}\sqrt{\dfrac{1}{\pi t}}$ $\qquad f_2(t) = 1$

By convolution theorem

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}\cdot u^{-1/2}\,du$$

Put $4u = x^2$

$du = \frac{x}{2}dx$

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\cdot\frac{\cancel{2}}{\cancel{x}}\cdot\frac{\cancel{x}}{\cancel{2}}\,dx = \frac{1}{2}\,\text{erf}\cdot 2\sqrt{t}$$

(iv) $F(s) = \dfrac{1}{s(s^2+a^2)}$

let $F_1(s) = \dfrac{1}{s^2+a^2}$ $\qquad F_2(s) = \dfrac{1}{s}$

$f_1(t) = \dfrac{1}{a}\sin at$ $\qquad f_2(t) = 1$

By convolution theorem

$$L^{-1}\{F(s)\} = \int_0^t \frac{1}{a}\sin au\,du = -\frac{1}{a^2}[\cos au]_0^t$$

$$= \frac{1}{a^2}(1-\cos at) \text{ Ans}$$

(v) $F(s) = \dfrac{1}{s^2(s^2+1)}$

$F_1(s) = \dfrac{1}{s^2+1}$ $\qquad F_2(s) = \dfrac{1}{s^2}$

$f_1(t) = \sin t$ $\qquad f_2(t) = t$

(i) By convolution theorem

$$L^{-1}\{F(s)\} = \int_0^t \sin u\,(t-u)\,du$$

$$= \left[(t-u)(-\cos u) - (-1)(-\sin u)\right]_0^t$$

$$= t - \sin t$$ Ans

(vi) $F(s) = \dfrac{(s+2)^2}{(s^2+4s+8)^2}$

$F_1(s) = \dfrac{s+2}{(s^2+4s+8)}$ $\quad$ $F_2(s) = \dfrac{s+2}{(s^2+4s+8)^2}$

$= \dfrac{s+2}{(s+2)^2+4}$ $\quad$ $= \dfrac{s+2}{(s+2)^2+4}$

$f_1(t) = e^{-2t}\cos 2t$ $\quad$ $f_2(t) = e^{-2t}\cos 2t$

$$L^{-1}\{F(s)\} = \int_0^t e^{-2u}\cos 2u\; e^{-2(t-u)}\cos 2(t-u)\,du$$

$$= e^{-2t}\int_0^t \cos 2u\cos 2(t-u)\,du$$

$$= \frac{e^{-2t}}{2}\int_0^t \left[\cos 2t + \cos(4u-2t)\right]du$$

$$= \frac{e^{-2t}}{2}\left[u\cos 2t + \frac{\sin(4u-2t)}{4}\right]_0^t$$

$$= \frac{e^{-2t}}{4}\left[t\cos 2t + \frac{\sin 2t}{4} + \frac{\sin 2t}{4}\right]$$

$$= \frac{e^{-2t}}{4}\left[\sin 2t + 2t\cos 2t\right]$$ Ans

(vii) $F_1(s) = \dfrac{1}{s^2+4s+13}$ $\quad$ $F_2(s) = \dfrac{1}{s^2+4s+13} = \dfrac{1}{}$

$= \dfrac{1}{(s+2)^2+9}$ $\quad$ $= \dfrac{1}{(s+2)^2+9}$

$f_1(t) = \dfrac{e^{-2t}}{3}\sin 3t$ $\quad$ $f_2(t) = \dfrac{e^{-2t}}{3}\sin 3t$

$$L^{-1}\{F(s)\} = \int_0^t \frac{e^{-2u}}{3}\sin 3u\;\frac{e^{-2(t-u)}}{3}\sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{18}\left(\frac{\sin 3t}{3} - t\cos 3t\right)$$ Ans

$= t \int e^{-(...)} (ut - u^2) \, du$

Differentiation of F(s)

$L\{f(t)\} = F(s)$ then

$L\{t f(t)\} = -F'(s)$

$L^{-1}\{F'(s)\} = -t f(t)$

Hence $L^{-1}\{F(s)\} = f(t) = -\frac{1}{t} L^{-1}\{F'(s)\}$

Q. find the inverse Laplace Transform of following function :.

i) $\log \frac{s+a}{s+b}$ (ii) $\log \frac{s^2+b^2}{s^2+a^2}$ (iii) $\log \sqrt{\frac{s-1}{s+1}}$

(iv) $\tan^{-1} \frac{2}{s}$ (v) $\tan^{-1}\left(\frac{s+a}{b}\right)$ (vi) $\cot^{-1}(s+1)$ (vii) $\log \sqrt{\frac{s^2+1}{s(s+1)}}$

Solⁿ i) $F(s) = \log \frac{s+a}{s+b} = \log(s+a) - \log(s+b)$

$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = -\frac{1}{t}\left(e^{-at} - e^{-bt}\right)$

(ii) $F(s) = \log \frac{s^2+b^2}{s^2+a^2} = \log(s^2+b^2) - \log(s^2+a^2)$

$F'(s) = \frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}$

$L^{-1}\{F(s)\} = -\frac{1}{t} L^{-1}\left\{\frac{2s}{s^2+b^2} - \frac{2s}{s^2+a^2}\right\} = -\frac{2}{t}(\cos bt - \cos at)$

$= \frac{2}{t}(\cos at - \cos bt)$ Ans

(iii) $F(s) = \log \sqrt{\frac{s-1}{s+1}} = \log\sqrt{s-1} - \log\sqrt{s+1} = \frac{1}{2}\left[\log(s-1) - \log(s+1)\right]$

$F'(s) = \frac{1}{2}\left[\frac{1}{(s-1)} - \frac{1}{s+1}\right]$

$L^{-1}F(s) = -\frac{1}{t} L^{-1} \frac{1}{2}\left[\frac{1}{s-1} - \frac{1}{s+1}\right]$

$= -\frac{1}{t}\left[\frac{e^{t} - e^{-t}}{2}\right] = -\frac{1}{t} \sinh t$ Ans

(iv) $F(s) = \tan^{-1}\left(\frac{2}{s}\right)$

$$F'(s) = \frac{1}{1+\frac{4}{s^2}}\left(-\frac{2}{s^2}\right) = -\frac{2}{s^2+4}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{2}{s^2+4}\right\} = \frac{1}{t}\sin 2t$$

(v) $F(s) = \tan^{-1}\left(\frac{s+a}{b}\right)$

$$F'(s) = \frac{1}{1+\left(\frac{s+a}{b}\right)^2}\cdot\left(\frac{1}{b}\right) = \frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{\frac{b}{(s+a)^2+b^2}\right\} = -\frac{1}{t}e^{-at}\sin bt$$

&

(vi) $F(s) = \cot^{-1}(s+1)$

$$F'(s) = -\frac{1}{1+(s+1)^2}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\} = \frac{1}{t}e^{-t}\sin t$$

&

(vii) $F(s) = \log\frac{s^2+1}{s(s+1)} = \log(s^2+1) - \log s - \log(s+1)$

$$F'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}\{F(s)\} = -\frac{1}{t}L^{-1}\{F'(s)\} = \frac{1}{t}\{e^{-t} + 1 - 2\cos t\}$$

&

Second shifting property (Heaviside unit step) => If $L\{f(t)\} = F(s)$

then $L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

Q. Find the inverse laplace transform of the following functions :-

(i) $\frac{e^{-2s}}{(s+4)^3}$ (ii) $\frac{e^{-3s}}{s^2+4}$ (iii) $\frac{e^{-2s}}{s^2+8s+25}$ (iv) $\frac{se^{-2s}}{s^2+2s+2}$

(v) $e^{-s}\left(\frac{1+\sqrt{s}}{s^3}\right)$ (vi) $\frac{e^{-\pi s}}{s^2-2s+2}$

Soln (i) $F(s) = \frac{1}{(s+4)^3}$

$L^{-1}\{F(s)\} = e^{-4t} L^{-1}\{\frac{1}{s^3}\} = e^{-4t} \cdot \frac{t^2}{2}$

$L^{-1}\{e^{-2s} F(s)\} = e^{-4(t-2)} \frac{(t-2)^2}{2} \cdot u(t-2)$ Ans

(ii) $F(s) = \frac{1}{s^2+4}$

$L^{-1}\{F(s)\} = \frac{1}{2}\sin 2t$

$L^{-1}\{e^{-3s} \cdot F(s)\} = \frac{1}{2}\sin 2(t-3)\, u(t-3)$

(iii) $F(s) = \frac{e^{-2s}}{s^2+8s+25}$

$L^{-1}\{\frac{1}{s^2+8s+25}\} = L^{-1}\{\frac{1}{(s+4)^2+9}\}$

$= e^{-4s} L^{-1}\{\frac{1}{s^2+9}\} = \frac{e^{-4s}}{3} \sin 3t$

$L^{-1}\{\frac{e^{-2s}}{s^2+8s+25}\} = \frac{1}{3} e^{-4(t-2)} \sin 3(t-2)\, u(t-2)$

(iv) $F(s) = \frac{s}{s^2+2s+2}$

$L^{-1}\{F(s)\} = L^{-1}\{\frac{(s+1)-1}{(s+1)^2+1}\} = e^{-t} L^{-1}\{\frac{s}{s^2+1} - \frac{1}{s^2+1}\}$

$= e^{-t}(\cos t - \sin t)$

$L^{-1}\{e^{-2s} F(s)\} = e^{-(t-2)}\{\cos(t-2) - \sin(t-2)\}\, u(t-2)$

(v) $F(s) = \frac{1+\sqrt{s}}{s^3}$

$L^{-1}\{F(s)\} = L^{-1}\{\frac{1}{s^3} + \frac{1}{s^{5/2}}\} = \frac{t^2}{2} + \frac{t^{3/2}}{\Gamma\frac{5}{2}} = \frac{t^2}{2} + \frac{4t^{3/2}}{3\sqrt{\pi}}$

$L^{-1}\{e^{-s} F(s)\} = \left[\frac{(t-1)^2}{2} + \frac{4}{3\sqrt{\pi}}(t-1)^{3/2}\right] u(t-1)$ Ans

(vi) $F(s) = \frac{1}{s^2-2s+2}$

$L^{-1}\{F(s)\} = L^{-1}\{\frac{1}{(s-1)^2+1}\} = e^{t} L^{-1}\{\frac{1}{s^2+1}\} = e^{t}\sin t$

$L^{-1}\{e^{-\pi s} F(s)\} = e^{(t-\pi)} \sin(t-\pi)\, u(t-\pi)$ Ans

Application of Laplace Transform →

Q. Solve $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$

Sol^n Taking L.T both sides

$$s\bar{y} - y(0) + 2\bar{y} = \frac{1}{s+3}$$

$$\bar{y}(s+2) = \frac{1}{s+3} + 1 = \frac{s+4}{s+3}$$

$$\bar{y} = \frac{s+4}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

on taking L^{-1}

$$y = 2e^{-2t} + e^{-3t} \quad \text{A}$$

Q. solve $y'' + 4y' + 8y = 1$, $y(0) = 0$ $y'(0) = 1$

Sol^n Taking L.T both sides

$$s^2\bar{y} - s\,y(0) - y'(0) + 4\{s\bar{y} - y(0)\} + 8\bar{y} = \frac{1}{s}$$

$$\bar{y}(s^2 + 4s + 8) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\bar{y} = \frac{s+1}{s(s^2+4s+8)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+8}$$

on solving

$$\bar{y} = \frac{1}{8s} - \frac{1}{8}\frac{s}{(s^2+4s+8)} + \frac{1}{2(s^2+4s+8)}$$

$$= \frac{1}{8s} - \frac{\{(s+2)-2\}}{8\{(s+2)^2+4\}} + \frac{1}{2}\{(s+2)^2+4\}$$

on taking L^{-1}

$$y = \frac{1}{8} - \frac{e^{-2t}}{8} \cdot L^{-1}\left\{\frac{s}{s^2+4} - \frac{2}{s^2+4}\right\} + \frac{1}{2}e^{-2t} L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{1}{8} - \frac{e^{-2t}}{8}(\cos 2t - \sin 2t) + \frac{e^{-2t}}{4}\sin 2t$$

$$= \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t \quad \text{A}$$

Q. solve $(D^2 + 2D + 5)y = e^{-t}\sin t$ $y(0) = 0$, $y'(0) = 1$

Sol^n taking laplace both sides

$\{s^2\bar{y} - sy(0) - y'(0)\} + 2\{s\bar{y} - y(0)\} + 5\bar{y} = \frac{1}{(s+1)^2+1}$

$(s^2+2s+5)\bar{y} = \frac{1}{s^2+2s+2} + 1$

$\bar{y} = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$

$\bar{y} = \frac{1}{3(s^2+2s+2)} + \frac{2}{3(s^2+2s+5)}$

$= \frac{1}{3\{(s+1)^2+1\}} + \frac{2}{3\{(s+1)^2+4\}}$

on ~~intever~~ inverse L.T

$y = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$ Ans

Q. solve $y'' + 4y = \delta(t)$, $y(0) = 0$, $y'(0) = 0$

Sol^n on L.T

$s^2\bar{y} - sy(0) - y'(0) + 4\bar{y} = 1$

$\bar{y} = \frac{1}{s^2+4}$

on L^{-1}

$y = \frac{1}{2}\sin 2t$

Q. solve $y'' + 3y' + 2y = t\,\delta(t-1)$, $y(0) = 0$, $y'(0) = 0$

Sol^n on L.T

$s^2\bar{y} - sy(0) - y'(0) + 3\{s\bar{y} - y(0)\} + 2\bar{y} = e^{-s}$

$\bar{y}(s^2+3s+2) = e^{-s}$

$\bar{y} = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = e^{-s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$

on L^{-1}

$y = e^{-(t-1)}u(t-1) - e^{-2(t-1)}u(t-1)$ Ans