Method Of Separation Of Variables

In this method, we assume that the dependent variable is the product of functions of independent variables.

Q using the method of separation of variables, solve.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \quad \text{where} \quad u(x,0) = 6e^{-3x}$$

Sol^n Let $u(x,t) = u = X(x) \times T(t) \rightarrow$ (A)

$$u = XT \quad (1)$$

wher X is the function of x only and T is the function of t only.

from (1)

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx}, \quad \frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

substituting in given Equation.

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

dividing both the sides by XT

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$$

this Equation is hold good If every side of Equation equal to any constant say 'c'

i.e $\frac{1}{X}\frac{dX}{dx} = c \Rightarrow \frac{dX}{dx} - cX = 0 \rightarrow (2)$

which O.D.E with constant coefficient

$$\frac{dX}{dx} - cX = 0 \qquad \frac{d}{dx} = D$$

$$DX - cX = 0 \Rightarrow (D-c)X = 0$$

$\underline{\underline{A.E}}$ $\quad D = m \quad m - c = 0 \Rightarrow m = c$ (one real root)

$$X = a\,e^{cx} \rightarrow (3)$$

again taking $\frac{2}{T}\frac{dT}{dt} + 1 = c \Rightarrow \frac{2}{T}\frac{dT}{dt} = c-1$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = \left(\frac{c-1}{2}\right) \Rightarrow \frac{dT}{dt} - \left(\frac{c-1}{2}\right)T = 0$$

which is again ODE with constant coefficient

$$\frac{d}{dt} = D \qquad DT - \left(\frac{c-1}{2}\right)T = 0 \Rightarrow \left[D - \left(\frac{c-1}{2}\right)\right]T = 0$$

$\underline{\underline{A.E}}$ $\quad D = m \quad m - \left(\frac{c-1}{2}\right) = 0 \Rightarrow m = \left(\frac{c-1}{2}\right)$ (real root)

$$T = b\,e^{\left(\frac{c-1}{2}\right)t} \rightarrow (4)$$

from (1)

$$u = XT \Rightarrow u = a\,e^{cx}\,b\,e^{\left(\frac{c-1}{2}\right)t}$$

$$u = ab\,e^{cx + \left(\frac{c-1}{2}\right)t} \Rightarrow u(x,t) = e^{cx + \left(\frac{c-1}{2}\right)t} \quad (5)$$

condition given $u(x,0) = 6e^{-3x}$.

$$u(x,t) = ab\,e^{cx + \left(\frac{c-1}{2}\right)t}$$

$$u(x,0) = ab\,e^{cx + 0}$$

$$6e^{-3x} = ab\,e^{cx}$$

comparing coefficien of exponent on both the sides

$$ab = 6$$

comparing power of exponent on both the sides-

$$-3x = cx \Rightarrow c = -3$$

substituting $ab=6$, $c=-3$ in (5) $\quad u(x,t) = 6e^{-3x + \left(\frac{-3-1}{2}\right)t}$

$$u(x,t) = 6e^{-3x - 2t}$$

Ans//

Solve by Separation of variables.

$$4\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 3u, \to ① \quad u = 3e^{-x} - e^{-5x}$$

$$u = XT$$

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt}, \quad \frac{\partial u}{\partial x} = T\frac{dX}{dx}. \quad \text{from } ①$$

$$4X\frac{dT}{dt} + T\frac{dX}{dx} = 3XT \quad \div XT$$

$$4\frac{1}{T}\frac{dT}{dt} + \frac{1}{X}\frac{dX}{dx} = 3$$

$$\frac{4}{T}\frac{dT}{dt} - 3 = -\frac{1}{X}\frac{dX}{dx} = p^2 \text{ (say)}$$

I

$$\frac{4}{T}\frac{dT}{dt} = p^2+3 \Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{p^2+3}{4}$$

$$\frac{dT}{T} = \left(\frac{p^2+3}{4}\right)dt$$

$$\log T = \left(\frac{p^2+3}{4}\right)\cdot t + \log C_1$$

$$\log T/C_1 = \left(\frac{p^2+3}{4}\right)t \Rightarrow T/C_1 = e^{\left(\frac{p^2+3}{4}\right)t}$$

$$T = C_1 e^{\left(\frac{p^2+3}{4}\right)t} \to ②$$

$$-\frac{1}{X}\frac{dX}{dx} = p^2$$

$$\frac{dX}{X} = -p^2 dx \Rightarrow \quad \log X = -p^2 x + \log C_2$$

$$\log\frac{X}{C_2} = -p^2 x$$

$$X = C_2 e^{-p^2 x} \to ③$$

$u = XT$

$$u = c_1 c_2 \, e^{\left(\frac{p^2+3}{4}\right)t} \, e^{-p^2 x}$$

$$u = \underline{c_1 c_2} \, e^{\left(\frac{p^2+3}{4}\right)t - p^2 x}$$

b_n

$$u = \sum_{n=1}^{\infty} b_n \, e^{\left(\frac{p^2+3}{4}\right)t - p^2 x} \rightarrow (4)$$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \, e^{-p^2 x}$$

$$3e^{-x} - e^{-5x} = b_1 e^{-p^2 x} + b_2 e^{-p^2 x} \text{ (cosp. up to Two term)}$$

$b_1 = 3, \; p^2 = 1, \qquad b_2 = -1, \; p^2 = 5$

by (4)

$$u = b_1 \, e^{\left(\frac{p^2+3}{4}\right)t + p^2 x} + b_2 \, e^{\left(\frac{p^2+3}{4}\right)t - p^2 x}$$

$$\boxed{u = 3e^{t-x} + e^{2t-5x}}$$

Solve by the method of separation of variables

$$4\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=3u\,,\quad u=3e^{-x}-e^{-5x}\ \text{ when } t=0$$

Sol^n: Let $u = XT$ —— ①

where X is a function of x only and T is a function of t only

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}(XT)=X\frac{dT}{dt}=XT'\ (\text{say})$$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(XT)\ T\frac{dX}{dx}=X'T\ (\text{say})$$

From the given equation

$$4XT'+X'T=3XT$$

$$\frac{4T'}{T}+\frac{X'}{X}=3$$

$$\frac{4T'}{T}-3=-\frac{X'}{X}=p^2\ (\text{say})$$

$$\frac{4T'}{T}-3=p^2$$

$$\frac{4T'}{T}=p^2+3$$

$$\frac{4}{T}\frac{dT}{dt}=p^2+3$$

$$\frac{dT}{T}=\left(\frac{p^2+3}{4}\right)dt \quad —— ②$$

integration gives

$$\log T=\left(\frac{p^2+3}{4}\right)t+\log c_1 \Rightarrow T=c_1e^{\left(\frac{3+p^2}{4}\right)t} \quad —— ③$$

...ce $-\frac{X'}{X} = p^2$

$$\frac{X'}{X} = -p^2$$

$$\frac{1}{X}\frac{dX}{dx} = -p^2$$

$$\frac{dX}{X} = -p^2\,dx$$

$$\log X = -p^2 x + \log c_2$$

$$\frac{X}{c_2} = e^{-p^2 x}$$

$$X = c_2\, e^{-p^2 x} \quad \text{--- (4)}$$

from (1), we get

$$u = XT = c_1 c_2 e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$$

or $\quad u(x,t) = b_n e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t}$

Most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-p^2 x + \left(\frac{3+p^2}{4}\right)t} \quad \text{--- (5)}$$

when $t=0$

$$u(x,0) = 3e^{-x} - e^{-5x} \text{(given)} = \sum_{n=1}^{\infty} b_n e^{-p^2 x}$$

comparing

when $p^2 = 1$, $b_1 = 3$ and when $p^2 = 5$, $b_2 = -1$

Hence from (5), the general solution is

$$u(x,t) = 3e^{-x+t} - e^{-5x+2t}$$

olve the partial differential equation by separation of variables method (3)

$$u_{xx} = u_y + 2u \quad , \quad u(0,y) = 0 \; , \; \frac{\partial}{\partial x}[u(0,y)] = 1 + e^{-3y}$$

Sol^n. $\quad u = XY$ ——— (1)

where X is a function of x only and Y is a function of y only

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(XY) = X\frac{dY}{dy} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(XY) = Y\frac{d^2X}{dx^2} = YX''$$

from the given equation

$$YX'' = XY' + 2XY$$

$$\frac{X''}{X} = \frac{Y' + 2Y}{Y}$$

$$\frac{X''}{X} = \frac{Y'}{Y} + 2 = k \text{ (say)} \quad \text{——— (2)}$$

$$\frac{X''}{X} = k$$

$$X'' - kX = 0$$

Auxiliary equ^n is

$$m^2 - K = 0$$

$$m = \pm\sqrt{K}$$

$$C.F. = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$$

$$P.I = 0$$

$$X = c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x} \quad \text{——— (3)}$$

$$\frac{Y'}{Y} + 2 = K$$

$$\frac{Y'}{Y} = K - 2$$

$$\frac{dY}{Y} = (K-2)\, dy$$

$$\log Y = (K-2)y + \log c_3$$

$$Y = c_3 e^{(K-2)y} \quad \text{———} \quad (4)$$

Hence from (1)

$$u(x,y) = \left(c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}\right) c_3 e^{(K-2)y} \quad \text{———} \quad (5)$$

$$u(0,y) = (c_1 + c_2) c_3 e^{(K-2)y} = 0$$

$$\Rightarrow \quad (c_1 + c_2) = 0 \Rightarrow c_2 = -c_1 \quad \text{———} \quad (6)$$

from (5), most general solution is

$$u(x,y) = \Sigma\, c_1 c_3 \left(e^{\sqrt{K}x} - e^{-\sqrt{K}x}\right) e^{(K-2)y}$$

$$\frac{\partial u}{\partial x} = \Sigma\, c_1 c_3 \sqrt{K} \left(e^{\sqrt{K}x} + e^{-\sqrt{K}x}\right) e^{(K-2)y}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 1 + e^{-3y} \text{ (given)} = \Sigma\, c_1 c_3 \sqrt{K}(1+1) e^{(K-2)y}$$

$$= \Sigma\, c_1 c_3\, 2\sqrt{K}\, e^{(K-2)y}$$

$$= \Sigma\, b_n e^{(K-2)y}$$

comparing the coefficients, we get

i) $b_1 = 1, \quad K - 2 = 0$

$2 c_1 c_3 \sqrt{K} = 1, \quad K = 2$

$c_1 c_3 = \frac{1}{2\sqrt{2}}$

ii) $b_3 = -1, \quad K - 2 = -3$

$2 c_1 c_3 \sqrt{K} = 1 \qquad K = -1$

$$2c_1c_3\sqrt{-1} = 1$$

$$c_1c_3 = \frac{1}{2\sqrt{-1}}$$

$$c_1c_3 = \frac{1}{2i}$$

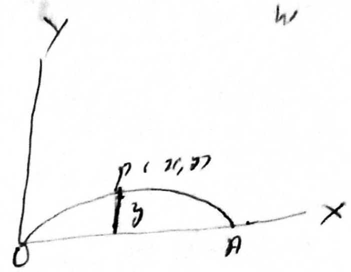
Hence from 6 the general solution is

$$u(x,y) = \frac{1}{2\sqrt{2}}\left(e^{\sqrt{2}x} - e^{-\sqrt{2}x}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)e^{-3y}$$

$$u(x,y) = \frac{1}{\sqrt{2}}\sinh\sqrt{2}x + e^{-3y}\sin x$$

One dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$



Q. Obtain the soln of the wave eqn. using the method of separation of variables.

Soln $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \rightarrow (1)$$

Let $y = XT$ be the soln of this eqn. where X is the function of x and T is the function of t.

$y = XT$

$$\frac{\partial y}{\partial x} = T \cdot \frac{dX}{dx}$$

$$\frac{\partial^2 y}{\partial x^2} = T \frac{d^2X}{dx^2}$$

$y = XT$

$$\frac{\partial y}{\partial t} = X \cdot \frac{dT}{dt}$$

$$\frac{\partial^2 y}{\partial t^2} = X \frac{d^2T}{dt^2}$$

$$X \frac{d^2T}{dt^2} = c^2 T \frac{d^2X}{dx^2}$$

$$\frac{1}{c^2 T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2}$$ by separating the variables dividing by XT

$$\frac{1}{c^2 T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2} = k$$ (say)

$$\frac{d^2T}{dt^2} - kc^2 T = 0$$ and $$\frac{d^2X}{dx^2} - kX = 0$$

A.E are

$m^2 - kc^2 = 0$

$m^2 = kc^2$

$m = \pm \sqrt{k}\, c$

and

$m^2 - k = 0$

$m^2 = k$

$m = \pm\sqrt{k}$

b

case I if $k>0$

$$T = c_1 e^{c\sqrt{k}t} + c_2 e^{-c\sqrt{k}t}$$

$$X = c_3 e^{\sqrt{k}x} + c_4 e^{-\sqrt{k}x}$$

case IInd if $k<0$ $m = \pm\sqrt{k}ci$ and $m = \pm\sqrt{k}i$

$$T = c_5 \cos c\sqrt{k}t + c_6 \sin c\sqrt{k}t$$

$$X = c_7 \cos\sqrt{k}x + c_8 \sin\sqrt{k}x.$$

case IIIrd if $k=0$

$$\frac{d^2T}{dt^2} = 0$$ Integrating Two times

$$T = c_9 t + c_{10}$$

$$X = c_{11} x + c_{12}$$

Here we are dealing with wave eqh. $k<0$

$$y = \cancel{x}TX$$

$$y = (c_5 \cancel{(\cos} \cos c\sqrt{k}t + c_6 \sin c\sqrt{k}t)(c_7 \cos\sqrt{k}x + c_8 \sin\sqrt{k}x)$$

$$y = (c_5 \cos c\sqrt{k}t + c_6 \sin c\sqrt{k}t)(c_7 \cos\sqrt{k}x + c_8 \sin\sqrt{k}x)$$

In General we take $k = -p^2$ for best solution.

[2]

Q 639.

(1)

A String of length l is initially at rest in equilibrium position and each of its points is given velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = b\sin^3\frac{\pi x}{l}$

find the displacement $y(x,t)$.

Sol^n → Since string is initially at rest in equilibrium position so that we have $y(0,t) = y(l,t) = 0$

$y(x,0) = 0$

$u = XT$

$= (c_1\cos px + c_2\sin px)(c_3\cos cpt + c_4\cos cpt)$ → (1)

Applying (1) condition

$y(0,t) = (c_1\cos 0 + c_2\sin 0)(c_3\cos cpt + c_4\cos cpt)$

$x=0,\ t=t$ $\quad 0 = c_1(c_3\cos cpt + c_4\cos cpt)$ $\quad \because u(0,t)=0$

$\Rightarrow c_1 = 0$

(1) becomes:

$y(x,t) = c_2\sin px\,(c_3\cos cpt + c_4\cos cpt)$ → (2)

Applying (2) condition

$x=l,\ t=t$ $\quad y(l,t) = c_2\sin pl\,(c_3\cos cpt + c_4\cos cpt)$.

$c_2 \neq 0 \quad \sin pl = 0 \quad \sin pl = \sin n\pi \Rightarrow pl = n\pi$

$p = n\pi/l$.

(2) becomes:

$y(x,t) = c_2\sin\frac{n\pi}{l}x\,(c_3\cos cpt + c_4\sin cpt)$.

→ (3)

$x=x,\ t=0$ $\quad y(x,0) \quad u(x,0) = c_2\sin\frac{n\pi}{l}x\,(c_3)$ $\quad \because u(x,0)=0$

$0 = c_2c_3\sin\frac{n\pi}{l}x. \Rightarrow c_3 = 0$.

(3) becomes:

$y(x,t) = c_2\sin\frac{n\pi}{l}x\; c_4\sin cpt$

$y(x,t) = c_2c_4\sin\frac{n\pi}{l}x\,\sin cpt$.

$$y(x,t) = b_n \sin\frac{n\pi}{l}x \sin c p t \quad (2) \qquad l_2 l_4 = b_n$$

$$y(x,t) = \sum b_n \sin\frac{n\pi}{l}x \sin\frac{nc\pi}{l}t \rightarrow (4)$$

$$\frac{\partial y}{\partial t} = \sum b_n \sin\frac{n\pi}{l}x \cos\frac{nc\pi}{l}t \times \frac{nc\pi}{l} \rightarrow (5)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum b_n \sin\frac{n\pi}{l}x \times \cos 0 \times \frac{nc\pi}{l}$$

$$b \sin^3\frac{\pi x}{l} = \sum b_n \sin\frac{n\pi}{l}x \frac{nc}{l}$$

$$\frac{b}{4}\left[3\sin\frac{\pi x}{l} - \sin 3\frac{\pi x}{l}\right] = \sum b_n \sin\frac{n\pi}{l}x \times \frac{nc\pi}{l}$$

$$n=1 \quad b_1 \sin\frac{\pi x}{l}\cdot\frac{c\pi}{l} + b_2 \sin\frac{2\pi}{l}x \frac{2c\pi}{l} + b_3 \sin\frac{3\pi x}{l} \times \frac{3c\pi}{l}$$

by putting $n = 1, 2, 3 ---$

$+ -----$

$\Rightarrow$ comparing of $\sin\frac{\pi x}{l}$ and $\sin\frac{3\pi x}{l}$.

coeff.

$$\frac{3b}{4} = b_1\frac{c\pi}{l} \Rightarrow b_1 = \frac{3lb}{4c\pi} \qquad \Bigg| \qquad -\frac{b}{4} = b_3\frac{3c\pi}{l} \Rightarrow b_3 = \frac{-bl}{12c\pi}$$

$$y(x,t) = b_1 \sin\frac{\pi x}{l}\sin\frac{c\pi}{l}t + b_3 \sin\frac{3\pi}{l}x \sin\frac{3c\pi}{l}t$$

$$= \frac{3lb}{4\pi c}\sin\frac{\pi x}{l}\sin\frac{c\pi}{l}t - \frac{bl}{12c\pi}\sin\frac{3\pi}{l}x\sin\frac{3c\pi}{l}t$$

$$y(x,t) = \frac{lb}{12c\pi}\left[9\sin\frac{\pi x}{l}\sin\frac{c\pi}{l}t - \sin\frac{3\pi x}{l}\sin\frac{3c\pi}{l}t\right]$$

$4\sin^3\theta = 3\sin\theta - \sin 3\theta$ use cond. firstly in which R.H.S in zero

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

A string is stretched and fastened to two point l apart. Motion is started by displacing the string in the form $y = A \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement at any point at a distance x from one end at time t is given by $y(x,t) = A \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}$.

Soln. Eqn of a string.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad (1)$$

boundary cond.

$$y(0,t) = 0, \qquad y(l,t) = 0$$

Initial condio $\frac{\partial y}{\partial t} = 0$ at $t = 0$

$$y(x,0) = A \sin \frac{\pi x}{l}.$$

$$y = XT$$

$$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$$

$$y(0,t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos p0 + c_4 \sin p0)$$

$$\Rightarrow \quad 0 = c_3 (c_1 \cos cpt + c_2 \sin cpt)$$

$$\Rightarrow \quad c_3 = 0$$

$$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$$

$$y(l,t) = 0 = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl$$

$$c_4 = c_3 = c_2 \neq 0 \qquad \sin pl = 0$$

$$\sin pl = \sin n\pi$$

$$p = \frac{n\pi}{l}$$

$$y(x,t) = c_4 \sin\frac{n\pi x}{l}\left(c_1 \cos\frac{n\pi ct}{l} + c_2 \sin\frac{n\pi ct}{l}\right)$$

$$\frac{\partial y}{\partial t} = c_4 \sin\frac{n\pi x}{l}\left(-c_1 \sin\frac{n\pi ct}{l} \times \frac{n\pi c}{l} + c_2 \frac{n\pi c}{l} \cos\frac{n\pi ct}{l}\right)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = c_4 \sin\frac{n\pi x}{l}\left(-c_1 0 + c_2 1\right)\frac{n\pi c}{l}$$

$$= c_4 \sin\frac{n\pi x}{l} \cdot \frac{n\pi c}{l} c_2 \qquad c_4 \neq 0 \quad \frac{n\pi x}{l} \neq 0$$

$$\Rightarrow \quad c_2 = 0$$

$$y(x,t) = c_4 \sin\frac{n\pi x}{l}\left(c_1 \cos\frac{n\pi ct}{l}\right)$$

$$= c_4 c_1 \sin\frac{n\pi x}{l} \cos\frac{n\pi ct}{l} \Rightarrow A$$

$$A \sin\frac{\pi x}{l} = y(x,0) = c_4 c_1 \sin\frac{n\pi x}{l} \times 1$$

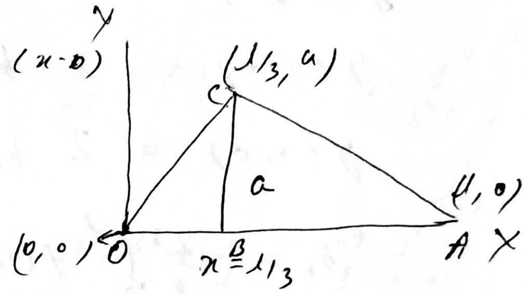
$c_4 c_1 = A$, compare the angle $n = 1$

$$y(x,t) = A \sin\frac{\pi x}{l} \cos\frac{n\pi ct}{l}$$ Hence prove.

(3) Tightly streetched violin string of length l ① and fixed at both Ends is plucked at $x = l/3$ and assumes initially the shape of a Triangle of hight 'a' Find the displacement y at any distance x and any time t after the string is released from rest

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

$$y(x,t) = (C_1 \cos cpt + C_2 \sin cpt)(C_3 \cos px + C_4 \sin px)$$



Eq^n of line OC is $y - 0 = \frac{a-0}{l/3 - 0}(x-0)$

$$y = \frac{3a}{l} x.$$

Eq^n of line CA

$$(y-a) = \frac{0-a}{l - l/3}(x - l/3)$$

$$y = \frac{3a}{2}(1 - x/l)$$

boundary cond. $y(0,t) = 0$, ① $y(l,t) = 0$ ②

Initial cond. $\frac{\partial y}{\partial t} = 0$ $t = 0$ ③

$$y(x,0) = \begin{cases} \frac{3ax}{l} & 0 < x < l/3 \\ \frac{3a}{2}(1 - x/l) & l/3 < x < l \end{cases} \quad ④$$

by ① $C_3 = 0$

$$y(x,t) = C_4 \sin px \,(C_1 \cos cpt + C_2 \sin cpt)$$

by ② $p = n\pi/l$.

by ③

$$y(x,t) = (C_1 \cos cpt + C_2 \sin cpt)\, C_4 \sin px \quad ⑤$$

where $p = \frac{n\pi}{l}$.

$$y(x,t) = \left(C_1 \cos \frac{n\pi ct}{l} + C_2 \sin \frac{n\pi ct}{l}\right) C_4 \sin \frac{n\pi x}{l}$$

$$\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-C_1 \sin \frac{n\pi ct}{l} + C_2 \cos \frac{n\pi ct}{l}\right) C_4 \sin \frac{n\pi x}{l}$$

$=0 \quad \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \frac{n\pi c}{l}\left[c_2 c_4 \sin\frac{n\pi x}{l}\right]$

$y = c_1 c_4 \cos\frac{n\pi ct}{l} \cdot \sin\frac{n\pi x}{l} = b_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \to (1)$

$$y(x,t) = \sum_{1}^{\infty} b_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l} \to (7)$$

$$y(x,0) = \sum b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l y(x,0) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\int_0^{l/3} \frac{3ax}{l} \sin\frac{n\pi x}{l}\, dx + \frac{2}{l}\int_{l/3}^{l} \frac{3a}{2}\left(1-\frac{x}{l}\right) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2 \cdot 3a}{l^2}\left[x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - 1\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^{l/3} + \frac{3a}{l}\left[\left(1-\frac{x}{l}\right)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\frac{1}{l}\right)\left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_{l/3}^{l}$$

$$= \frac{6a}{l^2}\left[\frac{l}{3}\left(-\cos\frac{n\pi}{3}\right)\frac{l}{n\pi} + \sin\frac{n\pi}{3}\frac{l^2}{n^2\pi^2}\right] + \frac{3a}{l}\left[+\frac{2}{3}\left(\cos\frac{n\pi}{3}\right)\frac{l}{n\pi} + \frac{1}{l}\sin\frac{n\pi}{3}\frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{6a}{l^2 n\pi} l^2\left[-\frac{1}{3}\cos\frac{n\pi}{3} + \frac{1}{n\pi}\sin\frac{\pi n}{3}\right] + \frac{3a}{n\pi l} \times l\left[\frac{2}{3}\cos\frac{n\pi}{3} + \sin\frac{n\pi}{3}\frac{1}{n\pi}\right]$$

$$= \frac{9a}{n^2\pi^2}\sin n\pi/3$$

Of eqn (7) become

$$y(x,t) = \frac{9a}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

(1)

Q. A tightly stretched string with fixed end point $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity $\lambda x(l-x)$. Find the displacement of the string at any distance x from one end at any time t

Soln

$$y = (c_1 \cos cpt + c_2 \sin cpt)\,(c_3 \cos px + c_4 \sin px)$$ ①

$y(0,t) = 0$, $y(l,t) = 0$

$y(x,0) = 0$ (Equilibrium position)

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x)$$

from ①

$$y = (c_1 \cos cpt + c_2 \sin cpt)\,(c_3 \cos px + c_4 \sin px)$$

using ① cond.

$$y(0,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_3 = 0$$

Eqn ① becomes.

$$y = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px$$ ②

using ② cond.

$$y(l,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin pl = 0$$

$$\sin pl = 0 \Rightarrow \sin pl = \sin n\pi$$

$$p = \frac{n\pi}{l}.$$

Two boundary cond. by velocity.

$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px)$ (2)

$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)\, c_4 \sin px \quad p = \frac{n\pi}{l} \to ①$

$y(x,0) = (c_1 \times 1 + c_2 \times 0)\, c_4 \sin px \quad p = \frac{n\pi}{l}$

$0 = c_1 c_4 \sin px \quad p = \frac{n\pi}{l}$

$\Rightarrow c_1 = 0$

$y(x,t) =$ ② becomes.

$c_2 \sin cpt\; c_4 \sin px. \quad n\pi/l = p.$

$y(x,t) = c_2 c_4 \sin cpt \sin px.$

$y(x,t) = \Sigma b_n. \sin cpt \sin px \to ③$

$\frac{\partial y}{\partial t} = \Sigma b_n \cos cpt \times cp \sin px.$

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \Sigma b_n\, cp \sin px.$

$\lambda(lx - x^2) = cp\, \Sigma b_n \sin px.$

$b_n \ne \frac{\lambda}{cp}(lx - x^2) = \Sigma b_n \sin px$

$b_n = \frac{\lambda}{cp} \cdot \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l}\, dx.$

$b_n = \frac{\lambda}{cp} \cdot \frac{2}{l} \Big[(lx - x^2)\left(-\cos \frac{n\pi x}{l}\right)\frac{l}{n\pi} - (l-2x)\left(-\sin \frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}$
$+ (-2)\left(\cos \frac{n\pi x}{l}\right)\frac{l^3}{n^3\pi^3} \Big]_0^l = \frac{\lambda}{cp} \cdot \frac{2}{l}\left[\frac{-2l \cos n\pi}{n^3\pi^3} + \frac{2l^3}{n^3\pi^3}\right]$

$= \frac{\lambda}{cp} \cdot \frac{2}{l}\left[\frac{8 l^2}{n^3\pi^3}\right]$ n is odd.

$= 0$ n is even.

by Eq² ③

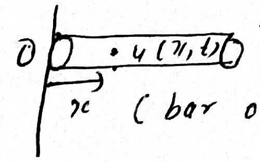
$y(x,t) = \Sigma b_n \sin cpt \sin px.$

$= \frac{\lambda}{cp} \Sigma \frac{8l^2}{n^3\pi^3} \sin cpt \sin px$

$= \frac{\lambda l}{c \cdot n\pi} \sum_{n \text{ is odd}} \frac{8l^2}{n^3\pi^3} \sin \frac{cn\pi t}{l} \sin \frac{n\pi x}{l} = \frac{8\lambda l^3}{c\pi^4} \sum_{n=1} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi x}{l} \sin \frac{(2n-1)\pi ct}{l}$

One Dimensional Heat Equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$$



where $u(x,t)$ be the temperature at a distance x from O. By the method of separation of variables.

$$u = XT \rightarrow (2)$$

where X is the function of x only and T is the function of t only.

from (1)

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx} \Rightarrow \frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt}$$

Substituting in (1)

$$X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{dT}{dt}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{dT}{dt} = p^2, -p^2 \text{ or } 0 \rightarrow (3)$$

Case I $\rightarrow$ $p^2 < 0 \Rightarrow p^2$ is negative $= -p^2$ (from (3))

$$\frac{1}{X}\frac{d^2X}{dx^2} = -p^2$$

$$\frac{d^2X}{dx^2} + p^2X = 0$$

for A.E $d/dx = D$ and $D = m$

$m^2 + p^2 = 0$

$m^2 = -p^2 \Rightarrow m = \pm pi$

$X = (c_1 \cos px + c_2 \sin px)$

$$\frac{1}{c^2T}\frac{dT}{dt} = -p^2 \Rightarrow \frac{dT}{dt} + p^2c^2T = 0$$

for A.E, $\frac{d}{dt} = D$ and $D = m$

$$m + p^2c^2 = 0 \Rightarrow m = -p^2c^2$$ (one root) and real also

$$T = c_3 e^{-p^2c^2t}$$

from (2)

$$u = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2c^2t}$$

Case II $p^2 > 0 \Rightarrow p^2$ is positive. (from (3))

$$\frac{1}{X}\frac{d^2X}{dx^2} = p^2 \Rightarrow \frac{d^2X}{dx^2} + p^2X = 0$$

for A.E. $\frac{d}{dx} = D$ and $D = m$

$$m^2 + p^2 = 0 \Rightarrow m^2 = +p^2 \Rightarrow m = \pm p$$

$$X = (c_4 e^{px} + c_5 e^{-px})$$

and $$\frac{1}{c^2T}\frac{dT}{dt} = p^2 \Rightarrow \frac{dT}{dt} + p^2c^2T = 0$$

for A.E. $\frac{d}{dt} = D$ and $D = m$

$$m + p^2c^2 = 0 \Rightarrow m = p^2c^2$$

$$T = c_6 e^{p^2c^2t}$$

from (2) $$u = (c_4 e^{px} + c_5 e^{-px})\, c_6 e^{p^2c^2t}$$

Case III $p^2 = 0$ (from (3))

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0$$ integrating Two times

$$X = (c_7 x + c_8)$$

and

$$\frac{1}{c^2 T}\frac{dT}{dt} = 0 \Rightarrow \frac{dT}{dt} = 0$$ integrating one time.

$$\Rightarrow T = c_9$$

from (2) $u = C(c_7 x + c_8)\, c_9$

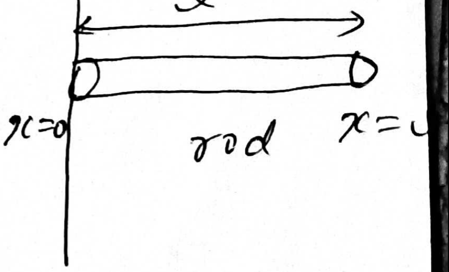
$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2 c^2 t}$$

is the most accurate solution.

Q Solve the following boundary value problem which arises in the heat conduction in a rod.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(l,t) = 0 \text{ and } u(x,0) = 100\frac{x}{l}$$

Solution (If there are words like rod or bar then we will solve the problem for one dimensional Heat Equation and for the one Dimensional Heat Equation we must have Two boundary conditions and one initial condition i.e at $t=0$)



In the given Question.

boundary conditions are.

1 $u(0,t) = 0$ 2. $u(l,t) = 0$

Initial condition is

$$u(x,0) = \frac{100\,x}{l}.$$

given Equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

most accurate solution is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-p^2c^2t} \rightarrow (1)$$

(by the method of separation of variables)

using the first boundary cond.: $x=0,\ t=t$, $u(0,t)=0$

$$u(0,t) = (c_1 \cos 0 + c_2 \sin 0)\, c_3 e^{-p^2c^2t}$$

$$0 = c_1 c_3 e^{-p^2c^2t}$$

$$\Rightarrow c_1 = 0$$

then Equation (1) becomes.

$$u(x,t) = c_2 \sin px\, c_3 e^{-p^2c^2t} \rightarrow (2)$$

using second boundary cond. $x=l,\ t=t$
$u(l,t)=0$

$$u(l,t) = c_2 \sin pl\, c_3 e^{-p^2c^2t}$$

$$0 = c_2 \sin pl\, c_3 e^{-p^2c^2t}$$

$$\Rightarrow \sin pl = 0$$

$$\Rightarrow \sin pl = \sin n\pi$$

$$pl = n\pi \Rightarrow p = \frac{n\pi}{l}$$

Equation (2) becomes.

$$u(x,t) = c_2 \sin px\, c_3 e^{-p^2c^2t} \rightarrow (3)$$

where $p = \frac{n\pi}{l}$

Now using the last and Initial cond. for the Equation (3)

$u(x,t) = c_2 \sin px \; c_3 \; e^{-p^2c^2t}$ (3)

where $p = \frac{n\pi}{l}$

$u(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \; e^{-(\frac{n\pi}{l})^2 c^2 t}$

$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \; e^{-(\frac{n\pi}{l})^2 c^2 t} \rightarrow$ (4)

where $c_2 c_3 = \sum_{n=1}^{\infty} b_n$

using Initial condition

$x = x, \; t = 0, \; u(x,0) = \frac{100x}{l}$

$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \; e^{0}$

$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

Now we have to calculate b_n from above relation. Above relation is like a Half Range sine series where $f(x) = \frac{100x}{l}$

then by the Half Range sine formula.

$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$.

$= \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} \, dx$

$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} \, dx$.

$= \frac{200}{l^2} \left[x \cdot \left(-\cos \frac{n\pi x}{l}\right) \cdot \frac{l}{n\pi} - 1\left(-\sin \frac{n\pi x}{l}\right) \frac{l^2}{n^2\pi^2} \right]_0^l$

$= \frac{200}{l^2} \left[\cdot l\,(-\cos n\pi)\,\frac{l}{n\pi} - 0 \right]$

$$b_n = -\frac{200}{l^2}\frac{l^2}{n\pi}\cos n\pi \qquad \because \cos n\pi = (-1)^n \quad (6)$$

$$b_n = \frac{200}{n\pi}(-1)^{n+1}$$

Substituting in Equation (4)

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(-1)^{n+1} \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t}$$

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2}{l^2}t}.$$

Q. Find the temperature in a bar of length 2 whose ends are kept at zero and lateral surface insulated if the initial temperature is $\sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2}$.

Ans

Let $u(x,t)$ be the initial temperature in the bar.

The boundary conditions are $u(0,t) = 0 = u(2,t)$ for any t. ---- I

The initial condition is.

$$u(x,0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} \quad \text{------ II}$$

One dimensional heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{------ III}$$

It's solution is $u(x,t) = (c_1 \cos px + c_2 \sin px) c_3 e^{-c^2 p^2 t}$ --- IV

$$u(0,t) = 0 = c_1 c_3 e^{-c^2 p^2 t} \quad [\text{Using I}]$$

$$c_1 = 0$$

From IV,

$$u(x,t) = c_2 c_3 \sin px \, e^{-c^2 p^2 t} \quad \text{------ V}$$

$$u(2,t) = 0 = c_2 c_3 \sin 2p e^{-c^2 p^2 t}$$

$$\sin 2p = 0 = \sin n\pi \quad \text{------} \quad [\text{Using I}]$$

$$p = \frac{n\pi}{2} ; n \in I$$

Hence from V

$$u(x,t) = b_n \sin \frac{n\pi x}{2} e^{-\frac{n^2 \pi^2 c^2 t}{4}} \quad \because c_2 c_3 = b_n$$

The most general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} e^{-\frac{n^2 \pi^2 c^2 t}{4}} \quad \text{------ VI}$$

$$u(x,0) = \sin\left(\frac{\pi x}{2}\right) + 3\sin\left(\frac{5\pi x}{2}\right) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$= b_1 \sin\left(\frac{\pi x}{2}\right) + b_2 \sin\left(\frac{2\pi x}{2}\right) + \dots + b_5 \sin\left(\frac{5\pi x}{2}\right) + \dots$$

Comparing, we get $b_1 = 1$ and $b_5 = 3$

Hence from $\overline{VI}$,

$$u(x,t) = \sin\left(\frac{\pi x}{2}\right) e^{\frac{-\pi^2 c^2 t}{4}} + 3\sin\left(\frac{5\pi x}{2}\right) e^{\frac{-25\pi^2 c^2 t}{4}}$$