Implementation of Parareal algorithm

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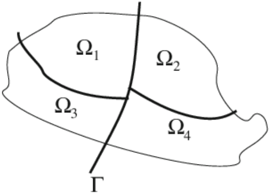
Parallelization in PDE solvers

Methods

- Spatial domain decomposition
- Time domain decomposition



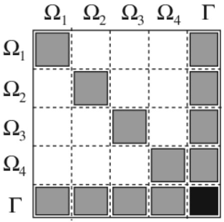


Figure: Spatial Domain Decompostion Illustration [1]

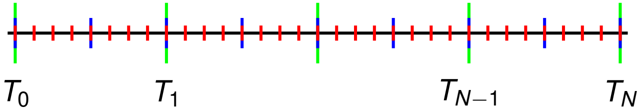
Parallelization in time - Motivation

$$\frac{\partial y(t)}{\partial t} + \mathcal{A}y(t) = 0, \;\; t \in [T_0, T_N]$$
$$y(T_0) = y_0$$

Run-time Consideration

- c : Time for computing one time step in numerical integration
- $T = M\,c$, total time for full simulation, M : Number of fine steps
- For $c = 0.04s$, $M = 1000,000$, $T = 40,000s \approx 11.1hrs$

Parallelization in time - Idea



Idea

- Split the time domain into a set of subdomains
- Solve the IVP on a coarse grid to get an approx. sol at $T_1, T_2, \ldots, T_{N-1}, T_N$
- This can be used as initial value for the IVP on each sub-domain
- Initial value is only serves as guess (coarse sol)
- Iteratively better solution with better initial guess for initial value in each iteration

Parareal Algorithm

Convention

- δt_c : coarse grid time step
- δt_f : fine grid time step
- $\mathcal{G}$: coarse grid integrator
- $\mathcal{F}$: fine grid integrator
- $Y_0, Y_1, \ldots Y_N$: numerical solution
- Yf_n^k, Yc_n^k : Fine and Coarse Integrator solutions

Construction

$$Y_n^k = \mathcal{F}(Yc_{n-1}^{k-1}) + \mathcal{G}(Y_{n-1}^k) - \mathcal{G}(Y_{n-1}^{k-1})$$

where

- k : iteration
- n : grid index

Parareal Algorithm

Algorithm 1 Parareal Algorithm

Iteration 0: $Y_0^0 = y_0$, given initial value
for $n = 1$ to N **do**
    $Y_n^0 = \mathcal{G}(Y_{n-1}^0)$
end
for $k = 1$ to N **do**
    parallel for $n = 1$ to N **do**
        $Yf_n^{k-1} = \mathcal{F}(Yc_{n-1}^{k-1})$
    end
    for $n = 1$ to N **do**
        $Yc_n^k = \mathcal{G}(Y_{n-1}^{k-1})$
    end
    $Y_n^k = Yc_n^k + Yf_n^{k-1} - Yc_n^{k-1}$
    if $|Y^k - Y^{k-1}| < \epsilon$ **then**
        break
    end if
end

Test problem

Problem 1

$$\frac{dy}{dt} = A(t)\, y(t)$$
$$y(0) = y_0, \quad t \in [0, 10]$$

where $A = -0.2$, $y_0 = 100$

Problem 2

$$\frac{dy}{dt} = \sin(t)\, y(t) + t$$
$$y(0) = 1, \quad t \in [0, 14]$$

Numerical Method

Classical Runge-Kutta scheme of 2^{nd} order is used for 1D ODE.

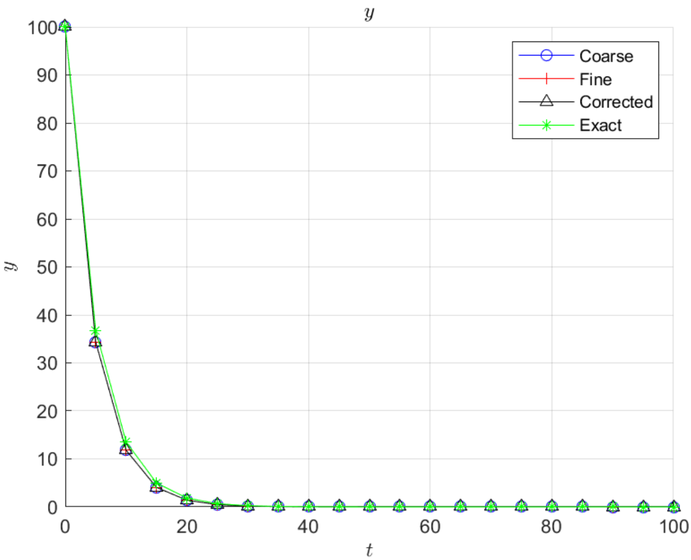
$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1) \\
k &= \frac{k_1 + k_2}{2} \\
y_{n+1} &= y_n + h\, k
\end{aligned}
$$

OpenMP implementation

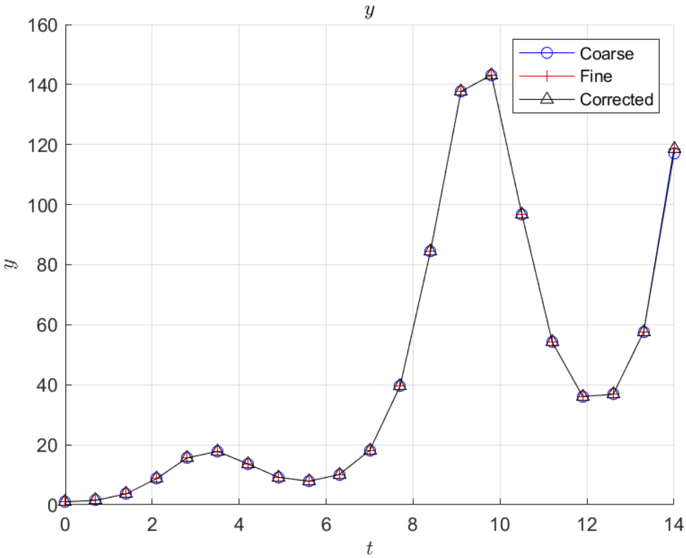
parareal.cpp > ...

```cpp
#include <stdio.h>
#include <omp.h>
#include <math.h>
// global variables
int main()
{
    // coarse grid integrator
    // ...
    while (err > tol and iter < max_iter)
    {
    #pragma omp parallel num_threads(NUM_THREADS) shared(t, y) private(i)
        {
        #pragma omp for schedule(dynamic, chunk)
            for (int i = 0; i < n_sub; i++)
            {
                //fine integrator
                //...
            }
        }
        // predictor: coarse integrator
        // corrector
        // error check
    }
    return 0;
}
```

Simulation : Problem 1



Simulation : Problem 2



Speedup

CPU	Cores	Multithreading	Num of threads	time serial	time parallel
i5 9500	6 cores	6 way	100	201.222	117.047
i5 8250u	4 cores	8 way	100	201.380	117.070
i3 5010u	2 cores	4 way	100	201.251	117.054

Speedup
1.72
1.72
1.72

Grasedyck, Lars and Kriemann, Ronald and Le Borne, Sabine
Domain decomposition based $\mathcal{H}$-LU preconditioning,
Volume 112, Numerische Mathematik, May 2009.

Allan S. Nielsen
Feasibility study of the parareal algorithm
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