17 MAKING COMPLEX DECISIONS

In which we examine methods for deciding what to do today, given that we may decide again tomorrow.

In this chapter, we address the computational issues involved in making decisions in a stochastic environment. Whereas Chapter 16 was concerned with one-shot or episodic decision problems, in which the utility of each action's outcome was well known, we are concerned here with **sequential decision problems**, in which the agent's utility depends on a sequence of decisions. Sequential decision problems incorporate utilities, uncertainty, and sensing, and include search and planning problems as special cases. Section 17.1 explains how sequential decision problems are defined, and Sections 17.2 and 17.3 explain how they can be solved to produce optimal behavior that balances the risks and rewards of acting in an uncertain environment. Section 17.4 extends these ideas to the case of partially observable environments, and Section 17.4.3 develops a complete design for decision-theoretic agents in partially observable environments, combining dynamic Bayesian networks from Chapter 15 with decision networks from Chapter 16.

SEQUENTIAL DECISION PROBLEM

The second part of the chapter covers environments with multiple agents. In such environments, the notion of optimal behavior is complicated by the interactions among the agents. Section 17.5 introduces the main ideas of **game theory**, including the idea that rational agents might need to behave randomly. Section 17.6 looks at how multiagent systems can be designed so that multiple agents can achieve a common goal.

17.1 SEQUENTIAL DECISION PROBLEMS

Suppose that an agent is situated in the 4×3 environment shown in Figure 17.1(a). Beginning in the start state, it must choose an action at each time step. The interaction with the environment terminates when the agent reaches one of the goal states, marked +1 or –1. Just as for search problems, the actions available to the agent in each state are given by $\text{ACTIONS}(s)$, sometimes abbreviated to $A(s)$; in the 4×3 environment, the actions in every state are *Up*, *Down*, *Left*, and *Right*. We assume for now that the environment is **fully observable**, so that the agent always knows where it is.

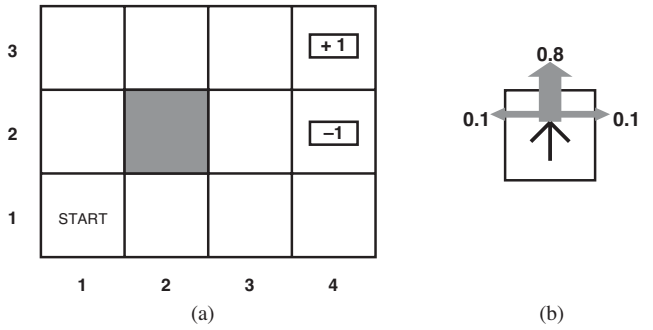


Figure 17.1 (a) A simple 4×3 environment that presents the agent with a sequential decision problem. (b) Illustration of the transition model of the environment: the "intended" outcome occurs with probability 0.8, but with probability 0.2 the agent moves at right angles to the intended direction. A collision with a wall results in no movement. The two terminal states have reward +1 and –1, respectively, and all other states have a reward of –0.04.

If the environment were deterministic, a solution would be easy: [*Up, Up, Right, Right, Right*]. Unfortunately, the environment won't always go along with this solution, because the actions are unreliable. The particular model of stochastic motion that we adopt is illustrated in Figure 17.1(b). Each action achieves the intended effect with probability 0.8, but the rest of the time, the action moves the agent at right angles to the intended direction. Furthermore, if the agent bumps into a wall, it stays in the same square. For example, from the start square (1,1), the action *Up* moves the agent to (1,2) with probability 0.8, but with probability 0.1, it moves right to (2,1), and with probability 0.1, it moves left, bumps into the wall, and stays in (1,1). In such an environment, the sequence $[Up, Up, Right, Right, Right]$ goes up around the barrier and reaches the goal state at (4,3) with probability $0.8^5 = 0.32768$. There is also a small chance of accidentally reaching the goal by going the other way around with probability $0.1^4 \times 0.8$, for a grand total of 0.32776. (See also Exercise 17.1.)

As in Chapter 3, the **transition model** (or just "model," whenever no confusion can arise) describes the outcome of each action in each state. Here, the outcome is stochastic, so we write $P(s' \mid s, a)$ to denote the probability of reaching state s' if action a is done in state s. We will assume that transitions are **Markovian** in the sense of Chapter 15, that is, the probability of reaching s' from s depends only on s and not on the history of earlier states. For now, you can think of $P(s' \mid s, a)$ as a big three-dimensional table containing probabilities. Later, in Section 17.4.3, we will see that the transition model can be represented as a **dynamic Bayesian network**, just as in Chapter 15.

To complete the definition of the task environment, we must specify the utility function for the agent. Because the decision problem is sequential, the utility function will depend on a sequence of states—an **environment history**—rather than on a single state. Later in this section, we investigate how such utility functions can be specified in general; for now,

REWARD

we simply stipulate that in each state s, the agent receives a **reward** $R(s)$, which may be positive or negative, but must be bounded. For our particular example, the reward is -0.04 in all states except the terminal states (which have rewards +1 and –1). The utility of an

environment history is just (for now) the *sum* of the rewards received. For example, if the agent reaches the +1 state after 10 steps, its total utility will be 0.6. The negative reward of –0.04 gives the agent an incentive to reach (4,3) quickly, so our environment is a stochastic generalization of the search problems of Chapter 3. Another way of saying this is that the agent does not enjoy living in this environment and so wants to leave as soon as possible.

MARKOV DECISION PROCESS

To sum up: a sequential decision problem for a fully observable, stochastic environment with a Markovian transition model and additive rewards is called a **Markov decision process**, or **MDP**, and consists of a set of states (with an initial state s_0); a set $\text{ACTIONS}(s)$ of actions in each state; a transition model $P(s' \mid s, a)$; and a reward function $R(s)$.[1]

POLICY

The next question is, what does a solution to the problem look like? We have seen that any fixed action sequence won't solve the problem, because the agent might end up in a state other than the goal. Therefore, a solution must specify what the agent should do for *any* state that the agent might reach. A solution of this kind is called a **policy**. It is traditional to denote a policy by π, and $\pi(s)$ is the action recommended by the policy π for state s. If the agent has a complete policy, then no matter what the outcome of any action, the agent will always know what to do next.

OPTIMAL POLICY

Each time a given policy is executed starting from the initial state, the stochastic nature of the environment may lead to a different environment history. The quality of a policy is therefore measured by the *expected* utility of the possible environment histories generated by that policy. An **optimal policy** is a policy that yields the highest expected utility. We use π^* to denote an optimal policy. Given π^*, the agent decides what to do by consulting its current percept, which tells it the current state s, and then executing the action $\pi^*(s)$. A policy represents the agent function explicitly and is therefore a description of a simple reflex agent, computed from the information used for a utility-based agent.

An optimal policy for the world of Figure 17.1 is shown in Figure 17.2(a). Notice that, because the cost of taking a step is fairly small compared with the penalty for ending up in (4,2) by accident, the optimal policy for the state (3,1) is conservative. The policy recommends taking the long way round, rather than taking the shortcut and thereby risking entering (4,2).

The balance of risk and reward changes depending on the value of $R(s)$ for the nonterminal states. Figure 17.2(b) shows optimal policies for four different ranges of $R(s)$. When $R(s) \leq -1.6284$, life is so painful that the agent heads straight for the nearest exit, even if the exit is worth –1. When $-0.4278 \leq R(s) \leq -0.0850$, life is quite unpleasant; the agent takes the shortest route to the +1 state and is willing to risk falling into the –1 state by accident. In particular, the agent takes the shortcut from (3,1). When life is only slightly dreary ($-0.0221 < R(s) < 0$), the optimal policy takes *no risks at all*. In (4,1) and (3,2), the agent heads directly away from the –1 state so that it cannot fall in by accident, even though this means banging its head against the wall quite a few times. Finally, if $R(s) > 0$, then life is positively enjoyable and the agent avoids *both* exits. As long as the actions in (4,1), (3,2),

[1] Some definitions of MDPs allow the reward to depend on the action and outcome too, so the reward function is $R(s, a, s')$. This simplifies the description of some environments but does not change the problem in any fundamental way, as shown in Exercise 17.4.

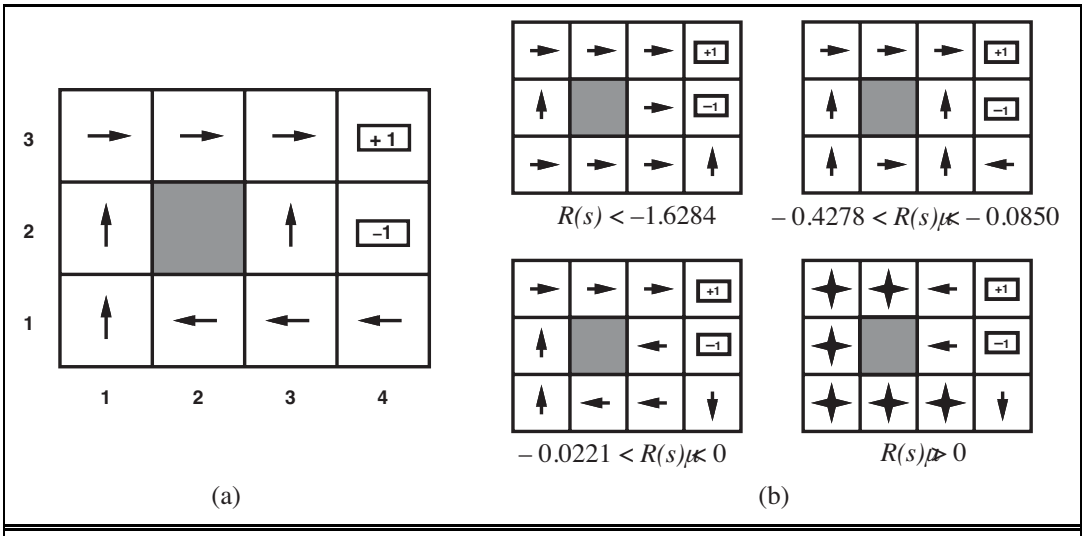


Figure 17.2 (a) An optimal policy for the stochastic environment with $R(s) = -0.04$ in the nonterminal states. (b) Optimal policies for four different ranges of $R(s)$.

and (3,3) are as shown, every policy is optimal, and the agent obtains infinite total reward because it never enters a terminal state. Surprisingly, it turns out that there are six other optimal policies for various ranges of $R(s)$; Exercise 17.5 asks you to find them.

The careful balancing of risk and reward is a characteristic of MDPs that does not arise in deterministic search problems; moreover, it is a characteristic of many real-world decision problems. For this reason, MDPs have been studied in several fields, including AI, operations research, economics, and control theory. Dozens of algorithms have been proposed for calculating optimal policies. In sections 17.2 and 17.3 we describe two of the most important algorithm families. First, however, we must complete our investigation of utilities and policies for sequential decision problems.

17.1.1 Utilities over time

In the MDP example in Figure 17.1, the performance of the agent was measured by a sum of rewards for the states visited. This choice of performance measure is not arbitrary, but it is not the only possibility for the utility function on environment histories, which we write as $U_h([s_0, s_1, \ldots, s_n])$. Our analysis draws on **multiattribute utility theory** (Section 16.4) and is somewhat technical; the impatient reader may wish to skip to the next section.

FINITE HORIZON
INFINITE HORIZON

The first question to answer is whether there is a **finite horizon** or an **infinite horizon** for decision making. A finite horizon means that there is a *fixed* time N after which nothing matters—the game is over, so to speak. Thus, $U_h([s_0, s_1, \ldots, s_{N+k}]) = U_h([s_0, s_1, \ldots, s_N])$ for all $k > 0$. For example, suppose an agent starts at (3,1) in the 4×3 world of Figure 17.1, and suppose that $N = 3$. Then, to have any chance of reaching the +1 state, the agent must head directly for it, and the optimal action is to go *Up*. On the other hand, if $N = 100$, then there is plenty of time to take the safe route by going *Left*. *So, with a finite horizon,*

NONSTATIONARY POLICY

STATIONARY POLICY

the optimal action in a given state could change over time. We say that the optimal policy for a finite horizon is **nonstationary**. With no fixed time limit, on the other hand, there is no reason to behave differently in the same state at different times. Hence, the optimal action depends only on the current state, and the optimal policy is **stationary**. Policies for the infinite-horizon case are therefore simpler than those for the finite-horizon case, and we deal mainly with the infinite-horizon case in this chapter. (We will see later that for partially observable environments, the infinite-horizon case is not so simple.) Note that "infinite horizon" does not necessarily mean that all state sequences are infinite; it just means that there is no fixed deadline. In particular, there can be finite state sequences in an infinite-horizon MDP containing a terminal state.

The next question we must decide is how to calculate the utility of state sequences. In the terminology of multiattribute utility theory, each state s_i can be viewed as an **attribute** of the state sequence $[s_0, s_1, s_2 \ldots]$. To obtain a simple expression in terms of the attributes, we will need to make some sort of preference-independence assumption. The most natural assumption is that the agent's preferences between state sequences are **stationary**. Stationarity for preferences means the following: if two state sequences $[s_0, s_1, s_2, \ldots]$ and $[s'_0, s'_1, s'_2, \ldots]$ begin with the same state (i.e., $s_0 = s'_0$), then the two sequences should be preference-ordered the same way as the sequences $[s_1, s_2, \ldots]$ and $[s'_1, s'_2, \ldots]$. In English, this means that if you prefer one future to another starting tomorrow, then you should still prefer that future if it were to start today instead. Stationarity is a fairly innocuous-looking assumption with very strong consequences: it turns out that under stationarity there are just two coherent ways to assign utilities to sequences:

STATIONARY PREFERENCE

ADDITIVE REWARD

1. **Additive rewards**: The utility of a state sequence is

$$U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + R(s_1) + R(s_2) + \cdots .$$

The 4×3 world in Figure 17.1 uses additive rewards. Notice that additivity was used implicitly in our use of path cost functions in heuristic search algorithms (Chapter 3).

DISCOUNTED REWARD

2. **Discounted rewards**: The utility of a state sequence is

$$U_h([s_0, s_1, s_2, \ldots]) = R(s_0) + \gamma R(s_1) + \gamma^2 R(s_2) + \cdots ,$$

DISCOUNT FACTOR

where the **discount factor** γ is a number between 0 and 1. The discount factor describes the preference of an agent for current rewards over future rewards. When γ is close to 0, rewards in the distant future are viewed as insignificant. When γ is 1, discounted rewards are exactly equivalent to additive rewards, so additive rewards are a special case of discounted rewards. Discounting appears to be a good model of both animal and human preferences over time. A discount factor of γ is equivalent to an interest rate of $(1/\gamma) - 1$.

For reasons that will shortly become clear, we assume discounted rewards in the remainder of the chapter, although sometimes we allow $\gamma = 1$.

Lurking beneath our choice of infinite horizons is a problem: if the environment does not contain a terminal state, or if the agent never reaches one, then all environment histories will be infinitely long, and utilities with additive, undiscounted rewards will generally be

infinite. While we can agree that $+\infty$ is better than $-\infty$, comparing two state sequences with $+\infty$ utility is more difficult. There are three solutions, two of which we have seen already:

1. With discounted rewards, the utility of an infinite sequence is *finite*. In fact, if $\gamma < 1$ and rewards are bounded by $\pm R_{\max}$, we have
$$U_h([s_0, s_1, s_2, \ldots]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \sum_{t=0}^{\infty} \gamma^t R_{\max} = R_{\max}/(1-\gamma)\ , \tag{17.1}$$
using the standard formula for the sum of an infinite geometric series.
2. If the environment contains terminal states *and if the agent is guaranteed to get to one eventually*, then we will never need to compare infinite sequences. A policy that is guaranteed to reach a terminal state is called a **proper policy**. With proper policies, we can use $\gamma = 1$ (i.e., additive rewards). The first three policies shown in Figure 17.2(b) are proper, but the fourth is improper. It gains infinite total reward by staying away from the terminal states when the reward for the nonterminal states is positive. The existence of improper policies can cause the standard algorithms for solving MDPs to fail with additive rewards, and so provides a good reason for using discounted rewards.

PROPER POLICY

AVERAGE REWARD

3. Infinite sequences can be compared in terms of the **average reward** obtained per time step. Suppose that square (1,1) in the 4×3 world has a reward of 0.1 while the other nonterminal states have a reward of 0.01. Then a policy that does its best to stay in (1,1) will have higher average reward than one that stays elsewhere. Average reward is a useful criterion for some problems, but the analysis of average-reward algorithms is beyond the scope of this book.

In sum, discounted rewards present the fewest difficulties in evaluating state sequences.

17.1.2 Optimal policies and the utilities of states

Having decided that the utility of a given state sequence is the sum of discounted rewards obtained during the sequence, we can compare policies by comparing the *expected* utilities obtained when executing them. We assume the agent is in some initial state s and define S_t (a random variable) to be the state the agent reaches at time t when executing a particular policy π. (Obviously, $S_0 = s$, the state the agent is in now.) The probability distribution over state sequences $S_1, S_2, \ldots$, is determined by the initial state s, the policy π, and the transition model for the environment.

The expected utility obtained by executing π starting in s is given by
$$U^\pi(s) = E\left[\sum_{t=0}^{\infty} \gamma^t R(S_t)\right]\ , \tag{17.2}$$
where the expectation is with respect to the probability distribution over state sequences determined by s and π. Now, out of all the policies the agent could choose to execute starting in s, one (or more) will have higher expected utilities than all the others. We'll use π_s^* to denote one of these policies:
$$\pi_s^* = \operatorname*{argmax}_\pi U^\pi(s)\ . \tag{17.3}$$

Remember that π_s^* is a policy, so it recommends an action for every state; its connection with s in particular is that it's an optimal policy when s is the starting state. A remarkable consequence of using discounted utilities with infinite horizons is that the optimal policy is *independent* of the starting state. (Of course, the *action sequence* won't be independent; remember that a policy is a function specifying an action for each state.) This fact seems intuitively obvious: if policy π_a^* is optimal starting in a and policy π_b^* is optimal starting in b, then, when they reach a third state c, there's no good reason for them to disagree with each other, or with π_c^*, about what to do next.[2] So we can simply write π^* for an optimal policy.

Given this definition, the true utility of a state is just $U^{\pi^*}(s)$—that is, the expected sum of discounted rewards if the agent executes an optimal policy. We write this as $U(s)$, matching the notation used in Chapter 16 for the utility of an outcome. Notice that $U(s)$ and $R(s)$ are quite different quantities; $R(s)$ is the "short term" reward for being in s, whereas $U(s)$ is the "long term" total reward from s onward. Figure 17.3 shows the utilities for the 4×3 world. Notice that the utilities are higher for states closer to the +1 exit, because fewer steps are required to reach the exit.

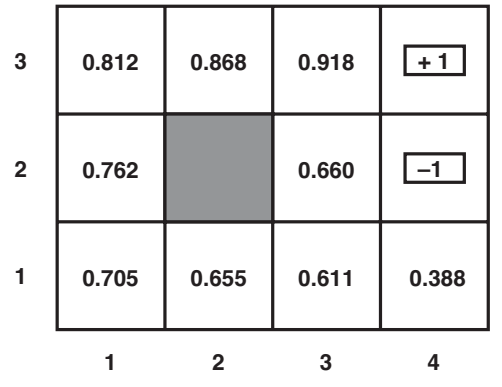


Figure 17.3 The utilities of the states in the 4×3 world, calculated with $\gamma = 1$ and $R(s) = -0.04$ for nonterminal states.

The utility function $U(s)$ allows the agent to select actions by using the principle of maximum expected utility from Chapter 16—that is, choose the action that maximizes the expected utility of the subsequent state:

$$\pi^*(s) = \operatorname*{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s') \ . \tag{17.4}$$

The next two sections describe algorithms for finding optimal policies.

[2] Although this seems obvious, it does not hold for finite-horizon policies or for other ways of combining rewards over time. The proof follows directly from the uniqueness of the utility function on states, as shown in Section 17.2.

17.2 VALUE ITERATION

VALUE ITERATION

In this section, we present an algorithm, called **value iteration**, for calculating an optimal policy. The basic idea is to calculate the utility of each state and then use the state utilities to select an optimal action in each state.

17.2.1 The Bellman equation for utilities

Section 17.1.2 defined the utility of being in a state as the expected sum of discounted rewards from that point onwards. From this, it follows that there is a direct relationship between the utility of a state and the utility of its neighbors: *the utility of a state is the immediate reward for that state plus the expected discounted utility of the next state, assuming that the agent chooses the optimal action.* That is, the utility of a state is given by

$$U(s) = R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U(s') \ . \tag{17.5}$$

BELLMAN EQUATION

This is called the **Bellman equation**, after Richard Bellman (1957). The utilities of the states—defined by Equation (17.2) as the expected utility of subsequent state sequences—are solutions of the set of Bellman equations. In fact, they are the *unique* solutions, as we show in Section 17.2.3.

Let us look at one of the Bellman equations for the 4×3 world. The equation for the state (1,1) is

$$\begin{aligned} U(1,1) = -0.04 + \gamma \max[\ & 0.8U(1,2) + 0.1U(2,1) + 0.1U(1,1), && (\mathit{Up}) \\ & 0.9U(1,1) + 0.1U(1,2), && (\mathit{Left}) \\ & 0.9U(1,1) + 0.1U(2,1), && (\mathit{Down}) \\ & 0.8U(2,1) + 0.1U(1,2) + 0.1U(1,1) \ ]. && (\mathit{Right}) \end{aligned}$$

When we plug in the numbers from Figure 17.3, we find that *Up* is the best action.

17.2.2 The value iteration algorithm

The Bellman equation is the basis of the value iteration algorithm for solving MDPs. If there are n possible states, then there are n Bellman equations, one for each state. The n equations contain n unknowns—the utilities of the states. So we would like to solve these simultaneous equations to find the utilities. There is one problem: the equations are *nonlinear*, because the "max" operator is not a linear operator. Whereas systems of linear equations can be solved quickly using linear algebra techniques, systems of nonlinear equations are more problematic. One thing to try is an *iterative* approach. We start with arbitrary initial values for the utilities, calculate the right-hand side of the equation, and plug it into the left-hand side—thereby updating the utility of each state from the utilities of its neighbors. We repeat this until we reach an equilibrium. Let $U_i(s)$ be the utility value for state s at the ith iteration. The iteration step, called a **Bellman update**, looks like this:

BELLMAN UPDATE

$$U_{i+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) U_i(s') \ , \tag{17.6}$$

function VALUE-ITERATION(mdp, ϵ) **returns** a utility function
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
    rewards $R(s)$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', vectors of utilities for states in S, initially zero
    δ, the maximum change in the utility of any state in an iteration

  repeat
    $U \leftarrow U'$; $\delta \leftarrow 0$
    for each state s **in** S **do**
      $U'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s' \mid s, a)\, U[s']$
      if $|U'[s] - U[s]| > \delta$ **then** $\delta \leftarrow |U'[s] - U[s]|$
  until $\delta < \epsilon(1-\gamma)/\gamma$
  return U

Figure 17.4 The value iteration algorithm for calculating utilities of states. The termination condition is from Equation (17.8).

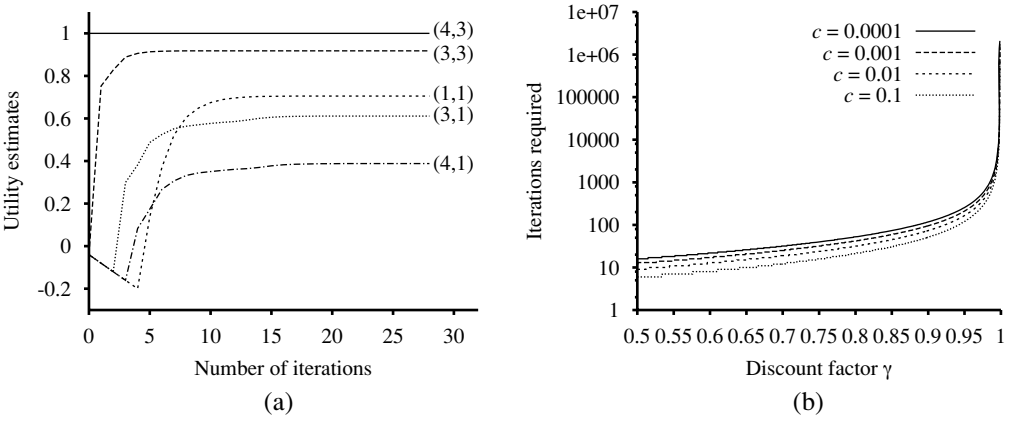


Figure 17.5 (a) Graph showing the evolution of the utilities of selected states using value iteration. (b) The number of value iterations k required to guarantee an error of at most $\epsilon = c \cdot R_{\max}$, for different values of c, as a function of the discount factor γ.

where the update is assumed to be applied simultaneously to all the states at each iteration. If we apply the Bellman update infinitely often, we are guaranteed to reach an equilibrium (see Section 17.2.3), in which case the final utility values must be solutions to the Bellman equations. In fact, they are also the *unique* solutions, and the corresponding policy (obtained using Equation (17.4)) is optimal. The algorithm, called VALUE-ITERATION, is shown in Figure 17.4.

We can apply value iteration to the 4×3 world in Figure 17.1(a). Starting with initial values of zero, the utilities evolve as shown in Figure 17.5(a). Notice how the states at differ-

ent distances from (4,3) accumulate negative reward until a path is found to (4,3), whereupon the utilities start to increase. We can think of the value iteration algorithm as *propagating information* through the state space by means of local updates.

17.2.3 Convergence of value iteration

We said that value iteration eventually converges to a unique set of solutions of the Bellman equations. In this section, we explain why this happens. We introduce some useful mathematical ideas along the way, and we obtain some methods for assessing the error in the utility function returned when the algorithm is terminated early; this is useful because it means that we don't have to run forever. This section is quite technical.

CONTRACTION

The basic concept used in showing that value iteration converges is the notion of a **contraction**. Roughly speaking, a contraction is a function of one argument that, when applied to two different inputs in turn, produces two output values that are "closer together," by at least some constant factor, than the original inputs. For example, the function "divide by two" is a contraction, because, after we divide any two numbers by two, their difference is halved. Notice that the "divide by two" function has a fixed point, namely zero, that is unchanged by the application of the function. From this example, we can discern two important properties of contractions:

- A contraction has only one fixed point; if there were two fixed points they would not get closer together when the function was applied, so it would not be a contraction.
- When the function is applied to any argument, the value must get closer to the fixed point (because the fixed point does not move), so repeated application of a contraction always reaches the fixed point in the limit.

Now, suppose we view the Bellman update (Equation (17.6)) as an operator B that is applied simultaneously to update the utility of every state. Let U_i denote the vector of utilities for all the states at the ith iteration. Then the Bellman update equation can be written as

$$U_{i+1} \leftarrow B\,U_i\ .$$

MAX NORM

Next, we need a way to measure distances between utility vectors. We will use the **max norm**, which measures the "length" of a vector by the absolute value of its biggest component:

$$||U|| = \max_s |U(s)|\ .$$

With this definition, the "distance" between two vectors, $||U - U'||$, is the maximum difference between any two corresponding elements. The main result of this section is the following: *Let U_i and U'_i be any two utility vectors. Then we have*

$$||B\,U_i - B\,U'_i|| \le \gamma\,||U_i - U'_i||\ . \tag{17.7}$$

That is, the Bellman update is a contraction by a factor of γ on the space of utility vectors. (Exercise 17.6 provides some guidance on proving this claim.) Hence, from the properties of contractions in general, it follows that value iteration always converges to a unique solution of the Bellman equations whenever $\gamma < 1$.

We can also use the contraction property to analyze the *rate* of convergence to a solution. In particular, we can replace U_i' in Equation (17.7) with the *true* utilities U, for which $B\,U = U$. Then we obtain the inequality

$$||B\,U_i - U|| \leq \gamma\,||U_i - U|| \;.$$

So, if we view $||U_i - U||$ as the *error* in the estimate U_i, we see that the error is reduced by a factor of at least γ on each iteration. This means that value iteration converges exponentially fast. We can calculate the number of iterations required to reach a specified error bound ϵ as follows: First, recall from Equation (17.1) that the utilities of all states are bounded by $\pm R_{\max}/(1-\gamma)$. This means that the maximum initial error $||U_0 - U|| \leq 2R_{\max}/(1-\gamma)$. Suppose we run for N iterations to reach an error of at most ϵ. Then, because the error is reduced by at least γ each time, we require $\gamma^N \cdot 2R_{\max}/(1-\gamma) \leq \epsilon$. Taking logs, we find

$$N = \lceil \log(2R_{\max}/\epsilon(1-\gamma))/\log(1/\gamma) \rceil$$

iterations suffice. Figure 17.5(b) shows how N varies with γ, for different values of the ratio $\epsilon/R_{\max}$. The good news is that, because of the exponentially fast convergence, N does not depend much on the ratio $\epsilon/R_{\max}$. The bad news is that N grows rapidly as γ becomes close to 1. We can get fast convergence if we make γ small, but this effectively gives the agent a short horizon and could miss the long-term effects of the agent's actions.

The error bound in the preceding paragraph gives some idea of the factors influencing the run time of the algorithm, but is sometimes overly conservative as a method of deciding when to stop the iteration. For the latter purpose, we can use a bound relating the error to the size of the Bellman update on any given iteration. From the contraction property (Equation (17.7)), it can be shown that if the update is small (i.e., no state's utility changes by much), then the error, compared with the true utility function, also is small. More precisely,

$$\text{if} \quad ||U_{i+1} - U_i|| < \epsilon(1-\gamma)/\gamma \quad \text{then} \quad ||U_{i+1} - U|| < \epsilon \;. \tag{17.8}$$

This is the termination condition used in the VALUE-ITERATION algorithm of Figure 17.4.

So far, we have analyzed the error in the utility function returned by the value iteration algorithm. *What the agent really cares about, however, is how well it will do if it makes its decisions on the basis of this utility function.* Suppose that after i iterations of value iteration, the agent has an estimate U_i of the true utility U and obtains the MEU policy π_i based on one-step look-ahead using U_i (as in Equation (17.4)). Will the resulting behavior be nearly as good as the optimal behavior? This is a crucial question for any real agent, and it turns out that the answer is yes. $U^{\pi_i}(s)$ is the utility obtained if π_i is executed starting in s, and the

POLICY LOSS

policy loss $||U^{\pi_i} - U||$ is the most the agent can lose by executing π_i instead of the optimal policy π^*. The policy loss of π_i is connected to the error in U_i by the following inequality:

$$\text{if} \quad ||U_i - U|| < \epsilon \quad \text{then} \quad ||U^{\pi_i} - U|| < 2\epsilon\gamma/(1-\gamma) \;. \tag{17.9}$$

In practice, it often occurs that π_i becomes optimal long before U_i has converged. Figure 17.6 shows how the maximum error in U_i and the policy loss approach zero as the value iteration process proceeds for the 4×3 environment with $\gamma = 0.9$. The policy π_i is optimal when $i = 4$, even though the maximum error in U_i is still 0.46.

Now we have everything we need to use value iteration in practice. We know that it converges to the correct utilities, we can bound the error in the utility estimates if we

stop after a finite number of iterations, and we can bound the policy loss that results from executing the corresponding MEU policy. As a final note, all of the results in this section depend on discounting with $\gamma < 1$. If $\gamma = 1$ and the environment contains terminal states, then a similar set of convergence results and error bounds can be derived whenever certain technical conditions are satisfied.

17.3 POLICY ITERATION

In the previous section, we observed that it is possible to get an optimal policy even when the utility function estimate is inaccurate. If one action is clearly better than all others, then the exact magnitude of the utilities on the states involved need not be precise. This insight suggests an alternative way to find optimal policies. The **policy iteration** algorithm alternates the following two steps, beginning from some initial policy π_0:

POLICY ITERATION

POLICY EVALUATION

- **Policy evaluation**: given a policy π_i, calculate $U_i = U^{\pi_i}$, the utility of each state if π_i were to be executed.

POLICY IMPROVEMENT

- **Policy improvement**: Calculate a new MEU policy π_{i+1}, using one-step look-ahead based on U_i (as in Equation (17.4)).

The algorithm terminates when the policy improvement step yields no change in the utilities. At this point, we know that the utility function U_i is a fixed point of the Bellman update, so it is a solution to the Bellman equations, and π_i must be an optimal policy. Because there are only finitely many policies for a finite state space, and each iteration can be shown to yield a better policy, policy iteration must terminate. The algorithm is shown in Figure 17.7.

The policy improvement step is obviously straightforward, but how do we implement the POLICY-EVALUATION routine? It turns out that doing so is much simpler than solving the standard Bellman equations (which is what value iteration does), because the action in each state is fixed by the policy. At the ith iteration, the policy π_i specifies the action $\pi_i(s)$ in

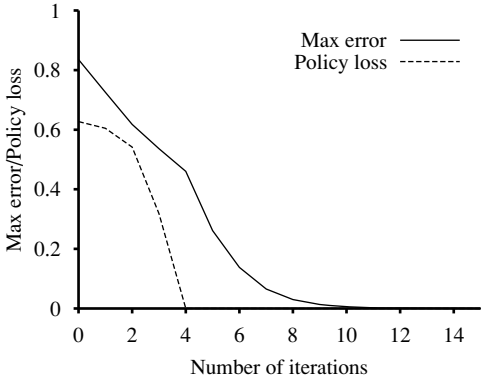


Figure 17.6 The maximum error $||U_i - U||$ of the utility estimates and the policy loss $||U^{\pi_i} - U||$, as a function of the number of iterations of value iteration.

state s. This means that we have a simplified version of the Bellman equation (17.5) relating the utility of s (under π_i) to the utilities of its neighbors:

$$U_i(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s') \; . \tag{17.10}$$

For example, suppose π_i is the policy shown in Figure 17.2(a). Then we have $\pi_i(1,1) = Up$, $\pi_i(1,2) = Up$, and so on, and the simplified Bellman equations are

$$\begin{aligned} U_i(1,1) &= -0.04 + 0.8U_i(1,2) + 0.1U_i(1,1) + 0.1U_i(2,1) \; , \\ U_i(1,2) &= -0.04 + 0.8U_i(1,3) + 0.2U_i(1,2) \; , \\ &\vdots \end{aligned}$$

The important point is that these equations are *linear*, because the "$\max$" operator has been removed. For n states, we have n linear equations with n unknowns, which can be solved exactly in time $O(n^3)$ by standard linear algebra methods.

For small state spaces, policy evaluation using exact solution methods is often the most efficient approach. For large state spaces, $O(n^3)$ time might be prohibitive. Fortunately, it is not necessary to do *exact* policy evaluation. Instead, we can perform some number of simplified value iteration steps (simplified because the policy is fixed) to give a reasonably good approximation of the utilities. The simplified Bellman update for this process is

$$U_{i+1}(s) \leftarrow R(s) + \gamma \sum_{s'} P(s' \mid s, \pi_i(s)) U_i(s') \; ,$$

MODIFIED POLICY ITERATION

and this is repeated k times to produce the next utility estimate. The resulting algorithm is called **modified policy iteration**. It is often much more efficient than standard policy iteration or value iteration.

function POLICY-ITERATION(mdp) **returns** a policy
  inputs: mdp, an MDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$
  local variables: U, a vector of utilities for states in S, initially zero
    π, a policy vector indexed by state, initially random

  repeat
    $U \leftarrow$ POLICY-EVALUATION(π, U, mdp)
    $unchanged? \leftarrow$ true
    for each state s **in** S **do**
      if $\max_{a \in A(s)} \sum_{s'} P(s' \mid s, a) \; U[s'] \; > \; \sum_{s'} P(s' \mid s, \pi[s]) \; U[s']$ **then do**
        $\pi[s] \leftarrow \operatorname{argmax}_{a \in A(s)} \sum_{s'} P(s' \mid s, a) \; U[s']$
        $unchanged? \leftarrow$ false
  until $unchanged?$
  return π

Figure 17.7 The policy iteration algorithm for calculating an optimal policy.

The algorithms we have described so far require updating the utility or policy for all states at once. It turns out that this is not strictly necessary. In fact, on each iteration, we can pick *any subset* of states and apply *either* kind of updating (policy improvement or simplified value iteration) to that subset. This very general algorithm is called **asynchronous policy iteration**. Given certain conditions on the initial policy and initial utility function, asynchronous policy iteration is guaranteed to converge to an optimal policy. The freedom to choose any states to work on means that we can design much more efficient heuristic algorithms—for example, algorithms that concentrate on updating the values of states that are likely to be reached by a good policy. This makes a lot of sense in real life: if one has no intention of throwing oneself off a cliff, one should not spend time worrying about the exact value of the resulting states.

ASYNCHRONOUS POLICY ITERATION

17.4 PARTIALLY OBSERVABLE MDPS

The description of Markov decision processes in Section 17.1 assumed that the environment was **fully observable**. With this assumption, the agent always knows which state it is in. This, combined with the Markov assumption for the transition model, means that the optimal policy depends only on the current state. When the environment is only **partially observable**, the situation is, one might say, much less clear. The agent does not necessarily know which state it is in, so it cannot execute the action $\pi(s)$ recommended for that state. Furthermore, the utility of a state s and the optimal action in s depend not just on s, but also on *how much the agent knows* when it is in s. For these reasons, **partially observable MDPs** (or POMDPs—pronounced "pom-dee-pees") are usually viewed as much more difficult than ordinary MDPs. We cannot avoid POMDPs, however, because the real world is one.

PARTIALLY OBSERVABLE MDP

17.4.1 Definition of POMDPs

To get a handle on POMDPs, we must first define them properly. A POMDP has the same elements as an MDP—the transition model $P(s' \mid s, a)$, actions $A(s)$, and reward function $R(s)$—but, like the partially observable search problems of Section 4.4, it also has a **sensor model** $P(e \mid s)$. Here, as in Chapter 15, the sensor model specifies the probability of perceiving evidence e in state s.[3] For example, we can convert the 4×3 world of Figure 17.1 into a POMDP by adding a noisy or partial sensor instead of assuming that the agent knows its location exactly. Such a sensor might measure the *number of adjacent walls*, which happens to be 2 in all the nonterminal squares except for those in the third column, where the value is 1; a noisy version might give the wrong value with probability 0.1.

In Chapters 4 and 11, we studied nondeterministic and partially observable planning problems and identified the **belief state**—the set of actual states the agent might be in—as a key concept for describing and calculating solutions. In POMDPs, the belief state b becomes a *probability distribution* over all possible states, just as in Chapter 15. For example, the initial

[3] As with the reward function for MDPs, the sensor model can also depend on the action and outcome state, but again this change is not fundamental.

belief state for the 4×3 POMDP could be the uniform distribution over the nine nonterminal states, i.e., $\langle \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, 0, 0 \rangle$. We write $b(s)$ for the probability assigned to the actual state s by belief state b. The agent can calculate its current belief state as the conditional probability distribution over the actual states given the sequence of percepts and actions so far. This is essentially the **filtering** task described in Chapter 15. The basic recursive filtering equation (15.5 on page 572) shows how to calculate the new belief state from the previous belief state and the new evidence. For POMDPs, we also have an action to consider, but the result is essentially the same. If $b(s)$ was the previous belief state, and the agent does action a and then perceives evidence e, then the new belief state is given by

$$b'(s') = \alpha\, P(e \mid s') \sum_s P(s' \mid s, a) b(s) \;,$$

where α is a normalizing constant that makes the belief state sum to 1. By analogy with the update operator for filtering (page 572), we can write this as

$$b' = \text{FORWARD}(b, a, e) \;. \tag{17.11}$$

In the 4×3 POMDP, suppose the agent moves *Left* and its sensor reports 1 adjacent wall; then it's quite likely (although not guaranteed, because both the motion and the sensor are noisy) that the agent is now in (3,1). Exercise 17.13 asks you to calculate the exact probability values for the new belief state.

The fundamental insight required to understand POMDPs is this: *the optimal action depends only on the agent's current belief state.* That is, the optimal policy can be described by a mapping $\pi^*(b)$ from belief states to actions. It does *not* depend on the *actual* state the agent is in. This is a good thing, because the agent does not know its actual state; all it knows is the belief state. Hence, the decision cycle of a POMDP agent can be broken down into the following three steps:

1. Given the current belief state b, execute the action $a = \pi^*(b)$.
2. Receive percept e.
3. Set the current belief state to $\text{FORWARD}(b, a, e)$ and repeat.

Now we can think of POMDPs as requiring a search in belief-state space, just like the methods for sensorless and contingency problems in Chapter 4. The main difference is that the POMDP belief-state space is *continuous*, because a POMDP belief state is a probability distribution. For example, a belief state for the 4×3 world is a point in an 11-dimensional continuous space. An action changes the belief state, not just the physical state. Hence, the action is evaluated at least in part according to the information the agent acquires as a result. POMDPs therefore include the value of information (Section 16.6) as one component of the decision problem.

Let's look more carefully at the outcome of actions. In particular, let's calculate the probability that an agent in belief state b reaches belief state b' after executing action a. Now, if we knew the action *and the subsequent percept*, then Equation (17.11) would provide a *deterministic* update to the belief state: $b' = \text{FORWARD}(b, a, e)$. Of course, the subsequent percept is not yet known, so the agent might arrive in one of several possible belief states b', depending on the percept that is received. The probability of perceiving e, given that a was

performed starting in belief state b, is given by summing over all the actual states s' that the agent might reach:

$$\begin{aligned} P(e|a,b) &= \sum_{s'} P(e|a,s',b)P(s'|a,b) \\ &= \sum_{s'} P(e \mid s')P(s'|a,b) \\ &= \sum_{s'} P(e \mid s') \sum_{s} P(s' \mid s,a)b(s) \ . \end{aligned}$$

Let us write the probability of reaching b' from b, given action a, as $P(b' \mid b,a))$. Then that gives us

$$\begin{aligned} P(b' \mid b,a) &= P(b'|a,b) = \sum_{e} P(b'|e,a,b)P(e|a,b) \\ &= \sum_{e} P(b'|e,a,b) \sum_{s'} P(e \mid s') \sum_{s} P(s' \mid s,a)b(s) \ , \end{aligned} \tag{17.12}$$

where $P(b'|e,a,b)$ is 1 if $b' = \text{FORWARD}(b,a,e)$ and 0 otherwise.

Equation (17.12) can be viewed as defining a transition model for the belief-state space. We can also define a reward function for belief states (i.e., the expected reward for the actual states the agent might be in):

$$\rho(b) = \sum_{s} b(s)R(s) \ .$$

Together, $P(b' \mid b,a)$ and $\rho(b)$ define an *observable* MDP on the space of belief states. Furthermore, it can be shown that an optimal policy for this MDP, $\pi^*(b)$, is also an optimal policy for the original POMDP. In other words, *solving a POMDP on a physical state space can be reduced to solving an MDP on the corresponding belief-state space.* This fact is perhaps less surprising if we remember that the belief state is always observable to the agent, by definition.

Notice that, although we have reduced POMDPs to MDPs, the MDP we obtain has a continuous (and usually high-dimensional) state space. None of the MDP algorithms described in Sections 17.2 and 17.3 applies directly to such MDPs. The next two subsections describe a value iteration algorithm designed specifically for POMDPs and an online decision-making algorithm, similar to those developed for games in Chapter 5.

17.4.2 Value iteration for POMDPs

Section 17.2 described a value iteration algorithm that computed one utility value for each state. With infinitely many belief states, we need to be more creative. Consider an optimal policy π^* and its application in a specific belief state b: the policy generates an action, then, for each subsequent percept, the belief state is updated and a new action is generated, and so on. For this specific b, therefore, the policy is exactly equivalent to a **conditional plan**, as defined in Chapter 4 for nondeterministic and partially observable problems. Instead of thinking about policies, let us think about conditional plans and how the expected utility of executing a fixed conditional plan varies with the initial belief state. We make two observations:

1. Let the utility of executing a *fixed* conditional plan p starting in physical state s be $\alpha_p(s)$. Then the expected utility of executing p in belief state b is just $\sum_s b(s)\alpha_p(s)$, or $b \cdot \alpha_p$ if we think of them both as vectors. Hence, the expected utility of a fixed conditional plan varies *linearly* with b; that is, it corresponds to a hyperplane in belief space.
2. At any given belief state b, the optimal policy will choose to execute the conditional plan with highest expected utility; and the expected utility of b under the optimal policy is just the utility of that conditional plan:

$$U(b) = U^{\pi^*}(b) = \max_p b \cdot \alpha_p \ .$$

If the optimal policy π^* chooses to execute p starting at b, then it is reasonable to expect that it might choose to execute p in belief states that are very close to b; in fact, if we bound the depth of the conditional plans, then there are only finitely many such plans and the continuous space of belief states will generally be divided into *regions*, each corresponding to a particular conditional plan that is optimal in that region.

From these two observations, we see that the utility function $U(b)$ on belief states, being the maximum of a collection of hyperplanes, will be *piecewise linear* and *convex*.

To illustrate this, we use a simple two-state world. The states are labeled 0 and 1, with $R(0) = 0$ and $R(1) = 1$. There are two actions: *Stay* stays put with probability 0.9 and *Go* switches to the other state with probability 0.9. For now we will assume the discount factor $\gamma = 1$. The sensor reports the correct state with probability 0.6. Obviously, the agent should *Stay* when it thinks it's in state 1 and *Go* when it thinks it's in state 0.

The advantage of a two-state world is that the belief space can be viewed as one-dimensional, because the two probabilities must sum to 1. In Figure 17.8(a), the x-axis represents the belief state, defined by $b(1)$, the probability of being in state 1. Now let us consider the one-step plans $[Stay]$ and $[Go]$, each of which receives the reward for the current state followed by the (discounted) reward for the state reached after the action:

$$\begin{aligned}
\alpha_{[Stay]}(0) &= R(0) + \gamma(0.9R(0) + 0.1R(1)) = 0.1 \\
\alpha_{[Stay]}(1) &= R(1) + \gamma(0.9R(1) + 0.1R(0)) = 1.9 \\
\alpha_{[Go]}(0) &= R(0) + \gamma(0.9R(1) + 0.1R(0)) = 0.9 \\
\alpha_{[Go]}(1) &= R(1) + \gamma(0.9R(0) + 0.1R(1)) = 1.1
\end{aligned}$$

The hyperplanes (lines, in this case) for $b \cdot \alpha_{[Stay]}$ and $b \cdot \alpha_{[Go]}$ are shown in Figure 17.8(a) and their maximum is shown in bold. The bold line therefore represents the utility function for the finite-horizon problem that allows just one action, and in each "piece" of the piecewise linear utility function the optimal action is the first action of the corresponding conditional plan. In this case, the optimal one-step policy is to *Stay* when $b(1) > 0.5$ and *Go* otherwise.

Once we have utilities $\alpha_p(s)$ for all the conditional plans p of depth 1 in each physical state s, we can compute the utilities for conditional plans of depth 2 by considering each possible first action, each possible subsequent percept, and then each way of choosing a depth-1 plan to execute for each percept:

$$[Stay;\ \textbf{if}\ Percept = 0\ \textbf{then}\ Stay\ \textbf{else}\ Stay]$$
$$[Stay;\ \textbf{if}\ Percept = 0\ \textbf{then}\ Stay\ \textbf{else}\ Go] \ldots$$

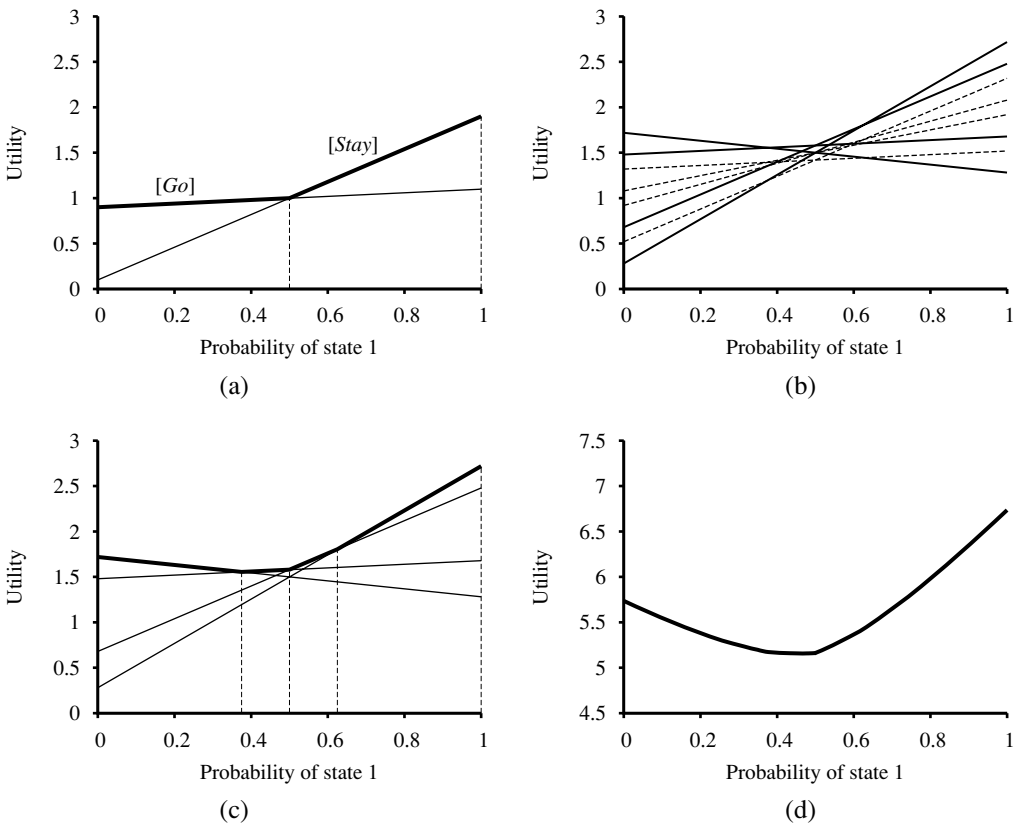


Figure 17.8 (a) Utility of two one-step plans as a function of the initial belief state $b(1)$ for the two-state world, with the corresponding utility function shown in bold. (b) Utilities for 8 distinct two-step plans. (c) Utilities for four undominated two-step plans. (d) Utility function for optimal eight-step plans.

There are eight distinct depth-2 plans in all, and their utilities are shown in Figure 17.8(b). Notice that four of the plans, shown as dashed lines, are suboptimal across the entire belief space—we say these plans are **dominated**, and they need not be considered further. There are four undominated plans, each of which is optimal in a specific region, as shown in Figure 17.8(c). The regions partition the belief-state space.

DOMINATED PLAN

We repeat the process for depth 3, and so on. In general, let p be a depth-d conditional plan whose initial action is a and whose depth-$d-1$ subplan for percept e is $p.e$; then

$$\alpha_p(s) = R(s) + \gamma \left(\sum_{s'} P(s' \mid s, a) \sum_{e} P(e \mid s') \alpha_{p.e}(s') \right) . \tag{17.13}$$

This recursion naturally gives us a value iteration algorithm, which is sketched in Figure 17.9. The structure of the algorithm and its error analysis are similar to those of the basic value iteration algorithm in Figure 17.4 on page 653; the main difference is that instead of computing one utility number for each state, POMDP-VALUE-ITERATION maintains a collection of

function POMDP-VALUE-ITERATION($pomdp, \epsilon$) **returns** a utility function
  inputs: $pomdp$, a POMDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
    sensor model $P(e \mid s)$, rewards $R(s)$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', sets of plans p with associated utility vectors α_p

  $U' \leftarrow$ a set containing just the empty plan $[\,]$, with $\alpha_{[\,]}(s) = R(s)$
  repeat
    $U \leftarrow U'$
    $U' \leftarrow$ the set of all plans consisting of an action and, for each possible next percept,
      a plan in U with utility vectors computed according to Equation (17.13)
    $U' \leftarrow$ REMOVE-DOMINATED-PLANS(U')
  until MAX-DIFFERENCE(U, U') $< \epsilon(1-\gamma)/\gamma$
  return U

Figure 17.9 A high-level sketch of the value iteration algorithm for POMDPs. The REMOVE-DOMINATED-PLANS step and MAX-DIFFERENCE test are typically implemented as linear programs.

undominated plans with their utility hyperplanes. The algorithm's complexity depends primarily on how many plans get generated. Given $|A|$ actions and $|E|$ possible observations, it is easy to show that there are $|A|^{O(|E|^{d-1})}$ distinct depth-d plans. Even for the lowly two-state world with $d = 8$, the exact number is 2^{255}. The elimination of dominated plans is essential for reducing this doubly exponential growth: the number of undominated plans with $d = 8$ is just 144. The utility function for these 144 plans is shown in Figure 17.8(d).

Notice that even though state 0 has lower utility than state 1, the intermediate belief states have even lower utility because the agent lacks the information needed to choose a good action. This is why information has value in the sense defined in Section 16.6 and optimal policies in POMDPs often include information-gathering actions.

Given such a utility function, an executable policy can be extracted by looking at which hyperplane is optimal at any given belief state b and executing the first action of the corresponding plan. In Figure 17.8(d), the corresponding optimal policy is still the same as for depth-1 plans: *Stay* when $b(1) > 0.5$ and *Go* otherwise.

In practice, the value iteration algorithm in Figure 17.9 is hopelessly inefficient for larger problems—even the 4×3 POMDP is too hard. The main reason is that, given n conditional plans at level d, the algorithm constructs $|A| \cdot n^{|E|}$ conditional plans at level $d+1$ before eliminating the dominated ones. Since the 1970s, when this algorithm was developed, there have been several advances including more efficient forms of value iteration and various kinds of policy iteration algorithms. Some of these are discussed in the notes at the end of the chapter. For general POMDPs, however, finding optimal policies is very difficult (PSPACE-hard, in fact—i.e., very hard indeed). Problems with a few dozen states are often infeasible. The next section describes a different, approximate method for solving POMDPs, one based on look-ahead search.

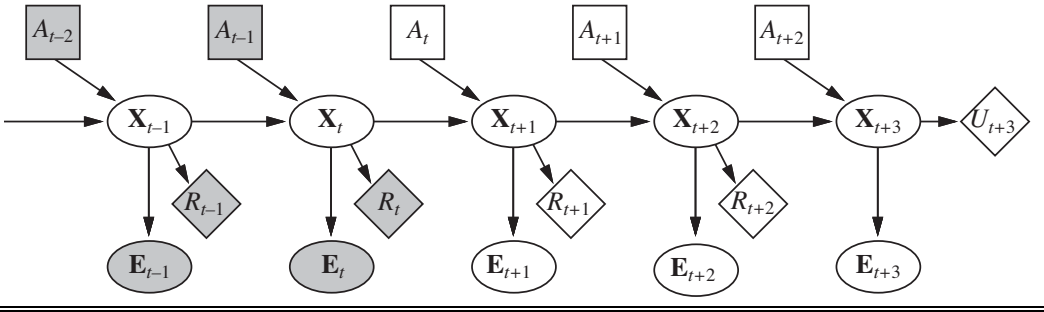


Figure 17.10 The generic structure of a dynamic decision network. Variables with known values are shaded. The current time is t and the agent must decide what to do—that is, choose a value for A_t. The network has been unrolled into the future for three steps and represents future rewards, as well as the utility of the state at the look-ahead horizon.

17.4.3 Online agents for POMDPs

In this section, we outline a simple approach to agent design for partially observable, stochastic environments. The basic elements of the design are already familiar:

- The transition and sensor models are represented by a **dynamic Bayesian network** (DBN), as described in Chapter 15.
- The dynamic Bayesian network is extended with decision and utility nodes, as used in **decision networks** in Chapter 16. The resulting model is called a **dynamic decision network**, or DDN.

DYNAMIC DECISION NETWORK

- A filtering algorithm is used to incorporate each new percept and action and to update the belief state representation.
- Decisions are made by projecting forward possible action sequences and choosing the best one.

DBNs are **factored representations** in the terminology of Chapter 2; they typically have an exponential complexity advantage over atomic representations and can model quite substantial real-world problems. The agent design is therefore a practical implementation of the **utility-based agent** sketched in Chapter 2.

In the DBN, the single state S_t becomes a set of state variables $\mathbf{X}_t$, and there may be multiple evidence variables $\mathbf{E}_t$. We will use A_t to refer to the action at time t, so the transition model becomes $\mathbf{P}(\mathbf{X}_{t+1}|\mathbf{X}_t, A_t)$ and the sensor model becomes $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$. We will use R_t to refer to the reward received at time t and U_t to refer to the utility of the state at time t. (Both of these are random variables.) With this notation, a dynamic decision network looks like the one shown in Figure 17.10.

Dynamic decision networks can be used as inputs for any POMDP algorithm, including those for value and policy iteration methods. In this section, we focus on look-ahead methods that project action sequences forward from the current belief state in much the same way as do the game-playing algorithms of Chapter 5. The network in Figure 17.10 has been projected three steps into the future; the current and future decisions A and the future observations

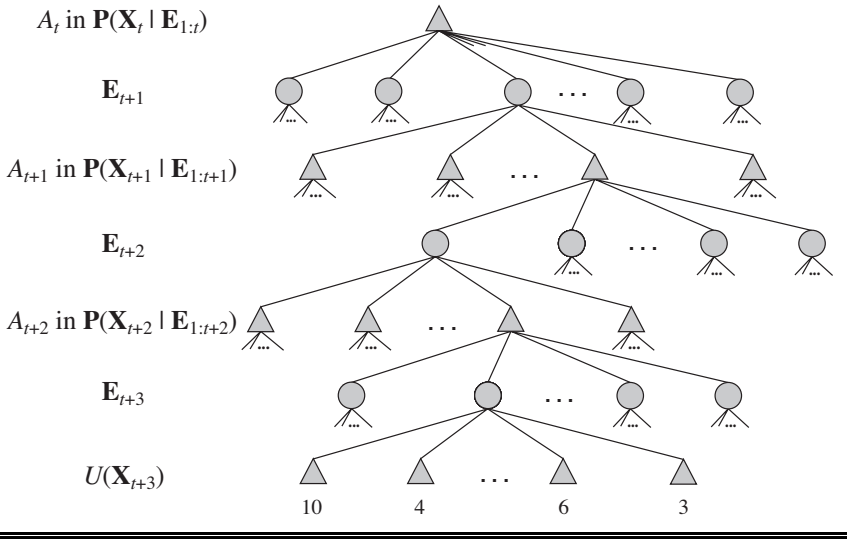


Figure 17.11 Part of the look-ahead solution of the DDN in Figure 17.10. Each decision will be taken in the belief state indicated.

E and rewards R are all unknown. Notice that the network includes nodes for the *rewards* for $\mathbf{X}_{t+1}$ and $\mathbf{X}_{t+2}$, but the *utility* for $\mathbf{X}_{t+3}$. This is because the agent must maximize the (discounted) sum of all future rewards, and $U(\mathbf{X}_{t+3})$ represents the reward for $\mathbf{X}_{t+3}$ and all subsequent rewards. As in Chapter 5, we assume that U is available only in some approximate form: if exact utility values were available, look-ahead beyond depth 1 would be unnecessary.

Figure 17.11 shows part of the search tree corresponding to the three-step look-ahead DDN in Figure 17.10. Each of the triangular nodes is a belief state in which the agent makes a decision A_{t+i} for $i = 0, 1, 2, \ldots$. The round (chance) nodes correspond to choices by the environment, namely, what evidence $\mathbf{E}_{t+i}$ arrives. Notice that there are no chance nodes corresponding to the action outcomes; this is because the belief-state update for an action is deterministic regardless of the actual outcome.

The belief state at each triangular node can be computed by applying a filtering algorithm to the sequence of percepts and actions leading to it. In this way, the algorithm takes into account the fact that, for decision A_{t+i}, the agent *will* have available percepts $\mathbf{E}_{t+1}, \ldots, \mathbf{E}_{t+i}$, even though at time t it does not know what those percepts will be. In this way, a decision-theoretic agent automatically takes into account the value of information and will execute information-gathering actions where appropriate.

A decision can be extracted from the search tree by backing up the utility values from the leaves, taking an average at the chance nodes and taking the maximum at the decision nodes. This is similar to the EXPECTIMINIMAX algorithm for game trees with chance nodes, except that (1) there can also be rewards at non-leaf states and (2) the decision nodes correspond to belief states rather than actual states. The time complexity of an exhaustive search to depth d is $O(|A|^d \cdot |\mathbf{E}|^d)$, where $|A|$ is the number of available actions and $|\mathbf{E}|$ is the number of possible percepts. (Notice that this is far less than the number of depth-d conditional

plans generated by value iteration.) For problems in which the discount factor γ is not too close to 1, a shallow search is often good enough to give near-optimal decisions. It is also possible to approximate the averaging step at the chance nodes, by sampling from the set of possible percepts instead of summing over all possible percepts. There are various other ways of finding good approximate solutions quickly, but we defer them to Chapter 21.

Decision-theoretic agents based on dynamic decision networks have a number of advantages compared with other, simpler agent designs presented in earlier chapters. In particular, they handle partially observable, uncertain environments and can easily revise their "plans" to handle unexpected evidence. With appropriate sensor models, they can handle sensor failure and can plan to gather information. They exhibit "graceful degradation" under time pressure and in complex environments, using various approximation techniques. So what is missing? One defect of our DDN-based algorithm is its reliance on forward search through state space, rather than using the hierarchical and other advanced planning techniques described in Chapter 11. There have been attempts to extend these techniques into the probabilistic domain, but so far they have proved to be inefficient. A second, related problem is the basically propositional nature of the DDN language. We would like to be able to extend some of the ideas for first-order probabilistic languages to the problem of decision making. Current research has shown that this extension is possible and has significant benefits, as discussed in the notes at the end of the chapter.

17.5 Decisions with Multiple Agents: Game Theory

This chapter has concentrated on making decisions in uncertain environments. But what if the uncertainty is due to other agents and the decisions they make? And what if the decisions of those agents are in turn influenced by our decisions? We addressed this question once before, when we studied games in Chapter 5. There, however, we were primarily concerned with turn-taking games in fully observable environments, for which minimax search can be used to find optimal moves. In this section we study the aspects of **game theory** that analyze games with simultaneous moves and other sources of partial observability. (Game theorists use the terms **perfect information** and **imperfect information** rather than fully and partially observable.) Game theory can be used in at least two ways:

GAME THEORY

1. **Agent design**: Game theory can analyze the agent's decisions and compute the expected utility for each decision (under the assumption that other agents are acting optimally according to game theory). For example, in the game **two-finger Morra**, two players, O and E, simultaneously display one or two fingers. Let the total number of fingers be f. If f is odd, O collects f dollars from E; and if f is even, E collects f dollars from O. Game theory can determine the best strategy against a rational player and the expected return for each player.[4]

[4] Morra is a recreational version of an **inspection game**. In such games, an inspector chooses a day to inspect a facility (such as a restaurant or a biological weapons plant), and the facility operator chooses a day to hide all the nasty stuff. The inspector wins if the days are different, and the facility operator wins if they are the same.

21 REINFORCEMENT LEARNING

In which we examine how an agent can learn from success and failure, from reward and punishment.

21.1 INTRODUCTION

Chapters 18, 19, and 20 covered methods that learn functions, logical theories, and probability models from examples. In this chapter, we will study how agents can learn *what to do* in the absence of labeled examples of what to do.

Consider, for example, the problem of learning to play chess. A supervised learning agent needs to be told the correct move for each position it encounters, but such feedback is seldom available. In the absence of feedback from a teacher, an agent can learn a transition model for its own moves and can perhaps learn to predict the opponent's moves, but *without some feedback about what is good and what is bad, the agent will have no grounds for deciding which move to make.* The agent needs to know that something good has happened when it (accidentally) checkmates the opponent, and that something bad has happened when it is checkmated—or vice versa, if the game is suicide chess. This kind of feedback is called a **reward**, or **reinforcement**. In games like chess, the reinforcement is received only at the end of the game. In other environments, the rewards come more frequently. In ping-pong, each point scored can be considered a reward; when learning to crawl, any forward motion is an achievement. Our framework for agents regards the reward as *part* of the input percept, but the agent must be "hardwired" to recognize that part as a reward rather than as just another sensory input. Thus, animals seem to be hardwired to recognize pain and hunger as negative rewards and pleasure and food intake as positive rewards. Reinforcement has been carefully studied by animal psychologists for over 60 years.

REINFORCEMENT

Rewards were introduced in Chapter 17, where they served to define optimal policies in **Markov decision processes** (MDPs). An optimal policy is a policy that maximizes the expected total reward. The task of **reinforcement learning** is to use observed rewards to learn an optimal (or nearly optimal) policy for the environment. Whereas in Chapter 17 the agent has a complete model of the environment and knows the reward function, here we assume no

prior knowledge of either. Imagine playing a new game whose rules you don't know; after a hundred or so moves, your opponent announces, "You lose." This is reinforcement learning in a nutshell.

In many complex domains, reinforcement learning is the only feasible way to train a program to perform at high levels. For example, in game playing, it is very hard for a human to provide accurate and consistent evaluations of large numbers of positions, which would be needed to train an evaluation function directly from examples. Instead, the program can be told when it has won or lost, and it can use this information to learn an evaluation function that gives reasonably accurate estimates of the probability of winning from any given position. Similarly, it is extremely difficult to program an agent to fly a helicopter; yet given appropriate negative rewards for crashing, wobbling, or deviating from a set course, an agent can learn to fly by itself.

Reinforcement learning might be considered to encompass all of AI: an agent is placed in an environment and must learn to behave successfully therein. To keep the chapter manageable, we will concentrate on simple environments and simple agent designs. For the most part, we will assume a fully observable environment, so that the current state is supplied by each percept. On the other hand, we will assume that the agent does not know how the environment works or what its actions do, and we will allow for probabilistic action outcomes. Thus, the agent faces an unknown Markov decision process. We will consider three of the agent designs first introduced in Chapter 2:

- A **utility-based agent** learns a utility function on states and uses it to select actions that maximize the expected outcome utility.
- Q-LEARNING Q-FUNCTION A **Q-learning** agent learns an **action-utility function**, or **Q-function**, giving the expected utility of taking a given action in a given state.
- A **reflex agent** learns a policy that maps directly from states to actions.

A utility-based agent must also have a model of the environment in order to make decisions, because it must know the states to which its actions will lead. For example, in order to make use of a backgammon evaluation function, a backgammon program must know what its legal moves are *and how they affect the board position*. Only in this way can it apply the utility function to the outcome states. A Q-learning agent, on the other hand, can compare the expected utilities for its available choices without needing to know their outcomes, so it does not need a model of the environment. On the other hand, because they do not know where their actions lead, Q-learning agents cannot look ahead; this can seriously restrict their ability to learn, as we shall see.

PASSIVE LEARNING ACTIVE LEARNING EXPLORATION We begin in Section 21.2 with **passive learning**, where the agent's policy is fixed and the task is to learn the utilities of states (or state–action pairs); this could also involve learning a model of the environment. Section 21.3 covers **active learning**, where the agent must also learn what to do. The principal issue is **exploration**: an agent must experience as much as possible of its environment in order to learn how to behave in it. Section 21.4 discusses how an agent can use inductive learning to learn much faster from its experiences. Section 21.5 covers methods for learning direct policy representations in reflex agents. An understanding of Markov decision processes (Chapter 17) is essential for this chapter.

21.2 PASSIVE REINFORCEMENT LEARNING

To keep things simple, we start with the case of a passive learning agent using a state-based representation in a fully observable environment. In passive learning, the agent's policy π is fixed: in state s, it always executes the action $\pi(s)$. Its goal is simply to learn how good the policy is—that is, to learn the utility function $U^\pi(s)$. We will use as our example the 4×3 world introduced in Chapter 17. Figure 21.1 shows a policy for that world and the corresponding utilities. Clearly, the passive learning task is similar to the **policy evaluation** task, part of the **policy iteration** algorithm described in Section 17.3. The main difference is that the passive learning agent does not know the **transition model** $P(s' \mid s, a)$, which specifies the probability of reaching state s' from state s after doing action a; nor does it know the **reward function** $R(s)$, which specifies the reward for each state.

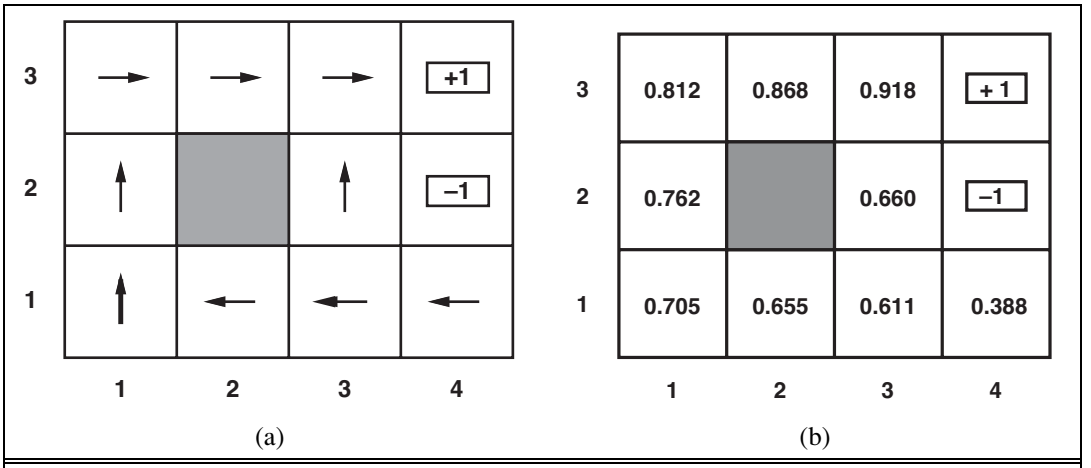


Figure 21.1 (a) A policy π for the 4×3 world; this policy happens to be optimal with rewards of $R(s) = -0.04$ in the nonterminal states and no discounting. (b) The utilities of the states in the 4×3 world, given policy π.

TRIAL

The agent executes a set of **trials** in the environment using its policy π. In each trial, the agent starts in state (1,1) and experiences a sequence of state transitions until it reaches one of the terminal states, (4,2) or (4,3). Its percepts supply both the current state and the reward received in that state. Typical trials might look like this:

$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
$$(1,1)_{\textbf{-.04}} \rightsquigarrow (1,2)_{\textbf{-.04}} \rightsquigarrow (1,3)_{\textbf{-.04}} \rightsquigarrow (2,3)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (3,2)_{\textbf{-.04}} \rightsquigarrow (3,3)_{\textbf{-.04}} \rightsquigarrow (4,3)_{\textbf{+1}}$$
$$(1,1)_{\textbf{-.04}} \rightsquigarrow (2,1)_{\textbf{-.04}} \rightsquigarrow (3,1)_{\textbf{-.04}} \rightsquigarrow (3,2)_{\textbf{-.04}} \rightsquigarrow (4,2)_{\textbf{-1}} .$$

Note that each state percept is subscripted with the reward received. The object is to use the information about rewards to learn the expected utility $U^\pi(s)$ associated with each nonterminal state s. The utility is defined to be the expected sum of (discounted) rewards obtained if

policy π is followed. As in Equation (17.2) on page 650, we write

$$U^{\pi}(s) = E\left[\sum_{t=0}^{\infty} \gamma^t R(S_t)\right] \tag{21.1}$$

where $R(s)$ is the reward for a state, S_t (a random variable) is the state reached at time t when executing policy π, and $S_0 = s$. We will include a **discount factor** γ in all of our equations, but for the 4×3 world we will set $\gamma = 1$.

21.2.1 Direct utility estimation

DIRECT UTILITY ESTIMATION
ADAPTIVE CONTROL THEORY
REWARD-TO-GO

A simple method for **direct utility estimation** was invented in the late 1950s in the area of **adaptive control theory** by Widrow and Hoff (1960). The idea is that the utility of a state is the expected total reward from that state onward (called the expected **reward-to-go**), and each trial provides a *sample* of this quantity for each state visited. For example, the first trial in the set of three given earlier provides a sample total reward of 0.72 for state (1,1), two samples of 0.76 and 0.84 for (1,2), two samples of 0.80 and 0.88 for (1,3), and so on. Thus, at the end of each sequence, the algorithm calculates the observed reward-to-go for each state and updates the estimated utility for that state accordingly, just by keeping a running average for each state in a table. In the limit of infinitely many trials, the sample average will converge to the true expectation in Equation (21.1).

It is clear that direct utility estimation is just an instance of supervised learning where each example has the state as input and the observed reward-to-go as output. This means that we have reduced reinforcement learning to a standard inductive learning problem, as discussed in Chapter 18. Section 21.4 discusses the use of more powerful kinds of representations for the utility function. Learning techniques for those representations can be applied directly to the observed data.

Direct utility estimation succeeds in reducing the reinforcement learning problem to an inductive learning problem, about which much is known. Unfortunately, it misses a very important source of information, namely, the fact that the utilities of states are not independent! *The utility of each state equals its own reward plus the expected utility of its successor states.* That is, the utility values obey the Bellman equations for a fixed policy (see also Equation (17.10)):

$$U^{\pi}(s) = R(s) + \gamma \sum_{s'} P(s' \mid s, \pi(s)) U^{\pi}(s') \; . \tag{21.2}$$

By ignoring the connections between states, direct utility estimation misses opportunities for learning. For example, the second of the three trials given earlier reaches the state (3,2), which has not previously been visited. The next transition reaches (3,3), which is known from the first trial to have a high utility. The Bellman equation suggests immediately that (3,2) is also likely to have a high utility, because it leads to (3,3), but direct utility estimation learns nothing until the end of the trial. More broadly, we can view direct utility estimation as searching for U in a hypothesis space that is much larger than it needs to be, in that it includes many functions that violate the Bellman equations. For this reason, the algorithm often converges very slowly.

```
function PASSIVE-ADP-AGENT(percept) returns an action
  inputs: percept, a percept indicating the current state s' and reward signal r'
  persistent: π, a fixed policy
              mdp, an MDP with model P, rewards R, discount γ
              U, a table of utilities, initially empty
              N_sa, a table of frequencies for state–action pairs, initially zero
              N_s'|sa, a table of outcome frequencies given state–action pairs, initially zero
              s, a, the previous state and action, initially null

  if s' is new then U[s'] ← r'; R[s'] ← r'
  if s is not null then
      increment N_sa[s, a] and N_s'|sa[s', s, a]
      for each t such that N_s'|sa[t, s, a] is nonzero do
          P(t | s, a) ← N_s'|sa[t, s, a] / N_sa[s, a]
  U ← POLICY-EVALUATION(π, U, mdp)
  if s'.TERMINAL? then s, a ← null else s, a ← s', π[s']
  return a
```

Figure 21.2 A passive reinforcement learning agent based on adaptive dynamic programming. The POLICY-EVALUATION function solves the fixed-policy Bellman equations, as described on page 657.

21.2.2 Adaptive dynamic programming

ADAPTIVE DYNAMIC PROGRAMMING

An **adaptive dynamic programming** (or ADP) agent takes advantage of the constraints among the utilities of states by learning the transition model that connects them and solving the corresponding Markov decision process using a dynamic programming method. For a passive learning agent, this means plugging the learned transition model $P(s' \mid s, \pi(s))$ and the observed rewards $R(s)$ into the Bellman equations (21.2) to calculate the utilities of the states. As we remarked in our discussion of policy iteration in Chapter 17, these equations are linear (no maximization involved) so they can be solved using any linear algebra package. Alternatively, we can adopt the approach of **modified policy iteration** (see page 657), using a simplified value iteration process to update the utility estimates after each change to the learned model. Because the model usually changes only slightly with each observation, the value iteration process can use the previous utility estimates as initial values and should converge quite quickly.

The process of learning the model itself is easy, because the environment is fully observable. This means that we have a supervised learning task where the input is a state–action pair and the output is the resulting state. In the simplest case, we can represent the transition model as a table of probabilities. We keep track of how often each action outcome occurs and estimate the transition probability $P(s' \mid s, a)$ from the frequency with which s' is reached when executing a in s. For example, in the three trials given on page 832, $Right$ is executed three times in (1,3) and two out of three times the resulting state is (2,3), so $P((2,3) \mid (1,3), Right)$ is estimated to be 2/3.

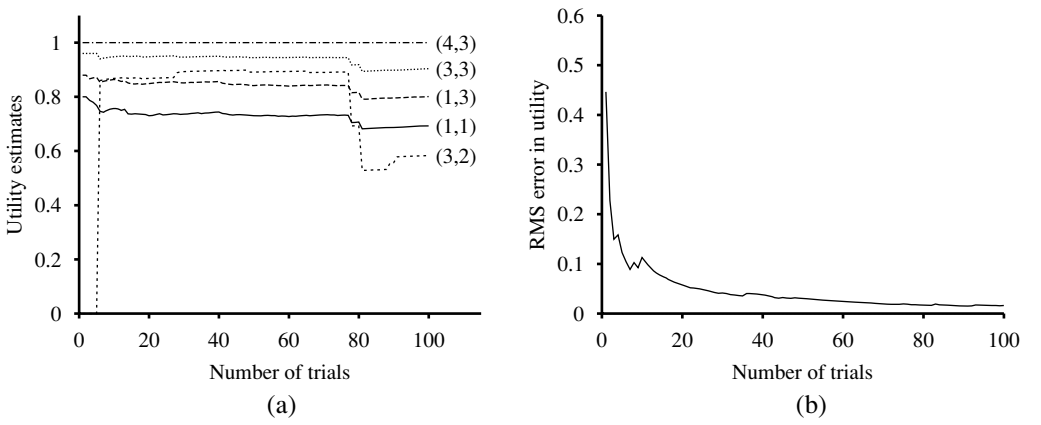


Figure 21.3 The passive ADP learning curves for the 4×3 world, given the optimal policy shown in Figure 21.1. (a) The utility estimates for a selected subset of states, as a function of the number of trials. Notice the large changes occurring around the 78th trial—this is the first time that the agent falls into the -1 terminal state at (4,2). (b) The root-mean-square error (see Appendix A) in the estimate for $U(1,1)$, averaged over 20 runs of 100 trials each.

The full agent program for a passive ADP agent is shown in Figure 21.2. Its performance on the 4×3 world is shown in Figure 21.3. In terms of how quickly its value estimates improve, the ADP agent is limited only by its ability to learn the transition model. In this sense, it provides a standard against which to measure other reinforcement learning algorithms. It is, however, intractable for large state spaces. In backgammon, for example, it would involve solving roughly 10^{50} equations in 10^{50} unknowns.

A reader familiar with the Bayesian learning ideas of Chapter 20 will have noticed that the algorithm in Figure 21.2 is using maximum-likelihood estimation to learn the transition model; moreover, by choosing a policy based solely on the *estimated* model it is acting *as if* the model were correct. This is not necessarily a good idea! For example, a taxi agent that didn't know about how traffic lights might ignore a red light once or twice without no ill effects and then formulate a policy to ignore red lights from then on. Instead, it might be a good idea to choose a policy that, while not optimal for the model estimated by maximum likelihood, works reasonably well for the whole range of models that have a reasonable chance of being the true model. There are two mathematical approaches that have this flavor.

BAYESIAN REINFORCEMENT LEARNING

The first approach, **Bayesian reinforcement learning**, assumes a prior probability $P(h)$ for each hypothesis h about what the true model is; the posterior probability $P(h \mid \mathbf{e})$ is obtained in the usual way by Bayes' rule given the observations to date. Then, if the agent has decided to stop learning, the optimal policy is the one that gives the highest expected utility. Let u_h^π be the expected utility, averaged over all possible start states, obtained by executing policy π in model h. Then we have

$$\pi^* = \operatorname*{argmax}_\pi \sum_h P(h \mid \mathbf{e}) u_h^\pi \ .$$

In some special cases, this policy can even be computed! If the agent will continue learning in the future, however, then finding an optimal policy becomes considerably more difficult, because the agent must consider the effects of future observations on its beliefs about the transition model. The problem becomes a POMDP whose belief states are distributions over models. This concept provides an analytical foundation for understanding the exploration problem described in Section 21.3.

ROBUST CONTROL THEORY

The second approach, derived from **robust control theory**, allows for a *set* of possible models $\mathcal{H}$ and defines an optimal robust policy as one that gives the best outcome in the *worst case* over $\mathcal{H}$:

$$\pi^* = \operatorname*{argmax}_{\pi} \min_{h} u_h^{\pi} .$$

Often, the set $\mathcal{H}$ will be the set of models that exceed some likelihood threshold on $P(h \mid \mathbf{e})$, so the robust and Bayesian approaches are related. Sometimes, the robust solution can be computed efficiently. There are, moreover, reinforcement learning algorithms that tend to produce robust solutions, although we do not cover them here.

21.2.3 Temporal-difference learning

Solving the underlying MDP as in the preceding section is not the only way to bring the Bellman equations to bear on the learning problem. Another way is to use the observed transitions to adjust the utilities of the observed states so that they agree with the constraint equations. Consider, for example, the transition from (1,3) to (2,3) in the second trial on page 832. Suppose that, as a result of the first trial, the utility estimates are $U^{\pi}(1,3) = 0.84$ and $U^{\pi}(2,3) = 0.92$. Now, if this transition occurred all the time, we would expect the utilities to obey the equation

$$U^{\pi}(1,3) = -0.04 + U^{\pi}(2,3) ,$$

so $U^{\pi}(1,3)$ would be 0.88. Thus, its current estimate of 0.84 might be a little low and should be increased. More generally, when a transition occurs from state s to state s', we apply the following update to $U^{\pi}(s)$:

$$U^{\pi}(s) \leftarrow U^{\pi}(s) + \alpha(R(s) + \gamma U^{\pi}(s') - U^{\pi}(s)) . \tag{21.3}$$

TEMPORAL-DIFFERENCE

Here, α is the **learning rate** parameter. Because this update rule uses the difference in utilities between successive states, it is often called the **temporal-difference**, or TD, equation.

All temporal-difference methods work by adjusting the utility estimates towards the ideal equilibrium that holds locally when the utility estimates are correct. In the case of passive learning, the equilibrium is given by Equation (21.2). Now Equation (21.3) does in fact cause the agent to reach the equilibrium given by Equation (21.2), but there is some subtlety involved. First, notice that the update involves only the observed successor s', whereas the actual equilibrium conditions involve all possible next states. One might think that this causes an improperly large change in $U^{\pi}(s)$ when a very rare transition occurs; but, in fact, because rare transitions occur only rarely, the *average value* of $U^{\pi}(s)$ will converge to the correct value. Furthermore, if we change α from a fixed parameter to a function that decreases as the number of times a state has been visited increases, then $U^{\pi}(s)$ itself will converge to the

```
function PASSIVE-TD-AGENT(percept) returns an action
  inputs: percept, a percept indicating the current state s' and reward signal r'
  persistent: π, a fixed policy
              U, a table of utilities, initially empty
              N_s, a table of frequencies for states, initially zero
              s, a, r, the previous state, action, and reward, initially null

  if s' is new then U[s'] ← r'
  if s is not null then
      increment N_s[s]
      U[s] ← U[s] + α(N_s[s])(r + γ U[s'] − U[s])
  if s'.TERMINAL? then s, a, r ← null else s, a, r ← s', π[s'], r'
  return a
```

Figure 21.4 A passive reinforcement learning agent that learns utility estimates using temporal differences. The step-size function $\alpha(n)$ is chosen to ensure convergence, as described in the text.

correct value.[1] This gives us the agent program shown in Figure 21.4. Figure 21.5 illustrates the performance of the passive TD agent on the 4×3 world. It does not learn quite as fast as the ADP agent and shows much higher variability, but it is much simpler and requires much less computation per observation. Notice that *TD does not need a transition model to perform its updates.* The environment supplies the connection between neighboring states in the form of observed transitions.

The ADP approach and the TD approach are actually closely related. Both try to make local adjustments to the utility estimates in order to make each state "agree" with its successors. One difference is that TD adjusts a state to agree with its *observed* successor (Equation (21.3)), whereas ADP adjusts the state to agree with *all* of the successors that might occur, weighted by their probabilities (Equation (21.2)). This difference disappears when the effects of TD adjustments are averaged over a large number of transitions, because the frequency of each successor in the set of transitions is approximately proportional to its probability. A more important difference is that whereas TD makes a single adjustment per observed transition, ADP makes as many as it needs to restore consistency between the utility estimates U and the environment model P. Although the observed transition makes only a local change in P, its effects might need to be propagated throughout U. Thus, TD can be viewed as a crude but efficient first approximation to ADP.

Each adjustment made by ADP could be seen, from the TD point of view, as a result of a "pseudoexperience" generated by simulating the current environment model. It is possible to extend the TD approach to use an environment model to generate several pseudoexperiences—transitions that the TD agent can imagine *might* happen, given its current model. For each observed transition, the TD agent can generate a large number of imaginary

[1] The technical conditions are given on page 725. In Figure 21.5 we have used $\alpha(n) = 60/(59 + n)$, which satisfies the conditions.

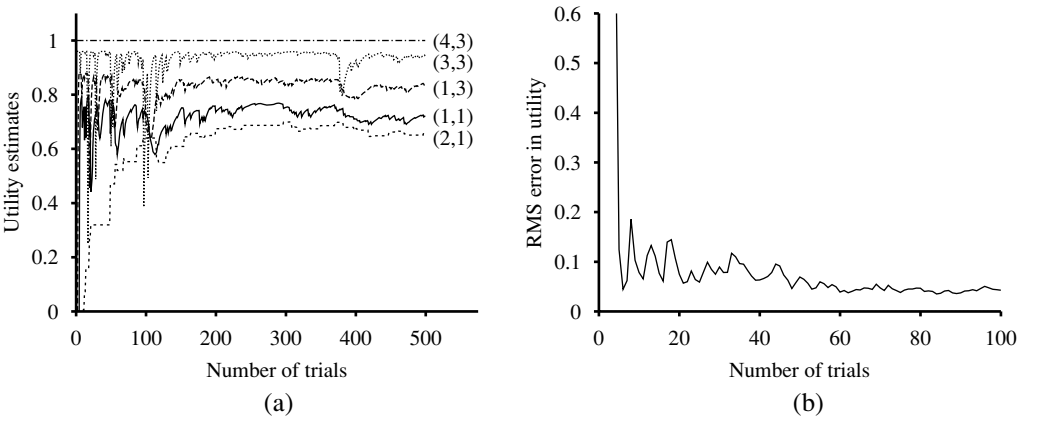


Figure 21.5 The TD learning curves for the 4×3 world. (a) The utility estimates for a selected subset of states, as a function of the number of trials. (b) The root-mean-square error in the estimate for $U(1,1)$, averaged over 20 runs of 500 trials each. Only the first 100 trials are shown to enable comparison with Figure 21.3.

transitions. In this way, the resulting utility estimates will approximate more and more closely those of ADP—of course, at the expense of increased computation time.

In a similar vein, we can generate more efficient versions of ADP by directly approximating the algorithms for value iteration or policy iteration. Even though the value iteration algorithm is efficient, it is intractable if we have, say, 10^{100} states. However, many of the necessary adjustments to the state values on each iteration will be extremely tiny. One possible approach to generating reasonably good answers quickly is to bound the number of adjustments made after each observed transition. One can also use a heuristic to rank the possible adjustments so as to carry out only the most significant ones. The **prioritized sweeping** heuristic prefers to make adjustments to states whose *likely* successors have just undergone a *large* adjustment in their own utility estimates. Using heuristics like this, approximate ADP algorithms usually can learn roughly as fast as full ADP, in terms of the number of training sequences, but can be several orders of magnitude more efficient in terms of computation. (See Exercise 21.3.) This enables them to handle state spaces that are far too large for full ADP. Approximate ADP algorithms have an additional advantage: in the early stages of learning a new environment, the environment model P often will be far from correct, so there is little point in calculating an exact utility function to match it. An approximation algorithm can use a minimum adjustment size that decreases as the environment model becomes more accurate. This eliminates the very long value iterations that can occur early in learning due to large changes in the model.

PRIORITIZED SWEEPING

21.3 ACTIVE REINFORCEMENT LEARNING

A passive learning agent has a fixed policy that determines its behavior. An active agent must decide what actions to take. Let us begin with the adaptive dynamic programming agent and consider how it must be modified to handle this new freedom.

First, the agent will need to learn a complete model with outcome probabilities for all actions, rather than just the model for the fixed policy. The simple learning mechanism used by PASSIVE-ADP-AGENT will do just fine for this. Next, we need to take into account the fact that the agent has a choice of actions. The utilities it needs to learn are those defined by the *optimal* policy; they obey the Bellman equations given on page 652, which we repeat here for convenience:

$$U(s) = R(s) + \gamma \max_a \sum_{s'} P(s' \mid s, a) U(s') \ . \tag{21.4}$$

These equations can be solved to obtain the utility function U using the value iteration or policy iteration algorithms from Chapter 17. The final issue is what to do at each step. Having obtained a utility function U that is optimal for the learned model, the agent can extract an optimal action by one-step look-ahead to maximize the expected utility; alternatively, if it uses policy iteration, the optimal policy is already available, so it should simply execute the action the optimal policy recommends. Or should it?

21.3.1 Exploration

Figure 21.6 shows the results of one sequence of trials for an ADP agent that follows the recommendation of the optimal policy for the learned model at each step. The agent *does not* learn the true utilities or the true optimal policy! What happens instead is that, in the 39th trial, it finds a policy that reaches the +1 reward along the lower route via (2,1), (3,1), (3,2), and (3,3). (See Figure 21.6(b).) After experimenting with minor variations, from the 276th trial onward it sticks to that policy, never learning the utilities of the other states and never finding the optimal route via (1,2), (1,3), and (2,3). We call this agent the **greedy agent**. Repeated experiments show that the greedy agent *very seldom* converges to the optimal policy for this environment and sometimes converges to really horrendous policies.

GREEDY AGENT

How can it be that choosing the optimal action leads to suboptimal results? The answer is that the learned model is not the same as the true environment; what is optimal in the learned model can therefore be suboptimal in the true environment. Unfortunately, the agent does not know what the true environment is, so it cannot compute the optimal action for the true environment. What, then, is to be done?

What the greedy agent has overlooked is that actions do more than provide rewards according to the current learned model; they also contribute to learning the true model by affecting the percepts that are received. By improving the model, the agent will receive greater rewards in the future.[2] An agent therefore must make a tradeoff between **exploitation** to maximize its reward—as reflected in its current utility estimates—and **exploration** to maxi-

EXPLOITATION

EXPLORATION

[2] Notice the direct analogy to the theory of information value in Chapter 16.

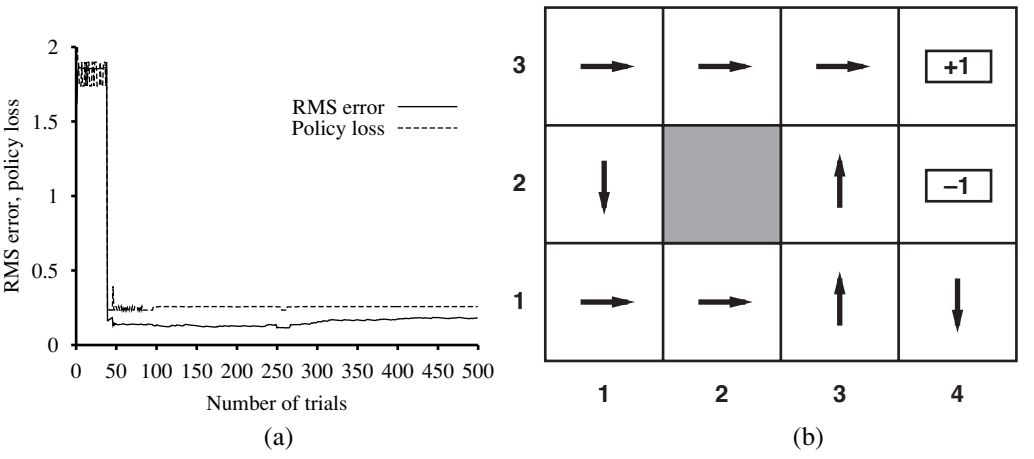


Figure 21.6 Performance of a greedy ADP agent that executes the action recommended by the optimal policy for the learned model. (a) RMS error in the utility estimates averaged over the nine nonterminal squares. (b) The suboptimal policy to which the greedy agent converges in this particular sequence of trials.

mize its long-term well-being. Pure exploitation risks getting stuck in a rut. Pure exploration to improve one's knowledge is of no use if one never puts that knowledge into practice. In the real world, one constantly has to decide between continuing in a comfortable existence and striking out into the unknown in the hopes of discovering a new and better life. With greater understanding, less exploration is necessary.

Can we be a little more precise than this? Is there an *optimal* exploration policy? This question has been studied in depth in the subfield of statistical decision theory that deals with so-called **bandit problems**. (See sidebar.)

BANDIT PROBLEM

Although bandit problems are extremely difficult to solve exactly to obtain an *optimal* exploration method, it is nonetheless possible to come up with a *reasonable* scheme that will eventually lead to optimal behavior by the agent. Technically, any such scheme needs to be greedy in the limit of infinite exploration, or **GLIE**. A GLIE scheme must try each action in each state an unbounded number of times to avoid having a finite probability that an optimal action is missed because of an unusually bad series of outcomes. An ADP agent using such a scheme will eventually learn the true environment model. A GLIE scheme must also eventually become greedy, so that the agent's actions become optimal with respect to the learned (and hence the true) model.

GLIE

There are several GLIE schemes; one of the simplest is to have the agent choose a random action a fraction $1/t$ of the time and to follow the greedy policy otherwise. While this does eventually converge to an optimal policy, it can be extremely slow. A more sensible approach would give some weight to actions that the agent has not tried very often, while tending to avoid actions that are believed to be of low utility. This can be implemented by altering the constraint equation (21.4) so that it assigns a higher utility estimate to relatively

EXPLORATION AND BANDITS

In Las Vegas, a *one-armed bandit* is a slot machine. A gambler can insert a coin, pull the lever, and collect the winnings (if any). An **n-armed bandit** has n levers. The gambler must choose which lever to play on each successive coin—the one that has paid off best, or maybe one that has not been tried?

The n-armed bandit problem is a formal model for real problems in many vitally important areas, such as deciding on the annual budget for AI research and development. Each arm corresponds to an action (such as allocating $20 million for the development of new AI textbooks), and the payoff from pulling the arm corresponds to the benefits obtained from taking the action (immense). Exploration, whether it is exploration of a new research field or exploration of a new shopping mall, is risky, is expensive, and has uncertain payoffs; on the other hand, failure to explore at all means that one never discovers *any* actions that are worthwhile.

To formulate a bandit problem properly, one must define exactly what is meant by optimal behavior. Most definitions in the literature assume that the aim is to maximize the expected total reward obtained over the agent's lifetime. These definitions require that the expectation be taken over the possible worlds that the agent could be in, as well as over the possible results of each action sequence in any given world. Here, a "world" is defined by the transition model $P(s' \mid s, a)$. Thus, in order to act optimally, the agent needs a prior distribution over the possible models. The resulting optimization problems are usually wildly intractable.

In some cases—for example, when the payoff of each machine is independent and discounted rewards are used—it is possible to calculate a **Gittins index** for each slot machine (Gittins, 1989). The index is a function only of the number of times the slot machine has been played and how much it has paid off. The index for each machine indicates how worthwhile it is to invest more; generally speaking, the higher the expected return and the higher the uncertainty in the utility of a given choice, the better. Choosing the machine with the highest index value gives an optimal exploration policy. Unfortunately, no way has been found to extend Gittins indices to sequential decision problems.

One can use the theory of n-armed bandits to argue for the reasonableness of the selection strategy in genetic algorithms. (See Chapter 4.) If you consider each arm in an n-armed bandit problem to be a possible string of genes, and the investment of a coin in one arm to be the reproduction of those genes, then it can be proven that genetic algorithms allocate coins optimally, given an appropriate set of independence assumptions.

unexplored state–action pairs. Essentially, this amounts to an optimistic prior over the possible environments and causes the agent to behave initially as if there were wonderful rewards scattered all over the place. Let us use $U^+(s)$ to denote the optimistic estimate of the utility (i.e., the expected reward-to-go) of the state s, and let $N(s,a)$ be the number of times action a has been tried in state s. Suppose we are using value iteration in an ADP learning agent; then we need to rewrite the update equation (Equation (17.6) on page 652) to incorporate the optimistic estimate. The following equation does this:

$$U^+(s) \leftarrow R(s) + \gamma \max_a f\left(\sum_{s'} P(s' \mid s,a)U^+(s'),\ N(s,a)\right) . \tag{21.5}$$

EXPLORATION FUNCTION

Here, $f(u,n)$ is called the **exploration function**. It determines how greed (preference for high values of u) is traded off against curiosity (preference for actions that have not been tried often and have low n). The function $f(u,n)$ should be increasing in u and decreasing in n. Obviously, there are many possible functions that fit these conditions. One particularly simple definition is

$$f(u,n) = \begin{cases} R^+ & \text{if } n < N_e \\ u & \text{otherwise} \end{cases}$$

where R^+ is an optimistic estimate of the best possible reward obtainable in any state and N_e is a fixed parameter. This will have the effect of making the agent try each action–state pair at least N_e times.

The fact that U^+ rather than U appears on the right-hand side of Equation (21.5) is very important. As exploration proceeds, the states and actions near the start state might well be tried a large number of times. If we used U, the more pessimistic utility estimate, then the agent would soon become disinclined to explore further afield. The use of U^+ means that the benefits of exploration are propagated back from the edges of unexplored regions, so that actions that lead *toward* unexplored regions are weighted more highly, rather than just actions that are themselves unfamiliar. The effect of this exploration policy can be seen clearly in Figure 21.7, which shows a rapid convergence toward optimal performance, unlike that of the greedy approach. A very nearly optimal policy is found after just 18 trials. Notice that the utility estimates themselves do not converge as quickly. This is because the agent stops exploring the unrewarding parts of the state space fairly soon, visiting them only "by accident" thereafter. However, it makes perfect sense for the agent not to care about the exact utilities of states that it knows are undesirable and can be avoided.

21.3.2 Learning an action-utility function

Now that we have an active ADP agent, let us consider how to construct an active temporal-difference learning agent. The most obvious change from the passive case is that the agent is no longer equipped with a fixed policy, so, if it learns a utility function U, it will need to learn a model in order to be able to choose an action based on U via one-step look-ahead. The model acquisition problem for the TD agent is identical to that for the ADP agent. What of the TD update rule itself? Perhaps surprisingly, the update rule (21.3) remains unchanged. This might seem odd, for the following reason: Suppose the agent takes a step that normally

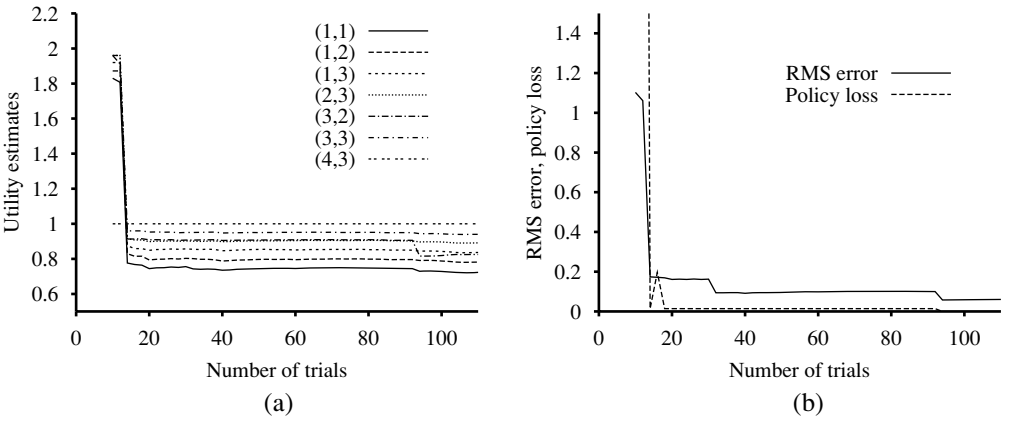


Figure 21.7 Performance of the exploratory ADP agent. using $R^+ = 2$ and $N_e = 5$. (a) Utility estimates for selected states over time. (b) The RMS error in utility values and the associated policy loss.

leads to a good destination, but because of nondeterminism in the environment the agent ends up in a catastrophic state. The TD update rule will take this as seriously as if the outcome had been the normal result of the action, whereas one might suppose that, because the outcome was a fluke, the agent should not worry about it too much. In fact, of course, the unlikely outcome will occur only infrequently in a large set of training sequences; hence in the long run its effects will be weighted proportionally to its probability, as we would hope. Once again, it can be shown that the TD algorithm will converge to the same values as ADP as the number of training sequences tends to infinity.

There is an alternative TD method, called **Q-learning**, which learns an action-utility representation instead of learning utilities. We will use the notation $Q(s, a)$ to denote the value of doing action a in state s. Q-values are directly related to utility values as follows:

$$U(s) = \max_a Q(s, a) \ . \tag{21.6}$$

MODEL-FREE

Q-functions may seem like just another way of storing utility information, but they have a very important property: *a TD agent that learns a Q-function does not need a model of the form $P(s' \mid s, a)$, either for learning or for action selection.* For this reason, Q-learning is called a **model-free** method. As with utilities, we can write a constraint equation that must hold at equilibrium when the Q-values are correct:

$$Q(s, a) = R(s) + \gamma \sum_{s'} P(s' \mid s, a) \max_{a'} Q(s', a') \ . \tag{21.7}$$

As in the ADP learning agent, we can use this equation directly as an update equation for an iteration process that calculates exact Q-values, given an estimated model. This does, however, require that a model also be learned, because the equation uses $P(s' \mid s, a)$. The temporal-difference approach, on the other hand, requires no model of state transitions—all

```
function Q-LEARNING-AGENT(percept) returns an action
  inputs: percept, a percept indicating the current state s' and reward signal r'
  persistent: Q, a table of action values indexed by state and action, initially zero
              N_sa, a table of frequencies for state–action pairs, initially zero
              s, a, r, the previous state, action, and reward, initially null

  if TERMINAL?(s) then Q[s, None] ← r'
  if s is not null then
     increment N_sa[s, a]
     Q[s, a] ← Q[s, a] + α(N_sa[s, a])(r + γ max_a' Q[s', a'] − Q[s, a])
  s, a, r ← s', argmax_a' f(Q[s', a'], N_sa[s', a']), r'
  return a
```

Figure 21.8 An exploratory Q-learning agent. It is an active learner that learns the value $Q(s,a)$ of each action in each situation. It uses the same exploration function f as the exploratory ADP agent, but avoids having to learn the transition model because the Q-value of a state can be related directly to those of its neighbors.

it needs are the Q values. The update equation for TD Q-learning is

$$Q(s,a) \leftarrow Q(s,a) + \alpha(R(s) + \gamma \max_{a'} Q(s',a') - Q(s,a)) \,, \tag{21.8}$$

which is calculated whenever action a is executed in state s leading to state s'.

The complete agent design for an exploratory Q-learning agent using TD is shown in Figure 21.8. Notice that it uses exactly the same exploration function f as that used by the exploratory ADP agent—hence the need to keep statistics on actions taken (the table N). If a simpler exploration policy is used—say, acting randomly on some fraction of steps, where the fraction decreases over time—then we can dispense with the statistics.

SARSA

Q-learning has a close relative called **SARSA** (for State-Action-Reward-State-Action). The update rule for SARSA is very similar to Equation (21.8):

$$Q(s,a) \leftarrow Q(s,a) + \alpha(R(s) + \gamma\, Q(s',a') - Q(s,a)) \,, \tag{21.9}$$

where a' is the action *actually taken* in state s'. The rule is applied at the end of each $s,\ a,\ r,\ s',\ a'$ quintuplet—hence the name. The difference from Q-learning is quite subtle: whereas Q-learning backs up the *best* Q-value from the state reached in the observed transition, SARSA waits until an action is actually taken and backs up the Q-value for that action. Now, for a greedy agent that always takes the action with best Q-value, the two algorithms are identical. When exploration is happening, however, they differ significantly. Because Q-learning uses the best Q-value, it pays no attention to the actual policy being followed—it is an **off-policy** learning algorithm, whereas SARSA is an **on-policy** algorithm. Q-learning is more flexible than SARSA, in the sense that a Q-learning agent can learn how to behave well even when guided by a random or adversarial exploration policy. On the other hand, SARSA is more realistic: for example, if the overall policy is even partly controlled by other agents, it is better to learn a Q-function for what will actually happen rather than what the agent would like to happen.

OFF-POLICY

ON-POLICY

Both Q-learning and SARSA learn the optimal policy for the 4×3 world, but do so at a much slower rate than the ADP agent. This is because the local updates do not enforce consistency among all the Q-values via the model. The comparison raises a general question: is it better to learn a model and a utility function or to learn an action-utility function with no model? In other words, what is the best way to represent the agent function? This is an issue at the foundations of artificial intelligence. As we stated in Chapter 1, one of the key historical characteristics of much of AI research is its (often unstated) adherence to the **knowledge-based** approach. This amounts to an assumption that the best way to represent the agent function is to build a representation of some aspects of the environment in which the agent is situated.

Some researchers, both inside and outside AI, have claimed that the availability of model-free methods such as Q-learning means that the knowledge-based approach is unnecessary. There is, however, little to go on but intuition. Our intuition, for what it's worth, is that as the environment becomes more complex, the advantages of a knowledge-based approach become more apparent. This is borne out even in games such as chess, checkers (draughts), and backgammon (see next section), where efforts to learn an evaluation function by means of a model have met with more success than Q-learning methods.

21.4 Generalization in Reinforcement Learning

So far, we have assumed that the utility functions and Q-functions learned by the agents are represented in tabular form with one output value for each input tuple. Such an approach works reasonably well for small state spaces, but the time to convergence and (for ADP) the time per iteration increase rapidly as the space gets larger. With carefully controlled, approximate ADP methods, it might be possible to handle 10,000 states or more. This suffices for two-dimensional maze-like environments, but more realistic worlds are out of the question. Backgammon and chess are tiny subsets of the real world, yet their state spaces contain on the order of 10^{20} and 10^{40} states, respectively. It would be absurd to suppose that one must visit all these states many times in order to learn how to play the game!

FUNCTION APPROXIMATION

One way to handle such problems is to use **function approximation**, which simply means using any sort of representation for the Q-function other than a lookup table. The representation is viewed as approximate because it might not be the case that the *true* utility function or Q-function can be represented in the chosen form. For example, in Chapter 5 we described an **evaluation function** for chess that is represented as a weighted linear function of a set of **features** (or **basis functions**) $f_1, \ldots, f_n$:

BASIS FUNCTION

$$\hat{U}_\theta(s) = \theta_1 f_1(s) + \theta_2 f_2(s) + \cdots + \theta_n f_n(s) .$$

A reinforcement learning algorithm can learn values for the parameters $\theta = \theta_1, \ldots, \theta_n$ such that the evaluation function $\hat{U}_\theta$ approximates the true utility function. Instead of, say, 10^{40} values in a table, this function approximator is characterized by, say, $n = 20$ parameters—an *enormous* compression. Although no one knows the true utility function for chess, no one believes that it can be represented exactly in 20 numbers. If the approximation is good

enough, however, the agent might still play excellent chess.[3] Function approximation makes it practical to represent utility functions for very large state spaces, but that is not its principal benefit. *The compression achieved by a function approximator allows the learning agent to generalize from states it has visited to states it has not visited.* That is, the most important aspect of function approximation is not that it requires less space, but that it allows for inductive generalization over input states. To give you some idea of the power of this effect: by examining only one in every 10^{12} of the possible backgammon states, it is possible to learn a utility function that allows a program to play as well as any human (Tesauro, 1992).

On the flip side, of course, there is the problem that there could fail to be any function in the chosen hypothesis space that approximates the true utility function sufficiently well. As in all inductive learning, there is a tradeoff between the size of the hypothesis space and the time it takes to learn the function. A larger hypothesis space increases the likelihood that a good approximation can be found, but also means that convergence is likely to be delayed.

Let us begin with the simplest case, which is direct utility estimation. (See Section 21.2.) With function approximation, this is an instance of **supervised learning**. For example, suppose we represent the utilities for the 4×3 world using a simple linear function. The features of the squares are just their x and y coordinates, so we have

$$\hat{U}_\theta(x, y) = \theta_0 + \theta_1 x + \theta_2 y \; . \tag{21.10}$$

Thus, if $(\theta_0, \theta_1, \theta_2) = (0.5, 0.2, 0.1)$, then $\hat{U}_\theta(1, 1) = 0.8$. Given a collection of trials, we obtain a set of sample values of $\hat{U}_\theta(x, y)$, and we can find the best fit, in the sense of minimizing the squared error, using standard linear regression. (See Chapter 18.)

For reinforcement learning, it makes more sense to use an *online* learning algorithm that updates the parameters after each trial. Suppose we run a trial and the total reward obtained starting at (1,1) is 0.4. This suggests that $\hat{U}_\theta(1, 1)$, currently 0.8, is too large and must be reduced. How should the parameters be adjusted to achieve this? As with neural-network learning, we write an error function and compute its gradient with respect to the parameters. If $u_j(s)$ is the observed total reward from state s onward in the jth trial, then the error is defined as (half) the squared difference of the predicted total and the actual total: $E_j(s) = (\hat{U}_\theta(s) - u_j(s))^2/2$. The rate of change of the error with respect to each parameter θ_i is $\partial E_j / \partial \theta_i$, so to move the parameter in the direction of decreasing the error, we want

$$\theta_i \leftarrow \theta_i - \alpha \frac{\partial E_j(s)}{\partial \theta_i} = \theta_i + \alpha \, (u_j(s) - \hat{U}_\theta(s)) \frac{\partial \hat{U}_\theta(s)}{\partial \theta_i} \; . \tag{21.11}$$

WIDROW–HOFF RULE
DELTA RULE

This is called the **Widrow–Hoff rule**, or the **delta rule**, for online least-squares. For the linear function approximator $\hat{U}_\theta(s)$ in Equation (21.10), we get three simple update rules:

$$\begin{aligned}
\theta_0 &\leftarrow \theta_0 + \alpha \, (u_j(s) - \hat{U}_\theta(s)) \; , \\
\theta_1 &\leftarrow \theta_1 + \alpha \, (u_j(s) - \hat{U}_\theta(s)) x \; , \\
\theta_2 &\leftarrow \theta_2 + \alpha \, (u_j(s) - \hat{U}_\theta(s)) y \; .
\end{aligned}$$

[3] We do know that the exact utility function can be represented in a page or two of Lisp, Java, or C++. That is, it can be represented by a program that solves the game exactly every time it is called. We are interested only in function approximators that use a *reasonable* amount of computation. It might in fact be better to learn a very simple function approximator and combine it with a certain amount of look-ahead search. The tradeoffs involved are currently not well understood.

We can apply these rules to the example where $\hat{U}_\theta(1,1)$ is 0.8 and $u_j(1,1)$ is 0.4. θ_0, θ_1, and θ_2 are all decreased by 0.4α, which reduces the error for (1,1). Notice that *changing the parameters θ in response to an observed transition between two states also changes the values of $\hat{U}_\theta$ for every other state!* This is what we mean by saying that function approximation allows a reinforcement learner to generalize from its experiences.

We expect that the agent will learn faster if it uses a function approximator, provided that the hypothesis space is not too large, but includes some functions that are a reasonably good fit to the true utility function. Exercise 21.5 asks you to evaluate the performance of direct utility estimation, both with and without function approximation. The improvement in the 4×3 world is noticeable but not dramatic, because this is a very small state space to begin with. The improvement is much greater in a 10×10 world with a +1 reward at (10,10). This world is well suited for a linear utility function because the true utility function is smooth and nearly linear. (See Exercise 21.8.) If we put the +1 reward at (5,5), the true utility is more like a pyramid and the function approximator in Equation (21.10) will fail miserably. All is not lost, however! Remember that what matters for linear function approximation is that the function be linear in the *parameters*—the features themselves can be arbitrary nonlinear functions of the state variables. Hence, we can include a term such as $\theta_3 f_3(x,y) = \theta_3\sqrt{(x-x_g)^2 + (y-y_g)^2}$ that measures the distance to the goal.

We can apply these ideas equally well to temporal-difference learners. All we need do is adjust the parameters to try to reduce the temporal difference between successive states. The new versions of the TD and Q-learning equations (21.3 on page 836 and 21.8 on page 844) are given by

$$\theta_i \leftarrow \theta_i + \alpha\,[R(s) + \gamma\,\hat{U}_\theta(s') - \hat{U}_\theta(s)]\frac{\partial \hat{U}_\theta(s)}{\partial \theta_i} \tag{21.12}$$

for utilities and

$$\theta_i \leftarrow \theta_i + \alpha\,[R(s) + \gamma\,\max_{a'}\hat{Q}_\theta(s',a') - \hat{Q}_\theta(s,a)]\frac{\partial \hat{Q}_\theta(s,a)}{\partial \theta_i} \tag{21.13}$$

for Q-values. For passive TD learning, the update rule can be shown to converge to the closest possible[4] approximation to the true function when the function approximator is *linear* in the parameters. With active learning and *nonlinear* functions such as neural networks, all bets are off: There are some very simple cases in which the parameters can go off to infinity even though there are good solutions in the hypothesis space. There are more sophisticated algorithms that can avoid these problems, but at present reinforcement learning with general function approximators remains a delicate art.

Function approximation can also be very helpful for learning a model of the environment. Remember that learning a model for an *observable* environment is a *supervised* learning problem, because the next percept gives the outcome state. Any of the supervised learning methods in Chapter 18 can be used, with suitable adjustments for the fact that we need to predict a complete state description rather than just a Boolean classification or a single real value. For a *partially observable* environment, the learning problem is much more difficult. If we know what the hidden variables are and how they are causally related to each other and to the

[4] The definition of distance between utility functions is rather technical; see Tsitsiklis and Van Roy (1997).

observable variables, then we can fix the structure of a dynamic Bayesian network and use the EM algorithm to learn the parameters, as was described in Chapter 20. Inventing the hidden variables and learning the model structure are still open problems. Some practical examples are described in Section 21.6.

21.5 POLICY SEARCH

POLICY SEARCH

The final approach we will consider for reinforcement learning problems is called **policy search**. In some ways, policy search is the simplest of all the methods in this chapter: the idea is to keep twiddling the policy as long as its performance improves, then stop.

Let us begin with the policies themselves. Remember that a policy π is a function that maps states to actions. We are interested primarily in *parameterized* representations of π that have far fewer parameters than there are states in the state space (just as in the preceding section). For example, we could represent π by a collection of parameterized Q-functions, one for each action, and take the action with the highest predicted value:

$$\pi(s) = \max_a \hat{Q}_\theta(s, a) \ . \tag{21.14}$$

Each Q-function could be a linear function of the parameters θ, as in Equation (21.10), or it could be a nonlinear function such as a neural network. Policy search will then adjust the parameters θ to improve the policy. Notice that if the policy is represented by Q-functions, then policy search results in a process that learns Q-functions. *This process is not the same as Q-learning!* In Q-learning with function approximation, the algorithm finds a value of θ such that $\hat{Q}_\theta$ is "close" to Q^*, the optimal Q-function. Policy search, on the other hand, finds a value of θ that results in good performance; the values found by the two methods may differ very substantially. (For example, the approximate Q-function defined by $\hat{Q}_\theta(s, a) = Q^*(s, a)/10$ gives optimal performance, even though it is not at all close to Q^*.) Another clear instance of the difference is the case where $\pi(s)$ is calculated using, say, depth-10 look-ahead search with an approximate utility function $\hat{U}_\theta$. A value of θ that gives good results may be a long way from making $\hat{U}_\theta$ resemble the true utility function.

One problem with policy representations of the kind given in Equation (21.14) is that the policy is a *discontinuous* function of the parameters when the actions are discrete. (For a continuous action space, the policy can be a smooth function of the parameters.) That is, there will be values of θ such that an infinitesimal change in θ causes the policy to switch from one action to another. This means that the value of the policy may also change discontinuously, which makes gradient-based search difficult. For this reason, policy search methods often use a **stochastic policy** representation $\pi_\theta(s, a)$, which specifies the *probability* of selecting action a in state s. One popular representation is the **softmax function**:

STOCHASTIC POLICY
SOFTMAX FUNCTION

$$\pi_\theta(s, a) = e^{\hat{Q}_\theta(s,a)} / \sum_{a'} e^{\hat{Q}_\theta(s,a')} \ .$$

Softmax becomes nearly deterministic if one action is much better than the others, but it always gives a differentiable function of θ; hence, the value of the policy (which depends in

a continuous fashion on the action selection probabilities) is a differentiable function of θ. Softmax is a generalization of the logistic function (page 725) to multiple variables.

POLICY VALUE

Now let us look at methods for improving the policy. We start with the simplest case: a deterministic policy and a deterministic environment. Let $\rho(\theta)$ be the **policy value**, i.e., the expected reward-to-go when π_θ is executed. If we can derive an expression for $\rho(\theta)$ in closed form, then we have a standard optimization problem, as described in Chapter 4. We can follow

POLICY GRADIENT

the **policy gradient** vector $\nabla_\theta \rho(\theta)$ provided $\rho(\theta)$ is differentiable. Alternatively, if $\rho(\theta)$ is not available in closed form, we can evaluate π_θ simply by executing it and observing the accumulated reward. We can follow the **empirical gradient** by hill climbing—i.e., evaluating the change in policy value for small increments in each parameter. With the usual caveats, this process will converge to a local optimum in policy space.

When the environment (or the policy) is stochastic, things get more difficult. Suppose we are trying to do hill climbing, which requires comparing $\rho(\theta)$ and $\rho(\theta + \Delta\theta)$ for some small $\Delta\theta$. The problem is that the total reward on each trial may vary widely, so estimates of the policy value from a small number of trials will be quite unreliable; trying to compare two such estimates will be even more unreliable. One solution is simply to run lots of trials, measuring the sample variance and using it to determine that enough trials have been run to get a reliable indication of the direction of improvement for $\rho(\theta)$. Unfortunately, this is impractical for many real problems where each trial may be expensive, time-consuming, and perhaps even dangerous.

For the case of a stochastic policy $\pi_\theta(s, a)$, it is possible to obtain an unbiased estimate of the gradient at θ, $\nabla_\theta \rho(\theta)$, directly from the results of trials executed at θ. For simplicity, we will derive this estimate for the simple case of a nonsequential environment in which the reward $R(a)$ is obtained immediately after doing action a in the start state s_0. In this case, the policy value is just the expected value of the reward, and we have

$$\nabla_\theta \rho(\theta) = \nabla_\theta \sum_a \pi_\theta(s_0, a) R(a) = \sum_a (\nabla_\theta \pi_\theta(s_0, a)) R(a) \ .$$

Now we perform a simple trick so that this summation can be approximated by samples generated from the probability distribution defined by $\pi_\theta(s_0, a)$. Suppose that we have N trials in all and the action taken on the jth trial is a_j. Then

$$\nabla_\theta \rho(\theta) = \sum_a \pi_\theta(s_0, a) \cdot \frac{(\nabla_\theta \pi_\theta(s_0, a)) R(a)}{\pi_\theta(s_0, a)} \approx \frac{1}{N} \sum_{j=1}^{N} \frac{(\nabla_\theta \pi_\theta(s_0, a_j)) R(a_j)}{\pi_\theta(s_0, a_j)} \ .$$

Thus, the true gradient of the policy value is approximated by a sum of terms involving the gradient of the action-selection probability in each trial. For the sequential case, this generalizes to

$$\nabla_\theta \rho(\theta) \approx \frac{1}{N} \sum_{j=1}^{N} \frac{(\nabla_\theta \pi_\theta(s, a_j)) R_j(s)}{\pi_\theta(s, a_j)}$$

for each state s visited, where a_j is executed in s on the jth trial and $R_j(s)$ is the total reward received from state s onwards in the jth trial. The resulting algorithm is called REINFORCE (Williams, 1992); it is usually much more effective than hill climbing using lots of trials at each value of θ. It is still much slower than necessary, however.

Consider the following task: given two blackjack[5] programs, determine which is best. One way to do this is to have each play against a standard "dealer" for a certain number of hands and then to measure their respective winnings. The problem with this, as we have seen, is that the winnings of each program fluctuate widely depending on whether it receives good or bad cards. An obvious solution is to generate a certain number of hands in advance and *have each program play the same set of hands.* In this way, we eliminate the measurement error due to differences in the cards received. This idea, called **correlated sampling**, underlies a policy-search algorithm called PEGASUS (Ng and Jordan, 2000). The algorithm is applicable to domains for which a simulator is available so that the "random" outcomes of actions can be repeated. The algorithm works by generating in advance N sequences of random numbers, each of which can be used to run a trial of any policy. Policy search is carried out by evaluating each candidate policy using the *same* set of random sequences to determine the action outcomes. It can be shown that the number of random sequences required to ensure that the value of *every* policy is well estimated depends only on the complexity of the policy space, and not at all on the complexity of the underlying domain.

21.6 APPLICATIONS OF REINFORCEMENT LEARNING

We now turn to examples of large-scale applications of reinforcement learning. We consider applications in game playing, where the transition model is known and the goal is to learn the utility function, and in robotics, where the model is usually unknown.

21.6.1 Applications to game playing

The first significant application of reinforcement learning was also the first significant learning program of any kind—the checkers program written by Arthur Samuel (1959, 1967). Samuel first used a weighted linear function for the evaluation of positions, using up to 16 terms at any one time. He applied a version of Equation (21.12) to update the weights. There were some significant differences, however, between his program and current methods. First, he updated the weights using the difference between the current state and the backed-up value generated by full look-ahead in the search tree. This works fine, because it amounts to viewing the state space at a different granularity. A second difference was that the program did *not* use any observed rewards! That is, the values of terminal states reached in self-play were ignored. This means that it is theoretically possible for Samuel's program not to converge, or to converge on a strategy designed to lose rather than to win. He managed to avoid this fate by insisting that the weight for material advantage should always be positive. Remarkably, this was sufficient to direct the program into areas of weight space corresponding to good checkers play.

Gerry Tesauro's backgammon program TD-GAMMON (1992) forcefully illustrates the potential of reinforcement learning techniques. In earlier work (Tesauro and Sejnowski, 1989), Tesauro tried learning a neural network representation of $Q(s, a)$ directly from ex-

[5] Also known as twenty-one or pontoon.

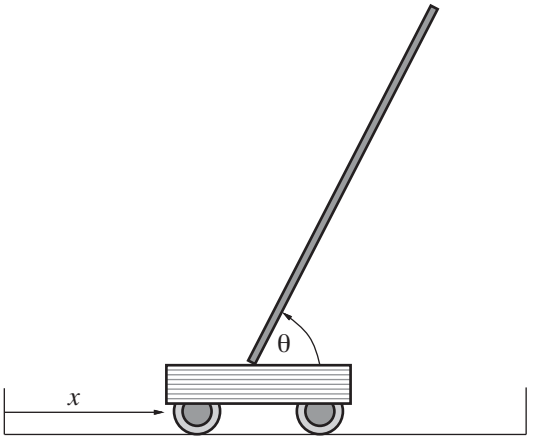


Figure 21.9 Setup for the problem of balancing a long pole on top of a moving cart. The cart can be jerked left or right by a controller that observes x, θ, $\dot{x}$, and $\dot{\theta}$.

amples of moves labeled with relative values by a human expert. This approach proved extremely tedious for the expert. It resulted in a program, called NEUROGAMMON, that was strong by computer standards, but not competitive with human experts. The TD-GAMMON project was an attempt to learn from self-play alone. The only reward signal was given at the end of each game. The evaluation function was represented by a fully connected neural network with a single hidden layer containing 40 nodes. Simply by repeated application of Equation (21.12), TD-GAMMON learned to play considerably better than NEUROGAMMON, even though the input representation contained just the raw board position with no computed features. This took about 200,000 training games and two weeks of computer time. Although that may seem like a lot of games, it is only a vanishingly small fraction of the state space. When precomputed features were added to the input representation, a network with 80 hidden nodes was able, after 300,000 training games, to reach a standard of play comparable to that of the top three human players worldwide. Kit Woolsey, a top player and analyst, said that "There is no question in my mind that its positional judgment is far better than mine."

21.6.2 Application to robot control

CART–POLE
INVERTED PENDULUM

The setup for the famous **cart–pole** balancing problem, also known as the **inverted pendulum**, is shown in Figure 21.9. The problem is to control the position x of the cart so that the pole stays roughly upright ($\theta \approx \pi/2$), while staying within the limits of the cart track as shown. Several thousand papers in reinforcement learning and control theory have been published on this seemingly simple problem. The cart–pole problem differs from the problems described earlier in that the state variables x, θ, $\dot{x}$, and $\dot{\theta}$ are continuous. The actions are usually discrete: jerk left or jerk right, the so-called **bang-bang control** regime.

BANG-BANG CONTROL

The earliest work on learning for this problem was carried out by Michie and Chambers (1968). Their BOXES algorithm was able to balance the pole for over an hour after only about 30 trials. Moreover, unlike many subsequent systems, BOXES was implemented with a

real cart and pole, not a simulation. The algorithm first discretized the four-dimensional state space into boxes—hence the name. It then ran trials until the pole fell over or the cart hit the end of the track. Negative reinforcement was associated with the final action in the final box and then propagated back through the sequence. It was found that the discretization caused some problems when the apparatus was initialized in a position different from those used in training, suggesting that generalization was not perfect. Improved generalization and faster learning can be obtained using an algorithm that *adaptively* partitions the state space according to the observed variation in the reward, or by using a continuous-state, nonlinear function approximator such as a neural network. Nowadays, balancing a *triple* inverted pendulum is a common exercise—a feat far beyond the capabilities of most humans.

Still more impressive is the application of reinforcement learning to helicopter flight (Figure 21.10). This work has generally used policy search (Bagnell and Schneider, 2001) as well as the PEGASUS algorithm with simulation based on a learned transition model (Ng *et al.*, 2004). Further details are given in Chapter 25.

Figure 21.10 Superimposed time-lapse images of an autonomous helicopter performing a very difficult "nose-in circle" maneuver. The helicopter is under the control of a policy developed by the PEGASUS policy-search algorithm. A simulator model was developed by observing the effects of various control manipulations on the real helicopter; then the algorithm was run on the simulator model overnight. A variety of controllers were developed for different maneuvers. In all cases, performance far exceeded that of an expert human pilot using remote control. (Image courtesy of Andrew Ng.)

21.7 SUMMARY

This chapter has examined the reinforcement learning problem: how an agent can become proficient in an unknown environment, given only its percepts and occasional rewards. Reinforcement learning can be viewed as a microcosm for the entire AI problem, but it is studied in a number of simplified settings to facilitate progress. The major points are:

- The overall agent design dictates the kind of information that must be learned. The three main designs we covered were the model-based design, using a model P and a utility function U; the model-free design, using an action-utility function Q; and the reflex design, using a policy π.
- Utilities can be learned using three approaches:
  1. **Direct utility estimation** uses the total observed reward-to-go for a given state as direct evidence for learning its utility.
  2. **Adaptive dynamic programming** (ADP) learns a model and a reward function from observations and then uses value or policy iteration to obtain the utilities or an optimal policy. ADP makes optimal use of the local constraints on utilities of states imposed through the neighborhood structure of the environment.
  3. **Temporal-difference** (TD) methods update utility estimates to match those of successor states. They can be viewed as simple approximations to the ADP approach that can learn without requiring a transition model. Using a learned model to generate pseudoexperiences can, however, result in faster learning.
- Action-utility functions, or Q-functions, can be learned by an ADP approach or a TD approach. With TD, Q-learning requires no model in either the learning or action-selection phase. This simplifies the learning problem but potentially restricts the ability to learn in complex environments, because the agent cannot simulate the results of possible courses of action.
- When the learning agent is responsible for selecting actions while it learns, it must trade off the estimated value of those actions against the potential for learning useful new information. An exact solution of the exploration problem is infeasible, but some simple heuristics do a reasonable job.
- In large state spaces, reinforcement learning algorithms must use an approximate functional representation in order to generalize over states. The temporal-difference signal can be used directly to update parameters in representations such as neural networks.
- Policy-search methods operate directly on a representation of the policy, attempting to improve it based on observed performance. The variation in the performance in a stochastic domain is a serious problem; for simulated domains this can be overcome by fixing the randomness in advance.

Because of its potential for eliminating hand coding of control strategies, reinforcement learning continues to be one of the most active areas of machine learning research. Applications in robotics promise to be particularly valuable; these will require methods for handling con-

tinuous, high-dimensional, partially observable environments in which successful behaviors may consist of thousands or even millions of primitive actions.

Bibliographical and Historical Notes

Turing (1948, 1950) proposed the reinforcement-learning approach, although he was not convinced of its effectiveness, writing, "the use of punishments and rewards can at best be a part of the teaching process." Arthur Samuel's work (1959) was probably the earliest successful machine learning research. Although this work was informal and had a number of flaws, it contained most of the modern ideas in reinforcement learning, including temporal differencing and function approximation. Around the same time, researchers in adaptive control theory (Widrow and Hoff, 1960), building on work by Hebb (1949), were training simple networks using the delta rule. (This early connection between neural networks and reinforcement learning may have led to the persistent misperception that the latter is a subfield of the former.) The cart–pole work of Michie and Chambers (1968) can also be seen as a reinforcement learning method with a function approximator. The psychological literature on reinforcement learning is much older; Hilgard and Bower (1975) provide a good survey. Direct evidence for the operation of reinforcement learning in animals has been provided by investigations into the foraging behavior of bees; there is a clear neural correlate of the reward signal in the form of a large neuron mapping from the nectar intake sensors directly to the motor cortex (Montague *et al.*, 1995). Research using single-cell recording suggests that the dopamine system in primate brains implements something resembling value function learning (Schultz *et al.*, 1997). The neuroscience text by Dayan and Abbott (2001) describes possible neural implementations of temporal-difference learning, while Dayan and Niv (2008) survey the latest evidence from neuroscientific and behavioral experiments.

The connection between reinforcement learning and Markov decision processes was first made by Werbos (1977), but the development of reinforcement learning in AI stems from work at the University of Massachusetts in the early 1980s (Barto *et al.*, 1981). The paper by Sutton (1988) provides a good historical overview. Equation (21.3) in this chapter is a special case for $\lambda = 0$ of Sutton's general TD(λ) algorithm. TD(λ) updates the utility values of all states in a sequence leading up to each transition by an amount that drops off as λ^t for states t steps in the past. TD(1) is identical to the Widrow–Hoff or delta rule. Boyan (2002), building on work by Bradtke and Barto (1996), argues that TD(λ) and related algorithms make inefficient use of experiences; essentially, they are online regression algorithms that converge much more slowly than offline regression. His LSTD (least-squares temporal differencing) algorithm is an online algorithm for passive reinforcement learning that gives the same results as offline regression. Least-squares policy iteration, or LSPI (Lagoudakis and Parr, 2003), combines this idea with the policy iteration algorithm, yielding a robust, statistically efficient, model-free algorithm for learning policies.

The combination of temporal-difference learning with the model-based generation of simulated experiences was proposed in Sutton's DYNA architecture (Sutton, 1990). The idea of prioritized sweeping was introduced independently by Moore and Atkeson (1993) and

Peng and Williams (1993). Q-learning was developed in Watkins's Ph.D. thesis (1989), while SARSA appeared in a technical report by Rummery and Niranjan (1994).

Bandit problems, which model the problem of exploration for nonsequential decisions, are analyzed in depth by Berry and Fristedt (1985). Optimal exploration strategies for several settings are obtainable using the technique called **Gittins indices** (Gittins, 1989). A variety of exploration methods for sequential decision problems are discussed by Barto *et al.* (1995). Kearns and Singh (1998) and Brafman and Tennenholtz (2000) describe algorithms that explore unknown environments and are guaranteed to converge on near-optimal policies in polynomial time. Bayesian reinforcement learning (Dearden *et al.*, 1998, 1999) provides another angle on both model uncertainty and exploration.

Function approximation in reinforcement learning goes back to the work of Samuel, who used both linear and nonlinear evaluation functions and also used feature-selection methods to reduce the feature space. Later methods include the **CMAC** (Cerebellar Model Articulation Controller) (Albus, 1975), which is essentially a sum of overlapping local kernel functions, and the associative neural networks of Barto *et al.* (1983). Neural networks are currently the most popular form of function approximator. The best-known application is TD-Gammon (Tesauro, 1992, 1995), which was discussed in the chapter. One significant problem exhibited by neural-network-based TD learners is that they tend to forget earlier experiences, especially those in parts of the state space that are avoided once competence is achieved. This can result in catastrophic failure if such circumstances reappear. Function approximation based on **instance-based learning** can avoid this problem (Ormoneit and Sen, 2002; Forbes, 2002).

CMAC

The convergence of reinforcement learning algorithms using function approximation is an extremely technical subject. Results for TD learning have been progressively strengthened for the case of linear function approximators (Sutton, 1988; Dayan, 1992; Tsitsiklis and Van Roy, 1997), but several examples of divergence have been presented for nonlinear functions (see Tsitsiklis and Van Roy, 1997, for a discussion). Papavassiliou and Russell (1999) describe a new type of reinforcement learning that converges with any form of function approximator, provided that a best-fit approximation can be found for the observed data.

Policy search methods were brought to the fore by Williams (1992), who developed the REINFORCE family of algorithms. Later work by Marbach and Tsitsiklis (1998), Sutton *et al.* (2000), and Baxter and Bartlett (2000) strengthened and generalized the convergence results for policy search. The method of correlated sampling for comparing different configurations of a system was described formally by Kahn and Marshall (1953), but seems to have been known long before that. Its use in reinforcement learning is due to Van Roy (1998) and Ng and Jordan (2000); the latter paper also introduced the PEGASUS algorithm and proved its formal properties.

As we mentioned in the chapter, the performance of a *stochastic* policy is a continuous function of its parameters, which helps with gradient-based search methods. This is not the only benefit: Jaakkola *et al.* (1995) argue that stochastic policies actually work better than deterministic policies in partially observable environments, if both are limited to acting based on the current percept. (One reason is that the stochastic policy is less likely to get "stuck" because of some unseen hindrance.) Now, in Chapter 17 we pointed out that

optimal policies in partially observable MDPs are deterministic functions of the *belief state* rather than the current percept, so we would expect still better results by keeping track of the belief state using the **filtering** methods of Chapter 15. Unfortunately, belief-state space is high-dimensional and continuous, and effective algorithms have not yet been developed for reinforcement learning with belief states.

Real-world environments also exhibit enormous complexity in terms of the number of primitive actions required to achieve significant reward. For example, a robot playing soccer might make a hundred thousand individual leg motions before scoring a goal. One

REWARD SHAPING

common method, used originally in animal training, is called **reward shaping**. This involves

PSEUDOREWARD

supplying the agent with additional rewards, called **pseudorewards**, for "making progress." For example, in soccer the real reward is for scoring a goal, but pseudorewards might be given for making contact with the ball or for kicking it toward the goal. Such rewards can speed up learning enormously and are simple to provide, but there is a risk that the agent will learn to maximize the pseudorewards rather than the true rewards; for example, standing next to the ball and "vibrating" causes many contacts with the ball. Ng *et al.* (1999) show that the agent will still learn the optimal policy provided that the pseudoreward $F(s, a, s')$ satisfies $F(s, a, s') = \gamma\Phi(s') - \Phi(s)$, where Φ is an arbitrary function of the state. Φ can be constructed to reflect any desirable aspects of the state, such as achievement of subgoals or distance to a goal state.

HIERARCHICAL REINFORCEMENT LEARNING

The generation of complex behaviors can also be facilitated by **hierarchical reinforcement learning** methods, which attempt to solve problems at multiple levels of abstraction—much like the **HTN planning** methods of Chapter 11. For example, "scoring a goal" can be broken down into "obtain possession," "dribble towards the goal," and "shoot;" and each of these can be broken down further into lower-level motor behaviors. The fundamental result in this area is due to Forestier and Varaiya (1978), who proved that lower-level behaviors of arbitrary complexity can be treated just like primitive actions (albeit ones that can take varying amounts of time) from the point of view of the higher-level behavior that invokes them. Current approaches (Parr and Russell, 1998; Dietterich, 2000; Sutton *et al.*, 2000; Andre and Russell, 2002) build on this result to develop methods for supplying an agent

PARTIAL PROGRAM

with a **partial program** that constrains the agent's behavior to have a particular hierarchical structure. The partial-programming language for agent programs extends an ordinary programming language by adding primitives for unspecified choices that must be filled in by learning. Reinforcement learning is then applied to learn the best behavior consistent with the partial program. The combination of function approximation, shaping, and hierarchical reinforcement learning has been shown to solve large-scale problems—for example, policies that execute for 10^4 steps in state spaces of 10^{100} states with branching factors of 10^{30} (Marthi *et al.*, 2005). One key result (Dietterich, 2000) is that the hierarchical structure provides a natural *additive decomposition* of the overall utility function into terms that depend on small subsets of the variables defining the state space. This is somewhat analogous to the representation theorems underlying the conciseness of Bayes nets (Chapter 14).

The topic of distributed and multiagent reinforcement learning was not touched upon in the chapter but is of great current interest. In distributed RL, the aim is to devise methods by which multiple, coordinated agents learn to optimize a common utility function. For example,

SUBAGENT

can we devise methods whereby separate **subagents** for robot navigation and robot obstacle avoidance could cooperatively achieve a combined control system that is globally optimal? Some basic results in this direction have been obtained (Guestrin *et al.*, 2002; Russell and Zimdars, 2003). The basic idea is that each subagent learns its own Q-function from its own stream of rewards. For example, a robot-navigation component can receive rewards for making progress towards the goal, while the obstacle-avoidance component receives negative rewards for every collision. Each global decision maximizes the sum of Q-functions and the whole process converges to globally optimal solutions.

Multiagent RL is distinguished from distributed RL by the presence of agents who cannot coordinate their actions (except by explicit communicative acts) and who may not share the same utility function. Thus, multiagent RL deals with sequential game-theoretic problems or **Markov games**, as defined in Chapter 17. The consequent requirement for randomized policies is not a significant complication, as we saw on page 848. What *does* cause problems is the fact that, while an agent is learning to defeat its opponent's policy, the opponent is changing its policy to defeat the agent. Thus, the environment is **nonstationary** (see page 568). Littman (1994) noted this difficulty when introducing the first RL algorithms for zero-sum Markov games. Hu and Wellman (2003) present a Q-learning algorithm for general-sum games that converges when the Nash equilibrium is unique; when there are multiple equilibria, the notion of convergence is not so easy to define (Shoham *et al.*, 2004).

Sometimes the reward function is not easy to define. Consider the task of driving a car. There are extreme states (such as crashing the car) that clearly should have a large penalty. But beyond that, it is difficult to be precise about the reward function. However, it is easy enough for a human to drive for a while and then tell a robot "do it like that." The robot then has the task of **apprenticeship learning**; learning from an example of the task done right, without explicit rewards. Ng *et al.* (2004) and Coates *et al.* (2009) show how this technique works for learning to fly a helicopter; see Figure 25.25 on page 1002 for an example of the acrobatics the resulting policy is capable of. Russell (1998) describes the task of **inverse reinforcement learning**—figuring out what the reward function must be from an example path through that state space. This is useful as a part of apprenticeship learning, or as a part of doing science—we can understand an animal or robot by working backwards from what it does to what its reward function must be.

APPRENTICESHIP LEARNING

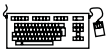
INVERSE REINFORCEMENT LEARNING

This chapter has dealt only with atomic states—all the agent knows about a state is the set of available actions and the utilities of the resulting states (or of state-action pairs). But it is also possible to apply reinforcement learning to structured representations rather than atomic ones; this is called **relational reinforcement learning** (Tadepalli *et al.*, 2004).

RELATIONAL REINFORCEMENT LEARNING

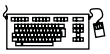
The survey by Kaelbling *et al.* (1996) provides a good entry point to the literature. The text by Sutton and Barto (1998), two of the field's pioneers, focuses on architectures and algorithms, showing how reinforcement learning weaves together the ideas of learning, planning, and acting. The somewhat more technical work by Bertsekas and Tsitsiklis (1996) gives a rigorous grounding in the theory of dynamic programming and stochastic convergence. Reinforcement learning papers are published frequently in *Machine Learning*, in the *Journal of Machine Learning Research*, and in the International Conferences on Machine Learning and the Neural Information Processing Systems meetings.

EXERCISES



21.1 Implement a passive learning agent in a simple environment, such as the 4×3 world. For the case of an initially unknown environment model, compare the learning performance of the direct utility estimation, TD, and ADP algorithms. Do the comparison for the optimal policy and for several random policies. For which do the utility estimates converge faster? What happens when the size of the environment is increased? (Try environments with and without obstacles.)

21.2 Chapter 17 defined a **proper policy** for an MDP as one that is guaranteed to reach a terminal state. Show that it is possible for a passive ADP agent to learn a transition model for which its policy π is improper even if π is proper for the true MDP; with such models, the POLICY-EVALUATION step may fail if $\gamma = 1$. Show that this problem cannot arise if POLICY-EVALUATION is applied to the learned model only at the end of a trial.

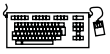


21.3 Starting with the passive ADP agent, modify it to use an approximate ADP algorithm as discussed in the text. Do this in two steps:

- **a**. Implement a priority queue for adjustments to the utility estimates. Whenever a state is adjusted, all of its predecessors also become candidates for adjustment and should be added to the queue. The queue is initialized with the state from which the most recent transition took place. Allow only a fixed number of adjustments.
- **b**. Experiment with various heuristics for ordering the priority queue, examining their effect on learning rates and computation time.

21.4 Write out the parameter update equations for TD learning with

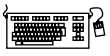
$$\hat{U}(x, y) = \theta_0 + \theta_1 x + \theta_2 y + \theta_3 \sqrt{(x - x_g)^2 + (y - y_g)^2} \, .$$



21.5 Implement an exploring reinforcement learning agent that uses direct utility estimation. Make two versions—one with a tabular representation and one using the function approximator in Equation (21.10). Compare their performance in three environments:

- **a**. The 4×3 world described in the chapter.
- **b**. A 10×10 world with no obstacles and a +1 reward at (10,10).
- **c**. A 10×10 world with no obstacles and a +1 reward at (5,5).

21.6 Devise suitable features for reinforcement learning in stochastic grid worlds (generalizations of the 4×3 world) that contain multiple obstacles and multiple terminal states with rewards of $+1$ or -1.



21.7 Extend the standard game-playing environment (Chapter 5) to incorporate a reward signal. Put two reinforcement learning agents into the environment (they may, of course, share the agent program) and have them play against each other. Apply the generalized TD update rule (Equation (21.12)) to update the evaluation function. You might wish to start with a simple linear weighted evaluation function and a simple game, such as tic-tac-toe.

21.8 Compute the true utility function and the best linear approximation in x and y (as in Equation (21.10)) for the following environments:

a. A 10×10 world with a single $+1$ terminal state at (10,10).

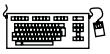
b. As in (a), but add a -1 terminal state at (10,1).

c. As in (b), but add obstacles in 10 randomly selected squares.

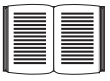
d. As in (b), but place a wall stretching from (5,2) to (5,9).

e. As in (a), but with the terminal state at (5,5).

The actions are deterministic moves in the four directions. In each case, compare the results using three-dimensional plots. For each environment, propose additional features (besides x and y) that would improve the approximation and show the results.



21.9 Implement the REINFORCE and PEGASUS algorithms and apply them to the 4×3 world, using a policy family of your own choosing. Comment on the results.



21.10 Is reinforcement learning an appropriate abstract model for evolution? What connection exists, if any, between hardwired reward signals and evolutionary fitness?