* Generalizing, solve following eq^n

$$a_0 t_n + a_1 t_{n-1} + \_\_ + a_k t_{n-k} = b^n p(n) \quad (1)$$

It is sufficient to use following char. poly.

$$(a_0 x^k + a_1 x^{k-1} + \_\_ + a_k)(x-b)^{d+1}$$

d is degree of Poly. p(n)

Ex:- Tower of Hanoi Problem

4

$$t(m) = \begin{cases} 0 & \text{if } m = 0 \\ 2t(m-1)+1 & \text{otherwise} \end{cases}$$

$\Rightarrow$ ~~are~~ written as,

$$t(m) - 2t(m-1) = 1 \quad (2)$$

Compare with General eq^n (1)

$b = 1$, $p(n) = 1$, $d = 0$

$a_0 = 1$, $a_1 = -2$ $\boxed{k=1}$

Characteristic Poly. is,

$$(x-2)(x-1)$$

$(x-2)$ Comes from left side of eq^n (2)

$(x-1)$ " " Right side

roots are $r_1 = 1$

$r_2 = 2$

Sol^n is,

$$t(m) = c_1 1^m + c_2 2^m \quad \text{—— (A)}$$

Initial Con^d,

$m=0, \quad t(0) = 0$

$m=1 \quad t(1) = 2t(1-1)+1$

$= 2t(0)+1$

$= 1$

Put $m=0$ & 1 in eq^n (A)

$c_1 + c_2 = 0$ if $m=0$

$c_1 + 2c_2 = 1$ if $m=1$

$c_1 = -1$ & $c_2 = 1$

Therefore, Sol^n is,

$$t(m) = -1^m + 2^m$$

$$= 2^m - 1^m$$

$$= 2^m - 1$$

$$\boxed{t(m) \in O(2^m)}$$

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Ex:- Consider recurrence,

$$\cancel{t(n) =}$$

$$t_n = 2t_{n-1} + n$$

→ rewritten,

$$t_n - 2t_{n-1} = n$$

Compare with eq^n (1)

$$a_0 = 1 \quad a_1 = -2 \quad K = 1$$
$$b = 1, \quad P(n) = n, \quad d = 1$$

The char. Poly. is,

$$(x-2)(x-1)^2$$

roots

$$r_1 = 2 \quad m_1 = 1$$
$$r_2 = 1 \quad m_2 = 2$$

Sol^n is,

$$t_n = C_1 2^n + C_2 1^n + C_3 n 1^n \quad \text{—— (A)}$$

Provided $t_0 \geq 0$ so $t_n \geq 0$ for $\forall n$,
we Conclude that $t_n \in O(2^n)$

If we substitute eq^n (A) into original recurrence,
we obtain,

$$n = t_n - 2t_{n-1}$$
$$= (C_1 2^n + C_2 + C_3 n) - 2(C_1 2^{n-1} + C_2 + C_3(n-1))$$
$$= C_1 2^n + C_2 + C_3 n - C_1 2^n - 2C_2 - 2C_3 n + 2C_3$$
$$= -C_2 - C_3 n + 2C_3$$

$n = (2C_3 - C_2) - C_3 n$

from which we read directly that,

$\cancel{2C_3 - C_2 = 0} \quad \cancel{\& -C_3 = 1}$

$0 + n = (2C_3 - C_2) - C_3 n \Rightarrow 2C_3 - C_2 = 0 \quad \& -C_3 n = n \Rightarrow C_3 = -1$

OR If $n = 0, \quad 2C_3 - C_2 = 0$

$n = 1, \quad 2C_3 - 2C_2 = 2$

$C_2 = -2 \Rightarrow 2C_3 + 2 = 0$

$\Rightarrow \cancel{C_3 = C_2}$

$\Rightarrow C_3 = -1$

$\cancel{2C_2 - C_2 = 0}$

Sol^n is,

$t_n = C_1 2^n - n - 2$

C_1 must be positive b'cz $t_n \geq 0$

Thus $t_n \in \Theta(2^n)$

Q6,

(N.A)

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Ex:- Consider recurrence

$$t_n = \begin{cases} 1 & \text{if } n=0 \\ 4t_{n-1} - 2^n & \text{otherwise} \end{cases}$$

→ First rewrite recurrence,

$$t_n - 4t_{n-1} = -2^n$$

Compare with eq^n (1),

$a_0 = 1$ $a_1 = -4$ $k=1$

$b = 2$ $p(n) = -1$ $d=0$

The char poly. is,

$$(x-4)(x-2)$$

roots $r_1 = 4$ $m_1 = 1$

$r_2 = 2$ $m_2 = 1$

Sol^n, $t_n = c_1 4^n + c_2 2^n$

Initial Con^d, $-2^n = t_n - 4t_{n-1}$

$= c_1 4^n + c_2 2^n - 4(c_1 4^{n-1} + c_2 2^{n-1})$

$= \cancel{c_1 4^n} + c_2 2^n - \cancel{c_1 4^n} - 2c_2 2^n$

$-2^n = -c_2 2^n$

$\boxed{c_2 = 1}$ doesn't tell that $c_1 > 0$

→ from initial Con^d, $n=0$ $1 = 4c_1 + c_2$

$1 = 4c_1 + 1 \Rightarrow \boxed{c_1 = 0}$

→ recurrences of form

$$a_0 t_n + a_1 t_{n-1} + \dots + a_k t_{n-k} = b_1^n P_1(n) + b_2^n P_2(n) + \dots$$

where b_i are distinct constants
$P_i(n)$ are polynomials in n
d_i degree of $P_i(n)$

Polynomial eqn is :

$$(a_0 x^k + a_1 x^{k-1} + \dots + a_k)(x-b_1)^{d_1+1}(x-b_2)^{d_2+1} + \dots$$

Ex :- Consider recurrence

$$t_n = \begin{cases} 0 & \text{if } n = 0 \\ 2t_{n-1} + n + 2^n & \text{otherwise} \end{cases}$$

→ rewrite recurrence

$$t_n - 2t_{n-1} = n + 2^n$$

2^{n-1}

n

$a_0 = 1$, $a_1 = -2$ $k = 1$
$b_1 = 1$ $P_1(n) = n$ $d_1 = 1$
$b_2 = 2$ $P_2(n) = 1$ $d_2 = 0$

The char. Poly. is,

$$(x-2)(x-1)^2(x-2)$$

$c_1 = 0$ $t_n = 2^n$

roots $r_1 = 1$ $m_1 = 2$
$r_2 = 2$ $m_2 = 2$

General Sol^n is,

$$t_n = C_1 1^n + C_2 n 1^n + C_3 2^n + C_4 n 2^n$$

Initial eq^n,

if $n=0$, $\quad C_1 + C_3 = t_0$

$n=1 \quad C_1 + C_2 + 2C_3 + 2C_4 = t_1$

$n=2 \quad C_1 + 2C_2 + 4C_3 + 8C_4 = t_2$

$n=3 \quad C_1 + 3C_2 + 8C_3 + 24C_4 = t_3$

Find $t_0 = ?$, $t_1 = ?$, $t_2 = ?$ $t_3 = ?$

$t_0 = 0$

$t_1 = 2t_0 + 1 + 2 = 3$

$t_2 = 2t_1 + 2 + 4 = 6+6 = 12$

$t_3 = 2t_2 + 3 + 8 = 24 + 11 = 35$

$C_1 + C_3 = 0$

$C_1 + C_2 + 2C_3 + 2C_4 = 3$

$C_1 + 2C_2 + 4C_3 + 8C_4 = 12$

$C_1 + 3C_2 + 8C_3 + 24C_4 = 35$

$C_1 = -2 \quad C_2 = -1 \quad C_3 = 2 \quad C_4 = 1$

$$t_n = -2\,1^n - n\,1^n + 2 \cdot 2^n + n\,2^n$$

$$= n \cdot 2^n + 2^{n+1} - n - 2$$

~~$t_n \in \Theta$~~ $t_n = O(n 2^n)$

In R.T. - each node represents the cost of single subproblem somewhere in the each level of tree to obtain a set of per level costs & the cost of all levels of recursion.

* Recursion Tree :-

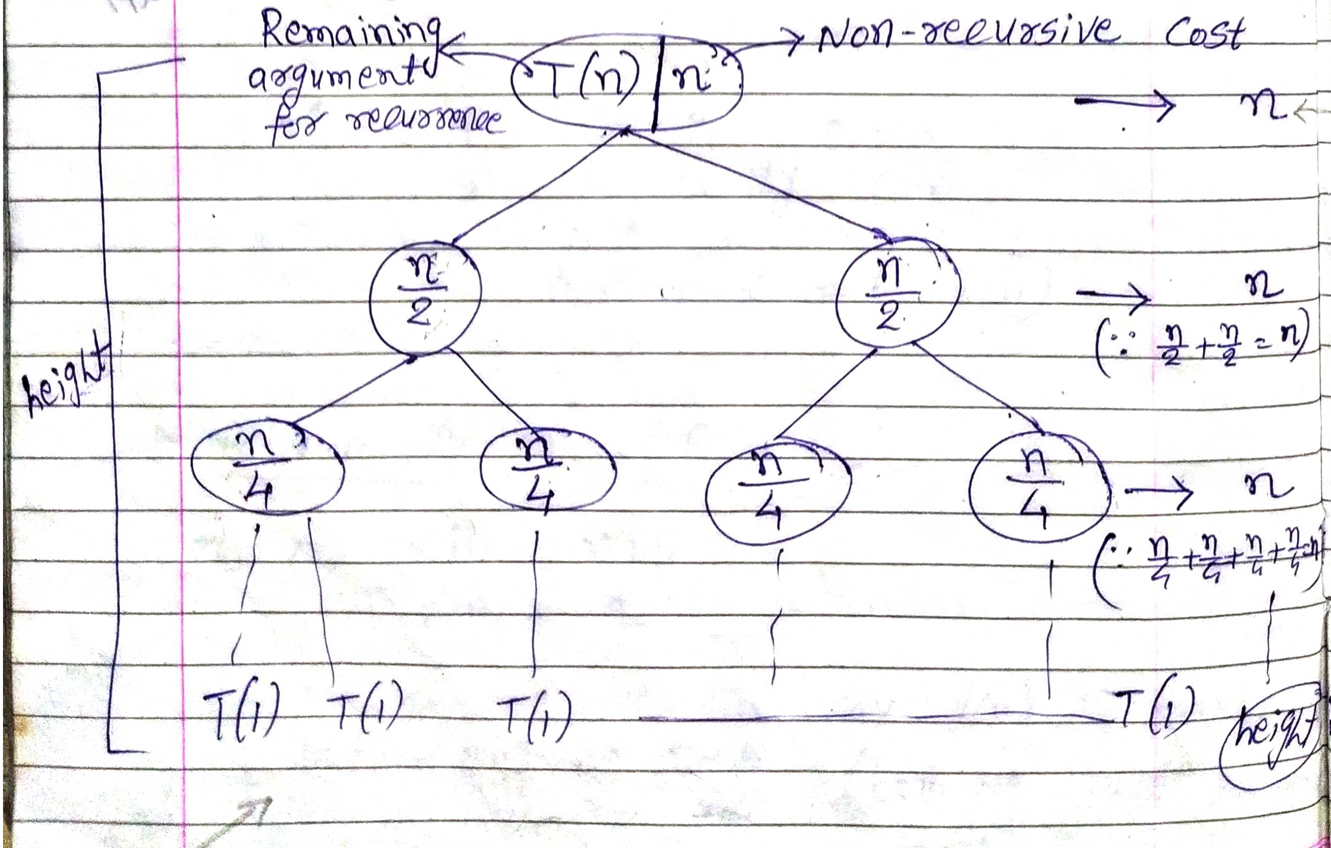
→ It is a Pictorial representation of recurrence which is in form of tree where each level nodes are expanded.

→ It is used to keep track of the size of remaining arguments in the recurrence & non-recursive costs.

ex:- Recurrence

(1) $T(n) = 2T(n/2) + n$

Constructing recursion tree for recurrence



fully expanded tree

here in the set of recursive funⁿ invocations, we sum the costs within
then we sum all the pre level costs to determine the total

To Calculate height
From 1^{st} tree take

So, Subprob. size for node at depth K.

$n = 2 \Rightarrow \frac{2}{2} = 1 \Rightarrow k = 1$

$n = 4 \Rightarrow$

each time Sub prob. size decrease by factor 2.
So it's time to find height means how far from the root do we reach one?

$n = 2 \Rightarrow 2^1 \Rightarrow k = 1$
$n = 4 = 2^2 \Rightarrow k = 2$
$n = 8 = 2^3 \Rightarrow k = 3$

In general $n = 2^k \Rightarrow \boxed{k = \log_2 n}$

Cost of top level of recursion $\boxed{\text{height} = \log_2 n}$

← further expanding node with $T(n/2)$

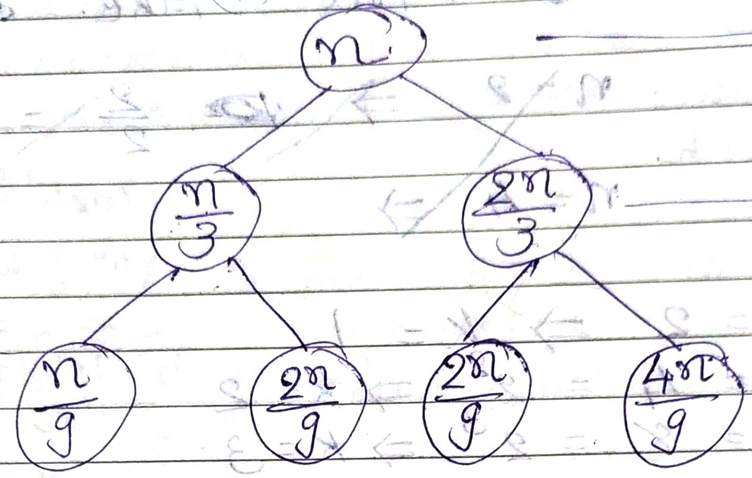
$T(n) = n + n + n + \cdots + \log n$ times

$\boxed{T(n) = \Theta(n \cdot \log n)}$

$= n(1 + 2 + 4 + \cdots \log$

ex:- ② $T(n) = T\left(\frac{n}{3}\right) + T\left(\frac{2n}{3}\right) + n$ Cost

$(\because b = 3/2)$ $\log_{3/2} n$



$\rightarrow n$

$\rightarrow n$

$\rightarrow n$

(∵ when tree is not balanced the longest path is rightmost one)   Total:

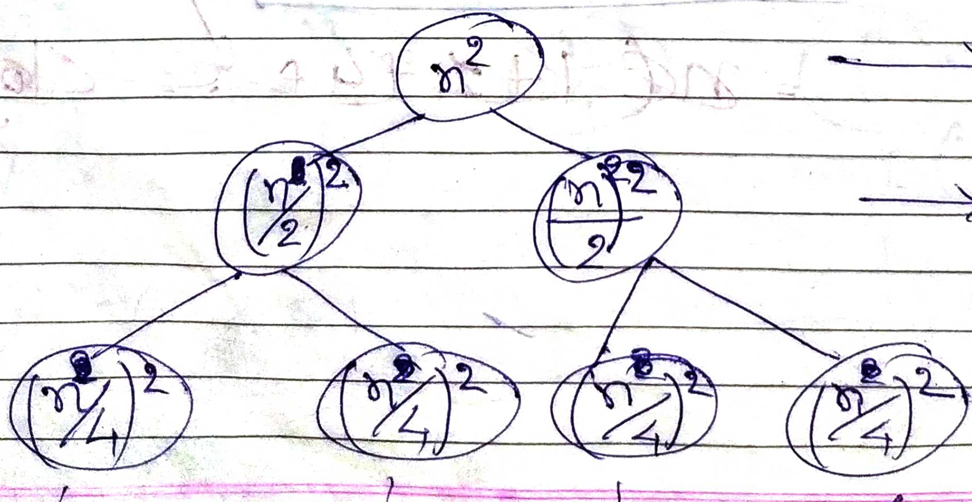
In both leaves, take highest one

$T(n) = n + n + n + \cdots + \log_{3/2} n$ times

$$\boxed{T(n) = O(n \log n)}$$

ex:- ③ $T(n) = 2T(n/2) + n^2$

$\log n$



$\rightarrow n^2$

$\rightarrow n^2/2$

$\rightarrow n^2/4$

$$\frac{\left(1-\frac{1}{2}\right)^{\log n+1}}{1-\frac{1}{2}} \Rightarrow (1)^{\log n+1}$$

we have,

$$T(n) = n^2 + \frac{n^2}{2} + \frac{n^2}{4} + \dots + \log n \text{ times}$$

$$= n^2\Big(\underbrace{1+\frac{1}{2}+\frac{1}{4}+\dots \log n \text{ times}}\Big)$$

$$\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$= n^2$$

$$\because \quad \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{}$$

$$= \sum_{i=1}^{\log n} \left(\frac{1}{2}\right)^i = \frac{\left(\frac{1}{2}-1\right)^{\log n+1}}{\frac{1}{2}-1}$$

$\because$ $1 + 0.5 + 0.25 + 0.125 \dots = 2$

Sum of this term is not greater than 2)

So, $T(n) = 2n^2$

$$\boxed{T(n) = \theta(n^2)}$$

Sol^n series $= 1+\frac{1}{2}+\frac{1}{4}+\dots \log_2 n$

$r = \frac{1}{2}$, $a_1 = 1$

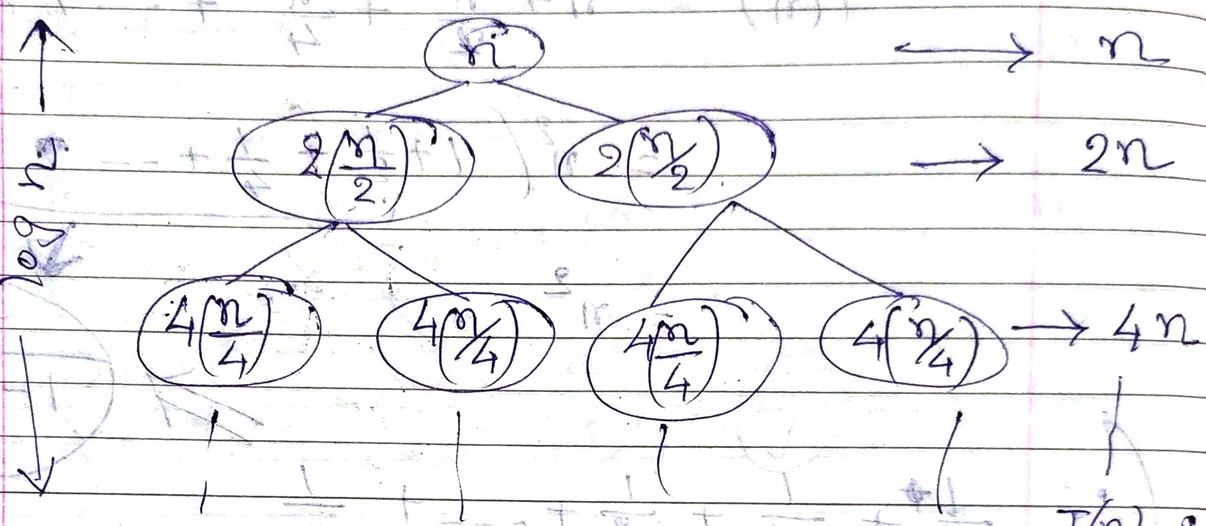
Geometric eq^n $= a_1\frac{(1-r^n)}{1-r}$

$$= \frac{1-\left(\frac{1}{2}\right)^{\log_2 n}}{1-\frac{1}{2}}$$

$$= 2\left(1-\frac{n^{\log_2 1}}{n^{\log_2 2}}\right) = 2\left(1-\frac{n^0}{n}\right) = 2\left(1-\frac{1}{n}\right)$$

ex :- $T(n) = 4T(n/2) + n$

4



we have,

$T(n) = n + 2n + 4n + \ldots + \log n$ times

$= n(1 + 2 + 4 + \ldots \log n \text{ times})$

$= n(n-1) = n^2$

here $r = 2$, $n = \log_2 n$

$= n\left(\frac{1 - 2^{\log_2 n}}{1-2}\right)$ $\cancel{= n \cdot \left(\frac{2^{\log_2 n} - 1}{2-1}\right)}$

$= n\left(\frac{n^{\log_2 2} - 1}{1}\right)$

$= n\left(\frac{1-n}{-1}\right)$

$= n(n-1)$

$= n(n-1)$

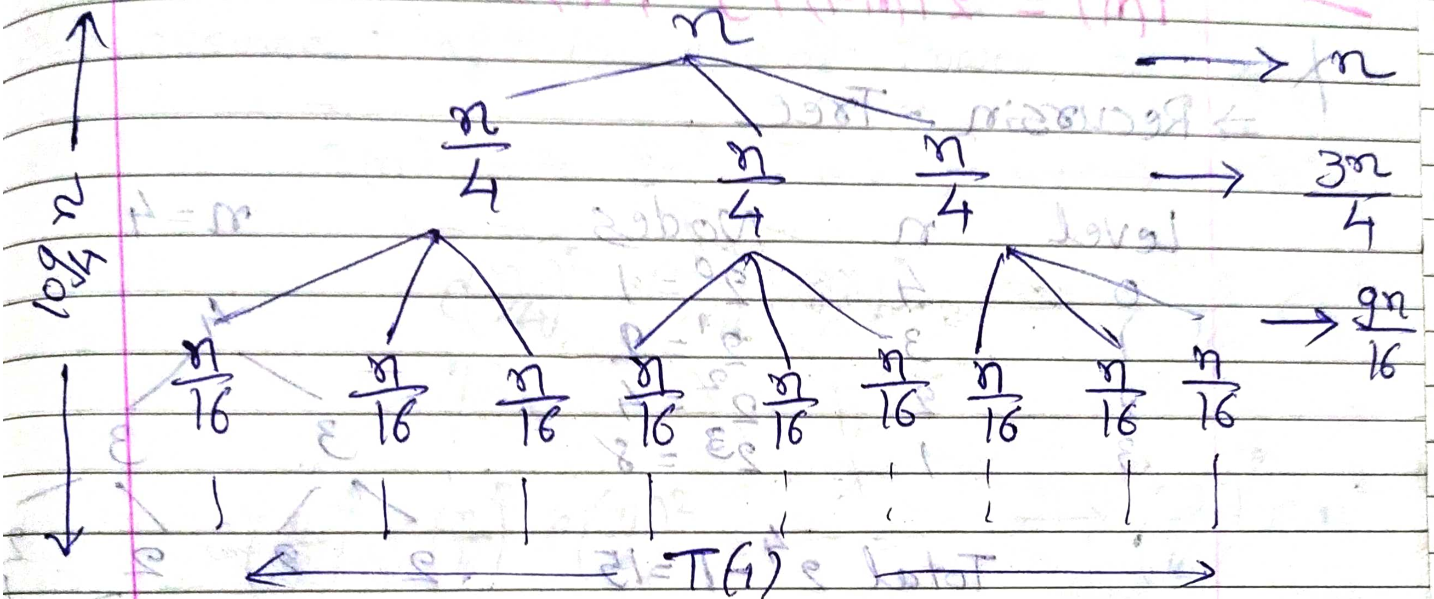
$= n^2 - n$

$\boxed{T(n) = \Theta(n^2)}$

$\because$ Geometric Series $(1 + r + r^2 + \ldots + r^{n-1})$

$= \frac{1 - r^n}{1 - r}$.

ex:- (5) $T(n) = 3T(n/4) + n$



we have,

$$T(n) = n + \frac{3n}{4} + \frac{9n}{16} + \cdots + \log_4 n \text{ times}$$

$$= n\left(1 + \frac{3}{4} + \frac{9}{16} + \cdots + \log_4 n \longrightarrow\right)$$

$$= n\left(\underbrace{1 + 0.75 + 0.563 + \cdots +}_{2}\right)$$

$$= 2n$$

$$\boxed{T(n) = \Theta(n)}$$

ex:- (7) $T(n) = 3T(n/4) + cn^2$ for some const c

$\rightarrow$ assume n is exact power of 4

$$T(n) \rightarrow cn^2 \longrightarrow cn^2$$

$$c\left(\frac{n}{4}\right)^2 \quad c\left(\frac{n}{4}\right)^2 \quad c\left(\frac{n}{4}\right)^2 \longrightarrow \frac{3}{16}cn^2$$

$$c\left(\frac{n}{16}\right)^2 \quad c\left(\frac{n}{16}\right)^2 \quad c\left(\frac{n}{16}\right)^2 \text{ ---} \longrightarrow \left(\frac{3}{16}\right)^2 cn^2$$

adding costs

$$T(n) = cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \text{---}$$

$$= cn^2\left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \text{---}\right)$$

if $n = 16$ or tree has depth at least 2 if $n \geq 16 = 4^2$.

For $n = 4^k$, $k = \log_4 n$

$$T(n) = cn^2 \sum_{i=0}^{\log_4 n} \left(\frac{3}{16}\right)^i$$

$$T(n) = 2cn^2 \quad \boxed{T(n) = O(n^2)}$$

apply geometric seri $S_n = \frac{1-r^n}{1-r}$

$\cancel{r = \frac{3}{16}}$

$$T(n) = cn^2 \frac{\left(\frac{3}{16}\right)^{\log_4 n + 1} - 1}{\frac{3}{16} - 1}$$

ex:- $T(n) = T\left(\frac{n}{4}\right) + T\left(\frac{n}{2}\right) + n^2$

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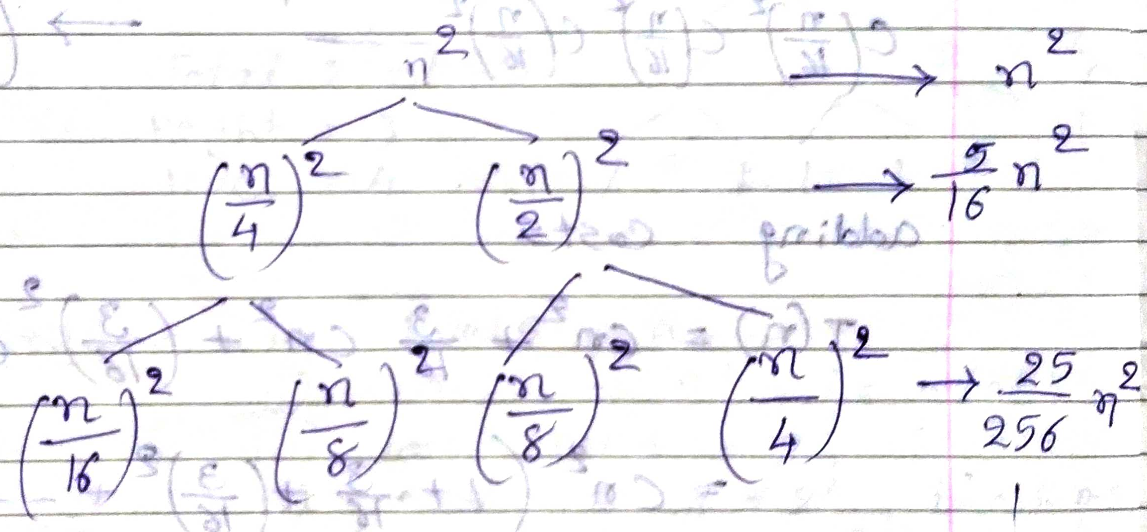

$$T(n) = n^2 + \frac{5}{16} n^2 + \left(\frac{5}{16}\right)^2 n^2 + \left(\frac{5}{16}\right)^3 n^2$$

$$= n^2 \left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \text{---}\right)$$

$$= \cancel{n^2 \frac{\left(\frac{5}{16}\right)^{\log_2 n} - 1}{\frac{5}{16} - 1}}$$

$$\simeq n^2$$

$T(n) = \Theta(n^2)$

$$= n^2 \underbrace{\left(1 + 0.3125 + 0.0976 + \dots\right)}_{2}$$

$$= 2n^2$$

$$\boxed{T(n) = \Theta(n^2)}$$