4. Membership functions

4.1. Types of Membership Functions

- Depending on the type of membership function, different types of fuzzy sets will be obtained.

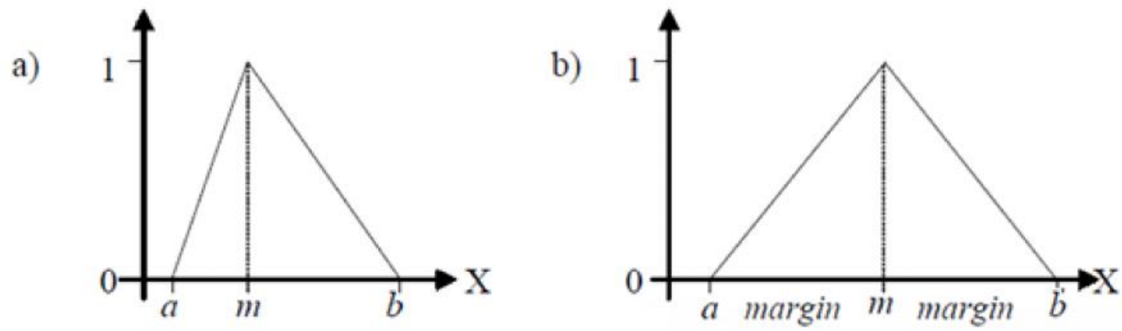
- Zadeh proposed a series of membership functions that could be classified into two groups: those made up of straight lines being "**linear**" ones, and the "**curved**" or "**nonlinear**" ones.

- However, the nonlinear functions increase the time of computation. Therefore, in practice, most applications use linear fit functions.

- We will now go on to look at some types of membership functions.

4.1.1. Triangular

Defined by its lower limit a, its upper limit b, and the modal value m, so that a < m < b. We call the value b-m margin when it is equal to the value m – a.

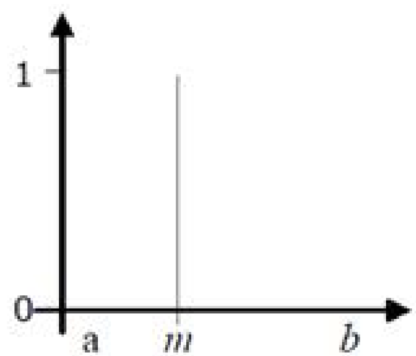


Triangular fuzzy sets: a) general and b) symmetrical.

$$A(x) = \begin{cases} 0 & \text{if } x \le a \\ (x-a)/(m-a) & \text{if } x \in (a, m] \\ (b-x)/(b-m) & \text{if } x \in (m, b) \\ 0 & \text{if } x \ge b \end{cases}$$

4.1.2. Singleton

- It takes the value 0 in all the universe of discourse except in the point m, where it takes the value 1.
- It is the representation of a crisp value.

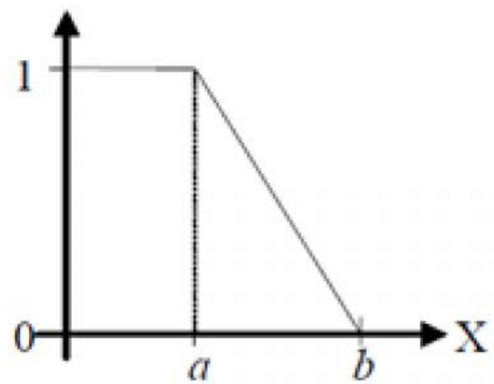


Singleton fuzzy set

$$SG(x) = \begin{cases} 0 & \text{if } \; x \neq m \\ 1 & \text{if } \; x = m \end{cases}$$

4.1.3. L-Function

This function is defined by two parameters **a** and **b**, in the following way:



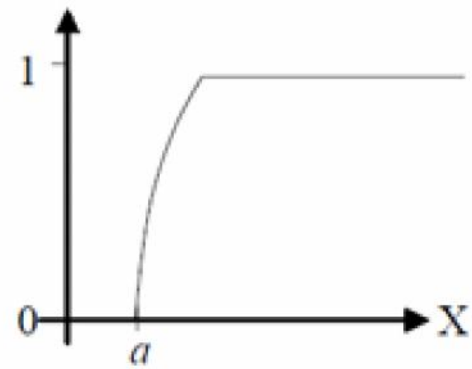
L fuzzy set

$$L(x) = \begin{cases} 1 & \text{if } x \le a \\ \dfrac{b-x}{b-a} & \text{if } a < x \le b \\ 0 & \text{if } x > b \end{cases}$$

4.1.4. Gamma Function

It is defined by its lower limit a and the value k>0.

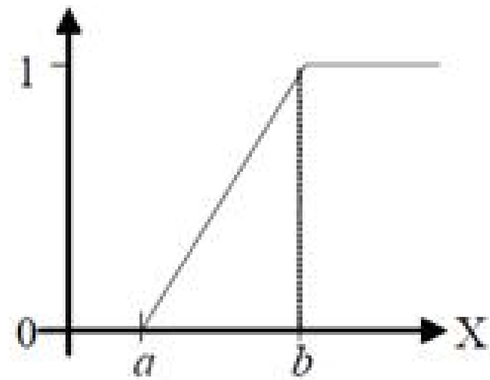
$$\Gamma(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{k(x-a)^2}{1+k(x-a)^2} & \text{if } x > a \end{cases}$$



- This function is characterized by a rapid growth starting from a.
- k determines the rate of growth
- It has a horizontal asymptote in 1.

The function can also be expressed in a linear way.

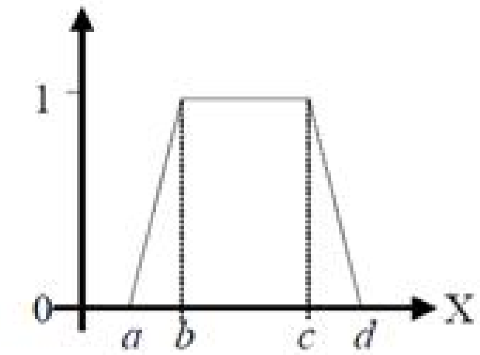
$$\Gamma(x) = \begin{cases} 0 & \text{if } x \le a \\ \dfrac{x-a}{b-a} & \text{if } a < x \le b \\ 1 & \text{if } x > b \end{cases}$$



4.1.5. Trapezoid Function

Defined by its lower limit a and its upper limit d, and the lower and upper limits of its nucleus, b and c respectively.

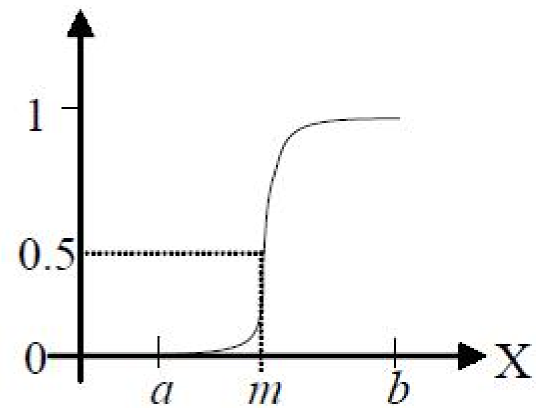
$$T(x)=\begin{cases} 0 & \text{if } (x \le a) \text{ o } (x \ge d) \\ (x-a)/(b-a) & \text{if } x \in (a,b] \\ 1 & \text{if } x \in (b,c) \\ (d-x)/(d-c) & \text{if } x \in (b,d) \end{cases}$$



4.1.6. S Function

Defined by its lower limit a, its upper limit b, and the value m or point of inflection so that a <m <b. A typical value is m = (a + b) / 2. Growth is slower when the distance a – b increases.

$$S(x) = \begin{cases} 0 & \text{if } x \le a \\ 2\{(x-a)/(b-a)\}^2 & \text{if } x \in (a, m] \\ 1 - 2\{(x-b)/(b-a)\}^2 & \text{if } x \in (m, b) \\ 1 & \text{if } x \ge b \end{cases}$$

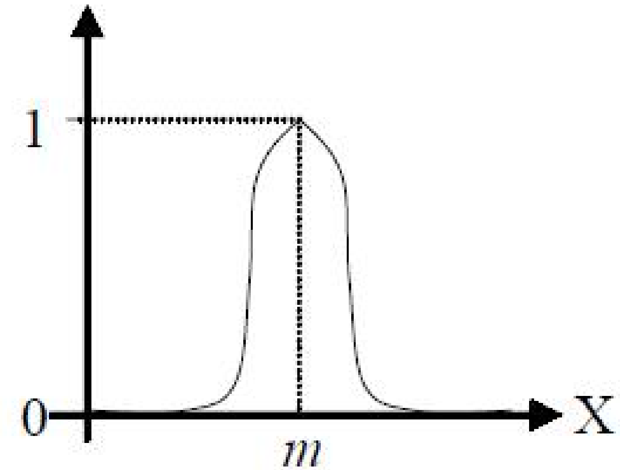


$$m = \frac{a+b}{2}$$

4.1.7. Gaussian Function

This is the typical Gauss bell, defined by its midvalue m and the value of σ > 0. The smaller σ is, the narrower the bell.

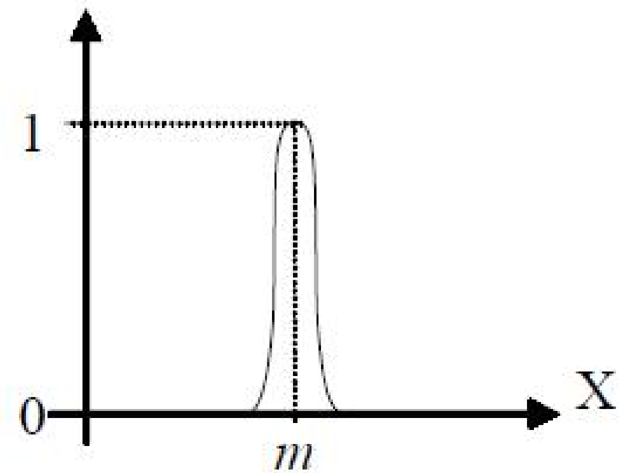
$$G(x) = \exp\left[\frac{-(x-\mathbf{m})^2}{2\sigma^2}\right]$$



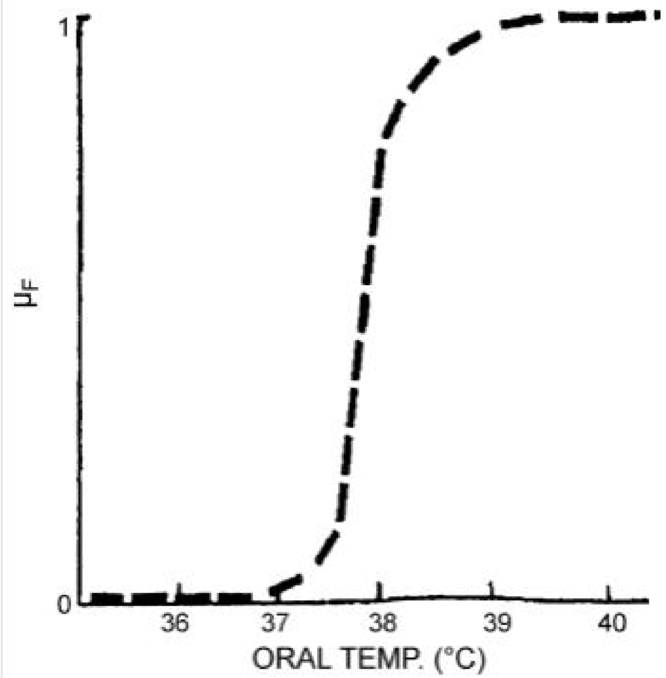
4.1.8. Pseudo-Exponential Function

Defined by its midvalue m and the value k > 1. As the value of k increases, the rate of growth increases, and the bell becomes narrower.

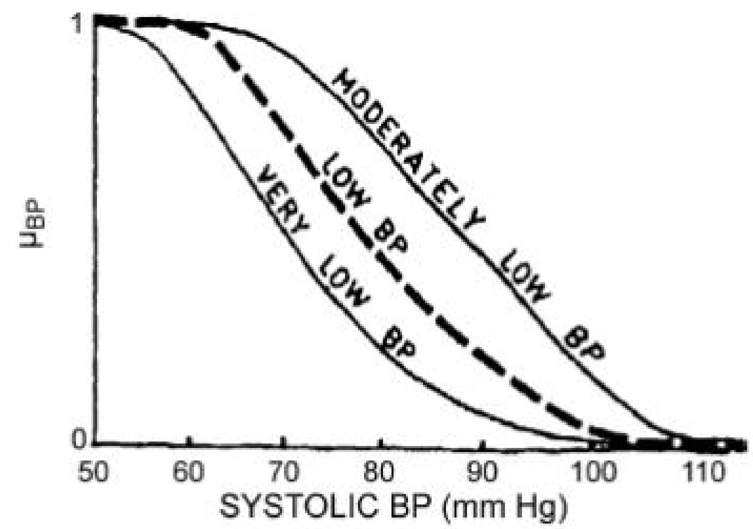
$$P(x) = \frac{1}{1 + k(x - m)^2}$$



So a Physician will choose the following membership functions for "Fever" and "Blood Pressure". With regard to medical diagnosis; the framework of fuzzy sets is very useful to deal with the absence of sharp boundaries of the sets of symptoms, diagnoses, and phenomena of diseases.



Membership function for "fever"

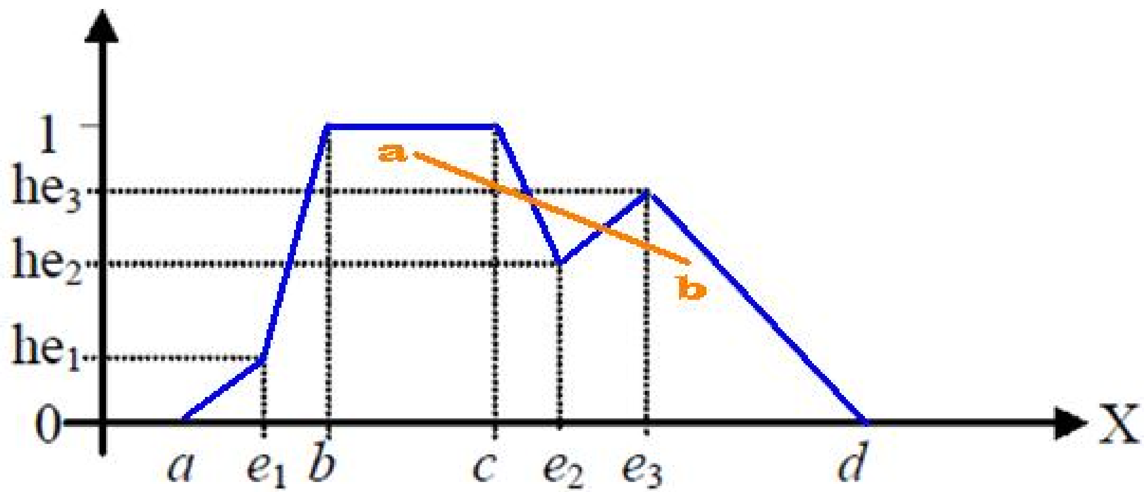


Membership function for "blood pressure"

4.1.9. Extended Trapezoid Function

Defined by the four values of a trapezoid (a, b, c, d) and a list of points between a and b and/or between *c and d, with its membership value (height) associated to each* of these points.

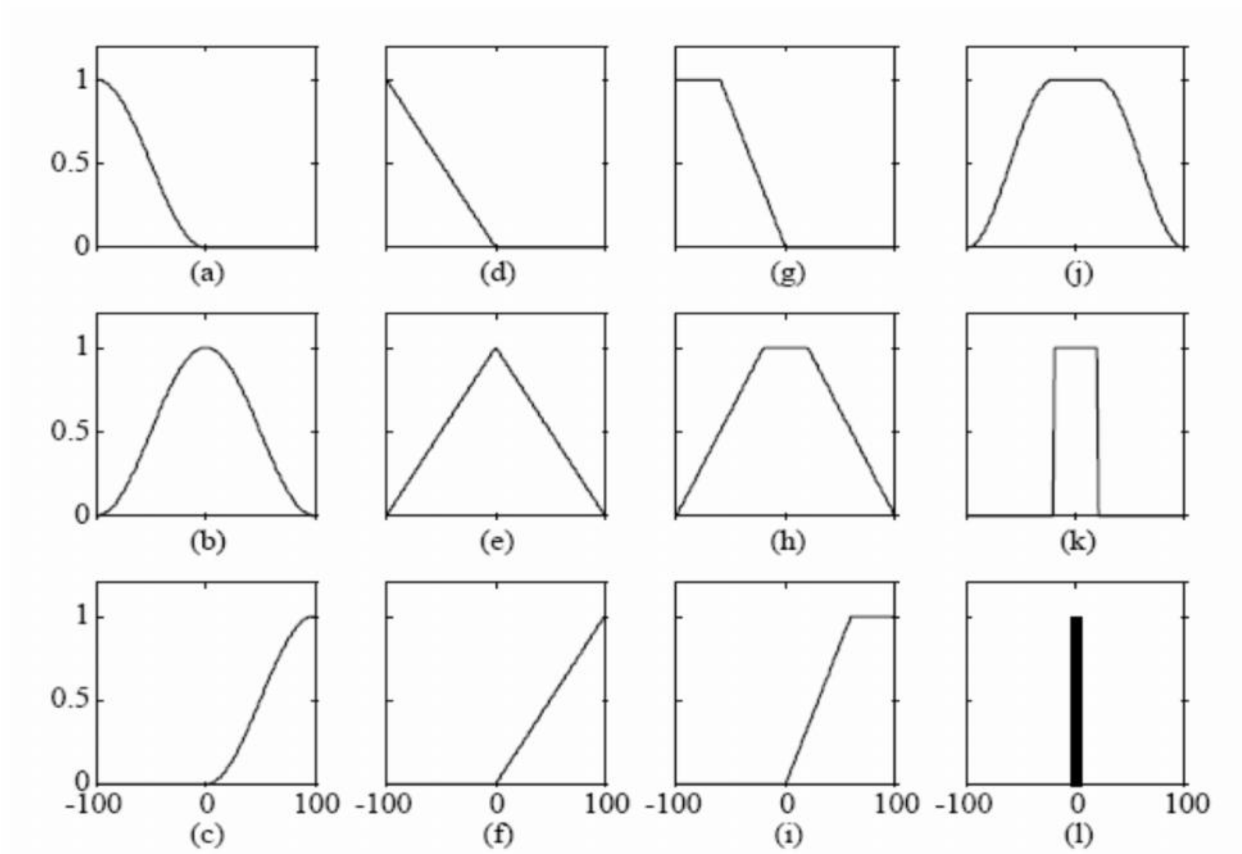
$(e_i\ ,\ h_{e\ i}\ )$.



This function is not **Convex** in the sense that every point on the line connecting two points a and b in A is also in A.

A ***Convex*** fuzzy set is described by a membership function whose membership values are strictly monotonically increasing, or whose membership values are strictly monotonically decreasing, or whose membership values are strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe. Said another way, if, for any elements x, y, and z in a fuzzy set A , the relation $x < y < z$ implies that

$$\mu A(y) \geq \min[\mu A(x), \mu A(z)]$$



Examples of membership functions. Read from top to bottom, left to right: (a) s_function, (b) π -function, (c) z_function, (d-f) triangular versions, (g-i) trapezoidal versions, (j) flat π - function, (k) rectangle, (l) singleton

4.1.10. Comments

- In general, the trapezoid function adapts quite well to the definition of any concept, with the advantage that it is easy to define, easy to represent, and simple to calculate.
- In specific cases, the extended trapezoid is very useful. This allows greater expressiveness through increased complexity.
- In general, the use of a more complex function does not give increased precision, as we must keep in mind that we are defining a fuzzy concept.
- Concepts that require a nonconvex function can be defined. In general, a nonconvex function expresses the union of two or more concepts whose representation is convex.

4.2. LINGUISTIC HEDGES

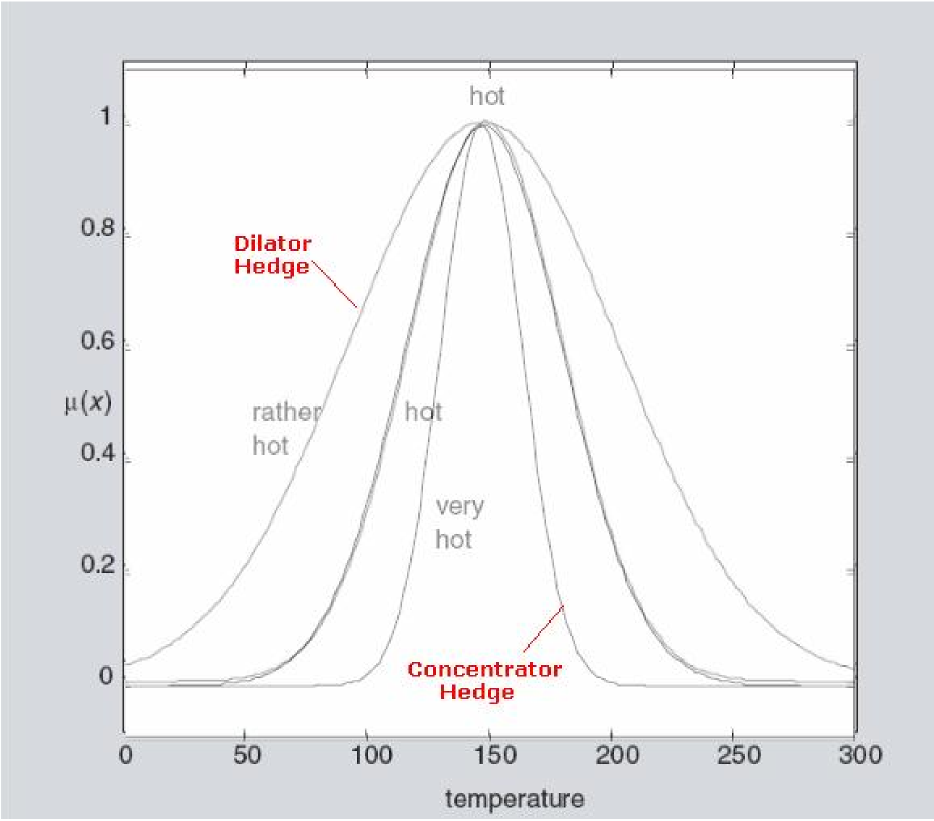
- Linguistic hedges (or simply hedges) are special linguistic terms by which other linguistic terms are modified. Linguistic terms such as very, more or less, fairly, or extremely are examples of hedges.
- Any linguistic hedge, H, may be interpreted as a unary operation, h, on the unit interval [0, 1].
- For example, the hedge very is often interpreted as the unary operation

$$h(a)=a^2$$

- while the hedge fairly is interpreted as

$$h(a)=\sqrt{a} \qquad (a \in [0,1])$$

- Let unary operations that represent linguistic hedges be called modifiers.

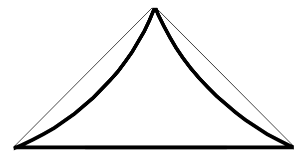
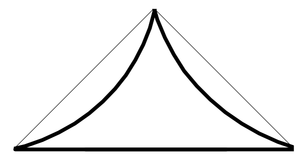
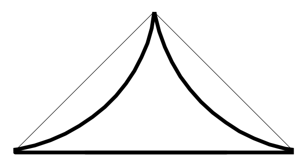
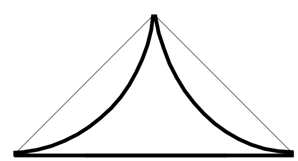


Example of hedges that can modify a fuzzy set

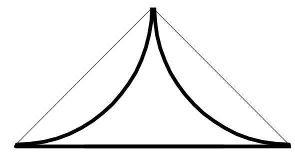
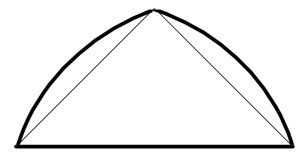
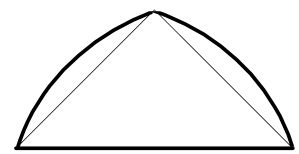
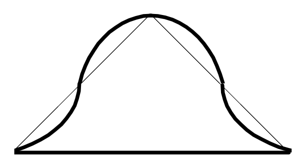
$h_\alpha(\mu(x)) = \mu(x)^\alpha$

Where α>1 is named a strong modifier (Concentrator)

and α <1 is named a weak modifier (Dilator)

Hedge	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

Examples of Strong Modifiers

Hedge	*Mathematical Expression*	*Graphical Representation*
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2\,[\mu_A(x)]^2$ if $0 \le \mu_A \le 0.5$ $1 - 2\,[1 - \mu_A(x)]^2$ if $0.5 < \mu_A \le 1$	

Examples of Strong/Weak Modifiers

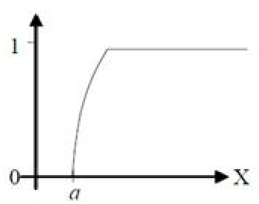
4.3. Membership Function Determination

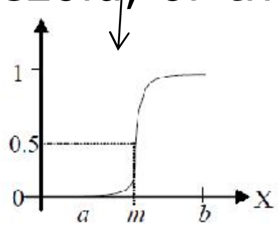
- In fuzzy control, for example, the aim is to express the notions of increase, decrease, and approximation, and in order to do so, the types of membership functions previously mentioned are used.

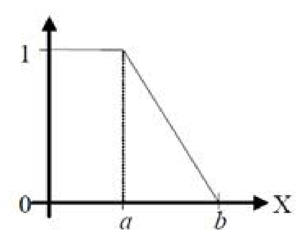
- The membership functions Gamma and S would be used to represent linguistic labels such as “tall” or “hot” in the dominion of height and temperature.

- Linguistic labels such as “small” and “cold” would be expressed by means of the L function.

- On the other hand, approximate notions are sometimes difficult to express with one word. In the dominion of temperature, the label would have to be “comfortable,” which would be expressed by means of the triangle, trapezoid, or the Gaussian function.







- Linguistic concepts are vague, and their meanings are almost context-dependent.

- For example/the concept of *large distance* has different meanings in the contexts of walking, driving, or air travel.

- The concepts *cheap, expensive, very expensive,* and so on, depend not only on the items to which they are applied (e.g., a house versus a vacation trip), but also on the influence of the buyer and a host of other circumstances.

- Concepts such as *beautiful, pleasing, painful,* or *talented* have many different meanings, which may differ from person to person even under the same circumstances.

- The scenario of the construction of fuzzy sets involves a specific knowledge domain of interest, one or more experts in this domain, and a knowledge engineer.

4.3.1. DIRECT METHODS WITH ONE EXPERT

an expert is expected to either:

1. Assign to some selected elements $x \in X$ a membership grade $\mu(x)$.

2. Or he may define a membership function completely in terms of a justifiable mathematical formula.

This may be facilitated by asking the expert questions of the form

"What is the degree of membership of x in A?"

or, alternatively,

"What is the degree of compatibility of x with LA ?"

where *LA* is the linguistic term (hedge) that we want to represent in a given context by fuzzy set A.

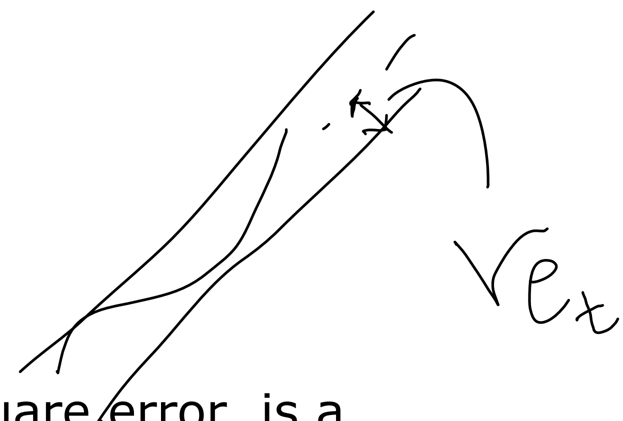
If desirable, the questions may be formulated in reverse form:

"Which elements x have the degree μ(x) of membership in A?"
"Which elements x are compatible with LA to degree A(x) ?"

These questions, regardless of their form, result in a set of pairs (x, μ(x)) . This set is then used for constructing the membership function A of a given shape (triangular, trapezoidal, S-shaped, bell-shaped, etc.) by an appropriate curve-fitting method.

For example, in pattern recognition of handwritten characters, the expert may define a fuzzy set of straight lines, S, in terms of the least square straight-line fitting with minimum error, e(x), for each given line x. Then, the function

$$S(x) = \begin{cases} 1 - e^{2}(x)/e_t & \text{when } e(x) < e_t \\ 0 & \text{otherwise,} \end{cases}$$



where e_t is the largest acceptable least square error, is a meaningful membership function that captures quite well the linguistic concept *straightness* in the context of handwritten character recognition.

4.3.2. DIRECT METHODS WITH MULTIPLE EXPERTS

Now *n* experts are asked to valuate the proposition "*x* belongs to A" as either true or false. The answers are crisp Y/N or 1/0. Then,

$$A(x) = \frac{\sum_{i=1}^{n} a_i(x)}{n}$$

may be viewed as a probabilistic interpretation of the constructed membership function.

It is, however, often useful to generalize this interpretation by allowing one to distinguish degrees of competence, ci, of the individual experts. This results in the formula

$$A(x) = \sum_{i=1}^{n} c_i \cdot a_i(x),$$

Where

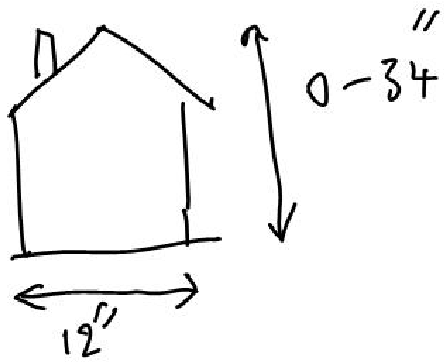
$$\sum_{i=1}^{n} c_i = 1.$$

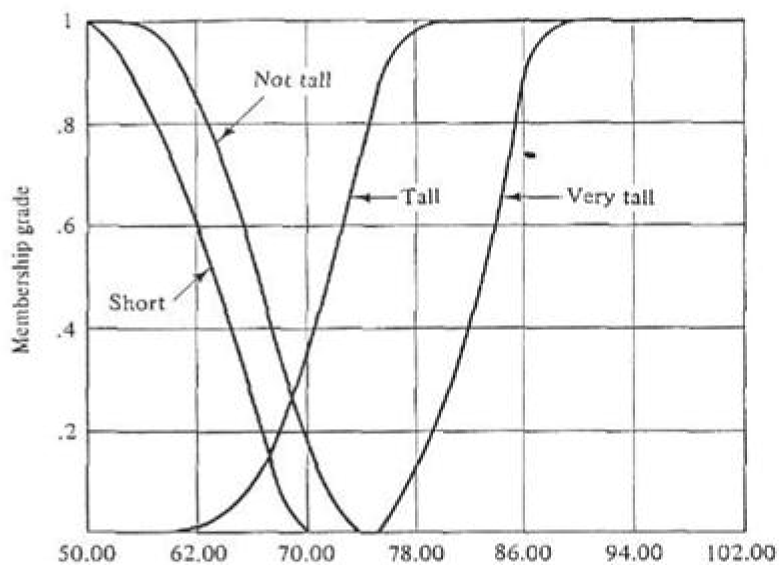
Examples

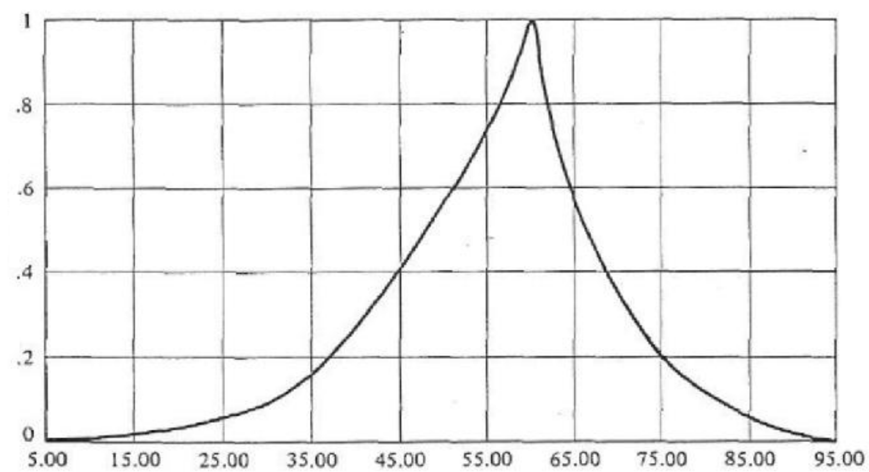
The following is a brief characterization of two experiments:

1. The first experiment involves the subjective perception of each participant of the notion of tall persons. It uses a life-sized wooden figure of adjustable height.

2. The second experiment involves the notion of aesthetically pleasing houses among one-story houses of fixed width and variable heights. A cardboard model of a house is used, which consists of a chimney and a triangular roof sitting on a rectangle of width 12 inches and of height adjustable from 0 inches to 34 inches.







Height in inches

Height in 1/120 of house width

4.3.3. INDIRECT METHODS

Direct methods have one fundamental disadvantage. They require the expert to give answers that are overly precise.

Indirect methods attempt to reduce the eventually resulting impreciseness by replacing direct estimates of membership grades with simpler tasks, as **pair wise comparisons**.

Instead of asking the expert to estimate values a_i directly, we ask him compare elements $x_1, x_2, .. x_n$ in pairs according to their relative weights of belonging to *A.* The pairwise comparisons are conveniently expressed by the square matrix $P = [p_{ij}]$.

Assume first that it is possible to obtain perfect values p_{ij} .

In this case $p_{ij} = a_i / a_j$;

and matrix P is consistent in the sense that

$$P_{ik} = P_{ij} P_{jk}$$

for all *i, j,* which implies that $p_{ii} = 1$ and $p_{ij} = 1/P_{ji}$.

4.3.4. CONSTRUCTIONS FROM SAMPLE DATA:

It is required to construct a membership function from samples of membership grades for **some** elements of the given universal set *X.* Now we have samples of degree of membership and we want to construct a membership function. We may use either curve fitting, which is exemplified here by the method of Lagrange:

Given the n sample data:

$$(x_1,a_1),(x_2,a_2), \ldots\ldots(x_n,a_n)$$

$$f(x) = a_1 L_1(x) + a_2 L_2(x) + \ldots + a_n L_n(x),$$

where

$$L_i(x) = \frac{(x - a_1)\ldots(x - a_{i-1})(x - a_{i+1})\ldots(x - a_n)}{(x_i - a_1)\ldots(x_i - a_{i-1})(x_i - a_{i+1})\ldots(x_i - a_n)}$$

for all $i \in \mathbb{N}_n$.

Since values of $f(x)$ can lie outside the range $[0,1]$, function f cannot be directly considered as the sought membership function . We may convert f to the required function by the formula

$$f^*(x) = \max[0, \min[1, f(x)]]$$

Example

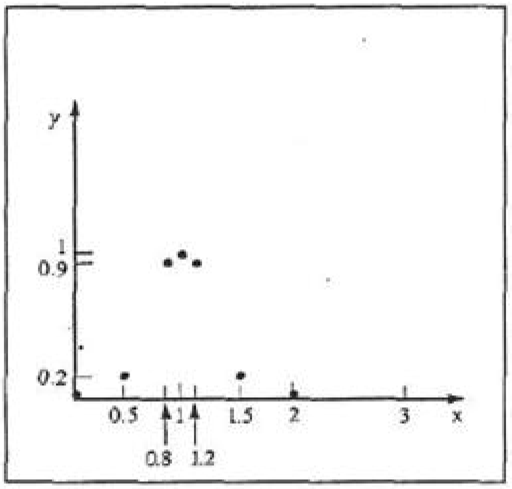
To realize some difficulties involved in using this method for constructing membership functions, let us consider the following sample data:

(0, 0), (.5, .2), (.8, .9), (1,1), (1.2, .9), (1.5, .2), (2, 0).

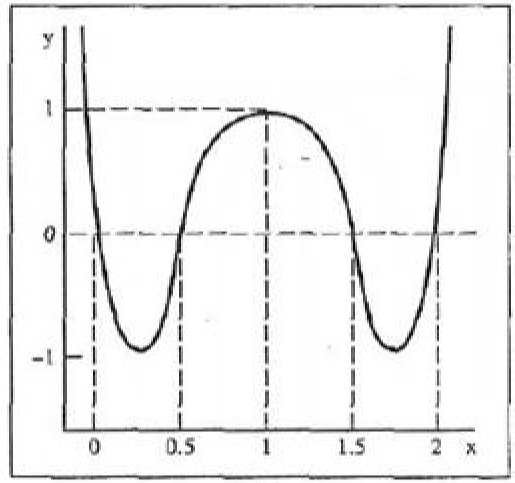
These data are shown graphically in Fig. a. Then:

$$f(x) = 6.53x^6 - 39.17x^5 + 92.69x^4 - 109.65x^3 + 64.26x^2 - 13.66x$$

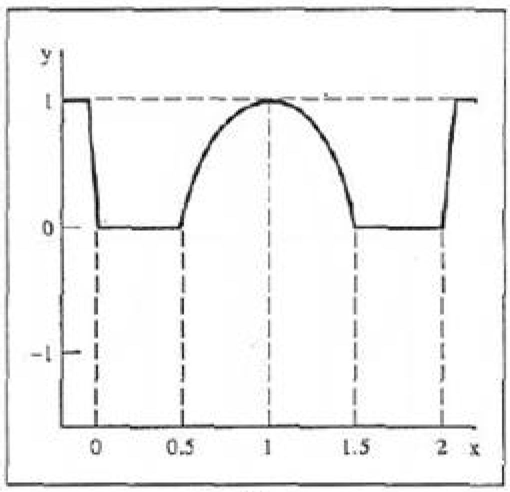
whose graph is shown in Fig. b. Applying now the modification to limit f(x) between in the range [0,1],we obtain function A whose graph is given in Fig. c. We can see on intuitive grounds that this function is not a reasonable representation of the data outside the interval [0, 2], It can be corrected by assuming that the estimated support of A is the interval [0, 2]. The corrected function is shown in Fig.d.


y
1
0.9
0.2
0.5
1
1.5
2
3
x
0.8
1.2

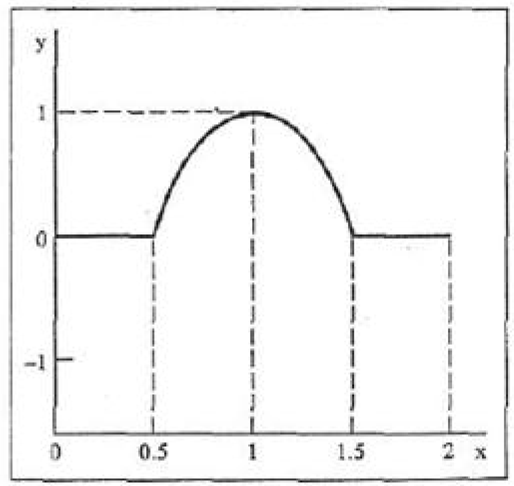
(a)


y
1
0
-1
0
0.5
1
1.5
2
x

(b)


y
1
0
-1
0
0.5
1
1.5
2
x

(c)


y
1
0
-1
0
0.5
1
1.5
2
x

(d)

Interval-valued membership function:

Suppose the level of information is not adequate to specify membership functions with precision. For example, we may only know the upper and lower bounds of membership grades for each element of the universe for a fuzzy set. Such a fuzzy set would be described by an *interval-valued membership function, such as the one shown in Fig. In this figure, for a* particular element, *$x = z$, the membership is a fuzzy set A*, i.e., *$\mu_A(z)$, would be expressed* by the membership interval [*α1, α2]. Interval-valued fuzzy sets can be generalized further* by allowing their intervals to become fuzzy. Each membership interval then becomes an ordinary fuzzy set.