Phase Portrait Analysis of Linear Regression with Two-Weights

$$h(x) = w_0 x_0 + w_1 x_1$$

where

$$x_0 = 1, x_1 = x$$

Cost Function is

$$J(w) = \frac{1}{2} \sum_{i=1}^{N} \left(y^{(i)} - h\left(x^{(i)}\right) \right)^2$$

Gradient Descent Update Rule

$$w_{j+1} = w_j - \alpha \frac{\partial}{\partial w_j} J(w)$$

$$\frac{\partial}{\partial w_j} J(w) = (h(x) - y) x_j$$

Therefore

$$w_{j+1} = w_j + \alpha \sum_{i=1}^{N} \left(y^{(i)} - h\left(x^{(i)}\right) \right) x_j^{(i)}$$

Approximating to Differential Equation

$$\dot{w}_j = \sum_{i=1}^{N} \left(y^{(i)} - h\left(x^{(i)}\right) \right) x_j^{(i)}$$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_0 \end{bmatrix} = - \begin{bmatrix} \sum_{i=1}^{N} {x^{(i)}}^2 & \sum_{i=1}^{N} x^{(i)} \\ \sum_{i=1}^{N} x^{(i)} & N \end{bmatrix} \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{N} x^{(i)} y^{(i)} \\ \sum_{i=1}^{N} y^{(i)} \end{bmatrix}$$

Let

$$\sum_{i=1}^{N} x^{(i)} = S(x)$$

$$\sum_{i=1}^{N} y^{(i)} = S(y)$$

$$\sum_{i=1}^{N} x^{(i)} y^{(i)} = S(xy)$$

$$\sum_{i=1}^{N} {x^{(i)}}^2 = S\left(x^2\right)$$

Therefore

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_0 \end{bmatrix} = - \begin{bmatrix} S\left(x^2\right) & S\left(x\right) \\ S\left(x\right) & N \end{bmatrix} \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} + \begin{bmatrix} S\left(xy\right) \\ S\left(y\right) \end{bmatrix}$$

1 STABILITY ANALYSIS

This is of the form

$$\dot{W} = AW + B$$

Here B is a constant with respect to w. Hence can be ignored for Stability Analysis.
Consider A

$$\tau = -\left[N + S\left(x^2\right)\right]$$

$$\Delta = NS\left(x^2\right) - \left[S\left(x\right)\right]^2$$

Here we can see that $\tau < 0$
Consider Δ

$$\Delta = \sum_{i=1}^{N} Nx_i\left[x_i - \mu_x\right] \qquad \textit{where} \quad \mu_x = \frac{S\left(x\right)}{N}$$

Let

$$\delta_i = Nx_i\left[x_i - \mu_x\right]$$

Hence

$$\Delta = \sum_{i=1}^{N} \delta_i$$

There are two cases .(1) $x_i > \mu_x$ (2) $x_i < \mu_x$. In Both the cases

$$S\left(x\right)\left[x_i - \mu_x\right] < \delta_i$$

Therefore

$$\sum_{i=1}^{N} S\left(x\right)\left[x_i - \mu_x\right] < \Delta$$

$$S\left(x\right)\left[S\left(x\right) - N\mu_x\right] < \Delta$$

$$0 < \Delta$$

Therefore No Saddle Points.
Consider $\tau^2 - 4\Delta$

$$= \left[N + S\left(x^2\right)\right]^2 - 4\left[NS\left(x^2\right) - \left[S\left(x\right)\right]^2\right]$$

$$= \left[N - S\left(x^2\right)\right] + 4\left[S\left(x\right)\right]$$

$$\tau^2 - 4\Delta > 0$$

Hence Fixed Point is Stable Node

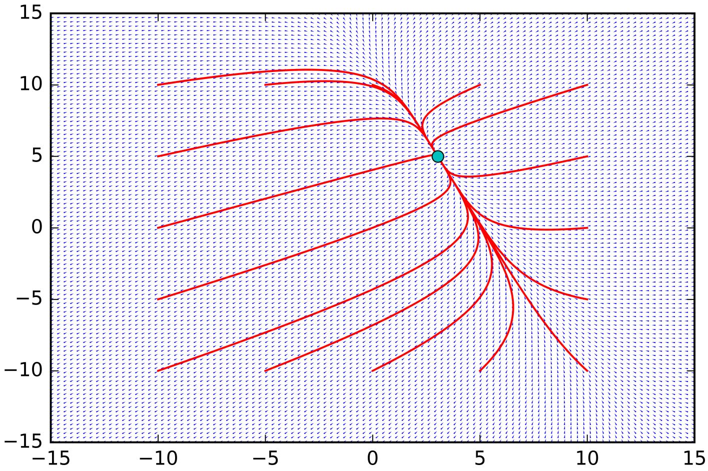

Figure: Phase Portrait showing the Stable Node.

Special Case : If

$$\frac{1}{N}S\left(x^2\right) = 1, S\left(x\right) = 0$$

$$A = -\begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$$

$$\tau = -2N \quad \tau < 0$$

$$\Delta = N^2 \quad \Delta > 0$$

$$\tau^2 - 4\Delta = \left(-2N\right)^2 - 4N^2 = 0$$

$$\tau^2 - 4\Delta = 0$$

Hence Fixed Points are stable stars

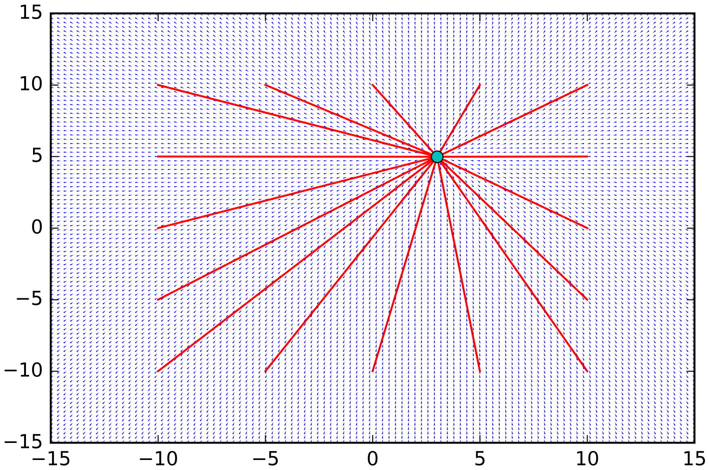

Figure: Phase Portrait showing the Stable Star.

2 FIXED POINT CALCULATION

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_0 \end{bmatrix} = - \begin{bmatrix} S\left(x^2\right) & S\left(x\right) \\ S\left(x\right) & N \end{bmatrix} \begin{bmatrix} w_1 \\ w_0 \end{bmatrix} + \begin{bmatrix} S\left(xy\right) \\ S\left(y\right) \end{bmatrix}$$

At Fixed Points

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_0 \end{bmatrix} = 0$$

Therefore

$$\begin{bmatrix} S\left(x^2\right) & S\left(x\right) \\ S\left(x\right) & N \end{bmatrix} = \begin{bmatrix} S\left(xy\right) \\ S\left(y\right) \end{bmatrix}$$

$$AX^* = B$$

Cases :

1) If $\Delta \neq 0$ A is Square-Invertible.

So Fixed Point is

$$X^* = A^{-1}B$$

$$X^* = \frac{1}{\Delta} \begin{bmatrix} N & -S\left(x\right) \\ -S\left(x\right) & S\left(x^2\right) \end{bmatrix} \begin{bmatrix} S\left(xy\right) \\ S\left(y\right) \end{bmatrix}$$

Special Case : If

$$\frac{1}{N} S\left(x^2\right) = 1, S\left(x\right) = 0$$

$$A = - \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$$

Therefore

$$X^* = \frac{1}{N} \begin{bmatrix} S\left(xy\right) \\ S\left(y\right) \end{bmatrix}$$

2) If $\Delta = 0$ Two Possibilities

i)A is a Rank Zero matrix. A=0

Then whole Plane is Full of Fixed Points.

ii)A is a Rank 1 matrix.

Nullspace of A $N\left(A\right)$ is a Line.

$$Ax_n = 0$$

Since

$$\begin{bmatrix} S\left(x^2\right) & S\left(x\right) \\ S\left(x\right) & N \end{bmatrix} \begin{bmatrix} N \\ S\left(x\right) \end{bmatrix} = 0$$

Therefore

$$x_n = \begin{bmatrix} N \\ S\left(x\right) \end{bmatrix}$$

If $\frac{N}{S(x)} = \frac{S(x)}{S(x^2)} \neq \frac{S(y)}{S(xy)}$ then No Fixed Point Exists.

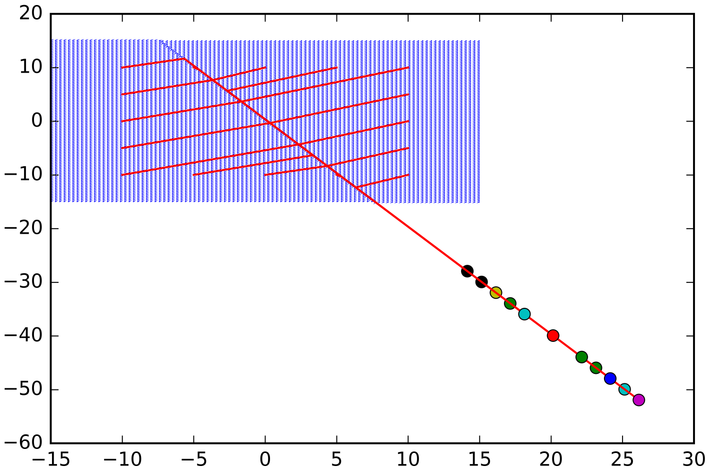

Figure: Phase Portrait showing absence of Fixed point and depicting the solutions running away to infinity.

Else if $\frac{N}{S(x)} = \frac{S(x)}{S(x^2)} = \frac{S(y)}{S(xy)}$ then there exists a Line of Fixed Points. The Line of Fixed Points is $x_p + cx_n$ where c is any constant

$$\begin{bmatrix} S\left(x^2\right) & S\left(x\right) \\ S\left(x\right) & N \end{bmatrix} \begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} S\left(xy\right) \\ S\left(y\right) \end{bmatrix}$$

$$x_p = \begin{bmatrix} z \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{S(y)}{S(x)} \\ 0 \end{bmatrix}$$

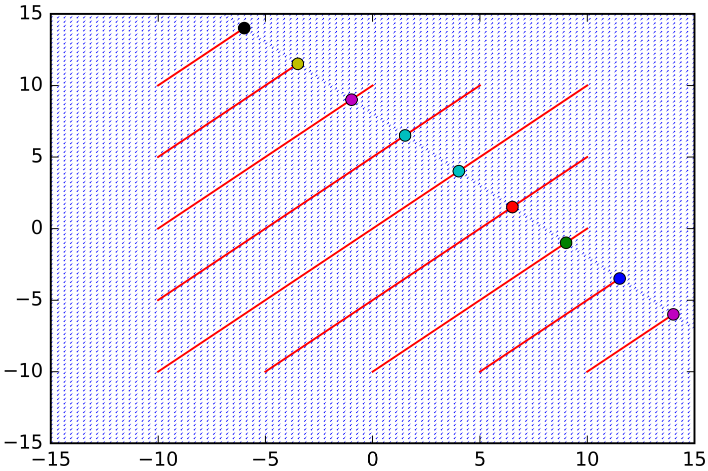

Figure: Phase Portrait showing the Line of Fixed Points.