Introductory Functional Analysis

by
Guru Kalyan Jayasingh

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Project Supervisor: **Dr. Sutanu Roy**
School of Mathematical Sciences
National Institute of Science Education and Research

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Chapter 1

Normed Vector Spaces

1.1 Vector Spaces

By a vector space we mean a nonempty set E with two operations:

1. (x,y) $\to$ x+y from E$\times$E into E called addition,

2. (λ, x)$\to$ λx from F$\times$E into E called multiplication by scalars

The main idea of vector space is that it is closed under linear combinations. **Examples**:-

1. 0 is a trivial vector space over reals, following 0+0 = 0 and $\lambda\times$ 0 = 0

2. Function spaces:- Let X be an arbitrary nonempty set and let E be a vector space. Denote by F the space of all functions from X into E. Then F becomes a vector space if the addition and multiplication by scalars are defined in the following natural way:

   a) (f+g)(x) = f(x)+g(x)

   b) (λ*f)(x) = λ*f(x)

3. l^pSpaces:- The space of all infinite sequences (z_n) of complex numbers such that $\Sigma_{n=1}^{\infty}|z_n|^p < \infty$ for $p \geq 1$

Examples- 1.The sequence $(x_n) = \dfrac{1}{n}$, $n \; \varepsilon \; \mathbb{N}$ is an l^2 sequence as it converges to the sum $\dfrac{\pi^2}{6}$ but not l^1.

2. The sequence (x_n)=$\dfrac{1}{n!}$ is an l^1 sequence.

Important fact:- l^p **is a vector space.**
Proof:- We need to show that if (x_n) and $(y_n) \; \epsilon \; l^p$, then $(x_n + y_n)\epsilon \; l^p$.

For this we use holder's inequality:- Let p>1 and q>1, and $\frac{1}{p}$+$\frac{1}{q}$=1.Then for positive numbers (a_i) and (b_i), then

$$(\sum a_i)^{\frac{1}{p}}(\sum b_i)^{\frac{1}{q}} \geq \sum a_i b_i$$

.

Now, applying that we see that(here x_i and y_i denotes the modulus/absolute value of corresponding terms in the sequence and hence can be treated as positive)

$$(x_i^p + y_i^p)^{\frac{1}{p}}(1+1)^{1-1/p} \geq x_i + y_i$$
$$(x_i^p + y_i^p)(2)^{p-1} \geq (x_i + y_i)^p$$

Now for a fixed p, we can sum both sides over to include all terms and see that $\Sigma_{i=1}^{\infty}(x_i + y_i)^p$ is bounded and hence convergent.

Problem:-If (x_i)be a real sequence and $lim_{n\to\infty} \; x_i \to 0$, does it always belong to an l^p space?
Answer:- Nope! Consider the sequence

$$(x_n) = \{1, 1/2, 1/2, 1/2, 1/2, 1/3, 1/3, 1/3, ...(3^3 times).....1/n, 1/n, 1/n...(n^n tim$$

This sequence converges to zero but consider the series sum, when raised to pth power

$$S = 1 + \frac{2^2}{2^p} + ... + \frac{n^n}{n^p}......$$

diverges, as for n>p , $\frac{n^n}{n^p} > 1$. **Problem:-** Give an example of a sequence

that is

a) l^1 $but\ not\ l^2$

b) $l^2 but\ not\ l^1$

Answer:-
1. This is not possible as we have

$$\sum_{n=1}^{\infty} x_i^2 \leq (\Sigma x_i)^2$$

where x_i refers to |a_i|.
So if a sequence is l^1, then we know that the rhs of the equality is finite and hence it is in l^2.
2. Trivial example- $x_n = \frac{1}{n}$, $n\ \varepsilon\ \mathbb{N}$.

1.2 Normed Spaces

Norm generalizes the notion of length on a vector space, defined axiomatically.

Definition:-A function x$\rightarrow ||x||$ from a vector space E to $\mathbb{R}$ is called a norm if it satisfies the following conditions:-

1. ||x||$\rightarrow 0$ implies x=0.
2. ||$\lambda x|| = |\lambda|||x||, \lambda\ \varepsilon \mathbb{F}$
3. ||x+y||$\leq$ ||x||+||y||, for every x,y ε E.

Examples:-
1.On C^N, the norm defined as

$$||z|| = \sqrt{\sum_{i=1}^{N} |z_i^2|}$$

is a valid norm.
Another norm(**Taxicab norm**) is defined as

$$||z|| = \sum_{i=1}^{N} |z_i|$$

is also a valid norm. As we will later see, these two norms are equivalent in some sense.
2.On C[0,1], the norm ||*f*|| = $max_{x \varepsilon [0,1]} |f(x)|$ is a valid norm.
3.For an l^p sequence, the norm defined by

$$||z|| = \sum_{i=1}^{N} |z_i^p|)^{1/p}$$

is a valid norm. The Minkowski inequality(stated below) is the triangle inequality for it.
If (x_n) and (y_n) ϵ l^p,we have

$$(\sum_{n=1}^{\infty} |x_n + y_n|^p)^{1/p} \ \leq \ (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} (\sum_{n=1}^{\infty} |y_n|^p)^{1/p}$$

1.2.1 Norms and convergence

Since ||x|| is a measure of length of x, ||x-y|| is a measure of distance between x and y, for x,y being elements of a vector space. Using this idea, we can define convergence.
Definition Let (E,||.||) be a normed space. We say that a sequence (x_n) of elements of E converges to some x ε E, if for every ϵ >0. there exists a number M such that for every n$\geq$ M, we have $||x_n - x|| < \epsilon$. In such a case we say that the sequence converges to x and we write $lim_{n \to \infty} \, x_n = \ x$.

Generally, we cannot always find a norm that will satisfy a given convergence.

Examples:-
Uniform convergence:-

Consider $\mathcal{C}[0,1]$ and sup norm. Define uniform convergence as follows:-
Let $\Omega \varepsilon \mathbb{R}$ be a closed bounded set and $\mathcal{C}(\Omega)$ be the set of all continuous functions from Ω to $\mathbb{R}$.
Then we say that a given sequence of functions $(f_n) \ \varepsilon \ \mathcal{C}(\Omega)$ **converges uniformly** to f, if for every $\epsilon > 0$, there exists an n_0 such that $|f_n(x) - f(x)| < \epsilon$ for all x ϵ Ω whenever n$\geq n_0$.
The important thing here is n_0 **does not depend on x**.
Now, if we consider the sup norm, we can clearly see that it gives way for uniform convergence as if for some n$\geq n_0 \ \varepsilon \ \mathbb{N}$, $max_{x \,\varepsilon\, \Omega} \ |f_n(x) - f(x)| < \epsilon$, means that for all $x \ \varepsilon \ \Omega$, we have $|f_n(x) - f(x)| < \epsilon$.
Therefore, it is also called uniform convergence norm.(i.e. convergence by sup norm also **guarantees** uniform convergence)

PROBLEMS:-

1. Show that
$$f_n(x) \ = \ \frac{x}{1+nx^2}$$
converges uniformly.

2. . Let f_n be a sequence of uniformly continuous functions. If f_n converges uniformly to f on [a, b] then f is continuous.Moreover,
$$\lim_{n\to\infty} \int_a^b \ f_n(x) \ dx \ = \ \int_a^b \ f(x) \ dx$$

Answers:-

1. The following proves that the given function converges to h(x)=0 uniformly.Clearly, for a fixed n, we have
$$f(0) = 0$$
$$f^{'}(x) \ = \ \frac{1-nx^2}{(1+nx^2)^2}$$
Thus maxima of f is $\dfrac{1}{2\sqrt{n}}$ (occurs at x=$\dfrac{1}{2\sqrt{n}}$).
Therefore, for any $\epsilon > 0$, we have
$|f_n(x) \ - \ 0| \leq \dfrac{1}{2\sqrt{n}} < \epsilon$. We choose $n_0 \ \geq \dfrac{1}{4\epsilon^2}$.

2. The following proof is an example of a more ubiquitous strategy of carefully choosing ϵ (the $\frac{\epsilon}{3}$ proof)

   Let N$\geq$ 1, be such that $||f_N - f||_\infty < \frac{\epsilon}{3}$. Now as f_n is uniformly continuous on [a,b], we see that $\exists\ \delta$ such that whenever |x-y|< δ, $|f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. Now $\forall\ x, y\ such\ that\ |x-y| < \delta,$
   $|f(x) - f(y)| \leq ||f(x) - f_N(x)||_\infty + |f_N(x) - f_N(y)| + ||f_N(y) - f(y)||_\infty$
   $= ||f_N - f||_\infty + \frac{\epsilon}{3} + ||f_N - f||_\infty$
   $= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$
   Now the last result implies from the following:-

   $$|\int_a^b f_n(x)\ dx - \int_a^b f(x)\ dx| \leq ||f_n - f||_\infty\ |b-a|$$

   However, it can be shown that there is no norm in $\mathcal{C}[0,1]$ that can guarantee pointwise convergence.

1.2.2 Equivalence of norms

Definition:- Two norms defined on the same vector space are called equivalent iff there exists real numbers $\alpha\ and\ \beta$ such that

$$\alpha||x||_1 \leq ||x||_2 \leq \beta||x||_1$$

Also we can see that if $||x||_1 \to 0$, we will have $||x||_2 \to 0$.

Fact: In a finite dimensional vector space, any two norms are equivalent.

PROBLEM:- Give an example of two norms in $\mathcal{C}[0,1]$ that are not equivalent.
Answer:- Consider the sequence $(f_n) = x^n$.
Using the uniform norm we have $||f_n||_\infty = 1\ \forall\ n\ \varepsilon\ \mathcal{N}$ i.e. the first norm is bounded.
But using the integral norm we have $\int_0^1 x^n\ dx = \frac{1}{n+1} \to 0\ as\ n \to \infty$.
Hence, even though, $||x|| \to 0$, we don't have $||x||_\infty \to 0$.

1.3 Banach Spaces

Definition:- A vector space E is called complete if every Cauchy sequence in E converges to an element of E. A complete normed space is called a Banach Space.

Every convergent sequence is Cauchy however, every Cauchy sequence is not convergent.

Examples-

1. Consider the sequence

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n},\ n \geq 1$$

   where $a_n \in \mathbb{Q}$. This series converges to $\sqrt{2}$ (irrespective of the value of a_0), which is not in the space of a_n i.e. space of rational numbers.

2. Let $\mathbb{P}[0,1]$ be the space of polynomials on [0,1] with the norm of convergence ||P||=$max_{[0,1]}|P(x)|$. Define

$$P_n(x) = 1 + x + x^2/2! + x^3/3! + ... + x^n/n!$$

   for n=1,2,3.... Then clearly (P_n) is a Cauchy sequence but it does not converge in $\mathbb{P}[0,1]$.

Examples of Banach spaces:- 1. $\mathbb{R}^n$ *and* $\mathbb{C}^n$ 2. l^p *spaces* 3. $\mathcal{C}$ wrt to the uniform norm.

PROBLEM:- Is $f_n = \dfrac{nx}{1+nx^2}$ a Cauchy sequence?

Answer:-It cannot be a Cauchy sequence wrt to the uniform norm as if it is, then it should converge uniformly to a continuous function, and also point-wise to a continuous function but we can clearly see that for a fixed x

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \frac{1}{x}$$

which is not continuous in [0,1].

1.3.1 Completion of spaces

Some spaces that appear in practice are not complete. But it turns out, we can always **enlarge** these to a complete space.

PROBLEM:- Can a incomplete space be always completed?

Answer:- Yes. It is always possible to complete an incomplete space. The following highlights the general completion strategy:- Define completion of a space (||.||,E) as ($||.||_1$,$\bar{E}$), defined as follows:-

1. There exists a one-one mapping $\Phi : \ E \rightarrow \bar{E}$, such that $\Phi(ax + by) \ = \ a \ \Phi(x) + b \ \Phi(y)$, x,y$\varepsilon$ E and a,b ε $\mathbb{F}$

2. ||x||= $||\Phi(x)||_1$,$\forall$ $x\varepsilon$ E

3. $\Phi(E)$ is dense in $\bar{E}$

4. $\bar{E}$ is complete.

Formally, $\bar{E}$ can be defined as the space of equivalent classes of Cauchy sequences of elements of E, where each two sequences (x_n) and (y_n) are called equivalent if $\lim_{n\rightarrow\infty} ||x_n - y_n|| \ = \ 0$.

Further, we define addition and scalar multiplication in this space as follows:-

[(x_n)]+[(y_n)]=[$(x_n + y_n)$] and λ[(x_n)]=[(λx_n)]. Further, define $\Phi(x)$ =[(x,x,x,...)] and norm by

||[x_n]$||_1$=$\lim_{n\rightarrow\infty} ||x_n||$. Now it can be shown that $\bar{E}$ is complete wrt to this norm.

1.4 Linear Mappings

A mapping from a space E_1 to E_2 is called a linear mapping if $L(ax + by) \ = \ a \ L(x) + b \ L(y)$, where x,y ε E_1 and a,b are scalars.

We can always assume that the domain of a linear map is a vector space as it can be uniquely extended to a linear mapping from the span $\mathcal{D}_L$(domain of L) to E_2.

1.4.1 Continuous maps

Def^n:- Let E_1 and E_2 be normed spaces. A mapping from E_1 to E_2 is called continuous at $x_0 \ \varepsilon \ E_1$ if for all sequences (x_n) ,

$$||x_n - x_0|| \to 0 \implies ||F(x_n) - F(x_0)|| \to 0$$

If F is continuous at each $x \ \varepsilon \ E_1$, then we say that F is continuous.

QUESTION Is the norm a continuous function from E to $\mathbb{R}$?

Answer:- Yes. It follows from the triangle inequality that

$$||x_n - x|| \geq |||x_n|| - ||x|||$$

and thus if $||x_n - x|| \to 0$, then the RHS also tends to zero.

PROBLEMS:- Let $F : E_1 \to E_2$. Then prove that the following are equivalent:-

1. F is continuous
2. The inverse image $F^{-1}(U)$ of any open subset U of E_2 is open in E_1.
3. The inverse image $F^{-1}(S)$ of any subset closed subset S is closed in E_1.

Answers:- 1. To show equivalence of 1 and 2, we observe,if f is continuous, take an open subset V of E_2. Now let U be the inverse image of V. Take any point x_0 in U, s.t. f(x_0) lies in V.

Now, if we construct a ball of radius ϵ around f(x_0)(V being open), there must exist a ball of radius δ around x_0 s.t. if x ε $\mathcal{B}(x_0, \delta)$, then $f(x) \varepsilon \ \mathcal{B}(f(x_0), \epsilon)$.

Now for the other direction, we need to show that if V$\subset$ E_2 contains f(p), we will have a $U \subset E_1$ containing p, s.t. $f(U)$ is contained in V.This is trivial, as we take any V containing f(p) and put U=$f^{-1}V$. Then it is open by assumption and f(U)=V is also contained in V.

However, **continuous maps may not necessarily take open sets to open sets**.

Examples:

1. Consider $f(x) = x^2$ on (-1,1).Here, $\mathcal{D}_f \to [0, 1)$ which is not open.

2. Consider $f(x) = e^(- |x|)$ on $(-\infty, \infty)$. Here, $\mathcal{D}_f \to (0, 1]$, which is not open.

- **Homeomorphisms**:- Continuous Bijections between two topological spaces that map open sets to open sets are called homeomorphisms.
  The last condition implies the continuity of f^{-1}.

However, **being continuous and bijective even may not ensure openness of maps**.
Examples:- Consider the map $f : [0, 1) \to S_1\ the\ unit; circle$, given by,

$$f(t) = (sin2\pi t, cos2\pi t)$$

Here we see that $f(1^-)$ is close to $f(0)$ but not close in the original domain. Hence, f^{-1} is not continuous and hence, open sets are not mapped to open sets.(the unit circle being a closed set)
Bounded linear map:- A linear map L : $E_1 \to E_2$ is called bounded if there exists a number $\alpha > 0$ such that $\|Lx\| \leq\ \alpha\ ||x||\ \forall$ x $\varepsilon\ E_1$.
The following are a few properties of linear maps:-

1. If L is a linear map and it is continuous at some $x_0\ \varepsilon\ E$, E being the vector space, it is continuous everywhere.

2. A linear map is continuous iff it is bounded.

Proof of (2):-
First assume L is bounded. Then we can see that as $\|x_n\| \to 0$, then $\|Lx\| \to 0$ and hence L is continuous.
Now if L is continuous but unbounded, then for each n $\varepsilon\ \mathcal{N}\ \exists\ (x_n) :\ ||Lx_n|| > n||x_n||$. Now consider the sequence:-

$$y_n = \frac{x_n}{n||x_n||}$$

We see that $||y_n|| \to 0$ but $||Ly_n||\ > 1$ hence doesn't vanish as $n \to 0$.

- A linear mapping between **FINITE** dimensional space is always bounded and hence is **always continuous**.

1.5 Banach Fixed Point theorem

$Def^n :-$ **Contraction Mappings**:- A mapping f from a subset A of a normed space E into E is called a contraction mapping if there exists a positive number α < 1, such that

$$||f(x)\ -\ f(y)|| <\ \alpha||x-y||$$

Clearly, it is a bounded mapping and hence continuous.

Theorem 1 (Banach Fixed Point Theorem)**.** *Let F be closed subset of a Banach space E and let f be a contraction mapping from F into F. Then there exists a unique z such that f(z)=z.*

Proof. We can see if there exists such a fixed point, then starting at any point, taking the image and iterating this continuously we will always converge to the fixed point. The proof here shows the existence of such a point:-

Let $x_0\ \varepsilon\ F$. Now let, $x_n =\ \ f(x_{n-1})$ for n$\geq$ 1 be a sequence of elements. We will show that this is a Cauchy sequence as follows:-

$||x_{n+1}-x_n|| \leq \alpha||x_n-x_{n-1}|| \leq ||x_{n-1}-x_{n-2}||.....\leq \alpha^n||x_1-x_0||$

Hence, for m,n ε N, for m<n, we have :-

$||x_n - x_m|| \leq ||x_n - x_{n-1}|| + ||x_{n-1} - x_{n-2}|| + ||x_{n-2} - x_{n-3}||..... + ||x_{m+1} - x_m||$

$\leq (\alpha^{n-1} + \alpha^{n-2} + \alpha^{n-3} + \alpha^{n-4}..... + \alpha^m)||x_1 - x_0||$

$\leq ||x_1 - x_0||\dfrac{\alpha^m}{1-\alpha}$

Thus (x_n) is a Cauchy sequence.Also as F is a closed subset, we will have the

$$\lim_{n\to\infty}(x_n) =\ z\ \varepsilon\ F$$

. $\square$

This result gives a practical method to compute and estimate the fixed point.

Example:- Consider the equation $x^3 - x - 1 = 0$. This has clearly 3 roots, one being in [1,2]. Now, we try to put the equation in the form Tx=x:-

Tx=x^3-1, Tx=$(1 + x)^{\frac{1}{3}}$, Tx=$\frac{1}{x^2-1}$

But we see that only Tx=$(1 + x)^{\frac{1}{3}}$ has a contraction in [1,2], as we can see from MVT:-

$$|Tx - Ty| =\ f^{'}(c)|x - y| \leq \frac{2^{\frac{1}{3}}}{6}|x - y|$$

where c ε (1,2).Hence we as this is a contraction mapping in [1,2], we can start at any point in [1,2] and iterate continuously to reach the fixed point. This is a workable,efficient and extremely simple method to obtain solutions of equations.

PROBLEMS:-

1. Show that the following sequence converges:-

$$a_1 = \frac{3}{2} \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \; n \geq 1$$

2. Prove that the following sequence converges and find it's limit:-

$$x_0 = 1 \quad x_{n+1} = \frac{1}{1 + x_n}, \; n \geq 1$$

3. **An accurate map of Mumbai is spread out flat on a table in Convocation Hall, in IIT Bombay. Prove that there is exactly one point on the map lying directly over the point it represents.**

4. Let g be a continuous real valued function on [0,1]. Prove that there exists a continuous real valued function f on [0,1] satisfying the equation:-

$$f(x) - \int_0^x f(x-t)e^{-t^2}dt \;=\; g(x)$$

5. Show that there is a unique continuous function f:[0,1]$\rightarrow$ R such that

$$f(x) = \; sinx \; + \; \int_0^1 \frac{f(y)}{e^{x+y+1}}$$

6. Let K be a continuous function on the unit square $0 \leq$ x,y$\leq$1 satisfying |K(x,y)| < 1for all x and y. Show that there is a continuous function f(x) on [0,1] such that we have

$$f(x) + \int_0^1 K(x,y)f(y)dy \;=\; e^{x^2}$$

Answers:-

1. We take the mapping $Tx \;=\; \frac{x}{2} + \frac{1}{x}$ in [1,2]. We can see that from MVT:- |Tx-Ty| = |f(x)-f(y)| = $|\frac{1}{2} - \frac{1}{c^2}| \; |x-y| \;\leq\; 1/2|x-y|$, where c ε [1,2].

Hence it is a contraction mapping and hence converges to a unique fixed point.
We can clearly see the limit is $\sqrt{2}$.

2. Consider the map $g(x) = \dfrac{1}{1+x}$ on [,]. Now, $|g^{'}(x)| = \dfrac{1}{(1+c)^2} \leq$ Now, for each x_0, we can choose a c s.t. the derivative is less than 1 and is a fixed number. Therefore, it is a contraction mapping.

3. The process of taking the map and laying it out is a contraction mapping and hence it has a fixed point. Also, the above process is a continuous mapping as nearby points in Mumbai are mapped to nearby points in the map.

4. Consider the mapping $T : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ defined by $T(f(x)) = \int_0^x f(x-t)e^{-t^2}\ dt\ +\ g(x).$
We see that :-
$||Tf - Tg|| = sup_{x\ \varepsilon[0,1]}|\int_0^x (f(x-t) - g(x-t))e^{-t^2}dt|$
$$\leq\ sup_{x\ \varepsilon[0,1]}|(\int_0^x e^{-t^2}dt)\int_0^x (f(t) - g(t)\ dt)|$$
$$\leq\ sup_{x\ \varepsilon[0,1]}|\frac{1}{e}\int_0^x (f(t) - g(t)\ dt)|$$
$$\leq\ \frac{1}{e}\ sup_{x\ \varepsilon[0,1]}|f(t) - g(t)|\ =\ \frac{1}{e}||f(t) - g(t)||$$
Therefore, it is a contraction mapping and hence there exists a **UNIQUE** function that satisfies these conditions.

5. Similarly as above we see that
$$||Tx - Ty||\ \leq\ \frac{1}{e}||x - y||$$
where x,y ε $\mathcal{C}[0,1]$. Hence it is a contraction map and has a unique fixed point.

6. Similar to the previous examples, we see that
$$||Tx - Ty||\ \leq\ M||x - y||$$

where x,y ε $\mathcal{C}[0,1]$ and M = $sup_{\,0\leq x,y\leq 1}|K(x,y)|$ < 1. (As K is continuous over the unit square, it attains it's supremum and infimum). Hence, it is a contraction mapping and has a unique fixed point.

- The conditions of the fixed point theorem are intricate and strictly require α to be less than 1.
  Example:- Consider f(x)=x+e^{-x}. Clearly, we see that |$f(x)-f(y)| < |x-y|$, but it has no fixed points as $\exists$ no α s.t. |$f(x)-f(y)| < \alpha|x-y|$ $\forall$ x,y ε $\mathbb{R}^{+}$.

PROBLEM Give an example of a non-continuous linear map between normed spaces.
Answer:- Consider the map:-

$$T(f)=f^{'}(0)$$

from $\mathcal{C}^{1}[0,1]$ with the uniform norm to $\mathbb{R}$ Now, consider the sequence of functions:-$f_n(x) = \dfrac{sin(n^2x)}{n}$
Clearly, $f_n \to 0$ but

$$T(f_n)=\frac{n^2cos(n^2.0)}{n} \to n$$

which diverges. Hence, even though $f_n \to 0$, $f_n \nrightarrow 0$. The catch here is the incompleteness of domain.
Another one- Consider the map from $\ell^2 \to \mathbb{R}$ as follows

$$f(e_i)=i$$

where $e_i = (0,0,0.....,1,....)$ with the one in the i^{th} place. Clearly, we see that f is not bounded and hence not continuous.

1.6 Appendix

Mise en Abyme

Mise en abyme is a term used in Western art history to describe a formal technique of placing a copy of an image within itself, often in a way that suggests an infinitely recurring sequence. In modern day cinema, this corresponds to the act of placing a story within a story. Here is an example :-

Figure 1.1: To infinity and beyond

Now a question to the reader:- **Does the above figure has a point that is present in all the images at the same place?**
The answer is YES!
This is precisely what Banach Fixed Point theorem says.
This fixed point,which at first sight is not at all intuitive to be present, has been well known in the world of photography as **Vanishing point**.
Artists use such ideas to highlight thematic aspect of various art forms, most widely recognized in the play *Hamlet* by Shakespeare.

A widely known phenomena is when we face mirrors(in a barber's shop, hopefully!) and find ourselves gazing at our infinitely recurring images.
Here are some more:-

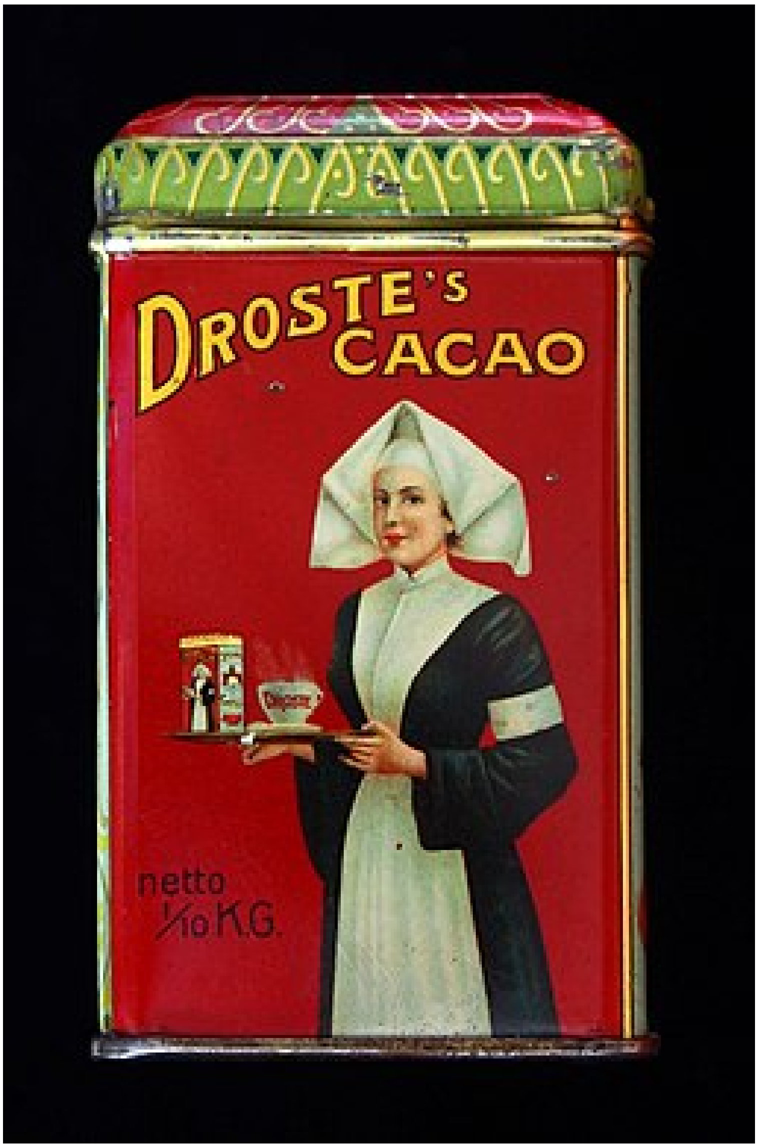

Figure 1.2: A lady holding a picture of herself

Figure 1.3: And I walk the lonely road

The last one's not for the weak hearted!!!!

Figure 1.4: You don't know it yet, but you've already crossed over to the Dark Side.

Chapter 2

Lebesgue Integrability and Hilbert spaces

2.1 Lebesgue Integrability

- Step Functions:- Characteristic function on an semi-open interval $[a_k, b_k)$ is a function that takes value 1 on $[a_k, b_k)$ and 0 elsewhere.
  We define a step function to be a finite linear combination of characteristic functions of semi open intervals[a,b)$\subset$ $\mathbb{R}$.

$$f \;=\; \Sigma_{k=1}^{\infty}\lambda_k f_k$$

- The collection of all step functions form a vector space and for two step functions, min(f, g) and max(f, g) also are step functions. Also, step functions are closed under translations.

- The integral of a step function is defined as

$$\int f \;=\; \lambda_1(b_1 - a_1) + \lambda_2(b_2 - a_2) + ... + \lambda_n(b_n - a_n)$$

- The following theorem looks quite intuitive but is not at all trivial to prove:-

  Theorem 2. *Let $[a_1, b_1)$,$[a_2, b_2)$.... be a partition of $[a, b)$, that is, the intervals $[a_1, b_1)$,$[a_2, b_2)$.... are disjoint and*

$$\bigcup_{n=1}^{\infty}[a_n, b_n) \;=\; [a, b)$$

Then

$$\sum_{n=1}^{\infty}(b_n - a_n) \;=\; b - a$$

- The significance of the above property can be seen by it's failure in the case of rationals $\mathbb{Q}$:-
  If a,b ε $\mathbb{Q}$, then $[a, b)_{\mathbb{Q}}$ =$\{x \;\varepsilon\; \mathbb{Q} : a \leq x \leq b\}$. Then we can prove that:-
  For any given $\epsilon > 0$, there are intervals $[a_n, b_n)_{\mathbb{Q}}$ such that
  $[0, 1)_{\mathbb{Q}} \subseteq \bigcup_{n=1}^{\infty}[a_n, b_n)_{\mathbb{Q}}$ and $\sum_{n=1}^{\infty}(b_n - a_n) < \epsilon$. This shows that some special properties of reals are essential here.

- Another result that gives a slight hint towards interchange of limit and integral is the following:-
  Let (f_n) be a non increasing sequence of non-negative step functions, such that $\lim_{n\to\infty} f_n \to 0$ for every xε $\mathbb{R}$. Then $\lim_{n\to\infty} \int f_n \;=\; 0$.

- **Definition**:-LEBESGUE INTEGRABLE FUNCTIONS
  A real valued function f defined on $\mathbb{R}$ is called Lebesgue Integrable if there exists a sequence of step functions (f_n) such that the following two conditions are satisfied:-

  1. $\sum_{n=1}^{\infty} \int |f_n| \;<\; \infty$
  2. $f(x) = \;\sum_{n=1}^{\infty} f_n(x)$ for every x ε $\mathbb{R}$ such that $\sum_{n=1}^{\infty} |f_n(x)| \;<\; \infty$

- The above definition, in order to be well defined, needs another imp. fact that the value of integral doesn't depend on the particular representation of f.

- Also consider the following case of the definition:-

  - The definition says that a Lebesgue integrable function will be equal to the value of the series only at points where the series is absolutely summable.We can show that the set of points where it fails to be so, is small in some sense.Let S be the set of points where this happens and let (a,b) $\subset$ S

- To show this, consider a Lebesgue integrable function $f = \sum_{n=1}^{\infty} f_n(x)$. Now consider a function g = 0 $\forall\ x\ \varepsilon\ \mathbb{R} \setminus (a,b)$, which vanishes outside (a,b).
- Now, define $h = f + g$.We can see that h = $\sum_{n=1}^{\infty} f_n(x)$.(because outside we see that g = 0, and inside (a,b) as the representation is invalid, this is vacuously true)
- Then by def^n, $\int h = \int f \implies \int g = 0$. Now, even if g happens to be the characteristic function, we will have $\int g = 0$, hence (a,b) must be small in some sense. These sets are called **null sets**.We can say that $\sum_{n=1}^{\infty} f_n(x)$ converges to f everywhere except on nulls sets. Such a convergence is called Convergence almost everywhere.

- The space of Lebesgue integrable functions is denoted by L^1 $(\mathbb{R})$. The L^1 norm is defined as
$$||f|| = \int |f|$$
.
  - This functional is not a norm! To make it a norm, we need to consider equivalence classes.
  - Two functions are called equivalent if they differ by a null function. The equivalence class of $f\ \varepsilon\ \mathbb{R}$, denoted by [f], is the set of all functions equivalent to f.
  - Now, on the space of equiv. classes, we define the L^1 norm as
$$||[f]|| = \int |f|$$
  - The space $\mathcal{L}^1(\mathbb{R})$ of equiv, classes is a Banach space with this norm.

- Also, we can prove that $L^p(\mathbb{R})$ is complete for all $1 \leq p < \infty$.

2.2 Hilbert Spaces

- Inner product space:-
An inner product space is a complex vector space. Let E be such a space.

A mapping <.,.>:$E \times E \to \mathbb{C}$ is called an inner product on this space if it satisfies the following properties:-

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
2. $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$
3. $\langle x, x\rangle \geq 0$(equality only in case of x=0)

- **Examples**:-

  1. Trivial example:- In space $\mathbb{C}^N$,it is easy to see that

  $$\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y_k}$$

  2. For l^2 space, we define inner product of sequences of complex numbers as follows:- $\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y_k}$, x = $(x_1, x_2.....)$ y =$(y_1, y_2, y_3, ...)$
  3. The space $\mathcal{C}[a, b]$ of all continuous complex valued functions on the interval [a,b] , with the inner product

  $$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}$$

  4.

- The inner product defines a norm given by

$$||x|| = \langle x, x\rangle$$

- Triangle inequality of the norm can be proved using the Schwarz inequality:-

$$|\langle x, y\rangle| \leq ||x||||y||$$

- A complete inner product space is called a Hilbert Space.

- If we have a inner product space, we can extend it to a normed space, however, the reverse might not always be true.

- For a normed space to be an inner product space,if and only if it ,satisfies the parallelogram law:-

$$||x+y||^2 \; + \; ||x-y||^2 \; = 2(||x||^2 + ||y||^2)$$

- **Examples:-**
  Consider the $\mathbb{R}^2$ with the taxicab norm. We see that it does not satisfy the parallelogram law. Hence , it cannot be an inner product space.

- Inner product spaces are the only normed spaces that satisfy the parallelogram law.

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