Notes on Spontaneous Symmetry Breaking

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1   Broad Overview of Symmetry Breaking

Definition:

When the state $|\Psi\rangle$ isn't invariant under the symmetry operator $\hat{U}$, s.t.
$$\left[\hat{H}, U\right] = 0$$
(i.e. it's a symmetry of the system), then the *__state__* is said to have spontaneously broken the symmetry.

Observation 1: For every such $|\Psi\rangle$, there exists a multitude of states $|\Phi\rangle$ s.t. they're degenerate. One can generate the set $|\Phi\rangle$ by the rule $|\Phi\rangle = \hat{U}|\Psi\rangle$. Check that $E_{|\Psi\rangle} = E_{|\Phi\rangle}$.

Order Parameter Operator: For the set of *broken* symmetry states, one can define an order parameter operator $\hat{O}$. The action of this operator is defined as:

- Each of the symmetry related states $|\Phi\rangle$ are the eigenstates of this operator with distinct non-zero eigenvalues.

- The expectation value $\hat{O}$ in a symmetric state is 0.

In general $\left[\hat{O}, \hat{H}\right] \neq 0$ (however there are *cases* where it happens to be true as well).

Observation 2: For the general case, we can see that $|\Psi\rangle$ will not be an Energy eigenstate. Moreover, it cannot be a thermal mixture of energy eigenstates. Proof: Taking $\left[\hat{O}, \hat{H}\right] \neq 0$, it can't be an eigenket since that would imply $\hat{H}$ and $\hat{O}$ commute. For the thermal mixture case, let

$$|\Psi\rangle = \sum e^{-\beta E_n} |n\rangle$$

$$\implies \hat{H}|\Psi\rangle = \frac{\partial |\Psi\rangle}{\partial \beta}$$

$$\implies \hat{O}\hat{H}\ket{\Psi} = \hat{O}\frac{\partial\ket{\Psi}}{\partial\beta} = \frac{\partial\hat{O}\ket{\Psi}}{\partial\beta}$$

$$\implies \hat{O}\hat{H}\ket{\Psi} = O\frac{\partial\ket{\Psi}}{\partial\beta}$$

$$\implies \left[\hat{H},\hat{O}\right] = 0$$

Conclusion: Clearly the symmetry broken state $\ket{\Psi}$ isn't in a thermal equilibrium! Then shouldn't these states be unstable? No. We do get to observe them in day-day life and these clearly unequivocally exist with large lifetimes(and hence are stable in the colloquial sense). How does this happen then? Ans - Due to the large singularity of thermodynamic limit.

At Thermodynamic limit($N \to \infty$, $V \to \infty$ with $\frac{N}{V} = constant$) :

- $< \left[\hat{H},\hat{O}\right] > = 0$

- SSB states become orthogonal to each other i.e. $\langle\Phi|\Psi\rangle = 0$

- SSB states become degenerate with Symmetric energy eigenstates, thus becoming eigenstates themselves. Hence they can occur in thermal equilibrium.

These states can therefore **exist** in the thermodynamic limit.
Thermodynamic limit, being qualitatively different as above from large finite N,V makes the limit itself a singular one(defined below). Ergo - Thermodynamic limit is an idealisation, only to serve as a guide and not a real description.

1.1   What happens for finite N?

Observation 3: Symmetric Hamiltonians (by symmetry here a global symmetry is implied) exhibit the following properties:

- The decompostion of $\hat{H} = \hat{H}_0 + \sum_{\vec{K}} \hat{H}_{\vec{K}}$ is possible, where the 1st part realizes $\vec{k} = 0$ part of the F.T. i.e. Centre of Mass part and the 2nd part describes internal degrees of freedom(basically a fourier decomposition into different non-zero and zero $\vec{k}$ parts).

- $\left[\hat{H}_0, \sum_{\vec{K}} \hat{H}_{\vec{K}}\right] = 0$

Question: What if the quantity $H_0 = \int \hat{H}(\vec{r_i}, \vec{p_i}) \, \Pi_{i=1}^N d^3\vec{r_i}$ diverges? Since energy is an extensive property of the system, won't it exclusively depend on the volume of the system as such (for e.g. set $\hat{H}_0 = \hat{H}_{SHO}$)?

Claim: The **global** symmetry breaking can be explained only by using $\hat{H}_0$ part of $\hat{H}$.

Example: For a solid, the exact hamiltonian looks like

$$\hat{H} = \sum_i \frac{p_i^2}{2m} + \sum_{i<j} V(\vec{r_i} - \vec{r_j})$$

Although the system has translational symmetry, when we observe daily life objects like a rock, they don't exhibit the symmetry of the hamiltonian i.e. they aren't delocalised over all space. Now for this, the collective part of the hamiltonian describes the C.O.M motion or the motion of N atoms on mass m moving in unison.
In free space, this motion corresponds to a free particle with M = mN and the lowest energy level spacing scales as $\frac{1}{N}$.

   Tower of States: These low lying energy levels of the solid makeup what's called the **tower of states**. Important to note that the states(which are eigenstates of $\hat{\vec{P}}_{total}$ and hence behave like a **delocalised wave** over the whole space) are collective excitations of the whole system and as such are non-local.

Non local in what sense? - Can't be written as $\bigotimes |\Psi\rangle_j$, for $|\Psi\rangle_j$ being single particle states(in a general basis that is). Because they're non-local, as the system size increases, they become increasingly unstable towards local interactions, hence not observed in daily life(**Why?**).

Question: Why is the GS of the true system only the eigenstate of the collective part?
(It seems plausible on energy grounds, since the collective part shall offer lowest energy, but the argument is motivated by a classical picture - basically net energy of the system $= E_{COM} + E_{system\ wrt\ COM}$, hence lowest energy contributions come from COM)

Now what happens to stable states in this analysis?
 - Clearly they aren't one of these low lying **symmetric**(i.e. wrt global symmetry of the system, for e.g. in case of a solid, states exhibiting

translational invariance) eigenstates of the collective hamiltonian.
This is from daily observation that these states are the normal SSB states.

- If not eigenstates, then these are superpositions of the low lying eigenstates (**Why?**).

- Now since low lying energy gap scales as $\dfrac{1}{N}$, for large N, they squeeze up and hence the energy uncertainty reduces for the stable SSB state.

- This makes them almost degenerate(i.e. $< E > \approx E_n$) in the large N limit and has energy close to exact GS of the hamiltonian(**Why?**).

- Overlap between SSB states now scales as $\exp\{-N\}$ and hence tunneling probability to another SSB state is suppressed.

- Why are they stable? Simply because they are local(unlike the symmetric states), can be written as $\bigotimes |\Psi\rangle_j$ and are stable against local perturbations.

Now for finite N, a symmetric H clearly cannot induce an SSB GS. One is forced to consider a **symmetry breaking perturbation** that singles out a particular SSB state out of the many degenerate,stable SSB states.
Why the Spontaneous(the 1st S) in SSB? - Large system are exceedingly sensitive to even a small perturbation. Latter suffices to single out a particular SSB state. Hence the termed "Spontaneus".

$\hat{H}_0 + \hat{H}_{sb\,perturb} =$ SSB GS of the full system!
In thermodynamic limit, the GS is an eigenstate of the order parameter operator (why? - Since here SSB states are orthogonal).
But without $\hat{H}_{sb\,perturb}$, the GS is symmetric for **any** system size. This clarifies that the thermodynamic limit is singular - adding or subtracting $\hat{H}_{sb\,perturb}$ changes the fate of GS itself, even qualitatively.

The Ergodicity Breakdown

- $\langle \Phi_{ssb} | \Psi_{ssb} \rangle \approx \exp(-N)$ - exponentially suppressed.

- Hence for all practical purposes, one can treat as if the system has only **1** SSB GS.

- All other SSB states are inaccessible with the entire dynamics taking place in the particular SSB state and it's excitations.

- Ergodicity breaks - System only lives("explores") a restricted Hilbert space. The rest of phase space isn't accessible on **experimental timescales**.

- However, if one wants to study physics of phase transitions, then clearly the system visits a different SSB GS.

- Moral - Global Thermal equilibrium isn't reached. Ex: Disordered Glasses

2 Singular Limits

Example of Singular Limit: $< section\ 2.3\ and\ 2.4\ >$

3 A Classical Example

4 Harmonic Crystal