Indian Institute of Technology Bombay

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# Introduction to Quantum Field Theory

Supervised Learning Project Report

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# ABSTRACT

Quantum Field theory(abbreviated as QFT) is one of the most successfully tested theories of the past century. Primarily initiated as an effort to reconcile the ideas of special relativity and quantum mechanics, it led to a fundamental change of viewpoint from particles to fields. We explore the formalism behind much of the existing framework in this domain. This is based on the canonical quantisation approach. Among others, the causal nature of free fields are described. We also highlight some important applications in topics such as scattering amplitudes and yukawa potential in the context of perturbation theory. Through the language of field operators, many ab-initio put in terminology in quantum mechanics for e.g. spin and exclusion principle can been shown to arise naturally in a field theory treatment of the quantum system.

# Contents

# Chapter 1

# Introduction

Quantum Field Theory(abbreviated as QFT) is needed primarily to reconcile Special Relativity and Quantum Mechanics.
An interesting consequence of this is existence of *anti-particles*. This can be reasoned as follows: In traditional quantum mechanics, uncertainty principle tells us that energy can fluctuate wildy over small intervals of time. In relativity, the dictum of "Mass-Energy" is oft quoted, stating that energy can be converted to mass. Combining these two ideas we see that wildly fluctuating energy can actually metamorphise into mass, that into new particles hitherto not present.

### 1.0.1 Motivation

Due to a large number of developments in the recent decades, quantum field theory has progressed our understanding of nature and shed light on high and low energy interactions. Despite that, quantum field theory provides for two prime predictions which are motivating to explore it as a theoretical framework in its own right. These are:(this has been referenced from D.Tong, *Lecture Notes on Quantum Field Theory*, part 1)

- The combination of quantum mechanics and special relativity implies that particle number is not conserved. Therefore particles aren't indestructible and haven't been here from early universe.
- All particles are of the same type
  Apriori, this statment is not trivial. For example a photon from a cosmic ray(coming from 8 billion light years away) and one freshly minted at LHC are exactly same in all aspects.Why aren't there errors in proton production? How can two objects, manufactured so far apart in space and time, be identical in all respects? One explanation that might be offered is that there's a sea of proton field filling the universe and when we make a proton we somehow tap into this proton field and extract a proton.

### 1.0.2 What is quantum field theory good for ?

There are numerous applications of this field, some of which are highlighted below:(This has been referenced from *QFT Lecture Notes* by Jared Kaplan and *Lectures on Quantum Field Theory*, part 1 by D.Tong)

1. **Particle Physics**

   - Standard Model
     A powerful framework for marrying Special relativity with Quantum Mechanics i.e. Standard Model.
   - Confinement
     The elementary excitations of quantum chromodynamics (QCD) are quarks. But they never appear on their own, only in groups of three (in a baryon) or with an anti-quark (in a meson). They are confined.

2. **Condensed Matter**

   - Many Body Systems
     Vibrations in solids can be visualised in terms of phonons, corresponding to the quanta of wave equation. Also, we know that in solids, energy levels available to electrons form bands. When an electron is kicked from a filled to an empty band, a *hole* is created, which enjoys the same freedom as a particle in its own right, until annihilated by an electron. This was first recognised by Dirac, who conceived the hole as a kind of "anti-electron".
   - Phase Transitions
     The behavior of systems near phase transitions, especially the universality of such systems. For example, systems of spins behave in the same way as water near its critical point – both are well-described by a simple quantum field theory.
   - Charge Fractionalization
     Although electrons have electric charge 1, under the right conditions the elementary excitations in a solid have fractional charge 1/N (where N $\varepsilon$ 2Z + 1). For example, this occurs in the fractional quantum Hall effect.

3. **Cosmology**

   - Inflation Models
     The perturbations in the energy density in the early universe, which seeded structure formation, seem to have arisen from quantum fluctuations of a quantum field that we refer to as 'the inflaton'.
     Interestingly enough, current studies of expansion of Bose-Einstein Condensates(a quantum fluid) seemed to follow the same dynamics as early universe. If this duality is worked out, we can possibly study the early universe evolution using cold-atoms. Promising indeed.
   - Emergent Space
     There are field theories in four dimensions which at strong coupling become

quantum gravity theories in ten dimensions. The strong coupling effects cause the excitations to act as if they're gravitons moving in higher dimensions. This is quite extraordinary and still poorly understood. It's called the AdS/CFT correspondence

More than that, areas like quantum electrodynamics has largely influenced the progress of fields like quantum optics, quantum information etc. and is among one of the most successfully tested theories.

## 1.1 Plan for this report

The broad idea of this report is to motivate the use of the basic formalism. Here's what this report broadly entails:-

1. *Classical Field theory*: We'll explore ideas of Lagrangian, EOM and least action principle. Noether's theorem and stress energy tensor are important ideas that will find use in later chapters.

2. *Canonical Quantization*: With the background of classical fields, we move onto Scalar Free Field theories. Taking example of Klein Gordon field we solve for the spectrum of the field Hamiltonian and recover particle states. Also, a discussion on causality pertinent to the propagators will useful in calculations of the following chapters.

3. *Interacting Fields*: Free fields are solvable but often uninteresting. We'll investigate weak coupling regime and understand the techniques for handling scattering processes. Feynman diagrams will be introduced as a computational tool and we'll explore the connection between classical yukawa potential and the corresponding field theory description.

# Chapter 2

# Classical Field Theory

A field is a quantity which is defined on all points of space and time. For example, temperature is a field when considered as a function of space and time. While classical mechanics deals with finite degrees of freedom, fields have infinite degrees of freedom, one associated with each spacetime point.
Classical field theory deals with the dynamics of a field and how evolution of field as a whole occurs. Due to an infinite domain, quantities of interest are often quoted in terms of densities. Much akin to classical mechanics, the Lagrangian for fields is a density and governs the dynamics as dictated by least action principle. The following discussion is based on *Lecture on Quantum Field Theory, Part 1*, by D.Tong.

### 2.0.1 Euler Lagrange

We state(without proof) the euler lagrange equations for fields:

$$\partial_\mu \frac{\partial L}{\partial \phi_{a,\mu}} = \frac{\partial L}{\partial \phi_a}$$

where $\phi_a$ refer to the set of fields and derivatives are taken over spacetime coordinates.

### 2.0.2 Noether's theorem and Symmetries

Every continuous symmetry of the lagrangian gives rise to a conserved current $j_\mu(x)$ such that the equations of motion imply

$$\partial_\mu j^\mu = 0$$

A conserved current implies a conserved charge Q, defined as

$$q = \int d^3x j^0$$

The existence of a current is a much stronger statement than the existence of a conserved charge because it implies that charge is conserved locally. This can be evident from the following calculation. Let

$$Q_v = \int_V d^3 \ j^0$$

Then

$$\frac{dQ_V}{dt} = \int_A -\mathbf{j}.\mathbf{dS}$$

where A = $\partial\, V$. This equation means that any charge leaving V must be accounted for by a flow of the current 3-vector **j** out of the volume. This kind of local conservation of charge holds in any *local* field theory. Locality is also intimately related to causality and is one of the reason fields are brought in to manifest causality in qft.

### 2.0.3 Stress-Energy Tensor

Implying the translational invariance of the field, we get the stress energy tensor as the follows:-

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}L$$

This arises out of invariance under 4 spacetime translations.
Also, from the lorentz invariance of the field, we can generate the angular momentum tensor

$$(J^\mu)^{ab} = x^a T^{\mu b} - x^b\, T^{\mu a}$$

which leads to 6 conserved current for $\mu,\ a,\ b\ =\ 1, 2, 3$. This satisfies the equation

$$\partial_\mu (J^\mu)^{ab} = 0$$

Such conserved currents play an important role when we look into quantum fields. Heuristically, since the associated charges are conserved classically, the corresponding quantum field operator is a constant of motion due to it's commutation with hamiltonian. This in turn provides us with extra symmetry operators that help distinguish eigenstates. With this background, we can now look at the description of a quantum field using the idea of lagrangian mechanics.

# Chapter 3

# Canonical Quantization

In the last chapter, we developed the essential preliminaries for analyzing a classical field. Now we explore the quantum picture which employs canonical quantization as a recipe to take a classical field hamiltonian to its quantum operator counterpart. Knowledge of the system is then equivalent to solving for the spectrum of the obtained hamiltonian operator. Once this is done, we can successfully extract out quantities of interest.
The material discussed here are based on Prof. David Tong's *Lectures on Quantum Field theory* and Chapter 2 of Peskin and Schroeder's book.

## 3.1 Idea

In canonical quantization, we take a classical theory and start by raising the status of the field and it's momenta to $\hat{X}$ and $\hat{P}$ variables respectively.
Analogous to standard quantum mechanics, there is a state of the system(field) governed by the Schrodinger equation:-

$$i\frac{\partial}{\partial t}\left|\Psi\right\rangle \;=\; H\left|\Psi\right\rangle$$

Here H is the field hamiltonian and $\left|\Psi\right\rangle$ is a wavefunctional, a function for every configuration of the field $\phi$.
Therefore, how do we exactly solve for $\left|\Psi\right\rangle$?

Conventionally, an opposite route has been preferred, wherein instead of solving for the schrodinger equation, we apriori start off with a solved system(this happens to be a quantum harmonic oscillator). Now we express the field in terms of these oscillators and generate the spectrum of the hamiltonian. Though counter intuitive, we present two examples to support this ansatz. First let's take a mechanical system of atoms interacting with each other via springs. A classical lagrangian for the system looks like

$$L \;=\; \sum_{1}^{n}\frac{1}{2}(m\dot{Q_r}^2 - m\omega_r^2 Q_r^2)$$

where $Q_r$'s are the normal coordinates, provided we take potential up to displacements in quadratic terms. Now we know that the each of these $Q_r$'s obey the EOM of a SHO and hence we have decoupled the dynamics into a sum of these modes. The net energy of each collective mode is the sum of energies of comprising individual normal modes.

### 3.1.1 Concrete Example: The Klein Gordon Field

The Klein Gordon field obeys the equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi \ = \ 0$$

Motivated by the classical analogy of normal modes, we do a fourier expansion of the field, hoping that we would be able to decouple the infinite degrees of freedom that appear thereof. This results in

$$\phi(\mathbf{x}, t) \ = \ \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}.\mathbf{x}} \ \phi(\mathbf{p}, t)$$

Putting this into the Equation of motion we get

$$(\frac{\partial^2}{\partial t^2} + (\mathbf{p}^2 + m^2))\phi(\mathbf{p}, t) \ = \ 0$$

which is SHO with frequency given by

$$\omega_p = \sqrt{p^2 + m^2}$$

Hence the most general solution to the KG field comes as a sum of *linear superposition of harmonic oscillators.*
Therefore to quantize this theory, we must simply quantize these harmonic oscillators.

## 3.2 Steps to Quantize A classical theory

Motivated by the above examples, we impose (without any justification for now) the following steps to quantize a classical theory:-

1. Take the classical Field theory keeping into account relativistic invariance. This should include the the field $\phi(x, t)$ and it's conjugate momenta $\pi(x, t)$.

2. This is a crucial part: *Solve for the general case of the EOM to the classical field.* For example, in case of the KG Lagrangian

   $$\mathcal{L} \ = \ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

   The EOM are

   $$(\Box \ + \ m^2)\phi(\mathbf{x}, t) \ = \ 0$$

   which yields the general solution

   $$\phi \ = \ A \ e^{i(kx - \omega t)} + \ B \ e^{-i(kx - \omega t)}$$

3. Now raise the status of field and conjugate momenta to operator valued distributions.

   $$\phi(\mathbf{x}, t) \ \rightarrow \ \hat{\Phi} \ (x)$$
   $$\pi(\mathbf{x}, t) \ \rightarrow \ \hat{\Pi} \ (x)$$

4. Now insert bosonic/fermionic(explained later) creation and annihilation operators to the field operator as a fourier series:-

$$\hat{\Phi} = \int \frac{d^3p}{(2\pi)^3}(a_p\, e^{-ipx} + a_p^\dagger\, e^{+ipx})$$

Annihilation(creation) operator goes with positive(negative) frequency solutions respectively.

5. Impose the equal time commutation relations such as

$$\left[\hat{\Phi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{3.1}$$

$$\left[\hat{\Phi}(\mathbf{x}), \hat{\Phi}(\mathbf{y})\right] = 0 \tag{3.2}$$

$$\left[\hat{\Pi}(\mathbf{x}), \hat{\Pi}(\mathbf{y})\right] = 0 \tag{3.3}$$

Now we have a desired field operator. Obtain the hamiltonian to get the spectrum.

## 3.3 The Quantum Klein Gordon Field

We expand the field in terms of the normal modes as follows:-

$$\Phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p\, e^{-i\mathbf{p}.\mathbf{x}} + a_p^\dagger\, e^{+i\mathbf{p}.\mathbf{x}})$$

$$\Pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{i\sqrt{\omega_p}}{\sqrt{2}}(a_p\, e^{-i\mathbf{p}.\mathbf{x}} + a_p^\dagger\, e^{+i\mathbf{p}.\mathbf{x}})$$

Now the equal time commutation relations translate into:

$$[a_p, a_q] = 0$$

$$\left[a_p^\dagger, a_q^\dagger\right] = 0$$

$$\left[a_p, a_q^\dagger\right] = (2\pi)^3\delta^3(\mathbf{p}\text{-}\mathbf{q})$$

Now the Lagrangian has become an operator. So, we can derive the corresponding hamiltonian also:

$$H = \frac{1}{2}\int d^3x(\pi^2 + (\nabla\phi)^2 + m^2\phi^2)$$

Putting all the operators in place, we see that the final form of the hamiltonian is

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_p[a_p a_p^\dagger + a_p^\dagger a_p]$$

which after using commutation relation can be written as

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_p[a_p^\dagger a_p + \frac{1}{2}(2\pi)^3\delta^3(0)]$$

The 2nd term in the above expression leads to divergences. This term is apparent when we look at the vacuum state $|0\rangle$, where a$|0\rangle = 0$.
This shows that the vacuum has infinite energy(due to infinite volume) and also that we are considering higher frequency modes, which leads to unbounded energies.
Hence we need to cutoff the integral at some higher frequency and consider only a small region of interest to cure these problems.

We redefine H as

$$H \ = \ \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^\dagger a_p$$

This can be done since we don't measure the actual energy, what we measure are energy differences.The difference between this hamiltonian and the previous one is just an ordering ambiguity. Had the classical hamitonian been

$$H \ = \ \frac{1}{2}(\omega q + ip)(\omega q - ip)$$

then we could've reached the above hamiltonian.
So we resolve this issue by using normal ordering.

**Definition 3.3.1.** A normal ordered string of operators $\phi(x_1), \phi(x_2), ...$ is given as : $\phi(x_1), \phi(x_2), ...$ : where we put all the $a^\dagger$'s to the left and $a$'s to the right.

Under normal ordering the new hamiltonian becomes

$$: H := \int \frac{d^3p}{(2\pi)^3}\omega_p a_p^\dagger a_p$$

## 3.4 Casimir Effect

As per the above section, we neglect the vacuum energy with the justification that we can only measure energy differences.However, there is atleast one case where the vacuum energy itself changes.
Consider a quantum field in the free space. We'll make the $x^1$ direction periodic with size L and impose the periodic boundary conditions such that

$$\phi(x + Ln) \ = \ \phi(x), \ where; n = (1, 0, 0)$$

This shall regulate the infrared divergences, though we'll need to calculate things per unit area.
We insert two reflecting plates, separated by a distance d<<L in the $x^1$ direction. The plates are such that they impose $\phi(x) = 0$ at the position of the plates.These plates impose boundary conditions that quantize the momenta inside as

$$\mathbf{p} \ = \ (\frac{n\pi}{d}, p_y, p_z) \ \ n \ \varepsilon \ Z$$

Setting m=0 we have the ground state energy between the plates is

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \sqrt{p_y^2 + p_z^2 + \frac{(n\pi)^2}{d^2}}$$

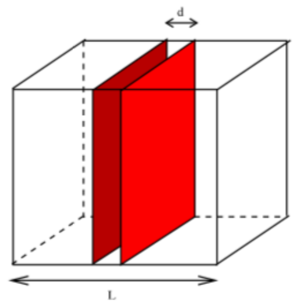


Figure 3.1: Two plates in free Vacuum

while the energy outside the plates is E(L - d). The total energy is therefore is E = E(d) + E(L-d).
Since the momenta inside the plates are quantized, we can expect a radiation pressure of lesser magnitude inside the plates as compared to the pair produced particles outside the plates. This would result in the inward pull of the plate towards each other, thus advocating the fact that *vacuum isn't empty.*

### 3.4.1 Calculation of the Casimir Force

The real world effect occurs due to the quantization of the electromagnetic field. But E(d)(energy) is infinite and presents a problem. We'll circumvent this problem by cutting off the integral at some momenta.
Physically one cannot expect plates to reflect arbitariyl high momenta. Mathematically, we want to find a way to neglect modes of momentum p >> $a^{-1}$ for some distance scale a << d, known as the ultra-violet (UV) cut-off. We change the integral as

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \sqrt{p_y^2 + p_z^2 + \frac{(n\pi)^2}{d^2}} \; e^{-a\sqrt{p_y^2+p_z^2+\frac{(n\pi)}{d^2}}}$$

This agrees with the normal integral at $lim_{a\to 0}$ sense. Hence we'll put a=0 at the end to regain the answer.
We can simplify the calculation in 1+1 d, which yields

$$E_{1+1d} \longrightarrow \frac{\pi}{2d} \sum_{n=0}^{\infty} n \; e^{-\frac{an\pi}{d}}$$

which can be evaluated by realising the sum as derivative of the an exp. series and the result finally comes as

$$E_{1+1d} \longrightarrow \frac{d}{2\pi a^2} - \frac{\pi}{24d} + ....$$

$$E \;=\; E(d) \;+\; E(L-d)$$

$$E \;=\; constant \;-\frac{\pi}{24}(\frac{1}{d} + \frac{1}{L-d}) + \; \mathcal{O}(a^2)$$

Hence the force goes by

$$F = -\frac{\partial E}{\partial a} = \frac{\pi}{24d^2}$$

which is independent of a i.e. the cutoff. Hence our ansatz was consistent and leads to a physical result - as the plates come closer, force increases and hence they attract even more.

## 3.5 Particles and Fock Space

We can now turn to the excitations of the field. They are obtained by the operation:

$$|p\rangle = a_p^\dagger |0\rangle$$

It's easy to verify that

$$H\ |p\rangle = \omega_p\ |p\rangle$$

Hence this is identified as the "single particle" state and similarly we can create multi-particle state. The set of these n particle states($n \geq 1$) is called the Fock Space.
The completeness relation for these states can be summarised as follows:

$$\boxed{1 = |0\rangle\langle 0| + |p\rangle\langle p| + |p,q\rangle\langle p,q| + .....} \tag{3.4}$$

Also, since $a_p^\dagger$ and $a_q^\dagger$ commute, we get bose statistics from this as this implies that

$$|p,q\rangle = |q,p\rangle$$

## 3.6 Complex Scalar Field

Consider a complex scalar field $\Psi(x)$ with Lagrangian

$$L = \partial_\mu \psi \partial^\mu \psi^* - M^2 \psi \psi *$$

These can be thought of as 2 uncoupled KG fields $\phi_1$ and $\phi_2$, when written as $\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.
Important Observation to make is the fact that L is invaraint under O(1) group action i.e. $\psi \longrightarrow \psi * e^{i\alpha}$, which leads to a conserved charge given by

$$Q = \int d^3x(\dot{\psi}^*\ \psi - \psi *\ \dot{\psi})$$

Classically, this is conserved as the system evolves. In QFT, this is an operator that should commute with H and hence is a symmetry operator when acts on the field $\psi$ through the action

$$\boxed{\psi \Longrightarrow e^{-i\alpha\hat{Q}}\ \hat{\psi}} \tag{3.5}$$

Now, we expand the field $\psi$ in terms of two sets of bosonic operators, $b_p$ and $c_p$.

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}}(b_p\ e^{i\mathbf{p}.\mathbf{x}} + c_p^\dagger\ e^{-i\mathbf{p}.\mathbf{x}})$$

$$\psi^\dagger(\mathbf{x}) \;=\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(c_p\, e^{i\mathbf{p.x}} + b_p^\dagger\, e^{-i\mathbf{p.x}})$$

and the corresponding conjugate momenta similarly. The commutation relations come out to be

$$\left[b_p, b_q^\dagger\right] \;=\; (2\pi)^3\, \delta^3(\mathbf{p\text{-}q})$$
$$\left[c_p, c_q^\dagger\right] \;=\; (2\pi)^3\, \delta^3(\mathbf{p\text{-}q})$$

with all the $b_p$'s commuting with $c_q^\dagger$ $\forall$ $p, q$. The hamiltonian looks like the sum of two KG fields and hence can be written as

$$H \;=\; \int \frac{d^3p}{(2\pi)^3}\omega_p\, (b_p^\dagger b_p \;+\; c_p^\dagger c_p)$$

And surpisingly the conserved operator $\hat{Q}$ takes the form

$$Q \;=\; \int \frac{d^3p}{(2\pi)^3}\, (c_p^\dagger c_p \;-\; b_p^\dagger b_p)$$

Hence in this theory, there are two types of particles: Particles(handled by $b_p$) and their corresponding antiparticles (handles by $c_p$). And Q counts their difference in quanta, with the evolution preserving the value of $\hat{Q}$.

# 3.7 Causality in Free field theories

## 3.7.1 Causality Violation in Quantum Mechanics

Consider the probability for a normal QM particle to propagate from $x_0$ to $\mathbf{x}$

$$U(t) \;=\; \langle x|e^{-iHt}|x_0\rangle$$

In non-rel qm, $H \;=\; \dfrac{p^2}{2m}$. Evaluating we have

$$U(t) \;=\; (\frac{m}{2\pi it})^{\frac{3}{2}}\, e^{\dfrac{im(x-x_0)^2}{2t}}$$

This expression is non-zero for arbitrary x and t, hence even non-zero for evolution of particles to states separated by space-like intervals, hence signifying the idea that quantum particles can travel faster than light.
Can we solve this problem by using a relativistic particle description?
We set $H \;=\; \sqrt{p^2+m^2}$(the relativistic dispersion relation). Calculating again we have

$$U(t) \;=\; \langle x|e^{-it\sqrt{p^2+m^2}}|x\rangle$$

Which on an approximation for $x^2 \;>>\; t^2$ leads to

$$U(t) \;\longrightarrow e^{-m\sqrt{x^2-t^2}}$$

This is again non - vanishing for a particle travelling across spacelike separated intervals. Hence the need for a field viewpoint is necessary, even from a causal perspective.

### 3.7.2 Does the Field viewpoint help?

For now, we'll focus on the problem of causality for Klein-Gordon theory. This provides some insight as follows:
Consider

$$\phi(x)\,|0\rangle \;=\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{2 * E_p}\; e^{-i\mathbf{p}.\mathbf{x}}\,|\mathbf{p}\rangle$$

which begs the comparison

$$|x\rangle \;=\; \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}.\mathbf{x}}\,|\mathbf{p}\rangle$$

just differing by a $E_p$ appearing below. Therefore, what we can think of the action of the operator $\phi(\mathbf{x})$ on $|0\rangle$ as creating a particle localised at position $\mathbf{x}$(this is not physical, since localising a particle beyond it's compton wavelength isn't possible).
In doing the comparison, we neglect the extra factor of $E_p$, making the analysis just for particles of small speeds.
Now we consider the heisenberg picture. In this case, the creation and annihilation operators evolve according to the relation:

$$a_p(t) \;=\; e^{iHt}\,a_p(0)\,e^{-iHt} \;=\; a_p\,e^{-iE_pt}$$

Here we demand that states be stationary and operators evolve wrt time. Though this may seem repetitive, it'll come in handy in the very next line.
Suppose we prepare a particle at a spacetime point y. What's the probability for it to travel to spacetime point x?

$$\begin{aligned} P &= \langle x|y\rangle \\ &= \langle 0|\phi(x)\phi(y)|0\rangle \end{aligned} \tag{3.6}$$

Now this "propagator" is evaluated to be complex number as evident from the free field calculations.

$$D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\;E_p}\; e^{-ip.(x-y)}$$

Since $D(x-y)$ called the( *Wightman Propagator*) depends only the separation x-y, we analyze by making cases on the separation nature:

1. For x = y
   $D(x-y)$ evaluates trivially to 1.

2. For (x-y) being time-like
   We perform a Lorentz transformation to a frame with in which $x-y \;=\; (x^0 - y^0,\; 0) \;=\; (t,\; 0)$. Now

$$D(x-y) \;=\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\;E_p}e^{-ip^0t}$$

Transforming into radial coordinates we have the integral:

$$\begin{aligned} D(x-y) &= \frac{1}{2\pi^2}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}} \\ &= \frac{1}{4\pi^2}\int_0^\infty dE\, \sqrt{E^2-m^2}\, e^{-iEt} \\ &\sim_{t\to\infty} e^{-imt} \end{aligned} \tag{3.7}$$

Therefore it doesn't vanish along time-like paths. A particle can propagate along time-like paths and doesn't violate causality.

3. For (x-y) being space-like,
Since (x - y) is spacelike separated, we make the Lorentz transformation to the frame where (x-y) = (0 , **x - y**) = (0, **r**). Now

$$\begin{aligned} D(x-y) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\,E_p} e^{-i\mathbf{p.r}} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\,\sqrt{p^2+m^2}} e^{-i\mathbf{p.r}} \\ &= \frac{-i}{2(2\pi)^2 r}\int_\infty^\infty \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}; \end{aligned} \tag{3.8}$$

Now we take the lim $_{r\to\infty}$ and use the the method of stationary phase to approximate the integral as

$$D(x-y) \;=\; e^{-mr}$$

And hence is *non-vanishing*!

Hence as per the above calculation, it appears that quantum particles can indeed travel faster than light.
As of now, this theory stands in direct contradiction to special relativity and presents a difficulty regarding the internal consistency of the framework.
We can escape this difficulty by circumventing the problem as follows: It's not correct to demand that particles shouldn't propagate across spacelike paths but rather focus on measurements of observables. After all only they can be measured and we would like to assert that information cannot be propagated across spacelike paths.
Hence we demand that *commutators of observables* across space-like paths should vanish.

**Why should we consider Commutators only?**

Consider two observables A and B with $[A, B] \neq 0$.
Then if me and my friend are separated by space like intervals, we can keep on making measurements of either A or B on a system. If I have only access to A, then I could (in principle) know whether my friend measured B by just noting the difference A values(since they don't commute, if my friend decides to measure B, then A's value shall change). This would mean instant flow of information across space like intervals, which would violate causality.

**What does $\langle 0|\phi^\dagger(x)\phi(y)|0\rangle$ mean then?**

The correlation between two spacelike points x,y is

$$\phi^\dagger(x)\phi(y) \neq 0$$

We infer that fields at two points(even spacelike) are correlated. These correlations were present per se and we didn't induce them by measuring either A or B. Such correlations arise due to the virtue of the dynamics of the field itself.
One of the simplest measurables is $\phi(x)$ itself and hence we focus on it's commutator relationship. It will ensure that any local function of $\phi($ derivatives e.g. momenta, polynomials etc.) are causal too. Checking for the commutator between fields at two spacetime points we have

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{-ip(x-y)} \; - \; e^{ip(x-y)}) \\ &= D(x-y) \; - \; D(y-x) \end{aligned} \tag{3.9}$$

For (x-y) separated space-like, we have (x-y) related to (y-x) by a lorentz transformation and hence the above two terms cancel. Hence local functions of fields respect causality.

### 3.7.3 Is the theory Causal now?

*No.* This does not yet prove that the Klein Gordon field theory is causal.
This just elucidates that for the two particular local function of observables $\phi(x)$ and $\phi(y)$, we have a causal relationship. We need to do this for every possible observable and only then can the theory be causal. Doing this for every theory is mathematically challenging, however for the KG-field it has been done (more generally, KG belongs to a class of *Algebraic Quantum Field theories* for which concrete mathematical framework exists, however it is beyond the scope of this discussion).
For the KG field, this problem can be dealt in 2 ways:-

1. We need to get a full unitary representation of the Poincare Group for this particular theory. This would guarantee that the theory is indeed causal.

2. We can simply declare that the observables that matter(i.e. can be experimentally obtained) are only local functions of field operators and none other. This is based on the idea of considering the S-matrix as the observables.

Both these methods can assure for the causality of scalar field theory. We end this chapter by briefly summarising the fact that free field theories can be solved exactly and degrees of freedom can be decoupled using fourier expansion. However, as we saw this leads to non-interacting modes, with $[H, N] = 0$ conserving the number of quanta. But to be of any assertable importance, we would like to introduce interactions between particles and describe them in a field theory setting. This treatment forms the basis for the next chapter.

# Chapter 4

# Interacting Field Theory

In the previous chapter we saw one of the simplest quantum field theories - scalar field theory. The free field behaviour of the same was evident in the fact that the field could be written as a sum over modes and the hamiltonian just turned out to be the sum of individual harmonic oscillator hamiltonians. The energies of the quanta just add up - they are non-interacting.

How can then we make the modes interact with each other? This can be traced to the fact that the harmonic oscillator(SHO) hamiltonian retained terms upto quadratic factor in $\phi$, which linearized the equations of motions. To make the modes interact, we retain terms of the order $\phi^3$, so that the resulting non-linear equation has solutions that cannot be obtained by superposing two independent solutions.

We then relate this to a theory in which numbers of different mode quanta are not conserved. It includes processes in which the no. of quanta that emerge finally as free particles is different from the no. that originally interacted. Archetypal of such a theory are the processes of particle scattering such as $e^+e^-$ scattering, $q\bar{q}$ annihilation etc.

Since these cases involve significant complications due to multi component nature of the fields, we employ a perturbative analysis of the field dynamics by exploring the ideas of Dirac picture and Feynman rules. The following discussion can also be referred to in Prof.David Tong's *Lectures on Quantum Field theory* and Chapter 4 of Peskin and Schroeder's book.

## 4.1 The Interaction Picture

The interaction picture, a contemporary to the Heisenberg and Schrodinger Picture, describes a useful way to look at time evolution of states in presences of a perturbation. Let the us add a perturbation $H_{int}$ to the unperturbed $H_0$, so the net hamiltonian is

$$H_{net} = H_0 + H_{int} \tag{4.1}$$

Now define

$$|\Psi(t)\rangle_I = e^{iH_0t} |\Psi\rangle_S \tag{4.2}$$

Calculation of the evolution equation then becomes

$$
\begin{aligned}
i\hbar\frac{\partial\left|\Psi(t)\right\rangle_I}{\partial t} &= -H_0(e^{iH_0t}\left|\Psi\right\rangle_S) \;+\; e^{iH_0t}[H_0+H_{int}]\left|\Psi\right\rangle_S \\
&= e^{iH_0t}H_{int}e^{-iH_0t}e^{iH_0t}\left|\Psi\right\rangle_S \\
&= H_{int}(t)\left|\Psi\right\rangle_I
\end{aligned} \tag{4.3}
$$

Hence in the interaction picture(or the Dirac Picture), evolution happens as follows:

$$O(t) = e^{iH_0t}O(t)e^{-iH_0t} \tag{4.4}$$

$$i\hbar\frac{\partial\left|\Psi(t)\right\rangle_I}{\partial t} = H_{int}(t)\left|\Psi\right\rangle_I \tag{4.5}$$

## 4.2 Dyson's Formula

Since the evolution equation is tougher to solve, we would like to search for a unitary operator $U(t,t_0)$ which evolves the state according to

$$\left|\Psi(t)\right\rangle_I(t) = U(t,t_0)\left|\Psi(t)\right\rangle_I(t_0) \tag{4.6}$$

Plugging this into the evolution equation we have

$$i\frac{\partial U}{\partial t} = H_{int}(t)U \tag{4.7}$$

Now, we attempt a formal solution of the form

$$U(t) = exp(-\frac{i}{\hbar}\int_{t_0}^{t}H_{int}(t')dt') \tag{4.8}$$

However, due to non-commutativity of $H_{int}(t_1)$ and $H_{int}(t_2)$, we run into a problem. Precisely, when we expand the solution at 2nd degree, we get two terms of the form

$$\frac{\partial U}{\partial t} = ... - \frac{1}{2}H_{int}(t)(\int_{t_0}^{t}H_{int}(t')dt') - \frac{1}{2}(\int_{t_0}^{t}H_{int}(t')dt')H_{int}(t) \tag{4.9}$$

Now the 2nd term above doesn't equal to the first one and hence invalidates the ansatz. To solve this problem, we explicitly time order the RHS of the equation above, in lieu of which, the 2nd term equals the first.

$$U(t) = T\ exp(-\frac{i}{\hbar}\int_{t_0}^{t}H_{int}(t')dt') \tag{4.10}$$

where T is the time ordering operator, which orders the latest operator to the left and subsequent timed operators to the right. This is called Dyson's formula. However, more relevant is the series form of $U(t)$, given by

$$U(t) = 1 - i(\int_{t_0}^{t}H_{int}(t')dt') - \frac{1}{2}T((\int_{t_0}^{t}H_{int}(t')dt')(\int_{t_0}^{t}H_{int}(t'')dt'')) + ...... \tag{4.11}$$

Simplifying the 2nd term, we have

$$T((\int_{t_0}^{t} H_{int}(t')dt')(\int_{t_0}^{t} H_{int}(t'')dt'')) = \int_{t_0}^{t} dt' H_{int}(t') \int_{t_0}^{t'} H_{int}(t'')dt'' + \int_{t_0}^{t} dt'' H_{int}(t'') \int_{t_0}^{t''} H_{int}(t')dt' \quad (4.12)$$

$$= 2\int_{t_0}^{t} dt' H_{int}(t') \int_{t_0}^{t'} H_{int}(t'')dt'' \quad (4.13)$$

since (4.10) is symmetric in $t'$ and $t''$. Hence, (4.9) becomes

$$U(t) = 1 - i(\int_{t_0}^{t} H_{int}(t')dt') + (-i)^2 \int_{t_0}^{t} dt' H_{int}(t') \int_{t_0}^{t'} H_{int}(t'')dt'' + ...... \quad (4.14)$$

## 4.3 Weakly Coupled Theories

In this report, focus has been on weakly coupled theories. These theories can be solved by considering these as a "small" perturbation to a free field. Examples include:

1. **$\phi^4$ theory**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad (4.15)$$

for $\lambda << 1$. On expanding $\phi$ as a normal KG operator(which is a perturbative approximation), we see that

$$V \sim a_p^\dagger a_p^\dagger a_p^\dagger a_q^\dagger + a_p^\dagger a_p^\dagger a_p^\dagger a_q \quad (4.16)$$

This renders $[H, N] \neq 0$. Hence in this theory, particle no. isn't conserved.

2. **Scalar Yukawa Theory**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \partial_\mu\psi\partial^\mu\psi^* - M^2\psi\psi * -g\psi^*\psi\phi \quad (4.17)$$

This theory entails the coupling between a complex and a real scalar field, via the $g\psi^*\psi\phi$ term( where g $\ll$M,m). Since $\mathcal{L}$ is symmetric under U(1) group operation on $\Psi$, we have $[H, Q] = 0$. The interaction hamiltonian has the form

$$H_{int} = \int d^3x\ \psi^*\psi\phi \quad (4.18)$$

Since $[H, N] \neq 0$, this theory allows for particles to morph into each other under the perturbation. This is because given an initial meson state( the $\phi$ particles), using dyson series we can see that

$$\begin{aligned} H_{int}(t) &= e^{iH_0t}(\psi^*\psi\phi)e^{-iH_0t} \\ &= \psi^*\psi\phi \end{aligned} \quad (4.19)$$

Now $H_{int}(t)$ contains terms like $b_p^\dagger c_p^\dagger a_q$, which destroys a meson and in turn creates a nucleon ($b_p$, $\psi$ particles) and anti-nucleon($c_p$, $\bar{\psi}$ particles). When expanded up to first order using Dyson series, we see that this would lead to non-vanishing amplitudes for $\phi \to \psi\bar{\psi}$ scattering (Meson - Nucleon Scattering). On a second order basis, this leads to processes like $\psi\bar{\psi} \to \phi \to \psi\bar{\psi}$ scattering(Nucleon Scattering).

## 4.4 Scattering Amplitudes

Scattering refers to processes in which a stream of particles are directed to a target. They scatter and by studying the outflux of particles through it as function of energy of particles and location of detector, inferences regarding the properties of the target can obtained. Specifically, we evaluate scattering amplitudes and cross sections which give an idea about the structure of the target( for e.g. X-Ray Diffraction).
As per the background developed till now, we'll try to explore scattering amplitudes in Scalar Yukawa theory as an example. In doing so we'll develop important computational tools like Feynman Diagrams.
To do scattering, we make one assumption for the scope of this discussion:

Initial and Final States are eigenstates of the free hamiltonian.

This assumption relies on the fact that at the initial setup, particles are free of interaction (at $t \to -\infty$). After the interaction, particles evolve independent of each other and hence the final $|f\rangle$ state is an eigenstate of the free hamiltonian.
We are interested in calculating the amplitude for the said process given by

$$A = \lim_{t+,-\to\infty} \langle f|U(t_+, t_-)|i\rangle \equiv \langle f|S|i\rangle \tag{4.20}$$

where S is aptly named the *S-matrix* (S for scattering) .
The eigenstates assumption has two major shortcomings outlined as follows

1. This cannot handle cases of bound states. For example the case of a proton and electron collision that leads to Hydrogen atom formation will not be plausible here.
2. A single particle is never generally alone in field theory. This is true even classically, since a charged particle sources an electromagnetic field and hence has photons surrounding it (the quanta of electromagnetic field).

## 4.5 Wick's Theorem

For calculation of S-matrix elements, we require computation of $\langle f|T(H_I(x_1)H_I(x_2)...H_I(x_n))|i\rangle$. However, all the $H_I's$ contain creation and annihilation operators and for ease of calculation, we would like to convert the time ordered product into a normal ordered one. Wick's theorem helps in precisely achieving the described goal.
To motivate this, we consider the time ordering of 2 scalar fields. Consider

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}a_p\; e^{-ip.x}$$
$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}a_p^\dagger\; e^{ip.x}$$

Now wlog we assume $x^0 \geq y^0$ and compute $T\phi(x)\phi(y)$.

$$\begin{aligned}T\phi(x)\phi(y) &= \phi(x)\phi(y)\\ &= (\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y))\\ &= \phi^+(x)\phi^+(y)+\phi^-(x)\phi^+(y)+\phi^-(x)\phi^-(y)+\phi^-(y)\phi^+(x)+\left[\phi^+(x),\phi^-(y)\right]\\ &= :\phi(x)\phi(y):+D_F(x-y)\end{aligned} \tag{4.21}$$

where $D_F(x-y)$ is the Wightman propagator discussed in the last chapter.
For the case $y^0 \geq x^0$, we will have

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : +D_F(y-x)$$

Hence, we see that

$$T(\phi(x)\phi(y)) =: \phi(x)\phi(y) : +\Delta_F(y-x) \tag{4.22}$$

where $\Delta_F(y-x)$ is the Feynman propagator(defined as $\langle 0|T\phi^\dagger(x)\phi(y)|0\rangle$).
Motivated by the above example, we define contraction of a pair of fields in a string of operators as the operations

$$\begin{aligned}\overbracket{\phi(x)\phi(y)} &= \Delta_F(x-y)\\ \overbracket{\psi(x)\psi^\dagger(y)} &= \Delta_F(x-y)\\ \overbracket{\psi(x)\psi\dagger(y)} &= 0\\ \overbracket{\psi^\dagger(x)\psi^\dagger(y)} &= 0\end{aligned}$$

where $\psi$ is a complex field. Now we state *Wick's Theorem* without proof.
**Wick's Theorem**:

$$T(\phi(x_1)\phi(x_2)..\phi(x_n)) = :\phi(x_1)\phi(x_2)..\phi(x_n): + :\textit{all possible pair}(s)\textit{ contractions}:$$

To clarify the implication of the last line, we look at an example. For $n=4$, we have

$$\begin{aligned}T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =&: \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) : + \overbracket{\phi(x_1)\phi(x_2)} : \phi(x_3)\phi(x_4) :\\ &+ \overbracket{\phi(x_1)\phi(x_3)} : \phi(x_2)\phi(x_4) : + 4\textit{ similar terms}\\ &+ \overbracket{\phi(x_1)\phi(x_2)}\overbracket{\phi(x_3)\phi(x_4)} + \overbracket{\phi(x_1)\phi(x_3)}\overbracket{\phi(x_2)\phi(x_4)}\\ &+ \overbracket{\phi(x_1)\phi(x_4)}\overbracket{\phi(x_2)\phi(x_3)}\end{aligned}$$

Though we won't cover the proof, the basic idea is to prove by induction for the cases of $n \geq 3$ (since n=2 case is true by definition).

## 4.6 Application of Wick's theorem: Nucleon Scattering

Let us consider a scattering process where we have two nucleons interacting via the Yukawa potential and the final result are two different nucleons, ones with different energies. For brevity, we'll call such a process as $\psi\psi \to \psi\psi$ scattering.
The initial and final states are

$$|i\rangle = \sqrt{2E_{p1}}\sqrt{2E_{p2}}\, b^\dagger_{p1} b^\dagger_{p2}\,|0\rangle \tag{4.23}$$
$$|f\rangle = \sqrt{2E_{p1'}}\sqrt{2E_{p2'}} b^\dagger_{p1'}\, b^\dagger_{p2'}\,|0\rangle \tag{4.24}$$

Now we calculate the S-matrix elements i.e. $\langle f|S-1|i\rangle$ (the -1 comes on account of the fact that we aren't interested in the process where no scattering occurs).
At first order, the amplitude is

$$A_1 = -ig\int \langle f|H_{int}|i\rangle\, dt \tag{4.25}$$
$$= -ig\int d^4x\,\langle f|\psi^*\psi\phi|i\rangle \tag{4.26}$$
$$\sim \langle f|F(c_p, b_p, c_p^\dagger, b_p^\dagger)(a_p + a_p^\dagger)|i\rangle \tag{4.27}$$
$$= 0 \;\; (\because\; \langle 0|a_p + a_p^\dagger|0\rangle = 0) \tag{4.28}$$

The first order amplitude being zero indicates the fact that nucleon scattering is a second order phenomena. Hence we proceed to 2nd order of the coupling constant g as follows

$$A_2 = \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2\; T(\psi^*(x_1)\psi(x_1)\phi(x_1)\psi^*(x_2)\psi(x_2)\phi(x_2)) \tag{4.29}$$

Evaluating the above integral by cascading time integrals over a simplex is quite a long and tedious process. However, Wick's theorem helps to avoid this problem and instead retains the form of the integral over 4 volume.
To have any non-zero contribution, we need to contract both the $\phi$ fields, since any mesonic operator lying around will make the amplitude zero(due to it's commutation with $b_p$ and $c_p$). Hence the only non-vanishing contributions comes from the term

$$:\psi^*(x_1)\psi(x_1)\psi^*(x_2)\psi(x_2):\,\phi(x_1)\phi(x_2) \tag{4.30}$$

Calculating the normal ordered part we get

$$\langle p_1', p_2'|:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):|p_1, p_2\rangle \tag{4.31}$$
$$= \langle p_1', p_2'|\psi^\dagger(x_1)\psi^\dagger(x_2)\psi(x_1)\psi(x_2)|p_1, p_2\rangle \tag{4.32}$$
$$= \langle p_1', p_2'|\psi^\dagger(x_1)\,|0\rangle\langle 0|\,\psi^\dagger(x_2)\psi(x_1)\psi(x_2)|p_1, p_2\rangle \tag{4.33}$$
$$= (e^{ip_1'x_1+ip_2'x_2} + e^{ip_1'x_2+ip_2'x_1})(e^{-ip_1x_1-ip_2x_2} + e^{-ip_1x_2-ip_2x_1}) \tag{4.34}$$

In the second line, we've used the fact that operators of interest are just $b_p$. For third line, the action of $\psi$ on particle eigenkets is as follows

$$\begin{aligned}\langle n|\psi|p\rangle &= \langle 0|b_n\psi b_p^\dagger|0\rangle &(4.35)\\ &= \langle 0|b_n\int d^3p\; b_\alpha e^{-i\alpha x}b_p^\dagger|0\rangle &(4.36)\\ &\sim \langle 0|b_n e^{-ipx}|0\rangle &(4.37)\\ &= 0,\; if\; n\neq 0. &(4.38)\end{aligned}$$

Hence we insert the $|0\rangle\langle 0|$ term in the middle. Similarly for the case of $\langle n|\psi(x)\psi(y)|i\rangle$. Putting in the factors, we have the net contribution to the amplitude at order $g^2$ as

$$A_2 = \frac{(-ig)^2}{2}\int d^4x_1 d^4x_2[e^{i\cdots}+e^{i\cdots}+(x_1\leftrightarrow x_2)]\int\frac{d^4k}{(2\pi)^4}\frac{ie^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon} \tag{4.39}$$

where $x_1x_2$ means the that terms are to be repeated with an interchange operation between $x_1$ and $x_2$. We integrate over $x_1, x_2$ to get delta functions and this simplifies to

$$A_2 = (-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i(2\pi)^8}{k^2-m^2+i\epsilon}[\delta^4(p_1'-p_1+k)\delta^4(p_2'-p_2-k)+\delta^4(p_2'-p_1+k)\;\delta^4(p_1'-p_2+k)] \tag{4.40}$$

which can be evaluated finally to

$$A_2 = i(-ig)^2[\frac{1}{(p_1-p_1')^2-m^2+i\epsilon}+\frac{1}{(p_1-p_2')^2-m^2+i\epsilon}]\delta^4(p_1+p_2-p_1'-p_2') \tag{4.41}$$

Now we can drop the $\epsilon$ since the denominator is never zero. This can be established by changing the reference frame to the centre of mass reference frame where $\mathbf{p_1} = -\mathbf{p_2}$. This and energy conservation imply that $\mathbf{k} = (0,\ \mathbf{p} - \mathbf{p}')$, so $k^2 < 0$. Therefore, the net amplitude evaluates to

$$A_2 = i(-ig)^2[\frac{1}{(p_1-p_1')^2-m^2}+\frac{1}{(p_1-p_2')^2-m^2}](2\pi)^4\delta^4(p_1+p_2-p_1'-p_2') \tag{4.42}$$

This calculation can be easily done with the help of Feynman diagrams as highlighted in the next section. Also, it provides for a more intuitive explanation of the scattering process in terms of particle-antiparticle exchange.

## 4.7 Feynman Diagrams

Feynman diagrams are a computational tool that help in calculation of the above said amplitudes( to various orders of coupling) by providing a one-one correspondence between terms of the Dyson series and certain diagrammatic figures with numbers( or precisely, integrals) associated with them. This helps as different schematic representations of the same process can be mapped to different terms of the series and hence can be accounted for easily.

There are certain rules that one has to adhere to while drawing a Feynman diagram. For the present discussion, we state them as follows:-

- We draw an external line for each particle in the initial $|i\rangle$ and final $|f\rangle$ state. External lines are lines that can be thought of extending to infinity and diverging from a given figure. Here we choose the convention of using dotted lines for mesons and solid ones for nucleons.

- For each external line, we assign a directed momentum $k$ to it. Further, add an arrow to solid lines to denote it's charge. Here we go with the choice of incoming arrow in the initial state for $\psi$ and outgoing for $\bar{\psi}$. For the final state, the reverse convention of outgoing arrow for the state $\psi$ is followed.

- Join the external lines together with trivalent vertices. Also allowed are joining vertices by using internal lines.

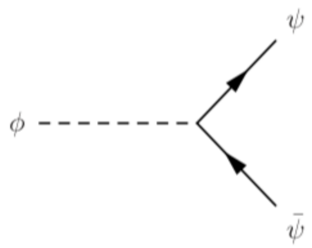


Figure 4.1: A simple Feynman Diagram

### 4.7.1 Feynman Rules

For each diagram that is presented, we need to account for it's contribution by including factors that are finally multiplied with each other. The rules for assigning factors to each diagram are as follows:-

- Add a different directed momentum k for each internal line.

- To each vertex of the diagram, we inlude a factor of

$$(-ig)(2\pi)^4\delta^4(\sum_i k_i)$$

  where $\sum_i k_i$ is the sum of all momenta flowing into the vertex. Momenta flowing outside are accounted for by including a negative sign.

- For each internal dotted line corresponding to a meson, we write down a factor of

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon}$$

  For an internal line corresponding to $\psi$ particle, we write down the same factors as above, with M replacing m.

### 4.7.2 Scattering Amplitudes with Feynman Diagrams

We consider the case of Nucleon Scattering and see the first application of Feynman Diagrams. At the order of $g^2$, the following two are the simple Feynman Diagrams:

Now we apply feynman rules to calculate the Scattering amplitudes. =

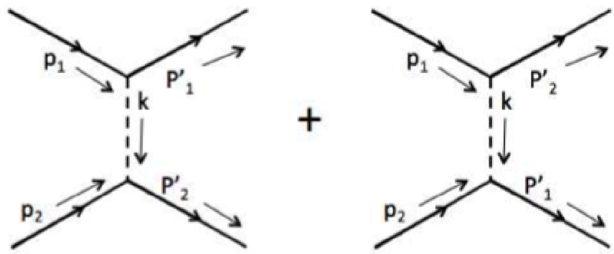


Figure 4.2: Feynman Diagrams for Nucleon Scattering

We get:

$$(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} 2\pi^8 \left[\delta^4(p_1 - p_1' - k)\delta^4(p_2 - p_2' + k)\right] \tag{4.43}$$

$$+(same) * \left[\delta^4(p_1 - p_2' - k)\delta^4(p_2 - p_1' + k)\right] \tag{4.44}$$

$$\tag{4.45}$$

$$= \; i(-ig)^2 [\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2}](2\pi)^4 \delta^4(p_1 + p_2 - p_1' - p_2')$$

This exactly matches the answer obtained by wick's theorem in Eqn. (4.42).
The above diagram leads to a more physical interpretation of the process: we can think of the second order process as happening due to exchange of a meson with $k = p - p'$. This meson, however, doesn't satisfy the dispersion relation $k^2 = m^2$ and therefore is called a *virtual particle*.
Heuristically, this particle has a lifetime that is too short to be measured. On the other hand, particles denoted by the legs of the vertices with solid lines, all denote nucleons who satisfy the condition: $k^2 = m^2$, and are said to be on-shell.

## 4.8 Yukawa Potential

The scattering amplitude computed in Eqn.(4.42) above is a bit abstract and has to be coarse grained to obtain quantitites that are measurable in experiments( for e.g. scattering cross sections and lifetimes of particles). Here we outline a construction to translate the amplitude fir a nucleon scattering into a non-relativistic potential of interaction between two particles.
To motivate the idea, let's consider a $\delta(x)$ static source for the klein gordon field. In it's presence, the field $\phi(x)$ evolves according to the static KG equation

$$-\nabla^2\phi + m^2\phi = \delta^3(x) \tag{4.46}$$

We can solve for $\phi$ as the green's function for the static KG field. In momentum space, expansion of $\phi(x)$ leads to the following:

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}.\mathbf{x}} \phi(\mathbf{k}) \tag{4.47}$$

Substituting in (4.46) and comparing the mode coefficients leads to the result

$$\phi(\mathbf{k}) = \frac{1}{\mathbf{k^2 + m^2}} \tag{4.48}$$

Therefore, plugging it back into (4.47), we have

$$\phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}.\mathbf{x}}}{k^2 + m^2} \tag{4.49}$$

which when done in radial coordinates transforms as

$$\phi(\mathbf{x}) = \frac{1}{4\pi r} e^{-mr} \tag{4.50}$$

Now an interesting query could be if this profile of $\phi(\mathbf{x})$ could be related to force between some particles. We argue for the possibility for such as follows:
In electrostatics, charges act as the source for the gauge potential $\nabla^2 A_0 = -\delta^3(x)$. The profile of $A_0$ then acts as the potential of interaction between two such test charges. Hence, it is plausible to assume that the $phi(x)$ profile obtained in (4.50) can act as potential of interaction between two of either $\phi$ particles or $\psi$ particles. As it happens to be, it can be given an interpretation in terms of the latter. We'll proceed here to work out the necessary details here, mostly by comparing the scattering amplitude to non-relativistic quantum mechanical scattering.
We first reduce the relativistic amplitude to non-relativistic amplitude by going into the centre of mass frame, where we define $p \equiv p_1 = -p_2$ and $p' \equiv p'_1 = -p'_2$. The scattering amplitude then becomes

$$A = g^2 [\frac{1}{(p - p')^2 + m^2} + \frac{1}{(p + p')^2 + m^2}] \tag{4.51}$$

For comparing this to a non-relativistic scattering amplitude, we apply born approximation. Here we make the case for scattering amplitude for a particle to scatter from momentum eigenket $\pm p$ to $\pm p'$ eigenket is given by

$$\langle p'|U(\mathbf{r})|p\rangle = -i \int U(\mathbf{r}) e^{-i(p-p').r} \tag{4.52}$$

Now we equate both the amplitudes(both differ by a factor of $(2M)^2$ and we redefine a new parameter $\lambda = g/2M$ to account for this) we have

$$\int d^3r \; U(r) \; e^{-i(p-p')r} = \frac{-\lambda^2}{(p - p')^2 + m^2} \tag{4.53}$$

Inverting this with a fourier series on the momentum space leads to

$$U(r) \ = \ -\lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{e^i p.r}{p^2 + m^2} \tag{4.54}$$

This upon evaluation in radial coordinates leads to the solution

$$U(r) = -\lambda^2 \frac{e^{-mr}}{4\pi r} \tag{4.55}$$

This is precisely the *Yukawa Potential*. The negative sign indicates the attractive nature of the force.
This provides for a new interpretation of the forces between nucleons: They can be thought to be inter mediated by the exchange of virtual particles(mesons). The force instead of being a fundamental concept, reduces to the idea of exchange of virtual particles. Similarly, coulomb forces are mediated by the exchange of virtual photons, typically dealt under the framework of QED.
Such predictions propounded for the field viewpoint and resolved other issues such as unboundedness of Dirac equation. We hereby end our discussion on Interacting field theories by summarising the main ideas of perturbation: We introduce small perturbations to free fields and work out the Dyson series that describes the evolution operator asymptotically. Using this, we can calculate scattering amplitudes and in certain cases, translate them into quantities of interest(namely scattering cross section, decay rates).

# Chapter 5

# Conclusion

So far we've covered the theory behind handling weakly coupled perturbations to a free field. This involved carefully calculating terms of the S-matrix. We saw how the whole ordeal can be avoided by the use of feynman diagrams. Finally, we explored how to turn these amplitudes into experimentally measurable relevant quantities such as scattering cross section and even into an interaction potential. Also the discussion on causality revealed how delicately the idea of commutator fits into the scheme of field theories. The programme outlined in this report can be continued to explore directions of other weakly coupled theories such as the $\phi^4$ theory. Also left unexplored is the quantization schemes describing fermions. The subtle differences between their statistics (i.e. Fermi - Dirac vs Bose Einstein) culminates in their commutation/anti-commutation relations followed by field operators. After quantizing scalar fields, dirac field offers a spinor description. This can also be extended quantization of vector fields such as the electromagnetic field. This forms the basis of Quantum electrodynamics, one of the most succesfully tested theories of the past century.

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