Notes on Electron-Phonon relaxation

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1   Introduction

We are interested in investigating the dynamics of a coupled electron-phonon system within the Schwinger Keldysh formalism, where both the electrons and phonons are allowed to evolve in time self-consistently, as opposed to the standard paradigm of one set of constituents forming a static bath for the other.

The electrons are assumed to be interacting (Coulombic?) and the phonons are taken to be noninteracting. As the initial condition of our problem, we envision the electrons being at temperature T_{el} and the phonons being at some other temperature $T_{\text{ph}} \neq T_{\text{el}}$. At $t = 0$ the interaction is switched on and the system is allowed to evolve while exchanging energy and momenta between the electrons and phonons. However, the total number of electrons doesn't change, which restricts the form of the interaction to involve only terms with equal number of fermionic creation and annihilation operators. The Hamiltonian of the entire system is thus posited to be

$$
\begin{aligned}
H &= H_{\text{el}} + H_{\text{ph}} + H_{\text{el-ph}} \\
&= \sum_k \varepsilon(k) \hat{c}_k^\dagger \hat{c}_k + 4 \text{ fermion interaction term of scale } u + \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k \\
&\quad + \lambda \sum_{k,p} \hat{c}_k^\dagger \hat{c}_{k+p} (\hat{a}_p^\dagger + \hat{a}_{-p})/\sqrt{2}
\end{aligned}
\tag{1}
$$

{eq:ham}

Integrating out the bosons would lead to a phonon mediated effective four-fermion interaction of $O(\lambda^2)$, whereas, integrating out the electrons would lead to a self-energy Σ_{ph} for the phonons.
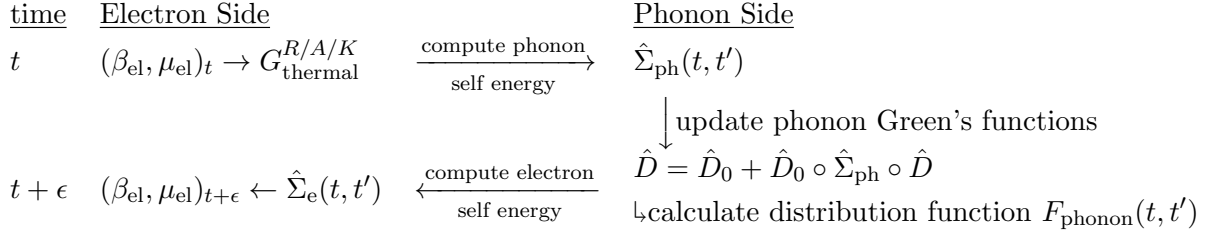
2   Approach

Instead of solving for the coupled electron-phonon problem fully, we make the assumption that at each time step t the electrons thermalize amongst themselves to some new temperature $T_{\text{el}}(t)$ and new chemical potential $\mu_{\text{el}}(t)$ owing to energy redistribution while respecting total number conservation of the electrons. This is motivated from the physical consideration that phonon-mediated electronic interactions operate on a much longer timescale than that of direct $e - e$ interactions. In terms of the parameters in Eq.(1), the above scenario is plausible in the regime $u \gg \lambda^2$ (dimensions?)

- the $e - e$ interaction, dressed or otherwise, also contributes to a finite spreading γ_{el} for the bare ε_k states through the electronic self-energy. For the purpose of this problem, we keep this γ_{el} as a book keeping variable without actually consistently calculating it, with the tacit understanding that we are interested in the limit where γ_{el} is small compared to the typical spacing δ_{el} of the electron spectrum (plausible assumption?). This set of approximations would imply that the *electron spectrum doesn't change with time evolution*, but only the population across the levels get redistributed.

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time	Electron Side		Phonon Side
t	$(\beta_{\mathrm{el}}, \mu_{\mathrm{el}})_t \to G_{\mathrm{thermal}}^{R/A/K}$	$\xrightarrow[\text{self energy}]{\text{compute phonon}}$	$\hat{\Sigma}_{\mathrm{ph}}(t,t')$

$\left\downarrow\right.$ update phonon Green's functions

$\hat{D} = \hat{D}_0 + \hat{D}_0 \circ \hat{\Sigma}_{\mathrm{ph}} \circ \hat{D}$

| $t+\epsilon$ | $(\beta_{\mathrm{el}}, \mu_{\mathrm{el}})_{t+\epsilon} \leftarrow \hat{\Sigma}_{\mathrm{e}}(t,t')$ | $\xleftarrow[\text{self energy}]{\text{compute electron}}$ | $\hookrightarrow$ calculate distribution function $F_{\mathrm{phonon}}(t,t')$ |

- Comment about electronic vertex corrections of mixed order $u\lambda^2$ vs those of $O(\lambda^4)$.

- electron quasiparticles near the Fermi energy have diverging lifetime and are not expected to thermalise. As a result $n_{\mathrm{ph}}(k,t)$ should show some nontrivial behaviour around $k = k_F$.

- When the electron side cools off, the available excitations in the system are largely clustered around the Fermi energy. In this situation, the approximation of fast thermalisation is not expected to hold.

3 Calculation

Instead of complex bosons, we choose to work with scalar ones. The action corresponding to the hamiltonian in Eq.(1) is given by

$$S_{\mathrm{el}}^{\pm} = \int \mathrm{d}t \sum_k \bar{\psi}_k^{\pm}(t) \left(\iota\partial_t - \varepsilon_k\right) \psi_k^{\pm}(t) + \sum_{q,k,k'} u(q)\bar{\psi}_{k+q}^{\pm}\bar{\psi}_{k'-q}^{\pm}\psi_{k'}^{\pm}\psi_k^{\pm}$$

$$S_{\mathrm{ph}}^{\pm} = \int \mathrm{d}t \sum_k \phi_k^{\pm}(t) \left(-\frac{1}{2}\partial_t^2 - \frac{1}{2}\omega_k^2\right) \phi_k^{\pm}(t) \tag{2}$$

$$S_{\mathrm{el\text{-}ph}}^{\pm} = \lambda \int \mathrm{d}t \sum_{k,p} \bar{\psi}_k^{\pm}(t) \, \psi_{k+p}^{\pm}(t) \, \phi_p^{\pm}(t)$$

We will put in exact forms of the dispersion relations later. We convert to the Keldysh rotated basis (Retarded-Advanced-Keldysh RAK basis) given by,

$$\phi_{\mathrm{cl}}(k,t) = \frac{1}{\sqrt{2}}(\phi_k^+(t) + \phi_k^-(t)) \qquad \psi_1(k,t) = \frac{1}{\sqrt{2}}(\psi_k^+(t) + \psi_k^-(t))$$

$$\phi_{\mathrm{q}}(k,t) = \frac{1}{\sqrt{2}}(\phi_k^+(t) - \phi_k^-(t)) \qquad \psi_2(k,t) = \frac{1}{\sqrt{2}}(\psi_k^+(t) - \psi_k^-(t))$$

$$\bar{\psi}_1(k,t) = \frac{1}{\sqrt{2}}(\bar{\psi}_k^+(t) - \bar{\psi}_k^-(t)) \tag{3}$$

$$\bar{\psi}_2(k,t) = \frac{1}{\sqrt{2}}(\bar{\psi}_k^+(t) + \bar{\psi}_k^-(t))$$

In the above basis, the action takes the form,

$$S = \int \mathrm{d}t\mathrm{d}t' \sum_{k,k'} [\phi_{\mathrm{cl}} \quad \phi_{\mathrm{q}}]_x \left[D_0^{-1}(x,x')\right] \begin{bmatrix} \phi_{\mathrm{cl}} \\ \phi_{\mathrm{q}} \end{bmatrix}_{x'} + \int \mathrm{d}t\mathrm{d}t' \sum_{k,k'} [\bar{\psi}_1 \quad \bar{\psi}_2]_x \left[G_0^{-1}(x,x')\right] \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{x'} + S_{\mathrm{int}} \tag{4}$$

The electron-phonon contribution to S_{int} looks like

$$S_{\mathrm{int}}^{\mathrm{el\text{-}ph}} = \int \mathrm{d}t \sum_{k,p} \frac{\lambda}{\sqrt{2}} \left[(\bar{\psi}_1(k,t)\psi_1(k+p,t) + \bar{\psi}_2(k,t)\psi_2(k+p,t))\phi_{\mathrm{cl}}(p,t) \right.$$

$$\left. + (\bar{\psi}_1(k,t)\psi_2(k+p,t) + \bar{\psi}_2(k,t)\psi_1(k+p,t))\phi_{\mathrm{q}}(p,t)\right] \tag{5}$$
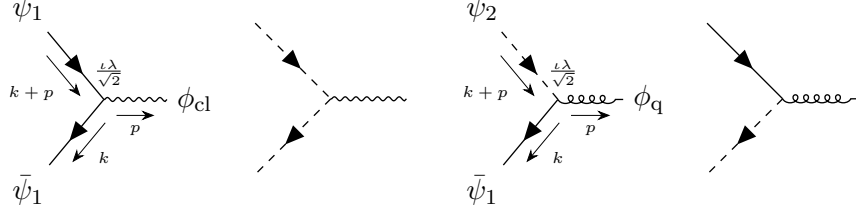
Figure 1: The 4 possible vertices in the $\psi_{1,2}$ and $\phi_{\text{cl,q}}$ basis. Each vertex has an amplitude of $\frac{\iota\lambda}{\sqrt{2}}$

3.1 Propagators and Vertex Structure

Propagators:

$$
\begin{array}{llll}
\iota D_0^R(x,x') & \phi_{\text{cl}}(x) \,\mathord{\sim\!\!\sim\!\!\sim\!\circ\circ\circ}\, \phi_{\text{q}}(x') & \quad \iota G_0^R(x,x') & \psi_1(x) \,\blacktriangleleft\!\!-\!\!\blacktriangleleft\, \bar\psi_1(x') \\[2mm]
\iota D_0^A(x,x') & \phi_{\text{q}}(x) \,\mathord{\circ\circ\circ\!\sim\!\!\sim\!\!\sim}\, \phi_{\text{cl}}(x') & \quad \iota G_0^A(x,x') & \psi_2(x) \,\text{-}\blacktriangleleft\text{-}\,\blacktriangleleft\text{-}\, \bar\psi_2(x') \\[2mm]
\iota D_0^K(x,x') & \phi_{\text{cl}}(x) \,\mathord{\sim\!\!\sim\!\!\sim\!\!\sim}\, \phi_{\text{cl}}(x') & \quad \iota G_0^K(x,x') & \psi_1(x) \,\blacktriangleleft\!\!-\,\blacktriangleleft\text{-}\, \bar\psi_2(x')
\end{array}
\tag{6}
$$

Full propagators are represented by double lines. In thermal equilibrium at (β,μ), the bare propagators are space and time translational invariant. In this case, we can write down their forms for electrons

$$
\begin{aligned}
G_0^R(\omega,\boldsymbol{k}) &= \frac{1}{\omega + \iota 0^+ - \varepsilon_{\boldsymbol{k}}} & \xrightarrow{F.T.} \; G_0^R(\boldsymbol{k},t,t') &= -\iota\Theta(t-t')\,\mathrm{e}^{-\iota\varepsilon_{\boldsymbol{k}}(t-t')} \\[2mm]
G_0^A(\omega,\boldsymbol{k}) &= \frac{1}{\omega - \iota 0^+ - \varepsilon_{\boldsymbol{k}}} & \xrightarrow{F.T.} \; G_0^A(\boldsymbol{k},t,t') &= \iota\Theta(t'-t)\,\mathrm{e}^{-\iota\varepsilon_{\boldsymbol{k}}(t-t')} \\[2mm]
G_0^K(\omega,\boldsymbol{k}) &= \tanh\frac{\beta_{\text{el}}(\omega-\mu_{\text{el}})}{2}[G_0^R - G_0^A] & \xrightarrow{F.T.} \; \iota G_0^K(\boldsymbol{k},t,t') &= \tanh\frac{\beta_{\text{el}}(\varepsilon_{\boldsymbol{k}}-\mu_{\text{el}})}{2}\,\mathrm{e}^{-\iota\varepsilon_{\boldsymbol{k}}(t-t')}
\end{aligned}
\tag{7}
$$

and for phonons[1]

$$
\begin{aligned}
D_0^R(\omega,\boldsymbol{k}) &= \frac{1}{(\omega+\iota 0^+)^2 - \omega_{\boldsymbol{k}}^2} & \xrightarrow{F.T.} \; D_0^R(\boldsymbol{k},t,t') &= -\Theta(t-t')\frac{1}{\omega_{\boldsymbol{k}}}\sin\omega_{\boldsymbol{k}}(t-t') \\[2mm]
D_0^A(\omega,\boldsymbol{k}) &= \frac{1}{(\omega-\iota 0^+)^2 - \omega_{\boldsymbol{k}}^2} & \xrightarrow{F.T.} \; D_0^A(\boldsymbol{k},t,t') &= \Theta(t'-t)\frac{1}{\omega_{\boldsymbol{k}}}\sin\omega_{\boldsymbol{k}}(t-t') \\[2mm]
D_0^K(\omega,\boldsymbol{k}) &= \coth\frac{\beta_{\text{ph}}\,\omega}{2}[D_0^R - D_0^A] & \xrightarrow{F.T.} \; \iota D_0^K(\boldsymbol{k},t,t') &= \frac{1}{\omega_{\boldsymbol{k}}}\coth\frac{\beta_{\text{ph}}\,\omega_{\boldsymbol{k}}}{2}\cos\omega_{\boldsymbol{k}}(t-t')
\end{aligned}
\tag{8}
$$

Phonons being lattice excitations, have no number conservation, and hence have no chemical potential. Some symmetries of the propagators are apparent, G_0^R and G_0^A being hermitian conjugate of each other and ιG_0^K being a hermitian matrix in the time arguments. In case of phonons, D_0^A is just the transpose of D_0^R and ιD_0^K is a symmetric matrix. These observations extend to the full propagators as well as general properties.

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[1]factor of 1/2 in Kamenev Chap 3 counterbalanced by different choice of $\phi_{\text{cl,q}}$ in terms of $\phi^{\pm}$; $\frac{1}{\sqrt{2}}$ instead of $\frac{1}{2}$

3.2 Dyson equation

The (forward) Dyson equation is given by

$$(\check{G}_0^{-1} - \check{\Sigma}) \circ \check{G} = \check{1} \Rightarrow \check{G} = \check{G}_0 + \check{G}_0 \circ \check{\Sigma} \circ \check{G} \tag{9} \quad \{\texttt{eq:dyson}\}$$

$$\check{G}_{\text{Phonon}} = \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix}, \quad \check{G}_{\text{Electron}} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix} \tag{10}$$

$$\check{\Sigma}_{\text{Phonon}} = \begin{pmatrix} 0 & \Sigma_{\text{ph}}^A \\ \Sigma_{\text{ph}}^R & \Sigma_{\text{ph}}^K \end{pmatrix}, \quad \check{\Sigma}_{\text{Electron}} = \begin{pmatrix} \Sigma_{\text{el}}^R & \Sigma_{\text{el}}^K \\ 0 & \Sigma_{\text{el}}^A \end{pmatrix} \tag{11}$$

Unpacking the content of (9) for phonons,

$$\begin{pmatrix} 0 & (D_0^A)^{-1} - \Sigma_{\text{ph}}^A \\ (D_0^R)^{-1} - \Sigma_{\text{ph}}^R & (D_0^{-1})^K - \Sigma_{\text{ph}}^K \end{pmatrix} \circ \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we get

$$\text{R/A:} \quad ((D_0^{R/A})^{-1} - \Sigma_{\text{ph}}^{R/A}) \circ D^{R/A} = 1$$

$$\{\texttt{eq:Gret}\} \qquad \Rightarrow D^{R/A} = D_0^{R/A} + D_0^{R/A} \circ \Sigma_{\text{ph}}^{R/A} \circ D^{R/A} \tag{12}$$

$$\text{K:} \quad D^K = -D^R \circ ((D_0^{-1})^K - \Sigma_{\text{ph}}^K) \circ D^A$$

$$\{\texttt{eq:Gkel}\} \qquad \Rightarrow \iota D^K = D^R \circ [-\iota (D_0^{-1})^K] \circ D^A + D^R \circ \iota \Sigma_{\text{ph}}^K \circ D^A \tag{13}$$

Here $-\iota(D_0^{-1})^K$ is the initial distribution function at $t = 0$. An alternate to (13) for expressing D^K is to start from the equivalent form in (9) and get

$$\{\texttt{eq:Gkelalt}\} \qquad \iota D^K = \iota D_0^K + \iota D_0^K \circ \Sigma_{\text{ph}}^A \circ D^A + D_0^R \circ \Sigma_{\text{ph}}^R \circ \iota D^K + D_0^R \circ \iota \Sigma_{\text{ph}}^K \circ D^A \tag{14}$$

Writing out the time coordinates explicitly in (13) and (14) we have

$$\{\texttt{eq:Dkdouble}\} \quad \iota D^K(t,t') = D^R(t,0) f_{\text{dist}}^0 D^A(0,t') + \int_0^t dt_1 \, D^R(t,t_1) \int_0^{t'} dt_2 \, \iota \Sigma^K(t_1,t_2) D^A(t_2,t') \tag{15}$$

$$\iota D^K(t,t') = \iota D_0^K(t,t') + \int_0^{t'} dt_1 \, \iota D_0^K(t,t_1) \int_{t_1}^{t'} dt_2 \, \Sigma^A(t_1,t_2) D^A(t_2,t') \tag{16}$$

$$+ \int_0^t dt_1 \, D_0^R(t,t_1) \int_0^{t_1} dt_2 \, \Sigma^R(t_1,t_2) \iota D^K(t_2,t') + \int_0^t dt_1 \, D_0^R(t,t_1) \int_0^{t'} dt_2 \, \iota \Sigma^K(t_1,t_2) D^A(t_2,t')$$

The equations for G^R, G^K are exactly the same and are omitted for brevity. The objective now is to evolve these greens functions in time for the coupled system.

3.3 Self Energies

Phonon self energies

$$-\iota \Sigma_{\text{ph}}^R(x,x') = \text{} + \text{} = -\left(\frac{\iota\lambda}{\sqrt{2}}\right)^2 \left[\iota G^K(x,x') \iota G^A(x',x) + \iota G^K(x',x) \iota G^R(x,x') \right]$$

$$\Rightarrow \Sigma_{\text{ph}}^R(x,x') = -\frac{\lambda^2}{2} \left[\iota G^K(x',x) G^R(x,x') + \iota G^K(x,x')(G^R(x,x'))^* \right] = -\lambda^2 \operatorname{Re}[\iota G^K(x,x')(G^R(x,x'))^*] \tag{17}$$

$$-\iota \Sigma_{\text{ph}}^K(x,x') = \text{} + \text{} + \text{}$$

$$= -\left(\frac{\iota\lambda}{\sqrt{2}}\right)^2 \left[\iota G^A(x,x') \iota G^R(x',x) + \iota G^R(x,x') \iota G^A(x',x) + \iota G^K(x,x') \iota G^K(x',x) \right]$$

$$\implies \iota \Sigma_{\text{ph}}^K(x,x') = \frac{\lambda^2}{2} \begin{cases} \left| G^R(x,x') \right|^2 - \left| \iota G^K(x,x') \right|^2 & \text{if } t > t' \\ \left| G^R(x',x) \right|^2 - \left| \iota G^K(x,x') \right|^2 & \text{if } t' > t \end{cases} \tag{18}$$

4

Electron self energies

$$-\iota\Sigma^R_{\text{el}}(x,x') = \text{[diagram]} + \text{[diagram]} = \left(\frac{\iota\lambda}{\sqrt{2}}\right)^2 \left[\iota G^R(x,x')\iota D^K(x,x') + \iota G^K(x',x)\iota D^R(x,x')\right]$$

$$\Rightarrow \Sigma^R_{\text{el}}(x,x') = \frac{\lambda^2}{2}\left[G^R(x,x')\iota D^K(x,x') + \iota G^K(x,x')D^R(x,x')\right] \qquad (19)$$

$$-\iota\Sigma^K_{\text{el}}(x,x') = \text{[diagram]} + \text{[diagram]} + \text{[diagram]}$$

$$= \left(\frac{\iota\lambda}{\sqrt{2}}\right)^2 \left[\iota G^R(x,x')\iota D^R(x,x') + \iota G^A(x,x')\iota D^A(x,x') + \iota G^K(x,x')\iota D^K(x,x')\right]$$

$$\implies \iota\Sigma^K_{\text{el}}(x,x') = -\frac{\lambda^2}{2}\begin{cases} G^R(x,x')D^R(x,x') - \iota G^K(x,x')\iota D^K(x,x') & \text{if } t > t' \\ G^R(x',x)^*D^R(x',x) - (\iota G^K(x',x))^* \iota D^K(x',x) & \text{if } t' > t \end{cases} \qquad (20)$$

A Deriving S_{int} in RAK basis

Electron phonon interaction term

$$\begin{aligned} S^{\text{el-ph}}_{\text{int}} &= \lambda \int dt \sum_{k,p} \bar{\psi}^+_k(t)\,\psi^+_{k+p}(t)\,\phi^+_p(t) - \bar{\psi}^-_k(t)\,\psi^-_{k+p}(t)\,\phi^-_p(t) \\ &= \lambda \int dt \sum_{k,p} \frac{1}{\sqrt{2}^3}\left[(\bar{1}+\bar{2})_k(1+2)_{k+p}(\text{cl}+\text{q})_p - (\bar{2}-\bar{1})_k(1-2)_{k+p}(\text{cl}-\text{q})_p\right] \\ &= \lambda \int dt \sum_{k,p} \frac{1}{\sqrt{2}}\left[(\bar{1}1+\bar{2}2)\text{cl} + (\bar{2}1+\bar{1}2)\text{q}\right] \\ &= \int dt \sum_{k,p} \frac{\lambda}{\sqrt{2}}\Big[[\bar{\psi}_1(k,t)\psi_1(k+p,t) + \bar{\psi}_2(k,t)\psi_2(k+p,t)]\phi_{\text{cl}}(p,t) \\ &\qquad\qquad + [\bar{\psi}_1(k,t)\psi_2(k+p,t) + \bar{\psi}_2(k,t)\psi_1(k+p,t)]\phi_{\text{q}}(p,t)\Big] \end{aligned} \qquad (21)$$

Electron-electron interaction term

$$\begin{aligned} S^{\text{el-el}}_{\text{int}} &= \sum_{q,k,k'} u(q)\left[\bar{\psi}^+_{k+q}\bar{\psi}^+_{k'-q}\psi^+_{k'}\psi^+_k - \bar{\psi}^-_{k+q}\bar{\psi}^-_{k'-q}\psi^-_{k'}\psi^-_k\right] \\ &= \sum_{q,k,k'} \frac{u(q)}{\sqrt{2}^4}\left[(\bar{1}+\bar{2})_{k+q}(\bar{1}+\bar{2})_{k'-q}(1+2)_{k'}(1+2)_k - (\bar{2}-\bar{1})_{k+q}(\bar{2}-\bar{1})_{k'-q}(1-2)_{k'}(1-2)_k\right] \\ &= \sum_{q,k,k'} \frac{u(q)}{4}\left[(\bar{1}\bar{1}+\bar{2}\bar{1}+\bar{1}\bar{2}+\bar{2}\bar{2})(11+21+12+22) - (\bar{1}\bar{1}-\bar{2}\bar{1}-\bar{1}\bar{2}+\bar{2}\bar{2})(11-21-12+22)\right] \\ &= \sum_{q,k,k'} \frac{u(q)}{2}\left[(\bar{2}\bar{1}+\bar{1}\bar{2})(11+22) + (\bar{1}\bar{1}+\bar{2}\bar{2})(21+12)\right] \end{aligned} \qquad (22)$$

B Dyson Iteration

The objective is to solve the Dyson equations for phonons and electrons as outlined in (13) and (12).

B.1 Phonon update

$$D^R(t,t') = D_0^R(t,t') + \int_{t'}^{t} dt_1\, D_0^R(t,t_1) \int_{t'}^{t_1} dt_2\, \Sigma^R(t_1,t_2)\, D^R(t_2,t') \qquad (23) \quad \texttt{\{eq:dreqn\}}$$

The space indices are suppressed for the time being. The limits in the time integrals are set by the theta function structure of each retarded component $(t > t_1 > t_2 > t')$. Instead of numerically evaluating the complicated double integration for each value of t, t' we want to find a method to incrementally find the value of D^R at subsequent grid points in terms of the values at earlier grid points, i.e find $D^R(t+\epsilon, t')$ in terms of $D^R(t,t')$ for $\epsilon > 0$.

We note that for the free greens function,

$$
\begin{aligned}
D_0^R(\boldsymbol{k}, t+\epsilon, t') &= -\Theta(t+\epsilon-t')\frac{1}{\omega_{\boldsymbol{k}}} \sin\omega_{\boldsymbol{k}}(t+\epsilon-t') \\
&= -\Theta(t+\epsilon-t')\frac{1}{\omega_{\boldsymbol{k}}} \left[\cos(\omega_{\boldsymbol{k}}\epsilon)\sin\omega_{\boldsymbol{k}}(t-t') + \sin(\omega_{\boldsymbol{k}}\epsilon)\cos\omega_{\boldsymbol{k}}(t-t') \right]
\end{aligned}
$$

{composition}
$$\implies D_0^R(\boldsymbol{k}, t+\epsilon, t') = \bar{D}_0^R(\boldsymbol{k}, t+\epsilon, t)D_0^R(\boldsymbol{k},t,t') + D_0^R(\boldsymbol{k},t+\epsilon,t)\bar{D}_0^R(\boldsymbol{k},t,t') \qquad (24)$$

where we have defined

$$\bar{D}_0^R(t,t') = -\Theta(t-t')\cos\omega_{\boldsymbol{k}}(t-t') \text{ and in general } \bar{D}^R(t,t') = \partial_t D^R(t,t') \qquad (25)$$

$\bar{D}_0^R$ also admits a decomposition like in (24)

$$
\begin{aligned}
\bar{D}_0^R(t+\epsilon, t') &= -\Theta(t-t')\left[\cos\omega_{\boldsymbol{k}}(t-t')\cos\omega_{\boldsymbol{k}}\epsilon - \sin\omega_{\boldsymbol{k}}(t-t')\sin\omega_{\boldsymbol{k}}\epsilon\right] \\
&= -\bar{D}_0^R(\boldsymbol{k},t+\epsilon,t)\bar{D}_0^R(\boldsymbol{k},t,t') + \omega_{\boldsymbol{k}}^2 D_0^R(\boldsymbol{k},t+\epsilon,t)D_0^R(\boldsymbol{k},t,t') \qquad (26)
\end{aligned}
$$

With this identification, taking a derivative with respect to t on both sides of (23) we find the evolution equation for $\bar{D}^R$. For notational convenience, call $F(t_1,t') = \int_{t'}^{t_1} dt_2\, \Sigma^R(t_1,t_2)\, D^R(t_2,t')$

$$\partial_t D^R(t,t') = \partial_t D_0^R(t,t') + \int_{t'}^{t} dt_1\, \partial_t D_0^R(t,t_1) F(t_1,t') + \underbrace{D_0^R(t,t)}_{=0} F(t,t')$$

$$\bar{D}^R(t,t') = \bar{D}_0^R(t,t') + \int_{t'}^{t} dt_1\, \bar{D}_0^R(t,t_1)F(t_1,t') \qquad (27)$$

Starting from (23) and using (24) we see (for clarity sake, we suppress the $\boldsymbol{k}$ index)

$$
\begin{aligned}
D^R(t+\epsilon, t') &= D_0^R(t+\epsilon, t') + \int_{t'}^{t+\epsilon} dt_1\, D_0^R(t+\epsilon,t_1)F(t_1,t') \\
&= \int_{t}^{t+\epsilon} dt_1\, D_0^R(t+\epsilon,t_1)F(t_1,t') + \bar{D}_0^R(t+\epsilon,t)\left[D_0^R(t,t') + \int_{t'}^{t} dt_1\, D_0^R(t,t_1)F(t_1,t')\right] \\
&\quad + D_0^R(t+\epsilon,t)\left[\bar{D}_0^R(t,t') + \int_{t'}^{t} dt_1\, \bar{D}_0^R(t,t_1)F(t_1,t')\right]
\end{aligned}
$$

$$\boxed{\therefore D^R(t+\epsilon,t') = \bar{D}_0^R(t+\epsilon,t)D^R(t,t') + D_0^R(t+\epsilon,t)\bar{D}^R(t,t') + \int_{t}^{t+\epsilon} dt_1\, D_0^R(t+\epsilon,t_1)F(t_1,t')}$$

$$(28)$$

In case of $\bar{D}^R$ we have

$$\bar{D}^R(t+\epsilon,t') = \bar{D}_0^R(t+\epsilon,t') + \int_{t'}^{t+\epsilon} dt_1\, \bar{D}_0^R(t+\epsilon,t_1)F(t_1,t')$$

$$= \int_t^{t+\epsilon} dt_1\, \bar{D}_0^R(t+\epsilon,t_1)F(t_1,t') + \omega_{\boldsymbol{k}}^2 D_0^R(t+\epsilon,t)\left[D_0^R(t,t') + \int_{t'}^t dt_1\, D_0^R(t,t_1)F(t_1,t')\right]$$

$$- \bar{D}_0^R(t+\epsilon,t)\left[\bar{D}_0^R(t,t') + \int_{t'}^t dt_1\, \bar{D}_0^R(t,t_1)F(t_1,t')\right]$$

$$\boxed{\therefore \bar{D}^R(t+\epsilon,t') = \omega_{\boldsymbol{k}}^2 D_0^R(t+\epsilon,t)D^R(t,t') - \bar{D}_0^R(t+\epsilon,t)\bar{D}^R(t,t') + \int_t^{t+\epsilon} dt_1\, \bar{D}_0^R(t+\epsilon,t_1)F(t_1,t')}$$

(29)

Similar to (24) we can also write a decomposition for D_0^K.

$$\iota D_0^K(\boldsymbol{k},t+\epsilon,t') = \frac{1}{\omega_{\boldsymbol{k}}}\coth\frac{\beta_{\mathrm{ph}}\,\omega_{\boldsymbol{k}}}{2}\left[\cos\omega_{\boldsymbol{k}}(t-t')\cos\omega_{\boldsymbol{k}}\epsilon - \sin\omega_{\boldsymbol{k}}(t-t')\sin\omega_{\boldsymbol{k}}\epsilon\right]$$

$$= -\bar{D}_0^R(\boldsymbol{k},t+\epsilon,t)\iota D_0^K(\boldsymbol{k},t,t') - D_0^R(\boldsymbol{k},t+\epsilon,t)\iota\bar{D}_0^K(\boldsymbol{k},t,t')$$

(30)

B.2 Electron update

For electrons the free propagators at subsequent timesteps are related to ones at present time step as

$$G_0^R(t+\epsilon,t') = \iota G_0^R(t+\epsilon,t)G_0^R(t,t'), \quad \iota G_0^K(t+\epsilon,t') = \iota G_0^R(t+\epsilon,t)\iota G_0^K(t,t')$$

(31)

This implies for the electron Dyson equation

$$G^R(t+\epsilon,t') = \iota G_0^R(t+\epsilon,t)G^R(t,t') + \int_t^{t+\epsilon} dt_1\, G_0^R(t+\epsilon,t_1)\int_{t'}^{t_1} dt_2\, \Sigma_{\mathrm{el}}^R(t_1,t_2)\, G^R(t_2,t') \quad (32)$$

$$\iota G^K(t+\epsilon,t') = \iota G_0^R(t+\epsilon,t)\iota G^K(t,t') + \int_t^{t+\epsilon} dt_1\, G_0^R(t+\epsilon,t_1)\int_0^{t_1} dt_2\, \Sigma_{\mathrm{el}}^R(t_1,t_2)\, \iota G^K(t_2,t')$$

$$+ \int_t^{t+\epsilon} dt_1\, G_0^R(t+\epsilon,t_1)\int_0^{t'} dt_2\, \iota\Sigma_{\mathrm{el}}^K(t_1,t_2)\, G^A(t_2,t')$$

(33)

Choosing an Euler approximation for $\int_t^{t+\epsilon} dt_1\, f(t) \approx \frac{\epsilon}{2}(f(t+\epsilon)+f(t))$, we have the iteration equations as

$$G^R(t+\epsilon,t') = \iota G_0^R(t+\epsilon,t)G^R(t,t') + \frac{\epsilon}{2}G_0^R(t+\epsilon,t)\int_{t'}^t dt_2\, \Sigma_{\mathrm{el}}^R(t,t_2)\, G^R(t_2,t') \qquad (34) \quad \text{\{eq:GRupdate}$$

$$\iota G^K(t+\epsilon,t') = \iota G_0^R(t+\epsilon,t)\iota G^K(t,t') + \frac{\epsilon}{2}G_0^R(t+\epsilon,t)\int_0^t dt_2\, \Sigma_{\mathrm{el}}^R(t,t_2)\, \iota G^K(t_2,t')$$

$$+ \frac{\epsilon}{2}G_0^R(t+\epsilon,t)\int_0^{t'} dt_2\, \iota\Sigma_{\mathrm{el}}^K(t,t_2)\, G^A(t_2,t') \qquad (35) \quad \text{\{eq:GKupdate}$$

However to get $\iota G^K(t+\epsilon,t+\epsilon)$ from this we would need $\iota G^K(t,t+\epsilon)$ among other things. One might imagine that we can find it by evaluating $\iota G^K(t+\epsilon,t)$ first and then take a complex conjugate. But recall that the Keldysh greens function is completely anti-hermitian in space-time coordinates and hence $\iota G^K(i,t;j,t') = [\iota G^K(j,t';i,t)]^* \neq [\iota G^K(i,t';j,t)]^*$ [2]. The other option is to use the analogue of (15) for electrons

$$\iota G^K(t,t) = D^R(t,0)f_{\mathrm{dist}}^0 D^A(0,t) + \int_0^t dt_1\, D^R(t,t_1)\int_0^t dt_2\, \iota\Sigma^K(t_1,t_2)D^A(t_2,t) \qquad (36) \quad \text{\{eq:GKupdate}$$

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[2] think about how it affects for $\boldsymbol{k}$ space blocks of ιG^K

and evaluate the double integrals at every time step, after generating the necessary G^R upto said time step. The caveat is that the double time integral quickly becomes more expensive with increasing number of time steps, and may restrict the accessible total time of simulation.

Steps:

1. Given $D^R(t+\epsilon, t')$ and $D^K(t+\epsilon, t')$ from the phonon side, evaluate $\Sigma_e^{\mathrm{R/K}} l$.

2. in all the calculations, G^R G^K are thermal greens functions at $[\beta, \mu]_{\mathrm{el}}(t)$.

3. update G^R and G^K through the Dyson equations. Objective: get $\iota G^K(t+\epsilon, t+\epsilon)$

option 1: Update $G^R(t+\epsilon, t')$ with (34), and then use the updated values to evaluate a double integral in (36) to find $\iota G^K(t+\epsilon, t+\epsilon)$

option 2: Figure out the appropriate $\boldsymbol{k}$ space relation between time swapped entries of ιG^K, and use (35) twice to go from

$$\iota G^K(t,t) \xrightarrow{(35)} \iota G^K(t+\epsilon, t) \xleftrightarrow{\text{identification}} \iota G^K(t, t+\epsilon) \xrightarrow{(35)} \iota G^K(t+\epsilon, t+\epsilon)$$

4. Extract mode occupations $n_{\boldsymbol{k}}(t+\epsilon) = \frac{1}{2}(1 - \iota G^K(\boldsymbol{k}, t+\epsilon, t+\epsilon))$

5. Evaluate total energy of the electrons $E(t+\epsilon) = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} n_{\boldsymbol{k}}(t+\epsilon)$ and the total number $N(t+\epsilon) = \sum_{\boldsymbol{k}} n_{\boldsymbol{k}}(t+\epsilon)$. We assume here that the electron dispersion $\varepsilon_{\boldsymbol{k}}$ doesn't change with time.

6. Find new temperature and chemical potential $[\beta, \mu]_{\mathrm{el}}(t+\epsilon) \equiv [\beta, \mu]_{\mathrm{el}}^{\mathrm{new}}$ by demanding number conservation and "instantaneous" energy redistribution from rapid thermalisation. Solve the following equations using Newton-Raphson (or gradient descent) with initial guess being the "old" values of the parameters β, μ.

$$\sum_{\boldsymbol{k}} \tanh \frac{\beta_{\mathrm{el}}^{\mathrm{new}}(\varepsilon_{\boldsymbol{k}} - \mu_{\mathrm{el}}^{\mathrm{new}})}{2} = N(t+\epsilon) \tag{37}$$

$$\sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} \tanh \frac{\beta_{\mathrm{el}}^{\mathrm{new}}(\varepsilon_{\boldsymbol{k}} - \mu_{\mathrm{el}}^{\mathrm{new}})}{2} = E(t+\epsilon) \tag{38}$$

7. use thermal electron greens functions at "new" value of parameters to generate Σ_{ph} for the next time step.