Physics 211C: Solid State Physics

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Lecture 14

Topic: Introduction to lattice gauge theory

where did we see gauge theories?

① Heavy fermions: $\vec{S} = f^+ \frac{\vec{\sigma}}{2} f$

$f_r \to e^{i\theta_r} f_r \quad S_r \to$ unchanged

$(f_r^+ f_{r'} \underbrace{e^{i a_{rr'}}}_{} + h.c.)$

② $\vec{S} = b^+ \frac{\vec{\sigma}}{2} b$

$\Rightarrow$ To account for such redudancy we employed gauge theories.

$(f_r^+ f_{r'} e^{i a_{rr'}} + h.c.)$
$+ (\nabla \times a)^2 + \cdot \; \cdots$

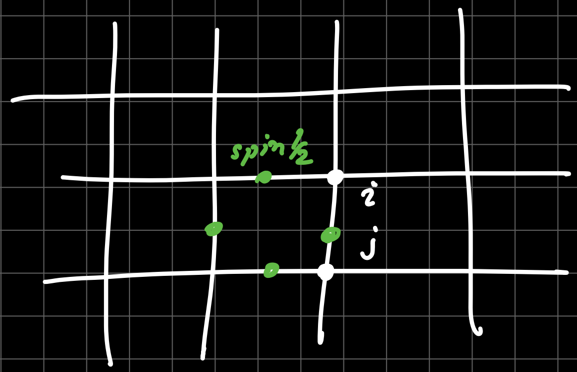
motivⁿ:- Enlarges the class of phases available to us.

Pure $\mathbb{Z}_2$ Gauge theory

Gauge group $\Rightarrow \mathbb{Z}_2$

$U(1)$ EM: $\mathcal{H} = E^2 + B^2$

$\left.\begin{array}{l} \nabla \cdot E = 0 \\ \nabla \cdot B = 0 \end{array}\right\}$ conditions $\Big\}$ one has to impose this in Hamiltonian picture



$\mathcal{H} = \bigotimes$ (spin ½ on bonds)

$\sigma_x \equiv X$

$\sigma_z \equiv Z$

$X_{ij} \equiv$ "E^2 term"

$E = -\frac{\partial A}{\partial t} \implies$ define $[E, A] = i$

$\nabla \times A = B$

$\iff e^{iA} E e^{-iA} = E + 1$

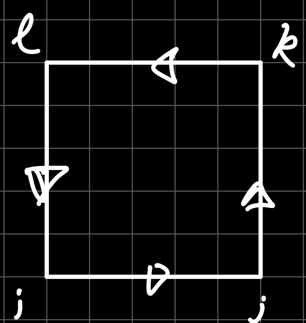
similar to

$b = \sqrt{n}\, e^{i\theta}$

$\iff e^{i\theta} n e^{-i\theta} = n + 1$

$e^{iA} \equiv Z_{ij}$

$Z_{ij} X_{ij} Z_{ij} = -X_{ij}$



$e^{i(A_{ij} + A_{jk} + A_{kl} + A_{li})} = e^{i(\nabla \times \vec{A})} + h.c.$

$A_{ij} = -A_{ji}$

$\therefore e^{i\oint \vec{A} \cdot d\vec{l}} \sim e^{iB}$

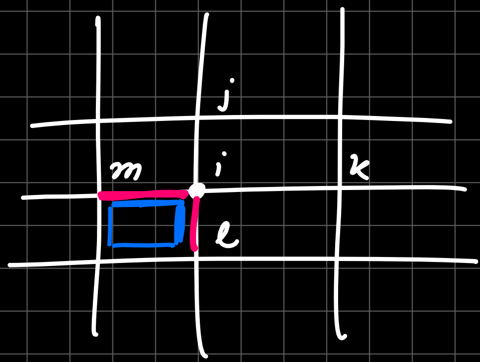
$B^2 = Z_{ij} Z_{jk} Z_{kl} Z_{li}$

$\therefore \mathcal{H} = -J \sum_{\langle ij \rangle} X_{ij} - K \sum_{\square} \prod_{\square} Z_{ij}$

nearest nbos

hold the links

still not complete? $\Rightarrow$ we can't declare our hilbert space yet



$X_{ij}\, X_{ik}\, X_{il}\, X_{im}$

$= \prod_{+\in i} X$ → commutes with H

$\Downarrow$

Too many conserved quantities

$$\Big[\prod_{+\in i} X,\ H\Big] = 0$$

$\Downarrow$

now we can define a gauge theory

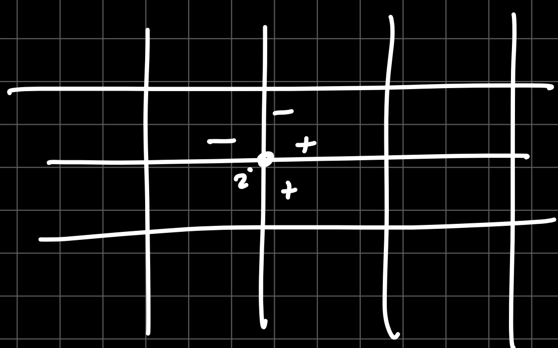
$$\left.\begin{aligned} \prod_{+\in i} X\, |\psi_1\rangle &= |\psi_2\rangle \\ |\psi_1\rangle &= |\psi_2\rangle \end{aligned}\right\}\ \text{call } \prod_{+\in i} X = \mathbb{1}$$

$\nabla\cdot E = 0$ } Analogous to Gauss law

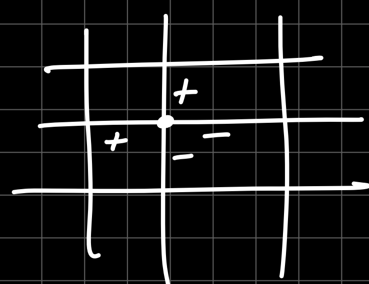
on physical hilbert space.

$\forall i$

Z → like gauge field



$\xrightarrow{\quad}$
$\prod_{+\in i}$



"H did not change"

- Operator that enters gauss low ($\hat{O} = \mathbb{1}$), $\hat{O}$ implements gauge transf.
- Physical states connected by a gauge transf. are one & the same.
- Gauge theory has no actual symmetry.

$\mathbb{Z}_2$ gauge theory with matter fields

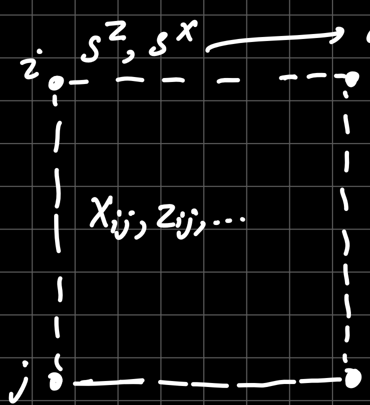
$\nabla \cdot \vec{E} = \rho$

matter + G. field coupling $\Rightarrow j_\mu A_\mu$

$$\left(\underbrace{c_i^\dagger c_j e}\,^{i A_{ij}} + h.c.\right)$$

$$i A_{ij} \left(\underbrace{c_i^\dagger c_j - c_j^\dagger c_i}_{j_{i\leftarrow j}}\right) \longrightarrow \text{capture } \vec{A}\cdot\vec{j} \text{ coupling}$$

* now allow charges to live on vertices. A, E live on links



$s^z, s^x \longrightarrow$ act on gauge charges

$\Downarrow$

Z_2 gauge charges (0 or 1)

$$-t \sum_{\langle ij \rangle} \underbrace{s_i^z s_j^z Z_{ij}}_{\text{hopping}}$$

$$-\mu \sum_i \underbrace{s_i^x}_{\text{Chemical potential}}$$

$$-J \sum X_{ij} - K \sum_{\square} \prod_{\square} Z_{ij}$$

$s^2 = 1 \quad z^2 = 1$

Gauge transf:- $\quad s_i^z \longrightarrow t_i\, s_i^z \qquad t_i = \pm 1$

$$Z_{ij} \longrightarrow t_i t_j Z_{ij}$$

$$\left(\prod_{+\in i} X\, s_i^x\right) \longrightarrow \text{1st term} = (-1) \times (-1)$$

$$\varphi_i \equiv \prod_{+\in i} X\, s_i^x = 1 \qquad \text{"}s^x \text{ is gauge invariant."}$$

$(\nabla \cdot E) = s_i^x \equiv \rho \longrightarrow$ why the 2nd term is gauge inv

$$\mathcal{Q}_i = 1 \quad \& \quad [\mathcal{Q}_i, \mathcal{H}] = 0$$

$\mathbb{Z}_2$ gauge theories with fermion charges

$$-t \sum_{\langle ij \rangle} c_i^\dagger c_j Z_{ij} - \mu \sum c_i^\dagger c_i - J \sum X_{ij} - \sum_{\square} \prod_{\square} Z_{ij}$$

$$Q_i = \prod_{+ \in i} X (-1)^{c_i^\dagger c_i} \rightarrow \text{commutes with } \mathcal{H}$$

$$-t\left(f_i^\dagger f_j e^{i a_{ij}} + h.c.\right) + \cos(\nabla \times \vec{a}) + e_{ij}^2$$

$$-\mu \sum_i f_i^\dagger f_i \qquad \left(\prod_{\square} e^{i a_{ij}} + h.c.\right)$$

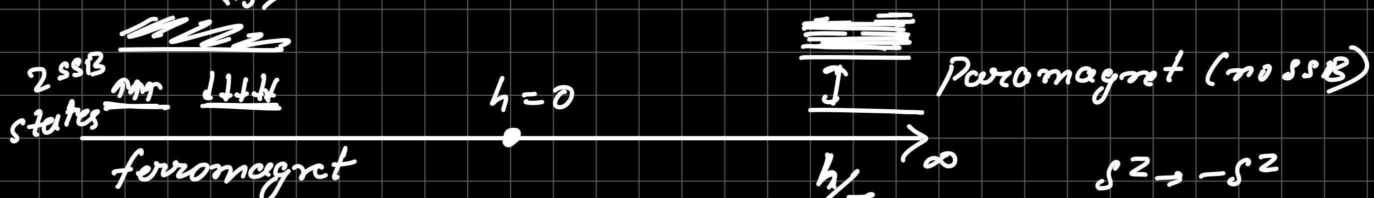
electric field

$$\nabla \cdot e_i = c_i^\dagger c_i$$

Phase diagram to pure $\mathbb{Z}_2$ gauge theory in 2+1-D

dual to Transverse field Ising model

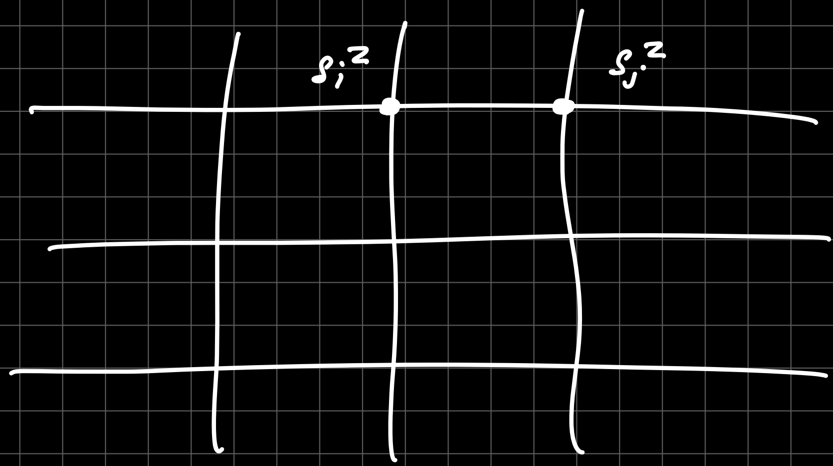
$$-J \sum_{\langle ij \rangle} S_i^z S_j^z - h \sum_i S_i^x$$



@ T = 0 that is.

will be a trivial / non-top ordered phase.

will be a topologically ordered phase in gauge theory (hosts anyons)



Chaikin Lubensky ch-8, 9