The Low Rank Hypothesis of Complex Systems

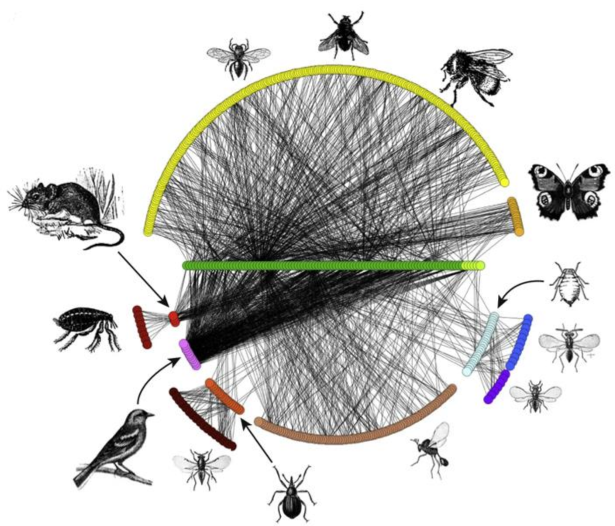
Authors: Vincent Thibeault, Antoine Allard, Patrick Desroisiers
Article: Nat. Phys. 20, 294–302 (2024).

Presenter: Guru Kalyan Jayasingh
Assisted by: Nigel Goldenfeld
Department of Physics
UCSD

Why is it possible to have simple descriptions of complex phenomena?

Low-rank hypothesis

- Complex systems, such as ecosystems, neural networks, and social structures, are often represented by high-dimensional non-linear differential equations.

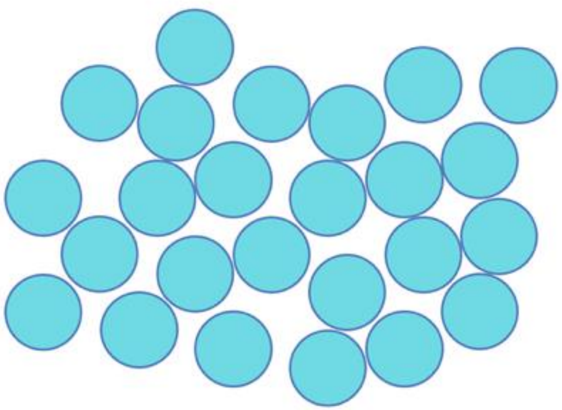


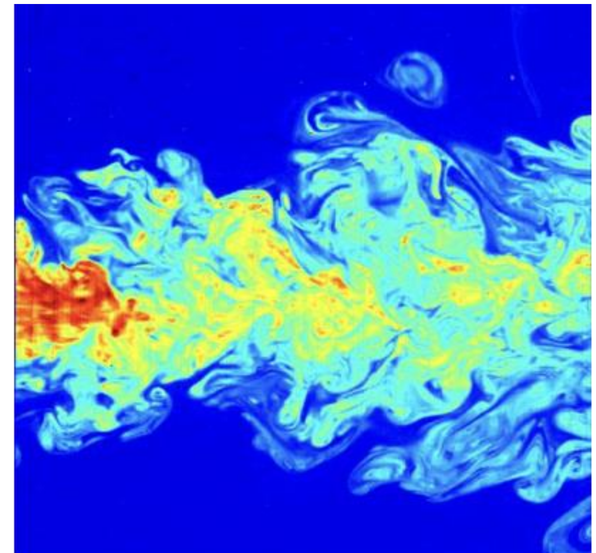
Credit: Quintessence Network DOI: 10.1016/j.tree.2015.12.003

- To make interpretable predictions about their large-scale behavior reduce dynamics to a few equations with small number of observables.

Low-rank hypothesis and dimension reduction

Collection of particles





Navier-Stokes equation

$$\rho\left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u\right) = \rho g - \nabla p + \mu \Delta u$$

- Dimension reduction
  - Reduced system of macrostates or observables
  - Small enough dimension to get an insightful description but large enough to preserve the phenomena of interest.

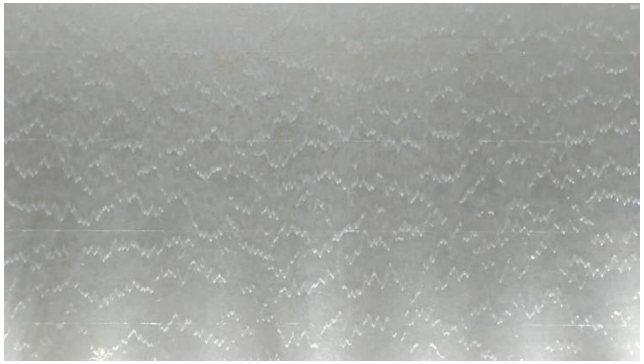
https://www.elevise.co.uk/gap3a.html
https://en.wikipedia.org/wiki/Turbulence

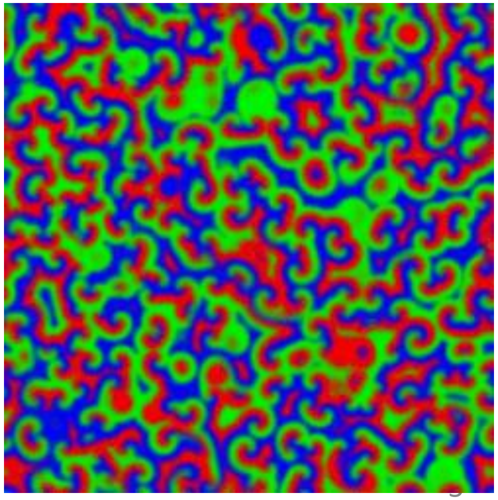
In Physics

- Kuramoto-Sivashinsky reaction

$$u_t + u_{xx} + u_{xxxx} + \frac{1}{2}u_x^2 = 0$$

  - For a system of size L, the long time behavior of solutions approach exponentially fast to a finite dimensional manifold *M.*
  - *Finite dimensional long time dynamics*





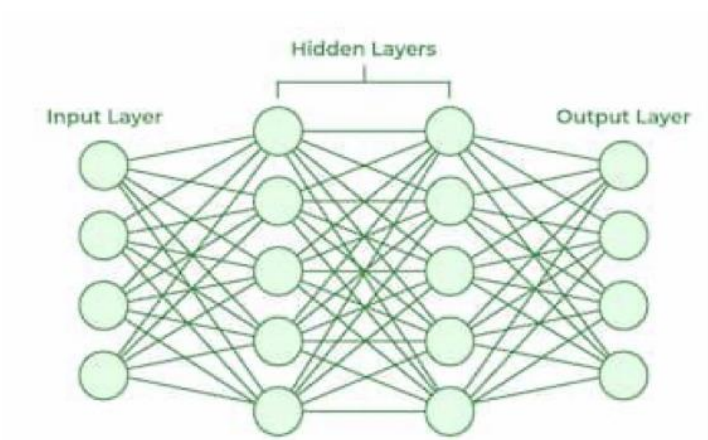
- Belousov–Zhabotinsky reaction
  - System of chemical reactions involving over 20 individual reactions amongst 18 chemical species.
  - Can be reduced to a system of 3 first order ODEs that reproduce the qualitative features.

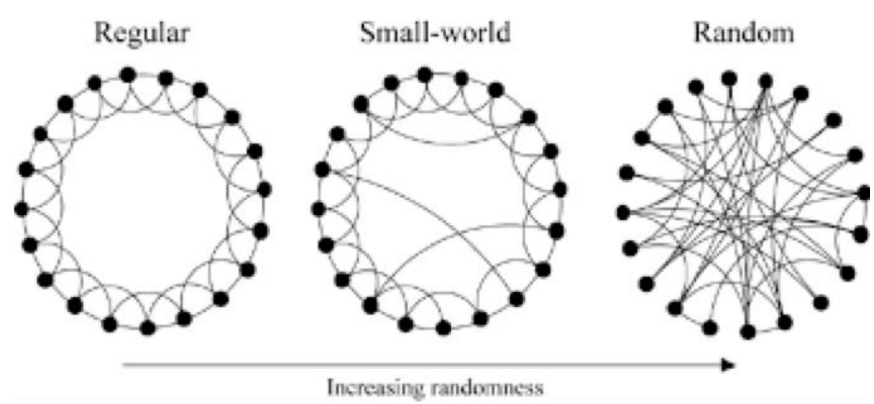
Pictures and videos from Nigel Goldenfeld (with permission)
https://en.wikipedia.org/wiki/Belousov%E2%80%93Zhabotinsky_reaction

Temam, Roger, and Xiao Ming Wang. "Estimates on the lowest dimension of inertial manifolds for the Kuramoto-Sivashinsky equation in the general case." (1994): 1095-1108.

Low rank hypothesis for networks

- In this paper, the authors
  - study the validity of this assumption for dynamics on networks and
  - study the effect of low-rank nature of these networks on dimension reduction of its network dynamics, for e.g. in RNNs.





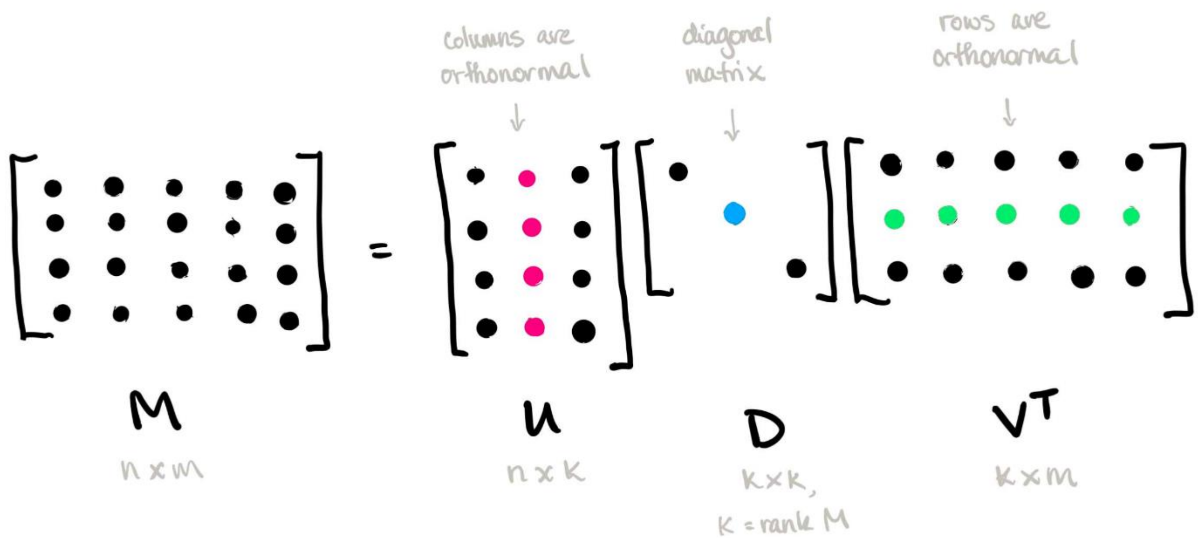
https://www.geeksforgeeks.org/artificial-neural-networks-and-its-applications/
https://www.researchgate.net/figure/Small-world-network-and-randomness-Windfinder-2019_fig2_363337974

Rank in low-rank

Singular Value Decomposition (SVD)

- What can it do?

$$M = U \quad D \quad V^T$$

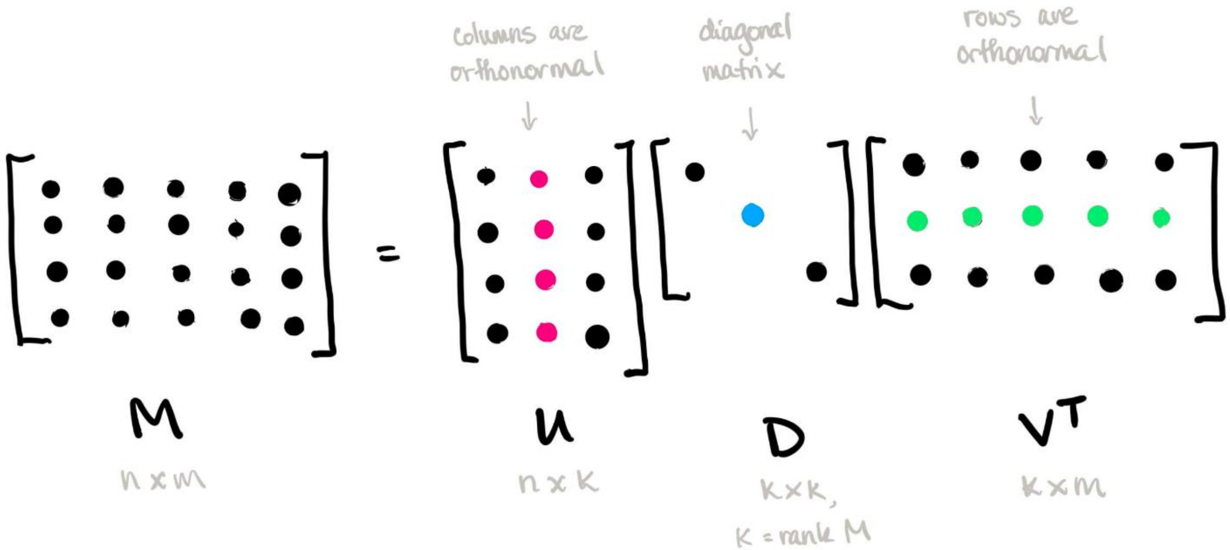


- U, D, V are special

$$UU^T = I \qquad VV^T = V^TV = I$$

https://www.math3ma.com/blog/understanding-entanglement-with-svd

- Diagonal entries are known as singular values. k is called the "rank" of the matrix.



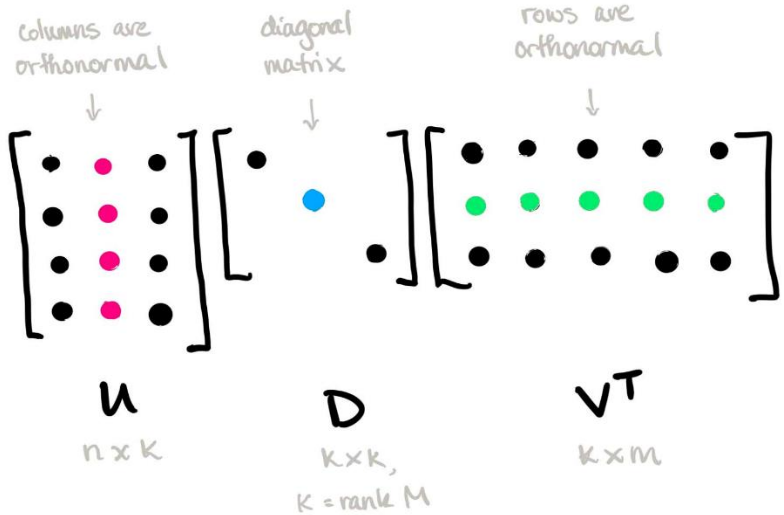
$$D_{11} \geq D_{22} \geq D_{33} \cdots \geq D_{kk}$$

- Can be done for any matrix (even non-square matrices).

https://www.math3ma.com/blog/understanding-entanglement-with-svd

- Useful? Say if we want to approximate the matrix M by a pair vectors. The best known approximation* that minimizes least squares error from M would be

$$M \sim U_1 D_{11} V_1^T$$



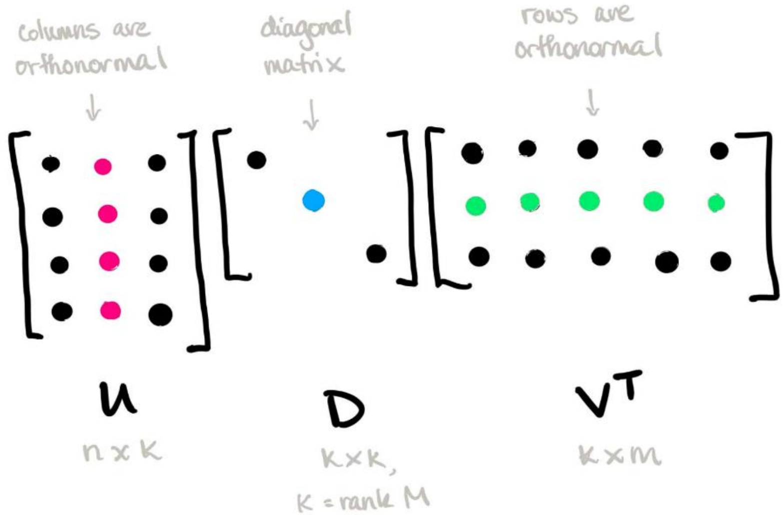
- How about improving with using 2 pairs of vectors? Or more? Just keep adding more terms

$$M \sim U_1 D_{11} V_1^T + U_2 D_{22} V_2^T + U_3 D_{33} V_3^T + \dots$$

* Wikipedia: https://en.wikipedia.org/wiki/Low-rank_approximation
https://www.math3ma.com/blog/understanding-entanglement-with-svd

- Useful? Say if we want to approximate the matrix *M* by a pair vectors. The best known approximation* that minimizes least squares error from M would be

$$M \sim U_1 D_{11} V_1^T$$



- How about improving with using 2 pairs of vectors? Or more? Just keep adding more terms

$$M \sim \underbrace{\boxed{U_1 D_{11} V_1^T}}_{\text{Rank 1}} + \underbrace{\boxed{U_2 D_{22} V_2^T}}_{\text{Rank 2}} + \underbrace{\boxed{U_3 D_{33} V_3^T}}_{\text{Rank 3}} + \dots$$

https://www.math3ma.com/blog/understanding-entanglement-with-svd

- How to get *U,V,D* ?

  Use

$$M \cdot M^T = UD^2U^T$$

- Columns of U are left eigenvectors of MM^T

  Eigenvalues are singular values squared

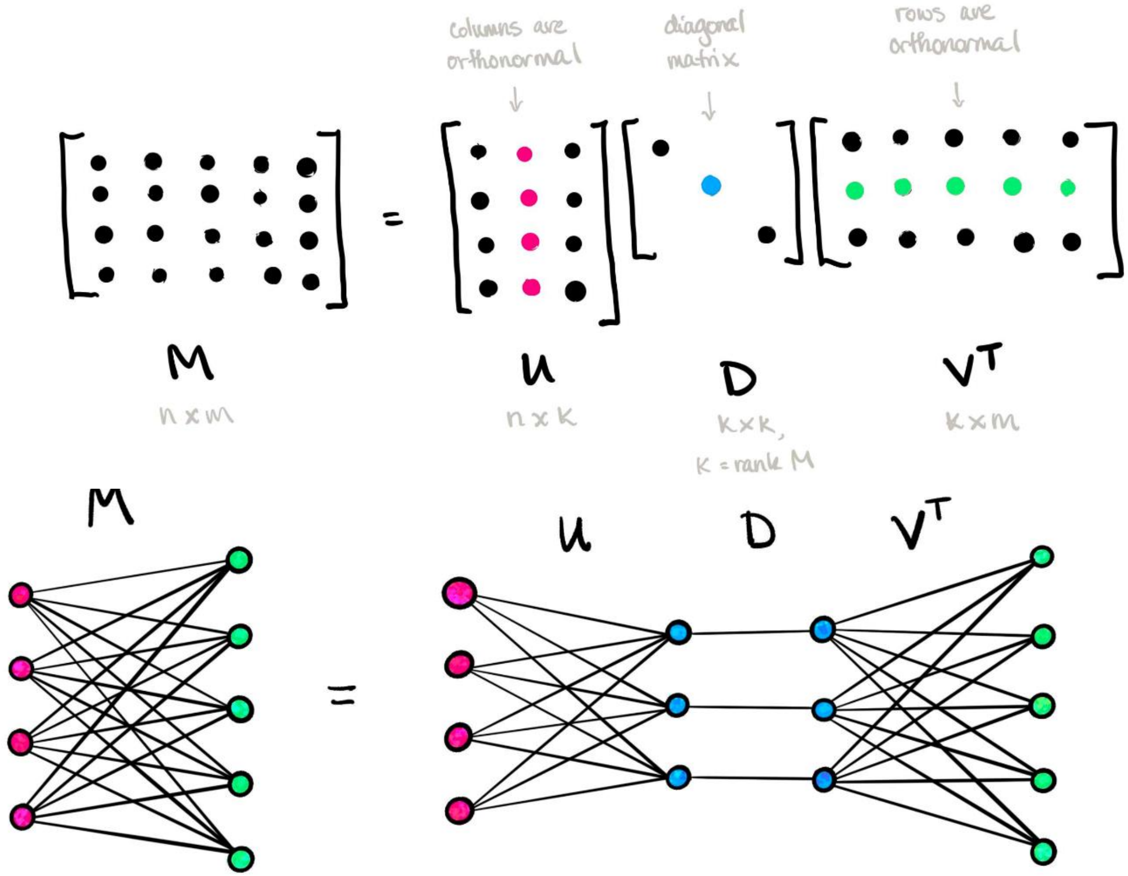
$$det(UD^2U^T - \lambda I) = 0$$

$$\implies \lambda_i = (\sigma_i)^2$$

- Repeat with $M^T M$ and compute the right eigenvector to get V.

SVD: Visualization

- The matrix D can be thought of as a bridge "joining " the rows to the columns. If rank of M is small, then it means there are a small number of such connections.

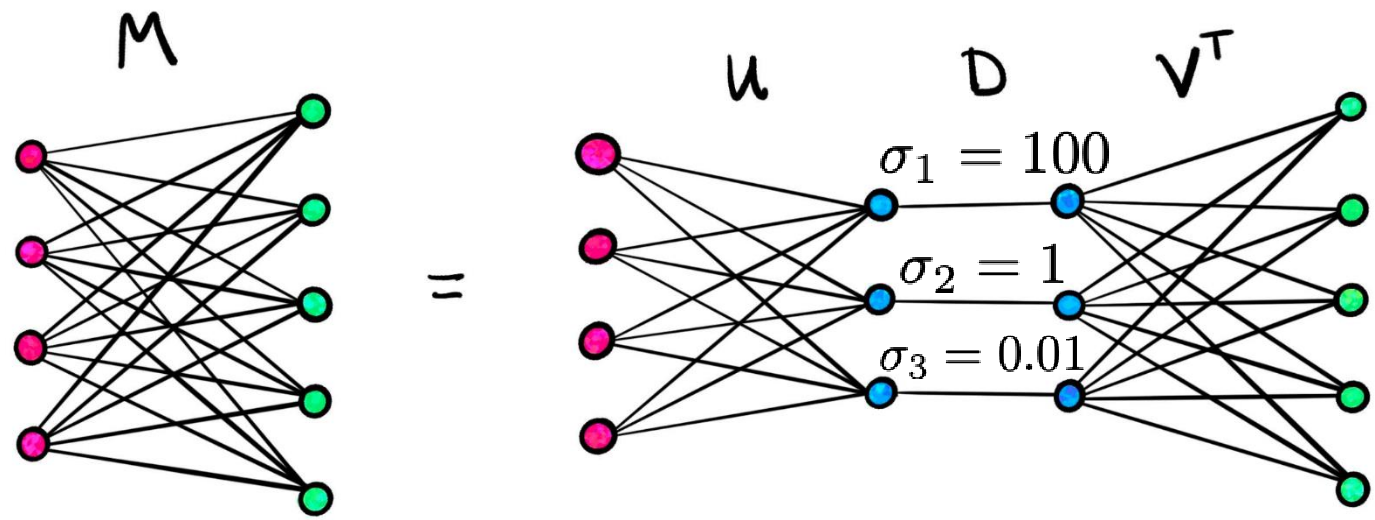


https://www.math3ma.com/blog/understanding-entanglement-with-svd

- Intuitively, singular values characterize “interaction” between *U (rows)* and *V (column)* spaces and how that contributes to total information contained in *M.*

- Singular values denote how much information is present in a particular bridge.

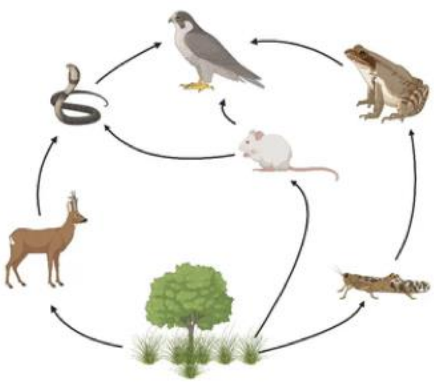
- A low rank implies that majority of information is concentrated in first few bridges.

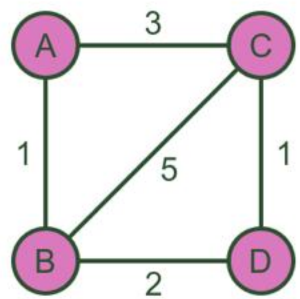


https://www.math3ma.com/blog/understanding-entanglement-with-svd

Complex systems and SVD

- Topology of interactions amongst different constituents of a complex system can be represented as a graph.





Weighted graph

	A	B	C	D
A	0	1	3	0
B	1	0	5	2
C	3	5	0	1
D	0	2	1	0

- Low-rank of a complex system → low rank of matrix
- Do we keep B & C? or C and A?
  - No! U,V have columns/rows that are superposition of basis vectors
  - Whole point of SVD : creates collective coordinates based on information content.

1. Picture credit (foodweb) :- https://biologynotesonline.com/food-web/
2. Picture credit (network and matrix)https://graphicmaths.com/computer-science/graph-theory/adjacency-matrices/

Testing low-rank hypothesis on random networks

- The authors test the low-rank hypothesis for several random networks.

$$W = \langle W \rangle + R$$

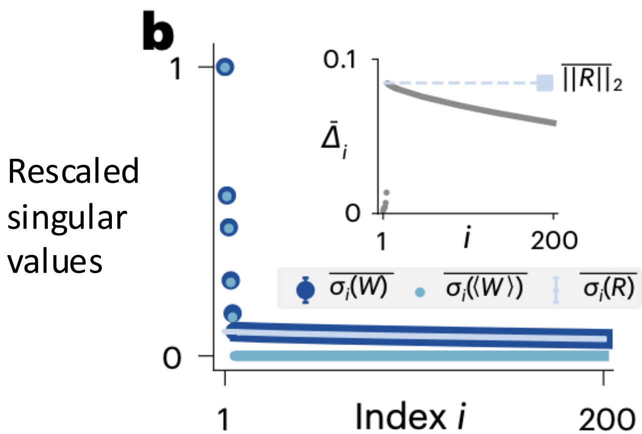
Where W is the weight matrix, $\langle W \rangle$ is the mean weight matrix and R is the noise matrix.

Rank-perturbed Gaussian

$\langle W \rangle = \phi(L) = L$

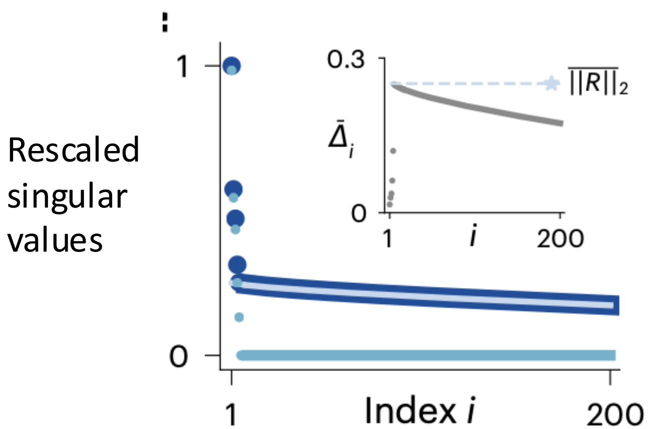
rank $(L) \ll N$

R: Gaussian

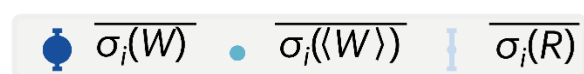


Small noise

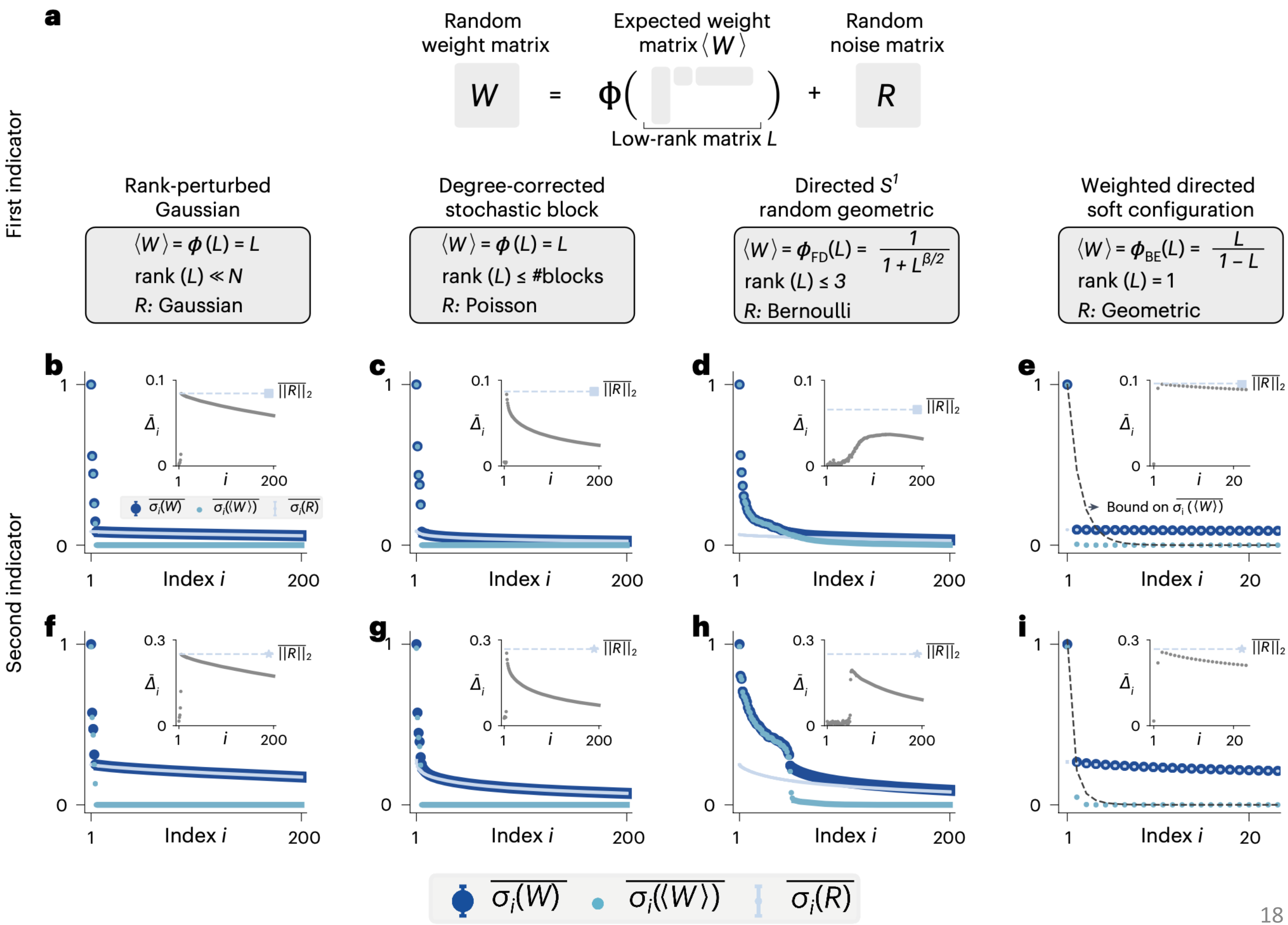
- Indicators of low-rank hypothesis
  - Avg weight matrix <*W*> is a function ϕ(L) of some low rank matrix *L.*
  - Few dominant singular values, quick decay.
  - Low rank: Effective rank should scale sub-linearly with system size.



Large noise



Singular values of random networks



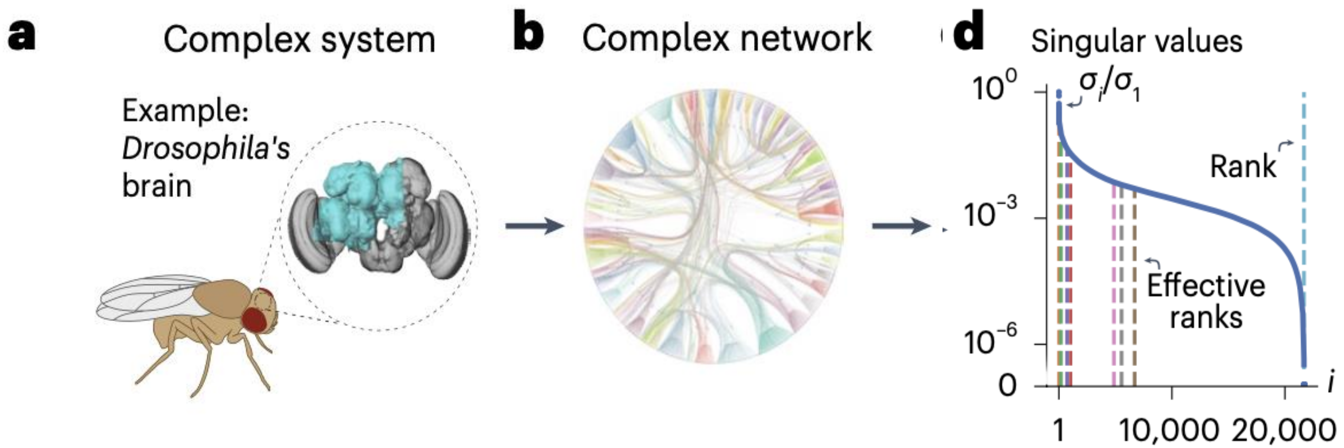
Different measures of effective ranks

Table 2 | Different effective ranks of a matrix of dimension $N \times N$ and of rank r expressed in terms of its singular values $\sigma_1 \geq \ldots \geq \sigma_N$

Abbreviation	Expression
srank	$\sum_{i=1}^{r} \sigma_i^2 / \sigma_1^2$
nrank	$\sum_{i=1}^{r} \sigma_i / \sigma_1$
energy	$\min\left[\arg\max_{\ell \in \{1,\ldots,N\}} \left(\sum_{i=1}^{\ell} \sigma_i^2 / \sum_{j=1}^{r} \sigma_j^2 > \tau\right)\right]$
elbow	$\frac{1}{\sqrt{2}} \arg\max_{i \in \{1,\ldots,N\}} \left\lvert \frac{i-1}{N-1} + \frac{\sigma_i - \sigma_N}{\sigma_1 - \sigma_N} - 1 \right\rvert - 1$
erank	$\exp\left[-\sum_{i=1}^{r} \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j} \log \frac{\sigma_i}{\sum_{j=1}^{r} \sigma_j}\right]$ e^(entropy)
thrank	$\#\left\{\sigma_i \mid i \in \{1, \ldots, N\} \text{ and } \sigma_i > \frac{4\sigma_{\text{med}}}{\sqrt{3}\mu_{\text{med}}}\right\}$
shrank	$\#\{s^*(\sigma_i) \mid i \in \{1, \ldots, N\} \text{ and } s^*(\sigma_i) > 0\}$

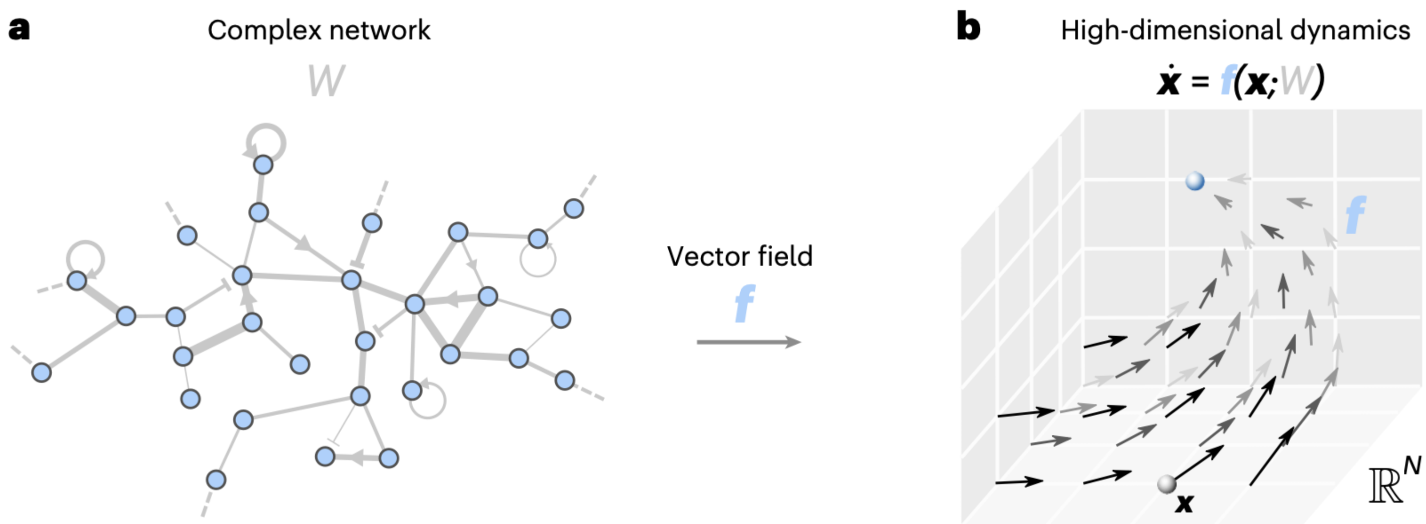
Testing the low rank hypothesis on real networks

- Looked at singular-value profile of the connectome of *Drosophila melanogaster* for 679 real networks from ten different origins.

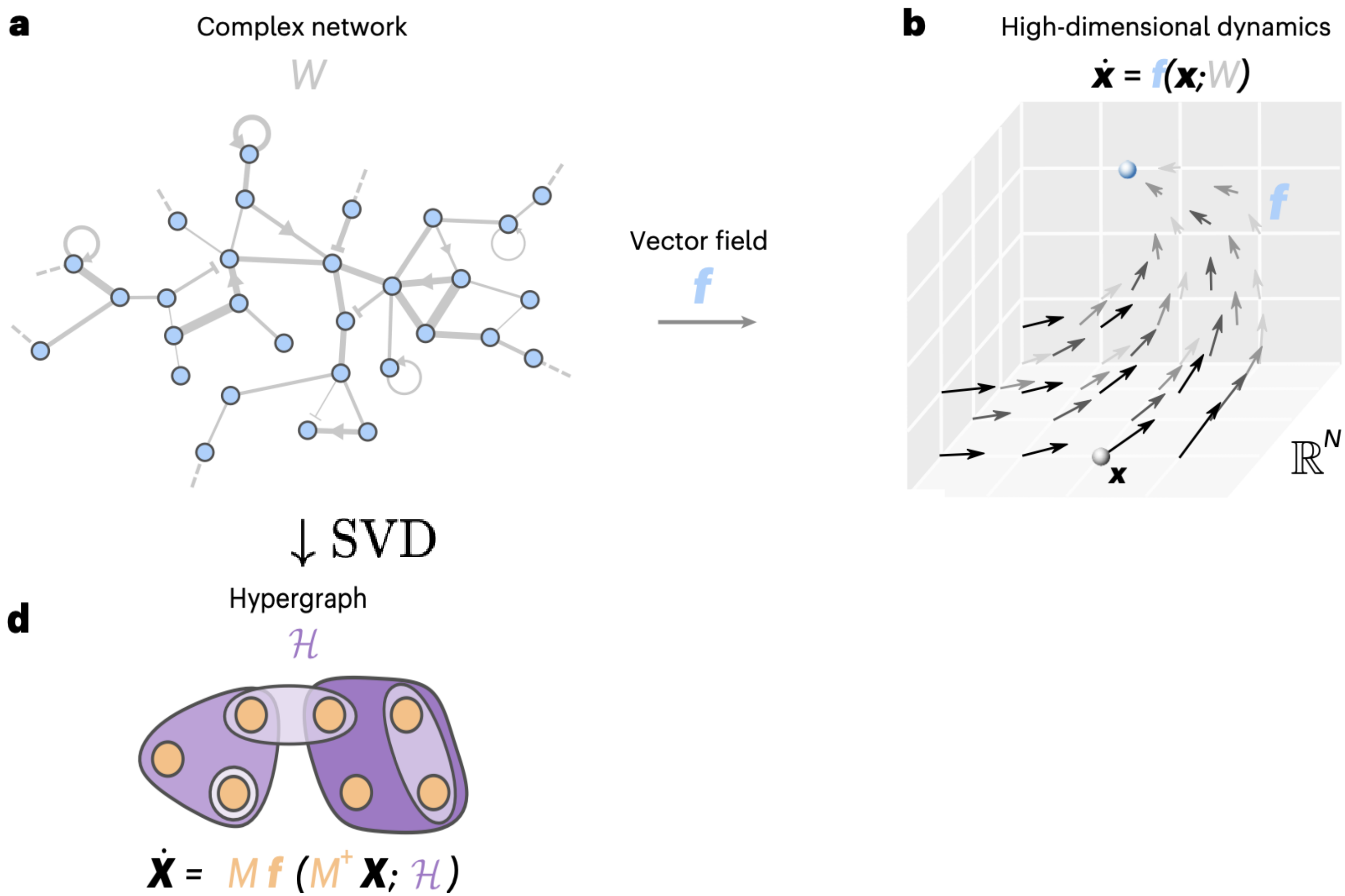


- Such observations seem to be widespread for big data matrices, but they remain puzzling.

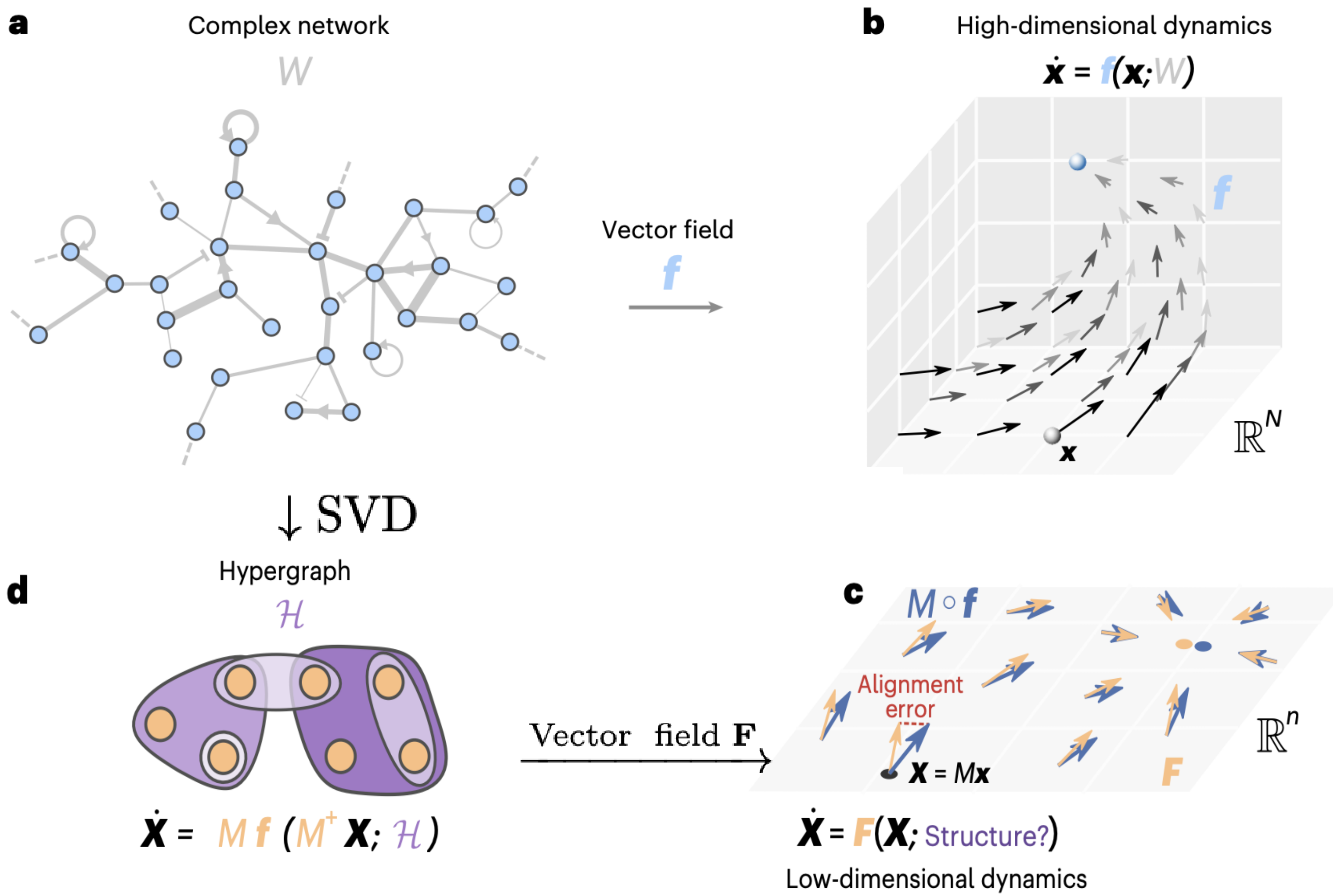
How does low rank affect dynamics?



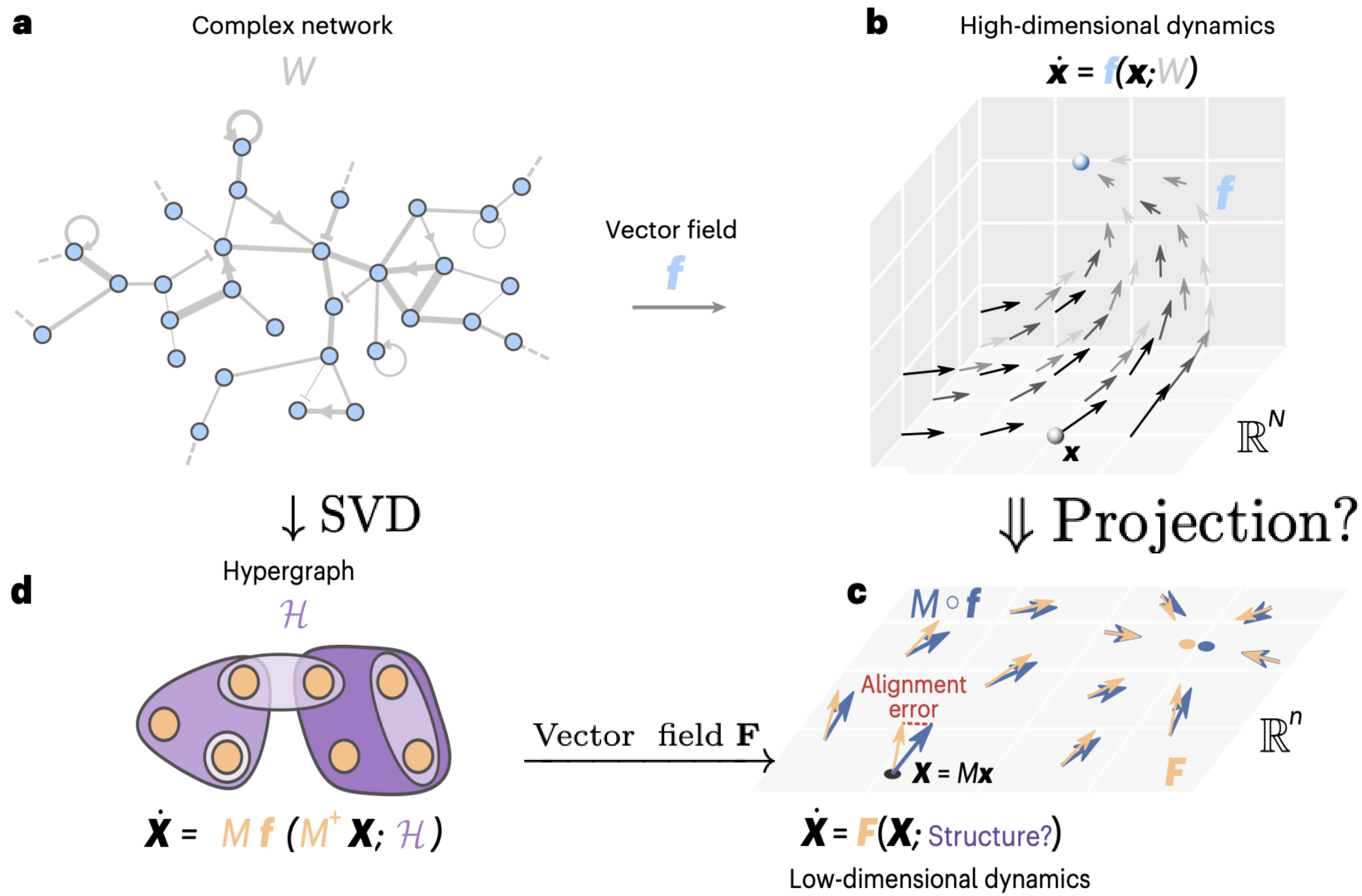
How does low rank affect dynamics?



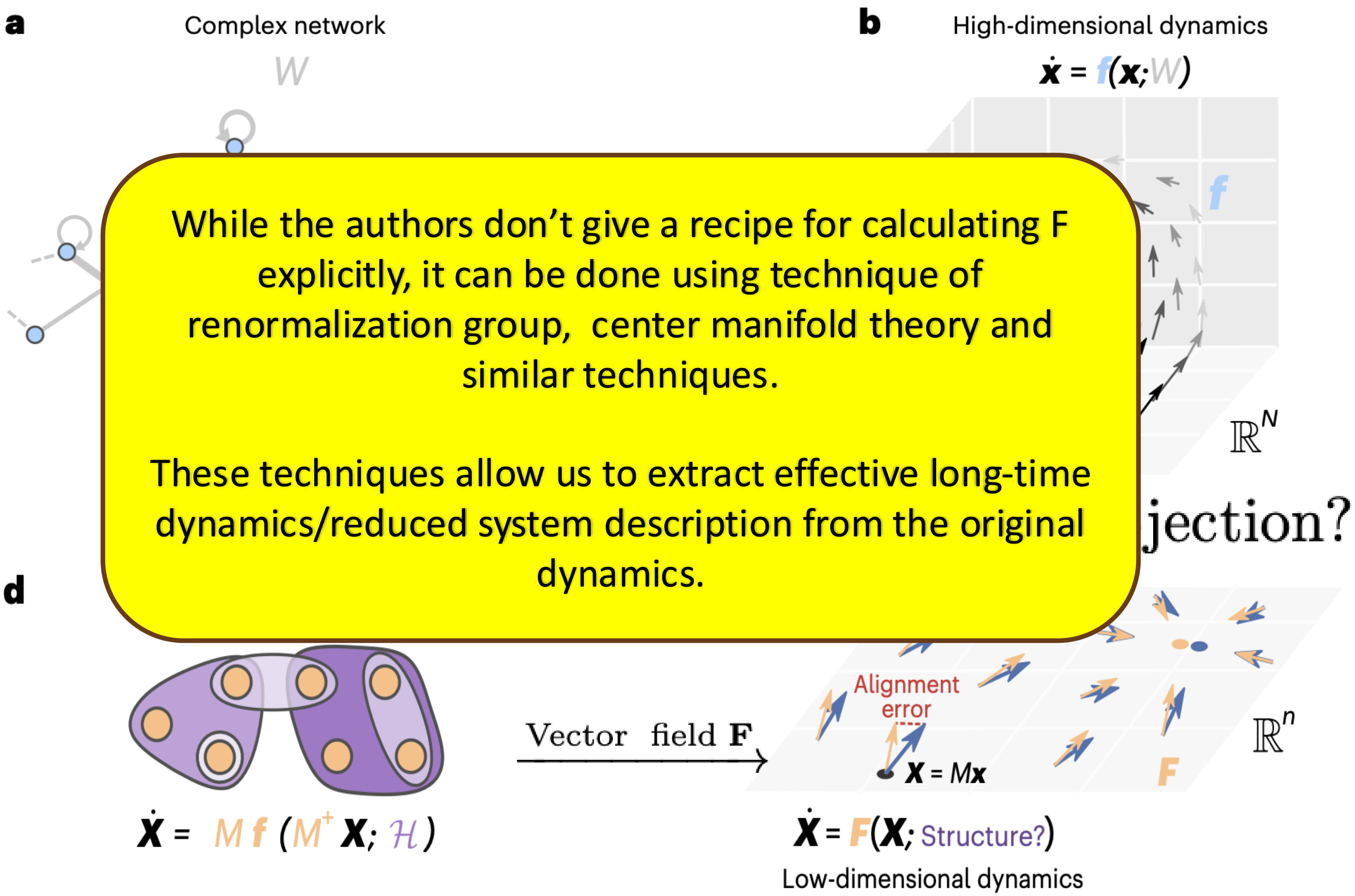
How does low rank affect dynamics?



How does low rank affect dynamics?



How does low rank affect dynamics?



While the authors don't give a recipe for calculating F explicitly, it can be done using technique of renormalization group, center manifold theory and similar techniques.

These techniques allow us to extract effective long-time dynamics/reduced system description from the original dynamics.

- Alignment error:
  Error when comparing observables from reduced dynamics and its higher dimensional representation.

- Good dimensional reduction?
  - Preserve qualitative/quantitative features depending on particular question

- Does low rank → low dimensions? No!
  - Depends on the observables you track.

- The authors consider dynamics of the kind

$$\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x}, W) \qquad \mathbf{g} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$$

  where *W* describes the $\mathrm{N} \times \mathrm{N}$ network. Calculate bounds on optimal alignment error.

Alignment error is bounded by singular values

The authors choose projection using SVD to study the effect of low-rank nature on reduced dynamics. Upper bound

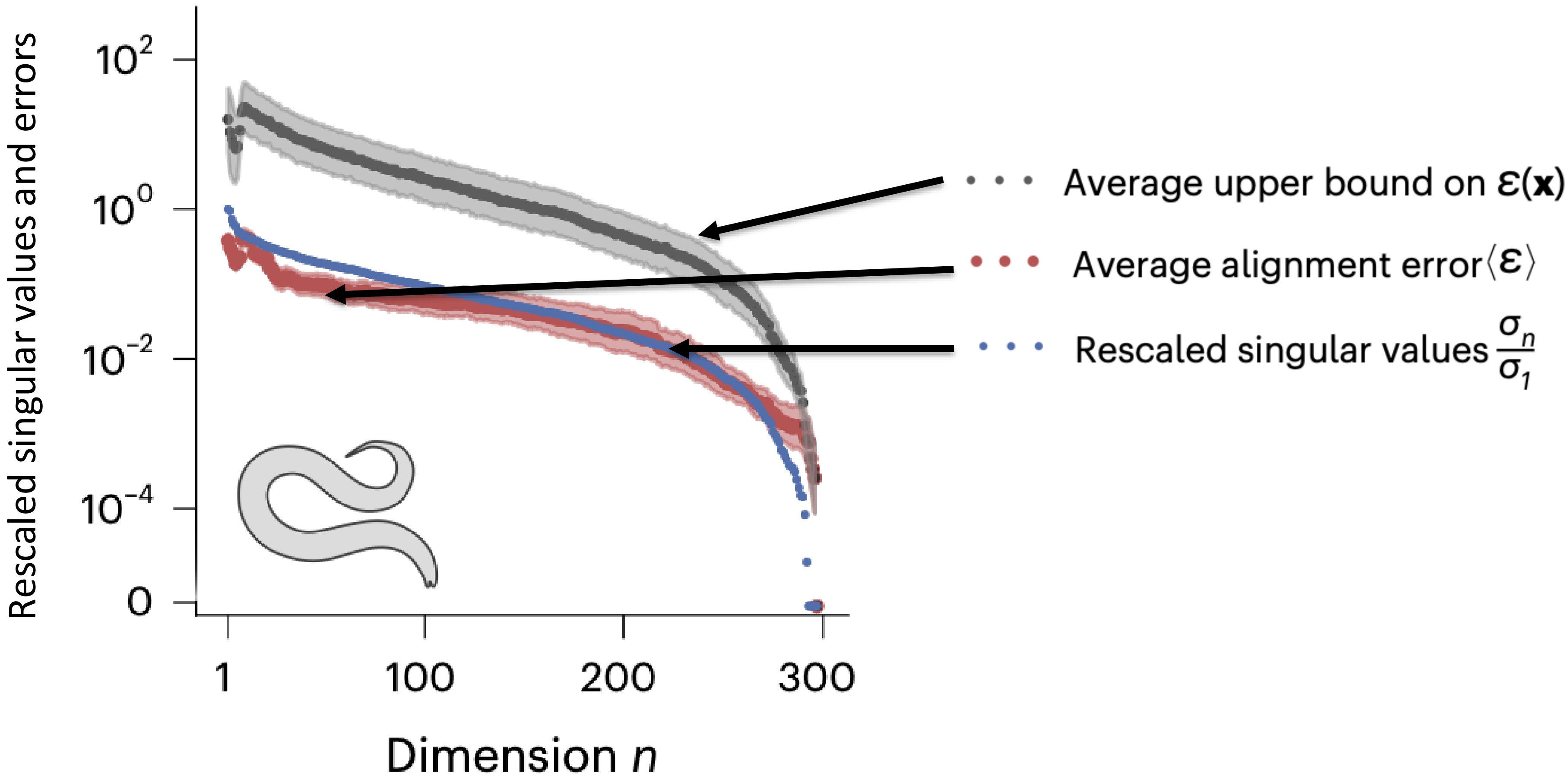
$$\sqrt{n}\,\mathcal{E}(\mathbf{x}) \leq \|V_n^\top J'_{\mathbf{x}}(I-P)\mathbf{x}\| + \boxed{\sigma_{n+1}}\|V_n^\top J'_{\mathbf{y}}\|_2\,\|\mathbf{x}\|$$

Alignment error

Next order singular value for a rank-n SVD

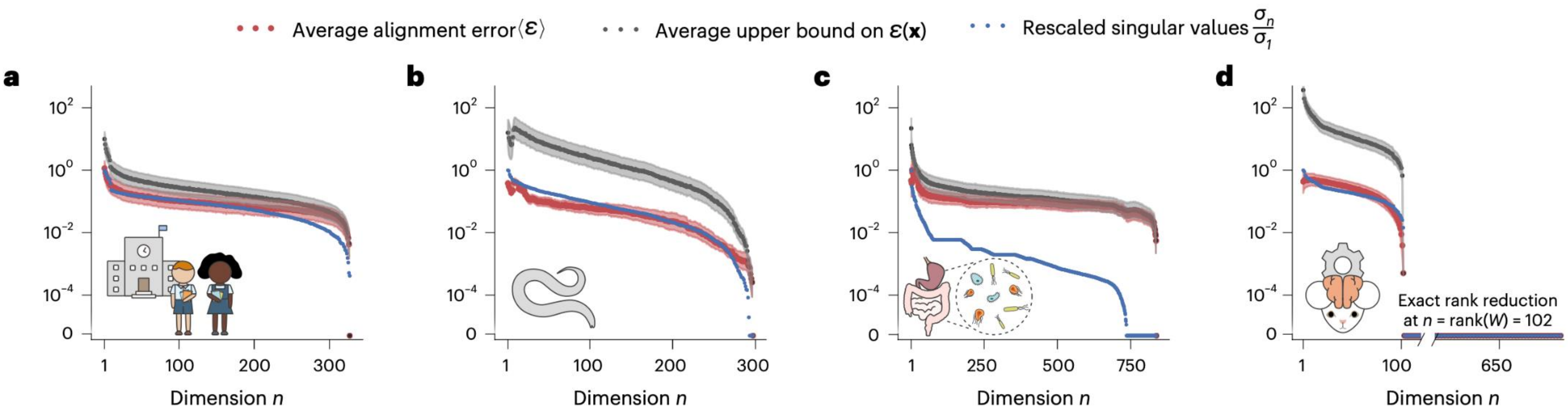
Vanishes for a class of dynamics (e.g. RNNs and neuronal dynamics). Dimension reduction is exact for these models.

Alignment error is bounded by singular values



Wilson-Cowan model of neuronal dynamics on the *C. elegans* connectome

Alignment error is bounded by singular values



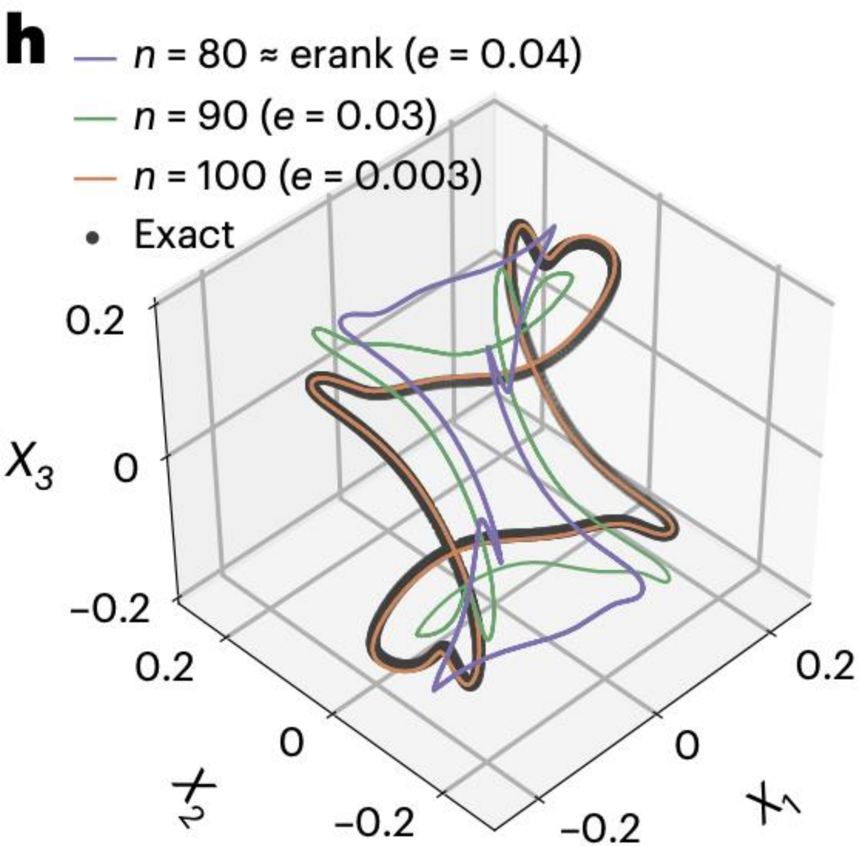
Susceptible-Infected-Susceptible Model of disease spreading On a high school network

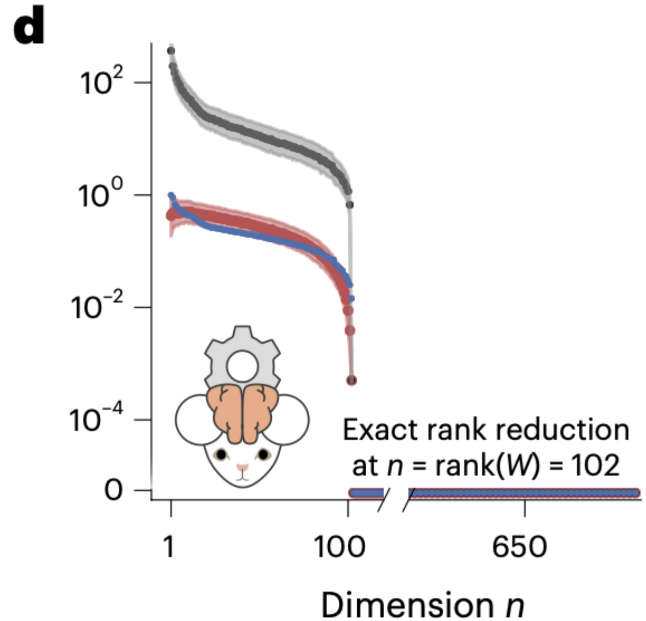
Wilson-Cowan model Of neuronal dynamics On the *C.elegans* connectome

Microbial population dynamics on a human gut microbiome network

RNN dynamics on a learned network (N = 669, signed, weighted, directed)

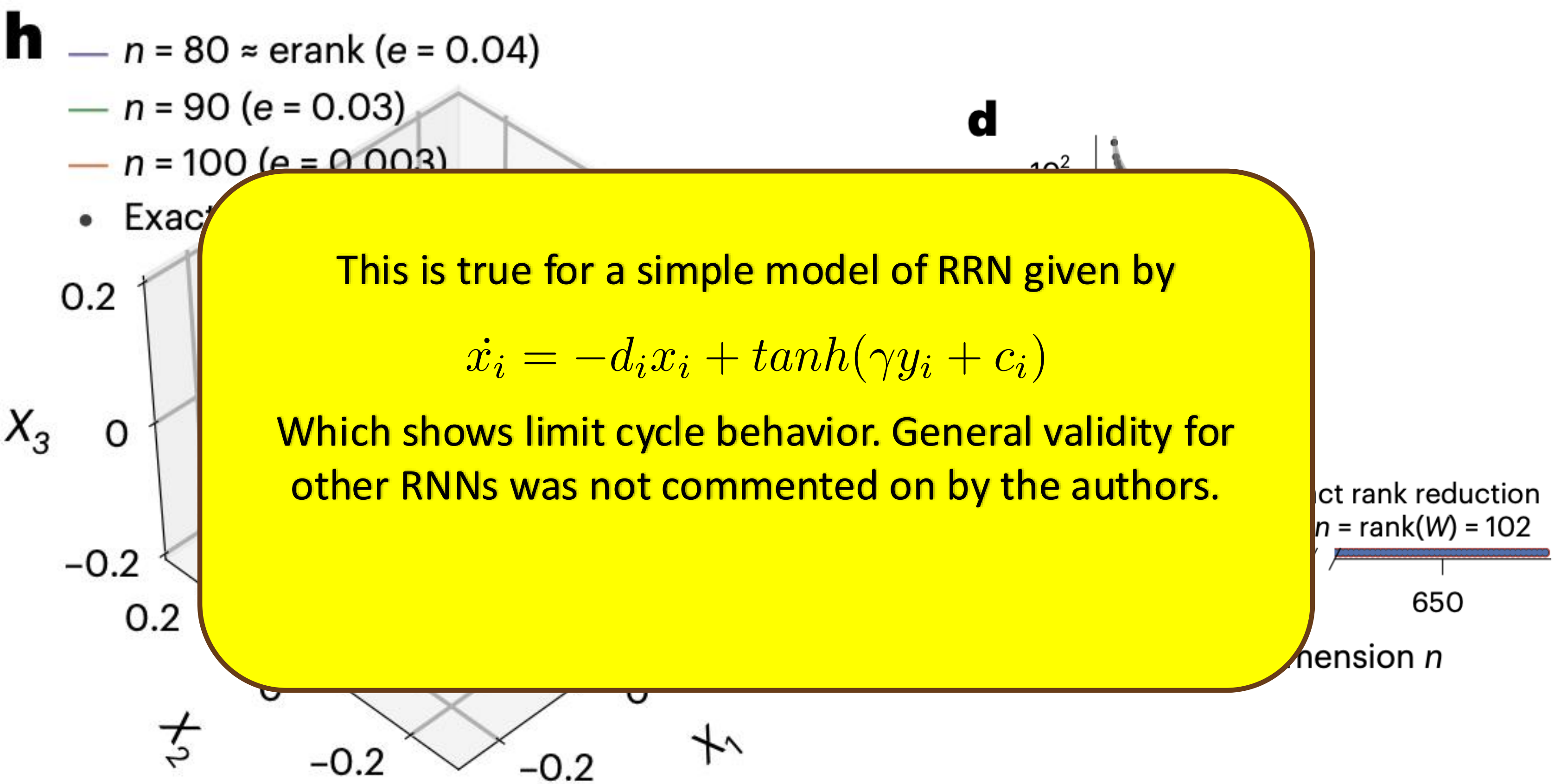
Dynamics is well captured by low-rank





Limit cycle of RNN dynamics on a learned network (N = 669, signed, weighted, directed)

Dynamics is well captured by low-rank



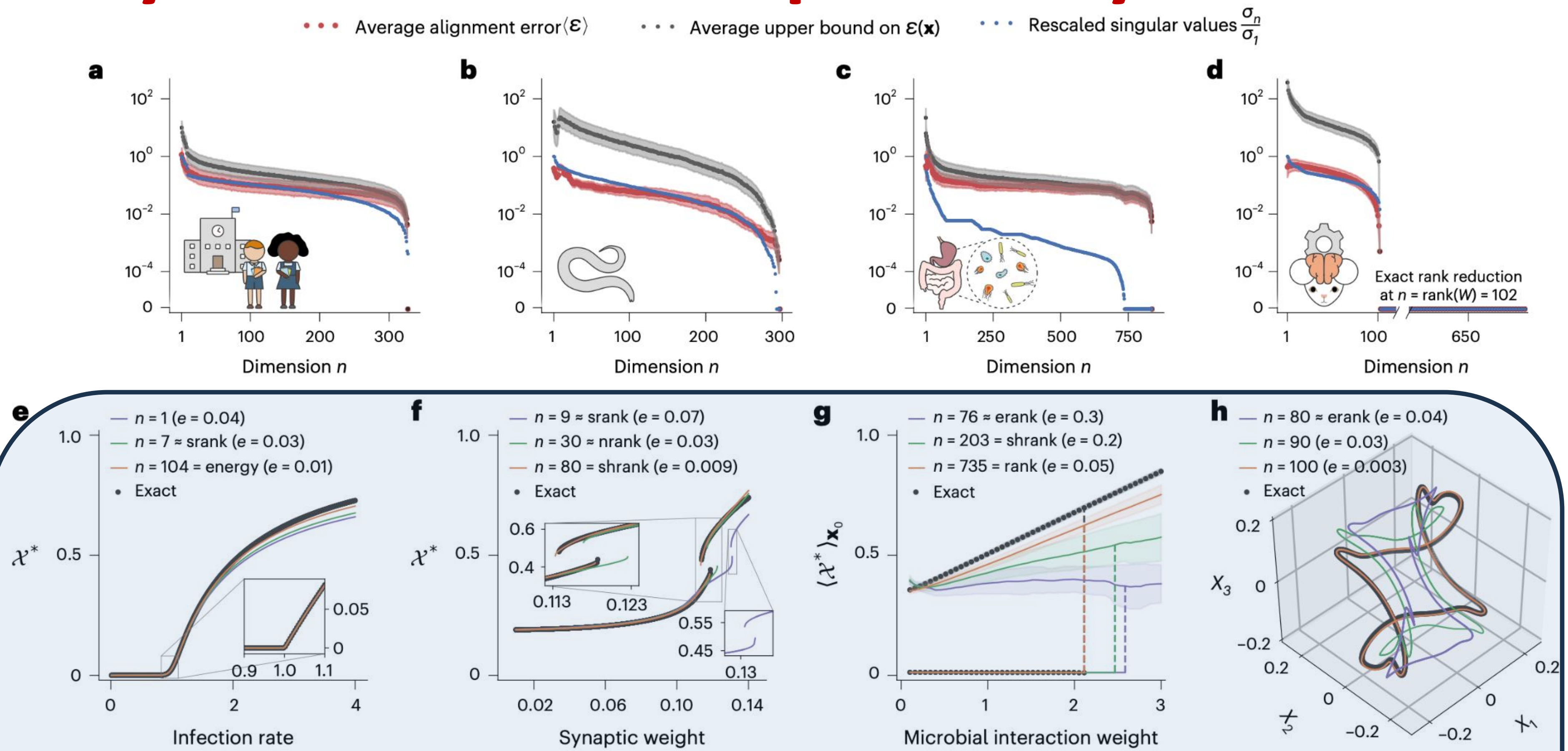
This is true for a simple model of RRN given by

$$\dot{x}_i = -d_i x_i + tanh(\gamma y_i + c_i)$$

Which shows limit cycle behavior. General validity for other RNNs was not commented on by the authors.

Limit cycle of RNN dynamics on a learned network (N = 669, signed, weighted, directed)

Dynamics is well captured by low-rank



Susceptible-Infected-Susceptible Model of disease spreading On a high school network

Wilson-Cowan model Of neuronal dynamics On the *C.elegans* connectome

Microbial population dynamics on a human gut microbiome network

RNN dynamics on a learned network (N = 669, signed, weighted, directed)

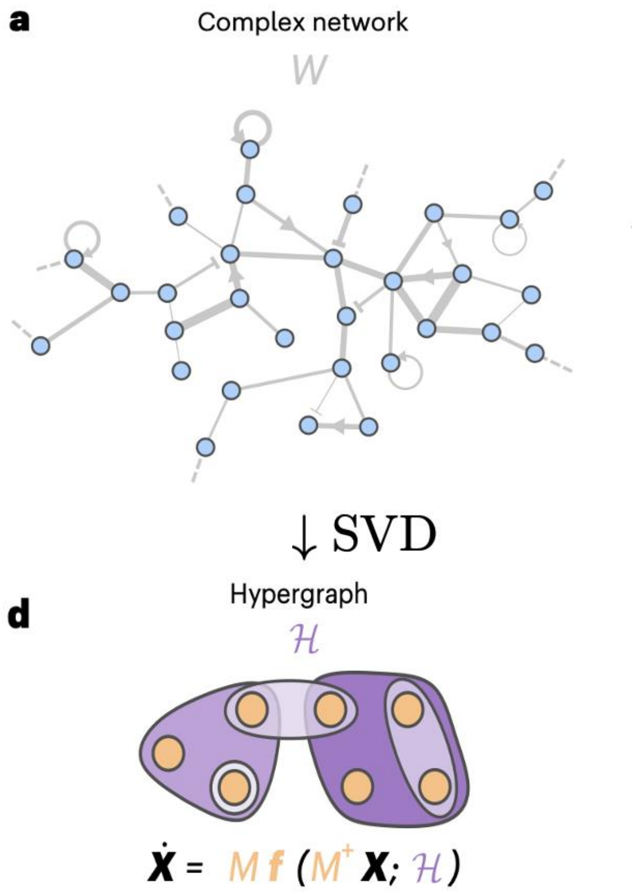
Emergence of higher order interactions

- Reducing the dimensionality produces a higher order interaction in the reduced system.

$$\dot{x}_i = -d_i x_i + \gamma(1 - x_i) y_i$$

$$\Downarrow$$

$$\dot{X}_\mu = \sum_{\nu=1}^{n} (\mathcal{D}_{\mu\nu} + \mathcal{W}_{\mu\nu}) X_\nu + \sum_{\nu,\kappa=1}^{n} \mathcal{T}_{\mu\nu\kappa} X_\nu X_\kappa$$



- Reason? Clustering: Reducing dimensions creates clusters of vertices from the original network. Observables depend on all vertices in a cluster.
- More generally, non-linear dynamics.

Are there systems which are not low-rank?

Yes! The brain

The brain: Criticality conjecture

- Case for brain at criticality
  - Claim: Brain operates near a critical point of a phase transition
  - Neural networks self-organize near a phase transition between subcritical (low activity) and supercritical (overactive) regimes.
  - Not low rank, need all scales.

- Evidence for
  - Neuronal Avalanches follow power-law distribution[1]
  - High sensitivity to perturbations, optimal information processing
  - Critical slowing down

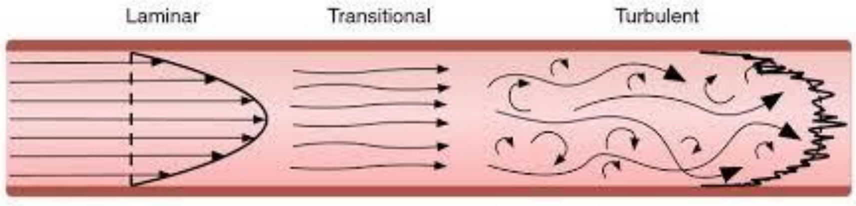
- Evidence against
  - Multiple neural states, transition b/w different regimes. Not at critical point
  - Non-power law fits, features could arise from different mechanisms
  - Local criticality, not global

1. *Beggs, J. M.; Plenz, D. (2003). "Neuronal avalanches in neocortical circuits". The Journal of Neuroscience.* ***23*** *(35): 11167–11177.*

Are there systems which are not low-rank?

- Phase transitions

- Fluid transition to turbulence:
  - Low rank approach: study most unstable modes and their linear instability as a function of Reynolds number.
  - Linear unstability
  - not only doesn't predict "ANY" transition, but in cases where it does, it predicts an incorrect critical Re[1].



- Economics

1. Shih, HY., Hsieh, TL. & Goldenfeld, N. Ecological collapse and the emergence of travelling waves at the onset of shear turbulence. Nature Phys 12, 245–248 (2016).
2. Lemoult, G. et al. Directed percolation phase transition to sustained turbulence in Couette flow. Nat. Phys. 12, 254–258 (2016).
Picture credit: Heritability of haemodynamics in the ascending aorta. Scientific Reports. 2020. 14356. 10.1038/s41598-020-71354-7.

My Perspective

- The authors demonstrate low-rank quantifiers and validate dimension reduction across various models and data but lack predictive tools for identifying low-rank systems during modeling.

- Their analysis is post-dictive, justifying low-rank structures retrospectively, but extending it to new problems requires additional signatures.

Conclusion

- Low-rank networks seem ubiquitous in nature. They allow for dimension reduction, modelling of reduced dynamics and are easier to interpret.

- The authors quantify metrics of effective ranks for networks and give sufficient (but not necessary) conditions for a complex network to be low rank.

- Give a mechanism for emergence of higher order interactions .
  As the system is probed at a coarser scale, these higher order interactions would likely manifest themselves.

- They show exact dimension reduction for models of RNNs, neuronal dynamics, disease spreading and microbial population dynamics.

Reserve Slides

- Good for approximation. What else?

- Used to extract relevant subspace of a dataset, commonly under the technique of principal component analysis (PCA).

- For a low rank matrix i.e. small k, it can impute missing data values of a matrix. Replace the missing value with the column mean, and compute the SVD up to required rank. Repeat this unto the entries stop changing.

- Also a number of other applications.

Name	Complete vector field $h_i(x_i, y_i)$	Reduced vector field $H_\mu(X_1, ..., X_n)$
Epidemiological	$-d_i x_i + \gamma(1 - x_i)\, y_i$	$\sum_{\nu=1}^{n} (\mathcal{D}_{\mu\nu} + \mathcal{W}_{\mu\nu})\, X_\nu + \sum_{\nu,\kappa=1}^{n} \mathcal{T}_{\mu\nu\kappa} X_\nu X_\kappa$
Microbial	$a - d_i x_i + b x_i^2 - c x_i^3 + \gamma\, x_i\, y_i$	$\mathcal{C}_\mu + \sum_{\nu=1}^{n} \mathcal{D}_{\mu\nu} X_\nu + \sum_{\nu,\kappa=1}^{n} \left(\mathcal{D}_{\mu(\nu,\kappa)} + \mathcal{T}_{\mu\nu\kappa}\right) X_\nu X_\kappa + \sum_{\nu,\kappa,\tau=1}^{n} \mathcal{D}_{\mu(\nu,\tau,\kappa)} X_\nu X_\kappa X_\tau$
Oscillator	$\mathrm{i}\omega_i x_i + \gamma\, e^{-\mathrm{i}\alpha}\, y_i - \gamma\, e^{\mathrm{i}\alpha}\, x_i^2\, \bar{y}_i$	$\sum_{\nu=1}^{n} (\mathcal{D}_{\mu\nu} + \mathcal{W}_{\mu\nu})\, X_\nu + \sum_{\nu,\kappa,\tau=1}^{n} \mathcal{T}_{\mu(\nu,\kappa)\tau} X_\nu X_\kappa \bar{X}_\tau$
RNN	$-d_i x_i + \tanh(\gamma y_i + c_i)$	$\sum_{\nu=1}^{n} \mathcal{D}_{\mu\nu} X_\nu + \sum_{i=1}^{N} M_{\mu i} \tanh\left(\gamma \sum_{\nu=1}^{n} \mathscr{W}_{j\nu} X_\nu + c_i\right)$
Neuronal	$-d_i x_i + (1 - a x_i)\, \mathcal{S}[b(\gamma\, y_i - c_i)]$	$\sum_{\nu=1}^{n} \mathcal{D}_{\mu\nu} X_\nu + \sum_{j=1}^{N} M_{\mu j} \left(1 - a \sum_{\nu=1}^{n} M_{j\nu}^{+} X_\nu\right) \mathcal{S}\left[b\left(\gamma \sum_{\kappa=1}^{n} \mathscr{W}_{j\kappa} X_\kappa - c_i\right)\right]$

Alignment error is bounded by singular values

- If one focuses on observables X_μ in a lower dimension R^n $(\mathrm{n} < \mathrm{N})$ then alignment error is defined (in a least squares sense)

$$\|F \circ X - M \circ g\|$$

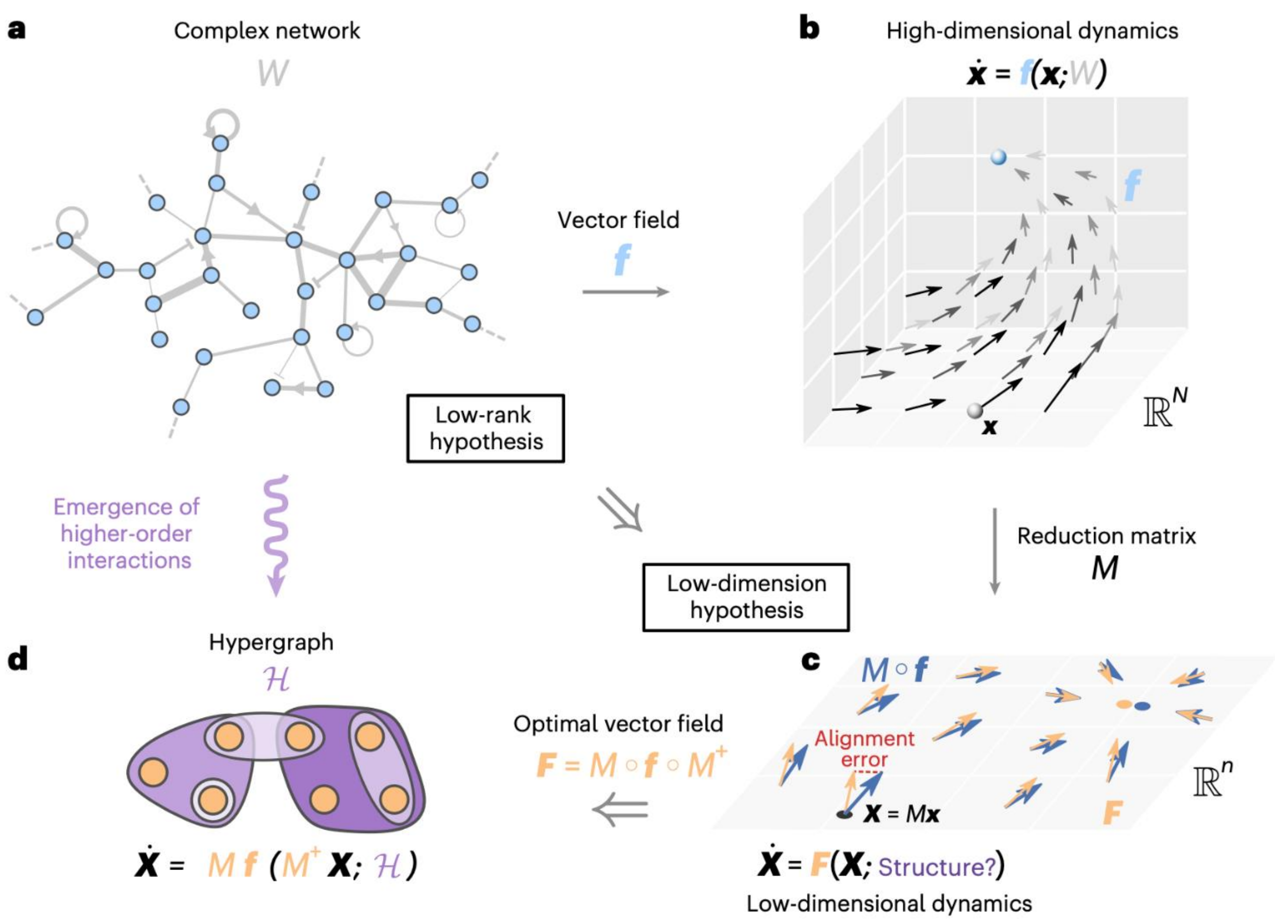
  where F denotes the new dynamics in the reduced dimensions and M denotes a mapping $R^N \rightarrow R^n$ between the complete system and the reduced system.

- The choice of (F, M) depends on the modeler's objective. The authors choose M as V^T using the SVD decomposition to study the effect of low-rank nature on reduced dynamics. This allows the

$$\sqrt{n}\,\mathcal{E}(\mathbf{x}) \leq \|V_n^\top J'_{\mathbf{x}}(I - P)\mathbf{x}\| + \sigma_{n+1}\|V_n^\top J'_{\mathbf{y}}\|_2\,\|\mathbf{x}\|$$

How does low rank affect dynamics?

- Intuitively, low rank would help ….. Reduced dimensions.



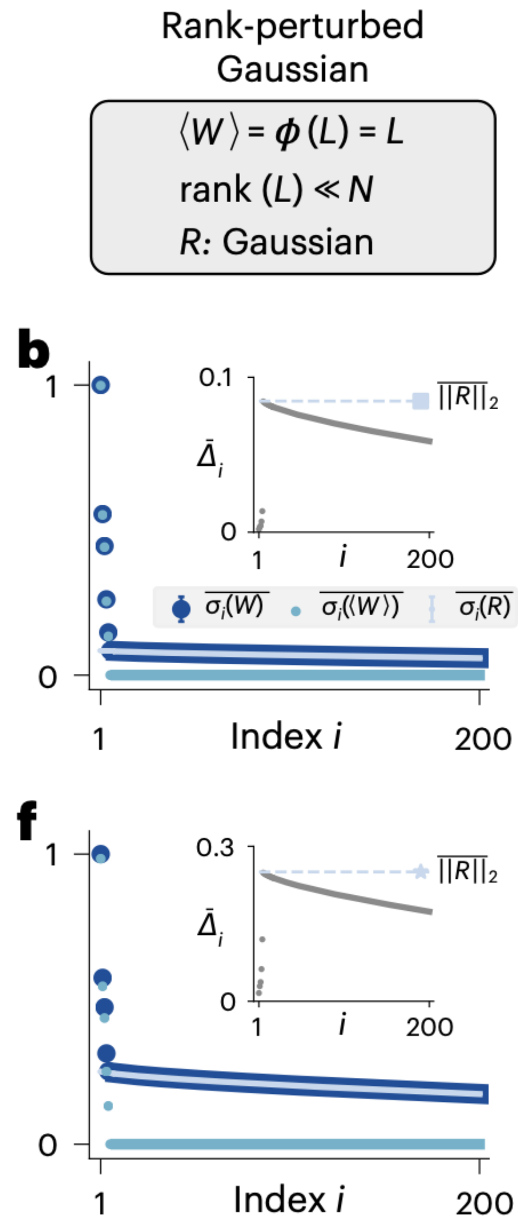
Prep

- What is the low-rank hypothesis?
  - Emphasize the complexity – simplicity duality
  - What do we know before?
- What is SVD and why?
- Network models are low rank?
  - Quantify low-ness
  - Verification for real networks
- Low rank dynamics:
  - Induced low dimensions from low rank graphs (paper)
  - Not low rank data but low rank learned dynamics (Litwin's paper)
- Emergence of higher order interactions
- Low rank can be incorrect: examples
- Another example of same theme: Sloppy models
- Takeaways and relevance for this class
  - RNNs, dimensional reduction and other things
  - What do we know after?
  - What did it set out to do and where does it lacks?

- Indicators of low-rank hypothesis
  - Avg weight matrix <*W*> is a function ϕ(L) of some low rank matrix *L.*
  - Few dominant singular values, quick decay, subdominant ones are affected by noise.
  - Should have a low effective rank i.e. at mo with large N (# of vertices) as

$$O(N^{1-\epsilon}) \qquad \epsilon > 0$$

  - Implies that effective rank over system size vanishes in the large N limit.



Small noise

Large noise

Dimension reduction in physics

- Sloppy models

- Renormalization group

- Procaccia-Grassberger algorithm