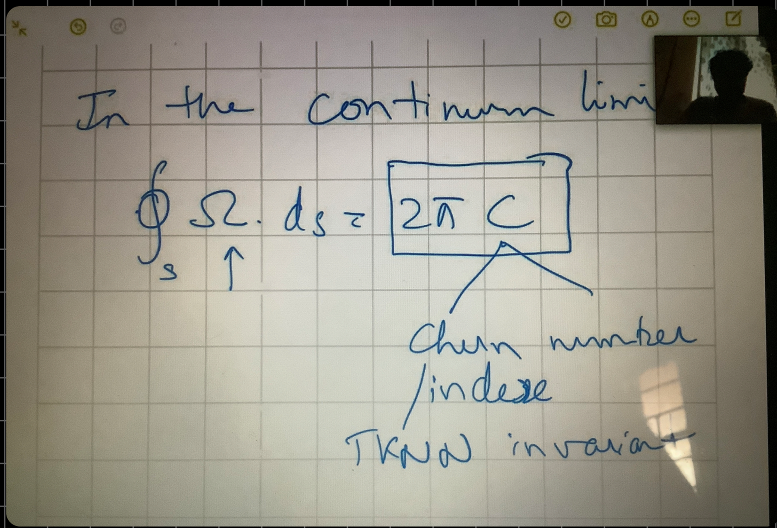
Review :- @ morning class had BC, def^n, mod 2π

#



If we have non-zero BC, then e^-'s in the band get an "anomalous" velocity.

new kids on the block $\Rightarrow$ fractional chern

$\Rightarrow$ due to interactions.

Chern # $\Rightarrow$ filling fraction in Q Hall systems

@ Bernevig

Examples of B. phase

Today's Goal

① $H = -\alpha \vec{S} \cdot \vec{B}$ $\rightarrow$ 3D B_x, B_y, B_z are parameters of H.

② $H = \beta \vec{\sigma} \cdot \vec{P}_{2D}$, $\vec{P}_{2D}$ is the parameter

③ $H = \beta \sigma_0 p + d\sigma_z$ $\rightarrow$ massive dirac eqn.

Ex 1 :- $H = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$ $\longrightarrow$ Eigenvalues $\pm\sqrt{\vec{B}\cdot\vec{B}}$

```
Eigenvectors[mat]
```

$$\left\{\left\{-\frac{\sqrt{x^2+y^2}}{x+i y},1\right\},\left\{\frac{\sqrt{x^2+y^2}}{x+i y},1\right\}\right\}$$

```
minus =
```

$$\left\{-\frac{-z+\sqrt{x^2+y^2+z^2}}{x+i y},1\right\}$$

$$\left\{-\frac{-z+\sqrt{x^2+y^2+z^2}}{x+i y},1\right\}$$

```
minusstar =
```

$$\left\{-\frac{-z+\sqrt{x^2+y^2+z^2}}{x-i y},1\right\}$$

$$\left\{-\frac{-z+\sqrt{x^2+y^2+z^2}}{x-i y},1\right\}$$

```
plus =
```

$$\left\{-\frac{-z-\sqrt{x^2+y^2+z^2}}{x+i y},1\right\}$$

$$\left\{-\frac{-z-\sqrt{x^2+y^2+z^2}}{x+i y},1\right\}$$

```
plusstar =
```

$$\left\{-\frac{-z-\sqrt{x^2+y^2+z^2}}{x-i y},1\right\}$$

$$\left\{-\frac{-z-\sqrt{x^2+y^2+z^2}}{x-i y},1\right\}$$

```
v1 = {minusstar.PauliMatrix[1].plus, minusstar.PauliMatrix[2].plus, minusstar.PauliMatrix[3].plus}
```

$$\left\{-\frac{-z-\sqrt{x^2+y^2+z^2}}{x+i y}-\frac{-z+\sqrt{x^2+y^2+z^2}}{x-i y},\ \frac{i\left(-z-\sqrt{x^2+y^2+z^2}\right)}{x+i y}+\frac{i\left(-z+\sqrt{x^2+y^2+z^2}\right)}{x-i y},\ -1+\frac{\left(-z-\sqrt{x^2+y^2+z^2}\right)\left(-z+\sqrt{x^2+y^2+z^2}\right)}{(x-i y)(x+i y)}\right\}$$

```
v2 = {plusstar.PauliMatrix[1].minus, plusstar.PauliMatrix[2].minus, plusstar.PauliMatrix[3].minus}
```

$$\left\{-\frac{-z-\sqrt{x^2+y^2+z^2}}{x-i y}-\frac{-z+\sqrt{x^2+y^2+z^2}}{x+i y},\ \frac{i\left(-z-\sqrt{x^2+y^2+z^2}\right)}{x-i y}-\frac{i\left(-z+\sqrt{x^2+y^2+z^2}\right)}{x+i y},\ -1+\frac{\left(-z-\sqrt{x^2+y^2+z^2}\right)\left(-z+\sqrt{x^2+y^2+z^2}\right)}{(x-i y)(x+i y)}\right\}$$

```
Cross[v1, v2]
```

$$\left\{-\frac{8 i x^3\sqrt{x^2+y^2+z^2}}{(x-i y)^2(x+i y)^2}-\frac{8 i x y^2\sqrt{x^2+y^2+z^2}}{(x-i y)^2(x+i y)^2},\ -\frac{8 i x^2 y\sqrt{x^2+y^2+z^2}}{(x-i y)^2(x+i y)^2}-\frac{8 i y^3\sqrt{x^2+y^2+z^2}}{(x-i y)^2(x+i y)^2},\ -\frac{8 i x^2 z\sqrt{x^2+y^2+z^2}}{(x-i y)^2(x+i y)^2}-\frac{8 i y^2 z\sqrt{x^2+y^2+z^2}}{(x-i y)^2(x+i y)^2}\right\}$$

```
FullSimplify[Cross[v1, v2]]
```

$$\left\{-\frac{8 i x\sqrt{x^2+y^2+z^2}}{x^2+y^2},-\frac{8 i y\sqrt{x^2+y^2+z^2}}{x^2+y^2},-\frac{8 i z\sqrt{x^2+y^2+z^2}}{x^2+y^2}\right\}$$

```
bc = FullSimplify[Cross[v1, v2] / ((minusstar.minus) * (plusstar.plus))]
```

$$\left\{-\frac{2 i x}{\sqrt{x^2+y^2+z^2}},-\frac{2 i y}{\sqrt{x^2+y^2+z^2}},-\frac{2 i z}{\sqrt{x^2+y^2+z^2}}\right\}$$

```
berrycurvature = Im[bc/((2) * Sqrt[z^2 + x^2 + y^2])^2]
```

$$\left\{-\frac{1}{2}\mathrm{Re}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right],-\frac{1}{2}\mathrm{Re}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right],-\frac{1}{2}\mathrm{Re}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right]\right\}$$

```
Integrate[r^2/(2 (r^2)), {t, 0, 4π}]
```

2π

H.W :- Supply a written calc. for this.

Like a source of B.Curv. at

$(x,y,z) = (0,0,0)$

note :- @MD :- Calculatⁿ of this for 3/2 spin / some other spin will in general be different.

Example:- Massless 2D dirac $\mathcal{H}$

$$\mathcal{H} = \vec{\sigma}\cdot\vec{p}_{2D}$$

$$\mathcal{H} = \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix} \qquad \underline{\underline{\pm\sqrt{p_x^2+p_y^2}}}$$

$$\langle u(\lambda)| d_\lambda u(\lambda)\rangle$$

$$\mathcal{H} = p\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

scales out

evs:- $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} \\ 1 \end{pmatrix}$ $\to |\uparrow\rangle$, $\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi} \\ -1 \end{pmatrix} \to |\downarrow\rangle$

$+p$ $\qquad$ $-p$

$\phi \in [0, 2\pi)$ or S^1 [this case corresponds to @ Vatsal's argument of $\oint A$ being topological since there's just one loop)

$$A^1_\phi = +i\langle\downarrow|\partial_\phi|\downarrow\rangle = \frac{1}{2} \qquad A^2_\phi = i\langle\uparrow|\partial_\phi|\uparrow\rangle = \frac{1}{2}$$

<u>Berry phase:</u> $e^{i\gamma} = \exp\left[i\oint A_\phi\, d\phi\right]$

$= \exp[i\pi]$ i.e. $\gamma = \pi$ (mod 2π ofc)

Berry phase is same for both states.

No Berry curvature though.

why?

$$\mathcal{H} = \vec{p}_{2D}\cdot\vec{\sigma}_{2D}$$

preserves sublattice symmetry if it was a lattice model like SSH

- TR invariant ~~yes~~ NO!
- Inv. symm. ✓

for TRS $\mathcal{H}_B(k) = \mathcal{H}_B^*(-k)$

$\to$ here no Berry curvature as such.

<u>why?</u>

① under inv. $\Omega(\vec{k}) = \Omega(-\vec{k})$

Proof:-

$A_n(\vec{k}) = i\langle u_{n,\vec{k}} | \nabla_{\vec{k}} | u_{n,\vec{k}}\rangle$ for a bloch state

under inv symm (if present).

now under inv. symmetry of lattice, from sch eqn, if

$$\left[-\frac{\hbar^2}{2m}\nabla_r^2 + V(r)\right]\psi_{n,k}(\vec{r}) = E_{nk}\,\psi_{nk}(\vec{r})$$

is a soln., then setting $r' = -r$ also satisfies the same eqn.

$\therefore\ \psi_{n,k}(-\vec{r})$ is also another eigenfn with same E_k.

But now suppose,

$$\psi_2 = S\psi_{n,k}(\vec{r}) = \psi_{n,k}(-\vec{r}) \Rightarrow T_R\,\psi_2 = \psi_{nk}(-\vec{r}-\vec{R})$$

$$= e^{+i(-k)\vec{R}}\,\psi_{n,k}(-\vec{r})$$

$$= e^{i(-\vec{k})\vec{R}}\,\psi_2(\vec{r})$$

$\therefore\ \psi_2$ is an eigenstate of T_R with $\exp(+i(-\vec{k})\cdot\vec{R})$ evalue.

$\therefore\ S\psi_{n,k} \rightarrow$ has same en. as $\psi_{n,k}(\vec{r})$

$\rightarrow$ momentum $(-\vec{k})$

$\therefore\ S\psi_{n,k} = \psi_{n,-\vec{k}}(\vec{r})$ [as $E(-\vec{k}) = E(\vec{k})$ under inversion]

$\therefore$ under inversion $|u_{n,k}\rangle \longrightarrow |u_{n,-k}\rangle$

now, $\vec{A}_n(\vec{k}) = i\langle u_{n,k}|\nabla_k|u_{n,k}\rangle$

under inversion,

$$\vec{A}' = i\langle u_{n,k}|I^{+}\,\nabla_k\, I|u_{n,k}\rangle$$

$$= i\int u_{n,-k}(\vec{r})\,\nabla_k\, u_{n,-\vec{k}}(\vec{r})\,d\vec{r}$$

$$= -\vec{A}_n(-\vec{k})$$

Consequently, $\gamma_\eta = \int d\vec{k}\, \vec{A}_\eta(\vec{k}) \longrightarrow \int d\vec{k}\, -\vec{A}_\eta(-\vec{k})$

$$= -\int d\vec{k}\, \vec{A}_\eta(\vec{k})$$

$$= -\gamma_\eta$$

Also,

$$\vec{\Omega}_\eta(\vec{k}) = \nabla_{\vec{k}} \times A_\eta(\vec{k}) \xrightarrow{I} \nabla_{\vec{k}} \times -A_\eta(-\vec{k})$$

$$= \nabla_{-\vec{k}} \times A_\eta(-\vec{k})$$

$$= \vec{\Omega}_\eta(-\vec{k})$$

$$\boxed{\therefore\ \vec{\Omega}_\eta(\vec{k}) = \Omega_\eta(-\vec{k})}$$

② Claim 2: under TRS,

$$\Omega(-\vec{k}) = -\Omega(\vec{k})$$

Proof:- • for a scalar particle, $\mathcal{T}\psi(\vec{r}) = \psi^*(\vec{r})$

• for spinors $\mathcal{T} = i\sigma_y K$

• Now suppose we have a crystalline system with TR invariance.

Then, $|\psi_{n,\vec{k}}\rangle$ is **degenerate** with TR partner at $-\vec{k}$ related by

$$\mathcal{T}|\psi_{n,k}\rangle = e^{i\varphi(n,\vec{k})}|\psi_{n,-\vec{k}}\rangle \quad \text{[equality upto a phase]}$$

$$\Rightarrow |u_{n,\vec{k}}\rangle = e^{i\varphi}|u_{n,-\vec{k}}\rangle$$

However, under TR transf.

$$\cancel{\mathcal{T}}\ u_{n,\vec{k}}(\vec{r}) = u_{n,-\vec{k}}\cancel{(\vec{r})^*} \quad \text{[whether or not the system TRI]}$$

$\therefore$ If a system is TRI,

$$u_{n,-k}(\vec{r})^* = e^{i\varphi}\, u_{n,-\vec{k}}(\vec{r})$$

Under TR transf

$$\mathcal{T}\vec{A}_n(\vec{k}) = i\langle u_{n,k}|\cancel{\mathcal{T}^\dagger \nabla_k \mathcal{T}}|u_{n,k}\rangle$$

$$= \int \cancel{u_{n,-\vec{k}}(\vec{r})\, \nabla_k\, u^*_{n,-\vec{k}}(\vec{r})}\, d\vec{r}$$

$$= \cancel{\int \nabla_k \left(\underbrace{u_{n,-\vec{k}}(\vec{r})\, u^*_{n,-\vec{k}}(\vec{r})}_{\text{indpt of } \vec{k}} \right] d^3\vec{r} - \int u^*_{n,-k}(\vec{r})\, \nabla_k u_{n,-\vec{k}}(\vec{r})\, d\vec{r}}$$

$$= \cancel{0 + \int d\vec{r}\, u^*_{n,-k}(\vec{r})\, \nabla_{-k} u_{n,-\vec{k}}(\vec{r})\, d\vec{r}}$$

$$\Rightarrow \cancel{\mathcal{T}\vec{A}_n(\vec{k}) = \vec{A}_n(-\vec{k})} \longrightarrow \text{take } B = \mathcal{T}A_n \text{ \& do } \underline{\underline{B^*}}$$

$$\therefore \quad \mathcal{T}\vec{\Omega}_n(\vec{k}) = \nabla_{\vec{k}} \times \mathcal{T}\vec{A}_n(-\vec{k}) = \nabla_{\vec{k}} \times \vec{A}_n(-\vec{k})$$

$$= -\vec{\Omega}_n(-\vec{k})$$

$$\boxed{\therefore \quad \mathcal{T}\vec{\Omega}_n(\vec{k}) = -\vec{\Omega}_n(-\vec{k})}$$ correct result but wrong method

$$\therefore \quad \underline{\underline{C = \int \vec{\Omega}_n(\vec{k}) \cdot d\vec{s}_{BZ} = 0}}$$

However, if a system is TRS, then

$$\mathcal{T}\vec{A}_n(\vec{k}) = i\int d\vec{r}\, u^*_{n,-\vec{k}}(\vec{r})\, e^{-i\varphi(n,k)}\, \nabla_k\, e^{i\varphi(n,k)}\, u_{n,-\vec{k}}(\vec{r})$$

$$= -\vec{A}_n(-\vec{k}) - \nabla_k \varphi(n,\vec{k}) \quad \checkmark$$

$$H_k u_k = \varepsilon_k u_k. \qquad THT$$

$$|\psi_k(r)\rangle = \sum C_G |k-G\rangle = e^{i\vec{k}\cdot\vec{r}} \sum_G C_G e^{-i\vec{G}\cdot\vec{r}} = \psi_k(\vec{r})$$

$$\therefore \quad |\psi_{k+\vec{R}}\rangle = |\psi_k\rangle$$

$$T_R\, \mathcal{T}\psi_{n,k} = e^{-ik(r+R)} \sum C^*_{k-G}\, e^{+iG(r+R)}$$

$$H\psi_{n,k} = \varepsilon_{n,k}(\psi_{n,k})$$

$$T_R \psi_2 = e^{-ikR}\psi_2$$

$$\psi_2 = e^{i\varphi}\underline{\underline{|\psi_{n,-k}\rangle}} \longrightarrow \begin{array}{c}\psi_2 \\ \varepsilon_n(k) = \varepsilon_n(-k)\end{array}$$

Physically :- $\langle u_{n,k} | e^{-ikr} \frac{\hat{p}}{m} e^{ikr} | u_{n,k} \rangle$

$$\tau \psi_{nk} = \psi_{n,k}^{*}$$

$$\Rightarrow \quad \tau u_{n,k} = u_{n,k}^{*}(\vec{r})$$

$$i \int u_{n,k}(r) \, \nabla_k \, u_{n,k}^{*}(\vec{r}) \, d\vec{r}$$

$$= -i \int u_{n,k}^{*}(\vec{r}) \, \nabla_k \, u_{n,k}(\vec{r}) \, d\vec{r} \longrightarrow -A_n(\vec{k})$$

$$\vec{k} = -\vec{z}$$

$$-A_n(\vec{k}) = - u_{n,-z}^{*}(\vec{r}) \, \nabla_{\vec{z}} \, u_{n,-z}(\vec{r})$$

$$\tau \Omega = \nabla_{\vec{k}} \times -A_n = -\Omega(\vec{k})$$

$$H_k u_k = E_k u_k \qquad T H T^{-1} = H$$

$$H_{-k}(T u_k) = E_k (T u_k)$$

$$\Downarrow$$

$u_{\vec{k}}$

$$E_{\vec{k}} = E_{-\vec{k}} \qquad T u_{\vec{k}} = u_{-\vec{k}}$$

$$\tau \Omega(\vec{k}) = \Omega(-\vec{k}) \quad i \int u_{-k}^{*} \, \nabla_k \, u_{-\vec{k}} \, d\vec{r}$$

$$-A(-\vec{k})$$

u

Clear argument

$$T\, u_{n,k}(\vec{r}) = u^*_{n,k}(\vec{r})$$

$\therefore$ new Berry phase is

$$A' = i\int d\vec{r}\, (u^*_{n,k})^* \,\partial_k\, (u_{n,k})^*$$

$$= -i\int d\vec{r}\; u^*_{n,k}\, \partial_k\, u_{n,k}$$

$$= -A(\vec{k}) \qquad -①$$

However, if the system has TRS, then

$$T u_{n,k} = e^{i\varphi}\, u_{n,-k}$$

[argument $T u_{n,k} = u^*_{n,k}$ has same en & momentum $= -k \;\therefore\; \psi_{n,-k}$]

calculating $\vec{A}$ from here

$$\vec{A}' = i\int u^*_{n,-k}\, e^{-i\varphi}\, \partial_k\, e^{i\varphi}\, u_{n,-k}$$

$$= -\vec{A}(-k) - \nabla\varphi \qquad -②$$

① = ②

$\Rightarrow \quad -\vec{A}(-k) - \nabla\varphi = -\vec{A}(\vec{k})$

$\Rightarrow \quad -\nabla_k \times \vec{A}(-k) = -\nabla_k \times \vec{A}(\vec{k})$

$\Rightarrow \quad \boxed{\vec{\Omega}(-\vec{k}) = -\vec{\Omega}(\vec{k})}$ ~~X~~

Remember:- for symmetry arguments, always rely on gauge invariant Quantities

$\therefore$ under TRS + IR $\rightarrow$ $\Omega(\vec{k}) = \Omega(-\vec{k})$

& $\Omega(\vec{k}) = -\Omega(-\vec{k})$

$$\boxed{\Rightarrow \Omega(\vec{k}) = 0}$$

- For a spinful system, TR operator $= i\sigma_y K = \mathcal{T}$

now $\mathcal{T}\, u_{k\uparrow}(\vec{r}) = -u^{*}_{k\downarrow}(\vec{r})$

$$\vec{A} \longrightarrow i\int \left(-u^{*}_{k\downarrow}\right)^{*} \partial_k \left(-u^{*}_{k\downarrow}\right) d\vec{r}$$

$$= -i\int u^{*}_{k\downarrow}\, \partial_k\, u_{k\downarrow}$$

$$= -\vec{A}_{\downarrow}(k)$$

$$\therefore\ \vec{\Omega}_{\uparrow}(\vec{k}) \xrightarrow{\ \mathcal{T}\ } \nabla_k \times \vec{A}_{\downarrow}(\vec{k}) = -\vec{\Omega}_{\downarrow}(\vec{k}) \quad -①$$

If the sys is symmetric under TRS, then

$$\mathcal{T}\, u_{k\uparrow}(\vec{r}) = e^{i\varphi}\, u_{-k\downarrow}(\vec{r})$$

then,

$$\vec{A} \longrightarrow i\int d\vec{r}\; u^{*}_{-k\downarrow}\, e^{-i\varphi}\, \partial_k\, e^{i\varphi}\, u_{-k\downarrow}(\vec{r})$$

$$= -\vec{A}_{\downarrow}(-\vec{k}) - \nabla\varphi$$

$$\vec{\Omega} \longrightarrow \vec{\Omega}_{\downarrow}(-\vec{k}) \quad -②$$

From ① & ②,

$$\boxed{\vec{\Omega}_{\downarrow}(-\vec{k}) = -\vec{\Omega}_{\downarrow}(\vec{k})}$$

Berry curvature formula

$$F_{ij} = -i\epsilon_{ij}\, \langle \partial_{k_i} u_n(k) | \partial_{k_j} u_n(k) \rangle$$

$$= -i\epsilon_{ij} \int d\vec{r}\; \frac{\partial}{\partial k_i} u^{*}_n(k)\, \frac{\partial}{\partial k_j} u_n(k)$$

TR spinless

$$-i\epsilon_{ij} \int d\vec{r}\; \frac{\partial}{\partial k_i} u^{*}_n\, \frac{\partial}{\partial k_j} u_n(k)$$

$$\Big\downarrow\, i \leftrightarrow j$$

$$= -F_{ij}(k)$$

$(n,\uparrow)$ TR spinful

$$-i\epsilon_{ij} \int d\vec{r}\; \frac{\partial}{\partial k_i} u^{*}_{n\downarrow}(-k)\, \frac{\partial}{\partial k_j} u_{n\downarrow}(-k)$$

Q. How do I prove $\vec{\Omega}_{k\uparrow} = \vec{\Omega}_{-k\downarrow}$???

↳ Change wavefunctions?

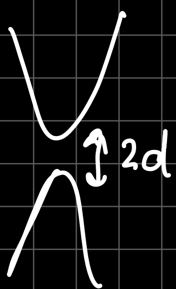
Do it for non-zero d.

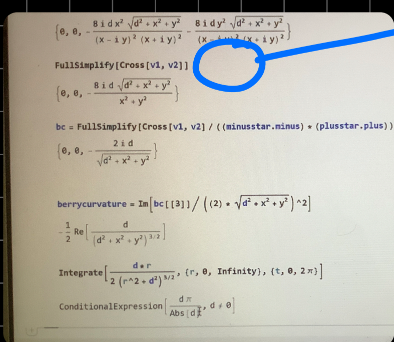
⇒ for 3D sphere, ∃ a source of Berry curvature source at origin.

H.W ⇒ Do it for a Rashba like hamiltonian.

↳ See if it changes.

③ Gapped dirac system :- Berry curvature points along z-direction.





$$\left\{0, 0, -\frac{8 i d x^2 \sqrt{d^2+x^2+y^2}}{(x-i y)^2 (x+i y)^2} - \frac{8 i d y^2 \sqrt{d^2+x^2+y^2}}{(x-i y)^2 (x+i y)^2}\right\}$$

```
FullSimplify[Cross[v1, v2]]
```

$$\left\{0, 0, -\frac{8 i d \sqrt{d^2+x^2+y^2}}{x^2+y^2}\right\}$$

```
bc = FullSimplify[Cross[v1, v2] / ((minusstar.minus) * (plusstar.plus))
```

$$\left\{0, 0, -\frac{2 i d}{\sqrt{d^2+x^2+y^2}}\right\}$$

```
berrycurvature = Im[bc[[3]] / ((2) * Sqrt[d^2 + x^2 + y^2])^2]
```

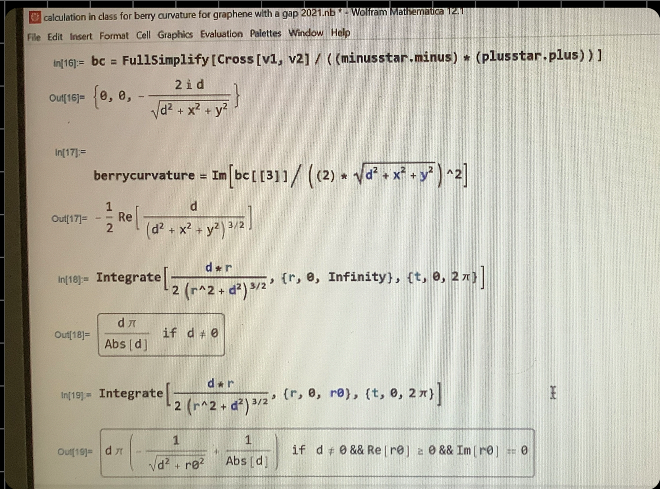
$$-\frac{1}{2} \mathrm{Re}\left[\frac{d}{(d^2+x^2+y^2)^{3/2}}\right]$$

```
Integrate[d*r / (2 (r^2 + d^2)^(3/2)), {r, 0, Infinity}, {t, 0, 2π}]
```

$$\mathrm{ConditionalExpression}\left[\frac{d\pi}{\mathrm{Abs}[d]}, d \neq 0\right]$$

sign of d determines the Berry phase

Comment:- depending on what area we capture the flux on, there'll be a deviation from π.   we'll discuss this soon



```
calculation in class for berry curvature for graphene with a gap 2021.nb * - Wolfram Mathematica 12.1
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help

In[16]:= bc = FullSimplify[Cross[v1, v2] / ((minusstar.minus) * (plusstar.plus))]
```

$$\text{Out[16]=}\ \left\{0,\ 0,\ -\frac{2\,i\,d}{\sqrt{d^2+x^2+y^2}}\right\}$$

```
In[17]:= berrycurvature = Im[bc[[3]]] / ((2) * Sqrt[d^2 + x^2 + y^2])^2
```

$$\text{Out[17]=}\ -\frac{1}{2}\,\mathrm{Re}\left[\frac{d}{(d^2+x^2+y^2)^{3/2}}\right]$$

$$\text{In[18]:= Integrate}\left[\frac{d*r}{2\,(r^{\wedge}2+d^2)^{3/2}},\ \{r, 0, \text{Infinity}\},\ \{t, 0, 2\pi\}\right]$$

$$\text{Out[18]=}\ \frac{d\,\pi}{\mathrm{Abs}[d]}\quad \text{if } d\neq 0$$

$$\text{In[19]:= Integrate}\left[\frac{d*r}{2\,(r^{\wedge}2+d^2)^{3/2}},\ \{r, 0, r0\},\ \{t, 0, 2\pi\}\right]$$

$$\text{Out[19]=}\ d\,\pi\left(-\frac{1}{\sqrt{d^2+r0^2}}+\frac{1}{\mathrm{Abs}[d]}\right)\quad \text{if } d\neq 0\ \&\&\ \mathrm{Re}[r0]\geq 0\ \&\&\ \mathrm{Im}[r0]==0$$

Check.

massless & massive

d — $x,y = 0,0$

d — as you come closer to $x,y=(0,0)$, then the B.phase will deviate.

$0 =$

Detailing:-

$$H = \left[\vec{\sigma}\cdot\vec{p}\right]_{2D} \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$H = \begin{pmatrix} d & p_x - ip_y \\ p_x + ip_y & -d \end{pmatrix} = \begin{pmatrix} d & pe^{-i\phi} \\ pe^{i\phi} & -d \end{pmatrix} \qquad p = \sqrt{p_x^2+p_y^2}$$

$$H \sim \sqrt{d^2+p^2}\begin{pmatrix} \frac{d}{\sqrt{d^2+p^2}} & \frac{pe^{-i\phi}}{\sqrt{d^2+p^2}} \\ \frac{pe^{i\phi}}{\sqrt{d^2+p^2}} & \frac{-d}{\sqrt{p^2+d^2}} \end{pmatrix} \sim \sqrt{d^2+p^2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$H = [\vec{\sigma} \cdot \vec{p}]_{2D} + d\sigma_z$ under TRS ✓

under inversion ??? $\longrightarrow$ in 3d, $\hat{P} = \pi\beta$

Parity operation :- $P = \pi\sigma_z$ } how???

$i = 1, 2$ or x, y

now $P^{\dagger} H P = \sigma_z \pi^{\dagger} p_i \sigma_i (\pi \sigma_z) + P^{\dagger} d\sigma_z P$

$= \sigma_z \pi^{\dagger} p_i \sigma_i \pi \sigma_z + d\sigma_z$

$= \sigma_z \; -p_i \pi^{\dagger} \sigma_i \pi \sigma_z = \underline{p_i \sigma_i + d\sigma_z}$

$= \underline{H}$

$\sigma_z \sigma_i \sigma_z \rightarrow -\sigma_i$ for $i = x, y$

$H = v\,\vec{p} \cdot \vec{\alpha} + \beta m \rightarrow$ Dirac eqn

s.t. $\alpha_i \beta = -\beta \alpha_i$, $\alpha_i^2 = 1 = \beta^2$

$\{\alpha_i, \alpha_j\} = 0$

1D

$\alpha_x, \beta =$ any 2 of 3 pauli matrices

e.g:- $H = v\sigma_x p_x + \sigma_z m$

2D

$\alpha_x = \sigma_x$

$\alpha_y = \sigma_y$

$\beta = \sigma_z$

3D

usual dirac eqn

↳ 4D matrices

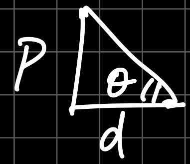
$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \sigma_x \otimes \sigma_i$

$\beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} = \sigma_z \otimes \sigma_0$

$\mathbb{1}$

$\therefore$ 2D dirac H is invariant under parity $P = \pi\sigma_z$

$\therefore H = \sqrt{d^2 + p^2} \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$ evs $= \begin{pmatrix} e^{-i\phi}\cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = |\eta\rangle$

where 

evs = $\therefore$ evs =

$$\begin{pmatrix} e^{-i\phi}\sqrt{\frac{1}{2}+\frac{d}{\sqrt{p^2+d^2}}\times\frac{1}{2}} \\ \frac{1}{2}-\frac{d}{\sqrt{p^2+d^2}} \end{pmatrix} \equiv |\uparrow\rangle$$

now

$$P|\uparrow\rangle = \pi\sigma_z|\uparrow\rangle$$

$$= \begin{bmatrix} e^{-i(\phi+\pi)}(\;) \\ -(\;) \end{bmatrix} = \underline{\underline{-|\uparrow\rangle}}$$

$$\Theta\vec{p}\,\Theta^{-1} = -\vec{p} \qquad \Theta\vec{J}\,\Theta^{-1} = -\vec{J}$$

$$\Theta = \underline{\underline{i\sigma_y K}} \qquad \Theta[\vec{\sigma}\cdot\vec{p}]_{2d}\,\Theta^{-1} = -[\vec{\sigma}\cdot\vec{p}]_{2D}$$

$$(i\sigma_y K)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} \qquad i\sigma_y K\begin{pmatrix} e^{i\phi}\cos\theta/2 \\ \sin\theta/2\end{pmatrix}$$

$$i\sigma_y K \quad i \quad -i\sigma_y K$$
$$= \quad K\,i\,K \quad = \underline{\underline{(-i)}}$$

$$i\sigma_y\begin{pmatrix}e^{-i\phi} \\ \; \end{pmatrix}$$

$$H = \begin{pmatrix}\cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & -\cos\theta\end{pmatrix} \qquad \begin{pmatrix}\sin\theta/2 \\ -e^{-i\phi}\cos\theta/2\end{pmatrix}$$

$\therefore$ $i\sigma_y|\uparrow\rangle = |\downarrow\rangle$ is <u>another</u> <u>ev.</u>

$$\mathcal{H}|\uparrow\rangle = +1|\uparrow\rangle$$

$$\Theta \mathcal{H} \Theta^{-1} \Theta^{-1} |\uparrow\rangle = \Theta|\uparrow\rangle$$

$$\boxed{\Theta\, \sigma_z\, \Theta = -\sigma_z} \qquad \sigma|\uparrow\rangle = +1|\uparrow\rangle$$

$$\Theta|\uparrow\rangle - \sigma\,\Theta|\uparrow\rangle = \Theta|\uparrow\rangle$$

$$\underbrace{\Theta\, \sigma\, \Theta^{-1}}_{-\sigma} \quad 0$$

$$\Theta\, \vec{p}\cdot\vec{\sigma}\, \Theta^{-1}$$

$$= \underbrace{\Theta\, \vec{p}\, \Theta^{-1} \cdot \Theta\, (\vec{\sigma})\, \Theta^{-1}}$$

$$= \vec{p}\cdot\vec{\sigma}$$

$$\Theta|\uparrow\rangle = |\downarrow\rangle$$

$\therefore$ For 2D Dirac eqn, $\mathcal{H} = \vec{\sigma}\cdot\vec{p}_{2D} + d_0\sigma_z$

$$|\uparrow\rangle = \begin{pmatrix} e^{-i\phi}\cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \to E_+ \qquad |\downarrow\rangle = \Theta|\uparrow\rangle = \begin{pmatrix} \sin\theta/2 \\ -e^{-i\phi}\cos\theta/2 \end{pmatrix} \to E_-$$

$$A_\phi^{(1)} = i\langle\uparrow|\frac{\partial}{\partial\phi}|\uparrow\rangle = \cos^2\theta/2 = \frac{1+\cos\theta}{2} = 0.5 + 0.5\times\frac{d}{\sqrt{d^2+p^2}}$$

new term for massive dirac particle

$$A_\phi^{(2)} = i\langle\downarrow|\frac{\partial}{\partial\phi}|\downarrow\rangle = \cos^2\theta/2 = A_\phi^{(1)}$$

$$\mathcal{H}_{massive} = \vec{\sigma}\cdot\vec{p} + d\sigma_z \to \text{not } \underline{TRS} \text{ but IR} \checkmark$$

as $\Theta\,\sigma_z\,\Theta^{-1} = \underline{-\sigma_z}$

Hence we get a non-zero <u>Berry curvature</u>.

$$A_p^{(1)} = i\langle \uparrow | \frac{\partial}{\partial p} | \uparrow \rangle = \frac{1}{2}\left[c\theta_{/2} \; -\sin\theta_{/2} + \sin\theta_{/2} \left(\frac{1}{2}\right) \cos\theta_{/2} \right] = \underline{\underline{0}}$$

$$A_p^{(2)} = i\langle \downarrow | \frac{\partial}{\partial p} | \uparrow \rangle = \underline{\underline{0}}$$

$$F_{\phi p}^{(1)} = \frac{\partial A_\phi^{(1)}}{\partial p} - \cancel{\frac{\partial A_p}{\partial \phi}}^{\to 0} = \frac{\partial A_\phi^{(1)}}{\partial p} = \left(\frac{d}{2}\right)\cdot\left(\frac{-1}{2}\right)\cdot\frac{2p}{\left(\sqrt{d^2+p^2}\right)^3}$$

$$= \left(\frac{-d}{2}\right)\cdot\frac{p}{(d^2+p^2)^{1.5}} \text{ along } \hat{z} \text{ axis}$$

Berry phase

$$A_\phi^{(1)} = 0.5 + 0.5\,\frac{d}{\sqrt{d^2+p^2}}$$

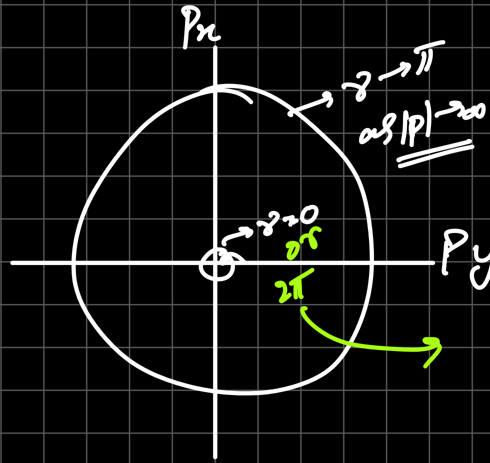
$$e^{i\gamma} = \exp\left[+i\oint A_\phi \, d\phi\right] = \exp\left[i\pi + i\pi\frac{d}{\sqrt{d^2+p^2}}\right]$$

$\therefore$ as $p \to \infty$, we recover same berry phase as massless.

$$e^{i\gamma} = e^{i\pi} \quad \therefore \gamma = \pi \pmod{2\pi}$$

as $p \to 0$, $e^{i\gamma} = \exp\left[i\pi + i\pi \,\mathrm{sgn}(d)\right]$

$d \neq 0$, $\gamma = 0$ or 2π



0 for d<0
2π for d>0

∘∘ therefore we see that Berry phase is definitely a path dependent quantity near the origin (i.e $p_x, p_y \to 0$), we have 0 or 2π phase diff depending upon sign of d.

As we move further from origin, we have the Berry phase $\underline{\underline{\pi}}$.