ASEN 5050 Spring 2021 HW 4 Solutions

Problem 1

given:

$\vec{r}_1 = -720000\hat{x} + 670000\hat{y} + 310000\hat{z}$ km

$\vec{v}_1 = 2.160\hat{x} - 3.360\hat{y} + 0.620\hat{z}$ km/s

$GM_{saturn} = 3.794\times10^7$ km^3/s^2

$r_{saturn} = 60268$ km

from HW3:

$\mathcal{E} = -28.62$ km^2/s^2

$a = 6.6278\times10^5$ km

$e = 0.9102$

$i = 62.1°$

$\Omega = 127.45°$

$\omega = -172.28°$

$h = 2.077\times10^6$

$\vec{h} = 1.457\times10^6\hat{x} + 1.116\times10^6\hat{y} + 9.72\times10^5\hat{z}$ km^2/s

$\theta_2^* = -13.17°$

$\theta_1^* = -167.83°$

note: E always in radians, $[-\pi, \pi]$

$\tan\left(\frac{E}{2}\right) = \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)$ sign check: if $\theta^* > 0$, $E > 0$

$M = n(t - t_p) = E - e\sin E$

$n = \sqrt{\mu/a^3}$

assume: 2BP, Saturn is a sphere, $GM_{s/c} \ll GM_{sat}$ $\therefore$ $\mu \approx GM_{saturn}$

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*}{2}\right)\right)$$

$E_1 = -2.2278$ rads $= 4.0554$ rad

$E_2 = -0.05$ rads $= 6.2331$ rad

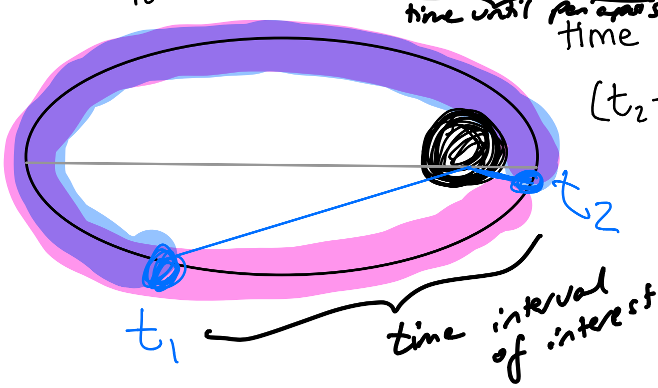
n = mean motion $= \sqrt{\frac{\mu_{sat}}{a^3}} = 1.1415\times10^{-5}$ rads/s

$(t_1 - t_p) = \frac{1}{n}(E_1 - e\sin E_1) = -1.3202\times10^5$ sec $= \overbrace{4.1839\times10^5 s}^{\text{time past periapsis}}$

$(t_2 - t_p) = \frac{1}{n}(E_2 - e\sin E_2) = \underbrace{-395.322 \text{ sec}}_{\text{time until periapsis}} = 5.5001\times10^5 s$

time for s/c to travel from 1 to 2:

$(t_2 - t_1) = (t_2 - t_p) - (t_1 - t_p)$



$t_2 - t_1 = 1.3162\times10^5$ sec $= 36.56$ hours

Problem 2

given:

$$\vec{r}_1 = -7.87701\times10^2\hat{x} - 8.81425\times10^2\hat{y} + 1.43864\times10^3\hat{z} \text{ km}$$
$$\vec{v}_1 = 0.9837\hat{x} + 0.7695\hat{y} + 1.01416\hat{z} \text{ km/s}$$
$$GM_{moon} = 4902.799 \text{ km}^3/s^2$$
$$r_{moon} = 1738 \text{ km}$$

assume: 2BP, Moon is a sphere, $GM_{s/c} \ll GM_{moon} \therefore \mu \approx GM_{moon}$

(a) how long is s/c above moon's equatorial plane?

$$r_1 = |\vec{r}_1| = \sqrt{r_1^2 + r_2^2 + r_3^2} = 1.862\times10^3 \text{ km}$$
$$v_1 = |\vec{v}_1| = \sqrt{v_1^2 + v_2^2 + v_3^2} = 1.6088 \text{ km/s}$$

$$\vec{h} = \vec{r}_1 \times \vec{v}_1 = -2.009\times10^3\hat{x} + 2.214\times10^3\hat{y} + 2.609\times10^2\hat{z} \quad \text{km}^3/s$$
$$h = |\vec{h}| = 2.9956\times10^3 \text{ km}^3/s$$

$$\mathcal{E} = \frac{v_1^2}{2} - \frac{\mu_{moon}}{r_1} = -1.3389 \text{ km}^2/s^2$$
$$a = -\frac{\mu_{moon}}{2\mathcal{E}} = 1.8309\times10^3 \text{ km}$$
$$\vec{e} = \frac{\vec{v}\times\vec{h}}{\mu} - \frac{\vec{r}}{r} = 6\times10^{-3}\hat{x} + 7.1\times10^{-3}\hat{y} - 1.44\times10^{-2}\hat{z}$$
$$e = |\vec{e}| = 0.0171$$

$$\vec{n} = \hat{z}\times\vec{h} = -2.214\times10^3\hat{x} - 2.0009\times10^3\hat{y} + 0\hat{z}$$
$$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{ne}\right) = \pm 122.6537^\circ \qquad \text{sign check: } \vec{e}\cdot\hat{z} = -1.44\times10^{-2} < 0 \;\therefore\; \omega < 0$$
$$\omega = -122.6537^\circ$$

find θ^* at the ascending and descending nodes:

$$\theta^*_{ascending} = -\omega = 122.65^\circ$$
$$\theta^*_{descending} = \theta^*_{ascend} + 180^\circ = 302.65^\circ = -57.346^\circ$$

find eccentric anomaly at each node:

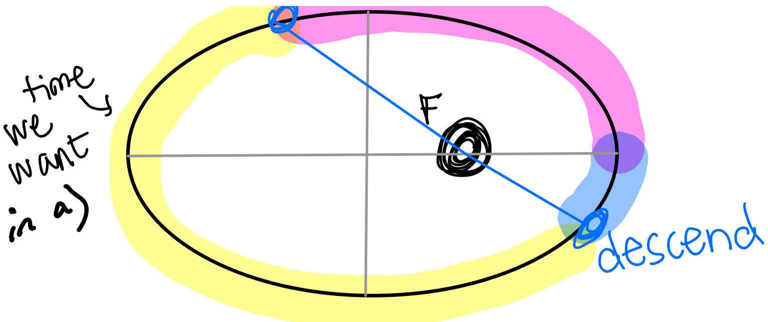
$$E_{ascend} = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*_{asc}}{2}\right)\right) = 2.1262 \text{ rads}$$
$$E_{desc} = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta^*_{des}}{2}\right)\right) = -0.9865 \text{ rads}$$

$$n = \sqrt{\frac{\mu}{a^3}} = 8.9378\times10^{-4} \frac{\text{rads}}{s}$$

$$(t-t_p)_{asc} = \frac{1}{n}(E_{asc} - e\sin E_{asc}) = 2.3627\times10^3 s \quad \text{time past periapsis}$$
$$(t-t_p)_{des} = \frac{1}{n}(E_{des} - e\sin E_{des}) = -1.0878\times10^3 s \quad \text{time until peri.}$$

$$= 5.9421\times10^3 s \quad (\text{time past peri})$$

ascend



Time above eq plane = Time from asc. node to desc. node.

Time below eq plane = Time from desc. node to asc. node

$$\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} = 7.0299\times10^3\,s$$

$$t_{pos} = (t-t_p)|_{desc} - (t-t_p)|_{asc}$$
$$= 3.5794\times10^3\,s$$
$$t_{pos} = 0.9943\text{ hrs}$$

(b) how long is the s/c below the equatorial plane?

$$t_{neg} = \mathbb{P} - t_{pos}$$
$$= 3.4505\times10^3\,s$$
$$= 0.9585\text{ hr}$$

(c) for this orbit, we expect t_{pos} and t_{neg} to be very similar b/c $e = 0.0171$ making it a relatively low eccentricity orbit. But, we also know that periapsis is below the orbital plane, so the s/c should spend less time below the orbital plane because it travels faster near periapsis. $\therefore t_{pos} > t_{neg}$

(d) here is one possible method:

use Newton's method to solve for E in rads using:

$$E_{i+1} = E_i - \frac{E_i - e\sin E_i - M}{1 - e\cos E_i}$$

M is used as the initial guess of E_0. It is important to set a stopping tolerance and/or a maximum number of iterations to stop Newton's Method in case it diverges. A good tolerance can be found by testing out multiple tolerances until you have enough signif digits. You can check that your E gives you the correct M.

(e) $E_1 = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta_1^*}{2}\right)\right)$

$\theta_1^* = \cos^{-1}\left(\frac{p/r - 1}{e}\right) = \pm 173.51°$ sign check $\bar{r}\cdot\bar{v} > 0 \therefore \theta_1^* > 0$

$\theta_1^* = 173.51°$

$E_1 = 3.0264$ rad

$(t_1 - t_p) = \frac{1}{n}(E_1 - e\sin E_1) = 3.3839 \times 10^3 s$

$= 56.3979 \text{ min}$

30 mins past $t_2 - t_p = 56.3979 + 30 = 86.3979$ min

$(t_2 - t_p) = 5.1839 \times 10^3 s$

using Kepler's equation and Newton's method from part (d)

$E_2 = 4.6162$ rads

$r_2 = a(1 - e\cos E_2) = 1.8339 \times 10^3$ Km

altitude $= r_2 - r_{moon} = 95.89$ Km

Since $E \in [\pi, 2\pi]$ the s/c is in the bottom of the orbit moving towards periapsis