ASEN 5050 Spring 2021 HW 2 Solutions

Problem 1

given: $p = 15345$ km, $\theta^*_b = 112°$ because the s/c is located at the top of the minor axis.

assume: 2BP, $m_{s/c} << m_\oplus$

a) let's calculate e first.

$e = -\cos(\theta^*_b)$

$= -\cos(112°)$

$e = 0.3746$

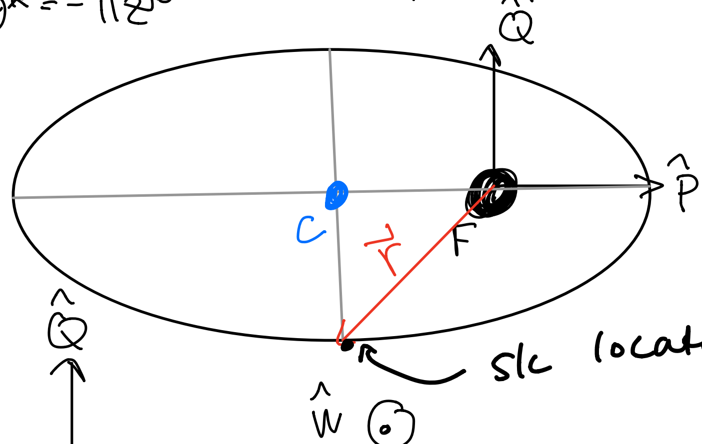
$p = a(1-e^2)$

$a = \dfrac{p}{(1-e^2)}$

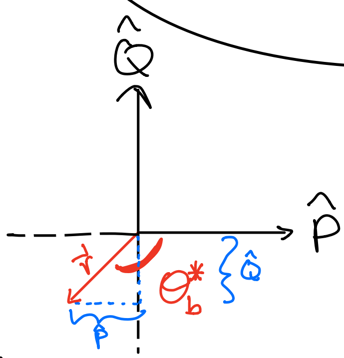
$a = 17849.87$ km

b) given $r = a$, so $r = r_b$, $\theta^* = \pm 112°$

"moving towards periapsis", so $\theta^* = -112°$



s/c located at bottom of the minor axis



position:

$$\vec{r} = r\sin(\theta^*_b + 90)\hat{P} - r\cos(\theta^*_b + 90)\hat{Q} + 0\hat{W}$$

$$= r\cos(\theta^*_b)\hat{P} + r\sin(\theta^*_b)\hat{Q} + 0\hat{W}$$

← this equation always works for $\hat{P}\hat{Q}\hat{W}$ frame!

$$\vec{r} = -6686.68\hat{P} - 16550.12\hat{Q} + 0\hat{W} \text{ km}$$

velocity in $\hat{r}\hat{\theta}\hat{h}$:

$v_r = \frac{\mu}{h} e\sin\theta^*$

$v_\theta = \frac{\mu}{h}(1 + e\cos\theta^*)$

we know $p = h^2/\mu$, so rearrange!

$\left(\frac{\mu}{h}\right)^2 = \frac{\mu^2}{h^2} = \mu\left(\frac{\mu}{h^2}\right) = \frac{\mu}{p}$

therefore if we take the square root,

$\frac{\mu}{h} = \sqrt{\frac{\mu}{P}}$ now we have values we can work with.

$v_r = \sqrt{\frac{\mu}{P}}\, e\sin\theta^*$, $V_\theta = \sqrt{\frac{\mu}{P}}(1+e\cos(\theta^*))$ 　　*use $\mu_\oplus$ for this problem

$\vec{V} = v_r\hat{r} + v_\theta\hat{\theta} + 0\hat{h}$

$\vec{V} = -1.7702\hat{r} + 4.3814\hat{\theta} + 0\hat{h}$ km/s

Problem 2

assumptions: regular 2BP

$m_{s/c} << m_{♂}$

$\vec{r} = 4981.75\hat{x} - 4121.90\hat{y} + 22.70\hat{z}$ km

$\vec{V} = -0.60359\hat{x} + 0.56812\hat{y} - 2.24093\hat{z}$ km/s

where $\hat{x}\hat{y}\hat{z}$ is an inertial frame

given that $\mu_{mars} = 4.305\times10^4$ km^3/s^2

$r_{mars} = 3397.2$ km

a) calculate $a, e, i, \Omega, \omega, \theta^*$

$|\vec{r}| = r = \sqrt{r_1^2+r_2^2+r_3^2} = \sqrt{4981.75^2 + (-4121.9)^2 + 22.7^2}$

$= 6.4659\times10^3$ km

$|\vec{v}| = v = \sqrt{v_1^2+v_2^2+v_3^2} = \sqrt{(-0.60359)^2 + (0.56812)^2 + (-2.24093)^2}$

$= 2.38932$ km/s

$\vec{h} = \vec{r}\times\vec{v}$

$= \underbrace{9223.993}_{h_1}\hat{x} + \underbrace{11150.05}_{h_2}\hat{y} + \underbrace{342.29}_{h_3}\hat{z}$ km^2/s

$h = |\vec{h}| = 1.4475\times10^4$ km^2/s

$\mathcal{E} = \frac{v^2}{2} - \frac{\mu_{♂}}{r} = \frac{(2.38932)^2}{2} - \frac{4.305\times10^4}{6.4659\times10^3}$

$= -3.8035$ km^2/s^2

$a = \frac{-\mu_{♂}}{2\mathcal{E}} = \frac{-4.305\times10^4}{2(-3.8035)}$

$a = 5659.20$ km

$i = \cos^{-1}\left(\frac{h_3}{h}\right) = \cos^{-1}\left(\frac{342.29}{1.4475\times10^4}\right)$

$= 1.547$ rads

$i = 88.64°$

$e = \sqrt{1+\frac{2h^2\mathcal{E}}{\mu_{♂}^2}} = \sqrt{1+\frac{2(1.4475\times10^4)^2(-3.8035)}{(4.305\times10^4)^2}}$

$e = 0.37415$

$\vec{n} = \hat{z} \times \vec{h} = [0 \;\; 0 \;\; 1] \times [9223.99 \quad 11150.1 \quad 342.29]$ in $\hat{x}\hat{y}\hat{z}$

$= -11150.05\,\hat{x} + 9223.99\,\hat{y} + 0\,\hat{z}$

$n = |\vec{n}| = 14470.86$

$\Omega = \cos^{-1}\left(\frac{\vec{n}\cdot\hat{x}}{n}\right) = \pm 140.4°$ sign check : $\vec{n}\cdot\hat{y} = 9223.99 > 0 \;\therefore\; \Omega(+)$

$\Omega = +140.4°$

$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{ne}\right) = \pm 48.83°$ sign check: $\vec{e}\cdot\hat{z} = -0.2816 < 0 \;\therefore\; \omega(-)$

$\omega = -48.83°$

$\theta^* = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{e}}{re}\right) = \pm 131.37°$ sign check: $\vec{r}\cdot\vec{v} = -5399 < 0 \;\therefore\; \theta^*(-)$

$\theta^* = -131.37°$

b) write $\vec{r}, \vec{v}$ in $(\hat{r}, \hat{\theta}, \hat{h})$

i) using direction cosine matrix

$\vec{r}_{r\theta h} = [C]^T \vec{r}_{xyz}$, $\theta = \theta^* + \omega = -180.2°$

$$[C] = \begin{bmatrix} C_\Omega C_\theta - S_\Omega C_i S_\theta & -C_\Omega S_\theta - S_\Omega C_i C_\theta & S_\Omega S_i \\ S_\Omega C_\theta + C_\Omega C_i S_\theta & -S_\Omega S_\theta + C_\Omega C_i C_\theta & -C_\Omega S_i \\ S_i S_\theta & S_i C_\theta & C_i \end{bmatrix}$$

$$[C] = \begin{bmatrix} 0.7705 & 0.0178 & 0.6372 \\ -0.6375 & 0.016 & 0.7703 \\ 0.0035 & -0.9997 & 0.0236 \end{bmatrix}$$

$\vec{r}_{r\theta h} = [C]^T \vec{r}_{xyz} = 6.4659 \times 10^3\,\hat{r} + 0\hat{\theta} + 0\hat{h}$ km

$$\vec{v}_{r\theta h} = [C]^T \vec{v}_{xyz} = \begin{bmatrix} 0.7705 & -0.6375 & 0.0035 \\ 0.0178 & 0.016 & -0.9997 \\ 0.6372 & 0.7703 & 0.0236 \end{bmatrix} \begin{bmatrix} -0.6036 \\ 0.5681 \\ -2.2409 \end{bmatrix}$$

$\vec{v}_{r\theta h} = -0.8351\,\hat{r} + 2.2386\,\hat{\theta} + 0\,\hat{h}$ km/s

2) $r = |\vec{r}| = 6.4659 \times 10^3$ km

$\vec{r}_{r\theta h} = r\hat{r}$

$\vec{r} = 6.4659 \times 10^3\,\hat{r} + 0\hat{\theta} + 0\hat{h}$ km

matches w/ 2b1 !

$\vec{v}_{r\theta h} = v_r\hat{r} + v_\theta\hat{\theta}$

$v_r = \frac{\mu}{h} e \sin(\theta^*)$

$= \left(\frac{4.305 \times 10^4}{1.4475 \times 10^4}\right)(0.3742)\sin(-131.37°)$

$= -0.8351$

$V_\theta = \frac{\mu}{h}(1 + e\cos\theta^*)$

$V_\theta = 2.2386$

$\vec{V}_{r\theta h} = -0.8351\,\hat{r} + 2.2386\,\hat{\theta}$ km/s

matches with part 2b1!

c) definition of ω is angle from $\vec{n}$ to $\vec{e}$ or $\vec{r}_{periapsis}$.
when s/c is at the ascending node, $\theta^* = -\omega$

we can get $\hat{r}\hat{\theta}\hat{h}$ frame and use DCM to put $\vec{r}$ and $\vec{v}$ in $\hat{x}\hat{y}\hat{z}$

2BP, so $\vec{h}, \vec{e}, \mathcal{E}$ are ~~[illegible]~~ constant

$$r = \frac{h^2/\mu_♂}{1 + e\cos\theta^*} = \frac{h^2/\mu_♂}{1 + e\cos(-\omega)} = 3.9051\times10^3 \text{ km}$$

$\vec{r} = 3.9051\times10^3\,\hat{r} + 0\,\hat{\theta} + 0\,\hat{h}$ km

$V_r = \frac{\mu_♂}{h} e\sin(-\omega) = 0.8377$ km/s

$V_\theta = \frac{\mu}{h}(1 + e\cos(-\omega)) = 3.7067$ km/s

$\vec{v} = 0.8377\,\hat{r} + 3.7067\,\hat{\theta} + 0\,\hat{h}$ km/s

$$[C] = \begin{bmatrix} C_\Omega C_\theta - S_\Omega C_i S_\theta & -C_\Omega S_\theta - S_\Omega C_i C_\theta & S_\Omega S_i \\ S_\Omega C_\theta + C_\Omega C_i S_\theta & -S_\Omega S_\theta + C_\Omega C_i C_\theta & -C_\Omega S_i \\ S_i S_\theta & S_i C_\theta & C_i \end{bmatrix}$$

$\theta = \theta^* + \omega = -\omega + \omega = 0°$

$$[C] = \begin{bmatrix} -0.7705 & -0.0151 & 0.6372 \\ 0.6374 & -0.0182 & 0.7703 \\ 0 & 0.9997 & 0.0236 \end{bmatrix}$$

$\vec{r}_{xyz} = [C]\,\vec{r}_{r\theta h}$

$\vec{r}_{xyz} = -3.009\times10^3\,\hat{x} + 2.4892\times10^3\,\hat{y} + 0\,\hat{z}$ km

$\vec{V}_{xyz} = [C]\,\vec{V}_{r\theta h}$

$\vec{V}_{xyz} = -0.7013\,\hat{x} + 0.4664\,\hat{y} + 3.7056\,\hat{z}$ km/s

Check: $\vec{r}_{xyz}\cdot\hat{z} = 0$
and $\vec{V}_{xyz}\cdot\hat{z} > 0$
$\therefore$ s/c at ascending node

d) Check your diagram using STK/GMAT –
were you able to draw it with approx. the right direction of $\vec{e}, \vec{h}, \vec{n}$?