4DM50 Dynamics and Control of cooperation
Problem set 1: Observers

Problem 1. One of the first examples of synchronization of chaotic systems was introduced in 1990 by Pecora and Carrol. Consider the Lorenz system:

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= -xz + rx - y\\ \dot{z} &= xy - bz\end{aligned}$$

Pecora and Carrol found that the following system ("receiver", or observer)

$$\begin{aligned}\dot{x}_1 &= \sigma(y_1 - x_1)\\ \dot{y}_1 &= -xz_1 + rx - y_1\\ \dot{z}_1 &= xy_1 - bz_1\end{aligned}$$

synchronizes with the Lorenz system.

1. Make a computer simulation to confirm this statement. Take $\sigma = 10, r = 28, b = 8/3$.

2. Using the Lyapunov function candidate

$$V = (x - x_1)^2/2\sigma + (y - y_1)^2/2 + (z - z_1)^2/2$$

prove the asymptotic stability of the synchronous regime.

Problem 2. Consider the Rössler system

$$\left\{\begin{aligned}\dot{x}_1 &= -x_2 - x_3\\ \dot{x}_2 &= x_1 + ax_2\\ \dot{x}_3 &= c + x_3(x_1 - b)\end{aligned}\right. \tag{0.1}$$

with positive parameters a, b, c.

1. Show that if the system initiated at a point where $x_3(0) > 0$ then for all t it follows that $x_3(t) > 0$. (Hint: consider the third equation).

2. Via computer simulation find an initial condition such that the corresponding solution of the Rössler system is bounded. Take $a = 0.2, b = 5.7, c = 0.2$.

3. The previous result suggests that the coordinate transformation

$$\eta_1 = x_1,\ \eta_2 = x_2,\ \eta_3 = \log x_3$$

is well defined for all t as soon as $x_3(0) > 0$. Rewrite the system in the coordinates η_1, η_2, η_3.

4. Notice that in the new coordinates the system has the form

$$\dot{\eta} = A\eta + f(\eta_3) \tag{0.2}$$

that allows to design a system coupled through the variable η_3 which synchronizes with (0.2). Complete the design using the concept of observability.

5. Perform computer simulations to verify the theoretical results.

Problem 3. Consider the Chua system:

$$\begin{cases} \dot{x}_1 = \alpha(-x_1 + x_2 - \varphi(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\beta x_2 \end{cases} \tag{0.3}$$

$$\varphi(s) = m_1 s + m_2(|s+1| - |s-1|),$$

with parameters

$$\alpha = 10, \quad \beta = 19.53, \quad m_1 = -0.783, \quad m_2 = -0.3247.$$

1. Find, using computer simulations, initial conditions $x(t_0)$ such that the corresponding solutions of the Chua system are bounded.
2. Consider x_1 as output of the Chua system, i.e. $y = x_1$. Design an observer that yields linear error dynamics and complete the observer design using the concept of observability.
3. Perform computer simulations to confirm your results.

Problem 4. Consider the unidirectionally coupled Chua systems:

$$\Sigma_m = \begin{cases} \dot{x}_1 = \alpha(-x_1 + x_2 - \varphi(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -\beta x_2 \end{cases} \tag{0.4}$$

$$\Sigma_s = \begin{cases} \dot{y}_1 = \alpha(-y_1 + y_2 - \varphi(y_1)) + k(x_1 - y_1) \\ \dot{y}_2 = y_1 - y_2 + y_3 \\ \dot{y}_3 = -\beta y_2 \end{cases} \tag{0.5}$$

$$\varphi(s) = m_1 s + m_2(|s+1| - |s-1|), \tag{0.6}$$

where constant parameter $k > 0$ is the *coupling strength*. The values of the other parameters are

$$\alpha = 10, \quad \beta = 19.53, \quad m_1 = -0.783, \quad m_2 = -0.3247.$$

System Σ_m is the so-called *master* system and system Σ_s is the *slave* system.

1. Consider the master system Σ_m. Find using computer simulations initial conditions $x(t_0)$ such that the corresponding solutions of the master system are bounded.
2. Let the initial conditions for the master system be such that its solutions are bounded. Show using a Lyapunov function of the form $V = (x-y)^\top P(x-y)$, P is a positive definite matrix, that there is a constant $\bar{k} > 0$ such that the slave system synchronizes with the master system for $k \geq \bar{k}$. (Note that there are constants C_1 and C_2 such that $C_1(x_1 - y_1) \leq \varphi(x_1) - \varphi(y_1) \leq C_2(x_1 - y_1)$ for all x_1, y_1.)
3. Make computer simulations to confirm your results. How conservative is your estimate $\bar{k}$?

4DM50 Dynamics and Control of cooperation
Problem set 1: CACC

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Assume a platooning situation with n vehicles driving on a single lane behind each other, where the dynamics of each vehicle are described the following vehicle model

$$\dot{q}_i(t) = v_i(t) \tag{1a}$$
$$\dot{v}_i(t) = a_i(t) \tag{1b}$$
$$\dot{a}_i(t) = -\frac{1}{\tau}a_i(t) + \frac{1}{\tau}u_i(t) \tag{1c}$$

with q_i, v_i, and a_i being the position, velocity, and acceleration of vehicle $i \in \{1, 2, \ldots, n\}$, respectively. The time constant τ represents the driveling dynamics. In the scope of Cooperative Adaptive Cruise Control (CACC), each vehicle should regulate the distance to its preceding vehicle to a desired value $d_{\mathrm{r},i}$, given by

$$d_{\mathrm{r},i}(t) = r + hv_i(t), \tag{2}$$

where r is the standstill distance and h the time gap. Defining the distance error e_i as

$$\begin{aligned} e_i(t) &= d_i(t) - d_{\mathrm{r},i}(t) \\ &= (q_{i-1}(t) - q_i(t) - L_i) - d_{\mathrm{r},i}(t), \end{aligned} \tag{3}$$

where L_i is the length of vehicle i, the control objective can be formulated as $\lim_{t\to\infty} e_i = 0$. A controller, which is known to be capable of asymptotically regulating e_i to zero, is given by

$$\dot{u}_i(t) = -\frac{1}{h}u_i(t) + \frac{1}{h}\left(k_p e_i(t) + k_d \dot{e}_i(t)\right) + \frac{1}{h}u_{i-1}(t-\theta), \tag{4}$$

where k_p and $k_s d$ are the controller parameters. Here, u_{i-1} can only be obtained by vehicle i through wireless communication, which is why a communication delay θ is involved.

Questions

i. A block scheme of the controlled i^{th} vehicle is given in Fig. 1. This scheme shows the transfer functions $G(s)$ of the vehicle, $K(s)$ of the controller, $D(s)$ of the communication delay, and $H(s)$ of the spacing policy. Given the aforementioned equations, can you provide these transfer functions?

ii. Derive an analytical expression for the String Stability Complementary Sensitivity (SSCS) $\Gamma_{i,j}(s)$, choosing the following inputs/outputs of interest:

a. $v_i(s) = \Gamma_{i,1}(s)v_{i-1}(s)$

b. $u_i(s) = \Gamma_{i,2}(s)u_{i-1}(s)$

c. $e_i(s) = \Gamma_{i,3}(s)e_{i-1}(s)$,

where $\cdot(s)$ denotes the Laplace transform of the corresponding time variable $\cdot(t)$.

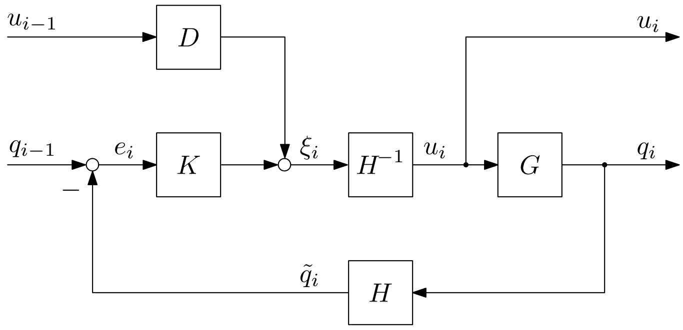

Figure 1: Block scheme of a vehicle equipped with CACC.

iii. Determine how the minimum $\mathcal{L}_2$ string-stable time gap $h_{\min}$ depends on the communication delay θ. Do this by writing a search algorithm in MATLAB to find the minimum time gap h for values $\theta = [0, 0.01, 0.02, \ldots, 0.1]$ for which the controlled system is on the bound of $\mathcal{L}_2$ string instability, using the following SSCS:

$$\Gamma(s) = \frac{1}{H(s)} \frac{D(s) + G(s)K(s)}{1 + G(s)K(s)}. \tag{5}$$

Plot the resulting values for $h_{min}(\theta)$ against θ.
Choose the following parameter values: $k_p = 0.2$, $k_d = 0.7$, and $\tau = 0.1$.
Hint: the command `norm` from MATLAB's Control System Toolbox to compute the $\mathcal{H}_\infty$ norm cannot handle 'internal' delays, which is why a (2$^{\text{nd}}$-order) Padé approximation should be applied to model the communication delay.