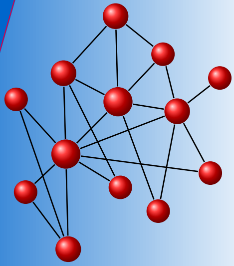
4DM50: Dynamics and Control of Cooperation

Network synchronization II



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Outline

Diffusively coupled systems

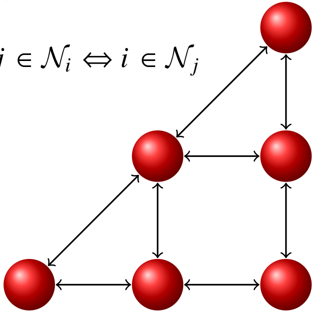
Systems of the form

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases}$$

with $CB > 0$ and diffusive coupling functions

$$u_i = \sum_{j \in \mathcal{N}_i} \gamma_{ij}(y_j - y_i)$$

where diffusion coefficients $\gamma_{ij} = \gamma_{ji} > 0$ and $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$



Coupling matrix

Diffusive coupling matrix

$$\Gamma = \begin{pmatrix} \sum_{j=2}^{N} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1N} \\ -\gamma_{21} & \sum_{j=1, j\neq 2}^{N} \gamma_{2j} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\gamma_{(N-1)N} \\ -\gamma_{N1} & \cdots & -\gamma_{N(N-1)} & \sum_{j=1}^{N-1} \gamma_{Nj} \end{pmatrix}$$

with $\gamma_{ij} = 0$ if (and only if) $j \notin \mathcal{N}_i$

- Γ is singular
- $\Gamma = \Gamma^T$ is positive semi-definite
- Eigenvalues of Γ (if and only if the network is connected):

$$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$$

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Strict semi-passivity

The systems

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That is: there exists $S : \mathbb{R}^n \to \mathbb{R}_+$ such that the dissipation inequality

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is strict for all x_i outside some ball $\mathcal{B} \subset \mathbb{R}^n$

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Result: networks of diffusively coupled strictly semi-passive systems have uniformly ultimately bounded solutions

Systems

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with CB (similar to) a positive definite matrix

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Exponentially convergent internal dynamics

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$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases}$$

with CB (similar to) a positive definite matrix in new coordinates

$$\begin{cases} \dot{z}_i = q(z_i, y_i) \\ \dot{y}_i = a(z_i, y_i) + CBu_i \end{cases}$$

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Assumption: There exists a $(n-m) \times (n-m)$ matrix $P = P^T > 0$ such that the eigenvalues of

$$P\left(\frac{\partial q}{\partial z}(z, w)\right) + \left(\frac{\partial q}{\partial z}(z, w)\right)^T P$$

are negative and separated away from zero for all $(z, w) \in \mathbb{R}^{n-m} \times \mathcal{W}$

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$\Rightarrow$ subsystem $\dot{z}_i = q(z_i, y_i)$ is exponentially convergent w.r.t. input y_i

Given a network of diffusively coupled systems

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases} \tag{1}$$

$$u_i = \sum_{j \in \mathcal{N}_i} \gamma_{ij}(y_j - y_i) \tag{2}$$

Synchronization of diffusively coupled systems

Given a network of diffusively coupled systems

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- system (8) is strictly semi-passive with a radially unbounded storage function S
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Result:

- solutions of the diffusively coupled systems are uniformly ultimately bounded
- there exists a positive constant $\bar{\lambda}$ such that for

$$\lambda_2 \geq \bar{\lambda} \quad (\lambda_2 \text{ is the smallest non-zero eigenvalue of } \Gamma)$$

Then the set $\mathcal{M} := \{x \in \mathbb{R}^{Nn} \,|\, x_1 = x_2 = \cdots = x_N\}$ contains a globally asymptotically stable subset, i.e. the systems synchronize

Extensions to

- Directed networks

Extensions to

- Directed networks
- Time-delay coupling

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}(y_j(t-\tau) - y_i(t))$$

and

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}(y_j(t-\tau) - y_i(t-\tau))$$

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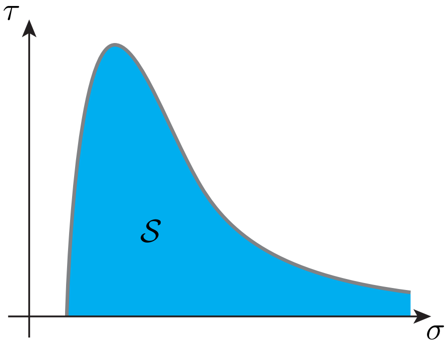

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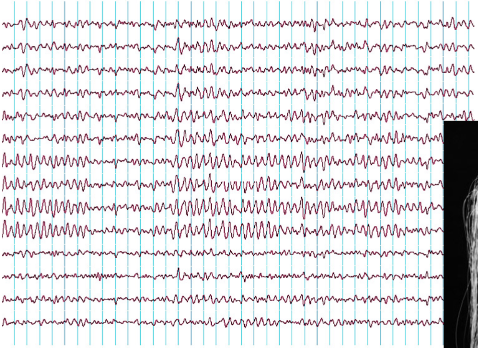
Partial synchronization = asymptotic match of the states of some, but not all systems

Example



Partial synchronization

Partial synchronization =

existence of partial synchronization manifold $\mathcal{P}$

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Questions:

- How to find $\mathcal{P}$?

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existence of partial synchronization manifold $\mathcal{P}$

+ asymptotic stability of $\mathcal{P}$

Questions:

- How to find $\mathcal{P}$?
- When is $\mathcal{P}$ stable?

Finding modes of partial synchronization

Look for symmetries

- in the coupling configuration (network)
- in the system dynamics

$$\dot{x}_i = f(x_i) + Bu_i$$
$$y_i = Cx_i$$
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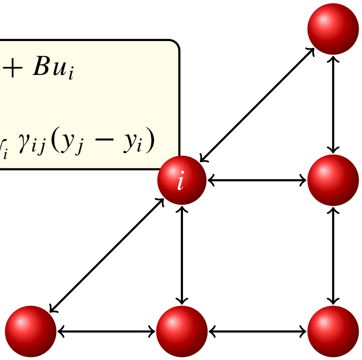

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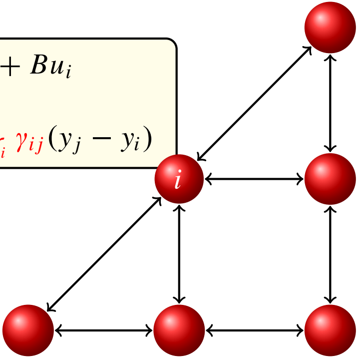
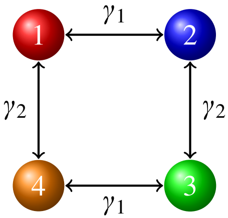
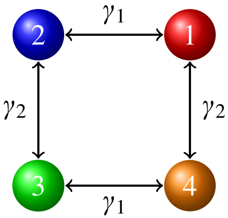


Rearrangement of some nodes of the network leave the network unchanged



Original network



Rearranged network

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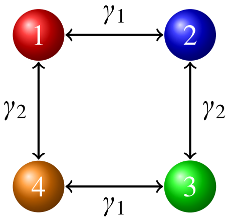
Network symmetries

Rearrangement of some nodes of the network leave the network unchanged



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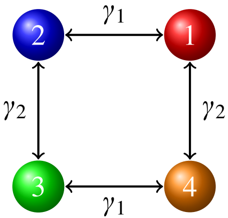

Rearranged network

Permutation $\pi : \{1, \ldots, N\} \to \{1, \ldots, N\}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) = 2 & \pi(2) = 1 & \pi(3) = 4 & \pi(4) = 3 \end{pmatrix}$$

Permutation matrix associated to π

$$\Pi = \left(\Pi_{ij}\right)$$

with

$$\Pi_{ij} = \begin{cases} 1 & \text{if } \pi(i) = j \\ 0 & \text{otherwise} \end{cases}$$

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Permutation matrices and network symmetries

Permutation matrix associated to π

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Result: if the permutation matrix Π and the coupling matrix Γ commute,

$$\Pi\Gamma = \Gamma\Pi,$$

then the set

$$\ker(I_{Nn} - \Pi \otimes I_n)$$

is invariant w.r.t. dynamics of the coupled systems

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Remarks

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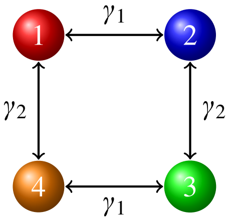
Π and L commute $\Rightarrow X = \Gamma$ is a solution

Example



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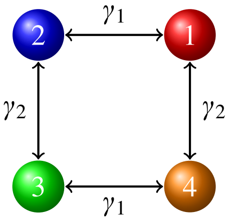

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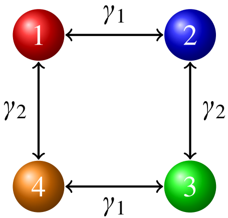

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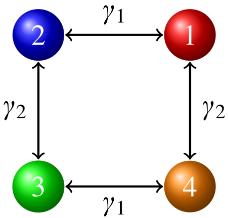

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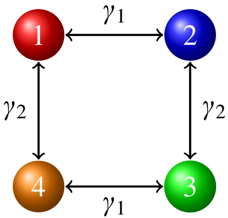
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \pi(1) = 2 & \pi(2) = 1 & \pi(3) = 4 & \pi(4) = 3 \end{pmatrix} \quad \Rightarrow \quad \Pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example



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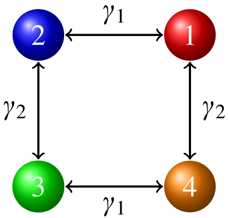
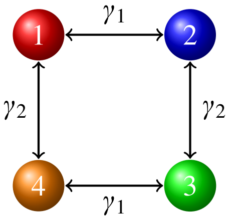


Rearranged network

$$\Pi\Gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_1 & 0 & -\gamma_2 \\ -\gamma_1 & \gamma_1+\gamma_2 & -\gamma_2 & 0 \\ 0 & -\gamma_2 & \gamma_1+\gamma_2 & -\gamma_1 \\ -\gamma_2 & 0 & -\gamma_1 & \gamma_1+\gamma_2 \end{pmatrix}$$



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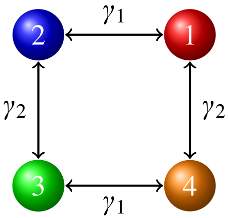
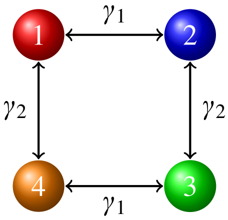


Rearranged network

$$= \begin{pmatrix} -\gamma_1 & \gamma_1+\gamma_2 & -\gamma_2 & 0 \\ \gamma_1+\gamma_2 & -\gamma_1 & 0 & -\gamma_2 \\ -\gamma_2 & 0 & -\gamma_1 & \gamma_1+\gamma_2 \\ 0 & -\gamma_2 & \gamma_1+\gamma_2 & -\gamma_1 \end{pmatrix} =$$

Example



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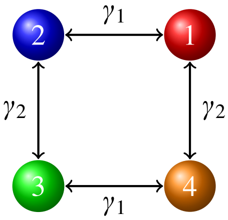

Rearranged network

$$\begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_1 & 0 & -\gamma_2 \\ -\gamma_1 & \gamma_1+\gamma_2 & -\gamma_2 & 0 \\ 0 & -\gamma_2 & \gamma_1+\gamma_2 & -\gamma_1 \\ -\gamma_2 & 0 & -\gamma_1 & \gamma_1+\gamma_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \Gamma\Pi$$

Other permutation matrices that commute with

$$\Gamma = \begin{pmatrix} \gamma_1 + \gamma_2 & -\gamma_1 & 0 & -\gamma_2 \\ -\gamma_1 & \gamma_1 + \gamma_2 & -\gamma_2 & 0 \\ 0 & -\gamma_2 & \gamma_1 + \gamma_2 & -\gamma_1 \\ -\gamma_2 & 0 & -\gamma_1 & \gamma_1 + \gamma_2 \end{pmatrix}$$

are:

$$\Pi_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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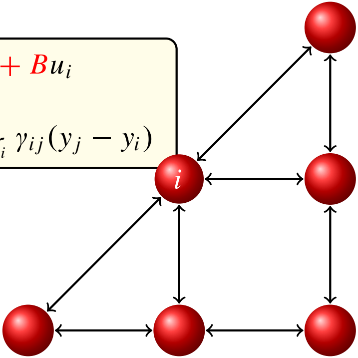

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is invariant w.r.t. dynamics of the coupled systems

System symmetries

Example: Lorenz system with input u and output $y = x_1$:

$$\begin{cases} \dot{x}_1 = \sigma\,(x_2 - x_1) + u \\ \dot{x}_2 = x_1(\rho - x_3) - x_2 \\ \dot{x}_3 = -\gamma\, x_3 + x_1 x_2 \end{cases}$$

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The set $\ker(I_{Nn} - \Pi \otimes J)$ is invariant for coupled Lorenz systems as

$$J(BC) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} = (BC)J$$

Partial synchronization

Network of diffusively coupled systems

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases} \tag{3}$$

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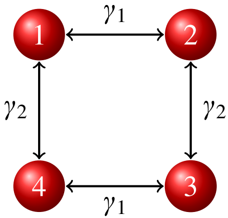
$\lambda' \geq \bar{\lambda}$ (λ' is the smallest eigenvalue of Γ with eigenvector in $\mathrm{range}(I_N - \Pi)$)

Then $\ker(I_{Nn} - \Pi \otimes I_n)$ contains a globally asymptotically stable subset

Example

$$\Gamma = \begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_1 & 0 & -\gamma_2 \\ -\gamma_1 & \gamma_1+\gamma_2 & -\gamma_2 & 0 \\ 0 & -\gamma_2 & \gamma_1+\gamma_2 & -\gamma_1 \\ -\gamma_2 & 0 & -\gamma_1 & \gamma_1+\gamma_2 \end{pmatrix}$$



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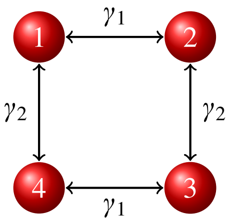
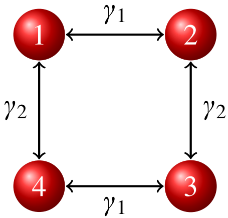


Recall: The following permutation matrices commute with Γ

$$\Pi_1 = \begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{pmatrix} \quad \Pi_2 = \begin{pmatrix} 0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0 \end{pmatrix} \quad \Pi_3 = \begin{pmatrix} 0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0 \end{pmatrix}$$

Example

$$\Gamma = \begin{pmatrix} \gamma_1+\gamma_2 & -\gamma_1 & 0 & -\gamma_2 \\ -\gamma_1 & \gamma_1+\gamma_2 & -\gamma_2 & 0 \\ 0 & -\gamma_2 & \gamma_1+\gamma_2 & -\gamma_1 \\ -\gamma_2 & 0 & -\gamma_1 & \gamma_1+\gamma_2 \end{pmatrix}$$



Eigenvalues and eigenvectors of Γ

$$\lambda_1 = 0 \qquad \lambda_2 = 2\gamma_1 \qquad \lambda_3 = 2\gamma_2 \qquad \lambda_4 = 2\gamma_1 + 2\gamma_2$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

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Example

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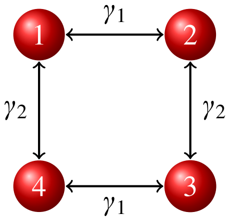

- $v_2 \in \mathrm{range}(I_n - \Pi_1), \mathrm{range}(I_n - \Pi_3)$
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hence

- Π_1: $\lambda' = 2\gamma_1$
- Π_2: $\lambda' = 2\gamma_2$
- Π_3: $\lambda' = \min(2\gamma_1, 2\gamma_2)$

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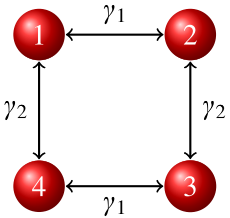

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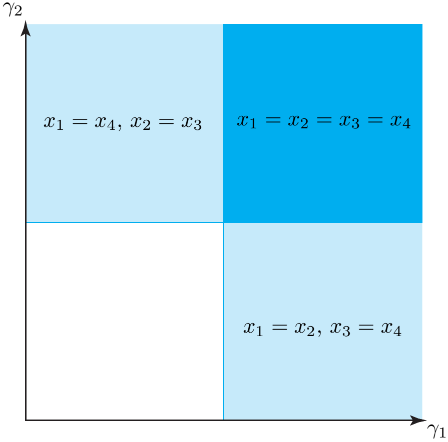
hence

- Π_1: $\lambda' = 2\gamma_1 \Rightarrow x_1 = x_2$ and $x_3 = x_4$ if $\gamma_1 > \gamma_2$ is suff. large
- Π_2: $\lambda' = 2\gamma_2 \Rightarrow x_1 = x_2$ and $x_3 = x_4$ if $\gamma_2 > \gamma_1$ is suff. large
- Π_3: $\lambda' = \min(2\gamma_1, 2\gamma_2) \Rightarrow$ full sync

Example



Recap

Partial synchronization

Partial synchronization with delays

Role of network topology

Diffusively time-delay coupled systems

Systems

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases} \tag{5}$$

with diffusive time-delay coupling (transmission delay)

$$u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t - \tau) - y_i(t)) \tag{6}$$

or (sensor/actuator delay)

$$u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t - \tau) - y_i(t)(t - \tau)) \tag{7}$$

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For now: Coupled systems (5), (7) with $a_{ij} = a_{ji}$ and $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$

(similar results are obtained for directed networks and/or coupled systems (5)-(6))

Coupling matrix

$$L = \begin{pmatrix} \sum_{j=2}^{N} a_{1j} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \sum_{j=1, j\neq 2}^{N} a_{2j} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{(N-1)N} \\ -a_{N1} & \cdots & -a_{N(N-1)} & \sum_{j=1}^{N-1} a_{Nj} \end{pmatrix}$$

with $a_{ij} = 0$ if (and only if) $j \notin \mathcal{N}_i$

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with $a_{ij} = 0$ if (and only if) $j \notin \mathcal{N}_i$

Note that

$$u(t) = -\sigma(L \otimes I_m) y(t - \tau)$$

(compare with the delay-free case!)

Finding modes of partial synchronization

Look for symmetries in the coupling matrix L

Finding modes of partial synchronization

Look for symmetries in the coupling matrix L

Result: if a permutation matrix Π and the coupling matrix L commute,

$$\Pi L = L \Pi,$$

then the set

$$\{\phi \in \mathcal{C} \,|\, \phi(\theta) \in \ker(I_{Nn} - \Pi \otimes I_n), \quad -\tau \leq \theta \leq 0\}$$

is forward invariant w.r.t. dynamics of the coupled systems

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Eigenvalues of L:

$$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$$

Partial synchronization

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Partial synchronization

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Given a permutation matrix Π, denote:

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- λ^* the largest eigenvalue of L with eigenvector in $\mathbf{range}(I_N - \Pi)$

Assumptions:

- system (5) is strictly semi-passive with a radially unbounded storage function S
- the subsystem $\dot{z}_i = q(z_i, w(t))$ associated to (5) satisfies the test for exponential convergence
- the permutation matrix Π that commutes with L

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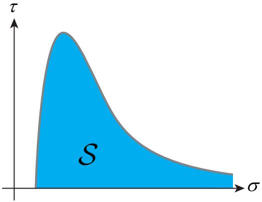
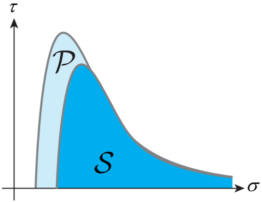
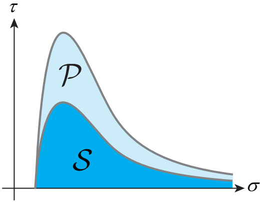
Partial synchronization result



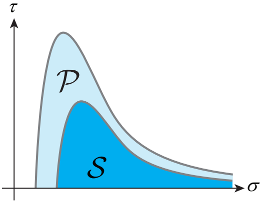
$\lambda' = \lambda_2$ and $\lambda^* = \lambda_N$



$\lambda' > \lambda_2$ and $\lambda^* = \lambda_N$



$\lambda' = \lambda_2$ and $\lambda^* < \lambda_N$



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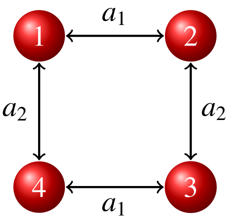
Example

$$L = \begin{pmatrix} a_1 + a_2 & -a_1 & 0 & -a_2 \\ -a_1 & a_1 + a_2 & -a_2 & 0 \\ 0 & -a_2 & a_1 + a_2 & -a_1 \\ -a_2 & 0 & -a_1 & a_1 + a_2 \end{pmatrix}$$



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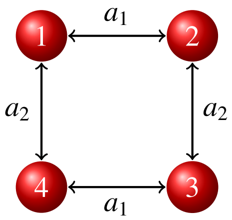

Eigenvalues and eigenvectors of L if $a_1 \leq a_2$

$$\lambda_1 = 0 \quad \lambda_2 = 2a_1 \quad \lambda_3 = 2a_2 \quad \lambda_4 = 2a_1 + 2a_2$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

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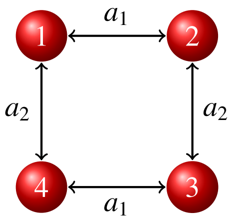

Recall: The following permutation matrix commutes with Γ

$$\Pi_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and $v_2, v_3 \in \mathrm{range}(I_n - \Pi_3)$

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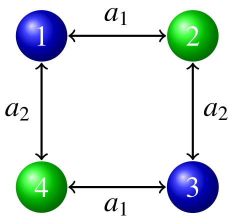

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and $v_2, v_3 \in \text{range}(I_n - \Pi_3) \Rightarrow \lambda' = 2a_1$ and $\lambda^* = 2a_2 < \lambda_4 = 2a_1 + 2a_2$

Recap

Partial synchronization

Partial synchronization with delays

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Network of diffusively time-delay coupled systems

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Diffusively time-delay coupled systems

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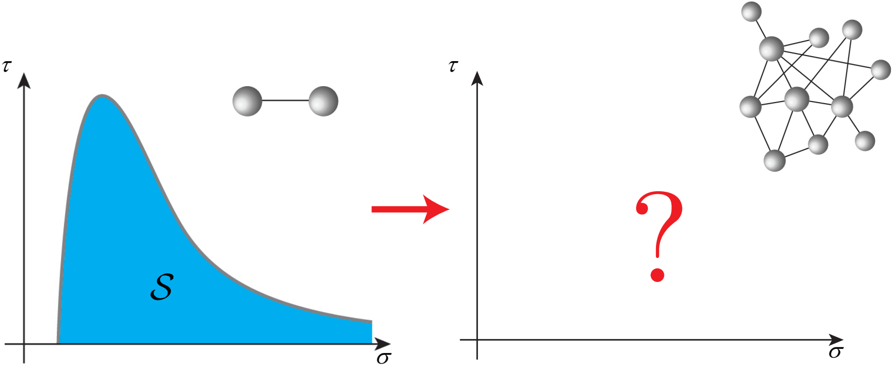
Scaling question



Coupling matrix

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with $a_{ij} = 0$ if (and only if) $j \notin \mathcal{N}_i$

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Scaling result

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with scaling factors the eigenvalues of L

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with scaling factors the eigenvalues of L

- Result: Network synchronizes if

$$(\sigma, \tau) \in \mathcal{S}_2^* \cap \mathcal{S}_N^*$$

Scaling result