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Control Systems 314
2018

Lecture 7-8:
Dynamic Response:
Effect of Pole Locations

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Lecture 7-8 Overview

MODELLING

Differential Equations

Laplace Transform

Transfer Functions	Block Diagrams	Poles and Zeroes

ANALYSIS

Stability	Steady State Error	Transient Response

CONTROL

Root Locus

TIME DOMAIN DESIGN

Bode Plot	Nyquist Plot	Nichols Chart

FREQUENCY DOMAIN DESIGN

Lecture 7-8 Overview

- Stability
- Effects of Poles on Stability

- Transient Response
- Effects of First and Second Order Poles on Transient Response
- First-Order Poles
  - Time constant
- Second-Order Poles
  - Damping ratio, Natural frequency
  - Underdamped
  - Critically damped
  - Overdamped

Lesing 7-8 Oorsig

- <u>Stabiliteit</u>
- Effek van Pole op Stabiliteit

- <u>Oorgangsverskynsels</u>
- Effek van Eerste en Tweede Orde Pole op Oorgangsverskynsels
- Eerste-Orde Pole
  - Tydkonstante
- Tweede-Orde Pole
  - Dempingsverhouding, Natuurlike Frekwensie
  - Ondergedemp
  - Krities gedemp
  - Oorgedemp

Context

- The performance of a control system is specified in terms of its stability, transient response and steady state error
- A good control system is stable, exhibits a fast transient response with little overshoot, and has a small steady state error
- The dynamic processes that we wish to control may be inherently unstable, and may have inherently slow and oscillatory transient response, possibly with large overshoot
- We must be able to quantify the speed of response and amount of oscillation in the transient responses of open-loop and closed-loop systems

Key Concepts

- The stability and transient response of a dynamic system is determined by the locations of its poles and zeros
- The transient response of a complex system can be expressed in terms of first-order and second-order responses
- A first-order system exhibits an exponential response
- A second-order system exhibits a damped oscillation, which may be underdamped, critically damped, or overdamped

We would like to…

- Determine the poles and zeroes of a transfer function
- Plot the poles and zeroes of a system in the s-plane

- Determine the stability of a system from its pole locations

- Calculate the time constant of a first-order system

- Calculate the damping ratio and undamped natural frequency of a second order system
- Classify a second-order system as underdamped, critically damped, or overdamped

- Sketch the expected dynamic response of a first-order or second-order system to a step input

Poles and Zeros (1)

- For a system of ordinary differential equations, the transfer function will always be a ratio of polynomials

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^n + b_2 s^{n-1} + \ldots + b_{n-1} s + b_n}{a_1 s^m + a_2 s^{m-1} + \ldots + a_{m-1} s + a_m}$$

- The **poles** of *H(s)* are the values of *s* for which *H(s)* becomes infinity
- The **zeros** of *H(s)* are the values of *s* for which *H(s)* becomes zero

- The **zeros** are the roots of the polynomial $b(s)$, i.e. the values of *s* that satisfy $b(s) = 0$
- The **poles** are the roots of the polynomial $a(s)$, i.e. the values of *s* that satisfy $a(s) = 0$

Poles and Zeros (2)

- The transfer function expressed as a ratio of polynomials

$$H(s)=\frac{b(s)}{a(s)}=\frac{b_1 s^n+b_2 s^{n-1}+\ldots+b_{n-1}s+b_n}{a_1 s^m+a_2 s^{m-1}+\ldots+a_{m-1}s+a_m}$$

is often factorised to explicitly show the poles and zeros

$$H(s)=\frac{b(s)}{a(s)}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_m)} \quad \text{with} \quad K=\frac{b_1}{a_1}$$

- The transfer function *H(s)* is therefore completely described by its poles and zeroes, except for a constant multiplier

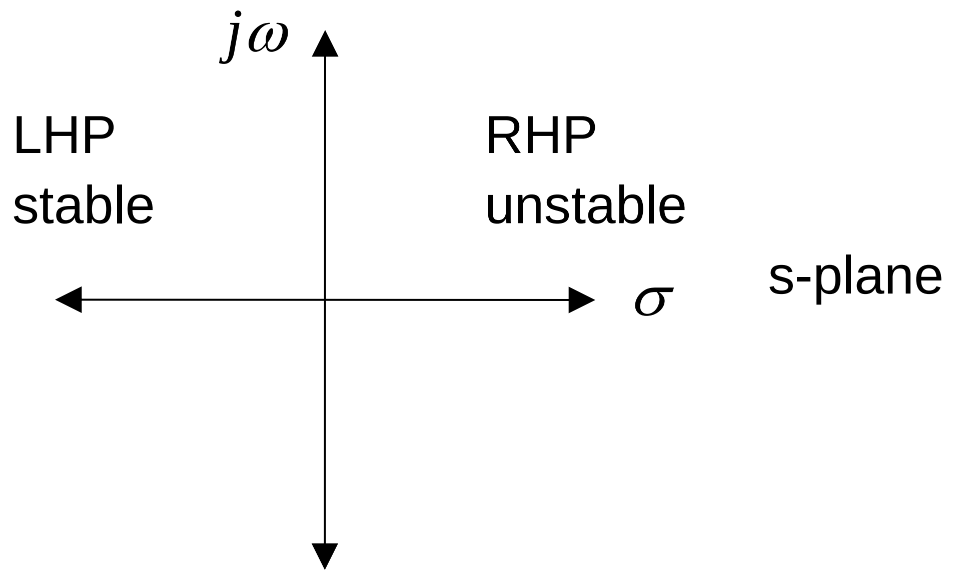
Stability

- A system is **stable** if its transient response eventually reaches equilibrium
- A system is **unstable** if its transient response grows monotonically or oscillates with increasing amplitude

- In practice, the response of a real physical system cannot increase indefinitely, and an unstable system eventually reaches a limiting value (e.g. a mechanical stop or output saturation) or damages itself and fails

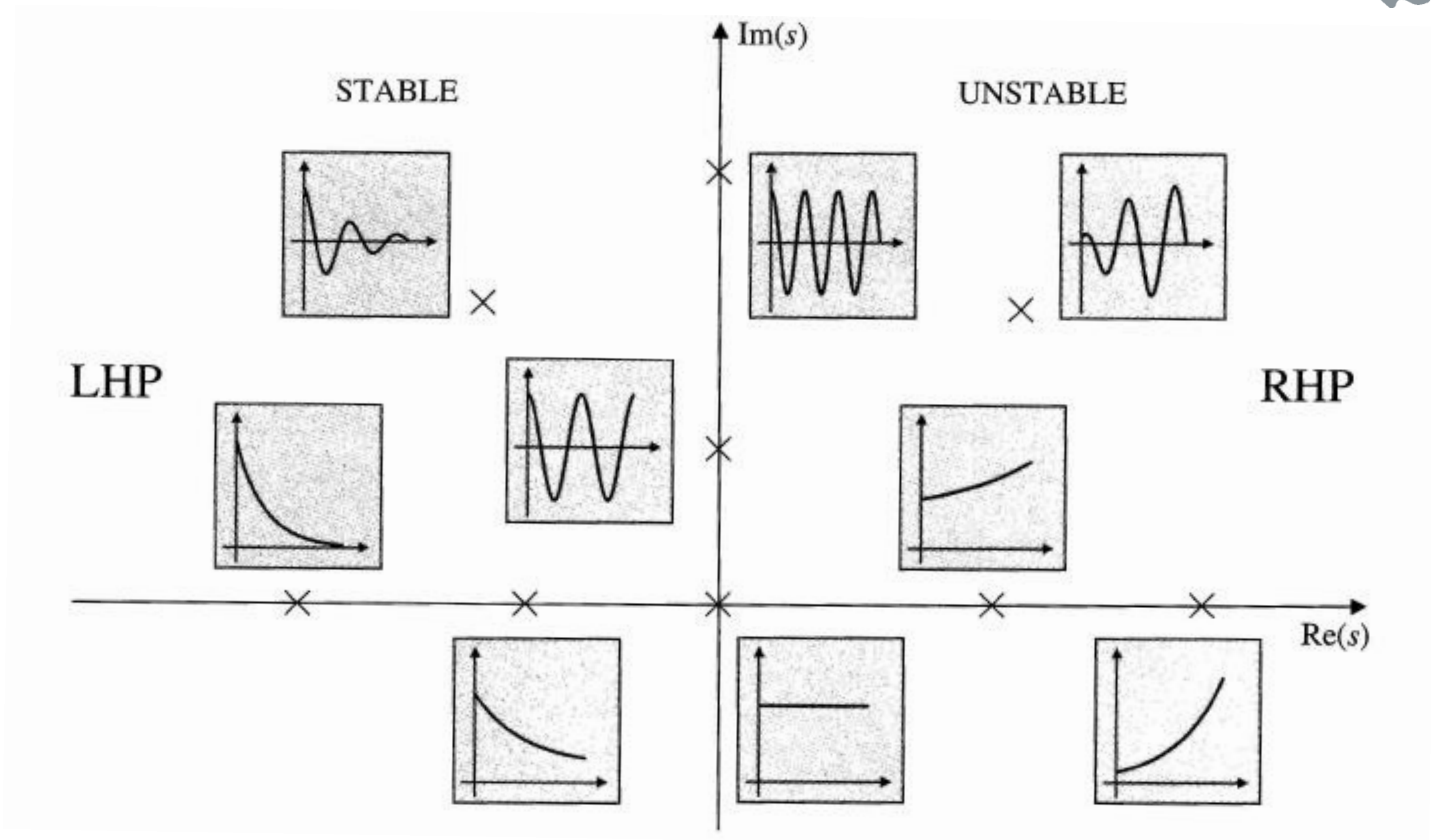
- Example: Pilot Induced Oscillation

Effect of Poles on Stability

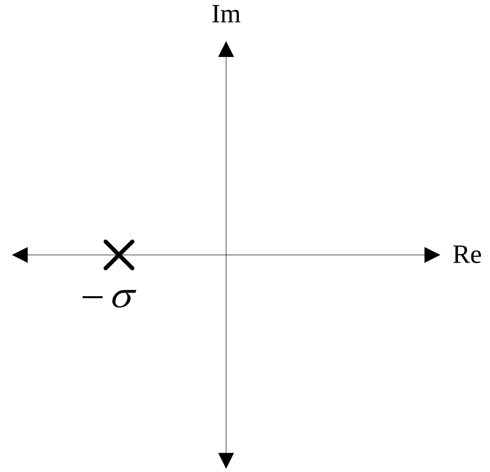


- If all poles of the system lie in the left half-plane (LHP), the system is **stable** (the real parts of all poles are **negative**)
- If any poles lie in the right half-plane (RHP), the system is **unstable** (the real parts of one or more poles are **positive**)
- If a pole lies on the imaginary axis (its real part is **zero**), the pole is said to be **marginally stable**, and the system is usually treated as if it is unstable

Effect of Poles on Transient Response



First-Order Pole



- For a first-order pole

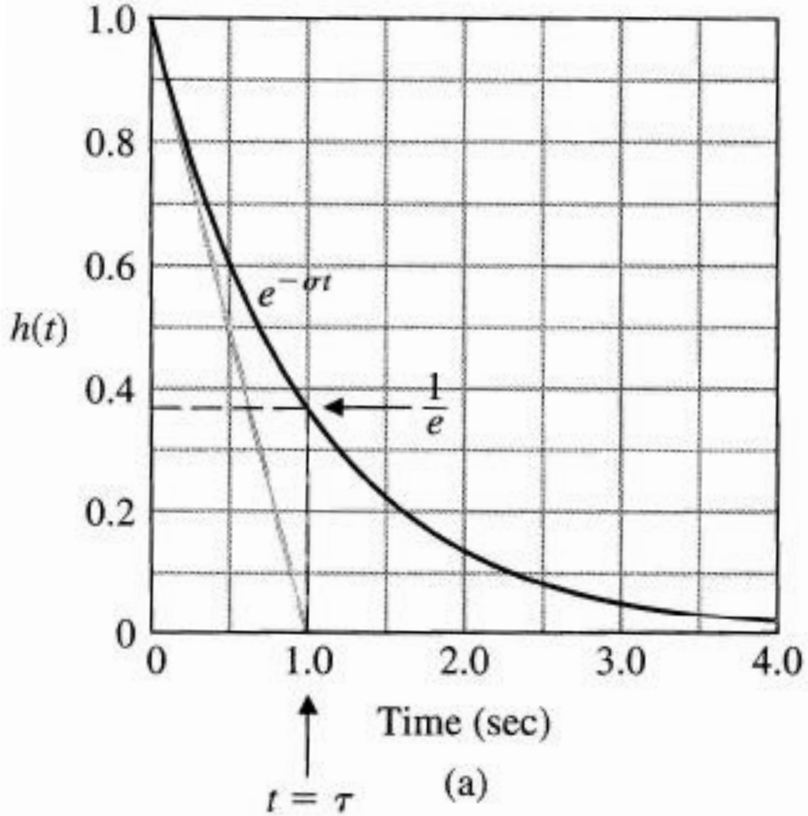
$$H(s) = \frac{1}{s+\sigma}$$

- the impulse response will be an exponential function

$$h(t) = e^{-\sigma t} \quad , t \geq 0$$

- A negative real pole $s < 0$, implies that $\sigma > 0$, and the exponential function decays to zero with time (**stable** impulse response)
- A positive real pole $s > 0$, implies that $\sigma < 0$, and the exponential function grows unbounded with time (**unstable** impulse response)

First-Order Pole – Impulse Response



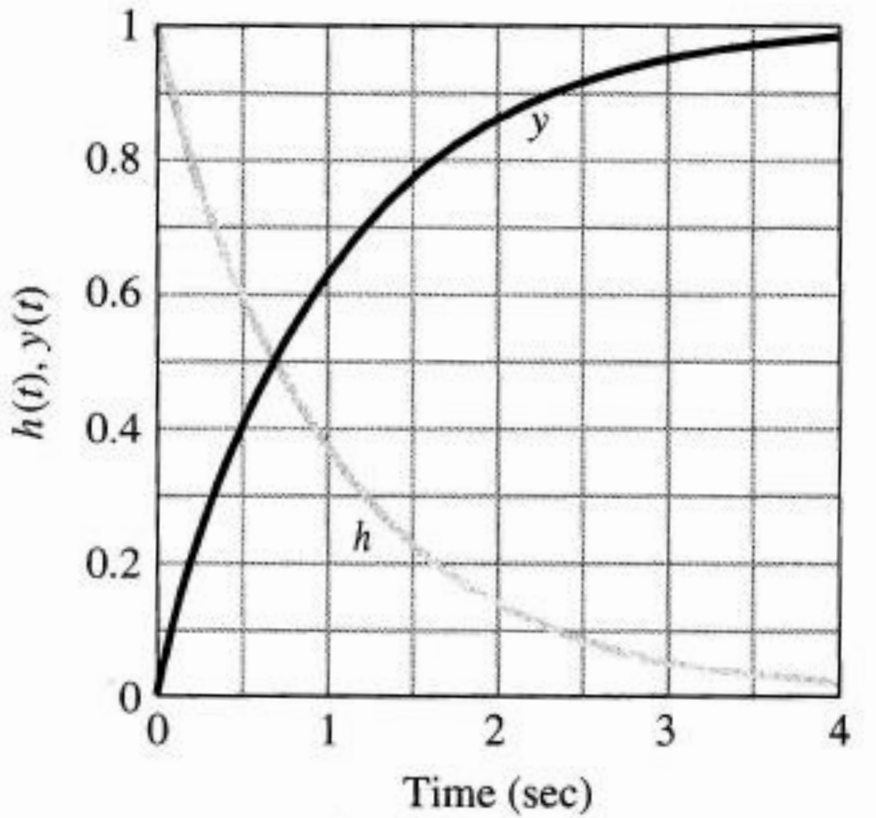
(a)

The **time constant** is related to the first-order pole by

$$\tau = \frac{1}{\sigma}$$

- The time constant is the time it takes for the impulse response to decay to 37% of its initial value
- (Remember: $e^{-\sigma\frac{1}{\sigma}} = e^{-1} = 0.368 \approx 37\%$)

First-Order Pole – Step Response



The **time constant** is related to the first-order pole by

$$\tau = \frac{1}{\sigma}$$

- The time constant is also the time it takes for the step response to reach 63% of its final value

Second-Order Poles

- A second-order transfer function may be written as

$$H(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- where ζ (zeta) is called the **damping ratio**
- and ω_n (omega-n) is called the **undamped natural frequency**

- The second-order poles are the roots of the denominator and can be written in terms of their real and imaginary parts

$$s=-\sigma\pm j\omega_d$$

- where ω_d is called the **damped natural frequency**

Second-Order Poles

- The damping ratio ζ and the natural frequency ω_n are related to the real and imaginary parts of the second-order poles

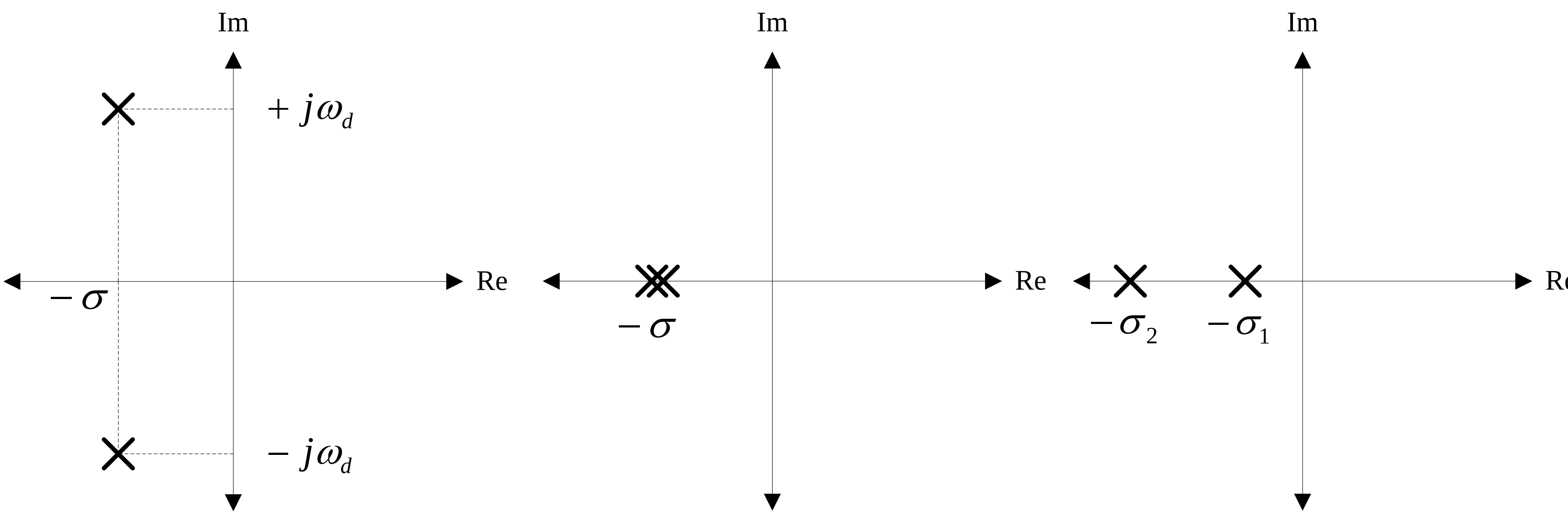
$$s = -\sigma + j\omega_d$$

- through

$$\sigma = \zeta\omega_n$$
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

- When $0 < \zeta < 1$, the poles are complex conjugates (underdamped)
- When $\zeta = 1$, the poles are real and repeated (critically damped)
- When $\zeta > 1$, the poles are real and distinct (overdamped)

Underdamped, Critically Damped, Overdamped



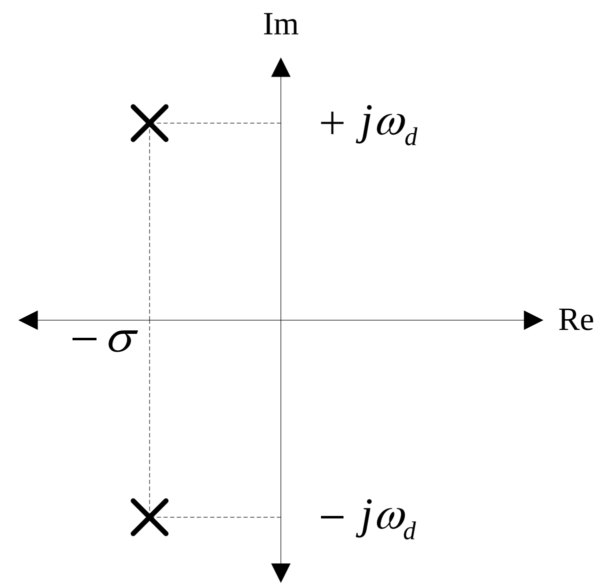
- Underdamped $0 < \zeta < 1$ | Critically Damped $\zeta = 1$ | Overdamped $\zeta > 1$
- Lower damping ratio implies more oscillation, higher damping ratio means less oscillation

Underdamped Poles

- Underdamped case: $0 < \zeta < 1$

- The second-order poles are complex conjugates

$$s = -\sigma + j\omega_d$$

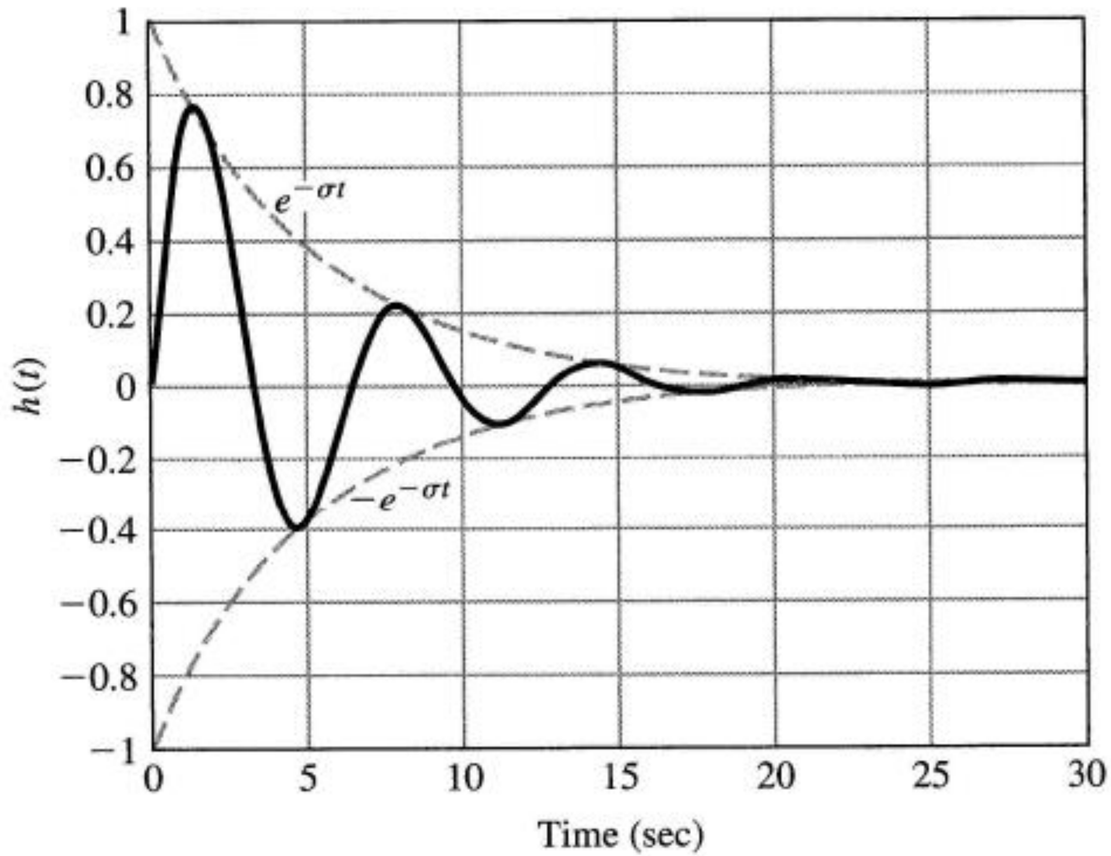


- the impulse response is an exponentially decaying oscillation

$$h(t) = \frac{K\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) \qquad , t \geq 0$$

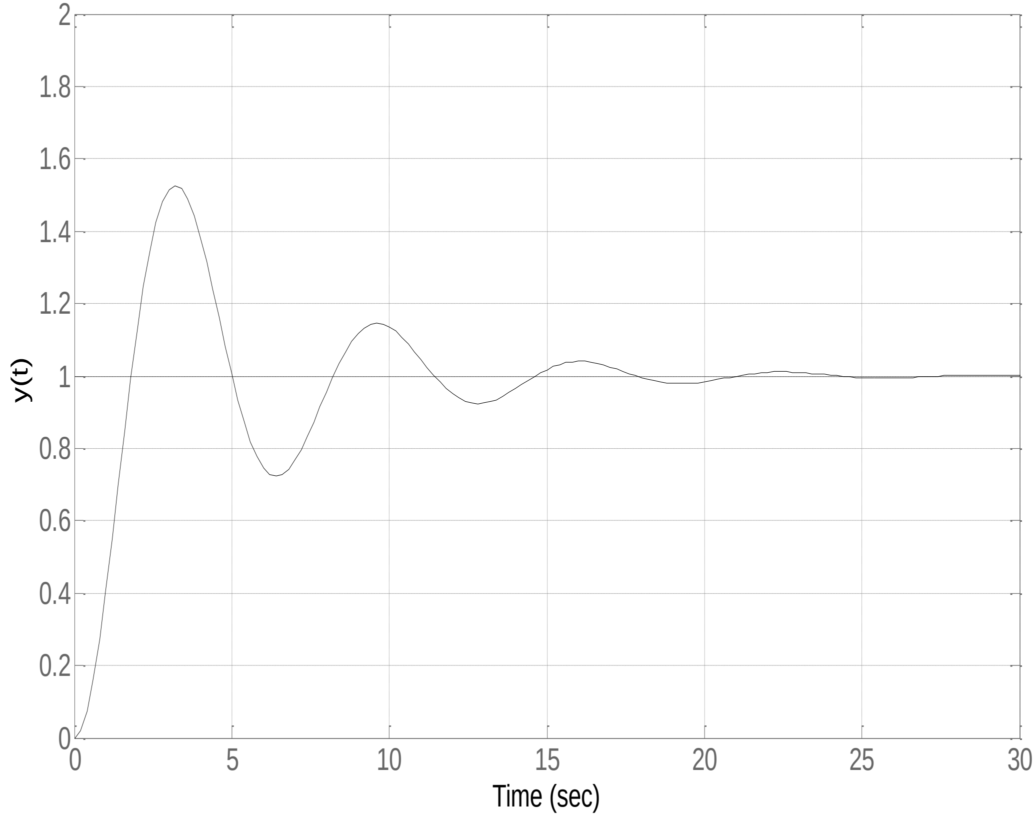
- where $1/\sigma$ is the time constant of the exponential decay envelope and ω_d is the frequency of the sinusoidal oscillation

Underdamped Poles – Impulse Response



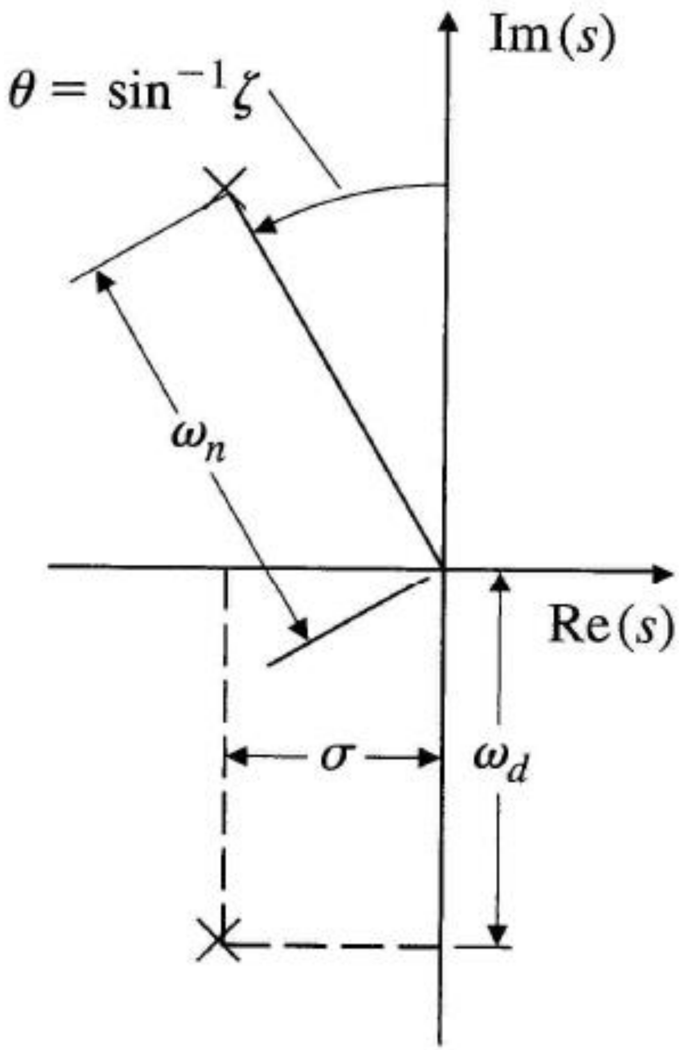
- $1/\sigma$ is the time constant of the exponential decay envelope
- ω_d is the frequency of the sinusoidal oscillation

Underdamped Poles – Step Response



- $1/\sigma$ is the time constant of the exponential decay envelope
- ω_d is the frequency of the sinusoidal oscillation

Underdamped Poles – zeta, wd, wn



- The poles are at an angle θ from the $j\omega$ axis

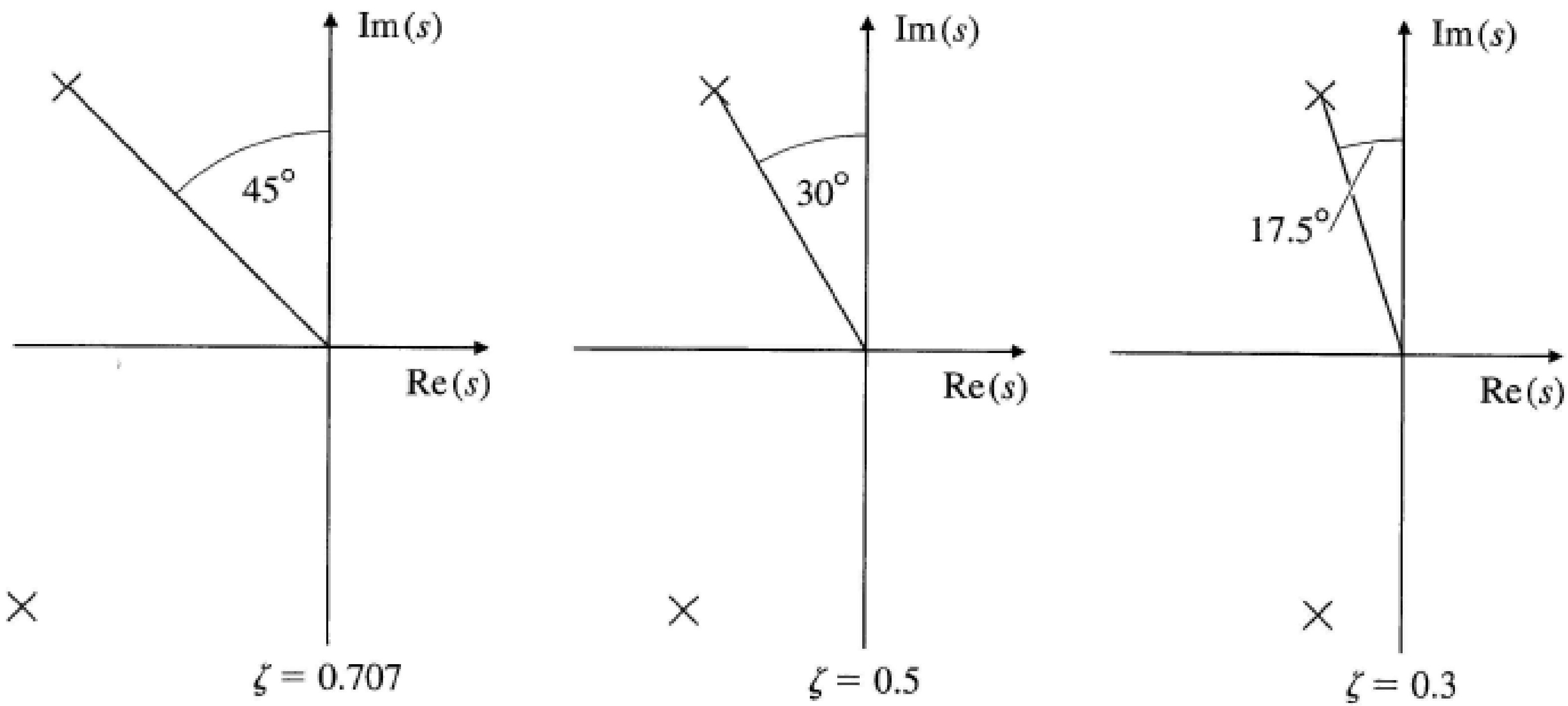
$$\tan\theta = \frac{\sigma}{\omega_d}$$

$$\zeta = \sin\theta$$

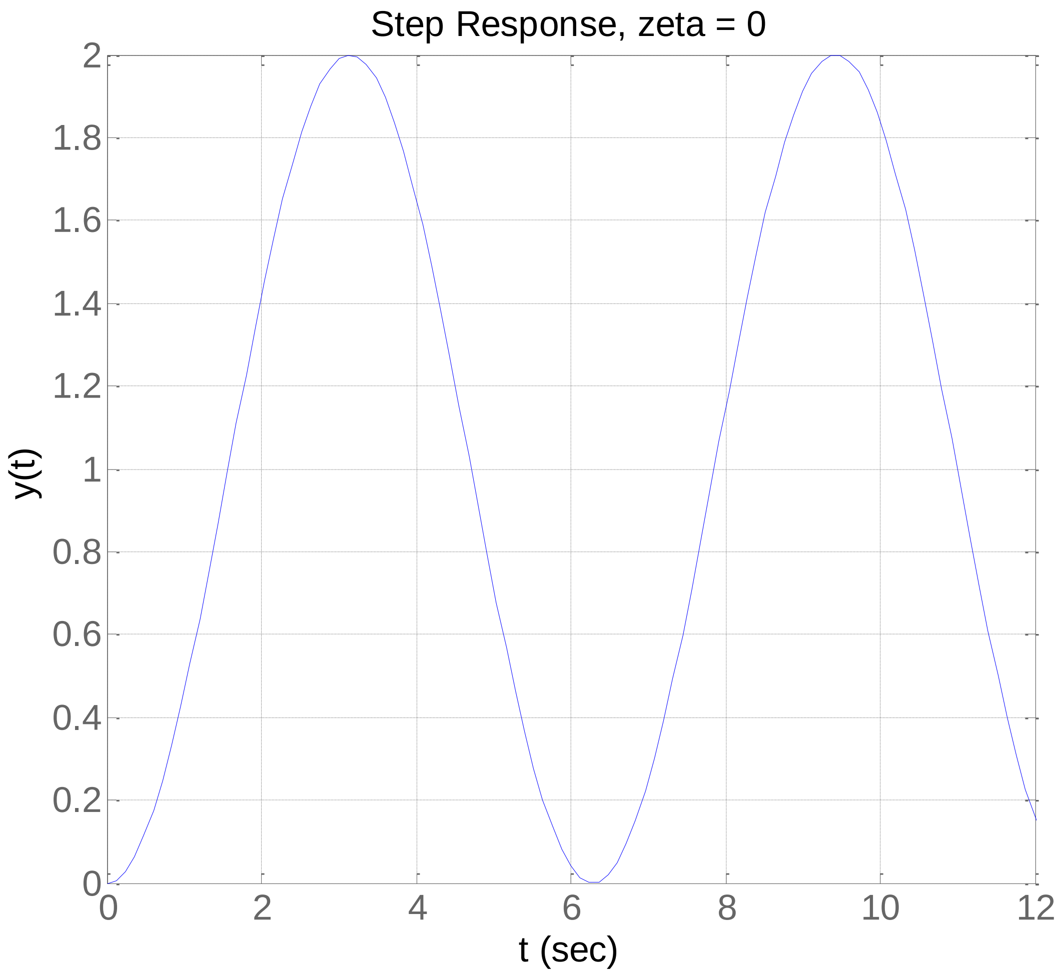
$$\omega_n = \sqrt{\sigma^2 + \omega_d^2}$$

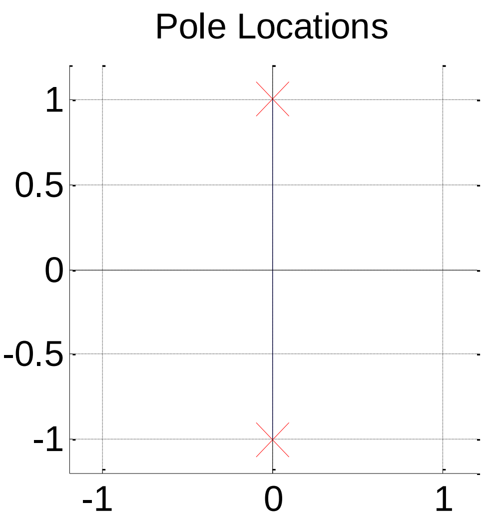
- $\theta = 0$ implies $\zeta = 0$ which means no damping and $\omega_d = \omega_n$

Underdamped Poles – Theta vs Zeta



Step Response vs Damping Ratio





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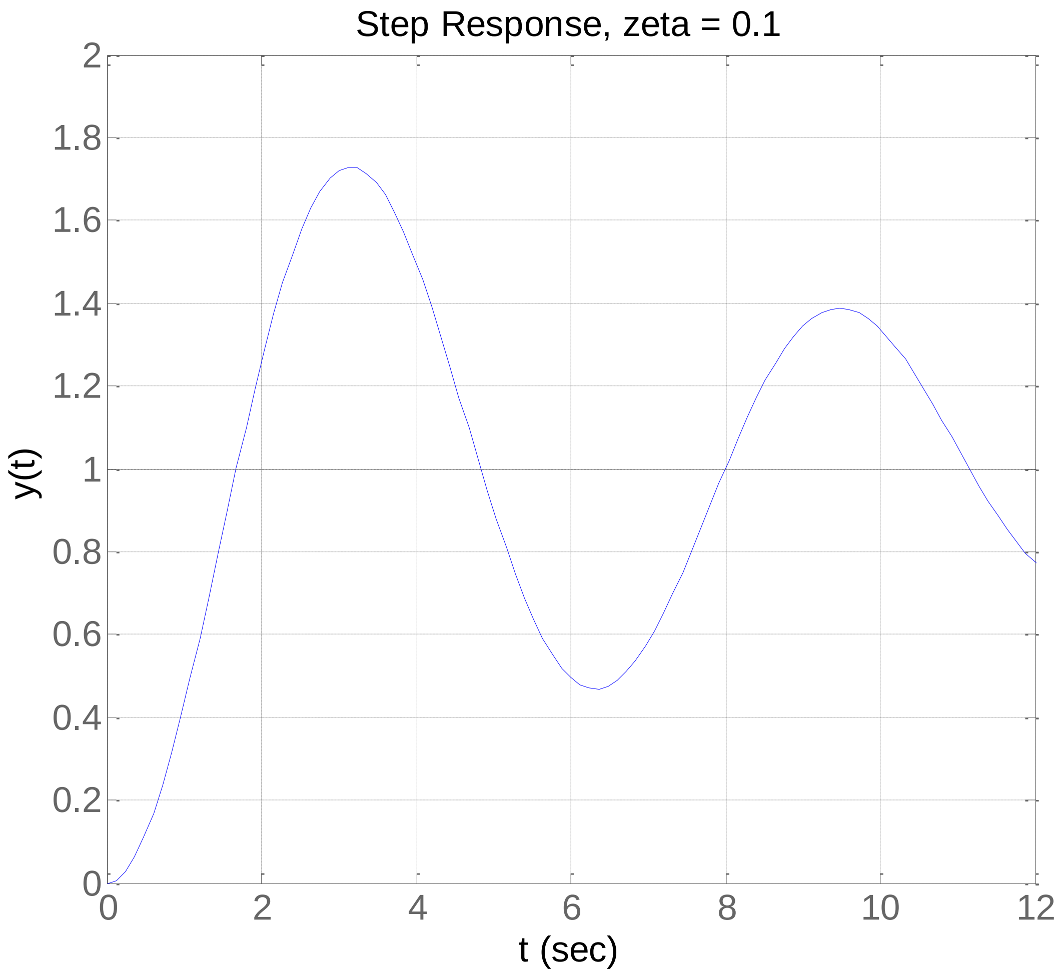
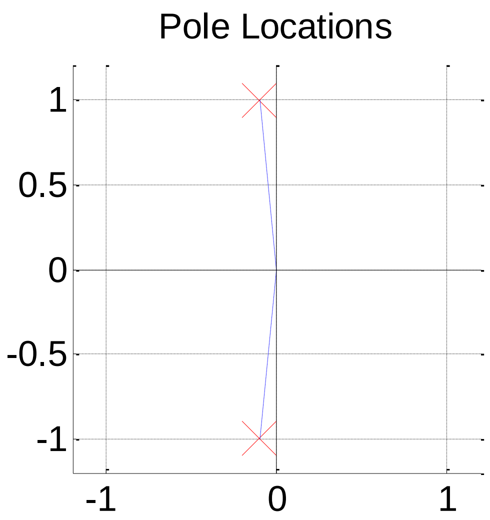
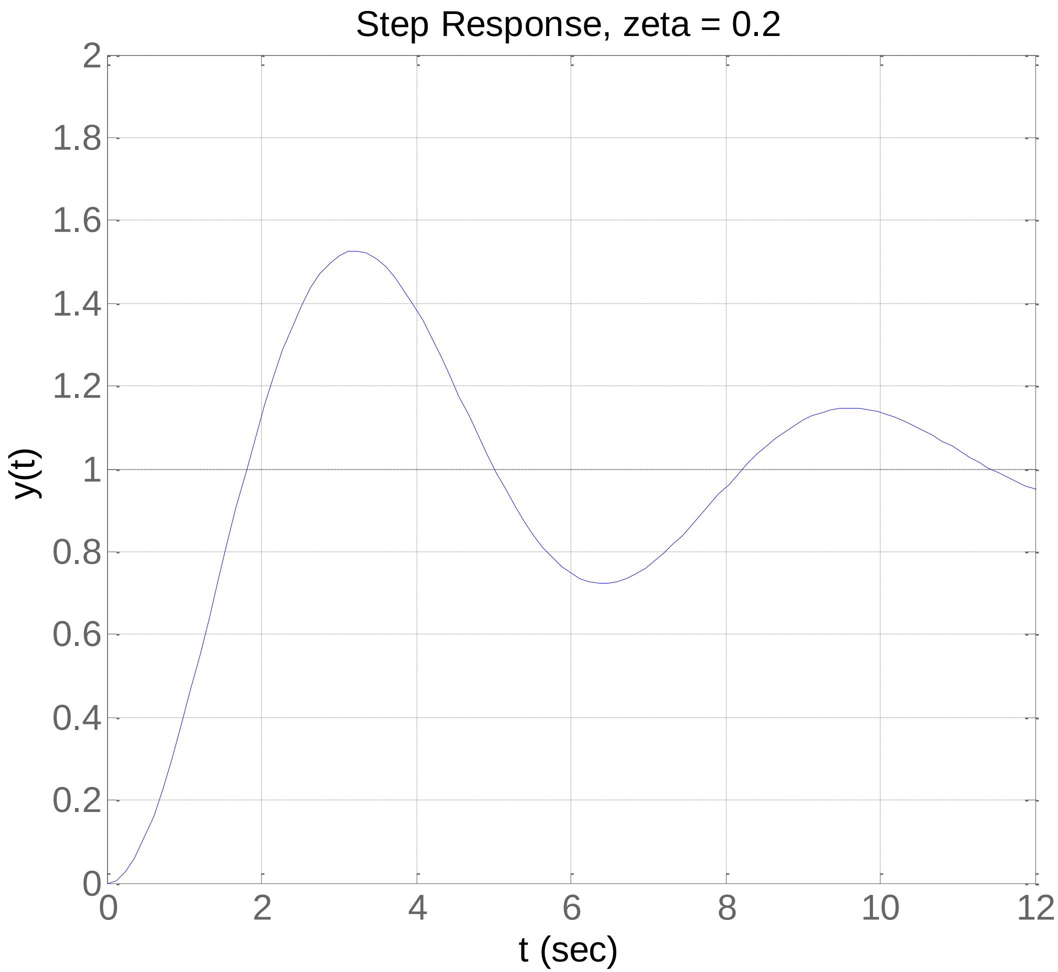
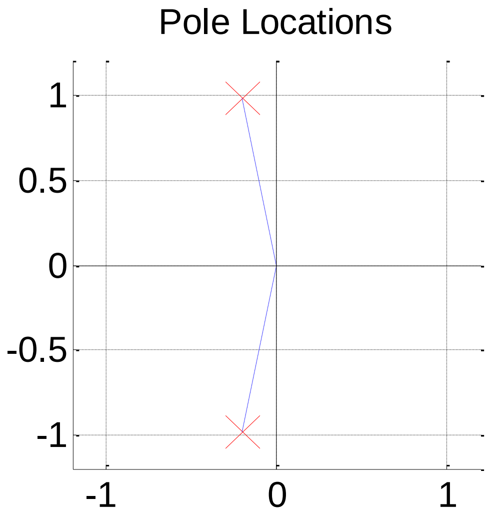


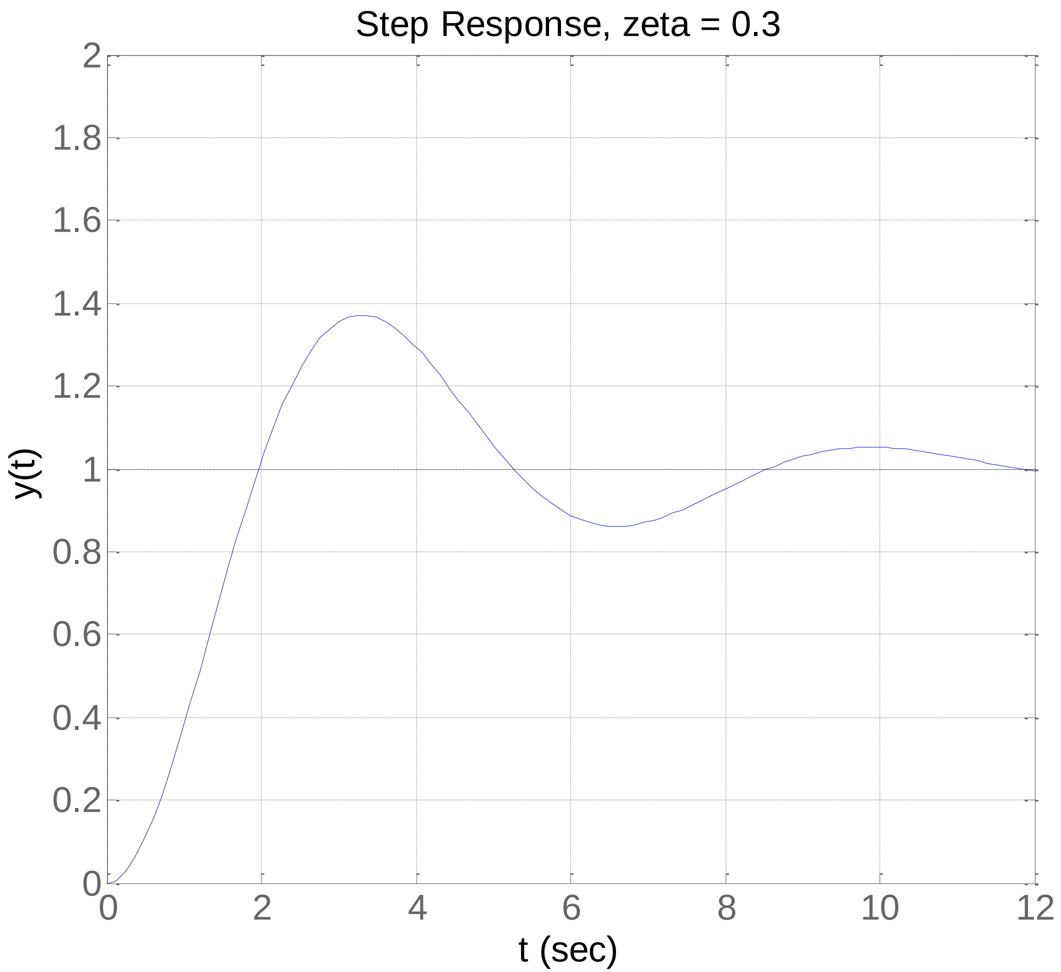


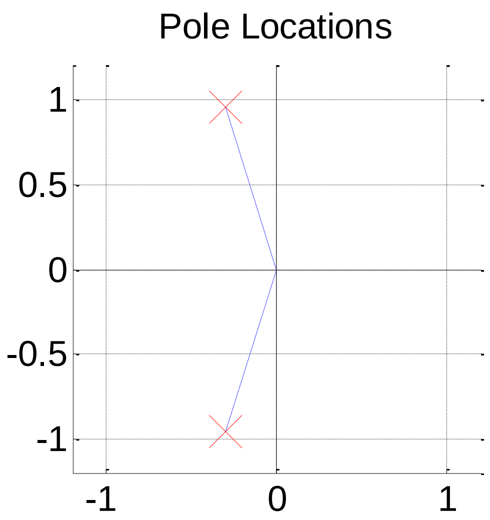
Step Response vs Damping Ratio



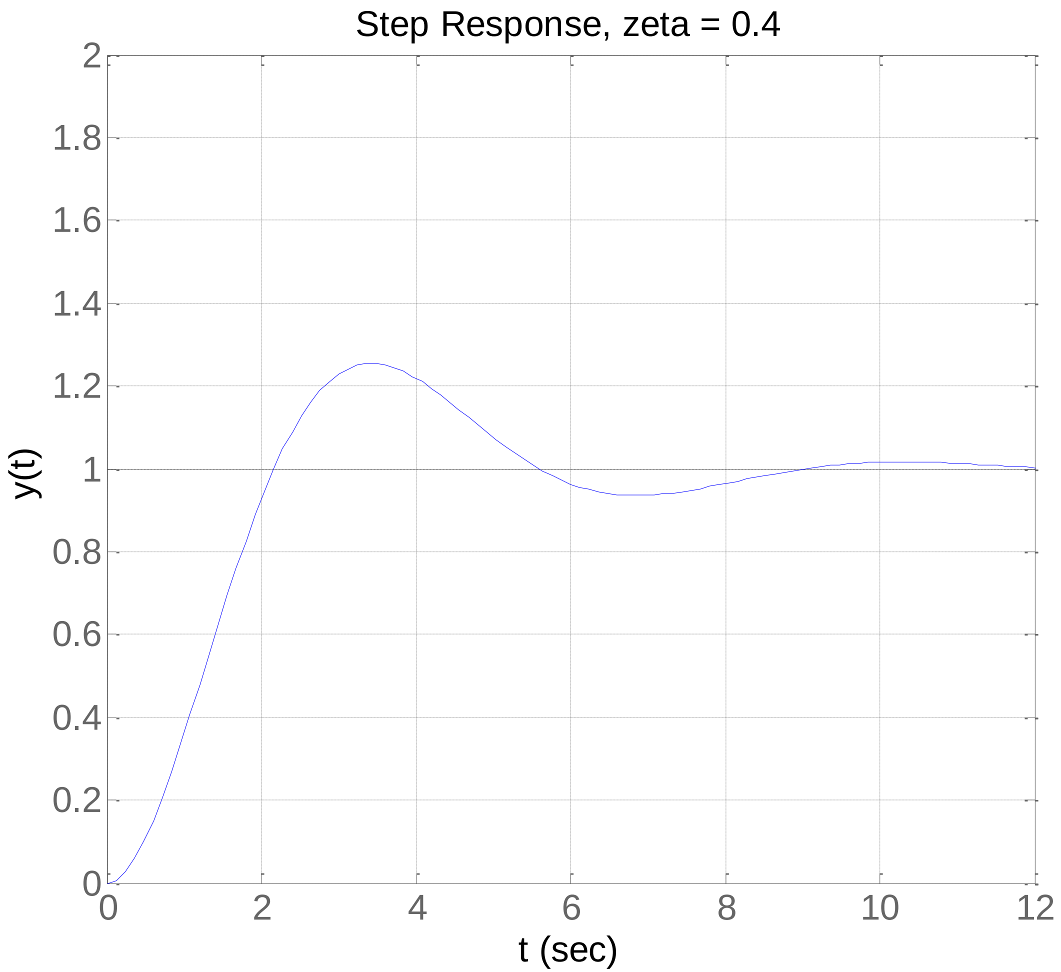


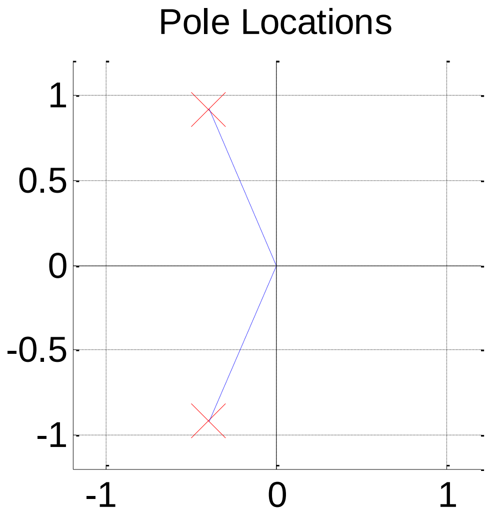
Step Response vs Damping Ratio





Step Response vs Damping Ratio





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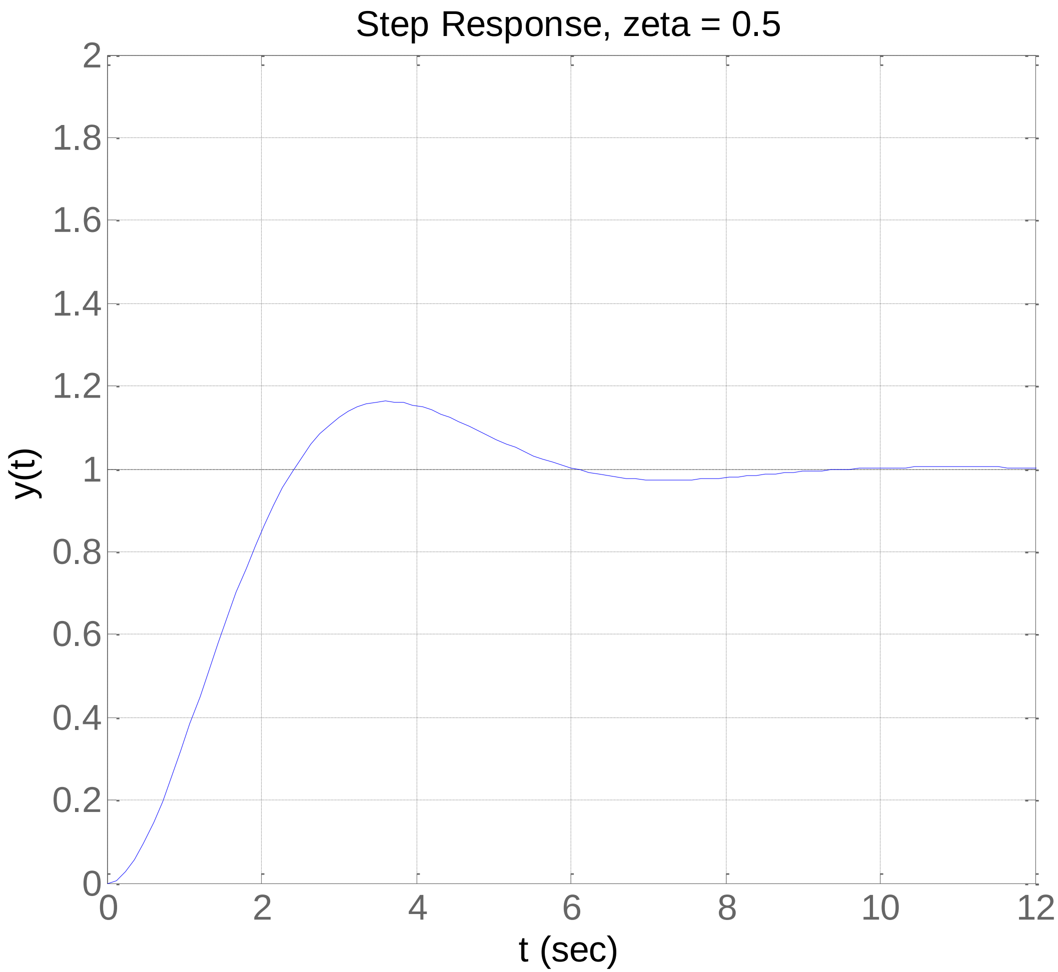
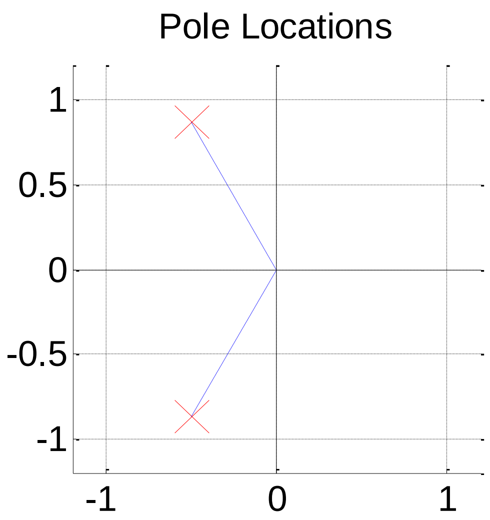
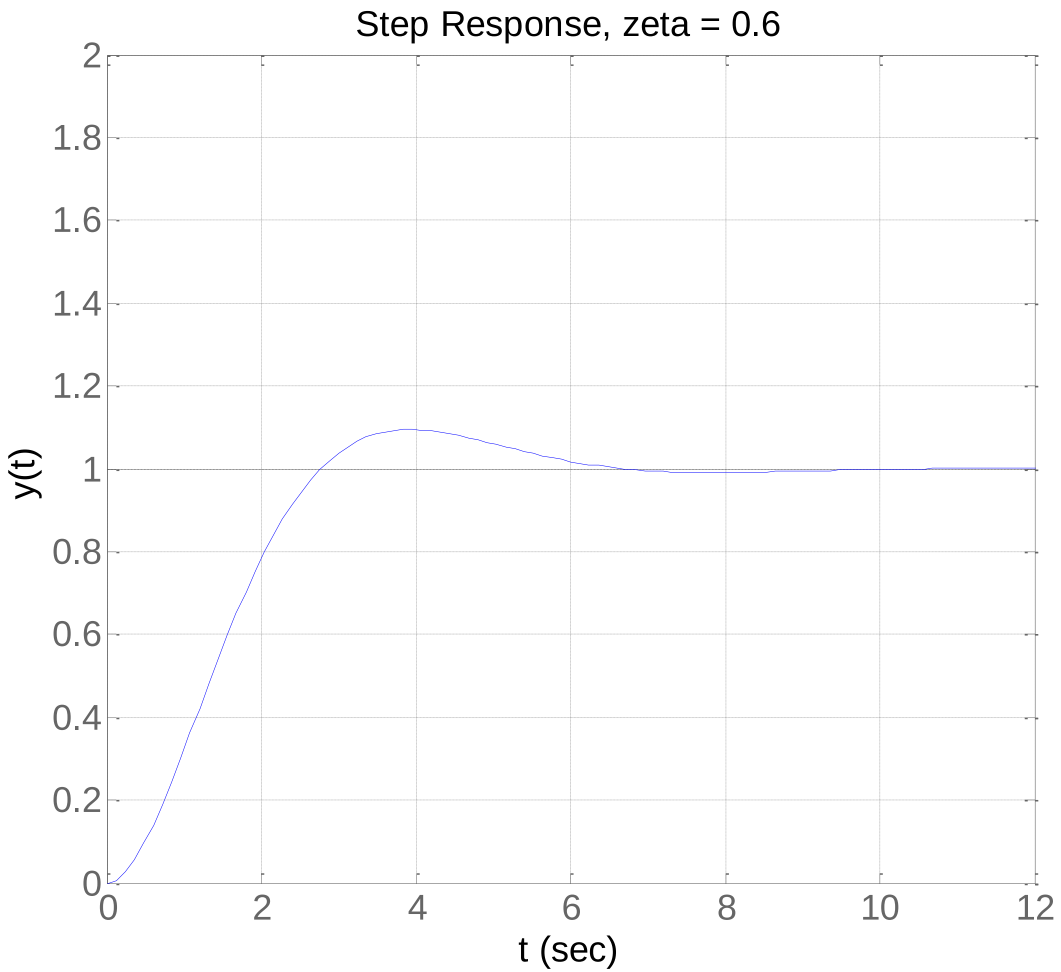
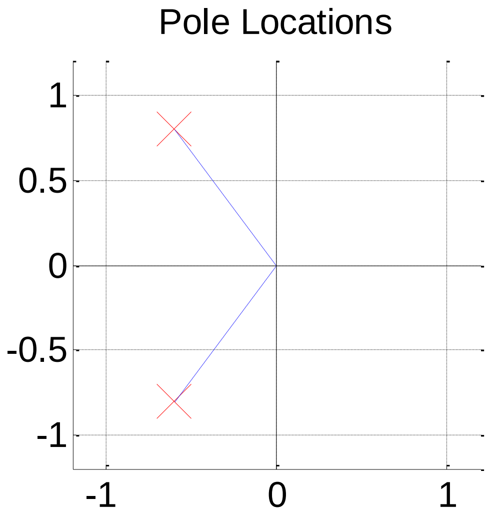


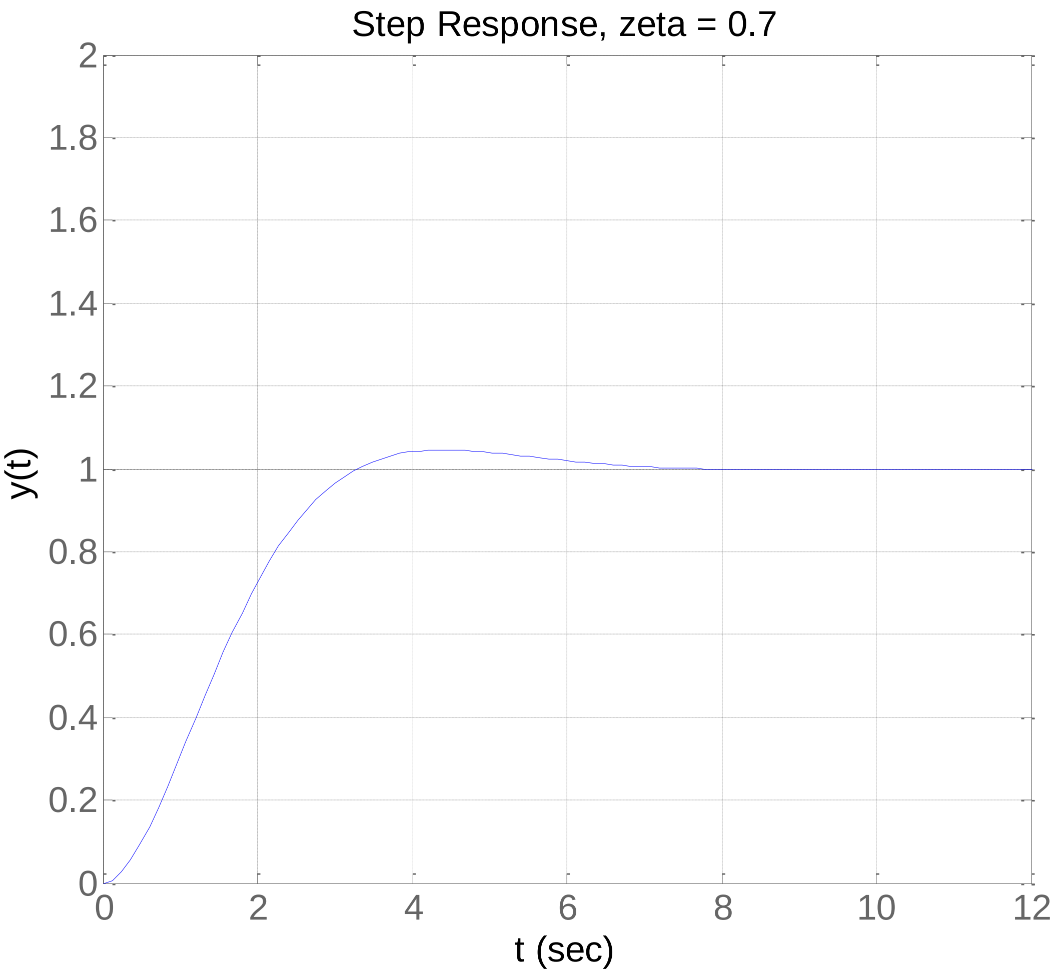


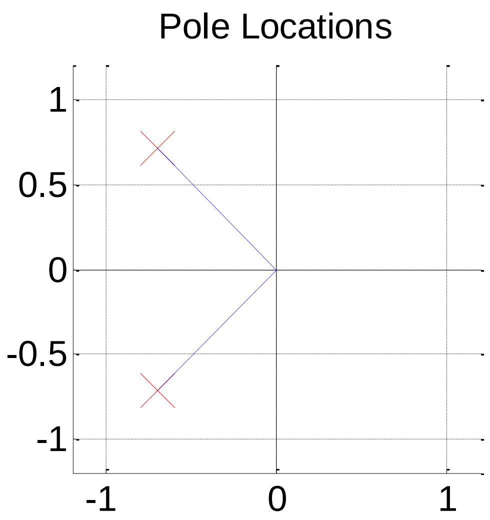
Step Response vs Damping Ratio





Step Response vs Damping Ratio





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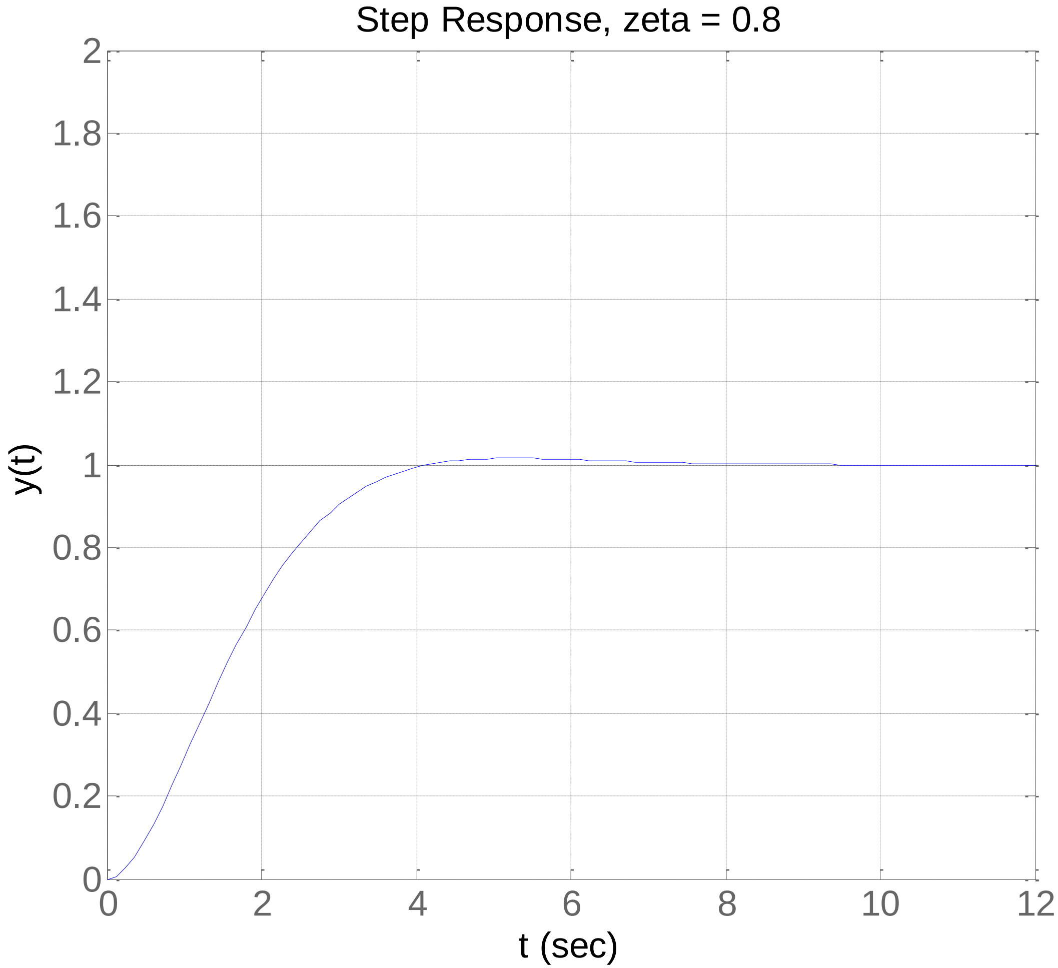
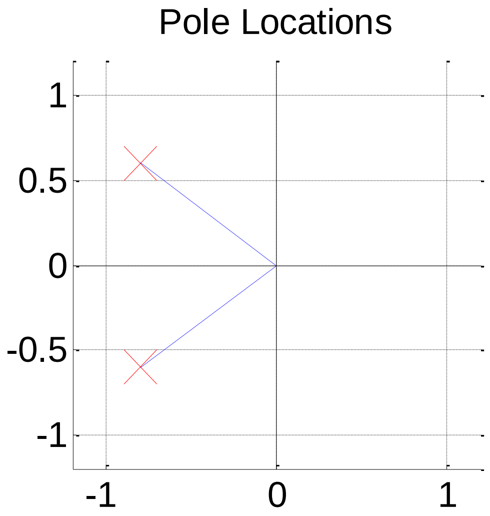
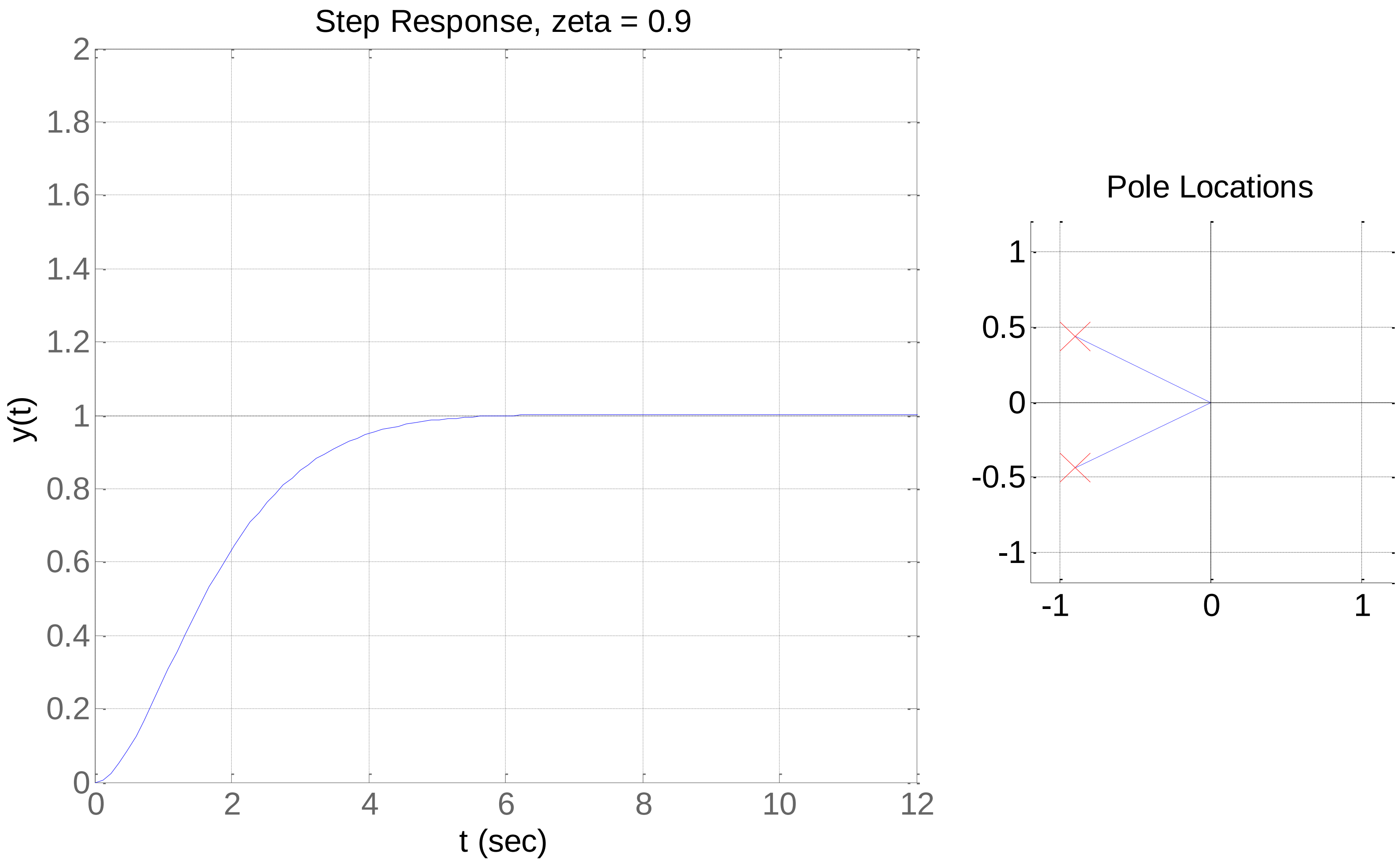




Step Response vs Damping Ratio



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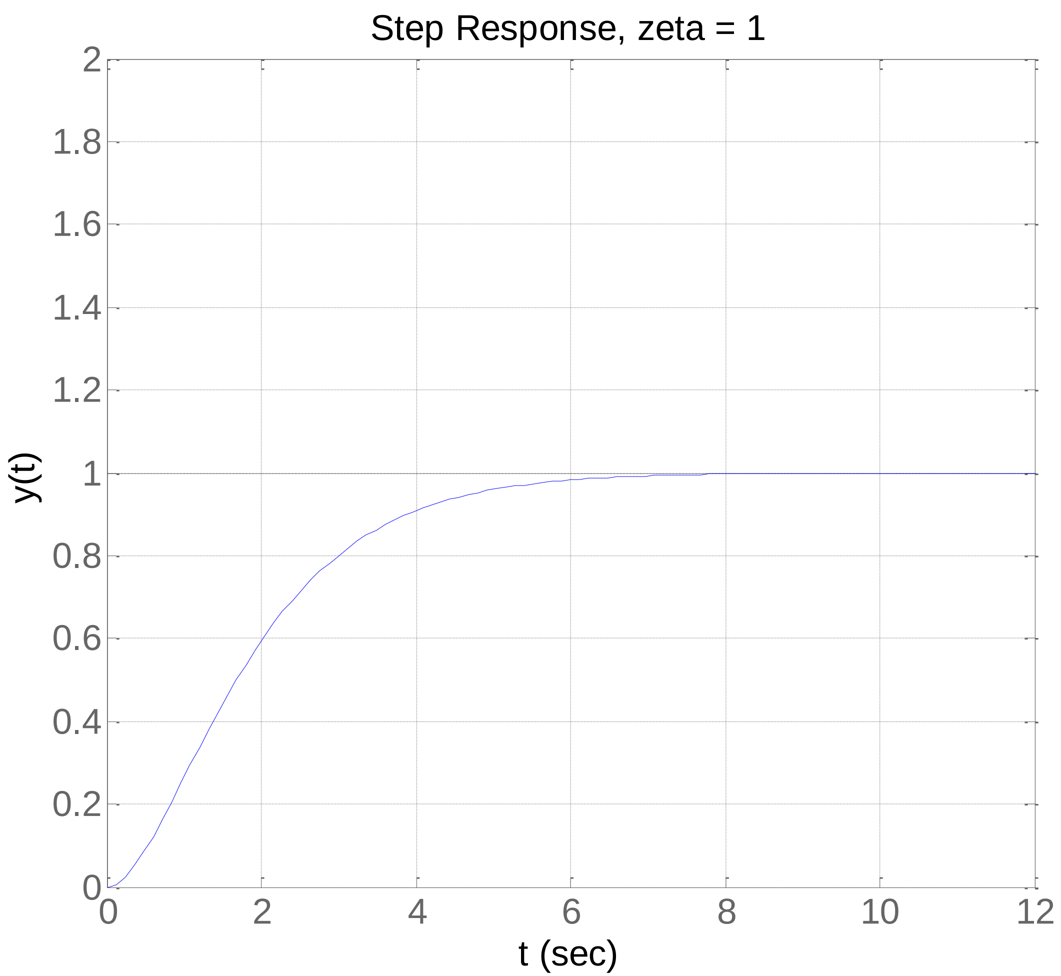
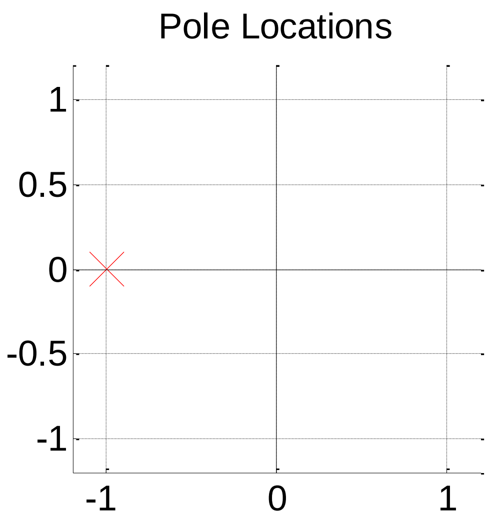
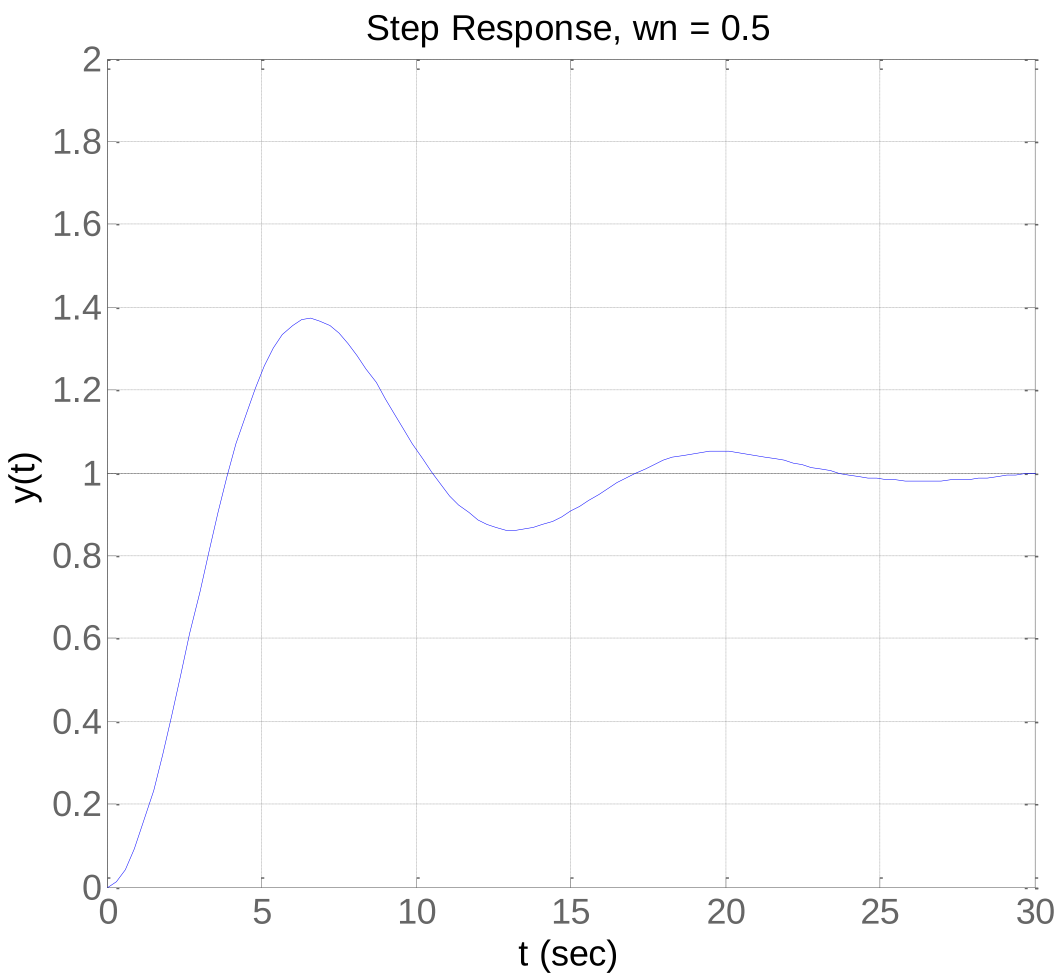
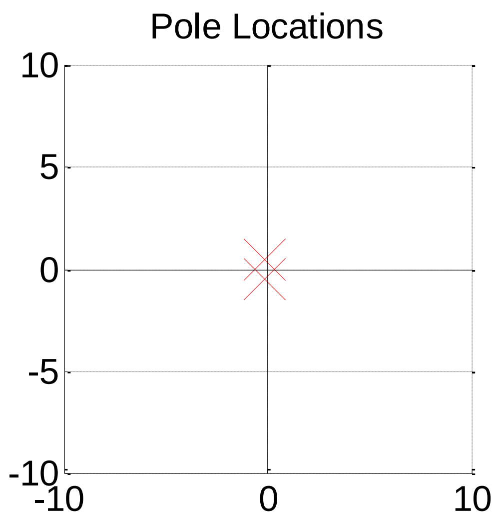


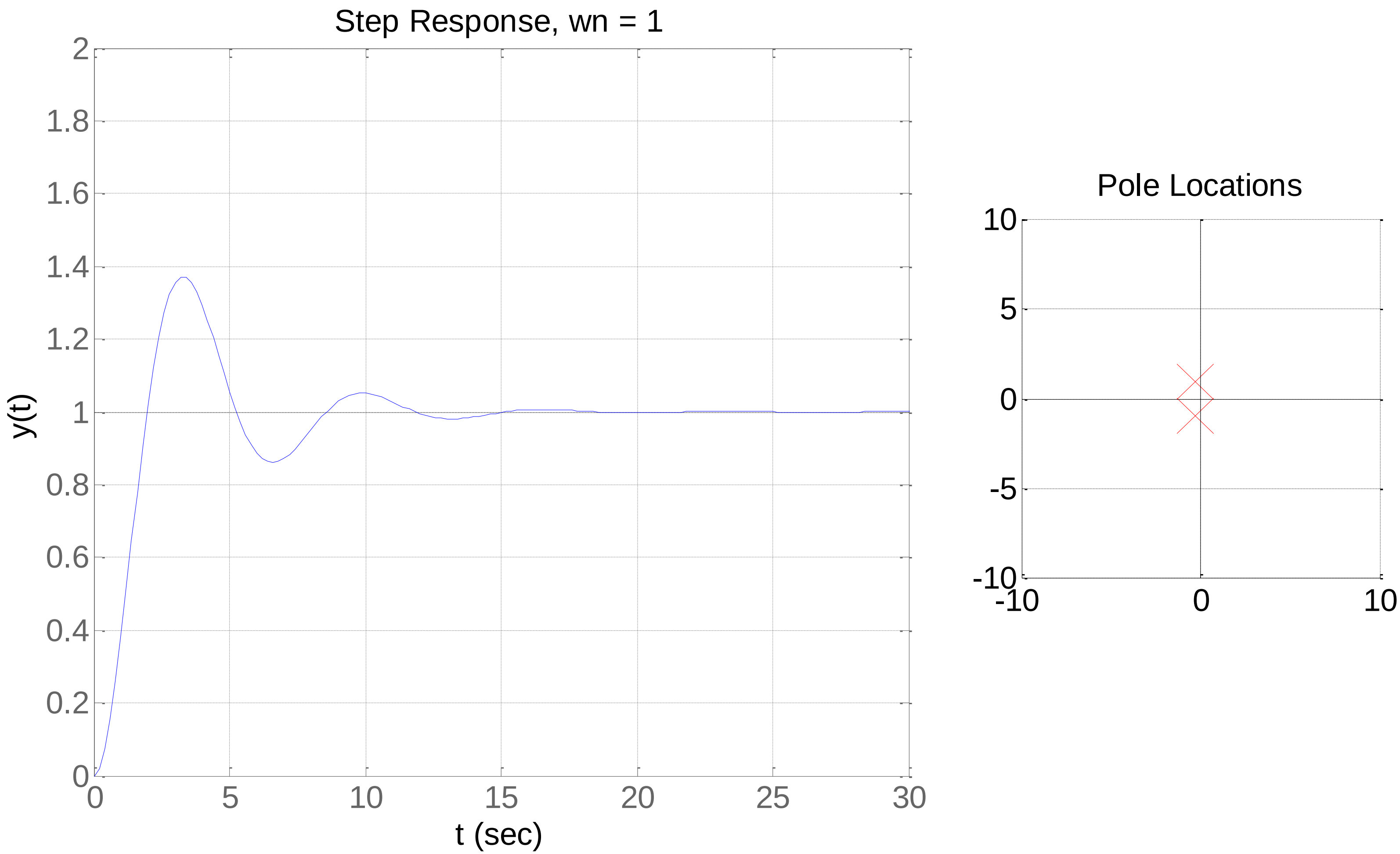


Step Response vs Natural Frequency

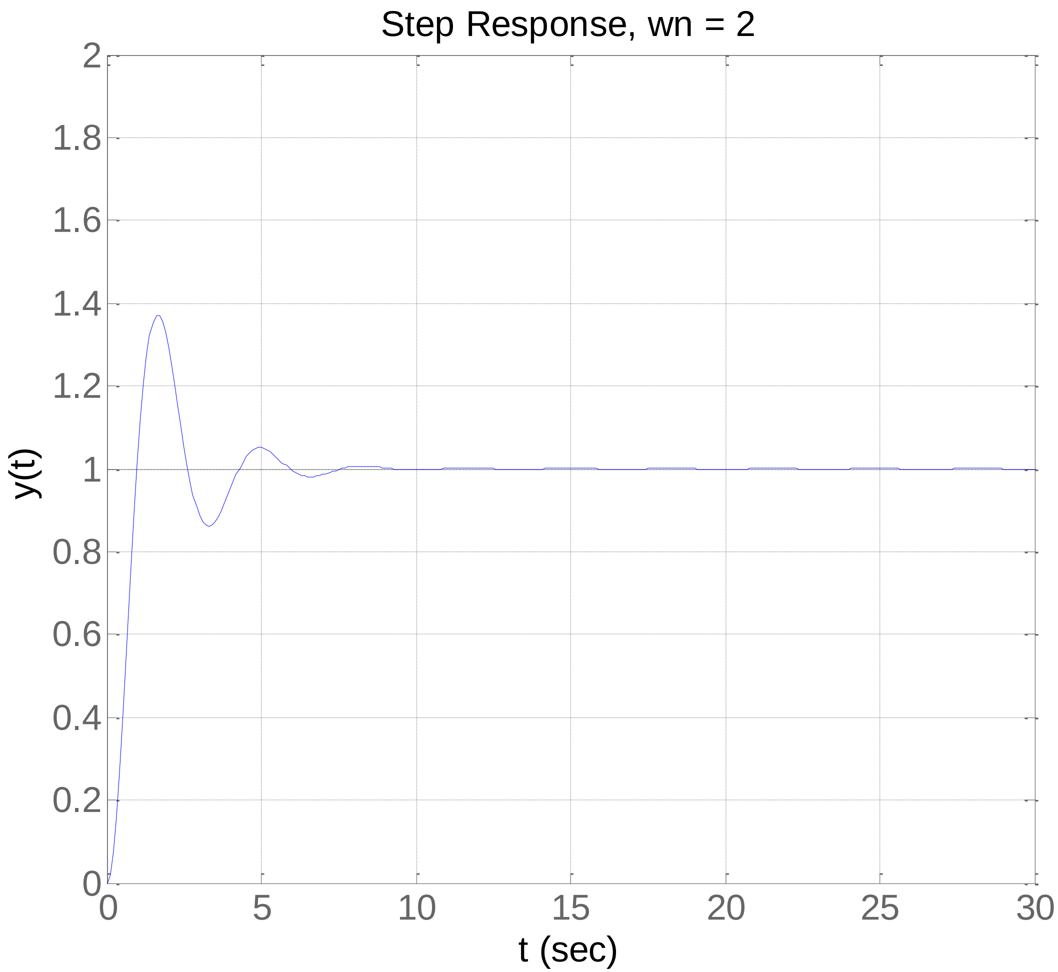


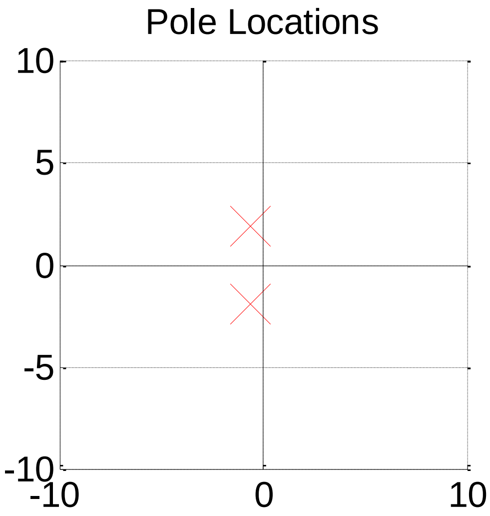


Step Response vs Natural Frequency

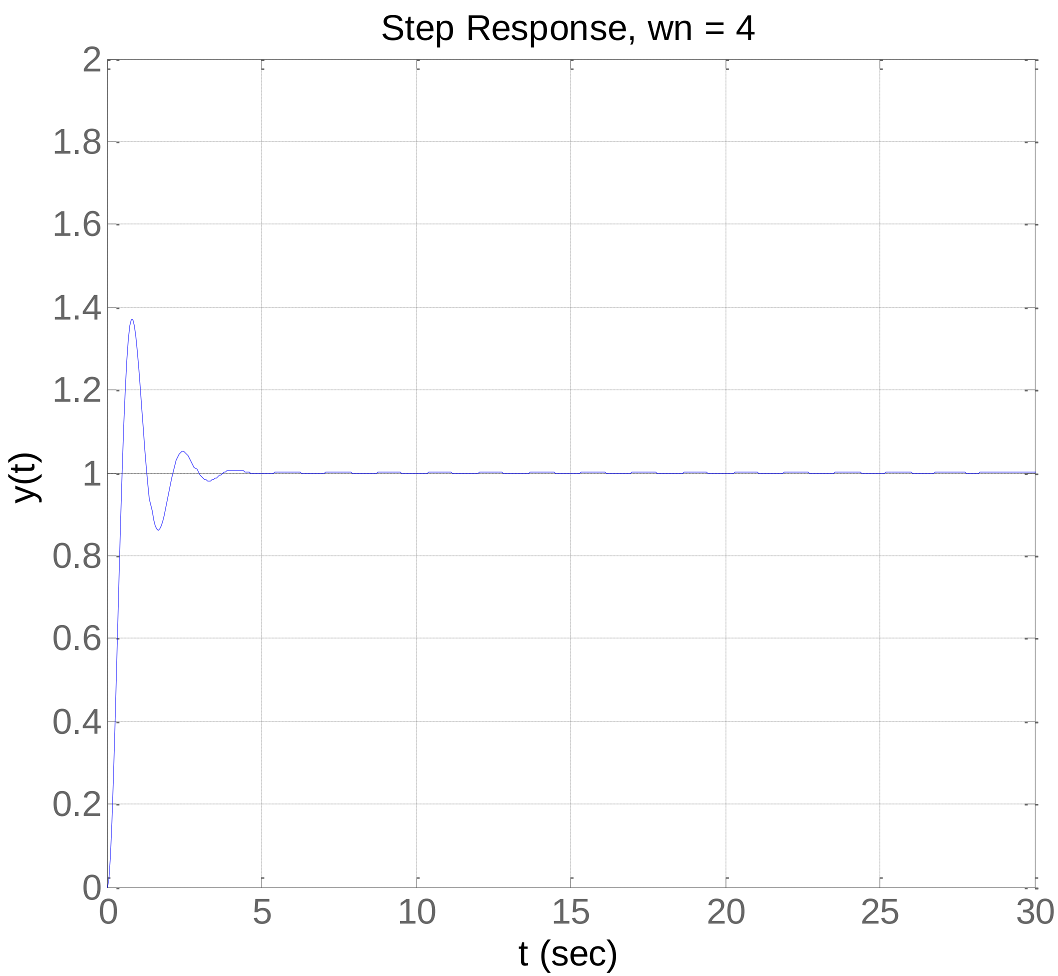


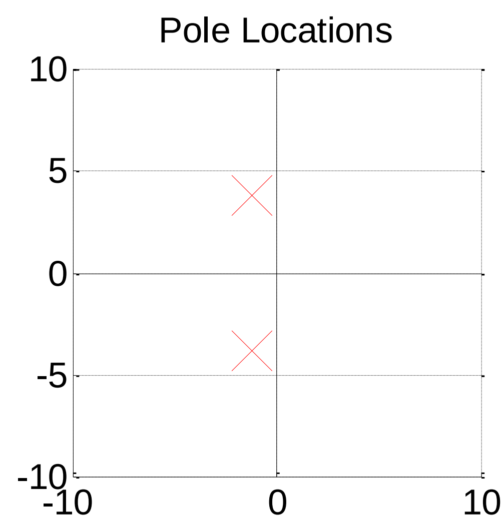
Step Response vs Natural Frequency



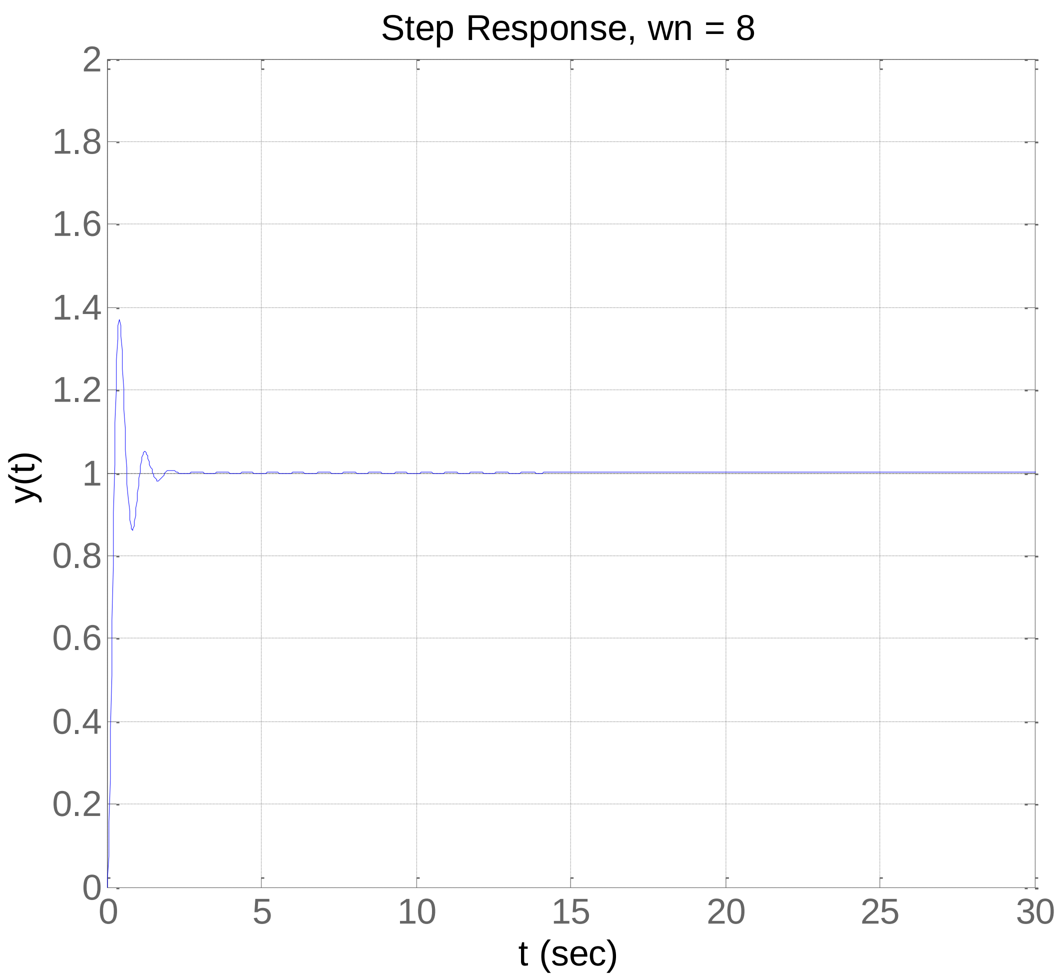


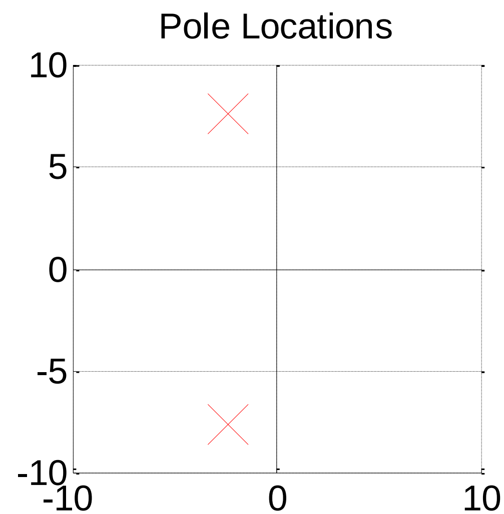
Step Response vs Natural Frequency





Step Response vs Natural Frequency



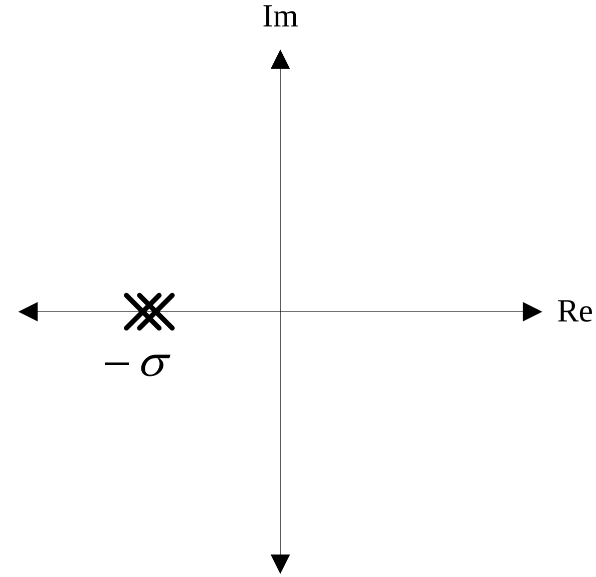


Critically Damped Poles

- <u>Critically damped case:</u> $\zeta = 1$

- The second-order poles are real and repeated

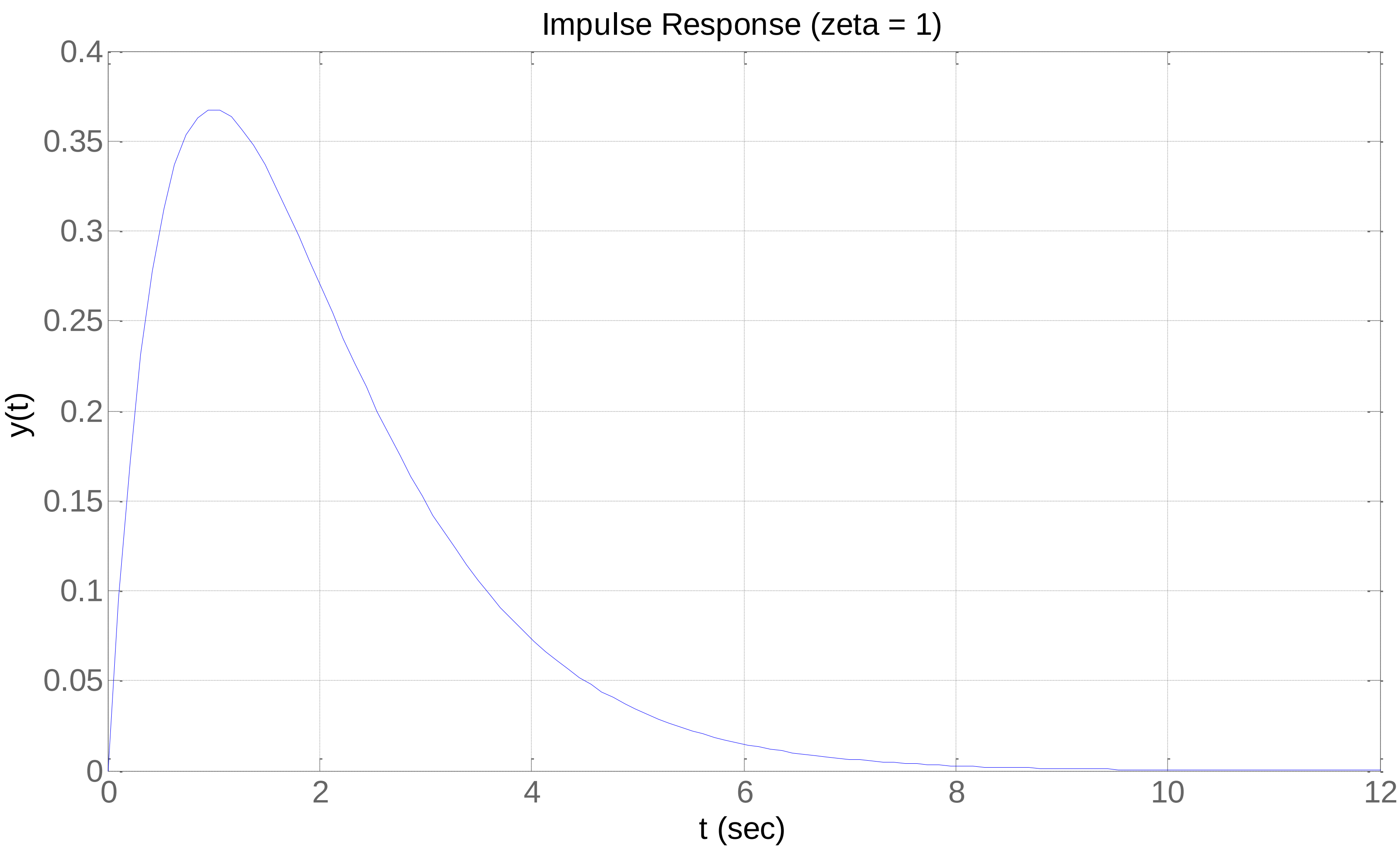
$$s_1, s_2 = -\sigma$$



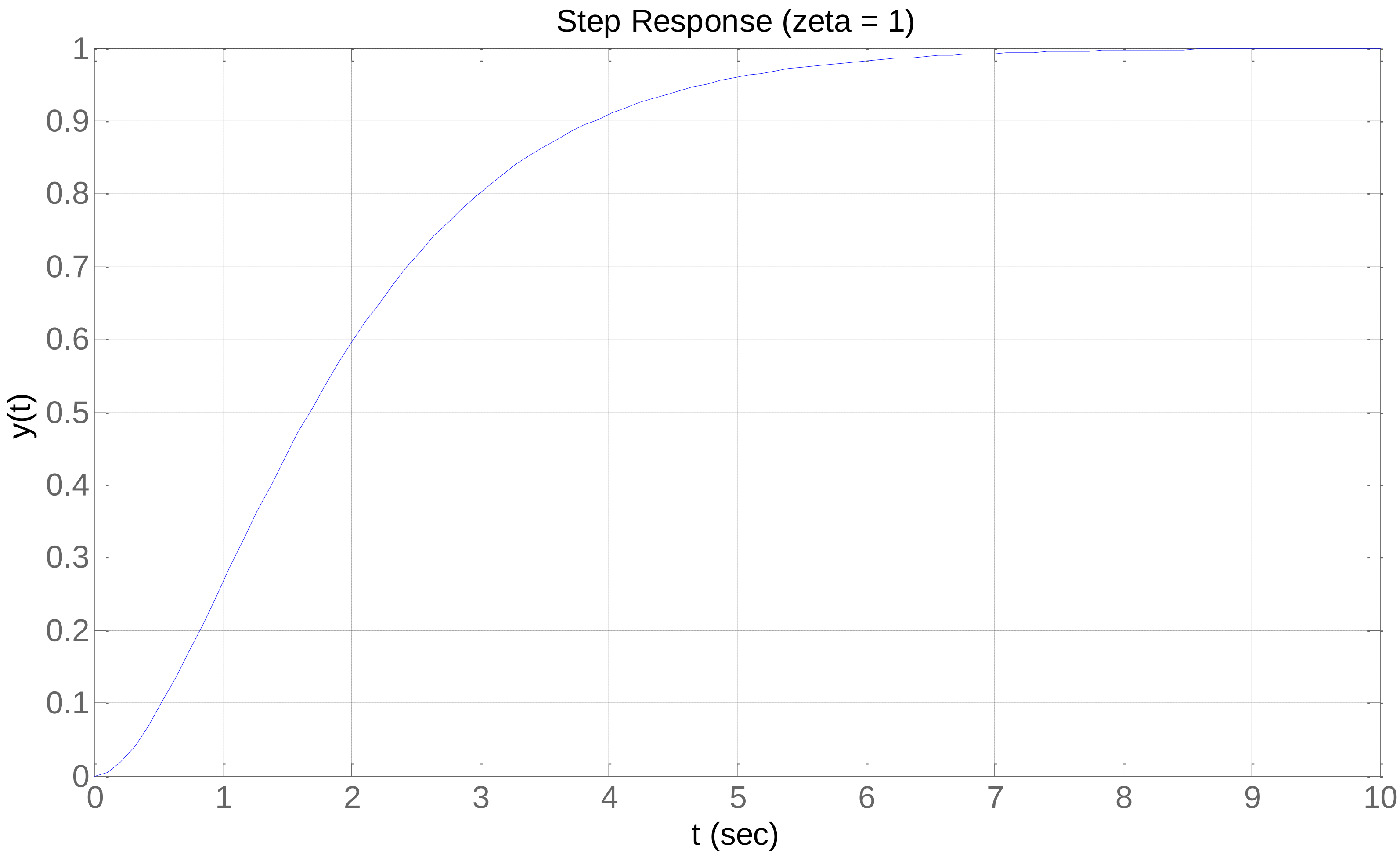
- At the critical damping ratio, the second-order poles change from complex conjugate poles to real poles

- The critically damped response is the fastest second-order response that exhibits no oscillation

Critically Damped Poles – Impulse Response



Critically Damped Poles – Step Response

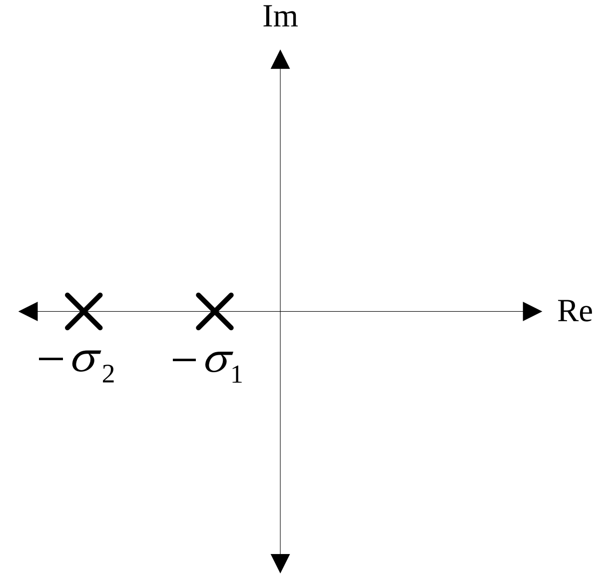


Overdamped Poles

- <u>Overdamped case:</u> $\zeta > 1$

- The second-order poles are real and distinct

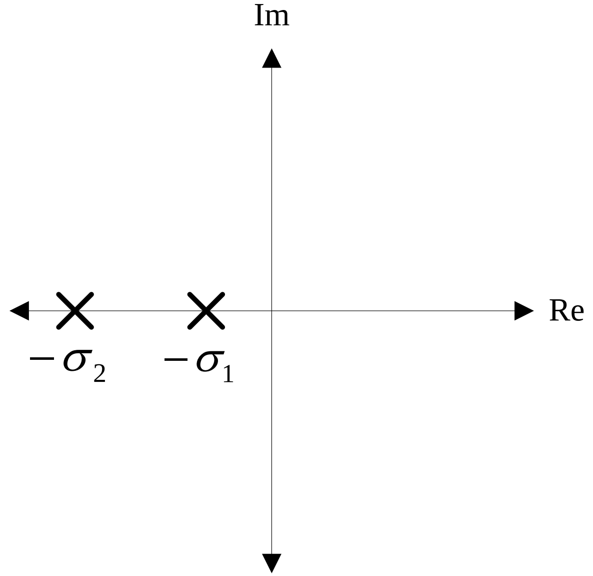
$$s_1 = -\sigma_1, s_2 = -\sigma_2$$



- The impulse response is the sum of two exponential functions

$$h(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t} \quad , \; t \geq 0$$

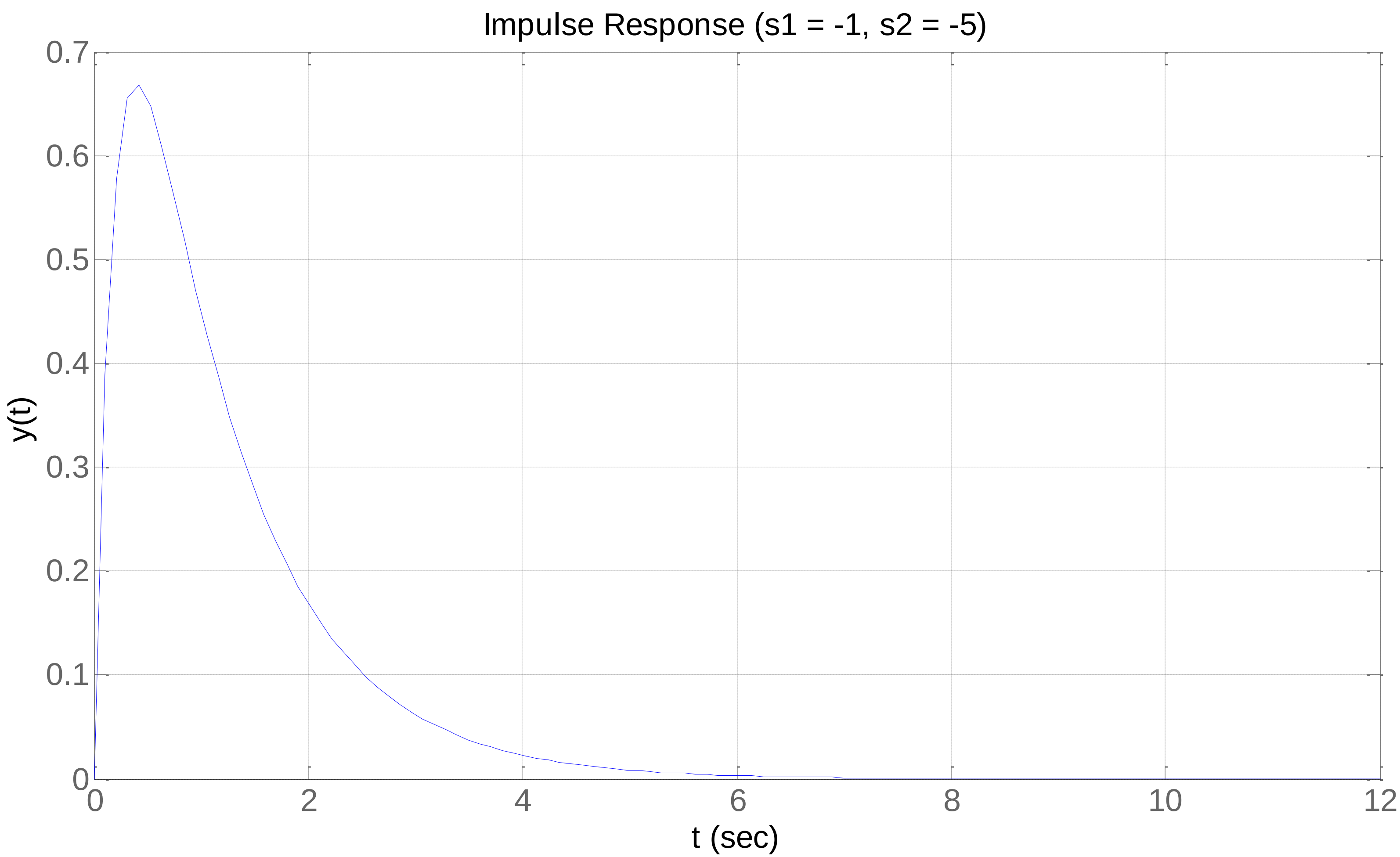
Overdamped Poles – Dominant Pole



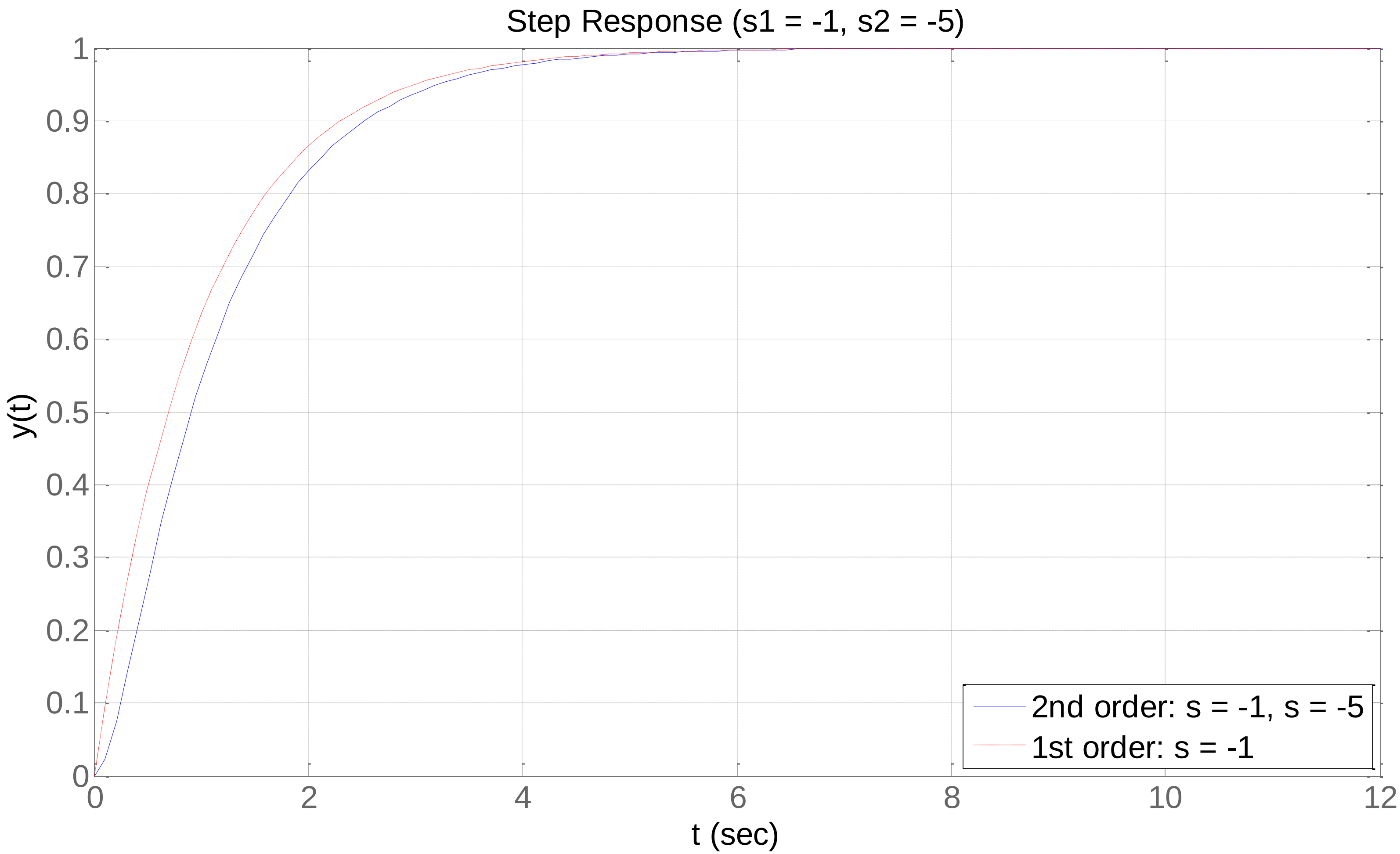
- The signal corresponding to the pole further to the left in the s-plane decays faster than the signal corresponding to the pole nearer to the origin, i.e. the pole at $-\sigma_2$ is “faster” than the pole at $-\sigma_1$

- If the pole at $-\sigma_1$ is significantly slower than the pole at $-\sigma_2$, then the response is dominated by the slower pole, and can be approximated as a first-order response with a single pole at $-\sigma_1$

Overdamped Poles – Impulse Response



Overdamped Poles – Step Response



Second-Order Poles

- Second-order poles are required “building blocks” to describe the dynamic response of higher order systems, because

  1. They are the simplest transfer functions that are able to model oscillatory behaviour (complex conjugate pole pair)

  2. The behaviour of higher-order systems are dominated by one or two pole pairs, and are therefore well approximated by second-order systems

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Reference:
Chapter 3.3