Assignment 4DM50

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Problem 1: Synchronization of mobile robots.

1. Given the reference signals $x_r(t)$ and $y_r(t)$ for the leader mobile robot $i = 1$ (the derivatives $\dot{x}_r(t)$, $\ddot{x}_r(t)$ and $\dot{y}_r(t)$, $\ddot{y}_r(t)$ are available too), show that for suitable gains k_p and k_d the law

$$\begin{aligned}\eta_{1,1}(t) &= \ddot{x}_r(t) + k_d(\dot{x}_r(t) - \dot{x}_1(t)) + k_p(x_r(t) - x_1(t)) \\ \eta_{1,2}(t) &= \ddot{y}_r(t) + k_d(\dot{y}_r(t) - \dot{y}_1(t)) + k_p(y_r(t) - y_1(t)),\end{aligned} \tag{1}$$

yield asymptotic tracking of the reference, i.e. $x_1(t) \to x_r(t)$ and $y_1(t) \to y_r(t)$ for $t \to \infty$.
Firstly, we need to provide error dynamics. Using linear dynamics represented in problem statement

$$\begin{aligned}\ddot{x}_1(t) &= \eta_{1,1}(t), \\ \ddot{y}_1(t) &= \eta_{1,2}(t)\end{aligned} \tag{2}$$

it is possible to rewrite (1)

$$\begin{aligned}\ddot{x}_1(t) - \ddot{x}_r(t) &= -k_d(\dot{x}_1(t) - \dot{x}_r(t)) - k_p(x_1(t) - x_r(t)), \\ \ddot{y}_1(t) - \ddot{y}_r(t) &= -k_d(\dot{y}_1(t) - \dot{y}_r(t)) - k_p(y_1(t) - y_r(t)).\end{aligned} \tag{3}$$

Assume

$$\begin{aligned}\ddot{e}_1(t) = \ddot{x}_1(t) - \ddot{x}_r(t),\ \dot{e}_1(t) = \dot{x}_1(t) - \dot{x}_r(t),\ e_1(t) = x_1(t) - x_r(t), \\ \ddot{e}_2(t) = \ddot{y}_1(t) - \ddot{y}_r(t),\ \dot{e}_2(t) = \dot{y}_1(t) - \dot{y}_r(t),\ e_2(t) = y_r(t) - y_1(t).\end{aligned} \tag{4}$$

Now we rewrite (3) using (4)

$$\begin{aligned}\ddot{e}_1(t) &= -k_d\, \dot{e}_1(t) - k_p e_1(t), \\ \ddot{e}_2(t) &= -k_d\, \dot{e}_2(t) - k_p e_2(t).\end{aligned} \tag{5}$$

Since equations of error dynamics have the same structure it will be enough to analyze one equation. Do the coordinate transformation to obtain

$$\begin{aligned}\dot{e}_1(t) &= z_1(t), \\ \dot{z}_1(t) &= -k_d\, z_1(t) - k_p e_1(t).\end{aligned} \tag{6}$$

Rewriting (6) in the matrix representation

$$\begin{bmatrix} \dot{e}_1(t) \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_p & -k_d \end{bmatrix} \begin{bmatrix} e_1(t) \\ z_1(t) \end{bmatrix} \tag{7}$$

To have system (7) asymptotically stable matrix A must be negative definite. For this trace of matrix must be negative and determinant positive.

$$\begin{aligned}tr(A) =& -k_d < 0 \Rightarrow k_d > 0, \\ \det(A) =& k_p > 0.\end{aligned}$$

Now we have lower boundary for gains. Choosing gains k_p and k_d positive, our systems become asymptotically stable so $e_1(t)$ and $e_2(t) \to 0$ for $t \to \infty$ which implies $x_1(t) \to x_r(t)$ and $y_1(t) \to y_r(t)$ for $t \to \infty$.

2. Use numerical simulations to test tracking controller for the reference signals

$$x_r(t) = 10\cos(0.1t),\ y_r(t) = 10\sin(0.1t), \tag{8}$$

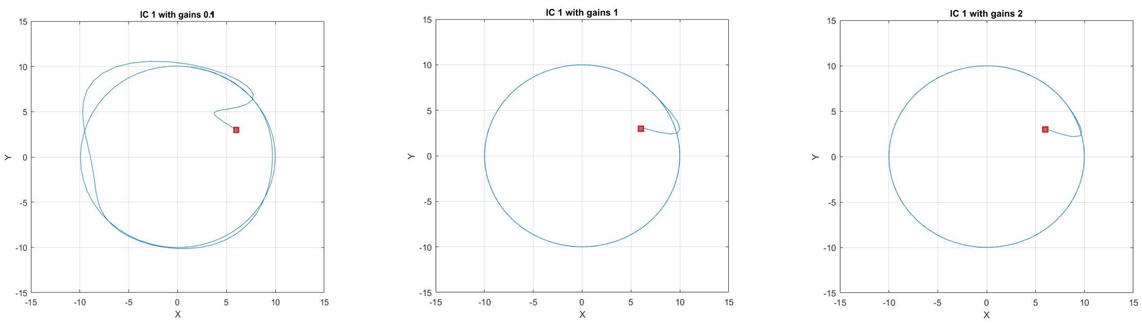
i.e., the robot drives along a circle of radius $R = 10$. Use various values of k_p and k_d. Plot for each simulation the signals $x_1(t)$, $y_1(t)$ and $v_1(t)$ and discuss results. Select the initial conditions and gains k_p and k_d with care such that the velocity $v_1(t)$ does never become zero! (Explain why it is

important that $v_1(\cdot)$ does not become zero?)
The Simulink environment is used to perform numerical simulations. The numerical simulations are done with different initial conditions and different gains k_p and k_d. All values are provided in the table below.

<table>
<tr><th rowspan="2">Gains</th><th colspan="4">Initial Conditions 1</th><th colspan="4">Initial Conditions 2</th><th colspan="4">Initial Conditions 3</th></tr>
<tr><th>x_0</th><th>$\dot{x}_0$</th><th>y_0</th><th>$\dot{y}_0$</th><th>x_0</th><th>$\dot{x}_0$</th><th>y_0</th><th>$\dot{y}_0$</th><th>x_0</th><th>$\dot{x}_0$</th><th>y_0</th><th>$\dot{y}_0$</th></tr>
<tr><td>0.1</td><td rowspan="3">6</td><td rowspan="3">-1.5</td><td rowspan="3">3</td><td rowspan="3">1</td><td rowspan="3">13</td><td rowspan="3">1</td><td rowspan="3">-12</td><td rowspan="3">2</td><td rowspan="3">-10</td><td rowspan="3">8</td><td rowspan="3">10</td><td rowspan="3">-14</td></tr>
<tr><td>1</td></tr>
<tr><td>2</td></tr>
</table>

Table 1: Initial values for simulations

From Table 1 we can see nine simulations which are spited into three sets. The figures bellow show the behavior of the first robot when we apply different gain values to initial conditions 1. We take simulation time equals to 140 seconds.



(a) Gain values = 0.1 (b) Gain values = 1 (c) Gain values = 2

Figure 1: Three simulation sets which present tracking of the reference (IC 1, Time = 140 sec)

These figures above show very well that bigger gain values imply faster convergence to the reference trajectory.
The following figures represent x, y and v for each gain values. The yellow signal, the blue signal and the red one refer to simulations with gains equal to 0.1, 1 and 2, respectively.

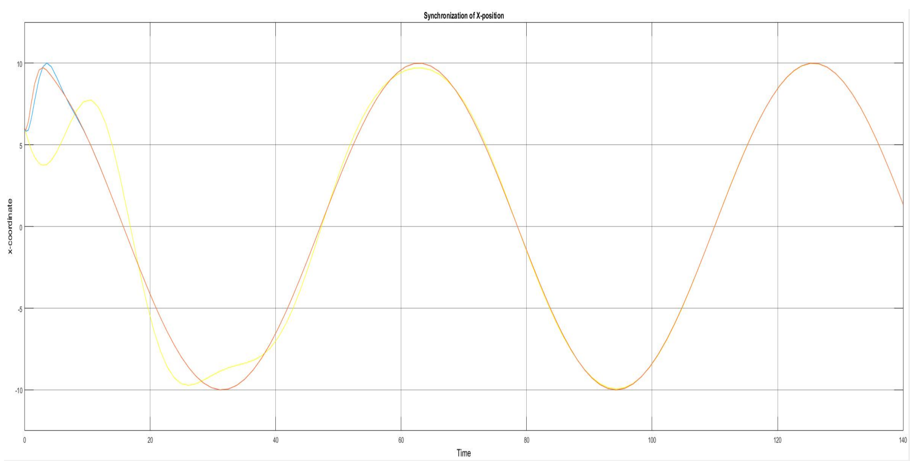


Figure 2: Convergence of X-position to the reference signal

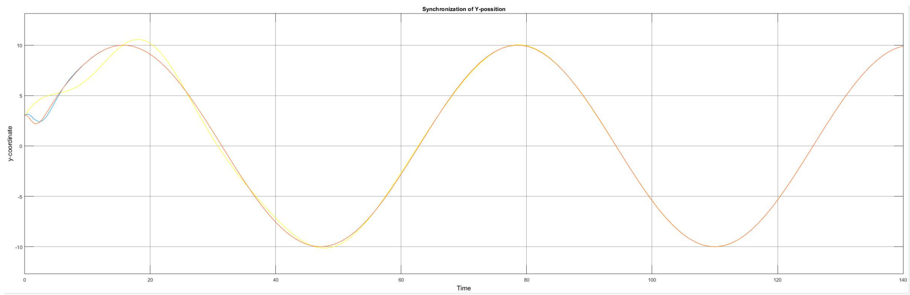


Figure 3: Convergence of Y-position to the reference signal

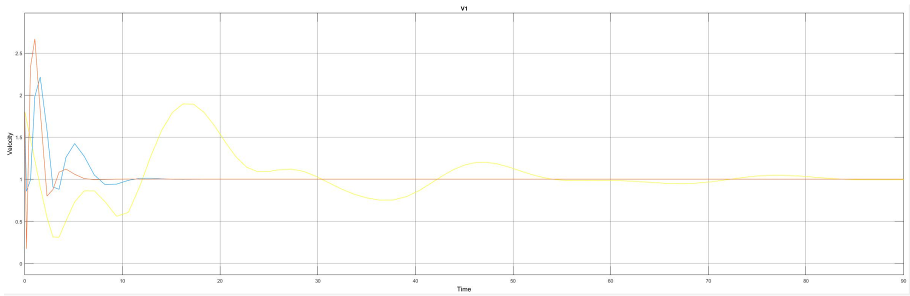
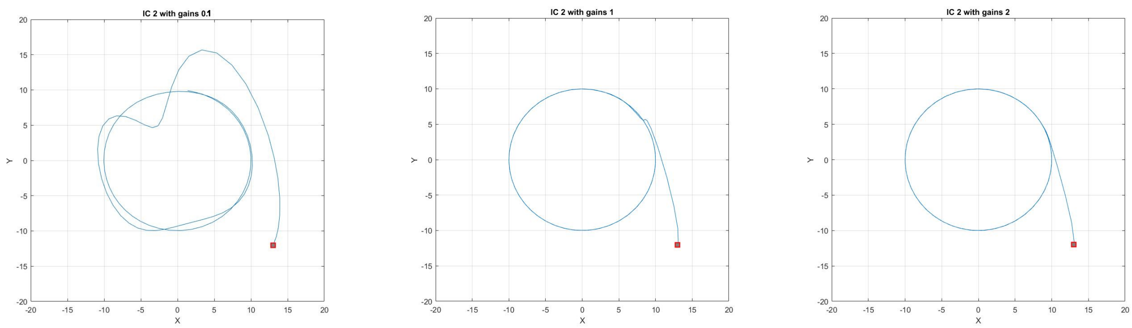


Figure 4: Convergence of V to the reference signal

Now the second simulation set is shown. For this set chosen time is also 140 seconds.



(a) Gain values = 0.1

(b) Gain values = 1

(c) Gain values = 2

Figure 5: Three simulation sets which present tracking of the reference (IC 2, Time = 140 sec)

We can see that convergence speed to the reference trajectory increases when we take bigger values of gain.
The following figures represent x, y and v separately. The yellow signal, the blue signal and the red one refer to simulations with gains equal to 0.1, 1 and 2, respectively.

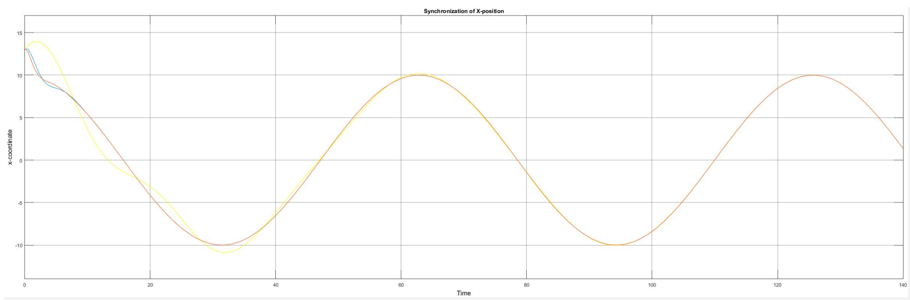


Figure 6: Convergence of X-position to the reference signal

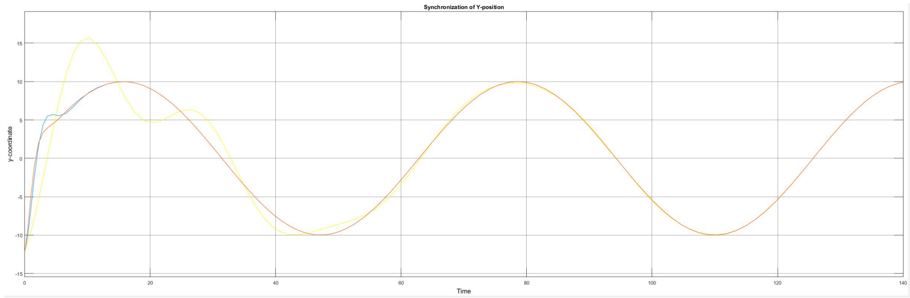


Figure 7: Convergence of Y-position to the reference signal

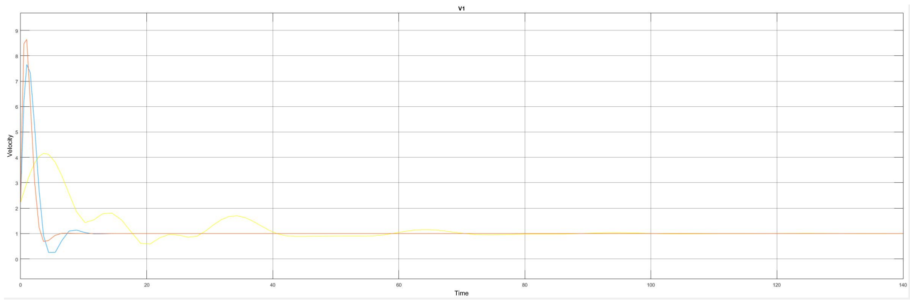
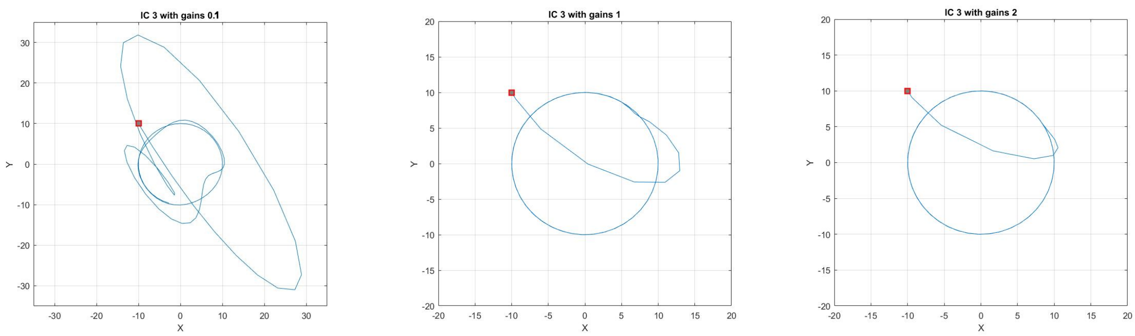


Figure 8: Convergence of V to the reference signal

The third simulation set is presented below. We need to take simulation time 170 seconds due to chosen initial conditions.



(a) Gain values = 0.1 (b) Gain values = 1 (c) Gain values = 2

Figure 9: Three simulation sets which present tracking of the reference (IC 3, Time = 170 sec)

It is easy to see that gain values have the same affect on the convergence rate as it was observed above.

The figures below show x, y and v separately. The yellow signal, the blue signal and the red one refer to simulations with gains equal to 0.1, 1 and 2, respectively.

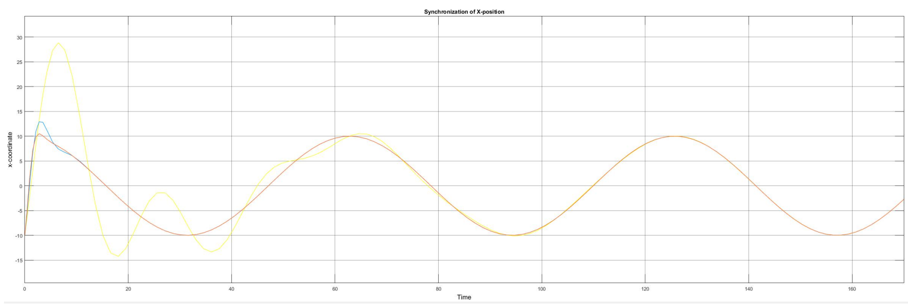


Figure 10: Convergence of X-position to the reference signal

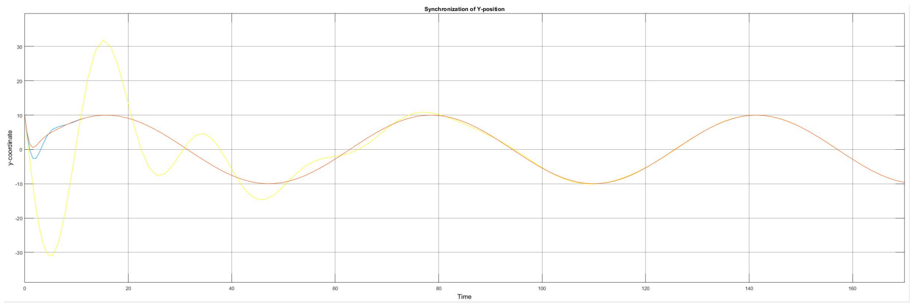


Figure 11: Convergence of Y-position to the reference signal

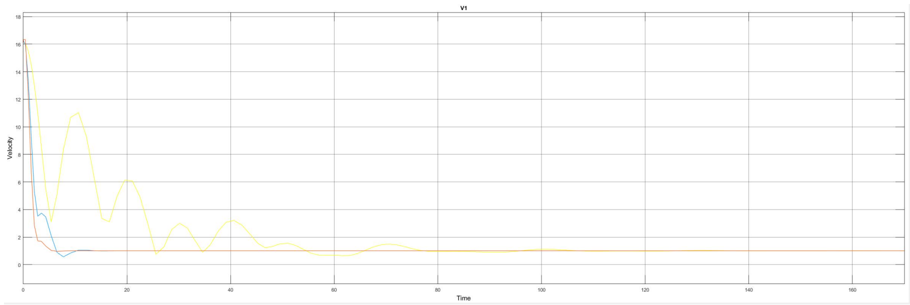


Figure 12: Convergence of V to the reference signal

Summarizing results we can conclude that our reference tracking controller has good performance. The first robot takes the reference trajectory if we choose very different initial conditions. We can control the rate of convergences to the reference trajectory by changing the gain values. Initial velocities $\dot{x}_0$ and $\dot{y}_0$ must be chosen such a way that common velocity v_1, which can be computed by formula $v_1 = \sqrt{\dot{x}_0^2 + \dot{y}_0^2}$, does never become 0. Otherwise we have both components of v_1 equals to 0 which means the robot does not move along x and y and the second equation of control law

$$\begin{aligned} a_i(t) &= \eta_{i,1}(t) cos\theta_i(t) + \eta_{i,2}(t)\sin\theta_i(t), \\ w_i(t) &= -\eta_{i,1}(t)\tfrac{\sin\theta_i(t)}{v_i(t)} + \eta_{i,2}(t)\tfrac{cos\theta_i(t)}{v_i(t)}. \end{aligned} \tag{9}$$

become undefined.

3. We now let the second and the third robot follow the trajectories of robot 1 and robot 2, respectively, at a fixed distance $\triangle R = R \triangle \phi = 1$on the circle. The desired behavior can be achieved using the control laws

$$\begin{aligned} \eta_{i,1}(t) &= \ddot{x}_{c,i-1}(t) + k_D(\dot{x}_{c,i-1}(t) - \dot{x}_i(t)) + k_P(x_{c,i-1}(t) - x_i(t)), \\ \eta_{i,2}(t) &= \ddot{y}_{c,i-1}(t) + k_D(\dot{y}_{c,i-1}(t) - \dot{y}_i(t)) + k_P(y_{c,i-1}(t) - y_i(t)) \end{aligned} \tag{10}$$

for $i = 2, 3$, where

$$\begin{aligned} x_{c,i-1}(t) &= x_{i-1}(t)\cos\Delta\phi + y_{i-1}(t)\sin\Delta\phi, \\ y_{c,i-1}(t) &= -x_{i-1}(t)\sin\Delta\phi + y_{i-1}(t)\cos\Delta\phi, \\ \dot{x}_{c,i-1}(t) &= \dot{x}_{i-1}(t)\cos\Delta\phi + \dot{y}_{i-1}(t)\sin\Delta\phi, \\ \dot{y}_{c,i-1}(t) &= -\dot{x}_{i-1}(t)\sin\Delta\phi + \dot{y}_{i-1}(t)\cos\Delta\phi, \\ \ddot{x}_{c,i-1}(t) &= \ddot{x}_{i-1}(t)\cos\Delta\phi + \ddot{y}_{i-1}(t)\sin\Delta\phi, \\ \ddot{y}_{c,i-1}(t) &= -\ddot{x}_{i-1}(t)\sin\Delta\phi + \ddot{y}_{i-1}(t)\cos\Delta\phi. \end{aligned} \tag{11}$$

Perform numerical simulations to test the controller. Select the initial conditions and gains k_P and $k_D$with care such that the velocities $v_i(t)$ do never become zero. Plot for each simulation the signals $x_i(t)$, $y_i(t)$ and $v_i(t)$ for $i = 1, 2, 3$ and discuss results.
In this section three simulations are presented. In the section 2 we showed that tracking problem can be solved for various initial conditions that is why in this section only gain values are subjects to change. In the table below data for the first simulation is provided. We set up weak gains for tracking the reference signal but strong gains for following previous robot.

	x_0	$\dot{x}_0$	y_0	$\dot{y}_0$	Gains	Time
Robot 1	6	-1.5	3	1	0.1	170
Robot 2	10	1.5	5	3	2	
Robot 3	-3	1	8	-5	2	

Table 2: Simulation 1

The figure below show behavior of three robots due to initial conditions which are given in the Table 2.

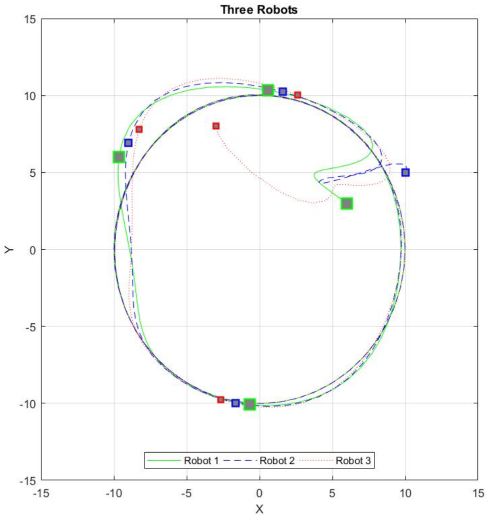


Figure 13: Three robots (Simulation 1)

Markers show position of each robot (the same color as a line) at identical time (0 sec; 30 sec; 40 sec; 60 sec). The following figures show x, y and v of the robots.

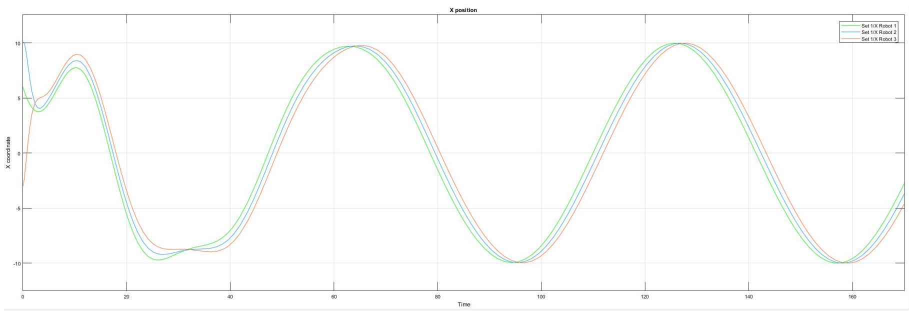


Figure 14: X position of the robots (Simulation 1)

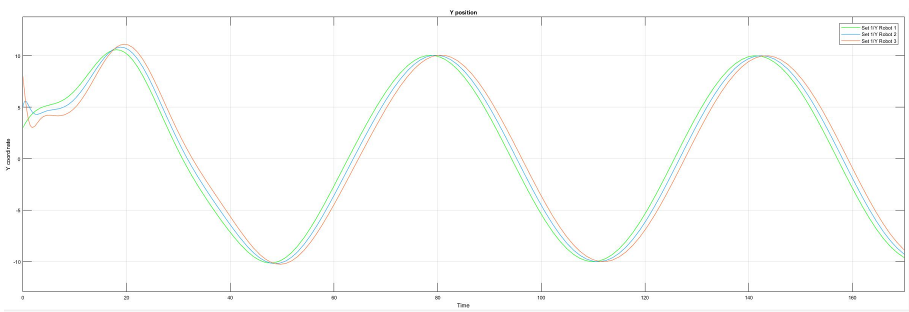


Figure 15: Y position of the robots (Simulation 1)

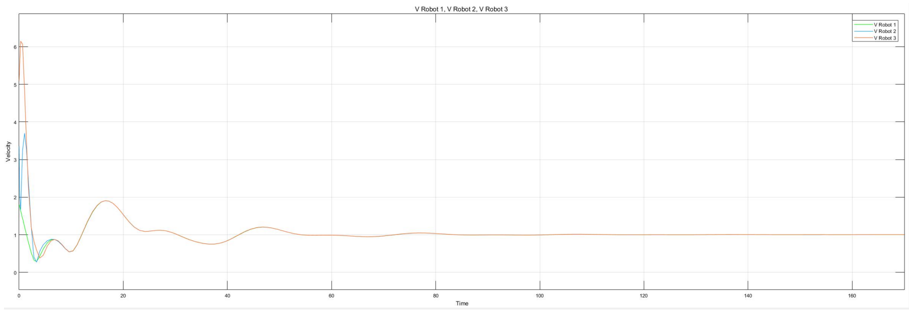


Figure 16: Velocities of robots (Simulation 1)

We can see that after some time three robots drive along a circle of radius $R = 10$. Also it is possible to observe that the robots $i = 2, 3$ follow the trajectories of robot 1 and robot 2, respectively, at a fixed distance $\triangle\phi = 0.1$ on the circle. The phase shift represented in the Figures 14 and 15 shows following at the fixed distance.
The Table 3 represents data for the second simulation. Here we set up moderate gain values for tracking the reference signal and weak gains for robots $i = 2, 3$.

	x_0	$\dot{x}_0$	y_0	$\dot{y}_0$	Gains	Time
Robot 1	6	-1.5	3	1	1	200
Robot 2	10	1.5	5	3	0.1	
Robot 3	-3	1	8	-5	0.1	

Table 3: Simulation 2

The figure below show behavior of three robots due to initial conditions which are given in the Table 3.

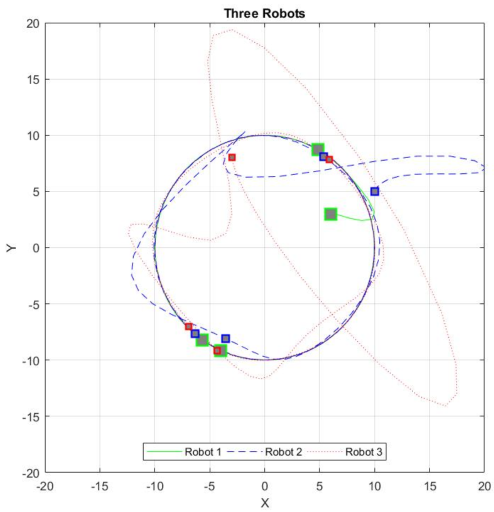


Figure 17: Three robots (Simulation 2)

Markers show position of each robot (the same color as a line) at identical time (0 sec; 40 sec; 60 sec; 80 sec). The following figures show x, y and v of the robots.

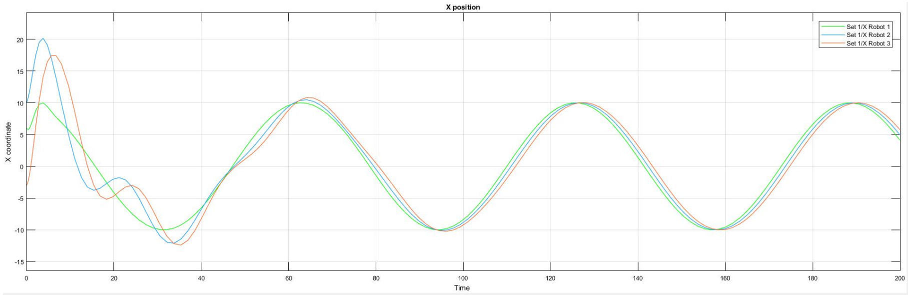


Figure 18: X position of the robots (Simulation 2)

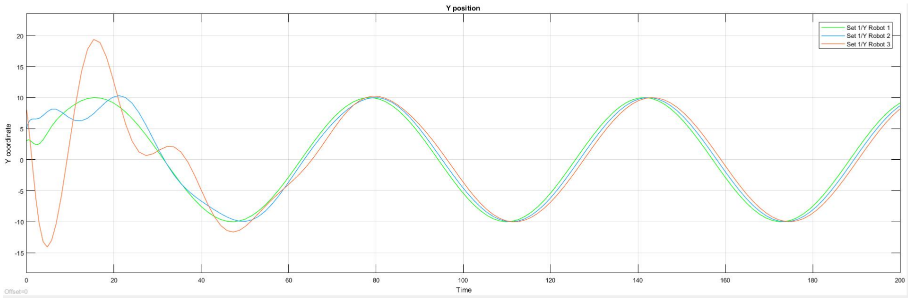


Figure 19: Y position of the robots (Simulation 2)

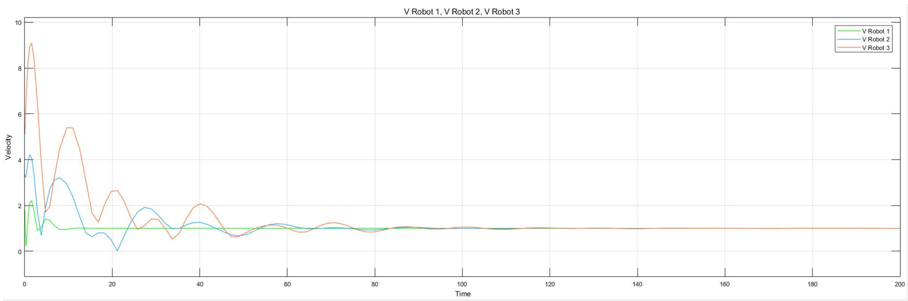


Figure 20: Velocities of robots (Simulation 2)

We can see that after some time three robots drive along a circle of radius $R = 10$. Also it is possible to observe that the robots $i = 2, 3$ follow the trajectories of robot 1 and robot 2, respectively, at a fixed distance $\triangle\phi = 0.1$ on the circle. The phase shift represented in the Figures 18 and 19 shows following at the fixed distance.
Now we set up moderate gain values for all robots $i = 1, 2, 3$. The Table 4 provides data for the third simulation.

	x_0	$\dot{x}_0$	y_0	$\dot{y}_0$	Gains	Time
Robot 1	6	-1.5	3	1	1	
Robot 2	10	1.5	5	3	1	140
Robot 3	-3	1	8	-5	1	

Table 4: Simulation 3

The figure below show behavior of three robots due to initial conditions which are given in the Table 4.

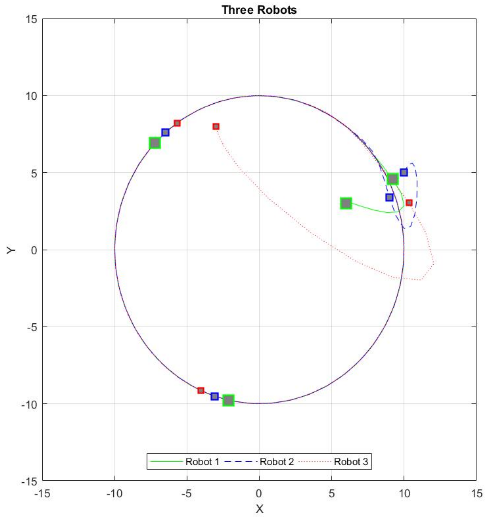


Figure 21: Three robots (Simulation 3)

Markers show position of each robot (the same color as a line) at identical time (0 sec; 15 sec; 30 sec; 45 sec). The following figures show x, y and v of the robots.

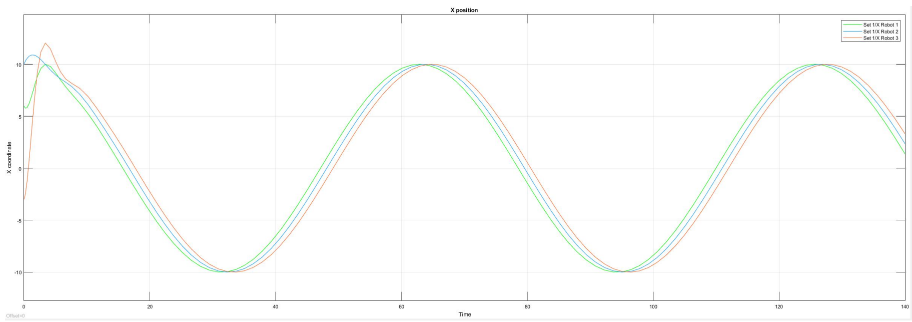


Figure 22: X position of the robots (Simulation 3)

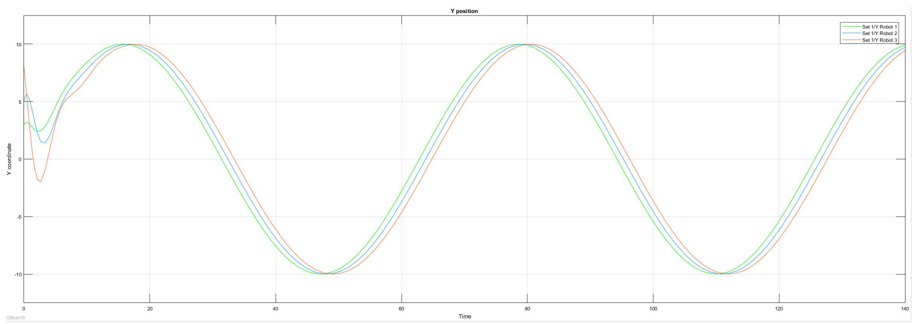


Figure 23: Y position of the robots (Simulation 3)

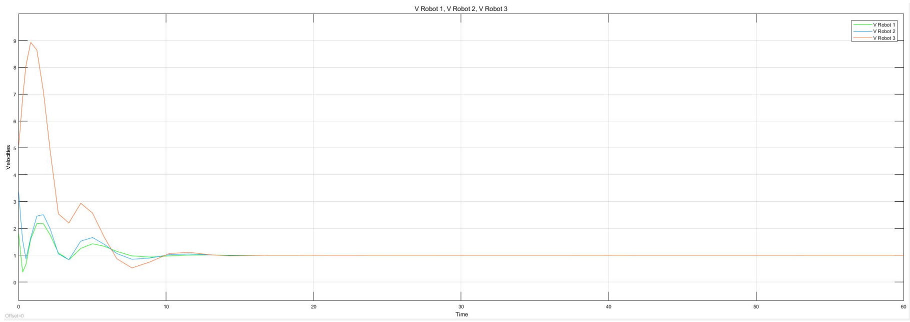


Figure 24: Velocities of robots (Simulation 3)

We can see that after some time three robots drive along a circle of radius $R = 10$. Also it is possible to observe that the robots $i = 2, 3$ follow the trajectories of robot 1 and robot 2, respectively, at a fixed distance $\triangle\phi = 0.1$ on the circle. The phase shift represented in the Figures 22 and 23 shows following at the fixed distance.
Talking about results of done simulations it is easy to understand that any solution converges to the reference signal since our control law is linear and asymptotically stable for any positive k_p and k_d. Choosing various initial conditions for the robots has affect only on the rate of convergence.

4. Let the reference trajectory for robot 1 now given by

$$x_r(t) = 3t,\; y_r(t) = 0, \tag{12}$$

i.e., robot 1 is moving in e_x-direction with constant velocity. Choose values of k_p and k_d such that robot 1 follows this reference. Next design control laws such that robot 2 follows robot 1 at a fixed distance $d = 1$ and robot 3 follows robot 2 at the same fixed distance $d = 1$. Prove that your control laws yield the desired behavior and perform numerical simulations to verify your results.
We can modify control laws (10) and use it to achieve desired behavior. Control laws for tracking new reference signal are given by

$$\begin{aligned} \eta_{i,1}(t) &= k_d(\dot{x}_r(t) - \dot{x}_i(t)) + k_p(x_r(t) - x_i(t)), \\ \eta_{i,2}(t) &= k_d(\dot{y}_r(t) - \dot{y}_i(t)) + k_p(y_r(t) - y_i(t)) \end{aligned} \tag{13}$$

where $\dot{x}_r(t) = 3$, $\dot{y}_r(t) = 0$ and $x_r(t)$, $y_r(t)$ are given in (12). For the following robots we need to take into account a desired fixed distance $d = 1$.

$$\begin{aligned} \eta_{i,1}(t) &= k_d(\dot{x}_{c,i-1}(t) - \dot{x}_i(t)) + k_p(x_{c,i-1}(t) - x_i(t)), \\ \eta_{i,2}(t) &= k_d(\dot{y}_{c,i-1}(t) - \dot{y}_i(t)) + k_p(y_{c,i-1}(t) - y_i(t)), \end{aligned} \tag{14}$$

for $i = 2, 3$, where

$$\begin{aligned} x_{c,i-1}(t) &= x_{i-1}(t) + d, \\ y_{c,i-1}(t) &= y_{i-1}(t), \\ \dot{x}_{c,i-1}(t) &= \dot{x}_{i-1}(t), \\ \dot{y}_{c,i-1}(t) &= \dot{y}_{i-1}(t). \end{aligned} \tag{15}$$

We drop all accelerations since we need robots to move with constant velocity. The Table below gives us initial conditions for numerical simulations.

	x_0	$\dot{x}_0$	y_0	$\dot{y}_0$	Gains	Time
Robot 1	6	-1	2	-0.5	1	30
Robot 2	8	2.5	1	0.2		
Robot 3	-4	0.5	-2.5	-0.24		

Table 5: Initial Conditions for three robots drive along the line

The following figures show convergence to desired behavior of three robots.

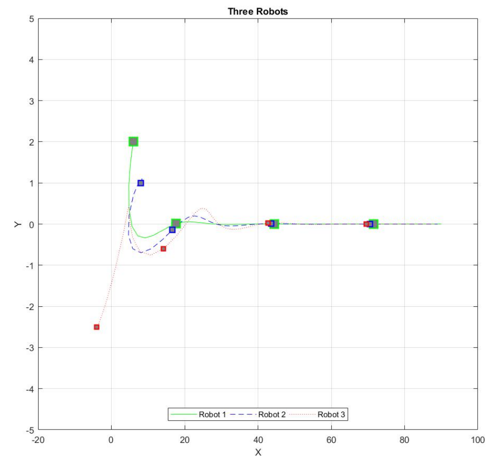


Figure 25: Three robots along the line XY representation

Markers show position of each robot (the same color as a line) at identical time (0 sec; 6 sec; 15 sec; 24 sec).

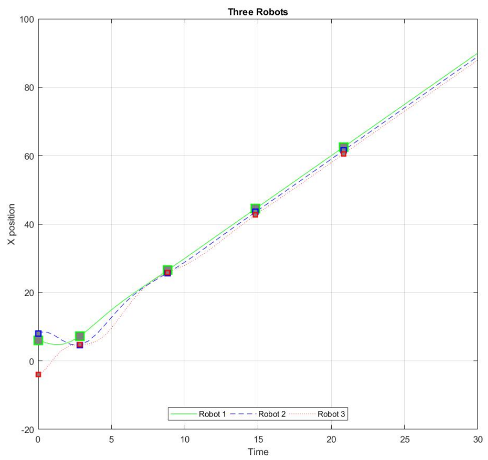


Figure 26: Three robots X direction

Markers show position of each robot (the same color as a line) at identical time (0 sec; 3 sec; 8 sec; 14 sec; 21 sec). We can see that the robots drive according to control signals. The figure below show verifies that desired velocity is reached.

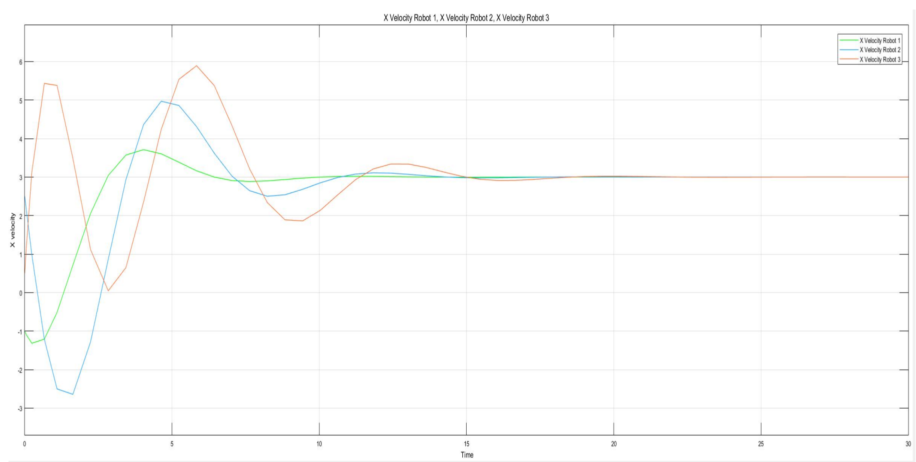


Figure 27: X velocity

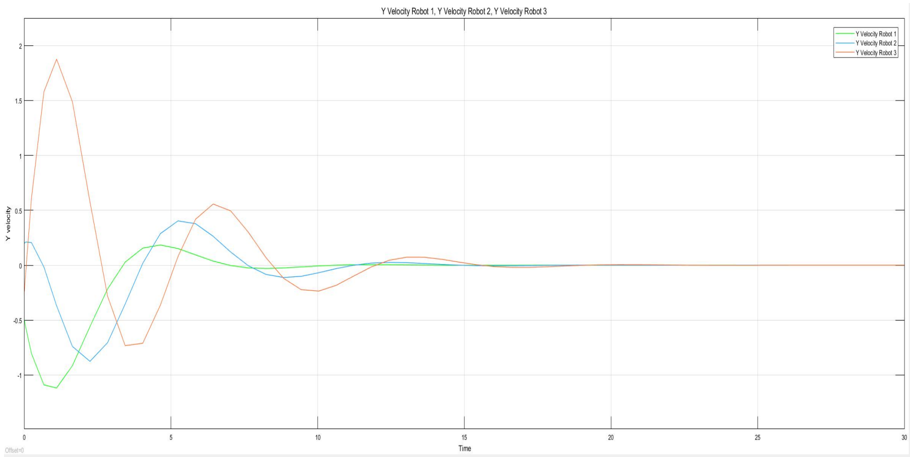


Figure 28: Y velocity

We can conclude that desired behavior is achieved and the robots drive according to the requirements.

Problem 2: Synchronization experiments.

1. Consider two coupled HR neurons with $a_{12} = a_{21} = 1$. For each $\tau = 0, 1, \ldots, 5$, determine a lower-bound and upper-bound for the coupling strength such that the two coupled HR neurons practically synchronized. Note that the maximum value of the coupling strength you can select is 10; In case the systems remain practically synchronous for $\sigma = 10$ you take 10 as upper-bound. The experimental measurements are done and results are provided in the following table.

τ	0	1	2	3	4	5
Lower-bounds	0.51	0.59	0.65	0.73	0.78	0.84
Upper-bounds	10	9.7	5.86	4.7	4.0	3.5

Table 6: Measurements for question 1

Using data from the Table 6 above we can provide $\sigma - \tau$ plot showing synchronization zone.

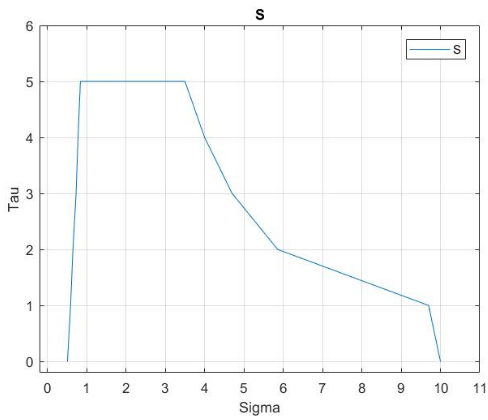


Figure 29: S plot for question 1

2. Consider the network with corresponding adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \tag{16}$$

(a) Use your results of question 1. to predict the values of and for which network practically synchronizes. In particular, determine the lower-bounds and upper-bounds of the coupling strength for $\tau = 0,\, 1,\, 2, \ldots$. (It might be that for higher values of the systems do not synchronize.)

We need to use method which is named Scaling to predict the lower-bounds and upper-bounds of the new network with 5 nodes. For this we recall coupling matrix L for the network

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}. \tag{17}$$

Now we obtain λ_2 and λ_N.

$$\lambda_2 = 1,\, \lambda_N = 5.$$

The sets $S_2$and S_N are computed by applying the given formula

$$S_j^* := \left\{ (\sigma, \tau) \in R_+ \times \overline{R_+} | \left(\frac{\lambda_j}{2}\sigma, \tau \right) \in S \right\} \tag{18}$$

and we look for intersection area to determine synchronization zone.

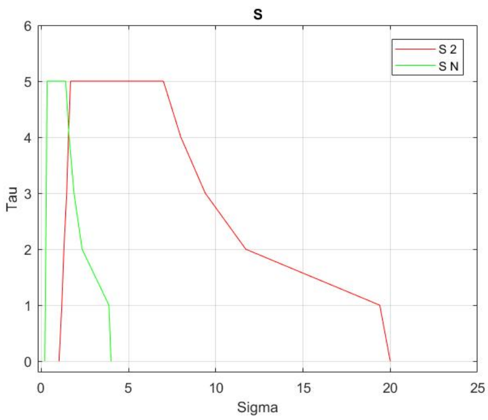


Figure 30: Results of scaling

The following table contents the lower-bounds and upper-bounds for the network (16).

τ	0	1	2	3	4	5
Lower-bounds	1.02	1.18	1.3	1.46	1.56	NaN
Upper-bounds	4	3.88	2.34	1.88	1.6	NaN

Table 7: Predicted boundaries

There is no synchronization for the 5 ms delay since there is no intersection. Also it is necessary to note that the upper-bound for $\tau = 0$ is not sufficient value since we have the limited maximum value of the coupling strength $\sigma = 10$.

(b) Verify your results by experiments with this network.

The experiment is done and measured values are given in the Table below.

τ	0	1	2	3	4	5
Lower-bounds	1.16	1.32	1.51	1.70	No Sync	No Sync
Upper-bounds	10	3.27	2.08	1.70	No Sync	No Sync

Table 8: Measured boundaries

Now we compare $\sigma - \tau$ plots

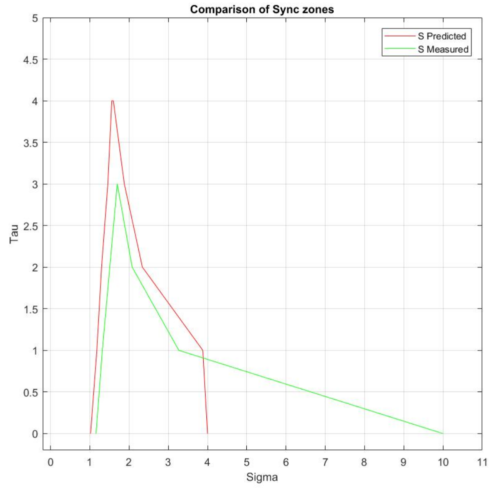
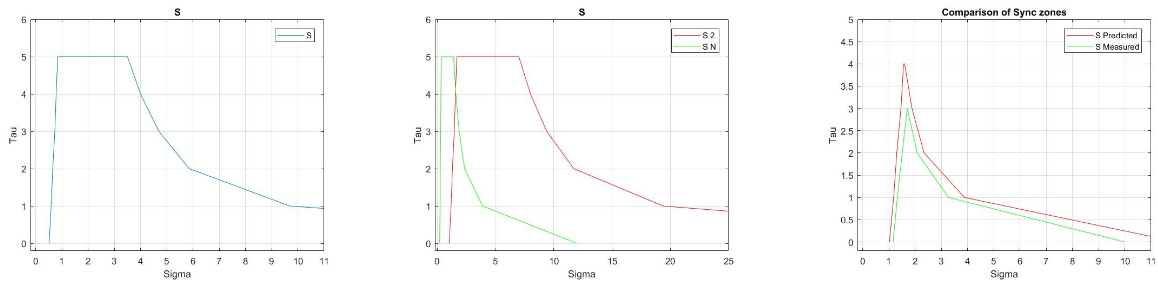


Figure 31: Comparison of theory and practice

It is important to note that predicted zone does not cover the whole measured zone due to the fact that for computation of the predicted value we had the limited coupling strength. Let us assume that we can take $\sigma > 10$, for instance, $\sigma = 30$ for $\tau = 0$. The plots taking into account our assumption describe situation more realistic.



(a) Sync zone for (16) (b) Scaling with assumption (c) Comparison of theory and practice

Figure 32: Sync zones with the assumption

Anyway we can see that the assumption only affects the upper-bound for time delay $\tau = 0$.

3. Consider the network of eight coupled neurons by the adjacency matrix

$$A = \begin{pmatrix} 0 & 2 & 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & 2 & 0 \end{pmatrix}. \tag{19}$$

(a) Take $\tau = 0$. Increase the coupling strength starting from 0 such that the whole network practically synchronizes. What coupling strength is needed for practical synchronization?
This network has two types of coupling: with normal strength and double strength. We experimentally discovered that the coupling strength $\sigma = 0.51$ is the lower-bound for the practical synchronization which we can observe in the figure below.

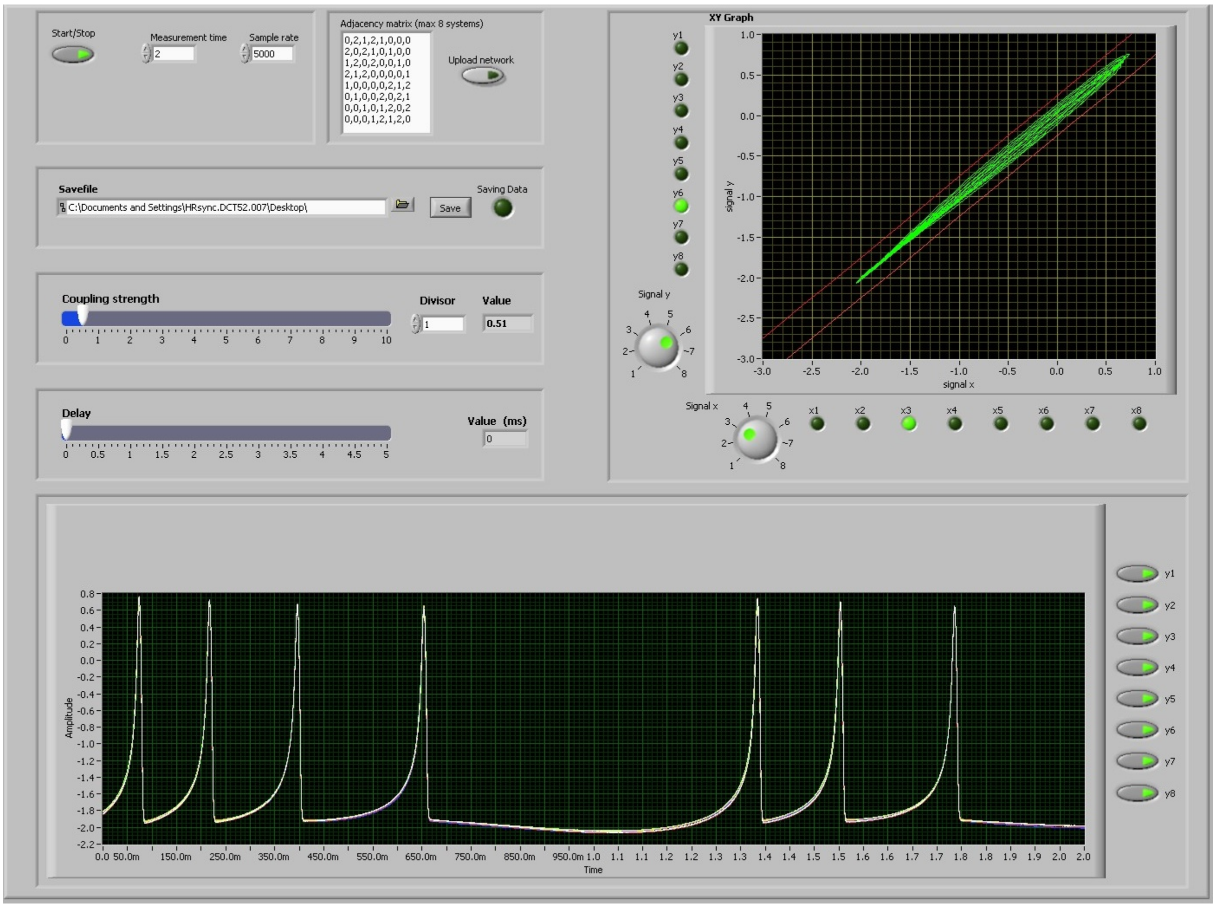


Figure 33: Screenshot of the equipment interface for 3a.

The following figures use data from experimental setup and show practical synchronization of the network. Note that the red line refers to the inequality given by

$$|y_i(t) - y_j(t)| \leq 0.25 \tag{20}$$

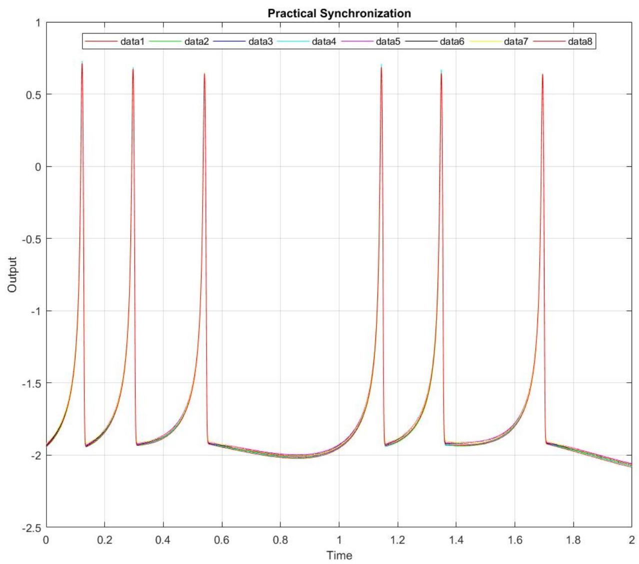


Figure 34: Output of the network with $\sigma = 0.51$.

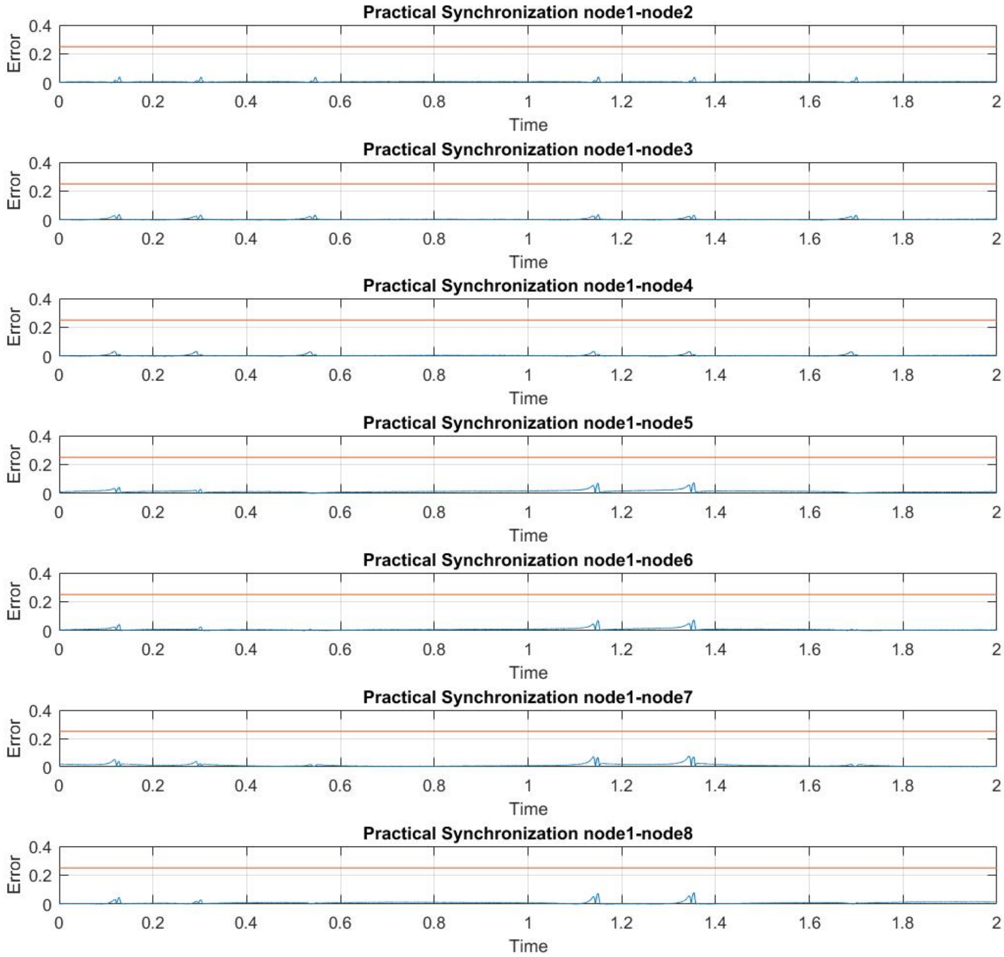


Figure 35: Errors computed with (20)

(b) Before the network becomes practically synchronous there is a mode of partial practical synchronization (for $\tau = 0$). Describe this mode of partial practical synchronization and specify what coupling strength is needed to enter this mode of partial practical synchronization.
Since we have different coupling gains in our network we can observe partial synchronization which is asymptotic match of the states of some, but not all systems. We can observe partial synchronization starting with the coupling strength $\sigma = 0.22$. The figure below presents it.

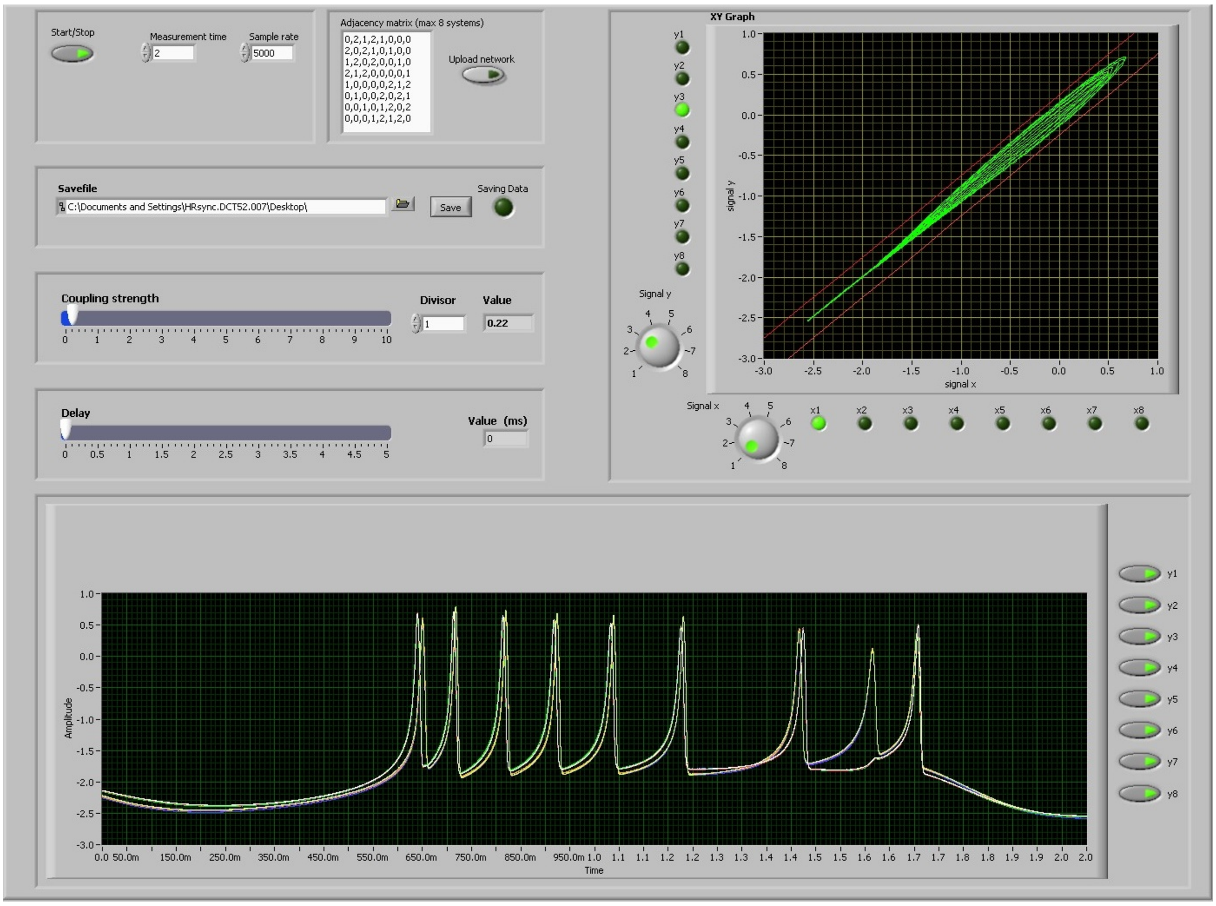


Figure 36: Screenshot of the equipment interface for 3b.

The following figures use data from experimental setup and show partial synchronization of the network.

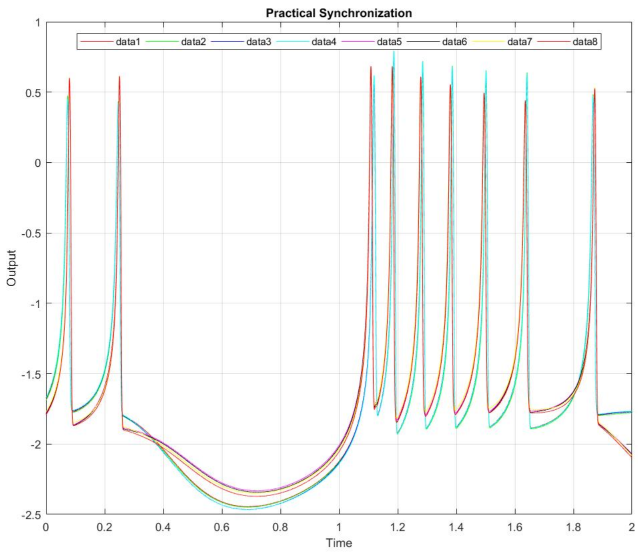


Figure 37: Output of the network with $\sigma = 0.22$.

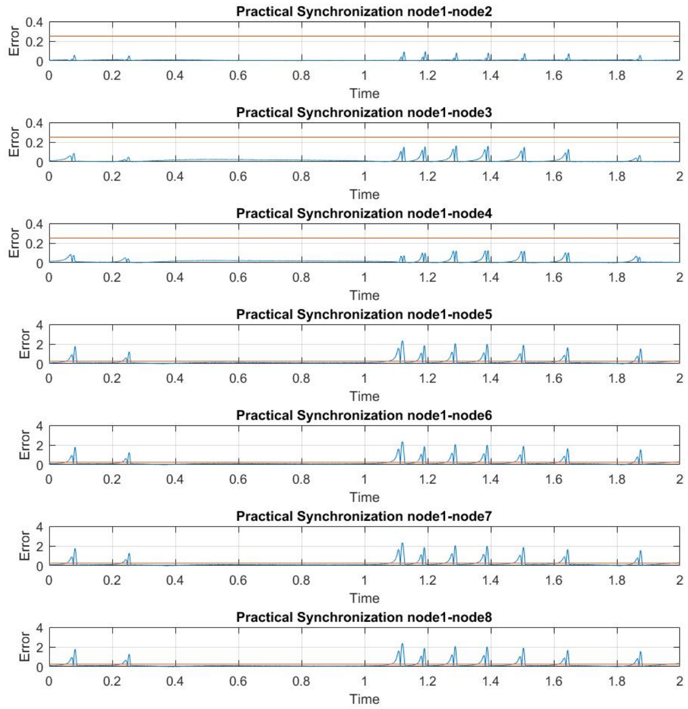


Figure 38: Sync of the first cluster

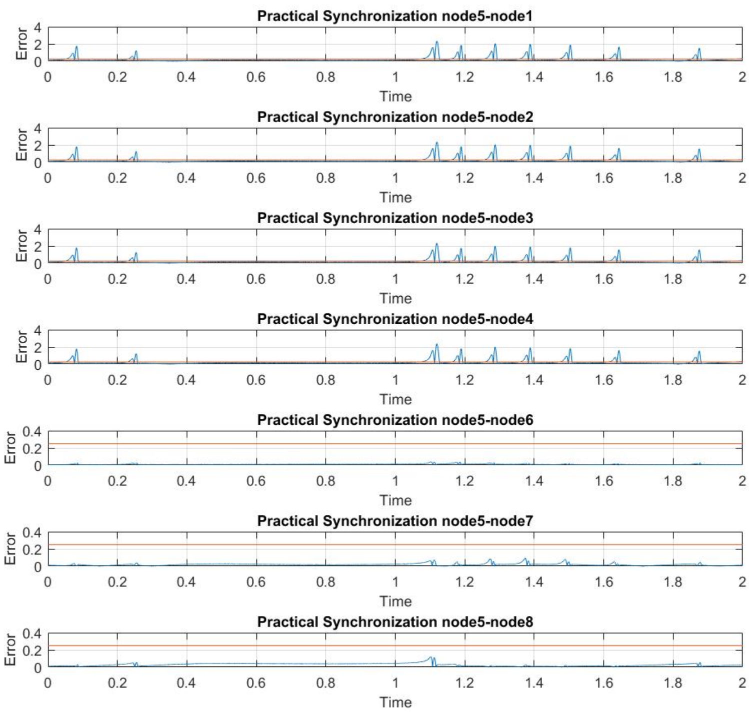


Figure 39: Sync of the second cluster

(c) Explain your results of 3b. using the theory of partial synchronization. (Hint: draw the network and identify its symmetries.)

From plots above we can conclude that there are two clusters showing similar behavior. The first cluster includes nodes 1, 2, 3 and 4 and the second one nodes 5, 6, 7 and 8. The network looks like

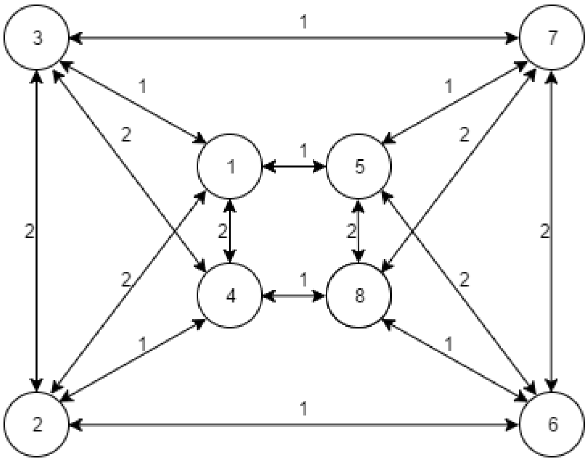


Figure 40: The network with adjacency matrix (19)

Now we define a coupling matrix for the network and a permutation matrix for this mode of partial synchronization.

$$L = \begin{pmatrix} 6 & -2 & -1 & -2 & -1 & 0 & 0 & 0 \\ -2 & 6 & -2 & -1 & 0 & -1 & 0 & 0 \\ -1 & -2 & 6 & -2 & 0 & 0 & -1 & 0 \\ -2 & -1 & -2 & 6 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 6 & -2 & -1 & -2 \\ 0 & -1 & 0 & 0 & -2 & 6 & -2 & -1 \\ 0 & 0 & -1 & 0 & -1 & -2 & 6 & -2 \\ 0 & 0 & 0 & -1 & -2 & -1 & -2 & 6 \end{pmatrix} \tag{21}$$

$$\Pi_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

This matrix $\Pi_1$commutes with matrix L. And since the permutation matrix Π_1 and the coupling matrix commute, we have the set

$$\{\phi \in C \,|\phi(\theta) \in \ker(I_{Nn} - \Pi_i \otimes I_n),\ -\tau \leq \theta \leq 0\} \tag{22}$$

which is forward invariant with respect to dynamics of the coupled systems. We know that eigenvalues of L and the given permutation matrix Π_1 denote:

$\lambda^{'}$the smallest eigenvalue of L with eigenvector in range$(I_{Nn} - \Pi_i)$;
$\lambda^{*}$the largest eigenvalue of L with eigenvector in range$(I_{Nn} - \Pi_i)$.

Now show the eigenvector space, the set of the eigenvalues $\lambda_{1...N} = \begin{bmatrix} 0 & 2 & 6 & 6 & 8 & 8 & 8 & 10 \end{bmatrix}$ and the range$(I_{Nn} - \Pi_1)$ to obtain $\lambda^{'}$ and λ^{*}.

$$V = \begin{pmatrix} -0.3536 & 0.3536 & 0.4267 & 0.2606 & -0.2974 & 0.4609 & -0.2723 & 0.3536 \\ -0.3536 & 0.3536 & -0.2606 & 0.4267 & -0.1280 & 0 & 0.5988 & -0.3536 \\ -0.3536 & 0.3536 & -0.4267 & -0.2606 & -0.1776 & -0.5316 & -0.2467 & 0.3536 \\ -0.3536 & 0.3536 & 0.2606 & -0.4267 & 0.6030 & 0.0707 & -0.0799 & -0.3536 \\ -0.3536 & -0.3536 & 0.4267 & 0.2606 & -0.1776 & -0.5316 & -0.2467 & -0.3536 \\ -0.3536 & -0.3536 & -0.2606 & 0.4267 & 0.6030 & 0.0707 & -0.0799 & 0.3536 \\ -0.3536 & -0.3536 & -0.4267 & -0.2606 & -0.2974 & 0.4609 & -0.2723 & -0.3536 \\ -0.3536 & -0.3536 & 0.2606 & -0.4267 & -0.1280 & 0 & 0.5988 & 0.3536 \end{pmatrix}$$

$$
(I_N - \Pi_1) = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}
$$

It is easy to see that both $\lambda^{'}$ and λ^{*} for our case are equal to 10 and $\lambda_2 = 2$, $\lambda_N = 10$. So we have the situation which is described in the figure below.

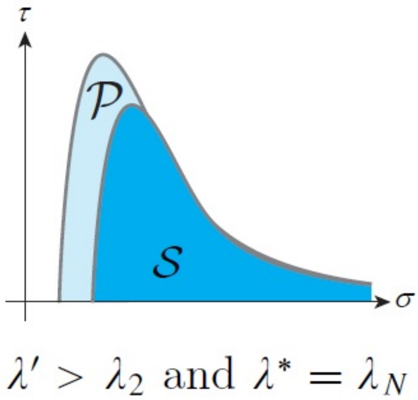


Figure 41: Partial synchronization for mode 1

It is necessary to say that this figure matches the behavior that we observed during the experiment.

(d) Take $\sigma = 2$ and start to increase the time-delay (starting from 0). For some value of you find that the practical synchronization is lost but the network exhibits partial practical synchronization. Describe this mode of partial practical synchronization and specify the value of the time-delay at the transition from practical synchronization to partial practical synchronization.
The another mode of partial synchronization can be observed if we take time delay $\tau = 0.6$. From the figure below we can see that there is no synchronous behavior between node 2 and 3 which implies that we have another two clusters which show partial synchronization.

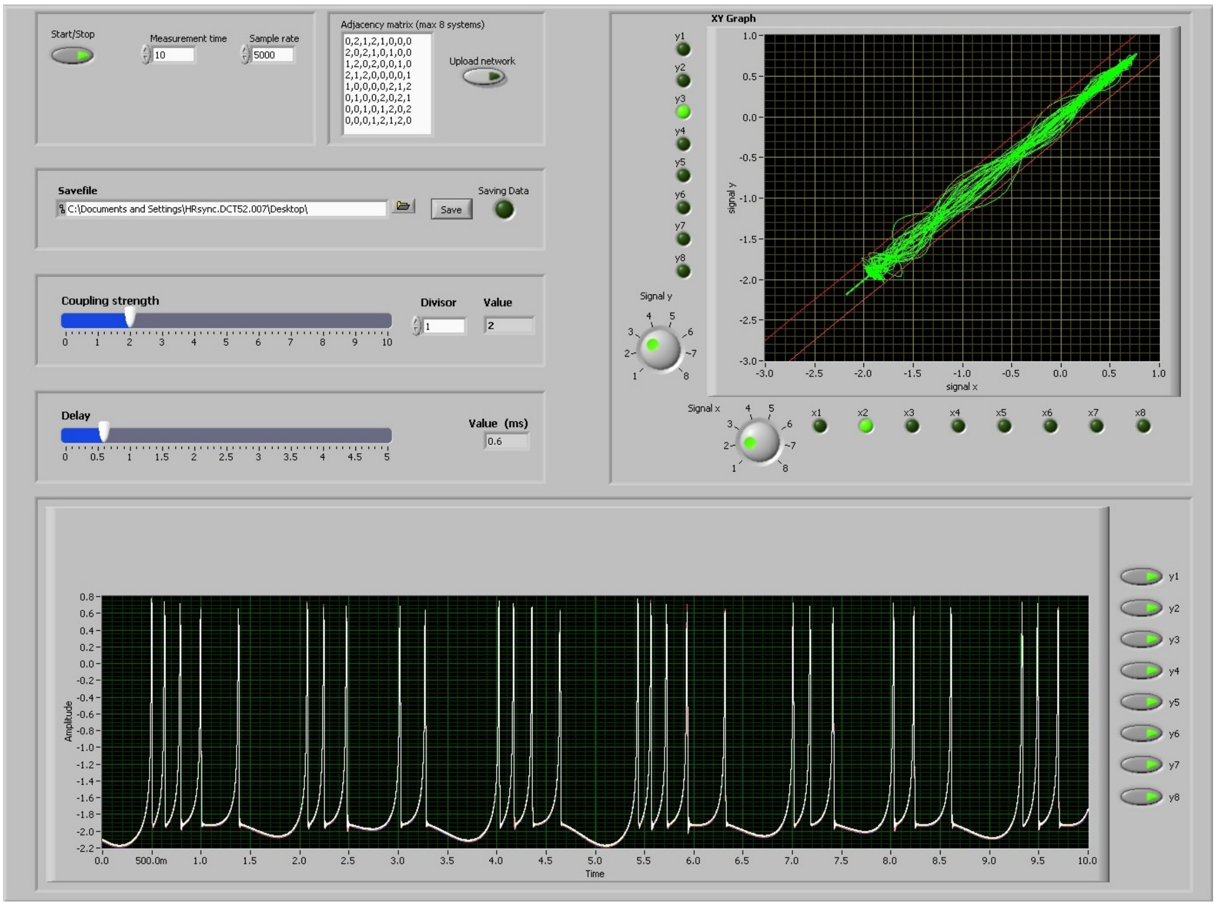


Figure 42: Screenshot of the equipment interface for 3d.

The following figures use data from experimental setup and show another mode of partial synchronization of the network.

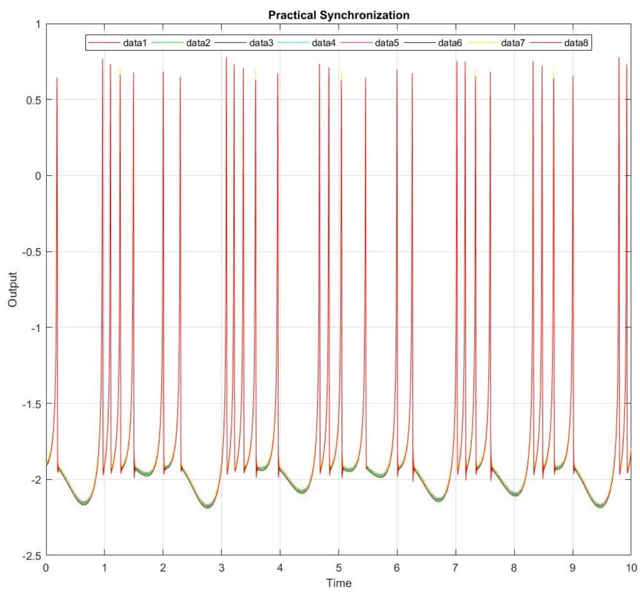


Figure 43: Output of the network with $\sigma = 0.22$.

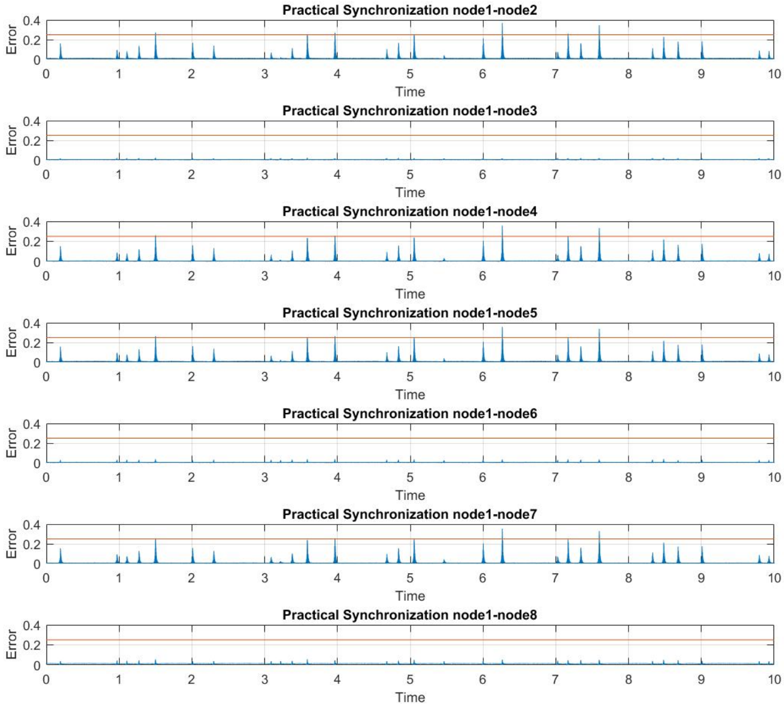


Figure 44: Sync of the first cluster

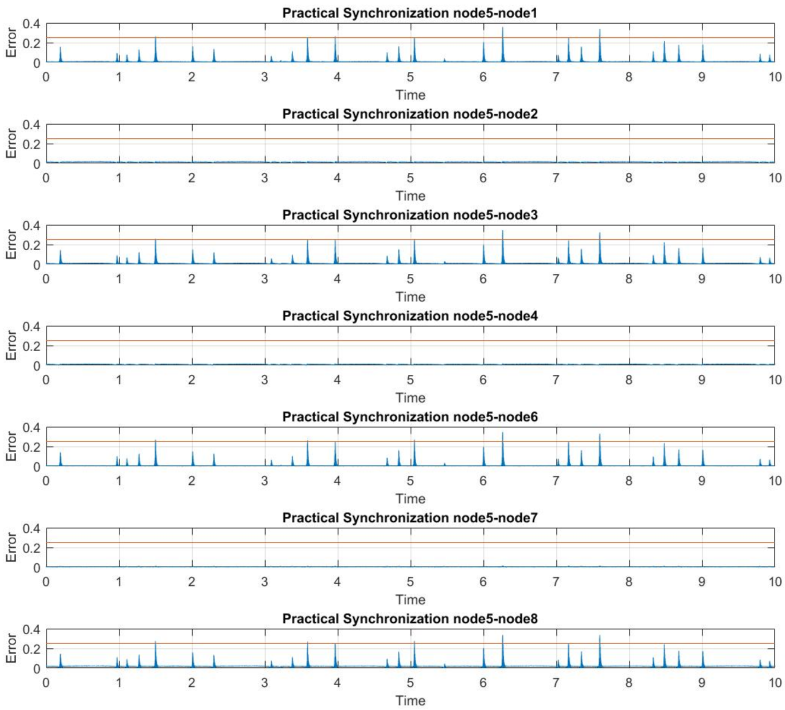


Figure 45: Sync of the second cluster

(e) Explain your results of 3e. using the theory of partial synchronization. (Hint: draw the network and identify its symmetries.)

From plots above we can conclude that here we have another two clusters showing similar behavior. The first cluster includes nodes 1, 3, 6 and 8 and the second one nodes 2, 4, 5 and 7. Now we recall a coupling matrix (21) for the network and provide a permutation matrix for this mode of partial synchronization. This mode of partial synchronization refers to the third permutation matrix we found.

$$\Pi_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We have the same eigenvector space

$$V = \begin{pmatrix} -0.3536 & 0.3536 & 0.4267 & 0.2606 & -0.2974 & 0.4609 & -0.2723 & 0.3536 \\ -0.3536 & 0.3536 & -0.2606 & 0.4267 & -0.1280 & 0 & 0.5988 & -0.3536 \\ -0.3536 & 0.3536 & -0.4267 & -0.2606 & -0.1776 & -0.5316 & -0.2467 & 0.3536 \\ -0.3536 & 0.3536 & 0.2606 & -0.4267 & 0.6030 & 0.0707 & -0.0799 & -0.3536 \\ -0.3536 & -0.3536 & 0.4267 & 0.2606 & -0.1776 & -0.5316 & -0.2467 & -0.3536 \\ -0.3536 & -0.3536 & -0.2606 & 0.4267 & 0.6030 & 0.0707 & -0.0799 & 0.3536 \\ -0.3536 & -0.3536 & -0.4267 & -0.2606 & -0.2974 & 0.4609 & -0.2723 & -0.3536 \\ -0.3536 & -0.3536 & 0.2606 & -0.4267 & -0.1280 & 0 & 0.5988 & 0.3536 \end{pmatrix},$$

the set of the eigenvalues $\lambda_{1 \dots N} = \begin{bmatrix} 0 & 2 & 6 & 6 & 8 & 8 & 8 & 10 \end{bmatrix}$ and the range$(I_{Nn} - \Pi_3)$ is given by

$$(I_N - \Pi_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

From here we have $\lambda^{'} = 2$ and $\lambda^{*} = 2$ and $\lambda_2 = 2$, $\lambda_N = 10$. That brings us to another partial synchronization mode that we observed during the experiments which is described by the following picture.

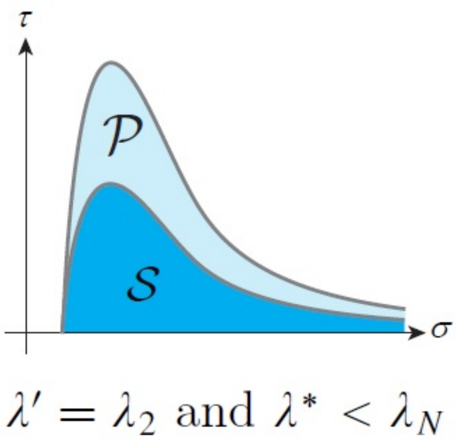


Figure 46: Partial synchronization for mode 2

Summary: The two problems of the assignment are accomplished. The synchronization of mobile robots was the first problem laying in the field of control. We developed the tracking controller which force robots follow the reference signal or previous robot. The experiments of different synchronous behavior in the network were the main part of the problem two. We observed practical synchronization and two different modes of practical partial synchronization and then explained results using the theory.