4DM50 Dynamics and Control of cooperation
Problem set 3

<u>Problem 1.</u> Consider the network with coupling matrix

$$\Gamma = \gamma \begin{pmatrix} 5 & -2 & -2 & -1 & 0 & 0 \\ -2 & 5 & -2 & 0 & -1 & 0 \\ -2 & -2 & 5 & 0 & 0 & -1 \\ -1 & 0 & 0 & 7 & -3 & -3 \\ 0 & -1 & 0 & -3 & 7 & -3 \\ 0 & 0 & -1 & -3 & -3 & 7 \end{pmatrix}$$

1. Draw the network corresponding to Γ.
2. One can show that there exist four permutation matrices (other than identity) that commute with Γ. Find these four permutation matrices.
3. Let the nodes of the network be Lorenz systems

$$\begin{cases} \dot{x}_{i,1} = \sigma(x_{i,2} - x_{i,1}) + u_i \\ \dot{x}_{i,2} = x_{i,1}(\rho - x_{i,3}) - x_{i,2} \\ \dot{x}_{i,3} = x_{i,1}x_{i,2} - \kappa x_{i,3} \end{cases}$$

with output $y_i = x_{i,1}$ and parameters

$$\sigma = 10, \quad \rho = 28, \quad \kappa = \frac{8}{3}.$$

Recall that the Lorenz system is strictly semipassive from u_i to y_i and, moreover, its internal dynamics are exponentially convergent. Thus the network of Lorenz systems synchronizes if γ (hence λ_2 of Γ) is large enough. Moreover, the result of the previous question suggests that one may also find partial synchronization in this network. Study by means of numerical simulations how synchronization and partial synchronization in the network of diffusively coupled Lorenz systems depends on the value of γ.

<u>Problem 2.</u> Consider the (modified) Chua system

$$\Sigma_i = \begin{cases} \dot{x}_{i,1} = \alpha(-x_{i,1} + x_{i,2} - \varphi(x_{i,1})) + u_i \\ \dot{x}_{i,2} = x_{i,1} - x_{i,2} + x_{i,3} \\ \dot{x}_{i,3} = -\beta x_{i,2} \end{cases} \tag{1}$$

$$\varphi(s) = m_1 s + m_2(|s+1| - |s-1|) + m_3(|s+10| - |s-10|), \tag{2}$$

with output $y_i = x_{i,1}$. The values of the parameters are

$$\alpha = 10, \quad \beta = 19.53, \quad m_1 = 10, \quad m_2 = -0.3247, \quad m_3 = \frac{-10.783}{2}.$$

1. Plot the nonlinearity $\varphi(s)$ for $-15 \leq s \leq 15$.
2. Consider the storage function

$$S = \frac{1}{2}x_{i,1}^2 + \beta x_{i,2}^2 - x_{i,2}x_{i,3} + x_{i,3}^2.$$

Show that S is positive definite and radially unbounded (i.e. $S \to \infty$ as $\|x_i\| \to \infty$).

3. Show that $\dot{S} < y_i u_i$ for $\|x_i\|$ outside some ball centered at $(0,0,0)$. Hint: there exist positive constants c_1, c_2 such that $x_{i,1}\varphi(x_{i,1}) \geq c_1 x_{i,1}^2 - c_2$ holds for all $x_{i,1} \in \mathbb{R}$; One may take, for instance, $c_1 = 5$ and $c_2 = c_1 11^2$.

4. Let $z_{i,1} = x_{i,2}$ and $z_{i,2} = x_{i,3}$ and write down the internal dynamics

$$\dot{z}_i = q(z_i, y_i).$$

5. Show that the internal dynamics are exponentially convergent with respect to "input" y_i.

<u>Problem 3.</u> In problem 2. you have shown that the modified Chua circuit

- is strictly semi-passive from input u_i to output y_i;
- has internal dynamics that are exponentially convergent.

Thus we know that a network of these modified Chua circuits will synchronize provided that the coupling is sufficiently strong.

1. Simulate the modified Chua circuit and plot the attractor for initial conditions $(1,1,1)$ and $(1,10,1)$. You should find two different attractors: a chaotic double scroll attractor and a closed orbit.

2. Consider two modified Chua circuits that are mutually coupled via the laws

$$u_1 = \gamma(y_2 - y_1)$$
$$u_2 = \gamma(y_1 - y_2)$$

Determine with the help of computer simulations the minimal coupling strength γ for which the two circuits synchronize on the chaotic double scroll attractor.

3. Consider a network of five all-to-all coupled modified Chua circuits. The couplings are in this case given by the equations

$$u_i = \gamma \sum_{j=1, j\neq i}^{5} (y_j - y_i), \quad i = 1, 2, \ldots, 5.$$

Determine with the help of computer simulations the minimal coupling strength γ for which the five circuits synchronize on the chaotic double scroll attractor.

4. Consider a network of five modified Chua circuits that are coupled in an un-directed ring:

$$u_i = \gamma(y_{i+1} - y_i) + \gamma(y_{i-1} - y_i), \quad i = 1, 2, \ldots, 5,$$

(with $y_0 = y_5$ and $y_6 = y_1$). Determine again with the help of computer simulations the minimal coupling strength γ for which the five circuits synchronize on the chaotic double scroll attractor.

5. How do the coupling strength required for synchronization you found in 1.–3. compare? Can you predict the coupling strength required for synchronization in any other network?