4DM50: Dynamics and Control of Cooperation

# Observers & Synchronization

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# Outline







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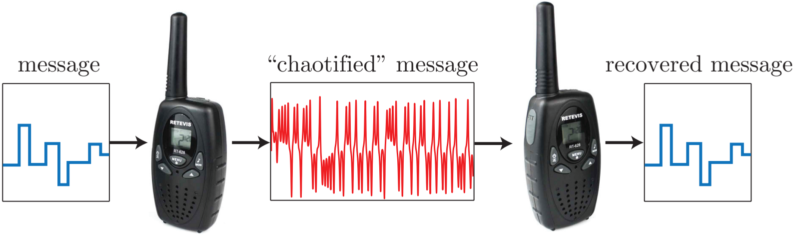
# Master-slave coupled Lorenz systems



Pecora and Carroll, 1990



# Master-slave coupled Lorenz systems

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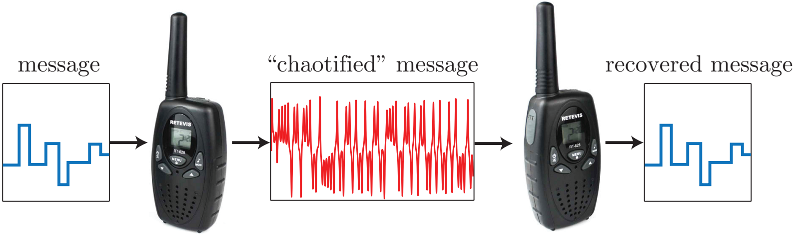


Message reconstruction possible if sender and receiver synchronize



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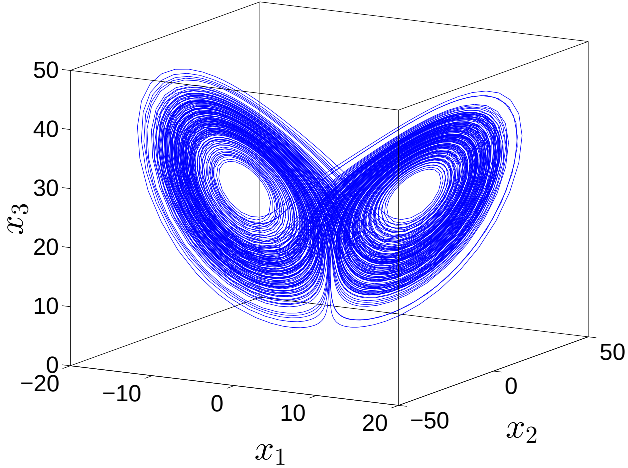
Pecora and Carroll, 1990

- Master system (transmitter)

$$\dot{x}_1 = \sigma(x_2 - x_1)$$
$$\dot{x}_2 = x_1(\rho - x_3) - x_2$$
$$\dot{x}_3 = x_1 x_2 - \gamma x_3$$

with parameters $\sigma = 10$, $\rho = 28$, $\gamma = \frac{8}{3}$ and output $x_1$

# Master-slave coupled Lorenz systems



# Master-slave coupled Lorenz systems



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\end{aligned}
$$

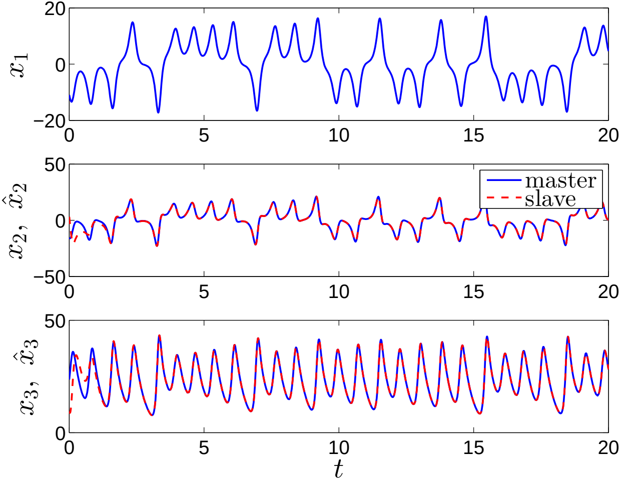
  with parameters $\sigma = 10$, $\rho = 28$, $\gamma = \frac{8}{3}$ and output $x_1$

- Slave system (receiver)

$$
\begin{aligned}
\dot{\hat{x}}_2 &= x_1(\rho - \hat{x}_3) - \hat{x}_2 \\
\dot{\hat{x}}_3 &= x_1 \hat{x}_2 - \gamma \hat{x}_3
\end{aligned}
$$

  No $\hat{x}_1$-dynamics, because $x_1$ is already known

# Master-slave coupled Lorenz systems



Proof of the result:

- $e_2 = x_2 - \hat{x}_2$ and $e_3 = x_3 - \hat{x}_3$

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- Lyapunov function

$$V(e_2, e_3) = e_2^2 + e_3^2$$

$$\dot{V}(e_2, e_3) = -2e_2^2 - 2e_3^2 < 0$$

$$(e_2, e_3) \to (0, 0) \quad \text{as} \quad t \to \infty$$


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- Control point of view: Slave is a reduced observer

- Slave system (receiver) with $\hat{x}_1$-dynamics

$$\begin{aligned}\dot{\hat{x}}_1 &= \sigma\,(\hat{x}_2 - \hat{x}_1)\\ \dot{\hat{x}}_2 &= x_1(\rho - \hat{x}_3) - \hat{x}_2\\ \dot{\hat{x}}_3 &= x_1\hat{x}_2 - \gamma\,\hat{x}_3\end{aligned}$$



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- Slave system (receiver) with $\hat{x}_1$-dynamics

$$\begin{aligned}
\dot{\hat{x}}_1 &= \sigma(\hat{x}_2 - \hat{x}_1) \\
\dot{\hat{x}}_2 &= {\color{red}x_1}(\rho - \hat{x}_3) - \hat{x}_2 \\
\dot{\hat{x}}_3 &= {\color{red}x_1}\hat{x}_2 - \gamma\hat{x}_3
\end{aligned}$$

- $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$ and $e_3 = x_3 - \hat{x}_3$



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# Master-slave coupled Lorenz systems



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$$\dot{\hat{x}}_1 = \sigma(\hat{x}_2 - \hat{x}_1)$$
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- $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$ and $e_3 = x_3 - \hat{x}_3$
- Lyapunov function

$$V(e_1, e_2, e_3) = \tfrac{1}{\sigma}e_1^2 + e_2^2 + e_3^2$$

$$\dot{V}(e_1, e_2, e_3) = -2e_1^2 + 2e_1e_2 - 2e_2^2 - 2e_3^2 < 0$$

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$$(e_1, e_2, e_3) \to (0, 0, 0) \quad \text{as} \quad t \to \infty$$

- Control point of view: Slave is a full observer

# Master-slave synchronization and observers

Transmitter system: $\dot{x} = f(t, x)$

$$\dot{x} = f(t, x) \xrightarrow{y(t) = h(t, x(t))} \dot{\hat{x}} = \hat{f}(t, \hat{x}, y)$$

Receiver system: $\dot{\hat{x}} = \hat{f}(t, \hat{x}, y)$

- Full observer: Given signal $y(t)$, reconstruct asymptotically the state $x(t)$ of the transmitter system



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# Master-slave synchronization and observers



**Transmitter system**

$$\dot{x} = f(t, x)$$

$$\xrightarrow{\;y(t) = h(t, x(t))\;}$$

**Receiver system**

$$\dot{\hat{x}} = \hat{f}(t, \hat{x}, y)$$

- Full observer: Given signal $y(t)$, reconstruct asymptotically the state $x(t)$ of the transmitter system
- Reduced observer: Given signal $y(t)$, reconstruct asymptotically the state $x(t)$ of the transmitter system modulo its output $y(t)$

# Master-slave synchronization and observers



Transmitter system | Receiver system

$$\dot{x} = f(t, x) \xrightarrow{\;y(t) = h(t, x(t))\;} \dot{\hat{x}} = \hat{f}(t, \hat{x}, y)$$

- Full observer: Given signal $y(t)$, reconstruct asymptotically the state $x(t)$ of the transmitter system
- Reduced observer: Given signal $y(t)$, reconstruct asymptotically the state $x(t)$ of the transmitter system modulo its output $y(t)$
- If the slave can be chosen freely, the master-slave synchronization problem and the (reduced) observer problem are equivalent

How to construct a (reduced) observer?

- Pecora and Carroll construction does not always work!

Example:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -ax_1 - bx_2, \quad y = x_1$$

with reduced observer

$$\dot{\hat{x}}_2 = -ay - b\hat{x}_2 = -ax_1 - b\hat{x}_2$$

Then for $e_2 = x_2 - \hat{x}_2$

$$\dot{e}_2 = -be_2$$

Example:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -ax_1 - bx_2, \quad y = x_1$$

with observer

$$\dot{\hat{x}}_1 = \hat{x}_2,$$
$$\dot{\hat{x}}_2 = -ay - b\hat{x}_2 = -ax_1 - b\hat{x}_2$$

Then for $e_2 = x_2 - \hat{x}_2$ and $e_1 = x_1 - \hat{x}_1$

$$\dot{e}_1 = e_2,$$
$$\dot{e}_2 = -be_2$$

How to construct a (reduced) observer?

- Pecora and Carroll construction does not always work!
- Use tools from (nonlinear) control theory
  - (nonlinear) observability
  - (nonlinear) detectability



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Linear system

$$
\begin{aligned}
\dot{x} &= Ax, \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n\times n},\\
y &= Cx, \qquad C \in \mathbb{R}^{m\times n}
\end{aligned} \tag{1}
$$



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Linear system

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\end{aligned}
\tag{1}
$$

Full observer for the linear system

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + L(\hat{y} - y(t)), \qquad \hat{x}(0) = \hat{x}_0, \qquad L \in \mathbb{R}^{n\times m}, \\
\hat{y} &= C\hat{x}
\end{aligned}
\tag{2}
$$

Linear system

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\hat{y} &= C\hat{x}
\end{aligned} \tag{2}
$$

Define the error $e = \hat{x} - x$ to obtain

$$\dot{e} = (A + LC)e$$

$\Rightarrow$ System (2) is an observer for (1) if and only if the matrix $A + LC$ is Hurwitz

$A + LC$ has arbitrary pole-placement if and only if the pair $(A, C)$ is observable:

$$\operatorname{rank}\mathcal{O} = n, \quad \mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Linear system

$$\begin{aligned} \dot{x} &= Ax, \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n\times n}, \\ y &= Cx, \qquad C \in \mathbb{R}^{m\times n} \end{aligned} \tag{1}$$


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Linear system

$$\begin{aligned} \dot{x} &= Ax, \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n\times n}, \\ y &= Cx, \qquad C \in \mathbb{R}^{m\times n} \end{aligned} \tag{1}$$

If $(A, C)$ is detectable, then there is a matrix $H \in \mathbb{R}^{(n-m)\times m}$ such that

$$\operatorname{rank}\begin{pmatrix} C \\ H \end{pmatrix} = n, \quad \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} C \\ H \end{pmatrix} x, \quad x = \begin{pmatrix} S & T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

and the matrix $HAT$ is Hurwitz

## Intermezzo

The linear system is detectable if and only if its dynamics restricted to $\ker(\mathcal{O})$ are asymptotically stable:

$$\mathrm{rank}\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} < n \quad \Rightarrow \quad \mathrm{real}(\lambda) < 0$$

# Linear systems: Reduced observer

## Intermezzo

The linear system is detectable if and only if its dynamics restricted to $\ker(\mathcal{O})$ are asymptotically stable:

$$\operatorname{rank}\begin{pmatrix}\lambda I - A\\ C\end{pmatrix} < n \quad \Rightarrow \quad \operatorname{real}(\lambda) < 0$$

$$\Updownarrow$$

there exists a matrix $L$ such that $A + LC$ is Hurwitz

Linear system

$$\begin{aligned} \dot{x} &= Ax, \qquad x(0) = x_0, \qquad A \in \mathbb{R}^{n\times n}, \\ y &= Cx, \qquad C \in \mathbb{R}^{m\times n} \end{aligned} \tag{1}$$

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and the matrix $HAT$ is Hurwitz

Dynamics of the partial state $z(t)$:

$$\dot{z} = HATz + HASy$$

Reduced observer:

$$\dot{\hat{z}} = HAT\hat{z} + HASy$$

# Output only nonlinearities



Nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(t,x) = Ax + g(t, Cx) \\ y &= Cx \end{aligned}$$

with time-varying measurable nonlinearity $g(t, Cx)$

Nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(t, x) = Ax + g(t, Cx) \\ y &= Cx \end{aligned}$$

with time-varying measurable nonlinearity $g(t, Cx)$

Candidate (reduced) observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + g(t, y(t)) + L(\hat{y} - y(t)) \\ \hat{y} &= C\hat{x} \end{aligned}$$



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Error dynamics with $e = \hat{x} - x$:

$$\dot{e} = (A + LC)e$$

$(A, C)$ is detectable (or observable) $\Rightarrow$ choose $L$ s.t. $A + LC$ is Hurwitz

Chua's system

$$\dot{x} = \begin{pmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix} x + \begin{pmatrix} -\alpha\phi(Cx) \\ 0 \\ 0 \end{pmatrix} = Ax + g(t, Cx)$$

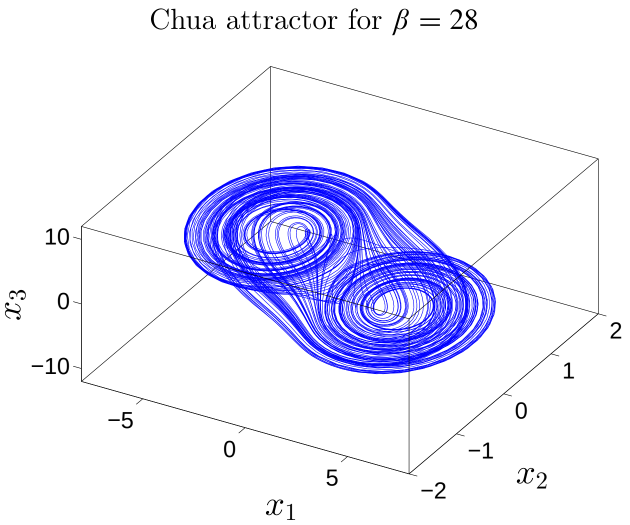
$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x = Cx$$

where

$$\phi(Cx) = \phi(x_1) = m_1 x + m_2(|x_1 + 1| - |x_1 - 1|)$$

with $m_1 = -\frac{5}{7}$, $m_2 = -\frac{3}{7}$, $23 < \beta < 31$, $\alpha = 15.6$

# Example: Chua's system



# Example: Chua's system (cont'd)



Chua's system

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$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x = Cx$$

Observer:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + g(t, y) + L(\hat{y} - y(t)) \\ \hat{y} &= C\hat{x} \end{aligned}$$

# Example: Chua's system (cont'd)



Chua's system

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Error dynamics $e = \hat{x} - x$:

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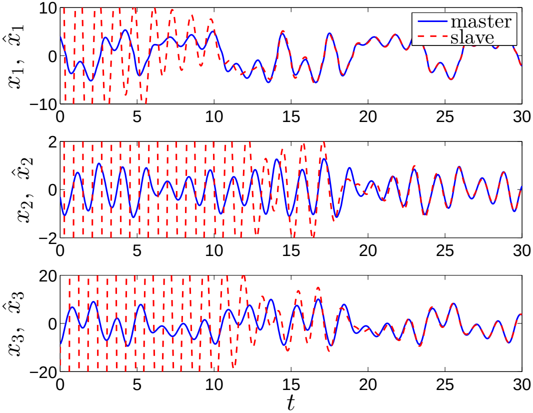
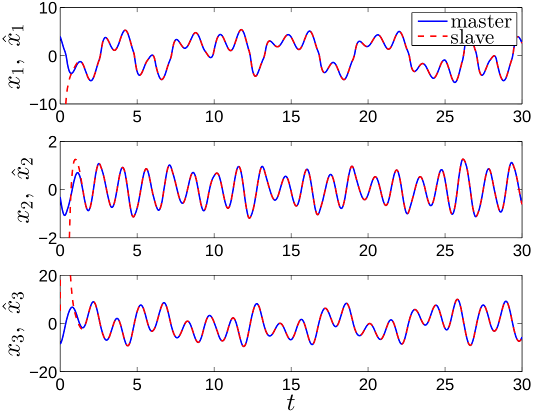
$(A, C)$ is observable $\Rightarrow$ arbitrarily fast convergence by choice of $L$




$$L = \begin{pmatrix} -16.0000 & -0.7622 & 0.7183 \end{pmatrix}^\top$$

(closed-loop poles at $-0.01$, $-0.02$ and $-0.03$)

# Example: Chua's system (cont'd)




$$L = \begin{pmatrix} -10.6000 & -0.4103 & -8.5897 \end{pmatrix}^\top$$

(closed-loop poles at $-1$, $-2$ and $-3$)

Observation: (reduced) observer design for system of the form

$$\begin{aligned} \dot{\xi} &= A\xi + g(t, C\xi) \\ \eta &= C\xi \end{aligned} \tag{3}$$

is relatively easy

# Change of coordinates



Observation: (reduced) observer design for system of the form

$$\begin{aligned} \dot{\xi} &= A\xi + g(t, C\xi) \\ \eta &= C\xi \end{aligned} \tag{3}$$

is relatively easy

General idea is to find a state space coordinates change $\xi = \Phi(x)$ and output transformation $\eta = \Psi(y)$ that transforms the nonlinear system

$$\begin{aligned} \dot{x} &= f(t, x) \\ y &= h(x) \end{aligned} \tag{4}$$

into (3)



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# Example: Rössler system



Rössler system ($a, b, c > 0$):

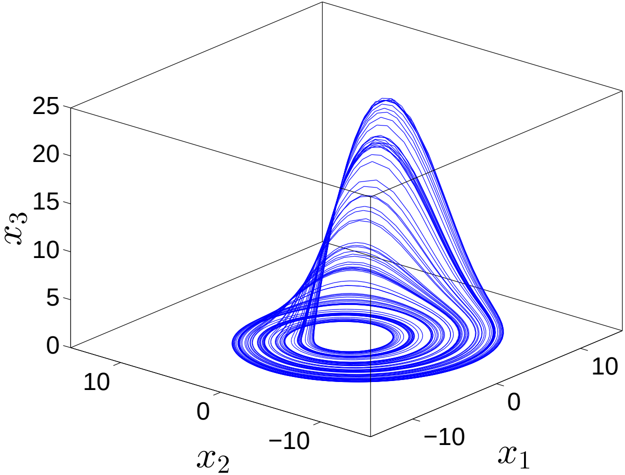
$$\begin{aligned}
\dot{x}_1 &= -x_2 - x_3 \\
\dot{x}_2 &= x_1 + a x_2 \\
\dot{x}_3 &= c + x_3(x_1 - b) \\
y &= x_3
\end{aligned}$$

Observation: $x_3(0) > 0$ implies $x_3(t) > 0$ for all $t \geq 0$

# Example: Rössler system



Rössler attractor for $a = 0.2$, $b = 5.7$, $c = 0.2$



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# Example: Rössler system



Rössler system $(a, b, c > 0)$:

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3\\ \dot{x}_2 &= x_1 + ax_2\\ \dot{x}_3 &= c + x_3(x_1 - b)\\ y &= x_3\end{aligned}$$

Observation: $x_3(0) > 0$ implies $x_3(t) > 0$ for all $t \geq 0$

$\Rightarrow$ the change of coordinates

$$\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}x_1\\ x_2\\ \ln x_3\end{pmatrix}$$

is well-defined

Rössler system in new coordinates:

$$\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{pmatrix} = \begin{pmatrix}-\xi_2 - e^{\xi_3}\\ \xi_1 + a\xi_2\\ \xi_1 - b + ce^{-\xi_3}\end{pmatrix} = \underbrace{\begin{pmatrix}0 & -1 & 0\\ 1 & a & 0\\ 1 & 0 & 0\end{pmatrix}}_{=A}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} + \underbrace{\begin{pmatrix}-e^{\xi_3}\\ 0\\ -b + ce^{-\xi_3}\end{pmatrix}}_{=g(t,\eta)}$$

with $\eta = \xi_3 = \begin{pmatrix}0 & 0 & 1\end{pmatrix}\xi = C\xi$

Rössler system in new coordinates:

$$\begin{pmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\end{pmatrix} = \begin{pmatrix}-\xi_2 - e^{\xi_3}\\ \xi_1 + a\xi_2\\ \xi_1 - b + ce^{-\xi_3}\end{pmatrix} = \underbrace{\begin{pmatrix}0 & -1 & 0\\ 1 & a & 0\\ 1 & 0 & 0\end{pmatrix}}_{=A}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} + \underbrace{\begin{pmatrix}-e^{\xi_3}\\ 0\\ -b + ce^{-\xi_3}\end{pmatrix}}_{=g(t,\eta)}$$

with $\eta = \xi_3 = \begin{pmatrix}0 & 0 & 1\end{pmatrix}\xi = C\xi$

Observer:

$$\begin{aligned}\dot{\hat{\xi}} &= A\hat{\xi} + g(t,\eta) + L(\hat{\eta} - \eta(t))\\ \hat{\eta} &= C\hat{\xi}\end{aligned}$$

$(A, C)$ is observable $\Rightarrow$ arbitrarily fast convergence by choice of $L$

# Example: van der Pol oscillator



Van der Pol oscillator with driving term ($\mu > 0$)

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 + q\cos(\omega t) \\
y &= x_1
\end{aligned}
$$

# Example: van der Pol oscillator

Van der Pol oscillator with driving term ($\mu > 0$)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu(1 - x_1^2)x_2 - x_1 + q\cos(\omega t) \\ y &= x_1 \end{aligned}$$

Let $z = x_2 - (1+\mu)y + \frac{\mu}{3}y^3$ to obtain

$$\dot{z} = -z - (2+\mu)y + \frac{\mu}{3}y^3 + q\cos(\omega(t))$$



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# Example: van der Pol oscillator

Van der Pol oscillator with driving term ($\mu > 0$)

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Let $z = x_2 - (1+\mu)y + \frac{\mu}{3}y^3$ to obtain

$$\dot{z} = -z - (2+\mu)y + \frac{\mu}{3}y^3 + q\cos(\omega(t))$$

Reduced observer:

$$\begin{aligned}
\dot{\hat{z}} &= -\hat{z}(t) - (2+\mu)y + \frac{\mu}{3}y^3 + q\cos(\omega(t)) \\
\hat{x}_2 &= \hat{z} + (1+\mu)y - \frac{\mu}{3}y^3
\end{aligned}$$



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Conditions for the existence of the state space transformation $\xi = \Phi(x)$ and output transformation $\eta = \Psi(y)$ exist, cf.

H. Nijmeijer and I. Mareels, "An observer looks at synchronization", IEEE trans. Circ. Syst. I, 44(10), pp 882–890, 1997

and the references therein

Introduction

Linear time-invariant systems

Nonlinear systems: Linear(izable) error dynamics

Nonlinear systems: High-gain observers

Effect of time-delays

# Nonlinear autonomous system



Nonlinear autonomous system

$$\dot{x} = f(x)$$
$$y = h(x)$$

- Dynamics are constrained to some compact simply connected set $\mathbf{\Omega}$
- $f$ is Lipschitz on $\mathbf{\Omega}$

# Example



Lorenz system with output $y = x$:

$$\begin{aligned}
\dot{x}_1 &= \sigma\,(x_2 - x_1) \\
\dot{x}_2 &= x_1(\rho - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - \gamma\, x_3
\end{aligned}$$

with parameters $\sigma = 10$, $\rho = 28$, $\gamma = \frac{8}{3}$

Compact domain

$$\mathbf{\Omega} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2^2 + (x_3 - \rho)^2 \leq \rho^2, |x_1| \leq \rho\right\}$$

is positively invariant w.r.t. the dynamics



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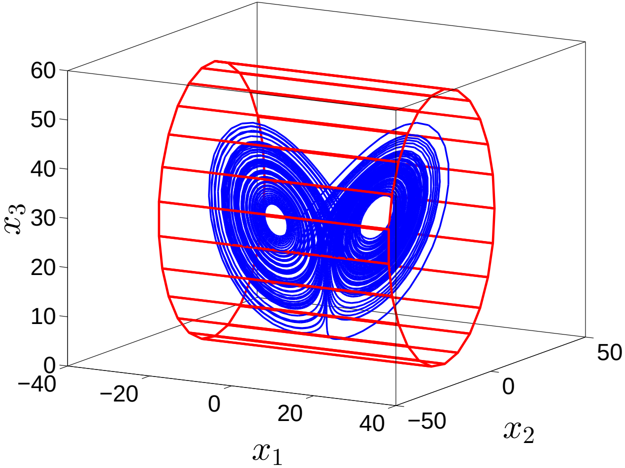
# Example





# High-gain observer



Nonlinear autonomous system

$$\dot{x} = f(x)$$
$$y = h(x)$$

Assumptions:

- Dynamics are constrained to some compact simply connected set $\mathbf{\Omega}$
- $f$ is Lipschitz on $\mathbf{\Omega}$

High-gain observer:

$$\dot{\hat{x}} = f(\hat{x}) + L(y - \hat{y})$$
$$\hat{y} = h(\hat{x})$$

"high-gain" $L \Rightarrow$ nonlinearities in the error-dynamics are dominated

High-gain observer for Lorenz system with output $y = x_1$:

$$\begin{aligned}\dot{\hat{x}}_1 &= \sigma(\hat{x}_2 - \hat{x}_1) + L_1(\hat{x}_1 - x_1)\\ \dot{\hat{x}}_2 &= \hat{x}_1(\rho - \hat{x}_3) - \hat{x}_2 + L_2(\hat{x}_1 - x_1)\\ \dot{\hat{x}}_3 &= \hat{x}_1\hat{x}_2 - \gamma\hat{x}_3 + L_3(\hat{x}_1 - x_1)\end{aligned}$$

Error dynamics $e_j = \hat{x}_j - x_j$:

$$\begin{aligned}\dot{e}_1 &= \sigma(e_2 - e_1) - L_1 e_1\\ \dot{e}_2 &= \rho e_1 - e_1 e_3 - x_1 e_3 - x_3 e_1 - e_2 - L_2 e_1\\ \dot{e}_3 &= e_1 e_2 + x_1 e_2 + x_2 e_1 - \gamma e_3 - L_3 e_1\end{aligned}$$

Error dynamics $e_j = \hat{x}_j - x_j$:

$$\begin{aligned}
\dot{e}_1 &= \sigma(e_2 - e_1) - L_1 e_1 \\
\dot{e}_2 &= \rho e_1 - e_1 e_3 - x_1 e_3 - x_3 e_1 - e_2 - L_2 e_1 \\
\dot{e}_3 &= e_1 e_2 + x_1 e_2 + x_2 e_1 - \gamma e_3 - L_3 e_1
\end{aligned}$$

Lyapunov function

$$V(e_1, e_2, e_3) = \frac{1}{2\sigma} e_1^2 + \frac{1}{2} e_2^2 + \frac{1}{2} e_3^2$$

$$\dot{V} = -e^T \begin{pmatrix} 1 + \frac{L_1}{\sigma} & \frac{1}{2}(L_2 - 1 - \rho + x_3) & \frac{1}{2}(L_3 - x_2) \\ \frac{1}{2}(L_2 - 1 - \rho + x_3) & 1 & 0 \\ \frac{1}{2}(L_3 - x_2) & 0 & \gamma \end{pmatrix} e$$

# Example (cont'd)



Error dynamics $e_j = \hat{x}_j - x_j$:

$$
\begin{aligned}
\dot{e}_1 &= \sigma(e_2 - e_1) - L_1 e_1 \\
\dot{e}_2 &= \rho e_1 - e_1 e_3 - x_1 e_3 - x_3 e_1 - e_2 - L_2 e_1 \\
\dot{e}_3 &= e_1 e_2 + x_1 e_2 + x_2 e_1 - \gamma e_3 - L_3 e_1
\end{aligned}
$$

Lyapunov function

$$
V(e_1, e_2, e_3) = \frac{1}{2\sigma} e_1^2 + \frac{1}{2} e_2^2 + \frac{1}{2} e_3^2
$$

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\dot{V} = -e^T \begin{pmatrix} 1 + \frac{L_1}{\sigma} & \frac{1}{2}(L_2 - 1 - \rho + x_3) & \frac{1}{2}(L_3 - x_2) \\ \frac{1}{2}(L_2 - 1 - \rho + x_3) & 1 & 0 \\ \frac{1}{2}(L_3 - x_2) & 0 & \gamma \end{pmatrix} e
$$

$\Rightarrow \; \dot{V} \leq -e^T Q e$ with $Q = Q^T > 0$ if $L_2 = L_3 = 0$ and $L_1$ large

# Systematic design of high-gain observers



Nonlinear autonomous system

$$\dot{x} = f(x)$$
$$y = h(x)$$

with scalar output $y \in \mathbb{R}$

Assumption: Dynamics are constrained to some compact simply connected set $\mathbf{\Omega}$

Let iterated Lie-derivatives define new coordinates on an open (simply connected) set containing $\mathbf{\Omega}$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = \Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}$$



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Dynamics in new coordinates

$$\begin{aligned}\dot{\xi} &= A\xi + B\psi(\xi) =: F(\xi)\\ y &= \xi_1\end{aligned}$$

with $\psi : \Phi(\Omega) \to \mathbb{R}$

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}^T$$

Dynamics in new coordinates

$$\dot{\xi} = A\xi + B\psi(\xi) =: F(\xi)$$
$$y = \xi_1$$

High-gain observer:

$$\dot{\hat{\xi}} = F(\hat{\xi}) + K_\theta(\hat{y} - y)$$
$$\hat{y} = \hat{\xi}_1$$

with

$$K_\theta = -\frac{1}{2} S_\theta^{-1} C^T$$

where, given $\theta > 0$, $S_\theta = S_\theta^T > 0$ solves

$$0 = \theta S_\theta^2 + A^T S_\theta + S_\theta A - C^T C$$

- Given $\theta > 0$ the equation

$$0 = \theta S_\theta^2 + A^T S_\theta + S_\theta A - C^T C$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}^T$$

has a positive definite solution $S_\theta$ because $(A, C)$ is observable

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$$0 = \theta S_\theta^2 + A^T S_\theta + S_\theta A - C^T C$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}^T$$

has a positive definite solution $S_\theta$ because $(A, C)$ is observable

- $A + K_\theta C$ with $K_\theta = -S_\theta^{-1} C^T$ is Hurwitz



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- Given $\theta > 0$ the equation

$$0 = \theta S_\theta^2 + A^T S_\theta + S_\theta A - C^T C$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}^T$$

has a positive definite solution $S_\theta$ because $(A, C)$ is observable
- $A + K_\theta C$ with $K_\theta = -S_\theta^{-1} C^T$ is Hurwitz
- For "large" $\theta$ the error-dynamics are dominated by the linear term

$$(A + K_\theta C)e$$

$\Rightarrow$ errors $e = \xi - \hat{\xi}$ converge exponentially to zero

Introduction

Linear time-invariant systems

Nonlinear systems: Linear(izable) error dynamics

Nonlinear systems: High-gain observers

Effect of time-delays

# Time-delays



Transmitter system

$$\dot{x} = f(t, x)$$

$$\xrightarrow{y(t-\tau)}$$

Receiver system

$$\dot{\hat{x}} = \hat{f}(t, \hat{x}, y(t-\tau))$$

time-delay $\tau > 0$



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Synchronization in presence of time-delays:

- perfect synchronization

$$\hat{x}(t) \to x(t) \quad \text{as} \quad t \to \infty$$

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- perfect synchronization

$$\hat{x}(t) \to x(t) \quad \text{as} \quad t \to \infty$$

- lag-synchronization

$$\hat{x}(t) \to x(t - \tau) \quad \text{as} \quad t \to \infty$$



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# Synchronization



Synchronization in presence of time-delays:

- perfect synchronization

$$\hat{x}(t) \rightarrow x(t) \quad \text{as} \quad t \rightarrow \infty$$

- lag-synchronization

$$\hat{x}(t) \rightarrow x(t-\tau) \quad \text{as} \quad t \rightarrow \infty$$

- lead-synchronization

$$\hat{x}(t) \rightarrow x(t+\tau) \quad \text{as} \quad t \rightarrow \infty$$

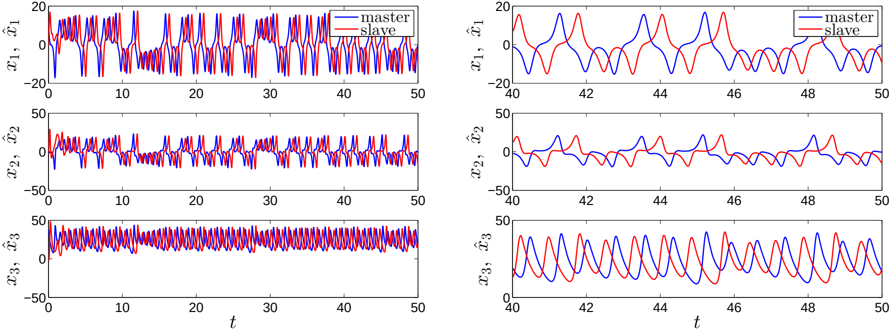
- Lorenz system as master

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1)\\ \dot{x}_2 &= x_1(\rho - x_3) - x_2\\ \dot{x}_3 &= x_1 x_2 - \gamma x_3\end{aligned}$$

with parameters $\sigma = 10$, $\rho = 28$, $\gamma = \frac{8}{3}$ and output $x_1$

- Slave system

$$\begin{aligned}\dot{\hat{x}}_1 &= \sigma(\hat{x}_2 - \hat{x}_1) + \kappa(x_1(t-\tau) - \hat{x}_1)\\ \dot{\hat{x}}_2 &= \hat{x}_1(\rho - \hat{x}_3) - \hat{x}_2\\ \dot{\hat{x}}_3 &= \hat{x}_1\hat{x}_2 - \gamma \hat{x}_3\end{aligned}$$




$$\kappa = 20 \text{ and } \tau = 0.5$$

# Perfect synchronization



- Lorenz system as master

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 \\ \dot{x}_3 &= x_1 x_2 - \gamma x_3 \end{aligned}$$

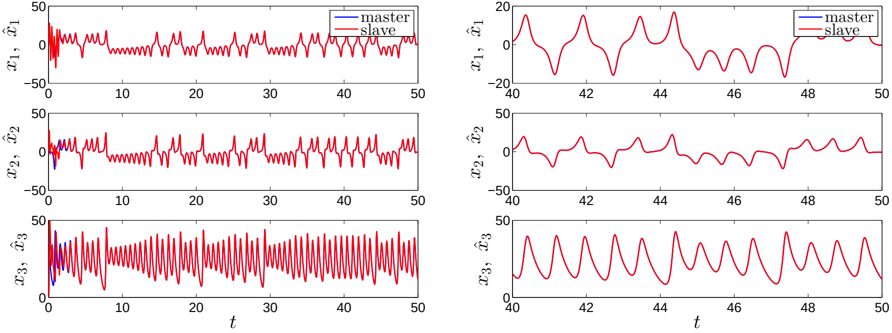
with parameters $\sigma = 10$, $\rho = 28$, $\gamma = \frac{8}{3}$ and output $x_1$

- Slave system

$$\begin{aligned} \dot{\hat{x}}_1 &= \sigma(\hat{x}_2 - \hat{x}_1) + \kappa(x_1(t-\tau) - \hat{x}_1(t-\tau)) \\ \dot{\hat{x}}_2 &= \hat{x}_1(\rho - \hat{x}_3) - \hat{x}_2 \\ \dot{\hat{x}}_3 &= \hat{x}_1 \hat{x}_2 - \gamma \hat{x}_3 \end{aligned}$$

Delay needs to be known!

# Perfect synchronization




$$\kappa = 20 \text{ and } \tau = 0.1$$

# Lead synchronization



- Lorenz system as master

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2 \\ \dot{x}_3 &= x_1 x_2 - \gamma x_3 \end{aligned}$$

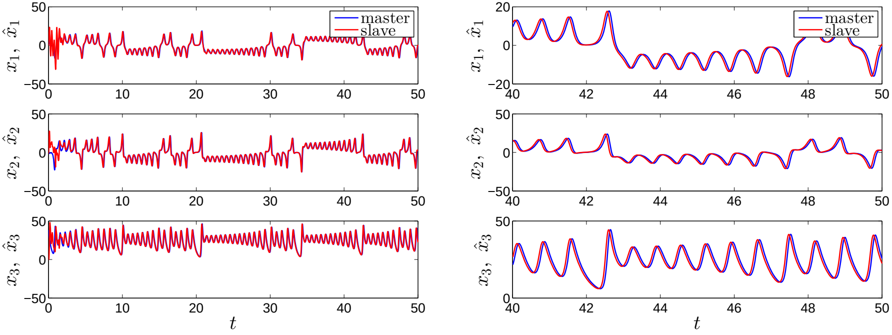
  with parameters $\sigma = 10$, $\rho = 28$, $\gamma = \frac{8}{3}$ and output $x_1$

- Slave system

$$\begin{aligned} \dot{\hat{x}}_1 &= \sigma(\hat{x}_2 - \hat{x}_1) + \kappa\big(x_1(t - \tau) - \hat{x}_1(t - \tau^*)\big) \\ \dot{\hat{x}}_2 &= \hat{x}_1(\rho - \hat{x}_3) - \hat{x}_2 \\ \dot{\hat{x}}_3 &= \hat{x}_1 \hat{x}_2 - \gamma \hat{x}_3 \end{aligned}$$

# Lead synchronization






$$\kappa = 20,\ \tau = 0.05,\ \tau^* = 0.1$$



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