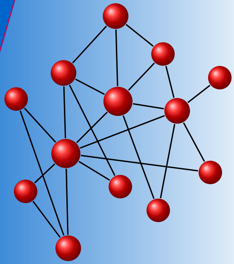
4DM50: Dynamics and Control of Cooperation

Network synchronization



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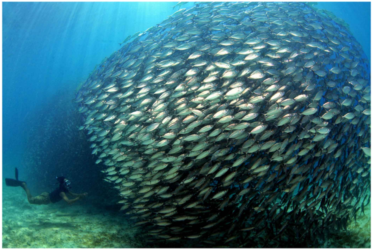
Outline

Cooperative dynamical systems in nature



Cooperative dynamical systems in engineering

Coupled systems with local interactions

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to model:

- spatially extended systems
- turbulence
- ...

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Simplified models of high-dimensional systems:

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Spatial structure

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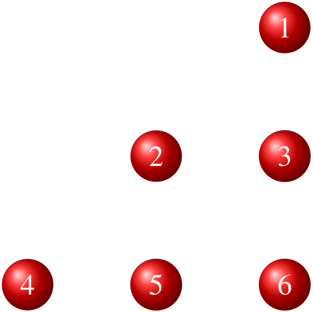
- a set of vertices or nodes $\mathcal{V}$

$$\mathcal{V} = \{1, 2, \ldots, N\}$$



Spatial structure

A network specifies which system interacts with which

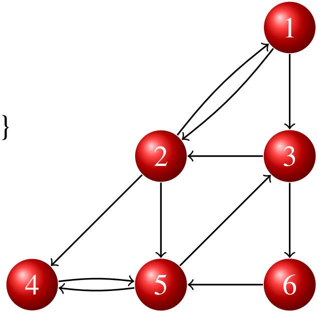
A network is

- a set of vertices or nodes $\mathcal{V}$

$$\mathcal{V} = \{1, 2, \ldots, N\}$$

- a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$

$$\mathcal{E} = \{(1, i), \ldots, (N, j)\}$$



Networks of dynamical systems

Each node $i \in \mathcal{V}$ is an open dynamical system

$$\begin{cases} \dot{x}_i = f(x_i, u_i) \\ y_i = h(x_i) \end{cases}$$

with

- state $x_i \in \mathbb{R}^n$
- input $u_i \in \mathbb{R}^m$
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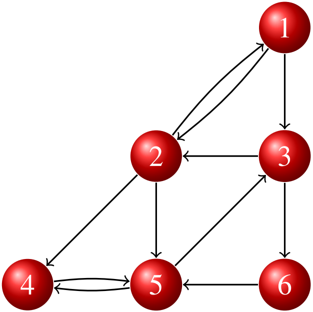

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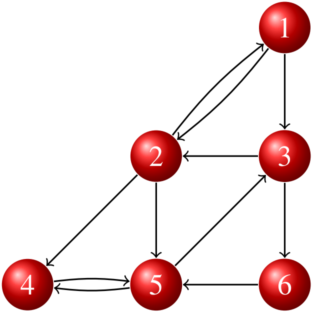
Interaction is defined via coupling functions

$$u_i = \sum_{j \in \mathcal{N}_i} g_{ij}(y_i, y_j)$$

with $\mathcal{N}_i$ the set of neighbors of i



In this series of lectures ...

- Synchronization in networks of diffusively coupled systems
- Partial synchronization in networks of diffusively coupled systems
- Effects of time-delays and network topology

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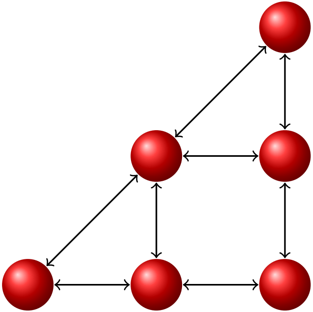
- A. M. Turing: Two ways to describe diffusion, PDE or ODE
- J. K. Hale: The dynamics of a system of reaction-diffusion equations (PDEs), with large diffusion and spatially independent reaction coefficients, is equivalent to the dynamics of the coupled systems of ODEs



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Problems:

- synchronization via diffusion
- pattern formation via diffusion
- generation of oscillations by means of diffusion
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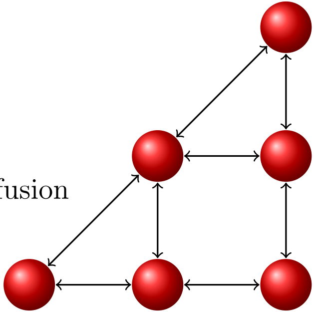

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Diffusively coupled systems

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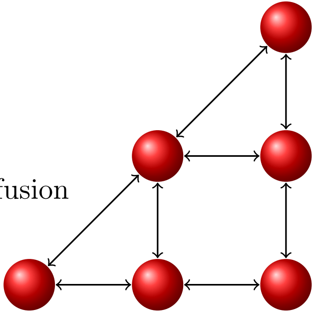

Setting

- Square input-output systems of the form

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases}$$

  with

  - state $x_i \in \mathbb{R}^n$
  - input $u_i \in \mathbb{R}^m$
  - output $y_i \in \mathbb{R}^m$
  - matrices $B, C^T \in \mathbb{R}^{n\times m}$
  - CB (similar to) a positive definite matrix

- Diffusive coupling functions

$$u_i = \sum_{j\in\mathcal{N}_i} \gamma_{ij}(y_j - y_i)$$

  with diffusion coefficients $\gamma_{ij} = \gamma_{ji} > 0$ (and $j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j$)

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Coupling matrix

Define the diffusive coupling matrix

$$\Gamma = \begin{pmatrix} \sum_{j=2}^{N} \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1N} \\ -\gamma_{21} & \sum_{j=1, j\neq 2}^{N} \gamma_{2j} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\gamma_{(N-1)N} \\ -\gamma_{N1} & \cdots & -\gamma_{N(N-1)} & \sum_{j=1}^{N-1} \gamma_{Nj} \end{pmatrix}$$

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- Γ is singular
- $\Gamma = \Gamma^T$ is positive semi-definite
- Eigenvalues of Γ (if and only if the network is connected):

$$0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$$

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Network equations

Dynamics of the diffusively coupled systems:

$$\begin{cases} \dot{x} = F(x) + (I_N \otimes B)u \\ u = -(\Gamma \otimes C)x \end{cases}$$

with $\otimes$ the Kronecker (tensor) product and

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad F(x) = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

Synchronization of diffusively coupled systems

The diffusively coupled systems

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Bounded solutions of diffusively coupled systems

Caution: systems with bounded solutions for $u_i \equiv 0$ may have unbounded solution in diffusive interaction

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Example: Two coupled linear systems

$$\begin{cases} \dot{x}_i = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -4 & 2 & -3 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_i \\ y_i = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x_i \end{cases}$$

$i = 1, 2,$ with coupling

$$u_1 = -u_2 = \gamma\,(y_2 - y_1)$$

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- for $0 \leq \gamma \leq 0.6513$ the origin of the coupled systems is exponentially stable

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- for $\gamma \geq 0.6514$ the origin of the coupled systems is unstable

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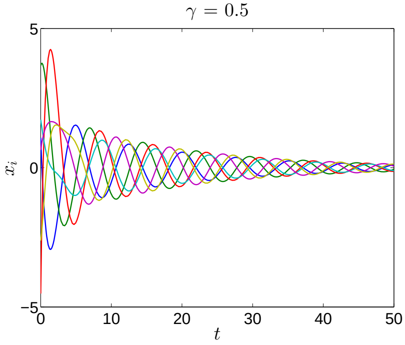

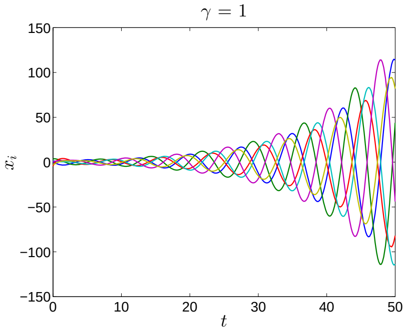

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Semi-passivity

A system

$$\begin{cases} \dot{x}_i = f(x_i, u_i) \\ y_i = h(x_i) \end{cases}$$

with associated positive semi-definite storage function $S : \mathbb{R}^n \to \mathbb{R}_+$ is

- passive if $\dot{S} \leq y^T u$

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- semi-passive if the dissipation inequality is satisfied outside some ball $\mathcal{B} \subset \mathbb{R}^n$:

  $$\dot{S} \leq y^T u - H(x)$$

  with $H(x) \geq 0$ for $|x| > R$

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Physical interpretation: a strictly semi-passive system has a limited amount of free energy

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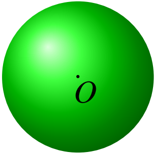

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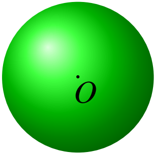

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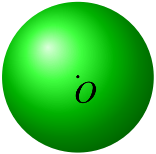

- solutions of strictly semi-passive systems with a radially unbounded storage function S are uniformly (ultimately) bounded if $y^T u \leq 0$
- for diffusive coupling:

$$y^T u = x^T (I_N \otimes C^T) u = -x^T (\Gamma \otimes C^T C) x \leq 0$$

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Semipassivity: example

Lorenz system with input u

$$\begin{cases} \dot{x}_1 = \sigma\,(x_2 - x_1) + u \\ \dot{x}_2 = x_1(\rho - x_3) - x_2 \\ \dot{x}_3 = x_1 x_2 - \gamma\, x_3 \end{cases}$$

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$$S(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + (x_3 - (\sigma + \rho))^2\right)$$

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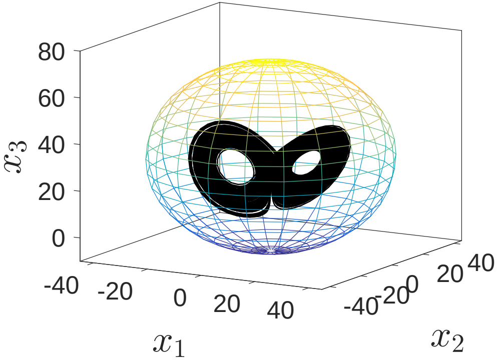
$$\dot{S}(x) = yu - H(x)$$

with

$$H(x) = \underbrace{\sigma x_1^2 + x_2^2 + \gamma\left(x_3 - \frac{\sigma + \rho}{2}\right)^2}_{\geq 0} - \gamma\frac{(\sigma + \rho)^2}{4}$$

Semipassivity: example

$H(x) > 0$ outside a ball of radius $R = \frac{\gamma^2(\sigma+\rho)}{4(\gamma-1)}$ and center $\left(0, 0, \frac{\gamma(\sigma+\rho)}{2(\gamma-1)}\right)$



$$\sigma = 10,\ \rho = 28,\ \gamma = \frac{8}{3}$$

Introduction

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Bounded solutions of diffusively coupled systems

Synchronization of diffusively coupled systems

Extensions

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System dynamics in new coordinates (normal form):

$$\begin{cases} \dot{z}_i = q(z_i, y_i) \\ \dot{y}_i = a(z_i, y_i) + CBu_i \end{cases}$$

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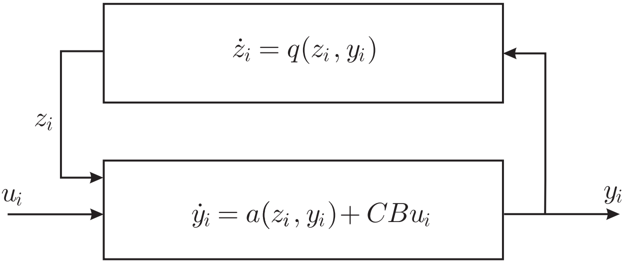

Internal dynamics $\dot{z}_i = q(z_i, y_i)$

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Dynamics given the constraint $y_1 \equiv y_2 \equiv \cdots \equiv y_N \equiv y_s$:

$$\dot{z}_1 = z_1 + y_s, \quad \dot{z}_2 = z_2 + y_s, \quad \ldots, \quad \dot{z}_N = z_N + y_s$$

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Error dynamics unstable $\Rightarrow$ no synchronization (in the sense that $x_1 \equiv x_2 \equiv \cdots \equiv x_N$)

Convergent systems (Demidovich, 1961)

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- there exists a unique bounded solution $\bar{z}_w(\cdot)$ defined on $(-\infty, \ \infty)$
- there exists constants $\mu, C > 0$ such that

$$\|z(t) - \bar{z}_w(t)\| \leq Ce^{-\mu(t-t_0)}, \quad \forall t \geq t_0$$

Convergent systems (Demidovich, 1961)

The system

$$\dot{z} = q(z, w(t)), \quad w(t) \in \mathcal{W}, \quad \mathcal{W} \text{ is compact}$$

Test for exponential convergence:

There exists a symmetric positive-definite $(n-m) \times (n-m)$ matrix P such that the eigenvalues of

$$P \left(\frac{\partial q}{\partial z}(z, w) \right) + \left(\frac{\partial q}{\partial z}(z, w) \right)^T P$$

are negative and separated away from zero for all $(z, w) \in \mathbb{R}^{n-m} \times \mathcal{W}$

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Example: Lorenz system

Lorenz system with input u and output $y = x_1$:

$$\begin{cases} \dot{x}_1 = \sigma\,(x_2 - x_1) + u \\ \dot{x}_2 = x_1(\rho - x_3) - x_2 \\ \dot{x}_3 = -\gamma\, x_3 + x_1 x_2 \end{cases}$$

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Dynamics in normal form:

$$\begin{cases} \dot{z}_1 = \rho y - z_1 - y z_2 \\ \dot{z}_2 = -\gamma z_2 + y z_1 \\ \dot{y} = \sigma(z_1 - y) + u \end{cases}$$

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Test for exponential convergence with $P = I$:

$$\begin{pmatrix} -1 & -y \\ y & -\gamma \end{pmatrix} + \begin{pmatrix} -1 & -y \\ y & -\gamma \end{pmatrix}^T = \begin{pmatrix} -2 & 0 \\ 0 & -2\gamma \end{pmatrix}$$

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Synchronization of diffusively coupled systems

Given a network of diffusively coupled systems

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases} \tag{1}$$

$$u_i = \sum_{j \in \mathcal{N}_i} \gamma_{ij}(y_j - y_i) \tag{2}$$

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Assumptions:

- system (1) is strictly semi-passive with a radially unbounded storage function S
- the subsystem

  $$\dot{z}_i = q(z_i, w(t))$$

  associated to (1) satisfies the test for exponential convergence

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Result:

- solutions of the diffusively coupled systems are uniformly ultimately bounded
- there exists a positive constant $\bar{\lambda}$ such that for

$$\lambda_2 \geq \bar{\lambda} \quad (\lambda_2 \text{ is the smallest non-zero eigenvalue of } \Gamma)$$

the set $\mathcal{M} := \{x \in \mathbb{R}^{Nn} \,|x_1 = x_2 = \cdots = x_N\}$ contains a globally asymptotically stable subset, i.e. the systems synchronize

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Whether or not a network of coupled Lorenz systems synchronizes depends on the coupling structure (λ_2)

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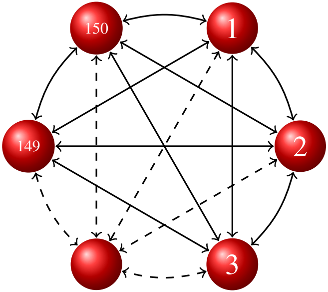
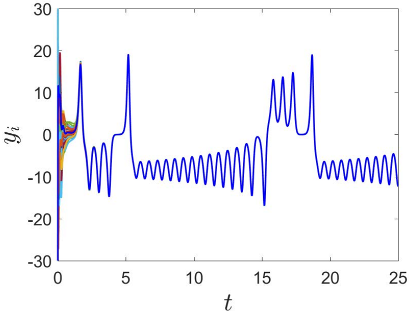
150 coupled Lorenz systems

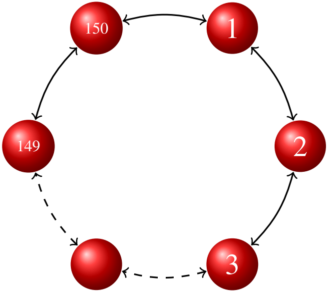
Globally coupled network ($\lambda_2 = 150$)

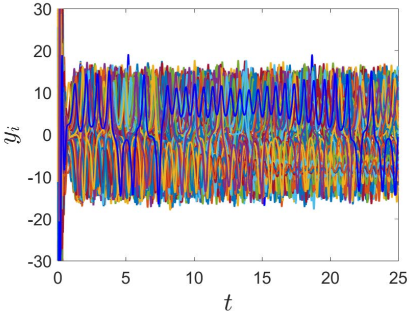




150 coupled Lorenz systems

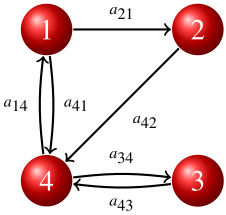
Ring network ($\lambda_2 = 0.0018$)





Extensions: directed networks

Networks with directed connections and/or asymmetric interaction weights



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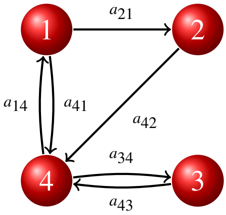

Assumption: the network is strongly connected

Diffusively coupled systems on strongly connected directed networks:

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases} \tag{3}$$

$$u_i = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}(y_j - y_i) \tag{4}$$

with coupling strength σ and $\max_i \sum_{j \in \mathcal{N}_i} a_{ij} = 1$

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Assumptions:

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- the subsystem

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  associated to (3) satisfies the test for exponential convergence

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Result:

- solutions of the diffusively coupled systems are uniformly ultimately bounded
- there exists a positive constant $\bar{\sigma}$ such that for

  $$\sigma \geq \bar{\sigma}$$

  the systems (3), (4) synchronize

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Diffusive coupling with time-delay

- Transmission delay

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}(y_j(t-\tau) - y_i(t))$$

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Extensions: time-delays

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Systems on strongly connected directed networks

$$\begin{cases} \dot{x}_i = f(x_i) + Bu_i \\ y_i = Cx_i \end{cases} \tag{5}$$

with diffusive time-delay coupling

$$u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t - \tau) - y_i(t)) \tag{6}$$

or

$$u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t - \tau) - y_i(t)(t - \tau)) \tag{7}$$

Extensions: time-delays

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Assumptions:

- system (5) is strictly semi-passive with a radially unbounded storage function S
- the subsystem $\dot{z}_i = q(z_i, w(t))$ associated to (5) satisfies the test for exponential convergence

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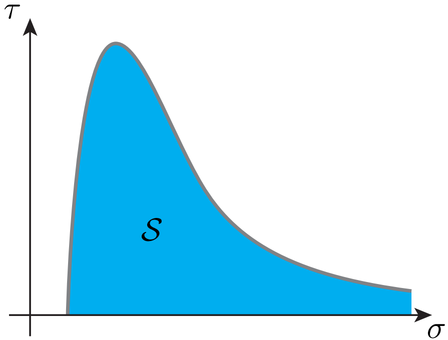
Result:

- solutions of the time-delay coupled systems are uniformly (ultimately) bounded (conditions apply in case of coupling (7))
- there exists a set $\mathcal{S}$ s.t. systems synchronize if $(\sigma, \tau) \in \mathcal{S}$



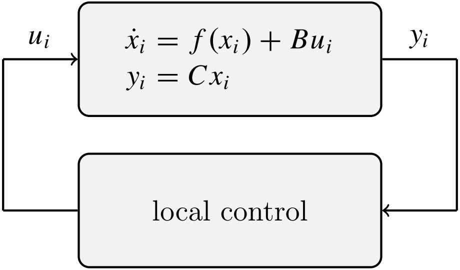
Extensions: local control

If the systems are not strictly semi-passive and/or internal dynamics are not exponentially convergent

$$u_i \longrightarrow \boxed{\begin{aligned} \dot{x}_i &= f(x_i) + Bu_i \\ y_i &= Cx_i \end{aligned}} \longrightarrow y_i$$

Extensions: local control

If the systems are not strictly semi-passive and/or internal dynamics are not exponentially convergent $\Rightarrow$ local control



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Extensions: local control

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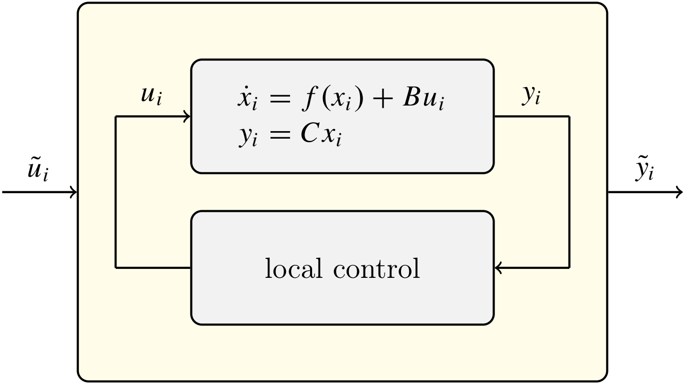

Strictly semi-passive from $\tilde{u}_i$ to $\tilde{y}_i$ and convergent internal dynamics
(see PhD Thesis Carlos G. Murguia Rendon, Eindhoven 2015)