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Abstract

We propose a microstructural modeling framework for studying optimal market making policies in a FIFO (first in first out) limit order book (LOB). In this context, the limit orders, market orders, and cancel orders arrivals in the LOB are modeled as Cox point processes with intensities that only depend on the state of the LOB. These are high-dimensional models which are realistic from a micro-structure point of view and have been recently developed in the literature. In this context, we consider a market maker who stands ready to buy and sell stock on a regular and continuous basis at a publicly quoted price, and identifies the strategies that maximize her P&L penalized by her inventory.

We apply the theory of Markov Decision Processes and dynamic programming method to characterize analytically the solutions to our optimal market making problem. The second part of the paper deals with the numerical aspect of the high-dimensional trading problem. We use a control randomization method combined with quantization method to compute the optimal strategies. Several computational tests are performed on simulated data to illustrate the efficiency of the computed optimal strategy. In particular, we simulated an order book with constant/ symmetric/ asymmetrical/ state dependent intensities, and compared the computed optimal strategy with naive strategies.

Keywords: Limit Order Book; pure-jump controlled process; High-Frequency Trading; Queuing model; High-dimensional Stochastic Control; Markov Decision Process; Markovian Quantization

1 Introduction

Most of the markets use a limit order book (LOB) mechanism to facilitate trade. Any market participant can interact with the LOB by posting either market orders or limit orders. In such type of markets, the market makers play a fundamental role by providing liquidity to other market participants, typically to impatient agents who are willing to cross the bid-ask spread. The profit made by a market making strategy comes from the alternation of buy and sell orders.

From the mathematical modeling point of view, the market making problem corresponds to the choice of an optimal strategy for the placement of orders in the LOB. Such a strategy should maximize the expected utility function of the wealth of the market maker up to a penalization of her inventory. In the recent litterature, several works focused on the problem of market making through stochastic control methods.

The seminal paper by Avellaneda and Stoikov [2] inspired by the work of Ho and Stoll [15] proposes a framework for trading in an order driven market. They modeled a reference price for the stock as a Wiener process, and the arrival of a buy or sell liquidity-consuming order at a distance δ from the reference price is described by a point process with an intensity in an exponential form decreasing with δ. They characterized the optimal market making strategies that maximize an exponential utility function of terminal wealth. Since this paper, other authors have worked on related market making

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problems. Gueant, Lehalle, and Fernandez-Tapia [13] generalized the market making problem of [2] by dealing with the inventory risk. Cartea and Jaimungal [5] also designed algorithms that manage inventory risk. Fodra and Pham [11] and [10] considered a model designed to be a good compromise between accuracy and tractability, where the stock price is driven by a Markov Renewal Process, and solved the market making problem. Guilbaud and Pham [14] also considered a model for the mid-price, modeled the spread as a discrete Markov chain that jumps according to a stochastic clock, and studied the performance of the market making strategy both theoretically and numerically. Cartea and Jaimungal [4] employed a hidden Markov model to examine the intra-day changes of dynamics of the order book. Very recently, Cartea, Penalva, and Jaimungal [7] and Gueant [12] published monographs in which they developped models for algorithmic trading in different contexts. Abergel and El Aoud [9] extended the framework of Avellaneda and Stoikov to the options market making. A common feature of all these works is that a model for the price or/and the spread is considered, and the LOB is then built from these quantities. This approach leads to models that predict well the long-term behavior of the LOB. The reason for this choice is that it is generally easier to solve the market making problem when the controlled process is low-dimensional.

Yet, some recent works have introduced accurate and sophisticated micro-structural order book models. These models reproduce accurately the short-term behavior of the market data. The focus is on conditional probabilities of events, given the state of the order book and the positions of the market maker. Abergel, Anane, Chakraborti, Jedidi, Muni Toke [1] proposed models of order book where the arrivals of orders in the LOB are driven by Poisson processes or Hawkes processes. Stoikov, Talreja, and Cont [8] also modeled the orders arrivals with Poisson processes. Lehalle, Rosenbaum and Huang [16] proposed a queue-reactive model for the order book. In this model the arrivals of orders are driven by Cox point processes with intensities that only depend on the state of the LOB (they are not time dependent). Other tractable dynamic models of order-driven market are available (see e.g. Stoikov, Talreja, and Cont [8], Rosu [20], Cartea, Jaimungal, Ricci [6]).

In this paper we adopt the micro-structural model of order book in [1], and solve the associated trading problem. The problem is formulated in the general framework of Piecewise Deterministic Markov Decision Process (PDMDP), see Bauerle and Rieder [3]. Given the model of order book, the PDMDP formulation is natural. Indeed, between two jumps, the order book remains constant, so one can see the modeled order book as a point process where the time becomes a component of the state space. As for the control, the market maker fixes her strategy as a deterministic function of the time right after each jump time. We prove that the value function of the market making problem is equal to the value function of an associated non-finite horizon Markov decision process (MDP). This provides a characterization of the value function in terms of a fixed point dynamic programming equation. Jacquier and Liu [17] recently followed a similar idea to solve an optimal liquidation problem.

The second part of the paper deals with the numerical simulation of the value functions. The computation is challenging because the micro-structural model used to model the order book leads to a high-dimensional pure jump controlled process, so evaluating the value function is computationally intensive. We rely on control randomization and Markovian quantization methods to compute the value functions. Markovian quantization has been proved to be very efficient for solving control problems associated with high-dimensional Markov processes. We first quantize the jump times and then quantize the state space of the order book. See Pages, Pham, Printemps [19] for a general description of quantization applied to controlled processes. The projections are time-consuming in the algorithm. Fast approximate nearest neighbors algorithms have been implemented to make it quicker (see [18]). We borrow the values of intensities of the arrivals of orders for the order book simulations in order to test our optimal trading strategies.

The paper is organized as follows. The model setup is introduced in section 2: we present the micro-structural model for the LOB, and show how the market maker interacts with the market. In section 3, we prove the existence and provide a characterization of the value function and optimal trading strategies. In section 4, we prove the convergence of the quantization method and present some results of numerical tests on simulated LOB.

2 Model setup

2.1 Order book representation

We recall the order book representation, as it has been introduced in chapter 6 of [1].
Fix $K \geq 0$. A LOB is supposed to be fully described by K limits on the bid side and K limits on the ask side. We use the pair of vectors $(\underline{a}_t, \underline{b}_t) = (a_t^1, ..., a_t^K, b_t^1, \ldots, b_t^K)$ to describe the order book.

- a_t^i is the number of shares available i ticks away from the best bid [1] at time t.

- $-b_t^i$ is the number of shares available i ticks away from the best ask [2] at time t.

$\underline{a}_t$ represents the ask side at time t, and $\underline{b}_t$ represents the bid side at time t. The quantities a_t^i, $1 \leq i \leq K$, live in the discrete space $q\mathbb{N}$ where $q \in \mathbb{N}^*$ is the minimum order size on each specific market (*lot size*). The quantities b_t^i, $1 \leq i \leq K$, live in the discrete space $-q\mathbb{N}$. By convention, the a^i are positive, and the b^i non-positive for $0 \leq i \leq K$.

Constant boundary conditions are imposed outside the moving frame of size $2K$. We assume that all the limits up to the K-th ones are equal to a_∞ in the ask side, and equal to b_∞ in the bid side. The bounding conditions make sure the order book is never empty.

We assume that market orders, limit orders and cancel orders can arrive to the order book at any time. The arrivals of orders are modeled as follows:

- the process for buy market orders arrivals is denoted M_t^+ and is modeled as a Cox process. We denote by λ^{M^+} its intensity.

- the process for sell market orders arrivals is denoted M_t^- and is modeled as a Cox process. We denote by λ^{M^-} its intensity.

- the process for sell orders arrivals at the ith limit on the ask side, $i \in \{1, ...K\}$, is denoted L_i^+ and is modeled by a Cox process with intensities denoted by $\lambda_1^{L^+}, \ldots, \lambda_K^{L^+}$.

- the process for buy orders arrivals at the ith limit on the bid side , $i \in \{1, ...K\}$, is denoted L_i^- and is modeled as a Cox process with stochastic intensities denoted by $\lambda_1^{L^-}, \ldots, \lambda_1^{L^-}$.

- the process for cancel of orders arrivals at the ith limit on the ask side, $i \in \{1, ...K\}$, is denoted C_i^+ and is modeled by a Cox processes with stochastic intensities denoted by $\lambda_1^{C^+}, \ldots, \lambda_K^{C^+}$.

- the process for cancel of orders arrivals at the ith limit on the bid side, $i \in \{1, ...K\}$, is denoted C_i^- and is modeled by a Cox processes with stochastic intensities denoted by $\lambda_1^{C^-}, \ldots, \lambda_K^{C^-}$.

The LOB is modeled as a pure jump process.
In the following figure 1, the reader can see an example of order book that may help to get more familiar with the notations.

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[1] highest price a participant in the market is willing to buy a stock

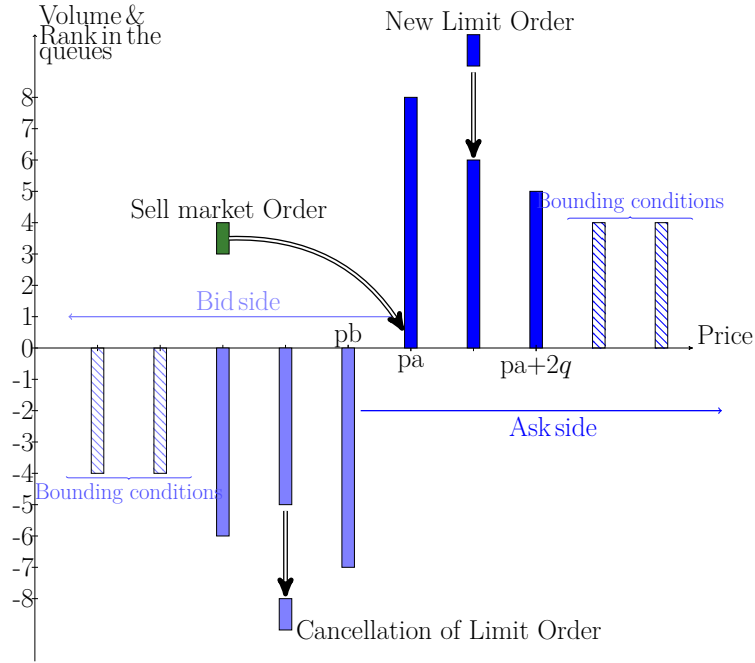[2] cheapest price a participant in the market is willing to sell a stock

Figure 1: Order book dynamics: in this example, $K = 3$, $q = 1$, $a_\infty = 4$, $b_\infty = -4$, $\underline{a} = (8, 6, 5)$, $\underline{b} = -7, -5, -6)$. The spread is equal to 1.
At any time, the order book can receive limit orders, market orders or cancel orders.

In the sequel we assume that the all the introduced intensities are at most linear with respect to the couple $(\underline{a}, \underline{b})$ and do not depend on the time.

2.2   market maker strategies

We assume that the market is governed by a *FIFO* (*First In First Out*) rule, which means that each limit of the order book is a queue, and the first order in the queue will be the first one to be executed. We consider a market maker who stands ready to send buy and sell limit orders on a regular and continuous basis at quoted prices. We make the assumption that, at any time, the total number of limit orders placed by the marker maker does not exceed one fixed (possibly large) number M. This restriction makes the control space compact, which will be useful for theoretical and numerical reasons, as the reader will see in the next sections.

2.2.1   Controls and strategies of the market maker

The market maker can choose at any time to keep some positions, cancel other positions, and take new positions in the LOB (as long as she does not hold more than M positions) The positions of the market maker in the LOB are fully described by the following $M-$dimensional vectors $\underline{ra}_t$, $\underline{rb}_t$ and $\underline{na}_t$, $\underline{nb}_t$ where $\underline{ra}, \underline{rb}$ record the limits in which the market maker's orders are located; and $\underline{na}, \underline{nb}$ record the ranks in the different queues of each market maker's orders.
The following figure 2 gives an example of a controlled order book.

Assumption1 We assume that the market maker does not cross the spread and we also assume that the other participants do not see her orders in the LOB.
This assumption is necessary if we want the market maker strategy to be predictable with respect to the natural filtration generated by the orders arrivals processes.
For consistency reasons, we also make the following natural assumption:
Assumption2 We assume that the market maker does not change her strategy when nothing happens in the LOB. In other words, each time something exogenous happens in the LOB (so to say one of the following orders arrivals processes $L^\pm, C^\pm, M^\pm$ jumps), the market maker first observes the new (updated) LOB, then makes a decision and keep it until the next exogenous jump of the LOB or until

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the terminal time is reached.

This is a mild assumption if the LOB jumps frequently.

Denote by $(T_n)_{n \in \mathbb{N}}$ the sequence of jump times of the order book. The set of the admissible strategies AS are the set of the predictable processes $(\underline{ra}_t, \underline{rb}_t)_{t \leq T}$ such that:

- $rb_*, ra_* > 0$

- for all $n \in \mathbb{N}$, $(\underline{ra}_t, \underline{rb}_t) \in \{0, ..., K\}^M \times \{0, ..., K\}^N$ are constant on $(T_n, T_{n+1}]$

where, for every vector $\underline{a}$: $a_* = \min_i a_i$; and: $a0 = \arg\min_i (\underline{a}_i)$. The control is the double vector of the positions of the M market maker's orders on the order book. By convention, we set: $ra_i(t) = -1$ if the ith order of the market maker is not placed on the LOB.
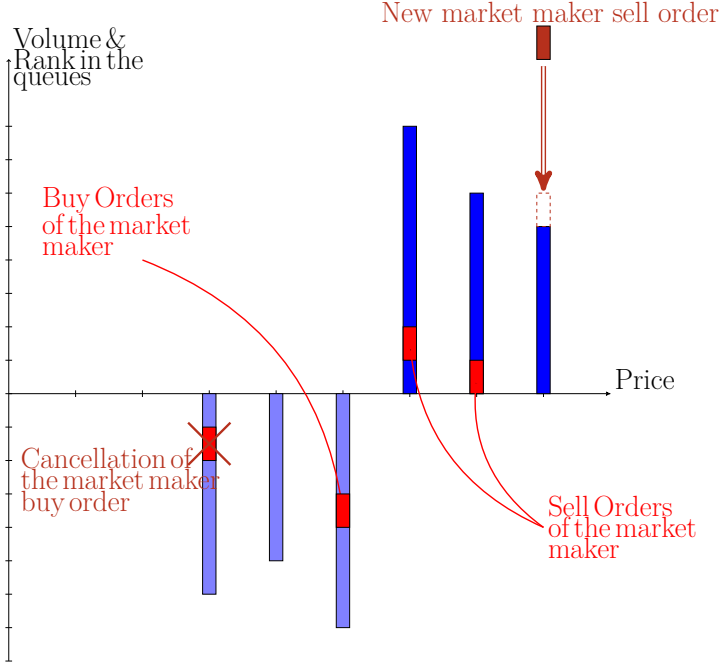

Figure 2: Positions and several available actions of the market maker on the order book. In this example: the positions of the market maker are $\underline{ra} = (0, 1, -1, ...)$, $\underline{rb} = (0, 2, -1...)$. The ranks vectors associated are $na = (2, 1, -1 ...)$ and $\underline{nb} = (3, 1, -1, ...)$. Each time an exogenous jumps of the LOB occurs, the market maker reacts by posting new orders, canceling some of her orders, or keeping her position.

2.2.2 Controlled order book

The controlled order book is described by the state process:

$$Z_t = \left(X_t, Y_t, \underline{a}_t, \underline{b}_t, \underline{na}_t, \underline{nb}_t, pa_t, pb_t, \underline{ra}_t, \underline{rb}_t \right)$$

where:

- X_t is the cash hold by the market maker on a zero interest account.

- Y_t is the inventory of the market maker. *ie* : it is the (signed) number of shares hold by the market maker at time t.

- $\underline{a}_t = (a_1(t), \ldots, a_K(t)), i \in \{1, \ldots, K\}$, $a_i(t)$ is the number of orders which are i ticks away from the bid price, at time t. We do not count the order of the market maker in a_t

- $\underline{b}_t = (b_1(t), \ldots, b_K(t))$ $i \in \{1, \ldots, K\}$, $b_i(t)$ is the number of buy orders which are i ticks aways from the ask-price at time t. We do not count the buy order of the market maker when computing the b_i.

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- For $i \in \{1, ..., M\}$, $\underline{na}_t(i) \in [-1, ..., |a| + M]$ is the rank of the i-th sell orders of the market maker in the queue in the LOB.
  Convention: we assume that $\underline{na}_t(i) = -1$ if the i-th order of the market maker is not placed on the order book.

- For $i \in \{1, ..., M\}$, $\underline{nb}_t(i) \in [-1, ..., |b| + M]$ is the rank of the i-th buy orders of the market maker in the queue.
  Convention: we assume that $\underline{na}_t(i) = -1$ if the i-th order of the market maker is not placed on the order book.

- pa_t is the ask price. *ie*: it is the cheapest price a participant is willing to sell the stock.

- pb_t is the bid price. *ie*: it is the highest price a participant is willing to buy the stock.

- For all $i \in \{1, ..., M\}$, $\underline{ra}_t(i)$ is the number of ticks between the i-th market maker sell order and the best opposite quote. $\underline{ra}_t(i) \in \{1, \ldots, K\} \cup \{-1\}$.
  Convention: we assume that $\underline{ra}_t(i) = -1$ if the i-th order of the market maker is not placed on the order book.

- For all $i \in \{1, ..., M\}$, $\underline{rb}_t(i)$ is the number of ticks between the i-th market maker buy orders and the best sell quote. $\underline{rb}_t(i) \in \{1, \ldots, K\} \cup \{-1\}$.
  Convention: we assume that $\underline{rb}_t(i) = -1$ if the i-th order of the market maker is not placed on the order book.

The dynamics of (Z_t) is simple since it is a pure jump Cox controlled process. However the corresponding equations are heavy to write. The reader can find the expressions of the dynamics of each components of (Z_t) in Appendix in the case where the set of admissible strategies is restricted to those where the market maker chooses to place orders only at the best limits. ie: at each jump time, she asks herself "should I post/keep/cancel an order at the best bid ? should I post/keep/cancel an order at the best ask ? "

3 Existence and Characterization of the optimal strategy

3.1 Definition and well-posedness of the value function

Let fix $T > 0$ a terminal time. The Value Function V is defined as the maximal wealth expectation of the market maker at time T among all the admissible strategies π, penalized by her terminal inventory. We have:

$$V(t, z) = \sup_{\pi \in AS} \mathbb{E}\left[g\left(Z_T^{t,z,\pi}\right) - \eta Y_T^{t,y^2}\right]$$

where:

- AS is the set of the admissible strategies that has been defined in section 2.2.1.

- η is the parameter that penalizes the inventory held by the market maker at terminal time. We can take $\eta = 0$ if we consider that the market maker is risk-neutral.

- $g : z \mapsto x + L(y)$ is the market maker's wealth function where L, defined below, measures the amount earned from the liquidation of the inventory based on the current state of the LOB:

We denote by L the liquidation function which is such that: for all $z = (x, y, \underline{a}, \underline{b}, \underline{na}, \underline{nb}, pa, pb, \underline{ra}, \underline{rb})$ (y being the (signed) market maker's inventory),

$$L(z) = \begin{cases} \sum_{k=0}^{j-1}\left[a_k(pa + kq)\right] + (y - a_0 - ... - a_{j-1})(pa + jq) & \text{if } y < 0 \\ -\sum_{k=0}^{j-1}\left[b_k(pb - kq)\right] + (y + b_0 + ... + b_{j-1})(pb - jq) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases}$$

with

$$
\jmath = \begin{cases} \arg\min \left\{ j \mid \sum_{i=0}^{j} a_i > -y \right\} & \text{if } y < 0 \\ \arg\min \left\{ j \mid \sum_{i=0}^{j} |b_i| > y \right\} & \text{if } y > 0 \end{cases}
$$

Our first result checks that the problem is well-posed.

Lemma 3.1. *Let $g : E \to \mathbb{R}$ be the terminal reward of the market maker. Then, we get:*

$$
\sup_{\pi} \mathbb{E}_z^{\pi} \left[g^+(Z_T^{\pi}) - \eta Y_T^{\pi 2} \right] < +\infty
$$

Proof. We assumed that the intensities of the orders arrivals are at most linear with respect to the couple $(\underline{a}, \underline{b})$. Denote by λ^L, λ^C, λ^M the constants such that for all state z of the order book: $\sum_{i=0}^{K} \lambda(L_i^{\pm})(z) \leq \lambda^L(|\underline{a}| + |\underline{b}|)$, $\sum_{i=0}^{K} \lambda(C_i^{\pm})(z) \leq \lambda^C(|\underline{a}| + |\underline{b}|)$, $\lambda(M^-)(z) + \lambda(M^+)(z) \leq \lambda^M(|\underline{a}| + |\underline{b}|)$. Define $(\bar{a}_t)_t$, the Cox process with intensity λ^L such that: $\bar{a}_0 = 2K(a_\infty \wedge \max_j(a_j) \wedge |b_\infty| \wedge \max_j(|b_j|) + M)$ (M, the maximum number of orders that the market maker is allowed to place on the LOB); $\bar{a}_0$ is the sum of the orders in the controlled order book.
Define the following market order arrival processes $\bar{M}_t$ as the Cox process with intensity $\lambda^M |\bar{a}|$.

Since we assumed that the limits further than K ticks are all non-empty, we have the following bounding for the expected penalized wealth of the market maker:

$$
\mathbb{E} \left[g(Z_T^{\pi}) - \eta Y_T^{t,y,\pi 2} \right] \leq \mathbb{E} \left[g(Z_T^{\pi}) \right] \leq M \mathbb{E} \left(\left[\bar{M}_T \right] + K \right) \tag{3.1}
$$

The r.h.s of (3.1) is bounded. Indeed, we have:

$$
\mathbb{E} \left[\bar{M}_T \right] \leq \lambda^M \int_0^T \mathbb{E} \left[\bar{a}_s \right] ds
$$

but $\mathbb{E} \left[\bar{a}_t \right] \leq \lambda^L \int_0^t \mathbb{E} \left[\bar{a}_s \right] ds$ so:

$$
\mathbb{E} \left[\bar{a}_t \right] \leq \bar{a} e^{\lambda^L t}
$$

This leads to:

$$
\mathbb{E} \left[\bar{M}_T \right] \leq \frac{\lambda^M}{\lambda^L} \bar{a} e^{\lambda^L T}
$$

Hence the right member of (3.1) is bounded and the value function of our market making problem is well defined. $\qquad \square$

We can now define the Value function of the trading problem:

$$
V(t, z) := \sup_{\pi} V_{\pi}(t, z), z \in E, t \in [0, T]
$$

where the supremum is taken over all the Markovian admissible policies π, and where:

$$
V_{\pi}(t, z) := \mathbb{E}_{t,z}^{\pi} \left[g(Z_T) - \eta Y_T^2 \right], z \in E, t \in [0, T]
$$

It holds that $V_{\pi}(T, z) = g(z) = V(T, z)$

In order to present the first main result of this article, we need to introduce some notations. These notations come from the new formulation of the market making problem as a Markov Decision Process (MDP). We followed the method of [3] and borrowed its notations to show existence and give a characterization of the value function.

3.2 Markov Decision Process formulation of the trading problem

The idea is to treat the continuous-time control problem as a discrete-time Markov Decision Process where we look at the jump times of the order book as state component.

We consider the Markov Decision Process (MDP) characterized by the following informations

$$\left(\underbrace{[0,T] \times E,}_{\text{state space}} \quad \underbrace{A_z}_{\text{market maker control}} \quad , \quad \underbrace{\lambda}_{\text{intensity of the jump}} \quad , \quad \underbrace{Q}_{\text{transitions kernel}} \quad , \quad \underbrace{r}_{\text{reward}} \right) \text{ such that:}$$

- E is the state space of the controlled order book (Z_t). The elements of E are denoted by $z \in E$. $z = (x, y, \underline{a}, \underline{b}, \underline{na}, \underline{nb}, \underline{ra}, \underline{rb}, pa, pb)$ where $\underline{na}, \underline{nb}$ are the M-dimensional vectors of the ranks of the market maker's orders in the queues ; and $\underline{ra}, \underline{rb}$ are the M-dimensional vectors of the number of ticks the M market maker's orders are from the best opposite quote in the order book.

- for every state $z \in E$ of the LOB, denote by A_z the control action space which are all the actions available for the market maker given the state z of the LOB.
  $A_z = \left\{ \underline{ra}, \underline{rb} \in \{0, ..., K\}^M \times \{0, ..., K\}^M \middle| rb_*, ra_* \geq a0 \right\}$, where, for every vector $\underline{c}$: $c_* = \min_i c_i$; and: $c0 = \arg\min_i (\underline{c}_i)$. The control is the double vector of the positions of the market maker's orders in the order book. The market maker is not allowed to cross the spread. She can chooses on which queues she wants to send orders, among the non empty ones.

- Given a strategy α of market making, the stochastic evolution is given by a marked point process (T_n, Z_n) where (T_n) is the increasing sequence of jump times of the controlled LOB pure jump process with intensity $\lambda(Z_{n-1})$. Just after a jump, at time T_n^+, the process can jump due to the new decision of the market maker. Then it remains constant on $]T_n, T_{n+1}[$. Between the jumps, the process is constant since the market maker does not change her strategy between two jumps.

  We denote by $\phi^\alpha(z) \in E$ the state of the LOB at time t such that $T_n < t < T_{n+1}$, given that $Z_{T_n} = z$ and the strategy α has been chosen by the market maker at time T_n.

- In the sequel, we denote by:
  $([0,T] \times E)^C := \left\{ (t, z, a) \in E \times \{0, \ldots, K\}^{2M} \middle| t \in [0, T], z \in E, a \in A_z \right\}$.
  $E^C := \left\{ (z, a) \in E \times \{0, \ldots, K\}^{2M} \middle| z \in E, a \in A_z \right\}$
  Q' is the stochastic kernel from E^C to E which describes the distribution of the jump goals. ie: $Q'(B|z, u)$ is the probability that the process jumps in the set B given that it was at state $z \in E$ right before the jump, and the control action $u \in A_z$ as been chosen right after the jump time. A admissible policy $\alpha = (\alpha_t)$ is entirely characterized by decision functions $f_n : [0, T] \times E \to A$ such that
  $$\alpha_t = f_n(T_n, Z_n) \text{ for } t \in (T_n, T_{n+1}]$$

In the sequel we write $\alpha = (\alpha_t) = (f_n)$.
The intensity of the controlled process (Z_t) is:

$$\lambda(z) := \lambda^{M^+}(z) + \lambda^{M^-}(z) + \sum_{1 \leq j \leq K} \lambda^{L_j^+}(z) + \sum_{1 \leq j \leq K} \lambda^{L_j^-}(z) + \sum_{1 \leq j \leq K} \lambda^{C_j^+}(z) + \sum_{1 \leq j \leq K} \lambda^{C_j^-}(z)$$

It does not depend on the strategy α chosen by the market maker since we assumed that the market does not "see" the market maker's orders in the LOB. The intensity of the order book process only depends on the vectors $\underline{a}$ and $\underline{b}$.

The transition kernel of the controlled order book, given a state z, is given by:

$$Q'(z'|z, u) = \begin{cases} \frac{\lambda^{M^+}(z)}{\lambda(z)} & \text{if } z' = e^{M^+}(\phi^u(z)) \\ \quad \vdots \\ \frac{\lambda^{C^+}(z)}{\lambda(z)} & \text{if } z' = e^{C_K^+}(\phi^u(z)) \end{cases}$$

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where $\phi^u(z)$ is the new state of the controlled order book when decision u as been taken and when the order book was at state z before the decision.

Also $e^{M^+}(z)$ is the new state of the LOB right after it received a buy market order, given that it was at state z before the jump; $e^{C_i^\pm}(z)$ is the new state of the LOB right after it received a cancel order on its ith ask/bid limit, given that it was at state z before the jump.

Given a policy $\alpha = (f_n)$ and an initial state $z \in E$, we notice:

$$
\begin{aligned}
\mathbb{P}(T_{n+1} - T_n \le t, Z_{n+1} \in B | T_0, Z_0, ... T_n, Z_n) &= \lambda(Z_n) \int_0^t e^{-\lambda(Z_n)s} Q'(B | Z_{T_n}, \alpha_{T_n}) ds \\
&= \lambda(Z_n) \int_0^t e^{-\lambda(Z_n)s} Q'(B | Z_{T_n}, f_n(Z_n)) ds
\end{aligned}
$$

So it is natural to define the stochastic kernel of the MDP as follows:

- For all Borel sets $B \subset \mathbb{R}_+, C \subset E$ and $(t, z) \in [0, T] \times E, \alpha \in A$, the stochastic kernel Q is given by:

$$
Q(B \times C | t, z, \alpha) := \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \mathbb{1}_B(t+s) Q'(C | \phi^\alpha(z), \alpha) ds + e^{-\lambda(z)(T-t)} \mathbb{1}_{T \in B, z \in C}
$$

- The reward function $r : [0, T] \times E^C \to \mathbb{R}$ is defined by:

$$
r(t, z, \alpha) := e^{-\lambda(z)(T-t)} \big(g(z) - \eta y^2\big) \mathbb{1}_{t \le T}
$$

We denote by $(T_n, Z_n)_{n \in \mathbb{N}}$ the corresponding state of the controlled Markov chain.

The cumulated reward function associated to the discrete-time Markov Decision Model for a policy (f_n) is defined as:

$$
V_{\infty,(f_n)}(t, z) = \mathbb{E}_{t,z}^{(f_n)} \left[\sum_{n=0}^{\infty} r(T_n, Z_n, f_n(T_n, Z_n)) \right]
$$

Its value function is defined as:

$$
V_{\infty}(t, z) = \sup_{(f_n) \in AS} V_{\infty(f_n)}(t, z), \quad (t, z) \in [0, T] \times E
$$

where the supremum is taken over all the sequences of borelian functions (f_n).

Proposition 3.1. *For a Markovian policy $\alpha = (f_n)$ we have:*

$$
V_\alpha(t, z) = V_{\infty(f_n)}(t, z), \quad (t, z) \in E'
$$

Proof. Let $H_n := (T_0, Z_0, ..., T_n, Z_n)$. We obtain:

$$
\begin{aligned}
V_\alpha(t, z) &= \mathbb{E}_{tz}^\alpha \left[\sum_{n=0}^{\infty} \mathbb{1}_{[T_n \le T < T_{n+1})} \big(g(Z_T) - \eta Y_T^2\big) \right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}_{tz} \Big[\mathbb{E}_{tz} \big[\mathbb{1}_{T_n \le T < T_{n+1}} \big(g(Z_T) - \eta Y_T^2\big) | H_n \big] \Big] \\
&= \sum_{n=0}^{\infty} \mathbb{E}_{tz}^{(f_n)} \Big[r(T_n, Z_n, f_n(T_n, Z_n)) \Big]
\end{aligned}
$$

$\square$

Property 3.1 implies that the MDP is equivalent to the Markov controlled process (Z_t). The value function of the market making problem coincides with the value function V_∞ of the discrete-time MDP, which is well-defined with regards to property 3.1. The maximal reward mapping for the non finite horizon MDP is given by:

$$
\begin{aligned}
(\mathcal{T}v)(t,z) &:= \sup_{a \in \mathbf{A}_z} \left\{ r(t,z,a) + \int v(t',z')Q(t',z'|t,\phi^a(z),a) \right\} \\
&= \sup_{a \in \mathbf{A}_z} \left\{ e^{-\lambda(z)(T-t)}(g(z) - \eta y^2)\mathbb{1}_{t \leq T} + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s,z')Q'(dz'|\phi^a(z),a)\,ds \right\}
\end{aligned}
$$
(3.2)
$$
= e^{-\lambda(z)(T-t)}(g(z) - \eta y^2)\mathbb{1}_{t \leq T} + \lambda(z) \sup_{\alpha \in \mathbf{A}_z} \left\{ \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s,z')Q'(dz'|\phi^a(z),a) \right\}ds
$$
(3.3)

where we recall that:

- y is the inventory of the market maker

- $g(z)$ is the wealth of the market maker once she closes out her positions.

- $\phi^\alpha(z)$ is the new state of the order book when the decision α is taken and the state of the order book was z before the decision was taken.

- $\lambda(z)$ is the intensity for the next exogenous event of the LOB given that the LOB is at state z..

We can now state the main result of this section:

Theorem 3.1. $\mathcal{T}$ *admits a unique fixed point v which coincides with the value function of the MDP. Moreover we have:*
$$
v = V_\infty = V
$$

Denote by f^* the maximizer of the operator $\mathcal{T}$. Then $(f^*, f^*, ...)$ is an optimal stationary (in the MDP sense) policy. ie : given a time t and a state of order book z, the optimal strategy for the market maker is given by $f^*(t,z)$.

The time dimension of the time-continuous optimal trading problem becomes a state variable when formulating the trading problem as an infinite horizon MDP. The optimal strategy is stationary for the MDP formulation of the problem, but it is not stationary for the original trading problem.

The next subsection is devoted to the proof of this theorem.

3.3 Proof of Theorem 3.1

Recall the following notations $E^C := \left\{ (z,a) \in E \times \{0,\ldots,K\}^{2M} | z \in E, a \in A_z \right\}$ and $([0,T] \times E)^C :=$ $\left\{ (t,z,a) \in [0,T] \times E \times \{0,\ldots,K\}^{2M} | t \in [0,T], z \in E, a \in A_z \right\}$

We first recall the definition of a bounding function of a MDP.

Definition 3.1. *A measurable function $b : E \to \mathbb{R}_+$ is called a* bounding function *for the controlled process (Z_t) if there exists constants c_g, $c_{Q'}, c_\phi$ such that:*

1. $|g(z) - \eta|y|^2| \leq c_g b(z)$ *for all z in E.*

2. $\int b(z')Q'(dz'|z,u) \leq c_{Q'}b(z)$ *for all $(z,u) \in E^C$.*

3. $b(\phi_t^\alpha(z)) \leq c_\phi b(z)$ *for all $(t,z,\alpha) \in ([0,T] \times E)^C$.*

Proposition 3.2. *Take g the wealth of the market maker. Then b, defined for every state z of the order book as: $b(z) := 1 + x^2 + y^2 + |\underline{a}|^2 + |\underline{b}|^2 + |\underline{ra}|^2 + |\underline{rb}|^2 + pa^2 + pb^2 + |\underline{na}|^2 + |\underline{nb}|^2$, is a bounding function for the controlled process (Z_t).*

Proof. Denote by z' the state of the order book after that an exogenous event occurred.

- Assertion 1 of definition 3.1 is obvious since the penalized wealth is sub-quadratic.

- Assertions 2. and 3. of definition 3.1 follow from the facts that:
    - $\underline{ra}, \underline{rb}$ are bounded by $\sqrt{N}K$ (K is the number of limits considered in the model of the order book).
    - $pa' \in B(pa, K), pb' \in B(pb, K)$ where $B(x, r)$ is the ball centered in x with radius $r > 0$
    - $|\underline{a'}| \leq |\underline{a}| + a_\infty K$
    - $\phi^\alpha(z) = z^\alpha$ can differ from z by the components pa, pb; $\underline{na}, \underline{nb}$, and $\underline{ra}, \underline{rb}$. pa^α, pb^α cannot be greater than $pa + K, pb + K$ whatever the decision taken by the market maker. We know that: $|\underline{na}| \leq \sqrt{M}(|\underline{a}| + M)$ and $|\underline{nb}| \leq \sqrt{M}(|\underline{b}| + M)$.
    $|\underline{ra}|, |\underline{rb}|$ are both smaller than $\sqrt{M}K$.

$\square$

Proposition 3.3. *We assumed that $\lambda^{M\pm}$ and $\lambda^{L\pm}$ are at most linear in $\underline{a}$ and $\underline{b}$.*
Let $\Lambda = \sup \{\frac{\lambda^{M\pm}}{|\underline{a}| + |\underline{b}|}, \frac{\lambda^{L\pm}}{|\underline{a}| + |\underline{b}|}, \frac{\lambda^{C\pm}}{|\underline{a}(z)| + |\underline{b}(z)|}\}$ If b is a bounding function for the controlled process, then

$$b(t, z) := b(z)e^{\gamma(z)(T-t)} \text{ with } \gamma(z) = \gamma_0(4K + 2)\Lambda.\left(1 + |\underline{a}| + |\underline{b}|\right) \text{ and } \gamma_0 > 0$$

is a bounding function for the discrete-time MDP. ie : there exists $C > 0$ such that for all $z \in E, t \in [0, T], a \in \mathcal{A}_z$:

$$|r(t, z, \alpha)| \leq c_g b(t, z) \tag{3.4}$$

$$\int b(s, z')Q(ds, dz'|t, z, \alpha) \leq b(t, z)c_\phi c_Q e^{C(T-t)}\frac{1}{1 + \gamma_0} \tag{3.5}$$

Proof. of prop 3.3
Let z' be the state of the order book right after an exogenous jump occurs given that it was at state z before the jump. $z' = (x', y', \underline{a'}, \underline{b'}, \underline{na'}, \underline{nb'}, \underline{ra'}, \underline{rb'})$.
Since $|\underline{a'}| \leq |\underline{a}| + a_\infty K$ and $|\underline{b'}| \leq |\underline{b}| - b_\infty K$, there exists $C > 0$ such that:

$$\gamma(z') \leq \gamma(z) + C \tag{3.6}$$

$$
\begin{aligned}
\int b(s, z')Q(ds, dz'|t, \phi^\alpha(z), \alpha) &= \lambda(z)\int_0^{T-t} e^{-\lambda(z)s}\int b(t + s, z')Q'(dz'|\phi_s^\alpha(z), \alpha)ds \\
&= \lambda(z)\int_0^{T-t} e^{-\lambda(z)s}\int b(z')e^{\gamma(z')(T-(t+s))}Q'(dz'|\phi_s^\alpha(z), \alpha)ds \\
&\leq \lambda(z)\int_0^{T-t} e^{-\lambda(z)s}\int b(z')e^{(\gamma(z)+C)(T-(t+s))}Q'(dz'|\phi_s^\alpha(z), \alpha)ds \\
&\leq \lambda(z)\int_0^{T-t} e^{-\lambda(z)s}e^{(\gamma(z)+C)(T-(t+s))}\int b(z')Q'(dz'|\phi_s^\alpha(z), \alpha)ds \\
&\leq \lambda(z)\int_0^{T-t} e^{-\lambda(z)s}e^{(\gamma(z)+C)(T-(t+s))}c_Q c_\phi b(z)ds \\
&\leq \frac{\lambda(z)}{\lambda(z) + \gamma(z) + C}c_Q c_\phi e^{(\gamma(z)+C)(T-t)}\left(1 - e^{-(T-t)(\lambda(z)+\gamma(z)+C)}\right)b(z) \\
&\leq c_Q c_\phi \frac{\lambda(z)}{\lambda(z) + \gamma(z) + C}e^{C(T-t)}\left(1 - e^{-(T-t)(\lambda(z)+\gamma(z)+C)}\right)b(t, z)
\end{aligned}
$$

We also have:

$$\frac{\lambda(z)}{\lambda(z) + \gamma(z) + C} = \frac{\lambda(z)}{\lambda(z)(1 + \gamma_0) + \gamma_0 \sum_{i \leq K, j \in \{L^{\pm}, C^{\pm}, M^{\pm}\}} \underbrace{\Lambda(|\underline{a}| + |\underline{b}|) - \lambda_i^j)}_{\geq 0} + C}$$

$$\leq \frac{1}{1 + \gamma_0}$$

which leads to the result. $\qquad\square$

Denote by $\|.\|_b$ the *weighted supremum norm* such that for all measurable function $v : E' \to \mathbb{R}$,

$$\|v\|_b := \sup_{(t,z) \in E'} \frac{|v(t,z)|}{b(t,z)}$$

and define the set:

$$\mathbb{B}_b := \left\{ v : E' \to \mathbb{R} | v \text{ is measurable and } \|v\|_b < \infty \right\}$$

Moreover let us define

$$\alpha_b := \sup_{(t,z,\alpha) \in E' \times \mathcal{R}} \frac{\int b(s,z') Q(ds, dz'|t, \phi^{\alpha}(z), \alpha)}{b(t,z)}$$

From the preceding considerations it follows that:

$$\alpha_b \leq c_Q c_\phi \frac{1}{1 + \gamma} e^{CT}$$

By taking: $\gamma = c_Q c_\phi e^{CT}$, we obtain: $\alpha_b < 1$.
From now, we assume that $\alpha_b < 1$.

One can easily check that

$$\|\mathcal{T}v - \mathcal{T}w\|_b \leq \alpha_b \|v - w\|_b \tag{3.7}$$

which means that the discrete-time MDP is contracting.

Let

$$\mathcal{M} := \left\{ v \in \mathbb{B}_b | v \text{ is continuous} \right\}$$

Since b is continuous, $\left(\mathcal{M}, \|.\|_b \right)$ is a Banach space.

$\mathcal{T}$ sends $\mathcal{M}$ to $\mathcal{M}$. Indeed, for all continuous function v in $\mathbb{B}_b$, $(t, z, \alpha) \mapsto e^{-\lambda(z)(T-t)}(g(z) - \eta y^2) \mathbb{1}_{t<T} + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s, z') Q'(dz'|\phi^{\alpha}(z), \alpha) ds$ is continuous on $[0, T] \times E^C$. A_z is finite, so we get the continuity of the application:

$$\mathcal{T}v : (t,z) \mapsto \sup_{\alpha \in A_z} \left\{ e^{-\lambda(z)(T-t)}(g(z) - \eta y^2) \mathbb{1}_{t<T} + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s, z') Q'(dz'|\phi^{\alpha}(z), \alpha) ds \right\}$$

Proposition 3.4. *There exists a maximizer for $\mathcal{T}$. ie: for each function v in $\mathcal{M}$ there exists a borelian function $f : [0, T] \times E \to A$ such that $\forall (t, z) \in E'$:*

$$\mathcal{T}v\Big(t, z, f(t,z)\Big) = \sup_{\alpha \in A} \left\{ e^{-\lambda(z)(T-t)}(g(z) - \eta y^2) \mathbb{1}_{t<T} + \lambda(z) \int_0^{T-t} e^{-\lambda(z)s} \int v(t+s, z') Q'(dz'|\phi^{\alpha}(z), \alpha) ds \right\}$$

Proof. $D^*(t,z) = \left\{ a \in A | \mathcal{T}_a v(t,z) = \mathcal{T}v(t,z) \right\}$ is finite, so it is compact. So $(t,z) \mapsto D^*(t,z)$ is a compact-valued mapping. Since the application $(t, z, a) \mapsto \mathcal{T}_a(t,z) - \mathcal{T}(t,z)$ is continuous, we get that $D^* = \left\{ (t,z,a) \in E'^C | \mathcal{T}_a v(t,z) = \mathcal{T}v(t,z) \right\}$ is borelian. Applying the measurable selection theorem yields to the existence of the maximizer. (see [3] p.352) $\qquad\square$

The last step before proving the theorem is to show that the MDP follows a convergence assumption.

Definition 3.2. *Convergence Assumption (C)*
We say that the non-finite horizon problem follows the Convergence assumption (C) if:

$$\lim_{n\to\infty} \sup_\alpha \mathbb{E}_z^\alpha \left[\sum_{k=n}^\infty |r(t_k, Z_k)| \right] = 0$$

The convergence assumption guarantees that the infinite horizon problem is an approximation of the finite horizon model.

Lemma 3.2. *It holds:*

$$\sup_\alpha \mathbb{E}_{tz}^\alpha \left[\sum_{k=n}^\infty |r(T_k, Z_k)| \right] \leq \frac{\alpha_b^n}{1-\alpha_b} b(t,z)$$

So taking $\alpha_b < 1$ yields to the convergence assumption.

Proof. of Lemma 3.2
By conditioning, one notices that for $k \in \mathbb{N}$, $\mathbb{E}\left[|r(T_k, Z_k)| \right] \leq c_g \alpha_b^k b(t,z)$. Summing leads to the result of the lemma. $\qquad\square$

We can now prove the main result of this paper.

Proof. of Theorem 3.1

- Equation (3.7) and property 3.3 shows that $\mathcal{T}$ is a stable and contracting operator defined on the Banach $\mathcal{M}$.
  By the Banach's fixed point theorem there exists a function $v \in \mathbb{M}$ such that $v = \mathcal{T}v$ and moreover we have $v = \lim_{n\to\infty} \mathcal{T}^n 0$.
  One can see that $\left(\mathcal{T}^n 0\right)_n$ is solution of the following Bellman equation:

$$\begin{cases} v_N &= 0 \\ v_n &= \mathcal{T}v_{n+1} \text{ for } n = 0, 1, ..., N-1 \end{cases} \tag{3.8}$$

  Applying theorem 2.3.7 p.22 in [3] yields to $\mathcal{T}^n 0 \geq \sup_f \mathbb{E}\left[\sum_{k=0}^{n-1} r(t_k, X_k) \right] =: J_n$ where J_n is the value function of the MDP with n stages associated to the previous Bellman Equation.
  Lemma 7.1.4 p.197 in [3] shows that $\left(J_n\right)_n$ converges as $n \to \infty$. We denote by J its limit. And by passing at the limit in the previous inequality we get: $\lim_{n\to\infty} \mathcal{T}^n 0 \geq J$. ie: $v \geq J$

- Let f be the maximizer of $\mathcal{T}$ associated to v. v is the fixed point of $\mathcal{T}$ so we get: $v = \mathcal{T}_f^n(v) \leq \mathcal{T}_f^n 0 + \mathcal{T}_o^n \delta$. Where $\mathcal{T}_o^n \delta = \sup_\alpha \mathbb{E}\left[\sum_{k=n}^\infty r(t_k, Z_k) \right]$ By Lemma 3.2, we know that $\mathcal{T}_o^n \delta \to 0$ as $n \to \infty$. So we get: $v \leq J_f$.

We showed that:

$$V_\infty \leq J \leq v \leq J_f \leq V_\infty$$

so all the inequalities are equalities, and we get the desired result.

$\qquad\square$

The theorem 3.1 characterizes the value function V of the market making problem as:

- a continuous function that belongs to $\mathcal{M}$.

- the unique fixed point of $\mathcal{T}$ in $\mathcal{M}$.

4 Numerical Algorithm

As the reader will see later, it is convenient for quantization reasons to rescale the components $\underline{na}$ and $\underline{nb}$ so that they become bounded. Indeed, this change of variables will make the reward function Lipschitz, which is important to show that the quantized Value function converges to the Value function of the trading problem. From now, we described the controlled order book as follows:

$$Z_t = \left(X_t, Y_t, \underline{a}_t, \underline{b}_t, \frac{na_t}{M(|\underline{a}|+M)}, \frac{nb_t}{M(|\underline{b}|+M)}, pa_t, pb_t, \frac{ra_t}{MK}, \frac{rb_t}{MK}\right)$$

The quantities $\frac{na_t}{M(|\underline{a}|+M)}$ and $\frac{nb_t}{M(|\underline{b}|+M)}$ are bounded by 1. So are the quantities $\frac{ra_t}{MK}$ and $\frac{rb_t}{MK}$.

In section 3.3, we proved that the value function V is characterized as solution of the equation (3.3).

Consider the following ansatz for the value function:

$$V(t,z) = g(z) + \rho(t, z') \tag{4.1}$$

where $z' = \left(y, \underline{a}, \underline{b}, pa, pb, \frac{na}{M(|a|+M)}, \frac{nb}{M(|b|+M)}, \frac{ra_t}{MK}, \frac{rb_t}{MK}\right)$ is z is the state of the controlled order book whose cash component has been omitted, and where ρ a function to determine.

Injecting the new expression of the value function to equation (3.3) leads to:

$$g(z) + \rho(t, z') = e^{-\lambda(z)(T-t)}(g(z) - \eta y^2)\mathbb{1}_{t \leq T} + \sup_{\alpha} \mathbb{E}\left[g(Z_1) + \rho(T_1, Z_1') \Big| T_0 = t, Z_0 = z, \alpha\right]$$

So (4.1) holds with ρ solution of the following equation:

$$\rho(t, z') = \sup_{a} \left\{r_\rho(t, z', a) + \mathbb{E}\left[\rho(T_1, Z_1') \Big| t, z', a\right]\right\}$$

where:

$$r_\rho(t, z', a) = -\eta e^{-\lambda(z')(T-t)}y^2\mathbb{1}_{t \leq T} + (1 - e^{-\lambda(T-t)})\left(\frac{\lambda^{M^+}}{\lambda}\mathbb{1}_{na^a=1, y^a>0}(pa - pb') + \frac{\lambda^{M^-}}{\lambda}\mathbb{1}_{nb^a=1, y^a<0}(pa' - pb)\right)$$

where pa' is the worst price the market maker buys a stock to liquidate her negative inventory, and pb' is the worse price the market maker sells a stock to liquidate her positive inventory.

As previously, one can show that ρ is limit of the sequence of functions $(\rho^{(N)})_{N \in \mathbb{N}}$, where for all $N \in \mathbb{N}$, $\rho^{(N)}$ is solution to the following N-stages problem (P_N):

$$(P_N): \quad \rho^{(N)}(t, z') \quad = \quad \sup_{f} \mathbb{E}\left[\sum_{k=0}^{N} r_\rho\left(T_k, Z_k', f_k(T_k, Z_k')\right) \Big| T_0 = t, Z_0' = z'\right] \tag{4.2}$$

and is characterized as solution of the following Bellman equation (B_N):

$$(B_N): \begin{cases} \rho_N^{(N)} & = \quad 0 \\ \rho_n^{(N)}(t, z) & = \quad r_\rho(t, z') + \sup_{\alpha}\left\{\mathbb{E}\left[\rho_{n+1}^{(N)}(t', Z') | t, z', \alpha\right]\right\} \text{ for } 0 \leq n < N \end{cases} \tag{4.3}$$

One has to compute the backward equation (B_N) to get a good approximation of the value function. (B_N) is slow to compute because the number of states of the MDP is non-countable: $\mathcal{L}(T_n|T_{n-1})$ is an exponential law, and the number of states of Z_n' is exploding when N (the number of exogenous jumps of the order book) goes to infinity. We use a markovian quantization method to compute the backward equation.

We start from the simple observation that $\left(\rho_n^{(N)}\right)_n$ is solution to the following Bellman backward equation:

$$(B_N'): \begin{cases} \rho_N^{(N)} & = \quad 0 \\ \rho_n^{(N)}(t, z) & = \quad \sup_{\alpha}\left\{r_\rho(t, z') + \mathbb{E}\left[\rho_{n+1}^{(N)}(T_{n+1}, Z_{n+1}')|T_n = t, Z_n' = z', I_n = \alpha\right]\right\} \text{ for } 0 \leq n < N \end{cases}$$
$$\tag{4.4}$$

where $(I_n)_n$ is a sequence of *iid* random variables such that $I_n \sim \mathcal{U}(A_z)$. In other words, we have replaced the control α by an exogenous randomized process (I_n) so that (t_n, Z_n, I_n) is a Markov Chain. We choose to take uniform distribution on the space of the available actions, when randomizing the control, because we do not know what is the good decision for the market maker to take.

Problem (B'_N) is a backward equation associated to a Markovian process. We just randomized the control. The next step is quantize the Markov process.

4.1 Quantization algorithm

We first build grids for the quantization. We do as follows:
Set: $\Gamma_0^E = \{z\}$ and $\Gamma_0^T = \{0\}$.
Simulate K processes to generate K couples of $(T_n^k, Z'^k_n)_{k \leq K}$ for every $0 \leq n \leq N$.

- For all $1 \leq n \leq N$, build the set $\Gamma_n^T = \{T_n^k, k \leq K\}$ (for the quantization of the nth jump time T_n).

- For all $1 \leq n \leq N$, build the set $\Gamma_n^E = \{Z'^k_n, k \leq K\}$ (for the quantization of the state Z'_n).

(t_n, Z'_n, I_n) is a Markov chain so there exists F and G (borelian functions), $(\epsilon_k)_k$ (the temporal noise: $\epsilon_k \sim \mathcal{E}(1)$) and $(d_k)_k$ (the state noise) such that $Z'_n = F(Z'_{n-1}, d_n, I_n)$ and $t_n = G(t_{n-1}, \epsilon_n, I_n)$.
Denote by $(\hat{t}_n, \widehat{Z'}_n)_{n \in \{0, N\}})$ the process such that: $\widehat{Z'}_0 = z$, and $\forall 1 \leq n \leq N$,

$$\hat{t}_n = Proj\Big(G(\hat{t}_{n-1}, \epsilon_n, I_n), \Gamma_n^T\Big), \qquad \widehat{Z'}_n = Proj\Big(F(\widehat{Z'}_{n-1}, d_n, I_n), \Gamma_n^E\Big)$$

We denote by $|\Delta_k|_2$ the L^2-error of quantization:

$$|\Delta_k|_2^2 = \left|F(\widehat{Z'}_{k-1}, d_k, I_{k-1}) - \widehat{Z'}_k\right|_2^2 + \left|G(\hat{t}_{k-1}, \epsilon_k, I_{k-1}) - \hat{t}_k\right|_2^2$$

By Zador theorem, it holds: $|\Delta_k|_2 \sim \frac{1}{K^{1/d}}$ where d is the dimension of the controlled order book.
$(\hat{t}_n, \widehat{Z}_n, I_n)_{n \in \{0, N\}})$ is a Markov chain. Its probability transition matrix at time $k = 1, ..., n$ reads:

$$\hat{p}_k^{ij}(a) = \mathbb{P}\Big[\hat{t}_k = t_k^j, \widehat{Z'}_k = z_k^j \Big| \hat{t}_{k-1} = t_{k-1}^i, \widehat{Z'}_{k-1} = z_{k-1}^i, I_k = a\Big] = \frac{\hat{\beta}_k^{ij}}{\hat{p}_{k-1}^i}, \quad i = 1, ..., N_{k-1}, j = 1, ..., N_k, a \in \mathcal{A}$$

where:

$$\hat{p}_{k-1}^i = \mathbb{P}[\hat{t}_{k-1} = t_{k-1}^i, \widehat{Z'}_{k-1} = z'^i_{k-1}] = \begin{cases} \mathbb{P}\big[F(\hat{t}_{k-2}, \widehat{Z'}_{k-2}, \epsilon_{k-1}, d_{k-1}) \in C_i(\Gamma_{k-1} \times \Gamma_{k-1}^E)\big] & \text{if} \quad k \geq 2 \\ 1 & \text{if} \quad k = 1 \end{cases}$$

$$\hat{\beta}_k^{ij} = \mathbb{P}\Big[\hat{t}_{k-1} = t_{k-1}^i, \widehat{Z'}_{k-1} = z_{k-1}^i, \hat{t}_k = t_k^j, \widehat{Z'}_k = z'^j_k\Big]$$

$$= \begin{cases} \mathbb{P}\big[F_k(\hat{t}_{k-2}, \widehat{Z'}_{k-2}, \epsilon_{k-1}, d_{k-1}) \in C_i(\Gamma_{k-1} \times \Gamma_{k-1}^E); F_k(\hat{t}_{k-1}, \widehat{Z'}_{k-1}, \epsilon_k, d_k) \in C_i(\Gamma_k \times \Gamma_k^E)\big] & \text{if } k \geq 2 \\ 1 & \text{if } k = 1 \end{cases}$$

and where, for all $i, 0 \leq i \leq K$, for all $k \in \mathbb{N}$, we denoted by $C_i(\Gamma_k \times \Gamma_k^E)$ the Voronoï cell associated to the point (t_k^i, z_k^i).

Consider $(\widehat{V}_n^{(N,K)})_{n \in [1,N]})$, the solution of the Bellman equation associated to the quantized problem:

$$(\widehat{B}_{N,K}) : \begin{cases} \widehat{\rho}_N^{(N,K)} & = \quad 0 \\ \widehat{\rho}_n^{(N,K)}(t, z') & = \quad r(t, z) + \sup_\alpha \left\{ \mathbb{E}\big[\rho_{n+1}^{(N,K)}(\widehat{T}_{n+1}, \widehat{Z'}_{n+1}) | \widehat{T}_n = t, \widehat{Z'}_n = z', I_{n+1} = \alpha\big]\right\} \end{cases} \text{ for } 0 \leq n < N$$

$$(4.5)$$

Before stating the main result of this section, we make the following remark.

Remark 4.1. *Let:* $h(t) = 1 - e^{-\lambda(T-t)}$ *and* $u(z') = \frac{\lambda^{M^+}}{\lambda} \mathbb{1}_{na^a=1,y^a>0}(pa - pb') + \frac{\lambda^{M^-}}{\lambda} \mathbb{1}_{nb^a=1,y^a<0}(pa' - pb)$.

Fix $N \geq 1$ the number of jumps of the order book considered for the approximation. Let $\mathcal{K}$ be the set of all the states that can take the controlled order book by jumping less than N times. The reward function u is bounded by K and Lipschitz on $\mathcal{K}$, with $|u|_L \leq K(|a| + |b| + N)$, where K is the number of limits that describes the bid and ask sides of the order book.

So r_ρ is Lipschitz on $\mathcal{K}$ and its Lischitz constant depends on N. Precisely:

$$|r_\rho|_L \leq \Lambda(|a| + |b| + N)\eta(|y| + N)^2 + 2\eta(|y| + N) + |u|_L + \Lambda(|a| + |b| + N)K$$

The following Theorem 4.1 is the main result of section 4. It states that the value function associated to the quantized process gets close to the value function of the market making problem when the number of points selected for the quantization goes to infinity.

Theorem 4.1. *Fix $N \geq 1$ and assume that the intensity function admits a non-negative global minimum.*

Then, the quantized value function converges to the value function of the trading problem as the number of points chosen for the quantization goes to infinity.

Denote $U_k = \rho_k^{(N)}(t_k, Z'_k)$ and $\widehat{U}_k = \widehat{\rho}_k^{(N,K)}(\widehat{t}_k, \widehat{Z'}_k)$. Then we get:

$$\left| U_k - \widehat{U}_k \right|_2 \leq |r_\rho|_L \sum_{j=k}^{N-1} |\widetilde{\Delta}_j|_2$$

where: $\forall k \leq N, \quad |\widetilde{\Delta}_k|_2 = \left| Z'_k - \widehat{Z'}_k \right|_2 + \left| t_k - \widehat{t}_k \right|_2$

Moreover one can find a constant C' such that, $\forall k \in \{0, ..., N\}$,

$$\left| \widetilde{\Delta}_k \right|_2 \leq \sum_{j=0}^{k} C'^{k-j} |\Delta_j|_2$$

where $|\Delta_k|_2 = \left| F_k(\widehat{Z'}_{k-1}, d_k) - \widehat{Z'}_k \right|_2 + \left| F_k(\widehat{t}_{k-1}, \epsilon_k) - \widehat{t}_k \right|_2$ *is the error of quantization.*

Finally:

$$\left| U_k - \widehat{U}_k \right|_2 \leq |r_\rho|_L \sum_{j=k}^{N-1} \sum_{i=0}^{j} C'^{j-i} |\Delta_i|_2$$

$$\leq \frac{|r_\rho|_L}{1 - C'} \sum_{i=0}^{N-1} \left(1 - C'^{N-i}\right) |\Delta_i|_2$$

By theorem 4.1, we get:

$$|\rho - \rho^{(N,K)}| \leq |\rho - \rho^{(N)}| + |\rho^{(N)} - \rho^{(N,K)}|$$

$$\lesssim \frac{\alpha^N}{1 - \alpha} b(t, z) + |r_\rho|_L N C^N \frac{1}{K^{1/d}}$$

One can choose K big enough to make the bound as small as necessary.

We now have an efficient way to compute the value function and compute the optimal strategies. The method is summarized in figure 3.

The next subsection is devoted to the proof of the theorem 4.1.

4.2 Proof of Theorem 4.1

$(Z'_k, I_k)_k$ is a Markov chain. As done previously, one can find F and a sequence of noises (b_k) such that:

$$Z'_k = F(Z'_{k-1}, b_k)$$

Let (b'_k) be the state noise associated to $\widehat{Z'}_k$ such that: $b'_k = b_k$ if $Z'_{k-1} = \widehat{Z'}_{k-1}$ and: $b'_k \perp\!\!\!\perp b_k$ otherwise.

We assumed that the intensity function admits a global minimum, and this function takes its value on a grid, so $z \mapsto \frac{1}{\lambda(z)}$ is Lipschitz. Denote by L the Lipschitz constant of this function.
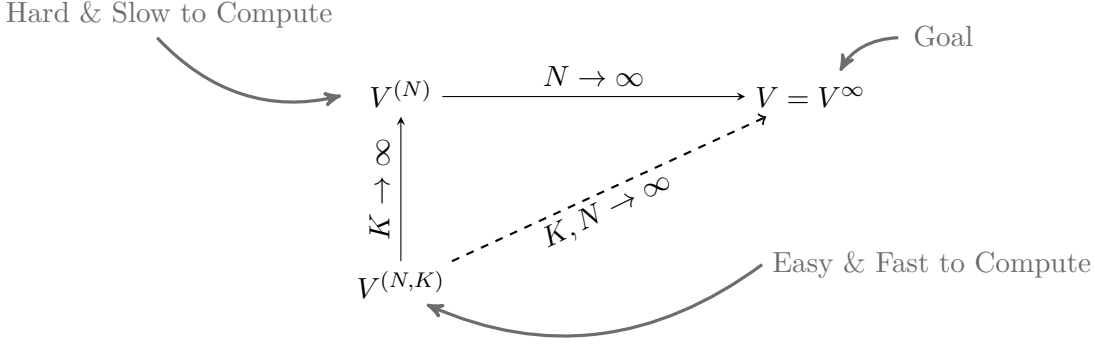
Figure 3: An efficient way to compute V and the optimal strategies

Lemma 4.1. *It holds:*

$$\left|\widetilde{\Delta}_k\right|_2 \leq \sum_{j=0}^{k} C'^{k-j}|\Delta_j|_2 \tag{4.6}$$

where $C' = b_\infty K + a_\infty K + 1 + \sqrt{2}L$ and $|\Delta_k|_2 = \left|F(\widehat{Z'}_{k-1}, b'_k) - \widehat{Z'}_k\right|_2 + \left|G(\widehat{T}_{k-1}, \epsilon_k) - \widehat{T}_k\right|_2$ is the error of quantization.

Proof.

$$\left|Z'_k - \widehat{Z'}_k\right|_2 \leq \left|F_k(Z'_{k-1}, b_k) - F_k(\widehat{Z'}_{k-1}, b'_k)\right|_2 + \left|F_k(\widehat{Z'}_{k-1}, b'_k) - Proj\left(F_k(\widehat{Z'}_{k-1}, b'_k), \Gamma_k\right)\right|_2$$

$$\leq (b_\infty K + a_\infty K + 2)\left|\widehat{Z'}_{k-1} - Z'_{k-1}\right|_2 + \left|F_k(\widehat{Z'}_{k-1}, b'_k) - Proj\left(F_k(\widehat{Z'}_{k-1}, b'_k), \Gamma_k\right)\right|_2 \tag{4.7}$$

Recall $\epsilon_k \sim \mathcal{E}(1)$ *iid* , so we have:

$$|t_k - \hat{t}_k|_2 = \left|t_{k-1} + \frac{1}{\lambda(Z'_{k-1})}\epsilon_k - Proj\left(\hat{t}_{k-1} + \frac{1}{\lambda(\hat{Z}'_{k-1})}\epsilon_k, \Gamma_k^T\right)\right|_2$$

$$\leq \left|t_{k-1} + \frac{1}{\lambda(Z'_{k-1})}\epsilon_k - \hat{t}_{k-1} - \frac{1}{\lambda(\hat{Z}'_{k-1})}\epsilon_k\right|_2 + \left|Proj\left(\hat{t}_{k-1} + \frac{1}{\lambda(\hat{Z}'_{k-1})}\epsilon_k, \Gamma_k^T\right) - \hat{t}_{k-1} - \frac{1}{\lambda(\hat{Z}'_{k-1})}\epsilon_k\right|_2$$

$$\leq |t_{k-1} - \hat{t}_{k-1}|_2 + \left|\frac{1}{\lambda(Z'_{k-1})} - \frac{1}{\lambda(\widehat{Z'}_{k-1})}\right|_2\sqrt{2} + \left|Proj\left(\hat{t}_{k-1} + \frac{1}{\lambda(\widehat{Z'}_{k-1})}\epsilon_k, \Gamma_k^T\right) - \hat{t}_{k-1} - \frac{1}{\lambda(\widehat{Z'}_{k-1})}\epsilon_k\right|_2$$

Then, we have:

$$|t_k - \hat{t}_k|_2 \leq |t_{k-1} - \hat{t}_{k-1}|_2 + \sqrt{2}L\left|\widehat{Z'}_{k-1} - Z'_{k-1}\right|_2 + \left|Proj\left(\hat{t}_{k-1} + \frac{1}{\lambda(\widehat{Z'}_{k-1})}\epsilon_k, \Gamma_k^T\right) - \hat{t}_{k-1} - \frac{1}{\lambda(\hat{Z}'_{k-1})}\epsilon_k\right|_2 \tag{4.8}$$

Combining (4.7) and (4.8), we get:

$$\left|\widetilde{\Delta}_k\right| \leq C'|\widetilde{\Delta}_{k-1}| + |\Delta_k|_2$$

By induction, we can complete the proof. $\square$

We can now proceed to the proof of the theorem 4.1.

Proof. Let $U_{n,N} = \sup\limits_{\alpha} \mathbb{E}_{\alpha,z}\Big[\sum_{k=n}^{N} r_\rho(T_k, Z'_k)\Big]$, and for all sequence of decisions α, let $U_{n,N}^\alpha = \mathbb{E}_{\alpha,z}\Big[\sum_{k=n}^{N} r_\rho(T_k^\alpha, Z'^\alpha_k)\Big]$. Also, let $\widehat{U}_{n,N} = \sup\limits_{\alpha} \mathbb{E}_{\alpha,z}\Big[\sum_{k=n}^{N} r_\rho(\widehat{T}_k, \widehat{Z'}_k)\Big]$, and for all sequence of Borelian decisions α, let: $\widehat{U}_{n,N}^\alpha = \mathbb{E}_{\alpha,z}\Big[\sum_{k=n}^{N} r_\rho(\widehat{T}_k, \widehat{Z'}_k)\Big]$

We have:

$$|U_{n,N}^\alpha - \widehat{U}_{n,N}^\alpha| = \mathbb{E}_{\alpha,z}\Big[\sum_{k=n}^{N} r_\rho(T_k, Z'_k) - r_\rho(\widehat{T}_k, \widehat{Z'}_k)\Big]$$

By taking the supremum over all the admissible strategies, we get:

$$|U_{n,N} - \widehat{U}_{n,N}| \leq 2|r_\rho|_L \sup\limits_{\alpha} \sum_{k=n}^{N} |\widetilde{\Delta}_k| \tag{4.9}$$

(4.6) and (4.9) lead to the result of theorem 4.1. $\qquad\square$

4.3 Numerical results

We tested the efficiency of the algorithm on simulated order books. The intensities have been taken constant or state dependent to simulate the order books. Although we decided not to controlled the intensities of the arriving processes for predictivity reasons, the intensities are controlled for the numerical tests. In other words, the oder book can see the orders of the market maker. The optimal trading strategies has been computed among two different classes of strategies. In section 4.3.1 we tested the optimal trading strategy among the strategies where the market maker allows herself to place orders only on the best limit on each side of the order book. The dynamics of the controlled order book are available in appendix for this class of control. In section 4.3.2, we computed the optimal trading strategy among the class of the strategies where the market maker allows herself to place orders on the two best limits on each side of the order book.

One can notice that the time-consuming part of the algorithm is the projection of the points to the grids for the quantization. To make it fast, we implemented the algorithm using the fast approximate nearest neighbors algorithm [18].

4.3.1 Case 1: The market maker can take positions on the best ask and best bid exclusively

We want to compare the optimal strategy with the naive strategy 11 "I am always placed at the bid and at the ask limit". We restrict the market maker to take positions only on the best bid and best ask queues of the order book.
In figure 7, we plotted the empirical histogram of the P&L of the market maker using the optimal strategy computed with $N, N \in \{1000, 10000, 100000, 1000000\}$ samples for the quantizations, and the empirical histogram of the P&L of the market maker using the strategy 11 ("I am always placed at the first buying and selling limit").
One can see that the taller the size of the grids are, the better is the computed optimal strategy.

In figure 4, we took constant intensities to model the limit orders processes and the market orders processes. We used linear intensity to model the cancellation orders processes. In this situation, the naive strategy consisting in always placing orders at the best limits of each side of the order book is a good strategy. Indeed, the market is symmetrical, so the market maker earns the spread in average. As we can see in the figure, the efficiency of the computed optimal strategy is similar to the one of the naive strategy.

In figure 8, we took intensities that depend on the size of the queues to model the orders arrivals processes. In this setting, the naive strategy does not perform well anymore. The computed optimal strategy does well.
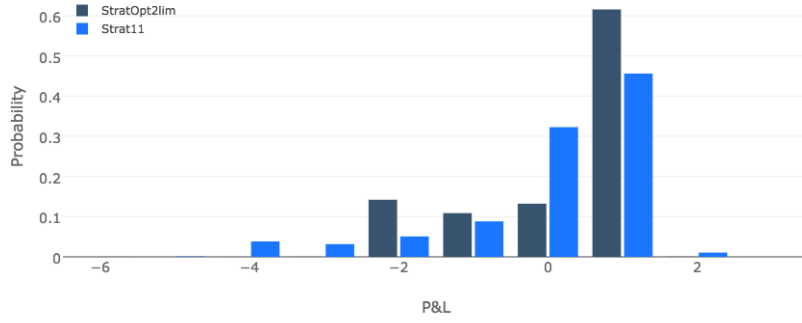
Figure 4: Symmetrical intensities. Short terminal time. The optimal strategy reduces the variance of the P&L. However, choosing 100000 points for the quantization, the expected wealth of the market maker following the computed optimal strategy is still a bit smaller than the one obtained when she follows the naive strategy.
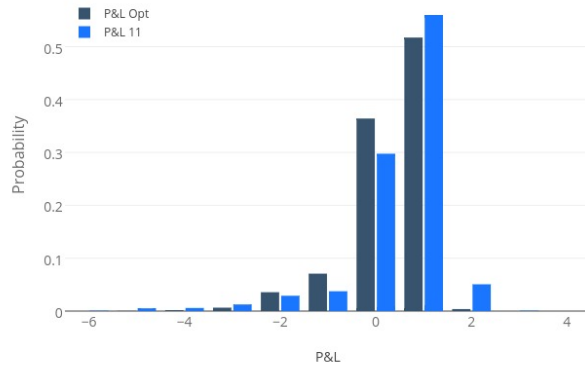

Figure 5: Symmetrical and constant intensities. 6000 points for the quantization. The computed optimal strategy is less efficient than the naive strategy.
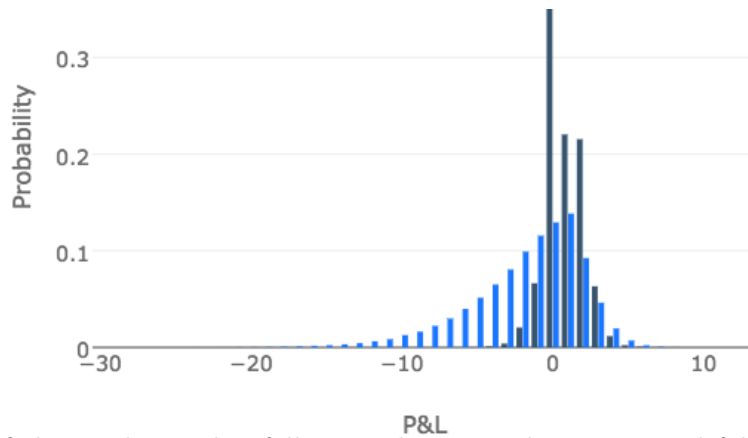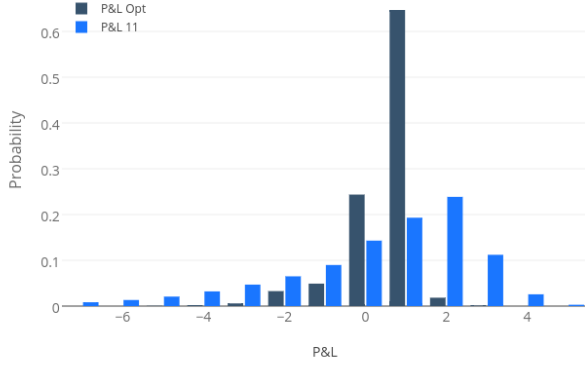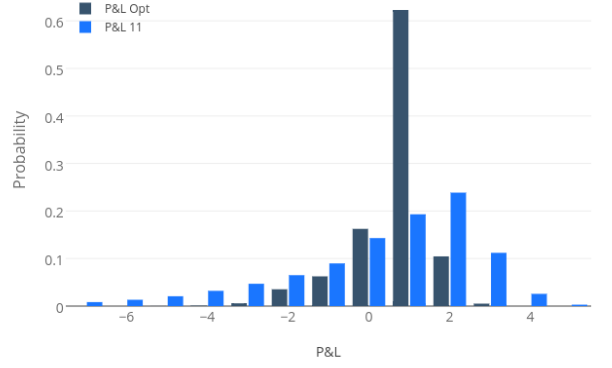

Figure 6: P&L of the market maker following the optimal strategy and following the naive strategy 11. Long Terminal Time. We notice that the computed optimal strategy does better than the naive strategy when the intensities are state dependent.
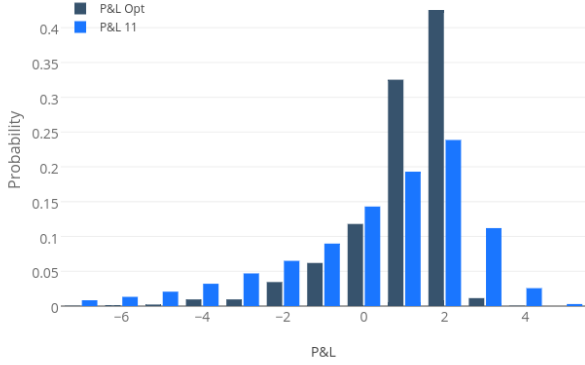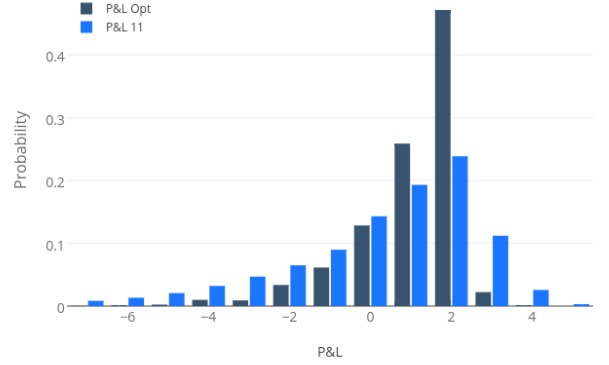
Figure 7: P&L when the intensities λ^M , λ_i^L and λ_i^C depend on the state of the order book. Figure (a) shows the P&L of the market maker when following the optimal strategy computed with 1000 points for the quantization. Figure (b) shows the P&L when following the optimal strategy computed with 9000 points for the quantization. Figure (c) shows the P&L when following the optimal strategy computed with 100000 points for the quantization. Figure (d) shows the P&L when following the optimal strategy computed with 1000000 points for the quantization.

The reader can see that the market maker increases her expected terminal wealth by taking more and more points for the quantization. Also, the naive strategy is beaten when the intensities are state dependent.
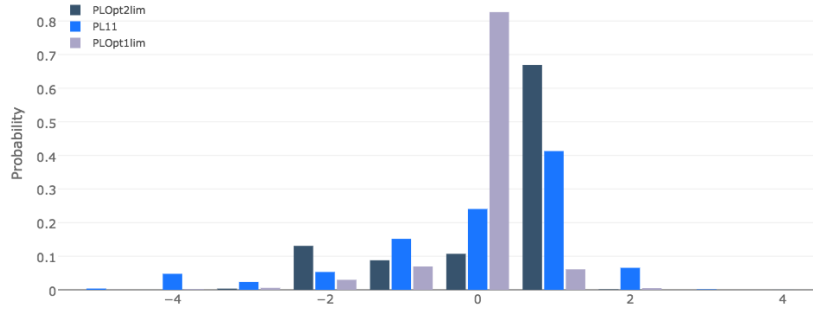
Figure 8: P&L of the market maker who follows optimal strategies and a naive strategy. Short Terminal Time. asymmetrical intensities for the market order arrivals: the intensity for the buying market order process is taken higher than the one for the selling market order process. The wealth of the market maker is greater when she places orders on the two first limits of each sides of the order book, rather than when she places orders only on the best limits at the bid and ask sides.
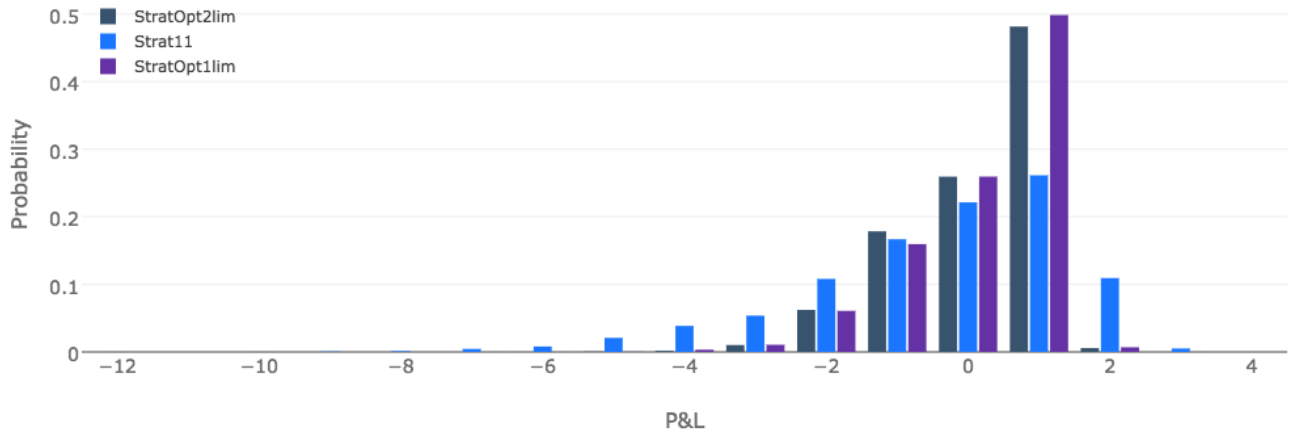

Figure 9: P&L of the market maker who follows the optimal strategy and following the naive strategy 11. Long Terminal Time. asymmetrical intensities for the arrival of market orders. 400000 points for the quantization. We notice that the optimal strategy of market making computed on the two first limits of the LOB (StratOpt2lim) still does not perform better than the one computed on the first limit of the bid side and ask side (StratOpt1lim).

4.3.2 Case 2: the market maker can take positions on the first two limits of the Orders Book

We simulated the LOB with state dependent intensities. We then tested different strategies of market making on these order books.

In figure 8, we plotted the empirical distribution of the P&L of the market maker who adopted three different strategies:

- "follow the optimal strategy to take positions in the two first limits of each side of the LOB" (PLOpt2lim)

- "follow the optimal strategy to take positions in the best ask and best bid queues of the LOB" (PLOpt1lim)

- "always take positions on the best bid and best ask queues" (PL11) (Naive Strategy)

One can see that the optimal P&L of the market maker is better when the market maker decides to send orders at other prices than the best ask or the best bid.
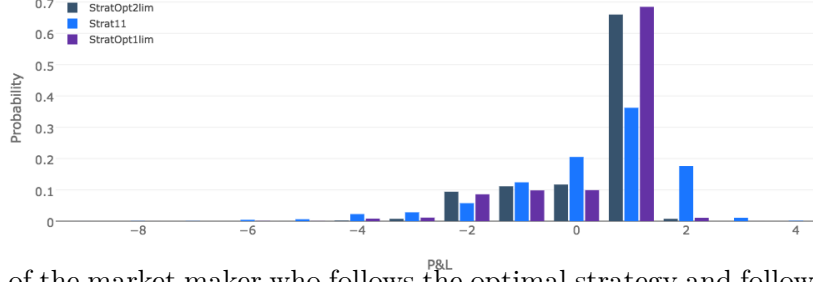
Figure 10: P&L of the market maker who follows the optimal strategy and following the naive strategy 11. Long Terminal Time. Constant and symmetrical intensities for the arrival of market orders. We notice that the optimal strategy computed on the 2 first limits is not more efficient than the one computed on the two best limits of the LOB exclusively. So placing orders in the best limit exclusively seems to be the the optimal strategy in this setting.

5   Appendix

5.1   Dynamics of the controlled order book (simplified version)

In this section, we give the expressions for the dynamics of the controlled order book process (Z_t). The market maker control has been simplified to a couple (la_t, lb_t), where $la = 1$ (resp. 0) if the market maker holds (does not hold) a sell order at the best ask limit. $lb = 1$ (resp. 0) if the market maker holds (does not hold) a buying order at the best bid limit. So to speak, the market maker considers to place orders at the best ask limit or at the best bid limit exclusively.

To do the numerical simulation, we also had to write the dynamic of the controlled order book for the set of generalized controls in which the market maker is allowed to post orders on the two first limits at the bid and at the ask side. The expression of the dynamics for the generalized controls are very similar to the ones for the simplified controls.

To understand the dynamic of the rank of the orders of the market maker, we need a model for the cancellation of orders.

Suppose for example that the market maker holds an order whose rank is na in the queue, with $na < a_{A^{-1}(0)}$. Suppose that the cancel process $L_{A^{-1}(0)}^{C^+}$ jumps. Then two scenarios can occur:

- If the rank of the canceled order is greater than the one of the market maker, then na_t stays constant.

- If the rank of the canceled order is smaller than the one of the market maker, then $na_t = na_{t^-} + 1$.

Model:

We consider a Bernoulli variable X^a with parameter: $\underbrace{\dfrac{na - 1}{a_{A^{-1}(0)}}}_{\alpha} \delta_1 + \underbrace{\dfrac{a_{A^{-1}(0)} + 1 - na}{a_{A^{-1}(0)}}}_{\beta} \delta_0$.

We assume that the canceled order is in front of the market maker's order in the queue if $X^a = 1$, and behind it if $X^a = 0$.

We proceed for the bid side as we just did for the ask side. We consider a random variable X^b following a Bernoulli law with parameter: $\underbrace{\dfrac{nb - 1}{|b_{B^{-1}(0)}|}}_{\alpha} \delta_1 + \underbrace{\dfrac{|b_{B^{-1}(0)}| + 1 - nb}{|b_{B^{-1}(0)}|}}_{\beta} \delta_0$.

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5.2 Dynamics of the processes in (Z_t)

5.2.1 Dynamics of X_t et Y_t

The dynamic of the amount hold by the market maker on a no-interest-bearing account $(X_t)_{t \in \mathbb{R}_+}$ is as follows:

$$dX_t = la_t pa_{t-} \mathbb{1}_{\{na_{t-}=1\}} dM_t^+ - lb_t pb_{t-} \mathbb{1}_{\{nb_{t-}=1\}} dM_t^-$$

The market maker's inventory (Y_t) follows the dynamic:

$$dY_t = -la_t \mathbb{1}_{\{na_{t-}=1\}} dM_t^+ + \mathbb{1}_{\{nb_{t-}=1\}} lb_t dM_t^-$$

where:

- $\hat{a} = sup\{a_i : \sum_{j=1}^{i-1} a_j = 0\}$ et $\hat{b} = sup\{b_i : \sum_{j=1}^{i-1} b_j = 0\}\}$

- $M_t^{\pm}$ are Cox processes with intensities $\lambda^{M^{\pm}}$

5.2.2 Dynamics of the a_i et b_i, $i \in \{1, \ldots, K\}$

We remind that a_i is the number of orders located i ticks away from the best buy order.
We denote by $\mathcal{J}$ the *shift* operator that re-index a side of the book when an event occurred on the opposite side.

$\mathcal{J}^{L_i^-}, i \in \{1, \ldots, B^{-1}(0)\}$ is the shift operator that shifts the bid side due to the jump of a L_i^+ for $i \in \{0, K\}$. We get:

$$\mathcal{J}^{L_i^-}(\underline{a}) = \left(a_{i+1}, ..., a_K, \underbrace{a_\infty, \ldots, a_\infty}_{i \text{ fois}} \right)$$

Dynamic of a_i:

$$
\begin{aligned}
da_i &= (1 - lb_t)dL_i^+ + lb_t dL_{i-(A^{-1}(0)-rb_{t-})}^+ + \left[(1 - lb_t) + lb_t \mathbb{1}_{\{nb_{t-}>1\}} \right] \left(\mathcal{J}^{M^-}(a_i) - a_i \right) dM^-(t) \\
&- (1 - lb_t)dC_i^+ - lb_t dC_{i-(A^{-1}(0)-rb_{t-})}^+ \\
&+ (1 - la_t) \Bigg[-\mathbb{1}_{\{i=A^{-1}(0)\}} dM_t^+ + (\mathcal{J}^{C^-}(a_i) - a_i) dC_{A^{-1}(0)}^- \\
&\qquad + (1 - lb_t) \sum_{j=1}^{A^{-1}(0)-1} (\mathcal{J}_{0,0}^{L_j^-}(a_i) - a_i) dL_j^-(t) \\
&\qquad + lb_t \sum_{j=1}^{A^{-1}(0)-1} (\mathcal{J}_{0,1}^{L_j^-}(a_i) - a_i) dL_j^-(t) \Bigg] \\
&+ la_t \Bigg[-\mathbb{1}_{\{na_{t-}>1\}} \mathbb{1}_{\{i=A^{-1}(0)\}} dM_t^+ + (\mathcal{J}^{C^-}(a_i) - a_i) dC_{ra_{t-}}^- \\
&\qquad + lb_t \sum_{j=1}^{ra_{t-}-1} (\mathcal{J}_{1,1}^{L_j^-}(a_i) - a_i) dL_j^-(t) \\
&\qquad + (1 - lb_t) \sum_{j=1}^{ra_{t-}-1} (\mathcal{J}_{1,0}^{L_j^-}(a_i) - a_i) dL_j^-(t) \Bigg]
\end{aligned}
$$

with $\mathcal{J}$ such that:

$$
\mathcal{J}^{C^-}(a_i) = \begin{cases}
a_\infty & \text{si } i > B^{-1}(1) - B^{-1}(0) + K \\
a_{i-\left(B^{-1}(1)-B^{-1}(0)\right)} & \text{si } i > (B^{-1}(1) - B^{-1}(0)) \\
0 & \text{si } i \leq B^{-1}(1) - B^{-1}(0)
\end{cases}
$$

$$\mathcal{J}^{M^-}(a_i) = \begin{cases} a_{i-\left(B^{-1}(1)-B^{-1}(0)\right)} & \text{si } i > \left(B^{-1}(1)-B^{-1}(0)\right) \\ 0 & \text{si } i \leq B^{-1}(1)-B^{-1}(0) \end{cases}$$

$$\mathcal{J}^{L_j^-}_{0,0}(a_i) = \begin{cases} a_{i+j} & \text{si } i+j \leq K \\ 0 & \text{si } i+j < K \end{cases}$$

$$\mathcal{J}^{L_j^-}_{0,1}(a_i) = \begin{cases} a_{i+rb_{t-}-j} & \text{si } i+rb_{t-}-j \leq K \\ a_\infty & \text{si } i+rb_{t-}-j > K \end{cases}$$

$$\mathcal{J}^{L_j^-}_{1,0}(a_i) = \begin{cases} a_{i+ra_{t-}-j} & \text{si } i+ra_{t-}-j \leq K \\ a_\infty & \text{si } i+ra_{t-}-j > K \end{cases}$$

$$\mathcal{J}^{L_j^-}_{1,1}(a_i) = \begin{cases} 0 & \text{si } i+rb_{t-}-j < 0 \\ a_{i+rb_{t-}-j} & \text{si } i+rb_{t-}-j \leq K \\ a_\infty & \text{si } i+rb_{t-}-j > K \end{cases}$$

We remind that b_i is the number of buy order located i ticks away from the best sell order.

<u>Dynamic of b_i:</u>

$$
\begin{aligned}
db_i =\ & -(1-la_t)dL_i^- - la_t dL_{i-(A^{-1}(0)-ra_{t-})}^- + \left[(1-la_t) + la_t \mathbb{1}_{\{na_{t-}>1\}}\right]\left(\mathcal{J}^{M^+}(b_i) - b_i\right)dM^+(t) \\
& + (1-la_t)dC_i^- + la_t dC_{i-(A^{-1}(0)-ra_{t-})}^- \\
& + (1-lb_t)\Bigg[\mathbb{1}_{\{i=A^{-1}(0)\}}dM_t^- + (\mathcal{J}^{C^+}(b_i) - b_i)dC_{A^{-1}(0)}^+ \\
& \qquad\qquad + (1-la_t)\sum_{j=1}^{B^{-1}(0)-1}(\mathcal{J}^{L_j^+}_{0,0}(b_i) - b_i)dL_j^+(t) \\
& \qquad\qquad + la_t\sum_{j=1}^{B^{-1}(0)-1}(\mathcal{J}^{L_j^+}_{1,0}(b_i) - b_i)dL_j^+(t)\Bigg] \\
& + lb_t\Bigg[\mathbb{1}_{\{nb_{t-}>1\}}\mathbb{1}_{\{i=A^{-1}(0)\}}dM_t^- + (\mathcal{J}^{C^+}(b_i) - b_i)dC_{rb_{t-}}^+ \\
& \qquad\qquad + la_t\sum_{j=1}^{rb_{t-}-1}(\mathcal{J}^{L_j^+}_{1,1}(b_i) - b_i)dL_j^+(t) \\
& \qquad\qquad + (1-la_t)\sum_{j=1}^{rb_{t-}-1}(\mathcal{J}^{L_j^+}_{0,1}(b_i) - b_i)dL_j^+(t)\Bigg]
\end{aligned}
$$

with $\mathcal{J}$ the shift operators:

$$\mathcal{J}^{C^+}(b_i) = \begin{cases} b_\infty & \text{si } i+A^{-1}(1)-A^{-1}(0) > K \\ b_{i-\left(A^{-1}(1)-A^{-1}(0)\right)} & \text{si } i > \left(A^{-1}(1)-A^{-1}(0)\right) \\ 0 & \text{si } i \leq A^{-1}(1)-A^{-1}(0) \end{cases}$$

$$\mathcal{J}^{M^+}(b_i) = \begin{cases} b_\infty & \text{si } i+A^{-1}(1)-A^{-1}(0) > K \\ b_{i-\left(A^{-1}(1)-A^{-1}(0)\right)} & \text{si } i > \left(A^{-1}(1)-A^{-1}(0)\right) \\ 0 & \text{si } i \leq A^{-1}(1)-A^{-1}(0) \end{cases}$$

$$\mathcal{J}^{L_j^+}_{0,0}(b_i) = \begin{cases} b_{i+j} & \text{si } i+j \leq K \\ 0 & \text{si } i+j > K \end{cases}$$

$$\mathcal{J}_{1,0}^{L_j^+}(b_i) = \begin{cases} b_{i-j+A^{-1}(0)} & \text{si } i-j+A^{-1}(0) \leq K \\ b_\infty & \text{si } i-j+A^{-1}(0) > K \end{cases}$$

$$\mathcal{J}_{0,1}^{L_j^+}(b_i) = \begin{cases} b_{i+rb_{t-}-j} & \text{si } i+rb_{t-}-j \leq K \\ b_\infty & \text{si } i+rb_{t-}-j > K \end{cases}$$

$$\mathcal{J}_{1,1}^{L_j^+}(b_i) = \begin{cases} 0 & \text{si } i+rb_{t-}-j < 0 \\ b_{i+rb_{t-}-j} & \text{si } i+rb_{t-}-j \leq K \\ b_\infty & \text{si } i+rb_{t-}-j > K \end{cases}$$

5.2.3 Dynamics of na_{t-} and nb_{t-}

<u>Dynamic of na_t:</u>

X^a has been introduced in part 5.1. It models whether the canceled order is behind or in front of the market maker's order in the queue.

We get:

$$
\begin{aligned}
dna_t = la_t \Bigg[&-X^a\Big((1-lb_t)dC_{A^{-1}(0)}^+(t) + lb_t dC_{rb_{t-}}^+\Big) + \Big(-\mathbb{1}_{\{na_{t-}>1\}} + (a_{A^{-1}(0)}+1-na_{t-})\mathbb{1}_{\{na_{t-}=1\}}\Big)dM_t^+ \\
&+ (2-na_{t-})\Big\{(1-lb_t)\sum_{i=1}^{ra_{t-}(t)-1} dL_i^+ + lb_t \sum_{i=1}^{ra_{t-}(t)-(A^{-1}(0)-rb_{t-})-1} dL_i^+\Big\}\Bigg] \\
&+ (1-la_t)\Bigg[\Big(a_{A^{-1}(0)}\mathbb{1}_{\{a_{A^{-1}(0)}>1\}} + (a_{A^{-1}(1)}+1)\mathbb{1}_{\{a_{A^{-1}(0)}=1\}} - na_{t-}\Big)dM_t^+ \\
&+ lb_t\Big[(2-na_{t-})\sum_{j=1}^{rb_{t-}-1} dL_j^+ + (a_{A^{-1}(0)}+2-na_{t-})dL_{rb_{t-}}^+ \\
&+ (a_{A^{-1}(0)}+1-na_{t-})\sum_{j=rb_{t-}+1}^{K}\Big(dL_j^+ + dC_j^+\Big) \\
&+ \Big(a_{A^{-1}(0)}\mathbb{1}_{\{a_{A^{-1}(0)}>1\}} + (a_{A^{-1}(1)}+1)\mathbb{1}_{\{a_{A^{-1}(0)}=1\}} - na_{t-}\Big)dC_{rb_{t-}}^+\Big] \\
&+ (1-lb_t)\Big[(2-na_{t-})\sum_{j=1}^{B^{-1}(0)-1} dL_j^+ + (a_{A^{-1}(0)}+2-na_{t-})dL_{A^{-1}(0)}^+ \\
&+ (a_{A^{-1}(0)}+1-na_{t-})\sum_{j=B^{-1}(0)+1}^{K}\Big(dL_j^+ + dC_j^+\Big) \\
&+ \Big(a_{A^{-1}(0)}\mathbb{1}_{\{a_{A^{-1}(0)}>1\}} + (a_{A^{-1}(1)}+1)\mathbb{1}_{\{a_{A^{-1}(0)}=1\}} - na_{t-}\Big)dC_{A^{-1}(0)}^+\Big] \\
&+ (a_{A^{-1}(0)}+1-na_{t-})\Big[dM_t^- + \sum_{j=1}^{K}\Big(dL_j^- + dC_j^-\Big)\Big]\Bigg]
\end{aligned}
$$

$$dna_t = (la_t == 0)(-1 - na_{t-})\left[dM_t^+ + dM_t^-\right.$$

$$+ (lb_t! = 1)\sum_{i=1}^{K}\left(dL_i^+ + dL_i^- + dC_i^+ + dC_i^-\right)$$

$$+ (lb_t == 1)\left[\sum_{i=0}^{K-(A^{-1}(0)-rb_{t-})}\left(dL_i^+ + dC_i^+\right) + \sum_{i=1}^{K}\left(dL_i^- + dC_i^-\right)\right]\right]$$

$$+ \mathbb{1}_{la_t=1}\left\{\left(\mathbb{1}_{na_{t-}=-1} + \mathbb{1}_{na_{t-}!=-1}\mathbb{1}_{ra_{t-}>A^{-1}(0)}\right)\left[\left(a_{A^{-1}(0)} - na_{t-}\right)\left[dM_t^+ + \mathbb{1}_{lb_t=1}dC_{rb_{t-}}^+ + \mathbb{1}_{lb_t!=1}dC_{A^{-1}(0)}^+\right]\right.\right.$$

$$+ \left(a_{A^{-1}(0)} + 1 - na_{t-}\right)\left[\mathbb{1}_{lb_t!=1}\sum_{i=1}^{K}\left(dL_i^+ + dC_i^+\right) + \mathbb{1}_{lb_t=1}\sum_{i=1}^{K-(A^{-1}(0)-rb_{t-})}\left(dL_i^+ + dC_i^+\right)\right.$$

Dynamic of nb_t:

$$dnb_t = lb_t\left[-X^b\left((1-la_t)dC_{B^{-1}(0)}^-(t) + la_t dC_{ra_{t-}}^-\right) + \left(-\mathbb{1}_{\{nb_{t-}>1\}} + (|b_{B^{-1}(0)}| + 1 - nb_{t-})\mathbb{1}_{\{nb_{t-}=1\}}\right)dM_t^-\right.$$

$$+ (2 - nb_{t-})\left\{(1-la_t)\sum_{i=1}^{rb_{t-}(t)-1}dL_i^- + la_t\sum_{i=1}^{rb_{t-}(t)-(B^{-1}(0)-ra_{t-})-1}dL_i^-\right\}\right]$$

$$+ (1 - lb_t)\left[\left(|b_{A^{-1}(0)}|\mathbb{1}_{\{|b_{B^{-1}(0)}|>1\}} + (|b_{B^{-1}(1)}| + 1)\mathbb{1}_{\{|b_{B^{-1}(0)}|=1\}} - nb_{t-}\right)dM_t^-\right.$$

$$+ la_t\left[(2 - nb_{t-})\sum_{j=1}^{ra_{t-}-1}dL_j^- + (|b_{A^{-1}(0)}| + 2 - nb_{t-})dL_{ra_{t-}}^-\right.$$

$$+ (|b_{A^{-1}(0)}| + 1 - nb_{t-})\sum_{j=ra_{t-}+1}^{K}\left(dL_j^- + dC_j^-\right)$$

$$+ \left(|b_{A^{-1}(0)}|\mathbb{1}_{\{|b_{B^{-1}(0)}|>1\}} + (|b_{B^{-1}(1)}| + 1)\mathbb{1}_{\{|b_{B^{-1}(0)}|=1\}} - nb_{t-}\right)dC_{ra_{t-}}^-\right]$$

$$+ (1 - la_t)\left[(2 - nb_{t-})\sum_{j=1}^{B^{-1}(0)-1}dL_j^- + (|b_{B^{-1}(0)}| + 2 - nb_{t-})dL_{A^{-1}(0)}^-\right.$$

$$+ (|b_{B^{-1}(0)}| + 1 - nb_{t-})\sum_{j=B^{-1}(0)+1}^{K}\left(dL_j^- + dC_j^-\right)$$

$$+ \left(|b_{A^{-1}(0)}|\mathbb{1}_{\{|b_{B^{-1}(0)}|>1\}} + (|b_{B^{-1}(1)}| + 1)\mathbb{1}_{\{|b_{B^{-1}(0)}|=1\}} - nb_{t-}\right)dC_{A^{-1}(0)}^-\right]$$

$$+ (|b_{A^{-1}(0)}| + 1 - nb_{t-})\left[dM_t^+ + \sum_{j=1}^{K}\left(dL_j^+ + dC_j^+\right)\right]$$

5.2.4 Dynamics of pa and pb

Dynamic of $(pa_t)_t$:

We denote by δ the tick.

$$dP_t^A = \delta(1 - la_t)\Bigg[\Big((A^{-1}(1) - ra_{t-})dM^+(t)$$

$$+ lb_t\Bigg[-\sum_{i=1}^{rb_{t-}-1}\Big[rb_{t-} - (A^{-1}(0) - ra_{t-}) - j\Big]dL_i^+(t) + (A^{-1}(0) - ra_{t-})\sum_{j=rb_{t-}}^{K}dL_i^+$$

$$+ (A^{-1}(1) - ra_{t-})dC_{rb_{t-}}^+ + \sum_{j=rb_{t-}+1}^{K}(A^{-1}(0) - ra_{t-})dC_j^+\Bigg]$$

$$+ \Big(1 - lb_t\Big)\Bigg[-\sum_{i=1}^{A^{-1}(0)-1}(ra_{t-} - j)dL_i^+(t) + (A^{-1}(0) - ra_{t-})\sum_{j=A^{-1}(0)}^{K}dL_i^+$$

$$+ (A^{-1}(1) - ra_{t-})dC_{A^{-1}(0)}^+ + \sum_{j=A^{-1}(0)+1}^{K}(A^{-1}(0) - ra_{t-})dC_j^+\Bigg]$$

$$+ \mathbb{1}_{\{ra_{t-}\neq A^{-1}(0)\}}(A^{-1}(0) - ra_{t-})\Big(dM_t^- + \sum_{j=1}^{K}dL_t^- + \sum_{j=1}^{K}dC_t^-\Big)\Bigg]$$

$$+ \delta la_t\Bigg[(A^{-1}(0) - r_t^A)\Big)dM^+(t)$$

$$- lb_t\sum_{i=1}^{rb_{t-}-(A^{-1}(0)-ra_{t-})-1}\Big(ra_{t-} - (j + A^{-1}(0) - rb_{t-})\Big)dL_i^+(t)$$

$$- \Big(1 - lb_t\Big)\sum_{i=1}^{ra_{t-}-1}(ra_{t-} - j)dL_i^+(t)\Bigg]$$

Dynamic of (pb_t):

$$dP_t^B = -\delta(1 - lb_t)\Bigg[\Big((B^{-1}(1) - rb_{t-})dM^-(t)$$

$$+ la_t\Bigg[-\sum_{i=1}^{ra_{t-}-1}\Big[ra_{t-} - (B^{-1}(0) - rb_{t-}) - j\Big]dL_i^-(t) + (B^{-1}(0) - rb_{t-})\sum_{j=ra_{t-}}^{K}dL_i^-$$

$$+ (B^{-1}(1) - rb_{t-})dC_{rb_{t-}}^- + \sum_{j=ra_{t-}+1}^{K}\big(B^{-1}(0) - rb_{t-}\big)dC_j^-\Bigg]$$

$$+ \Big(1 - la_t\Big)\Bigg[-\sum_{i=1}^{B^{-1}(0)-1}\big(rb_{t-} - j\big)dL_i^-(t) + \big(B^{-1}(0) - rb_{t-}\big)\sum_{j=B^{-1}(0)}^{K}dL_i^-$$

$$+ (B^{-1}(1) - rb_{t-})dC_{A^{-1}(0)}^- + \sum_{j=A^{-1}(0)+1}^{K}\big(B^{-1}(0) - rb_{t-}\big)dC_j^-\Bigg]$$

$$+ \mathbb{1}_{\{rb_{t-}\neq A^{-1}(0)\}}\big(A^{-1}(0) - rb_{t-}\big)\Big(dM_t^+ + \sum_{j=1}^{K}dL_j^+ + \sum_{j=1}^{K}dC_j^+\Big)\Bigg)\Bigg]$$

$$- \delta lb_t\Bigg[\big(B^{-1}(0) - r_t^B)\big)dM^-(t)$$

$$- la_t\sum_{i=1}^{ra_{t-}-(B^{-1}(0)-rb_{t-})-1}\Big(rb_{t-} - (j + B^{-1}(0) - ra_{t-})\Big)dL_i^-(t)$$

$$- \Big(1 - la_t\Big)\sum_{i=1}^{rb_{t-}-1}\big(rb_{t-} - j\big)dL_i^-(t)\Bigg]$$

5.2.5 Dynamics of ra and rb

We remind that ra_t denotes the number of ticks between the market maker's order and the best buy order in the order book. We assumed in this simplified control problem that the market maker is allowed to place no more than one order on the best ask and best bid limits. So ra and rb are vectors of size 1 here.

Dynamic of ra:

$$dra_t = la_t \left[\mathbb{1}_{\{na=1\}} \left(A^{-1}(0) - r_t^A \right) dM_t^+ \right.$$

$$+ (1 - lb_t) \sum_{i=1}^{ra_{t-}-1} \left(i - ra_{t-} \right) dL_i^+ + lb_t \sum_{i=1}^{rb_{t-}-(B^{-1}(0)-ra_{t-})-1} \left(i + B^{-1}(0) - rb_{t-} - ra_{t-} \right) dL_i^+$$

$$+ \sum_{i=1}^{ra_{t-}-1} (i - ra_{t-}) dL_i^- + (B^{-1}(1) - B^{-1}(0)) dC_{ra_{t-}}^-$$

$$\left. \left(\left((1 - lb_t) + lb_t \mathbb{1}_{\{nb_{t-}>1\}} \right) \left[B^{-1}(1) - B^{-1}(0) \right] \right) dM_t^- \right]$$

$$+ (l - la_t) \left[lb_t \left[\sum_{j=1}^{rb_{t-}-1} \left(j + B^{-1}(0) - rb_{t-} - ra_{t-} \right) dL_j^+ + (A^{-1}(0) - ra_{t-}) \sum_{j=rb_{t-}}^{K} dL_j^+ \right. \right.$$

$$\left. (A^{-1}(1) - ra_{t-}) dC_{rb_{t-}}^+ + (A^{-1}(0) - ra_{t-}) \sum_{j=rb_{t-}+1}^{K} dC_j^+ \right]$$

$$+ (1 - lb_t) \left[\sum_{j=1}^{B^{-1}(0)-1} \left(j - ra_{t-} \right) dL_j^+ + (A^{-1}(0) - ra_{t-}) \sum_{j=B^{-1}(0)}^{K} (dC_j^+ + dC_j^-) \right.$$

$$+ (A^{-1}(1) - ra_{t-}) dC_{A^{-1}(0)}^+ + \sum_{j=A^{-1}(0)+1}^{K} (A^{-1}(0) - ra_{t-}) dC_j^+$$

$$+ \sum_{j=1}^{A^{-1}(0)-1} (j - ra_{t-}) dL_j^- + \sum_{j=A^{-1}(0)}^{K} (A^{-1}(0) - ra_{t-}) dL_j^-$$

$$+ (B^{-1}(1) - ra_t) dC_{B^{-1}(0)}^- + (A^{-1}(0) - ra_{t-}) \sum_{j=A^{-1}(0)+1}^{K} dC_j^-$$

$$+ \left[\left((1 - lb_t) + lb_t \mathbb{1}_{\{nb_{t-}>1\}} \right) \left(B^{-1}(1) - ra_{t-} \right) + lb_t \mathbb{1}_{\{nb_{t-}=1\}} (B^{-1}(0) - ra_{t-}) \right] dM_t^-$$

$$\left. + (A^{-1}(1) - ra_{t-}) dM_t^+ \right.$$

We remind that rb_t is the number of ticks between the market maker's order and the best sell order in the order book.

Dynamic of rb:

$$
\begin{aligned}
drb_t = lb_t\Bigg[& \mathbb{1}_{\{nb=1\}}\Big(B^{-1}(0)-rb\Big)dM_t^- \\
& + (1-la_t)\sum_{i=1}^{rb_{t-}-1}\Big(i-rb_{t-}\Big)dL_i^- + la_t\sum_{i=1}^{ra_{t-}-(A^{-1}(0)-rb_{t-})-1}\Big(i+A^{-1}(0)-ra_{t-}-rb_{t-}\Big)dL_i^- \\
& + \sum_{i=1}^{rb_{t-}-1}(i-rb_{t-})dL_i^+ + (A^{-1}(1)-A^{-1}(0))dC_{rb_{t-}}^+ \\
& \Big(\big((1-la_t)+la_t\mathbb{1}_{\{na_{t-}>1\}}\big)\big[A^{-1}(1)-A^{-1}(0)\big]\Big)dM_t^+\Bigg] \\
+ (l-lb_t)\Bigg[& la_t\Bigg[\sum_{j=1}^{ra_{t-}-1}\Big(j+A^{-1}(0)-ra_{t-}-rb_{t-}\Big)dL_j^- + \sum_{j=ra_{t-}}^{K}(B^{-1}(0)-rb_{t-})dL_j^- \\
& (B^{-1}(1)-rb_{t-})dC_{ra_{t-}}^- + \sum_{j=ra_{t-}+1}^{K}(B^{-1}(0)-rb_{t-})dC_j^-\Bigg] \\
+ (1-la_t)\Bigg[& \sum_{j=1}^{A^{-1}(0)-1}\Big(j-rb_{t-}\Big)dL_j^- + \sum_{j=A^{-1}(0)}^{K}(B^{-1}(0)-rb_{t-})dL_j^- \\
& + (B^{-1}(1)-rb_{t-})dC_{B^{-1}(0)}^- + \sum_{j=B^{-1}(0)+1}^{K}(B^{-1}(0)-rb_{t-})dC_j^- \\
+ \sum_{j=1}^{B^{-1}(0)-1}&(j-rb_{t-})dL_j^+ + \sum_{j=B^{-1}(0)}^{K}\big(B^{-1}(0)-rb_{t-}\big)dL_j^+ \\
+ (A^{-1}(1)-rb_t)&dC_{A^{-1}(0)}^+ + (B^{-1}(0)-rb_{t-})\sum_{j=B^{-1}(0)+1}^{K}dC_j^+ \\
+ \Big[\big((1-la_t)&+la_t\mathbb{1}_{\{na_{t-}>1\}}\big)\big(A^{-1}(1)-rb_{t-}\big)+la_t\mathbb{1}_{\{na_{t-}=1\}}\big(A^{-1}(0)-rb_{t-}\big)\Big]dM_t^+ \\
+ (B^{-1}(1)&-rb_{t-})dM_t^-
\end{aligned}
$$

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