Outline

1. Algorithm Analysis
2. Growth rate functions
3. The properties of growth rate functions:
4. Importance of the growth rate
5. An example
6. Encryption

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- Commonly, n is the # of items in the input, if each item is of fixed size.
- This makes no difference in asymptotic analysis in most cases.

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Example 2

The input is one integer of k digits long. Since its size is not fixed (k can be arbitrarily large). The input size is **not** $n = 1$. It is $n = 4k$ bits long.

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- These are **not** basic instructions: input/output statement, $\sin(x), \exp(x) \ldots$. These actions are done by function calls, not by a single machine instruction.
- Knowing $T(n)$ and the machine speed, we can estimate the real runtime.
- Example 3: The machine speed is 10^8 ins/sec. $T(n) = 10^6$. The real runtime would be about $10^{-8} \times 10^6 = 0.01$ sec.

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- So $T(n) = an^2 + bn + c$ for some constants a, b, c.
- We say the growth rate of $T(n)$ is n^2. This is the sole property of the algorithm and is our main concern.

Outline

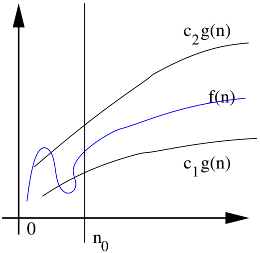
Growth rate functions

We want to define the precise meaning of growth rate.

Definition 1:

$$\Theta(g(n)) = \{f(n) \mid \exists c_1 > 0, c_2 > 0, n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$$

If $f(n) \in \Theta(g(n))$, we also write $f(n) = \Theta(g(n))$ and say: the growth rate of $f(n)$ is the same as the growth rate of $g(n)$.



Example 5

$f(n) = \frac{1}{12}n^2 + 60n - 4 \in \Theta(n^2)$ (or write $f(n) = \Theta(n^2)$.)

Proof: We need to find c_1 and n_0 so that $\forall n \geq n_0$,

$$c_1 n^2 \leq \frac{1}{12}n^2 + 60n - 4$$

Pick $c_1 = 1/12$, the above becomes: $0 \leq 60n - 4$. This is true for all $n \geq n_0 = 1$. We also need to find c_2 and n_0 so that $\forall n \geq n_0$,

$$\frac{1}{12}n^2 + 60n - 4 \leq c_2 n^2$$

For any $n \geq 1$, we have:

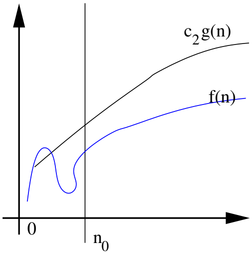
$$\frac{1}{12}n^2 + 60n - 4 < n^2 + 60n \leq n^2 + 60n^2 = 61n^2$$

So if $c_1 = 1/12, c_2 = 61$ and $n_0 = 1$, all the required conditions hold.

Definition 2:

$$O(g(n)) = \{f(n) \quad | \quad \exists c_2 > 0, n_0 \geq 0 \ \text{ so that } \ \forall n \geq n_0, 0 \leq f(n) \leq c_2 g(n)\}$$

If $f(n) \in O(g(n))$, we also write $f(n) = O(g(n))$ and say: the growth rate of $f(n)$ is at most the growth rate of $g(n)$.



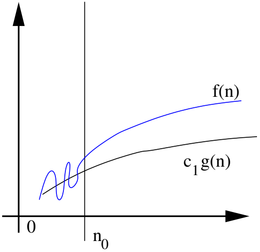
Example 6

$f(n) = 10n - 4 \in O(0.01n^2)$ (or write $f(n) = O(0.01n^2)$.)

Definition 3:

$$\Omega(g(n)) = \{f(n) \mid \exists c_1 > 0, n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq c_1 g(n) \leq f(n)\}$$

If $f(n) \in \Omega(g(n))$, we also write $f(n) = \Omega(g(n))$ and say: the growth rate of $f(n)$ is at least the growth rate of $g(n)$.



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$$o(g(n)) = \{f(n) \mid \forall c > 0, \exists n_0 \geq 0 \text{ so that } \forall n \geq n_0, 0 \leq f(n) \leq cg(n)\}$$

If $f(n) \in o(g(n))$, we also write $f(n) = o(g(n))$ and say: the growth rate of $f(n)$ is strictly less than the growth rate of $g(n)$.

Example:

$f(n) = 2n$ and $g(n) = n^2$. Then:
$f(n) = O(g(n)), f(n) = o(g(n))$, but $f(n) \neq \Theta(g(n))$,

Definition 5:

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If $f(n) \in \omega(g(n))$, we also write $f(n) = \omega(g(n))$ and say: the growth rate of $f(n)$ is strictly bigger than the growth rate of $g(n)$.

Outline

The properties of growth rate functions:

The meaning of these notations (roughly speaking):

if	the growth-rate is
$f(n) = \Theta(g(n))$	$=$
$f(n) = O(g(n))$	$\leq$
$f(n) = \Omega(g(n))$	$\geq$
$f(n) = o(g(n))$	$<$
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- Read Ch. 3 for more relations and properties.

Outline

Importance of the growth rate

The growth rate of the runtime function is the most important property of an algorithm. Assuming 10^9 instruction/sec, The real runtime:

$f(n)$	$n = 10$	30	50	1000
$\log_2 n$	3.3 ns	4.9 ns	5.6 ns	9.9 ns
n	10 ns	30 ns	50 ns	1 μs
n^2	0.1 μs	0.9μs	2.5 μs	1 ms
n^3	1 μs	27 μs	125 μs	1 sec
n^5	0.1 ms	24.3 ms	0.3 sec	277 h
2^n	1 μs	1 sec	312 h	3.4 $\cdot 10^{281}$ Cent

- If $T(n) = n^k$ for some constant $k > 0$, the runtime is polynomial.
- If $T(n) = a^n$ for some constant $a > 1$, the runtime is exponential.

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- Moore's law: CPU speed doubles every 18 months. Then, instead of solving the problem of size $n =$ say 100, we can solve the problem of size 101.
- An exponential time algorithm **cannot** be used to solve problems of realistic input size, no matter how powerful the computers are!

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- If $X = 117$, output ”no”.
- If $X = 456731$, output = ?

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Find-Factor(X)

```
if X is even then
    return "2 is a factor"
end if
for i = 3 to √X by +2 do
    test if X%i = 0, if yes, output "i is a factor"
end for
return "X is a prime."
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- Minor improvements can be (and had been) made. But basically, we have to perform most of these tests. No poly-time algorithm for Factoring is known.
- It is strongly believed, (but not proven), no poly-time algorithm for solving the Factoring problem exists.

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  - Even if Evil sees C, he doesn't know $S_A()$, so cannot recover M.

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- There are many different ways for 1-1 Encryption. It is not hard.
- However, Bob is dealing with many customers, and Alice is dealing with many banks, on-line accounts ...
- It would be a nightmare if we have to arrange a different key for each (Alice, Bob) pair.

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- The pair (X, e) is the public key. Bob makes it public.
- (x, y, d) is the secret key. Only Bob knows it.

Example

$x = 7$, $y = 29$. Then $X = 7 \cdot 29 = 203$, and $(x-1) \cdot (y-1) = 168$.
Pick $e = 11$ and $d = 107$, then $11 \cdot 107 = 1177 = 1 \pmod{168}$.
Thus $(203, 11)$ is the public key. $(7, 29, 107)$ is secret key.

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- Bob: Receiving $C(= 4)$. Recover the original message $M = S_A(C) \stackrel{\text{def}}{=} C^d \pmod{X}$. (In our example $4^{107} \pmod{203} = 100$).
- Because of the the choice of e, d, the number theory ensures the result M is the same as the original message M. (Namely $(M^e)^d = M \pmod{X}$ for all M.)

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- But he must factor a 256 digit number X. This requires about $\sqrt{10^{256}} = 10^{128} \approx 2^{426}$ divisions. This will need much much much .... longer time than the previous 2^{360} example!

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- In 2001, Agrawal, Kayal and Saxena found a poly-time algorithm for solving P2.
- Had they found a poly-time algorithm for solving P1 (Factoring), RSA system (and the entire computer security industry) would have collapsed overnight!