

# CSE431/531,                      Problem Set 1                      Due Wed 9/19, in class

**Note:** There are 6 problems with a total of 130 points. You are required to do all the problems. In the problems, the base of  $\log n$  is 2.

1. ( $7 \times 5 = 35$  points) Let  $f(n)$  and  $g(n)$  be two functions from  $N^+$  to  $R^+$ . Prove or disprove the following assertions. To disprove, you only need to give a counter example for functions  $f(n)$  and/or  $g(n)$  which make the assertion false.
  - (a)  $O(O(f(n))) = O(f(n))$
  - (b)  $O(\Theta(f(n))) = O(f(n))$
  - (c)  $\Theta(O(f(n))) = \Theta(f(n))$
  - (d)  $\Omega(O(f(n))) = O(\Omega(f(n)))$
  - (e) If  $f(n) = \Theta(h(n))$  and  $g(n) = \Theta(h(n))$ , then  $f(n) + g(n) = \Theta(h(n))$
  - (f) If  $f(n) = \Theta(g(n))$ , then  $2^{f(n)} = \Theta(2^{g(n)})$
  - (g)  $f(n) + g(n) = \Theta(\min(f(n) + g(n)))$
2. ( $2 \times 5 = 10$  points) Use mathematical induction to prove the following.
  - (a)  $\sum_{i=1}^n ir^{i-1} = \frac{1-r^{n+1}-(n+1)(1-r)r^n}{(1-r)^2}$  for all  $n \geq 1$ , where  $0 \leq r \leq 1$ .
  - (b) Every integer  $n \geq 1$  can be represented as the sum of distinct Fibonacci numbers, no two of which are consecutive in the Fibonacci sequence.
3. ( $4 \times 5 = 20$  points) Prove or disprove the following assertions.
  - (a)  $n! = O(n^n)$
  - (b)  $\sum_{i=1}^n i \log i = \Theta(n^2 \log n)$
  - (c) If  $n = 2^k$ , then  $\sum_{i=0}^k \log(n/2^i) = \Theta(\log^2 n)$
  - (d)  $n^n = O(2^n)$
4. (15 points) Rank the following functions in asymptotically increasing order based on  $O$ -notation and justify your ordering:  $n!$ ,  $(\lg n)^{\lg(\lg n)}$ ,  $[\lg(\lg n)]^{\lg n}$ ,  $2^{n^{0.001}}$ ,  $n^{1/\lg n}$ ,  $\lg^*(\lg n)$ ,  $2^{\sqrt{2 \lg n}}$ ,  $2^{2^n}$ ,  $n^5$ ,  $\sqrt{\lg n}$ .
5. ( $8 \times 5 = 40$  points) Find a closed form for each  $T(n)$ . Assume  $T(1) = 1$ .
  - (a)  $T(n) = T(n-1) + 2^n$
  - (b)  $T(n) = 4T(n/3) + n^2$
  - (c)  $T(n) = 6T(n/7) + n$
  - (d)  $T(n) = T(\sqrt{n}) + \log n$
  - (e)  $T(n) = 2 + \sum_{i=1}^{n-1} T(i)$
  - (f)  $T(n) = 3T(n/2) + n \log n$
  - (g)  $T(n) = 2T(n/2) + n/\log n$
  - (h)  $T(n) = \sqrt{n}T(\sqrt{n}) + n$
6. (10 points) The sequence  $\langle a_n \rangle$  is defined for  $n \geq 0$  by  $a_0 = 2$ ,  $a_1 = 5$ , and  $a_n = 5a_{n-1} - 6a_{n-2}$  for  $n > 1$ . The first few elements of the sequence are 2, 5, 13, 35, 97. Find a closed form for  $a_n$ .