What is Dynamic Programming

- Like DaC, Dynamic Programming is another useful method for designing efficient algorithms.
- Why the name?

Eye of the Hurricane: An Autobiography - A quote from Richard Bellman

I spent the Fall quarter (of 1950) at RAND. My first task was to find a name for multistage decision process. An interesting question is, Where did the name, dynamic programming, come from? The 1950s were not good years for mathematical research. We had a very interesting gentleman in Washington named Wilson. He was Secretary of Defense, and he actually had a pathological fear and hatred of the word, research. ... I felt I had to do something to shield Wilson and the Air Force from the fact that I was really doing mathematics inside the RAND Corporation. .... Thus, I thought dynamic programming was a good name. It was something not even a Congressman could object. So I used it as an umbrella for my activities.

General Description of Dynamic Programming

- Divide the problem into smaller subproblems (of the same type).
- Solve each subproblem.
- Combine the solutions of subproblems into the solution of the original problem.
- Looks familiar? It's identical to DaC!
- But they differ substantially in details.
- For DaC:
  - The sub-problems are independent, they do not overlap.
  - We solve sub-problems in a top-down fashion. Namely, solve the largest sub-problem first, then the second largest ...
  - Usually by recursive calls.
- For Dynamic Programming:
  - The sub-problems are intermingled, they do overlap.
  - We solve sub-problems in a bottom-up fashion. Namely, solve the smallest sub-problem first, then the second smallest ...

Example for DaC

MergeSort

Sort array $A[1..n]$
Divide it into two subproblems: Sort $A[1..n/2]$ and Sort $A[(n/2+1)..n]$

- The two sub-problems do not overlap. They are totally independent.
- We solve these two largest sub-problems, by recursive calls.
- Move to smaller problems Sort $A[1..n/4]$, Sort $A[(n/4+1)..n/2]$ etc.

Example for Dynamic Programming

$\text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2)$

- We divide Fib(n) into two sub-problems Fib($n-1$) and Fib($n-2$).
- They overlap, not totally independent: Fib($n-1$) contains Fib($n-2$).
- If we solve the largest sub-problems Fib($n-1$) and Fib($n-2$) first by using recursive calls, as we did before, we get exp time algorithm.

Fib(n)

```
Fib[0] = 0; Fib[1] = 1;
for i = 2 to n do
    Fib[i] = Fib[i − 1] + Fib[i − 2]
end for
output Fib[n]
```

- Solves the smallest sub-problem Fib[0], then Fib[1] ... bottom-up.
- It clearly takes $O(n)$ time.

Matrix Chain Product Problem

Matrix Chain Product Problem

Input: n matrices $A_1, A_2, \ldots, A_n$, the size of A_i is $p_{i-1} \times p_i$ for $i = 1, \ldots, n$.
Compute the chain product: $A_1 \times A_2 \times \cdots \times A_n$

- This is a basic operation in linear algebra.
- To calculate

$$p_{i-1} \left\{ \boxed{\begin{matrix} \leftarrow & p_i & \rightarrow \\ & A_i & \end{matrix}} \right. \times p_i \left\{ \boxed{\begin{matrix} \leftarrow & p_{i+1} & \rightarrow \\ & A_{i+1} & \end{matrix}} \right. = p_{i-1} \left\{ \boxed{\begin{matrix} \leftarrow & p_{i+1} & \rightarrow \\ & C & \end{matrix}} \right.$$

  We need to calculate $p_{i-1} \cdot p_{i+1}$ entries, and each entry takes p_i scalar multiplications. So it totally takes $p_{i-1} \cdot p_i \cdot p_{i+1}$ scalar multiplications.

- $\times$ is associative. So we can compute the chain product in different ways, with different total cost.

Example

$$100 \left\{ \boxed{\begin{array}{c} \leftarrow \quad 2 \quad \rightarrow \\ A_1 \end{array}} \right., \quad 2 \left\{ \boxed{\begin{array}{c} \leftarrow \quad 50 \quad \rightarrow \\ A_2 \end{array}} \right., \quad 50 \left\{ \boxed{\begin{array}{c} \leftarrow \quad 6 \quad \rightarrow \\ A_3 \end{array}} \right.$$

There are two ways to compute $A_1 \times A_2 \times A_3$:

- $(A_1 \times A_2) \times A_3$:
  - Calculate $X = A_1 \times A_2$ takes $100 \cdot 2 \cdot 50 = 10000$ ops.
  - Calculate $X \times A_3$ takes $100 \cdot 50 \cdot 6 = 30000$ ops.
  - Total cost is $10000 + 30000 = 40000$ ops.
- $A_1 \times (A_2 \times A_3)$:
  - Calculate $Y = A_2 \times A_3$ takes $2 \cdot 50 \cdot 6 = 600$ ops.
  - Calculate $A_1 \times Y$ takes $100 \cdot 2 \cdot 6 = 1200$ ops.
  - Total cost is $600 + 1200 = 1800$ ops.
- The total costs are very different.

Matrix Chain Product Problem

Input: n matrices $A_1, A_2, \ldots, A_n$, the size of A_i is $p_{i-1} \times p_i$ for $i = 1, \ldots, n$.
Find: The best way to compute the chain product: $A_1 \times A_2 \times \cdots \times A_n$ so that the total cost is minimum.

- A way to calculate the product is a parenthesization of the chain.
- A simple algorithm:

Brute-Force

1. Enumerate all parenthesizations of $A_1 \times \ldots A_n$.
2. For each, compute the total cost.
3. Pick the one with the lowest total cost.

Matrix Chain Product Problem

- But how many possible solutions?
- Let C_i be the number of different ways to put parenthesis into a n term chain.
- C_0 and C_1 are meaningless. For simplicity, define $C_0 = 0$ and $C_1 = 1$.
- It's easy to see: $C_2 = 1$ and $C_3 = 2$.
- $C_4 = 5$:
  $((A_1 \times A_2) \times A_3) \times A_4$, $(A_1 \times A_2) \times (A_3 \times A_4)$, $(A_1 \times (A_2 \times A_3)) \times A_4$, $A_1 \times ((A_2 \times A_3) \times A_4))$, $A_1 \times (A_2 \times (A_3 \times A_4))$

Matrix Chain Product Problem

We will show:

$$C_n = \frac{1}{n}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} = \frac{(2n-2)!}{n!(n-1)!}$$

- C_n is called the nth Catalan number.
- $C_4 = \frac{6!}{4! \cdot 3!} = 5$.
- $C_{10} = 4862, \quad C_{15} = 2,674,440$.
- By using Stirling's approximation, we have:

$$C_n = \Omega\left(\frac{4^n}{n^{3/2}}\right)$$

- Thus, the Brute-Force algorithm takes exponential time. This is not acceptable.

Generating Function Method

- Generating Function Method is a systematic way for solving recursive sequences (linear or non-linear)
- You may wondered why our method for solving linear recursive sequences works. Here we will see why.
- Let $\{f_0, f_1, f_2 \ldots\} = \{f_n\}_{n\geq 0}$ be a recursively defined sequence (linear or non-linear.)

Generating Function Method

1. Define a formal series $f(x) = \sum_{n=0}^{\infty} f_n x^n$
   $f(x)$ is called the generating function of $\{f_n\}_{n\geq 0}$.
2. Using the recursive definition of $\{f_n\}_{n\geq 0}$, try to get an equation that only involves $f(x)$ and x, without mentioning f_n.
   - This is the key step of this method.
   - For linear recursive sequences, we have an easy systematic way to do this.
   - For other cases, we will need good luck!
3. Solving this equation for $f(x)$ in terms of x.
4. Find the Taylor Series of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

5. Then

$$f_n = \frac{f^{(n)}(0)}{n!}$$

Examples

Fib numbers: $f_0 = 0,\ f_1 = 1,\ f_n = f_{n-1} + f_{n-2}$

Step 1: define $f(x) = \sum_{n=0}^{\infty} f_n x^n$

Step 2: Try to get an equation of $f(x)$ and x;

$$
\begin{array}{ccccccccccc}
f(x) & = & f_0 & + & f_1 x & + & f_2 x^2 & \cdots & f_n x^n & \cdots \\
xf(x) & = & & & f_0 x & + & f_1 x^2 & \cdots & f_{n-1} x^n & \cdots \\
x^2 f(x) & = & & & & & f_0 x^2 & \cdots & f_{n-2} x^n & \cdots \\
f(x) - xf(x) - x^2 f(x) & = & f_0 & + & (f_1 - f_0)x & + & (f_2 - f_1 - f_0)x^2 & \cdots & (f_n - f_{n-1} - f_{n-2})x^n & \cdots \\
 & = & x & & & & & & &
\end{array}
$$

The last line is because: $f_0 = 0, f_1 = 1, f_2 - f_1 - f_0 = 0$ and $f_n - f_{n-1} - f_{n-2} = 0$. This implies $f(x)(1 - x - x^2) = x$. Hence:

$$f(x) = \frac{x}{1 - x - x^2}$$

Examples

Step 3: Find the Taylor Series of $f(x)$. Because $f(x)$ is a fraction of polynomials, instead of using the formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}}{n!} x^n$, we have an easier way.

Partial Fraction

Let $p(z)$ be a polynomial of degree k. Let $q(z)$ be a polynomial of degree at most $k-1$. Let $\alpha_1, \alpha_1, \ldots, \alpha_k$ be the roots of the equation $p(z) = 0$.

- If all roots are distinct, then for some constants $a_1, a_2, \ldots a_k$:

$$\frac{q(z)}{p(z)} = \frac{a_1}{z-\alpha_1} + \frac{a_2}{z-\alpha_2} + \cdots + \frac{a_k}{z-\alpha_k}$$

Partial Fraction - Continued:

- If there are repeated roots, say $\alpha_1 = \alpha_2 = \ldots = \alpha_t$ repeat t times, then the portion in the above formula corresponding to the roots $\alpha_1 \ldots \alpha_t$ becomes:

$$\cdots \frac{a_1}{(z-\alpha_1)} + \frac{a_2 z^1}{(z-\alpha_1)^2} + \cdots + \frac{a_t z^{t-1}}{(z-\alpha_1)^t} + \cdots$$

- This is a fact from algebra.
- Extensively used in Calculus.

Examples

Back to the generating function for Fib numbers:

$$\begin{aligned}
f(x) &= \frac{x}{1-x-x^2} = \frac{1/x}{(1/x)^2-(1/x)-1} = \frac{z}{z^2-z-1} && \text{(here } z = 1/x\text{)} \\
&= \frac{a_1}{z-\alpha_1} + \frac{a_2}{z-\alpha_2} && \text{(Partial Fraction, } \alpha_1, \alpha_2 \text{ are the roots of } z^2 - z - 1\text{)} \\
&= \frac{a_1}{1/x-\alpha_1} + \frac{a_2}{1/x-\alpha_2} = \frac{a_1 x}{1-\alpha_1 x} + \frac{a_2 x}{1-\alpha_2 x} \\
&= a_1 x \sum_{n=0}^{\infty} (\alpha_1 x)^n + a_2 x \sum_{n=0}^{\infty} (\alpha_2 x)^n && \text{(sum of geometric series)} \\
&= \sum_{n=0}^{\infty} (a_1 \alpha_1^n + a_2 \alpha_2^n) \cdot x^{n+1}
\end{aligned}$$

This is the Taylor Series of $f(x)$. Thus $f_{n+1} = a_1(\alpha_1)^n + a_2(\alpha_2)^n$. Or

$$f_n = a_1(\alpha_1)^{n-1} + a_2(\alpha_2)^{n-1}$$

Linear Recursive Sequences

By using this method, we can find the solution of any linear recursive sequences. The result is the procedure we discussed before.

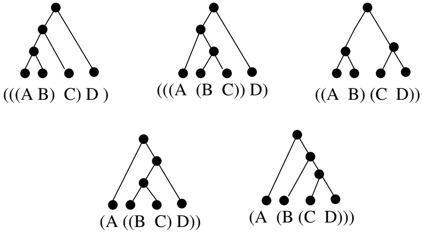
Catalan Number

- Let c_n be the nth Catalan number. Namely:
  c_n = the number of different parenthesizations of n terms.
- We have $c_0 = 0,\ c_1 = 1$ (these two are defined for convenience), $c_2 = 1,\ c_3 = 2,\ c_4 = 5 \cdots$.
- We want to show:

$$c_n = \frac{1}{n}\left(\begin{array}{c} 2n-2 \\ n-1 \end{array} \right) = \frac{(2n-2)!}{n!(n-1)!}$$

- Actually c_n is also the number of distinct n-leaf binary trees.
- This is because there is a 1-1 correspondence between the set of n-leaf binary trees and the set of parenthesizations of n terms.

Correspondence with Binary Trees



The case of $n = 4$.

Catalan Number

In order to find the solution for the sequence $\{c_n\}_{n\geq 0}$, first we need to find a recursive formula for c_n.

Consider a particular kind parenthesization:

$$(A_1A_2\cdots A_k)(A_{k+1}\cdots A_n)$$

where the last two pairs of parenthesis are $(A_1\cdots A_k)$ and $(A_{k+1}\cdots A_n)$

- The number of parenthesizations of $A_1\cdots A_k$ is c_k.
- The number of parenthesizations of $A_{k+1}\cdots A_n$ is c_{n-k}.
- For each parenthesization of $A_1\cdots A_k$ and each parenthesization of $A_{k+1}\cdots A_n$, we get a valid parenthesization of $A_1\cdots A_n$.
- So the number of parenthesization of $A_1\cdots A_n$ that satisfies this condition is $c_k\cdot c_{n-k}$.
- The possible values for k: $1\leq k<n$. Therefore:

Catalan Number

$$c_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} c_k \cdot c_{n-k} & \text{if } n > 1 \end{cases}$$

Check a few cases:

- $c_2 = c_1 \cdot c_1 = 1 \cdot 1 = 1$
- $c_3 = c_1 \cdot c_2 + c_2 \cdot c_1 = 1 \cdot 1 + 1 \cdot 1 = 2$
- $c_4 = c_1 \cdot c_3 + c_2 \cdot c_2 + c_3 \cdot c_1 = 1 \cdot 2 + 1 \cdot 1 + 2 \cdot 1 = 5$

Now we use the generating function method to find the solution for $\{c_n\}_{n\geq 0}$.

Step 1. Define $f(x) = \sum_{n=0}^{\infty} c_n x^n$.

Step 2: Try to get an equation involving only $f(x)$ and x. Since $\{c_n\}_{n\geq 0}$ is not a linear recursive sequence, this is much harder to do.

Catalan Number

$$\begin{aligned}
(f(x))^2 &= (c_1x + c_2x^2 + c_3x^3 \cdots) \cdot (c_1x + c_2x^2 + c_3x^3 \cdots) \\
&= (c_1 \cdot c_1)x^2 + (c_1 \cdot c_2 + c_2 \cdot c_1)x^3 + (c_1 \cdot c_3 + c_2 \cdot c_2 + c_3 \cdot c_1)x^4 + \\
&\quad \cdots + (\textstyle\sum_{k=1}^{n-1} c_k \cdot c_{n-k})x^n + \cdots \\
&= c_2x^2 + c_3x^3 + c_4x^4 + \cdots + c_nx^n + \cdots \\
&= f(x) - c_1x = f(x) - x
\end{aligned}$$

It's pure luck we get this simple equation!

Step 3. Solve this equation for $f(x)$ in terms of x (note: this is a quadratic equation for $f(x)$):

$$f(x) = \frac{1}{2}(1 - \sqrt{1 - 4x})$$

Step 4. Find Taylor series for $f(x)$. There is no short cut this time.

Catalan Number

$$
\begin{array}{rclrcl}
f(x) & = & \frac{1}{2}(1-\sqrt{1-4x}) & f(0) & = & 0 \\
f'(x) & = & \frac{1}{2}\frac{1}{2}(1-4x)^{-\frac{1}{2}}\cdot 4 & & & \\
 & = & (1-4x)^{-\frac{1}{2}} & f'(0) & = & 1 = 1\cdot 2^0 \\
f^{(2)}(x) & = & \frac{1}{2}(1-4x)^{-\frac{3}{2}}4 & f^{(2)}(0) & = & 1\cdot 2^1 \\
f^{(3)}(x) & = & \frac{1}{2}\frac{3}{2}(1-4x)^{-\frac{5}{2}}4^2 & f^{(3)}(0) & = & 1\cdot 3\cdot 2^2 \\
f^{(4)}(x) & = & \frac{1}{2}\frac{3}{2}\frac{5}{2}(1-4x)^{-\frac{7}{2}}4^3 & f^{(4)}(0) & = & 1\cdot 3\cdot 5\cdot 2^3 \\
f^{(5)}(x) & = & \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{7}{2}(1-4x)^{-\frac{9}{2}}4^4 & f^{(5)}(0) & = & 1\cdot 3\cdot 5\cdot 7\cdot 2^4 \\
\ldots & & & & & \\
f^{(n)}(x) & = & \frac{1}{2}\frac{3}{2}\cdots\frac{2n-3}{2}(1-4x)^{-\frac{2n-1}{2}}4^{n-1} & f^{(n)}(0) & = & 1\cdot 3\cdot 5\cdots(2n-3)\cdot 2^{n-1} \\
\ldots & & & & &
\end{array}
$$

Step 4: Hence:

$$\begin{aligned}
f_n &= \frac{f^{(n)}(0)}{n!} = \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\cdot 2^{n-1} \\
&= \frac{1\cdot 3\cdot 5\cdots(2n-3)}{n!}\cdot\frac{2\cdot 4\cdots(2n-2)}{2\cdot 4\cdots(2n-2)}\cdot 2^{n-1} \\
&= \frac{(2n-2)!}{n!(n-1)!2^{n-1}}\cdot 2^{n-1} \\
&= \frac{(2n-2)!}{n!(n-1)!}
\end{aligned}$$

Matrix Chain Product Problem

Recursive Formulation

- Let $A_{i..j}$ $(1 \le i \le j \le n)$ be the chain product: $A_i \times A_{i+1} \times \cdots \times A_j$.
- Let $m[i,j]$ be the minimum number of scalar multiplications needed to compute $A_{i..j}$.

- We derive a recursive formula to compute $m[i,j]$.
- For $i > j$, $m[i,j]$ is undefined and not needed.
- For $i = j$, $A_{i..j} = A_i$, we have nothing to compute. So $m[i,i] = 0$ for all $1 \le i \le n$.
- For $i < j$: Suppose that we know the optimal parenthesization in $A_i \times \cdots \times A_j$, that gives the minimum cost.

Matrix Chain Product Problem

- Further assume that

$$\underbrace{(A_i \times \cdots \times A_k)}_{A_{i..k}} \times \underbrace{(A_{k+1} \times \cdots \times A_j)}_{A_{(k+1)..j}}$$

  are the last two pairs of parenthesis in the optimal parenthesization.
- The minimum cost for calculating $A_{i..k}$ is $m[i,k]$ by definition.
- The minimum cost for calculating $A_{(k+1)..j}$ is $m[(k+1),j]$ by definition.
- The size of $A_{i..k}$ is $p_{i-1} \times p_k$. The size of $A_{(k+1)..j}$ is $p_k \times p_j$. The cost of calculating $A_{i..k} \times A_{(k+1)..j}$ is $p_{i-1} \cdot p_k \cdot p_j$.
- Thus the total cost is $m[i,k] + m[(k+1),j] + p_{i-1} \cdot p_k \cdot p_j$.
- Of course, we do not know where the last two pairs of parenthesis are located. So we consider all possible positions $i \le k < j$, and take the minimum.

Recursive Algorithm

$$m[i,j] = \begin{cases} \text{undefined} & \text{if } i > j \\ 0 & \text{if } i = j \\ \min_{i \le k < j}\{m[i,k] + m[(k+1),j] + p_{i-1} \cdot p_k \cdot p_j\} & \text{if } i < j \end{cases}$$

- $m[1,n]$ is the minimum cost for computing $A_1 \times \cdots \times A_n$.
- The following recursive top-down algorithm calculates $m[i,j]$. We call MCP-REC$(1,n)$ to solve our problem.

MCP-REC(i,j)

1. If $i > j$, return "undefined"
2. If $i = j$, return 0
3. If $i < j$, return $\min_{i \le k < j}\{$ MCP-REC(i,k)+ MCP-REC$(k+1,j)$ $+p_{i-1} \cdot p_k \cdot p_j\}$

The algorithm is simple. But is it efficient?

Recursive Algorithm

- Say we call MCP-REC$(1,5)$ to compute $m[1,5]$.
- It will need to solve each of the following recursively: $m[1,1], m[2,5], m[1,2], m[3,5], m[1,3], m[4,5], m[1,4], m[5,5]$.
- Each of these will make several recursive calls.
- Many of these subproblems overlap. And we make repeated calls to solve the same subproblem over and over again!
- It can be shown this algorithm takes $\Theta(C_n) = \Omega(4^n/n^{3/2})$ time. Not acceptable.
- However, there are only at most n^2 subproblems! Why would it take exp time?
- All we have to do is calculate the 2D array $m[1..n, 1..n]$ according to the above formula.
- Must make sure when calculating $m[i,j]$, all entries needed have been calculated already.
- This gives the dynamic programming algorithm.

Dynamic Programming Algorithm

DynamicProg

1. Fill all entries below main diagonal by $-$.
2. Fill all entries on the main diagonal (from $m[1,1]$ to $m[n,n]$) by 0.
3. Fill the 2nd main diagonal according to above formula.
4. Fill the 3rd main diagonal ...
5. Output $m[1,n]$.

- There are $n-1$ entries on the 2nd main diagonal, each is the min of 1 terms.
- There are $n-2$ entries on the 3rd diagonal, each is the min of 2 terms.
- .....
- Only one entry ($m[1,n]$) on the last (nth) diagonal, it is the min of $n-1$ terms.
- So the total runtime is Θ of $\sum_{i=1}^{n-1} i \cdot (n-i) = \Theta(n^3)$.

Dynamic Programming Algorithm

- This algorithm only calculates the min cost of the MCP. It doesn't tell us how to put parenthesis into the chain.
- In order to do so, we must keep additional information.
- Define a 2D array $S[1..n, 1..n]$ where, for $i \leq j$, $S[i,j] = k$ if k corresponds to the term in the min sign that gives the value of $m[i,j]$.
- Using $S[*,*]$, the following algorithm puts parenthesis into the product chain.

MCP-Multiply(A, S, i, j)

if $i < j$ **then**
  $X =$MCP-Multiply$(A, S, i, S[i,j])$
  $Y =$MCP-Multiply$(A, S, S[i,j]+1, j)$
  return $X \times Y$
else
  return A_i
end if

Example: $A_1 \times \cdots \times A_6$

The dimensions are given below.

i	0	1	2	3	4	5	6
p_i	10	20	1	40	5	30	15

The matrix $m[*, *]$ is as follows (the $S[i, j]$ value is given in ()):

m_{ij}	1	2	3	4	5	6
1	0	200(1)	600(2)	450(2)	850(2)	1150(2)
2	-	0	800(2)	300(2)	950(2)	1100(2)
3	-	-	0	200(3)	350(4)	800(5)
4	-	-	-	0	6000(4)	5250(4)
5	-	-	-	-	0	2250(5)
6	-	-	-	-	-	0

So the optimal way is: $(M_1M_2)(((M_3M_4)M_5)M_6)$, which needs 1150 scalar multiplications.

The calculations are given below:

$m_{11} = m_{22} = m_{33} = m_{44} = m_{55} = m_{66} = 0$

$m_{12} = p_0p_1p_2 = 10 \times 20 \times 1 = 200$
$m_{23} = p_1p_2p_3 = 20 \times 1 \times 40 = 800$
$m_{34} = p_2p_3p_4 = 1 \times 40 \times 5 = 200$
$m_{45} = p_3p_4p_5 = 40 \times 5 \times 30 = 6000$
$m_{56} = p_4p_5p_6 = 5 \times 30 \times 15 = 2250$

Example

$m_{13} = \min\{m_{11}+m_{23}+p_0p_1p_3 = 0+800+10\times20\times40 = 8800, m_{12}+m_{33}+p_0p_2p_3 = 200+0+10\times1\times40 = 600\} = 600$
$S_{13} = 2$

$m_{24} = \min\{m_{22}+m_{34}+p_1p_2p_4 = 0+200+20\times1\times5 = 300, m_{23}+m_{44}+p_1p_3p_4 = 800+0+20\times40\times5 = 4800\} = 300$
$S_{24} = 2$

$m_{35} = \min\{m_{33}+m_{45}+p_2p_3p_5 = 0+6000+1\times40\times30 = 7200, m_{34}+m_{55}+p_2p_4p_5 = 200+0+1\times5\times30 = 350\} = 350$
$S_{35} = 4$

$m_{46} = \min\{m_{44}+m_{56}+p_3p_4p_6 = 0+2250+40\times5\times15 = 5250, m_{45}+m_{66}+p_3p_5p_6 = 6000+0+40\times30\times15 = 24000\} = 5250$
$S_{46} = 4$

$m_{14} = \min\{m_{11}+m_{24}+p_0p_1p_4 = 0+300+10\times20\times5 = 1300, m_{12}+m_{34}+p_0p_2p_4 = 200+200+10\times1\times5 = 450,$
$m_{13}+m_{44}+p_0p_3p_4 = 600+0+10\times40\times5 = 2600\} = 450$
$S_{14} = 2$

$m_{25} = \min\{m_{22}+m_{35}+p_1p_2p_5 = 0+350+20\times1\times30 = 950, m_{23}+m_{45}+p_1p_3p_5 = 800+6000+20\times40\times30 = 30800,$
$m_{24}+m_{55}+p_1p_4p_5 = 300+0+20\times5\times30 = 3300\} = 950$
$S_{25} = 2$

$m_{36} = \min\{m_{33}+m_{46}+p_1p_2p_6 = 0+5250+1\times40\times15 = 5850, m_{34}+m_{56}+p_2p_4p_6 = 200+2250+1\times5\times15 = 2525,$
$m_{35}+m_{66}+p_2p_5p_6 = 350+0+1\times30\times15 = 800\} = 800$
$S_{36} = 5$

Example

$m_{15} = \min\{m_{11} + m_{25} + p_0p_1p_5 = 0 + 950 + 10 \times 20 \times 30 = 6950, m_{12} + m_{35} + p_0p_2p_5 = 200 + 350 + 10 \times 1 \times 30 = 850,$
$m_{13} + m_{45} + p_0p_3p_5 = 600 + 6000 + 10 \times 40 \times 30 = 18600, m_{14} + m_{55} + p_0p_4p_5 = 450 + 0 + 10 \times 5 \times 30 = 1950\} = 850$
$S_{15} = 2$

$m_{26} = \min\{m_{22} + m_{36} + p_1p_2p_6 = 0 + 800 + 20 \times 1 \times 15 = 1100, m_{23} + m_{46} + p_1p_3p_6 = 800 + 5250 + 20 \times 40 \times 15 = 18050,$
$m_{24} + m_{56} + p_1p_4p_6 = 300 + 2250 + 20 \times 5 \times 15 = 4050, m_{25} + m_{66} + p_1p_5p_6 = 950 + 0 + 20 \times 30 \times 15 = 9950\} = 1100$
$S_{26} = 2$

$m_{16} = \min\{m_{11} + m_{26} + p_0p_1p_6 = 0 + 1100 + 10 \times 20 \times 15 = 4100, m_{12} + m_{36} + p_0p_2p_6 = 200 + 800 + 10 \times 1 \times 15 = 1150,$
$m_{13} + m_{46} + p_0p_3p_6 = 600 + 5250 + 10 \times 40 \times 15 = 11850, m_{14} + m_{56} + p_0p_4p_6 = 450 + 2250 + 10 \times 5 \times 15 = 3450,$
$m_{15} + m_{66} + p_0p_5p_6 = 850 + 0 + 10 \times 30 \times 15 = 5350\} = 1150$
$S_{16} = 2$

Elements of Dynamic Programming

If a problem has the following properties, it's likely that dynamic programming technique will work.

Optimal Substructure Property

An optimal solution to the problem Q contains within it optimal solutions of subproblems.

Example

MCP: Let $(A_1 \cdots A_k) \times (A_{k+1} \cdots A_n)$ be the optimal parenthesization of $A_1 \cdots A_n$ where $(A_1 \cdots A_k)$ and $(A_{k+1} \cdots A_n)$ are the last two pairs of parenthesis in the optimal solution. Then

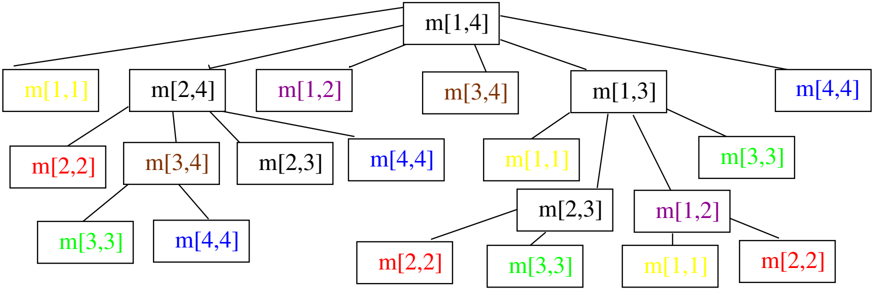
- The optimal solution restricted to $A_1 \cdots A_k$ is an optimal parenthesization of $A_1 \times \cdots \times A_k$.
- The optimal solution restricted to $A_{k+1} \cdots A_n$ is an optimal parenthesization of $A_{k+1} \times \cdots \times A_n$.

Elements of Dynamic Programming

Overlapping Subproblems Property

If the Optimal Substructure property holds, then the problem can be divided into sub-problems. If the sub-problems overlap, or depend on each other, we say the problem exhibit the Overlapping Subproblems Property.

MCP: Consider the recursion tree for computing $m[1, 4]$



Elements of Dynamic Programming

- The number of sub-problems is small (n^2).
- But the number of recursive calls is exp (because many repeated calls).
- In this case, we should use dynamic programming. Namely solve sub-problems bottom up, starting from the smallest sub-problems.
- We solve all possible sub-problems, even without knowing if we really need to solve them or not. (Do we really need to find $m[2,3]$? Have no idea!)
- But the total runtime is smaller!
- For MCP, the smallest problems are $m[i,i]$ for $1 \leq i \leq n$.
- For other problems, the smallest sub-problem might mean something else.
- Another way to do this is by Memorization:

Memorization

- First, fill the entire array $m[1..n, 1..n]$ by $-$.
- Then call the following recursive procedure.

MCP-Mem(i, j)

```
if m[i,j] ≠ − then
    return m[i,j]
else
    if i = j then
        m[i,j] = 0 and return 0
    else
        m[i,j] = min_{i≤k<j}{MCP-Mem(i,k) + MCP-Mem(k+1,j) + p_{i−1} · p_k · p_j}
        return m[i,j]
    end if
end if
```

- Once we have computed $m[i, j]$, it is memorized. So we will not make repeated call to solve it again.

Summary of Dynamic Programming

Matrix Chain Product is a representative of a group of other similar problems. We will discuss other versions in class.

Summary of Dynamic Programming

- Divide problem into sub-problems. Derive a recursive formula for getting the solution of the original problem from the solutions of sub-problems.
- Verify that the Optimal Substructure Property holds.
- Draw the recursion tree for several levels. If it shows the Overlap Subproblems Property, then the problem should be solved by dynamic programming.
- Select the proper order for solving subproblems: When solving a sub-problem, the solutions of all needed other sub-problems have been obtained.

0/1 Knapsack Problem

0/1 Knapsack Problem

Input: n *items*. Each item_i $(1 \leq i \leq n)$ has a *weight* $w[i]$ (pounds) and a *profit* $p[i]$ (dollars). We also have a *Knapsack* with *capacity* K (pounds).
Problem: Choose a subset of items and put them into the knapsack so that:

- The total weigh of the items we put into the knapsack is at most K.
- The total profit of the items we put into the knapsack is maximum.

Application 1:

You win a prize from your favorite candy shop. You are given a knapsack with capacity K pounds. You can fill the knapsack with any candy box you want. (But if the knapsack breaks, you get nothing). The box_i weighs $w[i]$ pounds and costs $p[i]$ dollars. How to pick boxes to maximize the total price?

0/1 Knapsack Problem

Application 2:

The knapsack is a super-computer at a computing center. On a particular day, there are K seconds computing time available for outside users. Each item is a user job. The job_i needs $w[i]$ seconds computing time, and the user will pay $p[i]$ dollars to the computing center, if job_i is run by the computing center. How should the computing center select the jobs, in order to maximize its revenue?

Application 3:

The knapsack is a hard disk drive, with K bytes capacity. Each item is a file. The size of the file_i is $w[i]$ bytes. The file owner will pay $p[i]$ cents to the disk owner if file_i is stored on the disk. How should the disk owner select the files in order to maximize his income?

Mathematical Description of the Problem:

Mathematical Description of the Problem:

Input: $2n+1$ positive integers, $p[1], \ldots, p[n], w[1], \ldots, w[n], K$.
Output: Find a 0/1 vector $(x_1, x_2, \ldots, x_n)$ such that:

1. $\sum_{i=1}^{n} x_i w[i] \leq K$;
2. $\sum_{i=1}^{n} x_i p[i]$ is maximized.

(Here, we put the item_i into the knapsack iff $x_i = 1$.)

The knapsack problem can be easily solved by the following algorithm.

Simple algorithm:

1. Enumerate all possible n-bit 0/1 vectors $(x_1, x_2, \ldots, x_n)$;
2. For each vector $(x_1, x_2, \ldots, x_n)$, calculate the total weight and the total profit of the subset of the items represented by the vector;
3. Select the vector with total weight $\leq K$ and maximum profit.

This algorithm works fine, except that there are 2^n n-bit 0/1 vectors and the algorithm must loop thru all of them. So the run time is at least $\Omega(2^n)$. This is not acceptable.

Dynamic Programming Algorithm:

- Define a 2D array $\text{Profit}[0..n][0..K]$.
- The value of the entries in $\text{Profit}[*][*]$ is defined as:

  $\text{Profit}[i][j] =$ the maximum profit if the knapsack capacity is j, and we can fill the knapsack only with a subset of $\text{item}_1, \text{item}_2, \ldots, \text{item}_i$.
- By this definition, the maximum profit for the original problem is $\text{Profit}[n][K]$.
- So, we only need to calculate this entry. To do so, use the following recursive formula.

Dynamic Programming Algorithm:

$$\text{Profit}[i][j] = \begin{cases} 0 & \text{if } i = 0 & (1) \\ 0 & \text{if } j = 0 & (2) \\ \text{Profit}[i-1][j] & \text{if } i \neq 0, j \neq 0 \text{ and } w[i] > j & (3) \\ \max\{\overbrace{\text{Profit}[i-1][j]}^{(4a)}, & \\ \underbrace{p[i] + \text{Profit}[i-1][j-w[i]]}_{(4b)}\} & \text{if } i \neq 0, j \neq 0 \text{ and } w[i] \leq j & (4) \end{cases}$$

Explanation:

(1) $i = 0$: we cannot put any item into the knapsack. So, the max profit is 0.

(2) $j = 0$: the capacity of the knapsack is 0. Thus we cannot put any item into it. So the max profit is again 0.

Dynamic Programming Algorithm:

(3) $w[i] > j$: We are allowed to use $\text{item}_1, \dots, \text{item}_{i-1}, \text{item}_i$. However, since $w[i] > j$, we cannot put item_i into the knapsack (its weight exceeds the knapsack capacity). Thus, we can actually only choose from $\text{item}_1, \dots, \text{item}_{i-1}$. Therefore, the max profit is $\text{Profit}[i-1][j]$.

(4) There are two choices: Either we put item_i into the knapsack, or we don't.

(4a) Do not put item_i into the knapsack. Then the capacity of the knapsack remains the same (j), and now we can only use $\text{item}_1, \dots, \text{item}_{i-1}$. So the max profit for this case is $\text{Profit}[i-1][j]$.

(4b) Put item_i into the knapsack. The remaining capacity is reduced by the weight of item_i so it becomes $j - w[i]$, and now we can only use $\text{item}_1, \dots,$ item_{i-1}. On the other hand, since we do put item_i into the knapsack, we gain its profit $p[i]$. So the max profit for this case is: $p[i] + \text{Profit}[i-1][j - w[i]]$.

Because we do not know which of the cases (4a) and (4b) gives larger profit, we take the maximum of the two cases.

Recursive Algorithm

RecKS(int i, int j)

1. **if** $((i = 0)$ or $(j = 0))$ return 0
2. **else if** $(w[i] > j)$ return RecKS$(i-1, j)$
3. **else** return **max** $\{$ RecKS$(i-1, j)$,$p[i]$+RecKS$(i-1, j-w[i])\}$

- This algorithm is very slow. The reason is that it makes many repeated recursive calls. It takes exponential time.
- The problem shows the Overlapping Sub-problem Property. So we should use dynamic programming.

Dynamic Programming algorithm

Input: $p[1..n]$, $w[1..n]$, K

1. **for** $(j = 0; j <= K)$ Profit$[0][j]$ = 0;
2. **for** $(i = 0; i <= n)$ Profit$[i][0]$ = 0;
3. **for** $(i = 1; i <= n)$
4. **for** $(j = 1; j <= K)$
5. **if** $(w[i] > j)$ Profit$[i][j]$ = Profit$[i-1][j]$;
6. **else** Profit$[i][j]$ = **max** (Profit$[i-1][j], p[i]+$Profit$[i-1][j-w[i]]$);
7. **output** Profit$[n][K]$;

Analysis

- We calculate the entries of Profit array row by row according to formula (1) - (4).
- When calculating an entry Profit$[i][j]$, it only depends on other entries that have been calculated already.
- Thus each entry needs $O(1)$ time to calculate.
- Since there are $(n+1)(K+1) = O(nK)$ entries in Profit array, the total run time of the algorithm is $O(nK)$.

Construct Solution Set

- This algorithm only calculates the max profit. It doesn't tell us the subset to be put into the knapsack.
- In order to do so, we need to keep additional information.
- We need another 2D array Dir$[1..n][1..K]$. The definition of Dir$[j][k]$ is given below and its calculation should be included in the lines (5) and (6) in the above algorithm.

$$\text{Dir}[i][j] = \begin{cases} 1 & \text{if Profit}[i][j] \text{ is set to Profit}[i-1][j]. \\ & \text{(It gets its value from above, i.e. from (3) or (4a).)} \\ 2 & \text{if Profit}[i][j] \text{ is set to } p[i]+\text{Profit}[i-1][j-w[i]]. \\ & \text{(It gets its value from upper left, i.e. from (4b).)} \end{cases}$$

- After calculating the arrays Profit$[*][*]$ and Dir$[*][*]$, the following code segment will print out items (in the reverse order) that should be included into the knapsack in order to achieve the max profit.

Construct Solution Set

Printout Items:

1. $j = K$;
2. **for** ($i = n$ **to** 1 **by** -1)
3.   **if** (Dir$[i][j] = 2$) $\{ j = j - w[i]$; and **print out** $\text{item}_i;\}$

- j keeps the remaining knapsack capacity, initialized to K.
- Start at the lower right corner Dir$[n][k]$.
- Dir$[i][j] = 2$: Profit$[i][j]$ gets its value from (4b) which means we put the item_i into the knapsack. So we print out this item, and reduce the capacity by its weight $w[i]$ (the line $j = j - w[i]$).
- Dir$[i][j] = 1$: Profit$[i][j]$ gets its value from formula (3) or (4a). In either case, item_i is not included in the knapsack, so we do nothing.

Example

Example

Input: $K = 13, n = 5, W[1] = 5, W[2] = 4, W[3] = 3, W[4] = 2, W[5] = 4, P[1] = 4, P[2] = 2, P[3] = 4, P[4] = 1, P[5] = 5$

Profit array:

	$j = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13
$i = 0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	4	4	4	4	4	4	4	4	4
2	0	0	0	0	2	4	4	4	4	6	6	6	6	6
3	0	0	0	4	4	4	4	6	8	8	8	8	10	10
4	0	0	1	4	4	5	5	6	8	8	9	9	10	10
5	0	0	1	4	5	5	6	9	9	10	10	11	13	13

Dir array:

	$j = 0$	1	2	3	4	5	6	7	8	9	10	11	12	13
$i = 0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	1	1	1	2	2	2	2	2	2	2	2	2
2	0	1	1	1	2	1	1	1	1	2	2	2	2	2
3	0	1	1	2	2	1	1	2	2	2	2	2	2	2
4	0	1	2	1	1	2	2	1	1	1	2	2	1	1
5	0	1	1	1	2	1	2	2	2	2	2	2	2	2

Items in the knapsack ("*" means the item is NOT in the knapsack).

Item	1	2	3	4	5
Weight	5	*	3	*	4
Profit	4	*	4	*	5

When printing out the items in the knapsack, we set $j = K = 13$ and start at the lower-right entry Dir$[5][13]$. Since this entry is 2, item_5 is in the knapsack. So we reduce j to $j - w[5] = 13 - 4 = 9$. Then we look at the entry Dir$[4][9]$ which is 1. So item_4 is not in the knapsack and j remains 9. Next we look at Dir$[3][9]$, and so on. (The red entries are visited by the printing algorithm.)

Longest Common Subsequence (LCS) Problem

- Let $\sum$ be an alphabet set. (Ex: $\sum = \{a, b, \ldots z\}$)
- $X = \langle x_1, x_2, \ldots, x_m \rangle$ is a sequence over $\sum$ (i.e. each $x_i \in \sum$.)
- $|X| = m$ denotes the length of X. $X[i] = x_i$ denotes the ith letter.
- $Z = \langle z_1, z_2, \ldots, z_k \rangle$ is another sequence of $\sum$.
- We say "Z is a subsequence of X" if Z can be obtained by deleting some letters from X.

Example

$Z = \langle BCDB \rangle$ is a subsequence of $X = \langle \underline{D}BC\underline{B}D\underline{C}B \rangle$

Longest Common Subsequence (LCS) Problem

Definition

Let X, Y and Z be three sequences. If Z is a subsequence of both X and Y, we say "Z is a common subsequence of X and Y".

Longest Common Subsequence (LCS) Problem

Input: Given $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$.
Find: a common subsequence Z of X and Y with maximum length.

Brute Force Approach

- Enumerate all subsequences of Y. (There are 2^n of them.)
- Check each of them to see if it is a subsequence of X.
- Pick a longest one.

This takes $\Omega(2^n)$ time.

Dynamic Programming Algorithm

Definition

A **prefix** of the sequence $X = \langle x_1 \dots x_m \rangle$ is $X_i = \langle x_1 \dots x_i \rangle$ $(1 \leq i \leq m)$

Theorem

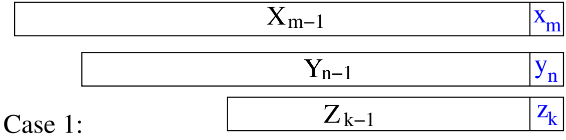
Optimal Substructure Property of LCS

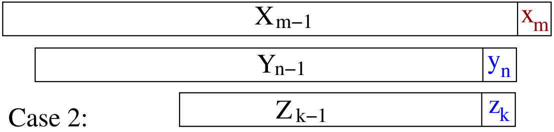
Let $X = \langle x_1, x_2, \dots, x_m \rangle$ and $Y = \langle y_1, y_2, \dots, y_n \rangle$. Let $Z = \langle z_1 \dots z_k \rangle$ be a LCS of X and Y.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is a LCS of X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$, then $z_k \neq x_m$ implies Z is a LCS of X_{m-1} and Y.
3. If $x_m \neq y_n$, then $z_k \neq y_n$ implies Z is a LCS of X and Y_{n-1}.

This theorem says: an LCS (Z) of X and Y contains within it an LCS of the prefixes of X and Y.

Recursive Formulation





Case 3 is similar.

Recursive Formulation

Define a 2D array $c[0..m,\ 0..n]$, where $c[i,j]$ is defined to be the length of the LCS of X_i and Y_j. Then:

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1,\ j-1] + 1 & \text{if } x_i = y_j \\ \max\{c[i,\ j-1],\ c[i-1,j]\} & \text{if } x_i \neq y_j \end{cases}$$

- If $i = 0$, $X_i = \emptyset$. The LCS$(\emptyset, Y_j) = \emptyset$. Its length is 0.
- If $j = 0$, $Y_j = \emptyset$. The LCS$(X_i, \emptyset) = \emptyset$. Its length is 0.
- if $x_i = y_j$, then we find the length of LCS(X_{i-1}, Y_{j-1}). This is $c[i-1,\ j-1]$. Since $x_i = y_j$ match, this is the last letter in LCS, so we add 1.
- If $x_i \neq y_j$, we find LCS(X_i, Y_{j-1}) (the length is $c[i,\ j-1]$) and LCS (X_{i-1}, Y_j) (the length is $c[i-1,\ j]$) and pick the longer one.

Dynamic Programming Algorithm

DynPro-LCS

- Fill the 2D array $c[*,*]$ row by row according to the recursive formula.
- Output $c[m,n]$.

- There are $(m+1)\cdot(n+1) = \Theta(nm)$ entries in $c[*,*]$.
- When we calculate $c[i,j]$, we may need the values $c[i-1,\ j-1]$, $c[i-1,\ j]$ and $c[i,\ j-1]$. They have been computed already. So each entry takes $O(1)$ time.
- So the algorithm takes $\Theta(nm)$ time.
- This algorithm only computes the length of the LCS, not the actual LCS. To do so, we need to keep more information.
- Define an array $b[1..m, 1..n]$ where $b[i,j] = \uparrow, \leftarrow$, or $\nwarrow$, pointing to the direction where $c[i,j]$ gets its value.

Dynamic Programming Algorithm

$X = \langle ABCBDAB \rangle$ and $Y = \langle BDCABA \rangle$

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	$\uparrow$0	$\uparrow$0	$\uparrow$0	$\nwarrow$1	$\leftarrow$1	$\nwarrow$1
2	B	0	$\nwarrow$1	$\leftarrow$1	$\leftarrow$1	$\uparrow$1	$\nwarrow$2	$\leftarrow$2
3	C	0	$\uparrow$1	$\uparrow$1	$\nwarrow$2	$\leftarrow$2	$\uparrow$2	$\uparrow$2
4	B	0	$\nwarrow$1	$\uparrow$1	$\uparrow$2	$\uparrow$2	$\nwarrow$3	$\leftarrow$3
5	D	0	$\uparrow$1	$\nwarrow$2	$\uparrow$2	$\uparrow$2	$\uparrow$3	$\uparrow$3
6	A	0	$\uparrow$1	$\uparrow$2	$\uparrow$2	$\nwarrow$3	$\uparrow$3	$\nwarrow$4
7	B	0	$\nwarrow$1	$\uparrow$2	$\uparrow$2	$\uparrow$3	$\nwarrow$4	$\uparrow$4

To construct LCS (in reverse order)

- Starts at $b[m, n]$, follow the arrows.
- For a $\nwarrow$, the corresponding letter is in LCS.
- For a $\leftarrow$ or a $\uparrow$, do nothing.
- Stop when reaching the first row or column.
- In the above, blue indicates the LCS, and the path for constructing it.

Summary on Using Dynamic Programming Algorithm

- Step 1: Analyze the structure of the problem, prove or convince yourself it has the Optimal Substructure Property.
- Step 2: Derive a recursive formulation of the problem. Namely, express the solution of the bigger problem in terms of the subproblems.
- Note: In most cases, we want to construct an optimal solution. However, it is often easier to concentrate on the value of the optimal solution. Once we have an alg. for computing the value, it's pretty easy to construct it. All me have to do is to memorize how the optimal value is obtained.
- Step 3: Write a recursive procedure according to the recursive formulation obtained in Step 2. Draw the recursion tree for a few levels. If the algorithm is making recursive calls to solve overlapping subproblems, or repeatedly solving the same subproblems, then you should use dynamic programming (i.e. bottom-up approach).
- Write a bottom up alg for solving subproblems. Pay attention to the order: When solving a subproblem, the solution of other subproblems needed by it have been obtained already.