Outline

1 Max-Flow Problems
2 Interpretation
3 Variations of Max-Flow Problem
4 Properties
5 Max-Flow Algorithm Outline
6 Residual Network and Augmenting paths
7 Ford-Fulkerson Algorithm
8 Max-Flow Min-Cut Theorem
9 Karp-Edmonds Algorithm
10 Maximum Matching
11 MM Problem for Bipartite Graphs
12 Connectivity Problems

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Note: The last condition is not essential. It is included here for convenience.

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$f(u, v)$ is called the net flow from u to v.

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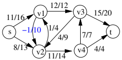

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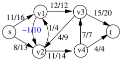

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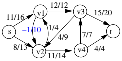

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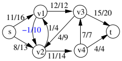

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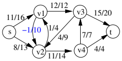

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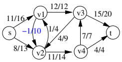

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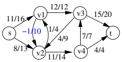

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- The flow value is: $|f| = 11 + 8 = 19$.

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- So we have: $f(v_2, v_1) = 1$ and $f(v_1, v_2) = -1$

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- $|f|$ is the total bandwidth of data flow from s to t through the network.

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- We discuss a few examples.

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Applications

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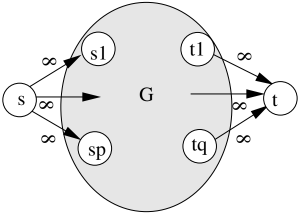

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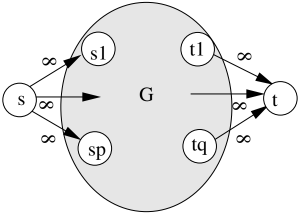

The max flow function of the converted network gives the answer of the original problem.

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Applications

- In the oil pipeline application, oil can flow in either direction in pipelines.
- In the internet communication application, data can be transmitted in either direction.

This problem can be easily converted to the basic problem: For each undirected edge $e = (u, v)$, replace e by a pair of directed edges $u \rightarrow v$ and $v \rightarrow u$. The capacity of these two edges are the same as the capacity of e.

Vertex Capacity

It is the same as the basic problem, except that:

- Each vertex $u \in V - \{s, t\}$ has a capacity $c(u) \geq 0$.

Max-Flow: Vertex Capacity

Vertex Capacity

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- Each vertex $u \in V - \{s,t\}$ has a capacity $c(u) \geq 0$.
- The total amount of flow going through u is at most $c(u)$. Namely:

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Application

In the oil pipeline application, each vertex u is an oil pumping station. $c(u)$ is the capacity of the pump.

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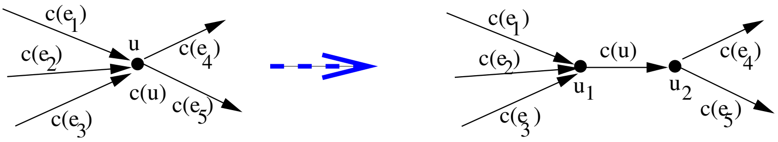
- For each edge $u \rightarrow w$, replace it by $u_2 \rightarrow w$ with the same capacity.



Max-Flow: Min-Cost Max-Flow Flow

Max-Flow: With Flow Cost

It is the same as the basic problem, except that:

- For each edge $u \to v$, in addition to capacity, there is a cost$(u \to v) \geq 0$. (Meaning: you must pay cost$(u \to v)$ in order to ship 1 unit flow through the edge $u \to v$.)
- The goal of the problem is to find a flow function $f(*)$ such that:
  - The flow value $|f|$ is maximum.

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$$\text{cost}(f) = \sum_{e=u\to v\in E} f(u \to v) \cdot \text{cost}(u \to v)$$

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- This problem cannot be easily converted to the basic problem.
- It can be solved by using similar idea.

Max-Flow: Multi-commodity Flow Problem

- We have a directed graph G. Each edge e has a capacity $c(e) \geq 0$.

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- We define a flow for commodity i: $f_i : V \times V \to R$. $f_i(*)$ must satisfy the Capacity, Skew Symmetry and Flow Conservation constraints.
- In addition, we require Aggregate Capacity constraint: For any edge $e = u \to v \in E$:

$$f(u,v) = \sum_{i=1}^{t} f_i(u,v) \leq c(u,v)$$

Max-Flow: Multi-commodity Flow Problem

Question:

Can we find flow functions $f_1, f_2, \ldots, f_t$ that satisfy all of these constraints, and also satisfy the demands for all commodities?

Application

In the Internet connetion network example, we must transmit d_i units data from the site s_i to the site t_i.

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This problem cannot be converted to the basic max-flow problems, and is significantly harder to solve.

Outline

1. Max-Flow Problems
2. Interpretation
3. Variations of Max-Flow Problem
4. Properties
5. Max-Flow Algorithm Outline
6. Residual Network and Augmenting paths
7. Ford-Fulkerson Algorithm
8. Max-Flow Min-Cut Theorem
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Definition

Let $X \subseteq V$ and $Y \subseteq V$ be two subsets of V. The total flow from X to Y is:

$$f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x \to y)$$

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- By the definition: $|f| = \sum_{v \in V} f(s,v) = f(s,V)$.
- The Flow Conservation Constraint is: For any $u \neq s,t$ we must have $f(u,V) = 0$.

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Lemma

Let $G = (V, E)$ be a flow network and f a flow function of G. Let X, Y and Z be any subsets of vertices of G. Then:

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1. $f(X, X) = 0$.
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3. If $X \cap Y = \emptyset$, then $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$.

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- Statement 1: the total flow from a subset into itself is 0.
- Statement 2: the total flow from X into Y is equal to the negative of the total flow from Y into X.
- Statement 3: If $X$and Y are disjoint, then the total flow from $X \cup Y$ to Z is the sum of the flow from X to Z and the flow from Y to Z.

Max-Flow: Properties

Proof.

2.

$$\begin{aligned} f(X,Y) &= \sum_{x\in X}\sum_{y\in Y} f(x,y) \quad \text{(by skew symmetry property)} \\ &= \sum_{x\in X}\sum_{y\in Y} -f(y,x) = -\sum_{y\in Y}\sum_{x\in X} f(y,x) \\ &= -f(Y,X) \quad \text{(by the definition of } f(Y,X)\text{.)} \end{aligned}$$

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1. Since statement 2 is true for any X and Y, plug in $Y = X$: $f(X,X) = -f(X,X)$. This implies $f(X,X) = 0$.

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4. Similar to 3.

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Fact

We can show $|f| = f(V,t)$ (i.e. the flow value is also the total amount of flow into the sink t.)

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Max-Flow: Algorithm

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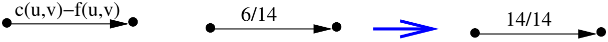
How an edge $e = u \to v$ can be used to increase the flow?

Case 1: $f(u \to v) \geq 0$. Then $c(u \to v) - f(u \to v)$ additional flow can go through e from u to v.



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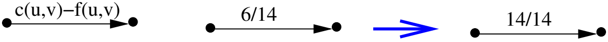

In this example, the capacity is $c(e) = 14$, the current flow is $f(e) = 6$. It is possible we can increase the flow by $c(e) - f(e) = 14 - 6 = 8$. (This is the maximum amount of additional flow that can be pushed through e without exceeding the capacity.)

Case 2: $f(u \to v) < 0$. Then $a = f(v \to u) = -f(u \to v) > 0$. We can do two things:

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Case 2: $f(u \to v) < 0$. Then $a = f(v \to u) = -f(u \to v) > 0$. We can do two things:

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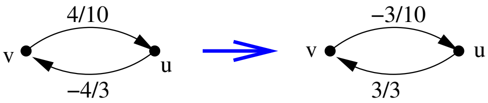
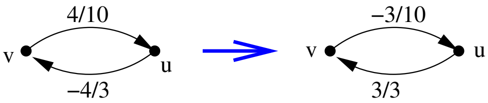
- The net effect is that we can push $c(u \to v) + a$ units flow from u to v. which is $c(u \to v) - f(u \to v)$.



Case 2: $f(u \to v) < 0$. Then $a = f(v \to u) = -f(u \to v) > 0$. We can do two things:

- Cancel the a units flow from v to u.
- Push $c(u \to v)$ units flow from u to v.
- The net effect is that we can push $c(u \to v) + a$ units flow from u to v. which is $c(u \to v) - f(u \to v)$.



In this example $f(u \to v)$ is changed from -4 to 3. The net change is $7 = 3 - (-4) = c(u \to v) - f(u \to v)$.

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Max-Flow: Algorithm

Residual Network

Let $G = (V, E)$ be a flow network and $f(*)$ a flow function of G. The residual network of G (with respect to f) is $G_f = (V, E_f)$ where:

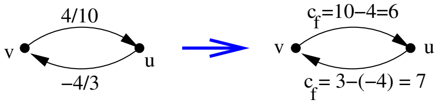
$$E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$$

where

$$c_f(u, v) = c(u, v) - f(u, v)$$

is called the residual capacity of (u, v).

Note: $c_f(u, v)$ is the maximum amount of additional flow that can be pushed from u to v.



Max-Flow: Algorithm

Augmenting Path

- Let $G = (V, E)$ be a flow-network, f a flow function of G.
- Let G_f be the residual network of G with respect to f.
- Let P be a path in G_f from s to t. P is called an augmenting path of G_f.
- Define: $c_f(P) = \min\{c_f(u,v) \mid (u,v)$ is an edge on $P\}$.
- Define a flow function $f_P(*)$ on G_f by:

$$f_P(u,v) = \begin{cases} c_f(P) & \text{if } (u,v) \text{ is on } P \\ -c_f(P) & \text{if } (v,u) \text{ is on } P \\ 0 & \text{otherwise} \end{cases}$$

- Define a new flow function $f'(*)$ of G by:

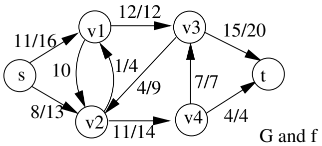
$$f'(e) = f(e) + f_P(e) \qquad \text{for all } e \in E$$

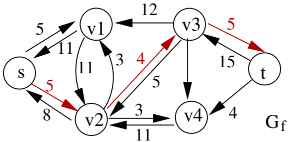
Max-Flow: Algorithm

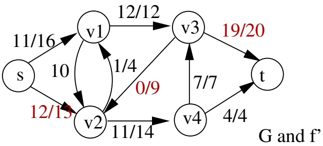
- Along the path P, we can push more flow from s to t.

Max-Flow: Algorithm

- Along the path P, we can push more flow from s to t.
- $c_f(P)$ is the maximum amount of additional flow we can push along P. This is because there is an edge (u, v) on P whose residual capacity is $c_f(P)$. This is the bottle neck edge.
- $f'(*)$ is a new flow function with $|f'| = |f| + |f_P| = |f| + c_f(P)$. In other words, we increased the flow value by the amount $c_f(P)$.







augmenting path P

|f| = 11+8 = 19

c_f(P) = min {5,4,5} = 4

|f'| = 11+12 = 23

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Ford-Fulkerson Algorithm

for each $e = (u, v) \in E$ **do**

  $f(u, v) = f(v, u) = 0$

while there's a path P from s to t in G_f **do:**

  $c_f(P) = \min\{c_f(u, v) \mid (u, v) \text{ is on } P\}$

  for each $e = (u, v)$ on P **do**

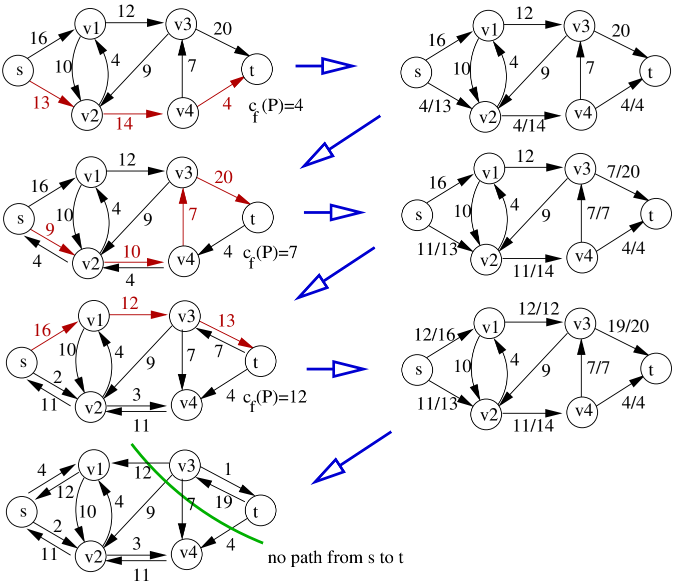
    $f(u, v) = f(u, v) + c_f(P)$

    $f(v, u) = -f(u, v)$

end while

return f

Max-Flow: Example



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Theorem

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Definition

Let $G = (V, E)$ be a flow network with source s and sink t. A cut of G is a partition of the vertex set V into two subsets S and T such that:

- $S \cap T = \emptyset$ and $S \cup T = V$.
- $s \in S$ and $t \in T$.
- The capacity of the cut is: $c(S, T) = \sum_{u \in S, v \in T, (u \to v) \in E} c(u, v)$
- The flow cross the cut is: $f(S, T) = \sum_{u \in S, v \in T, (u \to v) \in E} f(u, v)$

Proof of Correctness

Lemma (26.5)

Let G be a flow network with source s and sink t. Let f be a flow function of G. Let (S, T) be any cut of G. Then:

$$f(S, T) = |f|$$

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Proof.

$$\begin{array}{lcll} f(S,T) & = & f(S,V) - f(S,S) & \text{(because } S \cup T = V \ S \cap T = \emptyset\text{)} \\ & = & f(S,V) & \text{(because } f(S,S) = 0\text{)} \\ & = & f(s,V) + f((S - \{s\}), V) & \text{(because } S = \{s\} \cup (S - \{s\})\text{)} \\ & = & f(s,V) & \text{(because the flow conservation} \\ & & & \text{constraint, the 2nd term is 0)} \\ & = & |f| & \end{array}$$

□

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Proof.

$$\begin{array}{rcll}
|f| & = & f(S,T) & \text{(because Lemma 26.5)} \\
& = & \sum_{u\in S, v\in T, (u\to v)\in E} f(u,v) & \text{(by definition of } f(S,T)\text{)} \\
& \le & \sum_{u\in S, v\in T, (u\to v)\in E} c(u,v) & \text{(by capacity constraint)} \\
& = & c(S,T) & \text{(by definition of } c(S,T)\text{)}
\end{array}$$

□

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Max-Flow Min-Cut Theorem (26.6)

The following three statements are equivalent:

1. f is a max flow of G.
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(2) $\Rightarrow$ (3): Define: $S = \{v \in V \mid$ there is a path in G_f from s to $v\}$ and $T = V - S$. We show (S, T) is a cut of G:

- $s \in S$: This is trivial because s itself is a path from s to s.
- $t \in T$: Because the condition (2), $t \notin S$. Hence $t \in T$.
- By the definition of T, we have $S \cap T = \emptyset$ and $S \cup T = V$.

Hence (S, T) is a cut of G.

Proof of Correctness

Proof (continued): For each edge $u \to v$ with $u \in S$ and $v \in T$, we have $f(u,v) = c(u,v)$. This is because if $f(u,v) < c(u,v)$, then $u \to v$ would be an edge in G_f with residual capacity $c(u,v) - f(u,v) > 0$. That means there is a path $s \to u \to v$ in G_f. This contradicts to the fact that $v \in T$.
Therefore $|f| = f(S,T) = c(S,T)$.

Proof (continued): For each edge $u \rightarrow v$ with $u \in S$ and $v \in T$, we have $f(u, v) = c(u, v)$. This is because if $f(u, v) < c(u, v)$, then $u \rightarrow v$ would be an edge in G_f with residual capacity $c(u, v) - f(u, v) > 0$. That means there is a path $s \rightarrow u \rightarrow v$ in G_f. This contradicts to the fact that $v \in T$.
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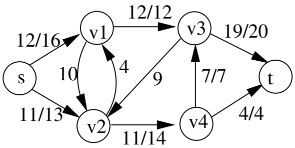

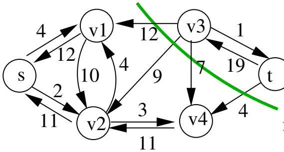

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Notes

- The equivalence of (1) and (3) says: For any flow network, the value of a max-flow is equal to the capacity of a min-cut. Hence, this theorem is called Max-Flow Min-Cut Theorem.

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- This theorem is a fundamental theorem in mathematics. It appears in different branches, in different forms, by different names.

Proof of Correctness

Now we can prove:

Theorem

When Ford-Fulkerson algorithm terminates, f is a max-flow function of G.

Proof.

The algorithm stops only when there is no $s \to t$ path in G_f can be found. By Max-Flow Min-Cut Theorem (the equivalence of the statements (1) and (2)), f is a max flow. $\square$

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No! There's a catch here: When the algorithm terminates ...
How do we know the algorithm will stop?

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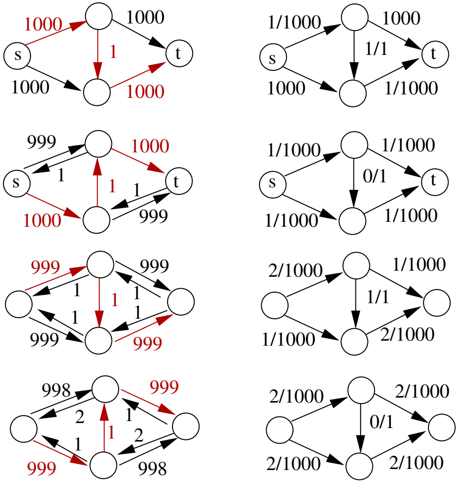
  - If we pick augmenting path as shown, each iteration increases flow by 1.
  - So it takes 2000 iterations.
- If the capacities can be real numbers, there are examples of flow network and sequence of bad choices of augmenting paths for which the algorithm will never stop!

A Bad Example



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Karp-Edmonds Theorem

The while loop in Karp-Edmonds Algorithm runs at most $|V| \cdot |E| = nm$ iterations.

Karp-Edmonds Algorithm: Analysis

Theorem

Karp-Edmonds Algorithm solves the max flow problem in $O(nm^2)$ time.

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  - Construct G_f. This takes $O(n+m)$ time.
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- So the total run time is $O(nm(n+m)) = O(nm^2)$. (This is because $n \leq m$).
- Note: Max-flow is a fundamental problem. Great efforts have been made to reduce the runtime. But for general cases, only small improvements have been made.

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- A matching M is a perfect matching if every vertex u in G is incident to an edge in M.

Application: Maximum Matching

- In Fig 1 below, the blue edges form a maximal matching of G. (If we add any edge $e \notin M$ into M, it will no longer be a matching.) But it is not a maximum matching.
- In Fig 2, the set of blue edges is a maximum matching. It is also a perfect matching of G.

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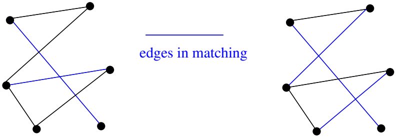

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- This problem is much harder than it looks. It can be solved in polynomial time, but quite complicated.
- If the problem is to find a maximal matching, it can be solved by the following easy algorithm:

Maximal Matching$(G = (V, E))$

1. $M \leftarrow \emptyset$
2. **while** $E \neq \emptyset$ **do**
3. pick any edge e in E, add e into M
4. delete from E all edges that share a common vertex with e
5. **output** M

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Maximum Matching

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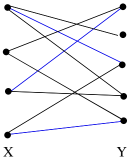
Maximum Matching for Bipartite Graph (MMBG) Problem

Given a bipartite graph $G = (X, Y, E)$, find a maximum matching of G.

MMBG Problem

The following Fig shows a bipartite graph $G = (X, Y, E)$ and a matching M of G. Is M maximum? It's not clear.



Application 1 of MMBG: Marriage Problem

Given a bipartite graph $G = (X, Y, E)$:

- $X = \{x_1, x_2, \ldots, x_p\}$ is the set of boys.

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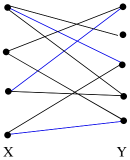

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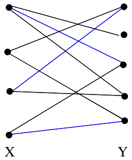

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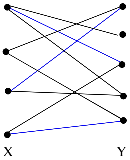

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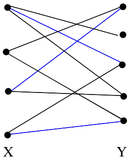

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- If you run an on-line match-making web-site, you need this algorithm.

MMBG Problem

Application 2 of MMBG: Distinct Representative Problem

- UB has p student clubs $C_1, C_2, \ldots, C_p$.
- A student may be a member of several clubs.
- Need to select a committee so that:
  - Each club has one representative in the committee.
  - Each student in the committee represents only one club (even if he/she is a member of multi-clubs.)

How to do this?

MMBG Problem

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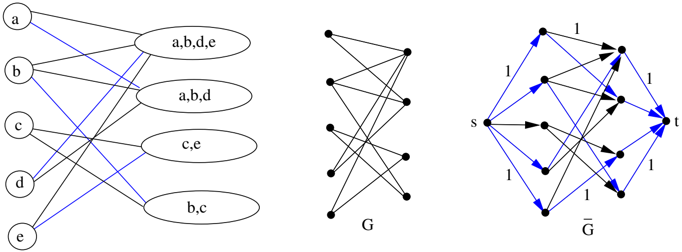
This is a MMBG problem. Define a bipartite graph $G = (X, Y, E)$:

- $X = \{C_1, C_2, \ldots, C_p\}$ (each element of X is a club.)
- $Y = \{y_1, y_2, \ldots, y_q\}$ (each element of Y is a student.)
- $(C_i, y_j) \in E$ if and only if y_j is a member of C_i.
- Find a MM M of G. If every C_i is incident to an edge in M, then we can select the committee. If not, this is impossible.

MMBG Problem



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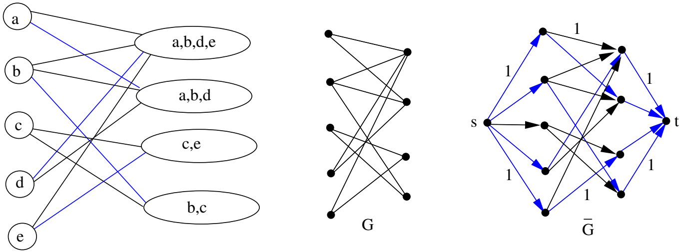

Converting MMBG to Max-Flow

Given an input instance $G = (X, Y, E)$ of MMBG, we construct a flow network $\bar{G}$ as follows:

- $V(\bar{G}) = X \cup Y \cup \{s, t\}$
- $E(\bar{G}) = \{s \to x \mid \forall x \in X\} \cup \{y \to t \mid \forall y \in Y\} \cup \{x \to y \mid \forall (x, y) \in E\}$
- All edges have capacity 1.

MMBG Problem

Lemma

Let M be a MM of $G = (V, E)$. Let f be a max-flow function of $\bar{G}$. Then $|M| = |f|$.

Proof.

Let M be a MM of G. Suppose $M = \{e_1, e_2, \ldots, e_k\}$, where $e_i = (x_i, y_i)$. Let $f_M(*)$ be defined as follows:

- $f_M(s \to x_i) = 1$ for all $1 \leq i \leq k$
- $f_M(y_i \to t) = 1$ for all $1 \leq i \leq k$
- $f_M(x_i \to y_i) = 1$ for all $1 \leq i \leq k$
- $f_M(e) = 0$ for all other edges.

It is easy to check f_M is a flow function of $\bar{G}$ and $|f_M| = k$. Since f is a max flow, we have $|f| \geq |f_M| = k = |M|$. □

MMBG Problem

Proof.

Conversely, let f be a max flow function of $\bar{G}$. Because all edge capacities of $E(\bar{G})$ are 0/1, it is easy to see that $f(e)$ is 0/1 for all edges e. Define:

$$M_f = \{(x,y) \mid x \in X, y \in Y \text{ and } f(x \to y) = 1\}$$

We show M_f is a matching of G. It's enough to show each vertex of G is incident to at most one edge in M_f. Suppose $M_f = \{(x_1,y_1),(x_2,y_2),\ldots,(x_t,y_t)\}$. Consider an edge $(x_i,y_i) \in M_f$.

- $(x_i,y_i) \in M_f$ is because $f(x_i \to y_i) = 1$.
- Because $f(e) = 0$ or 1 for all edges, and the flow conservation constraint at x_i, we must have: (a) $f(s \to x_i) = 1$ and (b) $f(x_i, y_j) = 0$ for all $y_j \neq y_i$.
- So $(x_i,y_j) \notin M$ for all $j \neq i$. Namely x_i is incident to exactly one edge in M_f
- Similarly, we can show y_i is incident to exactly one edge in M_f.

So M_f is a matching of G. But M is a MM of G. Thus $|M| \geq |M_f| = t = |f|$.

□

- The conversion from G to $\bar{G}$ can be done in $O(n+m)$ time.

MMBG Problem: Time Analysis

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- For this kind network, the max-flow algorithm runs much faster.
- Currently, the best algorithm for solving MMBG is Karp-Hopcroft algorithm, with runtime $O(n^{1/2}m)$.

Weighted MM and MMBG Problem

Weighted MM Problem

Input: An undirected graph $G = (V, E)$. Each edge $e \in E$ has a weight $w(e) \geq 0$.
Find: A matching M of G so that the total weight of M $w(M) = \sum_{e \in M} w(e)$ is maximum.

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Weighted MMBG Problem

The weighted version of the MMBG problem

Weighted MMBG Problem: Application

The Weighted MMBG Problem is also called:

Personnel Assignment Problem

A bipartite edge weighted graph $G = (X, Y, E)$ represents the following:

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A maximum weight matching M of G is the optimal worker assignment.

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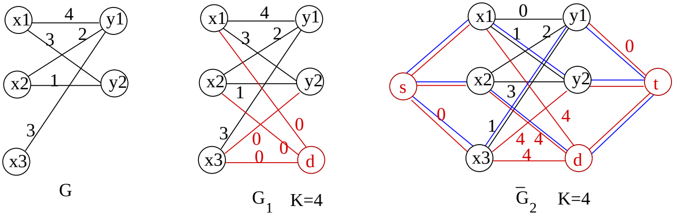
  - Let K be the largest $w(e)$ value for all edges e in G_1. For each edge e in G_1, define capacity $c(e) = 1$ and $cost(e) = K - w(e)$.

Weighted MMBG Problem: Application



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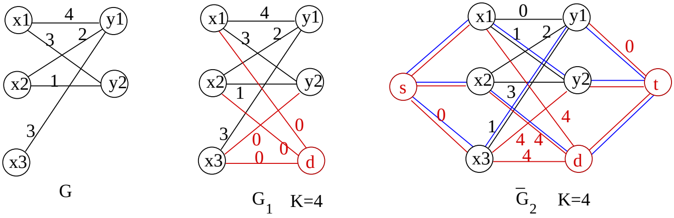

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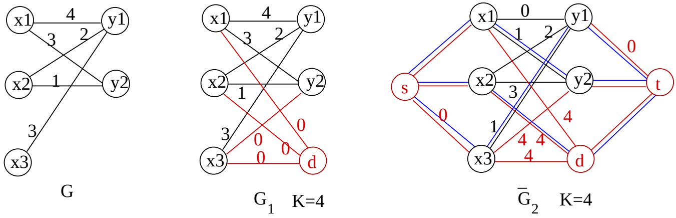

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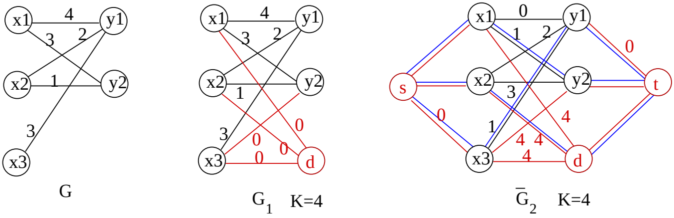

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- $\bar{G}_2$ is the flow-network constructed from G_1. All edges have capacity 1. The edges adjacent to s and t have $cost = 0$.. The cost of other edges are as marked. The flow on each of the three blue paths is 1. The flow on all other edges are 0. The corresponding assignment is: x_1 is assigned to y_2. x_3 is assigned to y_1. x_2 has no real assignment.

Weighted MMBG Problem: Application

We can argue this procedure indeed solves the personnel assignment problem. Let f be the min-cost max-flow function of G_2.

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- For each edge e in G_1, $cost(e) = K - w(e)$. So minimizing cost is the same as maximizing the profit (i.e. weight $w(e)$.)

Outline

1. Max-Flow Problems
2. Interpretation
3. Variations of Max-Flow Problem
4. Properties
5. Max-Flow Algorithm Outline
6. Residual Network and Augmenting paths
7. Ford-Fulkerson Algorithm
8. Max-Flow Min-Cut Theorem
9. Karp-Edmonds Algorithm
10. Maximum Matching
11. MM Problem for Bipartite Graphs
12. Connectivity Problems

Connectivity Problems

Vertex Connectivity

Let $G = (V, E)$ be an undirected graph. The vertex connectivity of G, denoted by $\kappa(G)$, is the minimum number of vertices that must be deleted from G in order to disconnect G.

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The vertex connectivity and the edge connectivity can also be defined for directed graphs with similar meaning.

Connectivity Problems

$\kappa(G)$ and $\kappa'(G)$ are two fundamental parameters for G. How to find them?

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- We can find if $\kappa(G) \geq 2$ by a much more complicated application of DFS in $O((n+m)\log n)$ time. (We do dot discuss details here).
- But for a general k, how do we determine if $\kappa(G) \geq k$ or not?

Brute-Force-Vertex-Connectivity$(G = (V, E))$

1. Enumerate all subsets $C \subset V$.
2. **for** each $C \subset V$ generated **do**
3. delete C from G
4. test if $G - C$ is connected
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However, there are 2^n vertex subsets. This would take $\Omega(2^n)$ time.

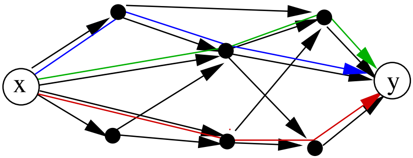
Connectivity Problems: Using Max-Flow

We consider the computation of $\kappa'(G)$ for directed graph G first.

Definition

Let $G = (V, E)$ be a directed graph and x, y two vertices of G.

$$\begin{aligned}\kappa'_G(x, y) &= \text{the minimum number of edges that must be deleted} \\ &\quad\ \text{from } G \text{ in order to disconnect } x \text{ from } y. \\ &= \text{the maximum number of edge disjoint paths from } x \text{ to } y.\end{aligned}$$



Note: The equivalence of the two definitions of $\kappa'_G(x, y)$ is another form of the max-flow min-cut theorem.

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$\kappa'_G(x,y)$ can be computed as follows:

- Treat $G=(V,E)$ as a flow network.
- x is the source and y is the sink.
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$$\kappa'(G) = \min_{x,y \in V, x \neq y} \kappa'_G(x, y)$$

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1. **for** each pair of vertices $x, y \in V$ **do**
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Fact

Let $T(n,m)$ be the run time for solving the max-flow problem for this special case. (Because of the special structure of the input, it is easier than the general max-flow problem). Then the edge connectivity problem can be solved in $\Theta(n^2 T(n,m))$ time.

Connectivity Problems

The edge connectivity problem for undirected graph is very similar.

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We can calculate $\kappa'_G(x, y)$ by using max-flow algorithm. The only difference is that $G = (V, E)$ is an undirected network with source x and sink y. Then we can change it to the basic max-flow problem as we discussed before.

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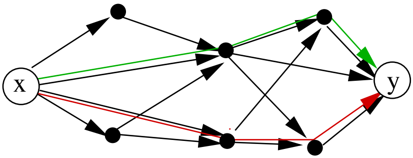

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- So this is the max-flow problem for directed network, with both edge and vertex capacities.
- It can be converted to the basic max-flow problem as discussed before.
- $\kappa_G(x, y)$ can be computed in $\Theta(T(n, m))$ time.
- $\kappa(G) = \min_{x,y \in V} \kappa_G(x, y)$ can be computed in $\Theta(n^2 T(n.m))$ time.
- The vertex connectivity problem for directed graph is almost identical