Outline

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- If $T_1(n) = \Theta(T_2(n))$, then the efficiency of the two algorithms are about the same (when n is large).
- If $T_1(n) = o(T_2(n))$, then the efficiency of the algorithm A_1 will be better than that of algorithm A_2 (when n is large).

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- We compare the growth rate of $T_1(n)$ and $T_2(n)$.
- If $T_1(n) = \Theta(T_2(n))$, then the efficiency of the two algorithms are about the same (when n is large).
- If $T_1(n) = o(T_2(n))$, then the efficiency of the algorithm A_1 will be better than that of algorithm A_2 (when n is large).
- By using the definitions, we can directly show whether $T_1(n) = O(T_2(n))$, or $T_1(n) = \Omega(T_2(n))$. However, it is not easy to prove the relationship of two functions in this way.

Outline

Limit Test

Limit Test is a powerful method for comparing functions.

Limit Test

Let $T_1(n)$ and $T_2(n)$ be two functions. Let $c = \lim_{n\to\infty} \frac{T_1(n)}{T_2(n)}$.

1. If c is a constant > 0, then $T_1(n) = \Theta(T_2(n))$.
2. If $c = 0$, then $T_1(n) = o(T_2(n))$.
3. If $c = \infty$, then $T_1(n) = \omega(T_2(n))$.
4. If c does not exists (or if we do not know how to compute c), the limit test fails.

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Proof of (1): $c = \lim_{n\to\infty} \frac{T_1(n)}{T_2(n)}$ means: $\forall \epsilon > 0$, there exists $n_0 \geq 0$ such that for any $n \geq n_0$: $\left|\frac{T_1(n)}{T_2(n)} - c\right| \leq \epsilon$; or equivalently: $c - \epsilon \leq \frac{T_1(n)}{T_2(n)} \leq c + \epsilon$. Let $\epsilon = c/2$ and let $c_1 = c - \epsilon = c/2$ and $c_2 = c + \epsilon = 3c/2$, we have

$$c_1 T_2(n) \leq T_1(n) \leq c_2 T_2(n)$$

for all $n \geq n_0$. Thus $T_1(n) = \Theta(T_2(n))$ by definition.

Example

Example 1

$$T_1(n) = 10n^2 + 15n - 60, \quad T_2(n) = n^2$$

$$\lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \lim_{n\to\infty} \frac{10n^2 + 15n - 60}{n^2} = \lim_{n\to\infty} (10 + \frac{15}{n} - \frac{60}{n^2}) = 10 + 0 - 0 = 10$$

Since 10 is a constant > 0, we have $T_1(n) = \Theta(T_2(n)) = \Theta(n^2)$ by the statement 1 of Limit Test (as expected).

Log function

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$$\lg = \log_2 n$$

$$\log n = \log_{10} n$$

$$\ln n = \log_e n$$

($\ln n$ is the log function with the natural base $e = 2.71828\ldots$).

Log base change formula

Log base change formula

For any $1 < a, b$, $\log_b n = \frac{\log_a n}{\log_a b} = \log_b a \cdot \log_a n$.

Proof: Let $k = \log_b n$. By definition: $n = b^k$.

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This implies: $\log_b n = k = \frac{\log_a n}{\log_a b}$.

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Let $n = a$ in this formula and note $1 = \log_a a$:

$$\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$$

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Let $n = a$ in this formula and note $1 = \log_a a$:

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This proves the second part of the formula.

Outline

1. Compare the growth rate of functions
2. Limit Test
3. L'Hospital Rule
4. Stirling Formula
5. Summations
6. Integration Method
7. Solving Linear Recursive Equations

L'Hospital Rule

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- If $\lim_{n\to\infty} f(n) = 0$ and $\lim_{n\to\infty} g(n) = 0$, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

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- If $\lim_{n\to\infty} f(n) = \infty$ and $\lim_{n\to\infty} g(n) = \infty$, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Example

Example 2

$T_1(n) = n^2 + 6$, $T_2(n) = n \lg n$. (Recall: $\lg n = \log_2 n$.)

$$\begin{aligned}
\lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} &= \lim_{n\to\infty} \frac{n^2+6}{n \lg n} = \lim_{n\to\infty} \frac{n+\frac{6}{n}}{\lg n} \\
&= \lim_{n\to\infty} \frac{1-\frac{6}{n^2}}{\frac{1}{\ln 2 \cdot n}} \text{ (by L'Hospital Rule)} \\
&= \ln 2 \lim_{n\to\infty} (n - \frac{6}{n}) = \ln 2(\infty - 0) = \infty
\end{aligned}$$

By Limit Test, we have $n^2 + 6 = \omega(n \lg n)$.

Example

Example 3

$T_1(n) = (\ln n)^k$, $T_2(n) = n^\epsilon$, where $k > 0$ is any (large) constant and $\epsilon > 0$ is any (small) constant. (Recall: $\ln n = \log_e n$.)

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \lim_{n\to\infty} \frac{(\ln n)^k}{n^\epsilon} \text{ (use L'Hospital Rule)} \\
= & \lim_{n\to\infty} \frac{k(\ln n)^{k-1} \times (1/n)}{\epsilon n^{(\epsilon-1)}} \\
= & \frac{k}{\epsilon} \lim_{n\to\infty} \frac{(\ln n)^{k-1}}{n^\epsilon} \text{ (use L'Hospital Rule again and simplify)} \\
= & \frac{k(k-1)}{\epsilon^2} \lim_{n\to\infty} \frac{(\ln n)^{k-2}}{n^\epsilon} \text{ (use L'Hospital Rule } k \text{ times)} \\
& \ldots\ldots \\
= & \frac{k(k-1)\cdots 2\cdot 1}{\epsilon^k} \lim_{n\to\infty} \frac{1}{n^\epsilon} = 0
\end{aligned}
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= \; & \frac{k(k-1)\cdots 2\cdot 1}{\epsilon^k} \lim_{n\to\infty} \frac{1}{n^\epsilon} = 0
\end{aligned}
$$

So by Limit Test, $(\ln n)^k = o(n^\epsilon)$ for any k and ϵ. For example, take $k = 100$ and $\epsilon = 0.01$, we have $(\ln n)^{100} = o(n^{0.01})$.

Example

Example 4

$T_1(n) = n^k$, $T_2(n) = a^n$, where $k > 0$ is any (large) constant and $a > 1$ is any constant bigger than 1.

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \lim_{n\to\infty} \frac{n^k}{a^n} \text{ (using L'Hospital Rule)} \\
= \; & \lim_{n\to\infty} \frac{k \cdot n^{k-1}}{\ln a \cdot a^n} = \frac{k}{\ln a} \lim_{n\to\infty} \frac{n^{k-1}}{a^n} \text{(using L'Hospital Rule } k \text{ times)} \\
= \; & \frac{k(k-1)\cdots 2\cdot 1}{(\ln a)^k} \lim_{n\to\infty} \frac{n^0}{a^n} \\
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\end{aligned}
$$

So by Limit Test, $n^k = o(a^n)$ for any $k > 0$ and $a > 1$. For example, take $k = 1000$ and $a = 1.001$, we have $n^{1000} = o((1.001)^n)$.

Example

Example 5

$T_1(n) = \log_a n$, $T_2(n) = \log_b n$, where $a > 1$ and $b > 1$ are any two constants bigger than 1.

By the Log Base Change Formula: $\log_b n = \log_b a \cdot \log_a n$

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Thus: $\lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \lim_{n\to\infty} \frac{\log_a n}{\log_b n} = \lim_{n\to\infty} \frac{\log_a n}{\log_b a \cdot \log_a n} = \frac{1}{\log_b a}$

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Since $\frac{1}{\log_b a} > 0$ is a constant, we have $\log_a n = \Theta(\log_b n)$ by Limit Test.

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So: **the growth rates of the $\log$ functions are the same for any base > 1.**

Example 6

$T_1(n) = a^n$, $T_2(n) = b^n$, where $1 < a < b$ are any two constants.

We have: $\lim_{n\to\infty} \frac{T_1(n)}{T_2(n)} = \lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} (\frac{a}{b})^n = 0$.

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Thus: $a^n = o(b^n)$ (for any $1 < a < b$) by Limit Test.

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The list of common functions:

The following list shows the functions commonly used in algorithm analysis, in the order of increasing growth rate (a, b, c, d, k, ϵ are positive constants, $\epsilon < 1$, $k > 1$, $d > 1$ and $a < b$):

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$$c, \log_d n, (\log_d n)^k, n^\epsilon, n, n^k, a^n, b^n, n!, n^n$$

in the sense that if $f(n)$ and $g(n)$ are any two consecutive functions in the list, we have $f(n) = o(g(n))$

Examples

Example 7

$T_1(n) = n!$ and $T_2(n) = a^n$ $(a > 1)$

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$$\frac{a^n}{n!} = \underbrace{\frac{a}{1} \cdot \frac{a}{2} \cdots \frac{a}{2\lceil a\rceil}}_{2\lceil a\rceil \text{ terms}} \cdot \underbrace{\frac{a}{2\lceil a\rceil + 1} \cdots \frac{a}{n}}_{(n-2\lceil a\rceil) \text{ terms}}$$

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The first part is a constant $c > 0$. In the second part, each term $< 1/2$. So:

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$$0 \le \lim_{n\to\infty}\frac{a^n}{n!} \le c\cdot\lim_{n\to\infty}\left(\frac{1}{2}\right)^{(n-2\lceil a\rceil)} = 0$$

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By Limit Test: $a^n = o(n!)$.

Outline

Stirling Formula

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

or:
$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

When $n = 10$;

- $n! = 3628800$.
- $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n = 3598696$, 99% accurate.

Example 7 (another solution)

$T_1(n) = n!$ and $T_2(n) = a^n$ $(a > 1)$

$$\lim_{n\to\infty} \frac{n!}{a^n} \geq \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{ae}\right)^n = \lim_{n\to\infty} \sqrt{2\pi n} \cdot \lim_{n\to\infty} \left(\frac{n}{ae}\right)^n$$

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Example 7 (another solution)

$T_1(n) = n!$ and $T_2(n) = a^n$ $(a > 1)$

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The first limit is ∞. The second limit goes to ∞^∞. So it's also ∞. Thus $\lim_{n\to\infty} \frac{n!}{a^n} = \infty$ and $n! = \omega(a^n)$ by limit test.

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Example 8

$T_1(n) = n^n$ and $T_2(n) = n!$

By using similar method as in Example 7, we can show: $n! = o(n^n)$

Outline

Summations

Consider the following simple program:

1: **for** $i = 1$ **to** n **do**
2: the loop body takes $\Theta(i^k)$ time
3: **end for**

What's the runtime of this program? It should be:

$$T(n) = \sum_{i=1}^{n} \Theta(i^k) = c\sum_{i=1}^{n} i^k \quad \text{(for some constant } c\text{)}$$

Thus, it is important to know the sum of the form $\sum_{i=1}^{n} i^k$.

Summation formulas

$$\sum_{i=1}^{n} i^1 = 1 + 2 \cdots + n = \frac{n(n+1)}{2} = \Theta(n^2) \quad (1)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \Theta(n^3) \quad (2)$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 \cdots + n^3 = \frac{n^2(n+1)^2}{4} = \Theta(n^4) \quad (3)$$

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In general, for any $k > 0$, the following is true.

$$\sum_{i=1}^{n} i^k = \Theta(n^{k+1}) \quad (4)$$

Summations:

By using these formulas, we can compute the runtime of nested loops.

Example

for $i = 1$ **to** n **do**
  for $j = i$ **to** n **do**
    for $k = i$ **to** j **do**
      $(\ldots$ loop body takes $\Theta(1)$ time.)
    end for
  end for
end for

Since the inner loop body takes $\Theta(1)$ time, we only need to count the number $D(n)$ of the inner loop iterations. Then $T(n) = D(n) \cdot \Theta(1) = \Theta(D(n))$.

$$D(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1 = \sum_{i=1}^{n} \sum_{j=i}^{n} (j - i + 1)$$

Calculate $D(n)$

To calculate the second sum, let $t = j - i + 1$. When $j = i$, $t = 1$. When $j = n$, $t = n - i + 1$. Thus

$$\sum_{j=i}^{n}(j - i + 1) = \sum_{t=1}^{n-i+1} t = 1 + 2 \cdots + (n - i + 1) = \frac{(n - i + 2)(n - i + 1)}{2}$$

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Next we calculate: $\sum_{i=1}^{n} \frac{(n-i+2)(n-i+1)}{2}$. Let $s = n - i + 1$. When $i = 1$, $s = n$. When $i = n$, $s = 1$. Thus:

$$\begin{aligned} \sum &= \sum_{s=1}^{n} \frac{(s+1)s}{2} = \frac{1}{2}\{\sum_{s=1}^{n} s^2 + \sum_{s=1}^{n} s\} \\ &= \frac{1}{2}\{\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\} = \Theta(n^3) \end{aligned}$$

More Summations:

The following summation formulas are useful.

$$\sum_{i=0}^{n} a^i = 1 + a + a^2 + \cdots + a^n = \begin{cases} \frac{1-a^{n+1}}{1-a} & = \Theta(1) & \text{if } 0 < a < 1 \\ n+1 & = \Theta(n) & \text{if } a = 1 \\ \frac{a^{n+1}-1}{a-1} & = \Theta(a^n) & \text{if } 1 < a \end{cases} \tag{5}$$

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$\sum_{i=0}^{n} a^i$ is called geometric series.

$$H(n) = 1 + 1/2 + 1/3 + \cdots + 1/n = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\ln n) \tag{6}$$

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$H(n)$ is called Harmonic series.
How to compute $H(n)$?

Outline

Integration Method

Let $f(x)$ be an increasing function. Then for any $a \leq b$:

$$\int_{a-1}^{b} f(x)dx \leq \sum_{i=a}^{b} f(i) \leq \int_{a}^{b+1} f(x)dx$$

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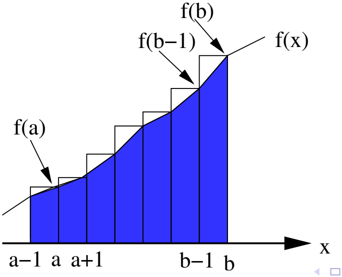
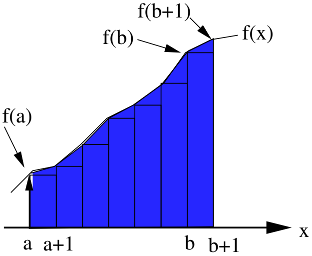
In the Fig, $\sum =$ the area of the staircase region. The first $\int =$ the area of the shaded region. Since $f(x)$ is increasing, the first $\leq$ holds.



In the Fig, $\sum$ is the area of the staircase region. The second $\int$ is the area of the shaded region. Since $f(x)$ is increasing, the second $\leq$ holds.



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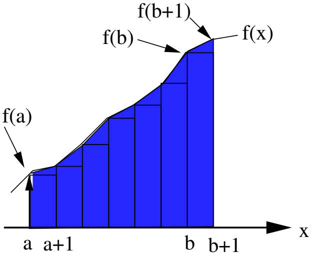

Similarly:

Let $f(x)$ be a decreasing function. Then for any $a \leq b$:

$$\int_{a-1}^{b} f(x)dx \geq \sum_{i=a}^{b} f(i) \geq \int_{a}^{b+1} f(x)dx$$

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For any $k > 0$, $f(x) = x^k$ is an increasing function. Let $a = 1$ and $b = n$.

$$\int_0^n x^k dx \leq \sum_{i=1}^{n} i^k \leq \int_1^{n+1} x^k dx$$

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By limit test, both lower and upper bounds $= \Theta(n^{k+1})$. Thus $\sum_{i=1}^{n} i^k = \Theta(n^{k+1})$.

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$f(x) = \frac{1}{x}$ is a decreasing function. Let $a = 1$ and $b = n$.

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Note: $\lim_{n\to\infty}(\ln n - \sum_{i=1}^{n} \frac{1}{i}) = c$, where $c = 0.577...$ is Euler constant.

Outline

Solving Linear Recursive Equations

Fibonacci number

$$\text{Fib}_0 = 0, \text{Fib}_1 = 1, \ldots, \text{Fib}_{n+2} = \text{Fib}_{n+1} + \text{Fib}_n$$

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Example 2: $f_0 = 1, f_1 = 2, f_2 = 4$ and for all $n \geq 0$, $f_{n+3} = 3f_{n+1} - 2f_n$
Then $\{f_n\}$ is a linear recursive sequence of order 3 where $c_2 = 0$, $c_1 = 3$ and $c_0 = -2$.

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- The characteristic equation of the linear recursive seq is:

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- Plug in the initial values $f_0, f_1, \ldots, f_{k-1}$, we get k equations. Solve them to find $a_1, a_2, \ldots, a_k$.

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- Solving this equation system, we get: $a_1 = \frac{1}{\sqrt{5}}$ and $a_2 = -\frac{1}{\sqrt{5}}$.

Example

Fibonacci number

$$F_0 = 0, F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n$$

- The characteristic equation is: $x^2 = x + 1$.
- The roots are: $\alpha_1 = \frac{1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{1-\sqrt{5}}{2}$
- The solution has the form: $F_n = a_1(\alpha_1)^n + a_2(\alpha_2)^n$
- Plug in initial value $F_0 = 0$ and $F_1 = 1$:

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- Solving this equation system, we get: $a_1 = \frac{1}{\sqrt{5}}$ and $a_2 = -\frac{1}{\sqrt{5}}$.
- Thus:

$$\mathrm{F}_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving linear recursive sequences

- The roots $\alpha_1, \ldots, \alpha_k$ of the characteristic equ are not distinct.

Solving linear recursive sequences

- The roots $\alpha_1, \ldots, \alpha_k$ of the characteristic equ are not distinct.
- Say $\alpha_1 = \alpha_2 = \ldots = \alpha_t$ repeats t times.
- Then in the solution formula, the portion

$$\cdots a_1(\alpha_1)^n + a_2(\alpha_2)^n + \cdots + a_t(\alpha_t)^n \cdots$$

  is replaced by:

$$\cdots a_1(\alpha_1)^n + a_2 n^1(\alpha_1)^n + \cdots + a_t n^{t-1}(\alpha_1)^n \cdots$$

- Other steps are the same.

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- To solve $f(x) = 0$, try $x = 1$. We find $x = 1$ is a root. So $(x - 1)$ is a factor of $f(x)$. Thus $f(x) = (x - 1)(x^2 + x - 2) = (x - 1)(x - 1)(x + 2)$.

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- The solution has the form: $F_n = a_1 \cdot 1^n + a_2 \cdot n \cdot 1^n + a_3 \cdot (-2)^n$. Plug in initial values:

$$\begin{aligned} 1 = F_0 &= a_1 + 0 + a_3 \\ 2 = F_1 &= a_1 + a_2 - 2a_3 \\ 4 = F_2 &= a_1 + 2a_2 + 4a_3 \end{aligned}$$

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- Solving this: $a_1 = 8/9$, $a_2 = 4/3$ and $a_3 = 1/9$
- Thus: $F_n = \frac{8}{9} + \frac{4}{3}n + \frac{1}{9}(-2)^n$