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- If $(u, v) \in E$, we say v is a neighbor of u.
- The degree $deg(u)$ of a vertex u is the number of edges incident to u.

Fact

$$\sum_{v \in V} deg(v) = 2m$$

This is because, for each $e = (u, v)$, e is counted twice in the sum, once for $deg(v)$ and once for $deg(u)$.

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$$\sum_{v \in V} deg_{in}(v) = \sum_{v \in V} deg_{out}(v) = m$$

This is because, for each $e = (u \rightarrow v)$, e is counted once ($deg_{in}(v)$) in the sum of in-degrees, and once ($deg_{out}(u)$) in the sum of out-degrees.

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- Because n and m are rather independent to each other, we usually use both parameters to describe the runtime of a graph algorithm. Such as $O(n + m)$ or $O(n^{1/2}m)$.

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Graph Representations

We mainly use two graph representations.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i,j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

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- For undirected graph, A is always symmetric.
- The Adjacency Matrix Representation for directed graph is similar.
  $A[i,j] = 1$ (or $w(v_i, v_j)$ if G has edge weights) iff $v_i \to v_j \in E$.

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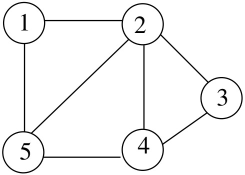
- For directed graphs, each edge has only one entry in the representation.

Example



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
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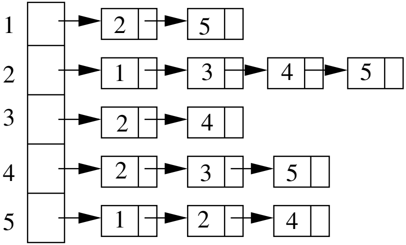

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- How easy to implement.

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- If we use Adj List, it is possible the algorithm can run in linear $\Theta(m+n)$ time.

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Before describing details, we need to pick a graph representation. Because we need to visit all neighbors of a vertex, it seems we need the neighbor listing operation. So we use Adj list representation.

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- $d[u]$: the distance from u to the starting vertex s.

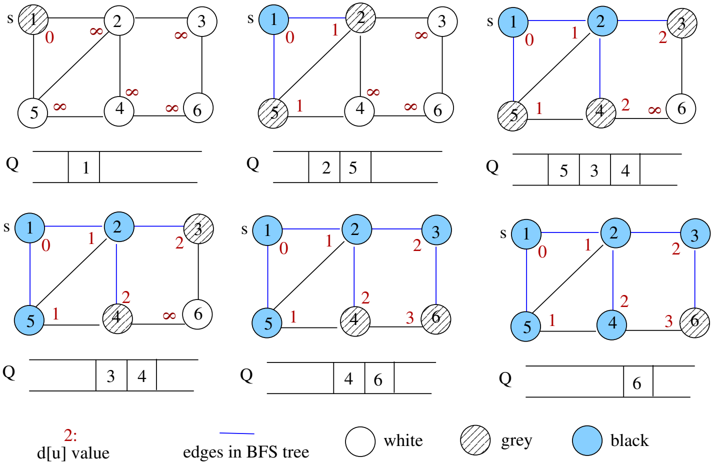
In addition, we also use a queue Q as mentioned earlier.

BFS: Algorithm

BFS(G, s)

$Q \leftarrow \emptyset$
for each $u \in V - \{s\}$ **do**
    $\pi[u] = \text{NIL}$; $d[u] = \infty$; color$[u]$=white
$d[s] = 0$; color$[s]$=grey; $\pi[s] = \text{NIL}$
Enqueue(Q, s)
while $Q \neq \emptyset$ **do**
    $u \leftarrow$**Dequeue**(Q)
    for each $v \in Adj[u]$ **do**
        if color$[v]$ = white
        then color$[v]$ = grey; $d[v] \leftarrow d[u] + 1$; $\pi[v] \leftarrow u$; **Enqueue**(Q, v)
    color$[u]$ = black

BFS: Example



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- If the order is different, then the progress of the BFS algorithm would be different. And the BFS tree T constructed by the algorithm would be different.
- However, regardless of which order we use, the properties of the BFS algorithm and BFS tree are always true.

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  - Since each queue operation takes $O(1)$ time, the total time needed for all queue operations is $\Theta(n)$.
  - Lines 8-10: Each item in $Adj[u]$ is processed once.
  - When an item is processed, $O(1)$ operations are needed.

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- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
  - Each vertex is enqueued and dequeued exactly once.
  - Since each queue operation takes $O(1)$ time, the total time needed for all queue operations is $\Theta(n)$.
  - Lines 8-10: Each item in $Adj[u]$ is processed once.
  - When an item is processed, $O(1)$ operations are needed.
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BFS algorithm takes $\Theta(n+m)$ time.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

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- Then v is in Q at that moment.
- By the previous observation, $d[u] = k$ for some k, and $d[v] = k$ or $k + 1$. Thus $d[v] - d[u] \leq 1$.

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Definition

Let $G = (V, E)$ be a graph and T a spanning tree of G rooted at the vertex s. Let x and y be two vertices. Let (u, v) be an edge of G.

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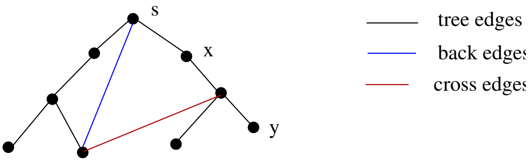

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Shortest Path Problem

Let $G = (V, E)$ be a graph and s a vertex of G. For each $u \in V$, let $\delta(s, u)$ be the length of the shortest path between s and u.
Problem: For all $u \in V$, find $\delta(s, u)$ and the shortest path between s and u.

BFS: Applications

Theorem

Let $d[u]$ be the value computed by BFS algorithm and T the BFS tree constructed by BFS algorithm. Then for each vertex $u \in V$,

- $d[u] = \delta(s, u)$.
- The tree path in T from u to s is the shortest path.

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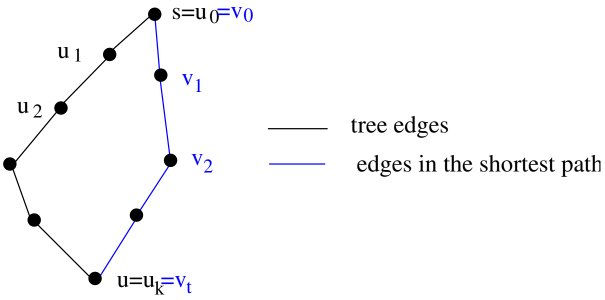
Proof: Let $P = \{s = u_0, u_1, \ldots, u_k = u\}$ be the path from s to u in the BFS tree T. Then: $d[u] = d[u_k] = k$, $d[u_{k-1}] = k - 1$,$d[u_{k-2}] = k - 2 \ldots$
Suppose $P' = \{s = v_0, v_1, v_2, \ldots, v_t\}$ is the shortest path from s to u in G. We need to show $k = t$.
Toward a contradiction, suppose $t < k$. Then there must exist $(v_i, v_{i+1}) \in P'$ such that $|d[v_i] - d[v_{i+1}]| \geq 2$. This is impossible.

Shortest Path Problem



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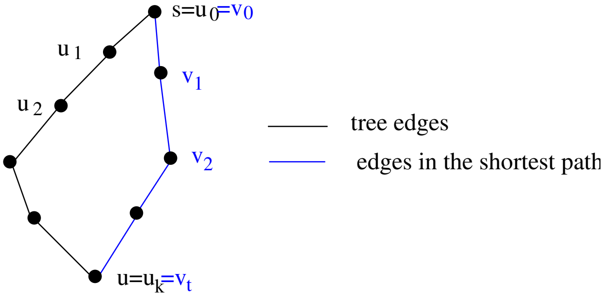

BFS algorithm solves the Single Source Shortest Path problem in $\Theta(n+m)$ time.

Connectivity Problem

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Connectivity Problem

Given $G = (V, E)$, is G a connected graph?
If not, find the connected components of G.

We can use BFS algorithm to solve the connectivity problem.

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In the **BFS** algorithm, delete the lines 2-3 (initialization of vertex variables).

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Connectivity$(G = (V, E))$

1. **for** each $i \in V$ **do**
2.   color$[i]$ =white; $d[i] = \infty$; $\pi[i] = nil$;
3. count = 0; (count will be the number of connected components)
4. **for** $i = 1$ **to** n **do**
5.   **if** color$[i]$ =white **then**
6.   **call BFS**(G, i); count = count+1
7. **output** count;
8. **end**

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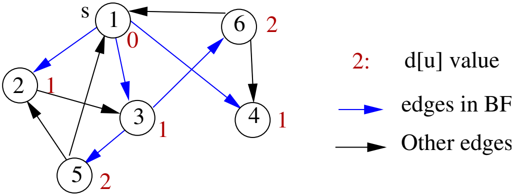
- If count$= 1$, G is connected. The algorithm also constructs a BFS tree.
- If count> 1, G is not connected. The algorithm also constructs a BFS spanning forest F of G. F is a collection of trees.
- Each tree corresponds to a connected component of G.

BFS for Directed Graphs

BFS algorithm can be applied to directed graphs without any change.



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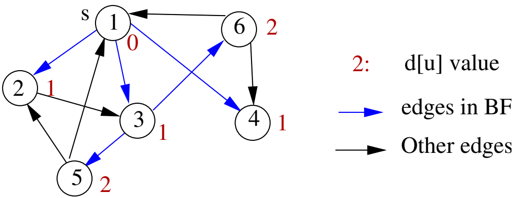

Definition

Let $G = (V, E)$ be a directed graph, T a spanning tree rooted at s. An edge $e = u \rightarrow v$ is called:

- tree edge if $e = u \rightarrow v \in T$.
- backward edge if u is a decedent of v.
- forward edge if u is an ancestor of v.
- cross edge if u and v are unrelated.

BFS for Directed Graphs: Property

Theorem

Let $G = (V, E)$ be a directed graph. Let T be the BFS tree constructed by BFS algorithm. Then there are no forward edges with respect to T.

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- The tree path in T from s to u is the shortest path.
- $d[u]$ = the length of the shortest path from s to u.

Outline

Depth First Search (DFS)

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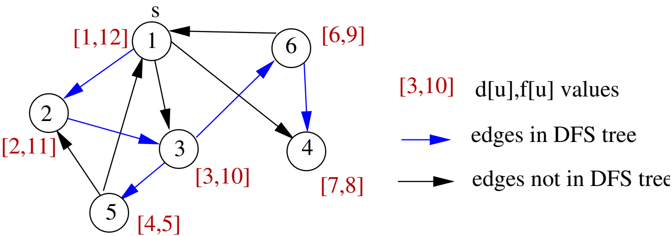
- Backtrack to a vertex that still has unvisited neighbors, and continue

DFS: Example


s
[1,12]
1
6
[6,9]
2
[2,11]
3
4
[3,10]
[7,8]
5
[4,5]
[3,10] d[u],f[u] values
edges in DFS tree
edges not in DFS tree

DFS: Recursive algorithm

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DFS: Recursive algorithm

- It is easier to describe the DFS by using a recursive algorithm.
- DFS also computes two variables for each vertex $u \in V$:
  - $d[u]$: The time when u is "discovered", i.e. pushed on the stack.
  - $f[u]$: the time when u is "finished", i.e. popped from the stack.
- These variables will be used in applications.

DFS: Recursive algorithm

DFS(G)

1. **for** each vertex $u \in V$ **do**
2. color$[u] \leftarrow$ white; $\pi[u] =$ NIL
3. $time \leftarrow 0$
4. **for** each vertex $u \in V$ **do**
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DFS: Properties

DFS: Properties

Let T be the DFS tree of G by DFS algorithm. Let $[d[u], f[u]]$ be the time interval computed by DFS algorithm. Let $u \neq v$ be any two vertices of G.

- The intervals of $[d[u], f[u]]$ and $[d[v], f[v]]$ are either disjoint or one is contained in another.
- $[d[u], f[u]]$ is contained in $[d[v], f[v]]$ if and only if u is a descendent of v with respect to T.

DFS: Properties

Classification of Edges

Let $G = (V, E)$ be a directed graph and T a spanning tree of G. The edge $e = u \rightarrow v$ of G can be classified as:

- tree edge if $e = u \rightarrow v \in T$.
- back-edge if $e \notin T$ and v is an ancestor of u.
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DFS: Properties

Classification of Edges

Let $G = (V, E)$ be a directed graph and T the spanning tree of G constructed by DFS algorithm. The classification of the edges can be done as follows.

- When $e = u \to v$ is first explored by DFS, color e by the color$[v]$.
- If color$[v]$ is white, then e is white and is a tree edge.
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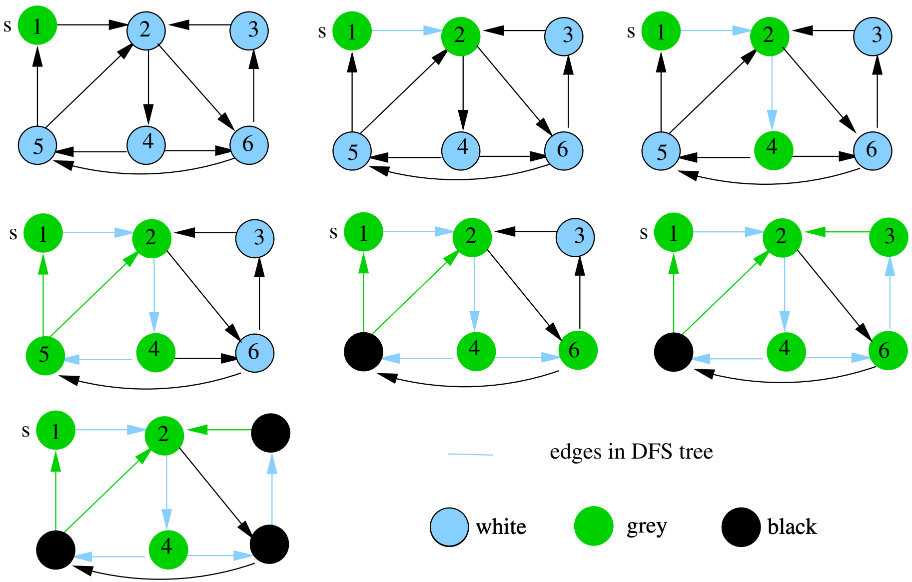
For DFS tree of directed graphs, all four types of edges are possible.

DFS: Example



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Theorem

Let G be a directed graph, and T the DFS tree of G. Then G is DAG $\Longleftrightarrow$ there are no back edges.

DFS: Applications

Proof: $\Longrightarrow$ Suppose $e = u \to v$ is a back edge. Let P be the path in T from v to u. Then the directed path P followed by $e = u \to v$ is a directed cycle.

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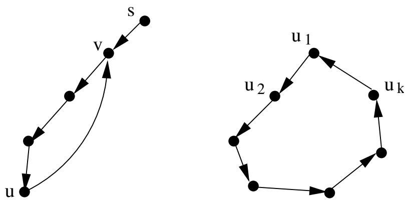

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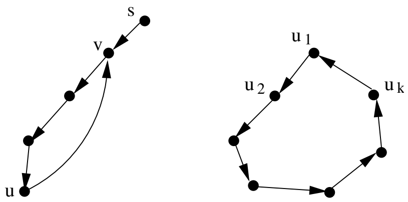
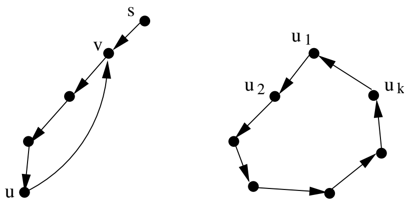


$\Longleftarrow$ Suppose $C = u_1 \to u_2 \to \cdots u_k \to u_1$ is a directed cycle. Without loss of generality, assume u_1 is the first vertex visited by DFS. Then, the algorithm visits $u_2, u_3, \ldots u_k$, before it backtrack to u_1. So $u_k \to u_1$ is a back edge.

DFS: Applications

Proof: $\Longrightarrow$ Suppose $e = u \to v$ is a back edge. Let P be the path in T from v to u. Then the directed path P followed by $e = u \to v$ is a directed cycle.



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DAG Testing in $\Theta(n+m)$ time

1. Run DFS on G. Mark the edges "white", "grey" or "black",
2. If there is a grey edge, report "G is not a DAG". If not "G is a DAG".

Outline

Topological Sort

Topological Sort

Let $G = (V, E)$ be a DAG. A topological sort of G assigns each vertex $v \in V$ a distinct number $L(v) \in [1..n]$ such that if $u \to v$ then $L(u) < L(v)$.

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Note: If G is not a DAG, topological sort cannot exist.

Application

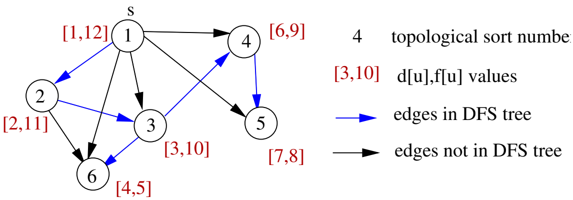
- The directed graph $G = (V, E)$ specifies a job flow chart.
- Each $v \in V$ is a job.
- If $u \to v$, then the job u must be done before the job v.
- A topological sort specifies the order to complete jobs.

Topological Sort

We can use DFS to find topological sort.

Topological-Sort-by-DFS(G)

1. Run DFS on G.
2. Number the vertices by decreasing order of $f[v]$ value. (This can be done as follows: During DFS, when a vertex v is finished, insert v in the front of a linked list.)



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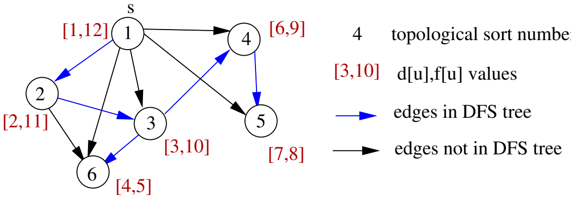

Clearly, this algorithm takes $\Theta(m+n)$ time.

Outline

Strong Connectivity

Definition

- A directed graph $G = (V, E)$ is strongly connected if for any two vertices u and v in V, there exists a directed path from u to v.
- A strongly connected component of G is a maximal subgraph of G that is strongly connected.

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Given a directed graph G, find the strongly connected components of G.

Note: G is strongly connected if and only if it has exactly one strongly connected component.

Application: Traffic Flow Map

- $G = (V, E)$ represents a street map.
- Each $v \in V$ is an intersection.
- Each edge $u \to v$ is a 1-way street from the intersection u to the intersection v.
- Can you reach from any intersection to any other intersection?
- This is so $\Longleftrightarrow$ G is strongly connected.
- All intersections within each connected component can reach each other.

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This problem can be solved by using DFS. Without it, it would be hard to solve efficiently.

Strong Connectivity

Strong-Connectivity-by-DFS(G)

1. Run DFS on G, compute $f[u]$ for all $u \in V$,
2. Order the vertices by decreasing $f[v]$ values.
3. Construct the transpose graph G^T, which is obtained from G by reversing the direction of all edges.
4. Run DFS on G^T, the vertices are considered in the order of decreasing $f[v]$ values.
5. The vertices in each tree in the DFS forest correspond to a strongly connected component of G.

Strong Connectivity

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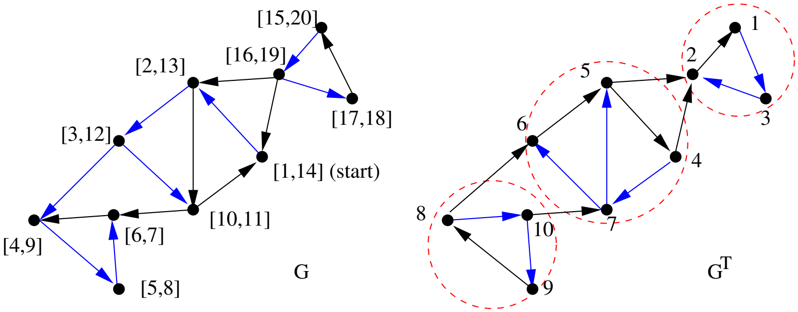
Analysis:

- Steps 1 and 2: $\Theta(n+m)$ (step 2 is a part of step 1.)
- Step 3: $\Theta(n+m)$ (how?)
- Step 4 and 5: $\Theta(n+m)$ (step 5 is part of step 4.)

Strong Connectivity: Example



Outline

DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.

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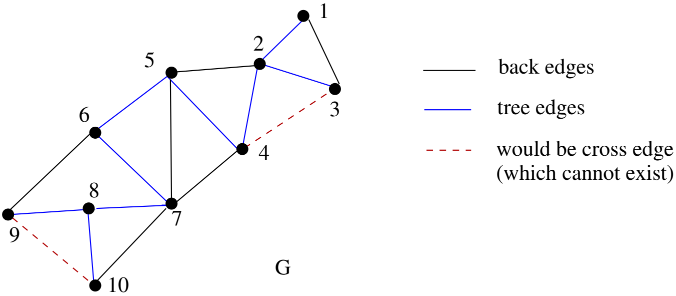
  - cross edges

Theorem

Let G be an undirected graph, and T the DFS tree of G constructed by DFS algorithm. Then there are no cross edges.

DFS for Undirected Graphs



Summary: Edge Types

For Directed Graphs

	Tree	Forward	Backward	Cross
BFS	yes	no	yes	yes
DFS	yes	yes	yes	yes

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	Tree	Forward	Backward	Cross
BFS	yes	no	yes	yes
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For Undirected Graphs

	Tree	Back-edge	Cross
BFS	yes	no	yes
DFS	yes	yes	no

Outline

Biconnectivity Problem

Definition

Let $G = (V, E)$ be an undirected connected graph.

- A vertex $v \in V$ is a cut vertex (also called articulation point) if deleting v and its incident edges disconnects G.
- G is biconnected if it is connected and has no cut vertices.
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Biconnectivity Problem

Given an undirected graph $G = (V, E)$, is G biconnected?
If not, find the cut vertices and the biconnected components of G.

Biconnectivity Problem

Application

- G represents a computer network.
- Each vertex is a computer site.
- Each edge is a communication link.
- If v is a cut vertex, then the failure of v will disconnect the whole network.
- The network can survive any single site failure if and only if G is biconnected.

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Simple-Biconnectivity(G)

1. **for** each vertex $v \in V$ **do**
2. delete v and its incident edges from G
3. test if $G - \{v\}$ is connected

This algorithm takes $\Theta(n) \times \Theta(n+m) = \Theta(n(n+m))$ time.

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- Let T be the DFS tree of G.
- Re-number the vertices by increasing $d[v]$ values.
- For each vertex v, define:

  low$[v]$ $=$ the smallest vertex that can be reached from v or a descendent of v through a back edge.
- If v is a leaf of T, then low$[v] = \min \left\{ \begin{array}{c} v \\ \{w \mid (v,w) \text{ is a back-edge}\} \end{array} \right\}$

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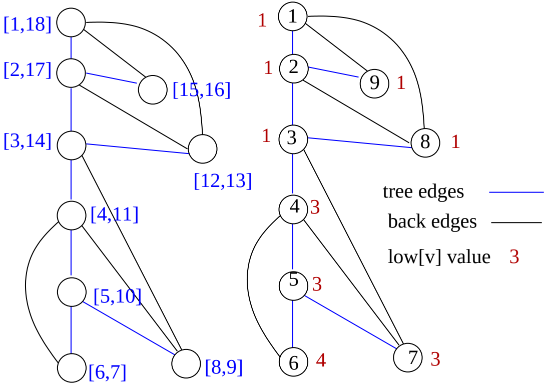

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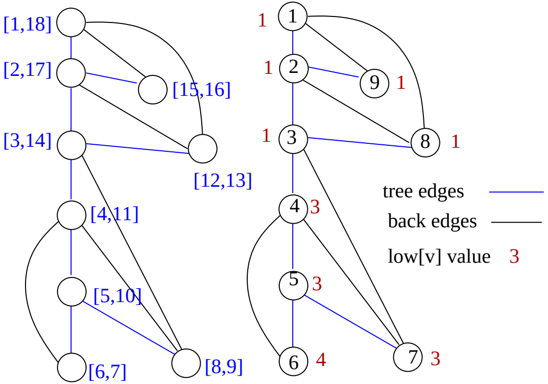
Biconnectivity Problem



The vertices have been renumbered by increasing d[v] values.

In this figure, low means closer to the root. So the root is the lowest vertex. $\text{low}[v]$ is the lowest vertex that can be reached from v or a descendent of v thru a single back edge.

Biconnectivity Problem

Theorem

Let T be the DFS of $G = (V, E)$ rooted at the vertex s.

1. s is a cut vertex $\Longleftrightarrow$ s has at least two sons in T.
2. A vertex $a \neq s$ is a cut vertex $\Longleftrightarrow$ a has a son b such that $low[b] \geq a$.

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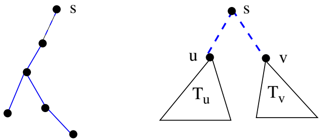
Proof of (1): Suppose s has only one son. After deleting s, all other vertices are still connected by the remaining edges of T. So s is not a cut vertex.



Suppose s has at least two sons u and v (there may be more). Let T_u be the subtree of T rooted at u and T_v be the subtree of T rooted at v. Because there are no cross edges, no edges connect T_u with T_v. So after s is deleted, T_u and T_v become disconnected. Hence s is a cut vertex.

Proof of (2): Let T_s be the subtree of T above the vertex a. Let $b, c \ldots$ be the sons of a. Let $T_b, T_c, \ldots$ be the subtree of T rooted at $b, c \ldots$

- Suppose a has a son b with low$[b] \geq a$. Because low$[b] \geq a$, no vertex in T_b is connected to T_s. Because there are no cross edges, no edges connect vertices in T_b and T_c. So after a is deleted, T_b is disconnected from the rest of G. So a is a cut vertex.

Biconnectivity Problem

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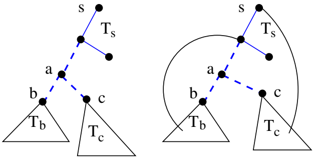

- Suppose for every son b of a we have low$[b] < a$. This means that there is a back edge connecting a vertex in T_b to a vertex in T_s. So after a is deleted, all subtrees $T_b, T_c, \ldots$ are still connected to T_s, and G remains connected. So a is not a cut vertex.

We can now describe the algorithm. For conceptual clarity, the algorithm is divided into several steps. Actually, all steps can be and should be incorporated into a single DFS run.

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Biconnectivity-by-DFS(G)

1. Run DFS on G
2. Renumber the vertices by increasing $d[*]$ values.
3. For all $u \in V$, compute low$[u]$ as described before.
4. Identify the cut vertices according to the conditions in the theorem.

Analysis

- Steps 1 and 2: takes $\Theta(m+n)$ time.

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  - 1 for u itself, we charge this to the edge between u and its parent.
  - So $k = deg(u)$.

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The Biconnectivity problem can be solved in $\Theta(n+m)$ time

Notes

- The DFS based Biconnectivity algorithm was discovered by Tarjan and Hopcroft in 1972. (See Problem 22-2, Page 558).
- They advocated the use of adjacent list representation over the adjacent matrix representation for solving complex graph problems in linear (i.e. $O(n+m)$) time.
- This DFS algorithm is a good example. Without using adjacent list representation, the problem would take at least $\Theta(n^2)$ time to solve.