Outline

Divide and Conquer Strategy

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- Divide the problem into smaller subproblems (of the same type).
- Solve each subproblem (usually by recursive calls).
- Combine the solutions of the subproblems into the solution of the original problem.

Merge Sort

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Output: Sort A into increasing order.

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- It sorts $A[p..r]$.
- In main program, we call MergeSort($A, 1, n$).

Merge Sort

MergeSort(A, p, r)

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if (p < r) then
    q = (p + r)/2
    MergeSort(A, p, q)
    MergeSort(A, q + 1, r)
    Merge(A, p, q, r)
else
    do nothing
end if
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- Divide $A[p..r]$ into two sub-arrays of equal size.
- Sort each sub-array by recursive call.
- Merge(A, p, q, r) is a procedure that, assuming $A[p..q]$ and $A[q+1..r]$ are sorted, merge them into sorted $A[p..r]$
- It can be done in $\Theta(k)$ time where $k = r - p$ is the number of elements to be sorted.

Analysis of MergeSort

Let $T(n)$ be the runtime function of MergeSort($A[1..n]$). Then:

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

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- $\Theta(n)$ is the time needed by Merge(A, p, q, r) and all other processing.

Outline

Master Theorem

For DaC algorithms, the runtime function often satisfies:

$$T(n) = \begin{cases} O(1) & \text{if } n \leq n_0 \\ aT(n/b) + \Theta(f(n)) & \text{if } n > n_0 \end{cases}$$

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- $T(n) = ?$

Master Theorem

Master Theorem (Theorem 4.1, Cormen's book.)

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and $af(n/b) \leq cf(n)$ for some $c < 1$ for sufficiently large n, then $T(n) = \Theta(f(n))$.

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Example: MergeSort

We have $a = 2, b = 2$, hence $\log_b a = \log_2 2 = 1$. So $f(n) = \Theta(n^1) = \Theta(n^{\log_b a})$.

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By statement (2), $T(n) = \Theta(n^{\log_2 2} \log n) = \Theta(n \log n)$.

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- Output: Find i such that $A[i] = x$, if no such i exists, output "no".

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BinarySearch(A, p, r, x**)**

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    if A[p] ≠ x return "no"
else
    q = (p + r)/2
    if A[q] = x return q
    if A[q] > x call BinarySearch(A, p, q − 1, x)
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- So $a = 1$, $b = 2$ and $f(n) = \Theta(n^0)$ time.
  Since $\log_b a = \log_2 1 = 0$, $f(n) = \Theta(n^{\log_b a})$.

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  Since $\log_b a = \log_2 1 = 0$, $f(n) = \Theta(n^{\log_b a})$.
- Hence $T(n) = \Theta(n^{\log_b a} \log n) = \Theta(\log n)$.

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We have $a = 4$, $b = 2$. So $\log_b a = \log_2 4 = 2$.
$f(n) = n^3 = \Theta(n^{\log_b a+1}) = \Omega(n^{\log_b a+0.5})$.
This is the case 3 of Master Theorem. We need to check the 2^{nd} condition:

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This is the case 3 of Master Theorem. We need to check the 2^{nd} condition:

$$a \cdot f(n/b) = 4\left(\frac{n}{2}\right)^3 = \frac{4}{8}n^3 = \frac{1}{2} \cdot f(n)$$

If we let $c = 1/2 < 1$, we have: $a \cdot f(n/b) \leq c \cdot f(n)$.
Hence by case 3, $T(n) = \Theta(f(n)) = \Theta(n^3)$.

Master Theorem

If $f(n)$ has the form $f(n) = \Theta(n^k)$ for some $k \geq 0$, We have the following:

A simpler version of Master Theorem

$$T(n) = \begin{cases} O(1) & \text{if } n \leq n_0 \\ aT(n/b) + \Theta(n^k) & \text{if } n > n_0 \end{cases}$$

1. If $k < \log_b a$, then $T(n) = \Theta(n^{\log_b a})$.
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Only the case 3 is different. In this case, we need to check the 2^{nd} condition. Because $k > \log_b a$, $b^k > a$ and $a/b^k < 1$:

$$a \cdot f(n/b) = a \cdot \left(\frac{n}{b}\right)^k = \frac{a}{b^k} \cdot f(n) = c \cdot f(n)$$

where $c = \frac{a}{b^k} < 1$, as needed.

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- The proof of Master Theorem is given in textbook.
- We'll illustrate two examples in class.

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Theorem

If $T(n) = aT(n/b) + f(n)$, where $f(n) = \Theta(n^{\log_b a} (\log n)^k)$, then $T(n) = \Theta(n^{\log_b a} (\log n)^{k+1})$.

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In the above example, $T(n) = \Theta(n \log^2 n)$

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Matrix Multiplication

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Definition

Matrix Addition

$$C = (c_{ij})_{1 \leq i,j \leq n} = A + B$$

is defined by $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i, j \leq n$.

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- Each entry takes $O(1)$ time.
- So matrix addition takes $\Theta(n^2)$ time.

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$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

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$$A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 \cdot 3 + (-1) \cdot 0 & 4 \cdot 1 + (-1) \cdot (-2) \\ 2 \cdot 3 + 1 \cdot 0 & 2 \cdot 1 + 1 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 6 & 0 \end{pmatrix}$$

Matrix Matrix Multiplication

MatrixMultiply(A, B)

for $i = 1$ **to** n **do**
    for $j = 1$ **to** n **do**
        $c_{ij} = 0$
        for $k = 1$ **to** n **do**
            $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$
        end for
    end for
end for

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for i = 1 to n do
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- This algorithm clearly takes $\Theta(n^3)$ time.

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- This algorithm clearly takes $\Theta(n^3)$ time.
- Since MM is an important operation, can we do better than this?

Outline

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It can be shown:

$$\begin{aligned} r &= a \times e + b \times g & \qquad s &= a \times f + b \times h \\ t &= c \times e + d \times g & \qquad u &= c \times f + d \times h \end{aligned}$$

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$$T(n) = 8T(n/2) + \Theta(n^2)$$

Thus: $a = 8, b = 2$ and $k = 2$. Since $\log_b a = \log_2 8 = 3 > 2$, we get: $T(n) = \Theta(n^{\log_2 8}) = \Theta(n^3)$.

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No better than the simple $\Theta(n^3)$ algorithm.

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We need 7 recursive calls, and a total 18 additions/subtractions of $n/2 \times n/2$ sub-matrices.

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- For larger n, Strassen's $\Theta(n^{2.81})$ algorithm is better.
- The break-even value is $20 \leq n \leq 50$, depending on implementation.
- In some Science/Engineering applications, the matrices in MM are sparse (namely most entries are 0.) In such cases, neither the simple, nor the Strassen's algorithm work well. Completely different algorithms have been designed.

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- Or, in a few cases, it's trivial.

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- Since the lower and upper bounds are the same, we get $C_{MA}(n) = \Theta(n^2)$.

Sorting (general purpose)

Given an array $A[1..n]$ of elements, sort A. (The only operations allowed for A: comparison between array elements).

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- Strassen's algorithm gives an upper bound: $C_{MM}(n) = O(n^{2.81})$.
- Currently, the best known upper bound for MM is $C_{MM}(n) = O(n^{2.376})$.
- A trivial lower bound: Any MM algorithm must write down the resulting matrix C, this alone requires at least $\Omega(n^2)$ time. Thus $C_{MM}(n) = \Omega(n^2)$.
- If $C_{MM}(n) = \Theta(n^{\alpha})$, we know $2 \le \alpha \le 2.376$.

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- If $C_{MM}(n) = \Theta(n^{\alpha})$, we know $2 \leq \alpha \leq 2.376$.
- Determining the exact value of α is a long-standing open problem in CS/Math.

Outline

1. Divide and Conquer Strategy
2. Master Theorem
3. Matrix Multiplication
4. Strassen's MM Algorithm
5. Complexity of a Problem
6. **Selection Problem**
7. Summary
8. Computational Geometry

Selection Problem

Three basic problems for ordered sets:

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- We will present a Linear Time Select algorithm.
- $\Omega(n)$ is a trivial lower bound for Select: we must look at each array element at least once, otherwise the answer could be wrong.
- This would give: $C_{\text{Select}}(n) = \Theta(n)$.

Selection Problem

This algorithm uses DaC. We need a function **partition(A,p,r)**.
The goal: rearrange $A[p..r]$ so that for some q ($p \leq q \leq r$),

$$\begin{aligned} A[i] &\leq A[q] \;\; \forall i = p, \ldots, q-1 \\ A[q] &\leq A[j] \;\; \forall j = q+1, \ldots, r \end{aligned}$$

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p $\quad$ $q-1$	q	$q+1$ $\quad$ r
$\leq A[q]$	$A[q]$	$\geq A[q]$

The following code partitions $A[p..r]$ around $A[r]$

Partition(A, p, r)

$x \leftarrow A[r]$ (x is "pivot".)
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
    if $A[j] \leq x$ **then**
        $i \leftarrow i + 1$
        swap $A[i]$ and $A[j]$
    end if
end for
swap $A[i+1]$ and $A[r]$
return $i + 1$

Example: ($x = 4$ is the pivot element.)

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...		3	1	8	5	6	2	7	**4**		...

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1. If $k \in [p, i]$, then $A[k] \leq x$
2. If $k \in [i+1, j-1]$, then $A[k] > x$
3. If $k = r$, then $A[k] = x$
4. If $k \in [j, r-1]$, then $A[k]$ is unrestricted.

p $\quad$ i	$i+1$ $\quad$ $j-1$	j $\quad$ $r-1$	r
$\leq x$	$> x$	unrestricted	x

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$\leq x$	$> x$	unrestricted	x

Before the 1^{st} iteration, $i = p - 1, j = p$.

- $[p, i] = [p, p-1] = \emptyset$, condition (1) is trivially true.
- $[i+1, j-1] = [p, p-1] = \emptyset$, condition (2) is trivially true.
- condition (3) and (4) are trivially true.

Partition

Case (a) $A[j] > x$: Before:

p $\quad$ i	$i+1$	....	j	$r-1$	r
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p	i	....		j $\quad$ $r-1$	r
$\leq x$	$b \leq x$	$> x$	$a > x$	unrestricted	x

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After the loop termination $j = r - 1$:

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- This is what we want.
- It is easy to see **Partition**$(A, p.r)$ takes $\Theta(n)$ time where $n = r - p + 1$ is the number of elements in $A[p..r]$.

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2: $x = A[r]$
3: **swap**$(A[r], A[r])$
4: $q =$ **Partition**(A, p, r)
5: $k = q - p + 1$
6: **if** $i = k$, **return** $A[q]$
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The runtime will be $\Theta(n + (n-1) + (n-2) \dots 2 + 1) = \Theta(n^2)$.

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- How to do this?

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Replace the line (2) by the following:

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We will show in class this modification will give a $\Theta(n)$ time selection algorithm.

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    - Random Sampling: randomly pick pivot x to partition the array. On average, the algorithm takes $\Theta(n)$ time.
    - Derandomization: Make a clever choice of x, remove the randomness.
- These ideas are used in other algorithms.

Outline

Summary on Using DaC Strategy

- When divide into subproblems, the size of the sub-problems should be n/b for some constant $b > 1$. If it is only $n - c$ for some constant c, and there are at least two subproblems, this usually leads to exp time.

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- When divide into subproblems, the size of the sub-problems should be n/b for some constant $b > 1$. If it is only $n - c$ for some constant c, and there are at least two subproblems, this usually leads to exp time.

Example:

Fib(n)

1: **if** $n = 0$ **return** 0
2: **if** $n = 1$ **return** 1
3: **else return** Fib$(n-1)$+Fib$(n-2)$

Summary on DaC Strategy

We have:

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 1 \\ T(n-1) + T(n-2) + O(1) & \text{if } n \geq 2 \end{cases}$$

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When $k = n/2$, we have: $T(n) \geq 2^{n/2}T(0) = \Omega((\sqrt{2})^n) = \Omega((1.414)^n)$.

Actually, $T(n) = \Theta(\alpha^n)$ where $\alpha = \frac{\sqrt{5}+1}{2} \approx$1.618.

Summary on DaC Strategy

We have:

$$T(n) = \begin{cases} O(1) & \text{if } n \leq 1 \\ T(n-1) + T(n-2) + O(1) & \text{if } n \geq 2 \end{cases}$$

Thus:

$T(n) \geq T(n-2)+T(n-2) = 2T(n-2) \geq 2^2 T(n-2\cdot 2) \geq \cdots \geq 2^k T(n-2\cdot k)$.

When $k = n/2$, we have: $T(n) \geq 2^{n/2} T(0) = \Omega((\sqrt{2})^n) = \Omega((1.414)^n)$.

Actually, $T(n) = \Theta(\alpha^n)$ where $\alpha = \frac{\sqrt{5}+1}{2} \approx$1.618.

We make two recursive calls, with size $n-1$ and $n-2$. This leads to exp time.

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- If $\log_b a = k$, then the cost of two parts are about same. To improve, we must reduce both. Quite often, when you reach this point, you have the best algorithm!

Outline

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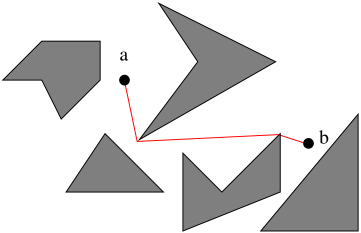
Motion Planing

Given a set of polygons in 2D plane and two points a and b, find the shortest path from a to b, avoiding all polygons.



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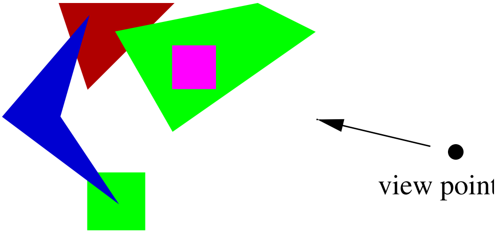

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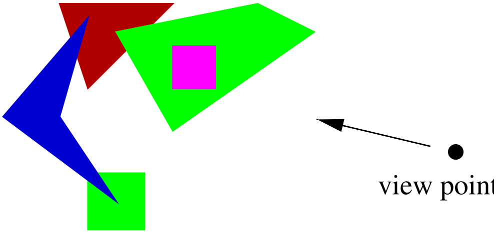

Application: Computer Graphics.

Closest Point Pair Problem

Input: A set $P = \{p_1, p_2 \ldots p_n\}$ of n points ($p_i = (x_i, y_i)$).

Find: $i \neq j$ such that dist$(p_i, p_j) \stackrel{\text{def}}{=} [(x_i - x_j)^2 + (y_i - y_j)^2]^{1/2}$ is the smallest among all point pairs.

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- By using DaC, we get a $\Theta(n \log n f(n))$ time algorithm.

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Note: Since X is already sorted, we can draw l between $x_{\lceil n/2 \rceil}$ and $x_{\lceil n/2 \rceil + 1}$.
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3: Recursively call **ClosestPair**(P_L). Let

- p_{Li}, p_{Lj} be the point pair with smallest distance in P_L.
- $\delta_L = \text{dist}(p_{Li}, p_{Lj})$.

Closest Point Pair Problem

4: Recursively call **ClosestPair**(P_R). Let

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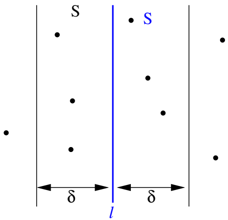
Note: Let S be the vertical strip with width 2δ centered at the line l. Then p_i and p_j must be in S. (Why?)


S
S
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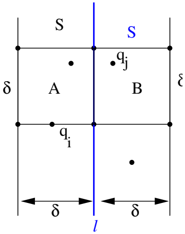
S
S
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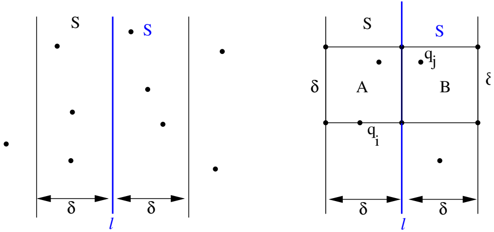
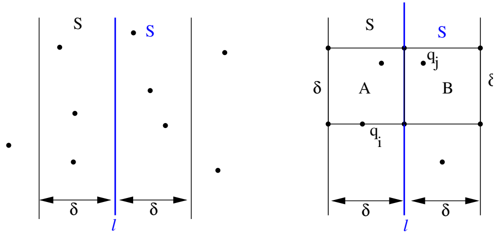


6.1: Let $P' = \{q_1, q_2, \ldots, q_t\}$ be the points in the 2δ wide strip S. Let Y' be the y-coordinates of points in P' in increasing order.



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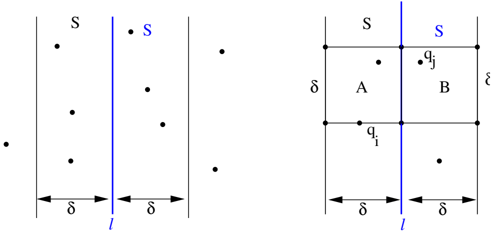
Note: Since Y is already sorted, we scan Y, and only include the points that are in the strip S. This takes $O(n)$ time.



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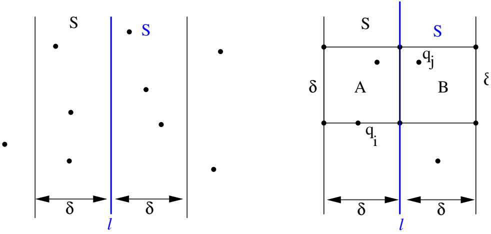

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Note: If (q_i, q_j) is the closest pair, then both must be in the region A or B (q_i is at the bottom edge). But any two points in A have inter-distance at least δ. A can contain at most 4 points. Similarly B can contain at most 4 points. So we only need to compare dist between q_i and next 7 points in P'!

Closest Point Pair Problem

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