INT201 Decision, Computation and Language

Lecture 12 – Review

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Summary of Summary

What can machines do and can not do?

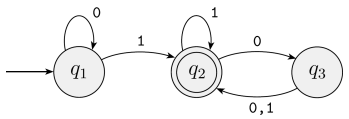
Regular Languages	Deterministic Finite Automata (DFA)
Regular Expression	Nondeterministic Finite Automata (NFA)
Context Free Languages	Nondeterministic Push-down Automata (PDA)
Turing-Decidable Languages	Turing Machine that Always Halts (Decider)
Turing-Recognizable Languages	Turing Machine
All Languages	Nothing that we can enumerate

Deterministic Finite Automata and Regular Language

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

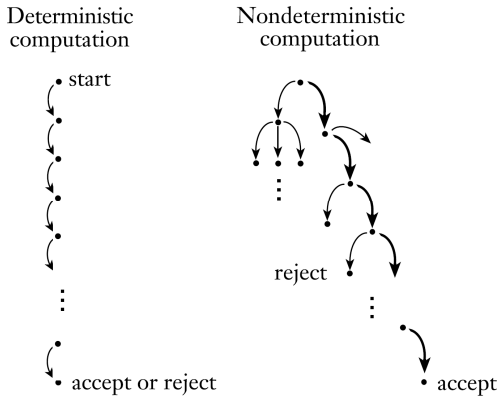
1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta: Q \times \Sigma \longrightarrow Q$ is the ***transition function***,[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.[2]

If A is the set of all strings that machine M accepts, we say that A is the ***language of machine*** M and write L(M) = A.



A language is called a ***regular language*** if some finite automaton recognizes it.

Nondeterministic Finite Automata



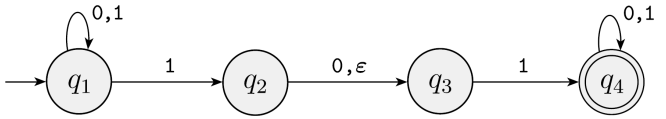
Accept as long as one branch can accept the string

Nondeterministic Finite Automata

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Multiple transition in the same state for the same string, defined by transition function has range in the power set.



Nondeterministic Finite Automata

Theorem:
Every nondeterministic finite automaton has an equivalent deterministic finite automaton.

Proof idea:

Construct DFA with states corresponds to the sets of states in NFA.

Regular Language Closure Properties

Theorem
The class of regular languages is closed under the **complement, union, intersection, concatenation, and Kleene star** operation.

Prove by constructing a new DFA/NFA to accept the language or relying on proven closure property.

Regular Expression

Say that R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

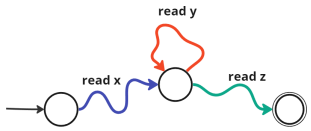
Theorem
A language is regular if and only if some regular expression describes it.

Proof idea:
Build NFA through its' closure property and build RE from generalized NFA whose transitions are REs.

Regular Language Pumping Lemma

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.



Proof idea:
DFA cannot recognize arbitrary long strings without repeating its state.

Application:
$\{0^n1^n | n \geq 0\}$ is non regular.

Context-free Language

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

Example:

G= ({S},{0,1},{S → 0S1 | ε}, S)

L(G)= $\{0^n1^n | n \geq 0\}$

Chomsky Normal Form

A context-free grammar is in ***Chomsky normal form*** if every rule is of the form

$$A \to BC$$
$$A \to a$$

where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

All variable generate two non-start variables or a terminal symbol.
Only start variable generates empty string.

CNF ruled out unnecessary production rules and constrained the derivation search pace.

Chomsky Normal Form is Sufficient

A context-free grammar is in ***Chomsky normal form*** if every rule is of the form

$$A \to BC$$
$$A \to a$$

where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

Theorem
Any context-free language is generated by a context-free grammar in Chomsky normal form.

Proof idea:
Non start variables that generate empty string can be avoided in the first place.
Long production rules can be decomposed.
Unary production rules can be eliminated.

Pushdown Automata

A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta: Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

(Standard) PDA is non-deterministic. They are NFA with a stack.
Deterministic PDA is weaker than PDA.

Pushdown Automata

A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

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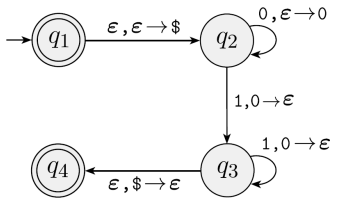


$L(M) = \{0^n 1^n | n \geq 0\}$

Pushdown Automata and CFG are equivalent

Theorem
A language is context free if and only if some pushdown automaton recognizes it.

Proof idea:

PDA simulate CNF by replacing symbols on the stack.

All PDA can be simplified so that they start and ends with empty stack, have a unique accepting state and have simple transitions.
CFG production rules can cover all states to states transitions that did not look at the stack context.

Context Free Language Closure Properties

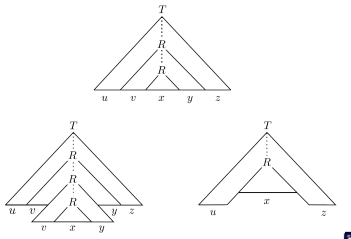
Theorem
The class of context free languages is closed under the **union, concatenation, and Kleene star** operation.

Proof idea:
Easy to see from the PDA perspective.

Pumping Lemma for Context Free Language

Pumping lemma for context-free languages If A is a context-free language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ satisfying the conditions

1. for each $i \geq 0$, $uv^ixy^iz \in A$,
2. $|vy| > 0$, and
3. $|vxy| \leq p$.



Proof idea:
CFG cannot generate arbitrary long strings without repeating its variable.

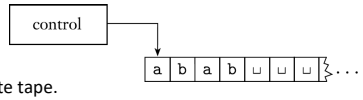
Application:
$\{ww | w \in \{0,1\}^*\}$ is non context free.

Take time to fill the MQ

Turing Machine

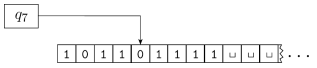
A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta: Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.



They are DFA with a writable infinite tape.
Accepts or rejects once it is in accept state or reject state.

Turing Machine Configuration



A Turing machine with configuration $1011q_701111$

$\delta\delta(q_i, b) = (q_{jj}, c, L)$ gives
$C_t = uaq_ibv$ yields $C_{t+1} = uq_{jj}acv$

A Turing machine M ***accepts*** input w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists, where
1. C_1 is the start configuration of M on input w,
2. each C_t yields C_{t+1}, and
3. C_k is an accepting configuration.

Recognizable Language and Decidable Language

Call a language A ***Turing-recognizable*** if some Turing machine M recognizes it in the sense that $\forall w \in \Sigma^*, w \in A$ if and only if M accepts w.

Call a language A ***Turing-decidable*** if some Turing machine (*decider*) M decides it in the sense that $\forall w \in \Sigma^*$, if $w \in A$, M accepts w, and if $w \notin A$, M rejects it.

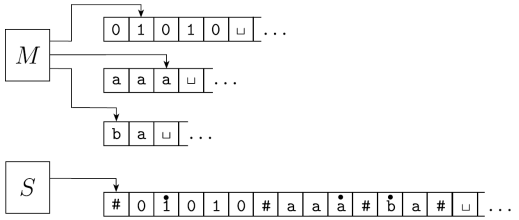
Clearly, all decidable languages are recognizable.

Multi-Tape Turing Machine

$$\delta: Q \times \Gamma^k \longrightarrow Q \times \Gamma^k \times \{\mathrm{L}, \mathrm{R}, \mathrm{S}\}^k,$$

Theorem
Every multi-tape Turing machine has an equivalent single-tape Turing machine.

Proof idea:

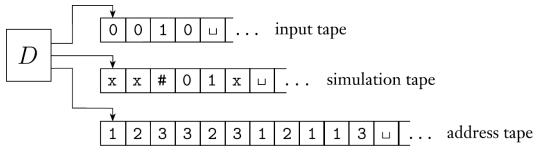


Nondeterministic Turing Machine

$$\delta: Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}).$$

Theorem
Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

Proof idea:



Check all branches of NTM through BFS over computation history recorded in the address tape.

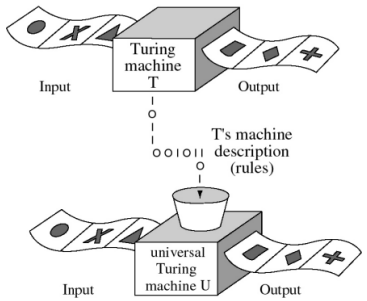
Existence of Non-Turing Recognizable Languages

Theorem
There exists non Turing-recognizable languages.

Proof:
There are uncountable many languages, but countable many Turing machines.

Universal Turing Machines

Theorem
There exists a TM U that takes a Turing machine description and input tape and simulate one step of that given Turing machine on the input tape.



Proof idea:

TM can simulate modern computer, following TM descriptions are purely mechanical.

Turing Machines High Level Description

A = {<G>| G is a connected undirected graph}.

The following is a high-level description of a TM M that decides A.

M = "On input $\langle G \rangle$, the encoding of a graph G:

1. Select the first node of G and mark it.
2. Repeat the following stage until no new nodes are marked:
3. For each node in G, mark it if it is attached by an edge to a node that is already marked.
4. Scan all the nodes of G to determine whether they all are marked. If they are, *accept*; otherwise, *reject*."

TM can perform computation on high level data structure that modern computers can perform.
In reality everything is encoded in binary after all.

Turing Language Closure Properties

Theorem
The class of Turing recognizable languages is closed under ?

Theorem
The class of Turing decidable languages is closed under ?

Left to you, as Ass2 has a question on this.
Think about whether they have the same closure properties.

Decidable Languages

- $L_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accept } w\}$
- $L_{CFG} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$
- $E_{CFG} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G)=\emptyset\}$

Intuitively, we consider decidable languages to be solvable by Turing machines.

Acceptance Problem is Recognizable but Undecidable

$A_{TM} = \{\langle M, w\rangle : M \text{ is a Turing machine that accepts the string } w\}$

Theorem

A_{TM} is Turing-recognizable but not decidable.

Proof idea:

UTM can simulate M on w, so it is Turing-recognizable.

If a decider H exists:

$D =$ "On input $\langle M \rangle$, where M is a TM:
1. Run H on input $\langle M, \langle M \rangle\rangle$.
2. Output the opposite of what H outputs. That is, if H accepts, *reject*; and if H rejects, *accept*."

D is paradoxical.

Unrecognizable Languages

Theorem

A language A is decidable if and only if it is Turing-recognizable and co-Turing-recognizable.

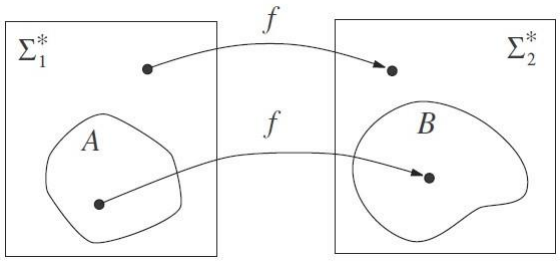
Proof idea:

One direction is easy.

Assume recognizable and co-recognizable, we simulate two recognizable in parallel to form a decider.

Corollary $\overline{A_{TM}}$ is not Turing-recognizable.

Mapping Reduction



$w \in A \Longleftrightarrow f(w) \in B$

We write A is mapping reducible to B as $A \leq_m B$

Mapping Reducibility and Halting Problem

Theorems

- If $A \leq_m B$ and B is decidable, then A is decidable.
- If $A \leq_m B$ and A is undecidable, then B is undecidable also.
- If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.
- If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable also.

$H_{TM} = \{\langle M, w\rangle : M \text{ is a Turing machine that halts on the string } w\}$

Theorem

H_{TM} is Turing-recognizable but not decidable.

Proof idea:

Reduce undecidable A_{TM} to H_{TM} shows H_{TM} is undecidable.

Non-trivial Properties and Rice's Theorem

- $E_{TM} = \{< M > \mid M \text{ is a TM and } L(M) = \emptyset\}$.
- $INFINITE_{TM} = \{< M > \mid M \text{ is a TM and } |L(M)| = \infty\}$.
- $LT_{TM} = \{< M > \mid M \text{ is a TM and } L(M) = L(T)\}$.
- $FINITE_{TM} = \{< M > \mid M \text{ is a TM and } \exists n \in N, |L(M)| = n\}$.
- $ALL_{TM} = \{< M > \mid M \text{ is a TM and } L(M) = \Sigma^*\}$

Turing machine descriptions of non-trivial properties P :

1. none empty set. i.e., $P \neq \emptyset$
2. none of them includes all Turing machines . i.e., $P \neq \{< M > \mid M \text{ is a TM}\}$
3. <M> is accepted iff L(M) satisfy some properties,

   i.e., P=$\{< M > \mid M \text{ is a TM and } L(M) \in LP \subset Power(\Sigma^*)\}$.

Rice's Theorem

Any non-trivial property of Turing machines is undecidable

Rice's Theorem and Applications

Let P be a subset of Turing machine descriptions such that

1. $P \neq \emptyset$,
2. P is a proper subset of Turing machine descriptions ,
3. for any two Turing machines M_1 and M_2 with $L(M_1) = L(M_2)$,

   (a) either both $\langle M_1 \rangle$ and $\langle M_2 \rangle$ are in P or

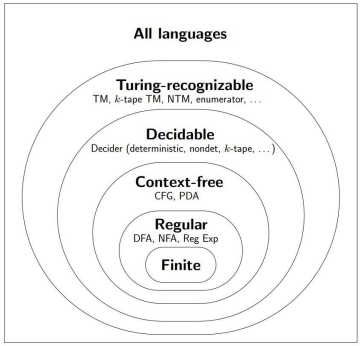
   (b) none of $\langle M_1 \rangle$ and $\langle M_2 \rangle$ is in P.

$INFINITE_{TM} = \{< M > \mid M \text{ is a TM and } |L(M)| = \infty\}$.

1. A TM accepts all inputs is in the set.
2. A TM rejects all inputs is not in the set.
3. If $L(M_1) = L(M_2)$, $|L(M_1)| = |L(M_2)|$ so they are both either infinite or finite, either both in the set or not in the set respectively.

Therefore $INFINITE_{TM}$ is undecidable by Rice's theorem

Language Hierarchy



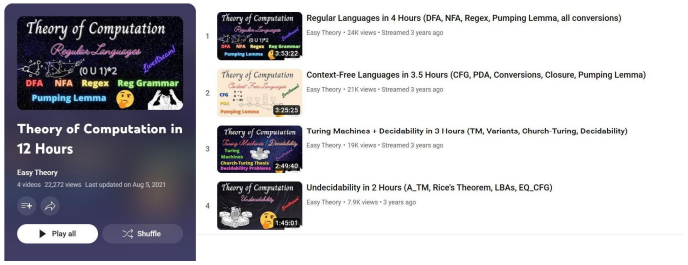
Except enumerator (check Sipser), you need to know them all.

You need to know their closure properties.

You need to know examples that distinguish the circles.

You need to know theorems that allow you to distinguish the circles.

Highly Recommended Revision Resources



Theory of Computation in 12 Hours at Easy Theory YouTube Channel

Email me at chunchuan.lyu@xjtlu.edu.cn at any time

TAs will be here for QA Thursday.


THANK YOU
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