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Int 201: Decision Computation and Language
Tutorial 11 Solution

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Question 1. Show if $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

Solution 1. Proof: Let TM_B be the TM that recognizes B, and the map be f. TM_A = "On input w:

1. Compute $f(w)$.
2. Run TM_B on $f(w)$ and output whatever TM_B outputs."

Clearly, if $w \in A$, then $f(w) \in B$ by the mapping reduction definition. Thus TM_A accepts w, as TM_B accepts $f(w)$. If $w \notin A$, $f(w) \notin B$ and TM_B either rejects or loops, and consequently TM_A either rejects or loops.

Question 2. Show $\mathrm{EQ_{TM}} = \{\langle M_1, M_2\rangle | M_1, M_2$ are TMs and $L(M_1) = L(M_2)\}$ is undecidable. Hint: reduction from the emptiness problem $\mathrm{E_{TM}}$.

Solution 2. Proof by contradiction: Suppose we have a decider EQ for $\mathrm{EQ_{TM}}$, we reduce $\mathrm{E_{TM}}$ to $\mathrm{EQ_{TM}}$. Let $TM_\emptyset$ be a Turing machine that accepts no string.

E = "On input $< M >$: Run EQ on $(\langle M\rangle,\langle TM_\emptyset\rangle)$ and output whatever TM_B outputs."

We claim E is a decider for $\mathrm{E_{TM}}$, as $\mathrm{EQ_{TM}}$ decides $L(M) = \emptyset$. As we know $\mathrm{E_{TM}}$ is undecidable, we have a contradiction. So, $\mathrm{EQ_{TM}}$ cannot be decidable.

Question 3. Show the two property conditions for the Rice's theorem are equivalent.

1. for any two TMs M_1 and M_2 with $L(M_1) = L(M_2)$, $\langle M_1\rangle \in P \iff \langle M_2\rangle \in P$.

2. $\exists p : \{L(M) | M \text{ is a TM}\} \rightarrow \{0, 1\}$ such that $\langle M \rangle \in P \iff p(L(M)) = 1$

Solution 3. To show $1 \implies 2$, we construct the p function as:

$$p(L) = \begin{cases} 1, & \text{if } \exists M \text{ such that } L(M) = L \text{ and } \langle M \rangle \in P \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Now we need to show $\langle M \rangle \in P \iff p(L(M)) = 1$. If $\langle M \rangle \in P$, then $p(L = L(M)) = 1$ as the existence of M such that $L(M) = L(M)$ and $\langle M \rangle \in P$ is given. If $p(L(M)) = 1$, then there exists M' such that $L(M') = L(M)$ and $\langle M' \rangle \in P$. By the condition 1, we have $\langle M \rangle \in P$.

To show $2 \implies 1$, we have the existence of p function and we consider M_1 and M_2 with $L(M_1) = L(M_2)$. First, if $\langle M_1 \rangle \in P$, then by the condition 2, we have $p(L(M_1)) = 1$. However, $p(L(M_2)) = p(L(M_1)) = 1$ as $L(M_1) = L(M_2)$. Apply the condition 2 again, we have $\langle M_2 \rangle \in P$. By symmetry, we also have $\langle M_2 \rangle \in P \implies \langle M_1 \rangle \in P$.

Question 4. Show $\text{FINITE}_{\text{TM}} = \{\langle M \rangle | M \text{ is a TM and } \exists n \in \mathcal{N}, |L(M)| = n\}$ is undecidable by using the Rice's theorem.

Solution 4. We need to check the three conditions of the Rice's theorem.

1. A Turing machine that only accepts empty tape, has L(M)=1, therefore it is in $\text{FINITE}_{\text{TM}}$ and this checks the non-emptiness.
2. A Turing machine that accepts all strings have $L(M) = \infty$, therefore it is not in $\text{FINITE}_{\text{TM}}$ and this checks the proper subset condition.
3. For any two TMs M_1 and M_2 with $L(M_1) = L(M_2)$, if $\langle M_1 \rangle \in P$, then $\exists n \in \mathcal{N}, |L(M_1)| = n$. Therefore, $|L(M_2)| = |L(M_1)| = n$, and $\langle M_2 \rangle \in P$. By symmetry, we have the only if part too. This checks the language property condition.

In all, we have shown that $\text{FINITE}_{\text{TM}}$ is a non-trivial property of TM descriptions. Therefore, by the Rice's theorem it is undecidable.