INT201 Decision, Computation and Language

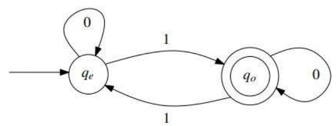
Lecture 3 – Nondeterministic Finite Automata

Dr Yushi Li

Xi'an Jiaotong-Liverpool University
西交利物浦大學

Nondeterministic Finite Automata (NFA)

A finite automata is **deterministic**, if the next state the machine goes to on any given symbol is uniquely determined.



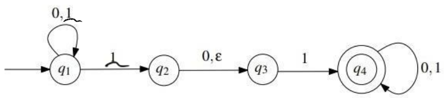
- DFA has exactly one transition leaving each state for each symbol.

每个symbol仅对应一条出路

Nondeterministic Finite Automata (NFA)

A finite automata is **nondeterministic**, if the machine allows for several or no choices to exist for the next state on a given symbol.

每个symbol可以存在不只一条出路

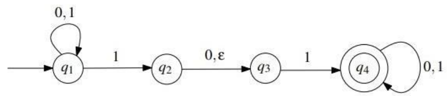


For a state q and symbol $s \in \Sigma$, NFA can have:

- Multiple edges leaving q labelled with the same symbol s; 一入多出
- No edge leaving q labelled with symbol s;
- Edge leaving q labelled with ϵ (without reading any symbol). ε 代表当前 q 不用读取任何信息就可以直接跳转到下一个状态.

ε 可以加在 DFA. 但是没有必要

Nondeterministic Finite Automata (NFA)

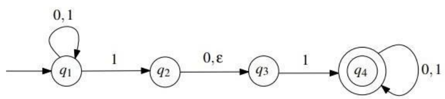


This NFA is in a state with multiple ways to proceed, e.g. state q_1 has two transition paths with 1.

The machine splits into multiple copies of itself (threads): 每个副本都独立于其他副本进行计算。

- Each copy proceeds with computation independently of others.
- NFA may be in a set of states, instead of a single state. NFA 可以有多种状态.
- NFA follows all possible computation paths in parallel. 允许并行的潜在计算
- If a copy is in a state and next input symbol doesn't appear on any outgoing edge from the state, then the copy dies or crashes. 如果在到达一个状态之后无法再接受接下来的输入则代表这个遇到了死胡同

Nondeterministic Finite Automata (NFA)

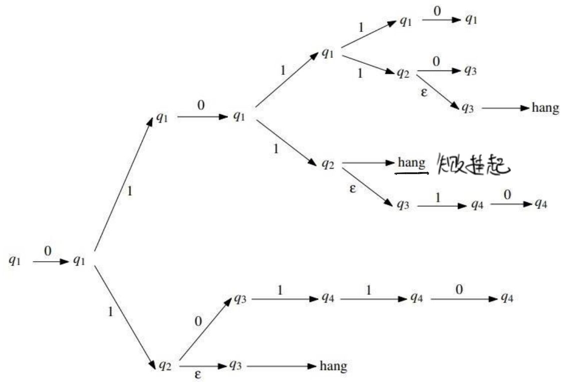


The NFA accepts the input string, if any copy ends in an accept state after reading the entire string. 只要有一组路线成立就可以

The NFA rejects the input string, if no copy ends in an accept state after reading the entire string. 如果所有路线都无法最终坐落于 accept state，那么就不成立

Nondeterministic Finite Automata (NFA)

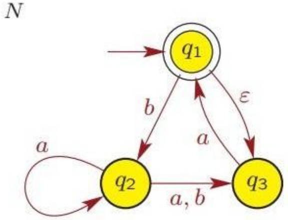
What can this automaton do when it gets the string 010110 as input?



Nondeterministic Finite Automata (NFA)

Example

NFA N



N accepts strings ϵ, a, aa, baa, $baba$, ...

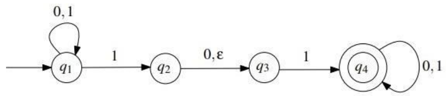
e.g. $aa = \varepsilon a \varepsilon a$

$\rightarrow q_1 \xrightarrow{\varepsilon} q_3 \xrightarrow{a} q_1 \xrightarrow{\varepsilon} q_3 \xrightarrow{a} q_1$

N does not accept (i.e. rejects) strings b, ba, bb, bbb, . . .

Nondeterministic Finite Automata (NFA)

Question



How about 010?

$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_1 \xrightarrow{0} q_1$

$q_1 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{\varepsilon} q_3 \longrightarrow$ hang

$q_2 \xrightarrow{0} q_3$

Formal Definition of NFA

For any alphabet Σ, we define Σ_ϵ to be the set:

$$\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$$

Recall the notion of a power set: For any set Q, the power set of Q, denoted by $P(Q)$, is the set of all subsets of Q:

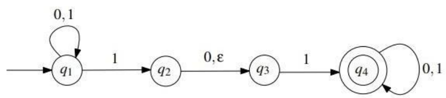
$$P(Q) = \{R : R \subseteq Q\}$$

A **nondeterministic finite automaton** (NFA) is a 5-tuple $M = (Q, \Sigma, \delta, q, F)$, where

1. Q is a finite set of **states**,
2. Σ is a finite set of symbols, called the **alphabet** of the automaton,
3. $\delta : Q \times \Sigma_\epsilon \rightarrow P(Q)$ is a function, called the **transition function**,
4. $q \in Q$ is called the **initial/start state**,
5. $F \subseteq Q$ is a set of **accepting/terminal states**.

Formal Definition of NFA

Example



Formal description of above NFA $M = (Q, \Sigma, \delta, q, F)$

1. $Q = \{q_1, q_2, q_3, q_4\}$
2. $\Sigma = \{0, 1\}$
3. $\delta : Q \times \Sigma_\epsilon \rightarrow P(Q)$

4. q_1 is the start state
5. $F = \{q_4\}$ is a set of accepting states.

Formal Definition of NFA

Let $M = (Q, \Sigma, \delta, q, F)$ be an NFA, and let $w \in \Sigma^*$. We say that M accepts w, if w can be written as $w = y_1 y_2 \ldots y_m$ where $y_i \in \Sigma_\epsilon$ for all i with $1 \leq i \leq m$, and there exists a sequence of states $r_1, r_2, \ldots, r_m$ in Q, such that:

- $r_0 = q$
- $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, 1, \ldots, m-1$
- $r_m \in F$

Otherwise, we say that M rejects the string w.

Difference between DFA and NFA

- DFA has transition function $\delta : Q \times \Sigma \rightarrow Q$
- NFA has transition function $\delta : Q \times \Sigma_\epsilon \rightarrow P(Q)$

Different transition function

✓ Returns a set of states rather than a single state.

✓ Allows for ϵ-transition because $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$.

✓ Note that every DFA is also an NFA.

Formal Definition of NFA

Extend the map δ to a map $Q \times \Sigma^* \to P(Q)$ by defining:

$$\delta(q, \epsilon) = \{q\} \quad \text{for all } q \in Q$$

$$\delta(q, wa) = \cup_{p\in\delta(q,\, w)} \delta(p, a) \quad \text{for all } \ q \in Q;\ w \in \Sigma^*;\ a \in \Sigma$$

Thus $\delta(q, w)$ is the set of all possible states that can arise when the input w is received in the state q. w is accepted provided that $\delta(q, w)$ contains an accepting state.

Notation: accepting/rejecting paths

Suppose, in a DFA, we can get from state p to state q via transitions labelled by letters of a word w. Then we say that the states p and q are connected by a path with label w.

If $w = abc$ and the 2 intermediate states are r_1 and r_2 we could write this as:

$$p \xrightarrow{a} r_1 \xrightarrow{b} r_2 \xrightarrow{c} q$$

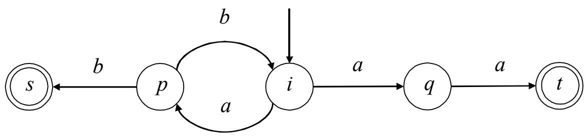
In a NFA, if $\delta(p, a) = \{q, r\}$ we could write:

$$\begin{array}{ll} & \xrightarrow{a} q \\ \to p & \\ & \xrightarrow{a} r \end{array} \quad = \quad \{p\} \xrightarrow{a} \{q, r\}$$

and this would be an **accepting path** if any state on the RHS is an accepting state, otherwise it would be **rejecting path**.

Nondeterministic Finite Automata (NFA)

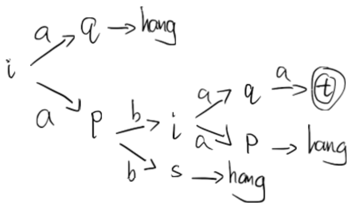
Exercise



Can input string *abaa* be accepted by this NFA? $i \xrightarrow{a} p \xrightarrow{b} i \xrightarrow{a} q \xrightarrow{a} t$

Possible paths of this string?

The accepting path of this string?



$\{i\} \xrightarrow{a} \{q,p\} \xrightarrow{b} \{i,s\} \xrightarrow{a} \{p,q\}$
$\xrightarrow{a} \{t\}$

Language accepted by NFA

Let $M = (Q, \Sigma, \delta, q, F)$ be an NFA. The language $L(M)$ accepted by M is defined as

$$L(M) = \{w \in \Sigma^* : M \text{ accepts } w\}.$$

Example

Let A be the language $A = \{w \in \{0, 1\}^* : w$ has a 1 in the third position from the right$\}$, design $M : L(M)$.

如果两个自动机能同时识别
同一个语言，那么这两个自动机相等

Equivalence of DFAs and NFAs)

Two machines (of any type) are **equivalent** if they recognize the same language.

DFA is a restricted form of NFA:

- Every NFA has an equivalent DFA.
- We can convert an arbitrary NFA to a DFA that accepts the same language.
- DFA has the same power as NFA

① Difference of NFA: $M = \{Q, \Sigma, \delta, q, F\}$
② Difference DFA, NFA
δ (DFA) $Q \times \Sigma = Q$
δ (NFA) : $Q \times \Sigma_\epsilon = P(Q)$

③ Relationship DFA, NFA
1. Equivalence of D and N
2. DFA -> NFA
3. NFA -> DFA

DFA to NFA

The formal conversion of a DFA to an NFA is done as follows: Let $M = (Q, \Sigma, \delta, q, F)$ be a DFA. Recall that δ is a function $\delta : Q \times \Sigma \rightarrow Q$. We define the function $\delta' : Q \times \Sigma_\epsilon \rightarrow P(Q)$ as follows. For any $r \in Q$ and for any $a \in \Sigma_\epsilon$,

NFA 转 DFA

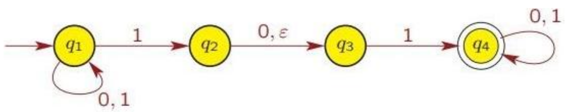
$$\delta'(r, a) = \begin{cases} \{\delta(r, a)\} & if\ a \neq \epsilon \\ \emptyset & if\ a = \epsilon \end{cases}$$

Then $N = (Q, \Sigma, \delta', q, F)$ is an NFA, whose behavior is exactly the same as that of the DFA M; the easiest way to see this is by observing that the state diagrams of M and N are equal. Therefore, we have $L(M) = L(N)$.

NFA to DFA

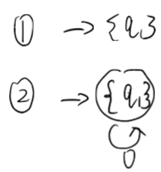
Example

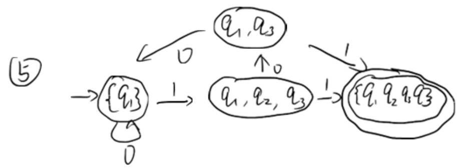
Convert NFA into equivalent DFA

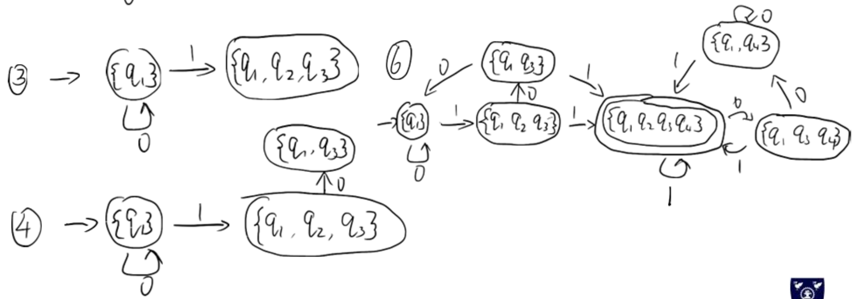


NFA to DFA

Conversion step







NFA to DFA

R is a set (can be ε)

E(R) = {可以从R中到达的q}

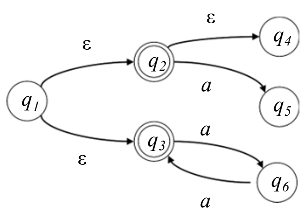
The ε-closure of a set of states $R \subseteq Q$:

$E(R) = \{\, q \mid q$ can be reached from R by travelling over zero or more ε transitions $\}$.

Example

$$E(\{q_1, q_2\}) = \{q_1, q_2, q_3\}.$$

Question



$E(q_1) = \{q_1, q_2, q_3, q_4\}$

$E(q_2) = \{q_2, q_4\}$

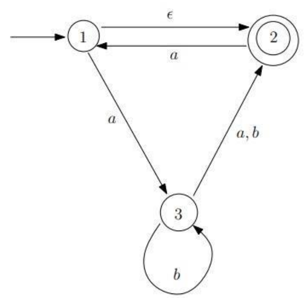
$E(q_1) = ?\quad E(q_2) = ?$

NFA to DFA

Example

Consider the NFA $M = (Q, \Sigma, \delta, q, F)$, where $Q = \{1, 2, 3\}$, $\Sigma = \{a, b\}$, $q = 1$, $F = \{2\}$, and δ is given by the following table:

	a	b	ϵ
1	$\{3\}$	$\emptyset$	$\{2\}$
2	$\{1\}$	$\emptyset$	$\emptyset$
3	$\{2\}$	$\{2,3\}$	$\emptyset$



NFA to DFA

Example

$M = \{Q, \Sigma, \delta, q, F\}$

How can we convert the above NFA to a DFA?

① $Q' = P(Q) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

② Σ' continue be $\{a, b\}$

③ $\delta' = Q' \times \Sigma' \to Q'$

④ $q' = E(\{1\}) = \{1,2\}$ 起始点，注意 ε

⑤ $F' = \{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

NFA to DFA

Example

How can we convert the above NFA to a DFA?