Random Matrix Theory problems (Work in progress).

Solutions by Gustavo Fuentes

Contents

1 Deterministic Matrices

Problem 1.1. Instability of eigenvalues of non-symmetric matrices

Consider the $N \times N$ square band diagonal matrix $\mathbf{M}_0$ defined by $[\mathbf{M}_0]_{ij} = 2\delta_{i,j-1}$:

$$\mathbf{M}_0 = \begin{bmatrix} 0 & 2 & 0 & ... & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 2 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}$$

(a) Show that $\mathbf{M}_0^N = 0$ and so all the eigenvalues of $\mathbf{M}_0$ must be zero. Use a numerical eigenvalue solver for non-symmetric matrices and confirm numerically that this is the case for $N = 100$.

(b) If $\mathbf{O}$ is an orthogonal matrix ($\mathbf{O}\mathbf{O}^\intercal = \mathbf{1}$), $\mathbf{O}\mathbf{M}_0\mathbf{O}^\intercal$ has the same eigenvalues as $\mathbf{M}_0$. Following Exercise 2, generate a random orthogonal matrix $\mathbf{O}$. Numerically find the eigenvalues of $\mathbf{O}\mathbf{M}_0\mathbf{O}^\intercal$. Do you get the same answer as in (a)?

(c) Consider $\mathbf{M}_1$ whose elements are all equal to those of $\mathbf{M}_0$ except for one element in the lower left corner $[\mathbf{M}_1]_{N,1} = (1/2)^{N-1}$. Show that $\mathbf{M}_1 = \mathbf{1}$; more precisely, show that the characteristic polynomial of $\mathbf{M}_1$ is given by $\det(\mathbf{M}_1 - \lambda\mathbf{1}) = \lambda^N - 1$, therefore $\mathbf{M}_1$ has N distinct eigenvalues.

(d) For N greater than about 60, $\mathbf{O}\mathbf{M}_0\mathbf{O}^\intercal$ and $\mathbf{O}\mathbf{M}_1\mathbf{O}^\intercal$ are indistinguishable to machine precision. Compare numerically the eigenvalues of these two rotated matrices.

Solution 1.1. (a) The elements of $[\mathbf{M}_0^N]_{ij}$ for $1 \leq i, j \leq N$ are given by the following product:

$$[\mathbf{M}_0^N]_{ij} = [\mathbf{M}_0\mathbf{M}_0^N]_{ij} = \sum_k [\mathbf{M}_0]_{ik}[\mathbf{M}_0^{N-1}]_{kj} =$$
$$= \sum_{k_1}\sum_{k_2}[\mathbf{M}_0]_{ik_1}[\mathbf{M}_0]_{k_1k_2}[\mathbf{M}_0^{N-1}]_{k_2j}$$

repeating the product N times, then

$$= \sum_{k_1,k_2,\dots,k_N}^{N} [\mathbf{M}_0]_{ik_1}[\mathbf{M}_0]_{k_1k_2}\cdots[\mathbf{M}_0]_{k_{N-1}k_N}[\mathbf{M}_0]_{k_Nj} =$$
$$= \sum_{k_1=2}^{N}\sum_{k_2=k_1+1}^{N}\cdots\sum_{k_N=k_1+N-1}^{N} 2^N\delta_{i,k_1-1}\delta_{k_1-1,k_2-2}\cdots\delta_{k_{N-1}-N,k_N-N}\delta_{k_N-N+1,j-N}$$

However, in order to obtain a non-zero value $j - N - 1 = i \Rightarrow j = i + N$ and this is out of bounds due to $1 \leq j \leq N$.

$\therefore [\mathbf{M}_0^N]_{ij} = 0, \forall i, j.$

The eigenvalues are given by:

$$\det(\mathbf{M}_0 - \lambda \mathbf{1}) = 0$$
$$\det\left(\begin{bmatrix} -\lambda & 2 & 0 & ... & 0 \\ 0 & -\lambda & 2 & \dots & 0 \\ & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 2 \\ 0 & 0 & 0 & ... & -\lambda \end{bmatrix}\right) = 0$$

This is a triangular matrix, then its determinant equals the product of the diagonal entries

$$\Rightarrow \prod_{k=1}^{N} (-1)^N \lambda_k = 0$$

then all its eigenvalues are $\lambda = 0$.

```
import numpy as np
M = 2*np.eye(100, k=1)
uM, _ = np.linalg.eig(M)
plt.hist(uM, bins='auto',ec = 'Black', color = 'slateblue', density = True)
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.xlim(-4,4)
plt.show()
```

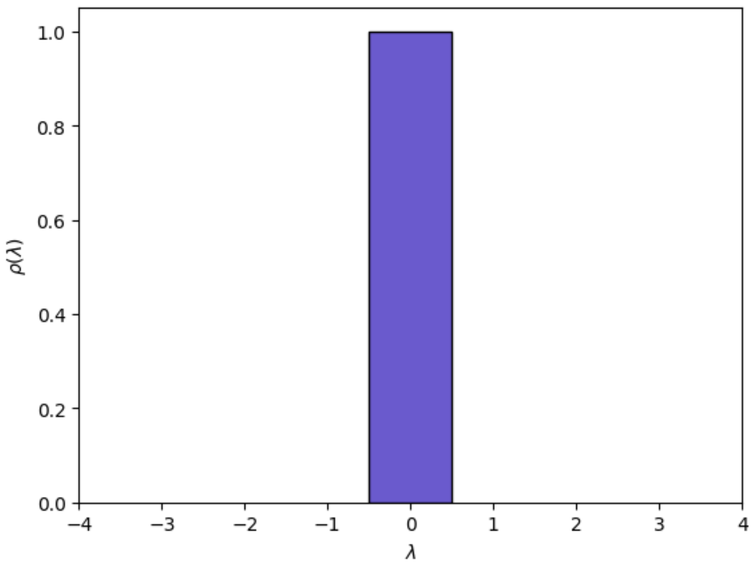


Figure 1: Delta distribution.

(b) The eigenvalues are not the same.

```
O = rmt(N = 100).orthogonal()
M0new = O@M@O.T
unew, _ = np.linalg.eig(M0new)
plt.hist(np.real(unew), bins=30,ec = 'Black', color = 'slateblue', density
                                        = True)
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.title('$OM_{0}O^{T}$ eigenvalues')
plt.show()
```

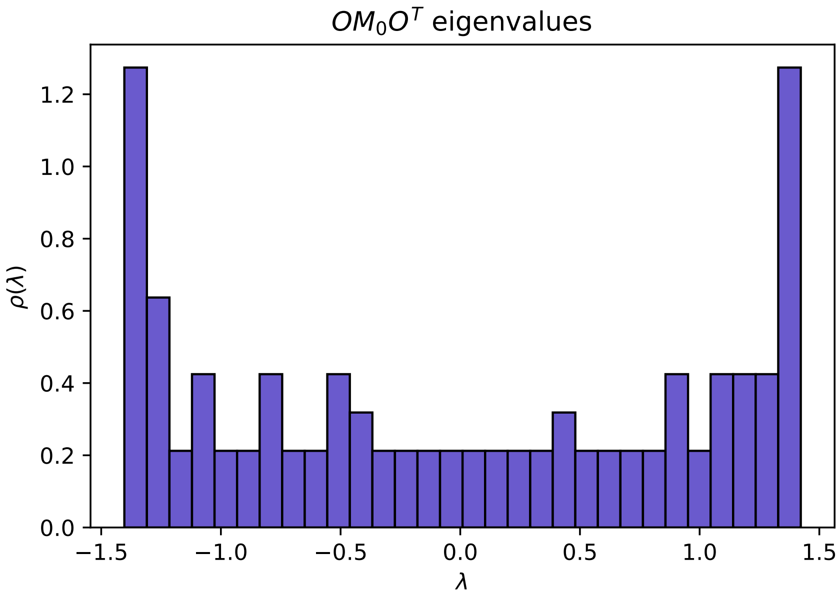


Figure 2: Eigenvalues of the product $\mathbf{OM_oO}^T$

(c) The non-zero elements of $\mathbf{M}_1$ are given by $[\mathbf{M}_1]_{i,i+1} = 2$, $[\mathbf{M}_1]_{N,1} = (1/2)^{N-1}$. The non-zero elements of $[\mathbf{M}_1^N]_{ij}$ are those for $i = j$ and that involves, other wise we have the same case as a) $[\mathbf{M}_1]_{N,1} = (1/2)^{N-1}$, this is:

$$[\mathbf{M}_1^N]_{i,i} = [\mathbf{M}_1]_{i,i+1}[\mathbf{M}_1]_{i+1,i+2}\cdots[\mathbf{M}_1]_{N-1,N}[\mathbf{M}_1]_{N,1}[\mathbf{M}_1]_{1,2}[\mathbf{M}_1]_{2,3}\cdots[\mathbf{M}_1]_{i-1,i} =$$
$$= 2\cdot 2\cdot 2\cdots\frac{1}{2^{N-1}}\cdot 2\cdots 2 = 1$$
$$\therefore \mathbf{M}_1^N = \mathbf{1}$$

The eigenvalues are given by:

$$\det(\mathbf{M}_1 - \lambda \mathbf{1}) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 2 & 0 & ... & 0 \\ 0 & -\lambda & 2 & \dots & 0 \\ & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 2 \\ \frac{1}{2^{N-1}} & 0 & 0 & ... & -\lambda \end{bmatrix}\right) = 0$$

$$= -\lambda(-\lambda)^{N-1} - (-1)^N \frac{1}{2^{N-2}} 2^{N-1} = \lambda^N - 1 = 0$$

$$\therefore \lambda_k = \cos\left(\frac{2\pi k}{N}\right)$$

$$\lambda_k = \exp\left(\frac{2\pi i k}{N}\right)$$

(d) For $N = 61$

```
def M1(N):
    M = 2*np.eye(N, k = 1)
    M[N-1,0] = 2**(-1*N+1)
    return M
M0 = 2*np.eye(61, k = 1)
O0 = rmt(N = 61).orthogonal()
M0new = O0@M0@O0.T
u0new, _ = np.linalg.eig(M0new)

M11 = M1(N = 61)
O1 = rmt(N = 61).orthogonal()
M1new = O1@M11@O1.T
u1new, _ = np.linalg.eig(M1new)

plt.figure(figsize = (8,8))
plt.subplot(2,1,1)
plt.hist(np.real(u0new), bins=30,ec = 'Black', color = 'slateblue',
                                                    density = True
     ,label = '$OM_{0}O^{T}$')
plt.legend(loc = 'upper center')
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')

plt.subplot(2,1,2)
plt.hist(np.real(u1new), bins=30,ec = 'Black', color = 'red', density =
                                                    True
     ,label = '$OM_{1}O^{T}$')
plt.legend(loc = 'upper center')

plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')

plt.show()
```

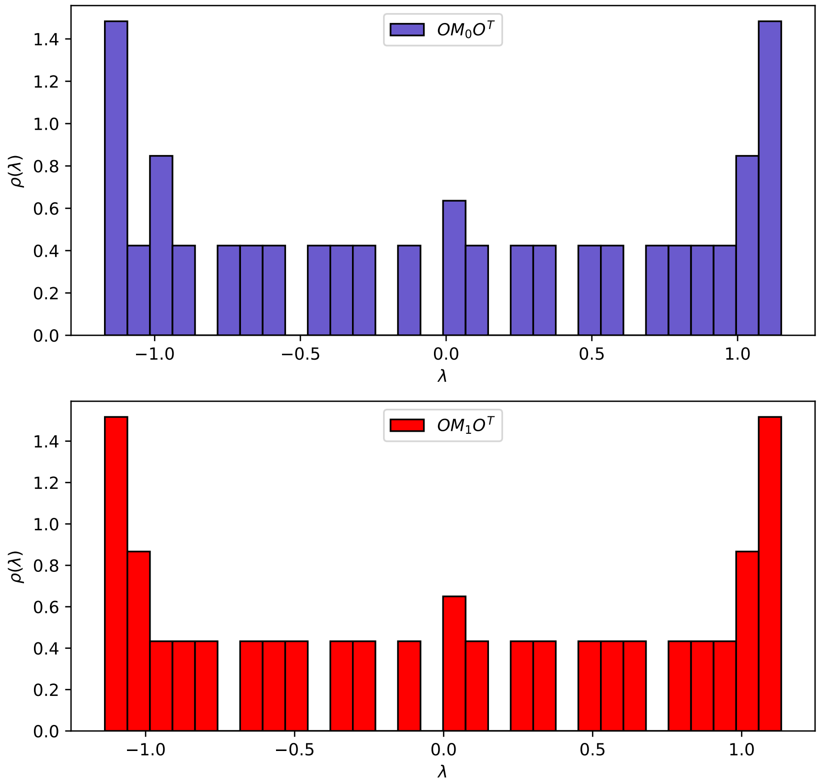


Figure 3: Eigenvalues comparision.

Problem 1.2. Gershgorin and Perron-Frobenius.

Show that the upper bound in

$$\min_j \sum_j \mathbf{A}_{ij} \leq \lambda_{\max} \leq \max_j \sum_j \mathbf{A}_{ij} \tag{1}$$

is a simple consequence of the Gershgorin theorem.

Solution 1.2. By the Gershgorin theorem $\exists \mathcal{D}_i : \lambda_{max} \in \mathcal{D}_i$
Then by the Perron-Frobenius theorem

$$\lambda_{\max} \leq \max_i \sum_j \mathbf{A}_{ij} \leq \sum_{i \neq j} \mathbf{A}_{ij} = R_i$$

due to the elements $\mathbf{A}_{ij} > 0$, then the left side is consequence of the Gershgorin theorem.

Problem 1.3. Sherman-Morrison.

Show that

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \tag{2}$$

is correct by multiplying both sides by $\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)$.

Solution 1.3. Left side:

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)\left(\mathbf{A} - \mathbf{u}\mathbf{v}^T\right)^{-1} = \mathbf{1} \tag{3}$$

Right side:

$$\begin{aligned}
&\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)\left(\mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}\right) = \\
&= \left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)\mathbf{A}^{-1} - \left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right)\frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} = \\
&= \mathbf{A}\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - \frac{\mathbf{A}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} = \\
&= \mathbf{1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - \frac{\mathbf{1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}}
\end{aligned}$$

Notice that $1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u} \in \mathbb{R}$, then:

$$\begin{aligned}
&= \mathbf{1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - \frac{\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} + \left(\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\right)\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} = \\
&= \mathbf{1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - \frac{\left(1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}\right)\mathbf{u}\mathbf{v}^T\mathbf{A}^{-1}}{1 + \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} = \\
&= \mathbf{1} + \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} - \mathbf{u}\mathbf{v}^T\mathbf{A}^{-1} = \mathbf{1}
\end{aligned} \tag{4}$$

Then left side (3) and right side (4) are equal.

Problem 1.4. Combining Schur and Sherman-Morrison.

The Schur complement, also called inversion by partitioning, relates the blocks of the inverse of a matrix to the inverse of blocks of the original matrix. Let $\mathbf{M}$ be an invertible matrix which we divide in four blocks as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \text{ and } \mathbf{M}^{-1} = \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix} \tag{5}$$

Where $[\mathbf{M}_{11}] = n \times n$, $[\mathbf{M}_{11}] = n \times n$, $[\mathbf{M}_{12}] = n \times (N-1)$,$[\mathbf{M}_{22}] = (N-1) \times (N-1)$ and $[\mathbf{M}_{22}]$ is invertible. The integer n can take any values from 1 to $N-1$.

For $n = 1$ and any $N > 1$, combine the Shur complement (5) and the Sherman-Morrison to show that:

$$\mathbf{Q}_{22} = (\mathbf{M}_{22})^{-1} + \frac{(\mathbf{M}_{22})^{-1}\mathbf{M}_{21}\mathbf{M}_{12}(\mathbf{M}_{22})^{-1}}{\mathbf{M}_{11} - \mathbf{M}_{12}(\mathbf{M}_{22})^{-1}\mathbf{M}_{21}}$$

Solution 1.4. Using the Sherman-Morrison formula (2) for $\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}\mathbf{M}_{12}$:

$$(\mathbf{M}_{22} - \mathbf{M}_{21}\mathbf{M}_{11}\mathbf{M}_{12})^{-1} = (\mathbf{M}_{22})^{-1} + (\mathbf{M}_{22})^{-1}\mathbf{M}_{21}\left(\mathbf{M}_{11} - \mathbf{M}_{12}(\mathbf{M}_{22})^{-1}\mathbf{M}_{21}\right)^{-1}\mathbf{M}_{12}(\mathbf{M}_{22})^{-1}$$

due to $\mathbf{M}_{11} = (\mathbf{M}_{11})^{-1}$ is a number.

$$\therefore \mathbf{Q}_{22} = (\mathbf{M}_{22})^{-1} + (\mathbf{M}_{22})^{-1}\mathbf{M}_{21}\left(\mathbf{M}_{11} - \mathbf{M}_{12}(\mathbf{M}_{22})^{-1}\mathbf{M}_{21}\right)^{-1}\mathbf{M}_{12}(\mathbf{M}_{22})^{-1}$$

2 Random Matrices.

Problem 2.1.

(i) Let $\mathbf{M}$ be a random real symmetric orthogonal matrix, that is an $N \times N$ matrix satisfying $\mathbf{M} = \mathbf{M}^\intercal = \mathbf{M}^{-1}$. Show that all the eigenvalues of $\mathbf{M}$ are ± 1.

(ii) Let $\mathbf{X}$ be a Wigner matrix, i.e. an $N \times N$ real symmetric matrix whose diagonal and upper triangular entries are iid Gaussian random numbers with zero mean and variance σ^2/N. You can use $\mathbf{X} = \sigma(\mathbf{H} + \mathbf{H}^\intercal)/\sqrt{2\sigma}$ where $\mathbf{H}$ is a non-symmetric $N \times N$ matrix with iid standard Gaussians.

(iii) The matrix $\mathbf{P}_+$ is defined as $\mathbf{P}_+ = \frac{1}{2}(\mathbf{M} + \mathbf{1})$. Convince yourself that $\mathbf{P}_+$ is the projector onto the eigenspace of $\mathbf{M}$ with eigenvalue $+1$. Explain the effect of the matrix $\mathbf{P}_+$ on eigenvectors of $\mathbf{M}$.

(iv) An easy way to generate a random matrix $\mathbf{M}$ is to generate a Wigner matrix (independent of $\mathbf{X}$), diagonalize it, replace every eigenvalue by its sign and reconstruct the matrix. The procedure does not depend on the σ used for the Wigner.

(v) We consider a matrix $\mathbf{E}$ of the form $\mathbf{E} = \mathbf{M} + \mathbf{X}$. To wit, $\mathbf{E}$ is a noisy version of $\mathbf{E}$. The goal of the following is to understand numerically how the matrix $\mathbf{E}$ is corrupted by the Wigner noise. Using the computer language of your choice, for a large value of N (as large as possible while keeping computing times below one minute), for three interesting values of σ of your choice, do the following numerical analysis.

(a) Plot a histogram of the eigenvalues of $\mathbf{E}$, for a single sample first, and then for many samples (say 100).

(b) From your numerical analysis, in the large N limit, for what values of σ do you expect a non-zero density of eigenvalues near zero.

(c) For every normalized eigenvector $\mathbf{v}_i$ of $\mathbf{E}$ compute the norm of the vector $\mathbf{P}_+\mathbf{v}_i$. For a single sample, do a scatter plot of $|\mathbf{P}_+\mathbf{v}_i|^2$ vs λ_i (its eigenvalue). Turn your scatter plot into an approximate conditional expectation value (using a histogram) including data from many samples.

(d) Build an estimator $\mathbf{\Xi}(\mathbf{E})$ of $\mathbf{M}$ using only data from $\mathbf{E}$. We want to minimize the error $\mathcal{E} = \frac{1}{N}||(\mathbf{\Xi}(\mathbf{E}) - \mathbf{M})||_F^2$ where $||\mathbf{A}||_F^2 = \mathrm{Tr}(\mathbf{A}\mathbf{A}^\intercal)$. Consider first $\mathbf{\Xi_1}(\mathbf{E}) = \mathbf{E}$ and then $\mathbf{\Xi_1}(\mathbf{E}) = 0$. What is the error $\mathcal{E}$ of these two estimators? Try to build an ad-hoc estimator $\mathbf{\Xi}(\mathbf{E})$ that has a lower error $\mathcal{E}$ than these two.

(e) Show numerically that the eigenvalues of $\mathbf{E}$ are not IID. For each sample $\mathbf{E}$ rank its eigenvalues $\lambda_1 < \ldots \lambda_N$. Consider the eigenvalue spacing $s_k = \lambda_k - \lambda_{k-1}$ for eigenvalues in the bulk $(.2N < k < .3N)$ and $(.7N < k < .8N)$. Make a histogram of $\{s_k\}$ including data from 100 samples. Make 100 pseudo-iid samples: mix eigenvalues for 100 different samples and randomly choose N from the 100N possibilities, do not choose the same eigenvalue twice for a given pseudo-iid sample. For each pseudo-iid sample, compute s_k in the bulk and make a histogram of the values using data from all 100 pseudo-iid samples. original data (not iid).

Solution 2.1. (i) Let $|\psi_i\rangle$ eigenvector of $\mathbf{M}$ with corresponding eigenvalue λ_i, i.e, $\mathbf{M}|\psi_i\rangle =$

$\lambda_i|\psi_i\rangle$, then

$$
\begin{aligned}
&(\mathbf{M}|\psi_i\rangle)^\intercal = \langle\psi_i|\mathbf{M}^\intercal = \mathbf{M}^\intercal\lambda_i^* \\
&\Rightarrow (\mathbf{M}|\psi_j\rangle)^\intercal\,(\mathbf{M}|\psi_i\rangle) = \langle\psi_j|\mathbf{M}^\intercal\mathbf{M}|\psi_i\rangle = \\
&= \langle\psi_j|\lambda_j^*\lambda_i|\psi_i\rangle = \lambda_j^*\lambda_i\langle\psi_j|\psi_i\rangle = \lambda_j^*\lambda_i\delta_{ij} = \lambda_i^*\lambda_i = 1 \\
&\Rightarrow |\lambda_i| = 1 \\
&\therefore \lambda_i = \pm 1
\end{aligned}
$$

(ii) A class for the problem

```
class rmt(object):
        def __init__(self,N,T = 1):
            self.N = N
            self.T = T

        def wigner(self):
            H = np.random.normal(0, 1, size = (self.N,self.N) )
            return (H + H.T)/np.sqrt(2*self.N)

        def wishart(self):
            H = np.random.normal(0,1, size = (self.N, self.T) )
            return (1/self.T)*H@H.T

        def orthogonal(self):
            H = rmt(self.N).wigner()
            u, v = np.linalg.eig(H)
            U = np.diag(np.sign(u))
            return v@U@v.T

X = rmt(N = 100).wigner()
uX,_ = np.linalg.eig(X)
plt.hist(uX, bins=30,ec = 'Black', color = 'slateblue', density = True
                                                )
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.title('Wigner semi circle')
plt.show()
```

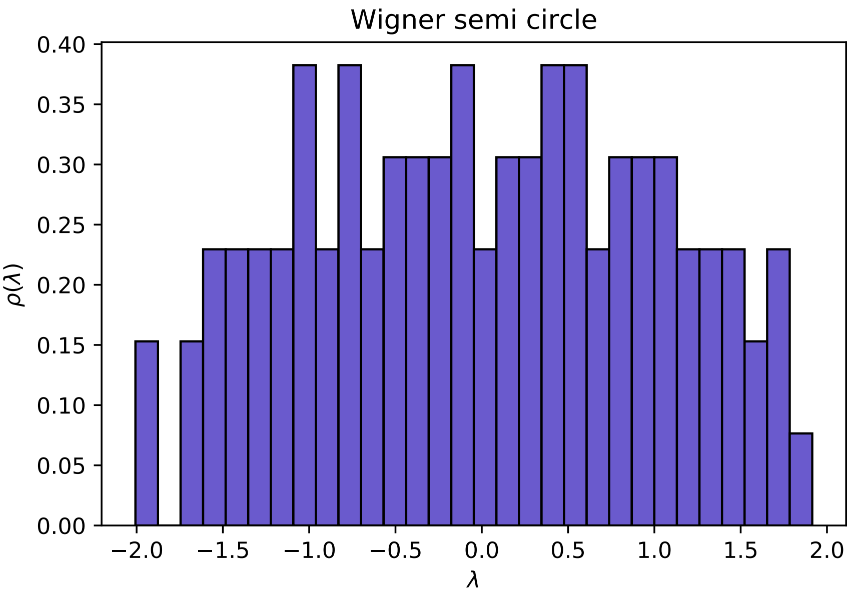


Figure 4: Wigner semi circle law.

(iii) Let $|\psi_i\rangle$ eigenvector of $\mathbf{M}$ with corresponding eigenvalue ± 1, i.e, $\mathbf{M}|\psi_i\rangle = \pm 1|\psi_i\rangle$, then the effect of the $\mathbf{P}_+$ is:

$$\mathbf{P}_+|\psi_i\rangle = \frac{1}{2}(\mathbf{M}+\mathbf{1})|\psi_i\rangle = \frac{1}{2}\left(\mathbf{M}|\psi_i\rangle + \mathbf{1}|\psi_i\rangle\right) = \frac{1}{2}\left(\lambda_i|\psi_i\rangle + |\psi_i\rangle\right)$$
$$\mathbf{P}_+|\psi_i\rangle = \frac{1}{2}(\lambda_i+1)|\psi_i\rangle$$
$$\mathbf{P}_+|\psi_i\rangle = \begin{cases} 1 & \text{if } \lambda_i = 1 \\ 0 & \text{if } \lambda_i = -1 \end{cases}$$

$\mathbf{P}_+$ projects $|\psi_i\rangle$ into itself space if its corresponding eigenvalues is 1 and otherwise project it to zero.

```
def proj(X):
    N = X.shape[0]
    return 0.5*(X + np.identity(N))
P = proj(M)
uP, vP = np.linalg.eig(P)
plt.hist(uP, bins=30,ec = 'Black', color = 'slateblue', density = True
    )
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.title('$ P$, eigenvalues')
plt.show()
```

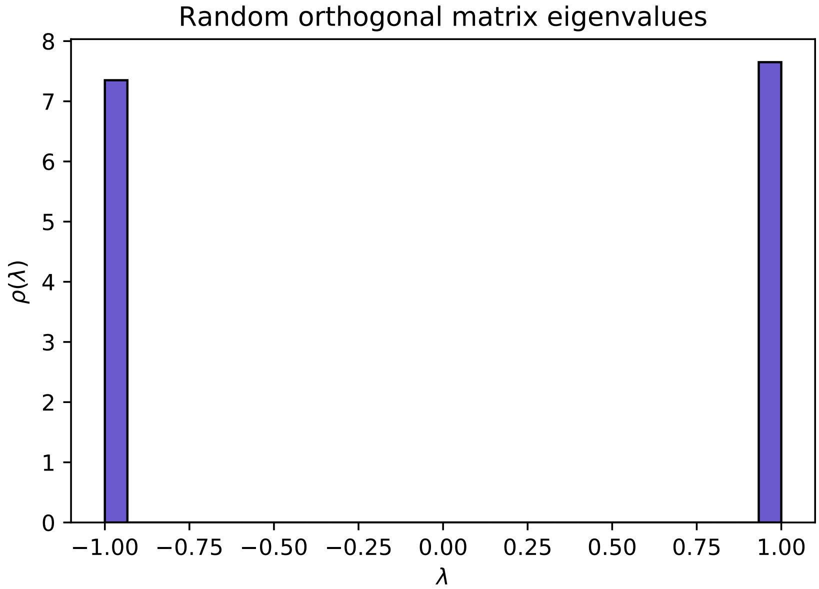


Figure 5: Graphical representation of the problem.

(iv) Orthogonal matrix using the class of ii)

```
M = rmt(N = 100).orthogonal()
uM, _ = np.linalg.eig(M)
plt.hist(uM, bins=30,ec = 'Black', color = 'slateblue', density = True
    )
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.title('Random orthogonal matrix eigenvalues')
plt.show()
```

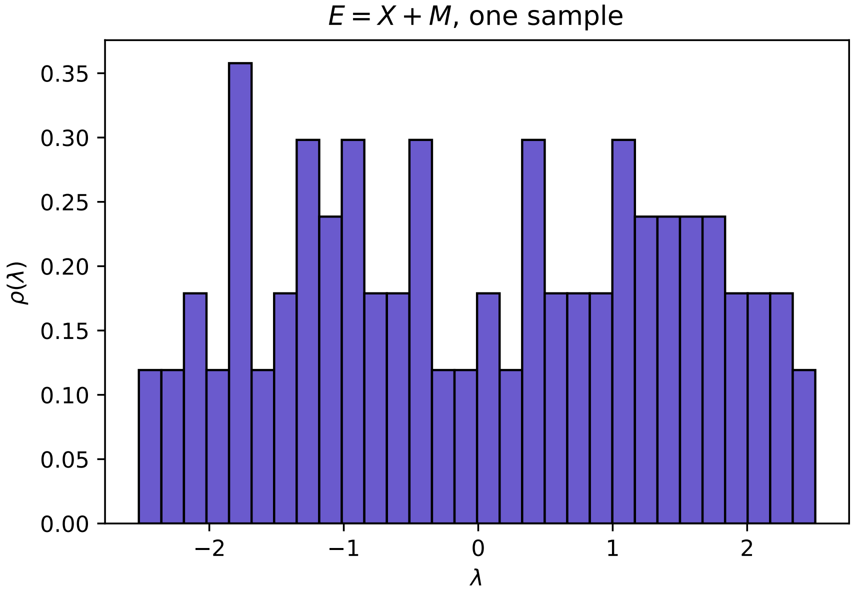


Figure 6: Eigenvalues corresponding to a orthogonal matrix plus a Wigner matrix.

(v) Define $\mathbf{E} = \mathbf{X} + \mathbf{M}$ given the previous indices.

```
E = X + M
```

(a) For one sample

```
uE, vE = np.linalg.eig(E)
plt.hist(uE, bins=30,ec = 'Black', color = 'slateblue', density = True
                                           )
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.title('$E = X +M $, one sample')
plt.show()
```

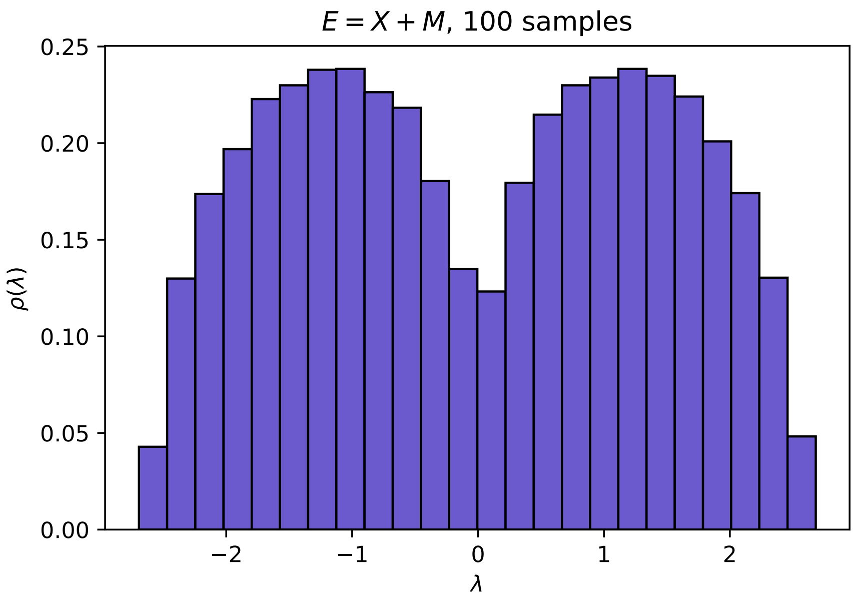


Figure 7: Eigenvalue distribution of a ensemble of 100 members.

For 100 samples

```
uall = []
pall = []
for i in range(100):
    Xi = rmt(N = 100).wigner()
    Mi = rmt(N = 100).orthogonal()
    Ei = Xi + Mi
    Pi = proj(Mi)

    uEi, vEi = np.linalg.eig(Ei)
    pivi = np.abs(np.diag(Pi@vEi))

    uall.append(uEi)
    pall.append(pivi)

uall= np.ravel(uall)

%plt.hist(uall, bins='auto',ec = 'Black', color = 'slateblue', density
                                           = True )
%plt.xlabel('$\lambda$')plt.ylabel('$\\rho(\lambda)$')
```

```
%plt.title('$E = X +M $, 100 samples')
%plt.show()
```

(b) For $\sigma >> 0$

(c) Using pall form the previous index:

```
pall = np.ravel(pall)**2
plt.scatter(pall, uall, color = 'slateblue', alpha = 0.4)
plt.title('$|Pv_{i}|^{2}$ vs $\lambda_{i}$')
plt.ylabel( '$\lambda_{i}$')
plt.xlabel('$|Pv_{i}|^{2}$')
plt.show()
```

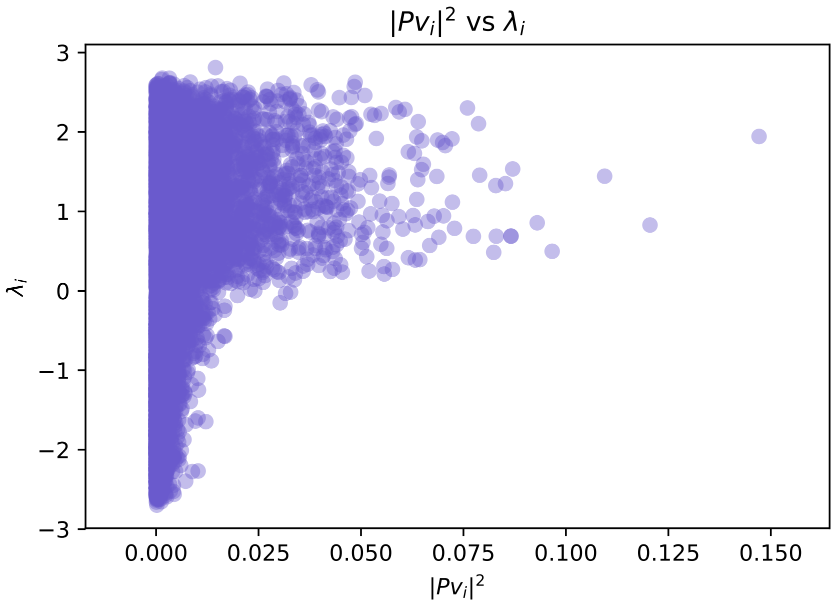


Figure 8: Scatter plot, the negative eigenvalues are projected to zero, while the positives remain the same.

(d) The following function as an estimator.

```
def error( Es, A):
    N = A.shape[0]
    return (1/N)*np.linalg.norm( Es - A, ord = 'fro')**2
Xi1 = error(E,M)
Xi0 = error(np.zeros_like(M), M)
Xieval = error(E/2, M)
```

$Xi1 = 0.98$, $Xi0 = 1$, $Xieval = 0.50$, hence $\Xi(\mathbf{E}) = \mathbf{E}/2$ is the better estimator for this case.

(e) For 100 samples

```
new = np.sort(uall.reshape(100,100),axis = 1)
s1 = new[:,19:29][:,1:]-new[:,19:29][:,:-1]
s2 = new[:,69:79][:,1:]-new[:,69:79][:,:-1]
sk = np.concatenate([s1.reshape(900), s2.reshape(900)], axis = 0)
plt.hist(sk,bins = 'auto', ec = 'black', color = 'slateblue',density =
                                                True)
plt.xlabel('$\lambda$')
```

```
        plt.ylabel('$\\rho(\lambda)$')
        plt.show()
```

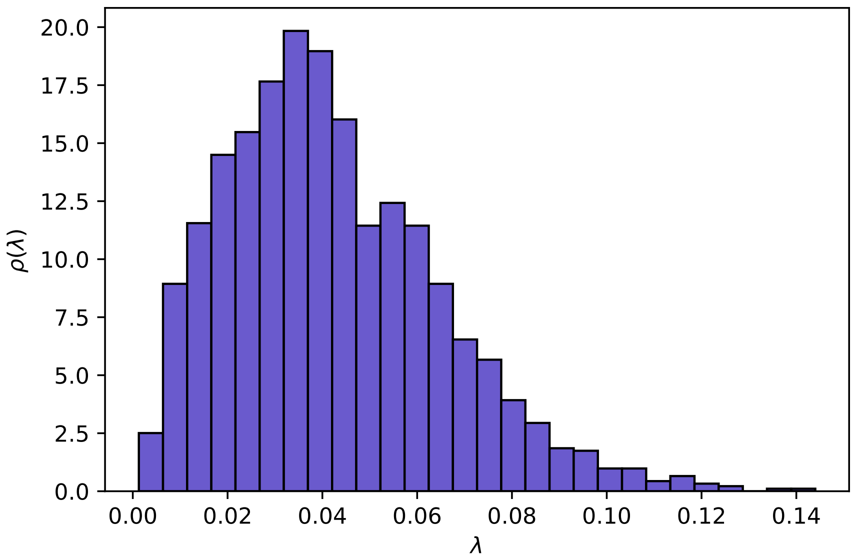


Figure 9: Spacing distribution, similar to the Wigner surmise, the eigenvalues in this case are iid.

```
         np.random.shuffle(uall)
        new2 = np.sort(uall.reshape(100,100),axis = 1)
        s1 = new2[:,19:29][:,1:]-new2[:,19:29][:,:-1]
        s2 = new2[:,69:79][:,1:]-new2[:,69:79][:,:-1]
        sk2 = np.concatenate([s1.reshape(900), s2.reshape(900)], axis = 0)
        plt.hist(sk2,bins = 'auto', ec = 'black', color = 'slateblue',density =
                                                    True)
        plt.xlabel('$\lambda$')
        plt.ylabel('$\\rho(\lambda)$')
        plt.show()
```

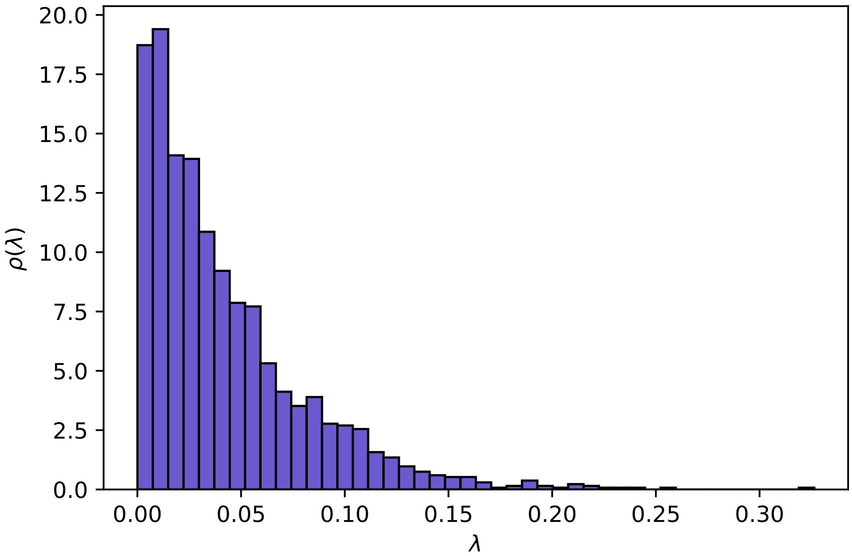


Figure 10: Spacing distribution similar to the Poisson distribution, in this case the eigenvalues are not idd.

3 Wigner ensemble and Semi-circle law.

Problem 3.1. Stieltjes transform for shifted and scaled matrices.

Let $\mathbf{A}$ be a random matrix drawn from a well-behaved ensemble with Stieltjes transform $g(z)$. What are the Stieltjes transforms of the random matrices $\alpha\mathbf{A}$ and $\mathcal{A}+\beta\mathbf{1}$ where α and β are non-zero real numbers and $\mathbf{1}$ the identity matrix?

Solution 3.1. This is equivalent to $g_{\alpha\mathbf{A}+\beta\mathbf{1}}(z)$, by definition:

$$
\begin{aligned}
& g_{\alpha\mathbf{A}+\beta\mathbf{1}}(z) = \frac{1}{N}\mathrm{Tr}\left(\frac{1}{z\mathbf{1}-\alpha\mathbf{A}-\beta\mathbf{1}}\right) = \frac{1}{N}\mathrm{Tr}\left(\frac{1}{(z-\beta)\mathbf{1}-\alpha\mathbf{A}}\right) = \\
& = \frac{1}{\alpha N}\mathrm{Tr}\left(\frac{1}{\left(\frac{z-\beta}{\alpha}\right)\mathbf{1}-\mathbf{A}}\right) = \frac{1}{\alpha}g_{\mathbf{A}}\left(\frac{z-\beta}{\alpha}\right) \\
& \therefore g_{\alpha\mathbf{A}+\beta\mathbf{1}}(z) = \alpha^{-1}g_{\mathbf{A}}\left(\frac{z-\beta}{\alpha}\right)
\end{aligned}
$$

Problem 3.2. Finite N approximation and small imaginary part.

$\mathrm{Im} g_N(x-i\eta)/\pi$ is a good approximation to $\rho(x)$ for a small positive η where $g_N(x)$ is the sample Stieltjes transform ($g_N(z)=(1/N)\sum_k 1/(z-\lambda_k)$). Numerically generate a Wigner matrix of size N and $\sigma^2=1$.

(a) For three values of η, $\left\{1/N, 1, 1/\sqrt{N}\right\}$ plot $\mathrm{Im} g_N(x-i\eta)/\pi$ and the theoretical $\rho(x)$ on the same plot for x between -3 and 3.

(b) Compute the error as a function of η where the error is $(\rho(x)-\mathrm{Im} g_N(x-i\eta)/\pi)^2$ summed for all values of x between -3 and 3 spaced by intervals of 0.01. Plot this error for η between $1/N$ and 1. You should see that $1/\sqrt{N}$ is very close to the minimum of this function.

Solution 3.2. On my github.

Problem 3.3. Stieltjes transform.

A large random matrix has moments $\tau(\mathbf{A}^k)=1/k$

(a) Write the Taylor series of $g(z)$ around infinity

(b) Sum the series to get a simple expression for $g(z)$.

(c) Where are the singularities of $g(z)$ on the real axis?

(d) Redo all the above steps for a matrix whose odd moments are zero and even moments are $\tau(\mathbf{A}^{2k})=1$.

Solution 3.3. (a) The Taylor series of $g(z)$ is given by

$$g(z)=\sum_{k=0}^{\infty}\frac{m_k}{z^{k+1}}$$

Where m^k are the moments of the random matrix $\mathbf{A}$, in this case:

$$g(z)=\sum_{k=0}^{\infty}\frac{m_k}{z^{k+1}}=\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\tau(\mathbf{A}^k)=\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\frac{1}{k}$$

We have a singularity for $k = 1$ then

$$g(z) = \sum_{k=1}^{\infty} \frac{1}{z^{k+1}} \frac{1}{k} = -\frac{1}{z} \sum_{k=1}^{\infty} -\frac{1}{kz^k}$$

(b) Remark, the Taylor series of $\log(x+1)$ is

$$\log(x+1) = \sum_{k=1}^{\infty} (-1)^{n+1} \frac{x^k}{k}$$

if $x = -\frac{1}{z}$ then

$$\log\left(\left(-\frac{1}{z}\right) + 1\right) = \sum_{k=1}^{\infty} (-1)^{n+1} \frac{(-1)}{kz^k} = \sum_{k=1}^{\infty} (-1)^{k+1+k} \frac{1}{kz^k}$$

$$\therefore \log\left(1 - \frac{1}{z}\right) = \sum_{k=1}^{\infty} -\frac{1}{kz^k}$$

Then the Stieltjes transform is:

$$g(z) = -\frac{1}{z} \log\left(1 - \frac{1}{z}\right)$$

(c) if $z = x - i\eta$ then the singularities on the real axis are $x = 0$ and $x = 1$, also the function is not defined when $x < 1$.

(d) In this case

$$\tau(\mathbf{A}^k) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Then the Stieltjes transform is

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \tau(\mathbf{A}^k) = \sum_{k=0}^{\infty} \frac{1}{z^{2k+1}} \tag{6}$$

but this is the geometric series,

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

for $a = 1$,$r = 1/z$

$$g(z) = \frac{1}{z} \frac{1}{1 - \frac{1}{z^2}}$$

$$\therefore g(z) = \frac{z}{z^2 - 1}$$

with singularities on the real axis when $x = \pm 1$.

4 More on Gaussian matrices.

Problem 4.1. Quaternionic matrices of size one.

The following four matrices

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{7}$$

can be thought of as the 2×2 complex representations of the four unit quaternions.

(a) Set $\mathbf{Z} = \mathbf{j}$ and compute $\mathbf{Z}^{-1}$

(b) Show that for all four matrices $\mathbf{Q}$, we have $\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{Q}^\dagger$ where the dagger here is the usual transpose plus complex conjugation.

(c) Convince yourself that, by linearity, any $\mathbf{Q}$ that is a real linear combination of the 2×2 matrices $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{1}$ must satisfy $\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{Q}^\dagger$.

(d) Give an example of a matrix $\mathbf{Q}$ that does not satisfy $\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{Q}^\dagger$.

Solution 4.1. (a) Let $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a simple formula for the inverse is:

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Then for $\mathbf{Z} = \mathbf{j}$

$$\Rightarrow \mathbf{Z}^{-1} = \mathbf{j}^{-1} = \frac{1}{-1(-1)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$\therefore \mathbf{Z}^{-1} = \mathbf{j}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Note that: $\mathbf{Z}^{-1} = \mathbf{j}^{-1} = \mathbf{j}^T$

(b) Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, 2×2 matrices, then the elements i, j of the product is given by:

$$[\mathbf{ABC}]_{ij} = \sum_{kj} A_{ik} B_{kl} C_{lj} =$$
$$= A_{i1}B_{11}C_{1j} + A_{i1}B_{12}C_{2j} + A_{i2}B_{21}C_{1j} + A_{i2}B_{22}C_{2j} \tag{8}$$

- Case $\mathbf{Q} = \mathbf{1} \Rightarrow \mathbf{Q}^T = \mathbf{1}$

$$\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{Z}\mathbf{1}\mathbf{Z}^{-1} = \mathbf{Z}\mathbf{Z}^{-1} = \mathbf{1}$$
$$\therefore \mathbf{1} = \mathbf{1}^\dagger = \mathbf{1}$$

- Case $\mathbf{Q} = \mathbf{i} \Rightarrow \mathbf{Q}^T = \mathbf{i}$

$$\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{Z}\mathbf{i}^T\mathbf{Z}^{-1} = \mathbf{Z}\mathbf{i}\mathbf{Z}^{-1}$$

Using eq. (8):

$$
\begin{aligned}
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{11} = \mathbf{Z}_{11}\mathbf{i}_{11}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{11}\mathbf{i}_{12}\mathbf{Z}^{-1}_{21} + \mathbf{Z}_{12}\mathbf{i}_{21}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{12}\mathbf{i}_{22}\mathbf{Z}^{-1}_{21}\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{11} = 0+0+0+1(-i)i = -i\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{12} = \mathbf{Z}_{11}\mathbf{i}_{11}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{11}\mathbf{i}_{12}\mathbf{Z}^{-1}_{22} + \mathbf{Z}_{12}\mathbf{i}_{21}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{12}\mathbf{i}_{22}\mathbf{Z}^{-1}_{22}\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{12} = 0+0+0+0 = 0\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{21} = \mathbf{Z}_{21}\mathbf{i}_{11}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{21}\mathbf{i}_{12}\mathbf{Z}^{-1}_{21} + \mathbf{Z}_{22}\mathbf{i}_{21}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{12}\mathbf{i}_{22}\mathbf{Z}^{-1}_{21}\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{21} = 0+0+0+0 = 0\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{22} = \mathbf{Z}_{21}\mathbf{i}_{11}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{21}\mathbf{i}_{12}\mathbf{Z}^{-1}_{22} + \mathbf{Z}_{22}\mathbf{i}_{21}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{22}\mathbf{i}_{22}\mathbf{Z}^{-1}_{22}\\
&[\mathbf{Z}\mathbf{i}\mathbf{Z}^T]_{22} = (-1)(i)(-1)+0+0+0 = i\\
&\Rightarrow \mathbf{Z}\mathbf{i}\mathbf{Z}^T = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = -\mathbf{i} = \mathbf{i}^\dagger\\
&\therefore \mathbf{Z}\mathbf{i}^T\mathbf{Z}^T = \mathbf{i}^\dagger
\end{aligned}
$$

- Case $\mathbf{Q} = \mathbf{j} \Rightarrow \mathbf{Q}^T = \mathbf{j}^{-1}$

$$
\begin{aligned}
&\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{j}\mathbf{j}^T\mathbf{j}^{-1} = \mathbf{j}\mathbf{j}^{-1}\mathbf{j}^{-1} = \mathbf{j}^{-1} = \mathbf{j}^T = \mathbf{j}^\dagger\\
&\therefore \mathbf{Z}\mathbf{j}^T\mathbf{Z}^{-1} = \mathbf{j}^\dagger
\end{aligned}
$$

- Case $\mathbf{Q} = \mathbf{k} \Rightarrow \mathbf{Q}^T = \mathbf{k}$

$$\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} = \mathbf{Z}\mathbf{k}^T\mathbf{Z}^{-1} = \mathbf{Z}\mathbf{k}\mathbf{Z}^{-1}$$

Using eq. (8):

$$
\begin{aligned}
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{11} = \mathbf{Z}_{11}\mathbf{k}_{11}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{11}\mathbf{k}_{12}\mathbf{Z}^{-1}_{21} + \mathbf{Z}_{12}\mathbf{k}_{21}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{12}\mathbf{k}_{22}\mathbf{Z}^{-1}_{21}\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{11} = 0+0+0+0 = 0\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{12} = \mathbf{Z}_{11}\mathbf{k}_{11}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{11}\mathbf{k}_{12}\mathbf{Z}^{-1}_{22} + \mathbf{Z}_{12}\mathbf{k}_{21}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{12}\mathbf{k}_{22}\mathbf{Z}^{-1}_{22}\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{12} = 0+0+1i(-1)+0 = -i\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{21} = \mathbf{Z}_{21}\mathbf{k}_{11}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{21}\mathbf{k}_{12}\mathbf{Z}^{-1}_{21} + \mathbf{Z}_{22}\mathbf{k}_{21}\mathbf{Z}^{-1}_{11} + \mathbf{Z}_{12}\mathbf{k}_{22}\mathbf{Z}^{-1}_{21}\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{21} = 0+(-1)i1+0+0 = -i\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{22} = \mathbf{Z}_{21}\mathbf{k}_{11}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{21}\mathbf{k}_{12}\mathbf{Z}^{-1}_{22} + \mathbf{Z}_{22}\mathbf{k}_{21}\mathbf{Z}^{-1}_{12} + \mathbf{Z}_{22}\mathbf{k}_{22}\mathbf{Z}^{-1}_{22}\\
&[\mathbf{Z}\mathbf{k}\mathbf{Z}^T]_{22} = 0+0+0+0 = 0\\
&\Rightarrow \mathbf{Z}\mathbf{k}\mathbf{Z}^T = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \mathbf{k}^\dagger\\
&\therefore \mathbf{Z}\mathbf{k}^T\mathbf{Z}^T = \mathbf{k}^\dagger
\end{aligned}
$$

(c) Let $\lambda_i \in \mathbb{R}\setminus\{0\}$ for $i = 1, ..., 4$. Let $\mathbf{M}$ a linear combination of (7), i.e,

$$
\begin{aligned}
&\mathbf{M} = \lambda_1\mathbf{1} + \lambda_2\mathbf{i} + \lambda_3\mathbf{j} + \lambda_4\mathbf{k}\\
&\Rightarrow \mathbf{Z}\mathbf{M}^T\mathbf{Z}^{-1} = \lambda_1\mathbf{Z}\mathbf{1}^T\mathbf{Z}^{-1} + \lambda_2\mathbf{Z}\mathbf{i}^T\mathbf{Z}^{-1} + \lambda_3\mathbf{Z}\mathbf{j}^T\mathbf{Z}^{-1} + \lambda_4\mathbf{Z}\mathbf{k}^T\mathbf{Z}^{-1} =\\
&= \lambda_1\mathbf{1} + \lambda_2\mathbf{i}^\dagger + \lambda_3\mathbf{j}^\dagger + \lambda_4\mathbf{k}^\dagger = \mathbf{M}^\dagger\\
&\therefore \mathbf{Z}\mathbf{M}^T\mathbf{Z}^{-1} = \mathbf{M}^\dagger
\end{aligned}
$$

(d) Let $\mathbf{Q} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then

$$\mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} =$$
$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$
$$\text{However, } \mathbf{Q}^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\therefore \mathbf{Z}\mathbf{Q}^T\mathbf{Z}^{-1} \neq \mathbf{Q}^\dagger$$

Problem 4.2. Three quarter-circle laws.

Let $\mathbf{H}$ be a (non-symmetric) square matrix of size N whose entries are iid Gaussian random variable of variance σ^2/N. Then as a simple consequence of the above discussion the following three sets of numbers are distributed according to the quarter-circle law $\rho(s)\frac{1}{\pi s^2}\sqrt{4\sigma^2 - s^2}$ in the large N limit. Define:

- $w_i = |\lambda_i|$ where $\{\lambda_i\}$ are the eigenvalues of $\frac{\mathbf{H}+\mathbf{H}^T}{\sqrt{2}}$,
- $r_i = 2|\text{Re}\lambda_i|$ where $\{\lambda_i\}$ are the eigenvalues of $\mathbf{H}$,
- $2_i = \sqrt{\lambda_i}$ where $\{\lambda_i\}$ are the eigenvalues of $\mathbf{H}\mathbf{H}^T$.

(a) Generate a large matrix $\mathbf{H}$ with say $N = 1000$ and $\sigma^1 = 1$ and plot the histogram of the three above sets.

(b) Although these three sets of numbers converge to the same distribution there is no simple relation between them. In particular they are not equal. For a moderate N (10 or 20) examine the three sets and realize that they are all different.

Solution 4.2. Using the following scrips in python

(a)
```
import numpy as np
import matplotlib.pyplot as plt

H = np.random.normal(0,1,size = (100,100))
W = (H + H.T)/np.sqrt(2)
S = H@H.T
uW, _ = np.linalg.eig(W)
wi = np.abs(uW)
uH, _ = np.linalg.eig(H)
ri = 2*abs(np.real(uH))
uS, _ = np.linalg.eig(S)
si = np.sqrt(uS)

plt.hist(ri,bins = 30, ec = 'black', color = 'red', histtype = 'stepfilled',
         alpha = 0.4, density = True, label = '$r_{i}$')
plt.hist(si,bins = 30, ec = 'black', color = 'green', histtype = 'stepfilled',
         alpha = 0.4, density = True, label = '$s_{i}$')
plt.hist(wi,bins = 30, ec = 'black', color = 'slateblue', histtype = '
                                     stepfilled',
         alpha = 0.8, density = True, label = '$w_{i}$')
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.legend(loc = 'best')
plt.show()
```

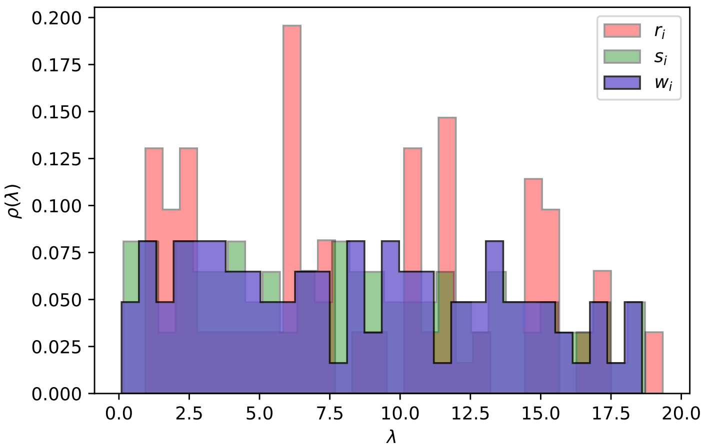


Figure 11: $N = 100$

(b)

```
H = np.random.normal(0,1,size = (20,20))
W = (H + H.T)/np.sqrt(2)
S = H@H.T
uW, _ = np.linalg.eig(W)
wi = np.abs(uW)
uH, _ = np.linalg.eig(H)
ri = 2*abs(np.real(uH))
uS, _ = np.linalg.eig(S)
si = np.sqrt(uS)

plt.hist(ri,bins = 20, ec = 'black', color = 'red', histtype = 'stepfilled',
         alpha = 0.4, density = True, label = '$r_{i}$')
plt.hist(si,bins = 20, ec = 'black', color = 'green', histtype = 'stepfilled',
         alpha = 0.4, density = True, label = '$s_{i}$')
plt.hist(wi,bins = 20, ec = 'black', color = 'slateblue', histtype = '
                                 stepfilled',
         alpha = 0.8, density = True, label = '$w_{i}$')
plt.xlabel('$\lambda$')
plt.ylabel('$\\rho(\lambda)$')
plt.legend(loc = 'best')
plt.show()
```

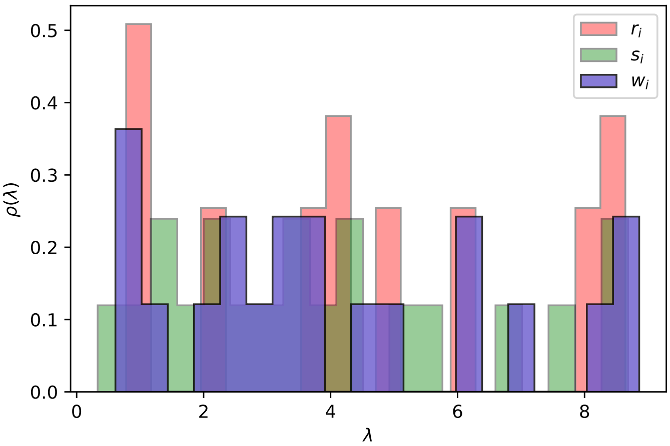


Figure 12: $N = 20$

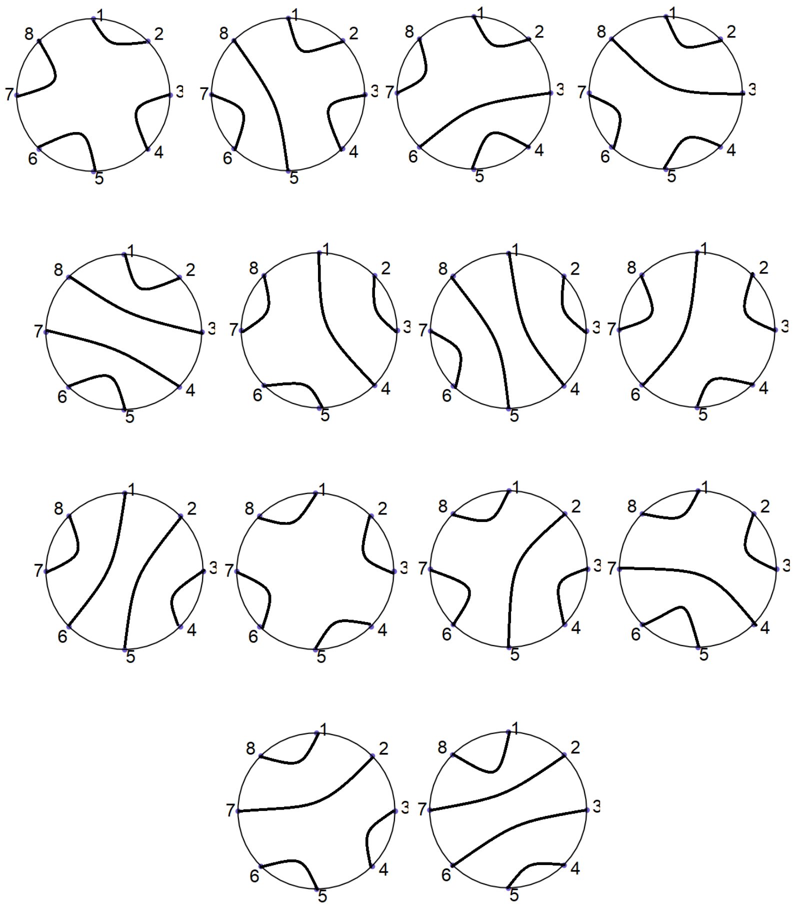
Problem 4.3. Non-crossing pair partitions of eight elements.

(a) Draw all the non-crossing pair partitions of eight elements.

(b) If $\mathbf{X}$ is a unit Wigner matrix, what is $\tau\left(\mathbf{X}^8\right)$?

Solution 4.3. (a) These are the 14 possible non-crossing pairs for 8 elements.

1.- (1357)(2)(4)(6)(8)
2.- (135)(2)(4)(68)(7)
3.- (137)(2)(46)(5)(8)
4.- (13)(2)(468)(5)(7)
5.- (13)(2)(48)(57)(6)
6.- (157)(24)(3)(6)(8)
7.- (15)(24)(3)(68)(7)
8.- (17)(246)(3)(5)(8)
9.- (17)(26)(35)(4)(8)
10.- (1)(2468)(3)(5)(7)
11.- (1)(268)(35)(4)(7)
12.- (1)(248)(3)(57)(6)
13.- (1)(28)(357)(4)(6)
14.- (1)(28)(37)(46)(5)



(b) For a Wigner matrix $\mathbf{X}$ its moments can be calculted by:

$$\tau\left(\mathbf{X}^{2k}\right) = C_k \sigma^{2k}$$

Where C_k are the Catalan numbers

$$C_k = \frac{1}{2k+1}\begin{pmatrix} 2k+1 \\ k \end{pmatrix}$$

For $2k = 8 \Rightarrow k = 4$ $C_4 = 14$.

$$\therefore \tau\left(\mathbf{X}^8\right) = C_4\sigma^8 = 14\sigma^8$$

5 Wishart ensemble and Marcenko–Pastur Distribution

6 Joint Distribution of Eigenvalues

7 Eigenvalues and Orthogonal Polynomials

8 The Jacobi Ensemble*

9 Addition of Random Variables and Brownian Motion