1.1. $A = \begin{pmatrix} -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$$\det(A-\lambda I) \overset{!}{=} 0 = (-0.5-\lambda)(-0.5-\lambda)(\overset{2}{\cancel{0.5}}-\lambda) - (2-\lambda)\cdot 0.5\cdot 0.5$$

$$= (0.25 + \lambda + \lambda^2)(2-\lambda) - 0.5 + 0.25\lambda$$

$$= \cancel{0.5} + 2\lambda + 2\lambda^2 - 0.25\lambda - \lambda^2 - \lambda^3 - \cancel{0.5} + \cancel{0.25\lambda}$$

$$= -\lambda^3 + \lambda^2 + 2\lambda = \lambda(-\lambda^2 + \lambda + 2)$$

$\downarrow$ $\lambda_1 = 0$

$\searrow$ $\lambda_{2,3} = \dfrac{-1 \pm \sqrt{9}}{-2}$

$\downarrow$

$\lambda_2 = -1$

$\lambda_3 = 2$

(i) $\lambda_1 = 0 \Rightarrow v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(ii) $\lambda_2 = -1 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(iii) $\lambda_3 = 2 \Rightarrow v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow x(t) = c_1 v_1 + c_2 e^{-1t} v_2 + c_3 e^{2t} v_3$$

2.1

$$\tau \dot{u}_1(t) = -u_1(t) - c[u_2 - \theta]_+ + s_1$$
$$\tau \dot{u}_2(t) = -u_2(t) - c[u_1 - \theta]_+ + s_2$$

$$\cancel{\tilde{\mu}_1 = \mu_1 - \theta} \quad \mu$$

$$\tilde{u}_1 = u_1 - \theta \qquad \tilde{u}_2 = u_2 - \theta$$

$$\tau \dot{\tilde{u}}_1(t) = -(\tilde{u}_1 + \theta) - c[\tilde{u}_2]_+ + \tilde{s}_1$$
$$\tau \dot{\tilde{u}}_2(t) = -(\tilde{u}_2 + \theta) - c[\tilde{u}_1]_+ + \tilde{s}_2$$

$$\tilde{s} = s - \theta \qquad \tilde{t} = t/\tau$$

$\cancel{\tau u_1}$

$$\Rightarrow \dot{\tilde{u}}_1(\tilde{t}) = -\tilde{u}_1(\tilde{t}) - c[\tilde{u}_2(\tilde{t})]_+ + \tilde{s}_1$$
$$\dot{\tilde{u}}_2(\tilde{t}) = -\tilde{u}_2(\tilde{t}) - c[\tilde{u}_1(\tilde{t})]_+ + \tilde{s}_2$$

2.3*

$$J = \begin{pmatrix} \frac{\partial \dot{u}_1}{\partial u_1} & \frac{\partial \dot{u}_1}{\partial u_2} \\ \frac{\partial \dot{u}_2}{\partial u_1} & \frac{\partial \dot{u}_2}{\partial u_2} \end{pmatrix} = \begin{pmatrix} -1 - \cancel{c\frac{\partial [u_2]_+}{\partial u_1}} & -c\frac{\partial [u_2]_+}{\partial u_2} \\ -c\frac{\partial [u_1]_+}{\partial u_1} & -1 - \cancel{c\frac{\partial [u_1]_+}{\partial u_2}} \end{pmatrix}$$

① $u_1 > 0,\ u_2 > 0 \Rightarrow \begin{pmatrix} -1 & -c \\ -c & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \Rightarrow \lambda_1 = -3,\ \lambda_2 = 1 \Rightarrow$ saddle point

② $u_1 > 0\ \ u_2 < 0 \Rightarrow \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \Rightarrow \lambda_{1,2} = -1 \Rightarrow$ stable point

③ $u_1 < 0,\ u_2 > 0$ ~~$\Rightarrow$ no fixed point in this quadrant $\Rightarrow$ calculation useless~~ $\Rightarrow \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda_{1,2} = -1 \Rightarrow$ stable point

~~④ = ② ⇒ stable point~~

④ fixed point outside of quadrant

* Computing Fixed points

① $u_1 > 0,\ u_2 > 0 \Rightarrow \begin{matrix} 0 = -u_1 - cu_2 + 1 \\ 0 = -u_2 - cu_1 + 1 \end{matrix} \Rightarrow \begin{matrix} u_1 = -2u_2 + 1 \\ 0 = 3u_1 - 1 \end{matrix} \Rightarrow \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix}$

② $u_1 > 0\ \ u_2 < 0 \Rightarrow \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

③ $u_1 < 0,\ u_2 > 0 \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

④ $u_1 < 0,\ u_2 < 0 \Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (outsid of quadrant)

Eigenvalues are always $\lambda_{1,2} = -1 \Rightarrow$ always stable

2.6

Computing fixed points: and Eigenvalus:

$$\begin{array}{lllll} & & & c=2 & c=-2 \\ ① \; u_1 > 0, \; u_2 > 0 \Rightarrow & \left.\begin{array}{l} 0 = -u_1 - c + 1 \\ 0 = -u_2 - c + 1 \end{array}\right\} & \begin{array}{l} u_1 = -c+1 \\ u_2 = -c+1 \end{array} & \begin{pmatrix} -1 \\ -1 \end{pmatrix} & \begin{pmatrix} 3 \\ 3 \end{pmatrix} \end{array}$$

$$\begin{array}{llll} & & c=2 & c=-2 \\ ② \; u_1 > 0 \;\; u_2 < 0 \Rightarrow & \begin{array}{l} u_1 = 1 \\ u_2 = -c+1 \end{array} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ 3 \end{pmatrix} \end{array}$$

$$\begin{array}{lll} & c=2 & c=-2 \\ ② \approx ③ \Rightarrow & \begin{pmatrix} -1 \\ 1 \end{pmatrix} & \begin{pmatrix} 3 \\ 1 \end{pmatrix} \end{array}$$

$$④ \; u_1 < 0 \;\; u_2 < 0 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ for all } c$$