1.1 Tipping points in climate model

Deadline: 24 Nov 23:59

(4 points)

One approach to complex systems like the climate uses comprehensive models with billions of variables, covering the state of the atmosphere, oceans, sea ice, ecosystems, and biogeochemical cycles. A complementary approach is simple conceptual models, which are easier to understand but still provide valuable insights.

The following is a simple energy-balance model for the Earth's mean surface temperature $T(t)$ (in degrees Kelvin) at time t

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = (1-\alpha(T))Q - esT^4 \,.$$

Here C, Q, e, and s are positive parameters, and α is a function of T. The left-hand side describes the change in atmospheric heat, where C is the heat capacity of the Earth. The right-hand side has a positive contribution from energy reaching the Earth from the Sun (proportional to the solar irradiance parameter Q) minus the energy emitted back out into the stratosphere (assumed to follow the Stefan-Boltzmann law with Stefan–Boltzmann constant s).

The parameter $0 \le \alpha \le 1$, called albedo, is the fraction of solar radiation the Earth reflects. Its value varies for ice, water, and land. Since ice is more reflective than land or open water, we expect a runaway ice-albedo feedback loop: the warmer the climate gets, the more ice melts, which means the Earth becomes less reflective, so even more solar radiation gets absorbed and the Earth will get even hotter. Conversely, the colder the planet gets, the more ice can form, increasing the reflectivity of the Earth and cooling it even more. We model this feedback by assuming that α decreases with rising temperature:

$$\alpha = \frac{4}{5} - \frac{7}{10}\frac{1}{1+\exp[-80(T-T_\mathrm{m})/T_\mathrm{m}])}$$

This function uses a sigmoid curve to smoothly interpolate the albedo from ice ($\alpha \approx 4/5$) for mean temperatures $T < 250$K, where Earth is completely frozen (snowball Earth), to ocean albedo ($\alpha \approx 1/10$) for $T > 300$K, where there is no ice. The parameter T_m lies between these temperature limits.

Finally, the parameter $0 < e < 1$, emissivity, accounts for the fraction of outgoing radiation absorbed by greenhouse gases in the atmosphere.

By a suitable transformation into the dimensionless variables (x, τ) with

$$x = \frac{T}{T_0} \qquad \text{and} \qquad \tau = \frac{t}{t_0}\,,$$

where T_0 and t_0 are positive constants, the dynamical system above (with $\alpha = a - bT$) can be transformed into

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{1}{5} + \frac{7}{10}\frac{1}{1+\exp[80(1-x)]} - rx^4 \, .$$

Here, r is a positive dimensionless parameter.

a) Give t_0 in terms of (C, Q, Tm, e, s)

(in terms of C,Tm,Q,s)

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b) Give T_0 in terms of (C, Q, Tm, e, s)

(in terms of Tm,C,Q,s)

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c) Give r in terms of (C, Q, Tm, e, s)

(in terms of s,Tm,Q,C)

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d) Using a computer, make a bifurcation diagram of x^* vs r, where x^* are the fixed points of the dimensionless system above. Use dotted lines to indicate unstable fixed points and solid lines to indicate stable fixed points. Make a arrows labeling the types of all bifurcations in the system. Upload the figure as .pdf or .png. *Hint: Since an analytical solution is difficult to find, solve the equations numerically. Make sure the temperature at the stable fixed points rises as the emissivity e decreases.*

e) Numerically determine the bifurcation point (tipping point) r_c and the fixed point x^* where the dynamics bifurcates to a snowball Earth. Answer as a vector [r_c,x^*] with three digits of precision

precision = 1%

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1.2 Imperfect transcritical bifurcation

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(4 points)

Consider the dynamical system $\dot{x} = f(x, h, r)$ with

$$f(x, h, r) = h + x(r - x)$$

for different values of the parameters h and r. For $h = 0$, the system undergoes a transcritical bifurcation at $r = 0$ along the r-direction. Define the fixed points $x^*(h, r)$ of the system by $f(x^*(h, r), h, r) = 0$.

a) Make a plot of the (h, r) plane. Label different regions according to the number and the types (stable or unstable) of fixed points for (h, r) in that region. The different regions are separated by a bifurcation curve that you are supposed to draw into your plot. Upload the plot as .pdf or .png.

b) Make a three-dimensional (h, r, x^*)-plot of the surface of fixed points, where $f(x^*, h, r) = 0$. Upload the plot (.pdf or .png).

c) Find an analytical expression for the bifurcation curve $[h_c(r), r]$. Write your result as a vector that depends on r.

(in terms of r)

d) At the bifurcation point $h = r = 0$, a transcritical bifurcation occurs along the r-direction. Similarly, for each point, $[h_c(r), r]$, on the bifurcation curve, a transcritical bifurcation occurs in one direction that depends on r. Find this direction analytically. Write your result as a vector in the $[h, r]$-plane that depends on r. For definiteness, normalise the vector to unity and make sure your solution is equal to $[0, 1]$ for $h = 0$.

(in terms of r)