3.2) $\dot{x} = \frac{1}{\varepsilon}[x - \frac{1}{3}x^3 - y + I]$
$\dot{y} = x + a - by$

$a \in \mathbb{R}$
$b, \varepsilon, I \in \mathbb{R} > 0$

a) $a = 4/9$
$b = 5/9$
$\varepsilon = 1$
$0 \leq I \leq 1$

$\Rightarrow \quad \dot{x} = x - \frac{1}{3}x^3 - y + I$
$\dot{y} = x + \frac{4}{9} - 5/9\, y$

$\dot{y} = \dot{x} = 0 \Rightarrow y = \frac{9}{5}x + \frac{4}{5}$

$\Rightarrow x - \frac{1}{3}x^3 - \frac{9}{5}x - \frac{4}{5} + I = 0 \Rightarrow x^3 + \frac{12}{5}x + (\frac{12}{5} - 3I)$

$-\frac{4}{5}x$

or $5x^3 + 12x + (12 - 15I) = 0$ (*)

$$\mathbb{J} = \begin{bmatrix} 1-x^2 & -1 \\ 1 & -5/9 \end{bmatrix}\Bigg|_{x^*, y^*}$$

$\Rightarrow \tau_r = \frac{4}{9} - x^{*2}$
$\Delta = (1 - x^{*2})(-5/9) + 1 = \frac{5}{9}x^{*2} - \frac{4}{9}$

Hopf occurs when $\Delta > 0$ & $T_r = 0$

$\tau_r = 0 \Rightarrow x^* = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$

$\Delta = \frac{5}{9}(\pm\frac{2}{3})^2 + \frac{4}{9} = \frac{20}{81} + \frac{4}{9} = \frac{59}{81}$

(*) $\Rightarrow I = \frac{5}{15}x^3 + \frac{12}{15}x + \frac{12}{15} = \frac{1}{3}x^3 + \frac{4}{5}x + \frac{4}{5}$

$x^* = \frac{2}{3} \Rightarrow I_c = \frac{8}{81} + \frac{8}{15} + \frac{4}{5} = \frac{40}{405} + \frac{216}{405} + \frac{324}{405} = \frac{580}{405} > 1$ not possible

$x^* = -\frac{2}{3} \Rightarrow I_c = -\frac{40}{405} - \frac{216}{405} + \frac{324}{405} = \frac{68}{405}$ $\begin{matrix} >0 \\ <1 \end{matrix}$

$I_c = \frac{68}{405}$

b) $$\mathbb{J} = \begin{bmatrix} \frac{1}{\varepsilon}(1 - x^2) & -\frac{1}{\varepsilon} \\ 1 & -b \end{bmatrix}$$

$\dot{x} = \frac{1}{\varepsilon}[x - \frac{1}{3}x^3 - y + I] = 0 \Rightarrow x - \frac{1}{3}x^3 - y + I \Rightarrow \dot{x} = x - \frac{1}{3}x^3 - \frac{x}{b} + \frac{a}{b} + I$
$\dot{y} = x + a - by = 0 \Rightarrow y = \frac{x+a}{b}$

$\tau = 0 \Rightarrow \frac{1}{\varepsilon}(1 - x^2) - b = 0 = 100 - 100x^2 - 1 = 0 \Rightarrow x = \pm\frac{\sqrt{99}}{10}$

$\Delta = -100(1 - x^2) + 100 = x^2 \Rightarrow > 0$

$\dot{x} = \frac{1}{\varepsilon}[x - \frac{1}{3}x^3 - y + I] = 100(x - \frac{1}{3}x^3 - \frac{x+a}{b} + I) = 100(\frac{x^3}{3} - 1 + I) = 0 \Rightarrow I = 1 - \frac{x^3}{3}$
$\dot{y} = x + a - by \Rightarrow y = \frac{x+a}{b}$

$I_1 = 1 - \frac{99\sqrt{11}}{1000} \approx 0.67165$
$I_2 = 1 + \frac{99\sqrt{11}}{1000} > 1$

$x^* = ? \Rightarrow 0 = x - \frac{1}{3}x^3 - x - 1 + 0{,}6 \Rightarrow x = \sqrt[3]{3 \cdot (-0{,}4)} \approx -1{,}06265$

$\dot{x} = 0 \Rightarrow y = x - \frac{1}{3}x^3 + I$

$\dot{y} = 0 \Rightarrow y = x + 1$

e) Consider $I = 1$ for simplicitiy

$$\Rightarrow \quad \dot{x} = \frac{1}{\varepsilon}\left[x - \frac{1}{3}x^3 - y + 1\right]$$
$$\dot{y} = x + 1 - y$$

nullclines $\Rightarrow$ $\dot{x} = 0 \Rightarrow y = x - \frac{x^3}{3} + 1$

$\dot{y} = 0 \Rightarrow y = x + 1$

On slow branch, assume we follow the nullcline $\dot{x} = 0$ & calculate $\dot{y}$

$$\dot{y} = \frac{dy}{dt} = x + 1 - x + \frac{x^3}{3} - 1 = \frac{x^3}{3} \Rightarrow dt = \frac{3}{x^3}\, dy \quad (1)$$

Derive nullclines $\dot{x} = 0$

$$\frac{dy}{dx} = 1 - x^2 \Rightarrow dy = (1 - x^2)\, dx \quad (2)$$

Set (2) in (1):

$$dt = \frac{3}{x^3}(1 - x^2)\, dx$$

Calculate time on slow branch w. $x_1 = 2$ & $x_2 = 1$

$$T_{slow} = \int_{x_1}^{x_2} dt = \int_2^1 \frac{3}{x^3}(1 - x^2)\, dx = \text{Through Mathematica}$$
$$= 3\left(\ln 2 - \frac{3}{8}\right)$$