2.2 $\quad \dot{x} = (\sigma+1)x + 3y$
$\dot{y} = -2x + (\sigma-1)y$

$$A = \begin{pmatrix} \sigma+1 & 3 \\ -2 & \sigma-1 \end{pmatrix}$$

a) $\quad \lambda_{1,2} = \frac{1}{2}\tau \pm \sqrt{\tau^2 - 4\Delta}$

$\tau = 2\sigma$
$\Delta = (\sigma^2-1) - (-6) = \sigma^2+5$

$$\lambda_{1,2} = \tfrac{1}{2}(2\sigma \pm \sqrt{4\sigma^2 - 4\sigma^2 - 20})$$
$$= \sigma \pm \tfrac{1}{2}\sqrt{-20} \quad \text{or} \quad \sigma \pm \tfrac{1}{2}i\sqrt{20}$$
$$= \sigma \pm \sqrt{-5} \quad \text{or} \quad \sigma \pm i\sqrt{5}$$

b) $\quad x(0) = u$
$y(0) = v$
Solutions on the form:

$$x(t) = \begin{cases} C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} & \text{if } \lambda_{1,2} \text{ real } \& \ \lambda_1 \neq \lambda_2 \\ C_1 e^{\mu t}(C_1 \cos(\omega t) + C_2 \sin(\omega t)) & \text{if } \lambda_{1,2} = \mu \pm i\omega \\ C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t} & \text{if } \lambda_1 = \lambda_2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\mu t}\begin{pmatrix} C_1 \cos(\omega t) + C_2 \sin(\omega t) \\ C_3 \cos(\omega t) + C_4 \sin(\omega t) \end{pmatrix} \quad \Bigg| \quad \begin{matrix} \mu = \sigma \\ \omega = \sqrt{5} \end{matrix}$$

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \mu e^{\mu t}\begin{pmatrix} u \cos(\omega t) + C_2 \sin(\omega t) \\ v \cos(\omega t) + C_4 \sin(\omega t) \end{pmatrix} + \omega e^{\sigma t}\begin{pmatrix} -u \sin(\omega t) + C_2 \cos(\omega t) \\ -v \sin(\omega t) + C_4 \cos(\omega t) \end{pmatrix}$$

$$= e^{\sigma t}\begin{pmatrix} (\sigma u + C_2\sqrt{5}) \cos(\sqrt{5}t) + (\sigma C_2 - u\sqrt{5}) \sin(\sqrt{5}t) \\ (\sigma v + C_4\sqrt{5}) \cos(\sqrt{5}t) + (\sigma C_4 - v\sqrt{5}) \sin(\sqrt{5}t) \end{pmatrix}$$

at $t=0$ $\quad \begin{pmatrix} (\sigma+1)u + 3v \\ -2u + (\sigma-1)v \end{pmatrix} = \begin{pmatrix} \sigma u + C_2\sqrt{5} \\ \sigma v + C_4\sqrt{5} \end{pmatrix}$

$$u + 3v = C_2\sqrt{5}$$
$$-2u - v = C_4\sqrt{5}$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\sigma t}\begin{pmatrix} u \cos(\sqrt{5}t) + \frac{u+3v}{\sqrt{5}} \sin(\sqrt{5}t) \\ v \cos(\sqrt{5}t) - \frac{2u+v}{\sqrt{5}} \sin(\sqrt{5}t) \end{pmatrix}$$

d) $\quad \omega = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}}$

e) Quadratic form:

$$x = \cos\sqrt{5}t + \frac{4}{\sqrt{5}}\sin t$$
$$y = \cos\sqrt{5}t - \frac{3}{\sqrt{5}}\sin t$$

$$(x-y) \Rightarrow \sin\sqrt{5}t = \frac{\sqrt{5}}{7}(x-y)$$

$$(x+y) \Rightarrow \cos\sqrt{5}t = \frac{1}{2}\left(x+y-\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{7}(x-y)\right) = \frac{6x+8y}{14}$$

$$\sin^2\sqrt{5}t + \cos^2\sqrt{5}t = 1 \Rightarrow \left(\frac{8x+6y}{14}\right)^2 + \left(\frac{\sqrt{5}(x-y)}{7}\right)^2 = 1$$

$$\Rightarrow 56x^2 + 56xy + 86y^2 = 196$$

$$= 2x^2 + 2xy + 3y^2 = 7 = (x \;\; y)\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}}_{Q}\begin{pmatrix} x \\ y \end{pmatrix} = 7$$

$$\det(Q - \lambda I) = 0 \Rightarrow \lambda^2 - 5\lambda + 5 = 0$$

$$\Rightarrow \lambda = \frac{\sqrt{5}}{2}(\sqrt{5} \pm 1)$$

$$\text{ratio} = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \sqrt{\frac{\sqrt{5}+1}{\sqrt{5}-1}}$$

f) $(Q - \lambda I)\mathbf{v} = 0$ $\qquad \lambda = \frac{5-\sqrt{5}}{2}$

$$\Rightarrow \begin{aligned} (2-\lambda)v_x + v_y &= 0 \\ v_x + (3-\lambda)v_y &= 0 \end{aligned}$$

set $v_x = 1$:

$$\begin{aligned} 2 - \frac{5-\sqrt{5}}{2} + v_y &= 0 \\ 1 + \left(3 - \frac{5-\sqrt{5}}{2}\right)v_y &= 0 \end{aligned} \Rightarrow v_y = \frac{5-\sqrt{5}}{2} - \frac{4}{2} = \frac{1-\sqrt{5}}{2}$$

$$\|\mathbf{v}\| = \sqrt{1^2 + \frac{1}{4}\cdot(1-2\sqrt{5}+5)} = \frac{1}{2}\sqrt{4+1+5-2\sqrt{5}} = \frac{\sqrt{16-2\sqrt{5}}}{2}$$

$$\Rightarrow \mathbf{v}_{norm} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} \qquad //$$