4.2

a) $\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + \nu r^2 \end{cases}$ $\overset{\dot{r}=0}{\Rightarrow}$ $r_0^* = 0$ (unstable)

$r_\pm^* = (\pm)\sqrt{\mu} \Rightarrow r_0 = \sqrt{\mu}$

$\dot{\theta} = \omega + \nu r_0^2$
$\dot{\theta} = \frac{2\pi}{T}$ $\Rightarrow$ $T = \frac{2\pi}{\omega + \nu\mu}$

b) $\begin{cases} X_1 = r\cos\theta \\ X_2 = r\sin\theta \end{cases} \sim \quad r = \sqrt{X_1^2 + X_2^2}$, $\theta = \arctan\left(\frac{X_2}{X_1}\right)$

$\begin{cases} \dot{X}_1 = \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta} \\ \dot{X}_2 = \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta} \end{cases} \Rightarrow$

$\Rightarrow$ $(\mu r - r^3)\cos\theta - r(\omega + \nu r^2)\sin\theta$
$(\mu r - r^3)\sin\theta + r(\omega + \nu r^2)\cos\theta$

$\begin{cases} \sin\theta = \frac{X_2}{r} \\ \cos\theta = \frac{X_1}{r} \end{cases} \Rightarrow$ $(\mu - r^2)X_1 - (\omega + \nu r^2)X_2 = \dot{X}_1$
$(\mu - r^2)X_2 + (\omega + \nu r^2)X_1 = \dot{X}$

$r = \sqrt{X_1^2 + X_2^2} \Rightarrow$ $(\mu - X_1^2 - X_2^2)X_1 - (\omega + \nu(X_1^2 + X_2^2))X_2$
$(\mu - X_1^2 - X_2^2)X_2 + (\omega + \nu(X_1^2 + X_2^2))X_1$

$$\Rightarrow \quad \begin{aligned} \dot{X}_1 &= \mu X_1 - \nu X_2^3 - X_1 X_2^2 - \nu X_1^2 X_2 - \omega X_2 - X_1^3 \\ \dot{X}_2 &= \omega X_1 + \mu X_2 + \nu X_1 X_2^2 + \nu X_1^3 - X_2^3 - X_1^2 X_2 \end{aligned}$$

Compare with

$$\begin{aligned} \dot{X}_1 = F_1(\mathbf{X}) &= \frac{1}{10}X_1 - X_2^3 - X_1X_2^2 - X_1^2X_2 - X_2 - X_1^3 \\ \dot{X}_2 = F_2(\mathbf{X}) &= X_1 + \frac{1}{10}X_2 + X_1X_2^2 + X_1^3 - X_2^3 - X_1^2X_2 \end{aligned}.$$

c) $\Rightarrow$ $\mu = 0.1$
$\omega = 1$
$\nu = 1$

d) $J = \frac{\partial F}{\partial x} = \begin{bmatrix} \mu - X_2^2 - 2\nu X_1 X_2 - 3X_1^2 & -3\nu X_2^2 - 2X_1X_2 - \nu X_1^2 - \omega \\ \omega + \nu X_2^2 + 3\nu X_1^2 - 2X_1X_2 & \mu + 2X_1X_2 - 3X_2^2 - X_1^2 \end{bmatrix}$

$M(0) = I \Rightarrow \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

g) $\dot{M}(t) = \mathbb{J}(t)\, M(t)$, $M(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M(T) = e^{\mathbb{J}(r^*)T} M(0)$ of polar coordinates, $T = \frac{2\pi}{1.1}$, $r^* = \sqrt{\mu}$

$$\mathbb{J}_{polar} = \begin{bmatrix} \mu - 3r^2 & 0 \\ 2\nu r & 0 \end{bmatrix}\Bigg|_{r^*} = \begin{bmatrix} -0.2 & 0 \\ 2\sqrt{0.1} & 0 \end{bmatrix}$$

$$\mathbb{J}T_{polar} = \begin{bmatrix} -\frac{0.4\pi}{1.1} & 0 \\ \frac{4\sqrt{0.1}\,\pi}{1.1} & 0 \end{bmatrix}$$

$$M_{cart} = J_{G}^{-1} M_{polar} J_{G}$$

$x_1 = r\cos\theta$
$x_2 = r\sin\theta$

$$J_G^{-1} = \begin{bmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\Bigg|_{r^*, \theta^*} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{0.1} \end{bmatrix}$$

$r = \sqrt{x_1^2 + x_2^2}$
$\theta = \arctan\left(\frac{x_2}{x_1}\right)$

$$J_G = \begin{bmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_2} \\ \frac{\partial \theta}{\partial x_1} & \frac{\partial \theta}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1 + x_2}} & \frac{x_2}{\sqrt{x_1 + x_2}} \\ -\frac{x_2}{x_1^2 + x_2^2} & \frac{x_1}{x_1^2 + x_2^2} \end{bmatrix}\Bigg|_{x_1^*, x_2^*} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{0.1}} \end{bmatrix}$$

$$M_{polar} = e^{\mathbb{J}T} = \begin{bmatrix} e^{-\frac{0.4\pi}{1.1}} & 0 \\ \sqrt{10}\left(1 - e^{-\frac{0.4\pi}{1.1}}\right) & 1 \end{bmatrix}$$

$$M_{cart} = J_G^{-1} M_{polar} J_G$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{0.1} \end{bmatrix} \begin{bmatrix} e^{-\frac{0.4\pi}{1.1}} & 0 \\ \sqrt{10}\left(1 - e^{-\frac{0.4\pi}{1.1}}\right) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{0.1}} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\frac{0.4\pi}{1.1}} & 0 \\ 1 - e^{-\frac{0.4\pi}{1.1}} & \sqrt{0.1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{0.1}} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-\frac{0.4\pi}{1.1}} & 0 \\ 1 - e^{-\frac{0.4\pi}{1.1}} & 1 \end{bmatrix} \text{ or } \begin{bmatrix} e^{-\frac{4\pi}{11}} & 0 \\ 1 - e^{-\frac{4\pi}{11}} & 1 \end{bmatrix}$$

h) $\sigma_i = \frac{1}{T} \ln(\text{eig } M) \Rightarrow \lambda_1 = e^{-\frac{0.4\pi}{1.1}}$, $\lambda_2 = 1$

$\sigma_1 = \frac{1.1}{2\pi} \ln\left(e^{-\frac{0.4\pi}{1.1}}\right) = \frac{1.1}{2\pi} \cdot -\frac{0.4\pi}{1.1} = -\frac{0.4}{2}$
$\sigma_2 = \frac{1.1}{2\pi} \ln(1) = 0$