FORMULAS FOR THE STIFFNESS OF COMPOSITES WITH PERIODIC MICROSTRUCTURE

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Abstract—In this paper, the mechanical behavior of composite materials with periodic micro-structure is analysed. The corresponding elastic problem is solved by using the Fourier series technique and assuming the homogenization eigenstrain to be piecewise constant. Then, the coefficients of the overall stiffness tensor of the composite material are expressed analytically in terms of the elastic properties of the constituents (fibers and matrix) and as a function of nine triple series which take into account the geometry of the inclusions. In the case of composite materials reinforced by long fibers, simple formulas for evaluating these series are proposed. Close-form expressions for the elastic moduli of the fiber reinforced composite with periodic microstructure and for the equivalent transversely isotropic material are obtained. Finally, several comparisons with experimental results are presented.

1. INTRODUCTION

Micromechanical models represent an efficient tool to estimate the overall stiffness of composite materials and a large number of results and comparisons with experimental data are available (Aboudi, 1991; Mura, 1987; Nemat-Nasser and Hori, 1993). The simplest model is the composite sphere or cylinder scheme which was proposed by Hashin (1962). In this method the composite material is modeled as a gradation of sizes of spherical or cylindrical inclusions embedded in a continuous matrix phase. Otherwise, in the self-consistent scheme (S-C), formulated by Budiansky (1965) and Hill (1965a, b), the fibers or the defects are considered as a typical micro inclusion embedded in an unbounded homogeneous elastic solid characterized by the unknown moduli of the composite. Then the overall elastic properties are computed by an iterative numerical procedure to take into account the interaction effects between the phases. For example, Budiansky and O'Connell (1976), Laws (1977), Laws et al. (1983), Laws and Brockenbrough (1987), Laws and Dvorak (1987) and Hoening (1979) used the S-C method to estimate the elastic properties of cracked composite materials. They analysed several cases such as: different geometries of cracks (ellipsoidal or cylindrical), isotropic and orthotropic matrix, two or more phase composite materials and obtained closed form solutions useful for engineering applications. Hori and Nemat-Nasser (1983) applied the S-C method for materials damaged by open and closed cracks and obtained the anisotropic response of the composite as a function of the load conditions and of loading path. Although the self-consistent method is simple to use, in the case of high volume fraction of the inclusions, it cannot be always applied for the analysis of multi-phase composite materials (Christensen, 1990). Conversely, the generalized self-consistent method, proposed by Christensen and Lo (1979, 1986), gives good results also in this case. They used this scheme to estimate the effective shear modulus and obtained physically realistic results for both spherical and cylindrical inclusions. On the other hand, for different geometries of the inclusions, many authors employed another micromechanical model, based on the Mori–Tanaka's theory (Mori and Tanaka, 1973).

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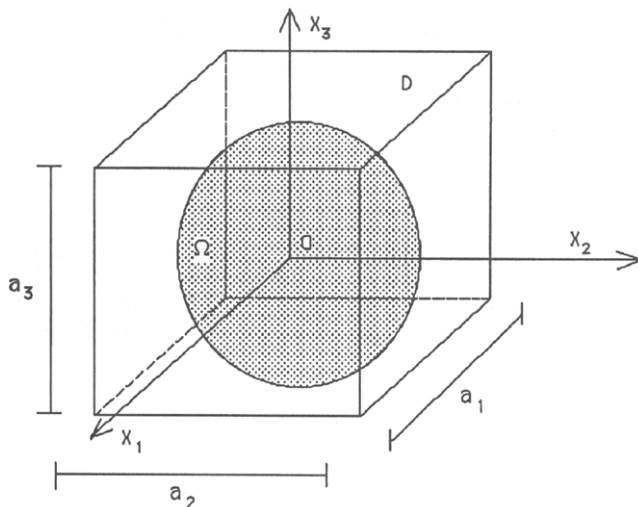
Fig. 1. Geometry of the unit cell D.

They considered isotropic, transversely isotropic and orthotropic matrix and ellipsoidal, cylindrical and ribbon fibers or cracks [see for example, Taya and Chou (1981), Weng (1984), Zhao et al. (1989), Tandon and Weng (1984) and Taya (1981)]. Recently, Benveniste proposed a mathematical justification of the Mori–Tanaka's method and, for composites with or without cracks, obtained estimates for the overall stiffness and compliance tensor (Benveniste, 1987). Finally the behavior of the advanced composites was analysed by Aboudi (1991), who proposed the method of cells, and by Iwakuma and Nemat-Nasser (1983) who formulated the linear elastic problem of composites with periodic microstructure. In Nemat-Nasser and Taya (1981, 1985) and Nemat-Nasser et al. (1982) the concept of a unit cell was introduced and the Fourier series technique was applied to estimate the overall elastic properties of materials with periodic distributed voids. Several approximations to the distribution of the homogenization eigenstrains were considered to solve the problem and in the hypothesis of piecewise constant eigenstrains, Nemat-Nasser et al. (1993) proposed analytical expressions to evaluate the coefficients of the stiffness tensor of cracked solids. Otherwise, for composites with periodic elastic inclusions, they proposed a procedure which entails considerable numerical efforts [see Iwakuma and Nemat-Nasser (1983)]. In the present paper, close-form expressions for the coefficients of the stiffness tensor and for technical elastic moduli of composites materials with periodically distributed elastic inclusions or voids are proposed. Moreover, analytical expressions are given for the elastic moduli of the transversely isotropic material equivalent to the solid reinforced by periodic long fibers. Finally, comparisons with available experimental data, numerical results obtained by Aboudi (1991) and results of the generalized self-consistent method (Christensen and Lo, 1979) are presented.

2. RELATION BETWEEN THE EIGENSTRAIN AND THE STRAIN INSIDE THE INCLUSION

Consider an infinitely extended linearly elastic solid represented by an assembly of unit cells. For simplicity, let each cell D be a parallelepiped with dimensions a_j (Fig. 1) in the direction of the coordinate axes x_j where $j = 1, 2, 3$, and let V be its volume. Then denote by Ω the part of D occupied by the inclusions, let $D - \Omega$ denote the matrix and let f be the volume fraction of Ω.

Next, the relation between the eigenstrain and the strain inside the inclusion is introduced. In order to simulate the inclusions inside the body, consider the homogenization eigenstrain ε^* defined in all D, which must be periodic for the particular geometry of the problem and different to zero only in Ω. Since the material is linear elastic, the actual stress tensor σ inside the unit cell can be expressed in terms of ε^* and the actual strain tensor ε in the following way:

$$\sigma = C(\varepsilon - \varepsilon^*) \quad \text{in} \quad D \tag{1}$$

where C is the elasticity tensor of the matrix. Then, assuming the body forces equal to zero, the tensor σ must satisfy the following equilibrium conditions:

$$\text{div } \sigma = 0 \quad \text{in} \quad D \tag{2}$$

where div denotes the divergence of a tensor field. Furthermore, since in a solid with periodic structure and suitable boundary conditions the displacement $\mathbf{u}$ are periodic, the following Fourier series representation of $\mathbf{u}$, ε and ε^* can be considered:

$$\mathbf{u}(x) = \sum_{\xi}^{\pm\infty} \bar{\mathbf{u}}(\xi) \exp(\mathrm{i}\xi x) \tag{3}$$

$$\varepsilon(x) = \text{sym}(\nabla \mathbf{u}(x)) = \sum_{\xi}^{\pm\infty} \bar{\varepsilon}(\xi) \exp(\mathrm{i}\xi x) \tag{4}$$

$$\varepsilon^*(x) = \sum_{\xi}^{\pm\infty} \bar{\varepsilon}^*(\xi) \exp(\mathrm{i}\xi x) \tag{5}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ with $\xi_j = 2\pi n_j / a_j$ $(n_j = 0, \pm 1, \pm 2, \ldots, j$ not summed, $j = 1, 2, 3)$ and:

$$\bar{\mathbf{u}}(\xi) = \int_D \mathbf{u}(x) \exp(-\mathrm{i}\xi x) \, \mathrm{d}x \tag{6}$$

$$\bar{\varepsilon}(\xi) = \frac{\mathrm{i}}{2} [\xi \otimes \bar{\mathbf{u}}(\xi) + \bar{\mathbf{u}}(\xi) \otimes \xi] \tag{7}$$

$$\bar{\varepsilon}^*(\xi) = \int_D \varepsilon^*(x) \exp(-\mathrm{i}\xi x) \, \mathrm{d}x. \tag{8}$$

Combination of eqns (1) and (2) gives:

$$\text{div}(C(\varepsilon - \varepsilon^*)) = 0 \quad \text{in} \quad D \tag{9}$$

then by using eqns (4), (7) and (5) in (9) the following expressions are obtained:

$$-\xi \cdot C(\xi \otimes \bar{\mathbf{u}}(\xi)) = \mathrm{i}\xi \cdot C\bar{\varepsilon}^*(\xi) \quad \text{for every} \quad \xi \neq 0 \tag{10}$$

where the symbols $\otimes$ and $\cdot$ represent the outer and the inner products, respectively (Spiegel, 1959). Thus, since C represents the elastic tensor of the matrix, the coefficients $\bar{\mathbf{u}}(\xi)$ are obtained uniquely in terms of the $\bar{\varepsilon}^*(\xi)$ in the following way:

$$\bar{\mathbf{u}}(\xi) = -\mathrm{i}(\xi \cdot C \cdot \xi)^{-1} \cdot \xi \cdot C\bar{\varepsilon}^*(\xi) \quad \text{for every} \quad \xi \neq 0 \tag{11}$$

and from eqn (7) the Fourier coefficients of the corresponding strain are:

$$\bar{\varepsilon}(\xi) = \text{sym}(\xi \otimes (\xi \cdot C \cdot \xi)^{-1} \otimes \xi) : C\bar{\varepsilon}^*(\xi) \quad \text{for every} \quad \xi \neq 0. \tag{12}$$

Finally denoting:

$$P'(\xi) = \text{sym}\,(\xi \otimes (\xi \cdot C \cdot \xi)^{-1} \otimes \xi) \tag{13}$$

obtain the actual strain inside the inclusion from eqn (12) using eqns (4) and (8) as:

$$\varepsilon(x) = \frac{1}{V} \sum_{\xi}^{\pm\infty'} P'(\xi) : C \int_D \varepsilon^*(x) \exp\left(-i\xi(x'-x)\right) dx' \tag{14}$$

where a prime on the sum indicates that $\xi = 0$ is excluded in the summation.

Now, note that the exact expression of the strain tensor $\varepsilon(x)$ is not necessary to obtain the overall elastic tensor C^* but only its volume average on Ω denoted by ($\bar{\varepsilon} = \int_\Omega \varepsilon(x)\,dx/V_\Omega$):

$$\bar{\varepsilon} = \frac{1}{V} \sum_{\xi}^{\pm\infty'} P'(\xi) : C\left(\frac{g_0(\xi)}{V_\Omega}\right) \int_D \varepsilon^*(x) \exp\left(-i\xi x'\right) dx' \tag{15}$$

where V_Ω is the volume of the inclusion and:

$$g_0(\xi) = \int_\Omega \exp\left(i\xi x\right) dx. \tag{16}$$

A good approximation of eqn (15) is obtained when a constant ε^* is considered in Ω, as shown in Nemat-Nasser et al. (1982). Then, replacing ε^* with its volume average $\bar{\varepsilon}^*$, eqn (15) becomes:

$$\bar{\varepsilon} = \frac{1}{V} \sum_{\xi}^{\pm\infty'} P'(\xi) : C\left(\frac{g_0(\xi)g_0(-\xi)}{V_\Omega}\right) \bar{\varepsilon}^* \tag{17}$$

or

$$\bar{\varepsilon} = f \sum_{\xi}^{\pm\infty'} \left(\frac{g_0(\xi)}{V_\Omega}\right)\left(\frac{g_0(-\xi)}{V_\Omega}\right) P'(\xi) : C : \bar{\varepsilon}^* \tag{18}$$

and by denoting

$$t(\xi) = f\left(\frac{g_0(\xi)}{V_\Omega}\right)\left(\frac{g_0(-\xi)}{V_\Omega}\right) \tag{19}$$

and

$$P = \sum_{\xi}^{\pm\infty'} t(\xi) P'(\xi) \tag{20}$$

the following expression holds:

$$\bar{\varepsilon} = P : C : \bar{\varepsilon}^* \quad \text{in } \Omega. \tag{21}$$

Note that eqn (21) represents the relation between the volume average of the strain inside the inclusion $\bar{\varepsilon}$ and the volume average of the eigenstrain $\bar{\varepsilon}^*$.

3. OVERALL STIFFNESS TENSOR

In order to obtain the homogenization eigenstrain which simulates the presence of the periodic inclusions inside the body, consider an average strain tensor $\overline{\varepsilon_0}$ in the unit cell, which is arbitrarily prescribed. In this hypothesis the following average consistency condition (equivalent eigenstrain method) can be used (Nemat-Nasser and Hori, 1993):

$$C' : (\overline{\varepsilon_0} + P : C : \overline{\varepsilon}^*) = C : (\overline{\varepsilon_0} + (P : C - I^{(4)}) : \overline{\varepsilon}^*) \tag{22}$$

where C' is the elastic tensor of the inclusion and $I^{(4)}$ is the identity fourth-order tensor. Observe that the tensor P takes into account the geometry of the inclusion and can be evaluated once and for all. Then from eqn (22), the equivalent average volume eigenstrain $\overline{\varepsilon}^*$ can be solved in terms of the tensors C', C, P and $\overline{\varepsilon_0}$ as:

$$\overline{\varepsilon}^* = [((C - C')^{-1} - P)C]^{-1}\overline{\varepsilon_0}. \tag{23}$$

Furthermore, since in this case the uniform overall stress σ_0 in the unit cell is:

$$C^* : \overline{\varepsilon_0} = C : (\overline{\varepsilon_0} - f\overline{\varepsilon}^*) \tag{24}$$

by using eqn (23) and noting that $\overline{\varepsilon_0}$ is arbitrary, the following expression of the overall stiffness tensor of the composite material is obtained:

$$C^* = C - f((C - C')^{-1} - P)^{-1}. \tag{25}$$

It is worth noting that evaluation of C^* [eqn (25)] involves the inversion of a symmetric tensor since P, C and C' are all symmetric tensors. In particular if the matrix is isotropic, denoting by $\overline{\xi} = \xi/|\xi|$, the tensor P is (Mura, 1987; Nemat-Nasser and Hori, 1993):

$$P = \frac{1}{\mu_0} \sum_{\xi}^{\pm\infty}{}' t(\xi)\left(\mathrm{sym}\,(\overline{\xi} \otimes I^{(2)} \otimes \overline{\xi}) - \frac{1}{2(1-v_0)}(\overline{\xi} \otimes \overline{\xi} \otimes \overline{\xi} \otimes \overline{\xi}) \right) \tag{26}$$

where μ_0 and v_0 are the shear modulus and the Poisson ratio of the matrix, respectively and $I^{(2)}$ is the identity second-order tensor. Hence, when the matrix and the inclusion are both isotropic, eqn (25) can be written:

$$C^* = \lambda_0 I^{(2)} \otimes I^{(2)} + 2\mu_0 I^{(4)} - f[(\lambda_0 - \lambda_1)I^{(2)} \otimes I^{(2)} + 2(\mu_0 - \mu_1)I^{(4)}]^{-1}$$

$$- \frac{1}{\mu_0} \sum_{\xi}^{\pm\infty}{}' t(\xi)\left[\left(\mathrm{sym}\,(\overline{\xi} \otimes I^{(2)} \otimes \overline{\xi}) - \frac{1}{2(1-v_0)}(\overline{\xi} \otimes \overline{\xi} \otimes \overline{\xi} \otimes \overline{\xi}) \right) \right]^{-1}. \tag{27}$$

Here μ_0, λ_0, μ_1 and λ_1 are the Lamé constants of the matrix and the inclusion, respectively. Then, defining the following series S_l (with $l = 1$–9) as:

$$S_1 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_1^2, \quad S_2 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_2^2, \quad S_3 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_3^2$$

$$S_4 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_1^4, \quad S_5 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_2^4, \quad S_6 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_3^4$$

$$S_7 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_2^2\overline{\xi}_3^2, \quad S_8 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_1^2\overline{\xi}_3^2, \quad S_9 = \sum_{\xi}^{\pm\infty}{}' t(\xi)\overline{\xi}_1^2\overline{\xi}_2^2 \tag{28}$$

the final expressions of the components of the tensor C^* different from zero can be written in the following way:

$$C_{11}^* = \lambda_0 + 2\mu_0 - f\left(\frac{S_3 S_2}{\mu_0^2} - \frac{S_5 S_3 + S_6 S_2}{\mu_0^2 g} - \frac{a(S_2 + S_3)}{2\mu_0 c} + \frac{S_6 S_5 - S_7^2}{\mu_0^2 g^2} \right.$$

$$\left. + \frac{a(S_5 + S_6) + 2bS_7}{2\mu_0 gc} + \frac{a^2 - b^2}{4c^2} \right) \Big/ D$$

$$C_{12}^* = \lambda_0 + f\left(\left(-\frac{S_9}{\mu_0^2 g} + \frac{b}{2c\mu_0}\right)S_3 + \frac{S_9 S_6 - S_8 S_7}{\mu_0^2 g^2} - \frac{b(S_6 - S_7) - bS_8 - aS_9}{2c\mu_0 g} - \frac{ba + b^2}{4c^2}\right)\Big/ D$$

$$C_{13}^* = \lambda_0 - f\left(\left(\frac{S_8}{\mu_0^2 g} - \frac{b}{2c\mu_0}\right)S_2 - \frac{S_8 S_5 - S_9 S_7}{\mu_0^2 g^2} + \frac{b(S_5 - S_7) - aS_8 - bS_9}{2c\mu_0 g} + \frac{ab + b^2}{4c^2}\right)\Big/ D$$

$$C_{22}^* = \lambda_0 + 2\mu_0 - f\left(\frac{S_3 S_1}{\mu_0^2} - \frac{S_4 S_3 + S_6 S_1}{\mu_0^2 g} - \frac{a(S_1 + S_3)}{2\mu_0 c} + \frac{S_6 S_4 - S_8^2}{\mu_0^2 g^2}\right.$$
$$\left. + \frac{a(S_4 + S_6) + 2bS_8}{2\mu_0 gc} + \frac{a^2 - b^2}{4c^2}\right)\Big/ D$$

$$C_{33}^* = \lambda_0 + 2\mu_0 - f\left(\frac{S_2 S_1}{\mu_0^2} - \frac{S_4 S_2 + S_5 S_1}{\mu_0^2 g} - \frac{a(S_1 + S_2)}{2\mu_0 c} + \frac{S_5 S_4 - S_9^2}{\mu_0^2 g^2}\right.$$
$$\left. + \frac{a(S_5 + S_4) + 2bS_9}{2\mu_0 gc} + \frac{a^2 - b^2}{4c^2}\right)\Big/ D$$

$$C_{23}^* = \lambda_0 + f\left(\left(-\frac{S_7}{\mu_0^2 g} + \frac{b}{2c\mu_0}\right)S_1 + \frac{S_7 S_4 - S_9 S_8}{\mu_0^2 g^2} - \frac{b(S_4 - S_8 - S_9) - aS_7}{2c\mu_0 g} - \frac{ab + b^2}{4c^2}\right)\Big/ D$$

$$C_{44}^* = \mu_0 - f\left(-\frac{S_2}{\mu_0} - \frac{S_3}{\mu_0} + (\mu_0 - \mu_1)^{-1} + \frac{4S_7}{\mu_0(2 - 2v_0)}\right)^{-1}$$

$$C_{55}^* = \mu_0 - f\left(-\frac{S_1}{\mu_0} - \frac{S_3}{\mu_0} + (\mu_0 - \mu_1)^{-1} + \frac{4S_8}{\mu_0(2 - 2v_0)}\right)^{-1}$$

$$C_{66}^* = \mu_0 - f\left(-\frac{S_1}{\mu_0} - \frac{S_2}{\mu_0} + (\mu_0 - \mu_1)^{-1} + \frac{4S_9}{\mu_0(2 - 2v_0)}\right)^{-1} \tag{29}$$

where:

$$D = -\frac{S_3 S_2 S_1}{\mu_0^3} + \frac{(S_6 S_2 + S_6 S_2 + S_6 S_2)S_1}{\mu_0^3 g} + \frac{a(S_1 S_2 + (S_1 + S_2)S_3)}{2\mu_0^2 c}$$

$$+ \frac{(S_5 S_4 + S_7^2)S_1 + (S_6 S_4 + S_8^2)S_2 + (S_5 S_4 + S_9^2)S_3}{\mu_0^3 g^2}$$

$$- \frac{(aS_5 + aS_6 + 2bS_7^2)S_1 + (aS_4 + aS_6 + 2bS_8^2)S_2 + (aS_4 + aS_5 + 2bS_9^2)S_3}{2\mu_0^2 gc}$$

$$+ \frac{(b^2 - a^2)}{4\mu_0 c^2}(S_1 + S_2 + S_3) + \frac{(S_5 S_6 - S_7^2)S_4 - S_8^2 S_5 - S_9^2 S_6 - 2S_8 S_9 S_7}{\mu_0^3 g^3}$$

$$+ \frac{(aS_5 + aS_6 + 2bS_7)S_4 - (aS_7 + 2bS_8 + 2bS_9)S_7 + (2bS_5 - aS_8 + 2bS_9)S_8}{2\mu_0^2 g^2 c}$$
$$\frac{- aS_9^2 + (2bS_9 + aS_5)S_6}{}$$

$$+ \frac{a(aS_4 + aS_5 + aS_6 + 2(bS_7 + bS_8 + bS_9))}{4\mu_0 gc^2} + \frac{d(2(S_7 + S_8 + S_9) - (S_4 + S_5 + S_6))}{4}$$

$$+ \frac{a^3 - 3ab^2 - 2b^3}{8c^3} \tag{30}$$

and

$$a = \mu_1 - \mu_0 - 2\mu_1 \nu_0 + 2\mu_0 \nu_1$$

$$b = -\mu_0 \nu_0 + \mu_1 \nu_1 + 2\mu_0 \nu_0 \nu_1 - 2\mu_1 \nu_0 \nu_1$$

$$c = (\mu_0 - \mu_1)(-\mu_0 + \mu_1 - \mu_0 \nu_0 - 2\mu_1 \nu_0 + 2\mu_0 \nu_1 + \mu_1 \nu_1 + 2\mu_0 \nu_0 \nu_1 - 2\mu_1 \nu_0 \nu_1)$$

$$d = b^2/(\mu_0 g c^2)$$

$$g = (2 - 2\nu_0). \tag{31}$$

Numerical values for the series S_i are given by Nemat-Nasser *et al.* (1982) and Iwakuma and Nemat-Nasser (1983) for several geometries of the inclusions. It is worth noting that the stiffness values presented by Nemat-Nasser *et al.* (1982) and Iwakuma and Nemat-Nasser (1983) can be obtained by using eqns (29) to (31).

4. UNIDIRECTIONAL COMPOSITE

In the case of composite material reinforced by long circular cylindrical fibers, five series are different from zero and only three are independent (Nemat-Nasser *et al.*, 1982). For unidirectional fibers aligned with the x_1-axis, the tensor $\varepsilon^*(x)$ is constant in the x_1-direction, therefore the Fourier series of $\varepsilon^*(x)$ in the x_1-direction reduces to a constant. Then, for the case of fibers aligned with the x_1-axis, we have:

$$S_1 = S_4 = S_8 = S_9 = 0$$

$$S_2 = S_3, \quad S_5 = S_6. \tag{32}$$

Therefore, the following formulas can be used to evaluate the stiffness tensor of a unidirectional composite with periodic microstructure:

$$C_{11}^* = \lambda_0 + 2\mu_0 - f\left[\frac{S_3^2}{\mu_0^2} - \frac{2S_6 S_3}{\mu_0^2 g} - \frac{aS_3}{\mu_0 c} + \frac{S_6^2 - S_7^2}{\mu_0^2 g^2} + \frac{aS_6 + bS_7}{\mu_0 g c} + \frac{a^2 - b^2}{4c^2}\right]\bigg/ D$$

$$C_{12}^* = \lambda_0 + fb\left[\frac{S_3}{2c\mu_0} - \frac{S_6 - S_7}{2c\mu_0 g} - \frac{a+b}{4c^2}\right]\bigg/ D$$

$$C_{23}^* = \lambda_0 + f\left[\frac{aS_7}{2\mu_0 g c} - \frac{ba + b^2}{4c^2}\right]\bigg/ D$$

$$C_{22}^* = \lambda_0 + 2\mu_0 - f\left[-\frac{aS_3}{2\mu_0 c} + \frac{aS_6}{2\mu_0 g c} + \frac{a^2 - b^2}{4c^2}\right]\bigg/ D$$

$$C_{44}^* = \mu_0 - f\left[-\frac{2S_3}{\mu_0} + (\mu_0 - \mu_1)^{-1} + \frac{4S_7}{\mu_0(2 - 2\nu_0)}\right]^{-1}$$

$$C_{66}^* = \mu_0 - f\left[-\frac{S_3}{\mu_0} + (\mu_0 - \mu_1)^{-1}\right]^{-1} \tag{33}$$

where:

$$D = \frac{aS_3^2}{2\mu_0^2 c} - \frac{aS_6 S_3}{\mu_0^2 gc} + \frac{a(S_6^2 - S_7^2)}{2\mu_0^2 g^2 c} + \frac{S_3(b^2 - a^2)}{2\mu_0 c^2}$$

$$+ \frac{S_6(a^2 - b^2) + S_7(ab + b^2)}{2\mu_0 gc^2} + \frac{(a^3 - 2b^3 - 3ab^2)}{8c^3} \quad (34)$$

and

$$a = \mu_1 - \mu_0 - 2\mu_1 v_0 + 2\mu_0 v_1$$

$$b = -\mu_0 v_0 + \mu_1 v_1 + 2\mu_0 v_0 v_1 - 2\mu_1 v_0 v_1$$

$$c = (\mu_0 - \mu_1)(-\mu_0 + \mu_1 - \mu_0 v_0 - 2\mu_1 v_0 + 2\mu_0 v_1 + \mu_1 v_1 + 2\mu_0 v_0 v_1 - 2\mu_1 v_0 v_1)$$

$$g = (2 - 2v_0) \quad (35)$$

where the series S_3, S_6, S_7 are given by Nemat-Nasser *et al.* (1982) in tabular form for several values of the volume fraction of the inclusions. However, the tabular data can be fitted with parabolic expressions using a least-square method. In the case of long fibers, the following expressions fit the data with a correlation coefficient $R = 1$:

$$S_3 = 0.49247 - 0.47603f - 0.02748f^2$$

$$S_6 = 0.36844 - 0.14944f - 0.27152f^2$$

$$S_7 = 0.12346 - 0.32035f + 0.23517f^2. \quad (36)$$

This procedure avoids the numerical evaluation of the series for each value of the fiber volume fraction, which entails significant computational effort, and also allows us to arrive at algebraic expressions for the elastic moduli.

5. TRANSVERSELY ISOTROPIC MATERIAL

Because of the periodicity of the microstructure, the stiffness tensor C^* for unidirectional composite represents an orthotropic material with square symmetry. In the case considered in the previous section, the directions x_2 and x_3 are equivalent and the stiffness tensor is unchanged by a rotation about x_1 of $n\pi/2$ ($n = 0, \pm 1, \pm 2, \ldots$). This implies that only six components are required to describe the tensor completely.

In order to obtain a transversely isotropic stiffness tensor, equivalent in average sense to the stiffness tensor with square symmetry, the following averaging procedure (Aboudi, 1991) is used. A rotation θ about the x_1-axis of the tensor C^* produces

$$B(\theta) = Q(\theta)C^* Q^{\mathrm{T}}(\theta) \quad (37)$$

where $Q(\theta)$ is the fourth-order orthogonal rotation tensor. Then the equivalent transversely isotropic tensor is obtained as:

$$\bar{B} = \frac{1}{\pi} \int_0^\pi B(\theta) \, d\theta. \quad (38)$$

Then, using the relations between the engineering constants and the components of the $\bar{B}$ tensor, the following expressions are obtained explicitly in terms of the coefficients of the tensor C^* [eqns (33)–(36)]:
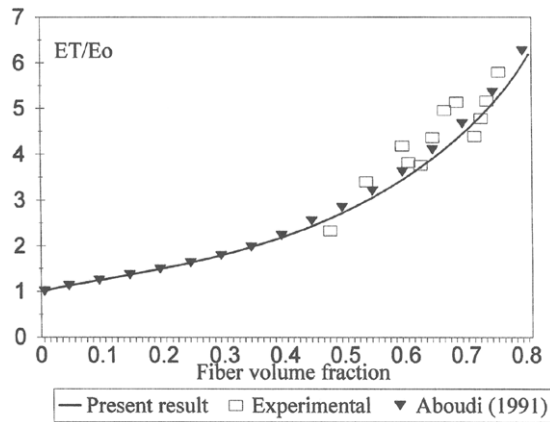
Fig. 2. Comparison with experimental results of transverse modulus E_T normalized with respect to the matrix modulus E_0.

$$E_A = C_{11}^* - \frac{2C_{12}^{*2}}{C_{22}^* + C_{23}^*}$$

$$E_T = \frac{(2C_{11}^* C_{22}^* + 2C_{11}^* C_{23}^* - 4C_{12}^{*2})(C_{22}^* - C_{23}^* + 2C_{44}^*)}{3C_{11}^* C_{22}^* + C_{11}^* C_{23}^* + 2C_{11}^* C_{44}^* - 4C_{12}^{*2}}$$

$$G_A = C_{66}^*$$

$$G_T = \frac{C_{22}^*}{4} - \frac{C_{23}^*}{4} + \frac{C_{44}^*}{2} = \frac{E_T}{2(1+v_T)}$$

$$v_A = \frac{C_{12}^*}{C_{22}^* + C_{23}^*}$$

$$v_T = \frac{C_{11}^* C_{22}^* + 3C_{11}^* C_{23}^* - 2C_{11}^* C_{44}^* - 4C_{12}^{*2}}{3C_{11}^* C_{22}^* + C_{11}^* C_{23}^* + 2C_{11}^* C_{44}^* - 4C_{12}^{*2}}. \tag{39}$$

In particular the transverse shear modulus G_T can be written in the following way:

$$G_T = \mu_0 - \frac{f}{4}\left[\left(-\frac{aS_3}{2\mu_0 c} + \frac{a(S_7+S_6)}{2\mu_0 gc} - \frac{ba+2b^2-a^2}{4c^2}\right)\bigg/ D \right.$$

$$\left. + 2\left(-\frac{2S_3}{\mu_0} + (\mu_0-\mu_1)^{-1} + \frac{4S_7}{\mu_0(2-2v_0)}\right)^{-1}\right] \tag{40}$$

where a, b, c, D and g are given in eqn (35) and S_3, S_6 and S_7 can be evaluated by eqn (36).

6. COMPARISONS WITH EXPERIMENTAL RESULTS

Comparisons with experimental results and the expressions proposed by other authors are presented in this section. Tsai and Hahn (1980) measured the transverse Young's modulus E_T and the axial shear modulus G_A of glass–epoxy composite for several values of the fiber volume fraction. The properties of the constituents are $v_0 = 0.38$, $v_1 = 0.22$, and $E_1/E_0 = 21.19$. The results obtained with eqns (39) and (36) (present result) are compared to the experimental data and to predictions using the method of cells (Aboudi, 1991) in Figs 2 and 3. Predicted values of the axial and transverse Poisson ratios are shown in Fig. 4. For the same properties of the constituents, the axial modulus predicted by the first of eqns (39) coincides with the rule of mixture estimate and the axial shear modulus obtained
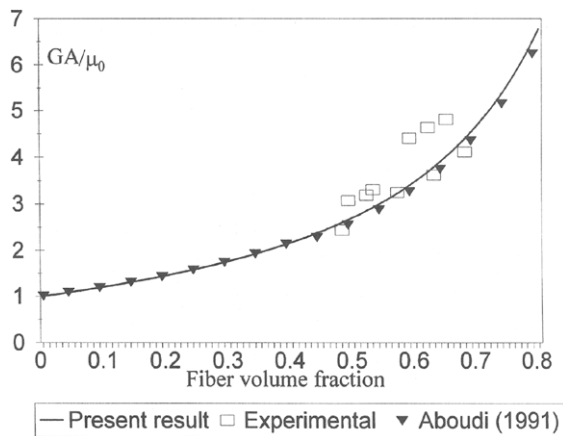
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Fig. 3. Comparison with experimental results of axial shear modulus G_A normalized with respect to the matrix modulus μ_0.
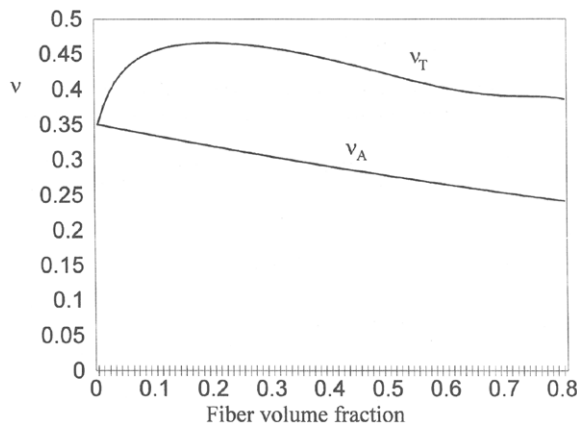

Fig. 4. Transverse and axial Poisson ratio (v_T and v_A) as a function of the fiber volume fraction for glass–epoxy composite.
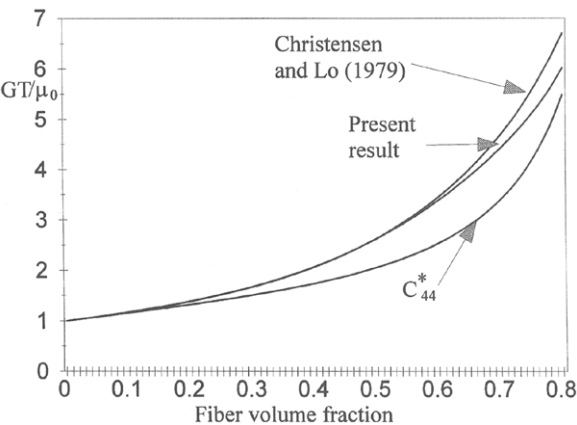

Fig. 5. Comparison with general self-consistent method of transverse shear modulus G_T normalized with respect to the matrix modulus μ_0.

from the third of eqns (39) gives the same values of the expressions proposed by Christensen and Lo (1979), for all values of the fiber volume fraction. Then, in Fig. 5 the transverse shear modulus obtained by eqn (40) is compared with the analytical expression proposed

by Christensen and Lo (1979) and with the transverse shear modulus in the material with square symmetry C_{44}^* [given by eqn (33)].

7. CONCLUSIONS

Simple formulas for the coefficients of the stiffness tensor of composite materials with general types of elastic inclusions or voids with periodic microstructure are presented. These formulas are reduced for the particular case of long fiber composites and the engineering properties of equivalent transversely isotropic materials are proposed. Good agreement with available experimental data is obtained. The interaction effects between the constituents are fully accounted for.

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