Tarea 5. Álgebra Matricial.
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1. Determinar la descomposición ortogonal de y con respecto a u.

a) $y = (2,-2)$; $u = (1,3)$

$$y = \text{proy}_u y + z$$

$\cdot$ $\text{proy}_u y = \frac{2-6}{10}(1,3) = -\frac{4}{10}(1,3) = -\frac{2}{5}(1,3)$

$\cdot$ $z = y - \text{proy}_u y = (2,-2) + \frac{2}{5}(1,3) =$
$= (2,-2) + (2/5, 6/5) = (12/5, -4/5)$

$$\boxed{\therefore\ y = (-2/5, -6/5) + (12/5, -4/5)}$$

b) $y = (3,2,-1)$; $u = (1,1,1)$

$$y = \text{proy}_u y + z$$

$\cdot$ $\text{proy}_u y = \frac{3+2-1}{3}(1,1,1) = \frac{4}{3}(1,1,1)$

$\cdot$ $z = y - \text{proy}_u y = (3,2,-1) - \frac{4}{3}(1,1,1)$
$= (5/3, 2/3, -7/3)$

$$\boxed{\therefore\ y = (4/3, 4/3, 4/3) + (5/3, 2/3, -7/3)}$$

c) $y = (4,-2,3)$; $u_1 = (1,2,1)$, $u_2 = (1,-1,1)$

- Verificamos si u_1 y u_2 forman una base ortogonal para $W = \text{gen}\{u_1, u_2\}$

$u_1 \cdot u_2 = 1 - 2 + 1 = 0 \Rightarrow$ Ortogonal ✓

- Descomposición

$$y = \hat{y} + z$$

$$\hat{y} = \text{proy}_{u_1} y + \text{proy}_{u_2} y = \frac{4-4+3}{5}(1,2,1) + \frac{4+2+3}{3}(1,-1,1)$$
$$= \tfrac{3}{5}(1,2,1) + 3(1,-1,1) = (\tfrac{3}{5}, \tfrac{6}{5}, \tfrac{3}{5}) + (3,-3,3)$$
$$= (18/5, -9/5, 18/5)$$

$$z = y - \hat{y} = (4,-2,3) - (18/5, -9/5, 18/5) =$$
$$= (2/5, -19/5, 33/5)$$

$$\boxed{\therefore y = (18/5, -9/5, 18/5) + (2/5, -19/5, 33/5)}$$

2. Construir una base ortogonal y una base ortonormal para las siguientes bases bajo el producto interno indicada

a) $v_1 = (1,2,-1)$, $v_2 = (4,7,1)$, $v_3 = (1,0,1)$

con $\langle X, Y\rangle = 2x_1y_1 + 3x_2y_2 + x_3y_3$

$X = (x_1, x_2, x_3)$; $Y = (y_1, y_2, y_3)$

• Base ortogonal -> Gram-Schmidt.

$u_1 = v_1 = (1,2,-1)$

$u_2 = v_2 - proy_{u_1} v_2 = (4,7,1) - \frac{2\cdot 4 + 3\cdot 14 - 1}{2+3\cdot 4+1}(1,2,-1)$

$= (4,7,1) - \frac{49}{15}(1,2,-1) = (4,7,1) - (\frac{49}{15}, \frac{98}{15}, -\frac{49}{15})$

$= (11/15, 7/15, 64/15)$

$u_3 = v_3 - proy_{u_1} v_3 - proy_{u_2} v_3 = (1,0,1) - \frac{1}{15}(1,2,-1) - \frac{86}{299}(11/15, 7/15, 64/15)$

$= (1,0,1) - (\frac{1}{15}, \frac{2}{15}, -\frac{1}{15}) - \frac{86}{299}(\frac{11}{5}, \frac{7}{15}, 64/15)$

$= (1348/4485, -80/299, -48/299)$

$\therefore B_o = \{(1,2,-1), (11/15, 7/15, 64/15), (1348/4485, -80/299, -48/299)\}$

forma una base ortogonal para $gen\{v_1, v_2, v_3\}$

Obtenemos la base ortonormal.

$\hat{u}_1 = u_1 \cdot \frac{1}{\sqrt{\langle u_1, u_1\rangle}} = u_1 \cdot \frac{1}{\sqrt{15}} = (1/\sqrt{15}, 2/\sqrt{15}, -1/\sqrt{15})$

$\hat{u}_2 = u_2 \cdot \frac{1}{\sqrt{\langle u_2, u_2\rangle}} = u_2 \cdot \sqrt{\frac{15}{299}} = (\frac{11}{\sqrt{4485}}, 7/\sqrt{4485}, 64/\sqrt{4485})$

$\hat{u}_3 = u_3 \cdot \frac{1}{\sqrt{\langle u_3, u_3\rangle}} = u_3 \cdot \frac{1}{0.699} \approx (0.4631, -0.4123, -0.247)$

$\therefore$ Base ortonormal $= \{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$

b) $v_1 = (4,1)$, $v_2 = \cancel{(4,7,1)} = (1,0)$

$\langle u, v \rangle = 3u_1v_1 + 2u_2v_2$.

• Base ortogonal → Gram-Schmidt

$u_1 = v_1 = (4,1)$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 = (1,0) - \frac{6}{25}(4,1)$$
$$= (1/25, -6/25)$$

$\therefore B = \{(4,1), (1/25, -6/25)\}$ es una base ortogonal.

• Base ortonormal

$$\hat{u}_1 = u_1 \frac{1}{\sqrt{\langle u_1, u_1 \rangle}} = \frac{1}{\sqrt{50}}(4,1) = (4/\sqrt{50}, 1/\sqrt{50})$$

$$\hat{u}_2 = u_2 \cdot (\langle u_2, u_2 \rangle)^{-1/2} = (1/25; -6/25)(3/25)^{-1/2}$$
$$= (1/25, -6/25)\, 5/\sqrt{3} = (1/5\sqrt{3}, -6/5\sqrt{3})$$

$\therefore \{(4/\sqrt{50}, 1/\sqrt{50}), (1/5\sqrt{3}, -6/5\sqrt{3})\}$ forma una base ortonormal.

c) $v_1 = (2,-1,4)$, $v_2 = (1,0,2)$, $v_3 = (3,-1,5)$
con $V = \mathbb{R}^3$.

• Base ortogonal.

$u_1 = v_1 = (2,-1,4)$

$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} \cdot u_1 = (1,0,2) - \frac{10}{21}(2,-1,4)$
$= (1/21, 10/21, 2/21)$

$u_3 = v_3 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 =$
$= (3,-1,5) - 3/5\,(1/21, 10/21, 2/21) - 9/7\,(2,-1,4)$
$= (2/5, 0, -1/5)$

$\therefore \{(2,-1,4), (1/21, 10/21, 2/21), (2/5, 0, -1/5)\}$
forma una base ortogonal de $\mathbb{R}^3$.

• Base ortonormal

$\hat{u}_1 = u_1 (\langle u_1, u_1 \rangle)^{-1/2} = u_1 (21)^{-1/2} = \frac{1}{\sqrt{21}}(2,-1,4)$
$= (2/\sqrt{21}, -1/\sqrt{21}, 4/\sqrt{21})$

$\hat{u}_2 = u_2 (\langle u_2, u_2 \rangle)^{-1/2} = u_2 (5/21)^{-1/2} = \sqrt{\frac{21}{5}}(1/21, 10/21, 2/21)$
$= (1/\sqrt{105}, 10/\sqrt{105}, 2/\sqrt{105})$

$\hat{u}_3 = u_3 (\langle u_3, u_3 \rangle)^{-1/2} = u_3 (1/5)^{-1/2} = \sqrt{5}(2/5, 0, -1/5)$
$= (2/\sqrt{5}, 0, -1/\sqrt{5})$

$\therefore \{(2/\sqrt{21}, -1/\sqrt{21}, 4/\sqrt{21}), (1/\sqrt{105}, 10/\sqrt{105}, 2/\sqrt{105}),$
$(2/\sqrt{5}, 0, -1/\sqrt{5})\}$
es una base ortonormal de $\mathbb{R}^3$

3. Encontrar una solución de mínimos cuadrados de $Ax=b$ mediante la factorización QR.

a) $A=\begin{pmatrix}-1 & 2\\ 2 & -3\\ -1 & 3\end{pmatrix}$, $b=\begin{pmatrix}4\\1\\2\end{pmatrix}$

$\Rightarrow$ Factorización QR.

- Obtenemos Q con Gram-Schmdt

$$v_1=\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$v_2=\begin{pmatrix}2\\-3\\3\end{pmatrix}-\frac{-11}{6}\begin{pmatrix}-1\\2\\-1\end{pmatrix}=\begin{pmatrix}2\\-3\\3\end{pmatrix}+\begin{pmatrix}-11/6\\11/3\\-11/6\end{pmatrix}$$

$$=\begin{pmatrix}1/6\\-2/3\\7/6\end{pmatrix}$$

Normalizamos.

$$\hat{v}_1=\frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}=\begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{pmatrix}$$

$$\hat{v}_2=\frac{6}{\sqrt{66}}\begin{pmatrix}1/6\\2/3\\7/6\end{pmatrix}=\frac{6}{\sqrt{66}}\begin{pmatrix}1/6\\2/3\\7/6\end{pmatrix}=\begin{pmatrix}1/\sqrt{66}\\4/\sqrt{66}\\7/\sqrt{66}\end{pmatrix}$$

$$\Rightarrow Q=\begin{pmatrix}-1/\sqrt{6} & 1/\sqrt{66}\\ 2/\sqrt{6} & 4/\sqrt{66}\\ -1/\sqrt{6} & 7/\sqrt{66}\end{pmatrix}$$

Para obtener R:

$$R = Q^t A = \begin{pmatrix} -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{66} & 4/\sqrt{66} & 7/\sqrt{66} \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 2 & -3 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{6} & -11/\sqrt{6} \\ 0 & 11/\sqrt{66} \end{pmatrix}$$

Resolvemos el sistema $R\hat{x} = Q^t b$

$$Q^t b = \begin{pmatrix} -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \\ 1/\sqrt{66} & 4/\sqrt{66} & 7/\sqrt{66} \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4/\sqrt{6} \\ 22/\sqrt{66} \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} \sqrt{6} & -11/\sqrt{6} & -4/\sqrt{6} \\ 0 & 11/\sqrt{66} & 22/\sqrt{66} \end{array}\right) (\sqrt{66}/11) \sim \left(\begin{array}{cc|c} \sqrt{6} & -11/\sqrt{6} & -4\sqrt{6} \\ 0 & 1 & 2 \end{array}\right) 11/\sqrt{6}$$

$$\sim \left(\begin{array}{cc|c} \sqrt{6} & 0 & 22/\sqrt{6} - 4/\sqrt{6} \\ 0 & 1 & 2 \end{array}\right)$$

$\boxed{x_2 = 2}$ $\wedge$ $\sqrt{6}\, x_1 = 18/\sqrt{6}$

$\boxed{x_1 = 18/6 = 3}$

$$\boxed{\therefore \hat{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

b) $A = \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}$; $b = \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$

• Para determinar Q -> Gram-Schmidt.

$$v_1 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \frac{8}{12}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 4/3 \\ -4/3 \\ 4/3 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 \\ 4/3 \\ 5/3 \end{pmatrix}$$

Unitarios:

$$\hat{v}_1 = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\hat{v}_2 = \begin{pmatrix} -1/3 \\ 4/3 \\ 5/3 \end{pmatrix} \cdot \frac{1}{\left(\frac{\sqrt{42}}{3}\right)} = \frac{3}{\sqrt{42}}\begin{pmatrix} -1/3 \\ 4/3 \\ 5/3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{42} \\ 4/\sqrt{42} \\ 5/\sqrt{42} \end{pmatrix}$$

$$\Rightarrow Q = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{42} \\ -1/\sqrt{3} & 4/\sqrt{42} \\ 1/\sqrt{3} & 5/\sqrt{42} \end{pmatrix}$$

• Hallamos R.

$$R = Q^t A = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{42} & 4/\sqrt{42} & 5/\sqrt{42} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2/\sqrt{3} & 4/\sqrt{3} \\ 0 & 14/\sqrt{42} \end{pmatrix}$$

Resolvemos $Rx = Q^t b$

$$Q^t b = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{42} & 4/\sqrt{42} & 5/\sqrt{42} \end{pmatrix} \begin{pmatrix} -5 \\ 8 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4/\sqrt{3} \\ 42/\sqrt{42} \end{pmatrix} = \begin{pmatrix} 4/\sqrt{3} \\ \sqrt{42} \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 2/\sqrt{3} & 4/\sqrt{3} & 4/\sqrt{3} \\ 0 & 14/\sqrt{42} & \sqrt{42} \end{array}\right) (\sqrt{42}/14)$$

$$\sim \left(\begin{array}{cc|c} 2/\sqrt{3} & 4/\sqrt{3} & 4/\sqrt{3} \\ 0 & 1 & 3 \end{array}\right) (-4/\sqrt{3})$$

$$\sim \left(\begin{array}{cc|c} 2/\sqrt{3} & 0 & -8/\sqrt{3} \\ 0 & 1 & 3 \end{array}\right)$$

$$\Rightarrow \boxed{x_2 = 3} \; ; \; 2/\sqrt{3}\, x_1 = -8/\sqrt{3}$$

$$\boxed{x_1 = -4}$$

$$\therefore \hat{x} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$