Tarea 6 - Álgebra Matricial
Gustavo Hernández Angeles

1. Determinar si las siguientes matrices son definidas positivas, definidas negativas, indefinidas, etc. Justificar.

a) $A = \begin{pmatrix} 4 & -3 \\ -3 & 12 \end{pmatrix}$

Veamos sus valores propios:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ -3 & 12-\lambda \end{vmatrix} = (4-\lambda)(12-\lambda) - 9$$
$$= 48 - 16\lambda + \lambda^2 - 9 = \lambda^2 - 16\lambda + 39 = 0$$
$$\lambda^2 - 16\lambda + 39 = (\lambda - 3)(\lambda - 13) = 0$$

$\Rightarrow \lambda_1 = 3, \ \lambda_2 = 13$

$\Rightarrow$ Sus valores propios son todos positivos

$\boxed{\therefore A \text{ es positiva definida.}}$

b) $A = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}$

· Valores propios

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 4 & 7-\lambda \end{vmatrix} = 7 - 8\lambda + \lambda^2 - 16 =$$
$$= \lambda^2 - 8\lambda - 9 = (\lambda - 9)(\lambda + 1)$$

$\Rightarrow \lambda_1 = 9 \ ; \ \lambda_2 = -1$

$\Rightarrow$ Sus valores propios son positivos y negativos

$\boxed{\therefore A \text{ es indefinida}}$

2. Encontrar una descomposición por valores singulares de las siguientes matrices.

a) $A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$

- Eigen valores de A^tA

$$A^tA = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$$

$$\det(A^tA - \lambda I) = \begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 52 - 17\lambda + \lambda^2 - 36$$

$$= \lambda^2 - 17\lambda + 16 = (\lambda - 16)(\lambda - 1) = 0$$

$\Rightarrow \lambda_1 = 16 \ ; \ \lambda_2 = 1$

$\Rightarrow \sigma_1 = 4 \ ; \ \sigma_2 = 1$ (1)

• Eigenvector de λ_1:

$$\left(\begin{array}{cc|c} 4-16 & 6 & 0 \\ 6 & 13-16 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -12 & 6 & 0 \\ 6 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 6 & -3 & 0 \end{array}\right) \Rightarrow \text{Para } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \ x_2 = 2x_1$$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (2)$$

• Eigenvector de λ_2:

$$\left(\begin{array}{cc|c} 4-1 & 6 & 0 \\ 6 & 13-1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & 6 & 0 \\ 6 & 12 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \text{Para } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 = -2x_2 \Rightarrow x = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\Rightarrow v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad (3)$$

De (1) podemos hacer a Σ y de (2) y (3) podemos hacer V. Para hacer U:

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8/\sqrt{5} \\ 4/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} ;$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Formamos las matrices

$\Rightarrow$

$$V = (v_1 . v_2) = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

$$D = \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$U = (u_1 \; u_2) = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

$$\boxed{\begin{aligned} \therefore A &= U \Sigma V^t \\ &= \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \end{aligned}}$$

b) $A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix}$

• Buscamos eigenvalores de A^tA

$$A^tA = \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 81 & -27 \\ -27 & 9 \end{pmatrix}$$

$$\Rightarrow \det(A^tA - \lambda I) = (81-\lambda)(9-\lambda) - 27^2$$
$$= \cancel{27^2} - 90\lambda + \lambda^2 - \cancel{27^2} = 0$$
$$\lambda^2 - 90\lambda = \lambda(\lambda - 90) = 0$$

$$\Rightarrow \boxed{\lambda_1 = 90 \ ; \ \lambda_2 = 0} \ ; \ \sigma_1 = \sqrt{90} \ ; \ \sigma_2 = 0$$

• Eigenvector con $\lambda_1 = 90$

$$\left(\begin{array}{cc|c} -9 & -27 & 0 \\ -27 & -81 & 0 \end{array}\right) \begin{array}{c} (3) \\ \lrcorner \end{array} \sim \left(\begin{array}{cc|c} -9 & -27 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \hat{v}_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

• Eigenvector con $\lambda_2 = 0$.

$$\left(\begin{array}{cc|c} 81 & -27 & 0 \\ -27 & 9 & 0 \end{array}\right) \begin{array}{c} 1/3 \\ \lrcorner \end{array} \sim \left(\begin{array}{cc|c} 81 & -27 & 0 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ; \ x_2 = 3x_1 \Rightarrow x = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \hat{v}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

De los valores propios:

$$\Sigma = \begin{pmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

De los vectores propios de A^tA:

$$V = \begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}$$

Ahora determinamos eigenvectores de AA^t, apoyandonos de los eigenvalores de A^tA.

Al ser más renglones que columnas A^tA tendrá un eigenvalor adicional, que será 0.

$$AA^t = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 10 & -20 & -20 \\ -20 & 40 & 40 \\ -20 & 40 & 40 \end{pmatrix}$$

Para obtener el eigenvector de AA^t con $\lambda_1 = 90$, podemos hacer uso de $\hat{v}_1$.

$$\hat{u}_1 = \frac{1}{\sigma_1} A\hat{v}_1 = \frac{1}{\sqrt{90}} \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{900}} \begin{pmatrix} 10 \\ -20 \\ -20 \end{pmatrix} = \frac{1}{3 \cdot 10} \begin{pmatrix} 10 \\ -20 \\ -20 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}$$

Para $\lambda_2 = 0$:

$$\left(\begin{array}{ccc|c} 10 & -20 & -20 & 0 \\ -20 & 40 & 40 & 0 \\ -20 & 40 & 40 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 10 & -20 & -20 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 = 2x_2 + 2x_3$$

$$\Rightarrow x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \; ; \; u_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Ortonormalizamos los vectores u_i

$$u_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{\overset{0}{\cancel{u_2 \cdot u_1}}}{u_1 \cdot u_1} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{\overset{0}{\cancel{u_3 \cdot u_1}}}{u_1 \cdot u_1} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} - \frac{u_3 \cdot u_2}{u_2 \cdot u_2} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{4}{5} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 - 8/5 \\ 0 - 4/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/5 \\ -4/5 \\ 1 \end{pmatrix}$$

- Normalizamos:

$$\hat{u}_1 = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \; ; \; \hat{u}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{5}}$$

$$\hat{u}_3 = \frac{5}{\sqrt{45}} \begin{pmatrix} 2/5 \\ -4/5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3\sqrt{5} \\ -4/3\sqrt{5} \\ \sqrt{5}/3 \end{pmatrix}$$

Formamos U.

$$U = (\hat{u}_1 \; \hat{u}_2 \; \hat{u}_3) = \begin{pmatrix} 1/3 & 2/\sqrt{5} & 2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ -2/3 & 0 & \sqrt{5}/3 \end{pmatrix}$$

Ahora, podemos formar la descomposición SVD.

$$\therefore \; A = U \Sigma_1 V^t$$

$$= \begin{pmatrix} 1/3 & 2/\sqrt{5} & 2/3\sqrt{5} \\ -2/3 & 1/\sqrt{5} & -4/3\sqrt{5} \\ -2/3 & 0 & \sqrt{5}/3 \end{pmatrix} \begin{pmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}$$

3. Encontrar la inversa de Moore-Penrose de la siguente matriz.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{3} & 0 \end{pmatrix}$$

Vemos que $\rho(A) = 2$, ya que las columnas son LI (no múltiplos)

$$\Rightarrow A^{+} = (A^{*}A)^{-1}A^{*}$$

- $$A^{*}A = \begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \sqrt{3} & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow (A^{*}A)^{-1} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow A^{+} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \sqrt{3} \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & \sqrt{3}/4 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{\therefore A^{+} = \begin{pmatrix} 1/4 & 0 & \sqrt{3}/4 \\ 0 & 1 & 0 \end{pmatrix}}$$

9. Encontrar una solución por mínimos cuadrados de $Ax=b$ a través de la inversa de Moore-Penrose.

$$A=\begin{pmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{pmatrix}_{m\times n},\quad b=\begin{pmatrix}3\\1\\-4\\2\end{pmatrix} \quad : \hat{x}=A^{+}b$$

Calcularemos A^{+}. Vemos que $\rho(A)=2=n$ ya que las columnas son LI.

$$\Rightarrow A^{+}=(A^{*}A)^{-1}A^{*}$$

$$=\left[\begin{pmatrix}1 & -1 & 0 & 2\\ -2 & 2 & 3 & 5\end{pmatrix}\begin{pmatrix}1 & -2\\ -1 & 2\\ 0 & 3\\ 2 & 5\end{pmatrix}\right]^{-1}\begin{pmatrix}1 & -1 & 0 & 2\\ -2 & 2 & 3 & 5\end{pmatrix}$$

$$=\begin{pmatrix}6 & 6\\ 6 & 42\end{pmatrix}^{-1}\begin{pmatrix}1 & -1 & 0 & 2\\ -2 & 2 & 3 & 5\end{pmatrix}$$

$$=\frac{1}{216}\begin{pmatrix}42 & -6\\ -6 & 6\end{pmatrix}\begin{pmatrix}1 & -1 & 0 & 2\\ -2 & 2 & 3 & 5\end{pmatrix}$$

$$=\frac{1}{216}\begin{pmatrix}54 & -54 & -18 & 54\\ -18 & 18 & 18 & 18\end{pmatrix}$$

$$=\begin{pmatrix}1/4 & -1/4 & -1/12 & 1/4\\ -1/12 & 1/12 & 1/12 & 1/12\end{pmatrix}$$

Podemos ahora obtener la solución: $\hat{x}$

$$\hat{x}=A^{+}b=\begin{pmatrix}1/4 & -1/4 & -1/12 & 1/4\\ -1/12 & 1/12 & 1/12 & 1/12\end{pmatrix}\begin{pmatrix}3\\1\\-4\\2\end{pmatrix}$$

$$=\begin{pmatrix}4/3\\ -1/3\end{pmatrix}$$

$$\therefore \hat{x}=\begin{pmatrix}4/3\\ -1/3\end{pmatrix}$$