Worst Case CVaR Portolio Optimization with Multidimensional Copulas

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Introduction and Motivation

- The seminal article *Portfolio Selection* published by Harry Markowitz (1952) introduces the foundation for modern portfolio theory (MPT) or mean-variance analysis.

- Markowitz considered the problem of an agent who wishes to find the maximum (expected) return for a given level of risk or minimize risk for a given level of return.

- By diversifying a portfolio among investments that have different return patterns, investors can build such an efficient portfolio.

Introduction and Motivation

- A well-known problem of the Markowitz model is its sensitivity to the input parameters.
- In practice, the implementation of strategies based on the risk-return trade-off remains a fundamental challenge in many areas of financial management, since estimation errors of the expected returns of the assets and the covariance matrix of these returns can significantly affect cause an impact in the asset allocations weights, no longer leading to an efficient portfolio.
- This issue can be overcome by employing robust optimization and worst case techniques (Zhu and Fukushima, 2009) in which assumptions about the distribution of the random variable are relaxed, and thus, we obtain the optimal portfolio solution by optimizing over a prescribed feasible set and possible densities.

- Akin to the classical Markowitz portfolio, in this approach we want to determine the weights that maximize the portfolio return for a specified VaR or CVaR at a given confidence level or minimize these quantiles for a given confidence level subject to a fixed portfolio return.

- Artzner *et al.*, 1999 show that VaR has undesirable properties such as lack of sub-additivity and thus it is not a coherent measure.

- Uryasev and Rockafellar, 2001 show that an outright optimization with respect to CVaR is numerically difficult due to the dependence of the CVaR on VaR.

- However, Rockafellar and Uryasev, 2000 show that VaR and CVaR can be computed simultaneously, introducing auxiliary risk measures.

- Their approach can be used in conjunction with scenario based optimization algorithms reducing the problem to a linear programming problem which allows us to optimize a portfolio with very large dimensions and stable numerical implementations.

- Kakouris and Rustem, 2014 show how copula-based models can be introduced in the Worst Case CVaR framework. This approach is motivated by an investor's desire to be protected against the worst possible scenario.

Goals

1. Evaluate the out-of-sample performance of the Worst Case Copula-CVaR and compare it to benchmark approaches in the long term in terms of wealth accumulation and downside risk.

2. Select a diversified set of assets that can be useful during crises and tranquil periods, i.e., that somehow involves hedging of decisions to protect the investors against any market conditions.

- Let Y be a stochastic vector standing for market uncertainties and F_Y be its distribution function, i.e., $F_Y(u) = P(Y \leq u)$. Let also $F_Y^{-1}(v) = \inf\{u : F_Y(u) \geq v\}$ be its right continuous inverse and assume that it has a probability density function represented by $p(y)$. Define the VaR_β as the β-quantile by

$$\begin{aligned} VaR_\beta(Y) &= \arg\min\{\alpha \in \mathbb{R} : P(Y \leq \alpha) \geq \beta\} \\ &= F_Y^{-1}(\beta), \end{aligned} \tag{1}$$

and the $CVaR_\beta$ as the solution to the following optimization problem:

$$CVaR_\beta(Y) = \inf\left\{\alpha \in \mathbb{R} : \alpha + \frac{1}{1-\beta} E[Y-\alpha]^+\right\}, \tag{2}$$

where $[t]^+ = \max(t, 0)$.

Conditional Value-at-Risk

- Uryasev and Rockafellar, 1999 have shown that the $CVaR$ is the conditional expectation of Y, given that $Y \geq VaR_\beta$, i.e.,

$$CVaR_\beta(Y) = \mathbb{E}\left(Y \,|Y \geq VaR_\beta(Y)\right). \tag{3}$$

- Let $f(x, y)$ be a loss function depending upon a decision vector x that belongs to any arbitrarily chosen subset $X \in \mathbb{R}^n$ and a random vector $y \in \mathbb{R}^m$.

- In a portfolio optimization problem, the decision vector x can be a vector of portfolios' weights, X be a set of feasible portfolios, subject to linear constraints and y a vector that stands for market variables that can affect the loss.

Conditional Value-at-Risk

- Using (3) we can write the $CVaR$ function, at confidence level β, by

$$CVaR_{\beta}(x) = \frac{1}{1-\beta}\int_{f(x,y)\geq VaR_{\beta}(x)} f(x,y)p(y)dy, \qquad (4)$$

- The optimization of $CVaR$ is difficult because of the presence of the VaR in its definition (it requires the use of the nonlinear function max).

- The main contribution of Rockafellar and Uryasev, 2000 was to define a simpler auxiliary function

$$F_{\beta}(x,\alpha) = \alpha + \frac{1}{1-\beta}\int_{f(x,y)\geq\alpha} (f(x,y)-\alpha)\,p(y)dy, \qquad (5)$$

  which can be used instead of $CVaR$ directly, without need to compute VaR first due to the following proposition (see Pflug, 2000):

Conditional Value-at-Risk

Proposition 1

The function $F_\beta(x, \alpha)$ is convex with respect to α. In addition, minimizing $F_\beta(x, \alpha)$ with respect to α gives $CVaR$ and VaR is a minimum point of this function with respect to α, i.e.

$$\min_{\alpha \in \mathbb{R}} F_\beta(x, \alpha) = F_\beta(x, VaR_\beta(x)) = CVaR_\beta(x) \tag{6}$$

- Moreover, we can use $F_\beta(x, \alpha)$ to find the optimal weights, $CVaR$ and VaR simultaneously over an allowable feasible set, i.e.,

$$\min_{x \in X} CVaR_\beta(x) = \min_{(x,\alpha) \in X \times \mathbb{R}} F_\beta(x, \alpha). \tag{7}$$

- Pflug, 2000 show that under quite general conditions $F_\beta(x, \alpha)$ is a smooth function. In addition, if $f(x, y)$ is convex with respect to x, then $F_\beta(x, \alpha)$ is also convex with respect to x.
- Hence, if the allowable set X is also convex, we then have to solve a smooth convex optimization problem.

Optimization Problem

- Assume that the analytical representation for the density $p(y)$ is not available, but we can approximate $F_\beta(x,\alpha)$ using J scenarios, y_j, $j = 1, ..., J$ which are sampled from the density function $p(y)$.

- We approximate

$$\begin{aligned} F_\beta(x,\alpha) &= \alpha + \frac{1}{1-\beta}\int_{f(x,y)\geq\alpha}(f(x,y)-\alpha)\,p(y)dy \\ &= \alpha + \frac{1}{1-\beta}\int_{y\in\mathbb{R}^m}(f(x,y)-\alpha)^+\,p(y)dy \qquad (8) \end{aligned}$$

by its discretized version

$$\widetilde{F}_\beta(x,\alpha) = \alpha + \frac{1}{(1-\beta)J}\sum_{j=1}^{J}(f(x,y_j)-\alpha)^+ .$$

Optimization Problem

- Assuming that the feasible set X and the loss function $f(x, y_j)$ are convex, we solve the following convex optimization problem:

$$\min_{x \in X, \alpha \in \mathbb{R}} \widetilde{F}_\beta\left(x, \alpha\right). \tag{9}$$

- In addition, if the loss function $f(x, y)$ is linear with respect to x, then the optimization problem (9) reduces to the following linear programming (LP) problem:

$$\min_{x \in \mathbb{R}^n, z \in \mathbb{R}^J, \alpha \in \mathbb{R}} \alpha + \frac{1}{(1-\beta)\, J} \sum_{j=1}^{J} z_j \tag{10}$$

$$\textit{subject to } x \in X, \tag{11}$$

$$z_j \geq f(x, y_j) - \alpha,\ z_j \geq 0,\ j = 1, ..., J, \tag{12}$$

where z_j are indicator variables.

CVaR Minimization with Finite Number of Scenarios

- By solving the LP problem we find the optimal decision vector, x^*, the optimal VaR, α^*, and consequently the optimal $CVaR$, $\widetilde{F}_\beta\left(x = x^*, \alpha = \alpha^*\right)$.
- Therefore, the optimization problem can be solved using algorithms that are capable of solving efficiently very large-scale problems with any distribution within reasonable time and high reliability as, for example, simplex or interior point methods.

The Worst Case CVaR

- Instead of assuming the precise distribution of the random vector r, Zhu and Fukushima, 2009 consider the case where the probability distribution π is only known to belong to a certain set, say $\mathcal{P}$, defined the worst-case CVaR (WCVaR) as the CVaR when the worst-case probability distribution in the set $\mathcal{P}$ occurs.

Definition 1

Given a confidence level β, $\beta \in (0, 1)$, the worst-case CVaR for fixed $w \in \mathcal{W}$ with respect to the uncertainty set $\mathcal{P}$ is defined as

$$\begin{aligned} WCVaR_\beta(\mathbf{w}) &\equiv \sup_{\pi \in \mathcal{P}} CVaR_\beta(\mathbf{w}) \\ &= \sup_{\pi \in \mathcal{P}} \min_{\alpha \in \mathbb{R}} F_\beta(\mathbf{w}, \alpha) \end{aligned} \tag{13}$$

- Zhu and Fukushima, 2009 further investigated the WCVaR risk measure with several structures of uncertainty in the underlying distribution. In particular, they consider the case where the distribution of the returns belong to a set of distributions consisting of all mixtures of some possible distribution scenarios, i.e.,

$$p(\cdot) \in \mathcal{P}_M \equiv \left\{ \sum_{i=1}^{d} \pi_i p^i(\cdot) : \sum_{i=1}^{d} \pi_i = 1,\ \pi_i \geq 0,\ i = 1, ..., d \right\}, \tag{14}$$

where $p^i(\cdot)$ denotes the j-th likelihood distribution and define

$$\begin{aligned} G_\beta(\mathbf{w}, \alpha, \pi) &= \alpha + \frac{1}{1-\beta} \int_{\mathbf{r} \in \mathbb{R}^n} (f(\mathbf{w}, \mathbf{r}) - \alpha)^+ \sum_{i=1}^{d} \pi_i p^i(\mathbf{r}) d\mathbf{r},\ j = 1, ..., J \\ &= \sum_{i=1}^{d} \pi_i G_\beta^i(\mathbf{w}, \alpha),\ i = 1, ..., d, \end{aligned} \tag{15}$$

where

$$G_\beta^i(\mathbf{w}, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\mathbf{r} \in \mathbb{R}^n} (f(\mathbf{w}, \mathbf{r}) - \alpha)^+ p^i(\mathbf{r}) d\mathbf{r},\ i = 1, ..., d. \tag{16}$$

The Worst Case CVaR

- Theorem 1 of Zhu and Fukushima, 2009 states that for each fixed $\mathbf{w} \in \mathcal{W}$,

$$\begin{aligned} WCVaR_\beta\left(\mathbf{w}\right) &= \min_{\alpha\in\mathbb{R}} \max_{\pi\in\Pi} G_\beta\left(\mathbf{w},\alpha,\pi\right) \\ &= \min_{\alpha\in\mathbb{R}} \max_{\pi\in\Pi} \sum_{i=1}^{d} \pi_j G_\beta^i\left(\mathbf{w},\alpha\right). \end{aligned} \tag{17}$$

- Thus, minimizing the worst-case CVaR over $w \in \mathcal{W}$ is equivalent to the following min-min-sup optimization problem:

$$\min_{\mathbf{w}\in\mathcal{W}} WCVaR_\beta\left(\mathbf{w}\right) = \min_{\mathbf{w}\in\mathcal{W}} \min_{\alpha\in\mathbb{R}} \sup_{\pi\in\Pi} G_\beta\left(\mathbf{w},\alpha,\pi\right). \tag{18}$$

- They also proved that WCVaR is a coherent risk measure and it clearly satisfies $WCVaR_\beta\left(\mathbf{w}\right) \geq CVaR_\beta\left(\mathbf{w}\right) \geq VaR_\beta\left(\mathbf{w}\right)$. Thus, $WCVaR_\beta\left(\mathbf{w}\right)$ can be effectively used as a risk measure.

Worst Case Copula-CVaR

- Kakouris and Rustem, 2014 extended their framework considering a set of copulas $C(\cdot) \in \mathcal{C}$.

- Let

$$C(\cdot) \in \mathcal{C}_M \equiv \left\{ \sum_{i=1}^{d} \pi_i C_i(\cdot) : \sum_{i=1}^{d} \pi_i = 1,\ \pi_i \geq 0,\ i = 1, ..., d \right\}, \qquad (19)$$

- Similarly to (15), for $i = 1, ..., d$

$$\begin{aligned} H_\beta(\mathbf{w}, \alpha, \pi) &= \alpha + \frac{1}{1-\beta} \int_{\mathbf{u} \in [0,1]^n} \left(\widetilde{h}(\mathbf{w}, \mathbf{u}) - \alpha \right)^+ \sum_{i=1}^{d} \pi_i c_i(\mathbf{u})\, d\mathbf{u} \\ &= \sum_{i=1}^{d} \pi_i H_\beta^i(\mathbf{w}, \alpha), \end{aligned}$$

$$H_\beta^i(\mathbf{w}, \alpha) = \alpha + \frac{1}{1-\beta} \int_{\mathbf{u} \in [0,1]^n} \left(\widetilde{h}(\mathbf{w}, \mathbf{u}) - \alpha \right)^+ c_i(\mathbf{u})\, d\mathbf{u}. \qquad (20)$$

Worst Case Copula-CVaR

- Theorem 1 of Zhu and Fukushima, 2009 states that for each fixed $\mathbf{w} \in \mathcal{W}$ the $WCVaR_\beta(\mathbf{w})$ with respect to the set $\mathcal{C}$ is represented by

$$
\begin{aligned}
WCVaR_\beta(\mathbf{w}) &= \min_{\alpha \in \mathbb{R}} \max_{\pi \in \Pi} H_\beta(\mathbf{w}, \alpha, \pi) \\
&= \min_{\alpha \in \mathbb{R}} \max_{\pi \in \Pi} \sum_{i=1}^{d} \pi_i H_\beta^i(\mathbf{w}, \alpha). \qquad (21)
\end{aligned}
$$

- Thus, the Worst Case Copula-CVaR with respect to $\mathcal{C}$ is the mixture copula that produces the greatest CVaR, i.e., the worst performing copula combination in the set $\mathcal{C}$.

- Minimizing the worst-case Copula-CVaR over $\mathbf{w} \in \mathcal{W}$ can be defined as the following optimization problem:

$$
\min_{\mathbf{w} \in \mathcal{W}} WCVaR_\beta(\mathbf{w}) = \min_{\mathbf{w} \in \mathcal{W}} \min_{\alpha \in \mathbb{R}} \sup_{\pi \in \Pi} H_\beta(\mathbf{w}, \alpha, \pi). \qquad (22)
$$

Worst Case Copula-CVaR

- Using the approach of Rockafellar and Uryasev, 2000 the integral in (20) is approximated by sampling realizations from the copulas $C_i(\cdot) \in \mathcal{C}$ using as inputs the filtered uniform margins.

- If the sampling generates a collection of values $\left(\mathbf{u}_i^{[1]}, \mathbf{u}_i^{[2]}, ..., \mathbf{u}_i^{[J]}\right)$, where $\mathbf{u}_i^{[j]}$ and S^i are the j-th sample drawn from copula $C_i(\cdot)$ of the mixture copula using as inputs the filtered uniform margins and its corresponding size, respectively, $i = 1, ..., d$, we can approximate $H_\beta^i(\mathbf{w}, \alpha)$ by

$$\widetilde{H}_\beta^i(\mathbf{w}, \alpha) = \alpha + \frac{1}{(1-\beta) S^i} \sum_{j=1}^{S^i} \left(\widetilde{h}(\mathbf{w}, \mathbf{u}_i^{[j]}) - \alpha\right)^+, i = 1, .., d \quad (23)$$

Optimization Problem

- Assuming that the allowable set $\mathcal{W}$ is convex and the loss function $\widetilde{h}\left(\mathbf{w},\mathbf{u}\right)$ is linear with respect to $\mathbf{w}$ then optimization problem

$$\min_{\mathbf{w}\in\mathcal{W},\alpha\in\mathbb{R}} \widetilde{H}_{\beta}\left(\mathbf{w},\alpha\right). \tag{24}$$

reduces to the following LP problem:

$$\min_{\mathbf{w}\in\mathbb{R}^{n},v\in\mathbb{R}^{m},\alpha\in\mathbb{R}} \alpha + \frac{1}{\left(1-\beta\right)S^{i}}\sum_{i=1}^{S^{i}} v_{i} \tag{25}$$

$$s.t.\ \mathbf{w}\in\mathcal{W}, \tag{26}$$

$$v_{i}^{[j]} \geq \widetilde{h}(\mathbf{w},\mathbf{u}_{i}^{[j]}) - \alpha,\ \ v_{i}^{[j]} \geq 0,\ \ j=1,...,J;\ \ i=1,...,d. \tag{27}$$

where v_i are auxiliary indicator (dummy) variables and $m=\sum_{i=1}^{d} S^{i}$.

- By solving the LP problem we find the optimal decision vector, $\mathbf{w}^{*}$, and at "one shot" the optimal VaR, α^{*}, and the optimal $CVaR$, $\widetilde{H}_{\beta}\left(\mathbf{w}=\mathbf{w}^{*},\alpha=\alpha^{*}\right)$.

Data

- Daily data of adjusted closing prices of all shares that belongs to the S&P500 market index from July 2nd, 1990 to December 31st, 2015. We obtain the adjusted closing prices from Bloomberg and the returns on the Fama and French factors from French's website.

- The data set sample period is made up of 6,426 days and includes a total of 1100 stocks over all periods. Only stocks that are listed during the formation period are included in the analysis, *i.e.*, around 500 stocks in each trading period.

Methodology

- We select, among all listed stocks in each formation period, a set of 50 stocks based on the ranked sum of absolute spreads (the 25 largest and the 25 smallest) between the normalized daily closing prices deviations (known as distance) of the S&P 500 index and all shares.
- We rebalance our portfolio every six months.
- Our optimization strategy adopt a sliding window of calibration of four years of daily data.
- We use day 1 to T, where T is the sliding window, to estimate the parameters of all models and determine portfolio weights for day T+1 and then repeat the process including the latest observation and removing the oldest until reaching the end of the time series.

1. We fit an AR(1)-GARCH(1,1) model with skew t-distributed innovations to each univariate time series selected from distance method.

2. Using the estimated parametric model, we construct the standardized residuals vectors given, for each $i = 1, ..., 50$ and $t = 1, ..., L - T - 1$, where L is the data set sample period, by

$$\frac{\widehat{\varepsilon}_{i,t}}{\widehat{\sigma}_{i,t}}.$$

   The standardized residuals vectors are then converted to pseudo-uniform observations $z_{i,t} = \frac{n}{n+1} F_i \left(\widehat{\varepsilon}_{i,t}\right)$, where F_i is their empirical distribution function.

3. Estimate the copula model, i.e., fits the multivariate Clayton-Frank-Gumbel (CFG) Mixture Copula to data that has been transformed to [0,1] margins by

$$C^{CFG}(\Theta,\mathbf{u}) = \pi_1 C^{C}(\theta_1,\mathbf{u}) + \pi_2 C^{F}(\theta_2,\mathbf{u}) + (1-\pi_1-\pi_2) C^{G}(\theta_3,\mathbf{u}) \quad (28)$$

where $\Theta = (\alpha,\beta,\delta)^{\top}$ are the Clayton, Frank and Gumbel copula parameters, respectively, and π_1, $\pi_2 \in [0,1]$. The estimates are obtained by the minimization of the negative log-likelihood consisting of the weighted densities of the Clayton, Frank and Gumbel copulas. Probability density function for multivariate Archimedean copula is computed as described in McNeil and Neshelova, 2009.

4. Use the dependence structure determined by the estimated copula for generating J scenarios. To simulate data from the three Archimedean copulas we use the sampling algorithms provided in Melchiori, 2006.

5. Compute t-quantiles for these Monte Carlo draws.

6. Compute the standard deviation $\widehat{\sigma}_{i,t}$ using the estimated GARCH model.

7. Determine the simulated daily asset log-returns, i.e., determine the simulated daily log-returns as $r_{i,t}^{sim} = \widehat{\mu}_t + \widehat{\sigma}_{i,t} z_{i,t}$.

8. Finally, use the simulated data as inputs when optimizing portfolio weights by minimizing CVaR for a given confidence level and a given minimum expected return.

- For each of the three copulas, we run 1000 return scenarios from the estimated multivariate CFG Mixture Copula model.
- The weights are recalibrated at a daily, weekly and monthly basis. We assume that the feasible set $\mathcal{W}$ attends the following linear constraints:
  - The sum of the weights should be 1, i.e., $\mathbf{w}^\top \mathbf{1} = 1$
  - No short-selling: $\mathbf{w} \geq 0$
  - The expected return should be bound below by an arbitrary value $\bar{r}$, i.e., $\mathbf{w}^\top \mathbb{E}(\mathbf{r}_{p,t+1}) \geq \bar{r}$, where $\bar{r}$ represents the target mean return.
- The confidence level β is set at $\beta = 0.95$.

Benchmarks

- Gaussian Copula Portfolio
- The equally-weighted naive portfolio or $1/N$ portfolio involves holding an equally-weighted portfolio $w_i = 1/N$, i.e. $\mathbf{w} = \left(\frac{1}{N}, ..., \frac{1}{N}\right)^\top$ for any rebalancing date t.
- We use the S&P 500 index as a proxy for market return.

Simulating from a Gaussian Copula

From Sklar's theorem the Gauss copula is given by

$$C_p\left(u_1, \ldots, u_d\right) = \mathbf{\Phi}_p\left(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)\right), \tag{29}$$

where Φ denotes the standard normal distribution function, and $\mathbf{\Phi}_p$ denotes the multivariate standard normal distribution function with correlation matrix P.

- Repeat the following steps n times.
    - Perform a Cholesky decomposition of P, and set A as the resulting lower triangular matrix.
    - Generate a vector $Z = (Z_1, \ldots, Z_d)'$ of independent standard normal variables.
    - Set $X = AZ$.
    - Return $U = (\Phi(X_1), \ldots, \Phi(X_d))'$.

Performance Measures

1. Portfolio mean excess return:

$$\widehat{\mu}_p = \frac{1}{L-T}\sum_{t=T+1}^{L-1}\widehat{r}_{p,t}, \tag{30}$$

$t = T+1, ..., L-1$, where

$$\widehat{r}_{p,t} = \mathbf{w}_{t-1}^{\top}\mathbf{r}_{p,t} - r_{f,t}, \tag{31}$$

2. Portfolio standard deviation

$$\widehat{\sigma}_p = \sqrt{\frac{1}{L-T-2}\sum_{t=T}^{L-1}\left(w_t r_{p,t+1} - \widehat{\mu}_p\right)^2}, \tag{32}$$

3. Sharpe Ratio

$$SR = \frac{\widehat{\mu}_p}{\widehat{\sigma}_p}. \tag{33}$$

4. The maximum drawdown on the interval $[0, T]$ is defined as

$$MaxDD\left(\mathbf{w}\right) = \max_{0 \leq \tau \leq t} \left\{D\left(\mathbf{w}, t\right)\right\}. \tag{34}$$

In other words, the maximum drawdown over a period is the maximum loss from worst peak to a trough of a portfolio drop from the start to the end of the period.

5 The Sortino's ratio (Sortino, 1994) is the ratio of the mean excess return to the standard deviation of negative asset returns, i.e.,

$$SoR = \frac{\widehat{\mu}_p}{\widehat{\sigma}_{p,n}}, \tag{35}$$

where

$$\widehat{\sigma}_{p,n} = \sqrt{\frac{1}{L-T-1} \sum_{t=T}^{L-1} \left(\min\left(0, \mathbf{w}_t^\top \mathbf{r}_{t+1} - \mathbf{r}_{MAR}\right)\right)^2}, \tag{36}$$

where $\mathbf{r}_{MAR}$ is the value of a minimal acceptable return (MAR), usually zero or the risk-free rate.

6. We define the portfolio turnover from time t to time $t+1$ as the sum of the absolute changes in the N risky asset weights, i.e., in the optimal values of the investment fractions:

$$Turnover = \frac{1}{L-T-1} \sum_{t=T}^{L-1} \sum_{j=1}^{N} \left(|w_{j,t+} - w_{j,t+1}| \right), \tag{37}$$

where $w_{j,t+}$ is the actual weight in asset j before rebalancing at time $t+1$, and $w_{j,t+1}$ is the optimal weight in asset j at time $t+1$.

- Turnover measures the amount of trading required to implement a particular portfolio strategy and can be interpreted as the average fraction of wealth traded in each period.
    - The higher the turnover, the higher the transaction cost that the portfolio incurs at each rebalancing day.

Performance Measures

7. Break-even transaction cost (Bessembinder and Chan, 1995) is the level of transaction costs leading to zero net profits, i.e., the maximum transaction cost that can be imposed before making the strategies less desirable than the buy-and-hold strategy. Following Santos and Tessari, 2012 we consider the average net returns on transaction costs, $\widehat{\mu}_{TC}$, given by

$$\widehat{\mu}_{TC} = \frac{1}{L-T}\sum_{t=T}^{L-1}\left[\left(1+\mathbf{w}_t^{\top}\mathbf{r}_{t+1}\right)\left(1-c\sum_{j=1}^{N}\left(|w_{j,t+1}-w_{j,t+}|\right)\right)-1\right], \tag{38}$$

where c is called breakeven cost when we solve $\widehat{\mu}_{TC} = 0$.

- Those portfolios that achieve a higher break-even cost are preferable, since the level required to make these portfolios non-profitable are higher.

9. Capital requirement loss function (CR)
   - Regulatory loss function to evaluate VaR forecasts

$$CR_t = \max\left[\left(\left(\frac{3+\delta}{60}\right)\sum_{i=0}^{59} VaR_{\beta,t-i}\right), VaR_{\beta,t}\right], \qquad (39)$$

where δ is a multiplicative factor that depends on the number of violations of the VaR in the previous 250 trading days.

   - In order to mitigate data-snooping problems we apply a test for superior predictive ability (SPA) proposed by Hansen, 2005 to determine which model significantly minimizes the expected loss function.

Table 1: Excess returns of Worst Case Copula-CVaR (WCCVaR), Gaussian Copula-CVaR (GCCVaR) and Equal Weights portfolios without a minimum expected return constraint

	WCCVaR	GCCVaR	1/N	S&P 500
Daily Rebalancing				
Mean Return (%)	11.29	10.59	10.52	6.61
Standard Deviation (%)	15.35	15.05	23.86	19.01
Sharpe Ratio	0.70	0.67	0.42	0.34
Sortino Ratio	1.14	1.09	0.68	0.54
Turnover	0.6868	0.6425	0.0001	
Break-even (%)	0.0618	0.0622	275.09	
MDD1 (%)	-13.20	-12.89	-18.00	-13.20
MDD2 (%)	-38.94	-41.66	-71.84	-52.61
$CVaR_{0.95}$	-0.0109	-0.0135		
$CR_{0.05}$ (%)	13.00***	15.63		
Weekly Rebalancing				
Mean Return (%)	10.57	10.04	10.52	6.61
Standard Deviation (%)	15.48	15.21	23.86	19.01
Sharpe Ratio	0.65	0.63	0.42	0.34
Sortino Ratio	1.06	1.02	0.68	0.54
Turnover	0.1535	0.1564	0.0001	
Break-even (%)	0.2596	0.2427	275.09	
MDD1 (%)	-12.59	-10.39	-18.00	-13.20
MDD2 (%)	-39.78	-41.32	-71.84	-52.61
$CVaR_{0.95}$	-0.0109	-0.0135		
$CR_{0.05}$ (%)	12.86***	15.57		

Empirical Results

Table 2: Excess returns of Worst Case Copula-CVaR (WCCVaR), Gaussian Copula-CVaR (GCCVaR) and Equal Weights portfolios without a daily minimum expected return constraint

	WCCVaR	**GCCVaR**	**1/N**	**S&P 500**
Monthly Rebalancing				
Mean Return (%)	9.22	9.42	10.52	6.61
Standard Deviation (%)	15.69	15.55	23.86	19.01
Sharpe Ratio	0.56	0.58	0.42	0.34
Sortino Ratio	0.91	0.94	0.68	0.54
Turnover	0.0447	0.0499	0.0001	
Break-even (%)	0.7829	0.7155	275.09	
MDD1 (%)	-13.21	-13.22	-18.00	-13.20
MDD2 (%)	-38.58	-46.13	-71.84	-52.61
$CVaR_{0.95}$	-0.0109	-0.0134		
$CR_{0.05}$ (%)	12.22***	14.67		

Note: Out-of-sample performance statistics between July 1994 and December 2015 (5414 observations). The rows labeled MDD1 and MDD2 compute the largest drawdown in terms of maximum percentage drop between two consecutive days and between two days within a period of maximum six months, respectively. Returns, standard deviation, Sharpe ratio and Sortino ratio are annualized. *** , **, * significant at 1%, 5% and 10% levels, respectively.

Cumulative Excess Returns

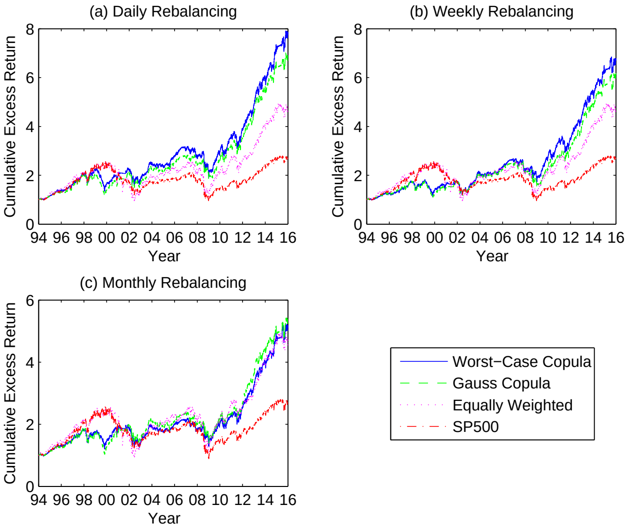

Figure 1: Cumulative excess returns of the portfolio strategies without daily target mean

Conclusions

- By selecting a diversified set of assets over a long-term period we found that copula-based approaches offer better hedges against losses than the 1/N portfolio.
- The WCCVaR approach generates portfolios with better downside risk statistics for any rebalancing period and it is more profitable than the Gaussian Copula-CVaR for daily and weekly rebalancing.

Highlights

- Worst-Case Copula Conditional Value-at-Risk Optimization using Linear Programming
- Risk management with high-dimensional copulas without pair-copula constructions
- We select a diversified set of assets that can be useful during any market conditions
- We compare the performance of Gaussian, Worst-Case and Equally Weighted portfolios
- Copula-based models show a superior performance after the subprime mortage crisis

Thanks! Any questions?