Math 536 Assignment 1

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1

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal distribution. The variance of each Y_i is σ^2, the same variance for all $i, i = 1, \ldots, n$. But the mean μ varies and is given by $\mu_i = \alpha + \beta X_i$. The X_i is the value of another variable that is given by the experiment.
Thus, $Y_i \sim N(\mu_i = \alpha + \beta X_i, \sigma^2)$. The normal model for each Y_i is then

$$f(Y_i, \alpha, \beta, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{1}{2\sigma^2}(Y_i - \alpha - \beta X_i)^2)$$

Let us assume that σ^2 is known.

A

Write the likelihood function

$$L(\alpha, \beta | X, Y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} exp(\frac{-1}{2\sigma^2}(Y_i - \alpha - \beta X_i)^2) = (2\pi\sigma^2)^{-n/2} \prod_{i=1}^{n} exp(\frac{-1}{2\sigma^2}(Y_i - \alpha - \beta X_i)^2)$$

B

Write the log-likelihood function.

$$l(\alpha, \beta | X, Y) = \frac{-n}{2} log(2\pi\sigma^2) - \sum_{i=1}^{n} (\frac{-1}{2\sigma^2}(Y_i - \alpha - \beta X_i)^2)$$

C

Find the maximum likelihood estimators of α and β.

For α, we have the following:

$$\begin{aligned}
\frac{\partial l}{\partial \alpha} &= \sum_{i=1}^{n} \frac{2(Y_i - \alpha - BX_i)}{2\sigma^2} = \sum_{i=1}^{n} \frac{Y_i - \alpha - \beta X_i}{\sigma^2} = 0 \\
&\longrightarrow \sum_{i=1}^{n} Y_i - n\alpha - \beta \sum_{i=1}^{n} X_i = 0 \\
&\longrightarrow n\bar{Y} - n\alpha - n\beta\bar{X} = 0 \\
&\longrightarrow \bar{Y} - \alpha - \beta\bar{X} = 0 \\
&\longrightarrow \hat{\alpha} = \bar{Y} - \beta\bar{X}
\end{aligned}$$

For β, we have the following

$$
\begin{aligned}
\frac{\partial l}{\partial \beta} &= \sum_{i=1}^{n} \frac{2(Y_i - \alpha - BX_i)(-X_i)}{2\sigma^2} \\
&\longrightarrow \sum_{i=1}^{n} Y_i X_i - \alpha \sum_{i=1}^{n} X_i - \beta(\sum_{i=1}^{n} X_i)^2 = 0 \\
&\longrightarrow \bar{Y}\bar{X} - \alpha\bar{X} - \beta\bar{X}^2 = 0 \\
&\longrightarrow \hat{\beta} = \frac{\overline{YX} - \alpha\bar{X}}{\bar{X}^2}
\end{aligned}
$$

We should note that the second derivative sign test should be performed in order to ensure that these estimates are indeed maximum values.

2

Suppose that a certain large population contains k different types of individuals ($k \geq 2$) and θ_i denote the proportions of individuals of type i, for $i = 1, 2, \ldots, k$. Here $0 \leq \theta \leq 1$, and $\theta_1 + \cdots + \theta_k = 1$. Suppose also that in a random sample of n individuals from this population, exactly n_i are type i, where $n_1 + \cdots + n_k = n$. Find MLE's of $\theta_1, \ldots, \theta_k$.

Our data follows a distribution with pdf:

$$f(x_i|\theta) = \theta_i^{n_i}$$

Here is the Likelihood function for our Multinomial Distribution:

$$L(\theta_1, \ldots, \theta_k | n_1, \ldots, n_k) = \prod_{i=1}^{k} \theta_i^{n_i}$$

Since, we know that $\theta_k = 1 - \theta_1 - \cdots - \theta_{k-1}$, The log-likelihood function is as follows:

$$l(\theta_1, \ldots, \theta_k | n_1, \ldots, n_k) = \sum_{i=1}^{k} n_i log(\theta_i) = \sum_{i=1}^{k-1} n_i log(\theta_i) + n_k log(1 - \sum_{i=1}^{k-1} \theta_i)$$

Thus, we have the following partial derivative with respect to θ_i:

$$\frac{\partial l}{\partial \theta_i} = \frac{n_i}{\theta_i} - \frac{n_k}{1 - \sum_{i=1}^{k-1} \theta_i} = 0$$

From this equality, we can obtain the following result:

$$
\begin{aligned}
\frac{n_i}{\theta_i} &= \frac{n_k}{1-\sum_{i=1}^{k-1}\theta_i} \\
\longrightarrow \frac{n_i}{\theta_i} &= \frac{n_k}{\theta_k} \\
\longrightarrow \hat{\theta}_i &= n_i\frac{\theta_k}{n_k} = n_i * K
\end{aligned}
$$

where K is a constant. Now, we know that $\sum_{i=1}^{n}\theta = 1$, so then $\sum_{i=1}^{n}\hat{\theta} = 1$. Thus, we have

$$\sum_{i=1}^{n} n_i K = 1 \rightarrow Kn = 1 \rightarrow k = \frac{1}{n} \rightarrow \hat{\theta}_i = \frac{n_i}{n} \text{ for } i = 1, \cdots, k$$

3

Note: All relevant code has been included in an apprendix at the end of the document.

A

Calculate the proportion of females in each of the 16 groups of progeny.

Here is a table that displays the proportions of females in each of the 16 groups of progeny:

females	males	total	proportion
18	11	29	0.6206897
31	22	53	0.5849057
34	27	61	0.5573770
33	29	62	0.5322581
27	24	51	0.5294118
33	29	62	0.5322581
28	25	53	0.5283019
23	26	49	0.4693878
33	38	71	0.4647887
12	14	26	0.4615385
19	23	42	0.4523810
25	31	56	0.4464286
14	20	34	0.4117647
4	6	10	0.4000000
22	34	56	0.3928571
7	12	19	0.3684211

B

Let Y_i denote the number of feamles and n_i the number of progeny in each group ($i = 1, \ldots, 16$). Suppose the $Y_i's$ are independent random variables with the Binomial distribution

$$f(y_i, \theta) = \binom{n_i}{y_i}\theta^{y_i}(1-\theta)^{n_i - y_i}$$

Find the maximum likelihood estimator of θ using calculus and evaluate it for these data.
Here, we have the Likelihood function defined as

$$L(\theta|y_i) \sim \theta^{\sum_{i=1}^n y_i}(1-\theta)^{\sum_{i=1}^n n_i - y_i}$$

and our log-likelihood follows as

$$l = \sum_{i=1}^n y_i log(\theta) + (\sum_{i=1}^n (n_i - y_i))log(1-\theta)$$

Taking our respective derivative, we have the following results:

$$\frac{\partial l}{\partial \theta} = \frac{\sum_{i=1}^n y_i}{\theta} - \frac{\sum_{i=1}^n n_i - y_i}{1-\theta} = 0$$

$$\rightarrow \frac{\sum_{i=1}^n n_i - y_i}{1-\theta} = \frac{\sum_{i=1}^n y_i}{\theta}$$

$$\rightarrow \theta \sum_{i=1}^n n_i - \theta \sum_{i=1}^n y_i = \sum_{i=1}^n y_i - \theta \sum_{i=1}^n y_i$$

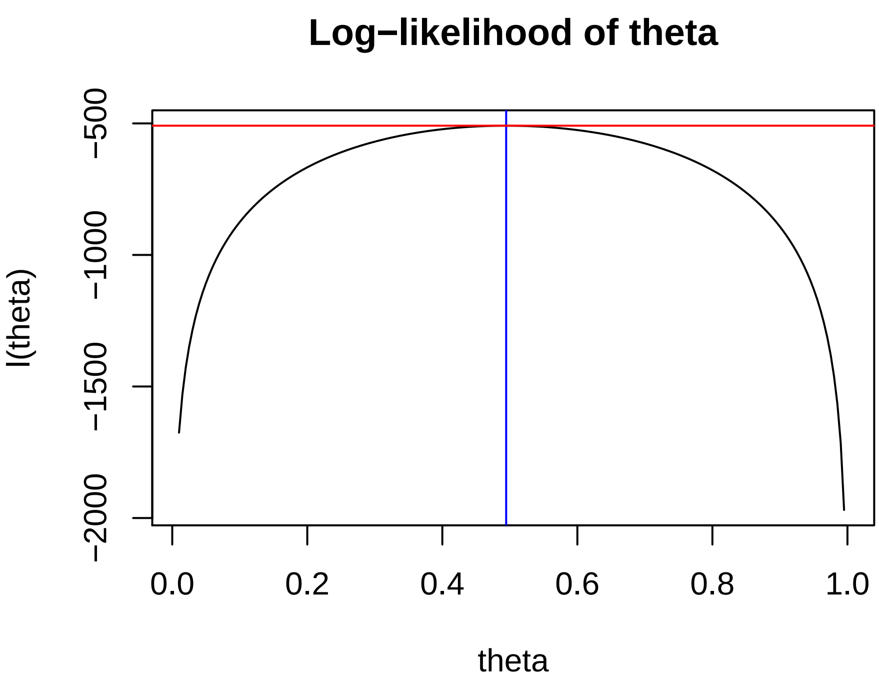
$$\rightarrow \theta \sum_{i=1}^n n_i = \sum_{i=1}^n y_i \rightarrow \hat{\theta} = \frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n n_i}$$

Thus, for this example, we have the following MLE estimate:

```
## MLE Estimate of theta =  0.4945504
```

C

Here, we will plot our log-likelihood function from B and estimate where the maximum value resides. Here is our result:



By adding a vertical line corresponding to the MLE estimate found in part B, we can see that the graphical estimate is essentially the same as the caluclus estimate.

4

Suppose that X_1 and X_2 are independent random variables, that X_1 has the binomial distribution with parameters n_1 and p, and X_2 is the binomial distribution with parameters n_2 and p. Prove that the conditional distribution of X_1 given $X_1 + X_2 = k$ is hypergeometric eith parameters n_1, n_2, and k.

Our conditional distribution can be defined as

$$p(x_1 = x|x_1 + x_2 = k) = \frac{p(x_1 = x, x_1 + x_2 = k)}{p(x_1 + x_2 = k)}$$

Provided our binomial distributions, we have the following results:

$$p(x_1 = x, x_1 + x_2 = k) = p(x_1 = x, x_2 = k - x) = \binom{n_1}{x} p^x (1-p)^{n_1 - x} \binom{n_2}{k-x} p^{k-x} (1-p)^{n_2 - k + x}$$

and

$$p(x_1 + x_2 = k) = \binom{n_1 + n_2}{k} p^k (1-p)^{n_1 + n_2 - k}$$

Using the definition of the conditional distribution, we have the following result

$$\frac{\binom{n_1}{x}\binom{n_2}{k-x}}{\binom{n_1+n_2}{k}} \frac{p^k(1-p)^{n_1+n_2-k}}{p^k(1-p)^{n_1+n_2-k}} = \frac{\binom{n_1}{x}\binom{n_2}{k-x}}{\binom{n_1+n_2}{k}} \sim hypergeo(n_1, n_2, k)$$

5

Suppose that X_1 and X_2 are independent random variables, that X_i has the poisson distribution with mean λ_i. Determine the conditional distribution of X_1 given $X_1 + X_2 = k$.

Again, using our conditional distribution definition we have the following results

$$p(x_1 = x, x_1 + x_2 = k) = p(x_1 = x, x_2 = k - x) = \frac{\lambda_1^x e^{-\lambda_1}}{x!} \frac{\lambda_2^{k-x} e^{-\lambda_2}}{(k-x)!}$$

and

$$p(x_1 + x_2 = k) \frac{(\lambda_1 + \lambda_2)^k e^{-(\lambda_1 + \lambda_2)}}{k!}$$

Using our conditional definition we have

$$\begin{aligned}
&p(x_1 = x|x_1 + x_2 = k) \\
&= \frac{k!}{(k-x)!x!} \frac{(\lambda_1)^x (\lambda_2)^{k-x}}{(\lambda_1 + \lambda_2)^x (\lambda_1 + \lambda_2)^{k-x}} \\
&= \frac{k!}{(k-x)!x!} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{k-x} \\
&\sim binom\left(k, \frac{\lambda_1}{\lambda_1 + \lambda_2}\right)
\end{aligned}$$

6

Let $\lambda_1, \lambda_2, \ldots$ be an increasing sequence with $\lambda_n \to \infty$ and let X_n be a sequence of Poisson random variables with the corresponding parameters. We know that $E(X_n) = V(X_n) = \lambda_n$. Let

$$Z_n = \frac{X_n - \lambda_n}{\sqrt{\lambda_n}}$$

We then have $E(Z_n) = 0$ and $V(Z_n) = 1$. Show that the mgf of Z_n converges to the mgf of the standard normal distribution.

We know the MGF is defined as follows:

$$M_x(t) = E(e^{tx})$$

We also know that the MGF for the Poisson distribution is as follows:

$$e^{\lambda_n(e^t - 1)}$$

Then, given Z_n, we have the following result:

$$\begin{aligned}
&M_{(x_n-\lambda_n)/\sqrt{\lambda_n}}(t) = E(e^{t*(x_n-\lambda_n)/\sqrt{\lambda_n}})\\
&= exp(-t\sqrt{\lambda_n})E\left[exp\Big(\frac{tx_n}{\sqrt{\lambda_n}}\Big)\right]\\
&= exp(-t\sqrt{\lambda_n})M_{x_n}\Big(\frac{t}{\lambda_n}\Big)\\
&= exp(-t\sqrt{\lambda_n})exp(\lambda_n(e^{t(\lambda_n)} - 1))
\end{aligned}$$

Using series expansion properties, our MGF is equivalent to the following

$$exp\Bigg(-t\sqrt{\lambda_n} + \lambda_n\Big(t(\lambda_n)^{-1/2} + t^2(\lambda_n)^{-1}/2 + t^3(\lambda_n)^{-3/2}/6 + \cdots\Big)\Bigg) = exp(\frac{t^2}{2} + \frac{t^3\lambda_n^{-1/2}}{6} + \cdots)$$

Taking the desired limit, we have the final result of

$$lim_{\lambda_n\to\infty} exp(\frac{t^2}{2} + \frac{t^3\lambda_n^{-1/2}}{6} + \cdots) = exp\Big(\frac{t^2}{2}\Big)$$

Which is the Moment Generating Function of the Standard Normal Distribution, as desired.

Appendix

3

A

Calculate the proportion of females in each of the 16 groups of progeny.

```
females = c(18,31,34,33,27,33,28,23,33,12,19,25,14,4,22,7)
males = c(11,22,27,29,24,29,25,26,38,14,23,31,20,6,34,12)
total = females + males

proportion = females/total
table = cbind(females,males,total,proportion)
```

Here is a table that displays the proportions of females in each of the 16 groups of progeny:

```
kable(table)
```

B

```
estimate = sum(females)/sum(total)
```

```
cat('MLE Estimate of theta = ',estimate,'' )
```

C

Here, we will plot our log-likelihood function from B and estimate where the maximum value resides. Here is our result:

```
loglike = function(theta){

  l = sum(females)*log(theta) + sum(males)*log(1-theta)
  return(l)
}
```

```
th=seq(0.01,1, length=200)

#Likelihood Graph
plot(th, sapply(X=th, FUN=function(th) loglike(th)),
type="l",xlab = "theta",ylab="l(theta)",main="Log-likelihood of theta")
abline(v=estimate,col="blue")
abline(h=loglike(estimate),col="red")
```