Entire Function

Next, we will discuss:

i) The zeros of entire function

ii) How zeros determine an entire function.

(1) Jensen's Formula:

Thm. $D(0,R) \subseteq \Omega \underset{open}{\subseteq} \mathbb{C}$. $f \in \mathcal{O}(\Omega)$. $f(0) \neq 0$.
$f(z) \neq 0$. $\forall z \in \partial D(0,R)$ If $\{z_k\}_1^n$ seq of zeros inside $D(0,R)$. Then $\log|f(0)|$

$$= \sum_1^n \log \frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta.$$

Pf: 1°) Note that $g(z) = f(z) / \prod_1^n (z-z_k)$

Redefine g at $\{z_k\}_1^n$ by series.

Then $g(z) \in \mathcal{O}(\Omega)$, $g(z) \neq 0$ in $\overline{D(0,R)}$

2°) For $g(z)$: $\exists\, h(z) \in \mathcal{O}(\Omega)$. s.t.

$g(z) = e^{h(z)}$. $\therefore |g(Re^{i\theta})| = e^{\mathrm{Re}\,h(Re^{i\theta})}$

By Mean value of harmonic $\mathrm{Re}\,h(z)$.

$\therefore \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta = \log|g(0)|$

3°) For $z - z_k$:

Prove: $\log|z_k| = \log\frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - z_k|\,d\theta$

Note $|Re^{i\theta} - z_k| = |R - z_k e^{-i\theta}| \neq 0$.

Similar method of 2°), we have

mean value of $R - z_k e^{-i\theta}$

4') Note that $\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta + \frac{1}{2\pi}\sum_1^N \int_0^{2\pi} \log|Re^{i\theta}-z_k|\,d\theta$$

Remark: From Jensen Formula, we can connect the growth of holomorphic $f(z)$ with its zeros number.

Def: $n(r)$ is the number of zeros of $f(z)$, which are inside $D(0,r)$.

Cor. $$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|.$$

Pf: Lemma $\int_0^R n(r)\frac{dr}{r} = \sum_1^N \log\left|\frac{R}{z_k}\right|$

Note that $n(r) = \sum_1^N \chi_{\{|z_k|<r\}}$

(2) Finite Orders:

$f \in \mathcal{O}(\mathbb{C})$. If there exists $\rho, A, B > 0$, s.t. $|f(z)| \le A e^{B|z|^\rho}$. Then we say the order of $f \le \rho$. Def $\rho_f = \inf \rho$

Thm. For $f \in \mathcal{O}(\mathbb{C})$, $\rho_f \triangleq \rho$.

i) $n(r) \le C r^\rho$, for some $C > 0$, r is large enough

ii) $\{z_k\}_1^\infty$ seq of zeros of $f(z)$, $z_k \ne 0$, $\forall k \in \mathbb{Z}^+$.

Then $\forall s > \rho$, we have: $\sum \frac{1}{|z_k|^s} < \infty$

Remark: The number of zeros is restricted by the order of entire function.

Pf: i) For applying Jensen Formula:

Consider $F(z) = f(z)/z^l$. l is multiple of zero $z=0$

$\therefore n_F(r) = n_f(r) - l$. $\rho_F = \rho_f$.

Note that $\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} |F(Re^{i\theta})| - \log|f(0)|$

We need $n(r)$ jump out of integration in LHS.

By monotone of $n(r)$. $\int_0^R n(r)\frac{dr}{r} \geq \int_{\frac{R}{2}}^R n(\frac{R}{2})\frac{dr}{r}$.

ii) From i):

$$\sum_{|z_k|\geq 1} \frac{1}{|z_k|^s} = \sum_k \sum_{2^i\leq |z_k| \leq 2^{i+1}} |z_k|^{-s} \leq \sum n(2^{i+1}) 2^{-si}$$

$$\leq \sum 2^{\rho(i+1)} \cdot 2^{-si} < \infty$$

(3) Infinite Products:

Lemma. $\{F_n\} \subseteq \mathcal{O}(\Omega)$. $\Omega \underset{open}{\subseteq} \mathbb{C}$. If $\exists \{c_n\} \subseteq \mathbb{R}^+$.

s.t. $|F_n - 1| < c_n$. $\sum c_n < \infty$. Then

i) $\prod_1^n F_k(z) \xrightarrow{u} F(z) \in \mathcal{O}(\Omega)$

ii) If $F_n(z) \neq 0$. $\forall n$. Then $\frac{F'(z)}{F(z)} = \sum_1^\infty \frac{F_k'(z)}{F_k(z)}$

Pf: i) $c_n \to 0$ $\therefore F_n(z) \neq 0$. When n is large enough.

$$\prod_N^M F_n(z) = e^{\sum_N^M \ln(1+F_n(z)-1)} \leq e^{\sum_N^M c_n}$$

$\therefore \prod F_n(z)$ converges.

ii) Note that $\sum_1^N \frac{F_n'(z)}{F_n(z)} = \frac{(\prod_1^N F_n)'}{\prod_1^N F_n}$

By i). we're done.

e.g. $F(z) = \pi \cot \pi z = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}$

Pf: Ideal: By Liouville Thm.

prove: $\Delta(z) = \pi\cot\pi z - \sum_{n\in\mathbb{Z}} \frac{1}{n+z}$ is bounded. entire.

1°) Obeservation:

i) $\Delta(z+1) = \Delta(z)$. when $z \notin \mathbb{Z}$.

ii) $F(z) = \frac{1}{z} + F_0(z)$. $F_0(z)$ is holomorphic near $z=0$.

iii) $F(z)$ has only simple isolated poles.

2°) $\Delta(z)$ is entire.

Since $z=0$ is removable by observation.

Use periodicity of $\Delta(z)$ $\therefore$ $z=k\in\mathbb{Z}$. are removable.

3°) $\Delta(z)$ is bounded.

using periodicity. prove $\Delta(z)$ is bounded

in $z \in \{|Re(z)| \le \frac{1}{2}\}$.

Condition on $|Im(z)| \le 1$ or $|Im(z)| > 1$.

Remark: Derive: $\frac{\sin \pi z}{\pi} = z \prod_{n\in\mathbb{Z}^+} (1 - \frac{z^2}{n^2})$.

Note that $\left(\frac{\sin\pi z}{\pi} \Big/ z\prod(1-\frac{z^2}{n^2}) \right)'$

$$= \frac{\sin\pi z}{\pi} \frac{\pi\cot\pi z - \sum \frac{1}{z+n}}{z\prod(1-\frac{z^2}{n^2})} = 0.$$

7

Thm. (Weierstrass Infinite Product)

$\{a_n\} \subseteq \mathbb{C}$. $|a_n| \to \infty$ $(n \to \infty)$ Then exists an entire function $f(z)$. s.t. $f(a_n) = 0$. $\forall n$.

$f(z) \neq 0$. when $z \neq a_n$.

Moreover, for any other one satisfies it has form: $f(z) e^{g(z)}$. $g(z) \in \mathcal{O}(\mathbb{C})$

Pf: 1°) Canonical functors:

$E_k(z) = (1-z) e^{\sum_1^k \frac{z^j}{j}}$. $E_0(z) = 1 - z$

Lemma. For $|z| < \frac{1}{2}$. $|1 - E_k| \leq c|z|^{k+1}$

Pf. By $|1 - e^w| \leq |w||e^w| \leq C|w|$.

2°) The ideal:

Insert $\prod(1 - \frac{z}{a_k})$ into $\{E_k(z)\}$.

Check $f(z) = z^n \prod_{k=1}^{\infty} E_k(\frac{z}{a_k})$ converges.

Besides. if f_1, f_2 satisfies the condition.

Then $f_1/f_2 \in \mathcal{O}(\mathbb{C})$. nonvanishes $\therefore \frac{f_1}{f_2} = e^{g(z)}$

Remark: We have a more general Thm:

Thm. (Hadamard)

For $k \leq \rho_f < k+1$. $\{a_n\}$ is seq of zeros of an entire function f. Then $f(z) = e^{p(z)} z^m \prod_1^{\infty} E_k(\frac{z}{a_n})$

m is order of zero $z=0$. $p(z)$ is a polynomial with degree $\leq k$.