Tangent Spaces

(1) Tangent vectors via curves:

① Tangent space:

i) Suppose $\tilde{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $\tilde{x} \in \tilde{U}$. A curve through $\tilde{x}$ is:

$\sigma = (\sigma_1, \sigma_2, \cdots \sigma_n) : (-\varepsilon, \varepsilon) \longrightarrow \tilde{U}$. $\sigma(0) = \tilde{x}$. $\sigma \in C^\infty$.

Derivate of σ at 0 is: $D\sigma|_0 : \mathbb{R} \to \mathbb{R}^n$.

$D\sigma|_0 = (\sigma_1'(0), \cdots \sigma_n'(0))$ means "velocity" of σ when it passes through $\tilde{x}$ (Then $(-\varepsilon, \varepsilon)$ means time interval).

Note that: σ and τ are tangent at $\tilde{x}$

$\Longleftrightarrow D\sigma|_0 = D\tau|_0$

Defines a map: $\Delta : \{\text{curve through } \tilde{x}\} / (\text{tangent at } \tilde{x}) \longrightarrow \mathbb{R}^n$

$[\sigma] \longmapsto D\sigma|_0$

Δ is also bijection:

$\forall \vec{v} \in \mathbb{R}^n$. Let $\sigma_v : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$

$t \longmapsto \tilde{x} + \vec{v}t$

$\therefore \Delta([\sigma_v]) = \vec{v}$. (injection by def)

ii) For X is n-dimension smooth manifold. $x \in X$.

Def: A curve through x is a smooth Func:

$$\sigma : (-\varepsilon, \varepsilon) \longrightarrow X. \ \sigma(0) = x.$$

As usual, we should see σ in chart:

For $(U, f) \in A_x$. $x \in U$. $\tilde{\sigma} = f \circ \sigma$. $D\tilde{\sigma}|_0 \in \mathbb{R}^n$.

However, for (U_1, f_1). $(U_2, f_2) \in A_x$. different charts.

$$\tilde{\sigma}_2 = \phi_{21} \circ \tilde{\sigma}_1. \quad \begin{cases} \tilde{\sigma}_1 = f_1 \circ \sigma \\ \tilde{\sigma}_2 = f_2 \circ \sigma \end{cases} \Rightarrow D\tilde{\sigma}_2|_0 = D\phi_{21}|_{f_1(x)} \cdot D\tilde{\sigma}_1|_0$$

$\therefore$ It depends on the choice of chart.

Remark: It make no sense to ask the tangent vector of σ.

From the relation: for curves σ, τ through x.

$$D\tilde{\sigma}_1|_0 = D\tilde{\tau}_1|_0 \Leftrightarrow D\tilde{\sigma}_2|_0 = D\tilde{\tau}_2|_0$$

Def: σ, τ two curves through x tangent at x

if $\exists (U_x, f) \in A_x$. s.t. $D(f \circ \sigma)|_0 = D(f \circ \tau)|_0$.

Remark: It's equivalent relation.

Def: Denote the set of all tangent vectors to x

by $T_x X = \{\text{curves through } x\} / (\text{tangency at } x)$.

call it tangent space to X at x.

If fix $(U, f) \in A_x$ through x. See in chart:

$$A_f : T_x X \longrightarrow \mathbb{R}^n \qquad A_f \text{ is well-def injection.}$$
$$[\sigma] \longmapsto D(f\circ\sigma)|_0$$

Besides. A_f is bijection:

$\forall v \in \mathbb{R}^n$. $\tilde{\sigma}_v : (-\varepsilon, \varepsilon) \longrightarrow \tilde{U}$ let $\sigma_v = f^{-1} \circ \tilde{\sigma}_v$

$$t \longmapsto f(x) + \vec{v} \cdot t$$

$\therefore A_f(\sigma_v) = v$. $\quad \therefore A_f : T_x X \xrightarrow{\sim} \mathbb{R}^n$.

<u>Remark</u>: For different charts (U_1, f_1), (U_2, f_2).

by chain rule: $A_{f_1} = D\phi_{12}|_{f_2(x)} \cdot A_{f_2}$

<u>Prop</u>. For n-dimension manifold X. The tangent space $T_x X$ is n-dimension vector space.

<u>Pf</u>. Pick $(U_x, f) \in A_x$ through x.

Define: $\begin{cases} [\sigma] + [\tau] = A_f^{-1}(A_f([\sigma]) + A_f([\tau])) \\ \lambda[\sigma] = A_f^{-1}(\lambda A_f([\sigma])) \end{cases}$

Check it's indept with choice of charts

by the relation: $A_{f_1} = D\phi_{12}|_{f_2(x)} \cdot A_{f_2}$.

② <u>Derivates</u>:

i) For $\tilde{U} \subseteq \mathbb{R}^n$, $\tilde{V} \subseteq \mathbb{R}^m$. $F : \tilde{U} \longrightarrow \tilde{V}$.

Pick $\tilde{x} \in \tilde{U}$. $\tilde{\eta} = F(\tilde{x}) \in \tilde{V}$.

If curves σ through $\tilde{x}$. Then $F\circ\sigma$ through $\tilde{\eta}$.

Besides. $D(F\circ\sigma)|_0 = DF|_{\tilde{x}} \cdot D\sigma|_0$

Therefore,

$$\{\text{Curves through } \tilde{x}\}/_{(\text{tangent at } \tilde{x})} \longrightarrow \{\text{Curves through } \tilde{\eta}\}/_{(\text{tangent at } \tilde{\eta})}$$

$$[\sigma] \longmapsto [F\circ\sigma]$$

is a well-def map.

ii) Genelize for manifolds X, Y. $\dim X = n$. $\dim Y = m$.

$F: X \longrightarrow Y$. Smooth

<u>prop.</u> For $x \in X$. $\eta = F(x)$. Then we have a well-def map: $DF|_x : T_xX \longrightarrow T_\eta Y$ $\quad DF|_x$ is linear.

$$[\sigma] \longmapsto [F\circ\sigma]$$

<u>Pf:</u> See in the charts:

$(U, f) \in A_X$. $(V, g) \in A_Y$. $x \in U$. $\eta \in V$.

1°) Denote $\tilde{F} = g\circ F\circ f^{-1}$. $\tilde{\sigma} = f\circ\sigma$.

$$\therefore \Delta_g(F\circ\sigma) = D(g\circ F\circ\sigma)|_0 = D(\tilde{F}\circ\tilde{\sigma})|_0$$

$$= D\tilde{F}|_{\tilde{x}}\cdot D\tilde{\sigma}|_0 = D\tilde{F}|_{\tilde{x}}\cdot\Delta_f(\sigma)$$

which is indept with choice of $\sigma \in [\sigma]$.

2°)
$$\begin{tikzcd} T_xX \arrow[r, "DF|_x"] \arrow[d, "\Delta_f"'] & T_\eta Y \arrow[d, "\Delta_g"] \\ \mathbb{R}^n \arrow[r, "D\tilde{F}|_{\tilde{x}}"] & \mathbb{R}^m \end{tikzcd}$$

$\therefore DF|_x = \Delta_g^{-1}\circ D\tilde{F}|_{\tilde{x}}\circ\Delta_f$

composition of LF's

$\therefore DF|_x$ is linear.

Thm. (IFT for manifolds)

$f: M \to N$. C^k map from C^k-manifolds M, N.

$\dim M = m \geq \dim N = n$. For $p \in M$.

If $Df|_p: T_pM \to T_{f(p)}N$ is surjection.

Then there $\exists$ $(g, V) \in A_Y$. $f(p) \in V$. $\exists$ U_p open

nbd of p. with $h: U_p \xrightarrow{\sim} \tilde{U}_p \subseteq \mathbb{R}^m$. C^k-

diffeomorphism. s.t. $g \circ f \circ h^{-1} = \pi: \tilde{U}_p \to \mathbb{R}^n$.

Pf: See in chart. Apply IFT.

(2) Tangent Spaces to submanifolds:

① For Z is submanifold of $\mathbb{R}^n$.

Since $\iota: Z \hookrightarrow \mathbb{R}^n$. For $z \in Z$.

$\therefore$ $D\iota|_z: T_zZ \to T_z\mathbb{R}^n$

$[\sigma] \mapsto [\iota\circ\sigma]$

Since ι is immersion. $\therefore$ $D\iota|_z$ is injection.

$\therefore$ T_zZ is subspace of $T_z\mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$.

② For Z is submanifold of n-dim manifold X.

Similarly: $D\iota|_z: T_zZ \hookrightarrow T_zX$.

We can view T_zZ as subspace of T_zX.

(Actually. $T_zZ \xrightarrow{\sim} \mathbb{R}^m$. if $\dim Z = m$)

Lemma. $F: X \to Y$. smooth. $\eta \in Y$ is regular value of F. $Z = F^{-1}\{\eta\}$. For $z \in Z$:

$T_z Z$ is kernel of: $DF|_z : T_z X \to T_\eta Y$.

Pf: Denote: $\dim X = n$. $\dim Y = k$. For $z \in Z$.

Apply IFT: $\exists (U_z, f) \in A_X$. $(V, g) \in A_Y$. $(\eta \in V$. $g(\eta) = 0)$

s.t. $\tilde{F} = g \circ F \circ f^{-1} = \pi : \tilde{U}_z \to \tilde{V}$. $\pi : \mathbb{R}^n \xrightarrow{\text{proj.}} \mathbb{R}^k$.

$\therefore D\tilde{F}|_{f(z)} = \pi : \mathbb{R}^n \to \mathbb{R}^k$.

Kernel is $T_z Z \subseteq T_x X$ in chart.

e.g. $h: \mathbb{R}^{n+1} \to \mathbb{R}^1$ $\quad h(\vec{x}) = 1$ is regular value.

$\vec{x} \mapsto \sum_0^n x_i^2$

$Dh = (2x_0, 2x_1, \cdots 2x_n)$. $\ker(Dh) = \{v \mid Dh \cdot v = 0\} = \{\vec{v} \mid \vec{x} \cdot \vec{v} = 0\}$

(3) Second Definition:

We can define the tangent vectors by translation law: $\Delta f_2(\sigma) = D\phi_{21}|_{f_1(x)} \cdot \Delta f_1(\sigma)$

Def: (From Physicist's)

For $x \in X$. n-dim manifold. A_x^X is all charts containing x. A tangent vector to x is Func:

$\delta : A_x^X \to \mathbb{R}^n$. s.t. $\delta f_2 = D\phi_{21}|_{f_1(x)} \cdot \delta f_1$

$(U, f) \mapsto \delta f$.

Denote the set of such δ at x by $T_x X$.

Remark: For $\delta, \hat{\delta} \in T_x X$. Define: $\delta + \hat{\delta}$ is:

$$\delta + \hat{\delta} : (U, f) \longrightarrow \delta_f + \hat{\delta}_f \in \mathbb{R}^n.$$

It's easy to see it satisfies Trans law.

Lemma: Fix $(U, f) \in A_x^*$. The Func "evaluate in (U, f)"

$$ev_f : T_x X \longrightarrow \mathbb{R}^n \quad \text{is linear isomorphism.}$$
$$\delta \longmapsto \delta_f$$

Pf: i) linear is easy to see

ii) Injection is from Transform law.

iii) Surjection: $\forall \vec{v} \in \mathbb{R}^n$.

Define: $\delta : (U_i, f_i) \longmapsto \delta_{f_i} = D\phi_{i0}|_{f_0(x)} \cdot v$

where $(U_0, f_0) = (U, f)$. Check $\delta \in T_x X$.

Remark: For two different charts:

$$ev_{f_2} = D\phi_{21}|_{f_1(x)} \cdot ev_{f_1}$$

Prop. There exist canonical linear isomorphism between $T_x X$ and $T_x X$.

Pf: Fix $(U, f) \in A_x$. $\exists \delta_\sigma \in T_x X$. $\delta_\sigma : (U, f) \longmapsto \Delta f(\sigma)$.

$$\begin{array}{ccccc} T_x X & \xrightarrow{\Delta f} & \mathbb{R}^n & \xrightarrow{ev_f^{-1}} & T_x X \\ [\sigma] & & \Delta f(\sigma) & & \delta_\sigma \end{array}$$

Composition of linear isomorphism.