Gambler's Ruin Problem.

Consider a gambler starts with i fortune. On each successive gamble, either win 1 or lose 1 fortune. Denote X_n is total fortune the gambler have after n^{th} game.

$X_n = i + \sum_1^n \Delta_k$. $p(\Delta_k = 1) = p$. $p(\Delta_k = -1) = q = 1-p$.

(Δ_k) i.i.d. r.v's.

<u>Denote</u>: $\tau_i^N = \{n \geq 1 \mid X_n \in \{0, N\}\}$, $S = \{0, 1 \cdots N\}$.

<u>prop.</u> $$P_i(N) =: p(X_{\tau_i^N} = N) = \begin{cases} (1-(q/p)^i)/(1-(q/p)^N), & p \neq q. \\ i/N & , p = q. \end{cases}$$

<u>Pf.</u> $P_i(N) = p(X_{\tau_i^N} = N \mid \Delta_1 = 1) \cdot p + p(X_{\tau_i^N} = N \mid \Delta_1 = -1) q$

$= p \cdot P_{i+1}(N) + q \cdot P_{i-1}(N)$.

<u>Cor.</u> $$P_i(\infty) = \lim_{N \to \infty} P_i(N) = \begin{cases} 0, & p \leq \frac{1}{2} \\ 1-(q/p)^i, & p > \frac{1}{2} \end{cases}$$

Consider $R_n = \sum_1^n \Delta_k$, $R_0 = 0$, i.e. the random walk starts at origin initially. For $a, b \in \mathbb{Z}^+$:

$p(a,b) =: p(R_n$ hits level a before hitting level $-b)$

Note it's equi. with: A gambler start at b fortune wishing to get target $N = a+b$ fortune.

$$S_0: \quad p(a.b) = \begin{cases} (1-(q/p)^b)/(1-(q/p)^{a+b}) & p \neq q. \\ b/a+b. & p = q. \end{cases}$$

Set $b \to \infty$. $\Rightarrow p(a.\infty) = p(\max_n R_n \geq a)$

$$\Rightarrow p(\max_{n \geq 0} R_n \geq a) = \begin{cases} (q/p)^a. & p < \frac{1}{2} \\ 1 & . \ p > \frac{1}{2} \end{cases}$$

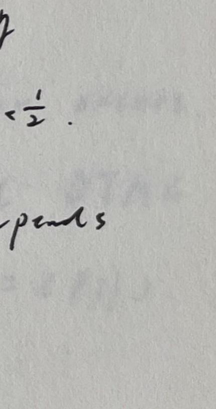
Also: $p(\max R_n = k) = (q/p)^k (1 - q/p)$. $\forall k \in \mathbb{Z}^+$.

Rmk: i) It's easy to obtain "$\min R_n$ case" by symmetry: $p(\min R_n \leq -b) = (p/q)^b$. if $q < \frac{1}{2}$.

ii) It's easy to interpret: Since it depends whether $E(\Delta) > 0$.

Consider $p = \frac{1}{2}$. Then it's equi. with one-dim Random Walk. It's recurrent: Note $E(N_0) = \sum P^n(0.0)$

$p^n(0,0) \neq 0$ if $n = 2m$. $p^{2m}(0,0) = \binom{2m}{m} 2^{-2m} \sim \frac{1}{\sqrt{m\pi}}$ by Stirling Formula. $\Rightarrow E(N_0) = \sum 1/\sqrt{m\pi} = \infty$.



Rmk: For $d = 2$. Project it on $y = x$:

$\Rightarrow$ It is composed of 2 indept RWs each moves $1/\sqrt{2}$ unit vertically or moves parapelly. to $y = x$. $\Rightarrow p^{2m}(\vec{0}, \vec{0}) = p^{2m}(0,0)\, p^{2m}(0,0) \sim 1/m\pi$

So: $E(N_{\vec{0}}) = \sum 1/m\pi = \infty$. $\Rightarrow$ It's recurrent.

For $d \geq 3$. analogously. it's transient. $p^{2m}(\vec{0}, \vec{0}) \sim (m\pi)^{-\frac{d}{2}}$