Nonlinear First Order PDEs.

· We will investigate: $F(Du, u, x) = 0$.

where $U \underset{open}{\subseteq} \mathbb{R}^n$. $u : \bar{U} \to \mathbb{R}^1$. $F(p, z, x) = F(p_1 \cdots p_n, z, x_1 \cdots x_n)$

$: \mathbb{R}^n \times \mathbb{R} \times \bar{U} \longrightarrow \mathbb{R}^1$. Usually subjects to:

$u = g$ on $\Gamma \subseteq \partial U$. $g : \Gamma \to \mathbb{R}^1$.

(1) <u>Complete Integrals and Envelops:</u>

① <u>Complete Integrals:</u>

· For $F(Du, u, x) = 0$. Suppose $A \underset{open}{\subseteq} \mathbb{R}^n$. $a = (a_1 \cdots a_n) \in A$ parameters. We have solution $u(x, a) \in C^2$.

<u>Denote:</u> $$(D_a u,\ D^2_{xa} u) = \left(\begin{array}{c|ccc} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_1 a_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{a_n} & u_{x_n a_1} & \cdots & u_{x_n a_n} \end{array}\right)_{n \times (n+1)}$$

<u>Def:</u> $u(x, a) \in C^2$ is called complete integral in $U \times A$ if it solves the equation for $\forall a \in A$. and $r(D_a u, D^2_{xa} u) = n$

Remark: $r((u_a\ u_{xa})) = n$ is guaranting $u(x,a)$ depends on all n independent parameters $\vec{a} = (a_1 \cdots a_n)$.

pf: If u depends on less than n parameters. i.e. suppose $B \subseteq \mathbb{R}^{n-1}$. $\forall b \in B$. $v(x,b)$ solve $F(Du,u,x)=0$. suppose $\varphi \in C^1(A,B)$. $v(x,\varphi(a)) \triangleq u(x,a)$.

Then $u_{a_i} = \sum_1^{n-1} v_{b_k}(x,\varphi(a))\,\varphi^k_{a_i}(a)$

$u_{x_i a_j} = \sum_1^{n-1} v_{x_i b_k}(x,\varphi(a))\,\varphi^k_{a_j}(a)$. we obtain:

$\forall$ $n\times n$ submatrix of $(D_a u\ D_{xa} u)$ has form:

$$\begin{pmatrix} v_{x_1 b_1} & \cdots & v_{x_1 b_{n-1}} \\ & \vdots & \\ v_{x_n b_1} & \cdots & v_{x_n b_{n-1}} \end{pmatrix}\begin{pmatrix} \varphi^1_{a_1} & \cdots & \varphi^1_{a_n} \\ & \vdots & \\ \varphi^{n-1}_{a_1} & \cdots & \varphi^{n-1}_{a_n} \end{pmatrix}$$ or replace some

row $\begin{pmatrix} v_{x_k b_1} \\ \vdots \\ v_{x_k b_{n-1}} \end{pmatrix}^T$ by $\begin{pmatrix} v_{b_1} \\ \vdots \\ v_{b_{n-1}} \end{pmatrix}^T$. rank $\leq n-1$

$\therefore r((u_a\ u_{xa})) < n$. Since these matrix have det 0.

e.g. Clairaut's equation:

$x \cdot Du + f(Du) = u$. $f: \mathbb{R}^n \to \mathbb{R}^1$.

One complete integral: $u(x,a) = a\cdot x + f(a)$.

② **New solution from envelop:**

· Next. we will show how to construct more solutions from $u(x,a)$. as the envelop of complete integral

Def: For $u(x,a) \in C^1$. $x \in U$. $a \in A$. Consider

$D_a u(x,a) = 0$. Solve $\vec{a} = \vec{\phi}(\vec{x})$. We have:

$D_a(u(x,\phi(x))) = 0$. Call $v(x) = u(x,\phi(x))$

is the envelop of $(u(x,a))_{a \in A}$. (Singular Integral)

Thm. If $u(x,a)$ solves $F(Du,u,x) = 0$. for $\forall a \in A$.

and $(u(x,a))_{a \in A}$ has envelop $v(x) \in C^1$.

Then v solve $F(Du,u,x) = 0$. as well.

Pf: $v_{x_i} = u_{x_i} + \sum_1^m u_{a_j}(x,\phi(x))\phi^j_{x_i} = u_{x_i}$

Since $D_a v(x) = D_a u(x,\phi(x)) = 0$.

Remark: Find such $u(x,a)$. Consider complete integral.

To generate more solution, consider $A' \subseteq \mathbb{R}^m$

$h : A' \longrightarrow \mathbb{R}^1$. St. $h \in C^1$. $G(h) \subseteq A$. Then:

$a = (a', h(a')) \in A$.

Def: General Integral depend on h is envelop $v'(x)$ of $u'(x,a') = u(x,a',h(a'))$

Remark: Find solutions depend on arbitrary h

$\not\Rightarrow$ Find all solutions. e.g. $F = F_1 \cdot F_2$.

u_1 is complete integral of F_1. We will miss solution of F_2. Since we just have F_1's.

(2) Characteristics:

· For $F(Du, u, x) = 0$ subject to $u = g$ on $\Gamma \subseteq \partial U$.

The ideal of method of characteristics is finding appropriate path $\vec{x}(s)$, connecting x ($f(x)$) and $x^0 \in \Gamma$. (since $g(x^0) = u(x^0)$). Calculate u on this path by solving an ODE.

① Procedure:

Def: $z(s) = u(x(s))$ record value of u along $x(s)$

$p(s) = Du(x(s))$ record gradient of u along $x(s)$

1°) Differentiate $F(Du, u, x)$ on x_i:

$$\sum_1^n F_{p_j}(Du, u, x) u_{x_j x_i} + F_z(Du, u, x) u_{x_i} + F_{x_i} = 0.$$

2°) For $u_{x_j x_i}$, differentiate $p^i(s) = u_{x_i}(x(s))$:

i.e. $\dot{p}^i(s) = \sum u_{x_i x_j}(x(s)) \dot{x}^j(s)$.

Let $\dot{x}^j(s) = F_{p_j}(p(s), z(s), x(s))$ which is for offsetting the equation 1°) when $x = x(s)$:

i.e. $\sum F_{p_j}(p(s), z(s), x(s)) \underset{x_j x_i}{u(x(s))} + F_z(p, z, x) + F_{x_i} = 0$

3°) Obtain $\dot{p}^i(s)$:

$$\dot{p}^i(s) = -F_z(p(s), z(s), x(s)) - F_{x_i}(p(s), z(s), x(s)).$$

4°) Obtain $z(s)$:

$$\dot{z}(s) = \sum u_{x_j}(x(s)) \dot{x}^j(s) = \sum p^i(s) F_{p_i}(p(s), z(s), x(s)).$$

$\Rightarrow$ Characteristic Equations: (CEs)

$$\begin{cases} \dot{x}(s) = D_p F(p(s), z(s), x(s)) \\ \dot{z}(s) = D_p F(p(s), z(s), x(s)) \cdot p(s) \\ \dot{p}(s) = -D_x F(p, z, x) - D_z F(p, z, x). \end{cases}$$

with $F(p(s), z(s), x(s)) = 0$.

<u>Thm.</u> (Structure of characteristic ODE)

$u \in C^2(U)$ solves $F(Du, u, x) = 0$ in U. If $x(s)$ solves the CEs. Then $p(s) = Du(x(s))$, $z(s) = u(x(s))$ solves CEs as well.

5°) After solving $x(s)$, $z(s)$.

We have: $x(s) = (x_1, x_2 \cdots x_n) = \vec{R}(x^0, s)$. $x^0 \in I$.

$z(s) = u(x(s)) = Q(z^0, s) = Q(g(x^0), s)$

Solve $x^0 = x^0(\vec{x})$, $s = s(\vec{x})$.

$\therefore u(\vec{x}) = u(x(s)) = z(s) = Q(g(x^0), s) = \bar{Q}(\vec{x})$.

② <u>Boundary Conditions:</u>

i) <u>straightening the boundary:</u>

Figure:

$$\begin{array}{ccc} \bar{V} & \underset{\phi}{\overset{\psi}{\rightleftarrows}} & \bar{U} \\ y & & x \\ \cup\!\!| & & \cup\!\!| \\ \Delta & \underset{\phi}{\overset{\psi}{\rightleftarrows}} & I \end{array}$$

Suppose we straighten ∂U locally at x^0, to ∂V.

$$\begin{cases} x = \psi(y) & \bar{U} = \psi(\bar{V}) \\ y = \phi(x) & \bar{V} = \phi(\bar{U}). \end{cases}$$

We obtain: $u(x) = u(\psi(\eta)) = v(\eta)$. $x \in \bar{U}$

$v(\eta) = v(\phi(x)) = u(x)$. $\eta \in \bar{V}$

$\therefore u_{x_i} = \sum v_{\eta_k}(\phi(x))\, \phi^k_{x_i}(x)$ i.e. $D_x u(x) = Dv(\eta) \cdot D_x\phi(x)$.

$F(Du, u, x) = F(Dv(\eta) \cdot D_x\phi(\psi(\eta)), v(\eta), \psi(\eta)) = 0$.

Denote it by $G(Dv(\eta), v(\eta), \eta) = 0$.

Besides, $v = h(\eta) = g(\psi(\eta))$ on $\Delta = \phi(\Gamma)$

ii) <u>Compatibility conditions:</u>

· Suppose $x^0 \in \Gamma$. Γ is flat near x^0, lying in $\{x_n = 0\}$.

For $x^0 = x(0)$, $p^0 = p(0)$, $z^0 = z(0) = u(x^0) = g(x^0)$.

$\therefore u_{x_i}(x^0) = g_{x_i}(x^0)$. The initial condition for p:

$$\begin{cases} p_i^0 = g_{x_i}(x^0) \\ F(p^0, z^0, x^0) = 0. \end{cases}$$

We call them compatibility condition. (p^0, z^0, x^0) are admissible.

<u>Remark</u>: p^0 satisfies it may not exist or be unique.

iii) <u>Noncharacteristic boundary data:</u>

· Suppose we have ascertained (p^0, z^0, x^0) have appropriate boundary conditions for CEs.

Ask: Can we perturb (p^0, z^0, x^0) slightly then the compatibility condition still retain.

For $\eta \in \Gamma$, closed to x^0. (Γ is straightened)

$\eta = (\eta_1 \cdots \eta_{n-1}, 0)$. $x^0 = (x_1^0 \cdots x_{n-1}^0, 0)$.

We intend to solve CEs with the initial condition: $p(0)=q(\eta),\ z(0)=g(\eta),\ x(0)=\eta$.

i.e. Find $q:\mathbb{R}^n \longrightarrow \mathbb{R}^n$. s.t. $q(x^0)=p^0$.

and $(q(\eta), g(\eta), \eta)$ are admissible.

$$\begin{cases} q^i(\eta) = g_{x_i}(\eta) \quad 1\le i\le n-1 \ (\eta_n = 0 = x_n^0) \\ F(q(\eta), g(\eta), \eta) = 0 \end{cases} \quad \forall \eta \in I. \text{ close to } x^0.$$

<u>Lemma</u> If $F_{p_n}(p^0, z^0, x^0) \neq 0$.(*) Then there exists unique $q(\eta)$ satisfies them.

(*) Call it by noncharacteristic condition.

<u>Pf:</u> Find $q^n(\eta)$: By Implicit Func. Thm.

<u>Remark:</u> Generally. I isn't flat near x^0.

Then the condition become:

$D_p F(p^0, z^0, x^0) \cdot \vec{\nu}(x^0) \neq 0$. $\vec{\nu}(x^0)$ is outer normal vector of ∂U. at x^0.

③ Local Solutions:

- Suppose (p^0, z^0, x^0) is admissible. noncharacteristic

- Then $\exists\, q(\eta)$. $p^0 = q(x^0)$. $(q(\eta), g(\eta), \eta)$ is admissible for $\forall \eta$ close to x^0. $\eta \in I$.

Denote $\begin{cases} p(s) = p(\eta, s) \\ z(s) = z(\eta, s) \\ x(s) = x(\eta, s) \end{cases}$ i.e. p, z, x depend on initial value η.

<u>Lemma</u>. (Local Invertibility)

$F_{p_n}(p^0, z^0, x^0) \neq 0$. Then $\exists I \subseteq \mathbb{R}^1$. neighbour of 0. $W \subseteq I \subseteq \mathbb{R}^{n-1}$. neighbour of x^0. and $V \subseteq \mathbb{R}^n$. neighbour of X^0. st.

$$W \times I \xrightarrow{x} V \quad \text{is } C^2\text{-homeomorphism}.$$
$$(\eta, s) \longmapsto x = x(\eta, s)$$

<u>Pf:</u> $x(x^0, 0) = x^0$. By Implicit Func Thm.

prove: $|D(x(x^0, 0))| \neq 0$.

$x(\eta, 0) = (\eta, 0)$. for $(\eta, 0) = x^0$.

$\therefore D_\eta x(\eta, 0) = \begin{pmatrix} I_{n-1} \\ 0 \end{pmatrix}$

$\because D_s x(\eta, s) = F_p(p(s), z(s), x(s))$.

$\therefore D x(x^0, 0) = \begin{pmatrix} I_{n-1} & * \\ 0 & F_{p_n}(p^0, z^0, x^0) \end{pmatrix}$.

<u>Thm</u>. Under the condition in the Lemma. We can solve $x = x(\eta, s)$ for $\eta = \eta(x)$. $s = s(x)$. define $u(x) = z(\eta(x), s(x))$. Then $u(x)$ solves $F(Du, u, x) = 0$ in V. $u(x) = g(x)$ on $\Gamma \cap V$. locally. $(V \subseteq U)$

<u>Pf:</u> 1°) For $\eta \in I$ close to x^0. we have:

$f(\eta, s) \triangleq F(p(\eta, s), z(\eta, s), x(\eta, s)) = 0$. $\forall s \in I$.

Note: $f(\eta, 0) = F(q(\eta), g(\eta), \eta) = 0$

$f_s(\eta, s) = 0$ (replace $\dot{p}, \dot{z}, \dot{x}$).

2°) By Lemma. $x(\eta, s) = x$. $\therefore F(p(x), u(x), x) = 0$

prove: $p(x) = Du(x)$. check $p^i(x) = D_{x_i} u$. directly.