Conformal Mappings

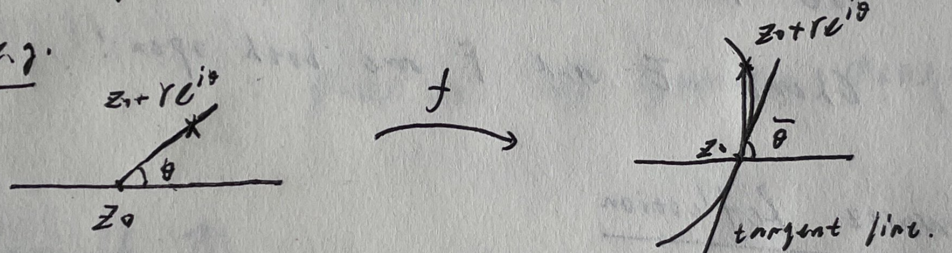
(1) <u>Def</u>:

$U \subseteq_{open} \mathbb{C}$. $f: U \to \mathbb{C}$. For $z_0 \in U$. if f is locally injective on $D(z_0, r)$. We say f is conformal at z_0 if

$$\lim_{r \to 0} e^{-i\theta} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{|f(z_0 + re^{i\theta}) - f(z_0)|} \text{ exists.}$$

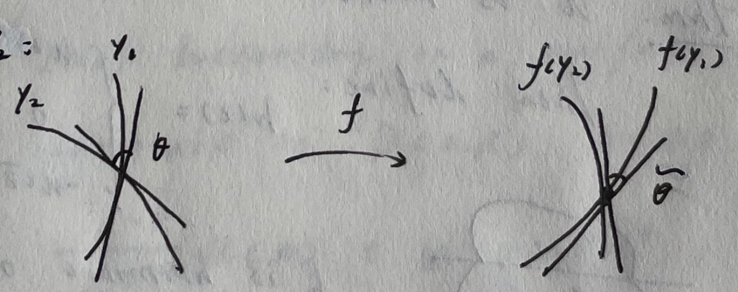
indept with θ.

<u>e.g.</u>



Then the limit is: $e^{-i\theta + i\tilde{\theta}}$. $\tilde{\theta}(r) \to \tilde{\theta}$ $(r \to 0)$

For γ_1, γ_2:



$\Rightarrow$ By the def. $\theta = \tilde{\theta}$.

<u>Remark</u>: That's because the curve "spin" a const angle.

Thm. $D \subseteq \mathbb{C}$. $f: D \to \mathbb{C}$. $z_0 \in D$.

i) $f(z) \in \mathcal{O}(D)$. If $f'(z_0) \neq 0$. Then f is conformal at z_0

ii) $f(z)$ is conformal at z_0. has a nonzero differential df. at z_0. Then $f'(z_0) \neq 0$.
f is differentiable at z_0

Pf: i) Suppose $z_0 = 0$. $f(z_0) = 0$. expand at $z=0$.

$$\lim_{r\to 0} e^{-i\theta} \frac{\sum a_k r^k e^{ik\theta}}{|\sum a_k r^k e^{ik\theta}|} = \frac{a_1}{|a_1|} \neq 0.$$

ii) Suppose $z_0 = f(z_0) = 0$.

$\therefore f(z) = \alpha z + \beta \bar{z} + o(|z|)$

$\therefore \lim \square = \dfrac{\alpha + \beta e^{-2i\theta}}{|\alpha + \beta e^{-2i\theta}|}$ $\therefore \beta = 0$.

Remark: Biholomorphic $\Leftrightarrow$ conformal.

(2) Schwartz Lemma:

① Thm. Denote $U = D(0,1)$ $f: U \to U$. holomorphic $f(0) = 0$. Then $|f| \leq |z|$. $|f'| \leq 1$.

If $\exists z_0 \in U$, s.t. $|f(z_0)| = |z_0|$ or $|f'(0)| = 1$.

Then $f = z \cdot e^{i\theta}$. $\theta \in [0, 2\pi]$.

Pf: Extend $f(z)/z$ to U by expansion at 0.

$\therefore f(z)/z \in \mathcal{O}(U)$.

$\forall\, 0 < r < 1$. $\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$. Let $r \to 1^-$

$\therefore |f(z)| \leq |z|$. Let $z \to 0$ $\therefore |f'(0)| \leq 1$.

By mmp of holomorphic. Conclude the latter.

General Form:

$f: U \to U$. holomorphic. For $\forall z_1, z_2 \in U$.

We have: $\left| \dfrac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \le \left| \dfrac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$

The "=" holds $\iff f(z) = e^{i\theta} \dfrac{z - a}{1 - \bar{a} z}$. $\exists \theta, a$.

Pf: Suppose $z_0 = f(z_1)$.

$e_1 = \dfrac{z - z_1}{1 - \bar{z}_1 z}$. $e_2 = \dfrac{z - z_0}{1 - \bar{z}_0 z}$.

$\therefore F = e_0 \circ f \circ e_1 : U \to U$. $F(0) = 0$.

Apply the thm above!

Cor. Since $\left| \dfrac{f(z_1) - f(z_2)}{z_1 - z_2} \right| \le \left| \dfrac{1 - \overline{f(z_1)} f(z_2)}{1 - \bar{z}_1 z_2} \right|$

Let $z_1 \to z_2$ $\therefore |f'(z_2)| \le \dfrac{1 - |f(z_2)|^2}{1 - |z_2|^2}$

Denote $w = f(z)$. We have differential form:

$$\frac{|dw|}{1 - |w|^2} \le \frac{|dz|}{1 - |z|^2}$$

② Cantor's proof:

$\Omega \subseteq \mathbb{C}$. open, bounded. $\varphi: \Omega \to \Omega$. holomorphic

If exists $z_0 \in \Omega$. St. $\varphi(z_0) = z_0$. $\varphi'(z_0) = 1$

Then φ is linear.

Pf. WLOG. Set $z_0 = \varphi(z_0) = 0$. $\therefore \varphi'(0) = 1$.

$\therefore \varphi(z) = z + a_m z^m + o(z^m)$. expand at $z=0$

where $a_m \neq 0$. m is the least integer.

$\varphi_k(z) = \underbrace{\varphi \circ \varphi \circ \cdots \varphi}_{k}(z) = z + k a_m z^m + o(z^m)$

By Cauchy Inequality $|\varphi_k^{(m)}(0)| \le \dfrac{m! \, \|\varphi_k\|_\infty}{r^m}$

$\therefore |a_m| \le \dfrac{\|\varphi_k\|_{L^\infty(U)}}{k r^m}$

Note that φ_k is uniformly bounded. Let $k \to \infty$.

$\therefore |a_m| = 0$. $\therefore \varphi(z) = z$.

③ Bieberbach's Conjecture:

$f \in \mathcal{O}(U)$. $f(0) = 0$. $f'(0) = 1$. Then for

expansion at 0: $\sum\limits_{1}^{\infty} a_n z^n$. we have $|a_n| \le n$.

④ Application:

Carathéodory Thm

$f \in \mathcal{O}(\bar{D}(0,R))$. for $A(r) = \max\limits_{\theta} \operatorname{Re} f(re^{i\theta})$, where

$0 < r < R$. we have: $|f(re^{i\theta})| \le |f(0)| + \dfrac{2r}{R-r}(A(R) - \operatorname{Re} f(0))$

Pf: Set $h(z) = f(z) - f(0)$ $\therefore h(0) = 0$

prove: $|h(re^{i\theta})| \le \dfrac{2r}{R-r} A_h(R)$. $0<r<R$.

Set $g(z) = \dfrac{h(z)}{2A_h(R) - h(z)}$. $|z| \le r < R$.

$\therefore g(Rz): U \longrightarrow U$. $g(0) = 0$. $|g(Rz)| \le |z|$

Remark: The real part dominates the whole function $f(z)$.

(3) Automorphism Group:

① $Aut(\mathbb{C}) = \{a+bz \mid a, b \in \mathbb{C}, b \neq 0\}$

Pf: $\forall f \in Aut(\mathbb{C})$, f is one-to-one, entire.

$\therefore f \in \mathcal{O}(\mathbb{C})$, $\forall z_0 \in \mathbb{C}$, $f = \sum_0^\infty a_n (z-z_0)^n$

Besides, $z=\infty$ can't be essential singular.

Suppose $f(z_0) = z_1$. By Open mapping Thm

$f(U(z_0)) = U(z_1)$. But if $\exists \{\tilde{z}_n\} \to \infty$, $f(\tilde{z}_n) \to z_1$

Then $\exists p \in U(z_1)$, has more than 2 preimages.

$\therefore f = \sum_1^k a_n (z-z_0)^n$. $f = w_0$ has k roots.

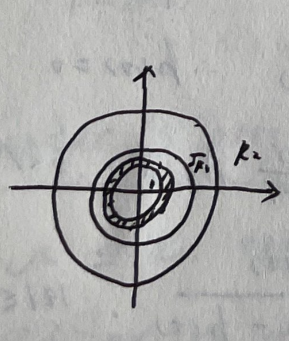
$\therefore f = a_0 + a_1(z-z_0)$.

② $Aut(A) = \{e^{i\theta} z, e^{i\theta} \frac{rR}{z}\}$, where $A = \{|z| \in (r, R)\}$.

Pf: **Lemma**. $A_1 = \{r_1 < |z| < R_1\}$, $A_2 = \{r_2 < |z| < R_2\}$

$$A_1 \cong A_2 \iff \frac{R_1}{r_1} = \frac{R_2}{r_2}$$

Pf: Assume $r_1 = r_2 = 1$. $R_1, R_2 > 1$.



($\Leftarrow$) It's trivial.

($\Rightarrow$). Suppose $A_1 \underset{f(z)}{\cong} A_2$.

Set $K = \{|z| = \sqrt{R_2}\}$ cpt

$A_\varepsilon = \{1 < |z| < 1+\varepsilon\}$.

ε is small enough st. $A_\varepsilon \cap f^{-1}(K) = \emptyset$.

where $f^{-1}(K)$ is cpt.

WLOG. suppose $f(A_1)$ fall in $\{1<|z|<\sqrt{R_2}\}$

Otherwise. set $g(z) = \frac{R_2}{f(z)}$. automorphis as well.

$\therefore |f(z_n)| \to 1$ when $|z_n| \to 1$ set $m = \frac{\log R_2}{\log R_1} > 0$.

$|f(z_n)| \to R_2$ when $|z_n| \to R_1$

We want to prove $m=1$.

Set $u(z) = 2\log|f(z)| - 2m\log|z|$. $\Delta u \equiv 0$

Since $u(1^+) = u(R_1^-) = 0$. extend to boundary

$\therefore u \equiv 0$ on $1 \le |z| \le R_1$

$\therefore \frac{\partial u}{\partial z} = \frac{f'}{f} - m\frac{1}{z} = 0$ i.e. $\frac{\partial}{\partial z}\left(\frac{f(z)}{z^m}\right) = 0$

$\therefore f(z) = cz^m$. $m=1$. Since one-to-one.

$m \in \mathbb{Z}^+$. Since $m = \oint_\gamma \frac{1}{2\pi i}\frac{1}{z}dz = \oint_\gamma \frac{1}{2\pi i}\cdot\frac{f'}{f} \in \mathbb{Z}^+$

$\Rightarrow$ suppose $f: A \to A$. biholomorphic

WLOG. set $|f(z_n)| \to r$ when $|z_n| \to r$

$|f(z_n)| \to R$ when $|z_n| \to R$

otherwise let $g(z) = \frac{Rr}{f(z)}$.

Annlogously $|\frac{f(z)}{z}| \equiv 0$ on A.

3 $\mathrm{Aut}(D) = \{e^{i\theta}\varphi_a \mid \theta \in [0, 2\pi], \varphi_a$ is Möbius Trans$\}$

Lemma: For $\varphi_a = \frac{z-a}{1-\bar{a}z}$

i) φ_a is one-to-one

ii) $\varphi_a^2 = id$.

iii) $\varphi_a : \partial U \to \partial U$. $\varphi_a(0) = a$. $\varphi_a(a) = 0$

$\Rightarrow \psi : \varphi_a \circ \varphi_\beta : \beta \mapsto \alpha$.

iv) $\varphi_a' = \frac{|a|^2-1}{(1-\bar{a}z)^2} \neq 0$.

Pf: If $f \in Aut(D)$. Suppose $f(\alpha)=0$

Then $f(\varphi_\alpha(z)) : U \to U$. $f\circ\varphi_\alpha(0)=0$

Apply Schwartz Lemma on $f\circ\varphi_\alpha$ and $(f\circ\varphi_\alpha)^{-1}$

④ $Aut(H) = \{ \frac{az+b}{cz+d} \mid ad-bc\neq 0, a,b,c,d \in \mathbb{R}\}$

H is the half upper plane.

Remark: Sometimes we will normalize $ad-bc$.

s.t. $ad-bc=1=\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Then:

$Aut(H) \cong SL(2,\mathbb{R})$.

Check: If $m=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $f_m = \frac{az+b}{cz+d}$ (Denote)

Then $f_{m_1}\circ f_{m_2} = f_{m_1m_2}$, retain the operation.

Pf: 1°) Note that: $\frac{i-z}{i+z} : H \xrightarrow{F} D$. $F\circ G = I_H$

$i\frac{1-z}{1+z} : D \xrightarrow{G} H$ $G\circ F = I_D$

$\Psi : Aut(D) \longrightarrow Aut(H)$. Ψ is Auto-!

$\varphi \longmapsto F^{-1}\circ\varphi\circ F$ $\therefore Aut(D)\cong Aut(H)$

2°) $\forall z,w \in H$. $\exists m \in SL(2,\mathbb{R})$. s.t. $f_m(z)=w$

3°) $F\circ f_{m_\theta}\circ F^{-1} = e^{-2i\theta}$. $M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

4°) $\forall f \in Aut(H)$.

suppose $f(\beta)=i$. $\exists f_N$. s.t. $f\circ f_N(i)=i$

$f\circ f_N \in Aut(H)$. $\therefore F\circ f\circ f_N\circ F^{-1}(0)=0$.

$F\circ f\circ f_N\circ F^{-1} \in Aut(D)$. $\therefore$ It's rotation.

Remark: Note that $f_m = f_{-m}$. Identify m with $-m$ in $SL(2,\mathbb{R})$. We obtain a new group: $PSL(\mathbb{R})$

(4) Riemann Mapping Thm:

① Montel's Thm:

Def: i) $\mathcal{F}$ is a normal family, if $\mathcal{F}$ is a family of holomorphic functions.

$\forall \{f_n\} \subseteq \mathcal{F}$, $\exists \{f_{n_k}\} \subseteq \{f_n\}$. s.t.

$f_{n_k} \xrightarrow{u.c.c} \text{some } f$ (may not $\in \mathcal{F}$)

ii) $\{K_n\}$ seq of cpt set is an exhaustion of Ω.

if $K_n \subseteq \operatorname{int} K_{n+1}$

$\forall K \underset{cpt}{\subseteq} \Omega$. $\exists l$. s.t. $K \subseteq K_l$.

$\cup K_n = \Omega$.

Remark: Every open set O has an exhaustion:

$K_n = \{ z \mid \operatorname{dist}(z, O^c) \geq \frac{1}{n}.\ |z| \leq n \}$.

Thm. $\mathcal{F}$ is a family of holomorphic func on Ω, $\Omega \underset{open}{\subseteq} \mathbb{C}$. If $\mathcal{F}$ is locally uniformly bounded on every cpt set $\subseteq \Omega$. Then.

i) $\mathcal{F}$ is equiconti on every cpt set

ii) $\mathcal{F}$ is normal family.

Pf: i) Easy to check by Cauchy Thm.

ii) For $\{K_n\}$ is exhaustion of Ω. $\forall \{f_k\} \subseteq \mathcal{F}$

By Ascoli. $\exists \{f_{1k}\} \subseteq \{f_k\}$. converges in K_1

$\exists \{f_{2k}\} \subseteq \{f_{1k}\}$. converges in K_2

$\vdots$

$\exists \{f_{nk}\} \subseteq \{f_{n-1,k}\}$ converges in K_n.

$\vdots$

Choose $\{f_{nn}\}$. it converge on every cpt set!

Cor. (Vitali Thm)

$D \underset{open}{\subseteq} \mathbb{C}$. $\{f_n\}$ is family of holomorphic functions on D. uniformly bounded. If f_n converges on a set of uniqueness

Then $\exists f$. s.t. $f_n \xrightarrow{u.c.c} f$.

Def: A is set of uniqness if for any $f, g \in \mathcal{O}(D)$. $f = g$ on A.

Then $f \equiv g$ on D. e.g. every set has a cumulation point.

Pf: By contradiction:

$\exists K \underset{cpt}{\subseteq} D$. $\varepsilon_0 > 0$. $\{f_{\alpha_k}\}$. $\{f_{\beta_k}\} \subseteq \{f_n\}$.

$\{z_n\} \subseteq D$. s.t. $|f_{\alpha_k}(z_k) - f_{\beta_k}(z_k)| \geq \varepsilon_0$

By Montel. select convergent subseq of $\{f_{\alpha_k}\}$. $\{f_{\beta_k}\}$. u.c.c to f, g. resp.

$\therefore f = g$ on A. set of uniqueness. $\therefore f \equiv g$.

Suppose $z_k \to z_0$. Then $|g(z_0) - f(z_0)| \geq \varepsilon_0$

which is a contradiction!

③ Riemann's Mapping Thm:

If $\Omega \subsetneqq \mathbb{C}$ simply connected. Then $\Omega \underset{f}{\cong} D \triangleq D(0,1)$

Moreover. There's unique f satisfies:

$f(z_0)=0$. and $f'(z_0)>0$ for some $z_0 \in D$.

Pf: 1°) $\Omega \cong D$

$\exists\, F \in \mathcal{O}(\Omega)$. $F: \Omega \to F(\Omega)$

bounded and one-to-one.

Pf: Consider $\mathcal{F} = \{f: \Omega \to D.\ f \in \mathcal{O}(\Omega).\ \text{injective}.\ f(z_0)=0\}$

i) Prove: $\mathcal{F} \neq \emptyset$. (weaken "biject" to "inject")

$\exists\, \alpha \in \mathbb{C}/\Omega$. Check $\varphi(z) = \sqrt{z-\alpha} \in \mathcal{O}(\Omega)$. injective

Find $g(z) = e^{i\theta} \frac{az+b}{cz+d}$. st. $g \circ f \in \mathcal{F}$.

Denote: $\lambda = \sup\limits_{f \in \mathcal{F}} |f'(z_0)| > 0$

$\lambda < \infty$. Since $|f| \leq 1$. By Cauchy Thm.

ii) Prove: $\exists\, f \in \mathcal{F}$. st. $|f'(z_0)| = \lambda$.

Since $\exists\, \{g_n\} \subseteq \mathcal{F}$. $g_n'(z_0) \to \lambda$.

By Montel on $\{g_n\}$. $\exists\, g_{n_k} \xrightarrow{\text{u.c.c.}} f$

$|f'(z_0)| = \lambda > 0$. $\therefore f \neq c$. $\therefore f \in \mathcal{F}$ by Hurwitz. Thm

iii) Prove: $f: \Omega \to D$ is automorphism.

If not. $\exists\, a \in D$. $f(z) \neq a$. $\forall z \in \Omega$.

Choose $\varphi_a = \dfrac{z-a}{1-\bar{a}z}$, $\sqrt{\varphi_a} \in \mathcal{O}(D)$, injective

Choose $\varphi_b = e^{i\arg b}\dfrac{z-b}{1-\bar{b}z}$. $b = \sqrt{\varphi_a(0)}$

$\therefore h(z) = \varphi_b \circ \sqrt{\varphi_a} \circ f \in \mathcal{F}$.

since $|(\varphi_b \circ \sqrt{\varphi_a})'(0)| = \dfrac{1+|b|^2}{2|b|} > 1$. $\therefore |h'(z_0)| > |f'(z_0)| = \lambda$

Which is a contradiction!

2) Uniqueness:

If F, G satisfies the condition.

Then $F \circ G^{-1} \in Aut(D)$. Fix origin

$\therefore F \circ G^{-1} = e^{i\theta} z$. $0 < \theta \leq 2\pi$.

$\theta = 2\pi$ since $(F \circ G^{-1})'|_{z=z_0} > 0$.

<u>Remark</u>: Consider $|f'(z_0)| = \sup_{f \in \mathcal{F}} |f'(z_0)|$ is for filling the Disc D as much as possible!

It's one-to-one eventually!

③ <u>Caratheodory Thm</u>:

$D \subsetneqq \mathbb{C}$. simply connected. If ∂D is a Jordan Curve. since $D \underset{\varphi}{\cong} U$. Then φ can be extended to: $\bar{D} \underset{\varphi}{\cong} \bar{U}$. homeo.

<u>Pf</u>: Note that in $D \underset{\text{open}}{\subsetneqq} \mathbb{R}$. bound.

If f is uniformly conti on D

Then f can be extended on $\bar{D}$.

1°) Prove: φ is uniformly conti on D.

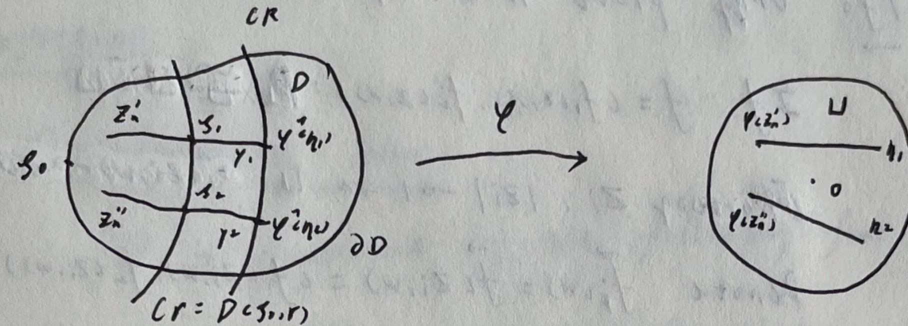
By Contradiction: $\exists \{z_n'\}, \{z_n''\} \subseteq D$.

s.t. $|\varphi(z_n') - \varphi(z_n'')| \geq \varepsilon_0$. $|z_n' - z_n''| < \frac{1}{n}$

Find subseq of $z_n' \to \zeta_0$. ζ_0 will $\in \partial D$.

Find subseq of $\varphi(z_n')$, $\varphi(z_n'')$ converges to

w_1, w_2. which will belong to ∂U. $|w_1 - w_2| \geq \varepsilon_0$



$\exists \eta_1, \eta_2 \in \partial U$. s.t. $d(\overline{\varphi(z_n')\eta_1}, \overline{\varphi(z_n'')\eta_2}) \geq \frac{\varepsilon_0}{2}$

Besides. $\varphi^{-1}(\overline{\varphi(z_n')\eta_1}) = \gamma_1$, $\varphi^{-1}(\overline{\varphi(z_n'')\eta_2}) = \gamma_2$

$$\frac{\varepsilon_0}{2} \leq |f(\zeta_1) - f(\zeta_2)| = \left|\int_{\zeta_1}^{\zeta_2} f(z)dz\right| \leq \int_{\theta_1}^{\theta_2} r|f(\zeta + re^{i\theta})|d\theta.$$

Estimate RHS. by Cauchy Inequilty, Contradict!

2°) Extend φ to $\partial D \to \partial U$.

Def: $\varphi(z_0) = \lim\limits_{z \in D \to z_0} \varphi(z)$. $z_0 \in \partial D$.

Check φ is homeomorphism!

④ Pioncáre Inequivalance Thm:

· In $\mathbb{C}^n$, $n > 1$. Riemann Mapping Thm doesn't hold any more.

Denote $B_n = \{ z = (z_1, \cdots z_n) \mid \sum_1^n |z_i|^2 < 1 \}$

$P(\vec{a}, \vec{r}) = \prod_1^n B(a_i, r_i)$. polydisc.

Then $Aut(B_n)$ is Unitary Group. nonabelian.

$Aut(P(0,1)) = \{ f \mid f: (z_1 \cdots z_n) \to (\varphi_{n_1}(z_{n_1}) \cdots \varphi_{n_n}(z_{n_n})) \}$.

is abelian Group. $\therefore Aut(B_n) \neq Aut(P(0,1))$

<u>Thm</u>. There's no biholomorphism between B_n and $P(0,1)$

<u>Pf</u>: Only prove $n=2$. By contradiction:

If $f = (f_1(z,w), f_2(z,w)) : U \times U \simeq B_2$.

For any z_i. $|z_i| \to 1$ in U. Then $(z_i, w) \to \partial U^2$

Denote $\vec{f_i}(w) = \vec{f}(z_i, w) = (f_1(z_i,w), f_2(z_i,w)) \to \partial B_2$

since $f_1(z_i,w) = f_{1i}$, $f_2(z_i,w) = f_{2i}$ uniformly bounded

$\therefore \exists\ \vec{f}_{n_k}(w)$ converges to $(g_1(w), g_2(w))$ on ∂B_2

$\therefore |g_1|^2 + |g_2|^2 = 1$. $\frac{\partial^2}{\partial z \partial \bar{z}}(|g_1|^2 + |g_2|^2) = 0$

i.e. $|g_1'|^2 + |g_2'|^2 = 0$ $\therefore g_1' = g_2' \equiv 0$

$\because \vec{f_i} \to (g_1, g_2)$ $\therefore \vec{f_i}' \to (g_1', g_2') = \vec{0}$

$\therefore \forall z \in \partial U$, $\frac{\partial}{\partial w} f_1(z,w) = \frac{\partial}{\partial w} f_2(z,w) = 0$

$\therefore \vec{f}(z,w)$ is indept with w. Contradict!

since by MMP. $\frac{\partial f}{\partial w} \equiv 0$. on U. fixed w

⑤ <u>For Dirichlet Problem:</u>

· We can extend the specific domain $D(a,r)$ to arbitary simply connected proper domain D

e.g. For g on ∂D. $\exists\ \varphi : \bar{D} \simeq \bar{U}$.

Then $P[g \circ \varphi^{-1}](z)$ is harmoric on D.