(2) General principle:

Consider X is polish space. $I: X \to \mathbb{R}^{\geq 0}$ is a proper rate func. i.e. $K_L = \{I(x) \leq L\}$ is cpt for $\forall L$. and I is l.s.c.

Def: For $(\mathbb{P}_n)$ seq of p.m. on X. it satisfy large deviation principle (LDP) on X at speed n. w.r.t. $I(\cdot)$ if:

i) $\overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{P}_n(C) \leq -\inf_C I(x)$. $\forall C \underset{\text{close}}{\subset} X$.

ii) $\underline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{P}_n(O) \geq -\inf_O I(x)$. $\forall O \underset{\text{open}}{\subset} X$

Rmk: i) Note for i.i.d seq case. it satisfies LDP.

ii) Note $\mathbb{P}_n(X) = 1 \Rightarrow \inf I(x)$ must be 0. for $(\mathbb{P}_n)$

Thm. For $(\mathbb{P}_n)$ satisfies LDP at speed n. w.r.t $I(\cdot)$. on X. Then. $\forall L < \infty$. $\exists D^L \underset{\text{cpt}}{\subset} X$. st. $\mathbb{P}_n(D^L) \geq 1 - e^{-nL}$. $\forall n$.

Pf: Consider $A^L = \{ I(x) \leq L+2 \}$. cpt

Suppose A_k is finite union of open balls with $r = 1/k$ cover A^L.

By LDP: $\overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{P}_n(A_k^c) \le -(l+2)$ $I \ge l+2$ on A_k^c

$\Rightarrow \exists\, n_0 = n_0(k).\ \forall n \ge n_0:\ \mathbb{P}_n(A_k^c) \le e^{-n(l+1)}$

WLOG. set $n_0(k) \ge k$. $\forall k \ge 1$.

$\Rightarrow \mathbb{P}_n(A_k^c) \le e^{-nl} \cdot e^{-k}$.

On the other hand. For each $k \ge 1$.

$\exists\, (B_{k,i})_{i \le n_0(k)}$ seq of cpt sets. s.t.

$\mathbb{P}_i(B_{k,i}^c) \le e^{-k} \cdot e^{-il}$. $\forall\, i \le n_0(k)$.

Let $D^l = \bigcap_k \left(\bar{A}_k \cup \left(\bigcup_{j=1}^{n_0(k)} B_{k,j} \right) \right)$

$\mathbb{P}_n((D^l)^c) \le \sum_{k\ge1} \mathbb{P}_n\left(A_k^c \cap (\cap B_{k,j}^c)\right) \le e^{-nl} \sum e^{-k}$

Since $\mathbb{P}_n\left(A_k^c \cap (\cap B_{k,j}^c)\right) \le \begin{cases} \mathbb{P}_n(A_k^c). & n \ge n_0(k) \\ \mathbb{P}_n(B_{k,n}^c) & n \le n_0(k) \end{cases}$

Thm (Product of LDP)

If $(\mathbb{P}_n)$. $(\mathbb{Q}_n)$ are two seq of p.m's satisfy LDP w.r.t. $I(\cdot)$ and $J(\cdot)$ on X and Y. Then $R_n =: \mathbb{P}_n \times \mathbb{Q}_n$. Satisfies LDP w.r.t $K(x,y) =: I(x) + J(y)$. on $X \times Y$.

Pf: i) Consider $z = (x,y) \in X \times Y$.

By l.s.c. of I. $\exists\, U_0$. nbd of x.

s.t. $I(x') \ge I(x) - \varepsilon$. $\forall x' \in U_0$

By separable. $\exists\, U_x$ open. $U_x \subset \bar{U}_x \subset U_0$.

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{P}_n(U_x) \le -I(x) + \varepsilon$, Similarly. $\exists V_y$. of y. Set $N_z = U_x \times V_y$.

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log R_n(N_z) \leq -K(z) + 2\varepsilon$.

2°) For $D \subset X \times Y$. cpt set. cover D by finite nbd's N_z (as above).

$R_n(\bigcup_1^N N_z) \leq N \max_{1\leq j\leq N} R_n(N_z)$

Note N will disappear. after taking logarithm and divide n. set $n\to\infty$

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log R_n(\bigcup_1^N N_z) \leq -\inf_D K(z) + \varepsilon$.

3°) For general $C \subset X \times Y$.

Note $\exists A^\varepsilon \underset{cpt}{\subset} X$. $B^\varepsilon \underset{cpt}{\subset} Y$. st.

$\mathbb{P}_n(A^\varepsilon)^c \leq e^{-n\varepsilon}$. $\mathbb{Q}_n(B^\varepsilon)^c \leq e^{-n\varepsilon}$.

Set $C^\varepsilon = A^\varepsilon \times B^\varepsilon$. $C = C \cap C^\varepsilon + C \cap C^{\varepsilon c}$.

$\Rightarrow \overline{\lim} \frac{1}{n} \log R_n(C) \leq \max(-\inf_{C^\varepsilon} K(z), -\varepsilon)$

$\leq \max\{-\inf_C K(z), -\varepsilon\} \xrightarrow[\varepsilon\to\infty]{} -\inf_C K(z)$

4°) For lower bound. Since it's local. find $U_x \times V_y =: N$. nbd of $z=(x,y)$. By LDP of $\mathbb{P}_n$. $\mathbb{Q}_n$. we have:

$\underline{\lim_{n\to\infty}} \frac{1}{n} \log R_n(N) \geq -K(z) \quad \forall z \in O$.

Thm: (Contraction principle)

$F: X \to Y$. is conti. between. polish spaces.

If $(\mathbb{P}_n)$ satisfies LDP with $I(\cdot)$ on X.

Then. $\mathbb{Q}_n =: \mathbb{P}_n \circ F^{-1}$ satisfies LDP on Y

w.r.t. $J(y) = \inf_{x \in F^{-1}(y)} I(x)$.

Pf: Note F^{-1} retains open/close. easy to check.

Thm (Varadhan's Thm)

If $(\mathbb{P}_n)$ satisfies LDP on X w.r.t. $I(\cdot)$.

$F \in C_B^{\mathbb{R}'}(X)$. Then $a_n =: \int_X e^{nF(x)} d\mathbb{P}_n(x)$.

satisfies $\lim_{n\to\infty} \frac{1}{n} \log a_n = \sup_x (F(x) - I(x))$.

Rmk: It's some kind of Laplace transf.

Pf: Procedure: nbd of point $\Rightarrow$ cpt set $\Rightarrow$ general.

1) $\forall x \in X$. $\exists N_x$. nbd of x. st.

$|F(x') - F(x)| \leq \varepsilon$. $I(x') + \varepsilon \geq I(x)$. $\forall x' \in N_x$.

Set $a_n(A) = \int_A e^{nF(x)} d\mathbb{P}_n(x)$

Note $a_n(N_x) \leq e^{n(F(x)+\varepsilon)} \mathbb{P}_n(\bar{N}_x)$

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log a_n(N_x) \leq (F(x) - I(x)) + 2\varepsilon$.

2) $\forall D$ cpt in X. similar as before

$\overline{\lim_{n\to\infty}} \frac{1}{n} \log a_n(D) \leq \sup_x (F(x) - I(x)) + 2\varepsilon$.

3') Find (K^l). s.t. $\mathbb{P}_n(K^l)^c \le e^{-nl}$

$\Rightarrow \overline{\lim_{n\to\infty}} \frac{1}{n} \log \lambda_n(K^l)^c \le \|f\|_\infty - l.$

So $\lim_{n\to\infty} \frac{1}{n} \log \lambda_n \le \max\{\|f\|_\infty - l,\ \sup(F(x)-I(x)) + 2\varepsilon\}$

$\xrightarrow[\varepsilon\to 0]{l\to\infty} \sup(F(x)-I(x))$

4') For lower bound. $\forall x \in X$. $\exists N_x$ nbd of x.

$\lambda_n \ge \lambda_n(N_x) \ge e^{n(F(x)-\varepsilon)} \mathbb{P}_n(N_x)$

$\Rightarrow \underline{\lim_{n\to\infty}} \frac{1}{n} \log \lambda_n \ge \sup_x (F(x)-I(x)) - \varepsilon.$

Rmk: Note bdd and u.s.c. are for upper bound. While l.s.c. is for lower.

Def: $Q_n(A) =: \dfrac{\int_A e^{nF(x)} d\mathbb{P}_n(x)}{\int_X e^{nF(x)} d\mathbb{P}_n(x)}$. for $A \in \mathcal{B}_X$.

Thm. For $(\mathbb{P}_n)$ satisfies LDP with rate function $I(\cdot)$ and $F \in C_B(X)$. Then (Q_n) also satisfies LDP on X. w.r.t $J(x) =: \sup_x (F-I) - (F(x)-I(x))$

Pf: Note that $Q_n(C) = \lambda_n(C) / \lambda_n$.

argue as before. $\Rightarrow \overline{\lim} \frac{1}{n} \log Q_n(C) \le -\inf_C J.$