

# Submanifolds

## (1) Pre:

For  $h: \mathbb{R}^n \rightarrow \mathbb{R}^k$ . Fix  $\vec{x} \in \mathbb{R}^n$ .  $Dh|_{\vec{x}}: \mathbb{R}^n \rightarrow \mathbb{R}^k$ .

where  $Dh|_{\vec{x}} \in M^{k \times n}(\mathbb{R})$ ,  $Dh|_{\vec{x}} = (\frac{\partial h_i}{\partial x_j}|_{\vec{x}})_{k \times n}$ .

Thm. (IFT).

$U \subseteq_{open} \mathbb{R}^n$ .  $F: U \rightarrow \mathbb{R}^n$  smooth. For  $x \in U$ .

If  $DF|_x: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is isomorphism (i.e.  $|DF|_x| \neq 0$ )

Then  $\exists V$  st.  $x \in V \subseteq_{open} U \subseteq \mathbb{R}^n$ .

$F: V \rightarrow F(V)$  is diffeomorphism.

Cor.  $F: U \rightarrow W$  smooth bijection. If  $|DF|_x| \neq 0$ .

$\forall x \in U$ . Then  $F^{-1}: W \rightarrow U$  is smooth.

## (2) Definitions:

① Def: i) An affine subspace of  $\mathbb{R}^n$  is a translation of a linear subspace. i.e.  $A = \{x + v \mid x \in W\}$ .

$W$  is subspace of  $\mathbb{R}^n$ .  $v \in \mathbb{R}^n$ . fix.

e.g. Standard affine subspace:

$$\{(x_1, \dots, x_m, 0, 0, \dots) \mid x_k \in \mathbb{R}\}, m \in [0, n].$$

i.e.  $W = \mathbb{R}^m$ ,  $v = \vec{0}$ .

ii) For  $X$  is smooth manifold.  $Z \subseteq X$ .

$Z$  is  $m$ -dimension submanifold of  $X$  if

$\forall z \in Z, \exists (U, f), z \in U$ . coordinate chart.

and  $\exists m$ -dimension affine subspace  $A \subseteq \mathbb{R}^n$ .

$f: U \xrightarrow{\sim} \tilde{U} \subseteq \mathbb{R}^n, f(U \cap Z) = A \cap \tilde{U}$ .

Remark: We can replace  $A$  by standard affine subspace  $\{(x_1, \dots, x_k, 0, 0, \dots, 0) \mid x_i \in \mathbb{R}\}$ .

Pf:  $\exists \tau: \mathbb{R}^n \rightarrow \mathbb{R}^n$ .  $\tau$  just exchanges the position of component.

replace  $(U, f)$  by  $(U, \tau \circ f)$ .

e.g. i) The interior of  $n$ -dimension manifold with boundary is  $n$ -dimension manifold. For  $\forall (U, f) \in A$ ,  $\tilde{U} \stackrel{\text{open}}{\subseteq} \mathbb{R}^n$ .

ii) The boundary  $\partial X$  of  $n$ -dimension manifold with boundary  $X$  is  $(n-1)$ -dimension. For  $(U, f) \in A$ .

$\Rightarrow f: U \cap \partial X \xrightarrow{\sim} \tilde{U} \cap \{x_1 = 0\}$ .

## ② Smooth Structure:

For  $X$  is  $n$ -dimension manifold.  $Z \subseteq X$  is  $m$ -dimension submanifold of  $X$ . ( $X$  is smooth)

Lemma:  $\forall (U_1, f_1), (U_2, f_2) \in A_X$  (map  $Z$  to standard subspace) induces chart  $(V_1, g_1), (V_2, g_2)$  of  $Z$ .

where  $V_i = U_i \cap Z, g_i: V_i \xrightarrow{\sim} \mathbb{R}^m \cap \tilde{U}_i = \tilde{V}_i$

Then  $g_{12} = g_1 \circ g_2^{-1}$  is smooth as well.

Pf: Let  $U = U_1 \cup U_2$ ,  $V = Z \cap U$ .

By assumption:  $f_1(U) = f_1(U) \cap \mathbb{R}^n$ ,  $f_2(V) = f_2(V) \cap \mathbb{R}^m$ .

Actually, the components of  $\phi_{12}$  equals:

the first  $m$  components of  $\phi_{12}|_{\mathbb{R}^n}$ .

Prop.  $Z$  is a  $m$ -dimension manifold carrying with a smooth structure induced from  $[Ax]$  of  $X$ .

Pf: Apply lemma at every point  $z \in Z$ .

Besides, the smooth structure is indent

with choice of  $Ax$  by compatibility.

(3) Level set:

Note that  $S^1 = \{x^2 + y^2 = 1\}$  is submanifold in  $\mathbb{R}^2$ .

generally, we can ask: (For  $h: \mathbb{R}^n \rightarrow \mathbb{R}^k$ ,

when is  $\{h(x) = c\} \subseteq \mathbb{R}^n$  a submanifold?

① Def: i)  $h: \mathbb{R}^n \rightarrow \mathbb{R}^k$ ,  $h \in C^\infty$ . For point  $x \in \mathbb{R}^n$

is called regular point if  $\text{rank } Dh(x) = k$ .

is called critical point if  $\text{rank } Dh(x) < k$ .

ii)  $h: \mathbb{R}^n \rightarrow \mathbb{R}^k$ ,  $h \in C^\infty$ . For value  $a \in \mathbb{R}^k$ ,

is called regular value if  $h^{-1}(a)$  is set

of regular point. Otherwise it's called

critical value.

iii) Standard projection:  $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^k$ , ( $k \leq n$ )

$$\pi(x_1, \dots, x_n) = (x_{n-k+1}, \dots, x_n), \quad \text{ker } \pi = \mathbb{R}^{n-k}.$$

Remark: Note that  $D\pi = (0|I_k) = \pi: \mathbb{R}^n \rightarrow \mathbb{R}^k$

Thm. (General IFT)

$k \leq n$ .  $U \subseteq \mathbb{R}^n$ ,  $h: U \rightarrow \mathbb{R}^k$ ,  $h \in C^1(U)$ . If  $z$

is a regular point of  $h$ . Then  $\exists V$  neighbour of

$z$ .  $\exists f: V \xrightarrow{\sim} \tilde{V} \subseteq \mathbb{R}^n$ . Diffeomorphism. s.t.

$$h \circ f^{-1} = \pi: \tilde{V} \rightarrow \mathbb{R}^k.$$

Remark: It said we can find  $f$  diffeomorphism.

$$\text{s.t. } f_{n-k+i} = h_i, \forall 1 \leq i \leq k.$$

Pf. Since  $\text{rank } Dh|_z = k$ . reorder  $(x_i)_i^n$ :

$$\text{s.t. } M = \frac{\partial (h_1, \dots, h_k)}{\partial (x_{n-k+1}, \dots, x_n)} \text{ has rank } k.$$

$$\text{let } f(x_1, \dots, x_n) = (x_1, \dots, x_{n-k}, h_1(\vec{x}), \dots, h_k(\vec{x}))$$

$$\text{since } Df|_z = \begin{pmatrix} I_{n-k} & 0 \\ 0 & M \end{pmatrix}. \text{ Apply IFT.}$$

Prop:

$h: \mathbb{R}^n \rightarrow \mathbb{R}^k$ . smooth. If  $k \leq n$ .  $a \in \mathbb{R}^k$  is regular value of  $h$ . Then the level set  $Z_a = h^{-1}(a) \subset \mathbb{R}^n$  is  $(n-k)$ -dim submanifold of  $\mathbb{R}^n$ .

Remark: i.e.  $Z_a$  has codim  $k$ . Intuitively.  $Z_a$  has  $n-k$  freedom.

Pf:  $\forall z \in h^{-1}(c)$ . By general IFT.  $\exists f: V \xrightarrow{\sim} \tilde{V}$ .

$$\therefore f(z_\alpha \cap V) = f(z_\alpha) \cap f(V) = \tilde{z}'^{\alpha} \cap \tilde{V}$$

It forms a chart at  $z$ . So an atlas.

Remark: i) We can generalize it by considering

$h: X \rightarrow \mathbb{R}^k$ .  $X \subseteq_{\text{open}} \mathbb{R}^n$ . (Since the procedure before is local). Then  $Z_\alpha$  is  $(n-k)$ -dim submanifold of  $X$ .

ii) For finding the chart of  $Z_\alpha$ :

Caption about "reorder". If  $\frac{\partial h_1 - h_k}{\partial x_{i_1} \dots x_{i_k}} = k$ .

generally. Let  $f_{ij} = h_{ij}$ .  $f_i = x_i$ .  $\forall i \neq (ij)_i^k$   
to obtain chart of  $\mathbb{R}^k/X$ .

② Sard's Thm:

For  $f: X \subseteq_{\text{open}} \mathbb{R}^k \rightarrow \mathbb{R}^k$ . Smooth. ( $m \leq k$ )

Then the critical value of  $f$  is a Lebesgue null set.

Pf: Proceed by induction on  $k$ :

$k=0$  it's trivial. Suppose it holds for  $k < n$

For the case  $k=n$ :

Denote:  $C = C_f$  (critical points of  $f$ )

$$C_n = \{x \in X \mid \frac{\partial^m f_i}{\partial x_{i_1} \dots \partial x_{i_m}}(x) = 0, 1 \leq j \leq m, 1 \leq i_j \leq n, (i_j)_j^m \subseteq \{1, 2, \dots, n\}\}.$$

$$\dots \dots C_2 \subseteq C_3 \dots \subseteq C_1 \subseteq C.$$

prove: i)  $f(C/C_1)$  ii)  $f(C_0/C_{0+1})$  iii)  $f(C_0), l > \frac{m}{n}-1$

are all  $L$ -null set.

$$\text{Then: } f(C) = \bigcup_{i=0}^p C f(C_i/C_{i+1}) \cup f(C_{p+1}), p > \frac{m}{n}$$

is a null set.

i)  $m=1$ . Then  $C_1 = C$ . trivial.

For  $m > 1$ .  $\forall a \in C/C_1$ . We can find  $V$  nbh of  $a$ .  $V \subset X$ .

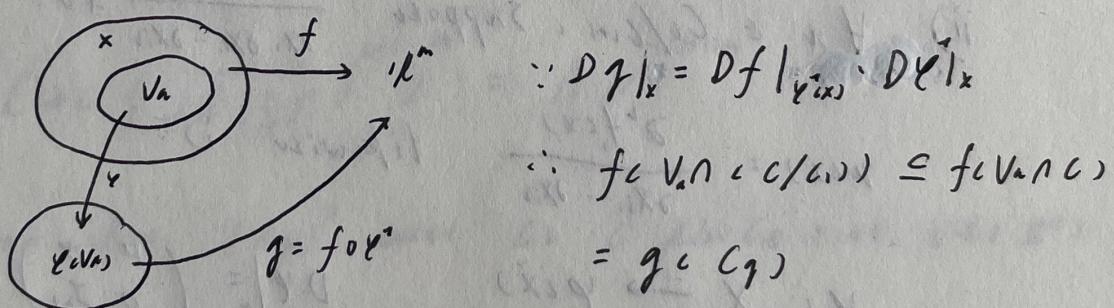
st.  $f(V \cap C/C_1)$  is  $L$ -null.

Then  $f(C/C_1) = \bigcup_{a \in C} f(C/C_1 \cap V_a)$  is  $L$ -null set.

WLOG. Suppose  $\frac{\partial f_i}{\partial x_1}(a) \neq 0$ .

Define:  $\varphi: X \rightarrow \varphi(x)$   $\therefore D\varphi(a) = \begin{pmatrix} \frac{\partial f_i}{\partial x_1}(a) & * \\ 0 & I_{n-1} \end{pmatrix}$   
 $(x_1, \dots, x_n) \mapsto (f_i(x_1), x_2, \dots, x_n)$

Apply IFT:  $\exists V_a \subseteq X$ . St.  $\varphi|_{V_a}$  is diffeomorphism.



For  $C_1$ :

$$\therefore f(t, x_2, \dots, x_n) = (t, q_1, \dots, q_m)$$

$$\therefore Df|_{(t, x)} = \begin{pmatrix} 1 & 0 \\ * & Dg \end{pmatrix}, \tilde{g} = (q_1, \dots, q_m)$$

$$\therefore (t, x) \in C_1 \iff x' \in \{(q_i) \mid \text{fix } t\}$$

$$\therefore C_1 = \bigcup \{t \} \times \{(q_i)\}$$

$$\therefore f(C_1) = \bigcup_{i \in \mathbb{N}} x_i \times \tilde{f}_i(C_{\tilde{f}_i})$$

By inductive assumption  $m(\tilde{f}_i(C_{\tilde{f}_i})) = 0$ .

Note that  $C_1 = \bigcup_{\substack{(j_1, j_2, \dots, j_m) \\ (1, 2, \dots, m)}} \{x \in \mathbb{R}^m \mid r \in Df|_{x_0}^{-1}(j_1, j_2, \dots, j_m)\} \subset m\}$ .

$\therefore g \in C^\infty$ .  $\therefore C_g$  is closed.

For  $V_n \subseteq \mathbb{R}^n$ . Since  $\mathbb{R}^n$  is  $\sigma$ -cpt.  $V = \bigcup_n V_n$ . cpt sets.

$$\therefore f(C_1) = g(C_1 \cap V_{C_1}) = g \left( \bigcup_n C_1 \cap V_{C_1} \cap V_n \right)$$

$= \bigcup_n g(C_1 \cap V_{C_1} \cap V_n)$  is Borel-measurable

Apply Fubini Thm:

$$m(g(C_1)) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \chi_{g(C_1)} = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} \chi_{V_{C_1}} \chi_{g(C_1 \cap V_n)} \right) dx_1 dt$$

$$= 0.$$

ii)  $\forall n \in \mathbb{N}/\{0\}$ . Suppose  $\frac{\partial^k f(x)}{\partial x_1 \partial x_2 \dots \partial x_k} \neq 0$ .

$$\mu(x) = \frac{\partial^k f(x)}{\partial x_1 \partial x_2 \dots \partial x_k} \quad \text{likewise } \nu =$$

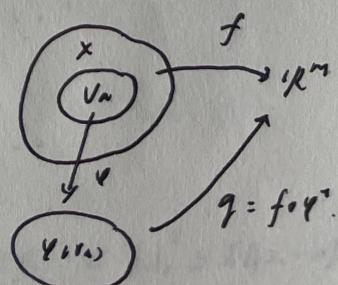
$$\varphi: X \rightarrow \varphi(X) \quad D\varphi|_x = \begin{pmatrix} I & * \\ 0 & I_{n-1} \end{pmatrix}$$

$$(x_1, \dots, x_n) \mapsto (\varphi_1, x_2, \dots, x_n)$$

$\exists V \subseteq X$ .  $\varphi|_V$  is diffeomorphism.

$$\therefore \varphi(V \cap C_0) \subseteq \{0\} \times \mathbb{R}^{n-1}$$

$$\text{Denote } \tilde{f}(y_1, y_2, \dots, y_n) = f(0, y_2, \dots, y_n)$$



$$\therefore f(\cup \cap_{\ell=1}^{\infty} C_{\ell}) \subseteq f(\cup_{\ell=1}^{\infty} (\{0\} \times C_{\ell})) = f(C)$$

$$\therefore \cup \cap_{\ell=1}^{\infty} C_{\ell} \subseteq \cup_{\ell=1}^{\infty} (\{0\} \times C_{\ell}) \quad \therefore \cup_{\ell=1}^{\infty} (\{0\} \times C_{\ell}) \subseteq \{0\} \times C$$

$$\Rightarrow f(\cup \cap_{\ell=1}^{\infty} C_{\ell}) = f(\cup_{\ell=1}^{\infty} (\{0\} \times C_{\ell})) = f(C)$$

By inductive assumption:  $m^*(f(C)) = 0 \quad \therefore m(f(\cup \cap_{\ell=1}^{\infty} C_{\ell})) = 0$

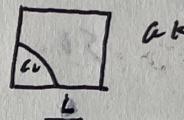
iii)  $\Omega$  is closed cube with width  $L$ .  $\Omega \subseteq X$ .

By Taylor expansion at  $a$ :  $|f(x) - f(a)| \leq C|x-a|^{l+1}$

for  $\forall x \in \Omega$ ,  $a \in \cap_{\ell=1}^{\infty} \Omega_{\ell}$ .

Subdivide  $\Omega$  into  $N^n$  closed identical cubes

Suppose  $(\Omega_k)$  touch  $C_0$



$f(\Omega_k)$  is contained in a close

cube with width:  $2C(\sqrt{\sum_1^l (\frac{L}{N})^2})^{l+1}$  (inside is diagonal length)

$\therefore$  Its volume is:

$$(2C(\sqrt{\sum_1^l (\frac{L}{N})^2})^{l+1})^n = AN^{-m(l+1)n}$$

Note that  $(\Omega_k)$  cover  $C_0$  ( $\dim C_0 \leq n$ ,  $\forall k \in \mathbb{Z}^+$ )

$\therefore f(\Omega \cap C_0)$  can be covered by  $r$  such cubes.

$$\therefore m^*(f(\Omega \cap C_0)) \leq ArN^{-m(l+1)} \leq A \cdot N^n \cdot N^{-m(l+1)}$$

$$\rightarrow 0 \quad (N \rightarrow \infty) \quad \therefore m(f(\Omega \cap C_0)) = 0$$

i.e.  $f(\Omega \cap C_0)$  is  $\mathcal{L}$ -null set for large  $\ell \in \mathbb{Z}^+$ .