

Discrete Martingales.

(1) Conditional Expectation:

① Def: In prob. space (Ω, \mathcal{F}, P) , $X \in \mathcal{F}$, σ -field $\mathcal{G} \subseteq \mathcal{F}$.

$E(X|X) < \infty$. The conditional expectation of X given \mathcal{G} is $Y = E(X|\mathcal{G})$, s.t.

$$\text{i)} Y \in \mathcal{G} \quad \text{ii)} \forall A \in \mathcal{G}, \int_A X dP = \int_A Y dP.$$

Any Y satisfies i), ii) is said a version of $E(X|\mathcal{G})$.

Rmk: Initially we consider $E(X|Y)$. Note for $w_1, w_2 \in \Omega$, $E(X|Y)$ only depends on value of $Y(w)$ but not w . So we only record the partition of Ω by different values of Y , i.e. $\sigma(Y)$.

Written in $E(X|\sigma(Y))$. More generally, consider $\mathcal{G} \subseteq \mathcal{F}$, σ -subfield.

Lemma: Version of $E(X|\mathcal{G}) = Y$ is integrable.

Pf: $A = \{Y > 0\} \in \mathcal{G}$. Then: by $\int_A Y = \int_A X$

$$\int_A Y dP \leq \int_A |X| dP \Rightarrow E(|Y|) \leq E(|X|)$$
$$-\int_{A^c} Y dP \leq \int_{A^c} |X| dP$$

Lemma. (Uniqueness)

If Y, Y' are versions of $E(x|g)$. Then. $Y = Y$. a.s.

Pf: Set $A = \{Y - Y' \geq \varepsilon > 0\}$.

$$0 = \int_A x - x \lambda p = \int_A Y - Y' \lambda p \geq \varepsilon p(A)$$

$\therefore p(A) = 0$. $\forall \varepsilon > 0$. By sym $\Rightarrow Y' = Y$. a.s.

Thm: If $X_+ = X_-$ on $B \in \mathcal{G}$. Then: $E(x_+|g) = E(x_-|g)$ on B

Pf: $A = \{E(x_+|g) - E(x_-|g) \geq \varepsilon > 0\}$. $A \cap B \in \mathcal{G}$.

Lemma (Existence)

Version of $E(x|g)$ always exist.

Pf: 1) For $x \geq 0$:

$$\text{Set } v(A) = \int_A x \lambda p. \quad \forall A \in \mathcal{G} \text{ by DCT. it's measure}$$

Then: $v < \infty$. By R-N Thm: $\frac{\lambda v}{\lambda p}$ exists. p-a.s unique.

$$\text{Set } E(x|g) = \frac{\lambda v}{\lambda p} \geq 0. \text{ integrable.}$$

2) For general x :

$$X = X^+ - X^- \text{. Set } Y_1 = E(X^+|g), Y_2 = E(X^-|g),$$

$Y_1 - Y_2 \in \mathcal{G}$. Integrable. satisfies i). ii)

② Examples:

i) If $X \in \mathcal{F}$. Then: $E(X|g) = X$. a.s.

Pf: Only need to check condition i).

ii) If X is indept with \mathcal{F} . Then: $E(X|g) = E(X)$.

Pf: $E(x) = \text{const} \in \mathcal{F}$. satisfies i)

$$\int_A x dP = E(X I_A) = E(X) E(I_A) = \int_A E(x) dP.$$

iii) If $\lambda = \sum \lambda_i$, $p_{\lambda(i)} > 0$, $\mathcal{F} = \sigma(\lambda_i)_{i \in \mathbb{Z}}$. Then:

$$E(X|g) = \sum \frac{E(X I_{\lambda_i})}{p_{\lambda(i)}} I_{\lambda_i}$$

Pf: It's same of "gross and verify".

Denote: $P(A|g) = E(I_A|g)$, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$\underline{\text{Rmk:}} \text{ So for } g = \sum \lambda_i, P(A|g) = \sum \frac{P(A \cap \lambda_i)}{p_{\lambda(i)}} I_{\lambda_i}$$

iv) If $f(x, y) \sim f(x, y)$, $\int f(x, y) dX > 0, \forall y, E(g(x)) < \infty$.

$$\text{Then: } E(g(x)|Y) = h(y) = \int g(x) f(x, y) dx / \int f(x, y) dx.$$

$$\underline{\text{Pf:}} \quad E(h(y) I_A) = \int_B \int h(y) f(x, y) dx dy$$

$$= \int_B \int g(x) f(x, y) dx dy$$

$$= E(g(x) I_B(Y)) = E(g(x) I_A)$$

where $A \in \sigma(Y)$, $A = \{Y \in B\}$, for $B \in \mathcal{B}_{X'}$.

Note $h(\eta)$ is measurable $\Rightarrow h(Y) \in \sigma(Y)$.

Rmk: To drop $\int f(x, \eta) \lambda x > 0$. Define h by:

$$h(\eta) \int f(x, \eta) \lambda x = \int g(x) f(x, \eta) \lambda x.$$

v) X indep with Y . $\varphi(x, Y) \in L^1$. Set $g(x) = \bar{E}(Y | \varphi(x, Y))$.

Then: $g(x) = \bar{E}(\varphi(x, Y) | X)$.

Pf. For $A \in \sigma(x)$. $A = \{X \in C\}$.

$$\begin{aligned}\int_A \varphi(x, Y) \lambda P &= \bar{E}(\varphi(x, Y) I_{C(x)}) \\ &= \iint \varphi(x, \eta) I_{C(x)} \nu(d\eta) M(dx) \\ &= \int I_{C(x)} g(x) M(dx) = \int_A g(x) \lambda P.\end{aligned}$$

③ Properties:

Thm. For $E|X|, E|Y| < \infty$.

i) $\bar{E}(ax + b | g) = a \bar{E}(x | g) + b$

ii) If $X \leq Y$. Then: $\bar{E}(X | g) \leq \bar{E}(Y | g)$

If $\bar{E}(X|A) \leq \bar{E}(Y|A)$, $\forall A \in \mathcal{F}$. Then: $\bar{E}(X | g) \leq \bar{E}(Y | g)$

iii) $X_n \geq 0$, $X_n \uparrow X$, $\bar{E}(X) < \infty$. Then: $\bar{E}(X_n | g) \uparrow \bar{E}(X | g)$

iv) If $\exists g \subseteq \mathcal{F}$, $\bar{E}(X | g) = 0$. Then: $\bar{E}(X) = 0$

Rmk: For iii). Rescale case by set $-X_n$.

Fatou's Lemma and DCT can be obtained by iii)

Pf: i) is trivial. Check from RHS.

ii) Set $A = \{E^c(X|g) - E^c(Y|g) \geq \epsilon > 0\} \in \mathcal{F}$.

$$\text{Note: } \int_A E^c(X|g) = \int_A X \leq \int_A Y = \int_A E^c(Y|g)$$

iii) Set $Y_n = X - X_n$. Show $E^c(Y_n|g) \downarrow 0$

Since $Y_n \downarrow 0$. Then $Z_n = E^c(Y_n|g) \downarrow$

$\forall A \in \mathcal{F}, \int_A Z_n dP = \int_A Y_n dP$. By DCT on Y_n .

iv) prove $E^c(E^c(X|g)) = E^c(X)$, i.e.

$$\int_n E^c(X|g) dP = \int_n X dP. \text{ Since } n \in \mathcal{F}.$$

Gr. i) Hölder Inequality: $E^c(|XY| | g) \leq E^c(|X|^p | g)^{\frac{1}{p}} E^c(|Y|^q | g)^{\frac{1}{q}}$

ii) $E^c(|X| \geq n | g) \leq E^c(X^2 | g) / n^2$ for $n > 0$.

Thm. (Bayes' Formula)

If $h \in \mathcal{G}$. $P(h|A) = \int_A P(A|h) dP / \int_n P(A|h) dP$.

$$\underline{\text{Pf: }} P(h|A) = \frac{P(A|h)}{P(A)} = \frac{\int_A E^c(h|g) dP}{\int_n E^c(h|g) dP} = \frac{\int_A P(A|g) dP}{\int_n P(A|g) dP}$$

Thm. (Jensen Inequality)

If φ is convex, $E|X|, E|\varphi(x)| < \infty$. Then: we have.

$$\varphi(E^c(X|g)) \leq E^c(\varphi(X) | g)$$

Pf: If φ isn't linear. Note: $\varphi(x) = \sup \{ax + b \mid (a, b) \in S\}$

where $S = \{(a, b) \mid a, b \in \mathbb{R}, ax + b \leq \varphi(x), \forall x\}$.

$$\therefore E^c(\varphi(x) | g) \geq a E^c(x | g) + b. \quad \forall (a, b) \in S. \text{ a.s.}$$

Rmk: $a, b \in Q$ come in because the last inequality hold for (a, b) except a null set $N_{a,b}$.

(*) is from φ'_+, φ'_- exists. $\varphi'_- \leq \varphi'_+$.
 $\exists (a_n) \subseteq Q \rightarrow \varphi'_-(x)$.

Thm. Conditional expectation is contraction in L^p . P31.

Pf: $|E(x|g)|^p \leq E(|x|^p|g)$ follows from Jensen.

$$\Rightarrow E|E(x|g)|^p \leq E(E(|x|^p|g)) = E|x|^p.$$

Thm. If $g \subset g$, $E(x|g) \in g$. Then $E(x|g) = E(x|g)$

Pf: $\forall A \in g$, $\int_A E(x|g) = \int_A x = \int_A E(x|g)$

Thm. If $g_1 \subset g_2$. Then $E(E(x|g_1)|g_2) = E(E(x|g_2)|g_1)$
 $= E(x|g_1)$.

Rmk: In other words. Smaller σ -field always wins.

Pf: $E(x|g_1) \in g_1 \subseteq g_2$. $\int_A E(x|g_2) = \int_A E(x|g_1)$, $A \in g_1$.

Rmk: generally, $E(E(x|g_1)|g_2) \neq E(E(x|g_2)|g_1)$

e.g. $\Omega = \{a, b, c\}$, $g_1 = \sigma\{a\}$, $g_2 = \sigma\{c\}$, $X = I_{\{b\}}$.

Thm. If $X \in g$, $Y, XY \in L'$. Then: $E(XY|g) = XE(Y|g)$

Pf: $X \in \mathcal{Y}(g) \in \mathcal{Z}$. Then to check condition ii).

Consider to approxi. X by simple func's $\in \mathcal{G}$.

$X = X^+ - X^-$. Note for $X = I_B$, $B \in \mathcal{J}$. ✓

Cor. For $X \in L^2$, $E(X|g) = \arg \min_{Y \in \mathcal{G}} E(X-Y)^2$.

i.e. $E(X|g) = P_{L(g)} X$.

Cor. If $\mathcal{G} \subseteq \mathcal{Z}$, $X \in L^2$. Then we have:

$$E(X-E(X|g))^2 + E(E(X|g)-E(X|g))^2 = E(X-E(X|g))^2.$$

Rmk: More information \Rightarrow Smaller MSE.

Cor. Set $\mathcal{G} = \{\alpha, \eta\}$. $\text{Var}(X) = E(\text{Var}(X|g)) + \text{Var}(E(X|g))$

Thm. $X, Y \in L^2$. If $E(Y|g) = X$, $E(X^2) = E(Y^2)$ Then: $X = Y$. a.s.

Pf: Check $E(X-Y)^2 = 0$. Note: $E(XY) = E(XE(Y|g)) = EX$.

Thm. For $X, Y \in L'$. If $E(X|Y) = Y$, $E(Y|X) = X$. Then: $X = Y$. a.s.

Pf: $E((X-Y)(\text{atan}(X) - \text{atan}(Y))) = 0$

Thm. For $Y \in L'$. $E(Y|g) \stackrel{d}{=} Y$. Then $Y = E(Y|g)$. a.s.

Pf: $E|E(Y|g) - c| = E|Y - c| = E(E(Y-c|g)) = E(Y-c)$

$\Rightarrow E|E(Y-c|g)| = E(E(Y-c|g))$

$\therefore |E(Y-c|g)| = E(|Y-c|g)$. a.s.

$\therefore E(Y-c|g) \text{sgn } E(Y|g) - c = E(Y-c) \text{sgn } Y - c|g|$

i.e. $\text{sgn } (E(Y|g) - c) = \text{sgn } (Y - c)$. a.s. ... (*)

If $P(X \in E | G) > Y > 0$. Since $\{E | Y(G) > Y\} = \bigcup_{z \in A} R_z$.

$R_z = \{E | Y(G) > z > Y\}$. However,

$\exists z_0$ s.t. $P(R_{z_0}) > 0$. Contradict with (*).

④ Regular conditional Probabilities:

Def: (Ω, \mathcal{F}, P) is a Prob. space. $X: (\Omega, \mathcal{F}) \rightarrow (S, \mathcal{G})$ is measurable map. $\mathcal{G} \subseteq \mathcal{F}$, σ -subfield. We say:

$M: \Omega \times S \rightarrow [0, 1]$ is regular conditional dist. (r.c.d.) of X on \mathcal{G} i) $\forall A \in \mathcal{G}$. $w \mapsto M(w, A)$ is version of $P(X \in A | G)$ if: ii) For a.e. w . $A \mapsto M(w, A)$ is p.m. on (S, \mathcal{G})

If $S = \Omega$. $X = \mathbb{1}_A$. Then M is called r.c.p.

e.g. $(X, Y) \sim f(x, y) > 0$. $M(y, A) = \int_A f(x) / \int f(x)$
 $M(Y(w), A)$ is r.c.d. of X on $\sigma(Y)$.

Thm: $M(w, A)$ is r.c.d. of X on \mathcal{G} . If $f: (S, \mathcal{G}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ has $E[|f(x)|] < \infty$. Then we have:

$$E[f(X) | \mathcal{G}] = \int f(x) M(w, dx), \text{ a.s.}$$

If $f = f^+ - f^-$. Approx. f by simple functions. by MCT.

Since $f = \mathbb{1}_A$. it holds by def.

Rmk: r.c.d. don't always exist. Note: For $\{A_n\}$ disjoint.

$$P(X \in \bigcup A_n | \mathcal{G}) = \sum P(X \in A_n | \mathcal{G}), \text{ a.s.}$$

But if S contains enough countable collusions.
then it may pile up.

Def: (Ω, \mathcal{F}) is nice if $\exists \varphi: S \rightarrow \mathbb{R}$. bijection.
st. φ, φ^{-1} are both measurable.

Thm. r.c.s exist if (S, \mathcal{S}) is nice.

Rmk: e.g. for $(S, \mathcal{S}) = (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

Thm. For X, Y take values in (S, \mathcal{S}) . nice space. $G = \sigma(Y)$.

Then $\exists M: S \times S \rightarrow [0, 1]$. st.

i) $\forall A, M(Y_{\text{low}}, A)$ is a version of $P(X \in A | G)$

ii) For a.e.w. $M(Y_{\text{low}}, A) = V(A)$ is P.m on (S, \mathcal{S}) .

Rmk: It's generalization of $P(X \in A | Y), A \in \mathcal{B}_{\mathbb{R}}$.

(2) Martingales:

Def: i) (\mathcal{F}_n) is filtration if $\mathcal{F}_n \uparrow, \mathcal{F}_n \subseteq \mathcal{F}$.

ii) (X_n) is adapted to (\mathcal{F}_n) if $X_n \in \mathcal{F}_n, \forall n$.

iii) $\sum \mathbb{E}(X_n)$ is martingale w.r.t. (\mathcal{F}_n) if:

(a) $X_n \in L^1, \forall n$.

(b) (X_n) is adapted to (\mathcal{F}_n)

(c) $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n, \forall n$

Replace (c) " $=$ " by " $<$ ". " $>$ ". it's super/sub martingale.

Rmk: If (X_n) is martingale w.r.t. (G_n) . Set :

$\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Then $\mathcal{F}_n \subseteq G_n$. (X_n) is martingale w.r.t. (\mathcal{F}_n) .

Pf: $X_k \in G_k \subseteq G_n$. $\forall 1 \leq k \leq n$.

① Examples:

i) (linear Mart.)

If $E(g_i) = m = 0$. Then (S_n) is mart. w.r.t. (\mathcal{F}_n) .

$S_n = \sum_0^n g_i$. $\mathcal{F}_n = \sigma(g_1, \dots, g_n)$. $\mathcal{F}_0 = \{\emptyset, \Omega\}$. and

g_i 's are indept. r.v.'s.

Rmk: If $m \leq 0$ or $m \geq 1$. it's super/submart.

Or apply on $\tilde{g}_i = g_i - m$. $S_n - nm$ is mart.

ii) (quadratic Mart.)

If g_i 's are indept. $E(g_i) = m = 0$. $\sigma^2 = \text{Var}(g_i) < \infty$.

Then: $S_n^2 - nr^2$ is martingale. w.r.t. (\mathcal{F}_n) .

iii) (Exponential Mart.)

If Y_n nonnegative i.i.d. $E(Y_n) = 1$. $\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)$.

Set $M_n = \prod_{k=1}^n Y_k$. Then M_n is mart. w.r.t. \mathcal{F}_n .

Thm. For (X_n) is submart. $\forall n \geq m$. $E(X_n | \mathcal{F}_m) \geq X_m$

Pf: $E(X_{m+k} | \mathcal{F}_m) = E(E(X_{m+k} | \mathcal{F}_{m+k}) | \mathcal{F}_m)$

Cor. (X_n) is submart. Then: $\forall n > m$. $E(X_n) \geq E(X_m)$

Thm. (X_n) is mart. w.r.t (\mathcal{F}_n) . φ is convex. s.t. $E|\varphi(X_n)| < \infty$

$\forall n$. Then $\varphi(X_n)$ is submart. w.r.t \mathcal{F}_n .

Pf: Apply Jensen Inequality.

Rmk: i) For concave \Rightarrow supermart.

ii) If $X_n \in L^p$. Then $|X_n|^p$ is submart. w.r.t \mathcal{F}_n .

Thm. (X_n) is submart w.r.t (\mathcal{G}_n) . φ is increasing, convex.

with $E|\varphi(X_n)| < \infty$. $\forall n$. Then: $\varphi(X_n)$ is submart.

Rmk: e.g. X_n submart. $\Rightarrow (X_n - a)^+$ submart.

X_n supermart. $\Rightarrow X_n \wedge a$ supermart.

Cor. If X_n, Y_n are submart. w.r.t. (\mathcal{F}_n) . Then:

So $X_n \vee Y_n$ is.

Pf: $X_n \vee Y_n = X_n I_{\{X_n \geq Y_n\}} + Y_n I_{\{X_n < Y_n\}}$

Def: (M_n) is predictable seq. w.r.t Filtration (\mathcal{F}_n)

if $M_n \in \mathcal{F}_{n-1}$, for $\forall n \geq 1$.

Thm. (X_n) is submart. If $H_n \geq 0$. predictable. b/w.

Then: $(H \cdot X)_n = \sum_{m=1}^n H_m (X_m - X_{m-1})$ is submart.

Rmk: The same holds for supermart. mart. If X is mart. " $H_n \geq 0$ " is unnecessary.

Def: r.v. N is stopping time w.r.t (\mathcal{F}_n) if

$$\{N=n\} \in \mathcal{F}_n, \forall n \in \mathbb{Z}^{\geq 0}.$$

Thm. N is stopping time. (X_n) is submart. Then:

so $X_{N \wedge n}$ is submart.

Pf: Set $H_n = I_{\{N \geq n\}} = I_{\{N \geq n\}^c} \in \mathcal{F}_{n-1}$. Then:

$$X_{N \wedge n} = X_0 + (H \cdot X)_n.$$

② Convergent Thm.

For $a < b$. Set $N_0 = -1$. $\begin{cases} N_{2k+1} = \inf \{m > N_{2k} \mid X_m \leq a\}, \\ N_{2k} = \inf \{m > N_{2k+1} \mid X_m \geq b\}. \end{cases}$
 $\Rightarrow N_j$ is stopping time. $\forall j$.

Note: $\{N_{2k+1} < m \leq N_{2k}\} = \{N_{2k+1} \leq m-1\} \cap \{N_{2k} \leq m-1\}^c \in \mathcal{F}_{m-1}$

$\Rightarrow H_n = I_{\{N_{2k+1} < m \leq N_{2k} \text{ for some } k\}}$ is predictable.

Since $X(N_{2k+1}) \leq a$, $X(N_{2k}) \geq b$.

So between $[N_{2k+1}, N_{2k}]$. X_n cross from a to b .

Set: $U_n = \sup \{k \mid N_{2k} \leq n\}$ numbers of upcrossing by time n .

Thm: (Upcrossing Inequality)

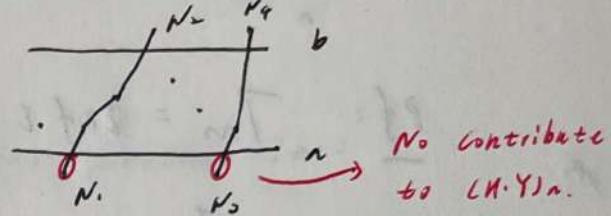
(X_n) is submart. Then $(b-a) E(U_n) \leq E(X_{n-a})^+ - E(X_{a-b})^+$.

Pf: Set $Y_n = n + (X_n - a)^+$. submart.

it crosses same number of times that X_n does.

$$\text{Note: } (b-a)U_n \leq (M \cdot Y)_n$$

$$\text{And } Y_n - Y_0 = (M \cdot Y)_n + (k \cdot Y)_n$$



where $k_n = 1 - M_n \geq 0$ $(k \cdot Y)_n$ is submart.

$$\begin{aligned} \text{So } E(Y_n - Y_0) &\geq E((M \cdot Y)_n) + E((k \cdot Y)_n) \\ &\geq (b-a) E(U_n). \end{aligned}$$

Thm. If (X_n) is submart. s.t. $\sup E(X_n^+) < \infty$.

Then $\exists X \in L'$, s.t. $X_n \xrightarrow{n \rightarrow \infty} X$, a.s. and $E(X) \geq E(X_0)$

Pf: Note: $E(U_n) \leq \frac{\ln 1 + E(X_n^+)}{b-a} < \infty$, $U_n \uparrow n \in \overline{\mathbb{R}}$.

By Fatou's: $E(U) < \infty \Rightarrow U < \infty$, a.s.

$$\Rightarrow \liminf_{n \rightarrow \infty} \{ \lim_{n \rightarrow \infty} X_n < n < b < \lim_{n \rightarrow \infty} X_n \} = 0.$$

i.e. $\lim X_n$ exists. a.s.

By Fatou again: $E(X^+) \leq \sup E(X_n^+)$.

$$\text{Besides: } E(X_0^-) = E(X_0^+) - E(X_0) \leq E(X_0^+) - E(X_0)$$

$$S_0 = E(X_0^-) \leq \sup E(X_n^+) - E(X_0), \text{ i.e. } X \in L'.$$

Rmk: For supermart.: $\sup E(X_n^-) < \infty \Rightarrow$

$$\exists X \in L'. X_n \rightarrow X, \text{ a.s. } (n \rightarrow \infty), E(X) = E(X_0)$$

Cor. (X_n) supermart. $X_n \geq 0$. Then $X_n \rightarrow X$ a.s.
with $E(X) \leq E(X_0)$.

Thm. X_n submart. $\sup_n X_n < \infty$. Set $S_n = X_n - X_{n-1}$. with
 $E(\sup_n S_n^+) < \infty$. Then: X_n converges a.s.

Pf.: $T_m = \inf\{n \mid X_n^+ > m\}$. stopping time.

$$X_{T_m \wedge n}^+ \leq M + S_n^+. \text{ And } X_{T_m \wedge n} \text{ is submart.}$$

$$\text{Note } \sup_n E(X_{T_m \wedge n}^+) \leq M + E(\sup_n S_n^+) < \infty$$

$\therefore X_n$ converges a.s. on $\{T_m = \infty\}$. $\forall m$.

i.e. $X_n I_{\{T_m = \infty\}} \rightarrow X I_{\{T_m = \infty\}}$. a.s. (except N_m).

Note: $\sigma/N = \bigcup_{m \in \mathbb{N}} \{T_m = \infty\}$ by $\sup_n X_n^+ < \infty$. $T_n \nearrow \infty$
 $\Rightarrow \forall w \notin N \cup \bigcup_{m \in \mathbb{N}} \{T_m = \infty\}$. $X_n(w) \xrightarrow{n \rightarrow \infty} X(w)$. $P(N \cup \bigcup_{m \in \mathbb{N}} \{T_m = \infty\}) = 0$.

Rmk: The convergence above won't be L^1 probability:

e.g. f_i i.i.d. $P(f_i = 1) = P(f_i = -1) = \frac{1}{2}$. $S_n = \sum_i^n f_i$

$$N = \inf\{n \mid S_n = 0\}. S_0 = 1. X_n = S_{N \wedge n} \geq 0$$

which is mart $\rightarrow X_\infty = 0$. But $E(X_n) = 1$.

Thm. $X_n, Y_n \geq 0$. integrable. adapted to \mathcal{F}_n . If $E(X_{n+1}/\mathcal{F}_n) \leq X_n + Y_n$. $\sum Y_n < \infty$. a.s. Then: $X_n \rightarrow X$ a.s. which is a finite limit.

Pf: $X_n - \sum_{k=1}^n Y_k$ is supermart. $N_m = \inf\{k \mid \sum_1^k Y_m > m\}$.

Denote by Z_n . $\therefore Z_{n \wedge N_m} \geq -m$.

Z_n converges to finite limit on $\cup\{N_m = \infty\}$, a.s.

Thm (Doob's Decomposition)

Any submart. X_n can be unique written in:

$X_n = M_n + A_n$. M_n is mart. A_n is predictable. $\int A_0 = 0$.

$$\underline{\text{Pf:}} \quad E(X_n | \mathcal{F}_{n-1}) = E(M_n | \mathcal{F}_{n-1}) + E(A_n | \mathcal{F}_{n-1})$$

$$= M_{n-1} + A_n = X_{n-1} - A_{n-1} + A_n.$$

$\therefore A_n$ must be $= \sum_{m=1}^n E(X_m - X_{m-1} | \mathcal{F}_{m-1})$. Then check condition

③ Bounded Increments:

Thm: X_n is mart. $|X_{n+1} - X_n| \leq M < \infty$. Set:

$C = \{\lim X_n \text{ exists, finite}\}, D = \{\overline{\lim} X_n = +\infty \text{ and } \underline{\lim} X_n = -\infty\}$.

$$\text{Then: } P(C \cap D) = 1.$$

Pf: WLOG. set $X_0 = 0$. $N_k = \inf\{t \mid X_t \leq -k\}$.

$-X_{n \wedge N_k} \geq -k - M$. mart. By convergent Thm:

So $\lim X_n + k + M$ exists on $\{N_k = \infty\}$.

$\Rightarrow \lim X_n$ exists on $\cup\{N_k = \infty\} = \{\lim X_n > -\infty\}$.

Apply on $-X_n$. We obtain: $\overline{\lim} X_n < \infty$.

Cor. (2nd Borel-Cantelli Lemma)

\mathcal{F}_n filtration. $B_n \in \mathcal{F}_n$. Then: we have:

$$\{B_n, i.o.\} = \left\{ \sum_{n=1}^{\infty} P(B_n | \mathcal{F}_{n-1}) = \infty \right\}$$

Pf: $X_n = \sum_1^n \mathbb{1}_{B_m}$ submrt. By decomposition:

$$A_n = \sum_1^n E(\mathbb{1}_{B_m} | \mathcal{F}_{m-1})$$

$$M_n = \sum_1^n (\mathbb{1}_{B_m} - E(\mathbb{1}_{B_m} | \mathcal{F}_{m-1})). |M_n - M_{n-1}| \leq 1.$$

Apply Thm: Consider on C.D.

Rmk: It's not enough if: $\sup_n |X_{n+1} - X_n| < \infty$ only

e.g. $P(X_{n+1} = -1 | X_n = 0) = P(X_{n+1} = 1 | X_n = 0) = \frac{1}{2}$

$$P(X_{n+1} = 0 | X_n \neq 0) = 1 - \frac{1}{n^2}. P(X_{n+1} = n^2 X_n | X_n \neq 0) = \frac{1}{n^2}$$

Then X_n is mart. But $P(X_n = n, i.o.) = 1$, $n = \pm 1, 0$.

④ Branching Process:

ζ_i^n i.i.d. nonnegative integer-valued. Set $Z_0 = 1$.

$$Z_n = \begin{cases} \zeta_1^n + \dots + \zeta_{Z_{n-1}}^n & \text{if } Z_{n-1} > 0 \\ 0 & \text{if } Z_{n-1} = 0 \end{cases} \quad P_k = P(\zeta_i^n = k), k \in \mathbb{Z}_{\geq 0}.$$

Rmk: i) 0 is absorbing

ii) Z_n means number of individuals in n^{th} generation.

iii) All states are transient. Since it's irreducible and "0" happens only once. if $p_0 > 0$, $p_0 + p_1 < 1$.

Then in this case: $Z_n \rightarrow \infty$ or get absorbed.

Lemma: $\mathcal{F}_n = \sigma(\mathcal{G}_i^m \mid i \geq 1, 1 \leq m \leq n)$. If $M = E(\mathcal{G}_i^m) \in (0, \infty)$

Then Z_n/M^n is mart. w.r.t. (\mathcal{F}_n) .

Pf: On $\{Z_n = k\} \in \mathcal{F}_n$, $E(Z_{n+1} \mid \mathcal{F}_n) = km = Z_n M$.

Thm. If $m < 1$. Then $Z_n = 0$ for n large enough. So $\frac{Z_n}{M^n} \rightarrow 0$

Pf: $E(Z_n/M^n) = E(Z_0) = 1$. $P(Z_n > 0) \leq E(Z_n) = M^n$.

$\therefore P(Z_n > 0, i.o.) = 0$ by Borel-Cantelli Thm.

Rmk: If each one on average give birth to less one child. Then the species will die out.

Thm. If $m = 1$, $p_i < 1$. Then $Z_n = 0$ for n large enough.

Pf: $Z_n \geq 0$ is mart. So $Z_n \rightarrow Z_\infty$ a.s. finite limit.

Note: $P(Z_{n+1} = k \mid Z_n = k) = \bar{p} < 1 \therefore P(Z_n = k, \forall n \geq N) = \bar{p}^\infty = 0$

$\Rightarrow Z_\infty = 0$, i.e. $\exists \bar{N}_1$, $\forall n > \bar{N}_1$, $Z_n < \epsilon$, i.e. $Z_n = 0$.

Set: $\ell = P(Z_n = 0, \text{ for some } n)$, $\ell = \sum_0^\infty P(Z_n = 0 \text{ for some } n)$

$| Z_1 = k \rangle P(Z_1 = k \mid Z_0 = 1) = \sum \ell^k p_k =: \varphi(\ell)$.

Actually $\varphi(s)$ is generating func. of \mathcal{G}_i^m .

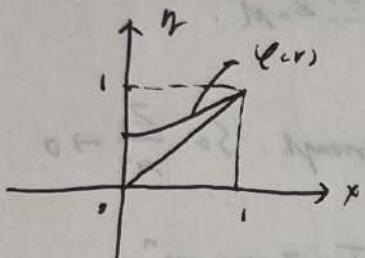
Lemma: (g.f. of Z_n)

$\psi_n(r) = E(r^{Z_n}) = \sum r^k P(Z_n = k)$. Then: $\psi_n = \varphi \circ \psi_1$

$$\text{Pf: } \varphi_n(r) = \sum p(z_1=i) E(r^{z_1} | z_1=i)$$

$$= \sum p(z_1=i) E^i(r^{z_{n+1}}) = \varphi_0 \varphi_{n+1}(r)$$

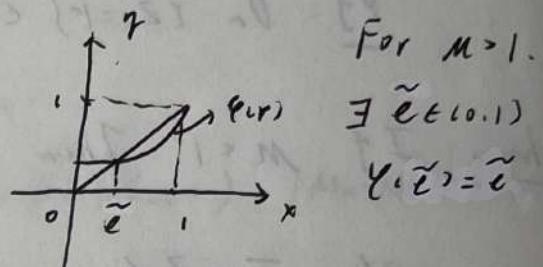
Note: $\varphi' > 0$. $\varphi(1) = m = E(\zeta) = \varphi(0)$. Then:



($m \leq 1$)

For $m \leq 1$. Then

$\varphi(c) = c$ has
unique solution c



($m > 1$)

Theorem. If $m > 1$. Then $\tilde{c} = c = p(z_n=0 \text{ for some } n)$

is the only solution of $\varphi(r)=r$ in $(0,1)$.

Pf: 1') $p_1=1$. is trivial case.

2') $p_1 \in (0,1)$.

Note: $\varphi(1) - \varphi(1-h) \sim mh$ ($\frac{\varphi(1) - \varphi(1-h)}{h} \rightarrow m > 1$)

∴ For h small enough. $\varphi(1-h) < 1-h$.

With $\varphi(1) = 1$. $\varphi(0) > 0$. So \tilde{c} exists.

For uniqueness:

Remark: c_0 is smallest: $\varphi(c_0) = c_0$.

But $x \in (c_0, 1)$. $\varphi(x) < x$. So c_0 is unique one.

For $c = c_0$: (existence)

$$\begin{aligned} \text{Set } \theta_n &= p(z_n=0) = \sum p(z_m=0 | z_i=k) p_k \\ &= \sum p_k \theta_m^k = \varphi(\theta_m) \end{aligned}$$

And $\theta_m \leq \theta_{m+1}$ increasing. $\theta_0 = 0 \leq c_0$

By induction: $\theta_m = \varphi(\theta_{m-1}) \leq \varphi(c_0) = c_0$

With: $\theta_m = \varphi(\theta_{m-1})$. Let $m \rightarrow \infty \Rightarrow \theta_\infty = c_0$

Thm. $\lim_n \frac{Z_n}{n^n}$ isn't $\equiv 0 \Leftrightarrow \sum_{k \geq 1} p_k k \log k < \infty$.

(3) Inequalities:

Thm. X_n is submart. N is stopping time. bdd. ($N \leq k$)

Thm: $E(X_0) \leq E(X_N) \leq E(X_k)$

Pf. 1) $X_{N \wedge n}$ is submart. $S_0 = E(X_0) \leq E(X_{N \wedge k}) = E(X_k)$

2) Set $k_n = \mathbb{1}_{\{N < n\}}$ predictable.

$S_0 (k \cdot X)_n$ is submart. $\Rightarrow E(k \cdot X)_k \geq E(k \cdot X)_0$.

Cor. X_n is submart. $M \leq N$. are stopping times. $N \leq k$. a.s.

Thm: $E(X_M) \leq E(X_N)$.

Pf. Set $Y_n = X_{n \wedge N} \Rightarrow E(Y_n) \leq E(Y_k)$

Rmk: Note " $X_M - X_N$ ". Set $k_n = \mathbb{1}_{\{M \leq n \leq N\}}$
can also prove it.

Rmk: For unbdd stopping time, it may not hold.

e.g. $S_n = \sum_1^n \beta_k$. random walk. $T = \inf \{n \mid S_n = 0\}$.

$$E(S_0) = 1 > 0 = E(S_T)$$

Thm. $M \leq N$ are stopping times. $N \leq k$. Then for submart.

X_n . we have: $X_m \leq E(X_N | \mathcal{F}_m)$.

Pf. Lemma. For $A \in \mathcal{F}_m$. Then $L = M I_A + N I_{A^c}$ is
stopping time. if $M \leq N$.

$$\begin{aligned} \text{Pf: } \{L \leq n\} &= \{M \leq n\} \cap A + \{N \leq n\} \cap A^c \\ &= \square + \{N \leq n\} \cap A^c \cap \{m \leq n\} \in \mathcal{F}_n. \end{aligned}$$

$$\Rightarrow L \leq N \quad (\because E(X_L) \leq E(X_N))$$

$$\text{i.e. } E(X_N I_A) \geq E(X_m I_A), \forall A \in \mathcal{F}_m.$$

Thm. (Doob's Inequality)

X_n is submart. $\bar{X}_n = \max_{0 \leq m \leq n} X_m^+$, $\lambda > 0$. $A = \{\bar{X}_n \geq \lambda\}$.

$$\text{Then: } \lambda P(A) \leq E(X_n I_A) \leq E(X_n^+ I_A)$$

Pf: $N = \inf \{m \mid X_m \geq \lambda \text{ or } m=n\} \leq n$. By Thm. above.

$$\therefore E(X_n) \leq E(X_n), \quad X_n = X_n I_{A^c} + X_n I_A$$

Cor. X_n is submart. $\lambda P(\max_{0 \leq m \leq n} |X_m| \geq \lambda) \leq 2E(|X_n|) + E(|X|)$.

$$\text{Pf: } \{\max_{0 \leq k \leq n} |X_k| \geq \lambda\} = \{\max_{0 \leq k \leq n} X_k \geq \lambda\} + \{\max_{0 \leq k \leq n} -X_k \geq \lambda\}.$$

Thm. (L^p maximum Inequality)

If X_n is submart. Then for $1 < p < \infty$. $E(\bar{X}_n^p) \leq \left(\frac{p}{p-1}\right)^p E(X_n^+)^p$

$$\underline{\text{Pf:}} \quad E(\bar{X}_n \wedge M)^p = \int_0^\infty p\lambda^{p-1} P(\bar{X}_n \wedge M > \lambda) \lambda d\lambda$$

$$\leq \int_0^\infty p\lambda^{p-2} \int X_n^+ I_{\{\bar{X}_n \wedge M > \lambda\}} \lambda p \lambda d\lambda$$

$$= \int X_n^+ \int_0^{\bar{X}_n \wedge M} p\lambda^{p-2} \lambda \lambda d\lambda$$

$$\text{Apply Hölder: } E(|\bar{X}_n \wedge M|^p) \leq \left(\frac{p}{p-1}\right)^p E(X_n^+)^p.$$

Set $M \rightarrow \infty$. Apply MCT.

Rmk: i) For supermart: $E(\max_{m \in \mathbb{N}} X_m^-)^p \leq \left(\frac{p}{p-1}\right)^p E(X_n^-)^p$.

For mart: $E(\max_{0 \leq m \leq n} |X_m|)^p \leq \left(\frac{p}{p-1}\right)^p E(|X_n|)^p$

ii) There's no L' inequality ($p=1$ doesn't hold)

Thm. (L' convergence Thm.)

If X_n is mart. $\sup E(|X_n|)^p < \infty$, for $p > 1$.

Then: $X_n \rightarrow X$ a.s. and in L' .

Pf: $\sup E(X_n^+)^p \leq \sup E(|X_n|)^p < \infty \Rightarrow \exists X. X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$

$E(\sup_{0 \leq m \leq n} |X_m|)^p \leq \left(\frac{p}{p-1}\right)^p E(|X_n|)^p \Rightarrow \sup |X_n| \in L^p$

Note $|X_n - X|^p \leq 2^p \sup |X_n|^p$. Apply DCT.

(4) Square Integrable Martingale:

Next, we suppose: X_n is mart. $X_0 = 0$. $X_n \in L^2$. $\forall n$.

① Decomposition:

Lemma: (Orthogonal of mart. increment)

If $m < n$, $Y \in \mathcal{F}_m$, $Y \in L^2$. Then: $E((X_n - X_m)Y) = 0$

Cor. $E(X_n - X_m)(X_m - X_l) = 0$ for $n \geq m \geq l$.

Lemma: $X_n, Y_n \in L^2$ mart's. Then, we have equation:

$$E(X_n Y_n) - E(X_0 Y_0) = \sum_{m=1}^n E(X_m - X_{m-1})(Y_m - Y_{m-1})$$

Pf. $E(X_n Y_n) = E(X_n Y_n) = E(X_m Y_{m-1})$

Theorem: (Conditional Variance Formula)

$$E((X_n - X_m)^2 | \mathcal{F}_m) = E(X_n^2 | \mathcal{F}_m) - X_m^2.$$

Cor. Apply on Branching Process of $m > 1$. $X_n = \frac{Z_n}{m^n}$.

Then: $E(X_n^2) = 1 + \sigma^2 \sum_{k=1}^{m^n} m^{-2k}$. So: $X_n \rightarrow X$ in L^2 .

$$E(X) = 1, X \neq 0, \{X > 0\} = \{Z_n > 0, \forall n\}.$$

Pf. Note: $\sup_n E(X_n^2) \leq 1 + \sum \sigma^2 m^{-2k} < \infty$.

$$\text{So } X_n \rightarrow X \text{ in } L^2 \Rightarrow X_n \rightarrow X \text{ in } L'$$

For the last: $\{X > 0\} \subseteq \{Z_n > 0, \forall n\}$.

Next show $p(X=0)$ is solution of $\varphi(r)=r$

So that $p(X > 0) = 1-e$. $\{X > 0\} = \{Z_n > 0, \forall n\}$.

Rmk: Return to (X_n) . Note X_n^2 is submart.

By Doob's decomposition: $X_n^2 = M_n + A_n$. where $M_0 = 0$.

$$A_n = \sum_{m=1}^n E(X_m^2 | \mathcal{F}_{m-1}) - X_{m-1}^2 = \sum_{m=1}^n E((X_m - X_{m-1})^2 | \mathcal{F}_{m-1}).$$

it can be thought of as path by path measure
of variance at time n . Since $A_n \uparrow$. Suppose
 $A_\infty = \lim_n A_n$. total variance in the path.

③ Convergence:

Thm. $E(\sup_n |X_n|^2) \leq 4E(A_\infty)$.

Pf. Note: $E(\sup_{m < n} |X_m|^2) \leq 4E(X_n) = 4E(A_n)$

since $E(M_n) = E(M_0) = 0$. Then by MCT.

Thm. $\lim X_n$ exists and finite a.s. on $\{A_\infty < \infty\}$

Pf. $N_n = \inf\{n \mid A_n > n^2\}$. is stopping time. by $A_n \in \mathcal{F}_n$.

$$\therefore E(\sup_n |X_{N_n \wedge n}|^2) \leq 4n^2. \text{ Apply } L^2 \text{ convergence.}$$

So $X_n \rightarrow X$. a.s. on $\{N_n = \infty\} = \{A_n < \infty\}$.

Thm. $f \geq c > 0$ and $f \nearrow \infty$. If $\int_0^\infty 1/f(x)^2 dt < \infty$. Then:

$X_n/f(A_n) \rightarrow 0$. a.s. on $\{A_n = \infty\}$.

Pf. $H_n = f(A_n)^{-1}$ is predictable. $Y_n = (H_n X)_n$ mart.

$$\text{Note } B_{n+1} - B_n =: E((Y_{n+1} - Y_n)^2 | \mathcal{F}_n) = (A_{n+1} - A_n)/f(A_n)$$

$$\text{So } B_n = \sum \frac{A_{n+1} - A_n}{f(A_n)^2} \leq \sum \int_{[A_n, A_{n+1}]} f^2(t) dt < \infty$$

By Thm. above. $Y_n \rightarrow Y_\infty$ finite.

So By Kronecker Lemma. $X_n/f(A_n) \rightarrow 0$.

Rmk: We call $A_n = \sum E((X_m - X_{m-1})^2 | \mathcal{F}_{m-1})$ by increasing process.

Cor. C Strengthened 2nd Borel-Cantelli

B_n is adapted to \mathcal{F}_n . $P_n = P(B_n | \mathcal{F}_{n-1})$.

Then: $\sum I_{B_m} / \sum P_m \rightarrow 1$. a.s. on $\{\sum P_m = \infty\}$.

Pf: We have had $X_n = \sum I_{B_m} - P(B_m | \mathcal{F}_{m-1})$ is martingale.

$$\text{So: } \sum I_{B_m} / \sum P_m = X_n / \sum P_m$$

Consider increasing process of X_n : $A_n = \sum (P_m - P_{m-1})$

$A_n - A_{n-1} = P_n - P_{n-1} \leq P_n$. $\{A_\infty = \infty\} = \{\sum P_n(1-P_n) = \infty\}$.
 $\subseteq \{\sum P_n = \infty\}$. Apply on $f(u) = tuV1$.

Rmk: The increasing steps are identical.

Thm. $E(\sup_n |X_n|) \leq 3E(A_\infty^{1/2})$.

Pf: Set $N_n = \inf \{n \mid A_{n+1} > n^2\}$. Stopping time.

$$P(\sup |X_n| \geq n) = P(\sup |X_n| \geq n, N < \infty) + P(\square, N = \infty)$$

$$\leq P(N < \infty) + P(\sup_m |X_{mn}| \geq n)$$

$$= P(A_\infty > n^2) + P(\sup_m |X_{mn}| \geq n)$$

$$P(\sup_m |X_{mn}| \geq n) \leq \frac{1}{n^2} E(X_{mn}^2) \quad (\text{Doob's inequality})$$

$$= n^{-2} E(A_{mn}) \leq n^{-2} E(A_\infty \wedge n^2)$$

$$\text{Then: Since } E(\sup |X_n|) = \int_0^\infty P(\sup |X_n| \geq a) da$$

By Fubini Thm. We obtain the estimate

$$\text{Rmk: } E(A_n^{\frac{1}{2}}) = \infty \Rightarrow \exists X \in L^1: X_n \xrightarrow{n \rightarrow \infty} X \text{ a.s. / in } L^1$$

$$\text{Since } \sup_n |X_n| \in L^1.$$

(5) Uniform Integrability:

① Converge for Conditional Expectation:

Thm. In (Ω, \mathcal{F}, P) . $X \in L^1$. Then $\{E(x|\mathcal{F})\}_{\mathcal{F}}$ is

σ -subfield $\subset \mathcal{F}_0$ is u.i.

Pf: choose M large st. $E(|x|_1)/m \leq \delta$.

$$\text{Note: } P(E(|x|_1/\mathcal{F}) \geq m) \leq E(|x|_1/m) \leq \delta.$$

$$E(E(|x|_1/\mathcal{F}) I_{\{|E(x|\mathcal{F})| \geq m\}}) \leq$$

$$E(E(|x|_1/\mathcal{F}) I_{\{E(|x|_1/\mathcal{F}) \geq m\}}) =$$

$$E(|x|_1 I_{A_m}) \leq \delta. \text{ Since } P(A_m) \leq \delta.$$

Lemma. For submart. i), ii), iii) are equi.

i) (X_n) is u.i. ii) It converges a.s. in L' .

iii) It converges in L' .

Pf: u.i. $\Rightarrow \sup_n E|X_n| < \infty$ in $L' \Rightarrow$ imp.

Lemma. If mart. $X_n \xrightarrow{L'} X$. Then $X_n = E(X|\mathcal{F}_n)$

Pf: $E(X_n I_A) = E(X_m I_A)$ if $n > m$, $A \in \mathcal{F}_m$.

Since $E(X_n | \mathcal{F}_m) = X_m$, set $n \rightarrow \infty$.

By condition $\therefore E(X I_A) = E(X_m I_A)$

$\therefore X_m = E(X | \mathcal{F}_m)$

Cor. For mart. (X_n) , i), ii), iii), iv). equi.

i) It's u.i. ii) It converges a.s. and in L' .

iii) It converges a.s. iv) $\exists X \in L'$, $X_n = E(X | \mathcal{F}_n)$

Thm. $\mathcal{F}_n \nearrow \mathcal{F} = \sigma(\cup_{n \geq 1} \mathcal{F}_n)$. Then: $E(X | \mathcal{F}_n) \xrightarrow{n \rightarrow \infty} E(X | \mathcal{F})$

a.s. and in L' , for $X \in L'$.

Pf: $Y_n = E(X | \mathcal{F}_n)$ is u.i. mart.

So $\exists Y_\infty \in L'$, $Y_n = E(Y_\infty | \mathcal{F}_n) = E(X | \mathcal{F}_n)$

i.e. $\int_A Y_\infty dP = \int_A X dP$, $\forall A \in \mathcal{F}_n, \forall n$. $\Rightarrow \forall A \in \mathcal{F}$

by Monotone Class Thm and DCT. $\Rightarrow Y_\infty = E(X | \mathcal{F})$

Cor. If $Z_n \uparrow Z_\infty$, $A \in \mathcal{F}_\infty$. Then: $E^c I_A | \mathcal{F}_n \rightarrow I_A$ a.s./L

Rmk: Set $\mathcal{F}_n = \sigma(X_j, j \geq n)$. (X_k) indept. Then:

$I_A = P(A) \in [0, 1]$. We have Kolmogorov 0-1 law

Thm. (Dominated Convergence for conditional Expectation)

If $Y_n \rightarrow Y$, a.s. $|Y_n| \leq Z$, $\forall n$. $Z \in L'$. $Z_n \uparrow Z_\infty$ ($n \rightarrow \infty$)

Thm: $E^c(Y_n | \mathcal{F}_n) \xrightarrow{n \rightarrow \infty} E^c(Y | \mathcal{F}_\infty)$, a.s.

Pf: Set $W_N = \sup \{ |Y_n - Y_m| \mid n, m \geq N \} \leq 2Z$

So $\overline{\lim} E^c(|Y_n - Y| | \mathcal{F}_n) \leq \lim_n E^c(W_N | \mathcal{F}_n) = E^c(W_N | \mathcal{F}_\infty)$

We have: $\begin{cases} |E^c(Y_n | \mathcal{F}_n) - E^c(Y | \mathcal{F}_n)| \leq E^c(|Y_n - Y| | \mathcal{F}_n) \rightarrow 0 \text{ a.s.} \\ |E^c(Y | \mathcal{F}_n) - E^c(Y | \mathcal{F}_\infty)| \rightarrow 0 \text{ a.s.} \end{cases}$

Thm. (L' convergence)

If $Z_n \uparrow Z_\infty$, $Y_n \xrightarrow{L'} Y \in L'$. Then $E^c(Y_n | \mathcal{F}_n) \xrightarrow{L'} E^c(Y | \mathcal{F}_\infty)$

Pf: $E^c(|E^c(Y_n | \mathcal{F}_n) - E^c(Y | \mathcal{F}_\infty)|) \leq$

$E^c(|E^c(Y_n | \mathcal{F}_n) - E^c(Y | \mathcal{F}_n)|) + E^c(|E^c(Y | \mathcal{F}_n) - E^c(Y | \mathcal{F}_\infty)|)$

$\leq \|Y_n - Y\|_{L'} + o(n) \rightarrow 0$

② Application in Backwards Mart's:

Def. It's mart. (X_n) index by negative integer $n \leq 0$.

Thm. $X_{-\infty} = \lim_{n \rightarrow -\infty} X_n$ exists a.s. and in L'

Pf: By upcrossing Inequality:

$$(b-a) E(X_n) \leq E(X_{0-n})^+ - E(X_{n-a})^+ \leq E(X_{0-n})^+$$

$$\Rightarrow E(X_n) < \infty. \exists X_{-\infty}, X_n \rightarrow X_{-\infty} \text{ a.s.}$$

Note: $X_n = E(X_0 | \mathcal{F}_n)$. a.s. so it's a.i.

$X_n \rightarrow X_{-\infty}$ in L' holds.

Rmk: It holds for backwards sub/supermart "a.s. case".

Thm. If $\lim_{n \rightarrow -\infty} X_n = X_{-\infty}$, $\mathcal{F}_{-\infty} = \cap \mathcal{F}_n$. Then: $X_{-\infty} = E(X_0 | \mathcal{F}_{-\infty})$

Pf: $\forall A \in \mathcal{F}_{-\infty} \subset \mathcal{F}_n$. $\int_A X_n dP = \int_A X_0 dP$.

So $\int_A X_0 dP = \int_A X_{-\infty} dP$. as $n \rightarrow -\infty$.

Thm. If $\mathcal{F}_n \downarrow \mathcal{F}_{-\infty}$ as $n \downarrow -\infty$. Then: $E(Y | \mathcal{F}_n) \rightarrow E(Y | \mathcal{F}_{-\infty})$

as and in L' . for $Y \in L'$.

Pf: $Y_n = E(Y | \mathcal{F}_n)$ is backward mart.

$$\Rightarrow Y_n \rightarrow Y_{-\infty} = E(Y_0 | \mathcal{F}_{-\infty}) = E(E(Y | \mathcal{F}_n) | \mathcal{F}_{-\infty}) = E(Y | \mathcal{F}_{-\infty})$$

Thm. If (X_n) has $X_0 \in L'$. Then: $X_n \rightarrow X_{-\infty}$ in L' .

Pf: By Doob's L^p inequality.

Thm. (DCT for Backward Mart.)

$Y_n \rightarrow Y_{-\infty}$ a.s. $|Y_n| \leq Z$ a.s. $Z \in L'$. If $\mathbb{F}_n \downarrow \mathbb{F}_{-\infty}$.

$\text{Thm. } E(Y_n | \mathbb{F}_n) \rightarrow E(Y_{-\infty} | \mathbb{F}_{-\infty})$ a.s.

Pf. Set $M_N = \sup \{ |Y_n - Y_m| \mid n, m < N\}$. $N \in \mathbb{Z}$. identical.

Thm. For backward martingales (X_n) . $\mathbb{F}_n \downarrow \mathbb{F}_{-\infty}$. Then:

$E(X_n | \mathbb{F}_n) \rightarrow E(X_{-\infty} | \mathbb{F}_{-\infty})$ in L' . automatically holds.

(b) Optional Stopping Thm.

① Thm. (X_n) is u.i. submart. For $\forall N$ stopping time. \Rightarrow

X_{Nnn} is u.i. as well.

Pf. X_n^+ is u.i. submart. And $\sup_n E(X_{n+1}^+) \leq \sup_n E(X_n^+) < \infty$

So: $X_{Nnn} \rightarrow X_n$ a.s. $X_n \in L'$

Thm. $E(X_{Nnn} | I_{\{X_{Nnn} > k\}} = E(I_{X_n} | I_{\{X_n > k\}}) +$

$E(I_{X_n} | I_{\{X_n > k\}}) \rightarrow 0$

follows from (X_n) u.i. and $X_n \in L'$.

Thm. $X_n \in L'$. $X_n I_{\{X_n > k\}}$ is u.i. Then: X_{Nnn} is u.i.

so if X_n is submart. then: $E(X_0) \leq E(X_n)$

Pf. Argue same as above. set $n \rightarrow \infty$ in $E(x_0) \leq E(X_{Nnn})$

Cor. If $L \leq m$, Stopping times. Y_{mn} is n.i.

Submart. Thm: $E(X_L) \geq E(Y_m)$. $X_L \geq E(Y_m | \mathcal{F}_L)$

Pf: i) $X_n = Y_{mn \wedge L} = Y_{mn}$ is n.i.

So from: $E(X_{mn}) \leq E(Y_{mn})$. Let $n \rightarrow \infty$.

ii) Set $N = L I_A + M I_{A^c} \leq m$. At \mathcal{F}_L .

Apply the argue i) on $N \leq m$.

Rmk: If we can prove $= \forall n$. Y_{mn} is n.i. The "optional stopping" works!

Thm. If X_n is n.i. submart. Then \forall stopping time N

We have: $E(X_0) \leq E(X_N) \leq E(X_\infty)$. $X_\infty = \lim_n X_n$.

Pf: Set $n \rightarrow \infty$ on $E(X_0) \leq E(X_{mn}) \leq E(X_n)$

Thm. If $X_n \geq Y \in L'$. (X_n) is supermart. N is stopping time.

Then: $E(X_0) \geq E(X_N)$.

Pf: Apply Fatou's Lemma.

Thm. X_n is submart. $E(|X_{n+1} - X_n| | \mathcal{F}_n) \leq B$, a.s. If N is stopping time with $E(N) < \infty$.

Then: $X_{N \wedge n}$ is n.i. $E(X_0) \leq E(X_N)$.

Pf: $|X_{n+1}| \leq \sum_i |X_{n+i} - X_n| I_{\{N>n\}}$. Provo: RNS $\in L'$

$$E(|X_{n+1} - X_n| I_{\{N>n\}}) = E(E(\square | \mathcal{F}_n) I_{\{N>n\}}) \leq B p_{m < N}$$

$$\therefore E(RNS) \leq B \mathbb{E} p_{m < N} = B E(N) < \infty.$$

Thm. $X_n \geq 0$. Supermart. Then: $p(\sup X_n \geq \lambda) \leq \frac{E(X_0)}{\lambda}$

Pf: $T = \inf \{n \mid X_n \geq \lambda\}$. LHS: $p(T < \infty) \leq \frac{E(X_T I_{\{T < \infty\}})}{\lambda} \leq \frac{E(X_0)}{\lambda}$

② Application in Random Walk:

Thm. (Wald's Equation)

i) g_i : i.i.d. $E(g_i) = m$. N is stopping time. St. $E(N) < \infty$.

Then: $E(S_N) = m E(N)$. $S_n = \sum_i^n g_i$

ii) g_i indept. $E(g_i) = 0$. $\text{Var}(g_i) = \sigma^2$. T is stopping time.

$E(T) < \infty$. Then: $E(S_T^2) = \sigma^2 E(T)$. $S_n = \sum_i^n g_i$.

Pf: i) $S_n - nM = X_n$ is mart. Check $E(|X_{n+1} - X_n| \mathcal{F}_n) \leq B$.

ii) $S_n^2 - n\sigma^2 = X_n$ is mart. Check the same things.

Thm. (Symmetric Simple Random Walk)

For g_i : i.i.d. $p(g_i=1) = p(g_i=-1) = \frac{1}{2}$, $S_0 = x$, $N = \min\{n \mid S_n \notin (a, b)\}$.

Then: i) $P_x(S_N=a) = \frac{b-x}{b-a}$, $P_x(S_N=b) = \frac{x-a}{b-a}$.

ii) $E_x(N) = -nb$, $E_x(N) = (b-x)(x-a)$.

Rmk: set $\begin{cases} T_a = \min\{n \mid S_n = a\} \\ T_b = \min\{n \mid S_n = b\} \end{cases} \Rightarrow \{T_a < T_b\} = \{S_N = a\}$.

Pf: i) $P(S_n < \infty) = 1$. (It's transient)

$$2) P(S_n > m \text{ if } n) \leq (1 - 2^{-n})^m = p^m \text{ (consecutive +1)}$$

So $E(S_n) < \infty$.

3') $S_n \in L'$. $S_n I_{\{n < N\}}$ is uni. (but actually)

$$\text{So } X = E(X|S_N) = aP(S_N=a) + b(1-P(S_N=a))$$

4') Consider $X_n = S_n^2 - n$ is mart.

Then: $0 = E_0(S_{N,n}^2) - E_0(N_{N,n})$. Let $n \rightarrow \infty$.

Thm. S_n is simple symmetric random walk. $S_0 = 0$.

$$T_1 = \min\{n | S_n = 1\}. \text{ Then } E(S^{T_1}) = \frac{1 - \sqrt{1-2s}}{s}$$

Pf: Set $X_n = e^{\theta S_n} / \phi(\theta)$. exponential mart.

$$\phi(\theta) = E(e^{\theta S_1}) = (e^\theta + e^{-\theta})/2.$$

Note $X_{T_1,n}$ is bdd $\Rightarrow 1 = E(X_{T_1})$

For convert: Set $\phi(\theta) = \frac{1}{s}$. solve $\theta = \theta(s)$.

Thm. (Asymmetric simple random walk)

For $P(S_i=1) = p \neq q = 1-p = P(S_i=-1)$. $\psi(x) = \left(\frac{1-p}{p}\right)^q$.

i) $\psi(S_n)$ is mart.

ii) $T_x = \inf\{n | S_n = z\}$. Then: $\forall a < x < b$.

$$P_x(T_a < T_b) = \frac{\psi(b) - \psi(x)}{\psi(b) - \psi(a)} \quad P_x(T_a > T_b) = \frac{\psi(x) - \psi(a)}{\psi(b) - \psi(a)}$$

Pf: $T = T_a \wedge T_b \Rightarrow \psi(S_{n \wedge T})$ is bdd. $\therefore \psi(x) = E(\psi(S_{n \wedge T}))$