

Preliminary

(1) Conformal mapping:

Def: For $\phi: D \xrightarrow{\sim} \bar{D}$. conformal isomorphism.

i) $\lambda_{\phi}(z, z') = |\phi(z) - \phi(z')|$. metric on D .

ii) \hat{D} is completion of D w.r.t λ_{ϕ} .

We say $\partial D = \hat{D}/D$ by martin boundary.

Rank: i) \hat{D} is independent of choice of ϕ .

ii) ϕ can be extended to: $\hat{D} \rightarrow \bar{D}$

uniquely. (a homeomorphism)

iii) If ∂D is Jordan. Then $\partial D = \delta D$.

Distortion estimate:

Def: $\mathcal{K} = \{ f: \bar{D} \rightarrow D \mid D \subset D, \text{ simply connected } D \neq \emptyset, f(0) = 0, f'(0) = 1 \}$.

Rank: For $f \in \mathcal{K} \Rightarrow f = z + \sum_{n \geq 2} \alpha_n z^n$.

Prop. If $f \in \mathcal{K}$. Then $|\alpha_2| \leq 2$.

Pf: Lemma. For $f \in \mathcal{N}$. $\exists h \in \mathcal{N}$. odd func.

$$\text{so. } f(z) = h^2(z).$$

Pf: extend $\tilde{f}(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$

$\Rightarrow \tilde{f}'(z) \neq 0$. conformal in ID .

$$\therefore \tilde{f}(z) = e^{\tilde{f}(z)} = \hat{f}^2(z).$$

$$\text{Let } h(z) = z\hat{f}(z), \text{ odd.}$$

It's easy to check $h \in \mathcal{N}$

$$\text{Note } h^2(z) = (z + a_3 z^3 + a_5 z^5 + \dots)^2 =$$

$$f(z) = (z + c_2 z^4 + c_3 z^6 + \dots)$$

$$\Rightarrow a_3 = c_2/2. \text{ Consider } \hat{f}(z) = h(z)/z.$$

$$\hat{f}(z) = z(1 - \frac{c_2}{2}z^{-2} + \dots) = z - \frac{c_2}{2}z^{-1} + \dots$$

The conclusion follows from the next prop.

Def: $\mathcal{H} = \{K \subset \mathbb{C} \mid K \text{ is opt. connected. } \mathbb{C}/K \text{ is connected.}$

$0 \in K. \lim_{z \rightarrow \infty} F(z)/z = 1$. where $F = \mathbb{C}/\bar{D}$

$\rightarrow \mathbb{C}/K$. conformal. isomorphism. } .

Rmk: i) $F(z) = 1/f(z/z_0)$. where $f: ID \xrightarrow{\sim} \mathbb{C}/\mathbb{C}/K$

$I(z) = \frac{1}{z}$. f is conformal. isomorphism.

ii) $F(z) = z + b_0 + \sum_{n \geq 1} b_n/z^n$.

Prop. Using notation above. If $K \in \mathcal{K}$. Then

$$\text{Area}(K) = \pi \left(1 - \sum_{n=1}^{\infty} n |b_n|^2 \right). \geq 0.$$

Pf: For $K_r = \text{For}(D)$. $y_{\text{co}} = \text{For}e^{i\theta}$

$$1) \frac{i}{2i} \int_Y \bar{z} \lambda z = \frac{i}{2i} \int_Y (x - iy) (dx + idy)$$

$$= \frac{i}{2i} \iint_{\text{For}(D)} 2i \lambda x \lambda y$$

$$= \text{Area}(\text{For}(D))$$

$$2) \frac{i}{2i} \int_Y \bar{z} \lambda z = \frac{i}{2i} \int_0^{2\pi} \overline{\text{For}e^{i\theta}} \text{For}e^{i\theta} \lambda \theta$$

$$= \pi r^2 - \sum n |b_n|^2 r^{-2n}.$$

Set $r \rightarrow 1^-$. (by Abel. Thm.)

Or. $\forall f \in \mathcal{K}$. $f = \sum_{n=1}^{\infty} a_n z^n$.

Then $\text{Area}(f(D)) = \pi \sum_{n=1}^{\infty} n |a_n|^2$.

Thm. $\alpha \text{Koebe} - \frac{1}{4}$)

If $f \in \mathcal{K}$. $0 < r \leq 1$. Then $B(0, r/4) \subseteq \text{for}(D)$.

Pf: WLOG. $r=1$. (Or set $f(z)/r$)

Suppose $f = z + \sum_{n=2}^{\infty} a_n z^n : D \rightarrow D$. Fix $z_0 \in D$.

Set $\tilde{f}(z) = z_0 f(z) / (z_0 - f(z)) \in \mathcal{K}$.

expand \tilde{f} . $\Rightarrow s_0 = |a_2| \leq 2$. $|a_2 + \frac{1}{z_0}| \leq 2$

(2) Brownian and Harmonic:

① Thm. (Conformal Variance of BM)

$\phi: D \xrightarrow{\sim} D'$ conformal isomorphism. Fix $z \in D$. $\phi(z) = z'$. Let (B_t) , (B'_t) are 2 complex BM. Starts at z, z' respectively.

$$T = \inf\{t \geq 0 \mid B_t \notin D\}, \quad T' = \inf\{t \geq 0 \mid B'_t \notin D'\}$$

$$\text{Set } \gamma: s \mapsto \int_0^s |\phi'(B_r)|^2 dr, \quad z(s) = \phi^{-1}(s).$$

Then: $(\gamma(t)), (\phi(B_{\gamma(t)}))_{t < \gamma(T)}, \sim^{h.s.} (T', (B'_t)_{t < T'})$

Pf: Only prove $(\phi(B_{\gamma(t)}))_{t < \gamma(T)} \sim^{h.s.} (B'_t)_{t < T}$.

$$\text{Set } \phi(z) = u(z) + iV(z).$$

$$\text{Consider } X_t = u(B_t), \quad Y_t = V(B_t).$$

are c.l.m. since $\Delta u = \Delta V = 0$.

$$\text{With: } \langle X, Y \rangle_t = 0, \quad [X]_t = [Y]_t = \int_0^t |\phi'(B_r)|^2 dr$$

Prop. For BM $(\vec{B}_t)_{t \geq 0}$ starts at x . Set $Z = \inf\{t \geq 0 \mid B_t \notin B(x, r)\}$. Then we have:

$B_Z \sim \sigma_{r(x, \cdot)}$. uniform dist. on $\partial B(x, r)$

Pf: Lemma. (characterization of uniform dist.)

If M is p.m. on $\partial B(x, r)$. st.

invariant under orthogonal linear transf.
on \mathbb{R}^n . Then $m = \delta_r(x)$.

\Rightarrow Note (B_t) is isotropic. as well.

prop. D is proper simply connected. fix $z \in D$.

(B_t) is complex BM starts at z . For

$$T(D) = \inf \{t \geq 0 \mid B_t \notin D\}. \Rightarrow P_z(T(D) < \infty) = 1.$$

Pf: $\phi: D \xrightarrow{\sim} \mathbb{D}$. conformal isomorphism.

$(\phi(B_t))_{t < T(D)}$ is time-change BM.

If $T(D) = \infty$. Note as $t \nearrow T(D)$.

$$|\phi(B_t)| \geq \frac{1}{2} \quad (\phi(\partial D) = \partial \mathbb{D})$$

But it's recurrent \Rightarrow visits $\sum |\phi| < \frac{1}{2}$

at an unbdd set of times!

Thm. (Kakutani's Formula)

u is harmonic on bdd domain \bar{D} . Fix $z \in D$

(B_t) is complex BM. start at z . $T(D) = \inf \{t \geq 0 \mid$
 $B_t \notin D\}$. Then $u(z) = \mathbb{E}_z(u(B_{T(D)}))$

Pf: By Itô's Formula on $u(B_t)$.

Lemma. (Estimate of Harmonic Func.)

u is harmonic in $D \Rightarrow |\frac{\partial u}{\partial x}(z)| \leq \frac{\text{Fill}_m}{z\lambda(z, D)}, \forall z \in D$.

② Harmonic Measure

Thm. (First Exit Distribution)

i) For $\tilde{z} \in \text{ID}$, the first exit dist. of complex BM starts from \tilde{z} at $e^{i\theta}$ is

$$h_{ID}(z, \theta) = \frac{1}{2\pi} \cdot \frac{1 - |\tilde{z}|^2}{|e^{i\theta} - \tilde{z}|^2}$$

ii) For $\tilde{z} = x + iy \in \text{IM}$, the first exit dist. of complex BM. starts from \tilde{z} at u is

$$h_{IM}(z, u) = \frac{1}{\pi} \cdot \frac{\eta^2}{(x-u)^2 + \eta^2}$$

Pf: i) $\phi: \text{ID} \xrightarrow{\sim} \text{ID}$ $\phi(0) = \tilde{z}$.
 $w \mapsto \frac{w+\tilde{z}}{1+w\bar{\tilde{z}}}$

Note that $m^*(B_{T(\text{ID})} \cap \tilde{z})$

$$= \frac{1}{2\pi} \lambda z = m^{\phi(0)}(\phi(B_{T(\text{ID})}) \cap \phi(\tilde{z}))$$

$$\stackrel{(*)}{=} m^{\tilde{z}}(B_{T(\text{ID})} \cap \phi(\tilde{z})) . \quad z = \phi'(e^{i\theta})$$

by conformal invariance of BM. on (*).

$$\Rightarrow h_{ID} \sim \frac{1}{2\pi} |\phi'(z)|^{-1} = \frac{1}{2\pi} |\phi'|_{e^{i\theta}}$$

ii) Set $\phi: \text{ID} \rightarrow \text{IM}$. s.t. $\phi(0) = \tilde{z}$.

Def: For D domain with martin boundary δD

As $t \nearrow T_{DD}$. $B_t \rightarrow \hat{B}_T \in \delta D$ in \hat{D} . we

Say $h_D(z, \cdot)$ is harmonic measure for D

Starts at z . if it's list. of \hat{B}_T on
 δD where B_t is complex B_m Starts at z .

Rmk: B_T kankutani's Formula:

$$n(z) = \mathbb{E}_z \cdot n(\hat{B}_T) = \int_{\delta D} u(s) h_D(z, ds)$$

prop. $h_D(z, \lambda w) \sim \frac{1}{2\pi} \cdot \frac{1}{|\phi'| |\phi''(w)|} \lambda w$

Pf: generalize the pf above.

(3) Green Function:

prop. For $(X_s)_{s \leq t}$ Brownian bridge from $x \neq y$
 in time t . Then $\mathbb{P}_D(t, x, y) = \mathbb{P}(X_s \in D,$

$\forall s \in [0, t]$) is symmetric and joint conti.
 on (x, y) .

Pf: Only prove sym:

For $(W_s)_{s \leq t}$ Brownian bridge in \mathbb{R}^2
 from 0 to 0 . in time t .

$$\Rightarrow X_s = (1 - \frac{s}{t})x + \frac{s}{t}y + \sqrt{t} W_{s/t}$$

Pf: i) Dirichlet heat kernel P_0 on $(0, \infty) \times D^2$ is

$$P_0(t, x, y) = P(t, x, y) \cdot \mathbb{P}_D(t, x, y), \text{ where}$$

$$P(t, x, y) = e^{-|x-y|^2/2t} / 2\pi t.$$

ii) Green func. $h_0(x, \eta) = \int_0^\infty p_0(t, x, \eta, dt)$.

Rmk: i) Note $\chi_{D(t, x, x)} \xrightarrow{t \rightarrow 0} 1 \Rightarrow h_0(x, x) = \infty$.

$$\text{ii) } \int_D h_0(x, \eta) f(\eta) \lambda_M(d\eta) = \int_D \int_0^\infty p_{D(t)} f(\eta) \lambda_M(d\eta)$$

$$= \mathbb{E}_x \left[\int_0^\infty f(B_t) \mathbb{I}_{\{t < T_D\}} dt \right]$$

$$= \mathbb{E}_x \left[\int_0^{T_D} f(B_t) dt \right], M \text{ is Lebesgue or } \mathbb{C}.$$

iii) $D \subseteq \mathbb{C}$ is Greenian if $h_0(x, \eta) < \infty$ for
 $\forall x, \eta \in D$.

prop. i) \mathbb{H} ball domain is Greenian.

ii) In Greenian domain D . h_0 is finite
 and conti. on $D \times D / \Delta$. $\Delta = \{(x, x) \mid x \in D\}$.

prop. (Conformal Invariance)

If $\phi: D \xrightarrow{\sim} \phi(D)$. conformal isomorphism.

Then $h_{\phi(D)}(\phi(x), \phi(\eta)) = h_0(x, \eta)$. $x, \eta \in D$.

Pf: Chark : $\int_D h_{\phi(D)}(\phi(x), \phi(\eta)) g(\eta) \lambda_M(d\eta) =$
 $\int_D h_0(x, \eta) g(\eta) \lambda_M(d\eta)$. $\forall g = (f \circ \phi)^{-1} \phi'$. $\forall f \in C_c^\infty$.

follow from Conformal invariance of B_m .

Rmk: Note $P_{IM}(t, x, \eta) = P(t, x, \eta) - P(t, \bar{x}, \eta)$ by reflec.
 princ. $\Rightarrow h_{IM}(x, \eta) = \frac{i}{\pi} \log \left| \frac{\eta - \bar{x}}{\eta - x} \right|$. $h_{IB}(0, \eta) = -\frac{\log |\eta|}{\pi}$