Harmonic Function

Thm (Extended MMP)

$D \underset{\text{bound}}{\subseteq} \mathbb{C}$, $f \in \mathcal{O}(D)$. If $\exists M > 0$. s.t. $\forall z_0 \in \partial D$.

$\lim\limits_{\substack{z \to z_0 \\ z \in D}} |f(z)| \le M$. Then $\sup\limits_{z \in D} |f(z)| \le M$.

Remark: There's no need f has def on ∂D.

Pf: If f is unbounded on D.

Then $\exists \{z_n\} \subseteq D$. $|f(z_n)| > n$.

suppose $\{z_{n_k}\} \subseteq \{z_n\}$. $z_{n_k} \to z_0$

But $z_0 \notin D$. ($f \in \mathcal{O}(D)$). or $z_0 \in \partial D$ (condition)

$\therefore$ f is bounded on D. Set $C = \sup\limits_{D} |f|$.

$\Rightarrow$ By open map Thm. f can't attain its max!

(1) Harmonic Conjugation:

· Given $f(z) \in \mathcal{O}(D)$. easy to check u, v are harmonic on D.

Conversely, given u is harmonic on D. Does there exist another harmonic $v(z)$. s.t. $u(z) + i v(z) \in \mathcal{O}(D)$?

We will call $v(z)$ is harmonic conjugation of $u(z)$.

To find $v(z)$:

Note that by C-R equation $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

$\Rightarrow v(x,y) = \int u_x(z)\,dy + \phi(x)$. (locally)

$v_x = \frac{\partial}{\partial x}\int u_x(z)\,dy + \phi'(x) = \frac{-\partial u}{\partial y}$. Then obtain $\phi(x)$.

Remark: i) Directly. $v(z) = \int_{z_0}^{z} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$.

ii) e.g. $u(z) = \ln|z|$. on $\{0<|z|<1\} = D$

but the only $v(z)$ should be $\arg(z)$.

s.t. $u+iv \in \mathcal{O}(D)$. $\arg(z)$ isn't conti on D.

(it's a "Func").

The problem is the punctual D. not simply connected.

Thm. $D \subseteq \mathbb{C}$. simply connected. $v = \int_{z_0}^{z} -u_y dx + u_x dy$

is harmonic conjation of harmonic $u(z)$.

Pf: 1) It's well-def. by Green Formula

2) $v(z), u(z) \in C^1(D)$. Satisfies C-R equation.

An easy method to calculate:

For $f \in \mathcal{O}(D)$. $\mathrm{Re} f = u$. we have: $\frac{\bar{f}+f}{2} = u$.

$\therefore f(z) = 2u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) - \bar{f}(z)$.

Note that locally: $f(z) = \sum a_n (z-z_0)^n$

$u(z) \in \mathbb{R}$.

$\therefore \bar{z}$ has been cancelled on RHS. (with $\bar{f}$)

$\therefore$ Replace $\bar{z}$ by any other fixed $\bar{z}_*$.

The equation still holds:

$$\therefore f(z) = 2u\left(\frac{z+\bar{z}_*}{2}, \frac{z-\bar{z}_*}{2i}\right) - \overline{f}(z_*)$$

$$= 2u\left(\frac{z+\bar{z}_*}{2}, \frac{z-\bar{z}_*}{2i}\right) - u(z_*) + i v(z_*)$$

$v(z_*) \in \mathbb{R}$. $\therefore$ $iv(z_*)$ is pure complex number

which will not influence the cancellation. Let $v(z_*)=0$.

$$\therefore f(z) = 2u\left(\frac{z+\bar{z}_*}{2}, \frac{z-\bar{z}_*}{2i}\right) - u(z_*).$$

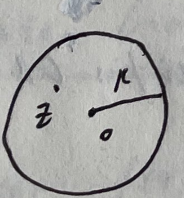
where z_* is seen as a fixed const. $\in \mathbb{C}$.

(2) Dirichlet Problem:

Given $g \in C(\partial D)$. (More generally, $g \in L(\partial D)$)

Does there exist $u \in C(\bar{D})$, s.t. $\Delta u = 0$, $u|_{\partial D} = g$?

① Possion kernel:



$$P_r(\theta - t) = \mathrm{Re}\left(\frac{Re^{it}+z}{Re^{it}-z}\right). \quad z = re^{i\theta}$$

where $0 < r < R$. $0 \le \theta \le 2\pi$.

Since $\dfrac{Re^{it}+z}{Re^{it}-z} \in \mathcal{O}(D)$ $\therefore$ $P_r(\theta - t)$ is harmonic.

<u>Thm.</u> $u(z)$ is harmonic on $D(0, R_1)$. For $0 < R < R_1$.

We have: $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} P_r(\theta - t)\, u(Re^{i\theta})\, d\theta.$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - |z|^2}{|Re^{it} - z|^2}\, u(Re^{i\theta})\, d\theta.$$

Pf: $\exists f \in \mathcal{O}(D(0,R))$. s.t. $\mathrm{Re} f = u$.

By Cauchy Thm: $\begin{cases} f(z) = \dfrac{1}{2\pi i} \oint_{|\zeta|=R} \dfrac{f(\zeta)d\zeta}{\zeta - z} \\ 0 = \dfrac{1}{2\pi i} \oint_{|\zeta|=R} \dfrac{f(\zeta)d\zeta}{\zeta - \frac{R^2}{\bar{z}}} \end{cases}$

$$\Rightarrow f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta}+z}{Re^{i\theta}-z} u(Re^{i\theta})d\theta + i\,\mathrm{Im} f(0).$$

Take the real part, we obtain it!

Remark: $f'(z) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} \frac{Re^{i\theta} u(Re^{i\theta})d\theta}{(Re^{i\theta}-z)^2}$

If $\lim_{z\to\infty} \mathrm{Re} f/z \to 0$. $f \in \mathcal{O}(\mathbb{C})$. Then $f \equiv \text{Const}$.

Def: $P[f](z) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathrm{Re}\left(\frac{Re^{it}+z}{Re^{it}-z}\right) f(Re^{it})dt$

where $|z| < R$. $f: \mathbb{C} \to \mathbb{R}$.

Since $\displaystyle\int_0^{2\pi} \frac{Re^{it}+z}{Re^{it}-z} f(Re^{it})dt \in \mathcal{O}(D)$.

$\therefore P[f](z) = \mathrm{Re}\left(\displaystyle\int_0^{2\pi} \frac{Re^{it}+z}{Re^{it}-z} \frac{f(Re^{it})}{2\pi}dt\right)$ is harmonic.

Thm. Def $F(z)$ on D (with nice property)

$F(z) = \begin{cases} P[g](z). & z \in D \\ g(z). & z \in \partial D \end{cases}$ Then $F \in C(\bar{D})$.

and solves Dirichlet Problem.

Remark: Condition of existence of solution:

$\forall p \in \partial D$. $\exists l$, a line segment. s.t.

p is one of l's endpoint. $l \subseteq \mathbb{C}\backslash D$.

Or we should require that:

$$P[f|_{\partial D_1}](z) = P[f|_{\partial D_2}](z) \cdots = P[f|_{\partial D_k}](z).$$

If D has different (Unpathconnected) boundary $\bigcup_{1}^{k} \partial D_i = \partial D$.

e.g. A counterexample:

$$\begin{cases} g(z) = 0, & |z|=1 \\ g(z) = 1, & z=0 \end{cases}$$

$\partial D = |z|=1 \cup \{0\}$. There's no solution.

② General Form of Mean Value Thm:

· Note that If $f \in \mathcal{O}(D)$, $D(z_0,r) \subseteq D$.

$$f(z_0) = \oint_{D(z_0,r)} \frac{1}{2\pi i} \frac{f(\zeta)d\zeta}{\zeta - z_0} = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0+re^{i\theta})ire^{i\theta}}{re^{i\theta}} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(z_0+re^{i\theta}) + iv(z_0+re^{i\theta}) d\theta$$

$$\therefore u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0+re^{i\theta}) d\theta.$$

Next, we introduce a general form:

Thm. $D \subseteq \mathbb{C}$. $f \in C(D)$. For any $a \in D$.

$$\exists \{r_n\} \to 0. \text{ s.t. } f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+r_n e^{i\theta}) d\theta$$

Then f is harmonic on D.

Pf: Harmonic is a local property

suppose $\overline{D(a,R)} \subseteq D$.

$h(z) = P[f|_{\partial D(a,R)}](z)$. is harmonic.

prove: $h(z) = f(z)$ on $\overline{D(a,R)}$.

Denote: $m = \max\limits_{\overline{D(a,R)}} g(z)$. $g(z) = h(z) - f(z)$

It suffices to prove: $m = 0$.

If $m > 0$. Set $E = \{g(z) = m\}$. closed by conti.

$E \cap \partial D(a,R) = \emptyset$. Since $g(z) = 0$ on $\partial D(a,R)$

Choose $p \in E$. st. $\mathrm{dist}(p,a) = \max\limits_{x \in E} \mathrm{dist}(x,a)$

For p. $\exists \{r_n\} \to 0$. $f(p) = \frac{1}{2\pi}\int_0^{2\pi} f(p + r_n e^{i\theta}) d\theta$.

But $\forall r_n > 0$. $D(p, r_n) \nsubseteq E$.

$\therefore g(p) = \frac{1}{2\pi}\int_0^{2\pi} h(p + e^{i\theta} r_n) - f(p + r_n e^{i\theta}) d\theta < m$.

which is a contradict! $\therefore m \le 0$.

By symmetry $\therefore m = 0$. $h(z) = f(z)$ on $\overline{D(a,R)}$.

Remark: i) A harmonic function is determined by its boundary value. from maximal module principle. i.e. $u(z) = P[u|_{\partial D}](z)$.

ii) From $u(z) = P[u|_{\partial D}](z)$. We can also know a harmonic function is real part of a holomorphic function.

③ Removable singularity of Harmonic Func:

Thm. For $u(z)$ harmonic on $D/\{p\}$. bounded.

It can be redefined to be a harmonic function on D.

Pf: Suppose $p=0$. $\overline{D(0,r)} \subseteq D$.

$h(z) = P[u|_{\partial D(0,r)}](z)$. $\phi(z) = h(z) - u(z)$.

prove: $\phi \equiv 0$ on $\overline{D(0,r)}$

Let $\phi_\varepsilon(z) = \phi(z) + \varepsilon \ln \frac{|z|}{r}$. $\therefore \phi_\varepsilon \equiv 0$ on $\partial D(0,r)$

(Control the boundary value. Then by MMP)

$\exists \delta > 0$. small enough. st. $\phi(z) + \varepsilon \ln \frac{\delta}{r} < 0$.

$\Rightarrow \phi_\varepsilon(z) \le 0$. on $0 < |z| \le r$. Let $\varepsilon \to 0^+$

$\therefore \phi(z) \le 0$. by symmetry $\phi(z) \equiv 0$.

④ Harnack Thm:

Lemma. (Harnack Inequality)

$\{u_n\}$ increasing on n. uniform with z.

harmonic on $D(0,R)$. $u_n \ge 0$. Then

$$\frac{R-r}{R+r} u(a) \le u(a + re^{i\theta}) \le \frac{R+r}{R-r} u(a)$$

Pf. $P_r(\theta - t) = \dfrac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} \in [\frac{R-r}{R+r}, \frac{R+r}{R-r}]$

Thm. $D \subseteq \mathbb{C}$. $\{u_n\}$ harmonic on D. simply connected.

i) If $u_n \xrightarrow{u.c.c} u$. Then u is harmonic

ii) If $u_1 \le u_2 \le u_3 \cdots \le u_n \le \cdots$ uniform with z.

Then either $u_n \xrightarrow{u.c.c} u$ or $\{u_n\}$ diverges to infinite for every point.

Pf: i) $u_n = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)\, u_n(a+Re^{i\theta})\, d\theta$

$\xrightarrow{u} \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-t)\, u(a+Re^{i\theta})\, d\theta = u$. on cpt set.

ii) WLOG. suppose $u_n \ge 0$. Or $u_n = u_n - u_1 \ge 0$.

Set $E = \{u_n \text{ coverges}\}$. $F = \Omega / E$.

By Harnack Inequality. for $r < \frac{R}{2}$.

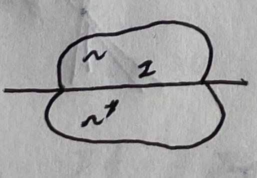
we have: $\frac{1}{3} u_n(a) \le u_n(z) \le 3 u_n(a)$. $|z-a| < r$.

Check E and F are both open!

(3) Schwartz Reflection for harmonic Func.

Thm. u is harmonic on Ω. $u(z) \in C(\Omega \cup I)$, $u|_I = 0$

Then define:
$$h(z) = \begin{cases} u(z). & z \in \Omega \\ 0, & z \in I \\ -u(\bar{z}). & z \in \Omega^* \end{cases}$$



h is harmonic on $\Omega \cup I \cup \Omega^*$.

Pf: Check the general mean value property.

For $a \in \Omega$ or Ω^*. it holds

For $a \in I$. $\forall r$. $0 = \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta})\, d\theta$.

$\therefore$ h is harmonic on $\Omega \cup I \cup \Omega^*$.

Remk: Ideal is from $\overline{f(\bar{z})} = -(-u(\bar{z}) + i v(\bar{z}))$