Linear Fractional Transform

· **Def.** $\varphi(z) = \dfrac{az+b}{cz+d}$. $a, b, c, d \in \mathbb{C}$. $ad-bc \neq 0$.

Then easy to check: $\overline{\mathbb{C}}_\infty \underset{\varphi}{\cong} \overline{\mathbb{C}}_\infty$

(1) Properties:

① Elementary Group:

φ is generated by the following four kinds elementary transformations:

i) $e^{i\theta}z$ ii) $z+a$

iii) $\frac{1}{z}$ iv) az.

② Fixed Points:

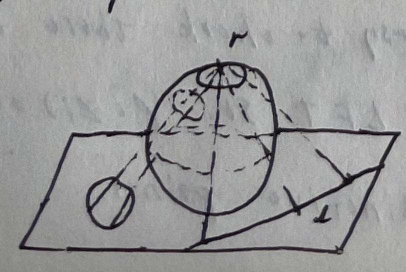
For $\varphi(z_0) = z_0$. z_0 is called fixed point.

φ has at most 2 fixed points except $\varphi = id = z$.

③ Generalized cycle:

We see a line pass ∞ as a circle.

Since it projects on Riemann Surface S^2 will produce a circle cross "N" on S^2.



Thm. C is a circle on $\overline{\mathbb{C}}_\infty$. Then φ will map C to another circle $\tilde{C}$ on $\overline{\mathbb{C}}_\infty$

Pf: Check 4 kinds of LFT in order.

(2) **Cross Ratio:**

Def: $[z_1, z_2, z_3, z_4] = \dfrac{z_1 - z_2}{z_1 - z_4} \Big/ \dfrac{z_3 - z_2}{z_3 - z_4}$

is cross ratio of distinctive $\{z_i\}_1^4$

Thm. $[z_1, z_2, z_3, z_4] = [\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)]$

For φ is LFT.

Pf: $f(z) = [z, \varphi(z_2), \varphi(z_3), \varphi(z_4)]$

For $h(z) = T \circ \varphi^{-1}$. $T(z) = [z, z_2, z_3, z_4]$.

Then $h \circ f^{-1}$ fix 3 points $\therefore h = f$.

$\therefore T(z) = [\varphi(z), \varphi(z_2), \varphi(z_3), \varphi(z_4)]$

Thm. $\{z_i\}_1^4$ distinctive, on a same circle

$\Leftrightarrow$ $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.

Pf: We can directly check $\arg(\text{Cross Ratio}) = \pi$ or 0.

By Thm above: $[z_1, z_2, z_3, z_4] = [\varphi(z_1), 0, 1, \infty]$

$\therefore [z_1, z_2, z_3, z_4] \in \mathbb{R} \Leftrightarrow \varphi(z_1) \in \mathbb{R} \Leftrightarrow z_1$ on $C(\{z_i\}_2^4)$

Remark: It easy to check there exists unique φ is LFT. s.t. $\varphi(z_i) = w_i$, $1 \le i \le 3$, distinctive points.

(3) Aut ($\overline{\mathbb{C}_\infty}$):

Aut ($\overline{\mathbb{C}_\infty}$) = { φ | φ is a LFT }.

Pf: 1°) For $\varphi(x) = \dfrac{az+b}{cz+d}$. $\varphi \in \theta(\mathbb{C})$. when $z \neq -\frac{d}{c}$

we only need to consider $\infty \to \infty$

or finite point $\to \infty$, or $\infty \to$ finite point.

Near ∞. choose coordinate: $(U(\infty), \frac{1}{z})$.

2°) $\forall \varphi \in$ Aut ($\overline{\mathbb{C}_\infty}$).

Suppose $\varphi: 0, 1, \infty \to \alpha, \beta, \gamma$.

Set $T(z) = [z, \alpha, \beta, \gamma]$. $\therefore T \circ \varphi \in$ Aut ($\overline{\mathbb{C}_\infty}$)

$T \circ \varphi: 0, 1, \infty \to 0, 1, \infty$

Since $T \circ \varphi|_{\mathbb{C}} \in$ Aut ($\mathbb{C}$). $\therefore T \circ \varphi|_{\mathbb{C}} = az+b$.

(4) Symmetry:

Thm. C is a circle on $\overline{\mathbb{C}_\infty}$. $z_2, z_3, z_4 \in C$. distinctive

Then z_1, z^* are symmetric w.r.t C $\Leftrightarrow$

$$[z_1, z_2, z_3, z_4] = \overline{[z^*, z_2, z_3, z_4]}$$

Pf: 1°) C is a line.

Note that translation and rotation won't change the relative position of z_1 and z^*. suppose C is X-axis.

2°) For C is a cycle. $|z-a|=r$.

Choose $\varphi(z) = z-a$. retain the cross ratio!

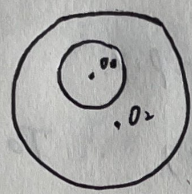
Cor. z is symmetric with z^*. w.r.t circle C

φ is LFT. Then $\varphi(z)$ is symmetric

with $\varphi(z^*)$. w.r.t $\varphi(C)$.

(5) Application:

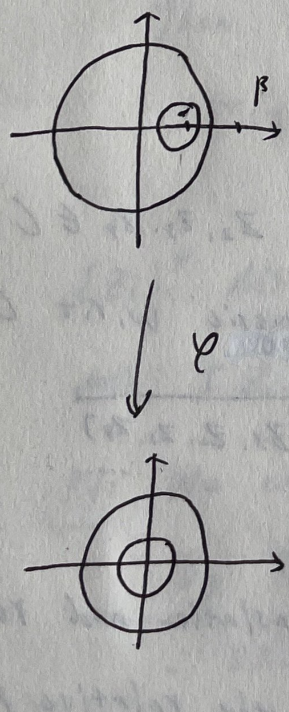
①



$\cong A$. A is some annulus.

Pf: WLOG. suppose $C_1: |z-a| \le r_1$. $C_2: |z| \le r_2$.

$r_2 > 2r_1$. $a < r_2 - r_1$. (By rotation, translation)



Find α, β on X-axis, s.t.

$\bar{\alpha}\beta = r_2^2$. $(\overline{\alpha - a})(\beta - a) = r_1^2$

Then α, β. sym. w.r.t C_1, C_2

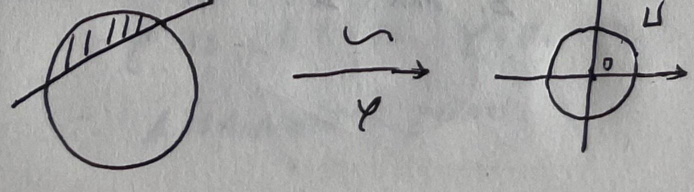
Note that $\varphi(z) = \dfrac{z-\alpha}{z-\beta}$ map

$\alpha \to 0$. $\beta \to \infty$. sym. w.r.t C_1, C_2

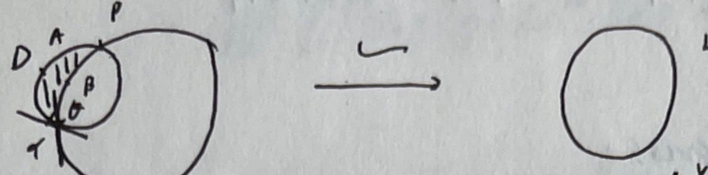
$\therefore \varphi(C_1), \varphi(C_2)$ are circles

with center at origin.

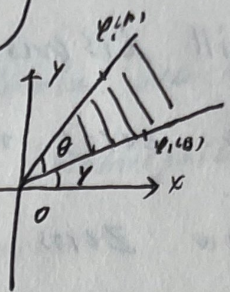
Remark: For C_1 is a line (Degenerate)



(2) Find a biholomorphism φ:



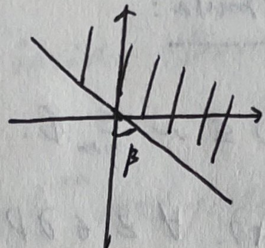
i) $\varphi_1(z) = \dfrac{z-\alpha}{z-\beta} : D \longrightarrow$



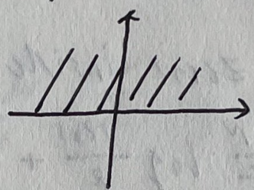
$(0 < \theta < \frac{\pi}{2})$

ii) Consider a holomophic branch : $\varphi_2(z) = z^{\frac{\pi}{\theta}} = e^{\frac{\pi}{\theta}\log z}$

$\varphi_2 : \varphi_1(D) \longrightarrow$



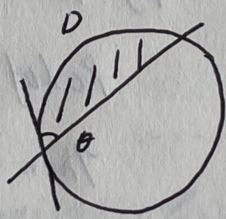
iii) $\varphi_3 : \varphi_2 \circ \varphi_1(D) \longrightarrow$



$\varphi_3 = e^{i(z-\beta)}$

iv) By Cayley Transform: $\varphi_4 = \dfrac{-z+i}{z+i} \longrightarrow U$.

__Remark:__ Degenerate Case:

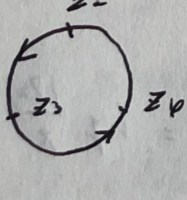


$\cong U$

(6) __Orientation:__

. For C is a cycle. $z_1, z_2, z_3 \in C$. distinct.

Then (z_2, z_3, z_4) defines an orientation of C.



Left side: $\{z \mid \operatorname{Im}[z, z_2, z_3, z_4] > 0\}$

Right side: $\{z \mid \operatorname{Im}[z, z_2, z_3, z_4] < 0\}$

__Oriental Principle:__

φ is LFT. w.r.t (z_2, z_3, z_4) of C. $(\varphi(z_2), \varphi(z_3), \varphi(z_4))$

of $\varphi(C)$. Then the left side of C correspond left side of $\varphi(C)$