Holomorphic Functions

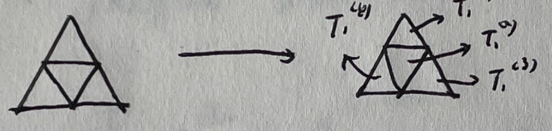
(1) Cauchy Thm:

① Goursat's Thm:

$\Omega \underset{open}{\subseteq} \mathbb{C}$. $f \in \mathcal{O}(\Omega)$. Then $\forall T$ triangle in Ω. we have: $\int_T f(z)\,dz = 0$.

Pf: By contradiction:

$\exists T_0 \subseteq \Omega$. $\int_{T_0} f(z) = c_0 \neq 0$.



T_n

By Drawer Theory we can construct:

$|\int_{T_n^{(k_n)}} f(z)\,dz| \geq \frac{1}{4^n} c_0$. $\exists z_0 \in \bigcap_1^\infty T_n^{(k_n)}$ (Nested Set)

Since $f \in \mathcal{O}(\Omega)$. $\therefore f(z) = f(z_0) + f'(z_0)(z - z_0) + \phi(z)$

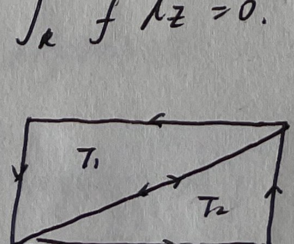
$\max_{z \in T_n^{(k_n)}} |\phi(z)| = \varepsilon_n \to 0$. when $n \to \infty$.

Then it will contradict by estimate c_0!

Cor. $f \in \mathcal{O}(\Omega)$. For any rectangle $R \subseteq \Omega$.

Then $\int_R f\,dz = 0$.

Pf:

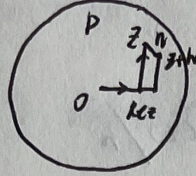


$R = T_1 \cup T_2$

② Local existence of primitive:

Thm. $f \in \mathcal{O}(D)$. Then f has a primitive in D.

Pf: Def: $F(z) = \int_{\gamma_z} f(z)\,dz$



where $\gamma_z = [0, \mathrm{Re}\,z] \times \{0\} \cup \{\mathrm{Re}\,z\} \times [0, \mathrm{Im}\,z]$

(WLOG. $\mathrm{Re}\,z, \mathrm{Im}\,z > 0$)

check : $F(z+h) - F(z) / h \to f(z)$. $h \to 0$.

$F(z+h) - F(z) = \int_\eta f(z)\,dz$, by cancellation of Goursat Thm, where η is segment of z to $z+h$

By conti. $f(w) = f(z) + \varphi(w)$, $\varphi(w) \to 0 \ (w \to z)$

Cor. (Cauchy Thm in Disc)

$f \in \mathcal{O}(D)$. γ is closed curve in D. Then $\oint_\gamma f\,dz = 0$

③ Cauchy's integral formula:

Thm $f \in \mathcal{O}(\Omega)$. $\bar{D} \subseteq \Omega$. $C = \partial D$ with positive orientation.

Then $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z}$, $\forall z \in D$.

Pf: $\frac{1}{2\pi i} \oint_C \frac{f(\zeta)\,d\zeta}{\zeta - z} - f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta$

$= \frac{1}{2\pi i} \oint_{D(z,\varepsilon)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta$. $\forall \varepsilon > 0$.

Since $\frac{f(\zeta) - f(z)}{\zeta - z} \in \mathcal{O}(D / D(z,\varepsilon))$. Let $\varepsilon \to 0$.

Remark: It can be extended C to any Jordan curve $\gamma \subseteq \Omega$. $\frac{1}{2\pi i} \oint_\gamma f(\zeta)/\zeta - z \, d\zeta = f(z)$

Cor. $f \in \mathcal{O}(\Omega)$. $f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}$. $\forall n \in \mathbb{Z}$

Pf: Induction on n.

Check on $f^{(n)}(z+h) - f^{(n)}(z)/h$

Cor. (Cauchy Inequality)

$f \in \mathcal{O}(\Omega)$. $\bar{D}(z_0, R) \subseteq \Omega$. $C = \partial D$. $\|f\|_C = \sup_{z \in C} |f(z)|$

Then we have: $|f^{(n)}(z_0)| \leq \frac{n! \|f\|_C}{R^n}$

Cor. (Liouville Thm)

$f \in \mathcal{O}(\mathbb{C})$. bounded. Then $f \equiv$ const.

Pf: $\forall z_0 \in \mathbb{C}$. $|f'(z_0)| \leq \frac{M}{R}$. Let $R \to \infty$.

④ Well-def primitive of holomorphic Func.:

Recall: Ω is simply connected $\Leftrightarrow$

$\forall$ curve $\gamma_0, \gamma_1 \subseteq \Omega$ s.t. $\gamma_0(a) = \gamma_1(a)$

$\gamma_0(b) = \gamma_1(b)$. on $[a,b]$. Then γ_0 is homotopic to γ_1 on $[a,b]$.

Thm. $f \in \mathcal{O}(\Omega)$. $\gamma_0, \gamma_1 \subseteq \Omega$. they're homotopic.

Then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$

Pf: There exist $\gamma_s(t) = F(s,t)$. $0 \le s \le 1$. $a \le t \le b$.

$\gamma_0 \xrightarrow{\text{conti}} \gamma_1$, when $s: 0 \to 1$. by def of homotopic

1°) Denote $K = F([0,1] \times [a,b])$ cpt.

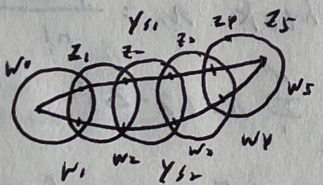
$\therefore$ dist$(K, \Omega^c) \triangleq d > 0$. Let $\varepsilon < \frac{d}{3}$

2°) Since exists $\delta > 0$. s.t. $|s_1 - s_2| < \delta$. Then:

$\sup |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \varepsilon$. By cpt of $[0,1]$.

prove: $\int_{\gamma_{s_1}} f(z)\,dz = \int_{\gamma_{s_2}} f(z)\,dz$

3°) Since $\gamma_{s_1}, \gamma_{s_2}$ are closed enough.



$\exists \{D_i\}_1^N$ with $r < \varepsilon$. covers $\gamma_{s_1}, \gamma_{s_2}$.

Note that on the intersection of D_i the primitive of $f(z)$ only differs by a constant.

(i.e. F_i, F_{i+1} is primitive on D_i, D_{i+1} respectively. Then $F_{i+1}(z) - F_i(z) = \text{constant}$.

for $\forall z \in D_i \cap D_{i+1}$)

$\therefore$ Partition $\gamma_{s_1}, \gamma_{s_2}$ into $\{z_i\}_0^N$. $\{w_i\}_0^N$

$z_i, w_i \in D_i \cap D_{i+1}$. $z_0 = w_0$. $z_N = w_N$.

Remark: It's well-def that let $F(z) = \int_\gamma f(z)\,dz$.

where γ is arbitrary curve from z_0 to z, lying in simply connected domain Ω.

(2) Expansion of Series:

Thm. $f \in \theta(\Omega)$. $\Leftrightarrow$ $f(z) \in A(\Omega)$.

Pf: ($\Rightarrow$). $\forall z_0 \in \Omega$. $\overline{D(z_0)} \subseteq \Omega$. $C = \partial D$.

Note that. $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)d\zeta}{\zeta - z}$

$= \frac{1}{2\pi i} \oint_C \frac{1}{\zeta - z_0} \frac{f(\zeta)d\zeta}{1 - \frac{z-z_0}{\zeta - z_0}}$

$= \frac{1}{2\pi i} \oint_C \frac{1}{\zeta - z_0} \sum \left(\frac{z-z_0}{\zeta - z_0}\right)^n f(\zeta)d\zeta.$

$\triangleq \sum a_n (z-z_0)^n$. check $a_n = \frac{f^{(n)}(z_0)}{n!}$

($\Leftarrow$) $f(z) = \lim_{N\to\infty} \sum_1^N a_n (z-z_0)^n$

Since $f_N(z) = \sum_1^N a_n (z-z_0)^n \in \theta(\Omega)$

Thm. (Uniqueness)

$f \in \theta(\Omega)$. If $\exists \{z_k\} \subseteq f^{-1}(0) \subseteq \Omega$

$\{z_k\} \to z_0$ in U open $\subseteq \Omega$ (Connected)

Then $f \equiv 0$. $\forall z \in \Omega$.

Pf: Expand f at $z_0 \in D(z_0, \varepsilon) \subseteq U$:

$f = \sum a_n (z-z_0)^n = a_m (z-z_0)^m (1 + g(z-z_0))$

where $a_m \neq 0$. (m is the least integer)

It's a contradiction. Since $\exists N,\ n>N,\ \{z_k\}_n \subseteq D(z_0,\varepsilon)$

But $f(z_k) = a_m(z_k - z_0)^m (1 + g(z_k - z_0)) \neq 0$.

(N satisfies: $|g(z_k - z_0)| < \frac{1}{2}$, $\forall k > N$. Since $g(z_k - z_0) \to 0$)

$\therefore f \equiv 0$ in $D(z_0, \varepsilon)$.

Let $\tilde{U} = \{f = 0\}$. it's open from above.

And $\tilde{U}$ is closed too. $\therefore \tilde{U} = \Omega$. Since $U \neq \emptyset$.

<u>Cor.</u> All zeros of analytic functions are isolated.

<u>Cor.</u> $f = g$ on a set with accumulation $\subseteq \Omega$.

Then $f \equiv g$, on Ω.

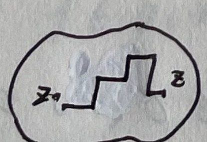
(3) <u>Applications:</u>

① <u>Morera Thm:</u>

$f \in C(\Omega)$. $\forall$ triangle $T \subseteq \Omega$, $\int_T f\,dz = 0$.

Then $f \in \mathcal{O}(\Omega)$.

<u>Pf:</u> It's easy to def $F(z) = \int_\gamma f(z)\,dz$.



where γ is consist of polylines

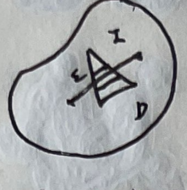
It's well-def. since $\int_R f\,dz = 0$.

Check: $F \in \mathcal{O}(\Omega)$, by $f \in C(\Omega)$

<u>Remark:</u> i) $\frac{1}{z}$ has no primitive. Since:

$$\oint_{\partial D(0,1)} \frac{1}{z} = 2\pi i \neq 0.$$

ii) For $f \in \theta(D/I)$ I is a segment.



$(f \in C(D))$

By Morera. Approxi by several triangles $\Rightarrow f \in \theta(D)$

① Limit Seq:

Thm. $\{f_n\} \subseteq \theta(\Omega)$. $f_n \xrightarrow{u.c.c} f$. Then $f \in \theta(\Omega)$

Moreover. $f_n' \xrightarrow{u.c.c} f'$

Pf: By Morera: $\int_T f_n \, dz \longrightarrow \int_T f \, dz = 0$.

(Because $f_n \xrightarrow{u} f$ on T. cpt set $\subseteq \Omega$)

By Cauchy Formula for the Latter.

Thm. $F(z,s) : \Omega \times [0,1] \longrightarrow \mathbb{C}$. $\Omega \underset{open}{\subseteq} \mathbb{C}$

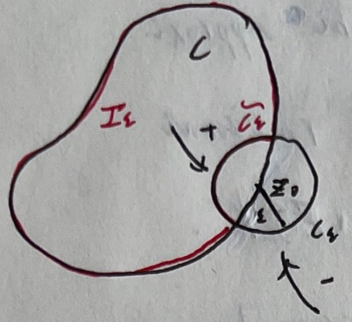
$F \in C(\Omega \times [0,1])$. $F(z,s) \in \theta(\Omega)$ for every $s \in [0,1]$. Then $\int_0^1 F(z,s) ds \in \theta(\Omega)$

Pf: $\frac{1}{n} \sum_1^n F(z, \frac{k}{n}) \in \theta(\Omega) \xrightarrow{u.c.c} \int_0^1 F(z,s) ds$

Remark: Not every $f \in C(\Omega)$ can be approximated by polynomials. Since $\sum_0^{\infty} a_n z^n \in \theta(\mathbb{C})$.

Then $\exists \tilde{\Omega}$. $\Omega \subseteq \tilde{\Omega}$. $f \in C(\tilde{\Omega})$

③ Sokhotski Formula:



∂C is C' Jordan curve. $f \in C(\bar{C})$

$$\tilde{f}(z) = \frac{1}{2\pi i}\int_{\partial C} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$\tilde{f}_+(z_0) = \lim_{z \to z_0} \tilde{f}(z), \quad z \in C.$$

$$\tilde{f}_-(z_0) = \lim_{z \to z_0} \tilde{f}(z), \quad z \in C^c.$$

Then. $\tilde{f}_+(z_0) = \tilde{f}_p(z_0) + \frac{1}{2} f(z_0)$. $\tilde{f}(z_0) = \tilde{f}_p(z_0) - \frac{1}{2} f(z_0)$

where $\tilde{f}_p(z_0) = \text{p.v. } \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} = \lim_{\varepsilon \to 0} \frac{1}{2\pi i}\int_{I_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z}.$

Pf: 1°) $f \in \mathcal{O}(\bar{C})$.

Then $\tilde{f} = f$. $\forall z \in C$. $\tilde{f} \equiv 0$. $\forall z \notin C$.

By conti. $\tilde{f}_+(z_0) = \tilde{f}(z_0)$

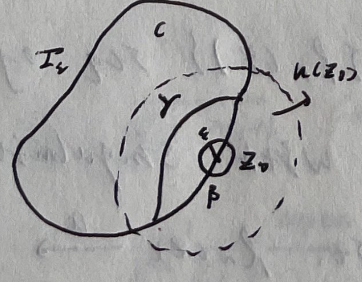
Calculate: $\tilde{f}_p = \text{p.v. } \int_{\partial C} \frac{1}{2\pi i} \cdot \frac{f(\zeta)\,d\zeta}{\zeta - z}$:

$$\int_{I_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z} = -\int_{C_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - z} \text{ by Cauchy.}$$

(Let $z = \varepsilon e^{i\theta}$. $\theta_1 \le \theta \le \theta_2$. $\theta_2 - \theta_1 \to 0$)

2°) $f \in \mathcal{O}(U(z_0))$ only.

$$\tilde{f}(z) = \int_{I_\varepsilon + \gamma} + \int_{\beta - \gamma}$$



Let $z \to z_0$. inside C.

The latter reduce to 1°)

Remark: For $f \in C^{0,\beta}(\bar{C})$. $\forall\, 0 < \beta \le 1$.

The conclusion still holds.

(4) Runge's Approximation Thm:

Thm. $\forall f \in \mathcal{O}(\Omega)$. $K \underset{cpt}{\subseteq} \Omega$. f can be approxi. uniformly on K by seq of rational functions whose singularities in K^c.

If K^c is connected. Then f can be approxi. uniformly by polynomials

Pf: ① $f \in \mathcal{O}(D)$. K cpt $\subseteq D$. Then exists $\{\gamma_i\}_1^n$ seq of Jordan curves. $\subseteq D$ st: $f = \frac{1}{2\pi i} \sum_1^{\tilde{n}} \oint_{\gamma_i} \frac{f(\zeta)d\zeta}{\zeta - z}$

Pf: Cover K by almostly disjoint cubes $\{Q_i\}_1^n$ with length $l < dist(K, \Omega^c) \cdot \frac{1}{\sqrt{2}}$.

$\therefore f = \frac{1}{2\pi i} \sum_1^{\tilde{n}} \oint_{\gamma_i} \frac{f(\zeta)d\zeta}{\zeta - z}$. $\forall z \in K$.

where. $\gamma_i = \partial Q_i$. $1 \le i \le N$.

② $f = \frac{1}{2\pi i} \sum_1^{\tilde{N}} \int_{\beta_i} \frac{f(\zeta)d\zeta}{\zeta - z}$. where β_i is segment $\subseteq D/K$. $1 \le i \le \tilde{N}$.

Pf: $\sum_1^{\tilde{N}} \partial Q_i = \sum_1^{\tilde{N}} \beta_i$. since they will cancel the segments fall in K.

③ Exists $\{R_n(z)\}$ seq of rational functions with singularities on β_i $\subseteq D/K$. st. $R_n(z) \xrightarrow{u} \int_{\beta_i} \frac{f(\zeta)d\zeta}{\zeta - z}$.

Pf: By dilation. Let $\beta_i : [0,1] \to \beta_i$

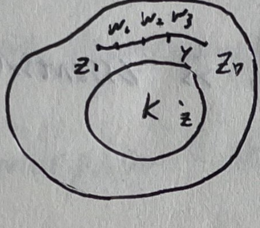
$$\therefore \int_{\beta_i} = \int_0^1 \frac{f(\beta_i(t))\beta_i'(t)dt}{\beta_i(t) - z}, \quad z \in K.$$

$f(\beta_i(t))\beta_i'(t) / \beta_i(t) - z \in \theta(K), \forall t \in [0,1]$.

Then approxi. $\frac{1}{2\pi i}\int_{\beta_i} \frac{f(s)ds}{s-z}$ by Riemann Sum.

④ If K^c is connected. $z_0 \notin K$. Then $\frac{1}{z-z_0}$ can be approxi. by polynomials on K. uniformly.

Pf:



Fix z_1. st. $|\frac{z}{z_1}| < 1$.

$\exists\ \gamma(t) : [0,1] \to \gamma$.

st. $\gamma(0) = z_0$. $\gamma(1) = z_1$

i) $$\frac{1}{z - z_1} = \frac{-1}{z_1} \cdot \frac{1}{1 - \frac{z}{z_1}} = -\frac{1}{z_1}\sum \left(\frac{z}{z_1}\right)^n$$

$\therefore \frac{1}{z-z_1}$ can be approxi. by polynomials

ii) Let $\varepsilon = \frac{1}{2}d(K, \gamma)$. $\{w_i\}_1^p$ on γ. cpt.

st. $|w_i - w_{i+1}| < \varepsilon$. $w_0 = z_1$. $w_{p+1} = z_0$

Note that $$\frac{1}{z - w_{i+1}} = \frac{1}{z - w_i} \cdot \frac{1}{1 - \frac{w_{i+1} - w_i}{z - w_i}}$$

$$= \frac{1}{z - w_i}\sum \left(\frac{w_{i+1} - w_i}{z - w_i}\right)^n$$

$\therefore \frac{1}{z - w_{i+1}}$ can be approxi. by $\frac{1}{z - w_i}$

$$\therefore \frac{1}{z - z_1} \xrightarrow{\text{Approx.}} \frac{1}{z - w_1} \longrightarrow \cdots \longrightarrow \frac{1}{z - z_0}$$

Remark: If K^c isn't connected. Then $\exists f \in \theta(U)$. $K \subseteq U \subseteq \Omega$.

st. f can't be approxi. by polynomials uniformly on K.