Radon Measures

(1) Pre:

- We will consider a measure acting like Lebesgue measure (in $\mathbb{R}^n$). But we will discuss it in Locally compact Hausdorff (LCH) space X.

① Thm. (Urysohn Lemma in LCH space)

If X is LCH space. $K \subset U \subset X$. K is cpt and U is open. Then exists $f(x) \in C(X, [0,1])$. st. $f \equiv 1$ on K. $\exists R$ cpt. $R \subset U$. $f(x) = 0$ on R^c.

Thm. (Tietze Thm in LCH space)

If X is LCH space. $K \underset{cpt}{\subset} X$. $f \in C(K)$. Then $\exists F \in C(X)$. st $F|_K = f$. $\exists R \underset{cpt}{\subset} X$. $F(x) = 0$. when $x \in R^c$.

Thm. (Partition of Unity in LCH space)

If X is LCH space. $K \underset{cpt}{\subset} X$. $\{U_k\}_1^n$ is open cover of K. Then exists POU on K subordinates to $\{U_k\}_1^n$, consisting of cpt-supp functions

Remark: In LCH space. Many Thm's conclusions is weakened to "on cpt set". "Finite elements" (e.g. PoU).

② Density:

Prop. If X is LCH space. $C_0(X) = \{f(x) \mid \lim_{x\to\infty} f = 0,\ f \in C(X)\}$

Then $\overline{C_c(X)} = C_0(X)$. (with uniform norm $\|\cdot\|_u$)

Pf: $\forall \{f_n\}$ converges in $C_c(X)$. suppose $f_n \rightrightarrows f \in C(X)$

i.e. $\forall \varepsilon > 0.\ \exists N.\ n > N.\ \|f_n - f\|_u < \varepsilon$.

$\therefore$ If $x \notin \operatorname{supp} f_n$. Then $\|f(x)\|_u < \varepsilon$.

$\therefore \lim_{x\to\infty} f(x) = 0$.

$\forall f \in C_0(X)$. Suppose $K_n = \{|f| \geq n^{-1}\}$. cpt.

By Urysohn. $\exists g_n$, st. $g_n \equiv 1$ on K_n.

Let $f_n = g_n f \in C_c(X) \longrightarrow f$.

Thm. (Stone - Weierstrass Thm)

X is cpt Hausdorff space. If A is closed subalgebra of $C^{\mathbb{R}}(X)$ separating points. Then

$A = C^{\mathbb{R}}(X)$ or $\{f \in C^{\mathbb{R}}(X) \mid f(x_0) = 0\}$ for some x_0

Cor. Suppose B is a subalgebra of $C^{\mathbb{R}}(X)$ separating points and containing const. Then $\overline{B} = C^{\mathbb{R}}(X)$.

(Or $\exists x_0 \in X$. s.t. $f(x_0)=0$. for $\forall f \in B$. then $\overline{B} = \{ f \in C^{\mathbb{R}}(X) \mid f(x_0)=0 \}$)

Remark: i) Recall: algebra: A vector space X satisfies:

$f, g \in X$. Then $fg \in X$.

A set $H \in C^{\mathbb{R}}(X)$ is separating: If $\forall$

$x \neq y \in X$. $\exists f \in H$. St. $f(x) \neq f(y)$

ii) In complex case. we require If $h \in B$.

Then $\overline{h} \in B \subseteq C^{\mathbb{C}}(X)$

iii) Common examples:

Bernstein polynomial: $\{ f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k} \}_{k \in \mathbb{Z}^+ \cup \{0\}}$.

where $f \in C^{[0,1]}(X)$.

Triangle polynomial: $\{ e^{i\theta t} \}_{\theta \in \mathbb{R}}$.

Pf: Lemma. i) $H \subseteq C^{\mathbb{R}}(X)$. satisfies: $\forall u, v \in H$. $\sup\{u, v\}$ and $\inf\{u, v\} \in H$. (We call it a lattice) $(*)$ $\longrightarrow$ Then $|h| \in H$. If $h \in H$.

Besides, $\forall x_1, x_2 \in X$. $a_1, a_2 \in \mathbb{R}$, $\exists f \in H$ st. $f(x_1)=a_1$, $f(x_2)=a_2$. Then $\overline{H} = C^{\mathbb{R}}(X)$

$\Downarrow$

ii) H is vector subspace of $C^{\mathbb{R}}(X)$. It's a separating lattice which contains const.

Then $\overline{H} = C^{\mathbb{R}}(X)$

(2) Positive linear Func on $C_c(X)$ and Representation:

① **Def**: I is positive linear function on $C_c(X)$, if $I(f) \geq 0$. whenever $f \geq 0$.

prop. For each $K \underset{cpt}{\subseteq} X$. $\exists\, C(K)$. const. st. $|I(f)| \leq C_K \|f\|_u$. where $f \in C_c(X)$. st. $\mathrm{supp} f \subseteq K$.

pf: By Urysohn. $\exists\, \phi$. st. $\phi = 1$ on K. Note that $|f| \leq \phi \|f\|_u$. put in $I(\cdot)$.

Rmk: Note that if μ is Borel measure on X st. $\forall K \underset{cpt}{\subseteq} X$. $\mu(K) < \infty$. Then $C_c(X) \in L^1(\mu)$

So $I: f \longmapsto \int_X f \, d\mu$ is a PLF.

Next, we will prove it's unique expression for some special measure.

② **Def**: μ is Borel measure on X: $E \in \mathcal{B}_X$.

μ is $\begin{cases} \text{outer regular if } \mu(E) = \inf\{\mu(U) \mid U \supseteq E. \text{ open}\} \\ \text{inner regular if } \mu(E) = \sup\{\mu(K) \mid K \subseteq E. \text{ cpt}\} \end{cases}$

μ is regular on all Borel sets, if it satisfies both on Borel sets.

$\Downarrow$

It will be a bit too much to ask for regular when X isn't σ-cpt

So we define radon measure:

It's a Borel measure satisfies finite on all cpt sets, inner regular on all open sets, outer regular on all Borel sets.

<u>Notation</u>: $U \underset{open}{\subseteq} X$. $f \in C_c(X)$. $f \prec U$. iff:

$0 \le f \le 1$. $\text{supp} f \subseteq U$.

<u>Thm.</u> (Riesz Rrepresentation)

If I is PLF on $C_c(X)$. Then there exists a unique random measure μ. st. $I(f) = \int f d\mu. \forall f \in C_c(X)$

Moreover, μ satisfies:

$$\begin{cases} \mu(U) = \sup\{I(f): f \prec U. f \in C_c(X)\}. \forall U \underset{open}{\subseteq} X & (A) \\ \mu(K) = \inf\{I(f): f \ge \chi_K. f \in C_c(X)\} \ \forall K \underset{cpt}{\subseteq} X. & (B) \end{cases}$$

<u>Pf</u>: 1°) Uniqueness:

prove: if μ is the random measure. st. $I(f) = \int f d\mu$.

Then it satisfies (A). (B).

(From def of random measure, take away "inf" and "sup". by approxi.!)

$\Rightarrow \mu$ is determined by $I(\cdot)$ on open sets.

extend it to Borel sets by outer regular.

2°) Existence:

The ideal is from uniqueness.

Def: $\mu(U) = \sup\{I(f) : f \prec U,\ f \in C_c(X)\}$ for U open

$\mu^*(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$. for every set $E \subseteq X$.

Then μ satisfies: $\mu(U) \geq \mu(V)$. If $U \supseteq V$. $\mu^*(U) = \mu(U)$

$\Rightarrow$ Prove: μ^* is outer measure and every open set

is μ^*-measurable. (Note: μ is premeasure)

a. $\Leftrightarrow$ $\inf\{\mu(U) : U \supseteq E, U \text{ open}\} = \inf\{\sum_1^\infty \mu(U_k) : E \subset \bigcup_1^\infty U_k, \text{open}\}$

Take away "inf", since $\mu(U) \geq \inf\{\sum \mu(U_k) : E \subset \bigcup U_k\}$

If $U = \bigcup U_k \supseteq E$. Next, check $\mu(U) \leq \sum_1^\infty \mu(U_k)$.

From def of μ. Apply POU to each U_k

to attain $\{g_k\}$. $\because f \prec U \Rightarrow f = \sum f g_k$

b. Check U open satisfies Caratheodory.

By def of μ^*: $E \overset{\varepsilon}{\leftarrow} V$ open.

Operate in $\mu(\cdot)$. $V \cap U \overset{\varepsilon}{\leftarrow} I(f)$, by def.

By Caratheodory extension Thm: $\mu^*|_{B_X}$ is a measure

So it's a Borel measure satisfies outer regular.

$\Rightarrow$ Prove: $\mu = \mu^*|_{B_X}$ satisfies (B).

We argue that: $I(f) \leq \mu(K)$. $\mu(K) \leq I(f)$.

By outer regular: $U \overset{\varepsilon}{\leftarrow} K$. $\mu(U) \overset{\varepsilon}{\leftarrow} I(f)$.

$\forall U \supseteq K$. By Urysohn. $\exists f \geq \chi_K$. $f \prec U$. $\therefore I(f) \leq \mu(U)$

$\forall f \geq \chi_K$. $f \in C_c(U)$. construct open set: $U_\varepsilon = \{f > 1-\varepsilon\}$.

$\forall g \prec U_\varepsilon$. $(1-\varepsilon)^{-1} f \geq g$. $\therefore (1-\varepsilon)^{-1} I(f) \geq I(g)$

i.e. $(1-\varepsilon)^{-1} I(f) \geq \mu(U_\varepsilon) \geq \mu(K)$. let $\varepsilon \to 0$

$\Rightarrow$ Inner regular is followed by:

$\mu(K) \le I(\chi_K) < \infty$. $\therefore \mu$ is finite on every cpt set.

$\forall \alpha$, $\alpha < \mu(U)$. $\exists f \in C_c(U)$, $f \prec U$. st. $\alpha < I(f) < \mu(U)$.

By Urysohn. $\exists g \ge \chi_K$, $g \prec U$. $K = \mathrm{supp} f$. $\therefore I(g) \ge I(f) > \alpha$.

Lastly, prove: $I(f) = \int f \, d\mu$.

Exhaust f: $K_i = \{ f > \frac{i}{N} \}$. Let $f_j = \begin{cases} 0. & x \notin K_{j-1} \\ f - \frac{1}{N}. & x \in K_{j-1}/K_j \\ \frac{1}{N}. & x \in K_j. \end{cases}$

$\therefore f = \sum_1^N f_i$.

Since $\frac{\chi_{K_j}}{N} \le f_j \le \frac{\chi_{K_{j-1}}}{N}$ $\therefore$ $\frac{\mu(K_j)}{N} \le \int f_j \, d\mu \le \frac{\mu(K_{j-1})}{N}$

By outer regular: we have $\frac{\mu(K_j)}{N} \le I(f_j) \le \frac{\mu(K_{j-1})}{N}$

$\therefore \begin{cases} \frac{1}{N} \sum_1^N \mu(K_j) \le \int f \, d\mu \le \frac{1}{N} \sum_0^{N-1} \mu(K_j) \\ \frac{1}{N} \sum_1^N \mu(K_j) \le I(f) \le \frac{1}{N} \sum_0^{N-1} \mu(K_j) \end{cases}$

Remark: The radon measure we obtain is a complete measure (since $\mu = \mu^*|_{B_X}$). And by outer regular:

$\mu^*(E) = \inf\{\mu(U) \mid U \supseteq E. \text{ open}\} = \inf\{\inf \mu(U) \mid E \subset B \subset U.\ B \in B_X\}$

$= \inf\{\mu(B) \mid E \subset B \in B_X\}$.

$\therefore \mu^*|_{B_X}$ is induce by (μ, B_X).

(3) Regularity and Approximation:

① Regular and σ-finite:

prop. Every Radon measure is inner regular on σ-finite set.

Cor. Every σ-finite Radon measure is regular.

Every Radon measure is regular on σ-cpt set X.

Pf: Since $E = \cup E_i$. $\mu(E_i) < \infty$. $\therefore E$ is μ-measurable

1°) E is finite μ-measured:

$E \overset{\varepsilon}{\subset} U \overset{\varepsilon}{\supset} K$ (First is outer measure. Second is inner regular)

2°) E is infinite μ-measured

Let $F_n = \bigcup_{1}^{n} E_i \uparrow E$. F_n is finite μ-measured!

Prop. μ is σ-finite Radon measure. $E \in \mathcal{B}_X$. Then:

i) $\forall \varepsilon > 0$, $\exists U$ open. F closed. $F \subset E \subset U$. st. $\mu(U/F) < \varepsilon$.

ii) $\exists F_\sigma$ set A, G_δ set B. $A \subset E \subset B$. st. $\mu(B/A) = 0$.

Pf: $E = \cup E_i$. $\mu(E_i) < \infty$. Suppose $\{E_i\}$ disjoint.

$E_i \overset{\varepsilon/2^i}{\subset} U_i$. replace by open sets.

Thm. X is LCH space where every open set is σ-cpt.
(e.g. X is C^2) Then every Borel measure μ on X
which is finite on cpt set is regular (so Radon).

Remark: It generalizes the prop. before.

Pf: $I(f) = \int f d\mu$ is PLF on $C_c(X)$.

By Riesz Thm. $\exists \nu$ Radon measure associated it.

Next. Consider to prove: $\mu(E) = \nu(E)$. $E \in \mathcal{B}_X$.

For U open. $U = \cup K_n$. By Urysohn on each K_n

$\exists f_n \nearrow \chi_U$. $f_n \geq \chi_{\bigcup_1^n K_i}$ $f_n \prec U$. $f_n \in C_c(U)$

By Monotone Convergence Thm.

$\mu(U) = \lim \int f_n \, d\mu = \lim \int f_n \, d\nu = \nu(U)$. $\forall U$ open.

Note $\exists F$ close. $\mu(U/F) = \nu(U/F) < \varepsilon$. $F \subseteq E \subseteq U$. $E \in B_X$. $\therefore U \overset{\mu\text{-}\varepsilon}{\frown} E$

By: $F \overset{\mu\text{-}\varepsilon}{\frown} E$. Besides: $\exists K_n \uparrow F$. in μ. K_n cpt. $\therefore \exists K_N$. cpt. $K_N \overset{\mu\text{-}\varepsilon}{\frown} E$

$\therefore \mu$ is regular. By Uniqueness. $\mu \equiv \nu$. a.e.

(2) Prop. μ is Radon measure on X. Then $\overline{C_c(X)} = L^p(\mu)$. $1 \le p < \infty$.

Pf: L^p Func. $\xleftarrow{\text{Approxi}}$ $\chi_E^{(E \in B_X)}$. We only need to approxi χ_E by $C_c(X)$

By Urysohn. and for $E \in B_X$. $\exists U$ open. F close. $U \supseteq E \supseteq F$.

We can obtain f. st. $\| f - \chi_E \|_p \le \mu^p(U/F) \le \varepsilon^p$

$\Downarrow$

Lusin's Thm: μ is Radon measure on X. $f: X \to \mathbb{C}$. μ-measurable.

vanishes outside a μ-finite-measure set. Then $\forall \varepsilon > 0$.

$\exists \phi \in C_c(X)$. st. $\phi \equiv f$ except a set of measure ε.

more-over. If $\|f\|_u < \infty$. Then $\exists \phi$. st. $\|\phi\|_u \le \|f\|_u$.

Pf: $E = \{f \ne 0\}$. If $\|f\|_u < \infty$. Then $f \in L^1(\mu)$.

$\therefore \exists g_n \in C_c(X) \longrightarrow f$ in L^1. $\therefore \exists g_{n_k} \to f$. a.e.

By Egorov. Thm. $\exists A \subset E$. $\mu(E/A) < \frac{\varepsilon}{3}$. st.

$g_{n_k} \rightrightarrows f$. on A. Refine A and by Tietze Thm

Obtain a $C_c(X)$ Function! Truncate it for $\|\phi\|_u \le \|f\|_u$.

If f is unbounded:

Since E is finite. $\exists A_n = [0 \le |f| \le n] \uparrow E$.

Apply the same argument on $A_n \overset{\varepsilon}{\frown} E$.

③ Integration of Semiconti. Func.:

· **Def.** i) f is lower semiconti. (LSC) if:

$f: X \to (-\infty, +\infty]$. $\{f > a\}$ is open for $\forall a \in \mathbb{R}$.

ii) f is upper semiconti (USC) if:

$f: X \to [-\infty, +\infty)$. $(f < a)$ is open for $\forall a \in \mathbb{R}$.

prop. i) $U \underset{\text{open}}{\subseteq} X$. $K \underset{\text{close}}{\subseteq} X$. χ_U. $-\chi_K$ are LSC.

ii). $c \geq 0$. f is LSC. Then cf is LSC

iii). $f = \sup\{g(x) : g \in G \subseteq \text{LSC Func}\}$.

Then f is LSC.

iv). $f_1 \cdot f_2$ are LSC. Then $f_1 + f_2$ is LSC.

v). If X is LCH space. $f \geq 0$. is LSC.

Then $f = \sup\{g(x) : g \in C_0(X). 0 \leq g \leq f\}$.

Pf: iii) $\{f > a\} = \bigcup_{g \in G} \{g > a\}$.

iv) $\forall x_0$. s.t. $f_1(x_0) + f_2(x_0) > a$. $\forall a \in \mathbb{R}$.

$\exists \varepsilon > 0$. s.t. $f_2(x_0) > a - f_1(x_0) + \varepsilon$

$\therefore \{f_1 + f_2 > a\} \supseteq \{f_1 > f_1(x_0) - \varepsilon\} \cup \{f_2 > a - f_1(x_0) + \varepsilon\}$

It's a neighbour of x_0 in $\{f_1 + f_2 > a\}$.

v). $\forall a < f(x)$, $a > 0$. $\{f > a\}$ is open. By LCH.

$\therefore \exists V \underset{\text{open}}{\subseteq} \{f > a\}$. contains x. By Urysohn.

$\exists g$. $g(x) = a$. $0 \leq g \leq a\chi_U \leq f$. $g \in C_c(X)$

$\therefore \exists g_n \to f(x)$. pointwise.

Thm. (Mono Converge for Net of LSC)

$\mathcal{G}$ is a family of nonnegative LSC on LCH space X directed by "$\le$". $f = \sup\{g \mid g \in \mathcal{G}\}$. If μ is Radon measure on X. Then $\int f d\mu = \sup\{\int g d\mu \mid g \in \mathcal{G}\}$.

Pf: Note that f is Borel-measurable (By LSC). $\int f d\mu \ge \sup \int g d\mu$.

For the reverse inequality:

return to def of f: Let $\phi_n = \frac{1}{2^n}\sum_1^{2^n} \chi_{U_{ni}}$. $U_{ni} = \{f > \frac{i}{2^n}\}$

refine U_{ni} by cpt set V_{ni}. from LCH.

Since $\phi_n \uparrow f$. $\int \phi_n d\mu = \sum \frac{\mu(U_{ni})}{2^n} \uparrow \int f d\mu$.

$\forall a$. $\int f d\mu > a > 0$. $\exists\ \psi_n = \frac{1}{2^n}\sum_1^{2^n} \chi_{V_{ni}}$. s.t. $\int \psi_n d\mu > a$. $\psi_n \le \phi_n$. $\psi_n \in C_c(X)$

Since $f = \sup g$. $\therefore \forall x \in X$. $\exists g_x > \psi_n$. And $g_x - \psi_n$ is LSC.

Note that $\{g_x - \psi_n > 0\}$ open. cover $\bigcup_1^n V_{ni}$ by such finite sets.

By direct: $\exists\ g > g_{x_k}$. corresspond such finite sets. $\therefore \int g d\mu > a$.

Cor. $\int f d\mu = \sup\{\int g d\mu \mid g \in C_c(X).\ 0 \le g \le f\}$. f is LSC.

Prop. μ is Radon measure on X. $f \ge 0$. Borel-measurable

Then $\int f d\mu = \inf\{\int g d\mu : g \text{ is LSC}.\ g \ge f\}$.

If $\{f > 0\}$ is σ-finite. Also: $\int f d\mu = \sup\{\int g d\mu \mid g \in USC.\ 0 \le g \le f\}$.

Pf: $\exists \sum_1^n a_j \chi_{E_j} \uparrow f$ pointwise. Refine E_j^n by open set U_j^n

Note that $\chi_{U_j^n}$ is LSC. $\sum_1^n a_j \chi_{U_j^n} \ge \sum_1^n a_j \chi_{E_j^n}$

For the second. $\forall a$. $\int f d\mu > a > 0$. By outer regular of

σ-finite E_j^n: $\exists K_{nj}$ cpt. $\sum a_j \mu(K_{nj}) > a$. $K_{nj} \subseteq E_j^n$.

Remark: It gives a way by LSC and USC to establish the corresspond between PLF with Radon measure

(4) <u>Dual of $C_0(X)$:</u>

① · Since $C_0(X) = \overline{C_c(X)}$ in LCH space X. We can extend $I(f) = \int f d\mu$ continuously from $C_c(X)$ to $C_0(X)$. For which Radon measure μ satisfies: $\mu(X) = \sup\{\int f d\mu \mid f \in C_c(X),\ 0 \le f \le 1\} < \infty$

Next, we will give a complete description of $C_0^*(X)$.

<u>Lemma:</u> (Jordan decomposition for Linear Fun on $C_0(X)$)

If $I \in C_0^*(X, \mathbb{R})$. Then exists PLF $I^{\pm} \in C_0^*(X, \mathbb{R})$

s.t. $I = I^+ - I^-$

<u>Pf:</u> Firstly def I^+ on $f \ge 0$:

$I^+(f) = \sup\{ I(g) \mid g \in C_0(X, \mathbb{R}),\ 0 \le g \le f\}$.

$\therefore\ 0 \le I^+(f) \le \|I\| \|f\|_u$. Check I^+ is linear on $C_0(X, [0,+\infty))$

Def: I^+ for general $f \in C_0(X, \mathbb{R})$: $I^+(f) = I^+(f^+) - I^+(f^-)$

Def: $I^- = I^+ - I$. Check I^+, I^- are linear on $C_0(X, \mathbb{R})$

<u>Def:</u> $M(X) = \{\mu = \mu_1^+ - \mu_1^- + i(\mu_2^+ - \mu_2^-) \mid \mu_1^{\pm}, \mu_2^{\pm}$ are radon on $X\}$.

with norm $\|\mu\| = |\mu|(X) < \infty$.

<u>Prop.</u> μ is complex Borel measure. Then $\mu \in M(X) \Leftrightarrow |\mu| \in M(X)$

<u>Remark:</u> We can show $\forall \mu_1, \mu_2 \in M(X)$. Then we have:

$\mu_1 + c\mu_2 \in M(X)$. $c \in \mathbb{R}$. $\therefore M(X)$ is linear span

<u>Thm.</u> (Riesz Representation)

X is LCH space. $\mu \in M(X)$. $I_\mu(f) = \int f d\mu$. $f \in C_0(X)$.

Then $I_\mu \xmapsto{\varphi} \mu$ is an isometric isomorphism from $C_0^*(X)$ to $M(X)$.

Pf: Only need to show it's isometry:

Note that $|I_\mu(f)| \le \int |f| d|\mu| \le \|f\|_u \|\mu\|$. i.e. $\|I_\mu\| \le \|\mu\|$.

For the reverse: Note: $\|\mu\| = |\mu|(X) = \int |h|^2 d|\mu|$. $h = \frac{d\mu}{d|\mu|}$

h transit $\int d|\mu|$ to $\int d\mu$. Since $\int |h|^2 d|\mu| = \int \bar{h} d\mu$

By Lusin's Thm. $\exists f$. $f = \bar{h}$. outside E. s.t. $\mu(E) < \frac{\varepsilon}{2}$.

refine supp f, Let it be $C_c(X)$. $\therefore \|\mu\| \overset{\varepsilon}{\sim} |\int f d\mu| \le \|I_\mu\|$.

Cor. If X is cpt Hausdorff space. Then $C^*(X) \underset{\text{isometry}}{\cong} M(X)$.

Pf: Since By Stone-Weirstrass Thm: $C(X) = \overline{C_c(X)}$

Remark: Another method to construct complex Radon measure:

Suppose μ is fixed positive Radon measure on X. $f \in L^1(\mu)$.

Then $dV_f = f d\mu \in M(X)$. with $\|V_f\| = \|f\|_{L^1(\mu)}$

Let $f \xmapsto{\varphi} V_f$. φ is isometry from $L^1(\mu)$ to $M \subseteq M(X)$.

$M = \{ V \in M(X) \mid V \ll \mu \}$. We can identify $L^1(\mu)$ as subset of $M(X)$.

④ Vague Convergence:

Prop. $\{\mu\} \cup \{\mu_k\}_{k \in N} \subseteq M(X)$. $F_n(x) = \mu_n(-\infty, x]$. $F(x) = \mu(-\infty, x]$.

$$F_n(x) \to F(x) \ \text{at continuities} \quad \underset{\substack{\sup |F_n(x)| \to 0 \\ (x \to -\infty).\ n \to 0}}{\overset{\sup_n \|\mu_n\| < \infty}{\rightleftarrows}} \quad \mu_n \xrightarrow{v} \mu.$$

Pf: The same way. since they belong to $M(X)$!

(5) Product of Radon Measure:

Thm. $B_X \otimes B_Y \subseteq B_{X \times Y}$. if X. Y are C^2. then $B_X \otimes B_Y = B_{X \times Y}$.

Moreover. In the latter case. if μ. ν are Radon measure on X. Y [resp.]. then $\mu \times \nu$ is Radon measure on $X \times Y$.

Pf: It's same as before. Just prove the last one:

Check $\mu\times\nu$ is finite on every cpt set K.

Since $K \subseteq \pi_1(K)\times\pi_2(K)$, $\pi_1(K), \pi_2(K)$ is finite!

Remark: When X or Y isn't C^2, then $\mu\times\nu$ isn't

Radon measure on $X\times Y$ certainly!

Next, we will construct product of Radon measure

on $X\times Y$. Denote: $g\otimes h(x,y) = g(x)h(y)$, on $X\times Y$:

Prop. $\mathcal{P} = \text{span}\{g\otimes h \mid g, h \in C_c(X), C_c(Y), \text{resp.}\}$

Then $\overline{\mathcal{P}} = C_c(X\times Y)$ in uniform norm.

Pf: It equals: $\forall f \in C_c(X\times Y)$, $\varepsilon>0$.

precpt open set $U\subseteq X$, $V\subseteq Y$. Containing

$\pi_x(\text{supp} f)$, $\pi_y(\text{supp} f)$, resp. Then exists

$F\in\mathcal{P}$, st. $\|F-f\|_u<\varepsilon$, $\text{supp}F\subset U\times V$.

1°) $\{g\otimes h \mid g\in C(\bar{U}), h\in C(\bar{V})\}$ is dense in $C(\bar{U}\times\bar{V})$.

Since it's cpt-Hausdorff, apply Weierstrass Thm.

2°) Refine the supp by Urysohn

Prop. i) $\forall f\in C_c(X\times Y)$ is $\mathcal{B}_X\otimes\mathcal{B}_Y$-measurable.

ii) If μ, ν is Radon measure on X, Y resp.

Then $C_c(X\times Y)\subseteq L^1(\mu\times\nu)$, satisfies:

$$\int f\, d(\mu\times\nu) = \int f\, d\mu d\nu = \int f\, d\nu d\mu.$$

Pf: $g\otimes h = (g\circ\pi_x)(h\circ\pi_y)$ on $X\times Y$.

It's $\mathcal{B}_X\otimes\mathcal{B}_Y$-measurable

By Approxi. Since $\overline{\mathcal{P}} = C_c(X\times Y)$.

Apply Fubini Thm to obtain the last one.

Remark: We attain: $I(f) = \int f \, d\mu\times\nu$. on $f \in C_c(X\times Y)$

By Riesz Thm. it determines Radon measure

$\mu \hat{\times} \nu$ on $C_c(X\times Y)$ (unique).

Note that: $\mu\hat{\times}\nu \neq \mu\times\nu$ in general.

Next, we will discover the domain of $\mu\hat{\times}\nu$.

Lemma. i) $E \in \mathcal{B}_{X\times Y} \Rightarrow E_x . E^y \in \mathcal{B}_Y . \mathcal{B}_X$. for $\forall x, y$. resp.

f is $\mathcal{B}_{X\times Y}$-measurable $\Rightarrow f_x . f^y$ is $\mathcal{B}_Y$-measurable.

$\mathcal{B}_X$-measurable for $\forall x, y$, resp.

ii) $f \in C_c(X\times Y)$. $\mu . \nu$ is Radon measure on $X . Y$.

Then $\int f_x \, d\nu . \int f^y \, d\mu$ is conti. on $X . Y$. resp.

Pf: i) open sets $\subseteq \{E \mid E_x . E^y \in \mathcal{B}_Y . \mathcal{B}_X \text{ resp}\} = \mathcal{M}$.

Check $\mathcal{M}$ is σ-algebra.

ii) By finite open cover of $\pi_Y(\operatorname{supp} f)$. cpt.

Thm. ($\mu\hat{\times}\nu$ on open sets)

μ, ν is Radon measure on $X\times Y$. $U \underset{\text{open}}{\subseteq} X\times Y$. Then

$\nu(U_x) . \mu(U^y)$ is $\mathcal{B}_X$-measurable . $\mathcal{B}_Y$-measurable. resp.

Besides: $\mu\hat{\times}\nu(U) = \int \nu(U_x) \, d\mu = \int \mu(U^y) \, d\nu$.

Pf: Since χ_U is LSC. By its Monotone Convergent Thm

on $\mathcal{I} = \{f \in C_c(X\times Y) \mid 0 \le f \le \chi_U\}$.

$\therefore$ We obtain the measurability of $\nu(U_x) . \mu(U^y)$.

Show $\int f \, d(\mu \hat{\times} \nu) = \int f^y d\mu d\nu = \int f_x d\nu d\mu$. for $f \in C_c(X \times Y)$.

$\therefore \mu \hat{\times} \nu (W) = \int \sup \{ \int f_x d\nu \} d\mu = \int \sup \{ \int f^y d\mu \} d\nu$.

Thm. ($\mu \hat{\times} \nu$ on Borel sets)

Suppose μ, ν are σ-finite Radon measure on X, Y. resp. If $E \in B_{X \times Y}$. Then $\nu(E_x)$, $\mu(E^y)$ are Borel-measurable on X, Y. resp. Besides.

$\mu \hat{\times} \nu (E) = \int \nu(E_x) d\mu = \int \mu(E^y) d\nu$.

Pf: ① Fixed open set $U, V \subseteq X, Y$. resp.
restrict on $U \times V$

② $\mathcal{M} = \{ E \in B_{X \times Y} \mid E$ satisfies the conclusions$\}$.

i) Open sets $\subseteq \mathcal{M}$.
ii) $E, F \in \mathcal{M} \Rightarrow E/F \in \mathcal{M}$. if $F \subseteq E$.
iii) $\{E_k\}_1^n$ disjoint $\subseteq \mathcal{M} \Rightarrow \bigcup_1^n E_k \in \mathcal{M}$.
} prop of measure

iv) $\mathcal{M}$ is closed under countable increase union and decrease intersection ← Mono Converge Thm

③ Let $\mathcal{E} = \{ A/B \mid A, B \underset{open}{\subseteq} X \times Y \}$. $\therefore \mathcal{E} \subseteq \mathcal{M}$.

Check it's an elementary family.

$\mathcal{A}$ is collection of finite union of elements in $\mathcal{E}$ (disjoint)

$\therefore \mathcal{A}$ is an algebra. And $\mathcal{A} \subseteq \mathcal{M}$.

$\therefore \sigma(\mathcal{A})$ = monotone class generated by $\mathcal{A}$.

By ② i) - iv). $\therefore \mathcal{M} \supseteq \sigma(\mathcal{A})$ $\therefore \mathcal{M} = B_{X \times Y}$.

④ $X = \bigcup U_n$. $Y = \bigcup V_n$. where $U_n \uparrow X$. $V_n \uparrow Y$.

For $E \in B_{X \times Y}$. $E \cap (U_n \times V_n)$ satisfies the conclusions for $\forall n$. Apply Mono-Converge Thm!

Remark: By Tonelli Thm. If $E \in B_X \times B_Y$. Then

$$\mu \hat{\times} \nu(E) = \int \nu(E_x) d\mu = \mu \times \nu(E).$$

Thm. (Fubini-Tonelli Thm for $\mu \hat{\times} \nu$)

Let μ, ν are σ-finite Radon measure on X, Y.

i) If f is Borel-measurable on $X \times Y$. Then f_x, f^y is Borel-measurable on X, Y, resp. $\forall x, y \in X, Y$

For $f \geq 0$. Then. $\int f_x d\nu$, $\int f^y d\mu$ is Borel-measurable on X, Y.

ii) $f \in L^1(\mu \hat{\times} \nu) \Rightarrow f_x \in L^1(\nu)$. for a.e. x. $f_y \in L^1(\mu)$ for a.e. y. $\int f_x d\nu \in L^1(\mu)$. $\int f^y d\mu \in L^1(\nu)$.

Besides: $\int f \, d(\mu \hat{\times} \nu) = \int f \, d\mu \, d\nu = \int f \, d\nu \, d\mu$.

Pf: Approxi. by χ_E. Apply Monotone Convergent Thm.

Extend to infinite products:

. Suppose $\{X_\alpha\}_{\alpha \in A}$ family of cpt Hausdorff spaces. μ_α is Radon measure on X_α, st. $\mu_\alpha(X_\alpha) = 1$.

Then $\prod_{\alpha \in A} X_\alpha$ is also cpt. Hausdorff

Def: μ Radon measure on X. for $E \subseteq X$

$$E = \prod_{\alpha \in A} E_\alpha. \; E_\alpha \in B_{X_\alpha}. \; E_\alpha = X_\alpha \text{ for all}$$

but finitely many α.

$$\mu(E) = \mu(\prod_{\alpha \in A} E_\alpha) = \prod_{\alpha \in A} \mu(E_\alpha)$$

<u>Thm.</u> In the space $\prod_{\alpha\in A} X_\alpha = X$ with measure $\{\mu_\alpha\}_{\alpha\in A}$ on $\{X_\alpha\}_{\alpha\in A}$

There exists a unique Radon measure μ

on X. s.t. for any $\{\alpha_k\}_{k=1}^n \subseteq A$. $\forall E \in \mathcal{B}_{\prod_1^n X_{\alpha_k}}$

we have: $\mu(\pi^{-1}_{(\alpha_1 \cdots \alpha_n)}(E)) = \mu_{\alpha_1} \hat{\times} \mu_{\alpha_2} \cdots \hat{\times} \mu_{\alpha_n}$

<u>Pf:</u> The point is extending μ

from elementary one to general:

1°) Denote $C_F = \{ f \in C_c(X) \mid f = g \circ \pi_{(\alpha_1, \alpha_2 \cdots \alpha_n)}$

where $g \in C(\prod_1^n X_{\alpha_i})$. for some $\{\alpha_i\}_1^n \subseteq A\}$.

Def: $I(f) = \int g \, d\mu_{\alpha_1} \hat{\times} \mu_{\alpha_2} \cdots \hat{\times} \mu_{\alpha_n}$

$= \int g \, d \hat{\bigotimes}_{\alpha\in A} \mu_\alpha$ (since $\mu_\alpha(X_\alpha) = 1$)

Check $I(\cdot)$ is PLF.

Since $C_F(X)$ is separating subalgebra $\therefore \overline{C_F(X)} = C(X)$

extend $I(f)$ continuly to $C(X)$.

By Riesz Thm. $\exists \mu$. unique Radon. correspond to $I(\cdot)$

2°) Denote $\mu_{(\alpha_1, \alpha_2 \cdots \alpha_n)} = \mu \circ \pi^{-1}_{(\alpha_1 \cdots \alpha_n)}$. it's Borel measure

satisfies: $\int g \, d\mu_{(\alpha_1 \cdots \alpha_n)} = \int g \circ \pi_{(\alpha_1 \cdots \alpha_n)} \, d\mu = \int g \, d\mu_{\alpha_1} \hat{\times} \cdots \hat{\times} \mu_{\alpha_n}$

We only need to check:

$\mu_{(\alpha_1 \cdots \alpha_n)}$ is Radon. (check regularity)

<u>Pf:</u> $\forall E \in \mathcal{B}_{\prod X_{\alpha_i}}$. Use regularity of μ.

$E \xrightarrow{\pi^{-1}} \pi^{-1}(E) \subseteq (X, \mu)$. $\therefore \exists K \underset{\text{cpt}}{\subseteq} \pi^{-1}(E)$.

s.t. $\mu(K) \geq \mu(\pi^{-1}(E)) - \varepsilon$. Then $\pi(K)$ cpt. is what we need!