Renewal Theory

(1) Recurrent Times:

① Forward:

<u>Def</u>: For $\varphi = (t_n)$ renewal process. $t_{N(t)} \le t \le t_{N(t)+1}$

the forward recurrent time is $A(t) =: t_{N(t)+1} - t$. (t_n t t_{n+1} $\to A(t)$)

<u>Rmk</u>: If φ is Poisson(λ) process. Then by memoryless property: $A(t) \sim Exp(\lambda)$.

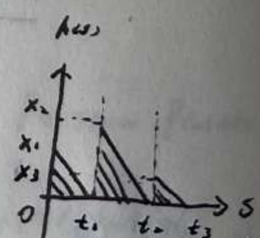
<u>Denote</u>: $X = (X_n)$ i.i.d interarrival times. $X \sim F$

<u>prop</u>. i) $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t A(s)\,ds = E(X^2)/2E(X)$. a.s.

ii) $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t I_{\{A(s)>a\}}\,ds = \lambda E(X-a)^+$. a.s.

iii) $\lim\limits_{t\to\infty} \frac{1}{t}\int_0^t P(A(s)>a)\,ds = \lambda E(X-a)^+$.

<u>Pf</u>: 1') Intuitively. $\int_0^t A(s)\,ds \approx \sum_1^{N(t)} \frac{X_k^2}{2}$.



Actually, we have:

$$\frac{1}{t}\sum_1^{N(t)} \frac{X_j^2}{2} \le \frac{1}{t}\int_0^t A(s)\,ds \le \frac{1}{t}\sum_1^{N(t)+1} \frac{X_j^2}{2}.$$

Then apply ERT. as before

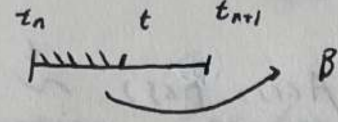
2') $\int_0^t I_{\{A(s)>a\}}\,ds \approx \sum_1^{N(t)} (X_k - a)^+$.

3') By DCT. in the expectation.

③ Backward:

Def: For $\mathcal{Y} = \{t_n\}$ renewal process. $t_{N(t)} \le t \le t_{N(t)+1}$.

$B(t) = t - t_{N(t)}$ is backward recurrence time.

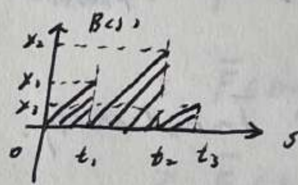


prop. i) $\lim_{t\to\infty} \frac{1}{t} \int_0^t B(s)\, ds = E(X^2)/2E(X)$. a.s.

ii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t I_{\{B(s) > a\}}\, ds = \lambda E(X-a)^+$. a.s.

iii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t p(B(s) > a)\, ds = \lambda E(X-a)^+$.

Pf: Similar as before:



(2) Distribution:

① Equilibrim dist.:

Denote: $\bar{F}(x) = 1 - F(x) = p(X > x)$. tail prob.

Rmk: $\lambda E(X-a)^+ = \lambda \int_a^\infty p(X > y)\, dy$

$= \lambda \int_a^\infty \bar{F}(y)\, dy$.

Def: As $a \ge 0$ varies. Rmk defines a tail prob. of a dist. We denote cdf of it by F_e. which is called equilibrim dist. of F.

$F_e(x) =: \lambda \int_0^x \bar{F}(y)\, dy$. Set $X_e \sim F_e$. r.v.

Rmk: X_e is conti. since density $= F_e' = \lambda \bar{F}$. exists.

Note that in a long term:

$$\lim_{t\to\infty}\int_0^t \frac{p(A(s)<x)}{t}ds = \lim_{t\to\infty}\int_0^t \frac{p(B(s)<x)}{t} = 1-\lambda E(X-x)^+$$

$$= \lambda\int_0^x \bar{F}(\eta)d\eta = F_e(x).$$

$\Rightarrow$ The stationary dist of $A(s)$. $B(s) \overset{d}{\sim} F_e$

③ Spread:

Def: Spread as length of interarrival time is

$$S(t) =: t_{N(t)+1} - t_{N(t)} = A(t) + B(t) = X_{N(t)+1}$$

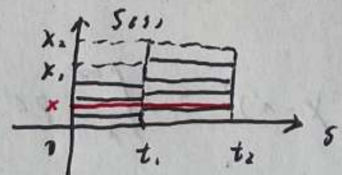
Rmk: Immediately. $\lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)ds = E(X^2)/E(X)$.

prop. i) $\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{\{S(s)>x\}}ds = \lambda E(X I_{\{X>x\}})$. a.s.

ii) $\lim_{t\to\infty}\frac{1}{t}\int_0^t p(S(s)>x)ds = \lambda E(X I_{\{X>x\}})$

Pf: $\frac{1}{t}\int_0^t I_{\{S(s)>x\}}ds \approx \frac{1}{t}\sum_1^{N(t)} X_j I_{\{X_j>x\}}$

by:



Rmk: In a long term:

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t P(S(s)<x)ds = \lambda E(X I_{\{X<x\}})$$

Denote: $\bar{F}_S(x) = \lambda E(X I_{\{X>x\}})$

$$= \lambda\int_x^\infty (t-x)^+ + x\, dF$$

$$= \lambda E(X-x)^+ + \lambda x\bar{F}(x).$$

$$= \bar{F}_e(x) + \lambda x\bar{F}(x).$$

by prop above. the stationary dist. of

$S(x) \sim 1-\bar{F}_S(x) = F_S(x)$. denote r.v. X_S.

$\Rightarrow$ X_S may not be conti. generally. (F' doesn't exist). $E(X_S) = E(X^2)/E(X)$.

(3) <u>Inspection Paradox:</u>

<u>prop.</u> $S(t) \geq_{stoch} X$. i.e. $p(X>x) \leq p(S(t)>x)$. $\forall t, x \geq 0$.

Moreover. $p(X_S \geq x) \geq p(X \geq x)$. $X_S \geq_{stoch} X$.

<u>Pf:</u> 1°) $p(S(t)>x \mid N(t)=n.\ t_n=s) = p(X_{n+1}>x \mid X_{n+1}>t-s)$

$$= \bar{F}(\max\{x, t-s\}) / \bar{F}(t-s)$$

$$\geq \bar{F}(x).$$

$$\Rightarrow E_{N, t_N}(p(S(t)>x \mid \square)) = p(S(t)>x) \geq \bar{F}(x)$$

2°) By $p(X_S \geq x) = \lim \frac{1}{t}\int_0^t p(S(s)>x)\,ds \geq p(X \geq x)$

<u>Rmk:</u> Note that: $E(S(t)) \geq E(X)$

$$E(X_S) \geq E(X)$$

$\Rightarrow$ The prop. above implies a paradox: Observing makes the expectation of lifetime be longer than usual. That's because if we observe at time t. then the part of lifetime $< t$ will be ignored.

<u>e.g.</u> A extreme case:

X is r.v. of lifetime of bulbs. $\begin{cases} p(X=0) = 0.9 \\ p(X=1) = 0.1 \end{cases}$

Then. all bulbs we observed has lifetime 1.

The right way to estimate lifetime of them is SLLN: $\frac{1}{n}\sum_{1}^{n} X_k \approx E(X)$. for large n.

(4) <u>Renewal Reward Thm:</u>

Suppose R_i is r.v. of reward between t_i, t_{i+1}.

$\Rightarrow R(t) = \sum_{1}^{N(t)} R_j$ Assume $(X_i, R_i)_{i \in \mathbb{Z}^+}$ i.i.d. but R_i can depend on X_i. (X, R) denote a cycle.

<u>Thm.</u> (RRT)

For a positive recurrent renewal process where a reward R_i is earned during a cycle with length X_i. If $E(|R_i|) < \infty$.

Then: $\lim_{t\to\infty} \frac{R(t)}{t} = E(R)/E(X)$. a.s.

$$\lim_{t\to\infty} \frac{E(R(t))}{t} = E(R)/E(X).$$

<u>Pf:</u> 1°) Suppose $R_i \geq 0$. or set $R_i = R_i^+ - R_i^-$.

Note: $\frac{1}{t}\sum_{1}^{N(t)} R_i \leq \frac{R(t)}{t} \leq \frac{1}{t}\sum_{1}^{N(t)+1} R_i$.

Apply ERT.

2°) $|R(t)|/t \leq Y(t) = \frac{1}{t}\sum_{1}^{N(t)+1} |R_i| \xrightarrow{a.s.} E|R|/E(X)$

$\Rightarrow |R(t)|/t$ is bdd a.s. By DCT.

Application: We can see some byproduct as "reward" to calculate its long time rate by RRT.

i) Set $R(t) = \int_0^t A(s)ds$. conti reward.

$$R_j = \int_{t_{j-1}}^{t_j} A(s)ds. = \int_{t_{j-1}}^{t_j} (t_j - s)ds$$

$$\overset{s=t_{j-1}+u}{=} \int_0^{X_j} (X_j - u)du = \frac{X_j^2}{2}$$

$$\Rightarrow \text{By RRT: } \frac{1}{t}\int_0^t A(s)ds \xrightarrow{a.s.} E(\tfrac{X^2}{2}) / E(X).$$

ii) Similarly. $R(t) = \int_0^t B(s)ds$.

iii) $R(t) = \int_0^t I_{\{A(s) > x\}} ds$. $R_i = \int_0^{X_i} I_{\{X_i - s > x\}} ds$

(5) Central Limit Thm:

Consider renewal process $\{t_n\}$ with interarrival times $X_n = t_n - t_{n-1}$. st. $E(X) = \frac{1}{\lambda}$. $Var(X) = \sigma^2$.

<u>Thm.</u> (CLT for counting process)

$$Z(t) = \frac{N(t) - \lambda t}{\sigma\sqrt{\lambda^3 t}} \xrightarrow[d]{t\to\infty} Z \sim N(0,1).$$

<u>Rmk:</u> It implies: $E(N(t)) \sim \lambda t$. $Var(N(t)) \sim \sigma^2\lambda^3 t$

<u>Pf:</u> The key is: $P(N(t) < n) = P(t_n > t)$.

set $n(t,x) = [\lambda t + x\sqrt{\sigma^2\lambda^3 t}]$.

$$\Rightarrow P(Z(t) < x) = P(N(t) < n(t,x))$$
$$= P(t_{n(t,x)} > t)$$

$$= P\left(\frac{t_{N(t,x)} - N(t,x)/\lambda}{\sigma\sqrt{N(t,x)}} > \frac{t - N(t,x)/\lambda}{\sigma\sqrt{N(t,x)}} \right)$$

Note by CLT. $t_n - n/\lambda \,/\, \sqrt{n}\,\sigma \xrightarrow{d} Z \sim N(0,1)$.

So : $P(Z(t) < x) \xrightarrow{t \to \infty} P(Z > -x) = P(Z < x)$

Rmk: We don't need "Renewal" actually, but require : $\psi = \{t_n\}$ satisfies CLT.

(6) Delayed Renewal Process:

Note that the counting process $N(t)$ of renewal process $\psi = \{t_n\}$ generally doesn't have stationary increment like poison process.

Def: ψ_s is a shift by time s version of point process $\psi = \{t_n\}$. which is $\{t_{n}(s)\}$. moving the origin to be $t = s$. with counting process:

$N_s(t) = N(s+t) - N(s)$.

Rmk: $\psi_s \overset{d}{\sim} \psi$. $\forall s \geq 0 \iff N(t)$ has stationary increment.

Def: A delayed renewal process is a renewal process where the first arrivial time $t_1 =$ X_1 indeptly has a different dist. $\sim F_1$

For $(X_n)_{n \geq 2}$ interarrival times. are i.i.d. $\sim F$

Rmk: i) ERT remains valid. F_1 doesn't need to have finite first moment.

ii) Ψ is stationary renewal process $\iff$ $t_1(s) = A(s)$ of Ψ_s has the same dist. $\forall s \geq 0$.

As $s \to \infty$. Ψ_s has a limiting dist. of the delayed version Ψ. denoted by $\Psi^* = \{t_n\}_{n\geq 1}$. $t_1^* \sim F_e$.

With its counting process $N^*(t)$. forward recurrent time process $\{A^*(t)\}_{t\geq 0}$.

<u>Def</u>: Ψ^* is stationary version of Ψ.

<u>Rmk</u>: Ψ^* is stationary renewal process:

$$p(A^*(u) \leq x) = \lim_{t\to\infty} \frac{1}{t}\int_0^t p(A(s+u) \leq x)\,ds$$

$$= \lim_{t} \frac{1}{t}\int_u^{t+u} p(A(s) \leq x)\,ds = F_e(x)$$

$$\Rightarrow A^* \sim F_e. \text{ i.e. } t_1^* \sim F_e. \ \Psi_s^* = \Psi^*.$$

<u>prop</u>. Stationary version Ψ^* of renewal process with rate $\lambda = E(X)^{-1}$ satisfies: $E(N^*(t)) = \lambda t$. and $\lambda = E(N^*(1))$.

<u>Pf</u>: Denote $u(t) = E(N^*(t))$ $\Rightarrow$ $u(n) = n u(1)$. $\forall n$.

So: $u(t) = t\,u(1)$. With $u(t)/t \to \lambda$. So $u(1) = \lambda$.

follows from ERT. ($\lim\limits_{t\to\infty}\lim\limits_{s\to\infty} \square = \lim\limits_{s\to\infty}\lim\limits_{t\to\infty} \square$)

(7) <u>Renewal Equations</u>:

For renewal process $\Psi = \{t_n\}$ with $p(X \leq y) = F(y)$.

Note: $p(A(t)>x) = p(A(t)>x \mid X_1>t)\, p(X_1>t) + \int_0^t p(\square \mid X_1=s)\, p(X_1=s) ds$

$$= p(X_1 - t > x) + \int_0^t p(A(t-s)>x)\, dF(s)$$

$$= \bar{F}(t+x) + \int_0^t p(A(t-s)>x)\, dF(s)$$

Denote: $\bar{F}(x+t) = a(t)$. $H(t) = p(A(t)>x)$.

$\Rightarrow H(t) = a(t) + H * F(t)$. renewal equation.

By Iteration: $H(t) = a + (a + H*F)*F$

$$= \cdots = a + \sum_{n\geq 1} a * F^{*n}.$$ H is unknown

prop. $m(t) = E(N(t))$ satisfies: $H(t) = a + a * m(t)$. $(*)$

Pf: $E(N(t)) = E(\sum_1^{\infty} I_{\{t \geq t_n\}}) = \sum F^{*n}$.

since $F^{*n}(t) = p(\sum_1^n X_i \leq t) = p(t_n \leq t) \xrightarrow{n\to\infty} 0$

Thm. (Key renewal Thm)

If the renewal equation holds for given non-lattice F with mean $1/\lambda$ and a is DRI (directly Riemann integrable, i.e. $\int_0^\infty a(t)dt$ exists)

Then solution for $(*)$ holds. $\lim_{t\to\infty} H(t) = \lambda \int_0^\infty a(t)dt$.

Rmk: There remain valid for delayed case:

in sense. $F_0 = F$. F_1 is delay X_1's dist. then.

$H_1 = a_1 + H_0 * F_1$. $H_0 = a_0 + a_0 * m_0$. $m_0 = E(N(t))$

With $H_1 = a_1 + a_0 * m_1$. $m_1 = E(N_1(t))$

And $\lim_{t\to\infty} a_0 * m_1 = \lim_{t\to\infty} a_0 * m_0 = \lambda \int_0^\infty a_0(t)dt$.

Pf: It's equi. with Blackwell's Thm