

Stochastic Integration

(1) Constructions:

Consider in on. \mathcal{F} , $(\mathcal{F}_t)_{t \geq 0}, P$. (\mathcal{F}_t) is complete

① For mart. bds in L^2 :

Denote: H^2 is span of all conti. martingale M bds in L^2 and $M_0 = 0$. Any two indistinguishable processes are identified. With inner product $\langle \cdot, \cdot \rangle_H$

$$\text{def by } \langle M, N \rangle_H = E(\langle M, N \rangle_\infty)$$

Rmk: i) $M \in H^2 \Leftrightarrow M_0 = 0, E(\langle M, M \rangle_\infty) < \infty$.

and M is c.l.m.

Moreover. $\exists M_\infty \in L^2, M_t = E(M_\infty | \mathcal{F}_t)$

$$\text{ii) Note: } \langle M, N \rangle_H = E(\langle M, N \rangle_\infty) = E(M_\infty N_\infty)$$

$$\langle M, M \rangle_H = 0 \Leftrightarrow M_\infty = 0 \Leftrightarrow M_t = 0, \forall t$$

It's a true inner product.

Prop. H^2 equipped with $\langle \cdot, \cdot \rangle_H$ is a Hilbert space.

Pf: Note: (M^n) is Cauchy seq in H^2 .

$\Leftrightarrow (M^n_t)$ converges in L^2 to a limit Z .

$$\text{By Doob: } E(\sup_t (M^n_t - M^m_t)^2) \rightarrow 0$$

$\Rightarrow (M^n_t)$ converges in L^2 . Denote M_t .

i) (M_t) has conti. sample paths.

$\exists \text{ const. subseq. } E^c \sum_i^\infty \sup_t |M_t^{n_k} - M_t^{n_{k+1}}| < \infty$ (By Fatou's)

$$\sum_i^\infty E^c \sup_t |M_t^{n_k} - M_t^{n_{k+1}}|^2 < \infty$$

$$\Rightarrow \sum_i^\infty \sup_t |M_t^{n_k} - M_t^{n_{k+1}}| < \infty \text{ a.s. } \int_0^{\cdot} M_t^{n_k} d\omega \xrightarrow{w} M_t \text{ a.s.}$$

M is adapted. Since (g_t) is complete.

$$2) M_t^{n_k} = E^c M_\infty^{n_k} | g_t). \text{ Let } k \rightarrow \infty \Rightarrow M_t = E^c Z | g_t)$$

$\Rightarrow M$ is conti. mart bdd in L^2 . $\therefore M \in H^2$.

3) $M_n = \lim M_\infty^{n_k} = Z \text{ a.s. by uniform converge of } M$.

So $M_n^n \rightarrow M_n$ in L^2 . i.e. $M^n \rightarrow M$ in H^2 .

Denote: i) \mathcal{P} is progressive σ -field on $\Omega \times \mathbb{R}_+$.

ii) For $M \in H^2$. $L^2(M) = L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, \lambda P \lambda \llcorner M, M \gg_s)$

Identify H and H' two progressive process

if $H = H'$. $\lambda \llcorner M, M \gg_s$ - a.e. P -a.s.

Rmk: i) $\lambda P \lambda \llcorner M, M \gg_s$ is a measure assign $A \in \mathcal{P}$

to $E^c \int_A \lambda \llcorner M, M \gg_s$. Its total mass is $E^c \llcorner M, M \gg_s = \|M\|_{H^2}$.

ii) $L^2(M)$ can be equipped with an inner product: $(H, k)_{L^2(M)} = E^c \int_0^\infty H_s k_s \lambda \llcorner M, M \gg_s$ which is a Hilbert space.

Def: A elementary process is a progressive process of

form $H_{\sigma(s)} = \sum_{i=0}^p H_{i(s)} I_{(t_i, t_{i+1}) \cap \sigma}$ where $\sigma = t_0 < t_1 < \dots < t_p$. $H_{i(s)} \in \mathcal{F}_{t_i}$. bdd. $\forall 0 \leq i \leq p-1$.

Rmk: The set of all elementary processes form
a linear subspace of $L^2(\Omega)$. Denote by \mathcal{E} .

prop. $\forall M \in H^2$. \mathcal{E} is dense in $L^2(\Omega)$.

Pf: prove: if $k \in L^2(\Omega)$, $k + \Sigma \Rightarrow k = 0$.

Let $X_t = \int_s^t k_n 1_{\{M_n > u\}}$ is FV process.

It makes sense: $E \left(\int_s^t |k_n| 1_{\{M_n > u\}} \right) \leq \|k_n\|_{L^2(\Omega)}^{\frac{1}{2}} \|M_n\|_{H^2}^{\frac{1}{2}}$

Besides, $\int_s^t |k_n| 1_{\{M_n > u\}} < \infty$. a.s. $\forall \omega \Rightarrow X_\omega$ is well-def.

For $H_r(\omega) = F(\omega) I_{[s,t]}(\omega)$, $F \in \mathcal{F}_s$. if $(H, k)_{L^2(\Omega)} = 0$

$\Rightarrow E \left(F \int_s^t k_n 1_{\{M_n > u\}} \right) = 0$. i.e. $E(F(X_t - X_s)) = 0$

By X is adapted. $X_t \in L'$ $\Rightarrow X$ is conti. mart.

Which implies $X = 0 \Rightarrow k = 0$. a.e. a.s.

Lemma: $M \in H^2 \Rightarrow M^T \in H^2$, for T stopping time.

Pf: $\langle M^T, M^T \rangle_\Omega = \langle M, M \rangle_T \leq \langle M, M \rangle_\Omega$.

Rmk: $H \in L^2(\Omega) \Rightarrow I_{[0, T(\omega)]}(s) H_s(\omega)$.

Thm. Construction for Sto-Integration

$M \in H^2$. $\forall N = \sum_i N_{i,i} I_{[t_i, t_{i+1}]} \in \mathcal{E}$. $(N \cdot M)_t = \sum_i N_{i,i} \Delta M_{t_i, t_{i+1}}$.

Defines a process $N \cdot M \in H^2$.

$N \mapsto N \cdot M$ can be extended to isometry from $L^2(\Omega) \rightarrow H^2$.

Moreover, $N \cdot M$ is unique mart in H^2 . satisfies: $\forall N \in H^2$.

$$\langle N \cdot M, N \rangle = N \cdot \langle M, M \rangle.$$

For T stopping time. $(I_{[0, T]})^* M = (M \cdot M)^T = M \cdot M^T$

Rmk: Denote $(H \cdot M)_t = \int_0^t H_s M ds$ and call it the star-Integration of H w.r.t. M .

Pf. 1) Pef of $H \cdot M$ for $H \in \mathcal{E}$ doesn't depend on the decomposition chosen of H .

2) $H \mapsto H \cdot M$ is linear. Next, prove: it's isometry.

Fix H . Set $M_t^i = H_{(i)} \Delta M_{\text{tint}}^{\text{tint}}$

check (M_t^i) is comp. mart. $\Rightarrow M_t^i \in H^2$.

$$\text{So: } H \cdot M = \sum_i M_t^i \in H^2.$$

Note (M_t^i) orthogonal in $\langle \cdot, \cdot \rangle$ bracket.

$$\langle H \cdot M, H \cdot M \rangle_t = \sum \langle M_t^i, M_t^i \rangle \quad (\text{by def of } \langle \cdot, \cdot \rangle)$$

$$= \sum H_{(i)}^2 \langle M, M \rangle_{\text{tint}} - \langle M, M \rangle_{\text{tint}}$$

$$= \int_0^t H_s^2 \lambda \langle M, M \rangle_s$$

$$\Rightarrow \|H \cdot M\|_{H^2}^2 = E \left(\int_0^t H_s^2 \lambda \langle M, M \rangle_s \right) = \|H\|_{L^2(\mu)}^2$$

3) Fix $N \in H^2$. if $H \in L^2(\mu)$, then by kw ineq.

$$E \left(\int_0^t |H_s|^2 \lambda \langle M, N \rangle_s \right) \leq \|H\|_{L^2(\mu)} \|N\|_{H^2} < \infty$$

So $(H \cdot \langle M, N \rangle)_\infty$ is well-def and in L' .

First. Consider $H \in \mathcal{E}$. Then:

$$\begin{aligned} \langle H \cdot M, N \rangle_t &= \sum \langle M^i, N \rangle_t = \sum H_{(i)} \Delta \langle M, N \rangle_{\text{tint}} \\ &= \int_0^t H_s \lambda \langle M, N \rangle_s = (H \cdot \langle M, N \rangle)_t \end{aligned}$$

By: $E(|\langle X, N \rangle|) \leq \|N\|_{H^2} \|X\|_{H^2}$ KW ineq.

$\Rightarrow X \mapsto \langle X, N \rangle_\infty$ is BLO from H^2 to L' .

For general $H \in L^2(\mu)$. consider $(H^n) \subset \mathcal{E} \xrightarrow{\lim} H$.

$$\Rightarrow \text{In } L' \quad \begin{cases} \langle H^n \cdot M, N \rangle_\infty \rightarrow \langle H \cdot M, N \rangle_\infty \text{ (by isometry)} \\ (H^n \cdot \langle M, N \rangle)_\infty \rightarrow (H \cdot \langle M, N \rangle)_\infty \text{ (by KW)} \end{cases}$$

$$\Rightarrow \langle H \cdot M, N \rangle_{\sim} = \langle H \cdot \langle M, N \rangle \rangle_{\sim}$$

$$\text{Replace } N \text{ by } N^t. \text{ So: } \langle H \cdot M, N \rangle_t = \langle H \cdot \langle M, N \rangle \rangle_t.$$

4') To characterize $H \cdot M$. i.e. its uniqueness.

$$\text{if } \langle H \cdot M - X, N \rangle = 0 \Rightarrow \text{set } N = H \cdot M - X \in H^2.$$

$$5) \langle (H \cdot M)^T, N \rangle_t = \langle H \cdot M, N \rangle_{t \wedge T} = \langle H \cdot \langle M, N \rangle \rangle_{t \wedge T}$$

$$= \langle I_{[0, T]} H \cdot \langle M, N \rangle \rangle_t = \langle (I_{[0, T]} H) \cdot M, N \rangle_t$$

Similarly for another statement.

Rmk: i) We can use $\langle H \cdot M, N \rangle = H \cdot \langle M, N \rangle, \forall N \in H^2$

to define $H \cdot M$. Note: $N \mapsto E(\langle H \cdot \langle M, N \rangle \rangle_{\sim})$ is

BLF on H^2 . (by kw inequality $\leq \|N\|_{H^2} \|H\|_{L^2(m)}$)

By Riesz Represent. \exists unique $H \cdot M \in H^2$ s.t.

$$\langle H \cdot M, N \rangle_{H^2} = E(\langle H \cdot \langle M, N \rangle \rangle_{\sim}) = \bar{E}(\langle H \cdot M, N \rangle_{\sim})$$

ii) By the notation of $H \cdot M$. We have:

$$\left\langle \int_0^{\cdot} H_s \lambda M_s, N \right\rangle_t = \int_0^t H_s \lambda \langle M, N \rangle_s \dots (*)$$

We interpret it by: " \int " commutes with " $\langle \cdot, \cdot \rangle$ ".

(it's Str-integration, not common integration)

Cor: For $m \in H^2$, $H \in L^2(m)$. Then: $\langle H \cdot M, H \cdot M \rangle$

$$= H \cdot (H \cdot \langle M, M \rangle) = H^2 \cdot \langle M, M \rangle.$$

$$\text{generally: } \langle H \cdot M, k \cdot N \rangle = H \cdot (k \cdot \langle M, N \rangle)$$

$$= (kH) \cdot \langle M, N \rangle, \text{ for } k \in L^2(N), N \in H^2.$$

prop. $H \in L^2(m)$. If k is a progressive process. Then:

$kH \in L^2(m) \Leftrightarrow k \in L^2(H \cdot M)$, if these hold.

$$\text{then, } (kH) \cdot M = k \cdot (H \cdot M).$$

$$\text{Pf. } E \left(\int_0^\infty k_s^2 N_s^2 \lambda \langle M, M \rangle_s \right) = E \left(\int_0^\infty k_s^2 \lambda \langle N \cdot M, N \cdot M \rangle_s \right)$$

For the latter, note that for $\forall N \in \mathbb{N}^2$.

$$\langle (kN) \cdot M, N \rangle = (kN) \cdot \langle M, N \rangle = \langle k \cdot (N \cdot M), N \rangle$$

Moments of Sto-Integration:

Suppose $M, N \in \mathbb{N}^2$, $N \in L^2(M)$, $k \in L^2(N)$. Then:

$$E \left(\int_0^t N_r \lambda M_r \right) = 0, \quad E \left(\int_0^t N_r \lambda M_r | \mathcal{F}_s \right) = 0, \quad \text{by mart. prop.}$$

$$E \left(\left(\int_0^t N_s \lambda M_s \right) \left(\int_0^t k_s \lambda N_s \right) \right) = E \left(\langle M \cdot M, k \cdot N \rangle_t \right)$$

$$= E \left(\int_0^t N_s k_s \lambda \langle M, N \rangle_s \right)$$

$$\Rightarrow \text{In particular, } E \left((N \cdot M)_t^2 \right) = E \left(\int_0^t N_s^2 \lambda \langle M, M \rangle_s \right)$$

② For Local mart:

i) Now we extend Sto-integration to c.l.m's. M .

Denote: $L_{loc}^2(M) = \{N \text{ is progressive} \mid \int_0^t N_s^2 \lambda \langle M, M \rangle_s < \infty, \forall t, n, s\}$

$L^2(M) = \{N \text{ is progressive} \mid \int_0^\infty N_s^2 \lambda \langle M, M \rangle_s < \infty\}$.

Thm. For M is c.l.m. $\forall N \in L_{loc}^2(M)$. \exists unique c.l.m with

initial value 0, denote by $N \cdot M$. So. $\forall N$ c.l.m.

$$\langle N \cdot M, N \rangle = N \cdot \langle M, N \rangle. \quad \text{Besides,}$$

$$\text{i) If } T \text{ is stopping time. Then: } (I_{[0,T]} M) \cdot M = (N \cdot M)^T$$

$$= N \cdot M^T.$$

$$\text{ii) } k \text{ is progressive. Then: } k \in L_{loc}^2(N \cdot M) \Leftrightarrow kN \in L_{loc}^2(M)$$

And then: $N \cdot (k \cdot M) = (Nk) \cdot M$, if those hold.

iii) If $M \in \mathbb{H}^2$, $H \in L^2(M)$. Then the def is consistent with def of $H \cdot M$ before.

Pf: i) Assume $M_0 = 0$. Or we can write $M = M_0 + M'$

$$\text{Set } H \cdot M = H \cdot M'.$$

$$\int_0^t H_s^2 d\langle M, M \rangle_s < \infty, \forall t \geq 0. \quad \forall w \in \mathbb{N}, \text{ by}$$

modify H . set $H=0$ on $N = I \cap \mathbb{Z} = \{0\}$.

$$\text{Int: } T_n = \inf \{t \geq 0 \mid \int_0^t (1 + H_s^2) d\langle M, M \rangle_s \geq n\}.$$

$$\Rightarrow M^{T_n} \in \mathbb{H}^2, \text{ and } \langle H \cdot M^{T_n}, H \cdot M^{T_n} \rangle_\infty \leq n, \text{ so: } H \in L^2(M^{T_n})$$

$$2) \text{ Note: for } m > n, H \cdot M^{T_n} = (H \cdot M^{T_m})^{T_n}.$$

$\Rightarrow \exists$ unique process denoted by $H \cdot M$. st.

$$(H \cdot M)^{T_n} = H \cdot M^{T_n}, \quad \forall n \in \mathbb{N}$$

$$H \cdot m = \lim_n H \cdot M^{T_n} \text{ conti and adapted.}$$

Since $(H \cdot M)^{T_n} \in \mathbb{H}^2 \Rightarrow H \cdot M$ is c.l.m.

$$3) \text{ Fix } N \text{ is c.l.m } N_0 = 0. \text{ Set } \widetilde{T}_n = \inf \{t \geq 0 \mid |N_t| \geq n\}$$

$$S_n = T_n \wedge \widetilde{T}_n. \quad \text{Check: } \langle H \cdot M, N \rangle_{S_n} = (H \cdot \langle M, N \rangle)^{S_n}$$

Let $n \rightarrow \infty$. For uniqueness. argue as before.

4) i), ii) is identical as before. follows from the characterization of $H \cdot M$.

$$\text{For iii): Note: } \langle H \cdot M, H \cdot m \rangle = H^2 \cdot \langle M, m \rangle$$

$\Rightarrow H \cdot M \in \mathbb{H}^2$. And the charac. show consistency.

Rmk: We denote $(H \cdot M)_t = \int_0^t H_s dm_s$ as before.

The formulas $(*)$ stay valid in a.s.s.

ii) Connection with

Wiener Integral:

For B is (\mathcal{F}_t) -SBM. $h \in L^2(\mathbb{R}^+, B_{\mathbb{R}^+}, \nu_t)$.

Def: Wiener integral $\int_0^t h(s) dB_s = G(h I_{[0,t]})$.

Prop. It coincides with Sti-integral $\langle h, B \rangle_t$.

Pf: It holds for $h = I_{[a,b]}$.

Note G is isometry. Approx. h by simple func's.

iii) Moments of integrals:

If M is c.l.m. $M \in L^2(\mu)$. for $t \in \mathbb{R}^+$. under condition $E \left(\int_0^t M_s^2 \lambda < M, M>_s \right) < \infty$. Then we obtain: $(M \cdot M)^t \in H^2$. As before. it satisfies:

$$\begin{cases} E \left(\int_0^t M_s \lambda M_s \right) = 0 \\ E \left(\left(\int_0^t M_s \lambda M_s \right)^2 \right) = E \left(\int_0^t M_s^2 \lambda < M, M>_s \right) \end{cases}$$

and mart property for $0 \leq s \leq t$:

$$E \left(\int_s^t M_s \lambda M_s \mid \mathcal{F}_s \right) = 0.$$

Rmk: If this condition doesn't hold. We still have:

$$E \left(\left(\int_0^t M_s dM_s \right)^2 \right) \leq E \left(\int_0^t M_s^2 \lambda < M, M>_s \right)$$

Since "if $RNs = \infty$ " is trivial.

③ For semimartingales:

Finally, we extend the sto-integral to conti. semimart.

Def: i) A progressive process H is locally bdd if:

$$\forall t \geq 0, \sup_{s \leq t} |H_s| < \infty \text{ a.s.}$$

Rmk: i) A conti. adapted process is locally bdd.

ii) If H is locally bdd. Then $H \cdot V$ FV process, we have: $\forall t \geq 0, \int_0^t |H_s| dV_s < \infty \text{ a.s.}$
and $H \in L_{loc}^2(\Omega)$ for m is c.t.m.

ii) For $X = M + V$ conti. semimart. $H \cdot X$ is locally bdd.

Sto-integral $H \cdot X$ is conti. semimart. defined by

$$H \cdot X = H \cdot M + H \cdot V. \text{ Define } H \cdot X = \int_0^{\cdot} H_s dX_s.$$

prop. i) $(H \cdot X) \mapsto H \cdot X$ is bilinear.

ii) $H \cdot (k \cdot X) = (Hk) \cdot X$ if H, k are locally bdd.

iii) For every stopping time. $(H \cdot X)^T = H \cdot X^T = (I_{[0,T]} H) \cdot X$.

iv) If X is c.l.m or FV process. Then: so does $H \cdot X$

v) If H is form: $H = \sum_{i=1}^{p+1} H_{i,i} I_{\{t_i \leq s < t_{i+1}\}}$, $H_{i,i} \in \mathcal{F}_{t_i}$. Then:

$$(H \cdot X)_t = \sum_{i=1}^p H_{i,i} (X_{t_{i+1} \wedge t} - X_{t_i \wedge t}).$$

Pf: i) - iv) follows from sto-integral of FV. c.l.m.

v) Enough to prove c.l.m case (FV have ref.)

Assume: $M_0 = 0$. stop it at suitable time. $\exists n \in \mathbb{N}$.

Set $T_n = \inf \{t \geq 0 \mid |H_t| \geq n\} = \inf \{t_i \mid |H_{i,i}| \geq n\}$.

which is for circumvent that $H_{i,i}$ isn't bdd.

$$\Rightarrow NI_{[0, T_n]} = \sum \hat{N}_{i,j} I_{[t_i, t_{i+1}]} \text{ (ss). } \hat{N}_{i,j} = N_{i,j} I_{[T_n > t_i]}$$

is elementary process. Since $|N_{i,j}| \leq n$.

$$\text{Besides, } (N \cdot M)_{[0, T_n]} = (NI_{[0, T_n]} \cdot M)_t = \sum \hat{N}_{i,j} \Delta M_i^{i+1}$$

Since $T_n \uparrow \infty$. set $n \rightarrow \infty$.

④ Convergence:

Thm. For $X = M + V$ conti. semimart. $(M^n)_{n \geq 1}$ is seq of locally bdd progressive process. κ is nonnegative progressive process. If following holds a.s.:

i) $N_s^n \xrightarrow{n \rightarrow \infty} N_s$, $\forall s \in [0, t]$, fix t .

ii) $|N_s^n| \leq k_s$, $\forall n$, $\forall s \in [0, t]$.

iii) $\int_0^t \kappa_s^2 \lambda < m, m > < \infty$, $\int_0^t \kappa_s | \lambda V_s | < \infty$.

Then: $\int_0^t N_s^n \lambda X_s \xrightarrow[n \rightarrow \infty]{P} \int_0^t N_s \lambda X_s$.

Rmk: i) if κ is locally bdd. then iii) holds automatically.

ii) Conditions i), ii) can be assumed: $s \in [0, t]$.

$\lambda < m, m >$, -a.e. and $|\lambda V_s|$ -a.e.

Pf: i) $\int_0^t N_s^n \lambda V_s \xrightarrow{n \rightarrow \infty} \int_0^t N_s \lambda V_s$ P-a.s. by DCT.

2) For a.l.m part:

Set $\bar{T}_p = \inf \{s \in [0, t] \mid \int_0^s \kappa_r^2 \lambda < m, m > \geq p\} \wedge t$.

$\Rightarrow (N^n - N) \cdot M^{\bar{T}_p} \in N^2$. by ii). So:

$$E \left(\left(\int_0^{\bar{T}_p} (N_s^n - N_s) \lambda M_s \right)^2 \right) = E \left(\int_0^{\bar{T}_p} (N_s^n - N_s)^2 \lambda < m, m > \right)$$

Let $n \rightarrow \infty$ by DCT. with $p \in T_p = t \xrightarrow{n \rightarrow \infty} 1$

prop. X is conti. semimart. H is adapted conti. process.

Then: $\forall t \geq 0$. $\forall 0 = t_0^n < t_1^n < \dots < t_{p^n}^n = t$ subdivision of $[0, t]$. When mesh $\rightarrow 0$. we have:

$$\lim_n \sum_{i=1}^{p^n-1} H_{t_i^n} (X_{t_{i+1}^n} - X_{t_i^n}) = \int_0^t H_s dX_s \text{ in prob.}$$

Pf: Set $H_s^n = H_s I_{s \leq t_0^n} + \sum_{i=0}^{p^n-1} H_{t_i^n} I_{s \in [t_i^n, t_{i+1}^n)}$. progressive.

Take $k_s = \max_{t \in [s, s+1]} |H_t|$. Dominated H_s . locally bdd

Then by DCT. Thm above. $\Rightarrow \int_0^t H_s^n dX_s \rightarrow \int_0^t H_s dX_s$

Rmk: The prop fails if $LHs = \sum H_{t_{i+1}^n} (X_{t_{i+1}^n} - X_{t_i^n})$

i.e. evaluate H at the right end rather than left end of $[t_i^n, t_{i+1}^n]$.

(2) Itô's Formula:

① Thm: (Itô's Formula)

X^1, \dots, X^P are conti. semimart. $F \in C^2_c(\mathbb{R}^P, \mathbb{R})$

Then $\forall t \geq 0$. P-a.s. $F(X_0^1, \dots, X_0^P) = F(X_t^1, \dots, X_t^P)$

$$\sum_1^P \int_0^t \frac{\partial F}{\partial x_i^j} (X_s^1, \dots, X_s^P) dX_s^i + \frac{1}{2} \sum_{i,j}^P \int_0^t \frac{\partial^2 F}{\partial x_i^j \partial x_i^j} (X_s^1, \dots, X_s^P) d\langle X^i, X^j \rangle_s$$

Pf: 1') $P=1$. $X=X'$.

Consider $0 = t_0^n < t_1^n < \dots < t_{p^n}^n = t$ subdivision of $[0, t]$. whose mesh $\rightarrow 0$.

$$\text{From: } F(X_t) = F(X_0) + \sum F(X_{t_{i+1}^n}) - F(X_{t_i^n})$$

By Taylor expansion on: $\theta \in [0, 1] \mapsto F(X_{t_i^n} + \theta \Delta X_i^{i+1})$

$$\Rightarrow F(X_{t_{i+1}^n}) - F(X_{t_i^n}) = F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n}) + \frac{1}{2} f_{nn} (\Delta X_i^{i+1})^2$$

By prop above. The first term $\xrightarrow{P} \int_0^t F'(X_s) dX_s$.

Select a subseq guarantee it holds a.s.

For the second term:

$$\text{prove: } \sum f_{n,i} \Delta^2 X_i^{i+1} \xrightarrow{P} \int_0^t F''(X_s) \lambda < X, X >_s$$

$$\text{where } f_{n,i} = F''(X_{t_i^*}) + c(X_{t_{i+1}^*} - X_{t_i^*}), c \in [0, 1].$$

$$\sup_i |f_{n,i} - F''(X_{t_i^*})| \leq \sup_i (\sup_{[X_{t_i^*}, X_{t_{i+1}^*}]} |F''(x) - F''(X_{t_i^*})|)$$

$\rightarrow 0$, a.s. by unif-conti of F'' .

$$\text{Combined with } \sum (X_{t_{i+1}^*} - X_{t_i^*})^2 \xrightarrow{P} < X, X >_t, n \rightarrow \infty$$

$$\Rightarrow |\sum f_{n,i} \Delta^2 X_i^{i+1} - \sum F''(X_{t_i^*}) \Delta^2 X_i^{i+1}| \xrightarrow{P} 0, n \rightarrow \infty.$$

$$\text{prove: } \sum F''(X_{t_i^*}) \Delta^2 X_i^{i+1} \xrightarrow{P} \int_0^t F''(X_s) \lambda < X, X >_s.$$

$$\text{LHS} = \int_{[0,t]} F''(X_s) M_n(s), M_n = \sum (X_{t_{i+1}^*} - X_{t_i^*}) \delta_{t_i^*}$$

Consider D countable lang in $[0,t]$. $(t_i^*) \subset D, \forall n$.

$$\forall r \in D, M_n([0,r]) \xrightarrow{n \rightarrow \infty} < X, X >_r.$$

By diagonal argument. $\exists n_k$. $M_{n_k}([0,r]) \xrightarrow{k \rightarrow \infty} < X, X >_r$. $\forall r \in D$

$$\Rightarrow \text{LHS} \xrightarrow{a.s.} \int_0^t F''(X_s) \lambda < X, X >_s.$$

2) For general case:

Apply Taylor on: $\theta \in [0,1] \mapsto F(X_{t_i^*} + \theta \Delta X_i^{i+1}, \dots, X_{t_i^*} + \theta \Delta \dots)$

Rmk: i) Most convergences we meet is converge in pr.

Sometimes, it doesn't matter. like in Itô's Formula
"n" doesn't involve in. So we can select subseq
to guarantee a.s. - convergence.

Sometimes, the seq is a.i. So we can imply
 L' -convergence.

ii) A special case of Itô's Formula is integration

by part = consider $\varphi = 1$ $F(x, y) = xy$.

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y \rangle_t$$

iii) Set $X = Y$ above. it shows: $X_t^2 - \langle X, X \rangle_t$
 $= X_0^2 + 2 \int_0^t X_s dX_s$. c.l.m. linear form.

iv) Note that we prove Itô's Formula locally.

if $F \in C^2(U)$, $U \subseteq \overset{\text{open}}{R^d}$. then $H \in C(U)$

open set. Let $T_V = \inf\{t \geq 0 \mid (X_t^1, \dots, X_t^P) \notin V\}$

and $\exists h = F$ in V . $h \in C^2(R^P)$.

\Rightarrow Apply Itô's Formula on $(X_{t \wedge T_V}^1, \dots, X_{t \wedge T_V}^P)$

If in addition, we know $(X_t^1, \dots, X_t^P) \in U$. a.s.

Let $V \supseteq U$. \Rightarrow we obtain Itô's Formula for $F(X_t)$

Stays valid. e.g. $F(x) = \log x$.

Part 1: A random process takes values in C is complex c.l.m
if its imaginary and real part are c.l.m's

prop. M is c.l.m. for $\lambda \in C$. Let $\Sigma(\lambda M)_t = e^{\lambda M_t - \frac{\lambda^2}{2} \langle M, M \rangle_t}$

Then $\Sigma(\lambda M)$ is complex c.l.m. having the form:

$$\Sigma(\lambda M)_t = e^{\lambda M_0} + \lambda \int_0^t \Sigma(\lambda M)_s dM_s.$$

Pf.: For $X_t = e^{\lambda t - \frac{\lambda^2}{2} r}$ $\in C$ satisfies $\begin{cases} \frac{\partial F}{\partial x} = \lambda F \\ \frac{\partial F}{\partial r} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} = 0 \end{cases}$

Consider $(\langle X, X \rangle_t, X_t)$.

Apply Itô's on $\operatorname{Im} F$. $\operatorname{Re} F$ separately.

(3) Applications of Itô's Formula:

① Lévy's Charac. of Bms:

Thm. For $X = (X^1, \dots, X^k)$ adapted. conti. Follows equi.:

i) X is λ -lim (\mathcal{F}_t) -BM.

ii) X^1, \dots, X^k are a.l.m.s and $\langle X^i, X^j \rangle_t = \delta_{ij} t$. $\forall i, j$.

Rmk. In particular, a.l.m. M is (\mathcal{F}_t) -BM.

$\Leftrightarrow \langle M, M \rangle_t = t, \forall t \geq 0 \Leftrightarrow M_t - t$ is a.l.m.

Pf. i) \Rightarrow ii) We have proved. For ii) \Rightarrow i).

For $\beta \in \mathbb{R}^k$. $\beta \cdot X_t = \sum \beta_i X_t^i$ is a.l.m.

with QV: $\langle \beta \cdot X, \beta \cdot X \rangle_t = \sum \beta_i \beta_j \langle X^i, X^j \rangle_t = |\beta|^2 t$

$\Rightarrow \mathbb{E} e^{i\beta \cdot X} = e^{i\beta \cdot X + \frac{1}{2}|\beta|^2 t}$ is complex. a.l.m. bdd

on every opt. interval. So it's true mart.

$\Rightarrow \mathbb{E} e^{i\beta \cdot (X_t - X_s)} | \mathcal{F}_s = e^{-\frac{1}{2}|\beta|^2(t-s)}$ by mart prop.

So. $\forall A \in \mathcal{F}_s$. $\mathbb{E} e^{i\beta \cdot (X_t - X_s)} | A = p(A) e^{-\frac{1}{2}|\beta|^2(t-s)}$

If $A = \Omega$. Then we have: $X_t - X_s \sim N(0, (t-s)I)$

Furthermore. if $p(A) > 0$, then $\mathbb{E} e^{i\beta \cdot (X_t - X_s)} | A$

$= e^{-\frac{1}{2}|\beta|^2(t-s)}$. i.e. $X_t - X_s | A \sim X_t - X_s, \forall A \in \mathcal{F}_s$

$\Rightarrow X_t - X_s$ is indep with \mathcal{F}_s .

② Conti. Mart. as

Time-changed BM:

Thm. (Dambis - Dubins - Schwartz)

If M is a.l.m. st. $\langle M, M \rangle_\infty = \infty$ a.s. Then \exists BM

$(\tilde{\mathcal{F}}_s)_{s \geq 0}$. st. a.s. $\forall t \geq 0$. $M_t = \tilde{P}_{\langle M, M \rangle_t}$.

Lemma: For M is c.l.m. We have a.s. for $0 \leq n < b$:

$$M_t = M_n \quad \forall t \in [n, b] \iff \langle M, M \rangle_b = \langle M, M \rangle_n$$

Pf: By conti of M . prove for $0 \leq a < b$. $a, b \in \mathbb{Q}$.

So we can fix a, b . prove: a.s.

$$\{M_t = M_n, \forall t \in [n, b]\} = \{\langle M, M \rangle_n = \langle M, M \rangle_b\}$$

" \subset " is trivial. for " \supset ":

$$\text{Set } N_t = M_t - M_{t \wedge n}. \quad T_\varepsilon = \inf\{t \geq 0 \mid \langle N, N \rangle_t \geq \varepsilon\}$$

$$\Rightarrow \langle N, N \rangle_t = \langle M, M \rangle_t - \langle M, M \rangle_{t \wedge n}$$

$$N^{T_\varepsilon} \in \mathbb{N}^2. \quad \text{So: } E(N_{t+T_\varepsilon}^2) = \bar{E}(\langle N, N \rangle_{t+T_\varepsilon}) \leq \varepsilon$$

$$\text{Note } A = \{\langle M, M \rangle_n = \langle M, M \rangle_b\} \subset \{T_\varepsilon \geq b\}.$$

$$\text{So for } t \leq b. \quad \bar{E}(N_{t+T_\varepsilon}^2 I_A) = \bar{E}(N_{t+T_\varepsilon}^2 I_A) \leq \varepsilon \rightarrow 0$$

$$\Rightarrow N_t = 0. \text{ a.s. for } n \leq t \leq b. \text{ on } A.$$

Rmk: It's intuitive that if TV of $F_{m,n} = 0$

then it's const. Consider $M_t - M_{t \wedge n}$ is

like set $\tilde{f} = f(x) - f(0)$. but we also
need to guarantee N_t is c.l.m.

Pf of Thm: 1) First we assume $M_0 = 0$.

$$\text{Set } Z_r = \inf\{t \geq 0 \mid \langle M, M \rangle_t \geq r\}. \quad Z_r < \infty. \quad \forall r.$$

by redefine $Z(w) = 1$ for $w \in \{\langle M, M \rangle_\infty < \infty\}$

since (g_t) is complete. Z_r is stopping time

2) $r \mapsto Z(rw)$ is left-conti \uparrow . (nondecreasing)

so it's has right-limit at $\forall r \geq 0$.

Denote by $Z_{r+} = \inf\{t \geq 0 \mid \langle M, M \rangle_t > r\}$.

$Z_{r+} = 0$ on $\{\langle M, M \rangle_\infty < \infty\}$.

3) Set $\beta_r = M_{2r}$ adapted to $\mathcal{F}_{2r} \stackrel{\Delta}{=} \mathcal{F}_r$. complete.

By 1°), 2°), conti of M , $r \mapsto \beta_{r(\omega)}$ is left-anti
and right-limit: $\beta_{r+} = \lim_{s \downarrow r} \beta_s = M_{2r+}$.

By Lemma: $\beta_{r+} = \beta_r$. Since $\langle M, M \rangle_{2r} = \langle M, M \rangle_{2r+} = r$
 \Rightarrow path of β is anti (refine it on null set)

4) Prove: β_s , $\beta_s^{2^n}$ -s are mart. wrt (\mathcal{F}_s) .

Note: M^{2^n} , $(M^{2^n})^2 - \langle M, M \rangle^{2^n}$ are u.i. marts.

implies DSSN. $E[\beta_s | \mathcal{F}_r] = \beta_r$... used by M^{2^n}

By Lévy charac. $\Rightarrow \beta$ is BM wrt \mathcal{F}_r .

5) Prove: $M_{2<0, n>t} = M_t$.

Since $\tau_{[m, M+t]} \leq t \leq \tau_{[m, M+t]}$. combined with:

$$\langle M, M \rangle_{20} = \langle M, M \rangle_{20+} \Rightarrow M_t = M_{20} \quad \forall t \in [20, 20+]$$

6°) General. for $M_0 \neq 0$. set $M = M_0 + \tilde{M}$.

Apply the argument on \tilde{M} . $M_t = \beta'_{<\tilde{M}, t>}$

$\Rightarrow \beta_s = M_0 + \beta'_s$ is also BM (β' indept with \mathcal{F}_0)

To remove " $\langle M, M \rangle_\infty = \infty$ " consider in a larger space:

Def. Enlargement of filtrated prob. space on $\tilde{\Omega}, (\tilde{\mathcal{F}}_t), \tilde{P}$

is $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P})$ with $\pi: \tilde{\Omega} \rightarrow \Omega$. s.t.

$$\pi^{-1}(\mathcal{F}_t) \subset \tilde{\mathcal{F}}_t \quad \forall t. \quad \tilde{P} \circ \pi^{-1} = P$$

Rmk: process X on Ω may be viewed as so on $\tilde{\Omega}$

by $\pi_t = X(\tilde{\omega}) = X(\omega)$ for $\pi(\tilde{\omega}) = \omega$.

Thm. Exists enlargement $(\widetilde{\mathcal{F}}, \widetilde{q}, (\widetilde{\mathbb{F}}_t), \widetilde{P})$ of $(\mathcal{F}, q, (\mathbb{F}_t), P)$ and BM \widetilde{B} on $\widetilde{\mathcal{F}}$. indept with M. c.l.m. st.

$$B_t = \begin{cases} M_{2t} & \text{if } t < \langle M, M \rangle_\infty \\ M_\infty + \widetilde{P}_{t-\langle M, M \rangle_\infty} & \text{if } t \geq \langle M, M \rangle_\infty \end{cases} \quad \text{is SBM.}$$

Besides. $W_t := M_{2t} I_{\{t < \langle M, M \rangle_\infty\}} + M_\infty I_{\{t \geq \langle M, M \rangle_\infty\}}$ is a $(\widetilde{\mathbb{F}}_t) - \text{BM}$ stopped at $\langle M, M \rangle_\infty$.

prop. For M. N. c.l.m's. st. $M_0 = N_0 = 0$. $M_t = \beta_{\langle M, M \rangle_t}$. $N_t = Y_{\langle N, N \rangle_t}$ where β, Y are BMs correspond M. N respectively.

- If i) $\langle M, M \rangle_t = \langle N, N \rangle_t \quad \forall t \geq 0$. a.s.
ii) $\langle M, M \rangle_\infty = \langle N, N \rangle_\infty = \infty$. a.s. iii) $\langle M, N \rangle_t = 0 \quad \forall t$. a.s.

Then β is indept with Y .

Pf. Note: $\begin{cases} \beta_r = M_{2r} & r = \inf \{t \geq 0 \mid \langle M, M \rangle_t \geq r\} \\ Y_r = N_{2r} & = \inf \{t \geq 0 \mid \langle N, N \rangle_t \geq r\}. \end{cases}$

By iii) $M+N \Rightarrow M+N_t$ is c.l.m. Then:

$M_t^{2n} + N_t^{2n}$ is n.i. mart. Apply Optional Stop Thm.

$$\Rightarrow E(\beta, Y_s \mid \mathcal{F}_r) = E(M_r^{2n}, N_r^{2n} \mid \mathcal{F}_r) = \beta_r Y_r.$$

So $\beta_r Y_r$ is (Y_r) -mart. $\Rightarrow \langle \beta, Y \rangle = 0$. a.s.

By Lévy charac.: (β, Y) is 2-lim BM.

③ BDG Inequality:

Result: For M. c.l.m. $M_t^* := \sup_{S \in t} |M_S|$

Thm. (Burkholder - Davis - Gundy)

$\forall p > 0$. $\exists C_p, c_p > 0$, s.t. for \forall a.l.m. M .

with $M_0 = 0$ and every stopping time T .

$$c_p E^{\mathbb{C}} \langle M, M \rangle_T^{\frac{p}{2}} \leq E^{\mathbb{C}} (M_T^*)^p \leq C_p E^{\mathbb{C}} \langle M, M \rangle_T^{\frac{p}{2}}$$

Pf: 1) Replace M by M^* . only need to prove it when $T = \infty$

Assume M is bad by replace M by

$$M^{T_n}, T_n = \inf \{t \geq 0 \mid |M_t| \geq n\} \Rightarrow M \in H^2.$$

Then set $n \rightarrow \infty$ by MCT.

2) Right side, $p \geq 2$.

Apply Ito's Formula on $|X|^p \in C^1(\mathbb{R})$

Then take expectation. By Hölder and Doob's

3) Left side, $p \geq 4$.

From $\langle M, M \rangle_t = M_t^2 - 2 \int_0^t M_s dM_s$. by C_p 's inequ.

$$\begin{aligned} \Rightarrow E^{\mathbb{C}} \langle M, M \rangle_{\infty}^{\frac{p}{2}} &\leq \alpha_p (E^{\mathbb{C}} (M_{\infty}^*)^p + E^{\mathbb{C}} (\int_0^{\infty} M_s dM_s)^{\frac{p}{2}}) \\ &\leq \alpha_p (E^{\mathbb{C}} (M_{\infty}^*)^p + E^{\mathbb{C}} (\int_0^{\infty} M_s^2 d\langle M, M \rangle_s)^{\frac{p}{2}}) \\ &\leq \alpha_p (E^{\mathbb{C}} (M_{\infty}^*)^p + (E^{\mathbb{C}} (M_{\infty}^*)^p) E^{\mathbb{C}} \langle M, M \rangle_{\infty}^{\frac{p}{2}})^{\frac{1}{2}} \end{aligned}$$

$$\text{Set } X = E^{\mathbb{C}} \langle M, M \rangle_{\infty}^{\frac{p}{2}} \stackrel{\frac{1}{2}}{=} \eta = E^{\mathbb{C}} (M_{\infty}^*)^{\frac{p}{2}}.$$

$$\text{Solve } x^2 - \alpha_p x \eta - \alpha_p \eta^2 \leq 0 \Rightarrow \alpha_p x \leq \eta.$$

For other parts, we introduce two Lemmas:

Def: (Dominated relation)

A positive. adapted. right-conti process X is dominated by an increasing process A if:

$$E^c(X_T | \mathcal{F}_0) \leq E^c(A_T | \mathcal{F}_0), \text{ for all stopping time } T.$$

Lemma: If X is dominated by A . conti. Then for $x, \eta > 0$

$$\text{we have: } p^c(X_n^* > x, A_n \leq \eta) \leq \frac{1}{x} E^c(A_n \wedge \eta).$$

Pf: It suffices to prove it when $p^c(A_0 \leq \eta) = 1$.

by replace p by $p' = p \cdot 1_{A_0 \leq \eta}$. Condition still holds.

Assume X_n exists by stopping X at proper time.

Then apply Fatou's Lemma to obtain conclusion.

$$\text{Set: } R = \inf \{t \geq 0 \mid A_t > \eta\}, \quad S = \inf \{t \geq 0 \mid X_t > x\}$$

$$\Rightarrow \{A_n \leq \eta\} = \{R = \infty\}. \quad \text{Then we have:}$$

$$p^c(X_n^* > x, A_n \leq \eta) = p^c(X_n^* > x, R = \infty)$$

$$\leq p^c(X_S \geq x, S < \infty, R = \infty)$$

$$\leq p^c(X_{S \wedge R} \geq x) \leq \frac{1}{x} E^c(X_{S \wedge R})$$

$$\leq \frac{1}{x} E^c(A_{S \wedge R}) \leq \frac{1}{x} E^c(A_n \wedge \eta)$$

Lemma: If X is dominated by A conti. Then: $\forall k \in \{0, 1\}$

$$\text{we have: } E^c(X_n^{*k}) \leq \frac{2-k}{1-k} E^c(A_n^k)$$

Pf: Set $F: \mathbb{R}^+ \rightarrow \mathbb{R}^+$. conti. $\int F(x) dx = 0$.

$$E^c(F(X_n^*)) = E^c \int_0^\infty I_{\{X_n^* > x\}} \lambda F(x) dx. \quad \text{by Fubini.}$$

$$\leq \int_0^\infty (p^c(X_n^* > x, A_n \leq x) + p^c(A_n > x)) \lambda F(x) dx$$

$$\begin{aligned}
 & \stackrel{\text{(Lemma)}}{\leq} \int_0^\infty \left(\frac{c}{x} E(A_n \wedge x) + P(A_n > x) \right) dF(x) \\
 & = \int_0^\infty \left(c \frac{1}{x} E(A_n I_{A_n \leq x}) + 2P(A_n > x) \right) dF(x) \\
 & \stackrel{\text{(Fatou)}}{=} 2E(F(A_n)) + E(A_n \int_{A_n} \frac{dF(x)}{x})
 \end{aligned}$$

Then set $F = x^k$, we obtain the result.

Return to the pf:

4) Right side. $0 < p < 2$.

$$\text{Set } X = M^2. A = \langle M, M \rangle. \Rightarrow E(M_T^2 | \mathcal{G}_0) = E(\langle M, M \rangle_T | \mathcal{G}_0)$$

$$\Rightarrow E(M_T^{2^k}) \leq \frac{2-k}{1-k} E(\langle M, M \rangle_T^k). k \in (0, 1)$$

5) Left side, $0 < p < 4$

$$\text{Set } X = \langle M, M \rangle^2. A = C_4 M_T^{*4}. \text{ Consider } M|A. A_t \in \mathcal{G}_t$$

$$\Rightarrow E(\langle M, M \rangle_T^2 | \mathcal{G}_0) \leq C_4 E(M_T^{*4} | \mathcal{G}_0) \text{ by (3)}.$$

$$S_0 = E(\langle M, M \rangle_\infty^2) \leq \frac{2-k}{1-k} C_4^k E(M_\infty^{*4k}). k \in (0, 1)$$

Cov. M is a.l.m. with $M_0 = 0$. Then:

$$E(\langle M, M \rangle_\infty^{\frac{1}{2}}) < \infty \Rightarrow M \text{ is u.i. mart.}$$

Pf: $M_\infty^{\frac{1}{2}} \in L^1$. dominates M .

Rmk: i) It's weaker than $E(\langle M, M \rangle_\infty) < \infty$
which implies: $M \in L^2$.

ii) Apply on Sto-integral: $\int_0^t M_s dM_s$.

If $E\left(\left(\int_0^t M_s dM_s\right)^{\frac{1}{2}}\right) < \infty$, $\forall t \geq 0$.

Then: $\int_0^t M_s dM_s$ is a mart.

($\forall (M \cdot M)^t$ is u.i. mart. Check mart prop)

(4) Representation of Martingales:

Suppose the filtration (\mathcal{F}_t) on Ω is complete canonical filtration of a SBM $(B_t)_{t \geq 0}$.

① Represent Thm:

Lemma: $V = \text{span} \{ e^{i \sum \lambda_i (B_{ti} - B_{ti-1})} \mid 0 = t_0 < t_1 < \dots < t_n \}$

$\lambda_i \in \mathbb{R}$, $1 \leq i \leq n$ is dense in $L^2(\Omega, \mathcal{F}_\infty, P)$

Pf: Prove: $\forall z \in L^2(\Omega, \mathcal{F}_\infty, P)$, st. $E^c z e^{i \sum \lambda_i (B_{ti} - B_{ti-1})} = 0$

for $\forall \lambda_i \in \mathbb{R}$, $i \leq n$. $\Rightarrow z = 0$, a.s.

By Weierstrass Approx. $(e^{i \sum \lambda_i x_i}) \xrightarrow{\text{DCT}} \forall f \in C_c(\mathbb{R})$

$\Rightarrow E^c z f(B_{t_0}, B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}}) = 0 \cdot \forall f \in C_c(\mathbb{R})$

$C_\theta \xrightarrow{\text{Approxi}} \text{Simple func. } I_A, A \in \sigma \subset B_{ti}, 1 \leq i \leq n$

$\Rightarrow E^c z I_A = 0 \cdot \forall A \in \sigma \subset B_t, t \geq 0 = \mathcal{F}_\infty \therefore z = 0$

follows from Monotone Class argument.

Thm. $\forall z \in L^2(\Omega, \mathcal{F}_\infty, P)$. \exists unique progressive process

$h \in L^2(\mathcal{B})$, st. $z = E^c(z) + \int_0^\infty h_s dB_s$

Pf: 1) Uniqueness of h :

$$E^c \left(\int_0^\infty (h_s - h'_s)^2 ds \right) = E^c \left(\int_0^\infty h_s dB_s - \int_0^\infty h'_s dB_s \right)^2 = 0$$

2) Set \mathcal{H} is L^2 of all $z \in L^2(\Omega, \mathcal{F}_\infty, P)$

which satisfies the statement

Note if $z \in \mathcal{H}$, then:

$$E^c(z^2) = E^c(z)^2 + E^c \int_0^\infty h_s^2 ds$$

$\Rightarrow \mathcal{H}$ is closed subspace of $L^2(\Omega, \mathcal{F}_\infty, P)$

3) Prove \mathcal{H} is dense in $L^2(\Omega, \mathcal{F}_\infty, P)$

Consider $(\lambda_i)_i \subset \mathbb{R}'$. $0 = t_0 < t_1 < \dots < t_n$.

Set $f(s) = \sum_i \lambda_i I_{(t_{i-1}, t_i]}$. $E_t^f = e^{i \int_0^t f(s) dB_s}$

By prop above: $e^{i \sum \lambda_i (B_{t_i} - B_{t_{i-1}})} \cdot e^{\frac{i}{2} \sum \lambda_i (t_i - t_{i-1})} =$

$$E_\infty^f = 1 + i \int_0^\infty E_s^f f(s) dB_s$$

$\Rightarrow e^{i \sum \lambda_i (B_{t_i} - B_{t_{i-1}})} \in \mathcal{H}$. By Lemma. \mathcal{H} is dense.

COR. i) If mart. M bdd in L^2 . Then \exists unique h

$\in L^2(B)$. const. $C \in \mathbb{R}'$. s.t. $M_t = C + \int_0^t h_s dB_s$

ii) If a.l.m M . Then \exists unique $h \in L^2_{loc}(B)$ and

const. $C \in \mathbb{R}'$. s.t. $M_t = C + \int_0^t h_s dB_s$.

Pf: i) Applying on $M_n \in L^2(\Omega, \mathcal{F}_n, P)$

$$M_n = E(M_n) + \int_0^\infty h_s dB_s. \quad E(M_n | \mathcal{F}_t) = M_t$$

ii) By Blumenthal Thm. $M_0 = C$. since $h_s \in \mathcal{F}_s$.

$$\text{set } T_n = \inf \{t \geq 0 \mid |M_t| \geq n\}.$$

$\Rightarrow M^{T_n}$ satisfies the conditions in i).

$$M_t^{T_n} = C + \int_0^t h_s^{(n)} dB_s. \quad \text{By uniqueness:}$$

$$h_s^{(n)} = I_{(0, T_n]} h_s^{(n)} \text{ for } n < n. \quad \text{a.s. a.e.}$$

$$\text{Set } h_s I_{(0, T_n]} = h_s^{(n)}. \quad \Rightarrow h_s \in L^2_{loc}(B).$$

Thm. (Multi Dimensional Extension)

Suppose (\mathcal{F}_t) is complete canonical filtration of

\cap λ -dim $SBM(\vec{B_t}) = (B'_t, \dots, B''_t)$

Then $\forall z \in L^2(\cap, \mathcal{F}_\infty, P)$. \exists unique $\vec{h} = (h_1, \dots, h_n)$

s.t. $h_i \in L^2(B)$. $\forall i \in I$. $z = \bar{E}(z) + \sum_{i=1}^n \int_0^\infty h_i s \lambda B_s^i$

Consequently, the identical results hold for

M. mart. bdd in L^2 or a.l.m.

Pf: Consider $\Sigma_t^{\vec{f}} = \Sigma \subset \bigcup_{i=1}^n \int_0^t f_k(s) \lambda B_s^k$

$f_k(s) = \sum_{i=1}^m I_{[c_{ti}, c_{ti+1}]}$ check \mathcal{H} is still closed dense.

(Note the prop. w.r.t Σ can be extended to
multidimension case: $\Sigma \subset \vec{\lambda} \cdot \vec{M}$). $F(\vec{r}, \vec{x}) = e^{\sum_{i=1}^n \lambda_i x_i - \frac{1}{2} \sum_{i=1}^n r_i^2}$)

② Applications:

i) The filtration (\mathcal{F}_t) is conti.

We have check $\mathcal{F}_t, \mathcal{F}_t^+, \mathcal{F}_t^-$ only differ a P -null set when learning B_m . Next, we use the represent theorem to check again.

If: For $z \in \mathcal{F}_t^+$. bnd $\exists h_s \in L^2(B)$. $z = \bar{E}(z) + \int_0^\infty h_s \lambda B_s$.

Note for $\mathbb{E}(z)$. $z = \bar{E}(z | \mathcal{F}_{t+\varepsilon}) = \bar{E}(z) + \int_0^{t+\varepsilon} h_s \lambda B_s$.

Set $\varepsilon \rightarrow 0$. RHS $\xrightarrow[n.s.]{} \bar{E}(z) + \int_0^t h_s \lambda B_s \in \mathcal{F}_t$.

Since (\mathcal{F}_t) is complete $\Rightarrow \mathcal{F}_t = \mathcal{F}_t^+$.

Conversely. for $z \in \mathcal{F}_t$. $\bar{E}(z | \mathcal{F}_{t-\varepsilon}) = \bar{E}(z) + \int_0^{t-\varepsilon} h_s \lambda B_s$

$z = \bar{E}(z | \mathcal{F}_t) = \bar{E}(z) + \int_0^t h_s \lambda B_s \uparrow \bar{E}(z | \mathcal{F}_{t-\varepsilon}), \varepsilon \rightarrow 0$

$\Rightarrow z = \bar{E}(z | \mathcal{F}_{t-})$. a.s. $\therefore \mathcal{F}_t = \mathcal{F}_t^-$. by complete.

ii) All mart. w.r.t (\mathcal{F}_t) has
a modification with conti paths:

For mart. M_t . WLOG. Suppose it's n.i.

Otherwise stop it at proper time. (M^*)

Then $\exists M_n$. $M_t = E[M_n | \mathcal{F}_t]$. $t \geq 0$.

Note: (\mathcal{F}_t) is conti. complete. $E[M_t]$ is conti

So M has a conti modification \tilde{M} .

$$\text{Set } M_n^{(n)} = \tilde{M}_n I_{\{\tilde{M}_n < n\}} + n I_{\{\tilde{M}_n \geq n\}} \cdot b/n$$

$$\text{since } |M_n^{(n)}| \leq |\tilde{M}| \in L^1. \Rightarrow M_n^{(n)} \xrightarrow{L^1} \tilde{M}_n \text{ by DCT.}$$

$$\text{Set mart. } M_t^{(n)} = E[M_n^{(n)} | \mathcal{F}_t] \text{ bdd in } L^2.$$

By represent Thm. $M^{(n)}$ are conti. a.s.

Apply Doob's c Require right-conti. That's why use \tilde{M} !)

$$\Rightarrow \forall \lambda > 0. \sup_t |M_t^{(n)} - \tilde{M}_t| > \lambda \leq \frac{3}{\lambda} E |M_1^{(n)} - \tilde{M}_1| \rightarrow 0$$

Select subseq unk). $\sup_t |M_t^{(n)} - \tilde{M}_t| \rightarrow 0$ a.s.

Since \tilde{M} is uniform limit of conti. mart. a.s

$\Rightarrow \tilde{M}$ has conti modification. so does M .

(5) Girsanov's Thm:

Suppose filtration (\mathcal{F}_t) is complete. right-conti

in L. $\mathbb{P}(\mathcal{F}_t), P$ filtered prob space.

O Thm:

prop. If α is another p.m. on (Ω, \mathcal{F}) . $\alpha \ll P$ on \mathcal{F}_∞ .

$\forall t \in \overline{\mathbb{R}^+}$. Set $D_t = \alpha/\lambda P|_{\mathcal{F}_t}$. R-N variate.

Then (D_t) is u.i. mart.

Pf: $\forall A \in \mathcal{F}_t$. $\alpha(A) = \mathbb{E}_\alpha(I_A) = \mathbb{E}_P(I_A D_\infty) = \mathbb{E}_P(I_A E^{(D_\infty | \mathcal{F}_t)})$
 $= \mathbb{E}_P(D_t I_A)$. $\Rightarrow D_t = E^{(D_\infty | \mathcal{F}_t)}$. a.s.

Cir. Under the conditions above. Then:

i) (D_t) has a càdlàg modification (\tilde{D}_t) .

ii) $\forall T$ stopping time. $D_T = \alpha/\lambda P|_{\mathcal{F}_T}$.

iii) if $P \ll \alpha$ on \mathcal{F}_∞ . then:

$$\inf_{t \geq 0} \tilde{D}_t > 0, \sup_{t \geq 0} \tilde{D}_t < \infty, P\text{-a.s. } \alpha\text{-a.s.}$$

Pf: i) Note (\mathcal{F}_t) is complete. right-conti.

ii) By optional stopping Thm.

iii) $\forall \varepsilon > 0$. Let $T_\varepsilon = \inf\{t \geq 0 \mid \tilde{D}_t < \varepsilon\}$. stop. time

$$[T_\varepsilon < \infty] \in \mathcal{F}_{T_\varepsilon} \Rightarrow \alpha([T_\varepsilon < \infty]) = \mathbb{E}_P(I_{[T_\varepsilon < \infty]} D_{T_\varepsilon}) \leq \varepsilon$$

follows from right-conti of \tilde{D}_t .

$$\Rightarrow \alpha(\cap [T_\frac{1}{n} < \infty]) = 0. \Rightarrow P(\cap [T_\frac{1}{n} < \infty]) = 0$$

By sym: $\lambda P/\alpha|_{\mathcal{F}_t} = 1/D_t$ holds as well.

prop. D is c.l.m. $D > 0$. P -a.s. Thm. \exists unique c.l.m L .

st. $D_t = \mathbb{E}(L)_t = e^{rt - \frac{1}{2}\sigma^2 t + L_r t}$. Actually, L has

form: $L_t = \log D_0 + \int_0^t \frac{\lambda D_s}{D_s} ds$

Pf: Uniqueness is trivial. Apply Itô's or $\log D_t$

follows from Rmk iv) of Itô's Thm.

Thm. (Girsanov's)

If $\rho < \alpha$, $\alpha < p$ on \mathcal{F}_∞ . $D_t = \lambda \alpha / \lambda P |_{\mathcal{F}_t}$ is a
conti. mart. L is the unique c.l.m. s.t. $D_t = L(L)_t$.

Then: M is c.l.m under $P \Rightarrow \tilde{M} = M - \langle M, L \rangle$
is c.l.m under α .

Lemma. X is anti. adapted process. T is stopping time.

$(XD)^T$ is mart. under $P \Rightarrow X^T$ is mart. under α .

Pf: $E_\alpha |X_{T \wedge t}| = E_P |X_{T \wedge t} D_{T \wedge t}| < \infty \Rightarrow X_t \in L(\alpha)$

$\forall A \in \mathcal{F}_S$. s.t. Then $A \cap \{T > S\} \in \mathcal{F}_S$.

$$\Rightarrow E_P (I_{A \cap \{T > S\}} X_{T \wedge t} D_{T \wedge t}) = E_P (I_{A \cap \{T > S\}} X_{T \wedge S} D_{T \wedge S})$$

Besides, $A \cap \{T > S\} \in \mathcal{F}_{T \wedge S} \subset \mathcal{F}_{T \wedge t}$

$$\Rightarrow E_\alpha (I_{A \cap \{T > S\}} X_{T \wedge t}) = E_\alpha (I_{A \cap \{T > S\}} X_{T \wedge S})$$

Combined with a trivial equation:

$$E_\alpha (I_{A \cap \{T > S\}} X_{T \wedge t}) = E_\alpha (I_{A \cap \{T > S\}} X_{T \wedge S})$$

Cor. (XD) is c.l.m under $P \Rightarrow X$ is c.l.m under α .

Return to the Pf:

$$\begin{aligned} \text{By Itô's: } \tilde{M}_t D_t &= M_0 D_0 + \int_0^t \tilde{M}_s \lambda ds + \int_0^t D_s \lambda M_s \\ &\quad - \int_0^t D_s \lambda \langle M, L \rangle_s + \langle M, D \rangle_t. \\ &= M_0 D_0 + \int_0^t \tilde{M}_s \lambda D_s + \int_0^t D_s \lambda M_s. \end{aligned}$$

$$\lambda \langle M, L \rangle_t = D_t^{-1} \lambda \langle M, D \rangle_t. \text{ by prop. above.}$$

$\Rightarrow \tilde{M} D$ is c.l.m under P . Then by Lemma.

\circledcirc Consequence:

i) If $P \ll \alpha \ll P$. Then $P - M_{\text{semi}}^{\text{conti}} = \alpha - M_{\text{semi}}^{\text{conti}}$

Pf: i) Role of P, α can be exchangeable.

$$\text{Consider } D_t^{-1} = \lambda P / d\alpha|_{S_t} = e^{-\frac{1}{2}\langle L, L \rangle_t - Lt}$$

$$\text{Set } \tilde{L} = L - \langle L, L \rangle \text{ c.l.m. on } \alpha. \Rightarrow D_t^{-1} = e^{c - \tilde{L} t}$$

$$\text{Besides. } \langle \tilde{L}, \tilde{L} \rangle = \langle L, L \rangle.$$

2) $\forall M$. c.l.m under $P \Rightarrow M = \tilde{M} + \langle M, L \rangle$. Semimart under α .

$$P - M_{\text{semi}}^{\text{conti}} = \{P - \text{c.l.m}'s + FV\} = \{\alpha - \text{semimart} + FV\}$$
$$\subset \{\alpha - \text{semimarts}\} = \alpha - M_{\text{semi}}^{\text{conti}}$$

Conversely. by symmetry of P, α .

Rmk: Set $\mathcal{Y}_\alpha^P = P - \text{c.l.m}'s \rightarrow \alpha - \text{c.l.m}'s$

$$M \longmapsto \tilde{M} = M - \langle M, L \rangle$$

$\mathcal{Y}_P^\alpha = \alpha - \text{c.l.m}'s \rightarrow P - \text{c.l.m}'s$

$$M \longmapsto \tilde{M} = M - \langle M, \tilde{L} \rangle$$
$$\Rightarrow \mathcal{Y}_\alpha^P \circ \mathcal{Y}_P^\alpha = \mathcal{Y}_P^\alpha \circ \mathcal{Y}_\alpha^P = I_A.$$

$\therefore \mathcal{Y}_\alpha^P, \mathcal{Y}_P^\alpha$ are bijections.

ii) Brackets of two conti semimarts X, Y are identical under P or α . if $P \ll \alpha \ll P$.

Pf. By approx. equation. converge in p.m. $P \Leftrightarrow \alpha$. Since $P \sim \alpha$.

Moreover. for M locally bdd progressive process. The sto-integration $M \cdot X$ is same under P or α .

Pf: Note that $X = M + A$, decomposition under P .

since $H \cdot A =: \int_0^{\cdot} H_s dA_s$ irrelevant with choice of p.m. It's FV process under P or α .

\Rightarrow prove: $(H \cdot M)_P = (H \cdot M)_{\alpha}$. M is c.l.m.

$$\begin{aligned} 1') \text{ Under } P: (H \cdot \tilde{M})_P &= (H \cdot M)_P - H \cdot \langle M, L \rangle \\ &= (H \cdot M)_P - \langle (H \cdot M)_P, L \rangle \\ &= (\tilde{H} \cdot \tilde{M})_P \text{ c.l.m. under } \alpha. \end{aligned}$$

2) $\forall N$ c.l.m on α . $\exists N'$ c.l.m. on P . s.t.

$$y_{\alpha}^P(N) = N \text{ i.e. } \tilde{N}' = N.$$

$$\begin{aligned} \langle (H \cdot \tilde{M})_P, N \rangle &= \langle (\tilde{H} \cdot \tilde{M})_P, \tilde{N}' \rangle = \langle (H \cdot M)_P, N' \rangle \\ &= H \cdot \langle M, N' \rangle = H \cdot \langle \tilde{M}, \tilde{N}' \rangle \\ &= H \cdot \langle \tilde{M}, N \rangle \end{aligned}$$

\Rightarrow By characterization. $(H \cdot \tilde{M})_P = (H \cdot \tilde{M})_{\alpha}$

So: By linearity, $(H \cdot M)_P = (H \cdot M)_{\alpha}$.

Kern: From 1). We have: $H \cdot y_{\alpha}^P(m) = y_{\alpha}^P(H \cdot m)$.

Cor. $y_{\alpha}^P, y_{\alpha}^{\alpha}$ are isometric isomorphisms under H^2 norm. Since $\langle \tilde{M}, \tilde{M} \rangle_P = \langle M, M \rangle_P = \langle M, M \rangle_{\alpha}$.

iii) B is $(\beta_0)-BM$ under $P \xrightarrow{P \cong \alpha} \tilde{B}$ is still $(\beta_0)-BM$ under α .

Pf: $\langle B, B \rangle_c = \langle \tilde{B}, \tilde{B} \rangle_c = t$. By Livg's characterization.

② Applications:

i) Construct P.m.:

Lemma. For M c.l.m. Then we have:

$$\{\lim_{t \rightarrow \infty} M_t \text{ exists, finite}\} = \{<M, M>_\infty < \infty\}, \text{ a.s.}$$

Pf: i) On $\{<M, M>_\infty = \infty\}$. By represent in B_M :

$$M_{zt} = \beta_z. \quad \left\{ \begin{array}{l} \overline{\lim}_{t \rightarrow \infty} \beta_t = +\infty \\ \underline{\lim}_{t \rightarrow \infty} \beta_t = -\infty \end{array} \right. \quad z \in \mathbb{Z}^\infty.$$

$$\Rightarrow \overline{\lim}_t M_{zt} = +\infty \leq \overline{\lim}_t M_t \quad . \quad \text{So: LHS} \subset \text{RHS.}$$

$$\underline{\lim}_t M_{zt} = -\infty \geq \underline{\lim}_t M_t$$

2) Conversely. assume $M_\infty = 0$. $T_n = \inf\{t \geq 0 | < M, M>_t \geq n\}$

$\Rightarrow M^{T_n}$ is bdd in L^2 . so $\lim_{t \rightarrow \infty} M_t^{T_n}$ exists. a.s.

But on $\{<M, M>_\infty = \infty\}$. $\exists n$. $T_n = \infty$. a.s.

Cor. For M . c.l.m. The followings equi.:

$$\text{i)} \overline{\lim}_{t \rightarrow \infty} M_t = +\infty \text{. a.s.} \quad \text{ii)} \underline{\lim}_{t \rightarrow \infty} M_t = -\infty \text{. a.s.}$$

$$\text{iii)} < M, M >_\infty = \infty \text{. a.s.}$$

Start from a.l.m L . $L_0 = 0$. $< L, L >_\infty < \infty$. a.s.

By Lemma. $\lim_{t \rightarrow \infty} L_t$ exists. a.s.

$\sum c(L)_t$ is nonnegative a.l.m. So a supermart.

which converges to $\sum c(L)_t = e^{L_0 - \frac{1}{2} < L, L >_\infty}$.

By Fatou's: $E^c E^c(L)_\infty \leq 1$.

Claim: $E^c \Sigma(L)_\infty = 1 \iff \Sigma(L)$ is u.i. mart.

Pf: \Leftarrow is trivial. For \Rightarrow :

$$\text{Note: } E^c \Sigma(L)_\infty = E^c \Sigma(L)_0 = E^c \Sigma(L)_t. \forall t.$$

$$\text{Check: } E^c \Sigma(L)_0 | \mathcal{F}_t = \Sigma(L)_t. \text{ u.s.}$$

follows from $\Sigma(L)$ is supermart.

$$\Rightarrow \text{Int } \lambda Q = \Sigma(L)_\infty(w) \lambda P(w). \quad D_t = \Sigma(L)_t. \quad \left(\begin{array}{l} E_p \subset \frac{\lambda Q}{\lambda P(w)} I_{\Omega} \\ = E_p \Sigma(L)_\infty(w) \end{array} \right)$$

Next, we introduce a Thm ensuring Claim holds.

Thm. L is c.l.m. with $L_0 = 0$.

i) (Norikov's Criteria) $E^c e^{\frac{1}{2} \langle L, L \rangle_\infty} < \infty$

ii) (Kazamaki's Criteria) L is u.i. mart. $E^c e^{\frac{1}{2} L_\infty} < \infty$.

iii) $\Sigma(L)$ is a u.i. mart.

Thm: i) \Rightarrow ii) \Rightarrow iii).

Pf: i) \Rightarrow ii): L is bdd in L^2 by $E^c \langle L, L \rangle_\infty < \infty$.

So it's u.i. mart.

$$\text{Besides: } e^{\frac{1}{2} L_\infty} = (\Sigma(L)_\infty)^{\frac{1}{2}} e^{\frac{1}{4} \langle L, L \rangle_\infty}.$$

ii) \Rightarrow iii): By Jensen inequi. $\forall T$ stopping time.

$$e^{\frac{1}{2} L_T} \leq E^c e^{\frac{1}{2} L_\infty} | \mathcal{F}_T =: N_T$$

N_T is u.i. since $e^{\frac{1}{2} L_\infty} \in L'$.

$\Rightarrow e^{\frac{1}{2} L_T}$ is u.i. w.r.t stopping time T .

i) Next. prove: $\forall \text{accl. } \Sigma(aL)_T$ is u.i. on

T . Stopping time.

$$\text{Set } Z_t^{(a)} = e^{\frac{aL}{1+a}}. \quad \Sigma(aL) = \Sigma(L)^{a^2} (Z_t^{(a)})^{1-a^2}.$$

$\forall I \in \mathcal{I}$. By Nölders Ineq:

$$\begin{aligned} E^c(I_T \Sigma(L)_T) &\leq E^c(\Sigma(L)_T)^{\frac{1}{n}} E^c(I_T \Sigma_T^{(n)})^{1-\frac{1}{n}} \\ &\stackrel{(Jensen)}{\leq} E^c(I_T e^{\frac{1}{2}L_T^{2n}(1-n)}) \end{aligned}$$

2) If T_n reduces $\Sigma(L)_t$. Then:

$$E^c(\Sigma(L)_{t \wedge T_n}) = \Sigma(L)_{S \wedge T_n}. \text{ Let } n \rightarrow \infty \text{ by a.i.}$$

So $\Sigma(L)_t$ is a.i. mart.

$$\Rightarrow I = E^c(\Sigma(L)_\infty) \leq E^c(L_\infty)^{\frac{1}{2}} E^c(e^{\frac{1}{2}L_\infty^{2n}(1-n)})$$

Then let $n \rightarrow 1$. $E^c(L_\infty) \geq 1$.

ii) Construct Solution

of SDEs:

If b is bdd measurable on $\mathbb{R}^+ \times \mathbb{R}$. $\exists g \in L^2(\mathbb{R}^+, \mathbb{R})$

s.t. $|b(t, x)| \leq g(t)$. $\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}$. B is $(g_t) - BM$.

Consider $L_t = \int_0^t b(s, B_s) dB_s$, a.l.m.

It satisfies Novikov's criterion. $E^c e^{\frac{1}{2}\langle L, L \rangle_\infty} < \infty$.

$\Rightarrow D_t = \Sigma(L)_t$ is a.i. mart. Let $\mathcal{Q} = D_\infty \cdot P$.

By Girsanov's Thm:

$B_t = B_t - \langle B, L \rangle_t = B_t - \int_0^t b(s, B_s) ds$ is $(g_t) - BM$ on \mathcal{Q} .

Rmk: Restate those above: Under p.m. \mathcal{Q} . $\exists (g_t) - BM$ β

s.t. $X = B$ solves SDE: $dX_t = \beta_t + b(t, X_t) dt$.

iii) Cameron - Martin Formula:

Def: \mathcal{H} is set of all func's hats. s.t. $\exists f \in L^2(\mathbb{R}^+, B_{\mathbb{R}^+}, \lambda_t)$, $h(t) =$

$h(t) = \int_0^t g(s) ds$, called Cameron-Martin space.

Rmk: By argument of ii). $Q = D_\alpha \cdot P = e^{\int_0^\infty h(s) d\lambda_s - \frac{1}{2} \int_0^\infty h^2(s) d\lambda_s} \cdot P$

is p.m. s.t. $B_t = B_t - h(t)$ is BM under Q .

Prop. (Cameron-Martin Formula)

$W(dw)$ is Wiener measure on $(C([0, t], \mathbb{R}))$, $h \in \mathcal{H}$.

Then, $\forall \phi$ nonnegative measurable on $(C([0, t], \mathbb{R}))$.

$$\int \phi(w+h) W(dw) = \int \phi(w) e^{\int_0^\infty h(s) \lambda_{ws} - \frac{1}{2} \int_0^\infty h^2(s) \lambda_{ws}} W(dw).$$

Pf: By Rmk above: $E_p \circ D_\alpha \phi((B_t)_{t \geq 0}) = E_Q \circ \phi((B_t)_{t \geq 0})$

$$= E_Q \circ \phi((B_t + h(t))_{t \geq 0}) = E_p \circ \phi((B_t + h(t))_{t \geq 0}).$$

Set $p = W$. \hat{h} is weak derivative of h)

Rmk: C-M Formula gives kind of "quasi-invariant" property of Wiener measure under translation $h \in \mathcal{H}$.

i.e. $\theta_h: w \mapsto w + h(t)$. $W \circ \theta_h^{-1}$ has density:

$$\frac{\lambda_{W \circ \theta_h^{-1}}}{\lambda_W}(dw) = \sum c \int_0^\infty h(s) \lambda_{ws} ds$$

Set $U_a = \inf \{t \geq 0 \mid B_t + ct = a\}$. hitting time with drift.

Apply C-M Formula on $h'(s) = c I_{\{s \leq t\}}$, $h(s) = c(s-t)$.

$$\phi(w) = I_{\max_{0 \leq s \leq t} w_s \geq a}. \text{ Then: } P(U_a < t) = \int_0^t \frac{n}{\sqrt{2\pi s^3}} e^{-\frac{(a-cs)^2}{2s}} ds$$

$$\Rightarrow P(U_a < \infty) = \begin{cases} 1 & \text{if } c \geq 0 \\ e^{2ac} & \text{if } c \leq 0 \end{cases}$$