Integration

For $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $h \in C^\infty(U)$. if $F: U \xrightarrow{\sim} V$. diffeomorphism.

We have : $\int_U h\, dx_1 \cdots dx_n = \int_V (h \circ F)\, |\mathrm{Det}\, DF|\, dx_1 \cdots dx_n$. ($U$ connected)

Written in n-form : $\int_V F^* \tau = \mathrm{sgn}(|DF|) \int_U \tau$.

For arbitrary manifold X :

Define : $\int_X \tau \in \mathbb{R}$ for $\tau \in \Lambda^n(X)$.

Pick $(U, f) \in A_X$. $f: U \cong \tilde{U}$. Then integrate it on $\tilde{U}$.

$\Rightarrow$ problems $\begin{cases} (1) \text{ Integral may not be coordinate indept.} \\ (2) \text{ Integral may not be convergent.} \end{cases}$

(1) <u>Oritentions</u> :

· Note that for $\alpha \in \Lambda^n(X)$. (U_1, f_1). $(U_2, f_2) \in A_X$.

WLOG. $U_1 = U_2 = U$. Then $\tilde{\alpha}_1 = \phi_{21}^* \tilde{\alpha}_2$. $\phi_{21}: \tilde{U}_1 \cong \tilde{U}_2$

$\therefore \int_{\tilde{U}_1} \tilde{\alpha}_1 = \mathrm{sgn}(|D\phi_{21}|) \int_{\tilde{U}_2} \tilde{\alpha}_2$

It's not be coordinate indept.

<u>Def</u>: $w \in \Lambda^n(X)$ is called volum form if $w \neq 0$.

If X has an volum form. Then say it's orientable.

Remark: i) by conti. It said $w > 0$ or $w < 0$

ii) If we have a volum form: w.

$h \in C^\infty(X)$. $h \neq 0$. Then so hw does.

prop. X is orientable n-dim manifold. $Z = h^{-1}(y) \subset X$

for $h \in C^\infty(X)$. level set at regular value y.

Then Z is orientable.

Pf: Suppose w is volum form on X.

Fix $z \in Z$. $\therefore w|_z \neq 0 \in \Lambda^n T_z^* X$.

1°) $Dh|_z : T_z X \longrightarrow T_z \mathbb{R} \cong \mathbb{R}^1$. surjective.

$\ker Dh|_z = T_z Z$. Choose $\vec{n} \in T_z X$. s.t. $Dh|_z \cdot \vec{n} = 1$.

Find basis $(e_k)_1^m$ for $T_z Z$.

$\therefore (e_k)_1^m \cup \{\vec{n}\}$ is basis of $T_z X$.

2°) Define: $w'|_z : (v_1, \cdots v_m) \longmapsto w|_z(v_1 \cdots v_m, n)$

$\forall v_k \in T_z Z$. Check: $w'|_z \in \Lambda^m T_z^* Z$.

It's indept with choice of $\vec{n}$:

since $w|_z(e_1, \cdots e_m, e_k) = 0$. $\forall 1 \le k \le m$.

$\therefore$ It makes no difference to consider: $\vec{n} + \ker Dh|_z$.

3°) $w'|_z \neq 0$. since $\{e_k\}_1^m \cup \{\vec{n}\}$ is basis of $T_z X$.

$\therefore w|_z(e_1 \cdots e_m, \vec{n}) = w'|_z(e_1 \cdots e_m) \neq 0$.

4°) w' is smooth $\in \Lambda^m T^* Z$.

WLOG. $y=0$. $\forall z \in Z$. Choose $(U, f) \in A_Z$. $z \in U$.

s.t. $\tilde{h} = h \circ f^{-1} = x_n$ in $\tilde{U}$. by IFT. ($h \circ f^{-1} = \tilde{h}$).

$\therefore f(Z \cap U) = f \circ h^{-1}\{0\} \cap \tilde{U} = \mathbb{R}^{n-1} \cap \tilde{U} = \{x_n = 0\} \cap \tilde{U}$.

Then choose $\vec{n} = (0 \cdots \partial/\partial x_n) \in T_z \tilde{U}$. $D\tilde{h}|_z \cdot \vec{n} = 1$. $\forall z \in Z$.

Written in coordinate: $\tilde{w} = \tilde{q}\, dx_1 \wedge \cdots dx_n$. $\tilde{q} \in C^\infty(\tilde{U})$.

$\Rightarrow \tilde{w}' = \tilde{q}|_{\{x_n=0\}}\, dx_1 \wedge \cdots dx_{n-1}$. smooth. $\square$

<u>Cor.</u> For $h: X \longrightarrow Y$. smooth between orientable manifolds X, Y. Then $Z = h^{-1}(y) \subset X$ is orientable for regular value $y \in Y$.

<u>Pf:</u> Fix z: $Dh|_z : T_z X \longrightarrow T_{h(z)} Y$. $\dim X = n$. $\dim Y = k$

$\ker Dh|_z = T_z Z$. Choose basis of $T_{h(z)} Y : (n_i)_1^k$.

With basis of $T_z Z : (e_i)_1^{n-k}$.

$\Rightarrow (e_i)_1^{n-k} \cup (n_i)_1^k$ is basis of $T_z X$.

Suppose $w \in \Lambda^n TX$. $w \neq 0$. Define:

$w'|_z : (v_1 \cdots v_{n-k}) \longmapsto w|_z (v_1 \cdots v_{n-k}, n_1 \cdots n_k)$.

Check $w'|_z \in \Lambda^{n-k} T_z^* Z$. $w' \neq 0$. smooth.

<u>Remark:</u> i) Not every submanifold of orientable manifold is orientable.

There exists nonorientable manifolds:

<u>e.g.</u> klein bottle. Mobiüs band.

ii) For application: $X = \mathbb{R}^{n+1}$ is orientable ($\bigwedge^{n} dx_k$)

$\therefore S^n, T^n$ are orientable. ($T^n \cong T^1 \times \cdots T^1$)

Def: $\omega \in \Lambda^n(X)$ is a volum form. For $(U, f) \in A_X$ is said oriented if: ω_0 is standard volum form of $\tilde{U}$. when write ω in chart: $\tilde{\omega} = h\omega_0$. $h > 0$. $\forall x \in \tilde{U}$.

Remark: It's not hard to find oriented chart: for U is connected. If $h < 0$. Then: Choose $F : (x_1 \cdots x_n) \longmapsto (-x_1, \cdots x_n)$. $\therefore (U, F \circ f)$ is oriented.

Def: An orientation of X is an equivalent class of volum forms on X. i.e. $\omega_1 \in [\omega] \Leftrightarrow$ $\exists g \in C^\infty(X)$. $g > 0$. s.t. $g\omega = \omega_1$.

If we fix an oriention on X. Then say X is oriented.

Remark: A manifold has two orientions or no oriention.

prop. X is oriented manifold. For (U_1, f_1). $(U_2, f_2) \in A_X$. oriented charts. Then $\forall \tilde{x} \in f_2(U_1 \cap U_2)$. We have: $D\phi_{12}|_{\tilde{x}} > 0$.

Pf: Pick $w \in [w]$. from orientation.

$\tilde{w}_1 = h_1 w_0 \in \Lambda^n(\tilde{U}_1)$. $\tilde{w}_2 = h_2 w_0 \in \Lambda^n(\tilde{U}_2)$.

$h_1, h_2 > 0$. By pull-back along ϕ_{12}:

$h_2|_{f_2(x)} = \mathrm{Det}(D\phi_{12}|_{f_2(x)})\, h_1|_{f_1(x)} \Rightarrow |D\phi_{12}|_{f_2(x)}| > 0$.

Remark: This solves problem (1). We only consider in oriented charts of oriented manifold.

(2) Integration:

① Bump Forms:

Def: $0 \le p \le n$. $\alpha \in \Lambda^p(X)$. We call α a bump form if $\exists\, (U, f) \in A_X$. some cpt set $W \subset U$. s.t. $\alpha \equiv 0$ outside W

Remark: For (U, f) is oriented. We can compose it with reflection: $F: (x_1, \cdots x_n) \longmapsto (-x_1, \cdots x_n)$

prop. (U_1, f_1). $(U_2, f_2) \in A_X$. two oriented charts s.t. $\exists\, W_1, W_2$ cpt set. α vanishes outside W_1, W_2. Then $\int_{\tilde{U}_1} \tilde{\alpha}_1 = \int_{\tilde{U}_2} \tilde{\alpha}_2$.

Pf: $U = U_1 \cap U_2$. $W = W_1 \cap W_2$. $\alpha \equiv 0$ on W^c.

$\therefore \int_{\tilde{U}_1} \tilde{\alpha}_1 = \int_{f_1(U)} \tilde{\alpha}_1 = \int_{f_2(U)} \tilde{\alpha}_2 = \int_{\tilde{U}_2} \tilde{\alpha}_2$.

since $\tilde{\alpha}_2 = \phi_{12}^* \tilde{\alpha}_1$

Remark: So for any bump form $\alpha \in \Lambda^n(X)$.

We have a well-def $\int_X \alpha$ by

using charts.

② For arbitrary n-forms:

Def. A partition of Unity on smooth manifold

X is set of func $\varphi_\circ = \{\varphi_i\}_{i\in I} \subset C^\infty(X)$.

s.t. i) φ_i is bump form. $\forall i \in I$.

ii) $\forall x \in X$. There're only finite $i \in I$. s.t. $\varphi_i(x) \neq 0$.

iii) $\sum_{i\in I} \varphi_i(x) = 1$. $\forall x \in X$.

Remark: Given any chart A_X. There exists a POU.

subordinate it for any manifold X.

prop. X is cpt manifold $\Leftrightarrow$ There exists a POU

$\varphi_\circ = \{\varphi_i . i \in I\}$ on X. where I is finite.

pf: ($\Leftarrow$) $X = \bigcup_i^n \operatorname{supp}(\varphi_i) = \bigcup_i^n W_i$. W_i's are cpt.

($\Rightarrow$) For any $x \in X$. Find bump Func. ψ_x.

s.t. $\exists$ nbd V_x of x. $\psi_x \equiv 1$ in V_x.

$\exists$ cpt $W \subseteq X$. $\psi_x \equiv 0$ outside W.

$X \subseteq \bigcup V_x \Rightarrow \exists (V_{x_i})_i^n$ cover X.

set $\varphi_i = \psi_{x_i} / \sum_i^n \psi_{x_k} \in C^\infty(X)$.

Remark: since $\alpha = \sum_i^{n} \varphi_i \alpha$. Then we denote:

$$\int_X^{\varphi.} \alpha = \sum_i^{n} \int_X \varphi_i \alpha .$$

prop. X is cpt. oriented manifold. $\varphi. = \{\varphi_i\}_1^r$. $\hat{\varphi}. = \{\varphi_i\}_1^s$ are two finite POU of X. Then for any $\alpha \in \Lambda^n(X)$. we have: $\int_X^{\varphi.} \alpha = \int_X^{\hat{\varphi}.} \alpha$.

Pf: 1°) For bump form β. $\varphi.$ is finite POU.

Then $\int_X \beta = \int_X^{\varphi.} \beta$:

Pf: For each $\varphi_i \beta$. it's bump form.

Written in charts:

$$\int_X \beta = \int_{\tilde{U}} \tilde{\beta} = \int_{\tilde{U}} \sum_i^{n} \tilde{\varphi}_i \tilde{\beta} = \sum_i^{n} \int_{\tilde{U}} \tilde{\varphi}_i \tilde{\beta} = \int_X^{\varphi.} \beta .$$

2°) It follows from the claim:

$$\int_X^{\varphi.} \alpha = \sum_i^{n} \int_X \alpha \varphi_i = \sum_i^{n} \int_X^{\hat{\varphi}.} \varphi_i \alpha = \sum_i^{\hat{n}} \sum_j^{n} \int_X \hat{\varphi}_j \varphi_i \alpha .$$

$$= \sum_1^{\hat{n}} \int_X^{\varphi.} \hat{\varphi}_i \alpha = \int_X^{\hat{\varphi}.} \alpha .$$

Remark: Then we can define a well-def integration: $\int_X \alpha = \int_X^{\varphi.} \alpha$ on every cpt. oriented manifold X.

(3) Stokes' Thm:

① For cpt oriented manifold X:

· Note that for $w \in \Lambda^n(X)$, a volum form.

Fix orientation $[w]$. Then $\int_X w > 0$.

$\therefore \int_X : \Lambda^n(X) \longrightarrow \mathbb{R}'$ defines a surjective

linear map. (Note: $\lambda w \in \Lambda^n(X)$, $\forall \lambda \in \mathbb{R}'$).

We claim: $d\alpha \in \ker \int_X$, $\forall \alpha \in \Lambda^{n-1}(X)$.

i) For $U \underset{op}{\subseteq} \mathbb{R}^n$:

Lemma. For any $\alpha \in \Lambda^{n-1}(U)$ vanishes outside

a cpt set $W \subseteq U$. Then $\int_U d\alpha = 0$.

Pf: Suppose $\alpha = \alpha_1 dx_2 \wedge \cdots \wedge dx_n$.

$\exists [-r,r]^n \subseteq U$. s.t. $W \subseteq [-r,r]^n$.

$\therefore \int_U d\alpha = \int_{[-r,r]^n} \frac{\partial \alpha_1}{\partial x_1} dx_1 \wedge \cdots dx_n$.

$= \int_{[-r,r]^{n-1}} (\alpha_1|_{x_1=r} - \alpha_1|_{x_1=-r}) = 0$.

Since $\alpha_1 \equiv 0$ outsider $[-r,r]^n$

ii) <u>X is cpt, oriented manifold:</u>

<u>Thm.</u> $\int_X d\alpha = 0$. for $\forall \alpha \in \Lambda^{n-1}(X)$. ($\dim X = n$).

<u>Pf:</u> $\exists$ POU. $\varphi. = \{\varphi_i\}_1^n$.

$\therefore d\alpha = \sum_1^n d(\alpha\varphi_i)$. sum of bump forms.

written in charts. Reduce to i).

$\Rightarrow$ Note that $H_{dR}^n(X) = \Lambda^n(X) / \{\text{exact } (n-1)\text{-forms}\}$.

Stokes's Thm said: $\int_X : H_{dR}^n(X) \longrightarrow \mathbb{R}'$ well-def.

<u>Remark:</u> For X is connected additionally.

Then: $\int_X : H_{dR}^n(X) \cong \mathbb{R}'$.

③ <u>Manifold with boundary:</u>

i) <u>For $U \underset{open}{\subseteq} \mathbb{R}_{\leq 0} \times \mathbb{R}^{n-1}$:</u>

$\partial U = U \cap \{X_1 = 0\}$ is its boundary.

Denote: $\iota : \partial U \hookrightarrow U$. inclusion.

Note that $T_z \partial U = \{X_1 = 0\} \subset \mathbb{R}^n = T_z U$.

For $\forall \alpha = \alpha_1 dx_2 \wedge \cdots dx_n + \alpha_2 dx_1 \wedge dx_3 \cdots + \cdots \in \Lambda^{n-1}(U)$

Pull back α along ι: $\iota^*\alpha = \alpha_1|_{\{x_1=0\}} dx_2 \wedge \cdots dx_n \in \Lambda^{n-1}(\partial U)$

<u>Claim:</u> $\int_U d\alpha = \int_{\partial U} \iota^*\alpha$. $\forall$ $\operatorname{supp}\alpha$ is cpt. $\subset U$.

Pf: 1°) Consider the integration on $[-r,0]\times[-r,r]^{n-1}$

for large enough $r \in \mathbb{R}^+$, rather than U.

2°) For $\alpha = \alpha_k \, dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \cdots dx_n$. $\iota^*\alpha = 0$.

$$\int_{[-r,0]\times[-r,r]^{n-1}} d\alpha = \int_{[-r,0]\times[-r,r]^{n-2}} \int_{-r}^{r} \left(\frac{\partial \alpha}{\partial x_k} dx_k\right) dx_1 \cdots = 0$$

3°) For $\alpha = \alpha_1 \, dx_2 \wedge dx_3 \cdots dx_n$.

$$\int_{[-r,0]\times[-r,r]^{n-1}} d\alpha = \int_{[-r,r]^{n-1}} \left(\int_{-r}^{0} \frac{\partial \alpha_1}{\partial x_1} dx_1\right) dx_2 \cdots dx_n$$

$$= \int_{[-r,r]^{n-1}} \alpha_1|_{\{x_1=0\}} \, dx_2 \cdots dx_n$$

$$= \int_{\partial U} \iota^*\alpha .$$

ii) For manifold-with-boundary X:

prop. If X is oriented manifold with boundary

Then there's a canonical orientation on

boundary ∂X, which is same as X.

Pf: For $w \in \Omega^n(X)$, volum form.

Under chart (U,f), $\partial \tilde{U} = \tilde{U} \cap \{x_1 = 0\}$.

$\tilde{w} = h \, dx_1 \cdots \wedge dx_n$, $h \in C^\infty(\tilde{U})$.

Choose $s = (s_1, \cdots s_n) : \partial \tilde{U} \longrightarrow \mathbb{R}^n$, $s_1 > 0$

s is a vector field on $\partial\tilde{U}$.

Contract $\tilde{\omega}$ with ξ: $i_\xi \tilde{\omega} = h\xi_1 dx_2 \wedge \cdots \wedge dx_n - h\xi_2 dx_1 \wedge dx_3 \cdots + \cdots$

$i_\xi \tilde{\omega} \in \Lambda^{n-1}(\tilde{U})$. Pull-back to $\partial \tilde{U}$: $\iota^*(i_\xi \tilde{\omega}) = h\xi_1 dx_2 \wedge \cdots dx_n$

$\therefore$ $\iota^*(i_\xi \tilde{\omega}) \in \Lambda^{n-1}(\partial \tilde{U})$. Volum form. induced by $f|_{\partial X}$

Besides. its orientation is indept with ξ.

For the whole ∂X: Construct ξ by POU.

<u>Remark</u>: For (U, f) is oriented chart. Then

$(\partial U, f|_{\partial U})$ on ∂U is also oriented.

<u>Thm</u>. (Full version)

X is cpt. oriented manifold with boundary.

$\dim X = n$. $\iota: \partial X \hookrightarrow X$ is inclusion. $\forall \alpha \in \Lambda^{n-1}(X)$.

$\int_X d\alpha = \int_{\partial X} \iota^* \alpha$.

<u>Pf</u>: Since we can obtain canonical orientation on ∂X.

By POU. Then see in charts.

Reduce to i).