Represent Solutions

(1) Separation of Variables:

· For some equations, the partial derivates are separated. e.g. $u_t - \Delta u = 0$. $u_t + H(Du) = 0$.

We can consider assume $u = w(t)v(x)$. or $u = w(t) + v(x)$. to simplify the equation.

It will split into 2 group equations:

e.g. $w'(t) = \Delta v(x)$. has form: $F(t) = G(x)$.

$$\therefore \begin{cases} F(t) = w'(t) = \text{const.} \\ G(x) = \Delta v(x) = \text{const.} \end{cases}$$

(2) Transform Methods:

① Fourier Transform:

· **Def:** i) For $u \in L^1(\mathbb{R}^n)$. $F(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(y)\,dy$

ii) For $u \in L^1(\mathbb{R}^n)$. $F^{-1}(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ix\cdot y} u(y)\,dy$

Remark: Denote $F(u) = \hat{u}$. $F^{-1}(u) = \check{u}$

Thm. (Plancherel's Thm)

For $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$

and $\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$

Pf. Only check $\|\hat{u}\|_{L^2}^2 = \|u\|_{L^2}^2$.

1') $\int \hat{v} w \, dx = \int v \hat{w} \, dx$. for $v, w \in L^1(\mathbb{R}^n)$.

and easy to check: $\check{u}, \hat{u} \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

2') $\widehat{e^{-\varepsilon|x|^2}} = e^{-|\eta|^2/4\varepsilon} / (2\varepsilon)^{\frac{n}{2}}$.

3') Denote $v(x) = \overline{u(-x)}$. then $\widehat{w}(x) \triangleq \widehat{u * v} = (2\pi)^{\frac{n}{2}} |\hat{u}|^2$.

since $\hat{w} = (2\pi)^{\frac{n}{2}} \hat{u} \cdot \hat{v}$, $\hat{v} = \overline{\hat{u}}$. $(w = u * v)$.

4') $\int \hat{w} \, dx = \int (2\pi)^{\frac{n}{2}} |\hat{u}|^2 dx = \int \hat{w} e^{-ix\cdot\eta}\big|_{\eta=0} dx$

$= (2\pi)^{\frac{n}{2}} w(0)$. $\therefore w(0) = \|\hat{u}\|^2$.

5') $w(0) = \int u(x) v(-x) \, dx = \|u\|^2$. $\therefore \|u\|^2 = \|\hat{u}\|^2$.

Remark: We can define Fourier Transform on $L^2(\mathbb{R}^n)$.

For $u \in L^2(\mathbb{R}^n)$. $\exists (u_n) \subseteq C_c^\infty(\mathbb{R}^n)$. $u_n \to u$ in L^2.

$\therefore \|u - u_n\|_{L^2} = \|\hat{u} - \hat{u}_n\|_{L^2} \to 0$. i.e. $\hat{u}_n \to \hat{u}$ in L^2.

$\therefore \exists (\hat{u}_{n_k}) \longrightarrow \hat{u}$ a.e. Define $F(u) = \hat{u}$.

properties:

i) $\langle u, v \rangle_{L^2} = \langle \hat{u}, \hat{v} \rangle_{L^2}$

ii) $(D^\alpha u)^\wedge = (i\eta)^\alpha \hat{u}$. for multindex. $D^\alpha u \in L^2(\mathbb{R}^n)$.

iii) If $u, v \in L^1 \cap L^2(\mathbb{R}^n)$. Then $\widehat{u * v} = \hat{u} \hat{v} (2\pi)^{\frac{n}{2}}$.

iv) $(\hat{u})^\vee = (\check{u})^\wedge = u$.

Cor. From iii). we have: $\widehat{uv} = \hat{u} * \hat{v} (2\pi)^{-\frac{n}{2}}$.

since $\widehat{uv} = \widehat{\hat{u}^\vee \hat{v}^\vee} = (2\pi)^{\frac{n}{2}} (\widehat{\hat{u} * \hat{v}})^\vee = (2\pi)^{-\frac{n}{2}} \hat{u} * \hat{v}$.

Pf: i) $\|u+\alpha v\|^2 = \|\hat{u}+\alpha\hat{v}\|^2$. Let $\alpha = 1, i$

ii) Approxi by $(u_\varepsilon)_{\varepsilon>0}$. Dominated Convergence Thm.

By Integration by part.

iv) Note that $(\bar{v})^\wedge = \check{v}$. $v \in L^2(\mathbb{R}^n)$.

$$\int (\hat{u})^\vee v = \int \hat{u}\, v^\vee = \int \hat{u}\, \overline{(\bar{v})^\wedge} = (L^2\text{-product})$$

$$\int u \bar{\bar{v}} = \int uv. \text{ for } u \in L^2(\mathbb{R}^n),\ \forall v \in L^2(\mathbb{R}^n).$$

$$\therefore \int [(\hat{u})^\vee - u]\, v\, dx = 0,\ \forall v \in C_c^\infty(\mathbb{R}^n).$$

Remark: Fourier Transform is powerful in solving linear, constant-coefficient PDE's.

e.g. $(-\Delta + 1)u = f$, $f \in L^2(\mathbb{R}^n)$ in $\mathbb{R}^n$.

Transform on $\vec{x}$: $(1+|y|^2)\hat{u} = \hat{f}$ (Cancel Δ)

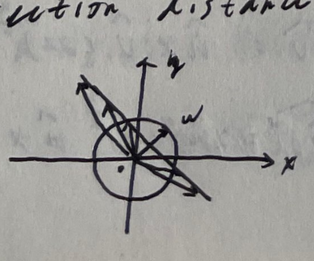
$\therefore \hat{u} = \hat{f}/(1+|y|^2) \Rightarrow u = (\hat{f}\cdot \frac{1}{1+|y|^2})^\vee$

i.e. $u(x) = \hat{f}^\vee * (\frac{1}{1+|y|^2})^\vee = f * (\frac{1}{1+|y|^2})^\vee$

② Radon Transformation:

Denote: $S^{n-1} = \partial B(0,1)$ in $\mathbb{R}^n$. For $w \in S^{n-1}$, $s \in \mathbb{R}^1$.

$\Pi(s,w) = \{y \in \mathbb{R}^n \mid y\cdot w = s\}$. It means the projection distance on w is s.



Def: For $u \in C_c^\infty(\mathbb{R}^n)$. $R(u) = \tilde{u}$. Radon Transform.

$$\tilde{u}(s,w) = \int_{\Pi(s,w)} u(y)\, dS(y).$$

Remark: If we choose orthonormal basis of $\Pi(0,w)$ is $(b_k)_1^{n-1}$. Then $(b_k)_1^{n-1} \cup (w)$ is orthonormal basis of $\mathbb{R}^n$. We can obtain:

$$\tilde{u}(s,w) = \int_{\mathbb{R}^{n-1}} u\left(\sum_1^{n-1} \eta_k b_k + sw\right) d\vec{\eta}.$$

Thm. (properties of Radon Transform)

For $u \in C_c^\infty(\mathbb{R}^n)$. Then:

i) $\tilde{u}(-s,-w) = \tilde{u}(s,w)$.

ii) $(D^\alpha u)^\sim = w^\alpha \dfrac{\partial^{|\alpha|}}{\partial s^{|\alpha|}} \tilde{u}$. for multiindex α.

iii) $(\Delta u)^\sim = \dfrac{\partial^2}{\partial s^2} \tilde{u}$.

iv) $u \equiv 0$. for some R. $|x| > R$.

Pf: ii) Consider $(b_k)_1^{n-1} \cup (w)$. Orthonormal basis.

By induction. Consider u_{x_i} firstly.

$$u_{x_i} = Du \cdot e_i = \left(\sum_1^{n-1} (Du \cdot b_k) b_k + (Du \cdot w) w\right) e_i$$

$$= \sum (b_k \cdot e_i)(Du \cdot b_k) + w_i\, Du \cdot w.$$

$\therefore \widetilde{u_{x_i}} = w_i \int_{\Pi(s,w)} Du \cdot w \, dS(y)$. Since $\int_{\Pi(s,w)} Du \cdot d(s, b_k) = 0$.

Note $\tilde{u}_s = \int_{\mathbb{R}^{n-1}} \frac{\partial u}{\partial s}\left(\sum \eta_k b_k + sw\right) = \int_{\Pi(s,w)} Du \cdot w \, dS$.

$\therefore \widetilde{u_{x_i}} = w_i \tilde{u}_s$. $n=1$ holds!

Thm. (Radon and Fourier Transform)

If $u \in C_c^\infty(\mathbb{R}^n)$. Then $\hat{\tilde{u}}(r,w) \triangleq F_s(\tilde{u})(r,w)\cdot(2\pi)^{\frac{1}{2}}$

$= F_x(u)(rw)(2\pi)^{n/2}$. F_s, F_x is FT on s, $\vec{x}$.

Pf: $F_s(\tilde{u})(r,w) = \int_{\mathbb{R}^1}\int_{\mathbb{R}^m} u(\sum_1^m \eta_k b_k + sw)\, e^{-irs}/(2\pi)^{\frac{1}{2}}\, d\eta\, ds$.

$\overset{x=\Sigma\square+\square}{=} \int_{\mathbb{R}^n} u(x)\, e^{-ir(x\cdot w)}/(2\pi)^{\frac{1}{2}}\, dx = F_s(u)(rw)\cdot(2\pi)^{\frac{m}{2}}$

Thm. (Inverting the Radon Transform)

i) $u(x) = \frac{1}{2(2\pi)^n}\int_{\mathbb{R}^1}\int_{S^{n-1}} \hat{\tilde{u}}(r,w)\, r^{n-1} e^{irw\cdot x}\, dS\, dr$.

ii) If $n=2k+1$. Then $u(x) = \int_{S^{n-1}} r(x\cdot w, w)\, dS(w)$

where $r(s,w) = \frac{(-1)^k}{2(2\pi)^{2k}}\frac{\partial^{2k}}{\partial s^{2k}}\tilde{u}(s,w)$.

Pf: i) is direct. by connection of Radon and Fourier.

ii) $(\frac{\partial^{2k}}{\partial s^{2k}}\tilde{u}(s,w))^\wedge = (ir)^{2k}(\tilde{u}(s,w))^\wedge$

$= \frac{(-1)^k r^{2k}}{(2\pi)^{\frac{1}{2}}}\hat{\tilde{u}}(r,w)$.

$\therefore \int_{S^{n-1}} r(x\cdot w, w) = \int_{S^{n-1}}\int_{\mathbb{R}^1}\frac{1}{2(2\pi)^n}\hat{\tilde{u}}\, r^{n-1} e^{ir(w\cdot x)}\, dr\, dS(w)$

$= u(x)$. by i)

Cor. If n is odd. $\tilde{u} = 0$ in $|s| \le R$.

Then $u \equiv 0$ in $B(0,R)$

③ Laplace Transform:

· Denote: $R_+ = \mathbb{R}^1_+ = (0, \infty)$

Def: For $u \in L^1(R_+)$, define: $\mathcal{L}u = u^\# = \int_0^\infty e^{-st} u(t) dt$.

Remark: It only defines on one-dimension $\mathbb{R}^1$.

properties:

i) $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$. $f, g \in L^1(R_+)$.

ii) $\mathcal{L}(\underbrace{\int_0^t dt \int_0^t dt \cdots \int_0^t}_{(n)} f(t) dt) = \frac{1}{s^n} \mathcal{L}(f(t))$

iii) $\mathcal{L}(f^{(n)}(t)) = s^n \mathcal{L}(f(t)) - \sum_0^{n-1} s^k f^{(n-1-k)}(0)$

iv) $f(t) = \mathcal{L}^{-1}(f^\#) = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{m-iR}^{m+iR} f^\#(s) e^{st} ds$.

$m \in \mathbb{R}^1$. large enough. s.t. poles of $f^\# \subseteq B(0, m)$

Pf: ii) Set $g(t) = \int_0^t \int_0^t \cdots \int_0^t f(t) d\vec{t}$ $\quad g(0) = 0$.

$\therefore \int_0^\infty e^{-st} g(t) dt = \frac{1}{-s} \int_0^\infty g(t) d e^{-st}$

$= \frac{1}{s} \int_0^\infty g'(t) e^{-st} = \cdots = \frac{1}{s^n} \int_0^\infty g^{(n)}(t) e^{-st} dt$.

iii) Integration by part.

Cor. $\mathcal{L}(\frac{f(t)}{t}) = \int_s^\infty \mathcal{L}(f(t)) ds$

Pf: Exchange the integration.