Vector Fields

(1) Definitions:

- A vector field on X is a Func: assigns $\forall x \in X$ to its corresponding tangent space $T_x X$.

① Vector fields:

i) For $X = U \underset{open}{\subseteq} \mathbb{R}^n$:

$\forall x \in X$. $T_x U \xrightarrow{\simeq} \mathbb{R}^n$. Def: $TU = \bigcup_{x \in U} T_x U \xrightarrow{\simeq} U \times \mathbb{R}^n$.

i.e. $TU = \{(x, v) \mid x \in U,\ v \in T_x U\}$.

Canonical Proj: $\pi : TU \longrightarrow U$. $\pi^{-1}(x) = T_x U$.

<u>Def:</u> vector field ξ on U is:

$$\xi : U \longrightarrow TU = U \times \mathbb{R}^n. \quad \pi \circ \xi = 1_U.$$
$$x \longmapsto (x, \xi|_x)$$

for some $\xi|_x \in T_x U$.

ii) For arbitrary n-dim manifold X:

$TX = \bigcup_{x \in X} T_x X = \{(x, v) \mid v \in T_x X\}$. tangent bundle of X.

$$\pi : TX \longrightarrow X \quad . \quad \pi^{-1}(x) = T_x X. \ \forall x \in X.$$
$$(x, v) \longmapsto x$$

<u>Remark:</u> TX may not be cross-product of X with some other set.

It naturally has structure of manifold.

So. $\dim(TX) = 2\dim(X)$.

<u>Def:</u> Vector field on X is Func:

$$\xi : X \to TX. \quad \pi \circ \xi = 1_X$$

② <u>Smooth Structure:</u>

i) For $\tilde{U} \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $\tilde{\xi} : \tilde{U} \to T\tilde{U}$.

It's surely that $\tilde{\xi}$ is smooth $\Leftrightarrow$ the second component of $\tilde{\xi}$ is smooth.

ii) For n-dim manifold X. $(U, f) \in \mathcal{A}_X$. ξ is vector field

$TU = \pi^{-1}(U) \subseteq TX$. See it in chart by:

$$F : TU \xrightarrow{\sim} T\tilde{U} = \tilde{U} \times \mathbb{R}^n$$
$$(x, v) \longmapsto (f(x), Af(v))$$

$$\Rightarrow F \circ \xi|_U \circ f^{-1} : \tilde{U} \longrightarrow T\tilde{U} \simeq \tilde{U} \times \mathbb{R}^n$$
$$f(x) \longmapsto (f(x), Af \circ \xi|_x \circ f^{-1}(f(x)))$$

i.e. $F \circ \xi|_U \circ f^{-1} = (1_{\tilde{U}}, Af \circ \xi|_x \circ f^{-1}) \triangleq (1_{\tilde{U}}, \tilde{\xi})$

$\tilde{\xi} : \tilde{U} \longrightarrow \mathbb{R}^n$.

<u>Remark:</u> So TX locally looks like an open subset of $\mathbb{R}^{2n}$.

<u>Def:</u> A vector field $\xi : X \to TX$ is smooth if $\forall (U, f) \in \mathcal{A}_X$. $F \circ \xi|_U \circ f^{-1} = (1_{\tilde{U}}, \tilde{\xi})$:

$\tilde{U} \longrightarrow \tilde{U} \times \mathbb{R}^n$. its 2^{nd}-component is smooth.

Remark: It's indept with the choice of charts:

For: $\tilde{F}_1 : F_1 \circ \xi|_U \circ f_1^{-1} = (I_{\tilde{U}}, \tilde{\xi}_1) : \tilde{U} \longrightarrow \tilde{U} \times \mathbb{R}^n$

$\tilde{F}_2 : F_2 \circ \xi|_U \circ f_2^{-1} = (I_{\tilde{U}}, \tilde{\xi}_2) : \tilde{U} \longrightarrow \tilde{U} \times \mathbb{R}^n$.

$U = U_1 \cap U_2$, Next, we define:

$$\varphi_{21} : \tilde{U} \times \mathbb{R}^n \rightleftharpoons \tilde{U} \times \mathbb{R}^n$$
$$(\vec{x}, \vec{v}) \longmapsto (\phi_{21}(\vec{x}), D\phi_{21}|_{\vec{x}} \cdot \vec{v})$$

$\therefore \tilde{F}_2 = \varphi_{21} \circ \tilde{F}_1$. Besides: $\tilde{\xi}_2|_{\phi_{21}(\vec{x})} = D\phi_{21}|_{\vec{x}} \cdot \tilde{\xi}_1|_{\vec{x}}$

(2) Vector Fields from Translation Law:

Def: ξ is a rule, s.t. $\xi : (U, f) \longmapsto \tilde{\xi}_f$, $\forall (U, f) \in A_X$.
$\tilde{\xi}_f : \tilde{U} \longrightarrow \mathbb{R}^n$. Smooth. Besides, $\forall (U_1, f_1), (U_2, f_2) \in A_X$, $\forall \vec{x} \in f_1(U_1 \cap U_2)$, $\tilde{\xi}_2|_{\phi_{21}(\vec{x})} = D\phi_{21}|_{\vec{x}} \cdot \tilde{\xi}_1|_{x_1}$

prop. ξ defines a smooth vector field on X.

Pf: Restrict ξ on A_x^X:

$$\xi_x = \xi|_{A_x^X} : A_x^X \longrightarrow \mathbb{R}^n \quad \text{this is a tangent vector}$$
$$(U_x, f) \longmapsto \tilde{\xi}|_{f(x)}$$

$\therefore \hat{\xi} : X \longrightarrow TX$ is a smooth vector field.
$x \longmapsto (x, \xi_x)$

since $\tilde{\xi}|_{f(x)}$ is smooth. (correspond $\xi|_X$, (U_x, f))

(3) Flow:

① Def. A Flow on X is a smooth map:

$F: (-\varepsilon, \varepsilon) \times X \longrightarrow X$. Besides. $\forall s \in (-\varepsilon, \varepsilon)$. fixed.

$F_s(x): X \longrightarrow X$ is C^∞ diffeomorphism. $F_0 = 1_X$.

Remark: $(-\varepsilon, \varepsilon) \times X$ is naturally a smooth manifold

with dimension: $\dim X + 1$. (Chart:

$(1 \times F, (-\varepsilon, \varepsilon) \times U)$).

e.g. $F_s(x): T^1 \longrightarrow T^1$. is flow of T^1.

$[x] \longmapsto [x+s]$

$\Rightarrow$ If we fix $x \in X$. $F_x(s): (-\varepsilon, \varepsilon) \longrightarrow X$.

$F_x(0) = F(0, x) = x$. it's a curve through x.

Repeat on all $x \in X$. We obtain a vector field

on X: $\xi^F: X \longrightarrow (\{x\} \times [F_x(s)])_{x \in X}$.

Remark: We can view vector field as the

infinitesimal version of flow.

② Procedure in charts:

$\cdot \tilde{F} = f \circ F \circ (1 \times f)^{-1} = (\tilde{F}_1, \tilde{F}_2, \cdots \tilde{F}_n): (-\varepsilon, \varepsilon) \times \tilde{U} \longrightarrow \tilde{U}$

and $\tilde{F}(0, x) = 1_{\tilde{U}}$. The associated vector field is:

$$\tilde{\xi}^F = \frac{\partial \tilde{F}}{\partial s}\Big|_{s=0} = \left(\frac{\partial \tilde{F}_1}{\partial s}\Big|_{s=0}, \cdots \frac{\partial \tilde{F}_n}{\partial s}\Big|_{s=0} \right): \tilde{U} \longrightarrow \mathbb{R}^n.$$

③ Existence:

If we have vector field ξ on X. cpt manifold. Then we can find flow F. s.t. $\xi^F = \xi$. If X is noncpt. ε may be $\rightarrow \infty$.