Cotangent Space.

(1) **Covectors:**

Denote: $R_x(X) \subseteq C^\infty(X)$. $R_x(X) = \{ f \in C^\infty(X) \mid$ rank of f is zero at $x \}$. Where $C^\infty(X) = C^\infty(X, \mathbb{R}^1)$.

Def: The cotangent space to X at x is:

$T_x^* X = C^\infty(X) / R_x(X)$. element in $T_x^* X$ is called covector.

① $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$ case:

prop. $\dim(T_x^* X) = n$

Pf:
$$C^\infty(X) \xrightarrow{F} \mathbb{R}^n$$
$$h \longmapsto Dh|_x = \left(\frac{\partial h}{\partial x_1}, \cdots \frac{\partial h}{\partial x_n}\right)\Big|_x$$

$\therefore \ker F = R_x X$. Besides. $\forall (a_1 \cdots a_n) \in \mathbb{R}^n$.

$\exists\, h = \sum_1^n a_k x_k$. s.t. $F(h) = \vec{a}$. F is surjection.

$\therefore C^\infty(X)/R_x X \xrightarrow{\sim} \mathbb{R}^n$.

Remark. Note: $C^\infty(X)$ is a infinite-dimension vector space.

② For general n-dim manifold X:

· For $h \in C^\infty(X)$. and $x \in X$. Find $(U_x, f) \in A_x$.

Then $\tilde{h} = h \circ f^{-1} : \tilde{U} \longrightarrow \mathbb{R}^1$. We can compute the rank of h at x (i.e. $D\tilde{h}|_{f(x)}$).

Fix (U, f) :

$\nabla_f : C^\infty(X) \longrightarrow \mathbb{R}^n$. $\nabla f(h) = D(h \circ f^{-1})|_{f(x)}$

$\ker(\nabla f) = R_x X$. ($\because R_x X$ is subspace of $C^\infty(X)$).

<u>Prop.</u> $T_x^* X \xrightarrow{\sim} \mathbb{R}^n$.

<u>Pf:</u> We only need to prove: ∇f is surjection.

$\tilde{h} \in C^\infty(\tilde{U})$. $\tilde{h} = \sum_1^n a_k x_k$. for $\vec{a} = (a_1, \cdots a_n) \in \mathbb{R}^n$.

Choose ψ is Bump $\begin{cases} \psi \equiv 1 & \text{in nbd of } x. \\ \psi \equiv 0 & \text{outside } U \end{cases}$

Let $h = \begin{cases} \psi \cdot \tilde{h}(f(x)). & x \in U \\ 0 & . \ x \in U^c \end{cases}$ $\therefore \nabla f(h) = a$. $h \in C^\infty(X)$.

<u>Remark:</u> By extension (using bump Func's).

We can prove there're lots of smooth vector field on X.

i.e. for $(U, f) \in \mathcal{A}_X$. Choose $\tilde{h} : \tilde{U} \to TX$.

s.t. $f^{-1}(\tilde{h}) = q$. (determines vector field)

Let $h = \begin{cases} \tilde{h}(f)\, \psi . & x \in U \\ 0 & . \ x \notin U \end{cases}$

$\therefore \Delta f \circ h \circ f^{-1} = Dq|_x$. locally in $x \in V \subseteq U$.

where $V = f^{-1}(B(x_0, r))$. $\psi(V) = \{1\}$.

③ Physicist's Definition:

· For $h \in C^\infty(X)$. Denote an element in T_x^*X by $dh|x$. i.e the equivalent class of h.

Since for $(U_1, f_1), (U_2, f_2) \in A_x$. $\begin{cases} \tilde{h}_1 = h \circ f_1^{-1} \\ \tilde{h}_2 = h \circ f_2^{-1} \end{cases}$

$\therefore \tilde{h}_1 = \tilde{h}_2 \circ \phi_{21}$. We obtain:

$\nabla f_1 (dh|x) = \nabla f_2 (dh|x) \cdot D\phi_{21}|_{f_1(x)}$.

Written in row vector: $\nabla f_1 = (D\phi_{21}|_{f_1(x)})^T \cdot \nabla f_2$.

<u>prop</u>. T_x^*X is the set collecting Func such Σ.

Then. $T_x^*X \xrightarrow{\sim} T_x^*X$.

<u>Pf</u>. Define: $ev_f : T_x^*X \longrightarrow \mathbb{R}^n$

$\Sigma \longmapsto \Sigma f$

There exists canonical linear isomorphism.

(2) <u>Third Definition of tangent vectors</u>:

<u>Claim</u>: $T_x^*X = (T_xX)^*$

① <u>$X \underset{open}{\subseteq} \mathbb{R}^n$</u>:

Since $T_xX \cong \mathbb{R}^n$. We can identify $\vec{v} \in T_xX$ with vector in $\mathbb{R}^n$.

consider the operation: take partial derivative at x in the direction $\vec{v}$:

$\partial_{x,\vec{v}}: C^{\infty}(X) \longrightarrow \mathbb{R}^1$. $\partial_{x,\vec{v}}(h) = Dh|_x \cdot \vec{v}$

It's easy to see $\partial_{x,\vec{v}}$ is linear.

Actually, replace $\vec{v}$ by $\sigma \in T_x X$: $D\sigma|_0 = \vec{v}$.

$\therefore \partial_{x,\sigma}(h) = D(h\circ\sigma)|_0 = Dh|_x \cdot D\sigma|_0$, vanish on $R_x X$.

$\Rightarrow \partial_{x,\sigma}: T_x^* X \longrightarrow \mathbb{R}^1$ is well-def. $\partial_{x,\sigma} \in (T_x^* X)^*$.

$$dh|_x \longmapsto \partial_{x,v}(h)$$

<u>Remark</u>: $\partial_{x,v}$ is simply: $\begin{array}{l} \mathbb{R}^n \longrightarrow \mathbb{R}^1 \\ u \longmapsto u\cdot v \end{array}$

Conversely, $\forall$ BLO $\delta: T_x^* X \longrightarrow \mathbb{R}^1$. since $T_x^* X \cong \mathbb{R}^n$, and $(\mathbb{R}^n)^* = \mathbb{R}^n$, by Riesz Thm: $\delta(dh|_x) = (Dh|_x, \vec{v})_2 = Dh|_x \cdot \vec{v}$ for some $\vec{v}$. $\therefore \delta = \partial_{x,\vec{v}}$, actually.

$\therefore (T_x^* X)^* = T_x X$.

② <u>For X is manifold:</u>

Define $\partial_{\sigma,x}: C^{\infty}(X) \longrightarrow \mathbb{R}^1$:

Fix $(U,f) \in A_x$. $\partial_{\sigma,x}(h) = D(h\circ\sigma)|_0 = \nabla_f(h) \cdot \Delta_f(\sigma)$

since $h\circ\sigma = (h\circ f^{-1})\circ(f\circ\sigma)$. And $\partial_{\sigma,x}$ is well-def. and chart-indept.

Prop. $\exists$ linear isomorphism: $T_xX \cong (T_x^*X)^*$.

Pf:

$$\begin{array}{ccc} T_xX & \xrightarrow{F} & (T_x^*X)^* \\ [\sigma] & & \partial_{\sigma,x} \\ \Delta f \downarrow \wr & & \downarrow \wr \\ \mathbb{R}^n & \xrightarrow{\cong} & (\mathbb{R}^n)^* \\ \Delta f(\sigma) & & \partial_{v,x} \end{array}$$

$v = \Delta f(\sigma)$.

Besides, $\partial_{v,x}(u) = u \cdot \Delta f(\sigma)$ for $u \in \mathbb{R}^n$.

Remark: i) Since $\dim T_xX = n < \infty$. $\therefore (T_xX)^* \cong T_x^*X$.

Explicitly: For $dh|_x \in T_x^*X$. define:

$\widehat{dh|_x} : T_xX \to \mathbb{R}^1$. $\widehat{dh|_x}([\sigma]) = \partial_{\sigma,x}(h)$.

$\therefore \widehat{dh|_x} \in (T_xX)^*$

ii) Note that: $\begin{cases} \Delta f : T_xX \xrightarrow{\sim} \mathbb{R}^n \\ \nabla f : T_x^*X \xrightarrow{\sim} \mathbb{R}^n \end{cases}$

$\therefore \Delta f$ is the dual BLO of ∇f^{-1}.

③ Derivation at x:

Def: For X is manifold. A derivation at x is BLO:

$\partial : C^\infty(X) \to \mathbb{R}^1$. s.t. $\partial(h_1 h_2) = h_1 \partial(h_2) + h_2 \partial(h_1)$.

for $\forall h_1, h_2 \in C^\infty(X)$. Denote the set by $\mathrm{Der}_x(X)$.

Prop. BLO $\partial : C^\infty(X) \to \mathbb{R}^1$ is a derivation at x

$\iff \partial$ vanishes on R_xX.

Remark: $\mathrm{Der}_x(X) \cong T_xX$.

Def: (Algebraist's).

A tangent vector to x on X is a derivation at x.

Remark: It only uses the fact: $C^\infty(X)$ is a ring.

(3) Vector Fields as Derivations:

① $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$:

· For $\tilde{\xi} : X \to TX$. We can define:

$\hat{\xi} : C^\infty(X) \to C^\infty(X)$. $\hat{\xi}(h) : x \longmapsto \partial_{x,\tilde{\xi}|_x}(h)$.

i.e. $\hat{\xi} = \sum \tilde{\xi}_i \frac{\partial}{\partial x_i}$ ($\tilde{\xi} = (\tilde{\xi}_1 \cdots \tilde{\xi}_n)$).

Remark: i) $(\frac{\partial}{\partial x_i})_1^n$ is Standard Basis

ii) Since $\partial_{x,\tilde{\xi}|_x} \in \mathrm{Der}_x(X)$

$\therefore \hat{\xi}(h_1 h_2) = h_1 \hat{\xi}(h_2) + h_2 \hat{\xi}(h_1)$.

② For X is arbitrary manifold:

· For $\xi : X \to TX$. vector field. (smooth)

Define: $\hat{\xi} : C^\infty(X) \to C^\infty(X)$.

$\hat{\xi}(h) : x \longmapsto \partial_{x,\xi|_x}(h)$

Check: $\hat{\xi}(h)$ is smooth. (see in charts)

For $(U,f) \in A_X$. $\xi(h)\circ f^{-1} : \tilde{x} \longmapsto \partial_{f^{-1}(\tilde{x})} \cdot \xi|_{f^{-1}(\tilde{x})}(h)$.

i.e. $\xi(h)(f^{-1}(\tilde{x})) = D(h\circ\xi|_{f^{-1}(\tilde{x})})|_0$

$= D(h\circ f^{-1}\circ f\circ\xi|f^{-1}(\tilde{x}))|_0$

$= D\tilde{h}|_{\tilde{x}} \cdot D\tilde{\xi}|_0 = \nabla f(\lambda h|_{f^{-1}(\tilde{x})}) \cdot \Delta f(\xi|_{f^{-1}(\tilde{x})})$

$\therefore$ $\xi(h)\circ f^{-1}$ is smooth. since ξ is smooth.

Def: A derivation on manifold X is LF:

$\mathcal{D} : C^\infty(X) \longrightarrow C^\infty(X)$. s.t. $\mathcal{D}(h_1 h_2) = h_1\mathcal{D}(h_2) + h_2\mathcal{D}(h_1)$

for $\forall h_1, h_2 \in C^\infty(X)$. Denote the set by $Der(X)$.

prop. $\forall\ \mathcal{D} \in Der(X)$ defines a smooth vector field.

pf. 1°) $\mathcal{D}|_x \in Der_x(X) = T_x X$

$\therefore$ $\xi : X \longrightarrow TX$. $\xi(x) = \mathcal{D}|_x$ is vector field.

2°) Check $\mathcal{D}$ is smooth.

For $(U,f) \in A_X$. $\tilde{\xi} = \Delta f\circ\xi|_x\circ f^{-1} : \tilde{U} \longrightarrow \mathbb{R}^n$.

Denote $\tilde{\xi} = (\tilde{\xi}_1, \cdots \tilde{\xi}_n)$.

$\therefore$ $\tilde{\xi} : C^\infty(\tilde{U}) \longrightarrow C^\infty(\tilde{U})$. $\tilde{\xi}(h) = \sum_1^n \tilde{\xi}_i \frac{\partial h}{\partial x_i}$. $\forall h \in C^\infty(\tilde{U})$.

3°) Check $\tilde{\xi}_i$ is smooth. $\forall 1\le i\le n$. (at $\forall \tilde{\eta} = f(\eta) \in \tilde{U}$).

Choose $\phi \in C^\infty(\tilde{U})$. $\phi \equiv 1$ in $\tilde{V}_{\tilde{\eta}}$.

Define: $\varphi_k = \begin{cases} (x_k\phi)\circ f & , x \in U \\ 0 & , x \notin U. \end{cases}$ $\therefore$ $\begin{matrix} \varphi_k \in C^\infty(X) \\ \varphi_k\circ f^{-1} \in C^\infty(\tilde{U}) \end{matrix}$

By def: $\mathcal{D}(\varphi_k)|_x = \sum_j \tilde{\xi}_j|_{f(x)} \frac{\partial\tilde{\varphi}_k}{\partial x_j}|_{f(x)} = \tilde{\xi}_k|_{f(x)}$

is smooth. $\forall f(x) \in \tilde{V}_{\tilde{\eta}}$.