

Smooth Functions

(1) Definition:

① For $h: \mathbb{R}^n \rightarrow \mathbb{R}^k$, it's clear what it means to say h is smooth.

② For $h: X \rightarrow \mathbb{R}^k$, we should see it in chart:

Def: h is smooth at $x \in X$. if $\exists (U_x, f) \in A_X$

st. $h \circ f^{-1}: \tilde{U}_x \rightarrow \mathbb{R}^k$ is smooth at $f(x)$.

Remark: It's independent with the choice of

charts: Since for $(U_1, f_1), (U_2, f_2)$,

$$\tilde{h}_1 = h_1 \circ f_1^{-1}, \quad \tilde{h}_2 = h_2 \circ f_2^{-1}$$

$$\Rightarrow \tilde{h}_1 = \tilde{h}_2 \circ \phi_{21}. \text{ Note: } \phi_{21} \text{ is smooth.}$$

③ Generalization: $H: X \xrightarrow{\text{anti}} Y$. X, Y are smooth manifolds. $\dim X = n$. $\dim Y = k$.

Def: H is smooth at $x \in X$. if $\exists (U_x, f)$

$\in A_X$. $(V, g) \in A_Y$. st. $H(U_x) \subseteq V$.

$g \circ H \circ f^{-1}: \tilde{U}_x \rightarrow \tilde{V}$ is smooth at $f(x)$.

Remark: i) Smooth Func is automatically conti:

Since $h = g \circ H \circ f^{-1}$ smooth. so anti

$\Rightarrow H = g^{-1} \circ h \circ f$ is anti.

ii) If H is conti. It's easy to find (U, f) .

(V, g) . St. $H(U) \subseteq V$:

Firstly find (U_x, f_x) of $x \in U_x$.

and $(V_{H(x)}, g)$ of $x \in U_x$. St. $H(x) \in V_{H(x)}$.

Restrict f on: $H^{-1}(V_{H(x)}) \cap U_x$, open sub.

Then: $(H^{-1}(V_{H(x)}) \cap U_x, f|_x)$. $(V_{H(x)}, g)$.

iii) It's indept with choice of charts:

For (U_1, f_1) , (V_1, g_1) and (U_2, f_2) , (V_2, g_2) .

$$g_1 \circ H \circ f_1 = g_{12} \circ (g_2 \circ H \circ f_2) \circ \phi_{21}$$

Since ψ_{12}, ϕ_{21} are smooth.

Def. For $H: X \rightarrow Y$. X, Y are smooth manifolds

with boundary. H is smooth at $x \in X$, if

$\exists (U, f)$, (V, g) of A_x, A_y . $x \in U$. $H(U) \subseteq V$.

And $\exists \tilde{U}, \tilde{V} \subseteq \mathbb{R}^n, \mathbb{R}^k$, resp. $\tilde{F}: \tilde{U} \rightarrow \tilde{V}$

smooth. St. $\tilde{U} \subseteq U$, $\tilde{V} \subseteq V$. $\tilde{F}|_{\tilde{U}} = g \circ H \circ f^{-1}$.

Remark: It's indept with charts and extension \tilde{F} .

since we can calculate derivatives at $\{x_i=0\}$. from $x_i \rightarrow 0^+$ or 0^- .

Lemma.

X, Y, Z are smooth manifolds. $H: X \rightarrow Y$.

$G: Y \rightarrow Z$. smooth. Func's. Then $G \circ H$ is

smooth as well.

Pf. Find $(U, f) \in A_x$, s.t. $x \in U$.

$(V, g) \in A_y$, s.t. $U \cap V \neq \emptyset$. \Rightarrow restrict on open nbd.

$(W, h) \in A_z$ s.t. $g \circ h|_{U \cap V} \in W$.

For $h \circ g \circ h^{-1} = h \circ g^{-1} \circ g \circ h^{-1}$ smooth.

Remark: There's a category

morphism: smooth Funcs
objects: smooth manifolds

Lemma: i) Z is submanifold of X . $i: Z \rightarrow X$ is inclusion.

Then i is smooth.

ii) $H: X \rightarrow Y$ smooth, between two manifolds.

$Z \subseteq Y$, submanifold. If $H(x) \subseteq Z$. Then

$H: X \rightarrow Z$ is smooth.

Pf. i) Find $(U \cap Z, \tilde{g})$, (U, f) .

$$f \circ i \circ \tilde{g}^{-1}: U \cap Z \xrightarrow{\text{id}} U \cap Z.$$

ii) $\exists (U, f), (V, g) \in A_x, A_y$.

By $g \circ H \circ f^{-1}$ is smooth.

Since $H(U) \subseteq Z$. It equals with:

$(V \cap Z, \tilde{g})$ and (U, f) . \tilde{g} is induced by $g|_{V \cap Z}$.

i.e. $g \circ H \circ f^{-1} = \tilde{g} \circ H \circ f^{-1}$ smooth.

Remark: From i) conclude:

$H: X \rightarrow Y$ is smooth. Z is submanifold of X . Then $H|_Z$ is smooth.

(2) Rank:

Def. $F: X \rightarrow Y$ smooth Func between X, Y

two smooth manifolds. The rank of F

at x is : $D\tilde{F}|_{f(x)}$, where $\tilde{F} = g \circ F \circ f^{-1}$.

$(U, f) \in A_x, (V, g) \in A_y, F(U) \subseteq V$.

Remark: It's indept with choices of charts.

$$\text{For } \begin{cases} \tilde{F}_1 = g_1 \circ F \circ f_1^{-1}, \text{ from } (U_1, f_1), (V_1, g_1) \\ \tilde{F}_2 = g_2 \circ F \circ f_2^{-1}, \text{ from } (U_2, f_2), (V_2, g_2) \end{cases}$$

$$\Rightarrow D\tilde{F}_2|_{f_2(x)} = Dg_2|_{g_1(f(x))} \circ D\tilde{F}_1|_{f_1(x)} \circ Df_2|_{f_2(x)}$$

\Rightarrow We can define regular points or critical points.

for $F: X \rightarrow Y$. (Note that x is regular point of $F \Leftrightarrow f(x)$ is regular point of \tilde{F} , for some chart)

Prop. $k \leq n$. $F: X \rightarrow Y$.

If y is regular value of F . Then the level

set $Z_y = F^{-1}(y)$ is k -codim submanifold of X .

Pf. For $(U, f) \in A_x, (V, g) \in A_y$. $\tilde{F} = g \circ F \circ f^{-1}$.

$x \in Z_y$. $f(x), g(y)$ is regular point/value of \tilde{F} .

$\therefore \tilde{F}^{-1}(g(y))$ is k -codim submanifold of \mathbb{R}^n .

$\exists (w, h) : f(x) \in w$. coordinate chart.

$$\text{i.e. } h \circ \tilde{F}^{-1}(g(y)) = \mathbb{R}^{n-k} \cap \tilde{w}.$$

$\therefore (f(x), h \circ f)$ is chart of Z_y at x .

(3) Special kinds of Smooth Func's:

① Def: For: $F: X \rightarrow Y$. Smooth

i) It's submersion if $\forall x \in X. \text{rank } D\bar{F}|_{f(x)} =$

$$= \dim Y.$$

ii) It's immersion if $\forall x \in X. \text{rank } D\bar{F}|_{f(x)} =$
 $\dim X.$

Where \bar{F} is F in chart.

Remark: Immersion or submersion do nothing
 with surjection or injection of F .

iii) It's diffeomorphism if \bar{F} is bijection and
 \bar{F}^{-1} is smooth as well

Remark: F is both immersion and submersion
 since $D\bar{F}^{-1} = (D\bar{F})^{-1}$ exists.

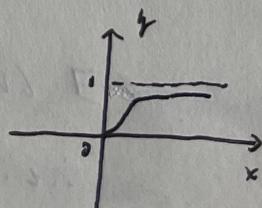
Lemma: For $\dim X = \dim Y$ equals rank of F .

If F is smooth bijection. Then F is diffeomorphism.

Pf: Apply IFT.

② Bump Function:

$$\text{Consider } \phi(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$$

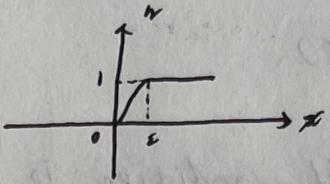


$\phi \in C^\infty(\mathbb{R}, \mathbb{R})$. (smooth). But ϕ isn't analytic

at $x=0$ (It only has Laurent Expansion)

$$\text{For } \psi^{\varepsilon}(x) = \frac{\phi(x)}{\phi(x) + \phi(-x)} \in C^\infty. \quad \begin{cases} \psi^\varepsilon = 0 \Leftrightarrow x \leq 0 \\ \psi^\varepsilon = 1 \Leftrightarrow x \geq 0. \end{cases}$$

Besides, $0 \leq \psi^\varepsilon \leq 1$.



For $\mathbb{R} \rightarrow \mathbb{R}$ case:

$$\text{Consider } \psi^{k,r}(x) = \frac{\phi(R - |x - \eta|)}{\phi(R - |x - \eta|) + \phi(|x - \eta| - r)} \in C^\infty(\mathbb{R}, \mathbb{R}), 0 < r < R.$$

$0 \leq \psi \leq 1$. Besides: $\psi = \begin{cases} 1 & \Leftrightarrow x \in B(\eta, r) \\ 0 & \Leftrightarrow x \in B(\eta, R)^c \end{cases}$ (looks like Bump)

i) Extension:

Bump Func's can be used to extend locally smooth functions defined in $U \subseteq X$:

Firstly, for X smooth manifold. $f \in C^\infty(U)$.

We create bump Func on the whole X :

Let $x \in X$. $(U, f) \in \mathcal{A}_x$. $x \in U$. Choose $\overline{B(0, r)} \subseteq \overline{U}$.

which is the largest ball. ($\forall 0 < r < R$)

$$\text{Def: } \tilde{\psi}(x) = \begin{cases} (\psi^{k,r} \circ f)(x), & x \in U \\ 0, & x \notin U \end{cases} \quad (\text{It's bump-like})$$

check $\tilde{\psi} \in C^\infty(X)$:

$\therefore \tilde{\varphi} \geq 0 \Leftrightarrow x \in f^{-1}(\overline{B(0, R)})$ opt. in X

$\therefore X$ is Hausdorff $\therefore f^{-1}(\overline{B(0, R)})$ is closed, too

$\therefore \tilde{\varphi}$ is smooth in U and $X/f^{-1}(\overline{B(0, R)})$ open, $\tilde{\varphi} \equiv 0$

$\therefore \tilde{\varphi} \in C^\infty(X)$.

Remark: Hausdorff condition is necessary:

1) Introduce topo on disjoint union: $X \sqcup Y$. (X, Y topo)

Generally, $\bigcup_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} (X_\alpha, \tau)$

$$\begin{aligned} ZC(\bigcup_{\alpha \in A} X_\alpha) &= \{V \subseteq \bigcup_{\alpha \in A} (X_\alpha, \tau) \mid V \cap X_\alpha \stackrel{\text{open}}{\subseteq} X_\alpha, \forall \alpha \in A\}, \\ &= \{V \subseteq \bigcup_{\alpha \in A} (X_\alpha, \tau) \mid \sigma_\alpha^{-1}(V) \stackrel{\text{open}}{\subseteq} X_\alpha, \forall \alpha \in A\}. \end{aligned}$$

$\sigma_\alpha: X_\alpha \longrightarrow \bigcup_{\alpha \in A} X_\alpha$ canonical projection.
 $x \longmapsto (x, \alpha)$

2) Consider ' $\mathbb{R}' \sqcup \mathbb{R}'$:

$$X = \mathbb{R}' \sqcup \mathbb{R}' / (x, 1) \sim (x, 2), \forall x \neq 0. \quad (\text{has 2 origins})$$

Then X isn't Hausdorff. since we can't separate $(0, 1)$ and $(0, 2)$. $\nexists U_{(0,1)} \cap U_{(0,2)} \neq \emptyset$. under the equivalent relation.

3) Consider $\varphi(x) \in C^\infty(\mathbb{R}')$. $\begin{cases} \varphi \equiv 1 & \text{in } (-r, r) \\ \varphi \equiv 0 & \text{in } [-m, m]^c. \end{cases}$

Extend φ on X :

$$\tilde{\varphi} = \begin{cases} \varphi, & x \in (\mathbb{R}', 1) \\ 0, & x \in (\mathbb{R}', 2) \end{cases} \quad \text{choose } (U, f) = (\mathbb{R}', 1)$$

Restrict $\tilde{\psi}$ on $C^1(\mathbb{R}, \mathbb{R})$: $\begin{cases} \tilde{\psi} = 1 & \text{in } (-r, 0) \cup (0, r), \\ \tilde{\psi} = 0 & \text{in } (-\infty, -r) \cup (r, \infty) \end{cases}$
 $\therefore \tilde{\psi}$ isn't conti at 0.

Secondly. Let $\hat{g} = \begin{cases} g\tilde{\psi} & x \in U \\ 0 & x \notin U \end{cases} \in C^\infty(X)$. $\hat{g}|_{f^{-1}(B(0, r))} = g$.

iii) Whitney Thm:

Thm. $\forall A \subseteq X$, smooth manifold. $\exists f \in C^\infty(X)$. St. $f|_{\partial A} = A$.

Pf. $U = X/A \stackrel{\text{open}}{\subseteq} X$. $U = \bigcup f_n^{-1}(B(x, r))$, $(U_x, f_x) \in Ax$.

Since X is C_2 $\therefore U = \bigcup_{n \in \mathbb{Z}} f_n^{-1}(B(x_n, r_n))$

B_n i). $\exists \phi_n \in C^\infty(X)$. $\phi_n \equiv 1$ in $f_n^{-1}(B(x_n, r_n))$. $\phi_n \equiv 0$ outside U

Choose ε_n . St. $\sum_n \sup_{x_i} \left| \frac{\partial^k \phi_n}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \leq \frac{1}{2^n}$. $\forall k \in \mathbb{N}$. ($\text{Supp } \phi_n$ is cpt)

Set $g_n(x) = \sum_i \varepsilon_n \phi_n \in C^\infty(X)$.

$\therefore \sum_i \left| \sum_n \frac{\partial^k \phi_n}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \leq \sum_i \varepsilon_i + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty$. $\forall k \in \mathbb{Z}^+$.

$\therefore g_n \in C^\infty(X) \xrightarrow{n} g = \sum \varepsilon_n \phi_n \in C^\infty(X)$. what we need.

Cor. $\forall A, B \subseteq X$. $\exists \phi: X \rightarrow [0, 1]$. St. $\phi|_{\partial A} = A$. $\phi|_{\partial B} = B$. $\phi \in C^\infty(X)$.

Pf. Choose $\phi_1, \phi_2 \in C^\infty(X)$. $\phi_1|_{\partial A} = A$. $\phi_2|_{\partial B} = B$. $\phi = \frac{\phi_1}{\phi_1 + \phi_2} \in C^\infty(X)$.

Remark: i) It's extension of Urysohn Thm.

ii) In particular, Let $X = \mathbb{R}^n$.