Preliminary

(1) Matrix Analysis:

Denotion: i) For $u: \mathbb{R}^n \to \mathbb{R}^m$.

$$Du = \nabla u = \begin{pmatrix} u^1_{x_1} & \cdots & u^1_{x_n} \\ & \vdots & \\ u^n_{x_1} & \cdots & u^n_{x_n} \end{pmatrix}$$

$$Div(u) = \nabla \cdot u = tr(Du) = \sum_1^n u^k_{x_k}. \text{ when } m=n.$$

ii) For $u: \mathbb{R}^n \to \mathbb{R}^1$

$Du = (u_{x_1} \cdots u_{x_n})$. $Div(u) = \nabla \cdot u = \sum_1^n u_{x_k}$

$D^2u = (u_{x_ix_j})_{n\times n}$. $\Delta u = tr(D^2u) = \sum_1^n u_{x_ix_i}$.

① Divergence:

Denote: $dV = \prod_1^n dx_i$. $dS_{x_i} = \prod_{k\neq i} dx_k$.

For $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$.

$$\int_V Div(\vec{F})\, dV = \int_{\partial V} \vec{F}\cdot\vec{\nu}\, ds, \quad \vec{\nu} = (\nu^1 \cdots \nu^n)$$

the outer pointing unit normal vector.

$(\vec{\nu}\, ds = (dS_{x_1}, dS_{x_2} \cdots dS_{x_n}))$.

interpretion: $Div(\vec{F}) = \lim_{|V|\to 0} \frac{1}{|V|} \int_{\partial V} \vec{F}\cdot\vec{\nu}\, ds$.

the flux in unit volum.

② Curling:

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$. $curl(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F^1 & F^2 & F^3 \end{vmatrix}$

$$\int_{\partial V} \mathrm{curl}(\vec{F})\cdot\vec{v}\, ds = \oint_{I} \vec{F}\, d\vec{r}. \quad d\vec{r} = (dx_1, dx_2, dx_3)$$

Interpretion: $\mathrm{curl}(\vec{F})\cdot\vec{v} = \lim_{|S|\to 0} \frac{1}{|S|}\oint_{I} \vec{F}\, d\vec{r}$

The circulation in unit area.

(2) n-dimensional Balls:

① $$V_{B_n(\vec{t},r)} = \int_{\sum_1^n (x_i - t_i)^2 \le r^2} dx_1 dx_2 \cdots dx_n = r^n \int_{\sum_1^n u_i^2 \le 1} du_1 \cdots du_n$$

$$= r^n V_{B_n(\vec{0},1)}. \quad \text{By recursion. } V_{B_n(\vec{0},1)} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

② $$S_{B_n(\vec{t},r)} = \int_{\sum_1^n (x_i - t_i)^2 = r^2} ds \quad \left(x_n - t_n = \pm\sqrt{r^2 - \sum_1^{n-1}(x_i - t_i)^2}\right)$$

$$= 2\int_{\sum_1^{n-1}(x_i - t_i)^2 \le r^2} \sqrt{1 + \sum_1^{n-1}\left(\frac{\partial x_n}{\partial x_k}\right)^2}\, dx_1 \cdots dx_{n-1}$$

$$= 2\int_{\sum_1^{n-1}(x_i - t_i)^2 \le r^2} \frac{r}{\sqrt{r^2 - \sum_1^{n-1}(x_i - t_i)^2}}\, dx_1 \cdots dx_{n-1}$$

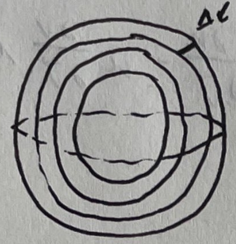
$$= 2\int_{\sum u_i^2 \le 1} \frac{r^{n-1}}{\sqrt{1 - \sum_1^{n-1} u_i^2}}\, du_1 \cdots du_{n-1}$$

$$= r^{n-1}\cdot S_{B_n(\vec{0},1)} = \frac{n r^{n-1}\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$$

③ Relation:

$$\frac{\partial}{\partial r} V_{B_n(\vec{x},r)} = S_{B_n(\vec{x},r)}$$

Pf: Consider volum of n-dimension ball is from the overlap of shells coming from inside with thickness Δl.



$$\therefore V_{B_n(\vec{x},r)} = \lim_{\Delta l \to 0} \sum_{1}^{N} S_{B_n(\vec{x},l_i)} \Delta l$$

$$\therefore V_{B_n(\vec{x},r)} = \int_0^r S_{B_n(\vec{x},l)} dl.$$

$$\therefore \frac{\partial V}{\partial r} = S. \quad (l_i \in [i\Delta l, (i+1)\Delta l])$$

(3) Calculus:

① Gauss-Green Formula:

U is open bounded. ∂U is C^1. For $u \in C^1(\bar{U}, \mathbb{R}^1)$:

$$\int_U u_{x_i} dV = \int_{\partial U} u \cdot \nu^i dS \quad , \quad \nu = (\nu^1 \cdots \nu^n)$$

$\vec{u} \in C^1(\bar{U}, \mathbb{R}^n)$. Then we also have:

$$\int_U \mathrm{Div}(\vec{u}) dV = \int_{\partial U} \vec{u} \cdot \vec{\nu} dS$$

② Cor. (Integration by part)

$u, v \in C^1(\bar{U}, \mathbb{R}^1)$. Then apply to uv:

$$\int_U u_{x_i} v + u v_{x_i} dV = \int_{\partial U} u v \nu^i dS$$

③ Green Formula:

$u, v \in C^2(\bar{U}, \mathbb{R}^1)$, Then (Denote $\frac{\partial u}{\partial \vec{v}} = Du \cdot \vec{v}$)

$$\int_U \Delta u \, dV = \int_{\partial U} \frac{\partial u}{\partial \vec{v}} dS$$. apply to uv:

$$\int_U Du \cdot Dv + u \cdot \Delta v \, dV = \int_{\partial U} u \frac{\partial v}{\partial \vec{v}} dS.$$

$\Rightarrow$ By symmetry:

$$\int_U v\Delta u - u\Delta v \, dV = \int_{\partial U} v \frac{\partial u}{\partial \vec{v}} - u \frac{\partial v}{\partial \vec{v}} dS$$

(4) Coarea Formular:

· It's an extension of Fubini Thm:

$u \in C^{Lip}(\mathbb{R}^n, \mathbb{R}^1)$. $[u = r]$ is smooth a.e. r.

$f \in C^1(\mathbb{R}^n, \mathbb{R}^1)$. Then:

$$\int_{\mathbb{R}^n} |Du| f \, dV = \int_{\mathbb{R}^1} \left(\int_{[u=r]} f \, dS \right) dr$$

Cor. (Polar coordinates)

Let $u = |x - x_0|$. Then

$$\int_{\mathbb{R}^n} f \, dV = \int_0^\infty \left(\int_{\partial B(x_0, r)} f \, dS \right) dr.$$

In particular, $\frac{\partial}{\partial r} \int_{B(x_0, r)} f \, dV = \frac{\partial}{\partial r} \int_0^r \left(\int_{\partial B(x_0, s)} f \, dS \right) ds$

$$= \int_{\partial B(x_0, r)} f \, dS$$