# Second - Order
# Elliptic Equations

(1) <u>Preliminaries:</u>

① Consider boundary - value problem:

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \qquad \begin{matrix} u: \bar{U} \longrightarrow \mathbb{R}^1 \\ f: U \longrightarrow \mathbb{R}^1. \end{matrix}$$

$L$ is an operator. defined by:

$$Lu = \begin{cases} -\sum_{i,j}^{n} (a^{ij}(x) u_{x_i})_{x_j} + \sum b^i(x) u_{x_i} + c(x) u. & \text{Divergence Form.} \\ -\sum_{i,j} a^{ij}(x) u_{x_i x_j} + \sum b^i(x) u_{x_i} + c(x) u. & \text{Nondivergence Form.} \end{cases}$$

<u>Remark:</u> Divergence Form is natural for energy method. Since it's convenient for integrating by part. Nondivergence form is fit for maximum principles.

<u>Def:</u> $L$ is uniformly elliptic if $\exists\, \theta > 0$. const.

s.t. $\sum a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$. $\forall \xi \in \mathbb{R}^n$. a.e. $x$.

<u>Remark:</u> It means $(a^{ij}(x))_{n \times n}$ is positive definite.

whose smallest eigenvalue $\geq \theta$.

<u>Cor.</u> $\sum_{i,j} \sum_{k,l} a^{ij}(x) a^{kl}(x) \xi_{ik} \xi_{jl} \geq \theta^2 \sum_{p,q} \xi_{pq}^2$

<u>Pf:</u> Fix $i, j$: $\sum_{k,l} a^{kl} \xi_{ik} \xi_{jl} = \xi^i A \xi^{jT}$

where $A = (a_{ij}(x))_{n\times n}$. $\xi^i = (\xi_{i1} \cdots \xi_{in})$

suppose $O$ is orthonormal, s.t. $OAO^T = \operatorname{diag}\{\theta_1 \cdots \theta_n\}$. $\theta_k \geq \theta > 0$.

Denote $\eta_i = \xi^i O^T$. $\therefore \xi^i A \xi^{iT} = \sum_k \theta_k \eta_{ik} \eta_{ik} \geq \theta \eta^i \eta^{iT}$

Repeat again. since $|\eta^i| = |\xi^i|$ $\therefore \sum \xi_{pq}^2 = \sum \eta_{pq}^2$.

Interpretion in Physics:

i) Second-order term $\sum a^{ij}(x) u_{x_i x_j}$ represents the diffusion of $u$ in $U$. $(a^{ij})$ describes anisotrophic, heterogeneous nature of medium.

ii) First-order term $\sum b^i(x) u_{x_i}$ represents transport in $U$.

iii) Zeroth-order term $c(x)u(x)$ describes increase or depletion.

② Weak Solutions:

· Suppose $a^{ij}(x), b^k(x), c(x) \in L^\infty(\bar{U})$, $f \in L^2(\bar{U})$.

For $Lu = f$. Consider $(Lu, v) = (f, v)$. test by $v \in C_c^\infty(U)$.

$$\Rightarrow \int_U \sum a^{ij} u_{x_i} v_{x_j} + \sum b^i u_{x_i} v + cuv = \int_U fv\, dx$$

since by approx. to $H_0^1(U)$, in $W^{1,2}(U)$, replace $C_c^\infty$ by $H_0^1$.

Def: $B[\cdot,\cdot]$ associated with divergence form $L$ is:

$$B[u,v] = \int_U \sum a^{ij}(x) u_{x_i} v_{x_j} + \sum b^i(x) u_{x_i} v + c(x) uv.$$

for $\forall u, v \in H_0^1(U)$.

We say $u$ is weak solution of $Lu = f$. if

$B[u,v] = (f, v)$. $\forall v \in H_0^1(U)$.

Remark: For other boundary conditions $\begin{cases} Lu = f & \text{in } U. \\ u = g & \text{on } \partial U. \end{cases}$

Find $w \in H^1(U)$. st. $w|_{\partial U} = g$.

solve $\begin{cases} L\tilde{u} = \tilde{f} & \text{in } U \\ \tilde{u} = 0. & \text{on } \partial U \end{cases}$. $\tilde{u} = u - w$. $\tilde{f} = f - Lw$

## (2) Existence of weak solutions:

### ① Energy Estimate:

Thm. There exists $\alpha, \beta > 0$. $\gamma \geq 0$. st.

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \quad \text{for } \forall u,v \in H_0^1(U).$$

$$\beta \|u\|^2_{H_0^1(U)} \leq |B[u,u]| + \gamma \|u\|^2_{L^2(U)}.$$

Pf. 1°) The first one directly by Cauchy Inequality.

2°) Apply Elliptic condition: (with Pióncare Ineq)

$$\theta \int_U |Du|^2 dx \leq B[u,u] + C \int_U |Du||u| + |u|^2)$$

$$\leq B[u,u] + C(\int \frac{\varepsilon}{2} |Du|^2 + \frac{1}{2\varepsilon} |u|^2 + |u|^2)$$

Thm. (First existence Thm for weak solutions)

There exists unique $u \in H_0^1(U)$. weak solution

for $\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$ where $\mu \geq \gamma$.

Pf: Let $B_\mu[u,v] = B[u,v] + \mu(u,v)$

$$\langle f, v \rangle = (f, v)_{L^2}$$

Remark: Note that $\forall (f^i)_0^n \subseteq L^2(\mathbb{R}^n)$.

since $\langle f, v\rangle = \int_U f^0 v + \sum f^i v_{x_i}$ is BLO on $H_0^1(U)$.

$\therefore \begin{cases} Lu + \mu u = f^0 - \sum_{i}^{n} f^i_{x_i} & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has unique solution $u$ in weak sense.

i.e. $L + \mu I : H_0^1 \xrightarrow{\sim} H^{-1}$ isomorphism.

② Fredholm Alternative:

Def: i) $L^* v = -\sum (a^{ij}(x) v_{x_j})_{x_i} - \sum b^i(x) v_{x_i} + (c - \sum b^i_{x_i}(x)) v$.

ii) $B^*[v, u] = (L^* v, u) = (v, Lu) = B[u, v]$.

iii) $v$ is weak solution for $\begin{cases} L^* v = f & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}$ if.

$B^*[v, u] = (f, u)$. $\forall u \in H_0^1(U)$.

Remark: It's from: $(Lu, v) = \sum a^{ij}(x) u_{x_i} v_{x_j} + \sum b^i(x) u_{x_i} v + cuv$

$= -\sum (a^{ij}(x) v_{x_j}(x))_{x_i} u - \sum b^i v_{x_i} u + (c - \sum b^i_{x_i}) uv$.

Thm. ( Second Existence Thm )

i) One of the following statements will hold:

(a) $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has unique weak solution for $\forall f \in L^2(U)$

(b) $\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ exists $u \not\equiv 0$, $u \in H_0^1(U)$ weak solution. Denote set $N$.

ii) $N^* \triangleq \{ v \mid L^* v = 0 \text{ in } U,\ v = 0 \text{ on } \partial U \}$. Then $\dim N^* = \dim N$.

iii) (a) $\Longleftrightarrow f \in N_*^{\perp}$. i.e. $(f, v) = 0$. $\forall v \in N_*$.

Pf: 1°) Choose $\mu = \gamma$. $L_\gamma u = Lu + \gamma u$. Correspond $B_\gamma[\cdot,\cdot]$.

$\forall g \in L^2(U)$. $\exists u \triangleq L_\gamma^{-1} g$ solves it.

Check $L_\gamma^{-1}$ is linear.

2°) $\because B[u,v] = (f,v) \Leftrightarrow u = L_\gamma^{-1}(\gamma u + f)$

Denote $Ku = \gamma L_\gamma^{-1} u$. $h = L_\gamma^{-1} f$. $u - Ku = h$.

3°) Check $K: L^2(U) \to L^2(U)$ is cpt operator.

prove: $K: L^2(U) \to H_0^1(U)$ is BLO. (use $B_\gamma[\cdot,\cdot]$)

Apply $H_0^1(U) \subset\subset L^2(U)$. attain subseq converges.

4°) Apply Fredholm Alternative on $u - Ku = h$.

$u - Ku = 0 \Leftrightarrow u - Lu = 0$. Similar as $u - K^* u = 0$

It has solution $\Leftrightarrow (h, v) = 0$. $\forall v \in N(I - K^*)$.

$\Leftrightarrow (f, v) = 0$. since $(h, v) = \frac{1}{\gamma}(f, v)$.

Remark: In this case. It holds when $\lambda = 0$.

for $\lambda I - K$. $K \in k(L^2(U))$.

Thm. (Third Existence Thm).

i) There exists an at most countable set $\Sigma \subset \mathbb{R}^1$.

s.t. $\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has unique weak solution

for $\forall f \in L^2(U)$. $\Leftrightarrow \lambda \notin \Sigma$.

ii) If $\Sigma$ is infinite. Then $\Sigma = (\lambda_k) \to +\infty$.

Denote: $\Sigma$ is spectral of $L$.

Pf: It has unique solution $\Leftrightarrow u\equiv 0$ is the only solution of $\begin{cases} Lu=\lambda u & \text{in } U \\ u=0 & \text{on } \partial U \end{cases}$

$\Leftrightarrow L_\gamma u = (\gamma+\lambda)u. \Leftrightarrow u = L_\gamma^{-1}(\gamma+\lambda)u = \frac{\gamma+\lambda}{\gamma} Ku.$

For $\lambda \le -\gamma$. Then it holds.

For $\lambda > -\gamma$. Then $\Leftrightarrow \frac{\gamma}{\gamma+\lambda}$ isn't eigenvalue of $K$.

Since $K$ is cpt. operator. Apply FA.

Thm. (Bounded inverse)

If $\lambda \notin \Sigma$. Then there exists cnst. $C$. st.

$\|u\|_{L^2(U)} \le C\|f\|_{L^2(U)}$ for $\begin{cases} Lu=\lambda u+f & \text{in } U \\ u=0 & \text{on } \partial U \end{cases}$

$f\in L^2(U)$. $u\in H_0^1(U)$ the unique weak solution.

Remark: It claims the boundness of $(L-\lambda I)^{-1}$ as well.

Pf. By contradiction: If $\exists (u_k)$. st. $\|u_k\|_{L^2}=1$.

$\begin{cases} Lu_k=\lambda u_k+f_k & \text{in } U \\ u_k=0 & \text{on } \partial U \end{cases}$ for some $f_k$. $\|u_k\|_2 > C\|f_k\|_2$.

Then since $\beta\|u_k\|_{H_0^1(U)} \le B[u_k,u_k] + \gamma\|u_k\|_2 \le \gamma + \|u_k\|_2\|f_k\|_2$

$\therefore (u_k)$ is bounded in $H_0^1(U)$ $\therefore \begin{cases} \exists (u_{nk}) \rightharpoonup u & \text{in } H_0^1(U) \\ u_{nk} \to u & \text{in } L^2 \end{cases}$

$\therefore \|u\|_{L^2(U)}=1$. And $\because f_k\to 0$ $\therefore Lu=\lambda u$. $u\equiv 0$. Since $\lambda\notin\Sigma$

Which is a contradiction.

(3) Regularity:

· Motivation:

Consider a case: $-\Delta u = f$. in $\mathbb{R}^n$.

Suppose $u \in C^\infty(\mathbb{R}^n)$. $u(x) \to 0$ $(|x| \to \infty)$

Note that: $\int f^2 = \int (\Delta u)^2 = \int |D^2 u|^2 dx$.

$\Rightarrow$ It means: second derivates of $u$ is dominated by $\|f\|_{L^2(\mathbb{R}^n)}$.

Replace $\tilde{u} = D^\alpha u$. $|\alpha| = m$. Then we obtain:

$(m+2)^{th}$- derivates of $u$ is controlled by $\|f\|_{H^m(\mathbb{R}^n)}$

① Interior Regularity:

· Suppose $U$ is open. bounded.

Thm.

If $a^{ij}(x) \in C^1(U)$. $b^i(x)$. $c(x) \in L^\infty(U)$.

$f \in L^2(U)$ and $u \in H^1(U)$ solve $Lu = f$ in $U$

weakly. Then $u \in H^2_{loc}(U)$. Besides.

$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$. $\forall V \subset\subset U$. $C = C(V, U, L)$.

pf: 1°) Fix $V \subset\subset U$. Find $W$ open. $V \subset\subset W \subset\subset U$.

Construct $\zeta \in C^\infty(U)$. $\zeta \equiv 1$ on $V$. $\zeta \equiv 0$ on $\mathbb{R}^n / W$

$0 \le \zeta \le 1$. which is for guarantee $u$ keep away from $\partial U$.

2°) From $B[u,v] = (f,v)$. $\forall v \in H_0^1(U)$.

Separate the second-order part : $\sum \int_U a^{ij} u_{x_i} v_{x_j} = \int_U \tilde{f} v \, dx$.

where $\tilde{f} = f - \sum b^i(x) u_{x_i} - c u(x)$.

Let $v = -D_k^{-h}(\zeta^2 D_k^h(u(x)))$. $|h|$ is sufficient small.

prove: $\| D_k^h D u \|_2 \le C$. $\forall k$.

( $\zeta^2$ is for retaining "$\zeta$" after differentiation ).

3°)

Recall $\begin{cases} \int_W v D_k^{-h} w = -\int_W w D_k^h v. \\ D_k^h(vw) = v^h D_k^h w + w D_k^h v \end{cases}$

$$\|v\|_{L^2(U)} \le C \int_W |D(\zeta^2 D_k^h u)|^2 \le C \int_W |D_k^h D u|^2 + |D_k^h u|^2$$

$$\le C \int_W |D_k^h D u|^2 + |Du|^2.$$

We obtain: $\int_V |D_k^h D u|^2 \le \int_W \zeta^2 |D_k^h D u|^2 \le C \int_W f^2 + u^2 + |Du|^2$.

$\therefore Du \in H_{loc}^1(U)$. and $\|u\|_{H^2(V)} \le C(\|f\|_{L^2(W)} + \|u\|_{H^1(W)})$

4°) Refine: $\|u\|_{H^1(W)} \le \|u\|_{L^2} + \|f\|_{L^2}$.

Choose $\zeta \in C^\infty(\mathbb{R}^n)$ : $\begin{cases} \zeta \equiv 1 \text{ on } W. \ \operatorname{supp} \zeta \subset U. \\ 0 \le \zeta \le 1. \end{cases}$

Let $v = \zeta u$. Apply elliptic condition:

$$\theta \int_W |Du|^2 \le \theta \int_U \zeta^2 |Du|^2 \le C \int_U f^2 + u^2.$$

<u>Remark</u>: i) Since we don't consider boundary of $U$.

There's no need: $u \in H_0^1(U)$.

ii) Since $u \in H_{loc}^2(U)$. Then $B[u,v] = (f,v)$

$= (Lu, v)$. $\forall v \in C_c^\infty(U)$. $\therefore Lu = f$. a.e. $U$.

**Thm.** (Higher order).

$m \in \mathbb{Z}/\mathbb{Z}^-$. If $a^{ij}, b^i, c \in C^{m+1}(U)$. $f \in H^m(U)$.

$u \in H^1(U)$ solves $Lu = f$ in $U$. weakly.

Then $u \in H^{m+2}_{loc}(U)$. Besides, $\|u\|_{H^{m+2}(V)} \le C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$.

where $\forall V \subset\subset U$. $C = C(U, V, L)$.

Pf. By induction on $m$.

1') $m=0$. it holds by the former thm.

2') Suppose $a^{ij}, b^i, c \in C^{m+2}(U)$. $f \in H^{m+1}(U)$.

By hypothesis: $u \in H^{m+2}_{loc}(U)$. with an estimation.

3') Consider $|\alpha| = m+1$. $\tilde{v} \in C_c^\infty(W)$. $V \subset\subset W \subset\subset U$.

Let $v = (-1)^{|\alpha|} D^\alpha \tilde{v}$. By integration by part:

$B[u, v] = (f, v) \Rightarrow B[\tilde{u}, \tilde{v}] = (\tilde{f}, \tilde{v})$. $\tilde{u} = D^\alpha u$.

$\tilde{f} = D^\alpha f - \sum_{\substack{\beta \le \alpha \\ \alpha \ne \beta}} \binom{\alpha}{\beta} [-\sum (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} + \cdots]$

$\|\tilde{f}\|_{L^2(W)} \le \|f\|_{H^{m+1}(U)} + \|u\|_{H^{m+2}(U)} \le C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)})$

4') Apply $m=0$ case on $\tilde{u}$. We have $u \in H^{m+3}(U)$.

$\|u\|_{H^{m+3}(U)} \le C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)})$

Cor. If $a^{ij}, b^i, c \in C^\infty(U)$. $f \in C^\infty(U)$. $u \in H^1(U)$

solves $Lu = f$ in $U$ weakly. Then $u \in C^\infty(U)$.

Pf: $u \in H^m_{loc}(U)$. $\forall m \in \mathbb{Z}^+$. Then for $\forall V \subset\subset U$.

$\Rightarrow u \in C^{m - [\frac{n}{2}] - 1, \gamma}(\bar{V})$. a.e. $\forall m \in \mathbb{Z}^+$.

$\therefore u \in C^\infty(U)$. a.e.

② Boundary Regularity:

Thm.

If $a^{ij} \in C^1(\bar{U})$, $b^i, c \in L^\infty(U)$, $f \in L^2(U)$, $u \in H_0^1(U)$.

solves $\begin{cases} Lu = f \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}$ weakly. $\partial U$ is $C^2$.

Then $u \in H_0^2(U)$. Besides, $\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$.

$C = C(U, V, L)$. (If $u$ is unique. Then $\|u\|_{H^2(U)} \le c\|f\|_2$. inverse is bounded)

Pf: 1°) Consider $U = B^\circ(0,1) \cap \mathbb{R}^n_+$ firstly. $V = B^\circ(0,\frac{1}{2}) \cap \mathbb{R}^n_+$.

Set $\zeta \in C_c^\infty(\mathbb{R}^n)$. $\zeta \equiv 1$ on $B^\circ(0,\frac{1}{2})$. $\zeta \equiv 0$ on $\mathbb{R}^n / B(0,1)$.

2°) Similarly, separate second-order part.

Let $v = -D_k^{-h}(\zeta^2 D_k^h u)$. $v \in H^1(U)$.

Besides, for $1 \le k \le n-1$: $v \equiv 0$ on $\partial U$. $\therefore v \in H_0^1(U)$.

3°) Prove: $\|D_k^h Du\|_{L^2(U)} \le C$. $\forall\, 1 \le k \le n-1$. $\therefore u_{x_k} \in H^1(V)$.

With. $\sum_{k+l<2n} \|u_{x_k x_l}\|_{L^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)})$

4°) Prove: $\|u_{x_n x_n}\|_{L^2(U)} \le C(\|f\|_2 + \|u\|_{H^1})$

Since $Lu = f$ a.e. in $U$. $\therefore a_{nn}(x) u_{x_n x_n} = \square$

Let $\xi = e_n$ $\therefore a_{nn} \ge \theta > 0$ $\therefore \theta \|u_{x_n x_n}\|_{L^2(U)} \le C(\|f\|_2 + \|u\|_{H^1(U)})$

5°) $\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)})$

$\le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$

Since under elliptic condition: $\|u\|_{H^1(U)}$ is controlled by $\|f\|_{L^2(U)}$, $\|u\|_{L^2(U)}$.

6°) "Straighten out" Argument:

WLOG. suppose $U \cap B^\circ(x_0, r) = \{x \in B^\circ(x_0, r) \mid x_n > \gamma(x')\}$.

$\gamma \in C^2(\mathbb{R}^{n-1})$. $\quad \underset{x}{U} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \underset{\eta}{\tilde{U}}$ (straightened). $\quad \phi^n = x_n - \gamma(x')$

choose $s$ small enough. s.t. $U' = B^\circ(0, s) \cap \{\eta_n > 0\} \subseteq \phi(U)$.

Set $V' = B^\circ(0, \frac{s}{2}) \cap \{\eta_n > 0\}$. $u'(\eta) \triangleq u(\psi(\eta)) = u(x)$.

7°) Check $u'(\eta) \in H_0^1(U')$. by approxi of $C^\infty(\bar{U})$

8°) Claim: $u'(\eta)$ is weak solution of $L'u' = f'$. in $U'$.

$f'(\eta) = f(\psi(\eta)) = f(x)$. $\quad c'(\eta) = c(\psi(\eta)) = c(x)$.

$$a'^{kl}(\eta) = \sum_{r,s} a^{rs}(\psi(\eta))\, \phi^k_{x_r}(\psi(\eta))\, \phi^l_{x_s}(\psi(\eta)).$$

$$L'u' = -\sum_{k,l} (a'^{kl} u'_{\eta_k})_{\eta_l} + \sum_k b'^k u'_{\eta_k} + c'u'.$$

It originates from:

$$\sum b_k(x) u_{x_k}(x) = \sum b_k(\psi(\phi(x)))\, u_{x_k}(\psi(\phi(x)))$$

$$= \sum_{k,i,l} b_k\, u_{x_i}\, \psi^i_{\eta_l}(\phi(x))\, \phi^l_{x_k} = \sum_{k,i,l} b_k(\psi(\eta))\, u_{x_i}\, \psi^i_{\eta_l}\, \phi^l_{x_k}$$

$$= \sum_l \Big(\sum_k b_k(\psi(\eta))\, \phi^l_{x_k}\Big)\Big(\sum_i u_{x_i} \psi^i_{\eta_l}\Big)$$

$\triangleq \sum_l b'_l(\eta)\, u_{\eta_l}(\psi(\eta))$. We obtain $b'_l(\eta) = \sum_k b_k(\psi(\eta))\, \phi^l_{x_k}$

Similar to obtain $a'_{ij}$, $c'$

It can be checked by $D\phi \cdot D\psi = I_n$. Conversely.

9°) Check $L'$ is uniformly elliptic.

Apply the half-ball case. And cover $\partial U$ by finite balls.

Thm. (Higher order)

$m \in \mathbb{Z}/\mathbb{Z}^-$. $a^{ij}, b^i, c \in C^{m+1}(\bar{U})$. $f \in H^m(U)$. $u \in H_0^1(U)$

Solves $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ weakly. $\partial U$ is $C^{m+2}$.

Then $u \in H_0^{m+2}(U)$. Besides. $\|u\|_{H_0^{m+2}(U)} \le C(\|f\|_{H^m(U)} + \|u\|_{L^2(U)})$

$C = C(U, L, m)$. Const. (If $u$ is unique solution. Then

we have: $\|u\|_{H^{m+2}(U)} \le C\|f\|_{H^m(U)}$).

Pf: 1°) By induction on $m$:

$m=0$ is proved by Thm above.

Now if $a^{ij}, b^i, c \in C^{m+2}(\bar{U})$. $f \in H^{m+1}(U)$. $\partial U \in C^{m+3}$

By inductive assumption: $u \in H_0^{m+2}(U)$ with estimation.

2°) For $|\alpha| = m+1$. $\alpha_n = 0$. (For $\tilde{u}|_{\{x_n=0\}} = 0$)

consider $\tilde{u} = D^\alpha u \in H_0^1(U)$. $L\tilde{u} = \tilde{f}$

(where it's from $D^\alpha Lu = D^\alpha f$ a.e.)

$\tilde{f} = D^\alpha f - \Sigma \binom{\alpha}{\beta} \boxed{\phantom{xxxx}}$. $\tilde{f} \in L^2(U)$

Apply $m=0$ case. $\therefore \tilde{u} \in H_0^2(U)$:

i.e. $\|D^\beta u\|_{L^2(U)} \le C(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)})$

for $|\beta| = m+3$. $\beta_n = 0, 1, 2$.

3°) For $|\beta| = m+3$. induction on $\beta_n = j$ again.

$j = 0, 1, 2$ we have proved.

If $\beta_n = j \in \{0, \dots m+2\}$ holds. for $\beta_n = j+1$.

Denote $\beta = \gamma + 2e_n$.

Since $Lu = f$. a.e. $U$. $\therefore D^\gamma Lu = D^\gamma f$. a.e.

$\therefore$ $D^\gamma f = a^{nn} D^\beta u$ + [ sum of terms involving at most

$j$ derivatives of $u$. w.r.t $x_n$ ]

$\because$ $a_{nn} \geq \theta > 0$ $\therefore$ $\|D^\beta u\|_{L^2(U)} \leq C\|f\|_{H^{m+j}(U)} + \|u\|_{L^2(U)}$

It follows from hypothesis. Then by straighten and cover.

<u>Cor.</u> If $a^{ij}, b^i, c \in C^\infty(\bar{U})$, $f \in C^\infty(\bar{U})$, $u \in H_0^1(U)$

solves $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ weakly, $\partial U$ is $C^\infty$.

Then $u \in C^\infty(\bar{U})$

<u>Pf:</u> $u \in H^m(U)$. $\forall m \in \mathbb{Z}^+$. $\Rightarrow u \in C^{m-[\frac{n}{2}]+1,\gamma}(\bar{U})$, $\forall m \in \mathbb{Z}^+$.

## (4) <u>Maximal Principle:</u>

- Suppose $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, bounded. For considering pointwise values of $Du$, $D^2u$. (Note that $u$ attains max at $x_0$ if $Du(x_0) = 0$, $D^2u(x_0) \leq 0$).

Suppose: $u \in C^2(U)$.

Consider $L$ in nondivergence form. And sym: $a^{ij} = a^{ji}$

Besides, $a^{ij}, b^i, c$ are conti.

### ① <u>Weak maximal Principle:</u>

For $u \in C^2(U) \cap C(\bar{U})$, and $c(x) \equiv 0$ in $L$.

i) If $Lu \le 0$ in $U$. Then $\max_{\bar{U}} u(x) = \max_{\partial U} u(x)$.

ii) If $Lu \ge 0$ in $U$. Then $\min_{\bar{U}} u(x) = \min_{\partial U} u(x)$.

<u>Pf:</u> Only prove i). Since for ii). let $\tilde{u} = -u$.

1°) Consider $u^\varepsilon(x) = u(x) + \varepsilon e^{\lambda x_1}$. Choose $\lambda$:

s.t. $Lu^\varepsilon(x) \le \varepsilon L e^{\lambda x_1} < 0$.

2°) Suppose $\exists x_0 \in U$. s.t. $u^\varepsilon(x_0) = \max_{\bar{U}} u^\varepsilon(x)$.

Then $Du^\varepsilon(x_0) = 0$. $D^2 u^\varepsilon(x_0) \le 0$ (negative definite)

3°) $\because A$. $D^2 u^\varepsilon$ are symmetric. $\therefore \exists\, O \in M^{n\times n}(\mathbb{R})$. orthonormal.

s.t. $OAO^T = \mathrm{diag}\{d_1 \cdots d_n\}$. $OD^2u^\varepsilon O^T = \mathrm{diag}\{p_1 \cdots p_n\}$.

$d_i \ge \theta > 0$. $\forall\, 1 \le i \le n$. $p_i \le 0$. $\forall\, 1 \le i \le n$.

For $u^\varepsilon(y) = u^\varepsilon(x_0 + O(x - x_0))$.

$D_x u^\varepsilon(y) = D_y u^\varepsilon \cdot O$. $D_x^2 u^\varepsilon = O^T D_y^2 u^\varepsilon O$ $\therefore \begin{cases} u^\varepsilon_{y_k y_l} = 0. & k \ne l. \\ u^\varepsilon_{y_k y_k} \le 0. \end{cases}$

4°) $\sum a^{ij} u^\varepsilon_{x_i x_j} = \sum d_k u^\varepsilon_{y_k y_k} \le 0$.

At $x = x_0$. $\because Du^\varepsilon(x_0) = 0$. $\therefore Lu^\varepsilon(x_0) \ge 0$. Contradict!

5°) Let $\varepsilon \to 0$. Attain $\max_{\bar{U}} u = \max_{\partial U} u$.

<u>Cor.</u> If $u \in C^2(U) \cap C(\bar{U})$. $c \ge 0$ in $L$ in $U$.

i) For $Lu \le 0$ in $U$. Then $\max_{\partial U} u^+ \ge \max_{\bar{U}} u$

ii) For $Lu \ge 0$ in $U$. Then $\max_{\partial U} u^- \ge \max_{\bar{U}} (-u)$

<u>Remark:</u> $Lu = 0 \Rightarrow \max_{\bar{U}} |u| = \max_{\partial U} |u|$.

Pf: Only prove i). ii) is from $\tilde{u}=-u$. $(-u)^+=u^-$

Consider $V=\{x\in U \mid u(x)>0\}$.

1') $V=\emptyset$. It's trivial. ("$\geq$" may be strict)

2') $V\neq\emptyset$. Since by $u\in C(\bar{U})$. $\therefore \partial V\cap U\subseteq\{u=0\}$.

$\therefore \partial V\cap\partial U\neq\emptyset$. For $Ku=Lu-cu$.

$Ku\leq -cu\leq 0$ in $V$. $\therefore$ By thm. $\max\limits_{\bar{V}} u=\max\limits_{\partial V} u$

$\max\limits_{\partial V} u=\max\limits_{\partial U} u^+$. $\max\limits_{\bar{V}} u=\max\limits_{\bar{U}} u(x)$. We're done.

Def: We say $L$ satisfies weak maximum principle if for $\forall u\in C^2(U)\cap C(\bar{U})$. and $\begin{cases} Lu\leq 0 & \text{in } U \\ u\leq 0 & \text{on } \partial U \end{cases}$

then $u\leq 0$ in $U$. (Denote WMP)

prop. If $\exists v\in C^2(U)\cap C(\bar{U})$. and $Lv\geq 0$. in $U$. $v>0$ on $\bar{U}$. Then $L$ satisfies WMP.

Pf: 1') Prove: $\exists M$. s.t. $M$ has no zeroth-order term. and $M(\frac{u}{v})\leq 0$. in $R=\{u>0\}$. Apply thm:

$\therefore \max\limits_{\bar{R}}\frac{u}{v}=\max\limits_{\partial R}\frac{u}{v}\leq 0$. $\therefore R=\emptyset$.

2') Suppose $Lu=-\sum a^{ij}u_{x_ix_j}+\sum b^iu_{x_i}+cu$.

Calculate: $-\sum a^{ij}(x)(\frac{u}{v})_{x_ix_j}=$ $(a_{ij}=a_{ji})$

$\frac{vLu-uLv}{v^2}-\frac{2}{v}\sum a_{ij}v_{x_i}(\frac{u}{v})_{x_j}+\vec{b}\cdot D(\frac{u}{v})$.

$\therefore$ Let $MW=-\sum a^{ij}W_{x_ix_j}+\frac{2}{v}\sum a_{ij}v_{x_i}W_{x_j}-\sum b^iW_{x_i}$

$\therefore M(\frac{u}{v})=\frac{vLu-uLv}{v^2}\leq 0$.

## ② Strong Maximum Principle:

### 1) Hopf's Lemma:

If $u \in C^2(U) \cap C(\bar{U})$. $c \equiv 0$ in $U$ of $L$. $Lu \le 0$ in $U$.

there exists $x_0 \in \partial U$. s.t. $u(x_0) > u(x)$. $\forall x \in U$. and

$U$ satisfies interior ball condition at $x_0$. (i.e. $\exists B \subseteq U$.

s.t. $x_0 \in \partial B$). Then: $\frac{\partial u}{\partial \nu}(x_0) > 0$. $\vec{\nu}$ is outer normal unit.

For $c \ge 0$. It holds when $u(x_0) \ge 0$.

Remark: If $\partial U$ is $C^2$. Then by formula of osculating ball. $U$ satisfies interior ball condition automatically.

Pf: 1') Denote $B = B^\circ(0,r)$. $R = B^\circ(0,r) / B(0,\frac{r}{2})$.

For $v(x) = e^{-\lambda|x|^2} - e^{-\lambda r^2}$. $Lv \le 0$ in $R$ for $\lambda$ large enough.

2') $\exists\, \varepsilon > 0$. so. $u(x_0) \ge u(x) + \varepsilon v(x)$ on $\partial B(0,\frac{r}{2})$.

$\therefore$ $u(x_0) \ge u(x) + \varepsilon v(x)$ on $\partial R$. ($v = 0$, $\forall x \in \partial B(0,r)$)

3') Since $L(u(x) - u(x_0) + \varepsilon v(x)) \le L(-u(x_0)) = -c\,u(x_0) \le 0$.

Apply thm in ①: $u(x) - u(x_0) + \varepsilon v(x) \le 0$ in $R$.

Besides. $u(x_0) - u(x_0) + \varepsilon v(x_0) = 0$. $\therefore$ $\frac{\partial u}{\partial \nu}(x_0) + \varepsilon \frac{\partial v}{\partial \nu}(x_0) \ge 0$.

$\Rightarrow \nu = \frac{x_0}{r}$. $\therefore$ $\frac{\partial u}{\partial \nu}(x_0) \ge 2\lambda \varepsilon r e^{-\lambda r^2} > 0$

### ii) Thm

If $u \in C^2(U) \cap C(\bar{U})$ and $c \equiv 0$ in $U$

where $U$ is connected.

i) For $Lu \le 0$ in $U$. $\exists x_0 \in U$. s.t. $u(x_0) = \max_{\bar U} u(x)$

Then $u \equiv$ const in $U$

ii) For $Lu \ge 0$ in $U$. $\exists x_0 \in U$. s.t. $u(x_0) = \min_{\bar U} u(x)$

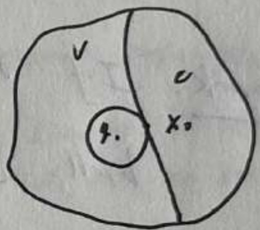
Then $u \equiv$ const. in $U$.

Pf. Denote $M = \max_{\bar U} u(x)$. $C = \{x \in U \mid u(x) = M\}$.

If $C \neq U$. set $V := \{x \in U \mid u < M\}$.

Since $U = C \cup V$. Choose $y \in V$. s.t. $d(y, C) < d(y, \partial U)$

with largest ball $B^{\circ}(y, r) \subseteq V$.




If $C \cap U = \emptyset$. $\exists x_0 \in C \cap U$.

s.t. $x_0 \in B(y, r)$. Apply Hopf Lemma.

$\therefore \frac{\partial u}{\partial \nu}(x_0) > 0$. contradict with $Du(x_0) = 0$

Cor. $u \in C^2(U) \cap C(\bar U)$. $c \ge 0$. $U$ is connected.

i) If $Lu \le 0$ in $U$. $\exists x_0 \in U$. s.t. $u(x_0) = \max_{\bar U} u(x) \ge 0$. Then $u \equiv$ const. in $U$.

ii) If $Lu \ge 0$ in $U$. $\exists x_0 \in U$. s.t. $u(x_0) = \min_{\bar U} u(x) \le 0$. Then $u \equiv$ const. in $U$.

Pf: There correspond the "$u(x_0) \ge 0$" part in Hopf's Lemma.

ii) is from : $\tilde u = -u$.

③ Harnack's Inequality:

**Thm.** If $u \geq 0$. $u \in C^2(U)$ solves $Lu = 0$ in $U$.

for $V \subset\subset U$. connected. Then $\exists$ Const. $C$

s.t. $\sup_V u \leq C \inf_V u$. $C = C(L, V)$

Pf: Only prove special case: $b^i \equiv c \equiv 0$. $a^{ij}$ are smooth

1°) Suppose $u > 0$. (other let $u = u + \varepsilon$. $\varepsilon \to 0$)

Let $v = \log u$. Suppose $V = B(x, r) \subset\subset U$.

prove: $\sup_V |Dv| \leq C$.

(Then $\forall x_1, x_2 \in V$. $v(x_1) - v(x_2) \leq r \sup_V |Dv| \leq C$

$\therefore u(x_1) \leq C u(x_2)$. $\Rightarrow \sup_V u \leq C \inf_V u$ )

2°) $\because Lu = 0$ $\therefore \sum a^{ij} v_{x_i x_j} + a^{ij} v_{x_i} v_{x_j} = 0$ in $U$.

Separate second-order term: $w \triangleq \sum a^{ij} v_{x_i} v_{x_j}$

$\therefore w = - \sum a^{ij} v_{x_i x_j}$

$$\begin{cases} w_{x_k x_l} = \sum_{i,j} (2a^{ij} v_{x_i x_k x_l} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j x_l}) + R \\ w_{x_i} = - \sum a^{lk} v_{x_i x_k x_l} + R. \end{cases}$$

where $|R| \leq \varepsilon |D^2 v|^2 + C(\varepsilon) |Dv|^2$

From $\sum\sum a^{ij} a^{lk} v_{x_i x_l} v_{x_j x_k} \geq \theta^2 |D^2 v|^2$. Choose $\varepsilon = \frac{\theta^2}{2}$

$\therefore - \sum a^{kl} w_{x_k x_l} + \sum b^k w_{x_k} \leq -\frac{\theta^2}{2} |D^2 v|^2 + c|Dv|^2$. $b^k = -2 \sum_l a^{kl} v_{x_l}$.

3°) Find $\zeta \in C^\infty(\mathbb{R}^n)$. $0 \leq \zeta \leq 1$. $\begin{cases} \zeta \equiv 1 \text{ in } V \\ \zeta \equiv 0 \text{ on } \partial U \end{cases}$

Let $z = \zeta^4 w$.

Since $z|_{\partial U} = 0$ $w \geq \theta |v|^2 > 0$.

$\therefore \exists x_0 \in U$. $z(x_0) = \max_{\bar{U}} z(x)$.

$\therefore \; \zeta w_{x_k} + 4\zeta_{x_k} w = 0$ at $x = x_0$.

Besides, at $x = x_0$, we have:

$0 \le -\sum a^{kl} z_{x_k x_l} + \sum b^k z_{x_k} \overset{\Delta}{=} \hat{L} z$.

Otherwise $\hat{L} z < 0$. by conti. $\therefore \hat{L} z < 0$ in $B(x_0, r)$.

Then $z \equiv z(x_0)$ in $B(x_0, r)$ $\therefore \hat{L} z \equiv 0$ contradict!

$\Rightarrow \; 0 \le \zeta^4 (-\sum a^{kl} w_{x_k x_l} + \sum b^k w_{x_k}) + \hat{R}$

where $|\hat{R}| \le C(\zeta^2 w + \zeta^3 |Dw|) = C\zeta^2 w$ (By $-\zeta w_{x_k} = 4\zeta_{x_k} w$)

Apply estimate in 2°):

$\zeta^4 |D^2 v|^2 \le C\zeta^4 |Dv|^2 + C\zeta^2 w$. From: $\theta |Dv|^2 \le w \le C |D^2 v|$

$\therefore \; z = \zeta^4 w \le C$ at $x = x_0$.

$\therefore \; |Dv|^2 \le C w \le C$.

4°) General case: Cover $V$ by balls $(B_n)$. □

(5) Eigenvalues:

① Symmetric Elliptic Operators:

· Consider $Lu = -\sum (a^{ij}(x) u_{x_i})_{x_j}$. $a^{ij} \in C^\infty(\bar{U})$.

Besides, $a_{ij} = a_{ji}$ $\therefore B[u, v] = (Lu, v) = (u, Lv) = B[v, u]$.

Thm:

For symmetric operator $L$.

i) Each eigen-value of $L$ is real.

ii) $\Sigma = (\lambda_n)$, where $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \cdots \le \lambda_n \le \cdots$

$\lambda_n \to \infty \ (n \to \infty)$

iii) There exists orthonormal basis $(w_n)$ of $L^2(U)$. s.t.

$w_k \in H_0^1(U)$. solves $\begin{cases} Lw_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$

<u>Remark</u>: $w_k \in C^\infty(U)$. What's more. if $\partial U \in C^\infty$. then

$w_k \in C^\infty(\bar{U})$

<u>Pf</u>: 1°) For $B[u,v] = (Lu, v)$. $\begin{cases} \theta \|u\|_{L^2}^2 \le B[u,u] \\ B[u,v] \le \|u\|_{L^2(U)} \|v\|_{L^2(U)} \end{cases}$

$\therefore L : L^2 \to L^2$ one-to-one.

$\therefore Lu = 0 \Leftrightarrow u \equiv 0$. Besides, $S = L^{-1}$ is BLD. cpt.

2°) Claim: $L$ is symmetric

For $f, g \in L^2(U)$. Suppose $\begin{cases} Lu = f & \text{in } U \\ Lv = g & \text{in } U \end{cases}$ $u, v \in H_0^1(U)$

$\therefore (Sf, g) = (u, g) = B[u, v]$

$= B[u, v] = (v, f) = (Sg, f)$

3°) Apply cpt. sym operator thm on $S$

Positive is from: $(Lu, u) \ge \theta |u|^2 > 0$.

$\therefore m : \min\limits_{\|u\|=1} (Lu, u) > 0$.

<u>Denition</u>: We call $\lambda_1 > 0$ principle eigenvalue of $L$.

Thm. (Variational principle for principle value)

i) $\lambda_1 = \min \{ B[u,u] \mid \|u\|_{L^2(U)} = 1,\ u \in H_0^1(U) \}$.

ii) $\exists\, w_1 \in H_0^1(U),\ \|w_1\|_{L^2(U)} = 1$. s.t. $\begin{cases} Lw_1 = \lambda_1 w_1 & \text{in } U \\ w_1 = 0 & \text{on } \partial U \end{cases}$

Besides, if $u$ is another solution, then $u = c w_1$ ( $\lambda_1$ is simple )

Pf: 1°) For $(w_k)$ is orthonormal basis in $L^2(U)$,

satisfies $\begin{cases} Lw_k = \lambda_k w_k & \text{in } U \\ w_k = 0 & \text{on } \partial U \end{cases}$

Claim: $( w_k / \lambda_k^{\frac{1}{2}} )$ is orthormal basis of $H_0^1(U)$ with inner product $B[\cdot,\cdot]$.

Since $\forall\, u \in L^2$, $u = \sum (w_k, u) u$.

$\therefore B[u, w_k/\lambda_k^{\frac{1}{2}}] = 0$, $\forall\, 1 \le k \Rightarrow u \equiv 0$.

2°) For $\|u\|_{L^2(U)} = 1$, since $u = \sum (w_k, u) w_k$

$\therefore \sum |(w_k, u)|^2 = 1$. $\therefore B[u,u] = \sum \lambda_k |(w_k, u)|^2 \ge \lambda_1$

"=" holds when $u = w_1$.

3°) Claim: For $u \in H_0^1(U)$, $\|u\|_{L^2(U)} = 1$.

$\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases} \iff B[u,u] = \lambda_1$

Denote $d_k = (w_k, u)$. $\therefore \sum d_k^2 = 1$.

If $B[u,u] = \lambda_1 \Rightarrow \lambda_1 \sum d_k^2 = \sum \lambda_k d_k^2$

$d_k = 0$ if $\lambda_k > \lambda_1$.

$\therefore u = \sum_1^{m} d_k w_k$, where $Lw_k = \lambda_1 w_k$.

4°) Prove: For $u \in H_0^1(U)$ solves $\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

$u \not\equiv 0$. Then $u > 0$ or $u < 0$ in $U$.

Lemma: $u \in W^{1,p}(U) \Leftrightarrow u^+, u^- \in W^{1,p}(U)$. Besides, we have:

$$Du^+ = \begin{cases} Du, & \text{in } \{u > 0\} \\ 0, & \text{on } \{u \le 0\}. \end{cases} \quad Du^- = \begin{cases} 0, & \text{in } \{u \ge 0\} \\ -Du, & \text{on } \{u < 0\}. \end{cases} \quad \text{a.e.}$$

Pf: $F_\varepsilon(r) = (\sqrt{r^2+\varepsilon^2} - \varepsilon)\chi_{\{r \ge 0\}} \in C^1(\mathbb{R})$

Besides, $F_\varepsilon'(r) \in L^\infty(\mathbb{R}^1)$. $F_\varepsilon(0) = 0$.

By Chain Rule: $\int_U F_\varepsilon(u) \frac{\partial \phi}{\partial x_i} = -\int_U F_\varepsilon'(u) \frac{\partial u}{\partial x_i} \phi$

By DCT. Let $\varepsilon \to 0^+$. Since $F_\varepsilon(u) \to |u| \chi_{\{u \ge 0\}} = u^+$

$\therefore \int_U u^+ \frac{\partial \phi}{\partial x_i} = -\int_U \frac{\partial u}{\partial x_i} \phi \chi_{\{u \ge 0\}}$

Apply on $\tilde{u} = -u$, obtain $u^-$ case.

$\Rightarrow$ WLOG. $\|u\|_{L^2(U)}^2 = 1 = \int_U (u^+)^2 + (u^-)^2$. $(u^+ u^- = 0)$

$\therefore \lambda_1 = B[u,u] = B[u^+,u^+] + B[u^-,u^-] \ge \lambda_1 (\|u^+\|_2 + \|u^-\|_2) = \lambda_1$

$\therefore \begin{cases} B[u^+,u^+] = \lambda_1 \|u^+\|_{L^2(U)} \\ B[u^-,u^-] = \lambda_1 \|u^-\|_{L^2(U)} \end{cases} \Rightarrow u^+, u^-$ solves $\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

$\because Lu^+ = \lambda_1 u^+ \ge 0$ $\therefore$ By SMP: $u^+ > 0$ in $U$ or $u^+ \equiv 0$ in $U$

Similiar for $u^-$. $\therefore u^+ > 0$ or $u^- > 0$ in $U$

5°) For $\tilde{u}$ is another solution. $\therefore \tilde{u} > 0$ or $< 0$ in $U$

$\therefore \int \tilde{u} \ne 0$. Suppose $\int \tilde{u} = c \int u$.

$\therefore \int \tilde{u} - cu = 0$. $\because \tilde{u} - cu$ is another solution.

$\therefore \tilde{u} \equiv cu$. Otherwise $\int \tilde{u} - cu > 0$ or $< 0$.

Thm. ( Courant minimax Principle )

For $\Sigma_L = (\lambda_k)$. We have: $\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u]$

$\Sigma_{k-1}$ is the collection of

all $(k-1)$-dimension subspace of $H_0^1(U)$.

Pf: Denote $A = L^{-1} : L^2(U) \to L^2(U)$. cpt BLO.

1°) Prove: $\lambda_k = \sup_{S \in E_{k-1}} \inf_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u]$.

$E_{k-1}$ collects all $(k-1)$-dimension subspaces of $L^2(U)$

Suppose $(e_k)$ is the correspond eigenfunctions.

Since $u = \Sigma (e_k, u) e_k$. $\therefore B[u,u] = \Sigma \lambda_k |(u, e_k)|^2$

2°) $\forall S \in E_{k-1}$. $\exists u_0 \in S^\perp \cap \text{span}\{e_i\}_1^k$. $u_0 = \sum_1^k \tau_i e_i$

$\therefore \inf B[u,u] \le B[u_0,u_0] = \sum_1^k \lambda_i a_i^2 \le \lambda_k$. $(\sum_1^k \tau_i^2 = 1)$

$\therefore \sup \inf B[u,u] \le \lambda_k$

3°) Pick $S_0 = \text{span}(e_i)_1^{k-1}$ $\therefore \inf_{u \in S_0^\perp} B[u,u] \ge \lambda_k$

$\therefore \sup \inf B[u,u] \ge \lambda_k$.

4°) Since $H_0^1(U) \subset\subset L^2(U)$. $\therefore \lambda_k \ge \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S \\ \|u\|_{L^2}=1}} B[u,u]$

conversely. Choose $\Sigma_{k-1}^0 = \text{span}\{e_i / \frac{1}{\lambda_i}\}_1^{k-1}$

② Nonsymmetric Case:

· For $Lu = -\Sigma a^{ij} u_{x_i x_j} + \Sigma b^i u_{x_i} + cu$. $a^{ij}, b^i, c \in C^\infty(\bar{U})$

$U$ is open. bounded. connected. $\partial U \in C^\infty$. $a^{ij} = a^{ji}$

$c \ge 0$ in $U$. for $u \in H_0^1(U)$.

**Thm.** (Principle eigenvalue)

i) There exists $\lambda_1 \in \Sigma_L$. $\lambda_1 \in \mathbb{R}$. st. $\forall \lambda \in \Sigma_L$.

$Re(\lambda) \geq \lambda_1$. Besides. $\lambda_1$ is simple

ii) There exists a corresponding eigenfunc. $w_1$

st. $w_1 > 0$ in $U$.

**Thm.** For principle eigenvalue $\lambda_1$. We have:

$$\lambda_1 = \sup \{ \inf_{x \in U} \frac{Lu(x)}{u(x)} \mid u \in C^\infty(\bar{U}), u > 0 \text{ in } U. u = 0 \text{ on } \partial U \}.$$

**Pf:** 1°) $\exists w_1 \in H_0^1(U)$. st. $Lw_1 = \lambda_1 w_1$.

Note that $\exists u_n \in C^\infty(\bar{U}) \to w_1$ in $H^1$.

$\therefore \sup \inf \frac{Lu}{u} \geq \inf \frac{Lu_n}{u_n} \to \lambda_1$.

2°) Prove: $\lambda_1$ is principle eigenvalue of $L^*$

Suppose $\lambda_1^*$ is. correspond $w_1^* > 0$

$\therefore (L^* w_1^*, w_1) = \lambda_1^* (w_1^*, w_1) = (w_1^*, Lw_1)$

$= \lambda_1 (w_1^*, w_1) \quad \therefore \lambda_1^* = \lambda_1$

3°) Conversely. prove:

$\inf \frac{Lu}{u} \leq \lambda_1$. for $\forall u \in C^\infty(\bar{U})$. $\begin{cases} u > 0 \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases}$

$\Leftrightarrow \inf_{x \in U} Lu - \lambda_1 u / u \leq 0 \Leftrightarrow \inf_{x \in U} Lu - \lambda_1 u \leq 0$

It follows from $(w_1^*, Lu - \lambda_1 u) = 0$.

But $w_1^* > 0$. $\therefore \inf Lu - \lambda_1 u \leq 0$.