SLE(4) and GFF

(1) Function Spaces:

Def: For D a simply connected domain.

i) $\mathcal{D}(D) =: C_c^\infty(D)$ with u.c.c topo.

ii) $\Sigma_D(\mu) =: \int_{D^2} G_D(x,y)\, d\mu(x) d\mu(y)$ for Borel measure μ.

$M_D =: \{\mu \text{ is Borel} \mid \Sigma_D(\mu) < \infty\}$.

prop. For $\phi: D \xrightarrow{\sim} D_0$. conformal isomorphism.

i) $f \mapsto f\circ\phi^{-1}$. is linear homeo of $\mathcal{D}(D_0) \to \mathcal{D}(D)$.

ii) For $u_0 \in \mathcal{D}^*(D_0)$. $u \in \mathcal{D}^*(D)$. s.t. $u(\ell) = u_0((\ell\circ\phi)|\phi'|^2)$. Then $u_0 \mapsto u$ is a linear homeo of $\mathcal{D}^*(D_0) \to \mathcal{D}^*(D)$.

prop. (Conformal Invariance)

$\phi: D_0 \xrightarrow{\sim} D$. conformal of proper simply connected domain. μ_0 is Borel measure on D_0.

$\Rightarrow \Sigma_D(\mu_0\circ\phi^{-1}) = \Sigma_{D_0}(\mu_0)$

Def: For bdd domain D. and $f, g \in \mathcal{D}(D)$.

$\langle f, g\rangle_{H'(D)} = \frac{1}{2}\int_D \langle \nabla f, \nabla g\rangle\, dx$. inner product

prop. (Poincaré Inequality)

$\|f\|_{L^2(D)} \le \sqrt{R}\, \|f\|_{H^1(D)}$. where $supp(f) \subseteq B(0,R)$.

Pf: $|f(re^{i\theta})|^2 = |\int_r^R \langle e^{i\theta}, \nabla f(te^{i\theta})\rangle dt|^2$

$\overset{\text{Bochner}}{\underset{\text{inequ:}}{\le}} \int_r^R |\nabla f(te^{i\theta})|^2 dt$

Replace in $\|f\|^2_{L^2(D)} = \int_0^{2\pi} \int_0^R |f(re^{i\theta})|^2 r dr d\theta$.

Rmk: Complete $\mathcal{D}(D)$ in $L^2(D)$. with the norm $\|\cdot\|_{H^1(D)}$ $\Rightarrow$ Obtain Space $H_0^1(D)$.

$H^{-1}(D) \triangleq \{ \varphi \in \mathcal{D}^*(D) \mid |\varphi(e)| \le c\|e\|_{H^1}.\ \forall e \in \mathcal{D}(D)\}$.

prop. $\phi: D_0 \xrightarrow{\sim} D$. conformal between bdd domains.

$\Rightarrow f \mapsto f \circ \phi^{-1}$ is also isometry of $H_0^1(D_0) \xrightarrow{\sim} H_0^1(D)$

Pf: $\int_D |\nabla f|^2 \circ \phi\, |\phi'|^2 dx = \int_{D_0} |\nabla f|^2 dx$.

prop. (Orthogonal Decomposition)

For $U \underset{\text{open}}{\subseteq} D$. $H_{supp} = H_0^1(U) \subseteq H_0^1(D)$. and

$H_{harm} = \{ f \in H_0^1(D) \mid f$ is harmonic in $U\}$.

$\Rightarrow H_0^1(D) = H_{supp} \oplus H_{harm}$.

Pf: 1) $\langle f, g\rangle_{H^1(D)} = -\frac{1}{2}\int_D f \cdot \Delta g \Rightarrow H_{supp} \perp H_{harm}$.

2) set $f_0 = P_{H_{supp}} f$. $g_0 = f - f_0 \in H_{harm}$.

Denote: For $(\lambda_n)\uparrow \subset \mathbb{R}^+$, $(f_n) \subset C^\infty(D)$ s.t.

$-\frac{1}{2}\Delta f_n = \lambda_n f_n$ and (f_n) is o.n.b in $H_0^1(D)$.

Set $e_n = \lambda_n^{\frac{1}{2}} f_n \Rightarrow (e_n)$ is o.n.b in $L^2(D)$

Prop. D is bdd domain. For $t \geq 0$, $x \in D$.

$$\Rightarrow f_n(x) = e^{\lambda_n t}\int_D p_D(t,x,y) f_n(y)\,dy = \lambda_n \int_D G_D(x,y) f_n(y)\,dy$$

Pf: WLOG. suppose $D \subseteq (0,1)^2$.

Set $M_t = e^{\lambda_n t} f_n(B_t)\, I\{t < T(D)\}$ $t \geq 0$.

Prove M_t is conti. mart. & it's c.lim

by Itô) $\Rightarrow$ Apply Optional Stopping Thm.

Rmk: Note by Markov property:

$$p_D(t,x,y) = \int_D p_D(t/2,x,z)\, p_D(\tfrac{t}{2},z,y)\,dz$$

$$\Rightarrow p_D(t,x,x) = \|p_D(t/2,x,\cdot)\|^2_{L^2(D)}$$

Since $\int_D p_D(t,x,y)\, e_n(y) = e^{-\lambda_n t} e_n(x)$

$$\Rightarrow \begin{cases} p_D(t,x,y) = \sum e^{-\lambda_n t} e_n(x) e_n(y). \\ G_D(x,y) = \sum e_n(x)e_n(y)/\lambda_n = \sum f_n(x) f_n(y). \end{cases}$$

So: $\int_D p_D(t,x,x)\,dx = \sum e^{-\lambda_n t}$.

$$\int_0^\infty \int_D t^{\alpha-1} p_D(t,x,x)\,dx\,dt = \Gamma(\alpha) \sum \lambda_n^{-\alpha}.\ \alpha > 0.$$

$\Rightarrow$ The second id. is increasing on D.

Cor. $\forall D \subseteq (0,\pi)^2$. $\alpha > 1$. $\sum \lambda_n^{-\alpha} \leq \sum_{m,n\geq 1} (m^2+n^2)^{-\alpha}$

Pf: $(\sin mx \sin ny)$ is o.n.b in $L^2(0,\pi)^2$

Prop: D is bdd domain. If $\mu \in M_D$. $f \in H_0^1(D)$.

Then: $f \mapsto \mu(f)$ is BLO on $H_0^1(D)$.

with norm $\Sigma_D(\mu)^{\frac{1}{2}}$

Pf: For $f = \sum a_n f_n \in H_0^1(D)$.

Note that $\Sigma_D(\mu) = \sum_n \int_{D\times D} f_n(x) f_n(y) \, d\mu(x) d\mu(y)$

$$= \sum \mu(f_n)^2$$

$$\Rightarrow \mu(f) = \sum a_n \mu(f_n)$$

$$\le (\sum a_n^2)^{\frac{1}{2}} (\sum \mu(f_n)^2)^{\frac{1}{2}}$$

$$= \|f\|_{H_0^1(D)} \Sigma_D(\mu)^{\frac{1}{2}}$$

(2) Gaussian Free Field:

① Motivation:

To extend the def of Gaussian variables:

For $\alpha_k \overset{i.i.d}{\sim} N(0,1)$.

i) For $x \in \mathbb{R}^n$. $h = \sum_1^n \alpha_k e_k$. (e_k) is o.n.b. of $\mathbb{R}^n$.

$\Rightarrow (h, x) \sim N(0, \|x\|^2)$. $\mathrm{cov}((h,x),(h,y)) = (x,y)$.

ii) For $f = \sum \beta_n f_n \in H_0^1(D)$. $h = \sum \alpha_k f_k$. where (f_n) is o.n.b. of $H_0^1(D)$.

$\Rightarrow \langle h, f \rangle_{H_0^1(D)} = \sum \alpha_n \beta_n \sim N(0, \|f\|^2_{H_0^1(D)})$.

$\mathrm{cov}(\langle h,f\rangle_{H_0^1(D)}, \langle h,g\rangle_{H_0^1(D)}) = \langle f, g\rangle_{H_0^1(D)}$

Rmk: We extend $\mathbb{R}^n$ to $H_0^1(D)$ by replacing (e_k) with (f_k).

$\Rightarrow$ GFF is family of random variables $\langle h, f\rangle_{H_0^1(D)}$ index by $H_0^1(D) \ni f$ with zero mean and cov given by $\langle \cdot, \cdot \rangle_{H_0^1(D)}$

Def: Fix D Greenian domain.

i) r.v. $I \in \mathcal{D}^*(D)$ is GFF in D with zero boundary values if $I(e)$ is Gaussian variable with zero-mean and var $\Sigma_D(e)$. $\forall e \in \mathcal{D}(D)$.

ii) For f on δD, bdd measurable, denote u is its harmonic extension in D

$\tilde{I}$, r.v. in $\mathcal{D}^*(D)$ is GFF with boundary value f if $\tilde{I} = u + I$.

Rmk: $\tilde{I}(e)$ is Gaussian variable with mean $H_D(f, e) = \int_{\delta D \times D} f(y) h_D(x, dy) e(x) dx$ and variance $\Sigma_D(e)$, for $e \in \mathcal{D}(D)$.

Thm. D is bdd domain. There exists a unique Borel p.m. on $\mathcal{D}^*(D)$, which is law of GFF on D with zero boundary values.

Pf: For $(X_n) \overset{i.i.d}{\sim} N(0,1)$. Set $S = \sum_n X_n^2/\lambda_n^2$.

$\Rightarrow \mathbb{E}(S) = \sum \lambda_n^{-2} < \infty$. $\therefore$ $S < \infty$ a.s.

Set $a_n = \int_D f_n \varphi$. $\varphi = \sum \lambda_n a_n f_n$. $\varphi \in D(D)$.

Def: $I(\varphi) =: \sum a_n X_n$

(i.e. $I(f_n) =: X_n/\lambda_n$)

Note that $\Sigma_D(\varphi) = \sum a_n^2$.

With $|I(\varphi)| \le (\sum \lambda_n^2 a_n^2)^{\frac{1}{2}} S^{\frac{1}{2}}$

$= \|\varphi\|_{H_0^1(D)} \sqrt{S} < \infty$ a.s.

$\Rightarrow I(\varphi) \sim N(0, \Sigma_D(\varphi))$. $I \in D^*(D)$

Uniqueness is from unique finite dimension law determined.

③ Properties:

Prop. $\varphi : D \xrightarrow{\sim} D'$. conformal between 2 Greenian domain. If I is GFF with zero b.c. on D Then $I \circ \varphi^{-1}$ is GFF with zero b.c on D'

If D is simply connected. I is GFF with b.c $\eta = f$ on D. Then $I \circ \varphi^{-1}$ is GFF with b.c. $\eta = f \circ \varphi^{-1}$ on D'.

Denote: $\mathcal{G}_2$ is CLS in L^2 of zero-mean Gaussian variables

Prop. D is bdd domain. $\mathcal{I}$ is GFF on D with zero b.c. $\Rightarrow$ There exists unique isometry : $\widetilde{\mathcal{I}} : H^{-1}(D) \to \mathcal{G}_0$. s.t. $\widetilde{\mathcal{I}}(e) = \mathcal{I}(e)$ for $\forall e \in \mathcal{D}(D)$.

Pf: In last prop. of (1) $\Rightarrow \|e\|^2_{H^{-1}(D)} = \Sigma_D(e)$

$\Rightarrow \mathcal{D}(D)$ is dense in $H^{-1}(D)$.

Then by unique extend to $H^{-1}(D)$.

Rmk: For $u \in H^{-1}(D)$. $u = \sum \lambda_n u(f_n) f_n$. in $H^{-1}(D)$.

where $u(f_n) = \int_D f_n u$.

Consider $(Y_n) \overset{i.i.d}{\sim} N(0,1)$. Then we have,

$\widetilde{\mathcal{I}}(u) = \sum u(f_n) Y_n$. a.s. given by $Y_n = \lambda_n \widetilde{\mathcal{I}}(f_n)$. by Doob's L^2 inequality.

Prop. (Extension)

$D \subsetneq \widetilde{D}$. simply connected. proper domain. If $\mathcal{I}$ is GFF on D with zero b.c. Then $\mathcal{I}$ extends uniquely to r.v. $\bar{\mathcal{I}} \in \mathcal{D}^*(\widetilde{D})$. s.t. $\bar{\mathcal{I}}(e) \sim N(0. \Sigma_D(e))$. for $\forall e \in \mathcal{D}(\widetilde{D})$

Pf: WLOG. set $\widetilde{D} = \mathbb{D}$. Set $\bar{f}_n = \begin{cases} f_n. & z \in D \\ 0. & z \in \mathbb{D}/D \end{cases}$

$\Rightarrow (\bar{f}_n)$ is o.n.b. in $H_0^1(\mathbb{D})$.

Set $e_n = e(f)$. $= \int_D f_n e = \int_{\mathbb{D}} \bar{f}_n e$.

$\Rightarrow \ell = \sum \lambda_n \ell_n f_n$. for $\forall \ell \in H_0^1(\mathbb{D})$

Def: $\bar{\Gamma}(\ell) = \sum \ell_n Y_n$. as before.

Note: $\bar{\Gamma}(\ell) \sim N(0, \sum \ell_n^2) = N(0, \Sigma_D(\ell))$

$$|\bar{\Gamma}(\ell)| \le (\sum \lambda_n^2 \ell_n^2)^{\frac{1}{2}} (\sum Y_n^2/\lambda_n^2)^{\frac{1}{2}}$$

$$\overset{\text{Bessel}}{\le} \|\ell\|_{H_0^1(\mathbb{D})} \cdot S^{\frac{1}{2}} < \infty.$$

$\Rightarrow \bar{\Gamma}$ has desired properties.

Uniqueness: $\Gamma^*(\ell) = \tilde{\Gamma}(\ell I_D) = \bar{\Gamma}(\ell)$.

for $\forall \ell \in D^*(\mathbb{D})$. $\Rightarrow \Gamma^* = \bar{\Gamma}$ on $H_0^1(\mathbb{D})$.

<u>prop.</u> (Markov Property)

$D \subsetneq \tilde{D}$. proper simply connected domain. For Γ^* is GFF on $\tilde{D}$. Then: Γ^* has unique decomposition $\Gamma^* = \bar{\Gamma} + \phi$. a.s. as sum of indept r.v. in $D^*(\tilde{D})$. St. ϕ is harmonic on D and $\bar{\Gamma}|_D$ is GFF on D with zero b.c.

<u>Pf</u>: Use the notations above. Set $\tilde{D} = \mathbb{D}$.

Def: $Y_n^* = \tilde{\Gamma}^*(\frac{-1}{2}\Delta f_n)$

$\bar{\Gamma}(\ell) = \sum \ell_n Y_n^*$. for $\ell = \sum \lambda_n \ell_n f_n \in D(\mathbb{D})$

$\Rightarrow \bar{\Gamma}$ is GFF with zero b.c. on D.

Set $\phi = \Gamma^* - \bar{\Gamma}$. Check: $\phi(\Delta\ell) = 0$. $\forall \ell \in D(D)$.

<u>Rmk</u>: Or directly, Note that $\Gamma^* \in D^*(\tilde{D}) \subset H_0^1(\tilde{D})$.

Suppose (f_n^1), (f_n^2) are o.n.b. of H_{harm}, H_{supp} and $(X_n^1) \cup (X_n^2) \overset{\text{i.i.d}}{\sim} N(0,1) \Rightarrow \Gamma^* = \sum X_n^1 f_n^1 + \sum X_n^2 f_n^2$.

(3) Angle Marts:

Def: i) $\varsigma_0(\pm x) = \pm 1$ for $x \in (0,\infty)$. left on $\partial\mathbb{H}$. and $\varsigma_0(0) = \varsigma_0(\infty) = 0$.

ii) σ_0 is harmonic extension of ς_0 in $\mathbb{H}$: $\sigma_0(z) = \int_{\partial\mathbb{H}} \varsigma_0(x) h_{\mathbb{H}}(z, dx)$

$= 1 - 2\arg(z)/\pi$.

Next fix γ is SLE(4) path. associated with $(g_t(z))_{t<\tau(z)}$ and $(\xi_t)_{t\geq 0}$.

Def: i) $\varsigma_t(x) = \varsigma_0(g_t(x) - \xi_t)$ on ∂H_t

ii) $\sigma_t(z) = \sigma_0(g_t(z) - \xi_t)$ on H_t.

Where $H_t = \{z \in \mathbb{H} \mid \tau(z) > t\}$.

prop. $\forall z \in \mathbb{H}$. $(\sigma_t(z))_{t<\tau(z)}$ is c.l.m. and $\tau(z) = \infty$ a.s.

Besides. $\forall w \in \mathbb{H}/\{z\}$. $(\sigma_t(z)\sigma_t(w) + 4G_{H_t}(z,w)/\pi)_{t<\tau(z)\wedge\tau(w)}$ is also. c.l.m

prop. (ch.f).

Set $\lambda = \sqrt{\pi/4}$. $\gamma^* = \{z \in \mathbb{H} \mid \tau(z) < \infty\}$. D^-, D^+ are left and right components of $\mathbb{H}/\gamma^*$.

Then: for all $\ell \in D(\mathbb{H})$. we have

$$e^{i\lambda \mathcal{M}_{\mathbb{H}}(S_0,\ell) - \frac{\Sigma_{\mathbb{H}}(\ell)}{2}} = \mathbb{E}\left(e^{i\lambda\ell(D^+) - \frac{\Sigma_{D^+}(\ell)}{2}} e^{-i\lambda\ell(D^-) - \frac{\Sigma_{D^-}(\ell)}{2}}\right)$$

__Thm.__ (Schramm - Sheffield)

As notated above. Condition on γ. Let Γ^-, Γ^+ be two indept GFF on D^-, D^+, with zero b.c.

Denote $\bar{\Gamma}^{\pm}$ is extension of $\Gamma^{\pm}$ in $D^{\pm}(\mathbb{H})$. Then:

$\Gamma =: (\bar{\Gamma}^+ + \lambda I_{D^+}) - (\bar{\Gamma}^- + \lambda I_{D^-})$ is GFF on $\mathbb{H}$.

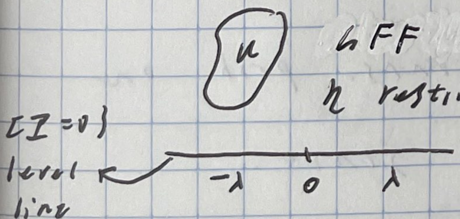
with b.c.: $\eta = \pm\lambda$ on the right and left half-line respectively.

__Pf:__ From prop. above: $\mathbb{E}(e^{i\Gamma(\ell)}) = e^{i\mathcal{M}_{\mathbb{H}}(\lambda S_0,\ell) - \frac{\Sigma_{\mathbb{H}}(\ell)}{2}}$

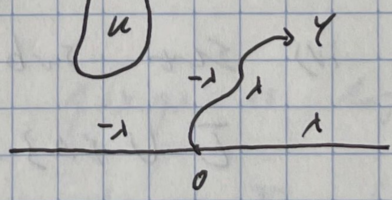
__Cor.__ For $\lambda^* = \frac{\pi}{2}$. $f_t = g_t - \xi_t$. $\eta(z) = \lambda^* - \frac{2\lambda^*}{\pi}\arg(z)$.

$U \underset{\text{open}}{\subseteq} \mathbb{H}$. $\tau =: \inf\{t \geq 0 \mid \gamma_t \in U\}$. Γ is GFF on $\mathbb{H}$. Then: $\Gamma \circ f_{t\wedge\tau} + \eta \circ f_{t\wedge\tau} \overset{d}{\sim} \Gamma + \eta$.

if restricted on U.



GFF in $\mathbb{H}$ + η restricted in U. $\overset{d}{\sim}$



GFF in $\mathbb{H}/\gamma[0,t\wedge\tau]$ + $\eta \circ f_{t\wedge\tau}$ restrict in U.

__Rmk:__ i) η is harmonic in $\mathbb{H}$ with b.c. $\pm\lambda$ on $\mathbb{R}_{\pm}$

ii) $\eta \circ f_{t\wedge\tau}$ is harmonic in $\mathbb{H}/\gamma[0,t\wedge\tau]$ with b.c. $\pm\lambda$ on $\mathbb{R}^{\pm}$ and right/left side of γ.

iii) γ is ridge line of GFF with jump 2λ.