Decision Theory and Bayesian Inference

· It's a unifying framework for theory of statistic including estimation and testing.

(1) Definition:

· Suppose "a" is an action, $a \in A$. the action space.

⇒ The choose of action "a" depends on:

{ i) Observations of r.v.: Data X where $X \in S$. sample space.

ii) d. the decision function. i.e. $d: S \longrightarrow A$. $d(\vec{X}) = a$

⇒ The prob dist of $\vec{X}$ depends on the parameter θ, called state of nature. $\theta \in \Lambda$. the space of parameters

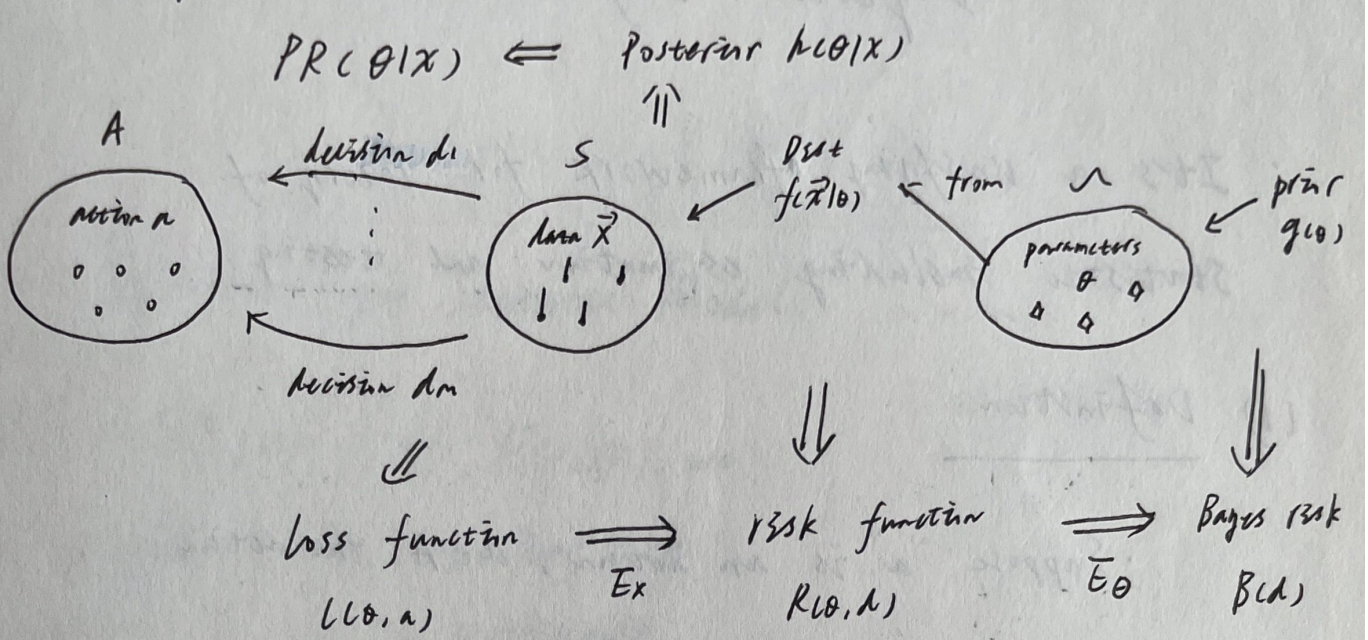
⇒ Then we can define a loss function $l(\theta, a)$ on $\Lambda \times A$. Since $a = d(\vec{X})$ ∴ $l(\theta, a) = l(\theta, d(\vec{X}))$

The expected loss of $d(\vec{X})$ is the risk function:

$R(\theta, d) = E_X(l(\theta, d(\vec{X})))$ (It depends on θ)

⇒ Our aim is to minimize $R(\theta, \delta)$ by

choosing a good decision $\delta(\vec{x})$!



e.g. estimate $\gamma(\theta)$. where $X_k \sim f(x|\theta)$. i.i.d. get data $\vec{x}$.

We choose $L(\theta, \delta(\vec{x})) = [\gamma(\theta) - \delta(\vec{x})]^2$. quadratic loss func

(2) Minimization:

· Difficulties of minimizing $R(\theta, \delta)$

i) $R(\theta, \delta)$ depend on unknown θ.

ii) For different θ_1, θ_2. It may happen:

$R(\theta_1, \delta_1) > R(\theta_1, \delta_2)$

$R(\theta_2, \delta_1) < R(\theta_2, \delta_2)$

. how to choose?

① Def of minimax rule:

· Consider the worst case: $\sup_{\theta \in \Theta} R(\theta, \delta)$

δ^* is the minimax rule. if $\sup_{\theta \in \Theta} R(\theta, \delta^*)$

$= \inf_{\delta} \sup_{\theta \in \Theta} R(\theta, \delta)$, δ^* may not exist!

Remark: It's very conservative to consider the worst case which isn't likely to occur.

② Bayesian Rule:

· We assume $\theta \in \Theta$. is random. with a prior dist. Then the Bayesian risk of decision function δ is $B(\delta) = E_{\theta}(R(\theta, \delta))$

Def: Bayesian rule is a decision func. δ^{**} attain the $\min_{\delta} (B(\delta))$

Remk: It can be interpreted as average of risk with weight form.

⇒ Posterior Analysis:

A method for finding Bayesian Rule:

· Suppose $g(\theta)$ prior dist of θ. $f_{X|\theta}(x)$ condition θ of X.

⇒ $f_{X,\theta} = g(\theta) f_{X|\theta}$. ⇒ Sum/Integration: f_X.

We obtain: $h_{\theta|X} = f_{X,\theta} / f_X$. posterior dist of θ.

Def: Posterior risk: $E_{\theta|X}(L(\theta, d(x))) = \rho(R(\theta|x))$

Remark: The observed data $X = x$ updates the p-risk!

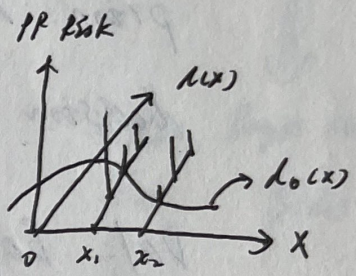
Thm. If $d_0(x)$ is a decision func. minimizes the posterior risk for each x. Then d_0 is Bayesian Rule.

Pf: $B(d) = E_\theta(R(\theta, d)) = E_\theta(E_x(L(\theta, d(x)|\theta))$

$$= \int \left[\int L(\theta, d(x)) f_{x|\theta}(x)\, dx \right] g_\theta(\theta)\, d\theta$$

$$= \int \left[\int L(\theta, d(x)) h_{\theta|x}(\theta)\, d\theta \right] f_x(x)\, dx$$

$$= \int E_{\theta|x}(L(\theta, d(x)) f_x(x)\, dx$$



Then $B(d)$ is minimized !

⇒ **Algorithm:**

1°) Caculate $h(\theta|x)$. for each x.

2°) Caculate $E_{\theta|x}(L(\theta, d(x)))$ for each x.

3°) Find $d(x) \in A$ minimizes every $PR(\theta|x_0)$. fixed x_0.

(3) Application of Decision Theory:

Estimation

①
- Action space $A \longrightarrow$ Parameter Space Ω.
- Decision Func. $d(x) \longrightarrow$ estimator of θ.
- $L(\theta, d(x)) = [\theta - d(x)]^2$ or $|\theta - d(x)|$

Thm. i) $E_{\theta|x}((\theta-\hat{\theta})^2|x) = Var_{\theta|x}(\theta|x) + [E_{\theta|x}(\theta|x) - \hat{\theta}]^2$

Then $\hat{\theta} = E_{\theta|x}(\theta|x)$ is the best predictor.

ii) $E_{\theta|x}(|\theta-\hat{\theta}||x)$ has the best predictor: median.

pf: For ii) $E_{\theta|x}(|\theta-\hat{\theta}||x) = \int |\theta-\hat{\theta}| h_{\theta|x}(\theta) d\theta$

$$= \int_{\hat{\theta}}^{\infty} (\theta-\hat{\theta}) h_{\theta|x} d\theta + \int_{-\infty}^{\hat{\theta}} (\hat{\theta}-\theta) h_{\theta|x} d\theta \triangleq f(\hat{\theta})$$

$$\frac{\partial f}{\partial \hat{\theta}} = 0 \Rightarrow -\int_{\hat{\theta}}^{\infty} h_{\theta|x}(\theta) d\theta + \int_{-\infty}^{\hat{\theta}} h_{\theta|x} d\theta = 0$$

$\therefore \hat{\theta}$ = median of $h_{\theta|x}$!

② **def:** i) d_1, d_2 two decision Functions $\in A$.

says d_1 dominates d_2 : $R(\theta, d_1) \le R(\theta, d_2)$, $\forall \theta \in \Theta$

ii) d: a decision Func. d is admissable. If d is not strictly dominated by any other decision Func.

Thm. i) If $\wedge$ is discrete. d^* is Bayesian rule w.r.t. prior pmf $g(\theta)$. where $g(\theta) > 0$. $\forall \theta$.

ii) If $\wedge$ is dense. d^* is Bayesian rule w.r.t prior pdf $g(\theta)$. $g(\theta) > 0$. $\forall \theta$. $R(\theta, d)$ is conti. of θ. $\forall d$.

Then d^* is admissible.

Remark: It claims the relation between Bayesian rule and admissible.

Pf: 对于 $\exists \delta'$. s.t. $R(\theta, \delta^*) \geq R(\theta, \delta')$. $\forall \theta$.

$\exists \theta_0, h$. $\theta \in (\theta_0 - h, \theta_0 + h) \Rightarrow R(\theta, \delta^*) > R(\theta, \delta') + \varepsilon$.

Then check: $B(\delta^*) - B(\delta') > 0$. contradict!

(4) Bayesian View for prob.:

(Personal Opinion)

① Bayesian prob is personal. It varies from person to person, embodying the beliefs of person. (Subjective)

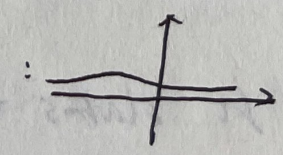
② Bayesian rule describes the prob. evolves with experience.

⇒ Difference between "B" and "F" Approach:

① Point Estimation:

- F: θ is fixed, unknown, not random → likelihood $L(\theta|x)$ → maximal: Find MLE.
- B: θ has prior dist $g(\theta)$ → Posterior dist $h(\theta|x)$ → Centering: Find $E(\theta|x)$

Remark: From $h(\theta|x) = \dfrac{f(x|\theta)g(\theta)}{\int f(x|\theta)g(\theta)d\theta}$ ∴ $h(\theta|x) \propto f(x|\theta)g(\theta)$

⇒ If θ is uniform:  almost const.

Then $g(\theta)$ has little effect. $h(\theta|x) \propto f(x|\theta) = L(\theta|x)$

② Interval Estimation:

- Frequentist: $P(\theta \in [\theta_L(\vec{X}), \theta_U(\vec{X})] \mid \theta) = 1-\alpha$. $\vec{X}$ is random, θ fixed.
  If $\vec{x}$ is observed data, then $P(\theta \in [\theta_L(\vec{x}), \theta_U(\vec{x})]) = 0$ or 1.
- Bayesian: $P(\theta \in [\theta_L(\vec{x}), \theta_U(\vec{x})] \mid \vec{X} = \vec{x}) = 1-\alpha$. θ is random.
  $\vec{x}$ is the observed data, fixed.

③ Testing:

- Frequentist: prob of Type I, II error.
- Bayesian: After observing data, compare posterior prob.

(5) Bayesian Inference for Normal Dist:

- Suppose $X|\mu \sim N(\mu, \sigma^2)$, $\mu \sim N(\mu_0, \sigma_0^2)$, σ is known.

$\Rightarrow$ Posterior dist of μ is $N(\mu_1, \sigma_1^2)$, where

$$\mu_1 = \frac{\tau_0}{\tau+\tau_0}\mu_0 + \frac{\tau}{\tau+\tau_0}x, \quad \sigma_1^2 = \frac{1}{\tau+\tau_0}$$

$$\tau_0 = \frac{1}{\sigma_0^2}, \quad \tau = \frac{1}{\sigma^2}.$$

Pf: Since the const. is for normalization.
We just care the ratio (about "μ"):

$$h(\mu|x) \propto f(x|\mu)g(\mu) \propto e^{-\frac{1}{2\sigma^2}(x-\mu)^2 - \frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}$$

$$= e^{-\frac{1}{2}\left(\mu^2\underbrace{\left[\frac{1}{\sigma^2}+\frac{1}{\sigma_0^2}\right]}_{=a} - 2\mu\underbrace{\left(\frac{x}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}\right)}_{=b} + \frac{x^2}{\sigma^2}+\frac{\mu_0^2}{\sigma_0^2}\right)}$$

$$\propto e^{-\frac{1}{2a^{-1}}\left(\mu-\frac{b}{a}\right)^2}$$

Cor. For n samples $\vec{X} = (x_1 \cdots x_n)$, i.i.d.

$$\mu | \vec{x} \sim N\left(\frac{\tau_0}{n\tau + \tau_0}\mu_0 + \frac{n\tau}{n\tau+\tau_0}\bar{x},\ \frac{1}{n\tau+\tau_0} \right)$$

Remark: i) Note that $\frac{\tau_0}{n\tau+\tau_0} + \frac{n\tau}{n\tau+\tau_0} = 1$. We mix up the prior and the data to generate the posterior. (Weighted Average!)

ii) $\frac{1}{n\tau+\tau_0} < \sigma_0^2$. Which means the dist of $\mu|\vec{x}$ is more concentrated. It carries more information (informative)

iii) If n is large enough. Then the data will dominate the prior dist!

iv) For objectiveness. the prior needs to be vague, non informative!

(6) Bayesian Inference

for Binomial Dist:

$\cdot X|p \sim B(n,p)$. $p \sim \text{Beta}(a,b)$

$\Rightarrow p|x \sim \text{Beta}(a+x, n+b-x)$

Similarly, $\mu_{post} = \frac{a+b}{a+b+n} \cdot \frac{a}{a+b} + \frac{n}{a+b+n} \cdot \bar{x}$

If $n \to \infty$, $\mu_{post} \longrightarrow \bar{x}$!

Remark: Define:

$$\begin{cases} G = \text{family of prior dist func for } \Theta \\ H = \text{family of conditional dist } f(x|\theta) \end{cases}$$

$\Rightarrow$ G is called conjugate prior to H:

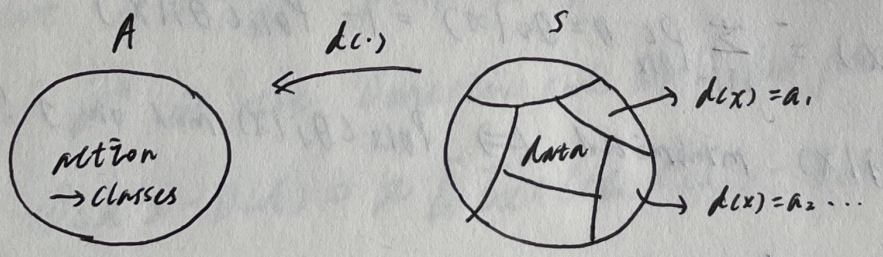
If the posterior of G under H also belongs to G:

e.g. G: Normal dist. H: Normal dist $\longrightarrow G|H = G$

H: Binomial dist G: Beta dist $\longrightarrow G|H = G$

(7) Application of Decision Theory:

① Classification:



Let $A = \{a_1 \in \text{Class } \theta_1,\ a_2 \in \text{Class } \theta_2,\ \cdots\ a_m \in \text{Class } \theta_m\}$.

Then $d(\cdot)$ is function to classify datas in S.

$\Rightarrow$ For parameter θ, let $A = \{\theta_i\}_1^m$.

$\lambda_i = P(\Theta = \theta_i)$, s.t. $\sum_1^m \lambda_i = 1$. $f(x|\theta_i)$ is known. $1 \le i \le m$

If l_{ij} is the loss in classifying class i to class j.

Then, we can find Bayesian Rule:

$$h(\theta_i|x) = \frac{\pi_i f(x|\theta_i)}{\sum \pi_k f(x|\theta_k)} = P(\Theta = \theta_i | X = x), \text{ let } d(x) = a_j$$

We obtain $PR(j|x) = \sum l_{ij} h(\theta_i|x)$. Posterior risk

$\Rightarrow$ Choose j to minimize $PR(j|x)$. for $\forall 1 \le i \le m$

Then $d: x \longmapsto a_j$ is a Bayesian Rule.

e.g. (0-1 loss)

$$l_{ij} = \begin{cases} 0, & i = j \\ 1, & i \neq j \end{cases}$$ (Because the classification is "qualitative", not "quantative")

$$R(\xi, d) = E_x(l(\theta_i, d(x))) = \sum_j l_{ij} P_{\theta_i}(d(x) = j)$$

$$= \sum_{j \neq i} P_{\theta_i}(d(x) = j) = 1 - P_{\theta_i}(d(x) = i)$$

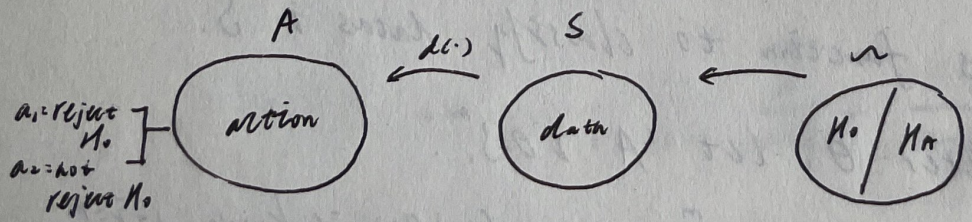
(From frequentist approach, minmax is)

$\Rightarrow$ For Bayesian approach:

$$PR(j|x) = \sum_{i \neq j} P_{\theta|x}(\theta = \theta_i | x) = 1 - P_{\theta|x}(\theta_j | x)$$

$\therefore PR(j|x)$ minimized $\Leftrightarrow P_{\theta|x}(\theta_j|x)$ max on j!

② Hypothesis Testing:



Then both type I and type II errors are misclassification errors.

$\Rightarrow$ Bayesian approach:

$p(H_0) = \pi$. $p(H_A) = 1-\pi$. then we obtain:

$$\frac{p(H_0|x)}{p(H_A|x)} = \frac{p(H_0)}{p(H_A)} \frac{p(x|H_0)}{p(x|H_A)} > 1 \quad (LHS = \frac{PR(H_0|x)}{PR(H_A|x)}, \text{ under 0-1 loss})$$

$\Leftrightarrow$ $\frac{p(x|H_0)}{p(x|H_A)} > \frac{1-\pi}{\pi} \triangleq c$. Which means:

assign $x \xrightarrow{d} H_0$ when $\frac{p(x|H_0)}{p(x|H_A)} > c$, get

$d(x)$ is Bayesian Rule under 0-1 loss.

Remark: For protecting H_0, we can assign large "π" to $p(H_0)$.

Alternative proof of N-P Lemma:

d^*: test accept $H_0 \Leftrightarrow \frac{f_0(x)}{f_1(x)} > c$, with level α^*.

Then d^* is the most power test of level $\alpha \leq \alpha^*$

Pf: let $c = \frac{\pi}{1-\pi}$. $p(H_0) = \pi$. $p(H_A) = 1-\pi$.

$\therefore$ d^* is the Bayesian rule with this prior, with 0-1 loss

$\therefore B(d^*) - B(d) = \pi [E_X(L(H_0, d^*(x))) - E_X(L(H_0, d(x)))]$

$+ (1-\pi) [E_X(L(H_A, d^*(x))) - E_X(L(H_A, d(x)))]$

$= \pi(\alpha^* - \alpha) + (1-\pi) [E_X(L(H_A, d^*(x))) - E_X(L(H_A, d(x)))] \leq 0$

$\therefore$ $E_X(L(H_A, d^*(x))) \leq E_X(L(H_A, d(x)))$

i.e. $\beta_{d^*} \geq \beta_d$.