Fourier Transform in $\mathbb{C}$

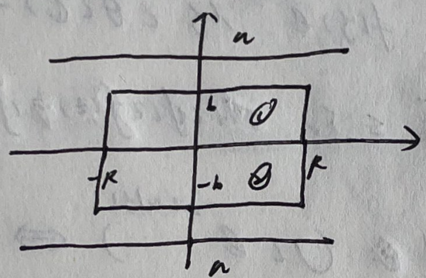
· **Def:** For $f \in \mathfrak{F}_a$ if i) $f \in \mathcal{O}(\{|\operatorname{Im} z| < a\})$

ii) $|f(z)| \le \frac{A}{1+x^2}$. $\forall z = x+yi \in \{|\operatorname{Im} z| < a\}$.

(1) Fourier Transform on $\mathfrak{F}$:

Thm. If $f \in \mathfrak{F}_a$. Then $|\hat{f}(\xi)| \le A e^{-2\pi b|\xi|}$. $\forall\, 0<b<a$.

Pf: Consider the contour:



when $\xi < 0$. use ①

$\xi \ge 0$ use ②

when $R \to \infty$. we obtain

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x \pm ib)\, e^{-2\pi i \xi (x \pm ib)}\, dx$$

Remark: For $f \in \mathfrak{F}$. $\hat{f}$ has rapid decay.

Thm.

i) (Inversion Formula)

For $f \in \mathfrak{F}$. $f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$.

ii) (Poisson Summation Formula)

For $f \in \mathfrak{F}$. $\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$

Pf: i) It's from $\hat{f}(\xi) = \int_{\mathbb{R}} f(x \pm ib)\, e^{-2\pi i \xi (x \pm ib)}\, d\xi$.

ii) Note that $\sum \hat{f}(n) = \sum \int_{\mathbb{R}} f(x)\, e^{-2n\pi i x}\, dx$

$= \int_{\mathbb{R}} f(x) \sum e^{-2n\pi x i}\, dx = \int_{\mathbb{R}} f(x) / e^{2\pi x i} - 1$

Apply Residue Formula. (only when $f \in \mathfrak{F}$)

(2) Paley-Wiener Thm.

- Suppose f is valid for Inversion Transf.

Next, we will discover what condition on f will lead to $\mathrm{supp}(\hat{f}(\xi)) = [-M, M]$.

Thm. If f satisfies: $|\hat{f}(\xi)| \le A e^{-2\pi a|\xi|}$ for some $A, a > 0$. Then $f \in \theta(S_b)$.

$S_b = \{ |\mathrm{Im}\, z| < b \}$. $\forall\, 0 < b < a$.

Pf: $f_n(z) = \int_{-n}^{n} \hat{f}(\xi) e^{2\pi z \xi i} d\xi \in \theta(\mathbb{C}) \to f(z)$

$\forall z \in \bar{S}_b \subseteq S_a$. $\therefore f(z) \in \theta(S_b)$.

Remark: $|\hat{f}(\xi)| = O(e^{-2\pi a|\xi|}) \Leftrightarrow f \in \mathcal{F}_a$.

Thm. If $f(x) \in C(\mathbb{R}) \cap M(\mathbb{R})$ Then f has an extension on $\mathbb{C}$, which is entire, satisfying: $f = O(e^{2\pi M|z|})$ $\exists M > 0$. $\Leftrightarrow$ $\mathrm{supp}(\hat{f}(\xi)) = [-M, M]$.

Pf: ($\Leftarrow$) It's easy to estimate.

($\Rightarrow$) 1°) For $f \in \theta(\mathbb{C})$. $|f| \le A \dfrac{e^{2\pi M|\eta|}}{1+x^2}$

By $\hat{f}(\xi) = \int_{\mathbb{R}} f(x+\eta i) e^{-2\pi i \xi (x+\eta i)} dx$

Then $|\hat{f}(\xi)| \le C e^{-2\pi\eta(|\xi| - M)}$. (let $\eta \to \infty$)

2°) For $f \in \theta(\mathbb{C})$. $|f| \le A e^{2\pi M|\eta|}$.

Approxi by $f_\varepsilon = f/(1+i\varepsilon z)^2$

(In fact. $\frac{1}{1+x^2}$ is only for converge!)

3°) Prove: $f(z)$ satisfies the condition in 2°)

<u>Lemma.</u> $f \in \mathcal{O}(S)$. $S = \{-\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$. $f \in C(\bar{S})$

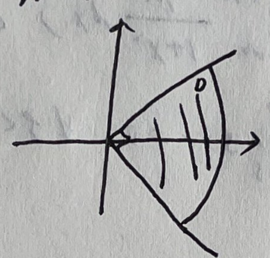
If $|f(z)| \le 1$ on ∂S. $|f(z)| \le A e^{c|z|}$ in S.

Then $|f(z)| \le 1$. $\forall z \in \bar{S}$.

<u>Pf</u>: For applying MMP. we want the domain can be bounded rather than consider $z = \infty$.

Let $f_\varepsilon(z) = f(z) e^{-\varepsilon z^{\frac{3}{2}}} \in \mathcal{O}(S)$

Then $|f_\varepsilon(z)| \to 0$ (when $z \to \infty$)



we only need to consider $f_\varepsilon(z)$ in D. where $|f_\varepsilon| \le 1$ outside D.

Then let $\varepsilon \to 0$.

$\Rightarrow$ Next, prove: $|f(x)| \le 1$ ($f \in M(\mathbb{R})$), $|f(z)| \le A e^{B|z|}$ $\Rightarrow |f(x+yi)| \le A e^{B|y|}$

Only consider in $Q = \{x>0, y>0\}$. Other three quadrants remains same. Let $F = f(z) e^{2\pi M i z}$

Then $|F(z)| \le 1$ when $z \in \partial Q$. $|F| \le A e^{B|z|}$

Rotate Q to $\{\frac{-\pi}{4} < \arg z < \frac{\pi}{4}\}$. Apply the lemma.

$\therefore |F(z)| \le 1$ on $\bar{Q}$. i.e. $|f(z)| \le e^{2\pi M|y|}$

<u>Thm</u> (The case $\hat{f}(\xi)$ vanishes on $\xi < 0$)

$f(x)$. $\hat{f}(x) \in M(\mathbb{R})$ Then $\hat{f}(\xi) = 0$. $\forall \xi < 0 \iff$

$f(x)$ can be extended continuously to $\bar{H}$. $|f| \le M$.

$\forall z \in \bar{H}$. with $f \in \mathcal{O}(H)$

Pf: ($\Rightarrow$). By Inversion Formular. check:

$$f(z) = \int_0^{\infty} \hat{f}(\xi) e^{-2\pi i \xi z} d\xi \text{ is conti. bounded.}$$

$$\text{and approx. by } \int_0^{n} \hat{f}(\xi) e^{-2\pi i \xi z} d\xi \in \mathcal{O}(H)$$

($\Leftarrow$) consider $f_{\varepsilon,\delta}(z) = \dfrac{f(z+i\delta)}{(1-i\varepsilon z)^2}$. satisfies

the condition as $f(z)$. $\varepsilon, \delta > 0$.

(δ is for $f_{\varepsilon,\delta} \in \mathcal{O}(\bar{H})$. ε is for converge)

By residue: $|\hat{f}_{\varepsilon,\delta}(\xi)| = |\int_{\mathbb{R}} f_{\varepsilon,\delta}(x+i\eta) e^{2\pi i \xi (x+i\eta)} dx|$

$$\le \left(\int_{-\infty}^{+\infty} \frac{M}{1+x^2} dx\right) e^{2\pi \xi \eta}. \ \forall \xi < 0. \text{ Let } \eta \to \infty.$$

$\therefore$ $\hat{f}_{\varepsilon,\delta}(\xi) = 0$. $\forall \xi < 0$. Let $\delta \to 0$. $\varepsilon \to 0$.

Prop. (Analogous Conclusion for Fourier coefficients)

For $f \in \mathcal{O}(D(z_0, R))$ with expansion at z_0:

$f(z) = \sum_{n \ge 0} a_n (z-z_0)^n$. Then $a_n = \dfrac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$

Pf. It's directly from $a_n = \dfrac{f^{(n)}(z_0)}{n!}$. by Cauchy Formula

which also states: $0 = \dfrac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$

$\forall n < 0$.