Meromorphic Func.

(1) Singularities:

① Def: i) Removable: If z_0 is removable

Then $f(z)$ is bounded on $U(z_0)$

ii) Pole: z_0 is a pole if $\lim_{z\to z_0}|f(z)|=\infty$

iii) Essential: $\lim_{z\to z_0} f(z)$ doesn't exist.

Then z_0 is essential singularity.

② Properties:

i) Thm $f\in \theta(\Omega/\{z_0\})$. f can be extended to Ω. $\Leftrightarrow \lim_{z\to z_0}(z-z_0)f(z)=0$.

Pf: ($\Leftarrow$) Def $g(z)=\begin{cases}(z-z_0)^2 f(z). & z\in\Omega/\{z_0\}.\\ 0. & z=z_0.\end{cases}$

Check $g\in\theta(\Omega)$. $g'(z_0)=0$.

$\therefore g(z)=\sum_2^{\infty}(z-z_0)^n a_n$. expand at z_0.

Def $\phi(z)=a_2+a_3(z-z_0)+\cdots$. $\phi|_{\Omega/\{z_0\}}=f$

($\Rightarrow$) $\exists\ \phi\in C(\Omega)$. $\phi|_{\Omega/\{z_0\}}=f$.

$\therefore \lim_{z\to z_0} f(z)=\phi(z_0)<\infty$.

ii) (Weierstrass Thm)

$f \in \mathcal{O}(D(z_0,r)/\{z_0\})$, where z_0 is essential singularity of f. Then $f(D(z_0,r)/\{z_0\})$ is dense.

<u>Pf</u>: By contradiction:

$\exists\, w_0 \in U(w_0)$. $f(D(z_0,r)/\{z_0\}) \cap U(w_0) = \emptyset$.

Let $g(z) = \dfrac{1}{f - w_0} \in \mathcal{O}(U(w_0))$.

(2) <u>Laurent Series</u>:

· <u>Def</u>: f is meromorphic on Ω if f is holomorphic except several poles on Ω.

<u>Lemma</u>. The poles of f are isolated.

<u>Pf</u>: If not. $\exists \{z_k\} \to z_0$. z_0 is a pole.

Then $\dfrac{1}{f}$ has an accumulation zero z_0.

Since $\{z_k\} \cup \{z_0\} \subseteq \text{int}\{f \neq 0\}$. $\therefore \dfrac{1}{f} \equiv C$ or 0. $\forall z \in \text{int}\{f \neq 0\}$.

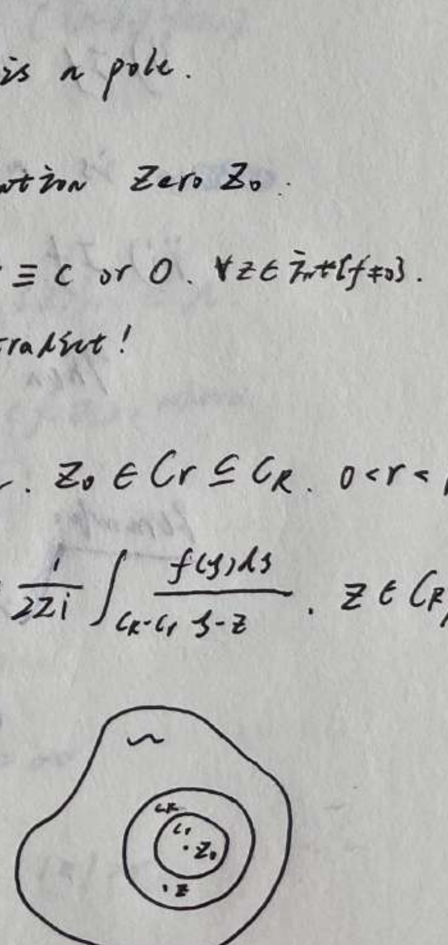
since $\dfrac{1}{f} \in \mathcal{O}(\text{int}\{f \neq 0\})$. Contradict!

For z_0 is an isolated singularity. $z_0 \in C_r \subseteq C_R$. $0 < r < R$.

where $f \in \mathcal{O}(C_R/\{z_0\})$. Then $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_{C_R - C_r} \frac{f(\xi)\,d\xi}{\xi - z}$. $z \in C_R/C_r$

i) $\displaystyle\int_{C_R} \frac{f(\xi)}{\xi - z}\,d\xi = \int_{C_R} \frac{1}{\xi - z_0} \frac{f(\xi)\,d\xi}{1 - \frac{z - z_0}{\xi - z_0}}$

$\displaystyle = \int_{C_R} \frac{1}{\xi - z_0} \sum_{0}^{\infty} \left(\frac{z - z_0}{\xi - z_0}\right)^n f(\xi)\,d\xi.$



since $\left|\dfrac{z - z_0}{\xi - z_0}\right| < 1$. $\xi \in \partial C_R$.

ii) $\int_{C_r} \frac{f(s)ds}{s-z} = \int_{C_r} \frac{1}{z-z_0} \frac{f(s)ds}{1-\frac{s-z_0}{z-z_0}}$

$= \int_{C_r} \frac{1}{z-z_0} \sum_0^{\infty} \left(\frac{s-z_0}{z-z_0}\right)^n f(s)ds.$

where $\left|\frac{s-z_0}{z-z_0}\right| < 1$. $s \in \partial C_r$

$\therefore f(z) = \sum_{n \in \mathbb{Z}} a_n (z-z_0)^n$. $z \in C_R / \{z_0\}$.

a_n is determined by above!

It's called Laurant Series.

Thm.

For Laurant series of $f(z)$ at singularity z_0. We have following criteria:

i) If $a_n = 0$. $\forall n < 0$. Then z_0 is removable

ii) If only finite $n < 0$. s.t. $a_n \neq 0$. Then z_0 is a pole

iii) If there exists infinite $n < 0$. s.t. $a_n \neq 0$. Then z_0 is essential singularity.

Remark: The criteria holds only when z_0 is a isolated singularity.

For $z_0 = \infty$. We can let $g(z) = f(\frac{1}{z})$ Expand $g(z)$ at $z=0$. Replace "$n>0$" with "$z<0$" in i). ii). iii).

(3) Residue:

① Zeros and poles:

Ω is connected

i) If $f \in \mathcal{O}(\Omega)$. $f(z_0)=0$. Then $\exists U(z_0)$ neighbour of z_0 st. $f(z) = (z-z_0)^n g(z)$. $n \geq 1$. $g \neq 0$ on $U(z_0)$

ii) If z_0 is a pole of f in Ω. Then $\exists U(z_0)$.

st. $f(z) = (z-z_0)^{-n} g(z)$. $n \geq 1$. $g \in \mathcal{O}(U(z_0))$.

$\therefore f(z) = \sum\limits_{k \geq -n} a_k (z-z_0)^k$. $\forall z \in U(z_0)/\{z_0\}$.

Note that $\int_C f(z)/_{2\pi i}\, dz = a_{-1}$. $z_0 \in C$.

we call it residue of f at z_0. Denote $Res(f, z_0)$.

Remark: $Res(f, z_0) = \frac{1}{(n-1)!} \lim\limits_{z \to z_0} \left(\frac{\partial}{\partial z}\right)^{n-1} \left((z-z_0)^n f(z) \right)$

② Residue Formula:

i) Thm. $f \in \mathcal{O}(\Omega/\{z_i\}_1^n)$. C contour $\{z_i\}_1^n \subseteq \Omega$.

Then $\frac{1}{2\pi i} \int_C f(s) ds = \sum\limits_1^N Res(f, z_i)$, where

$\{z_i\}_1^n$ are poles of f.

ii) On $\overline{\mathbb{C}}_\infty$:

Consider the reside at $z = \infty$.

$Res(f, \infty) = \frac{1}{2\pi i} \oint_{C^-} f(z) dz$. $C: |z| > r$.

where ∞ is the isolated singularity in $\mathbb{C}$.

Let $g(z) = f(\frac{1}{z})$. Laurant series of g at 0 is:

$g(z) = \sum a_n z^n$. $\therefore f(z) = \sum a_{-n} z^n$.

$Res(f, \infty) = -a_1$

<u>Remark</u>: When $z = \infty$ is removable. $Res(f, \infty)$ may not be zero. e.g. $Res(\frac{1}{z}, \infty) = -1$.

Actually, we can expand $f(z)$ at $z = \infty$: $f = \sum a_n z^n$. Then $Res(f, \infty) = -a_1$.

<u>Thm.</u> f defined on $\overline{\mathbb{C}}_\infty$, meromorphic. Then f has finite poles. Moreover, we have:

$$2\pi i \left(\sum_{1}^{r} Res(f, z_i) + Res(f, \infty) \right) = 0.$$

<u>Pf</u>: Note that $\overline{\mathbb{C}}_\infty$ cpt $\Rightarrow$ seq cpt.

The latter: $\int_{C - C} f(z)\, dz = 0$.

<u>prop.</u> $Res(f, \infty) = -Res(f(\frac{1}{z})\frac{1}{z^2}, 0)$

<u>Pf.</u> $Res(f, \infty) = \frac{1}{2\pi i} \oint_{C_r^-} f(z)\, dz$

$$= \frac{-1}{2\pi i} \int_0^{2\pi} f(re^{i\theta})\, ire^{i\theta}\, d\theta.$$

$$\overset{\theta = -\varphi}{=} \frac{-1}{2\pi i} \int_0^{2\pi} f\left(\frac{1}{\frac{1}{r}e^{i\varphi}}\right) i \frac{1}{\frac{1}{r}e^{i\varphi}}\, d\varphi.$$

$$= -\frac{1}{2\pi i} \int_{C_{\frac{1}{r}}} f(\frac{1}{z}) \frac{1}{z^2}\, dz$$

③ Integration Calculate:

i) For $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$. R is a rational Func.:

If $R(x,y) \neq \infty$ on $x^2+y^2=1$. Let $z = e^{i\theta}$.

Then $\begin{cases} \cos\theta = \frac{z^2+1}{2z} \\ \sin\theta = \frac{z^2-1}{2iz} \end{cases}$ $d\theta = \frac{dz}{iz}$. we obtain:

$$\int_0^{2\pi} R\, d\theta = \oint_{|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)/iz\, dz$$

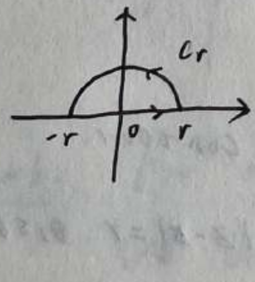
$$= 2\pi i \sum_{|z_k|<1} \operatorname{Res}(R/iz, z_k)$$

ii) For $\int_{-\infty}^{\infty} R(x)\, dx$. $R(x) = \frac{P(x)}{Q(x)}$. where P, Q are polynomials. st. $\deg Q \geq \deg P + 2$. $Q \neq 0$.

Lemma f conti. $\lim_{R\to\infty} z f(z) = \lambda$. when z is on C_R: $|z| = R$. $\theta_1 \leq \theta \leq \theta_2$. uniformly with θ. Then $\lim_{R\to\infty} \int_{C_R} f(z)\, dz = i(\theta_2-\theta_1)\lambda$.

Pf: $\exists R_0.\ \forall R > R_0.\ |zf(z) - \lambda| < \varepsilon$.

$\therefore \left|\int_{C_R} f(z)\, dz - i(\theta_2-\theta_1)\lambda\right| \leq \frac{\varepsilon l}{R}$. $\square$

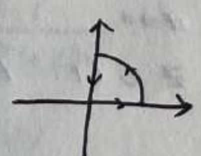
$\Rightarrow$ 

$$\oint_{C_r+[-r,r]} R(z)\, dz = 2\pi i \sum_{\substack{|z_k|<r \\ \operatorname{Im} z_k > 0}} \operatorname{Res}(R, z_k)$$

$$= \int_{-r}^{r} R(x)\, dx + \int_{C_r} R\, dz.$$

Let $r \to +\infty$. Since $zR(z) \to 0$.

$$\therefore \int_{-\infty}^{+\infty} R(x)\, dx = 2\pi i \sum \operatorname{Res}(R(z), z_k)$$

Remark: For $\int_0^{\infty}$: use . The ideal of Lemma is from: $\int_{C_R} f(z)\, dz = \int_{\theta_1}^{\theta_2} i f(z) z\, d\theta$

$= f(z_\xi) z_\xi \cdot i(\theta_2-\theta_1)$. mean value

Thm of Integral.

iii) For $\int_{-\infty}^{+\infty} R(x) e^{i\alpha x} dx$. $\alpha > 0$. $R = \frac{P}{Q}$, where

$P(x), Q(x)$ are polynomials. $\deg Q \geq \deg P + 1$.

and $Q(x) \neq 0$ $\forall x \in \mathbb{R}$.

<u>Jordan Lemma:</u>

g conti. $\lim_{R\to\infty} g(Re^{i\theta}) = 0$. uniformly with

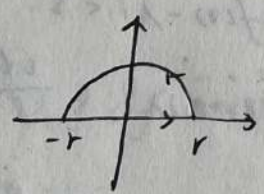
$\theta \in [\theta_1, \theta_2]$. Then $\int_{C_R} g(z) e^{i\alpha z} dz \to 0 \ (R\to\infty)$

<u>Pf:</u> Denote $M(R) = \max_{z\in C_R} |g(z)|$. $M(R) \to 0 \ (R\to\infty)$

$$\left|\int_{C_R} g(z) e^{i\alpha z} dz\right| \leq M(R) R \int_{\theta_1}^{\theta_2} e^{-\alpha R \sin\theta} d\theta.$$

$$\leq R M(R) \int_0^{\frac{\pi}{2}} e^{-\alpha \sin\theta \cdot R} d\theta. \leq C M(R) \to 0.$$

by $\sin\theta \geq \frac{2}{\pi}\theta$. when $0 \leq \theta \leq \frac{\pi}{2}$. □

$\Rightarrow$ 

$\int_{-\infty}^{+\infty} R(z) e^{i\alpha z} dz = 2\pi i \sum \mathrm{Res}(Re^{i\alpha z}, z_k)$

since $R(z) \to 0 \ (R\to\infty)$

iv) <u>A special case:</u>

when there's a pole on contour.

<u>Lemma.</u> f conti. C_r: $|z-a| = r$. $\theta_1 \leq \theta \leq \theta_2$

$\lim_{r\to a} (z-a) f(z) = \lambda$. uniformly with θ.

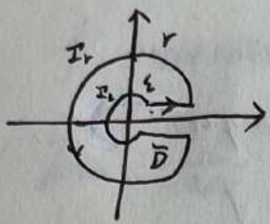
$z \in C_r$. Then $\lim_{r\to 0} \int_{C_r} f(z) dz = i(\theta_1-\theta_2)\lambda$

<u>Pf:</u> $\exists r_0$. $\forall r < r_0$. $|(z-a)f(z) - \lambda| < \varepsilon$.

Similar Argument.

v) For $\int_0^\infty \frac{R(x)}{x^\alpha}dx$. $\alpha \in (-1,1)$. $R(x)$ is rational function. $R = \frac{P}{Q}$

Case one: $\alpha \in (0,1)$. require $\deg Q \geq \deg P + 1$.



0 and ∞ are pivot.

Choose $z \geq 0$ to be partion line.

$C_{r,\varepsilon} = I_r \cup -I_\varepsilon \cup [\varepsilon, r] \cup [r, \varepsilon]$.

$$\therefore \oint_{C_{r,\varepsilon}} \frac{R(z)}{(z^\alpha)_0}dz = 2\pi i \sum \mathrm{Res}\left(R/(z^\alpha)_0, z_k\right)$$

where $(z^\alpha)_0 = e^{\alpha(\ln|z| + i\arg(z))}$

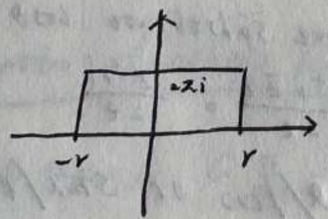
$$\Rightarrow \text{LHS} = \int_{I_r} - \int_{I_\varepsilon} + \int_\varepsilon^r \frac{R(x)dx}{x^\alpha} + e^{-i\alpha\cdot 2\pi}\int_r^\varepsilon \frac{R(x)dx}{x^\alpha}$$

Let $\varepsilon \to 0$. $r \to \infty$. Then $\int_{I_r - I_\varepsilon} \to 0$.

$$\therefore \int_0^\infty \frac{R\,dx}{x^\alpha} + e^{-i2\pi\alpha}\int_\infty^0 \frac{R(x)dx}{x^\alpha} = 2\pi i \sum \mathrm{Res}\left(R/(z^\alpha)_0, z_k\right)$$

Case two: $\alpha \in (-1,0)$. require: $\deg Q \geq \deg P + 2$.

Alternative method:



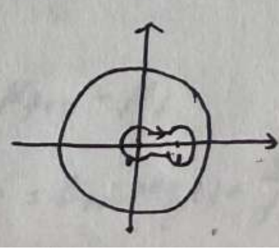
Let $z = e^u$.

$$\therefore \int R/x^\alpha = \int_{-\infty}^{+\infty} R(e^u)e^{u(1-\alpha)}du.$$

Remark: For multivalue functions. The contour we choose should detour the pivots and partion lines:

e.g. For $\frac{1}{x^\alpha(1+x)^\beta}$

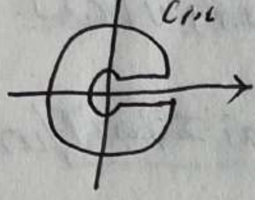
$\alpha, \beta \in (0,1)$



vi) For $\int_0^\infty R(x)(\ln x)^m dx$, $m \geq 1$. $R(x)$ is rational function. $R(x) = \frac{P}{Q}$. $\deg Q \geq \deg P + 2$. $Q \neq 0$.

For complexization:

Consider: $F(z) = R(z)(\ln z)^{m+1}$.



Choose $\ln z = \ln|z| + i\arg(z)$.

$$\therefore \oint_{C_{r,\varepsilon}} F(z)\,dz = \int_{\Gamma_r} - \int_{\Gamma_\varepsilon} + \int_\varepsilon^r R(x)(\ln x)^{m+1} + \int_r^\varepsilon R(x)(\ln x + i\cdot 2\pi)^{m+1}$$

$\Rightarrow$ Then $R(x)(\ln x)^{m+1}$ will be offsetted.

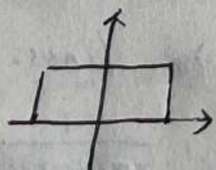
<u>Remark</u>: We should calculate $\int_0^\infty R(x)\ln x^k$. From $k=1$ to $m+1$ gradually.

vii) <u>Summary</u>:

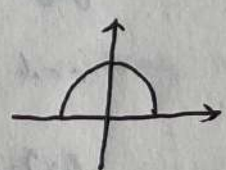
a. check the integrand is a meromorphic function first. That's why we substitute $\cos\theta = \frac{z+\bar{z}}{2}$ with $\frac{z^2+\bar{z}z}{2z} = \frac{z^2+1}{2z}$.

b. For $\cos x/R(x)$ or $\sin x/R(x)$. We only need to consider $e^{iz}/R(x)$. Figure its real and imagine part.

c.



For integral contain e^z.



For rational function.

(4) Argument Principle:

① Winding number:

Note that if $\Omega \underset{open}{\subseteq} \mathbb{C}$. $f \in \mathcal{O}(\Omega)$.

f nonvanishes and has no poles on $\partial\Omega$. $^{(*)}$ → If it holds. Then they may exist accumulation point on $\partial\Omega$. Then N or P will be ∞!

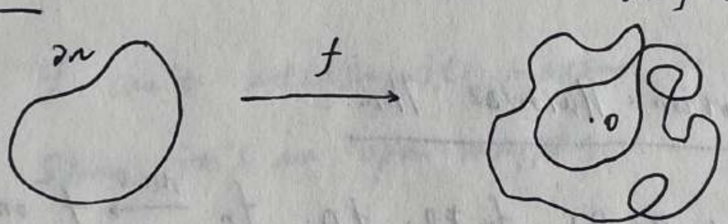
Then: $\frac{1}{2\pi i}\oint_{\partial\Omega} \frac{f'(z)\,dz}{f(z)} = N - P \triangleq n(f(\partial\Omega), 0)$

N is total number of zeros. P is for poles

Pf: Easy to check by expansion of series.

Use contours to surround poles and zeros

Intepretation:

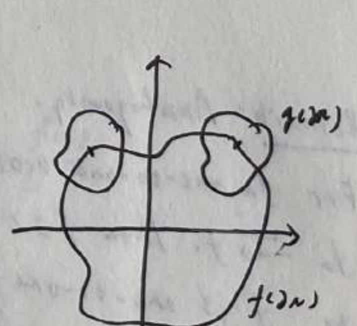


$$\oint_{\partial\Omega} \frac{f'(z)\,dz}{f(z)} = \oint \frac{dw}{w} = n(f(\partial\Omega), 0)$$

② Application: Rouché's Thm:

$f, g \in \mathcal{O}(\Omega)$. $C \underset{cycle}{\subseteq} \Omega$. If $|f| > |g|$ on C.

Then $N_{f+g} = N_f$.



Pf: Since $f+g$ and $f \in \mathcal{O}(\Omega)$.

Then $\Delta_{\partial\Omega} \arg\left(\frac{f+g}{f}\right)$

$$= \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{(f+g/f)'}{f+g/f}\,dz = N_{f+g} - N_f.$$

Note that $\Delta_{\partial\Omega} \arg(f+g/f) = \Delta_{\partial\Omega} \arg\left(1 + \frac{g}{f}\right)$

since $1+\frac{g}{f} \neq 0$, $\forall z \in \partial\Omega$. $\therefore \Delta_{\partial\Omega} \arg(1+\frac{g}{f}) = 0$.

Other Form:

$f, g \in \mathcal{O}(D)$. $\gamma \subseteq D$, Jordan curve.

If $|f|+|g| > |f+g|$, $\forall z \in \gamma$.

Then $N_f = N_g$.

Pf: Note that f or $g \neq 0$ on γ.

$\therefore 1 + |\frac{g}{f}| > |1+\frac{g}{f}|$.

$g/f \notin \mathbb{R}^+$. $\therefore \Delta_C \arg(\frac{g}{f}) = 0$.

since $\frac{g}{f}(c)$ won't wind around $z=0$ a whole circle.

because g/f won't walk through $\mathbb{R}^+$.

③ Application: Hurwitz Thm:

$f_n \in \mathcal{O}(D)$. $f_n \neq 0$. $\forall n$. $f_n \xrightarrow{n.l.u.} f$ on D.

Then $f \equiv 0$ or $f \neq 0$ on D.

Pf: Note that $\forall K \underset{cpt}{\subseteq} D$. $\frac{f}{f_n} \xrightarrow{n} 1$.

For $\varepsilon_0 > 0$. $\exists N_0$, s.t. $\forall n > N_0$. $|\frac{f}{f_n} - 1| < \varepsilon_0$

$\therefore N_f = N_{f-f_n+f_n} = \Delta_{\partial K} \arg(f-f_n+f_n)$

$= \Delta_{\partial K} \arg(f_n) + \Delta_{\partial K} \arg(\frac{f-f_n}{f_n}+1)$

for some $n > N_0$.

Remark: Analogously.
For f_n one-to-one, $\mathcal{O}(D)$
$f_n \xrightarrow{n.l.u.} f$. Then $f \equiv c$
or f is one-to-one!

since $|1+\frac{f-f_n}{f_n}| > 1-\varepsilon_0 > 0$.

$\therefore N_f = N_{f_n} = 0$. when $f \not\equiv c$.

When $f \equiv c$. it holds automatically.

④ <u>Opening mapping Thm</u>:

If $f \in \theta(\Omega)$, $f \not\equiv c$. then f is open mapping

<u>Pf</u>: $U \underset{open}{\subseteq} \Omega \xrightarrow{f} f(U) \ni w_0 = f(z_0)$

prove: the points surround w_0 with "ε" dist will have inverse image.

Note that $f(z) - w = f(z) - w_0 + w_0 - w$

Choose δ. ε : $|f - w_0| > \varepsilon > |w_0 - w|$. when $|z - z_0| = \delta$

Since : z_0 is isolated zero of $f - w_0 \in \theta(\Omega)$

<u>Cor.</u> (Maximal Modulus Principle)

$f \in \theta(\Omega)$. $f \not\equiv c$. conti on $\partial\Omega$. Then $\max\limits_{\bar{\Omega}} |f| = \max\limits_{\partial\Omega} |f|$.

<u>Pf</u>: f can't attain its maximal in Ω°.

Since it's an open mapping.

<u>Remark</u>: The minimal one won't hold if f has zeros. then $\frac{1}{f} \notin \theta(\Omega)$.

<u>Cor.</u> $f \in \theta(\Omega)$. $f \neq 0$ in Ω. If $f \equiv c$ on $\partial\Omega$.

Then $f \equiv c$ in $\bar{\Omega}$.

<u>Pf</u>: If $c = 0$. it holds. Otherwise. Let $g = \frac{f(z)}{c}$

(5) <u>m to 1 Functions</u>:

① <u>One to one</u>:

Thm. $f \in \theta(D)$. one-to-one. Then $f' \neq 0$ on D

Pf: WLOG. Suppose $f(0) = 0$. (By translation)

and $f'(0) = 0$. it will come into contrad.

Expand at $z=0$. $\therefore f = \sum_m^\infty a_n z^n = z^m \varphi(z)$. $m \geq 2$

Since $z=0$ is isolated zero of $f \cdot f'$

$\therefore \exists\ \overline{B}(0,\varepsilon)$. $f \cdot f' \neq 0$ on $\overline{B}(0,\varepsilon)$

If $a = \min_{\partial \overline{B}} |f|$. Then $\forall\ w < \frac{a}{2}$

$N_{f-w} = N_f = 2$. contradiction!

Remark: Converse is false. e.g. e^z

But it will hold locally.

Thm. $f \in \theta(D)$. $z_0 \in D$. If $f'(z_0) \neq 0$. Then

locally near z_0. f is one-to-one.

Pf: Expand f at $z = z_0$. then $a_1 \neq 0$:

$$f(z) = \sum_0^\infty a_n (z - z_0)^n.$$

Estimate $|f(z_1) - f(z_2)|$, when $\delta = |z_1 - z_2|$

is small enough.

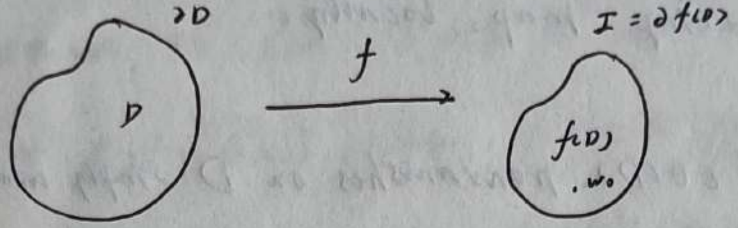
Thm. (Darboux-Picard Thm)

$f \in \theta(D)$. conti on ∂D. If f is one-to-one

on ∂D. Then f is one-to-one on $\overline{D}$.

(∂D is Jordan curve)

Pf: Note that $f(z_0) \in \text{int} f(D) \Leftrightarrow z_0 \in \text{int}(D)$ by opening map Thm.



$\forall\ w_0 \in \text{int} f(D).\ \exists\ z_0 \in \text{int} D.\ s.t.\ f(z_0) = w_0.$

$$\frac{1}{2\pi i} \oint_{\partial D} \frac{f'(z)dz}{f(z)-w_0} = \frac{1}{2\pi i} \oint_I \frac{dw}{w-w_0} = n(I, w_0) = 1$$

since f is one-to-one on $\partial f(D) \longleftrightarrow \partial D$. So when f walk around ∂D a circle. then $f(\partial D)$ contour w_0 a circle.

Remark: It holds when f is multicomplex (By proper map)

Def: $f: D \to f(D)$ is biholomorphic on D. when $f \in \mathcal{O}(D)$. one-to-one. Then:

$$f^{-1}: f(D) \to D. \quad (f^{-1}(z_0))' = \lim_{z \to z_0} \frac{f^{-1}(z) - f^{-1}(z_0)}{z - z_0}$$

$$= \lim_{w \to w_0} \frac{w - w_0}{f(w) - f(w_0)} = \frac{1}{f'(w_0)}, \quad w_0 = f^{-1}(z_0)$$

Thm. $f: \Omega \to f(\Omega)$, biholomorphic. If $D \subseteq \Omega$. simply connected. Then $f(D)$ is simply connected.

Pf: By Darboux - Picard Thm.

② m to one :

Thm. $f \in \mathcal{O}(D)$. Then f is a m-to-one covering map, locally.

Lemma. $f \in \mathcal{O}(D)$. nonvanishes on D simply connected.

Then exists $g \in \mathcal{O}(D)$. s.t. $f = e^g$

Pf: Def: $g(z) = \int_{z_0}^{z} \frac{f'}{f} dz + C_0 \in \mathcal{O}(D)$

$\therefore g'(z) = f'/f \Rightarrow (f e^{-g})' = 0$

$\therefore f = C e^g$. Let $e^{C_0} = f(z_0)$

$\therefore C = 1$. $f(z) = e^{g(z)}$ $\forall z \in D$.

$\Rightarrow$ Expand f at $z = z_0$. $f(z) = \sum a_n (z - z_0)^n$.

$\therefore f(z) = (z - z_0)^m \varphi(z)$. φ nonvanishes locally

$\therefore \varphi(z) = e^{g(z)}$. $\exists g \in \mathcal{O}(U(z_0))$.

$\therefore f(z) = \left((z - z_0) e^{\frac{g(z)}{m}} \right)^m$. ($m=0$. holds trivially suppose $m \geq 1$)

(b) Entire Func and Meromorphic on $\overline{\mathbb{C}_\infty}$:

① Entire Func on $\overline{\mathbb{C}_\infty}$:

Note that entire function has the unique singularity : $z = \infty$. it must be a pole or essential singularity (Otherwise it will degenerate to const.)

Only when $z=\infty$ is a pole, then f can be extend to $\overline{\mathbb{C}}_\infty$. (e.g. e^z can't)

$\Rightarrow f \in \mathcal{O}(\overline{\mathbb{C}}_\infty)$. Then f is a polynomial

(we call $f \in \mathcal{O}(\mathbb{C})$, but f isn't a polynomial by transcendental entire function)

<u>properties:</u>

i) <u>Thm.</u> $f \in \mathcal{O}(\mathbb{C})$. $\lim_{z\to\infty} \frac{f(z)}{z^n} = 0$. Then f is a polynomial with degree $< n$

<u>pf:</u> $\exists R$. $\forall |z|>R$. $\left|\frac{f(z)}{z^n}\right| < \varepsilon$.

By Cauchy Inequality.

<u>Remark:</u> It can be extended to $\overline{\mathbb{C}}_\infty$.

ii) <u>Thm</u> (Picard's little Thm)

$f \in \mathcal{O}(\mathbb{C})$. $f \not\equiv c$. Then f values all points $\in \mathbb{C}$ except one.

<u>Remark:</u> Picard's Great Thm:

z_0 is essential singularity of $f(z)$. Then $\forall U(z_0)$ of z_0. $f(U(z_0)/z_0)$ only misses at most one point.

iii) $f \in \theta(\mathbb{C})$. $\forall z_0 \in \mathbb{C}$. At least one coefficients is zero in its local expansion: $\sum_0^\infty a_n(z-z_0)^n$. Then f is polynomial.

Pf: When $a_n = \frac{f^{(n)}(z_0)}{n!} = 0$, then $f^{(n)}(z_0) = 0$

If f isn't a polynomial.

Then $f^{(n)}(z) \not\equiv 0$. $\forall n$.

For each n. $f^{(n)}(z) \in \theta(\mathbb{C})$. has countable zeros $^{(*)}$

Denote $Z_n = \{ z_n^k \mid f^{(n)}(z_n^k) = 0 \}$

$\therefore \bigcup_1^\infty Z_n$ is set of zeros of $\{f^{(n)}(z)\}$. countable.

But $\forall \Omega \underset{open}{\subseteq} \mathbb{C}$. $|\Omega| = 2^{\aleph_0} > |\bigcup_1^\infty Z_n| = \aleph_0$

which is a contradiction!

Remark: $(*)$: Since the zero is isolated.

which can correspond a neighbour.

② Memromorphic on $\overline{\mathbb{C}}_\infty$:

i) Thm. f is memromorphic on $\overline{\mathbb{C}}_\infty$. Then it's rational function.

Pf: Since $\overline{\mathbb{C}}_\infty$ is cpt. So poles of f are finite.

Expand f at each pole z_k.

$f(z) = f_k(z) + g_k(\frac{1}{z-z_k})$. g_k is polynomial

$f(\frac{1}{z}) = f_\infty(z) + g_\infty(z)$.

$\therefore f - g_\infty(\frac{1}{z}) - \sum g_k(\frac{1}{z-z_k}) \in \theta(\overline{\mathbb{C}}_\infty)$

$\therefore f(z) = g_\infty(\frac{1}{z}) + \sum g_k(\frac{1}{z-z_k}) + Const.$

Remark: A meromorphic isn't polynomial will be called transendental meromorphic func. e.g.

$\frac{1}{e^z+1}$. has infinite poles: $(2k+1)\pi i \to \infty$.

So it can't be extended to $\overline{\mathbb{C}}_\infty$.

ii) Thm. f is meromorphic on $\overline{\mathbb{C}}_\infty$. $f \not\equiv$ const.

The $|f^{-1}(p)|$ is indept with p.

Pf: $\forall p \in \overline{\mathbb{C}}_\infty$. $\exists z_0 \in \overline{\mathbb{C}}_\infty$. $\varepsilon > 0$. st.

$f(z) - p \in \mathcal{H}(\overline{D}(z_0, \varepsilon))$. $f(z) - p \neq 0$ on $\overline{D}(z_0, \varepsilon)$

$$\therefore \oint_{D(z_0,\varepsilon)} \frac{f'(z)\,dz}{f(z)-p} = 0 = \oint_{\overline{D}(z_0,\varepsilon)} \frac{f'(z)}{f(z)-p} = N - P.$$

$\therefore N = |f^{-1}(p)| = P$. poles of $f - p$.

$\therefore N_{f-p} = P_{f-p} = P_{f(z)}$.