

# Topological Manifolds

Generally speaking, a manifold is a space that "locally looks like ' $\mathbb{R}^n$ '". Actually, we often picture it as a subset of larger vector space.

## (1) Definitions:

① Def: A coordinate chart  $(U, \tilde{U}, f)$  of topo space  $X$  is :  $U \xrightarrow[\text{open}]{} X$ ,  $\tilde{U} \xrightarrow[\text{open}]{} \mathbb{R}^n$ , and  $U \xrightarrow[f]{\sim} \tilde{U}$ . homeomorphism.

Def: A topo space  $X$  is manifold if

- i) It's  $C_2$
- ii) It's Hausdorff
- iii)  $\forall x \in X$ , exists a coordinate chart at  $x$ .

Remark: i) For  $x, y \in X$ ,  $U_x \cap U_y \neq \emptyset$ . Assume :

$$U_x \xrightarrow{\sim} \tilde{U}_x \subseteq \mathbb{R}^{n(x)}, \quad U_y \xrightarrow{\sim} \tilde{U}_y \subseteq \mathbb{R}^{n(y)}.$$

Then  $n(x) = n(y)$ . it's called invariant of domain. From :

For  $\forall A, B \subseteq S^n$ . If  $A \xrightarrow{\sim} B$ , and

$$A \xrightarrow[\text{open}]{} S^n. \text{ Then } B \xrightarrow[\text{open}]{} S^n.$$

Cor.  $V \xrightarrow[\text{open}]{} \mathbb{R}^{n_1}$ ,  $V' \xrightarrow[\text{open}]{} \mathbb{R}^{n_2}$ ,  $V \xrightarrow{\sim} V'$ . Then  $n_1 = n_2$

If:  $V \xrightarrow[\text{open}]{} \mathbb{R}^{n_2}$  as well by above.

$$\therefore n_2 = n_1. \text{ Otherwise, } n_1 > n_2, \exists \pi_I k \xrightarrow[\text{open}]{} V.$$

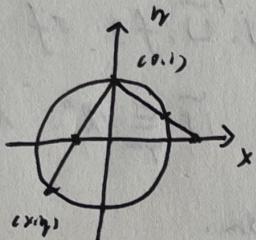
But  $\pi_I k \notin \mathbb{R}^{n_2}$ . contradict!

ii) If  $\forall x \in X, f(x) = n$ . We call it  $n$ -dimension manifold. (Next, our discussion restrict on it.)

e.g. i)  $S^1 : x^2 + y^2 = 1 \subset \mathbb{R}^2$ .

$$\text{Chart: } \begin{cases} U_1 = S^1 / (0, -1), f_1 = \frac{x}{1+y} \\ U_2 = S^1 / (0, 1), f_2 = \frac{x}{1-y} \end{cases}$$

It's called stereographic projection



$\therefore S^1$  is one-dimension manifold.

ii)  $S^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{k=0}^n x_k^2 = 1\}$ .

$$\text{Chart: } \begin{cases} U_1 = S^n / (\vec{0}, 1), f_1 = (\frac{x_0}{1-x_n}, \dots, \frac{x_{n-1}}{1-x_n}) \\ U_2 = S^n / (\vec{0}, -1), f_2 = (\frac{x_0}{1+x_n}, \dots, \frac{x_{n-1}}{1+x_n}) \end{cases}$$

(Consider projection on each  $x_n \neq 0, 0 \leq k \leq n$ )

## ② Manifold with boundary:

Def: A topo space  $X$  is  $n$ -dimension topo manifold with boundary if  $\forall x \in X, \exists x \in \bigcup_{\text{open}} \subseteq X$ , s.t.

$\bigcup_f \subseteq \mathbb{R}_{\leq 0} \times \mathbb{R}^n$ ,  $f$  is homeomorphism.

Remark: i) It's not a manifold. It locally looks like a half-space  $\{x_1 \leq 0\} \subseteq \mathbb{R}^n$ . So,  $\bigcup$  may not be open in  $\{x_1 \leq 0\}$ .

ii) For the point  $x \in X$ . s.t.  $f(x) \in \mathbb{R}^n \times \mathbb{R}^m$ .

We call it interior point.

For the point  $x \in X$ . s.t.  $f(x) \in \mathbb{R}^n \times \{0\}$ . We call it boundary point. Denote the set by  $\partial X$ .

e.g.  $X = \overline{B_n(0,1)}$ ,  $\partial X = S^{n-1}$ .

## (2) Atlas:

Now can we switch between the two coordinate systems? Note that

$$\begin{cases} f_1: U_1 \cap U_2 \xrightarrow{\sim} f_1(U_1 \cap U_2) \subseteq \tilde{U}_1 \\ f_2: U_1 \cap U_2 \xrightarrow{\sim} f_2(U_1 \cap U_2) \subseteq \tilde{U}_2 \end{cases}$$

Def: The transition function between  $(U_1, f_1)$  and  $(U_2, f_2)$  is  $\phi_{12} = f_1 \circ f_2^{-1}: f_2(U_1 \cap U_2) \xrightarrow{\sim} f_1(U_1 \cap U_2)$

Remark:  $\phi_{12}^{-1} = \phi_{21}$ .  $\phi$  depends on the order.

Def: For  $X$  is a topo manifold. An atlas for  $X$  is collection of coordinate charts:  $f_i: U_i \xrightarrow{\sim} \tilde{U}_i$ .

s.t.  $\bigcup_{i \in I} U_i = X$ .

An atlas is  $C^k$  if  $\forall (U_\alpha, f_\alpha), (U_\beta, f_\beta)$ ,

the transition func  $\phi_{\alpha\beta}$  is  $C^k$ .

Remark: i) For a smooth atlas. Then  $\cup \phi$

transition function is diffeomorphism.

(i.e.  $\phi, \phi^{-1}$  are smooth.)

ii) Smooth bijection may not be diffeomorphism.

e.g.  $f(x) = x^3: \mathbb{R} \rightarrow \mathbb{R}$

e.g. i)  $T' = \mathbb{R}/\mathbb{Z}$ . one-dimension torus.

Actually.  $T' = [0,1]/\partial[0,1]$ . Consider:

$q: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = T'$ . quotient map

1)  $\forall U \subseteq_{open} T'$ .  $q^{-1}(U) = \bigcup_{k \in \mathbb{Z}} U + k$  open

2)  $\forall W \subseteq_{open} \mathbb{R}$ .  $q^{-1}(q(W)) = W + k$  open

$\Rightarrow q$  is conti. open mapping.

Construct smooth atlas from  $q$ :

$$\begin{cases} \tilde{U}_1 = (0,1), U_1 = (0,1), f_1 = q^{-1} & \text{check } \phi_{12} \text{ and} \\ \tilde{U}_2 = (-\frac{1}{2}, \frac{1}{2}), U_2 = T'/\mathbb{Z}, f_2 = q^{-1} & \Rightarrow \phi_{21} \text{ are smooth.} \end{cases}$$

Remark:  $T' \xrightarrow[f]{\sim} S^1$ .  $f(x) = (\cos 2\pi x, \sin 2\pi x)$

ii) Generally. For  $T^n = \mathbb{R}^n/\mathbb{Z}^n$ . n-dim torus.

Consider  $\tilde{U}_i = \prod A_i$ .  $A_i = \begin{cases} (0,1) & \text{or} \\ (-\frac{1}{2}, \frac{1}{2}) \end{cases}$

with  $f_i = q^{-1}$ . check it's smooth atlas.

Remark:  $T^n \cong \prod T' \cong S^1 \times S^1 \times \dots \times S^1 \neq S^n$ .

(3) Smooth Structure:

① Compatible:

Def: For  $X$  is a Topo manifold.  $A$  is a smooth

atlas.  $(U, f)$  is another coordinate chart for  $X$ .

$(U, f)$  is compatible with  $A$  iff  $A \cup \{U, f\}$  is still a smooth atlas.

Remark: i) It's not important to know which cochart really in  $A$ . The important one is which one is compatible with  $A$ .

ii) Check whether  $(U, f)$  is compatible with  $A$ : only need to consider locality in  $U$ .

Lemma:

For  $(U, f)$  in a smooth atlas  $A$ .

i)  $\forall V \stackrel{\text{open}}{\subseteq} U$ .  $(V, f|_V)$  is compatible.

ii)  $\tilde{V} \stackrel{\text{open}}{\subseteq} \mathbb{R}^n$ .  $g: \tilde{U} \rightarrow \tilde{V}$ . Diffeomorphism.

Then  $(U, gof)$  is compatible.

pf: i)  $\varphi \circ (f|_V)^{-1} = \varphi \circ f^{-1}|_{f(V \cap U)}$ ,  $(f|_V) \circ \varphi^{-1} = f \circ \varphi^{-1}|_{\varphi(U \cap V)}$

ii)  $(gof) \circ \varphi^{-1} = g \circ (f \circ \varphi^{-1})$ ,  $\varphi \circ (gof)^{-1} = (\varphi \circ f^{-1} \circ g^{-1})$

Def: For smooth atlases  $A, B$  for  $X$ .  $A$  and  $B$  are compatible  $\Leftrightarrow A \cup B$  is still smooth atlas.

Lemma:

For  $A$  and  $B$  are compatible. If  $(U, f)$  is compatible with  $A$ . Then it's compatible with  $B$  as well. (Only need to check one in  $(A \cup B)$ )

Pf: For every  $(U_j, f_j) \in B$ . wlog.  $U_i \cap U_j \neq \emptyset$ .

Consider  $\phi_{j0} : f_j \circ f^{-1} : f_i(U_i \cap U_j) \xrightarrow{\sim} f_i(U_i \cap U_j)$

prove:  $\forall x \in f_i(U_i \cap U_j)$ ,  $\phi_{j0}|_{f_i(x)}$  is smooth.

Find  $(U_k, f_k) \in A$ . s.t.  $w = U_i \cap U_k \neq \emptyset$ .

$f_k(x) \in f_i(w)$ . Then we obtain:

$$\phi_{j0}|_{f_i(w)} = \phi_{ji}|_{f_i(w)} \circ \phi_{io}|_{f_i(w)}$$

It's easy to see:  $\phi_{j0}, \phi_{ji}$  are smooth.

Cor. Compatibility is an equivalence relation

## ② Smooth manifolds:

Def: A smooth manifold is a topo manifold  $X$  together with an equivalent class  $[A]$  of compatible smooth atlases (with its smooth structures) on  $X$ .

e.g.  $\mathbb{R}$  with  $[(\text{id}_{\mathbb{R}}, \mathbb{R})]$  is different with

$$\mathbb{R} \text{ with } [(\varphi, \mathbb{R})] \quad \varphi = \begin{cases} x & x \leq 0 \\ -x & x > 0 \end{cases}$$

Remark: i) Smooth atlases may not consist of smooth coordinate charts. Since transition function may not exist!

ii) There're  $C^\infty$ -diffeomorphism:

$$(\mathbb{R}, A_1) \xrightarrow{g^*} (\mathbb{R}, A_2) \text{ (see in chart)}$$

## Exotic Smooth Structure:

For a pathconnected topo manifold  $X$ :

$$S_x = \{M \mid M \text{ is smooth manifold with the underlying topo } X\} / \cong$$

where " $\cong$ " is equivalence under  $C^\infty$ -diffeomorphism.

Then:  $|S_x| \leq 1$  if  $\dim X = 1, 2, 3$

$|S_{X^n}| = 1$  if  $n \neq 4$ .  $|S_{S^n}| > 1$ .  $\forall n \in \mathbb{Z}^+$ .

$|S_{\mathbb{R}^4}|$  is uncountably infinite.

## (4) Pseudo-atlas:

- Actually, an smooth atlas doesn't just determine the smooth structure on  $X$ . It can also determine the underlying topology.

For  $X$ , a set merely. A pseudo-chart is:

$$f: U \subseteq X \xrightarrow{\sim} \tilde{U} \stackrel{\text{open}}{\subseteq} X \text{. bijection.}$$

Then a pseudo-atlas is  $A = \{(U_i, f_i)\}_{i \in I}$ , where

$$\bigcup_{i \in I} U_i = X. (U_i, f_i)$$
 is pseudo-chart.  $\forall i \in I$ .

Note that  $\phi_{ij}$  doesn't need to be smooth/conti

PROP. If for any two  $(U_1, f_1), (U_2, f_2)$  in

$A$ , satisfies:

$$i) f_i(U_i \cap U_2) \stackrel{\text{open}}{\subseteq} \tilde{U}_1, f_2(U_i \cap U_2) \stackrel{\text{open}}{\subseteq} \tilde{U}_2$$

ii)  $\phi_{21}$  is cont.

Then, there exists a unique topo on  $X$

st. Each  $(U_i, f_i)$  is co-ordinate chart.

Pf: 1) Uniqueness:

Firstly  $\{U_i\}_{i \in I}$  must be open. by def

If  $V \stackrel{\text{open}}{\subseteq} U$ . Then  $f_i(V \cap U_i) \stackrel{\text{open}}{\subseteq} \tilde{U}_i$ .  $\forall i \in I$ .

$\therefore V \cap U_i$  is open.  $\forall i \in I$ . ( $f_i$  is homeo)

Conversely. since  $V = \bigcup_{i \in I} (V \cap U_i)$ ,

if  $V$  satisfies the rule. then  $V$  is open.

2) Existence:

check:  $V \stackrel{\text{open}}{\subseteq} X \Leftrightarrow f_i(V \cap U_i) \stackrel{\text{open}}{\subseteq} \tilde{U}_i$

determines a topo structure.

e.g i)  $\mathbb{R}\mathbb{P}^1$  is the set of lines through origin

in  $\mathbb{R}^2$ . Any  $(x, y) \neq \vec{0}$  lies in a line.

We denote:  $\mathbb{R}\mathbb{P}^1 = \{x:y \mid (x, y) \neq \vec{0}, x:y \sim \lambda x:\lambda y, \forall \lambda \neq 0\}$

pseudo-chart:

$$\begin{cases} U_1 = \mathbb{R}\mathbb{P}^1 / (0:1), f_1(x:y) = y/x \rightarrow \mathbb{R}' \\ U_2 = \mathbb{R}\mathbb{P}^1 / (1:0), f_2(x:y) = x/y \rightarrow \mathbb{R}' \end{cases}$$

Note that  $\phi_{12}, \phi_{21}$  are smooth. determines a topo.

Remark:  $T' = \mathbb{R}'/\mathbb{Z}' \xrightarrow{f} \mathbb{RP}'$ .  $f(x) = \cos 2\pi x : \sin 2\pi x$ .

ii) Generally.  $\mathbb{RP}^n = \{x_0 : x_1 : \dots : x_n \mid (x_0, x_1, \dots, x_n) \neq 0 \in \mathbb{R}^{n+1}$

$x_0 : x_1 : \dots : x_n \sim x_0 : x_1 : \dots : x_n \cdot (\lambda + \alpha \in \mathbb{R})$ .

We have:  $U_i = \{x_0 : x_1 : \dots : x_n \mid x_i \neq 0\}$  with

$$f_i(x_0 : \dots : x_n) = \left( \frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \hat{1}, \dots, \frac{x_n}{x_i} \right).$$

check  $\phi_{ij}$ 's are smooth.

iii) Grassmannian Manifolds:

Denote  $\text{Gr}(k, n)$  is set of  $k$ -dimension subspaces of  $\mathbb{R}^n$ .

(e.g.  $\mathbb{RP}^n = \text{Gr}(1, n+1)$ )

Claim:  $\text{Gr}(k, n)$  has smooth structure of  $(k, n-k)$ -dim topo.

1)  $\forall S \in \text{Gr}(k, n)$ . Fix basis of  $S$ . determine a rank  $k$  matrix.  $M \in M^{k \times n}(\mathbb{R})$ . Note that it equals to  $P_M$ .  $P \in GL_k(\mathbb{R})$ . transitive matrix

$$\therefore \text{Gr}(k, n) = \{M \in M^{k \times n}(\mathbb{R}) \mid \text{rk}(M) = k\} / GL_k(\mathbb{R})$$

2) For  $m = (m' : m'') \in \text{Gr}(k, n)$ :

Consider  $U_J = \{M \in \text{Gr}(k, n) \mid \exists \tilde{m}. M \sim \tilde{m} \text{ s.t.}$

$$(\tilde{m}_1, \tilde{m}_2, \dots, \tilde{m}_k) = J_k\}$$
.  $J = \{j_1, j_2, \dots, j_k\} \subseteq \{1, 2, \dots, n\}$ .

$$f_J(m) = (\tilde{m}_{j_1}, \dots, \tilde{m}_{j_k}). \quad j_1 < j_2 < \dots < j_k. \quad (j_i) \subseteq \{1, 2, \dots, n\} / J.$$

e.g.  $U_J = \{M \in \text{Gr}(k, n) \mid M = (m' : m'') \sim (J_k | N)\}$ .

$$J = \{j_i\}^k. \quad f_J(m) = N \in M^{k \times (n-k)}(\mathbb{R}). \quad \text{where}$$

$f_J$  is bijection. Check the pseudo-atlas is

smooth. (determine by transform:  $M \rightarrow \tilde{m}$ .)