Differential Forms.

(1) One Forms:

Def: i) $T^*X = \bigcup_{x \in X} T_x^*X$ is called cotangent bundle.

with proj: $\pi : T^*X \longrightarrow X$. $\pi^{-1}(x) = T_x^*X$.

ii) A covector field (or say one-form) is:

$\alpha : X \longrightarrow T^*X$. s.t. $\pi \circ \alpha = I_X$.

i.e. $\alpha|_x \in T_x^*X$.

① Smooth Structure:

i) $U \underset{open}{\subseteq} \mathbb{R}^n$:

. Since $T_x^*U \cong \mathbb{R}^n$. $\forall x \in U$. $\therefore T^*U \cong U \times \mathbb{R}^n$.

$\alpha = (I_U, \tilde{\alpha})$. require $\tilde{\alpha}$ is smooth.

ii) X is arbitrary manifold:

. see in chart: $\tilde{\alpha} : \tilde{U} \longrightarrow \mathbb{R}^n$. $\tilde{\alpha}(\tilde{x}) = \Delta f(\alpha|_{f^{-1}(\tilde{x})})$

It's indept with choice of charts.

Since $\tilde{\alpha}_2|_{f_2(x)} = (D\phi_{12}|_{f_2(x)})^T \tilde{\alpha}_1|_{f_1(x)}$

It only needs to check along one atlas.

② For any smooth Func $h \in C^\infty(X)$. There exists an associated one-form on X. denote dh.

i) $X \underset{\text{open}}{\subseteq} \mathbb{R}^n$:

Fix $h \in C^\infty(X)$, $\exists\, dh|_x \in T_x^*X$. We obtain:

$dh : X \longrightarrow \mathbb{R}^n$. $dh|_x = \left(\frac{\partial h}{\partial x_1}\big|_x, \cdots \frac{\partial h}{\partial x_n}\big|_x \right)$.

Note that: $dx_i(x) = \vec{e_i} \in \mathbb{R}^n$.

$\Rightarrow (dx_k)_1^n$ is basis of T_x^*X. ($dh|_x = \sum_1^n \frac{\partial h}{\partial x_i}\big|_x dx_i$)

$\therefore \forall\, \alpha \in T^*X$. $\alpha = \sum_1^n \alpha_i dx_i$ (flexible each $x \in X$)

Remark: α may not be dh. for some $h \in C^\infty(X)$.

Since it should satisfy: $\frac{\partial \alpha_i}{\partial x_j} = \frac{\partial \alpha_j}{\partial x_i}$ firstly.

ii) X is arbitary manifold:

Define: $dh : X \longrightarrow T^*X$ check it's smooth.
$x \longmapsto dh|_x$

$\nabla f \circ dh|_x \circ f^{-1}(\tilde{x}) = \nabla f (dh|_{f^{-1}(\tilde{x})}) = p\tilde{h}|_{\tilde{x}}$.

③ One-forms behave nicely with : $F : X \longrightarrow Y$. Smooth.

i) For $DF|_x : T_xX \longrightarrow T_{F(x)}Y$.

We have dual linear map: $DF|_x^* : T_{F(x)}^*Y \longrightarrow T_x^*X$.

i.e. $DF|_x^* (dh|_{F(x)}) = d(h \circ F)|_x$

Choose $(U, f) \in A_X$. $(V, g) \in A_Y$. $\tilde{x} = f(x)$. $\tilde{y} = g \circ F(x)$

Written in chart: $D(\tilde{h}\circ\tilde{F})|_{\tilde{x}} = D\tilde{h}|_{\tilde{\eta}} \cdot D\tilde{F}|_{\tilde{x}}$

Written in column vector: $D(\tilde{h}\circ\tilde{F})|_{\tilde{x}} = D\tilde{F}|_{\tilde{x}}^{T} \cdot D\tilde{h}|_{\tilde{\eta}}$

i.e. $DF|_x^* = DF|_x^T$

ii) <u>Pull-back of one-form:</u>

<u>Def:</u> $F: X \longrightarrow Y$. smooth. α is one form on Y. The pull-back of α along F is:

$$F^*\alpha : x \longmapsto (DF|_x)^*(\alpha|_{F(x)}) \in T_x^* X.$$

<u>Remark:</u> In $X = U \underset{\text{open}}{\subseteq} \mathbb{R}^n$. $Y = V \underset{\text{open}}{\subseteq} \mathbb{R}^k$. $F: U \to V$.

$$F^*\alpha : z \longmapsto (DF|_z)^T \alpha|_{F(z)}$$

Test with basis $(d\eta_i)_1^k$ in $T_x^* V$.

$$F^* d\eta_i : z \longmapsto \left(\frac{\partial F_i}{\partial x_1} \cdots \frac{\partial F_i}{\partial x_n}\right)^T\Big|_z$$

i.e. $F^* d\eta_i = \sum_{k=1}^{n} \frac{\partial F_i}{\partial x_k} dx_k$

$\Rightarrow$ For general: $\alpha = \sum_i \alpha_i d\eta_i$. $\alpha_i \in C^\infty(U)$

$$F^*\alpha = \sum_1^k (\alpha_i \circ F)(F^* d\eta_i).$$

<u>Lemma.</u> For dh case: $F^* dh = d(h\circ F)$.

<u>Pf:</u> It's from i). Def of $DF|_x^*$.

<u>e.g.</u> Transition $\phi_{21}: f_1(U_1\cap U_2) \xrightarrow{\sim} f_2(U_1\cap U_2)$

Can induce pull-back: $\tilde{\alpha}_2 = \phi_{12}^* \tilde{\alpha}_1$.

i.e. $\tilde{\alpha}_2|_{f_2(x)} = D\phi_{12}|_{f_2(x)}^T \cdot \tilde{\alpha}_1|_{f_1(x)}$. Transform law.

(2) Wedge Products:

① 2-wedge:

i) For antisymmetric bilinear map on vector space V with finite dimension: $b(v, \hat{v}) = -b(\hat{v}, v)$. $\forall v, \hat{v} \in V$.

The set of such Func's is also a linear space.

Denote it by $\Lambda^2 V^*$:

Note that $\forall u, \hat{u} \in V^*$. We can define:

$u \wedge \hat{u} : V \times V \longrightarrow \mathbb{R}'$. $u \wedge \hat{u}(v, \hat{v}) = u(v)\hat{u}(\hat{v}) - u(\hat{v})\hat{u}(v)$.

$\Rightarrow u \wedge \hat{u} \in \Lambda^2 V^*$.

<u>prop.</u> $(e_k)_1^n$ is basis of V. correspond $(\varepsilon_k)_1^n$ basis of V^*.

Then $\{\varepsilon_i \wedge \varepsilon_j \mid i<j\}$ is basis of $\Lambda^2 V^*$.

i.e. $\dim \Lambda^2 V^* = \binom{n}{2}$.

<u>Pf:</u> $u = \sum_1^n \lambda_i \varepsilon_i$. $\hat{u} = \sum_1^n \mu_i \varepsilon_i$. By linearity:

$\therefore u \wedge \hat{u} = (\lambda_1\mu_2 - \lambda_2\mu_1)\varepsilon_1 \wedge \varepsilon_2 + \cdots + \square\, \varepsilon_{n-1} \wedge \varepsilon_n$.

ii) For $F: V \longrightarrow W$. BLO. induce: $F^*: W^* \longrightarrow V^*$.

We have: $\Lambda^2 F^* : \Lambda^2 W^* \longrightarrow \Lambda^2 V^*$. def by:

$\Lambda^2 F^*(b) : (v, \hat{v}) \longmapsto b(F(v), F(\hat{v}))$.

For another $G: W \longrightarrow U$. BLO. We have:

$\Lambda^2 (G \circ F)^* = \Lambda^2 F^* \circ \Lambda^2 G^*$ (contravariant Functor).

<u>Remark:</u> $\Lambda^2 F^*$ is a $\binom{n}{2} \times \binom{k}{2}$ matrix explicitly.

② p-Wedge:

i) Consider antisymmetric p-linear map on $V^{\times p}$:

$c(v_1 \cdots v_p) = -c(v_{\sigma(1)} \cdots v_{\sigma(p)})$. For transposition

$\sigma \in S_p$. Generally, for permutation $\sigma \in S_p$:

$c(v_1 \cdots v_p) = (-1)^{\sigma} c(v_{\sigma(1)} \cdots v_{\sigma(p)})$.

Denote the set of such Fun's by $\Lambda^p V^*$ (L.S).

Analogously, for $u_k \in V^*$, $1 \le k \le p$. Define:

$$u_1 \wedge u_2 \cdots \wedge u_p (v_1 \cdots v_p) = \sum_{\sigma \in S_p} (-1)^{\sigma} u_1(v_{\sigma(1)}) \cdots u_p(v_{\sigma(p)})$$

prop. $(e_k)_1^n \subseteq V$. basis. Correspond $(\varepsilon_k)_1^n \subseteq V^*$.

$\{\varepsilon_{i_1} \wedge \cdots \varepsilon_{i_p} \mid i_1 < i_2 < \cdots < i_p\} \subseteq \Lambda^p V^*$ is

set of basis. $\dim \Lambda^p V^* = \binom{n}{p}$.

Remark: $1 \le i_1 < \cdots < i_p \le n$ is "correctly-ordered".

Pf: 1°) Check it's l.i.

2°) Test by basis of $V^{\times p}$.

ii) Dimension:

· Extend the def to $p = 0$. $\Lambda^0 V^* = \mathbb{R}^1$. Then:

$\dim \Lambda^{n-p} V^* = \dim \Lambda^p V^*$.

Def: $u \in \Lambda^p V^*$ is decomposable if $u = F(v)$. $\exists v \in (V^*)^{\times p}$.

$(V^*)^{\times p} \xrightarrow{F} \Lambda^p V^*$. i.e. $u = u_1 \wedge u_2 \cdots \wedge u_p$.

Remark: For $\alpha \in \Lambda^p V^*$. It can be written in linear combination of decomposable elements

But the expression isn't unique.

iii) For $F: V \longrightarrow W$. linear. inducing:

$\Lambda^p F^*: \Lambda^p W^* \longrightarrow \Lambda^p V^*$. defined by:

$\Lambda^p F^*(\alpha): (v_1, \cdots v_p) \longmapsto \alpha(F(v_1), \cdots F(v_p))$.

Remark: For another BLO: $G: W \longrightarrow U$.

$\Lambda^p(G \circ F)^* = \Lambda^p F^* \circ \Lambda^p G^*$. (Contravariant Functor)

For decomposable element $\alpha = u_1 \wedge \cdots \wedge u_p$:

$\Lambda^p F^*(u_1 \wedge u_2 \cdots u_p) = (F^* u_1) \wedge \cdots (F^* u_p)$.

To generate explicit form of $F^*\alpha$:

Firstly. suppose $\dim V = \dim W = n$. $(e_k)_1^n$. $(f_i)_1^n$ are basis for V. W. $(\varepsilon_k)_1^n$. $(\phi_k)_1^n$ are dual bases for V^*. W^*

Then: $\Lambda^n F^*(\phi_1 \wedge \cdots \phi_n): (e_1, \cdots e_n) \longmapsto \phi_1 \wedge \cdots \wedge \phi_n (F(e_1) \cdots F(e_n))$

$$= \sum_{\sigma \in S_n} (-1)^\sigma \phi_1(F(e_{\sigma(1)})) \cdots \phi_n(F(e_{\sigma(n)}))$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma M_{1,\sigma(1)} \cdots M_{n,\sigma(n)} = \det(M).$$

where: $F(e_i) = \sum_k^n M_{k,i} f_k$. $M = (M_{i,j})_{n \times n}$.

$\therefore \Lambda^n F^*(\phi_1 \wedge \cdots \wedge \phi_n) = \det(M)\, \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$.

__Remark:__ generally, for $p < m$, $\dim V = n \neq m = \dim W$:

$$\Lambda^p F^*(\phi_{i_1} \wedge \cdots \wedge \phi_{i_p}) = \sum_{1 \le j_1 < \cdots < j_p \le n} \det\left(M\binom{i_1 \cdots i_p}{j_1 \cdots j_p}\right) \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_p}.$$

③ Extend Wedge Product to give:

$$\Lambda^p V^* \times \Lambda^q V^* \xrightarrow{F} \Lambda^{p+q} V^*.$$

__prop.__ There exists unique bilinear map F s.t.

$$\begin{tikzcd}
(V^*)^{\times p} \times (V^*)^{\times q} \arrow[r, "f"] \arrow[dr, "\varphi"'] & \Lambda^p V^* \times \Lambda^q V^* \arrow[d, "F"] \\
& \Lambda^{p+q} V^*
\end{tikzcd}$$

commutes.

__Pf:__ 1°) Uniqueness:

Choose basis $(\varepsilon_k)_1^n$ of V^*. Test by $\varepsilon_{i_1} \wedge \cdots \varepsilon_{i_p}$ and $\varepsilon_{j_1} \wedge \cdots \varepsilon_{j_q}$. It's determined by φ from commutativity.

2°) Existence:

Test with $(\varepsilon_{i_1} \cdots \varepsilon_{i_p}, \varepsilon_{j_1} \cdots \varepsilon_{j_q})$. Def: $F(c_1, c_2) = c_1 \wedge c_2$

Suppose $\sigma \in S_{p+q}$ corrects the order $\{i_k\} \cup \{j_k\}$ after mapping φ.

Since $\sigma = \sigma_1 \sigma_2 \sigma_3$.
$\begin{cases} \sigma_1 \text{ permutates inside } (i_k) \\ \sigma_2 \text{ permutates inside } (j_k) \\ \sigma_3 \text{ shuffle together.} \end{cases}$

Along vertical sides:

First "correct order" by σ_1, σ_2 after f.

Then "correct order" by σ_3 after F.

So it commutates since the sign coincides

prop. For $\omega \in \Lambda^p V^*$, $\hat{\omega} \in \Lambda^q V^*$, $\bar{\omega} \in \Lambda^r V^*$. We have:

i) $\omega \wedge (\hat{\omega} \wedge \bar{\omega}) = (\omega \wedge \hat{\omega}) \wedge \bar{\omega} \in \Lambda^{p+q+r} V^*$.

ii) $\omega \wedge \hat{\omega} = (-1)^{pq} \hat{\omega} \wedge \omega \in \Lambda^{p+q} V^*$.

iii) For linear map $F: U \to V$.

$$\Lambda^{p+q} F^* (\omega \wedge \hat{\omega}) = (\Lambda^p F^*(\omega)) \wedge (\Lambda^q F^*(\hat{\omega})).$$

pf: iii) Decompose into sum.

(3) **p-Forms:**

① **Smooth Structure:**

Def: i) $\Lambda^p T^* X = \bigcup_{x \in X} \Lambda^p T_x^* X$ is called p-th wedge power of cotangent bundle. with projection:

$\pi : \Lambda^p T^* X \to X$. $\pi^{-1}(x) = \Lambda^p T_x^* X$.

ii) A p-form on X is $\alpha : X \to \Lambda^p T^* X$. s.t. $\pi \circ \alpha = I_X$.

For $U \underset{open}{\subseteq} \mathbb{R}^n$. $\Lambda^p T^* U \cong U \times \Lambda^p (\mathbb{R}^n)^* \cong U \times \mathbb{R}^{\binom{n}{p}}$

So a p-form on U is just Func: $U \to \mathbb{R}^{\binom{n}{p}}$

For X is arbitary manifold:

Choose $(U, f) \in A_X$. See in chart:

Since $\Lambda^p (\Delta_f^x)^* : \Lambda^p T_x^* X \cong \Lambda^p (\mathbb{R}^n)^*$ $((\Delta_f^x)^* = \nabla f)$

$\therefore \tilde{\alpha} : \tilde{U} \to \Lambda^p (\mathbb{R}^n)^*$

$x \longmapsto \Lambda^p (\Delta_f^x)^* (\alpha \mid f^{-1}(x))$

Remark: It's indept with choice of charts:

There're related by:

$$\Lambda^p(D\phi_{12}|_{f(x)})^* : \Lambda^p(\mathbb{R}^n)^* \longrightarrow \Lambda^p(\mathbb{R}^n)^*.$$

It's $\binom{n}{p} \times \binom{n}{p}$ matrix of Smooth Func.

② Wedge Together:

For $\alpha \in \Lambda^p T^*X$. $\beta \in \Lambda^q T^*X$. Define:

$$\begin{aligned} \alpha\wedge\beta : X &\longrightarrow \Lambda^{p+q}T^*X \\ x &\longmapsto \alpha|_x \wedge \beta|_x \end{aligned}$$

α, β smooth $\Rightarrow \alpha\wedge\beta$ smooth.

Remark: It follows: $\alpha\wedge\beta = (-1)^{pq}\beta\wedge\alpha$.

$(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma)$ holds.

③ $F: X \to Y$. Smooth. inducing pull-back:

$$\begin{aligned} F^*\alpha : X &\longrightarrow \Lambda^p T^*X \\ x &\longmapsto \Lambda^p(DF|_x)^*(\alpha|_{F(x)}) \end{aligned}$$

for $\forall \alpha \in \Lambda^p T^*Y$.

check it's smooth: since $F^*(\alpha\wedge\beta) = F^*(\alpha)\wedge F^*(\beta)$

Besides. F is linear. F^*y is Smooth for $y \in T^*Y$.

e.g. Transition Law is pull back along transition function ϕ_{12}: $\tilde{\alpha}_2 = \phi_{12}^*\tilde{\alpha}_1$. $\alpha \in \Lambda^p T^*X$.

For $\beta \in \Lambda^n T^*X$. It's explicitly: $(\dim X = n)$

$$\tilde{\beta}_2|_{f_2(x)} = \det(D\phi_{12}|_{f_1(x)})\,\tilde{\beta}_1|_{f_1(x)}$$

(4) Exterior Derivate:

Denote: $\Lambda^p(X)$ is set of all smooth p-forms on X

Remark: It's infinite dimensional vector space.

For $p=0$. $\Lambda^0(X) = C^\infty(X)$.

① For $U \underset{open}{\subseteq} \mathbb{R}^n$:

· Suppose $x_1 \cdots x_n$ are coordinate Func's on U:

Define: linear operator: $d : \Lambda^p(U) \longrightarrow \Lambda^{p+1}(U)$.

s.t. for $\alpha = \alpha_{(i_1 \cdots i_p)} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$:

$$d\alpha = \sum_k^n \frac{\partial \alpha_{(\cdots)}}{\partial x_k} dx_k \wedge dx_{i_1} \cdots \wedge dx_{i_p}.$$

We call it by exterior derivate.

prop. i) $\forall \alpha \in \Lambda^p(U)$. $d(d\alpha) = 0 \in \Lambda^{p+2}(U)$.

ii) $\forall \alpha \in \Lambda^p(U)$. $\beta \in \Lambda^q(U)$.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta.$$

ii) For $V \subseteq \mathbb{R}^k$. $F : U \to V$ smooth.

$$d(F^*\alpha) = F^* d\alpha \text{. for } \forall \alpha \in \Lambda^p(U).$$

Pf: i). ii) are trivial. For iii): $\alpha = \tilde{\alpha}\, dx_{i_1} \wedge \cdots dx_{i_p}$.

$$F^*\alpha = (\tilde{\alpha} \circ F)(F^* dx_{i_1}) \wedge \cdots (F^* dx_{i_p}).$$

Since $F^* dx_{i_k} = d(x_{i_k} \circ F)$.

$$\therefore dF^*\alpha = d(\tilde{\alpha}\circ F)\wedge(F^*dx_{i_1})\cdots\wedge(F^*dx_{i_p})$$

$$= F^*d\tilde{\alpha}\wedge F^*dx_{i_1}\cdots\wedge F^*dx_{i_p}$$

$$= F^*(d\tilde{\alpha}\wedge dx_{i_1}\cdots\wedge dx_{i_p}) = F^*d\tau.$$

② <u>For arbitrary manifold X:</u>

<u>Lemma.</u> For $\alpha\in\Lambda^p(X)$. There exists a unique $(p+1)$-form $d\alpha\in\Lambda^{p+1}(X)$. St. $\forall (U,f)\in A_X$. Write $d\alpha$ in chart: $d\tilde{\alpha}\in\Lambda^{p+1}(U)$.

<u>Pf:</u> Check: $d\alpha: (U,f)\longmapsto d\tilde{\alpha}_f\in\Lambda^{p+1}(U)$.

Satisfies the transform law:

Since $\tilde{\alpha}_1=\phi_{21}^*\tilde{\alpha}_2$. $d\tilde{\alpha}_1=d(\phi_{21}^*\tilde{\alpha}_2)=\phi_{21}^*d\tilde{\alpha}_2$

③ <u>De Rham Cohomology:</u>

<u>Def:</u> $\alpha\in\Lambda^p(X)$. i) α is closed if $d\alpha=0$.

ii) α is exact if $\exists\beta\in\Lambda^{p-1}(X)$. $\alpha=d\beta$.

<u>Remark:</u> $\{$exact p-forms$\}\subseteq\{$closed p-form$\}\subseteq\Lambda^p(X)$.

There're actually subspaces.

<u>Def:</u> For $0\le p\le n$. p^{th}-De Rham cohomology group is $H^p_{dR}(X)=\{\text{closed } p\text{-forms}\}/\{\text{exact } p\text{-forms}\}$