

Lebesgue Measure

(1) Pre:

① An example: Cantor Set:

- Note firstly that:
 - i) $U \subseteq \bigcup_{n=1}^{\infty} I_n$: $U = \bigcup_{n=1}^{\infty} I_n$. disjoint intervals
 - ii) $U \subseteq \bigcup_{n=1}^{\infty} Q_n$, $1 > 1 = U = \bigcup_{n=1}^{\infty} Q_n$. almost disjoint cubes

The expression in the first case
is unique. But in the last case
it's not unique!

• Def. Cantor set C_g ($0 < g < 1$): It means remove
the central interval of length g . Denote
 C_g^k is the k^{th} stages of the operation:
Then $C_g = \bigcap_{k=1}^{\infty} C_g^k$, $m(\bigcap_{k=1}^{\infty} C_g^k) = (1-g)^n \rightarrow 0$.
 $\therefore m(C_g) = 0$.

Properties:

i) $|C_g| = 2^{-\frac{n}{g}}$

ii) C_g is totally disconnected (Nowhere is dense)

iii) C_g is cpt. No isolated point exists!

iv) C_g is perfect set ($[0,1]/C_g = \bigcup_{n=1}^{\infty} I_n$)

Remark: A more general case:

Cantor-like set \hat{C} : At k^{th} stage.
we will remove 2^{k^1} central intervals
of length ℓ_k in \hat{C}_k .

Then $\hat{C} = \bigcap_{n=1}^{\infty} \hat{C}_n$. satisfies prop. ii) - iv).

Note that $m(\hat{C}) = 1 - \sum_{k=1}^{\infty} 2^{k^1} \ell_k$

for ℓ_k is small enough. $m(\hat{C})$ may be > 0 !

Cantor-Lebesgue func.

Take $C_{\frac{1}{3}}$ for example:

$\forall x \in C_{\frac{1}{3}}$. x has unique representation:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}, \quad a_n = 0 \text{ or } 2. \quad (\text{Because: } \frac{1}{3^n} = \sum_{k=1}^{\infty} \frac{2}{3^{2k}})$$

$$F: C_{\frac{1}{3}} \longrightarrow [0,1] \quad . \quad \text{It's called the} \\ \sum_{n=1}^{\infty} \frac{a_n}{3^n} \longmapsto \sum_{n=1}^{\infty} \frac{b_n}{2^n}, \quad b_n = \frac{a_n}{2} \quad \text{Cantor-Lebesgue Func.}$$

F is conti, surjective. let $h = \sup_{\eta \leq x} F(\eta)$

Then $h: [0,1] \rightarrow [0,1]$. extend on $[0,1]$

$h(x)$ is monotone, conti.

(2) Exterior Measure:

- Def: $\forall E \subseteq \mathbb{R}^k. m^*(E) = \inf \{ \sum_{i=1}^{\infty} l(Q_i) \mid E \subseteq \bigcup_{i=1}^{\infty} Q_i \text{ and } Q_i \text{ are disjoint almost closed cubes} \}$

Remark: "Countably infinite" is necessary!

Properties:

$$\textcircled{1} \text{ Mono: } E_1 \subseteq E_2 \Rightarrow m^*(E_1) \leq m^*(E_2)$$

$$\textcircled{2} \quad m^*(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

$$\textcircled{3} \quad \text{If } \lambda(E_1, E_2) > 0. \text{ Then } m^*(E_1) + m^*(E_2) = m^*(E_1 \cup E_2)$$

(3) Lebesgue Measurable:

- E is L -measurable $\Leftrightarrow \forall \varepsilon > 0. \exists O \subseteq \bigcup_{i=1}^{\infty} Q_i. E \subseteq O.$

St. $m^*(O - E) < \varepsilon$. Denote $m^*(E) = m(E)$.

- Properties:
 - Open / closed sets are measurable
 - $m^*(E) = 0 \Rightarrow E$ is measurable
 - E is measurable $\Rightarrow E^c$ is measurable
 - $\bigcup_{i=1}^{\infty} E_i$ is measurable if E_i 's are measurable
 - For $\{E_n\}_{n=1}^{\infty}. E_i \cap E_j = \emptyset. i \neq j$. Then:

$$m(E \cup \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

$$\text{vi) } E_n \nearrow E. m(E) = \lim_{n \rightarrow \infty} m(E_n)$$

$$\text{vii) } E_n \downarrow E. \exists N. \forall n > N. m(E_n) < \infty. m(E) = \lim_{n \rightarrow \infty} m(E_n)$$

Thm. i) If E is \mathcal{L} -measurable Then exists $O \subseteq \mathbb{R}^n$ open
 $F \subseteq \mathbb{R}^n$ st. $F \subseteq E \subseteq O$. $m(O/F) < \varepsilon$.

ii) In case ii). If $m(E) < \infty$. Then exists opt
 set k . st. $m(E/k) < \varepsilon$. $K \subseteq E$. And it
 exists $F = \bigcup Q_j$. closed cubes. $m(E/F) < \varepsilon$.

Cor. From i). there exist Q_n set: $\cap Q_n$ open.

$E \subseteq Q_n \vdash \cap Q_n \supseteq E$. st. $m(\cap Q_n/E) = 0$.

and F_n set: $\cup K_n$. closed. $E \supseteq F_n \vdash E \supseteq \cup F_n$

st. $m(E/\cup K_n) = 0$.

② Techniques for check

L-measurable:

i) prop. If $m^*(A \Delta B) = 0$ Then $m^*(A) = m^*(B)$

Therefore. A is measurable $\Leftrightarrow B$ is measurable.

ii) prop. $A \subseteq E \subseteq B$. $m(A), m(B) < \infty$ If $m(A) = m(B)$

Then E is measurable

iii) prop. $m(E) < \infty$. $E = E_1 \cup E_2$ If $m(E) = m^*(E_1) + m^*(E_2)$

Then E_1, E_2 are measurable.

Pf: Only prove iii)

i), ii) are trivial.

Lemma. A is \mathcal{L} -measurable. Then for $\forall B \subseteq \mathbb{R}^d$.

$$\text{We have: } m(A) + m^*(B) = m^*(A \cup B) + m^*(A \cap B)$$

Pf: Apply Caratheodory Thm. on $B, A \cup B$.

It can be extended to general measure.

$$\Rightarrow \text{Since } \exists H_1, H_2 \text{ with form: } \bigcap_{n=1}^{\infty} (\bigcup_{i=1}^n Q_{ni}), H_i \supseteq E_i$$

$$\text{where } m^*(\bigcup_{i=1}^n Q_{ni}) \leq m^*(E) + \frac{1}{n}, E \subseteq \bigcup_{i=1}^n Q_{ni}, \forall n \in \mathbb{Z}^+$$

$$\text{i.e. } m(H) = m^*(E). \text{ (Denote: } H = \bigcap_n \bigcup_i Q_{ni})$$

$$\therefore m(H_1) + m(H_2) \geq m(H_1 \cup H_2) \geq m^*(E_1 \cup E_2)$$

$$= m^*(E_1) + m^*(E_2) = m(H_1) + m(H_2)$$

$$\therefore m(H_1 \cap H_2) = 0, m(H_1 - E_1 \cup E_2) = 0$$

$$\Rightarrow m(H_1 - E_1) = 0. \quad \because E_1 \text{ is } \mathcal{L}\text{-measurable. So, } E_1.$$

Remark: i) $\forall E \subseteq \mathbb{R}^d, \exists H$ is \mathcal{L} -measurable. st.

$$m(H) = m^*(E)$$

ii) If exists H is \mathcal{L} -measurable. st.

$$m(H/E) = 0 \text{ Then } E \text{ is } \mathcal{L}\text{-measurable.}$$

③ Open Interval problems:

i) $E \subseteq \mathbb{R}, m^*(E) > 0, \forall 0 < \alpha < 1, \exists I$ interval.

$$\text{st. } m^*(E \cap I) \geq \alpha m^*(I)$$

Pf: $\exists \bigcup Q_n \supseteq E, m^*(\bigcup Q_n) \leq m^*(E) + \varepsilon, \exists I_n$ st.

$$m^*(I_n) \leq m^*(Q_n) + \frac{\varepsilon}{2^n}, Q_n \subseteq I_n \text{ (refine)}$$

$\therefore \cup I_n \supseteq E$. $m^*(\cup I_n) \leq m^*(E) + 2\varepsilon$.

If $m^*(I_n \cap E) < \alpha m(I_n)$, $\forall n$. Then it will contradict!

ii) Steinhaus Thm:

$m(E), m(F) > 0$. $E, F \subseteq \mathbb{R}$. Then $E+F$ contains an interval.

Pf: $\chi_E * \chi_F : \mathbb{R} \rightarrow \mathbb{R}$. Conti.

$$\int_{\mathbb{R}^2} \chi_E(\eta) \chi_F(x-\eta) dm = m(E)m(F) > 0.$$

$\therefore m\{\chi_E * \chi_F > 0\} > 0$. $\{\chi_E * \chi_F > 0\}$ is open.

$\therefore \exists I \subseteq \{\chi_E * \chi_F > 0\}$.

When $x \in I \Rightarrow x-y \in F$, $y \in E$. $\therefore y \in E+F$

i.e. $I \subseteq E+F$.

(4) Invariant property

of Lebesgue measure:

For E is \mathcal{L} -measurable. $m(E+h) = m(E)$.

$$m(\delta E) = \frac{n}{h} \delta h m(E). E \subseteq \mathbb{R}^n. \delta E = \{(x_k + h_k) / (x_k) \in E\}.$$

Pf: Directly from cover of cubes!

(4) \mathcal{L} -measurable Functions:

Def: f is \mathcal{L} -measurable $\Leftrightarrow \{f < a\}$ is \mathcal{L} -measurable
for $\forall a \in \mathbb{R}$.

Properties: i) f is \mathcal{L} -measurable $\Leftrightarrow \forall O$ open, F closed.

$f'(O), f'(F)$ is measurable

ii) $\{f_n\}$ measurable. Then $\sup f_n, \inf f_n$.

$\overline{\lim} f_n, \underline{\lim} f_n$ are measurable.

iii) f, g measurable $\Rightarrow f^k, f+g, fg$ are measurable!

iv) f conti. ϕ is \mathcal{L} -measurable $\Rightarrow f \circ \phi$ is \mathcal{L} -measurable.

① Approximation:

i) $f \geq 0$, \mathcal{L} -measurable. Then exists seq of simple functions $\{\phi_n\}$ s.t. $\phi_n \nearrow f$.
A $x \in \mathbb{R}^d$, pointwise.

$$\Downarrow f = f^+ - f^-$$

ii) f is \mathcal{L} -measurable. Then exists seq of simple func. $\{\phi_n\}$ s.t. $|\phi_n| \leq |\phi_{n+1}|$, $\phi_n \rightarrow f$ pointwise for $\forall x \in \mathbb{R}^d$, s.t. $|\phi_n| \leq |f|$.

$$\Downarrow \begin{array}{l} \text{Step } \xrightarrow{\text{approx}} \\ \text{Simple func.} \end{array}$$

iii) f is \mathcal{L} -measurable. Then exists seq of step functions $\{\phi_n\}$, $\phi_n \rightarrow f$, a.e. x .

Pf: Use simple functions to exhaust f .

Are for $X_E = \exists \{R_j\}$ rectangles.

s.t. $\cup R_j \subseteq E$. refine R_j to obtain step func.

(*) Simple Func χ_E require E is \mathcal{L} -measurable!

② Littlewood's Three Principles:

- i) Every measurable set is nearly a finite union of intervals.
- ii) Every convergent seq of Func. is nearly uniform.

Thm (Egorov)

{ f_n } seq of measurable func on E .

where $m(E) < \infty$. $f_n \rightarrow f$ a.e. on E .

Then for $\forall \varepsilon > 0$. $\exists A_\varepsilon \subseteq E$, closed. St.

$f_n \xrightarrow{n} f$ in A_ε . $m(E/A_\varepsilon) < \varepsilon$.

- iii) Every measurable function is nearly conti.

Thm (Lusin)

$f < \infty$. measurable on E . $m(E) < \infty$.

Then for $\forall \varepsilon > 0$. $\exists A_\varepsilon \subseteq E$, closed. St.

$f|_{A_\varepsilon}$ is conti. $m(E/A_\varepsilon) < \varepsilon$.

Pf: For iii) The point is:

$$E_k^n = \{x \in E \mid |f_j - f| < \frac{1}{n}, j \geq k\} \nearrow E \text{ (as } n \rightarrow \infty)$$

exhaust the convergent domain.

For each n . $\exists E_{k_n}^n \overset{\frac{\varepsilon}{2^n}}{\subset} E$. Then intersect them to obtain $A'_\varepsilon = \cap E_{k_n}^n$.

For iii): Since we have $\{q_n\}$ step functions.

$\phi_n \rightarrow f$. a.e. by ii). Obtain A_ε :

$\phi_n \xrightarrow{u} f$ on A_ε .

Refine each $\text{supp } \phi_n$ to let ϕ_n be conti. on E_n .

$$E_n \subseteq \text{supp } \phi_n. m(\text{supp } \phi_n / E_n) < \frac{\varepsilon}{2^n}$$

Refine $A_\varepsilon - \cup E_n$ to obtain closed set!

Gr. Every measurable function is the limit a.e. of conti. functions.

Pf. Use $B_n = (n, n+1]^k$ to approx. \mathbb{R}^k .

By Lusin's Thm. $\exists E_n \subseteq B_n$. St.

$$m(B_n / E_n) < \frac{\varepsilon}{2^n}. f|_{E_n} \text{ is conti.}$$

By Tietze Extension. Let:

F_n is extension of $f|_{E_n}$ on \mathbb{R}^k .

$\therefore F_n \rightarrow f$ outside $\overline{\bigcup B_n / E_n}$.

$$m(\overline{\bigcup B_n / E_n}) = 0. \therefore F_n \rightarrow f. \text{ a.e.}$$

Integration Theory

(1) Establishment:

① Stage one:

• For simple functions: $\varphi(x) = \sum_{k=1}^N c_k \chi_{E_k}$, on \mathbb{R}^d

$$\text{Def: } \int_{\mathbb{R}^d} \varphi dm = \sum_{k=1}^N c_k m(E_k)$$

② Stage two:

• For bounded functions support on finite measure set.

f should be L -measurable firstly.

Then $\exists \varphi_n$ simple func. $\rightarrow f$. $|\varphi_n| \leq |f|$

$$\text{Def: } \int_{\mathbb{R}^d} f dm = \lim_n \int_{\mathbb{R}^d} \varphi_n dm.$$

(From Egorov Thm. claim $\{\int_{\mathbb{R}^d} \varphi_n\}$ converges)

Remark: It's similar with Riemann integral

$$\text{Since } \int f dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x_i$$

③ Stage Three:

• For $f \geq 0$, L -measurable. Def:

$$\int_{\mathbb{R}^d} f dm = \sup \left\{ \int_{\mathbb{R}^d} g dm \mid 0 \leq g \leq f, \|g\|_1 < \infty \right\}$$

$$m(\text{Supp } g) < \infty \right\}.$$

For $\int_{\mathbb{R}^d} f dm < \infty$. We say it's L -integrable.

(4) Stage four:

For general functions (L -measurable)

$$\text{Let } f = f^+ - f^-. \text{ Def: } \int f dm = \int f^+ dm - \int f^- dm$$

$\Rightarrow f$ is L -integrable $\Leftrightarrow \int |f| dm$ is finite.

Properties:

$$i) \exists F. m(F) < \infty. \text{ st. } \int_F |f(x)| dm < \varepsilon.$$

$$ii) \forall \varepsilon > 0. \exists \delta > 0. \text{ st. } \int_E |f| dm < \varepsilon. \text{ whenever } m(E) < \delta.$$

Pf: By approx: $f \cdot \chi_E$ is integral.

(2) Limit Thm:

Fatou's lemma:

If $\{f_n\}$ seq of L -measurable functions. $f_n \geq 0$.

$$\text{Then } \int \liminf_n f_n dm \leq \liminf_n \int f_n dm.$$

Cor. (Monotone Convergence Thm)

$f_n \geq 0. f_n \uparrow f. a.e.$ Then: $\int \lim f_n = \lim \int f_n dm.$

Remark: " $f_n \geq 0$ " can be removed. Since let

$$F_n = f_n - f_1 \geq 0. \text{ Apply on } F_n \text{'s.}$$

Cor. C Dominated Convergence Thm)

$g(x)$ is L -integrable. $|f_n| \leq g$. $\forall n$.

If $f_n \rightarrow f$ a.e. Then $\lim_n \int f_n = \int \lim_n f_n dm$

$f_n \rightarrow f$ in L^1 .

Pf: Return to the def: operate on \mathcal{I} .

$0 \leq g \leq f$. $\|g\|_1 < \infty$. $m(\text{supp } g) < \infty$.

Let $g_n = \min\{f_n, g\}$.

For the last one: Separate $E_N = \{g \leq N\}$.

By Egorov Thm.

(3) $L^1(\mathbb{R}^n)$ Space:

$L^1(\mathbb{R}^n)$ is a linear space, with a complete norm $\|f\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f| dm$.

Pf: For $\epsilon > 0$. $\exists N$. St. $n, m > N$. $\|f_n - f_m\|_{L^1} < \epsilon$.

1) Find a subseq has limit function:

Choose $\{n_k\}$. St. $\|f_{n_k} - f_{n_{k+1}}\|_{L^1} \leq \frac{1}{2^k}$

$\therefore m(\{f_{n_k} - f_{n_{k+1}} > \frac{1}{2^k}\}) \leq \frac{C}{2^k}$

$f = f_1 + \sum f_{n_k} - f_{n_{k+1}}$ converge a.e.

2) Since $\|f_n - f\|_{L^1} \leq \|f_n - f_{n_k}\|_{L^1} + \|f_{n_k} - f\|_{L^1}$

$\therefore f_n \rightarrow f$ in L^1 . (Check $f \in L^1(\mathbb{R}^n)$, first)

Remark: $f_n \xrightarrow{L^1} f \Rightarrow \exists f_{n_k} \rightarrow f$ a.e.

① Density:

$M = \{f | f \text{ is simple function on } \mathbb{R}^d\}$.

$N = \{f | f \text{ is step function on } \mathbb{R}^d\}$.

We have: $\overline{M} \supseteq L^1(\mathbb{R}^d)$, $\overline{N} \supseteq L^1(\mathbb{R}^d)$, $\overline{C_c(\mathbb{R}^d)} \supseteq L^1(\mathbb{R}^d)$

② Invariant properties:

$$\|f_h\|_{L^1(\mathbb{R}^d)} = \|f(x+h)\|_L = \|f\|.$$

$$\|f(\delta x)\|_{L^1(\mathbb{R}^d)} = \frac{1}{|\delta x|} \|f(x)\|_{L^1(\mathbb{R}^d)}$$

(Check on simple functions)

Prop. $\|f_h - f\|_{L^1(\mathbb{R}^d)}, \|f(\delta x) - f\|_{L^1(\mathbb{R}^d)} \rightarrow 0$ ($h \rightarrow 0, \delta \rightarrow 1$)

whenever $f \in L^1(\mathbb{R}^d)$.

Pf: Approx by $C_c(\mathbb{R}^d)$.

(4) Tonelli-Fubini Thm.

① Fubini Thm:

$f(\vec{x}, \vec{y}) \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Then:

i) $f^* \in L^1(\mathbb{R}^{d_2})$, a.e. \vec{y} , $f_x \in L^1(\mathbb{R}^{d_1})$, a.e. x .

ii) $\int f d(\mu_{d_1} \times \mu_{d_2}) = \int f d\mu_{d_2} d\mu_{d_1} = \int f d\mu_{d_1} d\mu_{d_2}$

If: let $F = \{f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) | f \text{ satisfies}$
the conclusion\}

- 1') F is closed under linear combination
and monotone limit. (Monotone Convergence Thm)
- 2') Check on χ_E . E is (a set. $m(E) < \infty$)
- For $E = \alpha_1 \times \alpha_2$ bounded open cube in \mathbb{R}^n .
 - For E is subset of boundary of closed cube in \mathbb{R}^n .
 - For $E = \bigcup_{i=1}^k \alpha_i$. finite union of almost disjoint closed cubes in \mathbb{R}^n .
- Pf: By i), ii) and linearity.
- iv) For E is open. $m(E) < \infty$.
- Pf: By iii). $\chi_E \leftarrow \sum_{i=1}^n \chi_{\alpha_i}$ (mono-limit)
- $\Rightarrow E = \bigcap_{i=1}^n O_i$. choose O_0 open. $E \subseteq O_0$
 $m(O_0) < \infty$. $O_0 \cap (\bigcap_{i=1}^n O_i)$ is decreasing seq.
From monotone limit $\therefore \chi_E \in F$.
- 3') Check on χ_E . $m(E) = 0$
- Pf: Since $\exists G$. G set. $E \subseteq G$. $m(G) = 0$.
apply 2') on G . obtain $\chi_E \in F$.
- 4') Check on χ_E . $m(E) < \infty$.
- Pf: $E = G \cup N$. $m(N) = 0$. Apply 2'), 3')

5°) $\forall f \in L^1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, $f \in F$.

Pf: $f = f^+ - f^-$, f^\pm are increasing limit
of sets of simple functions.

② Applications:

i) Tonelli Thm:

$f(\vec{x}_1, \vec{x}_2) \geq 0$, belongs to L -measurable func. on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Then: f^x, f_y is L -measurable on $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$ for
a.e. y and a.e. x .

$\int_{\mathbb{R}^{d_1}} f^x dx, \int_{\mathbb{R}^{d_2}} f_y dy$ is L -measurable
on $\mathbb{R}^{d_2}, \mathbb{R}^{d_1}$, a.e. y and a.e. x .

Pf: Approx. by L -measurable function:

$$f_N = \begin{cases} f(x, y), & |(x, y)| < N, |f| < N \\ 0, & \text{otherwise} \end{cases} \rightarrow f$$

Apply Fubini Thm on $\{f_N\}$.

ii) Product measure:

Note firstly that from i). $\forall E$ is measurable
in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, so E_x, E_y are measurable in
 $\mathbb{R}^{d_2}, \mathbb{R}^{d_1}$. Next, we can more general case.

Prop. $E = E_1 \times E_2$ is measurable on $\mathbb{R}^n \times \mathbb{R}^{k_2}$

If $m^*(E_1) > 0$. Then E_2 is measurable

Pf: By Fubini Thm. E_2 is measurable on F .

St. $m(F^c) = 0$. Since $m^*(E_1) > 0 \therefore E_1 \cap F \neq \emptyset$.

Select $x \in E_1 \cap F$. $E_2 = E_x$ is measurable

Remark: Note $\{0\} \times N \subseteq \mathbb{R}^n \times \mathbb{R}^k$. null set.

$m(\{0\}) = 0$. N is Vitali set. Unmeasurable!

Lemma. $m^*(E_1 \times E_2) \leq m^*(E_1)m^*(E_2)$

Pf: easy to check by def of cover.

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prop. For. E_1, E_2 are measurable. Then:

$$m(E_1)m(E_2) = m(E_1 \times E_2)$$

Pf: Only need to check $E_1 \times E_2$ is measurable

Then apply Fubini on it.

Check $m^*(h_1 \times h_2 / E_1 \times E_2) = 0$. where h_1, h_2

is as set of E_1, E_2 . St. $m(h_1/E_1), m(h_2/E_2) = 0$.

Apply the lemma!

prop. $f(x)$ is \mathcal{L} -measurable on \mathbb{R}^k . Then $f(x-y)$

$= \tilde{f}(x,y)$ is \mathcal{L} -measurable on $\mathbb{R}^k \times \mathbb{R}^k$

Pf. Check: If $E = \{f < a\}$, measurable set.

Then $E - E = \{x - y \mid x, y \in E\}$, measurable.

Lemma. G is open. Then $O - G$ is open.

Pf. $\forall x \in O - G$. $x = \eta_1 - \eta_2$, $\eta_1, \eta_2 \in O$

$\therefore \exists U_1, U_2$, open. $\eta_1 \in U_1$, $\eta_2 \in U_2$

$\therefore x \in U_1 - U_2 \subseteq O - G$, open neighbor

Lemma. $m(F) = 0$. Then $F - F$ is null set

Pf. Denote $F_k = F \cap B_k$, $B_k = \{|x| < k\}$, check $m(F_k - F_k) = 0, \forall k$.

$\exists O_{nk}$ open. $F_k \subseteq O_{nk}$, $m(O_{nk}) < \frac{1}{n}$. Note that:

$$m((O_{nk} - O_{nk}) \cap B_k) = \int \chi_{(O_{nk} - O_{nk}) \cap B_k} = \int \chi_{O_{nk}}(x - \eta) \chi_{B_k}(\eta) d\mu$$

$\therefore \text{LHS} = m(O_{nk}) m(B_k) \rightarrow 0 \therefore m(F_k - F_k) = 0$. $\therefore F - F$ is null set.

\Rightarrow Since $E = \bigcap O_n / N$. $E - E = \bigcap (O_n - O_n) / N$

$\bigcap (O_n - O_n)$ is a lis set as well!

Cor. If f is measurable on \mathbb{R}^d . Then $f(x, \cdot) = f(x)$

is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

Pf. $\{f < a\} = \{f < a\} \times \mathbb{R}^{d_2}$

Remark: The projection is measurable.

Cor. $f \geq 0$ on \mathbb{R}^d . $A = \{(x, \eta) \mid 0 \leq \eta \leq f(x)\}$. Then:

f is measurable $\Leftrightarrow A$ is measurable.

$\int f d\mu = m(A)$. If one side holds.

Pf. From above: $F(x, \eta) = \eta - f(x)$, measurable on \mathbb{R}^{d+1} .

$A = \{F \geq 0\} \cap \{\eta \geq 0\}$. Apply Tonelli Thm.

Differentiation Theory

(1) Differentiation of integral:

Consider average problem:

$$\lim_{\substack{m(B) \rightarrow 0 \\ x \in B}} \frac{1}{m(B)} \int_B f dm = f(x), \text{ a.e. } x \text{ hold?}$$

① Def: Littlewood maximal function:

$$f \in L^1(\mathbb{R}^n), f^*(x) = \sup_{\substack{B \\ x \in B}} \frac{1}{m(B)} \int_B |f| dm$$

- properties:
- i) f^* is measurable ($f^* > \alpha$ is open by def)
 - ii) $m\{f^* > \alpha\} \leq \frac{3^n}{\alpha} \|f\|_{L^1(\mathbb{R}^n)}$
 - iii) $f^* < \infty$, a.e. x. (Let $\alpha \rightarrow \infty$ in i))

Vitali's Lemma.

$U = \bigcup_{\alpha \in A} B_\alpha$. $\forall c < m(U)$. There exists disjoint

$$\{B_i\}_{i=1}^k, \text{ s.t. } \sum_{i=1}^k m(B_i) \geq \frac{c}{3^n}.$$

Pf: By cpt set $k \in U$, $m(k) > 0$.

Operate on these finite many balls over k.

For ii): Choose $B_k = \{x \mid \frac{1}{m(B_k)} \int_{B_k} |f| dm > \alpha\}$

Since $\{f^* > \alpha\}$ can be cover by UB_k .

Lebesgue differential

Theorem:

. For $f \in L^1_{loc}(R^d)$, $\lim_{\substack{n \rightarrow \infty \\ B \ni x}} \frac{1}{m(B)} \int_B |f(\eta) - f(x)| d\eta = 0$. a.e.x.

Pf: Approx. f by $C_c(R^d)$ function $g(x)$. firstly.

For conti. Func. the conclusion holds.

$$\Rightarrow \text{check } E_\alpha = \{ (f-g)^* + |f-g| > 2\alpha \}$$

has measure 0 for $\forall \alpha$.

$$\text{where } (f-g)^* = \overline{\lim_{\substack{n \rightarrow \infty \\ B \ni x}}} \frac{1}{m(B)} \int_B |g(\eta) - f(\eta)| d\eta$$

$$E_\alpha \subseteq \{ (f-g)^* > \alpha \} \cup \{ |f-g| > \alpha \}.$$

By Chebyshev, easy to prove!

Remark: replace $\{B(r, r)\}_{r>0}$ with general sets:

Σ_{basic} is family of sets shrink regularly

nt $x, y \in \mathbb{R}^d, \forall \alpha \in A, \exists B \text{ a ball. st. } y \in B$.

$$m(B) \geq c m(B)$$

$$\Rightarrow \overline{\lim_{\substack{n \rightarrow \infty \\ B \ni x}}} \frac{1}{m(B)} \int_B |f(\eta) - f(x)| d\eta = 0, \text{ a.e.x.}$$

for $L^1_{loc}(R^d)$ function f .

iv) $f^*(x) \geq |f(x)|, \forall x$.

Pf: $f^*(x) = \frac{1}{m(B)} \int_B |f(\eta)| d\eta \rightarrow |f(x)|, x \in B, m(B) \rightarrow 0$.

v) $f \in L_{loc}^1(x^n)$, $f \neq 0$. Then $f^*(x) \geq \frac{c}{|x|^d}$
 for some $c > 0$, and $\forall |x| \geq 1$

Pf: $f^*(x) \geq \frac{1}{m(B)} \int_B |f(y)| dm. \quad \forall x \in B$

Note that $B(0, R) \subseteq B(x, |x|+R) = \tilde{B}$

(It's for marking the integral
 in kept with x)

$$\therefore f^*(x) \geq \frac{1}{m(\tilde{B})} \int_{\tilde{B}} |f| dm \geq \frac{1}{m(\tilde{B})} \int_{B(0, R)} |f| dm$$

$$m(\tilde{B}) = (R+|x|)^d V_d \leq (R+1)^d |x|^d V_d.$$

② Application:

Def: Lebesgue density of E :

For E is measurable, x is Lebesgue

density of E . If $\lim_{\substack{m(B \cap E) \\ m(B) \rightarrow 0 \\ x \in B}} \frac{m(B \cap E)}{m(B)} \rightarrow 1$

$$\Rightarrow \text{Actually } \frac{1}{m(B)} m(B \cap E) = \frac{1}{m(B)} \int_B x_E dm$$

$x_E \in L_{loc}^1(x^n)$ $\therefore \forall x \in E$, x is Lebesgue
 density. (for. a.e. x)

Def: Lebesgue set E for f :

$x \in E$. Then $f(x)$ is finite and

$$\lim_{\substack{m(B) \rightarrow 0 \\ x \in B}} \frac{1}{m(B)} \int_B |f(y) - f(x)| dm = 0.$$

\Rightarrow If $f \in L^1_{loc}(\mathbb{R}^d)$. Then almost every $x \in E$.

(2) Differentiation of functions:

Next, we wish find condition for F . st:

$$F(b) - F(a) = \int_a^b F'(x) dx. \quad (\text{in } \mathbb{R}^d \text{ only!})$$

(*) In \mathbb{R} : the
open set O
 $= U$ in. unique
A strong property

① Bounded Variation:

Recall that for a curve y , parametrized by
 $Z(t) = (x(t), y(t))$ in \mathbb{R}^2 . We say it's rectifiable:

if $\exists M > 0$. for any: $a = t_0 < t_1 < \dots < t_n = b$. $M \in \mathbb{Z}^+$.

then $\sum |Z(t_i) - Z(t_{i-1})| \leq M$.

Def: length(y) = $\sup_{\{t_i\}} \sum |Z(t_i) - Z(t_{i-1})|$

Thm.
 $y \in BV[a, b]$
 $\Leftrightarrow y$ is recti-
-fiable!

\Rightarrow We say Function $F(x)$ is bounded variation

if for any partition $\{t_i\}^n$ of $[a, b]$.

$\sum |F(t_i) - F(t_{i-1})| \leq m$ (m adapt of $\{t_i\}^n$)

Def: i) Total variation $T_F(a, x)$

$$= \sup_{\{t_i\}} \sum |F(t_i) - F(t_{i-1})|, x \leq b.$$

$\{t_i\}$ is partition of $[a, x]$.

ii) Negative part: $N_F(a, x) = \sup_{\{t_i\}} \sum_{(t_i)} -(F(t_i) - F(t_{i-1}))$

iii) Positive part: $P_F(a, x) = \sup_{\{t_i\}} \sum_{(t_i)} (F(t_i) - F(t_{i-1}))$

$$\text{Prop. } T_F(a, x) = N_F(a, x) + P_F(a, x), a \leq x \leq b$$

$$F(x) = F(a) + P_F(a, x) - N_F(a, x), a \leq x \leq b$$

Lemma F is bound variation on $[a, b]$
 $\Leftrightarrow F$ is difference of two increasing func.

Pf: Increasing functions on $[a, b]$
are bound variation definitely.

Thm. F is bound variation. Then $F(x)$ is
differentiable a.e. x.

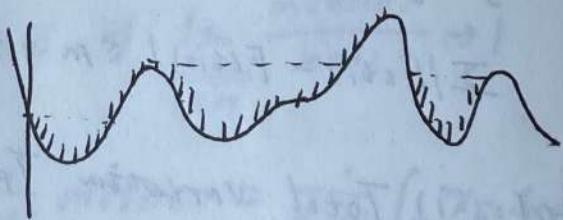
Pf: Rising Sun Lemma:

$h: \mathbb{R} \rightarrow \mathbb{R}$. conti. $E = \{x \in \mathbb{R} \mid h(x+h) > h(x)\}$

for some $\delta = h(x) > 0\}$. If $E \neq \emptyset$. Then E
is open, can be written in countable
disjoint union of intervals: $E = \bigcup_{k=1}^{\infty} (a_k, b_k)$

For $b_k < \infty$, $a_k > -\infty$. we have: $h(a_k) = h(b_k)$

Pf:



easy to check in the figure

Sometimes in the end point: $h(a_k) < h(b_k)$

1°) Consider F is conti.

By Lemma. consider F is increasing

$$\Delta_h F(x) = \frac{F(x+h) - F(x)}{h}$$

$$\underline{\text{Def}}: \text{Dim Number} = D^+(F)(x) = \lim_{h \rightarrow 0^+} A_h(F), \quad D_-(F) = \lim_{h \rightarrow 0^+} A_h(F)$$

$$D^-(F)(x) = \lim_{h \rightarrow 0^-} A_h(F), \quad D_0(F) = \lim_{h \rightarrow 0} A_h(F)$$

\Rightarrow We will show : i) $D^+(F) < \infty$. a.e.x
ii) $D^+(F) \leq D_-(F)$. a.e.x

Apply it on $-F(-x)$. Then : $D^+ \leq D_- \leq D \leq D_0 \leq D^+ < \infty$, a.e.x

Then $\lim_{h \rightarrow 0} A_h(F)(x)$ exists, finite.

For i) : Consider $E_y = \{D^+(F) > y\}$

Since $E_y = \bigcup_{n \in \mathbb{Z}} \left\{ \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}} > y \right\} \therefore E_y \text{ is measurable.}$

Apply Lemma on $F-yx$. $\therefore E_y \subseteq U(a_k, b_k)$

where $F(b_k) - F(a_k) \geq y(b_k - a_k)$

$\therefore E_y \subseteq \bigcup_{k \in \mathbb{Z}} (a_k, b_k) \leq \frac{1}{y} \sum F(b_k) - F(a_k) \leq \frac{F(b) - F(a)}{y}$

Let $y \rightarrow \infty$. $\therefore E_\infty$ is null set.

For ii) : Set $E_{k,r} = \{D^+(F) \geq r > D^-(F)\}, \quad R, r \in \mathbb{Q}$.

prove $m(E_{k,r}) > 0$. Then $E = \{D^+(F) > D^-(F)\} = \bigcup_{k,r} E_{k,r}$

E is null set!

a. First Note that $E_{k,r}$ is measurable

b. Suppose $m(E_{k,r}) > 0$.

Approx. $m(E_{k,r})$ by \cup open st. $E \subseteq O$. $m(O) \leq \frac{R}{f} m(E)$

$O = \bigcup I_n$. disjoint intervals.

Apply Lemma on $-F(-x) + rx$ on each I_n

then $E = \bigcup_{k,r} (a_k, b_k) \subseteq I_n$.

Apply Lemma on $F(y) - rx$, on each (a_k, b_k)

then $O_n = \bigcup_k (a_{k,n}, b_{k,n}) \subseteq (a_n, b_n)$

(i.e. Define $G = V_{\mathbb{Z}_+}$. $I_n \rightarrow (a_n, b_n) \rightarrow (c_{n,1}, d_{n,1})$)

\Rightarrow Find relation of O_n, I_n . Come into contradiction!

Cor. F is anti-monotone. Then F' exists. a.e.x.

St. $\int_N^b F(x) dx \leq F(b) - F(a)$

Pf: Apply Fatou's Lemma on $\frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}}$

Remark: " $<$ " won't hold strictly. e.g. F is

Cantor-Lebesgue Function of $\frac{1}{3}$

$$\text{since } m(C(\frac{1}{3})) = 0. \therefore F(x) = 0. \text{ a.e.x}$$

but $\int_0^1 F(x) dm = 0 < F(1) - F(0) = 1$

2) For general $F(x)$ isn't wrt.

Def: Jump function $J_F = \sum a_n j_n(x)$

of $F(x)$, where $\{x_n\}$ is discontinuities

of F . $F(x_n) = F(x_n) - \theta_n q_n$, $0 < \theta_n \leq 1$.

$$F(x_n) = F(x_n) - a_n.$$

$$j_n(x) = \begin{cases} 0 & x < x_n \\ 1 & x > x_n \\ \theta_n & x = x_n \end{cases}$$

$\Rightarrow F = F_0 + J_F$. We will prove $J_F = 0$. a.e.x.

② Absolute conti. Func:

Def: F is absolutely conti if $\forall \varepsilon > 0, \exists \delta > 0$

St. $\sum_1^n |F(b_k) - F(a_k)| < \varepsilon$. When $\sum (b_k - a_k) < \delta$

Remark: i) F is absolutely conti $\Rightarrow F$ is uniformly conti
 But converse doesn't hold. since we can let
 $N \rightarrow \infty$. (e.g. Lebesgue-Cantor Func is conti
 on \mathbb{R} , but not absolutely conti)

ii) Absolutely conti \Rightarrow bound variation

moreover. T_F is conti.

iii) $AC[a,b] = \{f: [a,b] \rightarrow \mathbb{R} \mid f \text{ is absolutely conti}\}$. It forms a linear space.

iv) $F(x) = \int_a^x f(t)dt \in AC[a,b]$.

So absolutely conti is necessarily

$$\text{for: } F(b) - F(a) = \int_a^b F'(x)dx$$

Thm. $F(x) \in AC[a,b]$. Then $F'(x)$ exist. a.e.x.

if $F'(x) = 0$. a.e.x. then $F(x) \equiv c$. $\forall x \in [a,b]$.

for c is a constant.

Cor. $F \in AC[a,b]$. Then $F(b) - F(a) = \int_a^b F'(x)dx$

(Apply Thm on $\int_a^x F'(t) - F(x)$)

Pf: Lemma. $m(E) < \infty$. $\forall \delta > 0$. $\exists \{B_k\}' \subseteq \mathcal{B}$

finitely of Vitali cover, which are disjoint

$$\text{St. } \sum_i m(B_k) \geq m(E) - \delta.$$

Def: \mathcal{B} is a Vitali cover. if $\forall x \in E$.

$\forall s > 0$. $\exists B \in \mathcal{B}$. St. $x \in B$. $m(B) < s$.

Pf: WLOG. suppose $m(E) > \delta$. $\exists k \subseteq E$ s.t. $m(k) \geq \delta$.

We will exhaust E in the following:

Apply Vitali covering Lemma:

$\exists \{B_i\}_{i=1}^N$, st. disjoint. $\sum_{i=1}^N m(B_i) \geq \frac{\delta}{3^d}$

If $\sum_{i=1}^N m(B_i) > m(E) - \delta$. Then we're done.

Otherwise, let $E_1 = E - \bigcup B_i$. $m(E_1) > \delta$.

(*) $\mathcal{B}_1 = \{B \in \mathcal{B} \mid B \cap B_i = \emptyset, \forall 1 \leq i \leq N\}$ is Vitali cover of E_1 .

Apply Vitali Lemma on E_1 . \mathcal{B}_1 again:

obtain $\{B_i\}_{i=1}^{k_1} \subseteq \mathcal{B}_1 / \{B_i\}_{i=1}^N$. $\sum_{i=1}^{k_1} m(B_i) > \frac{2\delta}{3^d}$

Since $\frac{k_1 \delta}{3^d} \rightarrow 0$ ($k_1 \rightarrow \infty$). At N^{th} stage, the conclusion holds!

Remark: i) (*) is because $\forall x \in E_1, \exists B(x, r) \in \mathcal{B}$.

St. $r < \min_{1 \leq i \leq N} \text{dist}(x, \bar{B}_i)$. Since $\bigcup B_i$ closed!

ii) Gr. $\exists \{B_i\}_{i=1}^n \subseteq \mathcal{B}$, st. $m(E / \bigcup B_i) \leq 2\delta$.

Pf: $O \supseteq E, E \subseteq O, m(E) + \delta > m(O)$

restrict \mathcal{B} in O :

$\mathcal{B}' = \{B \in \mathcal{B} \mid B \subseteq O\}$, still V -cover of E .

\Rightarrow For $\forall a, b' \in [a, b]$. We will prove: $F(a') = F(b')$

If we have: $F(x) > 0$. a.e.x. WLOG. Let $a = a', b = b'$

$\text{Since } E = \{x \mid F(x) > 0\} = \{x \mid \lim_{h \rightarrow 0} \left| \frac{F(x+h) - F(x)}{h} \right| > 0\}$

$m(E) = b - a$. $\forall x \in E, \forall \varepsilon > 0, \exists \delta > 0$, st.

$|F(x+h) - F(x)| < \varepsilon |h|$, when $|h| < \delta$.

Such interval $(x, x+h)$ collected will form a Vitali cover of E . Denote $\{I_{jk}\} = \mathcal{I}$.

Apply the Lemma : we obtain $\{\mathcal{I}_{jk}\}'' = \{(a_j, b_j)\}_j''$.

$$\text{s.t. } \sum_j m(I_{jk}) > m(E) - \delta. \quad \{\mathcal{I}_{jk}, p_j\}_j'' = [a_j, b_j] / \{I_{jk}\}_j''.$$

$$\therefore \sum_j m(a_j, b_j) < \delta. \quad \{I_{jk}, p_j\} \text{ disjoint.}$$

$$|F(b) - F(a)| \leq \sum |F(b_j) - F(a_j)| + \sum |F(b_j) - F(a_i)|$$

$$\text{Part one} \leq \sum |b_j - a_j| \varepsilon. \quad \text{by ref of } \mathcal{I}_k.$$

$$\text{Part two} \leq \varepsilon. \quad \text{by absolutely conti of } F.$$

Remark: $|F(b) - F(a)| \leq \sum |F(t_i) - F(t_{i+1})|$

Separate the partition into 2 part.

one part F changes almost nothing $\sim \varepsilon$.

Another part the interval $\sim \varepsilon$.

prop. F satisfies Lipschitz condition \Rightarrow

$F \in AC$ and $|F'(x)| \leq M$ for a.e. x

prop. $F \in BV[a, b]$. Then : $\int_a^b |F'(x)| dx \leq T_F(a, b)$,

" holds when $F \in ACC[a, b]$.

$$\underline{Pf:} \quad \text{Since } F(x) = F(a) + P_F(a, x) - N_F(a, x)$$

$$\therefore \int_a^b |F'(x)| dx = \int_a^b |P'_F + (-N'_F)|$$

$$\leq \int_a^b |P'_F| + |N'_F| = \int_a^b T'_F = T_F(a, b)$$

If $F \in AC[a, b]$. Then $\int_{a_i}^{b_i} F' = f(b_i) - f(a_i)$

P_F, N_F are increasing.

so, belong to $BV[a, b]$.

P'_F, N'_F exists!

Note that Δ partition $I(t_i)$

$$\sum |F(t_i) - F(t_{i+1})| = \sum \left| \int_{t_{i+1}}^{t_i} F'(t) dt \right|$$

$$\leq \sum \int_{t_{i+1}}^{t_i} |F'(t)| dt = \int_a^b |F'(x)| dx$$

Take supremum of $|F'(t)|$

$$\therefore T_F(a, b) \leq \int_a^b |F'(x)| dx.$$

If $\int_a^b |F'(x)| dx = T_F(a, b)$

$$\therefore \sum |F(x_i) - F(x_{i+1})| \leq \sum T_F(x_i, x_{i+1}) = \int_{[x_i, x_{i+1}]} |F'| dm$$

$\therefore F \in AC[a, b]$. by property of L -integrable.

Remark: We obtain: $F \in AC[a, b] \Leftrightarrow \int_a^b |F'| = T_F(a, b)$

Thm. (Variation Formula)

$F \in AC[a, b]$. one-to-one. $f \in L^1[A, B]$

where $[A, B] = [F(a), F(b)]$. Then:

$$\int_a^b f(x) dx = \int_a^b f(F(x)) |F'(x)| dx$$

Pf: Approx. by simple function.

choose χ_E , $E \subseteq [a, b]$.

Lemma: $m(O) = \int_{F^{-1}(O)} F(x) dx$. for open set O .

Pf: $O = \bigcup (a_k, b_k)$. Separate (a_k, b_k)

into monotone intervals of F .

(There're finite since $[a, b]$ is opn)

$$\begin{aligned} \therefore \int_A \chi_E dm &= m(E) = \int_{F^*(E)} F(x) dx \\ &= \int_A \chi_{F(E)} F(x) dx = \int_A \chi_E(F(x)) F(x) dx. \end{aligned}$$

Suppose E is open.. firstly. Then $E \subseteq F$

For general measurable set F :

We have it by monotone converge Thm.

③ Jump Functions :

Thm. $J_F'(x) = 0$, a.e.x.

Pf: Set $E_\varepsilon = \{x \in [a, b] \mid \limsup_{h \rightarrow 0} \frac{|J(x+h) - J(x)|}{h} \geq \varepsilon\}$.

$$1) E_\varepsilon = \bigcap_{k=1}^{\infty} \{x \mid \sup_{\frac{1}{k} < h < \frac{1}{k}} |J(x+h) - J(x)| / h \geq \varepsilon\} \cap [a, b].$$

$\therefore E_\varepsilon$ is measurable.

2) Suppose $m(E) > \delta > 0$.

$\forall \eta > 0$. $\exists N$. s.t. $\sum_{k=N}^{\infty} \tau_k < \eta$. Since $J_F = \sum \tau_k j_k$ converges.

Denote: $J_0(x) = \sum_{k=N}^{\infty} \tau_k j_k(x)$.

$\therefore E_\varepsilon(J_F) / E_\varepsilon(J_0) = \{\chi_k\}_1^N$. i.e. $m(E_\varepsilon \cap J_0) > \delta > 0$.

$\therefore k \text{ opt } \cup E_\varepsilon(J_0)$. $k \cup \bigcup_{n=1}^N I_n = \bigcup_{n=1}^N (a_n, b_n)$

$\therefore \eta \geq J_0(b) - J_0(a) \geq \sum J_0(b_n) - J_0(a_n)$

$\geq \sum (b_n - a_n) \varepsilon$. Let $n \rightarrow \infty$, contradiction!

Remark: The common way is refining a measurable set until the property of the ε -difference set is nice!

Prop. (Decomposition of monotone Function)

F is increasing on $[a,b]$. Then $F = F_A + F_C + F_J$

where $F_A \in A\mathcal{C}[a,b]$, F_C conti. $F_C' = 0, a.e.x$

F_J is jump function.

Moreover, the decomposition is unique.

up to a const.

Pf: $F = F - J_F + J_F$. $F - J_F$ is anti.

$\therefore F - J_F \in BV[a,b]$. Let $F_J = J_F$

Let $F_A = \int_a^x (F - J_F)' dm \in A\mathcal{C}[a,b]$

$F_C = F - J_F - F_A$. $F_C' = 0, a.e.x$.

$$If \quad F_A' + F_C' + F_J' = F_A^2 + F_C^2 + F_J^2.$$

$$\text{Then } F_J' - F_J^2 = F_A^2 + F_C^2 - F_A' - F_C'$$

LHS should be conti. $\therefore F_J' - F_J^2 \equiv \text{const}$

Differentiate both sides $\therefore F_J'' = F_A' + \text{const}$

(3) Application:

① Rectifiable Curves:

When will $L = \int_a^b (x'^2 + y'^2)^{\frac{1}{2}} dt$ hold?

e.g. When $x(t) = y(t) = f(t)$, Lebesgue-Cantor Function, it doesn't hold!

Thm. For curve $\gamma = (x(t), y(t))$, $t \in [a, b]$.

If $x, y \in AC[a, b]$. Then $L(\gamma) = \int_a^b |\gamma'(t)| dt$.

Pf: This is from : Let $F(t) = x(t) + iy(t)$

$\therefore F(t) \in AC[a, b]$. we obtain:

$$\int_a^b |F'(t)| dm = T_F(a, b) = \text{length}(\gamma).$$

It has been proved before!

Remark: Any curve can be realized in

different parametrizations:

let $s = s(t) = L(a, t)$, the length of
curve $z(t)$ in $[a, t]$.

The arc-length parametrization:

$$\tilde{z}(s) = \tilde{z}(s(t)) = z(t), \quad \tilde{x}(s) = \tilde{x}(s) + i\tilde{y}(s)$$

Thm. $z(t) = (x(t), y(t))$ & rectifiable of length L .

Consider the arc-length parametrization: $\tilde{z}(s)$

Then $\tilde{x}, \tilde{y} \in AC[a, b]$. $|\tilde{z}'(s)| = 1$. a.e. $x \in \mathbb{R}$

and $L = \int_0^L |\tilde{z}'(s)| ds$.

Pf: Geometrically observe: $|\tilde{z}(s_1) - \tilde{z}(s_2)| \leq |s_1 - s_2|$

$\therefore \tilde{z}(s) \in AC[a, b]$. and $|\tilde{z}'(s)| \leq 1$.

$$\text{since } \int_0^L |\tilde{z}'(s)| ds = T_{\tilde{z}}(0, L) = \text{length}(z) = L.$$

Besides $L \geq \int_0^L |\tilde{z}'(s)| ds = L$. $\therefore |\tilde{z}'(s)| = 1$. a.e.

② Minkovsky Content

b f curves:

Def: $M(k) = \lim_{\delta \rightarrow 0} \frac{m(k^\delta)}{2\delta}$, $k^\delta = \{x \mid \text{dist}(x, k) < \delta\} \subseteq \mathbb{R}^2$.

Geometrically:  : That's why $\frac{1}{2}$ exists

Def: For $Z(t) = (x(t), y(t))$, $a \leq t \leq b$.

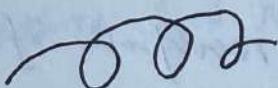
i) $Z(t)$ is simple curve if $Z(t)$ is injective

ii) $Z(t)$ is simple closed curve (Jordan curve) if $Z(t)$ is injective on $[a, b]$.

$$Z(a) = Z(b)$$

iii) $Z(t)$ is quasi-simple curve if:

$Z(t)$ is injective on $[a, b] / \{x_i\}$.

e.g.: 

Thm. For a quasi-simple curve $Y = Z(t) = (x(t), y(t))$

Its Minkovsky content exists \Leftrightarrow

It's rectifiable. When it holds, then:

$$M(Y) = \text{Length } c_Y$$

Pf: Def: $M^*(c_Y) = \overline{\lim}_{\delta \rightarrow 0} \frac{m(c_Y^\delta)}{2\delta}$ prove:

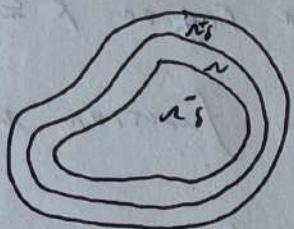
$$M^*(c_Y) = \underline{\lim}_{\delta \rightarrow 0} \frac{m(c_Y^\delta)}{2\delta} \quad M^*(c_Y) \leq L \leq M^*(c_Y)$$

(3) Isometric Inequalities:

Thm. $\Omega \subseteq \text{open } \mathbb{R}^2$ bounded. $\gamma = \bar{\gamma}/\pi$, with length ℓ .

$$\text{Then: } m(\Omega) \leq \frac{\ell^2}{4\pi}.$$

Pf: By approx. of Minkovsky content:



By Minkovsky Inequality^(*)

$$\text{Since } m(n_\delta^+ / n_\delta) = m(\gamma^\delta)$$

$$\text{estimate: } m(n_\delta^+) \cdot m(\Omega)$$

$$\text{Since } \Omega \subseteq n_\delta + D(0, \delta), n_\delta^+ \subseteq \Omega + D(0, \delta)$$

Remark: (*) : Minkovsky Inequality for L -measure:

$$(m(A)^\lambda + m(B)^\lambda)^{\frac{1}{\lambda}} \leq m(A+B)$$

ii) Note that " $=$ " holds when Ω is a ball!

iii) For general case, we have:

$$m(\Omega) \leq \frac{m(\gamma)^2}{4\pi}$$