Estimating the duration of random walks

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Objectives

The Gambler's Ruin is a random walk, one of the most classical models of a Markovian stochastic process. In this model, a gambler wagers money on each round, either gaining or losing varying amounts according to a probabilistic rule. Such models have many applications in statistical modeling and computational algorithms [5]. Their behavior has been thoroughly studied in the simplest form — the gambler wins $\$1.00$ with probability p and loses $\$1.00$ with probability $1-p$ — particularly concerning the probability of ruin. [4]. This project's aim is to investigate the expected game duration in more complex random walks by proposing estimators and evaluating their performance through computational approaches. We hope through this approach to shine light on interesting and computationally cheap ways to estimate random walks.

Materials and methods

We analyzed the expected value and variance of the duration of the classical Gambler's Ruin problem [1] using finite difference equations systems. We developed a heuristic estimator based on an analogy with average velocity in Physics. This estimator yielded accurate approximations for most random walks, both in the simple case and in models where the gambler may win or lose amounts different from $\$1.00$ per round. In the fair game case, that is, when the expected gain per round is zero, the estimator deviated significantly from the theoretical and simulated values.

For random walks with more general rules, where analytical solutions are not available, we evaluated this estimator using Monte Carlo simulations, based on methods proposed by Ritter [3], using the Python programming language. As we considered broader rules that greatly extended the simulation times and required more computational power, we migrated to the Julia language, as recommended by Godoy [2]. We further refined the estimator to account for ties within a round.

Having identified the limitations of this estimator in one dimension, we extended our study to random walks in $\mathbb{Z}^d$. By solving systems of equations computationally with symbolic libraries, we obtained analytical results for unit-step walks with different probabilistic rules. New estimators were analyzed for the simple and more complex random walks. Parallel programming enabled quick simulation of random walks in high dimensions.

Results

The proposed estimator showed efficacy in predicting the duration of one-dimensional walks, as depicted by Figure 1, and adapts successfully when the transition probability depends on the gambler's current fortune.

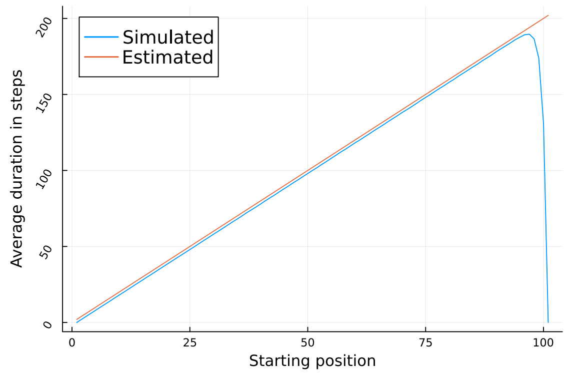

Figure 1: Comparing the estimator with $100\,000$ simulations per starting position

In higher dimensions, the expected duration of the random walk must be obtained numerically due to the system of finite different equation's exponential growth, rendering a closed form solution intractable. A surprising result was observed in this environment: once the dimensionality exceeds roughly $d = 100$, the average time to game completion increases almost linearly with each additional dimension as shown in Figure 2.

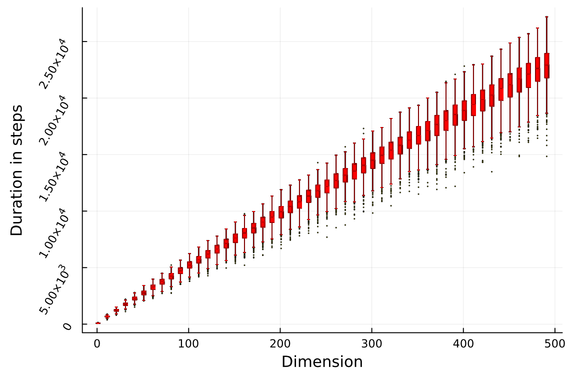

Figure 2: Game duration in $1\,000$ simulations by dimension.

In the particular model where the gambler cannot lose his money, we've obtained analytic expressions for the random and non-random variants. The complexity of the equations that describe the game's duration grows proportionally to the state space's size. In all cases, the estimated values show great proximity to simulated ones, although greater discrepancies show when the game is close to fair - that is, when the gambler's expected earnings per round are close to 0 - or when the starting position is close to a boundary value.

Conclusions

The Gambler's Ruin is a classical math problem proposed almost four centuries ago. Through intuition, undergraduates can formulate fresh questions and hypothesis, explore them computationally, and finally developg rigorous mathematics. It is our hope that this work encourages other undergraduates to explore and reexplore classical math problems in search of solutions to modern day challenges.

The authors declare no conflict of interests. Eduardo Yukio conceived the theoretical framework behind the estimators and derived the paper's analytical results. Gustavo Garone provided computational support and implemented the simulations used for improving the estimators.

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