O erro quadrático médio (EQM) do Estimador $T(X_1, \ldots, X_n)$ com respeito a $g(\theta)$ é definido por

$$\mathrm{EQM}(T, g(\theta)) = E_\theta((T(X_1, \ldots, X_n) - g(\theta)^2)$$

Obs.:

Se $T(X_1, \ldots, X_n)$ for não viciado para $g(\theta)$, então $\mathrm{EQM}(T, g(\theta)) = \mathrm{Var}_\theta(T(X_1, \ldots, X_n)) \forall \theta \in \Theta$

Propriedades do EQM

Seja $T(X_1, \ldots, X_n)$ um estimador para $g(\theta)$, seja $\mu_t = E_\theta(T(X_1, \ldots, X_n))$

$$\begin{aligned}
&\mathrm{EQM}(T, g(\theta)) \\
&= E_\theta[(T(X_1, \ldots, X_n) - \mu_t + \mu_t - g(\theta))^2] \\
\\
&= E_\theta[((T(X_1, \ldots, X_n) - \mu_t) + (\mu_t - g(\theta)))^2] \\
&= E_\theta[(T(X_1, \ldots, X_n) - \mu_t)^2 + 2(T(X_1, \ldots, X_n) - \mu_t)(\mu_t g(\theta)) + (\mu_t - g(\theta))^2] \\
&= \overbrace{E_\theta[(T(X_1, \ldots, X_n) - \mu_t)^2]}^{\mathrm{Var}_\theta(T(X_1, \ldots, X_n))} + 2(\mu_t - g(\theta))\cancelto{0}{E_\theta(T(X_1, \ldots, X_n) - \mu_t)} + (\mu_t - g(\theta))^2 \\
&= \mathrm{Var}_\theta(T(X_1, \ldots, X_n)) + (\mu_t - g(\theta))^2
\end{aligned}$$

Portanto,

$$\mathrm{EQM}(T, g(\theta)) = \mathrm{Var}_\theta(T(X_1, \ldots, X_n)) + (\mu_t - g(\theta))^2$$

Viés

Denotamos de viés de $T(X_1, \ldots, X_n)$ com respeito a $g(\theta)$ por

$$\text{Viés}(T, g(\theta)) = E_\theta(T(X_1, \ldots, X_n)) - g(\theta), \forall \theta \in \Theta$$

Dessa forma, temos que

$$\mathrm{EQM}(T, g(\theta)) = \mathrm{Var}_\theta(T(X_1, \ldots, X_n)) + [\text{Viés}(T, g(\theta))]^2$$

Exemplo

Seja $(X_1, \ldots, X_n)$ uma amostra aleatória, ou seja, independentes e identicamente distribuídas (iid), de $X \sim \mathrm{Ber}(\theta)$ em que $\theta \in \Theta = (0, 1)$. Calcule o viés e o EQM de $\bar{X}_n$ com respeito a $g(\theta) = P_\theta(X = 1)$

O estimador é, então, $T(X_1, \ldots, X_n) = \bar{X}_n = \frac{X_1 + \cdots + X_n}{n}$ para $g(\theta) = P_\theta(X = 1) = \theta$ (pelo modelo de Bernoulli).

$$E_\theta(\bar{X}_n) = E_\theta\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E_\theta(X_i) \overset{id.dist.}{\Rightarrow}$$

$$E_\theta(\bar{X}_n) = \frac{1}{n}\sum_{i=1}^{n} E_\theta(X), \forall\theta \in \Theta$$

$$= \frac{n}{n}\theta = \theta, \forall\theta \in \Theta$$

Portanto, $\bar{X}_\theta$ é não enviesado para $g(\theta) = \theta$.

$$\Rightarrow \text{Viés}(\bar{X}_n, g(\theta)) = 0, \forall\theta \in \Theta$$

Para o EQM,

$$\text{EQM}(\bar{X}_n, g(\theta)) = \text{Var}_\theta(\bar{X}_n) - 0^2 = \text{Var}_\theta\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{Var}_\theta\left(\sum_{i=1}^{n} X_i\right)$$

$$\overset{\text{ind}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}_\theta(X_i) \overset{\text{ind. dist.}}{=} \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}_\theta(X), \forall\theta \in \Theta$$

$$= \frac{n\theta(1-\theta)}{n^2} = \frac{\theta(1-\theta)}{n^2}, \forall\theta \in \Theta$$