

## Mathematics 309 — Spring 2004 — Fourth homework

1. Find the principal planes and focal length of a sphere of radius  $R$ .

In order to find the principal planes of any lens system, the first step is to calculate the transfer matrix of the system. Here it is

$$\begin{bmatrix} 1 & -P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \delta & 1 \end{bmatrix} \begin{bmatrix} 1 & -P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - P\delta & -2P - P^2\delta \\ \delta & 1 - P\delta \end{bmatrix}$$

where  $P = (n - 1)/R = 1/2R$ ,  $\delta = 4R/3$ . So  $P\delta = 2/3$  and this is

$$\begin{bmatrix} 1/3 & -2/3R \\ 4R/3 & 1/3 \end{bmatrix}$$

The principal planes are those in between which the transfer matrix is that of a thin lens, so we must solve

$$\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} = \begin{bmatrix} 1 & -P \\ 0 & 1 \end{bmatrix}$$

and this gives us

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \begin{bmatrix} 1 & -P \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix}$$

or

$$P = c, \quad t = \frac{a - 1}{P}, \quad s = \frac{d - 1}{P}.$$

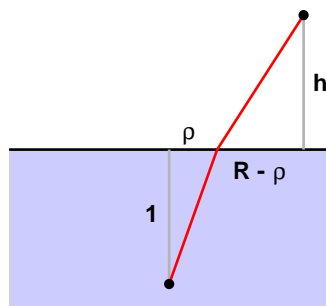
In our case  $s = t = -R$  and  $P = 2/3R$ . So the principal planes are at the centre of the sphere. The focal length is  $f = 1/P = 3R/2$ ; this is the distance of the focal planes out from the centre of the sphere, which is  $R/2$  in front of and behind it. This can be checked directly.

2. Consider a single convex (bulging out on both sides) lens, radius  $R_1$  on one side,  $R_2$  on the other,  $d$  in between. Find its principal planes and focal length.

I just mention that the way to simplify the algebra is to set  $P_i = (n - 1)/R_i$ ,  $\delta = d/n$ .

3. If you place a square on the bottom of a still pool and look at it from above the surface of the water, does it always look like a square? This problem is not trivial. You might begin by realizing that with binocular vision, every point you see is actually a point or a virtual point situated in space. So the figure you see when you look at the square is in fact some figure in space, not only a flat diagram. Start with an easier question: does a straight line at the bottom of the pool look straight? Explain. Include some carefully done examples.

The key to understanding this problem is to know how to calculate the ray that goes from a point under water to the eye. Suppose the point at (say) depth 1, the eye at height  $h$  and a horizontal distance  $R$  away. When seen in a top view, the ray looks like it goes in a straight line from the point to the eye, or in other words the ray lies in the plane containing the eye, the point, and the point on the surface of the water just above the point. To know the ray exactly, all that has to be done is to determine at what horizontal distance from the point the ray breaks the surface.



Snell's Law asserts that

$$\sin i = \frac{(R - \rho)}{\sqrt{(R - \rho)^2 + h^2}} = n \sin r = \frac{n\rho}{\sqrt{\rho^2 + 1}}.$$

This equation must be solved for  $\rho$ , which is not easy to do. The only way I know of is to use an approximate iterative solution like Newton's method to find a root of  $f(\rho) = 0$ , with

$$f(\rho) = \frac{(R - \rho)}{\sqrt{(R - \rho)^2 + h^2}} - \frac{n\rho}{\sqrt{\rho^2 + 1}}.$$

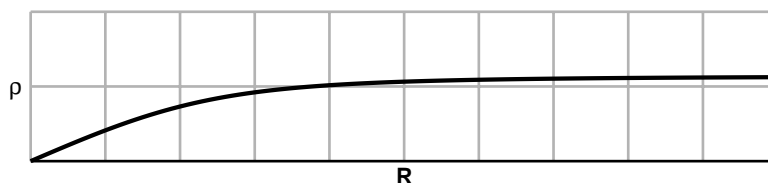
The solution is at any rate unique, since

$$\frac{(R - \rho)}{\sqrt{(R - \rho)^2 + h^2}}$$

decreases steadily from a positive value at  $\rho = 0$  to zero at  $\rho = R$ , while

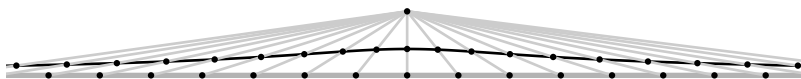
$$\frac{n\rho}{\sqrt{\rho^2 + 1}}$$

rises steadily from 0 to a positive value. So each value of  $R$  gives rise to a unique  $\rho$ . For example if  $h = 1$ , here is the graph of  $\rho$  versus  $R$ .



Notice that as  $R$  increases to  $\infty$ ,  $\rho$  is asymptotically equal to the tangent of the maximum refraction angle, which is  $\arcsin(1/1.3307)$ .

Suppose the eye located at  $(0, 1, 1)$ , the line under the water where  $y = 0, z = -1$ . Here is a top view of the path of points on the surface of the water where the rays to the eye break the surface, as well as a few top views of the rays themselves. The curve, as seen from above, is asymptotic to the straight line itself, since the point where the ray breaks the water approaches a fixed finite distance from the point of origin on the line. That finite distance is the horizontal distance from the object at which the rays traveling off to infinity break the surface.-



Now if the line were seen as a straight line, all the rays coming to the eye from the line would lie in a plane. The points where those lines break out of the water would also lie in that plane, hence on the intersection of the plane with the water surface. But that intersection would then be a straight line, which it is not. So the line does not appear as a line, but as some kind of curve.

This reasoning does not require any explicit calculation at all, in fact. It is the simplest argument I can think of why a square—a large one, at least—will not appear as a square.

4. In class I "proved" that a magnifying lens always has the image upside down. But if you place two lenses in succession, each focussing on where the previous one forms its image, the image will be doubly reversed. Where does my "proof" fit in? Write down my proof and then explain what goes wrong in this case. Give a numerical example. Draw some pictures to explain things clearly.

The proof goes like this

The start is to calculate how to find conjugate planes for a thin lens. If the power is  $P = 1/f$  and the first plane is at distance  $s$  on the left, the conjugate plane will be at distance  $t$  on the right where

$$s + t - st/f = 0, \quad t = -\frac{s}{1 - s/f}$$

A point at height  $y$  on the left plane will be transformed to that at height  $(1 - s/f)y$  so the magnification is  $1 - s/f$ . Now the sign of  $t$  will be the same as that of  $-(1 - s/f)$ , so the magnification will be positive if and only if  $t$  is positive. The paradox, therefore, is because in my 'proof' I assumed  $t$  positive, whereas for the system of two lenses I describe it must be negative. This means only that the conjugate plane must lie inside the principal plane at the right, no more and no less.

5. Place two thin lenses of power  $P_1$  and  $P_2$  in sequence along the optical axis, so the left focal plane of the second is the same as the right focal plane of the first, you will get a telescopic lens system. Find a formula for the magnification of the system. Include some diagrams to explain what's going on.

The transfer matrix is

$$\begin{bmatrix} 1 & -1/f_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/f_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -f_2/f_1 & 0 \\ f_1 + f_2 & -f_1/f_2 \end{bmatrix}.$$

Any two rays coming in at an angular separation of  $\theta$  will change to rays with an angular separation of  $-f_1/f_2$ , so the magnification is  $-f_1/f_2$ . Direction is reversed.

6. Consider a hemi-spherical lens of radius 1. Find its focal planes. Find the plane conjugate to the plane 1 unit to the left of the left focal plane. Find the linear image of a white spot on that plane, centered at  $y = 0.5$ , width 0.1 on the conjugate plane. Then find an approximate graph of the real image. (I'll explain more about this in class. You should trace about 25 rays in all departing from the spot.)

The transfer matrix is

$$\begin{bmatrix} 1 & -1/2R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ R/n & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/2R \\ 2R/3 & 1 \end{bmatrix}$$

so the principal plane data are  $P = 1/2R$ ,  $t = -2R/3$ ,  $s = 0$ , focal length  $f = 2R$ .

The second part is postponed.

7. Find out the indices of refraction of red and violet light in water. Calculate using calculus the angles at which the red and violet bands of primary and secondary rainbows appear in the sky.

The exit angle after  $N$  internal reflections is

$$180N + 2(1 + N)r - 2i = 180N + 2(1 + N) \arcsin(y/n) - 2 \arcsin(y)$$

and its derivative is

$$\frac{2(1 + N)}{n\sqrt{1 + (y/n)^2}} - \frac{2}{\sqrt{1 + y^2}}.$$

this will be 0, giving the location of a rainbow, when

$$y = \sqrt{\frac{(N + 1)^2 - n^2}{(N + 1)^2 - 1}}.$$

I get about  $42^\circ$  for one reflection, about  $50^\circ$  for two. (Be careful, there are two possible values of  $y$ , plus and minus each other. Only one gives a visible rainbow. For the secondary bow, the rainbow rays  $y < 0$ .)