Chapter 2
Finite Element Equations for Heat Transfer

Abstract Solution of heat transfer problems is considered. Finite element equations are obtained using the Galerkin method. The conductivity matrix for a triangular finite element is calculated.

2.1 Problem Statement

Let us consider an isotropic body with temperature-dependent heat transfer. A basic equation of heat transfer has the following form [15]:

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + Q = \rho c \frac{\partial T}{\partial t}. \tag{2.1}$$

Here, q_x, q_y and q_z are components of heat flow through the unit area; $Q = Q(x,y,z,t)$ is the inner heat-generation rate per unit volume; ρ is material density; c is heat capacity; T is temperature and t is time. According to Fourier's law the components of heat flow can be expressed as follows:

$$q_x = -k\frac{\partial T}{\partial x},$$

$$q_y = -k\frac{\partial T}{\partial y}, \tag{2.2}$$

$$q_z = -k\frac{\partial T}{\partial z},$$

where k is the thermal-conductivity coefficient of the media. Substitution of Fourier's relations gives the following basic heat transfer equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + Q = \rho c \frac{\partial T}{\partial t}. \tag{2.3}$$

It is assumed that the *boundary conditions* can be of the following types:

1. Specified temperature

   $T_s = T_1(x, y, z, t)$ on S_1 .

2. Specified heat flow

   $q_x n_x + q_y n_y + q_z n_z = -q_s$ on S_2 .

3. Convection boundary conditions

   $q_x n_x + q_y n_y + q_z n_z = h(T_s - T_e)$ on S_3 ,

4. Radiation

   $q_x n_x + q_y n_y + q_z n_z = \sigma \varepsilon T_s^4 - \alpha q_r$ on S_4,

where h is the convection coefficient; T_s is an unknown surface temperature; T_e is a convective exchange temperature; σ is the Stefan–Boltzmann constant; ε is the surface emission coefficient; α is the surface absorption coefficient, and q_r is the incident radiant heat flow per unit surface area. For transient problems it is necessary to specify an initial temperature field for a body at the time $t = 0$:

$$T(x, y, z, 0) = T_0(x, y, z). \tag{2.4}$$

2.2 Finite Element Discretization of Heat Transfer Equations

A domain V is divided into finite elements connected at nodes. We shall write all the relations for a finite element. Global equations for the domain can be assembled from finite element equations using connectivity information.

Shape functions N_i are used for interpolation of temperature inside a finite element:

$$T = [N]\{T\},$$
$$[N] = \begin{bmatrix} N_1 & N_2 & \ldots \end{bmatrix}, \tag{2.5}$$
$$\{T\} = \{T_1 \ T_2 \ \ldots\}.$$

Differentiation of the temperature-interpolation equation gives the following interpolation relation for temperature gradients:

$$\begin{Bmatrix} \dfrac{\partial T}{\partial x} \\ \dfrac{\partial T}{\partial y} \\ \dfrac{\partial T}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \ldots \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \ldots \\ \dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_2}{\partial z} & \ldots \end{bmatrix} \{T\} = [B]\{T\}. \tag{2.6}$$

Here, $\{T\}$ is a vector of temperatures at nodes, $[N]$ is a matrix of shape functions, and $[B]$ is a matrix for temperature-gradient interpolation.

Using the Galerkin method, we can rewrite the basic heat transfer equation in the following form:

$$\int_V \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - Q + \rho c \frac{\partial T}{\partial t} \right) N_i dV = 0. \tag{2.7}$$

Applying the divergence theorem to the first three terms, we arrive at the relations:

$$\int_V \rho c \frac{\partial T}{\partial t} N_i dV - \int_V \left[\frac{\partial N_i}{\partial x} \ \frac{\partial N_i}{\partial y} \ \frac{\partial N_i}{\partial z} \right] \{q\} dV$$

$$= \int_V Q N_i dV - \int_S \{q\}^T \{n\} N_i dS, \tag{2.8}$$

$$\{q\}^T = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix},$$

$$\{n\}^T = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix},$$

where $\{n\}$ is an outer normal to the surface of the body. After insertion of boundary conditions into the above equation, the discretized equations are as follows:

$$\int_V \rho c \frac{\partial T}{\partial t} N_i dV - \int_V \left[\frac{\partial N_i}{\partial x} \ \frac{\partial N_i}{\partial y} \ \frac{\partial N_i}{\partial z} \right] \{q\} dV$$

$$= \int_V Q N_i dV - \int_{S_1} \{q\}^T \{n\} N_i dS \tag{2.9}$$

$$+ \int_{S_2} q_s N_i dS - \int_{S_3} h(T - T_e) N_i dS - \int_{S_4} (\sigma \varepsilon T^4 - \alpha q_r) N_i dS.$$

It is worth noting that

$$\{q\} = -k[B]\{T\}. \tag{2.10}$$

The discretized finite element equations for heat transfer problems have the following form:

$$[C]\{\dot{T}\} + ([K_c] + [K_h] + [K_r])\{T\}$$

$$= \{R_T\} + \{R_Q\} + \{R_q\} + \{R_h\} + \{R_r\}, \tag{2.11}$$

$$[C] = \int_V \rho c [N]^T [N] dV,$$

$$[K_c] = \int_V k [B]^T [B] dV,$$

$$[K_h] = \int_{S_3} h [N]^T [N] dS,$$

$$[K_r]\{T\} = \int_{S_4} \sigma \varepsilon T^4 [N]^T dS,$$

$$\{R_T\} = -\int_{S_1} \{q\}^T \{n\} [N]^T dS, \qquad (2.12)$$

$$\{R_Q\} = \int_V Q [N]^T dV,$$

$$\{R_q\} = \int_{S_2} q_s [N]^T dS,$$

$$\{R_h\} = \int_{S_3} h T_e [N]^T dS,$$

$$\{R_r\} = \int_{S_4} \alpha q_r [N]^T dS.$$

Here, $\{\dot{T}\}$ is a nodal vector of temperature derivatives with respect to time.

2.3 Different Type Problems

Equations for different types of problems can be deducted from the above general equation:

Stationary linear problem

$$([K_c] + [K_h])\{T\} = \{R_Q\} + \{R_q\} + \{R_h\}. \qquad (2.13)$$

Stationary nonlinear problem

$$([K_c] + [K_h] + [K_r])\{T\}$$
$$= \{R_Q(T)\} + \{R_q(T)\} + \{R_h(T)\} + \{R_r(T)\}. \qquad (2.14)$$

Transient linear problem

$$[C]\{\dot{T}(t)\} + ([K_c] + [K_h(t)])\{T(t)\}$$
$$= \{R_Q(t)\} + \{R_q(t)\} + \{R_h(t)\}. \tag{2.15}$$

Transient nonlinear problem

$$[C(T)]\{\dot{T}\} + ([K_c(T)] + [K_h(T,t)] + [K_r(T)])\{T\}$$
$$= \{R_Q(T,t)\} + \{R_q(T,t)\} + \{R_h(T,t)\} + \{R_r(T,t)\}. \tag{2.16}$$

2.4 Triangular Element

Calculation of element conductivity matrix $[k_c]$ and heat flow vector $\{r_q\}$ is illustrated for a two-dimensional triangular element with three nodes. A simple triangular finite element is shown in Figure 2.1. The temperature distribution $T(x,y)$ inside the triangular element is described by linear interpolation of its nodal values:

$$T(x,y) = N_1(x,y)T_1 + N_2(x,y)T_2 + N_3(x,y)T_3,$$
$$N_i(x,y) = \alpha_i + \beta_i x + \gamma_i y. \tag{2.17}$$

Interpolation functions (usually called shape functions) $N_i(x,y)$ should satisfy the following conditions:

$$T(x_i,y_i) = T_i, \quad i = 1,\ 2,\ 3. \tag{2.18}$$

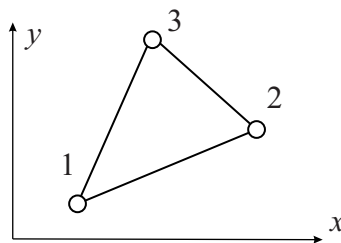Solution of the above equation system provides expressions for the shape functions:
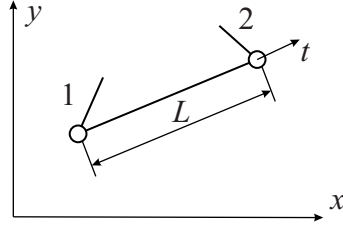

Fig. 2.1 Triangular finite element

Fig. 2.2 Integration along an element side

$$N_i = \frac{1}{2\Delta}(a_i + b_i x + c_i y),$$

$$a_i = x_{i+1}y_{i+2} - x_{i+2}y_{i+1},$$

$$b_i = y_{i+1} - y_{i+2}, \tag{2.19}$$

$$c_i = x_{i+2} - x_{i+1},$$

$$\Delta = \frac{1}{2}(x_2 y_3 + x_3 y_1 + x_1 y_2 - x_2 y_1 - x_3 y_2 - x_1 y_3),$$

where Δ is the element area.

The conductivity matrix of the triangular element is determined by integration over element area A (assuming that the element has unit thickness),

$$[k_c] = \int_A k[B]^{\mathrm{T}}[B]dxdy. \tag{2.20}$$

The temperature differentiation matrix $[B]$ has expression

$$[B] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \\[2mm] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \end{bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}. \tag{2.21}$$

Since the temperature differentiation matrix does not depend on coordinates, integration of the conductivity matrix is simple;

$$[k_c] = \frac{k}{4\Delta}\begin{bmatrix} b_1^2 + c_1^2 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_1 b_2 + c_1 c_2 & b_2^2 + c_2^2 & b_2 b_3 + c_2 c_3 \\ b_1 b_3 + c_1 c_3 & b_2 b_3 + c_2 c_3 & b_3^2 + c_3^2 \end{bmatrix}. \tag{2.22}$$

The heat-flow vector $\{r_q\}$ is evaluated by integration over the element side, as shown in Figure 2.2

$${r_q} = - \int_L q_s [N]^T dL = - \int_0^1 q_s [N_1 \ N_2]^T L dt. \qquad (2.23)$$

Here, integration over an element side L is replaced by integration using variable t ranging from 0 to 1. Shape functions N_1 and N_2 on element side 1–2 can be expressed through t:

$$N_1 = 1 - t, \quad N_2 = t. \qquad (2.24)$$

After integration with substituting integration limits, the heat-flow vector equals

$${r_q} = -q_s \frac{L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \qquad (2.25)$$

Element matrices and vectors are calculated for all elements in a mesh and assembled into the global equation system. After application of prescribed temperatures, solution of the global equation system produces temperatures at nodes.
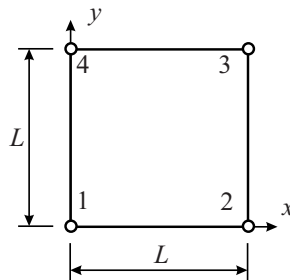
Problems

2.1. Calculate matrix $[k_h]$ describing convection boundary conditions

$$[k_h] = \int_L h[N]^T [N] dL$$

for a side of a triangular element (see Figure 2.2).

2.2. Obtain shape functions N_1, N_2, N_3 and N_4 for the square element shown below.

Assume that its size is $L = 1$ and that shape functions can be represented as $N_i = a_1(a_2 + x)(a_3 + y)$.

2.3. For the square element of the previous problem, estimate the heat-generation vector

$${r_Q} = \int_V Q[N]^T dV.$$

Use the shape functions obtained in the previous problem.