Assignment #10
Selected topics in LGT

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The expression for which we want to integrate out the fermionic terms is:

$$I = \int \left\{ \prod_{k=0}^{N} \mathrm{d}\bar{\psi}_k \mathrm{d}\psi_k \right\} \exp(-\bar{\Psi} M \Psi + \bar{\eta}\Psi + \bar{\Psi}\eta) \tag{1}$$

Let us take for now as a given (and to-prove later on) the following expression:

$$\psi_i{}' = \sum_{j=0}^{N} M_{ij}\psi_j \Rightarrow \Psi' = M\Psi \Rightarrow \prod_{k=0}^{N} \mathrm{d}\psi_k = \det(M) \prod_{k=0}^{N} \mathrm{d}\psi_k{}' \tag{2}$$

Then, we can transform equation 1 into the following form:

$$I = \det(M) \int \left\{ \prod_{k=0}^{N} \mathrm{d}\bar{\psi}_k \mathrm{d}\psi_k{}' \right\} \exp(-\bar{\Psi}\Psi' + \bar{\eta}M^{-1}\Psi' + \bar{\Psi}\eta) \tag{3}$$

To decompose the exponential in the integrand, note that (taking $a = \bar{\psi}_i \psi_i{}'$ and $b = \bar{\psi}_j \psi_j{}'$):

$$ab = \bar{\psi}_i \psi_i{}' \bar{\psi}_j \psi_j{}' = \bar{\psi}_j \psi_j{}' \bar{\psi}_i \psi_i{}' = ba \tag{4}$$

and the same applies for the other two terms being added in the argument of the exponential in equation 3 (i.e. $\bar{\eta}_i (M^{-1})_{ij} \psi_j'$ and $\bar{\psi}_i \eta_i$), which means that all the terms being added in the argument of that exponential commute. Therefore, we can separate the exponential in equation 3 in the following way (using the fact that $e^A e^B = e^{A+B}$ **if** $[A, B] = 0$):

$$\begin{aligned}
&\exp(-\bar{\Psi}\Psi' + \bar{\eta}M^{-1}\Psi' + \bar{\Psi}\eta) = \exp(-\bar{\Psi}\Psi')\exp(\bar{\eta}M^{-1}\Psi')\exp(\bar{\Psi}\eta) = \\
&\left(\prod_{k=0}^{N} \exp(-\bar{\psi}_k \psi_k') \right) \left(\prod_{k=0}^{N} \exp(\textstyle\sum_j \bar{\eta}_k (M^{-1})_{kj} \psi_j') \right) \left(\prod_{k=0}^{N} \exp(\bar{\psi}_k \eta_k) \right) = \\
&\left(\prod_{k=0}^{N} \exp(-\bar{\psi}_k \psi_k') \right) \left(\prod_{j=0}^{N} \exp(\textstyle\sum_k \bar{\eta}_k (M^{-1})_{kj} \psi_j') \right) \left(\prod_{k=0}^{N} \exp(\bar{\psi}_k \eta_k) \right) = \\
&\left(\prod_{k=0}^{N} (1 - \bar{\psi}_k \psi_k') \right) \left(\prod_{j=0}^{N} \left(1 + \textstyle\sum_k \bar{\eta}_k (M^{-1})_{kj} \psi_j'\right) \right) \left(\prod_{k=0}^{N} (1 + \bar{\psi}_k \eta_k) \right)
\end{aligned} \tag{5}$$

Now, to avoid cumbersome expressions, let us focus on integrals of the form (note that the exponentials can be safely moved around without carrying any -1 sign, as we saw above; the only sign change is coming from moving the $\mathrm{d}\bar{\psi}_k$ to the right to perform each integration):

$$I_k = \int \mathrm{d}\bar{\psi}_k \mathrm{d}\psi_k{}' f_k = \int \mathrm{d}\bar{\psi}_k \mathrm{d}\psi_k{}' (1 - \bar{\psi}_k \psi_k') \left(1 + \sum_j \bar{\eta}_j (M^{-1})_{jk} \psi_k' \right) (1 + \bar{\psi}_k \eta_k) \tag{6}$$

Expanding the integrand (i.e. f_k) in equation 6 (and displaying the terms relevant to the integration only):

$$f_k = ... = -\bar{\psi}_k \psi_k' + \left(\sum_m \bar{\eta}_m (M^{-1})_{mk} \psi_k' \right) \bar{\psi}_k \eta_k + ... \tag{7}$$

and then:

$$I_k = 1 + \left(\sum_m \bar{\eta}_m (M^{-1})_{mk} \right) \eta_k = \exp \left(\sum_m \bar{\eta}_m (M^{-1})_{mk} \eta_k \right) \tag{8}$$

With all of these reductions, the integral I takes the form:

$$I = S \ \det(M) \ \prod_{k=0}^{N} I_k = S \ \det(M) \ \prod_{k=0}^{N} \exp \left(\sum_m \bar{\eta}_m (M^{-1})_{mk} \eta_k \right) \tag{9}$$

where S is the *sign* factor.

Reducing equation 9 some more we finally obtain:

$$I = S \ \det(M) \ \exp \left(\sum_{m,k} \bar{\eta}_m (M^{-1})_{mk} \eta_k \right) = S \ \det(M) \ \exp \left(\bar{\eta} M^{-1} \eta \right) \tag{10}$$

1 Proving that $\mathbf{d}\Psi = \mathbf{det}(M)\mathbf{d}\Psi'$

...

2 Expression for S

Every time we want to *form* an integral I_k, we have to drag the necessary $\mathrm{d}\bar{\psi}_k$ and $\mathrm{d}\psi'_k$ differentials to the far right.

Now, the trick to simply things for the overall integration is to start by computing the integral I_k for $k = N$, and then proceed downwards towards $k = 0$ (in steps of 1 for each change in k); this is a simple strategy due to the possibility of moving the exponentials in the integrand around and without making any new sign changes pop-up.

Because the differentials $\mathrm{d}\bar{\psi}_k$ and $\mathrm{d}\psi'_k$ come in pairs, then there is no need to move any of those differential factors around, and therefore:

$$S = 1 \tag{11}$$