1. Spectrum of the fermion operators

In the lecture you have seen that the action for Dirac fermions is, except for term that vanishes naïvely in the continuum limit,

$$S_E = \sum_{\vec{n}\in\Lambda} \left(\frac{1}{2}\bar{\Psi}(\vec{n})\gamma_0 \left(\Psi(\vec{n}+\hat{t}) - \Psi(\vec{n}-\hat{t})\right) - \frac{\mathrm{i}}{2}\bar{\Psi}(\vec{n})\gamma_1 \left(\Psi(\vec{n}+\hat{x}) - \Psi(\vec{n}-\hat{x})\right) + m\bar{\Psi}(\vec{n})\Psi(\vec{n}) \right). \tag{1}$$

Here

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

a) Write this action in the form $S_E = -\bar{\Psi}_i M_{ij} \Psi_j$ where the multi-indices i and j label all Grassmann numbers. Write down an expression for M.

b) For a given $N_t \times N_s$, for example on a 12×12 lattice, construct this matrix and calculate numerical the eigenvalue spectrum of the Matrix M. Plot the eigenvalues in the complex plane.

c) Repeat tasks a) and b), but add a term of the form

$$\sum_{\vec{n}\in\Lambda} \left(-\frac{1}{2}\bar{\Psi}(\vec{n})(\Psi(\vec{n}+\hat{t}) + \Psi(\vec{n}-\hat{t}) - 2\Psi(\vec{n})) - \frac{1}{2}\bar{\Psi}(\vec{n})(\Psi(\vec{n}+\hat{x}) + \Psi(\vec{n}-\hat{x}) - 2\Psi(\vec{n})) \right) \tag{3}$$

to the action. What changes in the eigenvalue spectrum.

d) The Fermion operator can be couple to gauge fields using minimal coupling, that is making the replacement

$$\Psi(\vec{n})\Psi(\vec{n}) \to \Psi(\vec{n})\Psi(\vec{n}) \tag{4}$$

$$\Psi(\vec{n})\Psi(\vec{n}+\hat{\mu}) \to \Psi(\vec{n})U_\mu(\vec{n})\Psi(\vec{n}+\hat{\mu}) \tag{5}$$

$$\Psi(\vec{n})\Psi(\vec{n}-\hat{\mu}) \to \Psi(\vec{n})U_\mu^\dagger(\vec{n}-\vec{n})\Psi(\vec{n}+\hat{\mu}) \tag{6}$$

Construct M for this operator for some $U(1)$ gauge fields generated by your $U(1)$ code developed in previous exercises. Plot the spectrum of M for configurations with different topological charges.